SIAM J. Optim.
Vol. 15, No. 1, pp. 185–209

PROBABILISTIC COMBINATORIAL OPTIMIZATION: MOMENTS, SEMIDEFINITE PROGRAMMING, AND ASYMPTOTIC BOUNDS*

DIMITRIS BERTSIMAS†, KARTHIK NATARAJAN‡, AND CHUNG-PIAW TEO§

Abstract. We address the problem of evaluating the expected optimal objective value of a 0-1 optimization problem under uncertainty in the objective coefficients. The probabilistic model we consider prescribes limited marginal distribution information for the objective coefficients in the form of moments. We show that for a fairly general class of marginal information, a tight upper (lower) bound on the expected optimal objective value of a 0-1 maximization (minimization) problem can be computed in polynomial time if the corresponding deterministic problem is solvable in polynomial time. We provide an efficiently solvable semidefinite programming formulation to compute this tight bound. We also analyze the asymptotic behavior of a general class of combinatorial problems that includes the linear assignment, spanning tree, and traveling salesman problems, under knowledge of complete marginal distributions, with and without independence. We calculate the limiting constants exactly.

Key words. combinatorial optimization, probabilistic analysis, convex optimization, moments problem

AMS subject classifications. 90C22, 90C27

DOI. 10.1137/S1052623403430610

1. Introduction. We analyze the optimal objective value of the generic 0-1 optimization problem with random objective coefficients. Let $\boldsymbol{X} = \{1, \ldots, n\}$ and P be a nonnegative integer that denotes the number of feasible solutions to the combinatorial optimization problem. Let $\{B_p,\, p = 1, \ldots, P\}$ be nonempty subsets of $\boldsymbol{X}$ that correspond to the set of feasible solutions. Without loss of generality, we assume that the problem does not contain redundant variables, namely, for each $i \in \boldsymbol{X}$, there exists at least one feasible solution that contains i and at least one feasible solution that does not contain i. The nominal 0-1 optimization problem in maximization form is

$$Z^*_{\max}(\mathbf{c}) = \max_{p=1,\ldots,P} \sum_{i \in B_p} c_i. \tag{1.1}$$

Traditional models solve the nominal 0-1 optimization problem under the assumption that the objective vector $\mathbf{c}$ is completely deterministic. However, such models ignore the inherent data uncertainty that may affect the quality of the output solution. To overcome this shortcoming, an increasing research effort has focused on addressing uncertainty in the objective function. The standard approach to tackle such uncertainty is to assume an underlying distribution for the objective vector $\mathbf{c}$. Let θ represent this prescribed multivariate distribution. For example, when θ is specified to be uniform on $[0, 1]$ and i.i.d. for each coefficient, it reduces to the classical

*Received by the editors June 24, 2003; accepted for publication (in revised form) March 25, 2004; published electronically November 17, 2004. This research was partially supported by the Singapore-MIT Alliance.

http://www.siam.org/journals/siopt/15-1/43061.html

†Boeing Professor of Operations Research, Sloan School of Management and Operations Research Center, Massachusetts Institute of Technology, E53-363, Cambridge, MA 02139 (dbertsim@mit.edu).

‡High Performance Computation for Engineered Systems, Singapore-MIT Alliance, Singapore 119260 (karthik_natarajan@yahoo.com).

§Department of Decision Sciences, NUS Business School, National University of Singapore, Singapore 117591 (bizteocp@nus.edu.sg).

model studied in probabilistic combinatorial optimization [28]. Examples of problems that have been studied under this model are the linear assignment [1], quadratic assignment [6], and spanning tree problems [10]. The primary focus of such techniques is to estimate the expected optimal objective value:

$$E_\theta[Z^*_{\max}(\mathbf{c})]. \tag{1.2}$$

Evaluating this expectation exactly is nontrivial. Moreover, computing bounds on this expected optimal value is already quite challenging.

In this paper, we study this problem under a more general setting. Specifically, we consider a class of feasible distributions Θ, and our objective is to find a tight upper bound on the expected optimal objective value of the nominal 0-1 maximization problem:

$$\begin{aligned} Z^*_{\max} = \sup\nolimits_\theta \quad & E_\theta[Z^*_{\max}(\mathbf{c})] \\ \text{s.t.} \quad & \theta \in \Theta. \end{aligned} \tag{1.3}$$

By a tight upper bound we refer to a valid upper bound on the expected objective value that is achieved either exactly or asymptotically by a set of feasible distributions. Problem (1.3) arises naturally in applications where the distribution for the objective coefficients is not completely known. As an example, consider the problem of estimating an upper bound on the expected time of completion of a project that consists of activities with precedence relationships for which we have information on the expected duration and variability of individual activities but no information on their correlation (see also section 4). Specifying the exact distribution in such problems is a potentially difficult task. Most traditional approaches make the additional assumption of independence and try to estimate the expected optimal value. We, however, drop this assumption of independence completely and provide insights into the performance of the problem under dependence. Obtaining good lower bounds on $E_\theta[Z^*_{\max}(\boldsymbol{c})]$ is relatively easier by using Jensen's inequality. Our results can be easily extended to the analogous problem of finding a tight lower bound $Z^*_{\min}$ on the expected optimal objective value of the 0-1 minimization problem by setting $\mathbf{c} := -\mathbf{c}$.

1.1. Structure of data uncertainty. The feasible set of multivariate distributions Θ we consider is characterized by known information on the marginal distributions of each objective coefficient. Let $F_i(c) = \mathcal{P}(c_i \le c)$ denote the complete marginal distribution function of c_i. We let $\Theta(F_1, \dots, F_n)$ denote the set of multivariate distributions compatible with the marginal distributions F_i. Note that this model does not assume that objective coefficients are independently distributed, and hence Θ is not uniquely specified. Under complete knowledge of marginal distributions F_i, problem (1.3) is formulated as

$$\begin{aligned} Z^*_{\max} = \sup\nolimits_\theta \quad & E_\theta[Z^*_{\max}(\boldsymbol{c})] \\ \text{s.t.} \quad & \theta \in \Theta(F_1, \dots, F_n). \end{aligned} \tag{1.4}$$

Motivated by an application in project management, problem (1.4) was studied by Meilijson and Nadas [21] in the context of computing the longest path in a directed acyclic graph. Their main result shows that $Z^*_{\max}$ can be computed by solving a convex minimization reformulation of this problem. Weiss [31] observed that this approach generalizes to the maximum flow and shortest route problem.

In this paper, we generalize this result to problems with limited information on the distribution F_i. We assume that the objective coefficient c_i takes value in Ω_i (i.e., the support of the random variable is possibly a subset of Ω_i). The distribution is also assumed to satisfy the moment equality constraints on real-valued functions of c_i in the form $E_{F_i}[f_{ik}(c_i)] = m_{ik}$, $k = 1, \ldots, k_i$. In the classical case, with power functions $f_{ik}(c_i) = c_i^k$, this reduces to knowing the first k_i moments of c_i. Given this information, the feasible set of marginal distributions for coefficient c_i is represented as

$$\mathbb{F}_i = \Big\{ F_i \;\Big|\; E_{F_i}[f_{ik}(c_i)] = m_{ik},\ k = 1, \ldots, k_i,\quad E_{F_i}[\mathbb{I}_{\Omega_i}] = 1 \Big\}, \tag{1.5}$$

where $\mathbb{I}_S(c_i) = 1$ if $c_i \in S$ and 0 otherwise. We assume that the feasible region $\mathbb{F}_i$ is nonempty and $E_{F_i}\big[|c_i|\big] < \infty$ to guarantee a finite value for $Z^*_{\max}$. Under the marginal moment information model, problem (1.3) is formulated as

$$\begin{array}{rll} Z^*_{\max} = \sup_{\theta, F_1, \ldots, F_n} & E_\theta[Z^*_{\max}(\mathbf{c})] & \\ \text{s.t.} & F_i \in \mathbb{F}_i, & i = 1, \ldots, n, \\ & \theta \in \Theta(F_1, \ldots, F_n). & \end{array} \tag{1.6}$$

Remark. We make no assumptions on cross moment information in Θ. As the size of the problem increases, the number of estimates needed to capture dependent structures grows exponentially in the dimension of the problem. Instead of trying to characterize this exact structure, we optimize over all dependent structures with the given marginal moment information.

1.2. Contributions. In this paper, building on the work of Bertsimas and Popescu [4] connecting moment problems and semidefinite optimization, we generalize the approach by Meilijson and Nadas [21] and develop techniques to compute $Z^*_{\max}$ and $Z^*_{\min}$ for general 0-1 optimization problems. Our main contributions are as follows:

(a) We provide a general optimization formulation to compute $Z^*_{\max}$ under limited information on the marginal distributions of the objective coefficients. This formulation has an exponential number of constraints in general.

(b) Given limited marginal moment information for the objective coefficients, we show that an upper bound on $Z^*_{\max}$, first proposed in [21] in the context of the longest path problem in a directed acyclic graph, is tight for general combinatorial problems. Note that the tightness of the bound even in the context of the longest path problem has not been known to hold in general; see the discussion in [5, p. 844].

(c) For piecewise polynomial functions f_{ik}, we show that $Z^*_{\max}$ can be computed in polynomial time if the nominal 0-1 maximization problem can be solved in polynomial time. We provide an efficiently solvable semidefinite programming formulation with which to compute the $Z^*_{\max}$ in this case.

(d) We characterize the asymptotic behavior of the proposed bounds for a general class of combinatorial problems under knowledge of complete and identical marginal distributions. No assumptions on the independence of distributions is made. For this model, we show that for the minimum cost $N \times N$ linear assignment problem, the spanning tree, and the traveling salesman problem on N vertices, with the distribution function of the cost coefficients satisfying $F_\alpha(c) = \rho_\alpha c^\alpha$ as $c \downarrow 0$ for $\alpha \geq 0$, the tight lower bound $Z^*_{\min}$ scales as

$$\lim_{N \to \infty} \left(\frac{Z^*_{\min}}{N^{(\alpha-1)/\alpha}} \right) = C^*_\alpha,$$

Table 1.1
Scaling constants with distribution function of form $F(c) = \rho_\alpha c^\alpha$ as $c \downarrow 0$.

0-1 Optimization problem	$\lim_{N\to\infty}\left(\frac{Z^*_{\min}}{N^{(\alpha-1)/\alpha}}\right)$
$N \times N$ Linear assignment	$\left(\frac{\alpha}{\alpha+1}\right)\rho_\alpha^{-1/\alpha}$
Spanning tree on N node complete graph	$\left(\frac{\alpha}{\alpha+1}\right)\rho_\alpha^{-1/\alpha}2^{1/\alpha}$
Traveling salesman on N node complete graph	$\left(\frac{\alpha}{\alpha+1}\right)\rho_\alpha^{-1/\alpha}2^{1/\alpha}$

where C^*_α is explicitly computed in Table 1.1. It is interesting that the asymptotic bounds exhibit the same scaling behavior (although with different constants) under the assumption of independence and that the limiting constants for the minimum spanning tree and the traveling salesman problems are the same.

1.3. Structure of the paper. In section 2, we provide a general duality-based approach with which to compute the tight upper bound on the expected optimal objective value of a 0-1 maximization problem under marginal moment information. Unfortunately, this formulation has exponentially many constraints in general. In section 3, we obtain our first set of results based on the reformulation in [21] for the problem. We introduce a semidefinite programming approach for solving a class of such problems. In section 4, we provide numerical comparisons with other methods for a particular application in project management. In section 5, we develop the bound under complete and identical marginal distributions and provide new asymptotic bounds for classical combinatorial problems. In section 6, we summarize our conclusions.

2. General approach. Given limited marginal information for each objective coefficient c_i, we are interested in computing the tight upper bound on the expected optimal objective value of a 0-1 maximization problem. Given the feasible set of marginal distributions in (1.5), we can rewrite problem (1.6) as

$$
\begin{aligned}
Z^*_{\max} = \sup_\theta \quad & E_\theta[Z^*_{\max}(\mathbf{c})] \\
\text{s.t.} \quad & E_\theta[f_{ik}(c_i)] = m_{ik}, \quad k = 1, \dots, k_i,\ i = 1, \dots, n, \\
& E_\theta[\mathbb{I}_{\mathbf{\Omega}}] = 1,
\end{aligned}
\tag{2.1}
$$

where $\mathbf{\Omega}$ denotes $\Omega_1 \times \cdots \times \Omega_n$. We construct the dual of this formulation by introducing a dual variable y_{ik} for each moment equality constraint and y_{00} for the probability mass constraint. Since the primal problem is bounded, the dual problem is formulated as

$$
\begin{aligned}
Z^*_D = \min & \left(y_{00} + \sum_{i=1}^{n}\sum_{k=1}^{k_i} y_{ik} m_{ik}\right) \\
\text{s.t.} \quad & y_{00} + \sum_{i=1}^{n}\sum_{k=1}^{k_i} y_{ik} f_{ik}(c_i) \geq Z^*_{\max}(\boldsymbol{c}) \quad \forall \boldsymbol{c} \in \mathbf{\Omega}.
\end{aligned}
\tag{2.2}
$$

THEOREM 2.1 (see Isii [14]). (a) *The primal and dual formulations are related by weak duality as* $Z^*_{\max} \le Z^*_D$.

(b) *Furthermore, if the moments lie interior to the set of feasible moment vectors for arbitrary multivariate distributions, then strong duality holds, i.e.,* $Z^*_{\max} = Z^*_D$.

The strong duality condition in Theorem 2.1(b) is a Slater-type regularity condition that requires the moment vector to lie in the interior of the moment space. Under marginal moment specification, the multivariate moment space is defined as the product of univariate moment spaces. Checking for the interior point condition hence reduces to checking for the interior point condition for the univariate moment spaces. Such univariate conditions are easier to verify (as positive definiteness conditions on the moments matrix [15]). From this point on, we assume that this regularity condition is satisfied and hence the optimum objective value of problem (2.2), $Z^*_D = Z^*_{\max}$.

We now express the right-hand side of the constraint in formulation (2.2) explicitly. Since the objective function $Z^*_{\max}(\boldsymbol{c})$ is a piecewise linear convex function, we can rewrite the dual formulation as

$$
\begin{aligned}
Z^*_{\max} = \min & \left(y_{00} + \sum_{i=1}^{n} \sum_{k=1}^{k_i} y_{ik} m_{ik} \right) \\
\text{s.t.} \quad & y_{00} + \sum_{i=1}^{n} \sum_{k=1}^{k_i} y_{ik} f_{ik}(c_i) - \sum_{i \in B_p} c_i \ge 0 \quad \forall \boldsymbol{c} \in \boldsymbol{\Omega},\ p = 1, \dots, P.
\end{aligned} \tag{2.3}
$$

Each constraint in formulation (2.3) is equivalent to the nonnegativity of a multivariate function over a subset of $\boldsymbol{\Re^n}$. For the simplest case with $\boldsymbol{\Omega}$ defined by a discrete set of values, problems (2.1) and (2.3) reduce to the standard linear programming primal and dual problems. In general, there is an exponential number of such constraints due to the exponential number of feasible solutions to the nominal problem (1.1). For the simplest case with $f_{ik}(c_i) = c_i^k$ and $\boldsymbol{\Omega} = \boldsymbol{\Re^n}$, each constraint reduces to the nonnegativity of a multivariate polynomial over $\boldsymbol{\Re^n}$. A simple sufficient condition to ensure this is a sum of squares decomposition for the polynomial. This condition can be expressed by a positive semidefiniteness constraint on matrices of increasing size in the dual variables (cf. [15] and [25]). As the order of the relaxation increases, the size of these matrices increases drastically. Moreover, tractable necessary conditions to ensure the nonnegativity of a multivariate polynomial are in general not known. These reasons seem to suggest that formulation (2.3) is difficult to solve even for simple $f_{ik}(\cdot)$, regardless of the difficulty of the nominal problem.

3. Convex reformulation.

In this section, we provide a reformulation for computing $Z^*_{\max}$ that uses the marginal information structure of the objective vector based on a formulation originally proposed in [21] for a specific combinatorial problem under complete marginal distributions. Specifically, we apply the approach to general 0-1 optimization problems.

3.1. Complete marginal distribution information.

We first focus on the case with completely known marginal distributions F_i for each objective coefficient, and we outline the approach of Meilijson and Nadas [21]. We denote by $x^+ = \max(0, x)$. For each feasible solution to problem (1.1) indexed by p and for an arbitrary vector $\boldsymbol{d} \in \Re^{\boldsymbol{n}}$, we have

$$\begin{aligned}\sum_{i\in B_p} c_i &= \sum_{i\in B_p} d_i + \sum_{i\in B_p}(c_i - d_i),\\ &\le \max_{p=1,\dots,P}\sum_{i\in B_p} d_i + \sum_{i=1}^n [c_i - d_i]^+,\\ &= Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^n [c_i - d_i]^+.\end{aligned}$$

Since the right-hand side of the inequality is independent of any particular feasible solution of problem (1.1), we have

$$Z^*_{\max}(\boldsymbol{c}) \le Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^n [c_i - d_i]^+.$$

Taking expectations with respect to F_i and the minimum over $\boldsymbol{d} \in \Re^{\boldsymbol{n}}$ we obtain

$$E_\theta[Z^*_{\max}(\boldsymbol{c})] \le \min_{\boldsymbol{d}\in\Re^{\boldsymbol{n}}}\left(Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^n E_{F_i}[c_i - d_i]^+\right)\ \forall\theta \in \Theta(F_1,\dots,F_n).$$

Furthermore, a joint probability distribution is constructed in [21] such that the upper bound with given marginal distributions F_i is tight. This brings them to their central result that solving a convex minimization problem in n variables yields the tight upper bound:

$$\sup_{\theta\in\Theta(F_1,\dots,F_n)} E_\theta[Z^*_{\max}(\boldsymbol{c})] = \min_{\boldsymbol{d}\in\Re^{\boldsymbol{n}}}\left(Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^n E_{F_i}[c_i - d_i]^+\right). \tag{3.1}$$

3.2. Partial marginal distribution information. In this section, we consider the case where the marginal distribution function F_i is not assumed to be known exactly but to lie in the set $\mathbb{F}_i$. Optimizing over this set of feasible marginal distributions, and using (3.1), the tight upper bound $Z^*_{\max}$ is obtained by solving

$$Z^*_{\max} = \sup_{F_i\in\mathbb{F}_i, i=1,\dots,n}\ \min_{\boldsymbol{d}\in\Re^{\boldsymbol{n}}}\left(Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^n E_{F_i}[c_i - d_i]^+\right).$$

By interchanging the order of sup and min, we obtain an upper bound on $Z^*_{\max}$:

$$Z^*_{\max} \le \min_{\boldsymbol{d}\in\Re^{\boldsymbol{n}}}\left(Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^n \sup_{F_i\in\mathbb{F}_i} E_{F_i}[c_i - d_i]^+\right). \tag{3.2}$$

Klein Haneveld [13] showed that if the optimal marginal distribution $F_i^* \in \mathbb{F}_i$ for the inner subproblem is the same for all d_i, that is,

$$\sup_{F_i\in\mathbb{F}_i} E_{F_i}[c_i - d_i]^+ = E_{F_i^*}[c_i - d_i]^+\ \forall d_i \in \Re,\ i = 1,\dots,n, \tag{3.3}$$

then bound (3.2) is tight. We next give some examples of marginal moment information under which condition (3.3) is satisfied.

(a) Given bounded range $\Omega_i = [\underline{c}_i, \overline{c}_i]$, since the function $[c_i - d_i]^+$ is a nondecreasing function in c_i, the optimal marginal distribution F_i^* is

$$\mathcal{P}_{F_i^*}(c_i = \overline{c}_i) = 1.$$

Solving formulation (3.2) yields the tight upper bound as

$$Z^*_{\max} = Z^*_{\max}(\overline{\boldsymbol{c}}). \tag{3.4}$$

(b) Given bounded range $\Omega_i = [\underline{c}_i, \overline{c}_i]$ and first moment μ_i for each coefficient c_i, the optimal marginal distribution F_i^* is known to be [20]

$$\mathcal{P}_{F_i^*}(c_i = \underline{c}_i) = \frac{\overline{c}_i - \mu_i}{\overline{c}_i - \underline{c}_i}, \qquad \mathcal{P}_{F_i^*}(c_i = \overline{c}_i) = \frac{\mu_i - \underline{c}_i}{\overline{c}_i - \underline{c}_i}.$$

Since this distribution is independent of d_i, solving formulation (3.2) yields the tight bound $Z^*_{\max}$. The decision vector $\boldsymbol{d}$ can be restricted to lie in the range $\boldsymbol{\Omega} = [\underline{\boldsymbol{c}}, \overline{\boldsymbol{c}}]$ without affecting the optimal value. Hence the tight upper bound is the solution to the linear program:

$$Z^*_{\max} = \min_{\boldsymbol{d} \in [\underline{\boldsymbol{c}}, \overline{\boldsymbol{c}}]} \left(Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^n \left(\frac{\mu_i - \underline{c}_i}{\overline{c}_i - \underline{c}_i} \right) (\overline{c}_i - d_i) \right). \tag{3.5}$$

(c) Given semi-infinite range $\Omega_i = [\underline{c}_i, \infty)$ and first moment μ_i, Birge and Maddox [5] extended the previous result to obtain

$$Z^*_{\max} = Z^*_{\max}(\underline{\boldsymbol{c}}) + \sum_{i=1}^n (\mu_i - \underline{c}_i). \tag{3.6}$$

However, with additional second moment information, the optimal two-atom distribution F_i^* is dependent on the variable d_i [5]. Hence formulation (3.2) was not known to be tight for higher order moment information. However, we next show that it is in fact tight for any prescribed set of marginal moment information.

3.3. The main results. Let Z_1^* denote the optimum objective value of the right-hand side of formulation (3.2). With prescribed marginal information in (1.5), this bound is computed as

$$\begin{aligned} Z_1^* = \min & \left(Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^n \ \sup\nolimits_{F_i} E_{F_i}[c_i - d_i]^+ \right) \\ \text{s.t.} \quad & E_{F_i}[f_{ik}(c_i)] = m_{ik}, \quad k = 1, \dots, k_i, \ i = 1, \dots, n, \\ & E_{F_i}[\mathbb{I}_{\Omega_i}] = 1, \quad i = 1, \dots, n. \end{aligned} \tag{3.7}$$

Since no cross moment information is known, the ith inner subproblem is written as

$$\begin{aligned} \sup\nolimits_{F_i} \quad & E_{F_i}[c_i - d_i]^+ \\ \text{s.t.} \quad & E_{F_i}[f_{ik}(c_i)] = m_{ik}, \quad k = 1, \dots, k_i, \\ & E_{F_i}[\mathbb{I}_{\Omega_i}] = 1. \end{aligned}$$

To formulate the dual of the ith subproblem, we introduce variables $\tilde{y}_{i0}$ for the probability mass constraint and $\tilde{y}_{ik}$ for the moment equality constraints. The dual of the univariate subproblem is then written as

$$\begin{aligned}
&\min \left(\tilde{y}_{i0} + \sum_{k=1}^{k_i} \tilde{y}_{ik} m_{ik} \right) \\
&\text{s.t.} \quad \tilde{y}_{i0} + \sum_{k=1}^{k_i} \tilde{y}_{ik} f_{ik}(c_i) \geq c_i - d_i \quad \forall c_i \in \Omega_i, \\
&\qquad\ \ \tilde{y}_{i0} + \sum_{k=1}^{k_i} \tilde{y}_{ik} f_{ik}(c_i) \geq 0 \qquad\qquad \forall c_i \in \Omega_i.
\end{aligned}$$

Under the strong duality assumption, we substitute the dual of each subproblem into formulation (3.7) to obtain the equivalent formulation:

$$\begin{aligned}
Z_1^* = \min &\left(Z_{\max}^*(\boldsymbol{d}) + \sum_{i=1}^{n} \tilde{y}_{i0} + \sum_{i=1}^{n} \sum_{k=1}^{k_i} \tilde{y}_{ik} m_{ik} \right) \\
\text{s.t.} \quad & \tilde{p}_{i1}(c_i) := \tilde{y}_{i0} + d_i + \sum_{k=1}^{k_i} \tilde{y}_{ik} f_{ik}(c_i) - c_i \geq 0 \quad \forall c_i \in \Omega_i,\ i = 1, \ldots, n, \\
& \tilde{p}_{i2}(c_i) := \tilde{y}_{i0} + \sum_{k=1}^{k_i} \tilde{y}_{ik} f_{ik}(c_i) \geq 0 \qquad\qquad \forall c_i \in \Omega_i,\ i = 1, \ldots, n.
\end{aligned} \tag{3.8}$$

We denote by $\tilde{p}_{i1}(c_i)$ and $\tilde{p}_{i2}(c_i)$ the two univariate functions of c_i that are nonnegative over Ω_i. There is a total of $2n$ such functions in formulation (3.8).

We next prove the equivalence of formulation (3.8) with the original dual (2.3), rewritten here for clarity:

$$\begin{aligned}
Z_{\max}^* = \min &\left(y_{00} + \sum_{i=1}^{n} \sum_{k=1}^{k_i} y_{ik} m_{ik} \right) \\
\text{s.t.} \quad & y_{00} + \sum_{i=1}^{n} \sum_{k=1}^{k_i} y_{ik} f_{ik}(c_i) - \sum_{i \in B_p} c_i \geq 0 \quad \forall \boldsymbol{c} \in \boldsymbol{\Omega},\ p = 1, \ldots, P.
\end{aligned} \tag{3.9}$$

THEOREM 3.1. *The upper bound on the expected optimal objective value of the* 0-1 *maximization problem* Z_1^* *obtained by solving formulation* (3.8) *for a prescribed set of marginal moment information is tight, i.e.,* $Z_1^* = Z_{\max}^*$.

Proof. Clearly, formulation (3.8) provides an upper bound on the solution obtained by solving formulation (3.9). Hence $Z_{\max}^* \leq Z_1^*$.

To show the reverse inequality, consider an optimal solution to formulation (3.9) represented by variables y_{00}^*, y_{ik}^*. We now generate a feasible solution to formulation (3.8) in the following manner. We set $\tilde{y}_{ik} = y_{ik}^*$ for all k and i with $k, i \geq 1$. Having fixed the variables $\tilde{y}_{ik}$, we solve problem (3.8) to optimality for the remaining variables. Let $\tilde{y}_{i0}^*$ and d_i^* be the corresponding optimal values for the remaining variables.

We first prove a minimality property of the nonnegative univariate functions in formulation (3.8) that will be used later in the proof. We show that we can find an optimal solution to this problem such that the value $\tilde{y}_{i0}^*$ is minimal for $\tilde{p}_{i2}(c_i)$ to be nonnegative over Ω_i. To see why, suppose that there is an $\epsilon > 0$ such that we decrease $\tilde{y}_{i0}^*$ by ϵ and $\tilde{p}_{i2}(c_i)$ remains nonnegative over Ω_i. Then we can increase d_i^* by ϵ such that $\tilde{p}_{i1}(c_i)$ remains unchanged. Since the objective function is an increasing function in $\tilde{y}_{i0}$, the modification in $\tilde{y}_{i0}$ decreases the objective function by ϵ. Now, since $Z^*_{\max}(\boldsymbol{d}) = \max_p(\sum_{i\in B_p} d_i)$, the modification in d_i changes the objective function by at most ϵ. Hence the above modification will not increase the objective function.

Let $\tilde{y}^*_{i0}$ be chosen such that it is the minimal value for $\tilde{p}_{i2}(c_i)$ to be nonnegative over the specified support. Similarly, having fixed $\tilde{y}^*_{i0}$, we can decrease d^*_i as long as $\tilde{p}_{i1}(c_i)$ is nonnegative over Ω_i since the objective is an increasing function in d_i. Hence we can restrict our attention to optimal values of $\tilde{y}^*_{i0}$ and d^*_i that are minimal for the nonnegative functions $\tilde{p}_{i1}(c_i)$ and $\tilde{p}_{i2}(c_i)$ over Ω_i.

In general, given n such univariate functions $\tilde{p}_i(c_i) = \sum_{k=1}^{k_i} a_{ik} f_{ik}(c_i) + a_{i0}$ with a_{i0} at the minimal value for $\tilde{p}_i(c_i)$ to be nonnegative over Ω_i, the minimal value of a_{00} for the multivariate function $\tilde{p}(\boldsymbol{c}) = \sum_{i=1}^{n}\sum_{k=1}^{k_i} a_{ik} f_{ik}(c_i) + a_{00}$ to be nonnegative over $\boldsymbol{\Omega}$ is $\sum_{i=1}^{n} a_{i0}$. To see this, we start with

$$\tilde{p}_i(c_i) = \sum_{k=1}^{k_i} a_{ik} f_{ik}(c_i) + a_{i0} \geq 0 \quad \forall c_i \in \Omega_i,\ i = 1,\ldots,n,$$

and add the n functions to obtain

$$\sum_{i=1}^{n}\sum_{k=1}^{k_i} a_{ik} f_{ik}(c_i) + \sum_{i=1}^{n} a_{i0} \geq 0 \quad \forall \boldsymbol{c} \in \boldsymbol{\Omega}.$$

From the minimality of a_{i0} for the univariate function $\tilde{p}_i(c_i)$, we know that there exists an $\tilde{c}_i \in \Omega_i$ such that

$$\tilde{p}_i(\tilde{c}_i) = \sum_{k=1}^{k_i} a_{ik} f_{ik}(\tilde{c}_i) + a_{i0} = 0, \quad i = 1,\ldots,n.$$

Adding these equalities, we obtain a vector $\tilde{\boldsymbol{c}} = (\tilde{c}_1,\ldots,\tilde{c}_n)$ that lies in $\boldsymbol{\Omega}$, which satisfies

$$\sum_{i=1}^{n}\sum_{k=1}^{k_i} a_{ik} f_{ik}(\tilde{c}_i) + \sum_{i=1}^{n} a_{i0} = 0.$$

Thus, the minimal value for a_{00} such that the function $\tilde{p}(\boldsymbol{c})$ is nonnegative over $\boldsymbol{\Omega}$ is $a_{00} = \sum_{i=1}^{n} a_{i0}$.

Now consider any feasible solution to problem (1.1) indexed by p. Then, from formulation (3.8) we obtain

$$\begin{aligned}
&\forall\ i \text{ such that } i \in B_p, \quad \tilde{y}^*_{i0} + d^*_i + \sum_{k=1}^{k_i} y^*_{ik} f_{ik}(c_i) - c_i \geq 0 \quad && \forall c_i \in \Omega_i,\\
&\forall\ i \text{ such that } i \notin B_p, \quad \tilde{y}^*_{i0} + \sum_{k=1}^{k_i} y^*_{ik} f_{ik}(c_i) \geq 0 && \forall c_i \in \Omega_i.
\end{aligned}$$

Summing up these n constraints, we obtain

$$\sum_{i=1}^{n} \tilde{y}^*_{i0} + \sum_{i\in B_p} d^*_i + \sum_{i=1}^{n}\sum_{k=1}^{k_i} y^*_{ik} f_{ik}(c_i) - \sum_{i\in B_p} c_i \geq 0 \quad \forall \boldsymbol{c} \in \boldsymbol{\Omega},\ p = 1,\ldots,P.$$

From the minimality of the univariate functions, we know that $\sum_{i=1}^n \tilde{y}_{i0}^* + \sum_{i\in B_p} d_i^*$ is the minimal value for this function to be nonnegative over $\boldsymbol{\Omega}$. Comparing this with the multivariate function in formulation (3.9), we obtain

$$y_{00}^* \geq \sum_{i=1}^n \tilde{y}_{i0}^* + \sum_{i\in B_p} d_i^*, \quad p = 1,\ldots,P,$$

which reduces to

$$y_{00}^* \geq \sum_{i=1}^n \tilde{y}_{i0}^* + Z_{\max}^*(\boldsymbol{d^*}). \tag{3.10}$$

From formulation (3.8) we obtain

$$\begin{aligned} Z_1^* &\leq Z_{\max}^*(\boldsymbol{d^*}) + \sum_{i=1}^n \tilde{y}_{i0}^* + \sum_{i=1}^n \sum_{k=1}^{k_i} y_{ik}^* m_{ik} \\ &\leq y_{00}^* + \sum_{i=1}^n \sum_{k=1}^{k_i} y_{ik}^* m_{ik} \qquad \text{(from (3.10))}, \\ &= Z_{\max}^*. \qquad \square \end{aligned}$$

By setting $\boldsymbol{c} := -\boldsymbol{c}$ and $\boldsymbol{d} := -\boldsymbol{d}$, an equivalent result is obtained for $Z_{\min}^*$.

COROLLARY 3.2. *The tight lower bound on the expected optimal objective value of a* 0-1 *minimization problem for a prescribed set of marginal moment information is obtained by solving*

$$Z_{\min}^* = \max_{\boldsymbol{d}\in\Re^n} \left(Z_{\min}^*(\boldsymbol{d}) + \sum_{i=1}^n \inf_{F_i\in\mathbb{F}_i} E_{F_i}[\min(0, c_i - d_i)] \right). \tag{3.11}$$

3.4. Solution techniques. Since Theorem 3.1 implies that $Z_{\max}^* = Z_1^*$, we focus on solving formulation (3.8). We show that for a fairly general class of piecewise polynomial functions f_{ik}, this formulation can be solved as a semidefinite program. Some examples of such functions are $(c_i - a)^+$ or $\mathbb{I}_{(-\infty,a]}$. The power moment functions $f_{ik}(c_i) = c_i^k$ are included in this class.

For each coefficient c_i, the constraints in formulation (3.8) imply the nonnegativity of two univariate functions, $\tilde{p}_{i1}(c_i)$ and $\tilde{p}_{i2}(c_i)$ over Ω_i. For piecewise polynomial functions f_{ik}, we can decompose the support set Ω_i to intervals Ω_{ij}, $j = 1,\ldots,l_i$, with $\cup_j \Omega_{ij} = \Omega_i$ such that $\tilde{p}_{i1}(c_i)$ and $\tilde{p}_{i2}(c_i)$ are polynomials over each of these intervals. We can now rewrite formulation (3.8) as

$$\begin{aligned} Z_{\max}^* = \min & \left(Z_{\max}^*(\boldsymbol{d}) + \sum_{i=1}^n \tilde{y}_{i0} + \sum_{i=1}^n \sum_{k=1}^{k_i} \tilde{y}_{ik} m_{ik} \right) \\ \text{s.t.} \quad & \tilde{p}_{i1}(c_i) \geq 0 \qquad \forall c_i \in \Omega_{ij},\ j = 1,\ldots,l_i,\ i = 1,\ldots,n, \\ & \tilde{p}_{i2}(c_i) \geq 0 \qquad \forall c_i \in \Omega_{ij},\ j = 1,\ldots,l_i,\ i = 1,\ldots,n, \end{aligned} \tag{3.12}$$

where the constraints now correspond to the nonnegativity of univariate polynomials over Ω_{ij}. If Ω_{ij} is a finite set of atoms, this reduces to linear constraints.

The key observation is that although it is difficult to express exactly the nonnegativity of a multivariate polynomial, we can use positive semidefinite constraints to express the nonnegativity of a univariate polynomial over an interval. This simplification arises due to the equivalence of the sum of squares representation and nonnegativity of a polynomial in the univariate case; see [23] and [4]. We focus on three specific intervals, namely, the entire real line $\Omega = (-\infty, \infty)$, the positive ray $\Omega = [0, \infty)$, and the segment $\Omega = [0, 1]$. The semidefinite representation for all other intervals of $\Re$ can be obtained from simple affine transformations of these three cases.

PROPOSITION 3.3 (see Nesterov [23] and Bertsimas and Popescu [4]). (a) *A univariate polynomial* $f(c) = \sum_{r=0}^{2k} a_r c^r$ *is nonnegative over the interval* $\Omega = (-\infty, \infty)$ *if and only if there exists a positive semidefinite matrix* $\boldsymbol{Y} = [Y_{ij}]_{i,j=0,\dots,k}$ *such that*

$$\begin{aligned} a_r &= \sum_{i,j:\ i+j=r} Y_{ij}, \quad r = 0, \dots, 2k, \\ \boldsymbol{Y} &\succeq \mathbf{0}. \end{aligned}$$

(b) *A univariate polynomial* $f(c) = \sum_{r=0}^{k} a_r c^r$ *is nonnegative over the interval* $\Omega = [0, \infty)$ *if and only if there exists a positive semidefinite matrix* $\boldsymbol{Y} = [Y_{ij}]_{i,j=0,\dots,k}$ *such that*

$$\begin{aligned} 0 &= \sum_{i,j:\ i+j=2r-1} Y_{ij}, \quad r = 1, \dots, k, \\ a_r &= \sum_{i,j:\ i+j=2r} Y_{ij}, \quad r = 0, \dots, k, \\ \boldsymbol{Y} &\succeq \mathbf{0}. \end{aligned}$$

(c) *A univariate polynomial* $f(c) = \sum_{r=0}^{k} a_r c^r$ *is nonnegative over the interval* $\Omega = [0, 1]$ *if and only if there exists a positive semidefinite matrix* $\boldsymbol{Y} = [Y_{ij}]_{i,j=0,\dots,k}$ *such that*

$$\begin{aligned} 0 &= \sum_{i,j:\ i+j=2r-1} Y_{ij}, \quad r = 1, \dots, k, \\ \sum_{l=0}^{r} \binom{k-l}{r-l} a_l &= \sum_{i,j:\ i+j=2r} Y_{ij}, \qquad r = 0, \dots, k, \\ \boldsymbol{Y} &\succeq \mathbf{0}. \end{aligned}$$

Clearly, from Proposition 3.3, we can replace the condition that a univariate polynomial is nonnegative over an interval of $\Re$ by a requirement on a matrix to be positive semidefinite and a set of linear equalities that must be satisfied. The specific form depends on the nature of the interval Ω. For example, from Proposition 3.3(a), for a univariate polynomial to be nonnegative over $\Re$, the degree of the polynomial must be even.

From Proposition 3.3, we can rewrite formulation (3.12) as

$$\begin{aligned} Z^*_{\max} = \min &\left(Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^{n} \tilde{y}_{i0} + \sum_{i=1}^{n} \sum_{k=1}^{k_i} \tilde{y}_{ik} m_{ik} \right) \\ \text{s.t.} \quad & \boldsymbol{W_{ij}}(\tilde{y}_{i0}, \dots, \tilde{y}_{ik_i}, -1, d_i, \Omega_{ij}) \succeq \mathbf{0}, \quad j = 1, \dots, l_i,\ i = 1, \dots, n, \\ & \boldsymbol{X_{ij}}(\tilde{y}_{i0}, \dots, \tilde{y}_{ik_i}, \Omega_{ij}) \succeq \mathbf{0}, \qquad\qquad j = 1, \dots, l_i,\ i = 1, \dots, n, \end{aligned} \tag{3.13}$$

where $\boldsymbol{W_{ij}}$ and $\boldsymbol{X_{ij}}$ denote the semidefinite constraint matrix $\boldsymbol{Y}$ obtained from Proposition 3.3. The entries of the semidefinite matrix depends on the nature of the interval Ω_{ij} and the coefficients of the polynomial.

THEOREM 3.4. *Given marginal distribution information as moments of piecewise polynomial functions, the tight upper (lower) bound on the expected optimal objective value of a* 0-1 *maximization (minimization) problem can be computed in polynomial time if the nominal problem can be solved in polynomial time.*

Proof. Formulation (3.13) can be reformulated as

$$
\begin{aligned}
&Z^*_{\max} = \min \left(t + \sum_{i=1}^{n} \tilde{y}_{i0} + \sum_{i=1}^{n} \sum_{k=1}^{k_i} \tilde{y}_{ik} m_{ik} \right) \\
&\text{s.t.} \quad \boldsymbol{W_{ij}}(\tilde{y}_{i0}, \ldots, \tilde{y}_{ik_i}, -1, d_i, \Omega_{ij}) \succeq \mathbf{0}, \quad j = 1, \ldots, l_i, \ i = 1, \ldots, n, \\
&\qquad \boldsymbol{X_{ij}}(\tilde{y}_{i0}, \ldots, \tilde{y}_{ik_i}, \Omega_{ij}) \succeq \mathbf{0}, \quad j = 1, \ldots, l_i, \ i = 1, \ldots, n, \\
&\qquad t \geq Z^*_{\max}(\boldsymbol{d}),
\end{aligned} \tag{3.14}
$$

where t is an additional variable. We have a polynomial number of polynomial-size semidefinite matrices in this formulation. One approach to solve this problem is to disaggregate the term $Z^*_{\max}(\boldsymbol{d})$ in terms of its feasible solutions. However, the number of linear constraints in such a formulation would be exponentially large. This implies that solving formulation (3.14) is still difficult in general. For the class of polynomially solvable 0-1 maximization problems, we can, however, solve this efficiently by considering the separation version of the problem. Since the semidefinite constraints are polynomial sized, this essentially reduces to the separation problem: Given t and $\boldsymbol{d}$, verify if $t \geq Z^*_{\max}(\boldsymbol{d})$, and if not find a violated inequality.

To solve the separation problem, we solve the nominal discrete optimization problem with objective vector $\boldsymbol{d}$. If this can be solved in polynomial time, then we can find the optimal decision vector indexed by $p^* \in \{1, \ldots, P\}$ and $Z^*_{\max}(\boldsymbol{d})$ in polynomial time. Then we simply test if $t \geq Z^*_{\max}(\boldsymbol{d})$, and if not an index p^* is found such that $\sum_{i \in B_{p^*}} d_i > t$. The desired result follows from the equivalence of separation and optimization [11]. ☐

Remarks. (a) Theorem 3.4 implies that the tight lower bounds for the expected optimal objective values of combinatorial problems such as the minimum spanning tree, assignment, matching, and shortest path problems are computable in polynomial time under the marginal moment model.

(b) The previously known results (3.4), (3.5), and (3.6) can be viewed as special cases of our Theorems 3.1 and 3.4.

(c) The results hold even if we only know upper bounds on moments on $E[f_{ik}(c_i)]$. The only difference is that the corresponding dual variables are nonnegative.

(d) The variables d_i in formulation (3.14) can be restricted to lie in $[\underline{c}_i, \overline{c}_i]$, where $\underline{c}_i$ and $\overline{c}_i$ define lower and upper bounds on the range of Ω_i.

The solution method described thus far is a general cutting plane algorithm that solves the nominal problem (1.1) at each step in conjunction with semidefinite constraints. We now provide a compact semidefinite reformulation for a class of 0-1 optimization problems that may be useful computationally.

THEOREM 3.5. *For* 0-1 *optimization problems with compact linear programming formulations, the tight upper (lower) bound on the expected optimal objective value of a* 0-1 *maximization (minimization) problem under the marginal moment model can be computed as a compact semidefinite program.*

Proof. Any 0-1 optimization problem can be reformulated as a linear programming problem over the convex hull of its feasible region:

$$
\begin{aligned}
Z^*_{\max}(\boldsymbol{d}) = \max \quad & \boldsymbol{d}'\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},
\end{aligned} \tag{3.15}
$$

where $(\boldsymbol{A}, \boldsymbol{b})$ provides the convex hull representation. Using linear programming duality we have

$$
\begin{aligned}
Z^*_{\max}(\boldsymbol{d}) = \min \quad & \boldsymbol{p}'\boldsymbol{b} \\
\text{s.t.} \quad & \boldsymbol{p}'\boldsymbol{A} = \boldsymbol{d}', \\
& \boldsymbol{p} \ge \boldsymbol{0}.
\end{aligned} \tag{3.16}
$$

Substituting this dual into formulation (3.13), we obtain

$$
\begin{aligned}
Z^*_{\max} = \min & \left(\boldsymbol{p}'\boldsymbol{b} + \sum_{i=1}^{n} \tilde{y}_{i0} + \sum_{i=1}^{n}\sum_{k=1}^{k_i} \tilde{y}_{ik} m_{ik} \right) \\
\text{s.t.} \quad & \boldsymbol{W_{ij}}(\tilde{y}_{i0}, \dots, \tilde{y}_{ik_i}, -1, d_i, \Omega_{ij}) \succeq \boldsymbol{0}, \qquad j = 1, \dots, l_i, \ i = 1, \dots, n, \\
& \boldsymbol{X_{ij}}(\tilde{y}_{i0}, \dots, \tilde{y}_{ik_i}, \Omega_{ij}) \succeq \boldsymbol{0}, \qquad j = 1, \dots, l_i, \ i = 1, \dots, n, \\
& \boldsymbol{p}'\boldsymbol{A} = \boldsymbol{d}', \\
& \boldsymbol{p} \ge \boldsymbol{0}.
\end{aligned} \tag{3.17}
$$

The convex hull $(\boldsymbol{A}, \boldsymbol{b})$ representation could be exponentially large or not known explicitly. However, for a class of problems this linear representation is compact. For such problems, formulation (3.17) is a compact semidefinite program that can be solved efficiently in polynomial time. □

Examples of 0-1 optimization problems with compact linear programming representation include the assignment and network flow problems such as the shortest path problem. The longest path problem on acyclic graph is also an example of 0-1 optimization problem with compact LP formulation, although the longest path problem is NP-hard in general.

4. Application in project management. In this section, we provide an application of the techniques developed to the problem of computing the longest path in a directed acyclic graph. Such problems arise in project management and scheduling.

4.1. Description of the problem. A project is specified as a set of activities that has to be completed given certain precedence relationships. We represent a project in an activity-on-arc framework with an acyclic directed graph $G(V \cup \{s, t\}, E)$, where $|E| = n$ arcs. An arc in this graph represents an activity, and a node represents the completion of all activities leading to this node. Node s represents the start of the project while node t represents the completion of the project. The precedence constraints in the graph imply that if there exist arcs (i, j) and (j, k) in the graph, then job (i, j) must be performed before job (j, k). Such a graph constructed is necessarily acyclic, otherwise it would lead to an inconsistent ordering of jobs. We let the nonnegative random arc lengths c_{ij} denote the time required to complete each individual activity. Given this setup, the parameter of importance is the time required to complete the project measured by the longest path from the start node s to the end node t in the graph. The longest path in the graph $Z^*_{\max}(\boldsymbol{c})$ is computed by solving

$$
\begin{aligned}
Z^*_{\max}(\boldsymbol{c}) = \max \quad & \sum_{(i,j)\in E} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} 1 & \text{if } i = s, \\ -1 & \text{if } i = t, \\ 0 & \text{if } i \in V, \end{cases} \\
& x_{ij} \in \{0,1\} \ \forall (i,j) \in E.
\end{aligned}
\tag{4.1}
$$

In fact, we will focus on a more general measure of project performance, called project tardiness. Assume that we are specified a due date T for the project. Project tardiness is defined as a linear cost that is incurred only if the project completion time $Z^*_{\max}(\boldsymbol{c})$ exceeds the due date. If the project is completed before the deadline, then no cost is incurred. Mathematically, it is expressed as a piecewise convex function of project completion time:

$$\text{Project tardiness G(T)} := [Z^*_{\max}(\boldsymbol{c}) - T]^+.$$

It is clear that for $T = 0$, $G(0)$ reduces to $Z^*_{\max}(\boldsymbol{c})$.

Computing the longest path in a general graph is NP-hard, while it is polynomially time solvable for acyclic graphs [16]. However, if the arc lengths are random variables that are distributed independently and restricted to two possible values, Hagstrom [12] has shown that the computation of the expected longest path is #P-hard. Furthermore, the expected longest path cannot be computed in time polynomial in the number of values the completion time takes unless $P = NP$. This suggests that in general it is difficult to compute the exact expected project tardiness under uncertainty in activity duration, hence motivating the interest in bounds and approximations.

Robillard and Trahan [26] studied the expected completion time $E[Z^*_{\max}(\boldsymbol{c})]$ under assumptions of complete knowledge of distribution of activity durations and independence. Assuming independence, but only limited moment information, Devroye [8] computed upper bounds on the first moment and second moments of the completion time. However, Levy and Wiest [18] argued that activity durations are often dependent due to resource limitations. It is, however, not practical to estimate this complete multivariate joint distribution. Hence there has been effort focused on estimating the expected case tardiness under knowledge of limited moment information of the individual activities.

One approach to this problem deals with approximating the expected project tardiness. Under specific assumptions on the moments, the central limit theorem has been used to approximate the distribution of the project completion time; see [2], [27], and [19]. Under this method, the completion times obtained from different paths in the network are assumed to obey a multivariate normal distribution. Then, evaluating the expected project tardiness involves calculating the maximum of correlated normal distributions, a generally nontrivial calculation. Furthermore, if the various activities are correlated, one needs to impose restrictions on the distribution of arc lengths for the central limit theorem to apply. We do not make any such assumptions in this paper. Finally, even if the central limit theorem can be applied, it will not be a good approximation for smaller networks.

An alternative approach to the problem focuses on computing bounds on the expected project tardiness. A simple lower bound can be computed by applying

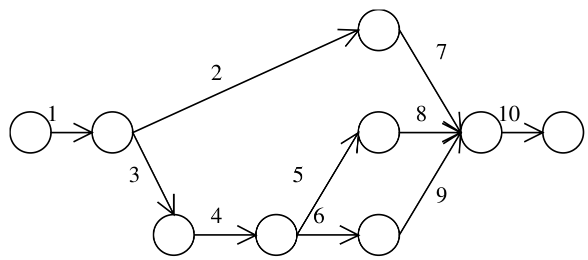

FIG. 4.1. *Project network in Example* 1.

Jensen's inequality to the convex tardiness function. Computing tight upper bounds is more challenging and more important as it provides an estimate of the worst case performance. Klein Haneveld [13] provided a tight upper bound on $E[G(T)]$ when the first moments of activity durations are given. With additional second moment information, Birge and Maddox [5] used the formulation in [21] to compute the upper bound on expected tardiness. However, as the computational results indicate, an approximation of the original objective function is used to compute this upper bound. The lack of precision from using such a linearization approach for solving a nonlinear problem demonstrates an additional advantage of using the semidefinite optimization approach. It should be noted that while formulation (3.13) can be directly used to compute the worst case expected project completion time, it could be easily extended to the tardiness objective. Specifically, we need to replace the term $Z^*_{\max}(\boldsymbol{d})$ in the objective function by $[Z^*_{\max}(\boldsymbol{d})-T]^+$ and solve the corresponding semidefinite program to compute the worst case expected tardiness denoted as $G^*(T)$:

$$
\begin{aligned}
G^*(T) = \min & \left([Z^*_{\max}(\boldsymbol{d}) - T]^+ + \sum_{i=1}^{n} \tilde{y}_{i0} + \sum_{i=1}^{n}\sum_{k=1}^{k_i} \tilde{y}_{ik} m_{ik} \right) \\
\text{s.t.} \quad & \boldsymbol{W_{ij}}(\tilde{y}_{i0},\ldots,\tilde{y}_{ik_i},-1,d_i,\Omega_{ij}) \succeq \boldsymbol{0}, \ j=1,\ldots,l_i, \ i=1,\ldots,n, \\
& \boldsymbol{X_{ij}}(\tilde{y}_{i0},\ldots,\tilde{y}_{ik_i},\Omega_{ij}) \succeq \boldsymbol{0}, \qquad j=1,\ldots,l_i, \ i=1,\ldots,n.
\end{aligned} \tag{4.2}
$$

Since $Z^*_{\max}(\boldsymbol{d})$ can be computed in polynomial time, formulation (4.2) is solvable in polynomial time.

4.2. Computational examples. We consider two sample projects taken from [5]. The specified data are as follows:

(a) The minimum time required to complete each activity is known. We assume that the maximum time required to complete an activity is infinity.

(b) For each activity, the first and second moments of the time to complete it are known. In addition for the first project, third moment information is assumed to be known. We add in this information to check the value of the extra information on the upper bound.

Under this marginal moment information, we solve formulation (4.2) with SeDuMi version 1.03 developed by Sturm [29]. An interior point method is used by the software to solve the semidefinite program. The computations were conducted on a Pentium II (550 MHz) Windows 2000 platform and the total computation time was less than a minute.

Example 1. The first project depicted in Figure 4.1 consists of 10 activities that are distributed over 3 paths. The marginal distribution information for the activities

Table 4.1
Activity duration data for Example 1.

Arc	Range	First mom.	Second mom.	Third mom.
1	$[1, \infty)$	1.0	1.0	1.0
2	$[4, \infty)$	3.0	9.333	30.0
3	$[3, \infty)$	2.0	4.333	10.0
4	$[3, \infty)$	2.5	6.333	16.25
5	$[7, \infty)$	5.0	26.333	146.0
6	$[5, \infty)$	4.0	16.333	68.0
7	$[5, \infty)$	3.0	10.333	40.0
8	$[5, \infty)$	4.5	20.333	92.25
9	$[2, \infty)$	1.5	2.333	3.75
10	$[6, \infty)$	5.0	25.333	130.0

Table 4.2
Upper bounds on expected tardiness for Example 1.

Information	Bound/ T	0	15	18.33	21.67	25
Range, first, second mom.	$G^*(T)$ reported in [5]	20.35	5.35	2.98	1.27	0.73
Range, first, second mom.	$G^*(T)$ from eq. (4.2)	20.29	5.29	2.66	0.94	0.47
Range, first, second, third mom.	$G^*(T)$ from eq. (4.2)	20.10	5.10	2.53	0.75	0.18

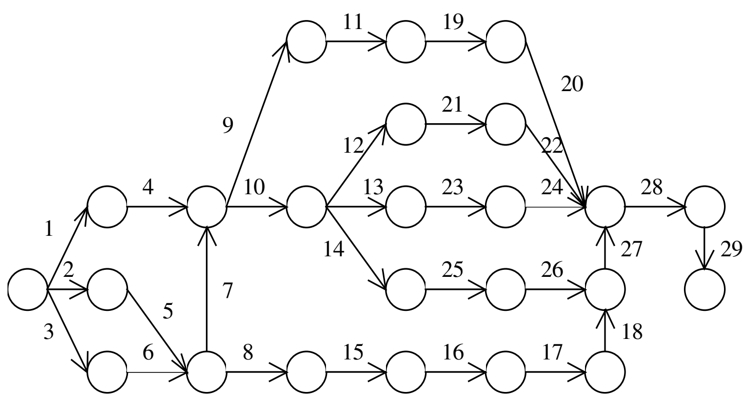

Fig. 4.2. *Project network in Example 2.*

is provided in Table 4.1. We consider five different deadline times T. First, we compute the worst case expected tardiness with range, first, and second moment information and compare it with the results reported in [5]. The results are provided in Table 4.2. In light of Theorem 3.1, it is clear that in [5] a heuristic method is used to calculate the bound. We next calculate the tight upper bound with additional third moment information. It is noted from Table 4.2 that the bounds are considerably tighter, especially for larger deadline dates T, indicating the value that additional information has on these estimates. It is important to note that our semidefinite programming formulation provides an easy way to incorporate third and higher order moment information, whereas previous techniques could not handle this.

Example 2. The second project is a larger one with 29 activities that are distributed over 14 paths. The activity-on-arc graph representation of the project is provided in Figure 4.2 and the data are provided in Table 4.3. The deadline time T is varied from 45.10 to 85 approximately in steps of 10, and the worst case expected

TABLE 4.3
Activity duration data for Example 2.

Arc	Range	First mom.	Second mom.	Arc	Range	First mom.	Second mom.
1	$[6, \infty)$	8.0	66.778	16	$[3, \infty)$	4.0	16.111
2	$[14, \infty)$	17.0	298.0	17	$[8, \infty)$	10.0	100.444
3	$[14, \infty)$	17.0	298.0	18	$[1, \infty)$	2.0	4.111
4	$[2, \infty)$	4.0	17.0	19	$[6, \infty)$	8.0	65.0
5	$[1, \infty)$	2.0	4.444	20	$[1, \infty)$	2.0	4.111
6	$[1, \infty)$	2.0	4.111	21	$[6, \infty)$	8.0	65.0
7	$[0.4, \infty)$	0.6	0.370	22	$[1, \infty)$	2.0	4.111
8	$[0.4, \infty)$	0.6	0.370	23	$[6, \infty)$	8.0	66.778
9	$[2, \infty)$	3.0	9.250	24	$[1, \infty)$	2.0	4.111
10	$[2, \infty)$	3.0	9.250	25	$[4, \infty)$	6.0	36.444
11	$[3, \infty)$	4.0	16.111	26	$[1, \infty)$	2.0	4.111
12	$[3, \infty)$	4.0	16.111	27	$[0.1, \infty)$	0.5	0.257
13	$[3, \infty)$	4.0	16.111	28	$[2, \infty)$	4.0	16.444
14	$[3, \infty)$	4.0	16.111	29	$[1, \infty)$	2.0	4.111
15	$[2, \infty)$	3.0	9.250				

TABLE 4.4
Upper bounds on expected tardiness for Example 2.

Information	Bound/ T	45.10	55	65	75	85
Range, first, second mom.	$G^*(T)$ reported in [5]	6.77	2.51	1.72	1.04	0.73
Range, first, second mom.	$G^*(T)$ from eq. (4.2)	6.52	2.38	1.38	0.97	0.73

project tardiness is computed for these times. The computed values are provided in Table 4.4 along with the values reported in [5]. Clearly these results indicate that the semidefinite programming method provides a general tractable approach for computing the worst case expected tardiness under moment information.

5. Asymptotic performance. In this section, we study the properties of the bounds asymptotically as the size of the problem increases to infinity. We make the following assumptions:

(A) Each objective coefficient is a random variable with a common prespecified distribution function F. For simplicity, we assume that F is a continuous distribution. No assumption of independence is made.

The complete knowledge of F is an important one in this setting to obtain interesting scaling results. Under this setting, the tight upper bound on the expected optimal value of a 0-1 maximization problem is computed as

$$Z^*_{\max} = \sup_\theta \quad E_\theta[Z^*_{\max}(\boldsymbol{c})] \qquad \text{s.t.} \quad \theta \in \Theta(F, \dots, F). \tag{5.1}$$

We restrict ourselves to a fairly general class of a 0-1 optimization problems that satisfy the following two properties:

(B) All feasible solutions to problem (1.1) indexed by $p \in \{1, \dots, P\}$ have the same cardinality. If we denote this common cardinality as K_1, we have

$$K_1 = |i : i \in B_p| \qquad \forall p = 1, \dots, P,$$

where $|S|$ denotes the cardinality of set S.

(C) Every element of the ground set $\boldsymbol{X}$ is contained in the same number of feasible solutions. If we denote this common value as K_2, we have

$$K_2 = |p : i \in B_p| \qquad \forall i = 1, \dots, n.$$

TABLE 5.1
Parameters for three classical combinatorial optimization problems.

0-1 Optimization problem	n	K_1	P	K_2
$N \times N$ Linear assignment	N^2	N	$N!$	$(N-1)!$
Spanning tree on N node complete graph	$\frac{N(N-1)}{2}$	$N-1$	N^{N-2}	$2N^{N-3}$
Traveling salesman on N node complete graph	$\frac{N(N-1)}{2}$	N	$\frac{(N-1)!}{2}$	$(N-2)!$

Clearly $K_1 \le n$ and $K_2 \le P$. In conjunction, these assumptions imply that K_1 and K_2 are related by

$$K_1 P = K_2 n.$$

Three classical combinatorial problems that satisfy properties (b) and (c) are as follows:

(a) Linear assignment problem: given a set $\{1, 2, \ldots, N\}$ and an $N \times N$ objective coefficient matrix $[c_{ij}]$, find a permutation ϕ of the set that optimizes the sum $\sum_{i=1}^{N} c_{i\phi(i)}$.

(b) Spanning tree problem on a complete graph: given a complete undirected graph $G(V, E)$ with N vertices and an $N \times N$ symmetric objective coefficient matrix $[c_{ij}]$, find a spanning tree with the optimal sum of the edge coefficients.

(c) Symmetric traveling salesman problem on a complete graph: given a complete undirected graph $G(V, E)$ with N vertices and an $N \times N$ symmetric objective coefficient matrix $[c_{ij}]$, find a cyclic permutation ϕ of the set of N vertices that optimizes $\sum_{i=1}^{N} c_{i\phi(i)}$.

The parameters for these three problems are summarized in Table 5.1.

Remarks. (a) While the first two problems are solvable in polynomial time, the traveling salesman problem is difficult to solve. All three problems, however, share structural properties (B) and (C). The asymptotic results in this section in fact do not depend on the complexity of the nominal optimization problem.

(b) An example of a problem that does not satisfy properties (B) and (C) is the Steiner tree problem.

THEOREM 5.1. *Under assumptions* (A)–(C)*, the tight upper bound on the expected optimal objective value of a* 0-1 *maximization problem is*

$$Z^*_{\max} = K_1 E_F \left[c \;\middle|\; c \ge F^{-1}\left(1 - \frac{K_1}{n}\right) \right], \tag{5.2}$$

where F^{-1} is the inverse distribution function for F.

Proof. From the previous sections, we know that tight bound $Z^*_{\max}$ on the expected value of a 0-1 maximization problem given marginal distribution F is obtained by solving formulation (3.1). Problem (5.1) can then be solved as

$$Z^*_{\max} = \min_{\boldsymbol{d} \in \Re^n} \left(Z^*_{\max}(\boldsymbol{d}) + \sum_{i=1}^{n} E_F[c_i - d_i]^+ \right). \tag{5.3}$$

Formulation (5.3) is a convex minimization problem in n variables. The Karush–Kuhn–Tucker conditions provide the necessary and sufficient optimality conditions for this problem. These conditions are

(i) $\lambda_p \ge 0$ for all $p = 1, \ldots, P$,

(ii) $\sum_{p=1}^{P} \lambda_p = 1$,

(iii) $Z^*_{\max}(\boldsymbol{d}) \geq \sum_{i \in B_p} d_i$ and $\lambda_p = 0$ whenever $Z^*_{\max}(\boldsymbol{d}) > \sum_{i \in B_p} d_i$ for all $p = 1, \dots, P$, and

(iv) $1 - \sum_{p: i \in B_p} \lambda_p = F(d_i)$ for all $i = 1, \dots, n$.

We now generate a set of primal d_i and dual variables λ_p that satisfy conditions (i)–(iv).

Set the dual multipliers to $\lambda_p = 1/P, p = 1, \dots, P$. Clearly this satisfies conditions (i) and (ii). From (C), condition (iv) then reduces to

$$1 - \frac{K_2}{P} = F(d_i) \qquad \forall i = 1, \dots, n.$$

If F^{-1} denotes the inverse of the distribution function, then the primal variables are set to

$$d_i = F^{-1}\left(1 - \frac{K_2}{P}\right) \qquad \forall i = 1, \dots, n$$

or equivalently

$$d_i = F^{-1}\left(1 - \frac{K_1}{n}\right) \qquad \forall i = 1, \dots, n.$$

Assumption (B) implies that $Z^*_{\max}(\boldsymbol{d}) = K_1 d$, satisfying condition (iii). Thus we have the optimal primal and dual variables to formulation (5.3). Substituting the optimal value of d_i into the objective function yields

$$\begin{aligned} Z^*_{\max} &= K_1 F^{-1}\left(1 - \frac{K_1}{n}\right) + nE_F\left[c - F^{-1}\left(1 - \frac{K_1}{n}\right)\right]^+, \\ &= K_1 F^{-1}\left(1 - \frac{K_1}{n}\right) + nE_F\left[c - F^{-1}\left(1 - \frac{K_1}{n}\right) \,\middle|\, c \geq F^{-1}\left(1 - \frac{K_1}{n}\right)\right] \frac{K_1}{n}, \end{aligned}$$

which reduces to the required result. $\square$

COROLLARY 5.2. *Under assumptions* (A)–(C)*, the tight lower bound on the expected optimal objective value of a* 0-1 *minimization problem is*

$$Z^*_{\min} = K_1 E_F\left[c \,\middle|\, c \leq F^{-1}\left(\frac{K_1}{n}\right)\right]. \tag{5.4}$$

5.1. Applications. We evaluate the explicit expected optimal objective value for a class of distributions next. Note that in the case $K_1/n \downarrow 0$ as $n \uparrow \infty$, $c \geq 0$, and from (5.4), the asymptotic performance of $Z^*_{\min}$ depends only on the distribution function of the cost coefficient $F(c)$ as $c \downarrow 0$. We thus focus on 0-1 minimization problems with the assumption that the distribution function of the cost coefficients F_α satisfies

$$F_\alpha(c) = \rho_\alpha c^\alpha \text{ as } c \downarrow 0. \tag{5.5}$$

The density function f_α is respectively assumed to satisfy

$$f_\alpha(c) = \rho_\alpha \alpha c^{\alpha - 1} \text{ as } c \downarrow 0. \tag{5.6}$$

Distributions of this type have been extensively studied in the asymptotic analysis of combinatorial problems [1], [3]. Models with distributions that satisfy (5.5) include the following:

(a) Uniformly distributed cost coefficients $U[0,1]$ with $\alpha = 1$ and $\rho_\alpha = 1$;
(b) Euclidean model with F_α defined as the distribution of the distances between independently and uniformly distributed points in dimension α; here ρ_α represents the volume of a sphere of unit radius in dimension α; and
(c) Independent model with the cost coefficients independently distributed with distribution F.

We generalize these models by allowing the cost coefficients to be dependent. Under this distributional assumption we obtain the following result.

THEOREM 5.3. *Under assumptions* (A)–(C) *with cost distribution satisfying* (5.5)*, the tight lower bound on the expected optimal objective value of a* 0-1 *minimization problem that satisfies* $K_1 = \mathbf{\Theta}(\sqrt{n})$ *scales asymptotically to a (positive) constant* C_α *as*

$$\lim_{n\to\infty}\left(\frac{Z^*_{\min}}{n^{(\alpha-1)/2\alpha}}\right) = C_\alpha, \tag{5.7}$$

where the asymptotic constant C_α *depends on the nominal problem.*

Proof. For cost coefficients with distribution function in (5.5), we obtain

$$F_\alpha^{-1}(c) = \left(\frac{c}{\rho_\alpha}\right)^{1/\alpha} \quad \text{as } c \downarrow 0.$$

The tight lower bound $Z^*_{\min}$ from Corollary 5.2 is then computed as

$$\begin{aligned} Z^*_{\min} &= K_1 E_{F_\alpha}\left[c \;\middle|\; c \le \left(\frac{K_1}{\rho_\alpha n}\right)^{1/\alpha}\right], \\ &= K_1\left[\int_0^{\left(\frac{K_1}{\rho_\alpha n}\right)^{1/\alpha}} \rho_\alpha \alpha c^\alpha dc\right] \Bigg/ \left[\int_0^{\left(\frac{K_1}{\rho_\alpha n}\right)^{1/\alpha}} \rho_\alpha \alpha c^{\alpha-1} dc\right] \quad \text{as } n \uparrow \infty. \end{aligned}$$

Note that with $K_1 = \mathbf{\Theta}(\sqrt{n})$, the term $(K_1/\rho_\alpha n)^{1/\alpha} \downarrow 0$ as $n \uparrow \infty$. Evaluating the integrals explicitly, we obtain

$$Z^*_{\min} = \left(\frac{\alpha}{\alpha+1}\right) K_1 \left(\frac{K_1}{n\rho_\alpha}\right)^{1/\alpha} \quad \text{as } n \uparrow \infty, \tag{5.8}$$

which brings us to the scaling result. ∎

COROLLARY 5.4. *The tight lower bound on the expected optimal objective value of the linear assignment, spanning tree, and traveling salesman problem with the marginal distribution satisfying* (5.5) *scales asymptotically to a (positive) constant* C^*_α *as*

$$\lim_{N\to\infty}\left(\frac{Z^*_{\min}}{N^{(\alpha-1)/\alpha}}\right) = C^*_\alpha, \tag{5.9}$$

where the asymptotic constant C^*_α *depends on the nominal problem.*

Proof. Using (5.8) and $N = \mathbf{\Theta}(\sqrt{n})$ for the three combinatorial problems of interest, we obtain the desired scaling result. ∎

We evaluate this asymptotic behavior for three specific problems in more detail next.

TABLE 5.2
Asymptotic scaling constant for the linear assignment problem.

Parameter α	With independence: $\tilde{C}^*_\alpha$	Without independence: C^*_α
2	0.7535	0.6667
3	0.8953	0.7500
4	0.9474	0.8000
5	0.9602	0.8333
6	0.9742	0.8571

Minimum cost linear assignment problem. (a) Uniform distribution. The random minimum cost linear assignment problem has been extremely well studied under the uniform distribution model on $[0,1]$. Under the additional assumption of independence, Walkup [30] proved an upper bound of 3 on the expected objective value. The first proven lower bound for this expected value was 1.368 [17]. These bounds have respectively been strengthened to 1.94 in [7] and 1.51 in [24]. Aldous [1] more recently proved a conjecture of Mezard and Parisi [22] that under this model the expected optimal objective value approaches $\pi^2/6 \approx 1.645$ as the size approaches ∞. In fact, this limiting value holds for any nonnegative continuous distribution such that the density of independent costs is 1 at 0. All of the above results hold under the strict assumption of independence.

Without this assumption, by setting $\alpha = 1$, $\rho_\alpha = 1$, $K_1 = N$, and $n = N^2$ in (5.8), we obtain the tight lower bound as

(5.10) For the linear assignment problem, $Z^*_{\min} = 0.5 \quad \forall N.$

Thus, by dropping the assumption of independence, the lower bound for the linear assignment problem becomes $C^*_1 = 0.5$.

(b) Power density. The linear assignment problem has been studied under the assumption that the cost coefficients are distributed on $[0,1]$ with density $\alpha c^{\alpha-1}$ for $\alpha > 1$. Under the assumption of independence, Donath [9] showed that asymptotically the expected optimal objective value is bounded as

$$\tilde{C}^*_\alpha \leq \lim_{N\to\infty} \left(\frac{E[Z^*_{\min}(\boldsymbol{c})]}{N^{(\alpha-1)/\alpha}} \right) \leq \tilde{C}^*_\alpha \left(\frac{\alpha-1}{\alpha} \right)$$

for some constant $\tilde{C}^*_\alpha$. The values of this constant for small values of α are provided by the same author.

By setting $K_1 = N$ and $n = N^2$ in (5.8), we obtain

(5.11) For the linear assignment problem, $\left(\dfrac{Z^*_{\min}}{N^{(\alpha-1)/\alpha}} \right) = \left(\dfrac{\alpha}{\alpha+1} \right) \ \forall N.$

Clearly, the tight lower bound scales with respect to $N^{(\alpha-1)/\alpha}$ under our model, too. This is similar to the scaling behavior observed under the independent model. In Table 5.2, we compare the constant $\tilde{C}^*_\alpha$ reported from [9] with $C^*_\alpha = \alpha/(\alpha+1)$ for five values of α. As should be expected, C^*_α is smaller than $\tilde{C}^*_\alpha$.

For these two particular distributions it can be verified that our results for the linear assignment problem hold for all N and are not just an asymptotic result. This is in contrast to the independent distribution model.

Minimum spanning tree and traveling salesman problems. (a) Uniform distribution. The random minimum spanning tree problem on a complete graph has been studied under the uniform distribution model. Under the additional assumption of independence, Frieze [10] showed that the expected length of the minimum spanning tree asymptotically converges to $\zeta(3) \approx 1.202$.

Without making the assumption of independence on arc length distribution, we obtain with $K_1 = N - 1$ and $n = N(N-1)/2$ from (5.8):

$$Z^*_{\min} = 1 - \frac{1}{N} \quad \text{as } N \uparrow \infty.$$

As the size of the problem N approaches infinity, the tight lower bound clearly converges to $C_1^* = 1$:

$$\text{For the spanning tree problem, } \lim_{N\to\infty} Z^*_{\min} = 1. \tag{5.12}$$

This value is surprisingly close to the expected optimal objective value under independence. For the traveling salesman problem, we similarly obtain

$$Z^*_{\min} = 1 + \frac{1}{1-N} \quad \text{as } N \uparrow \infty$$

and the same asymptotic constant of 1,

$$\text{For the traveling salesman problem, } \lim_{N\to\infty} Z^*_{\min} = 1. \tag{5.13}$$

(b) Euclidean and independent model. For the independent model with distribution function satisfying (5.5), the expected optimal objective value of the spanning tree asymptotically scales as [3]

$$\lim_{N\to\infty} \left(\frac{E[Z^*_{\min}(\boldsymbol{c})]}{N^{(\alpha-1)/\alpha}} \right) = \frac{1}{\alpha \rho_\alpha^{1/\alpha}} \sum_{k=1}^{\infty} \frac{\Gamma(k + \frac{1}{d} - 1)}{k! k^{1/d+1}}.$$

Using the same model but without assuming independence, we obtain the tight lower bound for the spanning tree problem as

$$\lim_{N\to\infty} \left(\frac{Z^*_{\min}}{N^{(\alpha-1)/\alpha}} \right) = \left(\frac{\alpha}{\alpha+1} \right) \rho_\alpha^{-1/\alpha} 2^{1/\alpha}. \tag{5.14}$$

Under the Euclidean model we can set the value of $\rho_\alpha = \pi^{d/2}/\Gamma(d/2+1)$. The values of the limiting constants are provided in Table 5.3. Interestingly, the traveling salesman problem exhibits the same asymptotic scaling constant but the rates of convergence are different.

5.2. Limited marginal information. The previous section showed that without the assumption of independence, the asymptotic performance for a general class of combinatorial optimization problems exhibits interesting behavior under the model of completely known marginal distributions. We now show that under the marginal moment model such problems in fact may exhibit a different kind of scaling behavior.

Consider probabilistic combinatorial optimization problems that satisfy assumptions (B) and (C), and the following:

(A′) Each objective coefficient is a random variable with common first and second moments μ and $\mu^2 + \sigma^2$.

TABLE 5.3
Asymptotic scaling constant for the spanning tree problem.

Parameter α	With independence: $\tilde{C}^*_\alpha$	Without independence: C^*_α
2	0.5684	0.5319
3	0.6094	0.5862
4	0.6558	0.6383
5	0.7010	0.6867
6	0.7440	0.7317

(D) The ground set can be partitioned into disjoint collection of feasible solutions, i.e., there is a set $\{0,\dots,\tilde{P}-1\}$ such that

$$\boldsymbol{X} = \cup_{p=0,\dots,\tilde{P}-1} B_p, \quad B_i \cap B_j = \emptyset \quad \forall i \neq j, i, j = 0,\dots,\tilde{P}-1.$$

Note that we change the indexing of the feasible solutions here slightly for ease of developing the result.

Assumption (D) is valid for a wide variety of combinatorial optimization problem. For instance, for the assignment problem, (D) follows from the well-known fact that the edges of a complete bipartite graph $K_{N\times N}$ can be partitioned into a collection of N disjoint perfect matching. For the spanning tree and traveling salesman problems, assumption (D) is obtained for an even and an odd number of vertices N, respectively.

Under these assumptions, we show that the tight lower bound $Z^*_{\min} = 0$ for these problems. To see this, we construct an instance of the problem, and a corresponding distribution satisfying all the conditions, such that the expected minimum cost solution is zero.

Let $\{B_0, B_1, \dots, B_{\tilde{P}-1}\}$ be the feasible solutions stipulated by assumption (D). We define

$$m = \left(\frac{\mu^2}{\mu^2+\sigma^2}\right)\tilde{P},$$

$$S_k = \cup_{i=k}^{k+m-1} B_i, \quad k = 0,\dots,\tilde{P}-1,$$

where the addition is taken modulo $\tilde{P}$. Consider the cost function $\boldsymbol{c}^k$, with

$$c_i^k = \begin{cases} \left(\dfrac{\mu^2+\sigma^2}{\mu}\right) & \text{if } i \in S_k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for a particular objective function $\boldsymbol{c}^k$, since there is a feasible solution B_l, $l \notin S_k$, with zero optimal objective value, $\boldsymbol{c}_k(B_l) = 0$. Construct a probability distribution with $\boldsymbol{c} = \boldsymbol{c}_k$ with probability $1/\tilde{P}$. For this distribution it can be verified that the first two moments are

$$E[c_i] = \frac{m}{\tilde{P}}\left(\frac{\mu^2+\sigma^2}{\mu}\right) = \mu, \qquad i = 1,\dots,n,$$

$$E[c_i^2] = \frac{m}{\tilde{P}}\left(\frac{\mu^2+\sigma^2}{\mu}\right)^2 = \mu^2+\sigma^2, \qquad i = 1,\dots,n.$$

Furthermore, for each realization of $\boldsymbol{c}$, the optimum solution value is 0, so we have $Z^*_{\min} = 0$.

6. Conclusions and extensions. In this paper, we addressed the problem of computing tight bounds on the expected value of a 0-1 optimization problem under uncertainty in the objective coefficients. The feasible multivariate distributions were characterized by known moment information on the marginal distributions of the objective coefficients. We showed that given moment information on piecewise polynomial functions of the objective coefficients, the tight upper (lower) bound on the expected optimal objective value of the 0-1 maximization (minimization) problem can be computed in polynomial time if the deterministic problem is solvable in polynomial time. We provided an efficiently solvable semidefinite program to compute this tight bound. Under an extension of this model with completely known and identical marginal distributions we analyzed the asymptotic bounds for a class of combinatorial problems.

REFERENCES

[1] D. Aldous, *The $\zeta(2)$ limit in the random assignment problem*, Random Structures Algorithms, 18 (2001), pp. 381–418.

[2] K. P. Anklesaria and Z. Drezner, *A multivariate approach to estimating the completion time for PERT networks*, J. Oper. Res. Soc., 40 (1986), pp. 811–815.

[3] F. Avram and D. Bertsimas, *The minimum spanning tree constant in geometrical probability and under the independent model: A unified approach*, Ann. Appl. Probab., 2 (1992), pp. 113–130.

[4] D. Bertsimas and I. Popescu, *Optimal inequalities in probability: A convex optimization approach*, SIAM J. Optim., to appear.

[5] J. R. Birge and M. J. Maddox, *Bounds on expected project tardiness*, Oper. Res., 43 (1995), pp. 838–850.

[6] R. E. Burkard and U. Fincke, *Probabilistic analysis of some combinatorial optimization problems*, Discrete Appl. Math., 12 (1985), pp. 21–29.

[7] D. Coppersmith and G. Sorkin, *Constructive bounds and exact expectations for the random assignment problem*, Random Structures Algorithms, 15 (1999), pp. 113–144.

[8] L. P. Devroye, *Inequalities for the completion times of stochastic PERT networks*, Math. Oper. Res., 4 (1979), pp. 441–447.

[9] W. E. Donath, *Algorithms and average-value bounds for assignment problems*, IBM J. Res. Dev., 13 (1969), pp. 380–386.

[10] A. M. Frieze, *On the value of a minimum spanning tree problem*, Discrete Appl. Math., 10 (1985), pp. 47–56.

[11] M. Grotschel, L. Lovasz, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer, Berlin, Heidelberg, 1988.

[12] J. H. Hagstrom, *Computational complexity of PERT problems*, Networks, 18 (1988), pp. 139–147.

[13] W. K. Klein Haneveld, *Robustness against dependence in PERT: An application of duality and distributions with known marginals*, Math. Programming Stud., 27 (1986), pp. 153–182.

[14] K. Isii, *On the sharpness of Chebyshev-type inequalities*, Ann. Inst. Statist. Math., 14 (1963), pp. 185–197.

[15] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim., 11 (2001), pp. 796–817.

[16] E. L. Lawler, *Combinatorial Optimization: Networks and Matroids*, Holt, Rinehart, and Winston, New York, 1976.

[17] A. J. Lazarus, *Certain expected values in the random assignment problem*, Oper. Res. Lett., 14 (1993), pp. 207–214.

[18] F. K. Levy and J. D. Wiest, *A Management Guide to PERT/CPM: With GERT/PDM/DPCM and Other Networks*, 2nd ed., Prentice–Hall, Englewood Cliffs, NJ, 1977.

[19] A. Ludwig, R. H. Mohring, and F. Stork, *A computational study on bounding the makespan distribution in stochastic project networks*, Ann. Oper. Res., 102 (2001), pp. 49–64.

[20] A. Madansky, *Inequalities for stochastic linear programming problems*, Management Sci., 6 (1960), pp. 197–204.

[21] I. Meilijson and A. Nadas, *Convex majorization with an application to the length of critical*

path, J. Appl. Probab., 16 (1979), pp. 671–677.

[22] M. Mezard and G. Parisi, *On the solution of the random link matching problems*, J. Phys., 48 (1987), pp. 1451–1459.

[23] Y. Nesterov, *Squared function systems and optimization problems*, in High Performance Optimization, H. Frenk et al., eds., Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000, pp. 405–440.

[24] B. Olin, *Asymptotic Properties of Random Assignment Problems*, Ph.D. thesis, Division of Optimization and Systems Theory, Department of Mathematics, Royal Institute of Technology, Stockholm, Sweden, 1992.

[25] P. A. Parillo, *Structured Semidefinite Programs and Semi-algebraic Geometry Methods in Robustness and Optimization*, Ph.D. thesis, California Institute of Technology, 2000.

[26] P. Robillard and M. Trahan, *The completion times of PERT networks*, Oper. Res., 25 (1976), pp. 15–29.

[27] D. Sculli and Y. W. Shum, *An approximate solution to the PERT problem*, Comput. Math. Appl., 21 (1991), pp. 1–7.

[28] J. M. Steele, *Probability Theory and Combinatorial Optimization*, CBMS-NSF Regional Conf. Ser. in Appl. Math. 69, SIAM, Philadelphia, 1997.

[29] J. F. Sturm, *SeDuMi v.* 1.03. Available at http://fewcal.kub.nl/sturm/software/sedumi.html.

[30] D. W. Walkup, *On the expected value of a random assignment problem*, SIAM J. Comput., 8 (1979), pp. 440–442.

[31] G. Weiss, *Stochastic bounds on distributions of optimal value functions with applications to PERT, network flows and reliability*, Oper. Res., 34 (1986), pp. 595–605.