CLASSICAL THEOREMS IN REVERSE MATHEMATICS AND HIGHER RECURSION THEORY

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DECLARATION

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Li Wei 31 May, 2013

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Contents

Summary

In this thesis, we study classical theorems of recursion theory, effective descriptive set theory and analysis from the view point of reverse mathematics and higher recursion theory. Here we consider reverse recursion theory as a part of reverse mathematics and study problems in two areas of higher recursion theory — hyperarithemtic theory and α-recursion.

In Chapter 1, we give a brief review of the history and background of the research areas involved in this thesis and summarize results in Chapter 3 to Chapter 5.

In Chapter 2, we review the basic notions, properties and theorems that will be needed in subsequent chapters.

In Chapter 3, we study the structure of Turing degrees below $\mathbf{0}'$ in the theory that is a fragment of Peano arithmetic without Σ_1 induction, with special focus on proper d-r.e. degrees and non-r.e. degrees. We prove

(1) $P^- + B\Sigma_1 + \mathrm{Exp} \vdash$ "There is a proper d-r.e. degree."

(2) $P^- + B\Sigma_1 + \mathrm{Exp} \vdash I\Sigma_1 \leftrightarrow$ "There is a proper d-r.e. degree below $\mathbf{0'}$."

(3) $P^- + B\Sigma_1 + \mathrm{Exp} \nvdash$ "There is a non-r.e. degree below $\mathbf{0'}$."

Here all the English sentences can be expressed in the language of PA.

In Chapter 4, we investigate the existence of a Friedberg numbering in fragments of Peano arithmetic and initial segments of Gödel's constructible hierarchy L_α, where α is Σ_1 admissible. We prove that

(1) Over $P^- + B\Sigma_2$, the existence of a Friedberg numbering is equivalent to $I\Sigma_2$,

and

(2) For L_α, there is a Friedberg numbering if and only if the tame Σ_2 projectum of α equals the Σ_2 cofinality of α.

In Chapter 5, we study continuous functions f on $[0,1]$, the Kechris-Woodin derivative and the Kechris-Woodin kernel of f. We show that

(1) The set $\hat{D} = \{e : \Phi_e \text{ describes an everywhere differentiable function on } [0,1]\}$ is Π^1_1 complete.

(2) For any continuous function f on $[0,1]$, if f has a recursive description, then the Kechris-Woodin rank of f is less than or equal to ω_1^{CK}.

(3) For any everywhere differentiable function f on $[0,1]$, if f has a recursive description, then the Kechris-Woodin rank of f is less than ω_1^{CK}, and conversely, for any $0 < \alpha < \omega_1^{CK}$, there is an everywhere differentiable function f on $[0,1]$ such that the Kechris-Woodin rank of f is α and f has a recursive description.

(4) Suppose f is continuous on $[0,1]$. If the Kechris-Woodin kernel of f is nonempty, then ATR_0 suffices to show the existence of a non-empty subset P of the Kechris-Woodin kernel of f. Over ACA_0, the existence of the Kechris-Woodin kernel for any continuous function on $[0,1]$ is equivalent to Π^1_1 comprehension.

In Chapter 6, we discuss some open questions left unanswered by the results of this thesis.

Chapter 1

Introduction

The study of the properties of the set of natural numbers has a long history, going back to Euclid and continued in the hands of Fermat, Euler and modern figures such as Hilbert, Cantor, Gödel, von Neumann, etc. Yet, the investigation of the computation properties of subsets of natural numbers is relatively new. It was initiated by Gödel in his famous Incompleteness Theorem [16] in 1931 which launched a new area of mathematical logic. Nowadays, the study of computational aspects of numbers and sets of numbers, is known as *Recursion (Computability) Theory*, a subject which developed rapidly over the last eighty years.

With the development of mathematical logic, the notion of computation, as well as notions from other branches of mathematics, was generalized to models other than the standard model $\langle \omega, \mathcal{P}(\omega) \rangle$, where ω is the set of natural numbers and $\mathcal{P}(\omega)$ is the power set of ω. The motivations for this were multifolded. One was the desire to capture essential features of a computation. The basic notions such as computation, finiteness, relative computation and effectiveness, which lie at the heart of recursion theory, should not be confined to the consideration of ω alone (Chong, [2]). In other words, the key properties of a computation should not depend solely on the underlying structure of the standard model. Therefore, it is necessary and possible to consider notions of computation in a more general setting. Another motivation was from the study of the foundations of mathematics. In foundations of mathematics, a major problem concerns the appropriate axiom systems for mathematics other than set theory. Given an axiom system, a theorem that is derived from the system shows the sufficiency of the system to prove the theorem, but it does not demonstrate the necessity of the system for the theorem. To establish the latter, one needs to show that the axiom system is satisfied in every model in which the theorem is true. In this sense, models could not be limited to the standard one.

Reverse mathematics (including reverse recursion theory) and higher recursion theory are typical areas in which the generalization of notions to models other than $\langle\omega, \mathcal{P}(\omega)\rangle$ play a central role. This thesis is devoted to the study of classical theorems from the view point of these two areas. First we study properties in recursion theory (Chapters 3 and 4) and then investigate the effectiveness of some particular theorems in analysis and descriptive set theory (Chapter 5). Chapters 3, 4 and 5 are relatively independent, but they are connected by the analysis of models of computation different from $\langle\omega, P(\omega)\rangle$. In this chapter, we briefly recall the history of reverse mathematics, reverse recursion theory and higher recursion theory, and introduce results in this thesis.

1.1 Reverse Mathematics

In reverse mathematics, a basic question concerns set existence axioms that are needed to prove theorems in ordinary mathematics. By ordinary mathematics, we mean areas such as number theory, analysis, countable algebra, geometry, combinatorics, etc. that developed independently of set theory. In ordinary mathematics, the objects considered are either countable (e.g. the set of natural numbers) or subsets of a separable structure (in the sense of a topological space). The weakest language appropriate to the study of these topics is the language of second order arithmetic. So reverse mathematics is investigated in the setting of second order arithmetic.

The program of reverse mathematics was started by Harvey Friedman [18] in the 1970's. Many researchers have since contributed to this area and a major systematic developer as well as expositor of the subject has been Stephen Simpson [44]. The study of reverse mathematics has proven to be a great success in classifying theorems of ordinary mathematics. Five subsystems of second order arithmetic of strictly increasing strength (in terms of the strength of set existence assumption) emerged as the core systems by which many theorems in ordinary mathematics are classified. The five subsystems are usual axioms for Peano Arithmetic (with Σ_1 induction) plus

Recursive Comprehension Axiom (RCA_0), Weak König's Lemma (WKL_0), Arithmetical Comprehension Axiom (ACA_0), Arithmetical Transfinite Recursion (ATR_0) and Π^1_1 Comprehension Axiom (Π^1_1-CA_0) respectively. RCA_0, ACA_0, and Π^1_1-CA_0 are systems that restrict the comprehension axiom to Δ^0_1, arithmetic and Π^1_1 formulas. WLK_0 asserts the compactness theorem in the Cantor space 2^ω, and ATR_0 permits transfinite induction. Specifically, a mathematical statement belongs to one of these five systems if it is provably equivalent to that system. A classical introduction to this subject can be found in Simpson [44].

1.1.1 Reverse recursion theory

An area that developed from the general study of reverse mathematics is the classification of the strength of mathematical induction required in the proof of mathematical theorems. Reverse recursion theory is a nice example of such a study. The general question it asks is: What is the strength of mathematical induction that is necessary (and sufficient) to prove theorems in classical recursion theory over a base theory? Since in classical recursion theory many of the objects studied are arithmetically definable, we investigate reverse recursion theory in the context of first order arithmetic. In particular, we use the first order language of arithmetic and the base theory will usually be a fragment of the axioms of Peano arithmetic (PA). A detailed introduction to the reverse recursion theory is given in [6, 8].

The theoretical foundation of subsystems of PA (also called fragments of PA) was established by Paris and Kirby [36] in the late 1970's. To set the stage, let P^- denote the axioms of PA concerning rules governing the standard arithmetic operations such as the associative law of "+", the distributive law with respect to "+" and "$\cdot$", etc, excluding the induction scheme. Paris and Kirby [36] defined fragments of PA by restricting the induction scheme to instances of bounded logical complexity and showed the relative logical strengths of the resulted theories. For $n \geq 1$, let $I\Sigma_n$ (Σ_n induction) denote the restriction of the induction scheme to Σ_n formulas, and let $B\Sigma_n$ (Σ_n bounding) be the statement saying that every Σ_n function maps a finite set in the sense of the model onto a finite set. It is known that $I\Sigma_n$ is strictly stronger than $B\Sigma_n$, and $B\Sigma_{n+1}$ is strictly stronger than $I\Sigma_n$, over the base theory $P^- + I\Sigma_0 + \text{Exp}$ ("Exp" says that $x \mapsto 2^x$ is a total function, and is a theorem of $P^- + I\Sigma_1$). It is possible to develop a theory of computation within a weak system of arithmetic. In fact, all the notions of classical recursion theory concerning primitive recursive functions, partial and total recursive functions, recursively enumerable (r.e.) sets etc. studied by Kleene and Post have their analogs in the system $P^- + B\Sigma_1 + \mathrm{Exp}$. The research area in which we analyze the strength of induction required to establish theorems in recursion theory is called reverse recursion theory.

A Turing degree is r.e. if it contains an r.e. set. The degree of a complete r.e. set is denoted $\mathbf{0}'$. In the 1980's, S. Simpson first proved (unpublished) the Friedberg-Muchnik Theorem (the existence of a pair of incomparable r.e. degrees, originally proved in the standard model of PA using the $\mathbf{0}'$-priority method) within the system $P^- + I\Sigma_1$. Slaman and Woodin [46] then studied Post's problem in models of the weaker theory $P^- + B\Sigma_1 + \mathrm{Exp}$. They provided examples of models of $P^- + B\Sigma_1 + \mathrm{Exp}$ where the Sacks Splitting Theorem failed. Thus, $P^- + B\Sigma_1 + \mathrm{Exp}$ is not strong enough for the implementation of the $\mathbf{0}'$-priority method involving the Sacks preservation strategy. Mytilinaios [34] continued the study and proved that $I\Sigma_1$ suffices to prove the Sacks Splitting Theorem. Later, Chong and Mourad [6] showed (without using the priority method) that the Friedberg-Muchnik Theorem is provable in $P^- + B\Sigma_1 + \mathrm{Exp}$. In general, any construction which is priority-free or involves not more than the use of a $\mathbf{0}'$-priority argument may be successfully implemented in a model of $P^- + I\Sigma_1$. Similarly, the $\mathbf{0}''$-priority method is applicable in models of $P^- + I\Sigma_2$ (see [8, 34, 35, 46]). It is reasonable to conjecture, in view of the success story concerning the Friedberg-Muchnik Theorem, that all theorems proved by using the $\mathbf{0}'$-priority method with effective bounds on the number of injuries for each requirement (a hallmark of the construction of a pair of r.e. sets with incomparable Turing degrees for the Friedberg-Muchnik Theorem) remain valid in models of $P^- + B\Sigma_1 + \mathrm{Exp}$, even if the original methods of proof do not carry over in the new setting. This conjecture is, however, false. The existence of a nonrecursive low set, originally proved using a $\mathbf{0}'$-priority construction with effective bounds, is known to be equivalent to $I\Sigma_1$ over $P^- + B\Sigma_1 + \mathrm{Exp}$ (see Chong and Yang [10]).

Also, the insights about the inductive principles needed to prove theorems in ordinary mathematics and recursion theory have been applied to other branches of reverse mathematics. In reverse mathematics, methods of reverse recursion theory have been used to tackle problems that are of a purely combinatorial nature. For instance, Cholak, Jockusch and Slaman [1] proved that over RCA_0, Ramsey's theorem of finite colorings for Pairs is strictly stronger than Ramsey's theorem of 2-coloring for Pairs by showing that the former could prove Σ_3 bounding (Σ_n bounding is equivalent to the inductive principle of Δ_n formulas for every n, see [45]), but not the latter. Further examples of this nature can be found in [1, 7, 19, 43].

1.2 Higher Recursion Theory

In the 1960's, Kreisel suggested the idea of generalizing the syntactic aspects of classical recursion theory, building on the earlier works of Church, Gandy, Kleene, Spector and Kreisel himself. Sacks pursued this idea and developed recursion theory on admissible ordinals [39]. Higher recursion theory includes four main parts — hyperarithmetic theory, metarecursion, α-recursion and E-recursion theory. In this thesis, we focus our study on the first and third part.

The study of hyperarithmetic theory began with the work of Church and Kleene on notation systems and recursive ordinals (see Church-Kleene [14], Church [13], Kleene [25]). Hyperarithmetic sets are defined by iterating the Turing jump though recursive ordinals. Kleene's theorem states that hyperarithmetic sets are exactly Δ^1_1 sets. It rises a construction process and hierarchy for the class of Δ^1_1 sets and constitutes the first real breakthrough into second order logic. Correspondingly, $\boldsymbol{\Delta^1_1}$ sets (called bold face Δ^1_1 sets), which are known as Borel sets, have a parallel construction hierarchy in descriptive set theory. In fact, hyperarithmetic theory is often regarded as the source of effective descriptive set theory.

Another approach to generalize recursion theory is α-recursion theory, which studies the theory of computation over initial segments L_α of Gödel's constructible hierarchy. The core of classical recursion theory is the notion of an effective construction (and its relativization). From the set theoretical point of view, an effective construction is a Σ_1 operator definable over the structure of the standard model. An ordinal α is Σ_1 admissible if L_α is a model that is closed under Σ_1 definable operators. In particular, ω is Σ_1 admissible.

The generalization of recursion theory to ordinals was introduced by Takeuti [50] and its set theoretical framework in the context of admissible sets was introduced by

Kripke [28] and Platek [38]. Kreisel and Sacks [27] initiated the study of the structure of recursively enumerable (r.e.) sets over the first admissible ordinal greater than ω. In general, admissible ordinals lack certain combinatorial properties that come with the standard model ω and crucial to the construction of r.e. sets. This results in constructions which are sometimes much more intricate than those for ω, and in certain cases, the failure of the combinatorial property leads to a negative conclusion. A key feature in the study of α-recursion theory is the fruitful application of ideas and methods from Jensen's work [21] on the fine structure of the constructible universe. The interplay between fine structure theory and recursion theory provides many new insights not available previously. Hence the study of generalized recursion theory elucidates the essence of an effective construction and the nature of notions that are fundamental to a theory of computation. In 1972, Sacks and Simpson [40] solved Post's problem for every Σ_1 admissible ordinal. Their proof uses a combination of the priority method and the fine structure theory of L. Lerman [30] gave a more recursion theoretic proof by reducing the use of fine structure theory. Both of the two approaches have proven to be of wide applications in the study of α-recursion theory (see [39]). In [41], Shore proved the splitting theorem which relies heavily on his method of Σ_2 blocking. Shore's blocking method has also been applied successfully in reverse recursion theory. (For instance, Mytilinaios [34] proved Sacks' splitting theorem in Σ_1 induction.) Shore [42] also showed the density theorem remains valid for all $\Sigma_1$admissible ordinals. It is an example of a Σ_3 argument of classical recursion theory lifted to all Σ_1 admissible ordinals.

1.3 Results

1.3.1 Chapter 3 – Δ_2 degrees

In Chapter 3, we consider problems about non-r.e. sets in the system $P^-+B\Sigma_1+\text{Exp}$. In particular, we study the structure of degrees below $\mathbf{0}'$. In classical recursion theory, i.e. in the standard model of PA, these degrees are precisely those which contain as members only sets that are Δ_2 definable, but in models of $P^-+B\Sigma_1+\text{Exp}$, the situation may be different.

For any two r.e. sets A and B, $A \setminus B$ is said to be a d-r.e. set (*difference* of two r.e. sets). A degree is d-r.e. if it contains a d-r.e. set. The degree is called *proper* d-r.e., if it is d-r.e. but not r.e. Clearly every r.e. degree is d-r.e., and every d-r.e. set in a model of $P^- + B\Sigma_1 + \mathrm{Exp}$ is Δ_2 definable. Furthermore, in classical recursion theory, we have the following result.

Theorem 1.1 (Cooper [12]). *There is a proper d-r.e. degree.*

In Chapter 3, we first investigate the existence of a proper d-r.e. degree from the point of view of reverse recursion theory. By the general observation on the $\mathbf{0}'$-priority method described above, Cooper's proof of the existence of a proper d-r.e. degree may be carried out in models of of $P^- + I\Sigma_1$. This result was shown by Kontostathis [26] in 1993. The situation becomes particularly interesting when working with a model that precludes the use of a priority construction, such as in a model where Σ_1 induction fails, and so the $\mathbf{0}'$-priority method fails in general. We show that in a model of $P^- + B\Sigma_1 + \mathrm{Exp}$ where $I\Sigma_1$ fails (called a $B\Sigma_1$ model), by adopting a new approach, we can still construct a proper d-r.e. degree. The key to the new approach is to exploit the definition of Turing reducibility in the setting of $B\Sigma_1$ models. In a model of weak induction, finite sets in the sense of the model are used in place of singletons in the definition of Turing reducibility to ensure the transitivity of $\leq_T$. This fine difference in the definition of reducibility enables one to construct a d-r.e. degree $\mathbf{d}$ that does not lie below $\mathbf{0}'$.* Such a $\mathbf{d}$ is not r.e., since every r.e. degree is Turing reducible to $\mathbf{0}'$. In fact, the existence of a proper d-r.e. degree not below $\mathbf{0}'$ is not accidental. In any $B\Sigma_1$ model, we show that every d-r.e. degree below $\mathbf{0}'$ is r.e. Beyond this, we also exhibit a $B\Sigma_1$ model in which every degree below $\mathbf{0}'$ is r.e. The conclusion one draws from these results is that in the absence of Σ_1 induction, the structure of Turing degrees below $\mathbf{0}'$ presents a relatively neater picture. The fact that it is possible for $\mathbf{0}'$ to bound only r.e. degrees also looks intriguing and calls for further investigation.

1.3.2 Chapter 4 – Friedberg numbering

The idea of coding information using numbers was introduced by Kurt Gödel. In the proof of his famous Incompleteness Theorem [16], Gödel effectively assigned to

*In a $B\Sigma_1$ model, a d-r.e. degree may not be below $\mathbf{0}'$, yet is still r.e. in $\mathbf{0}'$. Thus, any d-r.e. degree is reducible to $\mathbf{0}''$.

each formula a unique natural number. Generally, any map from ω onto a set of objects, such as formulas, is called a *numbering* of the objects. For example, one can follow Gödel to effectively list all Σ_1 formulas, hence all r.e. sets, which we shall refer to as the *Gödel numbering* of r.e. sets. In Chapter 4, we focus on numberings f of recursively enumerable (r.e.) sets such that $\{(x, e) : x \in f(e)\}$ is r.e.

A universal numbering is a recursive list of all r.e. sets. Gödel numbering is universal. Yet, Gödel numbering is not one-one, as two Σ_1 formulas may define the same r.e. set. A natural question was raised by S. Tennenbaum: "Is there a recursive list of all r.e. sets without repetition?" Essentially, the question asks for an effective choice function of r.e. sets. Friedberg [15] gave an affirmative answer to Tennenbaum's question for the standard model ω of natural numbers. Thus, a one-one universal numbering is said to be a Friedberg numbering. In [29], Kummer simplified Friedberg's proof by a priority-free argument. Kummer's proof and Friedberg's proof both use the method of effective approximation to search for the least index of an r.e. set and obtain as a result a one-one enumeration of r.e. sets.

Our purpose in Chapter 4 is to investigate the existence of Friedberg numbering in different models of computation: models of fragments of PA and initial segments L_α of Gödel's constructible universe, where α is Σ_1 admissible.

An intuitive approach to analyzing the existence of a Friedberg numbering in models of fragments of PA or L_α is illustrated in the following paragraphs.

Let $\{W_e\}$ be a Gödel numbering in such a model. Then e is the least index of W_e if

$$\forall i < e\,(W_i \neq W_e). \tag{1.1}$$

(1.1) is a Σ_2 sentence preceded by a bounded quantifier. A careful examination of known proofs shows that $P^- + I\Sigma_2$ and α satisfying Σ_2 replacement suffice to prove the existence of a Friedberg numbering in the model. The most interesting situation is then when $I\Sigma_2$ or Σ_2 replacement fails.

Though no priority method is required to construct a Friedberg numbering, interestingly, we will show that $I\Sigma_2$ is in fact necessary for the existence of a Friedberg numbering in models that satisfy $P^- + B\Sigma_2$. Observe that $B\Sigma_2$ reduces (1.1) to a Σ_2 formula as in the standard model ω. However, in a model satisfying $B\Sigma_2$ but not $I\Sigma_2$, for an r.e. set W, there may not be an e satisfying (1.1) such that $W_e = W$. Therefore, the straightforward extension of known proofs does not work. In the other direction, if e is the least index, $B\Sigma_2$ suffices to establish an upper bound of the least differences between W_e and all W_i, $i < e$. That property provides a possible way to do a diagonalization argument to show that no one-one numbering is universal, so that there is no Friedberg numbering.

For an L_α not satisfying Σ_2 replacement, the lifting of the construction from ω to α has another complication. Because of the failure of Σ_2 replacement, (1.1) is in fact Π_3 and not Σ_2. Hence the least index of an α-r.e. set, while it exists, may not be effectively approximated. An analysis of this situation leads to different outcomes. We give two examples to illustrate this point by way of the ordinals: ω_1^{CK} and $\aleph_\omega^L$. Though $L_{\omega_1^{CK}}$ does not satisfy Σ_2 replacement, the collection of α-r.e. sets can be arranged in order type ω through a Σ_1 projection from ω_1^{CK} into ω. Then the construction may be carried out in the new ordering and yields the existence of a Friedberg numbering. The second example $\aleph_\omega^L$, however, does not have the advantage of a Σ_1 projection into a smaller ordinal, as $\aleph_\omega^L$ is a cardinal of L. Here the lack of Σ_2 admissibility and a Σ_1 projection to a suitably smaller regular ordinal results in the nonexistence of a Friedberg numbering for $L_{\aleph_\omega^L}$. Our plan is to extend the diagonalization argument in $B\Sigma_2$ models to $L_{\aleph_\omega^L}$. Since $L_{\aleph_\omega^L}$ does not satisfy Σ_2 replacement, in general, for W_e from (1.1), the least upper bound of the least differences of W_e and all W_i, $i < e$, may be $\aleph_\omega^L$. Nevertheless the situation is different when W_e is α-finite. Suppose W_e is an α-finite set satisfying (1.1), then $\zeta = \sup W_e < \aleph_\omega^L$. Therefore for every $i < e$, if $W_i \not\supseteq W_e$, then the least difference between W_i and W_e is less than ζ. If $W_i \supsetneq W_e$, then there exists a large enough $\aleph_n^L > \zeta$ such that $W_{i,\aleph_n^L} \supsetneq W_e$, since for every $m < \omega$, $\langle L_{\aleph_m^L}, \in\rangle$ is a Σ_1 elementary substructure of $\langle L_{\aleph_\omega^L}, \in\rangle$. Also, note that the Π_1 function: $n \mapsto \aleph_n^L$, allows an arrangement of α-r.e. sets in blocks of length $\aleph_0^L, \aleph_1^L, \ldots$. By considering α-finite sets, the diagonalization strategy for $B\Sigma_2$ models may be extended to $L_{\aleph_\omega^L}$ block by block. The argument for $L_{\aleph_\omega^L}$ can be generalized to an arbitrary Σ_2 inadmissible cardinal α. A further analysis leads to the characterization in Chapter 4 of the existence of a Friedberg numbering in terms of the notions of tame Σ_2 projectum (a Σ_1 projection is also tame Σ_2) and Σ_2 confinality of α (denoted as $t\sigma 2p\,(\alpha)$ and $\sigma 2cf(\alpha)$ respectively). The notion of $t\sigma 2p\,(\alpha)$ was introduced by Lerman [30] and $\sigma 2cf(\alpha)$ was introduced by Jensen [21] in his study of the fine structure theory of Gödel's L. The precise definitions of $t\sigma 2p\,(\alpha)$ and $\sigma 2cf(\alpha)$ are given in Section 4.2. In the two examples shown here, $t\sigma 2p\,(\omega_1^{CK}) = \sigma 2cf(\omega_1^{CK}) = \omega$, and $t\sigma 2p\,(\aleph_\omega^L) = \aleph_\omega^L > \sigma 2cf(\aleph_\omega^L) = \omega$. They give some hints about the characterization of the existence of a Friedberg numbering in L_α.

1.3.3 Chapter 5 – Recursive aspects of everywhere differentiable functions

In Chapter 5, we apply results in hyperarithmetic theory and reverse mathematics to analyze the complexities of everywhere differentiable functions on the closed interval $[0, 1]$.

Let $\mathbf{C}[0, 1]$ be the set of continuous functions on $[0, 1]$ and $\mathbf{D} \subset \mathbf{C}[0, 1]$ be the collection of everywhere differentiable functions in $\mathbf{C}[0, 1]$. Mazurkiewicz [33] (see also [24]) proved that $\mathbf{D}$ is $\mathbf{\Pi}_1^1$ complete. In a general sense, his method of proof is effective. In Chapter 5, we apply his method to show $D = \{e < \omega : \Phi_e$ *describes* an everywhere differentiable function on $[0, 1]\}$ is Π_1^1 complete (for subsets of ω). The precise definition of "describe an everywhere differentiable function on $[0, 1]$" is in Section 5.3.

The rank of an everywhere differentiable function in the context of descriptive set theory was investigated by Kechris and Woodin [24]. They defined a natural rank which associates each function in $\mathbf{D}$ with a countable ordinal. We call this ordinal the *Kechris-Woodin rank*. Kechris-Woodin rank was given two descriptions — in terms of well founded trees and in terms of Cantor-Bendixson type analysis. Ranks defined in these two descriptions are essentially the same. For any non-liner function f in $\mathbf{D}$, the Kechris-Woodin rank of f in the sense of the first description is ω times of the rank in the sense of the second description. In Chapter 5, we adopt the latter description and denote the Kechris-Woodin rank of f by $|f|_{\mathrm{KW}}$. Also, we extend this rank definition so that it applies to every function f in $\mathbf{C}[0, 1]$.

Before stating the results, let us review the Cantor-Bendixson analysis of a tree. Consider the Cantor space $2^{<\omega}$ and let $T \subseteq 2^{<\omega}$ be a tree. Let $[T] = \{x \in 2^\omega : \forall n\,(x \upharpoonright n \in T)$, i.e. x is a path in $T\}$. The Cantor-Bendixson derivative of T, denoted as $\mathrm{CB}(T)$, is

$$\mathrm{CB}(T) = \{\sigma \in 2^{<\omega} : \exists x, y \in [T]\,(x \neq y \text{ extend } \sigma)\}.$$

We may iteratively apply the Cantor-Bendixson derivative through the ordinals, i.e. let $T_0 = T$ and for every $\alpha > 0$, let $T_\alpha = \bigcap_{\beta<\alpha} \mathrm{CB}(T_\beta)$. Using this hierarchy, it is shown that any tree T in the Cantor Space, $[T]$ is either countable, or contains a perfect subset. This result is called the Cantor-Bendixson theorem. $\bigcap_\alpha T_\alpha$ is called the Cantor-Bendixson kernel of T and denoted as $\mathrm{Ker}_{\mathrm{CB}}(T)$, which is the largest perfect subset of T. The least ordinal α such that $T_\alpha = \mathrm{Ker}_{\mathrm{CB}}(T)$ is the Cantor-Bendixson rank of T, denoted as $|T|_{\mathrm{CB}}$. In descriptive set theory and hyperarithmetic theory, we have the following results.

(i) For every $\alpha < \aleph_1$, there is a tree T such that $[T]$ is countable and $|T|_{\mathrm{CB}} = \alpha$; if $\alpha < \omega_1^{CK}$, then the tree T can be made recursive.

(ii) For every tree T with $[T]$ countable, $|T|_{\mathrm{CB}} < \aleph_1$; if T is recursive, then $|T|_{\mathrm{CB}} < \omega_1^{CK}$.

(iii) For every tree T, $|T|_{\mathrm{CB}} < \aleph_1$; if T is recursive, then $|T|_{\mathrm{CB}} \leq \omega_1^{CK}$.

Given a continuous function f in $\mathbf{C}[0,1]$, the Kechris-Woodin rank $|f|_{\mathrm{KW}}$ is defined in a similar manner. In [24], for every $\varepsilon \in \mathbb{Q}^+$ and closed set $P \subseteq [0,1]$, the Kechris-Woodin derivative $P'_{f,\varepsilon}$ of P is defined according to the derivative property of f (see Chapter 5). We may iterate this operation as follows.

$$
\begin{aligned}
&P^0_{f,\varepsilon} = [0,1] \\
&P^\alpha_{f,\varepsilon} = \textstyle\bigcap_{\beta<\alpha} (P^\beta_{f,\varepsilon})'_{f,\varepsilon}, \quad \alpha > 0
\end{aligned}
$$

Let $\alpha_f(\varepsilon)$ be the least α such that $P^\alpha_{f,\varepsilon} = \bigcap_\beta P^\beta_{f,\varepsilon}$ and its rank $|f|_{\mathrm{KW}} = \sup_\varepsilon \alpha_f(\varepsilon)$. The Kechris-Woodin kernel of f, $\mathrm{Ker}_{\mathrm{KW}}(f) = \bigcup_\varepsilon \bigcap_\alpha P^\alpha_{f,\varepsilon}$. As for Cantor-Bendixson rank, the Kechris-Woodin rank satisfies the following properties.

(i) For any $\alpha < \aleph_1$ not zero, there is a function $f \in \mathbf{D}$ such that $|f|_{\mathrm{KW}} = \alpha$; if $\alpha < \omega_1^{CK}$, then the function f can be constructed so that f has a recursive description.

(ii) For any function $f \in \mathbf{D}$, $|f|_{\mathrm{KW}} < \aleph_1$; if f has a recursive description, then $|f|_{\mathrm{KW}} < \omega_1^{CK}$.

(iii) For any function $f \in \mathbf{C}[0,1]$, $|f|_{\mathrm{KW}} < \aleph_1$; if f has a recursive description, then $|f|_{\mathrm{KW}} \leq \omega_1^{CK}$.

In Chapter 5, we discuss the hyperarithmetical aspects of these properties, their descriptive set theoretic aspect was investigated by Kechris and Woodin [24].

The correspondence between Cantor-Bendixson derivative and Kechris-Woodin derivative is not coincidental. Clearly, whenever a derivative operation is defined on a countable structure, the descriptive set theoretic aspects of properties (ii)–(iii) hold. We prove that if the operation of derivative is hyperarithmetic, then the hyperarithmetic aspects of properties (ii)–(iii) also hold (see Proposition 2.2.4). On the other hand, the validity of (i) depends on the operator itself.

In reverse mathematics, it was shown that the existence of $\mathrm{Ker}_{\mathrm{CB}}(T)$ for every tree T in a second order arithmetic model is equivalent to Π^1_1-CA_0. We end Chapter 5 by showing that a similar result for $\mathrm{Ker}_{\mathrm{KW}}(f)$ is true.

Chapter 2

Preliminaries

In this chapter, we give a summary of the background material involved in this thesis.

2.1 First Order Arithmetic

Here we recall some useful facts about first order arithmetic. More details can be found in [9, 22, 34, 36].

2.1.1 Fragments of Peano arithmetic

The language of first order arithmetic $\mathcal{L}(0, 1, +, \cdot, <)$ consists of variables $x_1, x_2, x_3 \ldots$ constants $0, 1$, and functions $+$ (plus), $\cdot$ (times).

Atomic formulas are $t = s$ and $t < s$, where t and s are number theoretic terms. Formulas are built up from atomic formulas, propositional connectives and quantifiers. In formulas, we also use $t \leq s$ to denote $(t < s) \vee (t = s)$.

A formula of $\mathcal{L}(0, 1, +, \cdot, <)$ is Σ_n (Π_n respectively) if it is of the form $\exists x_1 \forall x_2 \ldots \theta$ ($\forall x_1 \exists x_2 \ldots \theta$ respectively), where $\exists x_1 \forall x_2 \ldots$ ($\forall x_1 \exists x_2 \ldots$ respectively) is n alternative blocks of quantifiers, and θ is a formula containing only bounded quantifiers. A formula is Δ_n if it is both Σ_n and Π_n.

P^- consists of the usual axioms on arithmetical operations without induction as follows.

$$\forall x, y, z\,((x+y)+z = x+(y+z)) \qquad \forall x, y\,(x+y = y+x)$$
$$\forall x, y, z\,((x \cdot y) \cdot z = x \cdot (y \cdot z)) \qquad \forall x, y\,(x \cdot y = y \cdot x)$$
$$\forall x, y, z\,(x \cdot (y+z) = x \cdot y + x \cdot z)$$
$$\forall x\,((x+0 = x) \wedge (x \cdot 0) = 0) \qquad \forall x\,(x \cdot 1 = x)$$
$$\forall x\,\neg(x < x)$$
$$\forall x, y, z\,((x < y \wedge y < z) \rightarrow x < z) \qquad \forall x, y\,(x < y \vee x = y \vee y < x)$$
$$\forall x, y, z\,(x < y \rightarrow x+z < y+z) \qquad \forall x, y, z\,(0 < z \wedge x < y \rightarrow x \cdot z < y \cdot z)$$
$$\forall x, y\,(x < y \rightarrow \exists z\, x+z = y)$$
$$0 < 1 \wedge \forall x\,(x > 0 \rightarrow x \geq 1) \qquad \forall x\,(x \geq 0)$$

An induction principle may have different forms of expression. One of them, called *induction scheme*, is the following:

$$\forall x\,[(\forall y < x\,\varphi(y)) \rightarrow \varphi(x)] \rightarrow \forall x\,\varphi(x),$$

for every φ, possibly with parameters.

Another forms are the *bounding scheme*:

$$\forall x\,(\forall y < x\,\exists w\,\varphi(y, w) \rightarrow \exists b\,\forall y < x\,\exists w < b\ \varphi(y, w)),$$

and the *least number scheme*

$$\exists w\,\psi(w) \rightarrow \exists w\,(\psi(w) \wedge \forall v < w\,(\neg\psi(v))),$$

for any φ, possibly with parameters.

The Σ_n *induction* (Σ_n *bounding*, Σ_n *least number principle* respectively), denoted by $I\Sigma_n$ ($B\Sigma_n$, $L\Sigma_n$, respectively), is the induction scheme (bounding scheme, the least number principle, respectively) restricted to Σ_n formulas.

Theorem 2.1. *Let $n \geq 1$. Assume $P^- + I\Sigma_0 + \mathrm{Exp}$, where Exp says that $\forall x\,\exists y\,(y = 2^x)$.*

(1) (Paris and Kirby) The following implications hold:

(a) $B\Sigma_n \Leftrightarrow B\Pi_{n-1}$.

(b) $I\Sigma_n \Leftrightarrow I\Pi_n \Leftrightarrow L\Sigma_n \Leftrightarrow L\Pi_n$.

(c) $B\Sigma_{n+1} \Rightarrow I\Sigma_n \Rightarrow B\Sigma_n$. *Furthermore, the arrows do not reverse.*

(2) (Gandy) $B\Sigma_n \Leftrightarrow L\Delta_n$.

(3) (Slaman) $B\Sigma_n \Leftrightarrow I\Delta_n$.

By Clause (1) of Theorem 2.1, the hierarchy of restricted induction

$$\cdots \Rightarrow I\Sigma_2 \Rightarrow B\Sigma_2 \Rightarrow I\Sigma_1 \Rightarrow B\Sigma_1 \tag{2.1}$$

does not collapse. Other forms of restricted induction, by Theorem 2.1 again, can be reduced to the ones in (2.1).

2.1.2 Models of fragments of PA

Sets

Let $\mathcal{M}$ be a model of $P^- + I\Sigma_0 + \text{Exp}$. A subset of $\mathcal{M}$ is *r.e.*, if it is Σ_1 definable. If the complement of an r.e. set is also Σ_1 definable, then the set is *recursive*. The set difference of two r.e. sets is called *d-r.e.* (difference of r.e. sets). In general, an n-r.e. set is a set of the form $A \setminus D$, where A is r.e. and D is $(n-1)$-r.e., and $n \geq 2$ (d-r.e. sets are 2-r.e. and r.e. sets are 1-r.e.).

A set $D \subset \mathcal{M}$ is *bounded*, if there is a $b \in \mathcal{M}$ such that $D \subseteq [0, b)$. A bounded set is $\mathcal{M}$*-finite*, if it is represented by the binary expansion of some element in $\mathcal{M}$*. A set is *regular* if its intersection with any $\mathcal{M}$-finite set is $\mathcal{M}$-finite. To distinguish between sets and numbers, in this chapter, we use lower case letters to denote numbers and capital letters to denote sets.

Lemma 2.2 (H. Friedman). *Let $n \geq 1$ and $\mathcal{M} \models P^- + I\Sigma_n$. Then any Σ_n subset of $\mathcal{M}$ is regular, and any partial Σ_n function maps a bounded set to a bounded set. In particular, if $\mathcal{M} \models P^- + I\Sigma_1$, then all r.e. sets and d-r.e. sets of $\mathcal{M}$ are regular.*

Given an r.e. set A, let $A_s \subseteq A$ be the collection of numbers enumerated into A by stage s. Then A_s is $\mathcal{M}$-finite for any s. For any d-r.e. set D, D_s is defined similarly.

*In general, a bounded set may not be $\mathcal{M}$-finite. For instance, in any nonstandard model, the set ω is bounded but not $\mathcal{M}$-finite (and not regular).

Suppose $f : \mathcal{M}^2 \to \mathcal{M}$ is a partial function. We define its *limit* at x as follows:

$$\lim_s f(s,x) = y \leftrightarrow \exists t\, \forall s > t\, (f(s,x) \downarrow = y).$$

Clearly, for every $x \in \mathcal{M}$ and r.e. (d-r.e.) set F, $\lim_s F_s(x) = F(x)$.

Computation and degrees

Fix a Δ_0 bijection $\langle \cdot, \cdot \rangle : \mathcal{M}^2 \to \mathcal{M}$ such that

(i) $\langle a, b \rangle \geq \max\{a, b\}$ for all $a, b \in \mathcal{M}$, and

(ii) $\langle \cdot, \cdot \rangle$ is strictly increasing with respect to each component.

By Σ_1 induction, we define

$$\langle z_0, z_1, \ldots, z_{n+1} \rangle = \langle \langle z_0, z_1, \ldots, z_n \rangle, z_{n+1} \rangle,$$

for every $n \in \mathcal{M} \setminus \{0\}$ and $z_0, z_1, \ldots, z_{n+1} \in \mathcal{M}$. (Without Σ_1 induction, functions $\langle z_0, z_1, \ldots, z_{n+1} \rangle$ are defined for every $n < \omega$.)

An r.e. set Φ is a *Turing functional* if it satisfies the universal closure of the following conditions:

(i) $\langle X, z, P, N \rangle \in \Phi \to ((z = 0 \vee z = 1) \wedge (P \cap N = \emptyset))$,

(ii) $(\langle X, z, P, N \rangle \in \Phi \wedge P' \cap N' = \emptyset \wedge P' \supseteq P \wedge N' \supseteq N \wedge X' \subseteq X)$
$\to \langle X', z, P', N' \rangle \in \Phi$,

(iii) $(\langle X, z, P, N \rangle, \langle X, z', P, N \rangle \in \Phi) \to z = z'$.

Here, $\mathcal{M}$-finite sets X, P, N etc. are identified with their representations of binary expansion. Intuitively, for a Turing functional Φ, $\langle X, z, P, N \rangle \in \Phi$ means the program Φ with input X produces output z, whenever P is some positive part of the oracle and N is some negative part of the oracle.

Let $\{W_e\}_{e \in \mathcal{M}}$ be an effective enumeration of all r.e. sets. Any W_e and its enumeration could be modified uniformly and recursively to produce an r.e. Turing functional Φ_e such that:

(i) If W_e is a Turing functional, then $\Phi_e = W_e$.

(ii) For every stage s and computation $\langle X, z, P, N\rangle \in \Phi_{e,s}$, the $\mathcal{M}$-finite sets X, P, N are subsets of $[0, s)$.

(iii) Φ_e satisfies the *local downward closer property with respect to* $\Phi_{e,s}$:

For any stage s and computation $\langle X, z, P, N\rangle \in \Phi_{e,s}$, if Y is an $\mathcal{M}$-finite subset of X, then $\langle Y, z, P, N\rangle \in \Phi_{e,s}$.

Note that the modification could be uniformly recursive, so the enumeration of all r.e. Turing functionals $\{\Phi_e\}_{e\in\mathcal{M}}$ is recursive.

Given $A, B \subseteq \mathcal{M}$, A is said to be *Turing reducible* (or *reducible*, for short) to B, denoted by $A \leq_T B$, if there is an r.e. Turing functional Φ such that for every $\mathcal{M}$-finite set X,

$$\begin{aligned} X \subseteq A \quad &\Leftrightarrow \exists P\, \exists N\, (P \subseteq B \,\wedge\, N \subseteq \overline{B} \,\wedge\, \langle X, 1, P, N\rangle \in \Phi), \\ X \subseteq \overline{A} \quad &\Leftrightarrow \exists P\, \exists N\, (P \subseteq B \,\wedge\, N \subseteq \overline{B} \,\wedge\, \langle X, 0, P, N\rangle \in \Phi). \end{aligned}$$

In the above definition, if $\Phi = \Phi_e$, then we say $A \leq_T B$ *via* Φ_e (in symbols $A = \Phi_e^B$). Turing degrees, r.e. degrees, etc, are defined as usual.

Turing reducibility is also called *strong reducibility* or *setwise reducibility.* They are defined so against the notion of *weak reducibility* or *pointwise reducibility* (denoted by $\leq_p$), which is obtained by substituting an element "x" for an $\mathcal{M}$-finite set "X" in the definition of Turing functional and Turing reducibility. Turing reducibility is transitive and stronger than weak reducibility, but weak reducibility needs not be transitive.

Now we fix the following notations. Suppose Φ_i is a Turing functional, W_e is an r.e. set. Then for any stage s,

$$\Phi_i^{W_e}[s] = \{\langle X, z, P, N\rangle \in \Phi_{i,s} : P \subseteq W_{e,s},\ N \subseteq \overline{W}_{e,s}\}.$$

That is, $\Phi_i^{W_e}[s]$ is a collection of computations consistent with W_e from the view of stage s. Since Φ_i is a Turing functional, $\Phi_i^{W_e}[s]$ is also *self consistent*, i.e. the universal closure of the following formula holds:

$$\langle X, z, P, N\rangle \in \Phi_i^{W_e}[s] \,\rightarrow\, \langle X, 1-z, P', N'\rangle \notin \Phi_i^{W_e}[s].$$

Note that $\Phi_i^{W_e}$ also satisfies the *local downward closure property* with respect to $\Phi_i^{W_e}[s]$.

Now suppose $\mathcal{M}$ is a model of $B\Sigma_1$. If $\langle X, z, P, N\rangle \in \Phi_i$ such that $P \subseteq W_e$ and $N \subseteq \overline{W}_e$, then $\langle X, z, P, N\rangle \in \Phi_i^{W_e}[s]$ for all large enough stages s. If $\mathcal{M}$ also satisfies $I\Sigma_1$, then we can define $\Phi_i^D[s]$ similarly. Here, $I\Sigma_1$ is required to ensure that whenever $\langle X, z, P, N\rangle \in \Phi_i$, $P \subseteq D$, $N \subseteq \overline{D}$ and s is large enough, we have $P \subseteq D_s$ and $P \subseteq \overline{D_s}$ so that $\langle X, z, P, N\rangle \in \Phi_i^D[s]$.

$B\Sigma_n$ model

Let $n \geq 1$. A model $\mathcal{M} \models P^- + I\Sigma_0 + \mathrm{Exp}$ is said to be a $B\Sigma_n$ *model*, if $\mathcal{M} \models B\Sigma_n$ and $\mathcal{M} \not\models I\Sigma_n$. Clause (1) of Theorem 2.1 asserts that there exists a $B\Sigma_n$ model.

An analysis of $B\Sigma_n$ models is needed to study the relationship between fragments of PA and theorems in recursion theory proved under $I\Sigma_n$. A theorem is equivalent to $I\Sigma_n$ over $B\Sigma_n$, if it is provable by $I\Sigma_n$ but fails in every $B\Sigma_n$ model.

A subset I of $\mathcal{M}$ is a *cut*, if I is a nonempty proper initial segment of $\mathcal{M}$ and closed under successor. A partial function on $\mathcal{M}$ is *cofinal* if its range is unbounded in $\mathcal{M}$.

Lemma 2.3 ([5]). *Let $\mathcal{M} \models P^- + B\Sigma_n + \mathrm{Exp}$. Then $\mathcal{M}$ is a $B\Sigma_n$ model if and only if there exists a Σ_n cut I with a Δ_n function $f : I \to \mathcal{M}$ such that f is strictly increasing and cofinal.*

Assume $A \subseteq \mathcal{M}$. A set $G \subseteq A$ is said to be *coded on A* if there is an $\mathcal{M}$-finite set X such that $X \cap A = G$. Let $n \geq 1$. A set $G \subseteq A$ is Δ_n *on A* if G and $A \setminus G$ are both Σ_n.

Lemma 2.4 (Chong and Mourad [5]). *Suppose $\mathcal{M} \models P^- + B\Sigma_n + \mathrm{Exp}$ and $A \subseteq \mathcal{M}$. Then every set bounded and Δ_n on A is coded on A. In particular, any Δ_n set of $\mathcal{M}$ is regular and any bounded Δ_n set is $\mathcal{M}$-finite.*

The above lemma makes more sense for $B\Sigma_n$ models. In a $B\Sigma_n$ model, as its induction principle is weak, classical proofs of a theorem usually do not work. Nevertheless, by Lemma 2.4, some information, which is Δ_n on a Σ_n cut I, is coded on I. Such a code is employed as a parameter in a proof of either the theorem or its negation. An example is Lemma 2.5, which states an induction principle on a Σ_n cut. More examples are seen in Sections 3.1, 3.2 and 4.1.

To fix notations, we use $[a, b]$ ($[a, b)$ respectively), where $a < b \in \mathcal{M}$, to denote the set $\{x \in \mathcal{M} : a \leq x \leq b\}$ ($\{x \in \mathcal{M} : a \leq x < b\}$ respectively). We use 2^I to represent the set $\{x \in \mathcal{M} : x < 2^i \text{ for some } i \in I\}$. If f is a function, we will use $\mathrm{dom}(f)$ to denote the domain of f and use $\mathrm{ran}(f)$ to denote the range of f. (The notations of $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ will have the same meaning for functions f in other sections and chapters).

A number z is said to *code a partial function* if it codes an $\mathcal{M}$-finite set D and D is the graph of a partial function.

Lemma 2.5. *Suppose $\mathcal{M}$ is a $B\Sigma_n$ model with $n \geq 2$, $I \subset \mathcal{M}$ is a Σ_n cut, $a_0 \in \{0, 1\}$, and $h : I \times 2^I \to \{0, 1\}$ is total on $I \times 2^I$ and Σ_n definable. Let $G \subseteq I$ be defined by iterating h:*

$$\begin{aligned} G(0) &= a_0 \\ G(i+1) &= h(i, G \restriction [0, i]), \text{ if } i \in I \text{ and } G \restriction [0, i] \text{ is } \mathcal{M}\text{-finite.} \end{aligned}$$

Then for every $i \in I$, $G(i)$ is uniquely defined. Thus, G is Δ_n on I and coded on I.

Proof. It follows immediately from the definition that

$$\begin{aligned} G(i) = y \leftrightarrow [(i \in I) \wedge \exists z\, (&z \text{ codes a partial function} \wedge \\ &z(0) = a_0 \wedge \forall j < i\, (z(j+1) = h(j, z \restriction [0, j])) \wedge z(i) = y)]. \end{aligned}$$

Therefore, G is Σ_n definable[†] and $\mathrm{dom}(G) \subseteq I$ is a Σ_n cut of $\mathcal{M}$. By $I\Sigma_1$ (which follows from $B\Sigma_n$, for $n \geq 2$), $G(i)$ is unique for any $i \in \mathrm{dom}(G)$.

To see that $\mathrm{dom}(G) = I$, choose an arbitrary $i \in I$, and we only need to show $\mathrm{dom}(G) \supseteq [0, i]$. For any $j \leq i$,

$$\begin{aligned} G(j) = y \leftrightarrow \exists z < 2^{i+1}\, (&z \text{ codes a partial function} \wedge \\ &z(0) = a_0 \wedge \forall k < j\, (z(k+1) = h_i(j, z \restriction [0, k])) \wedge z(j) = y), \end{aligned} \tag{2.2}$$

where $h_i = h \restriction [0, i] \times [0, 2^{i+1}]$. The function h_i is total on $[0, i] \times [0, 2^{i+1}]$, so h_i is Δ_n definable. In addition, h_i is bounded. Lemma 2.4 implies that h_i is $\mathcal{M}$-finite. Then the right hand side of (2.2) is Σ_0. Thus, $\mathrm{dom}(G) \supseteq [0, i]$. □

[†] G may not be Δ_n definable. In fact, $\mathrm{dom}(G) = I$ as we see in the rest of the proof. Therefore, $G(i)$ may be equal to 1 for all $i \in I$ (i.e. as a set, G may be equal to I), in which case G is not Σ_n definable.

Computation and cut

Now we suppose $\mathcal{M}$ is a $B\Sigma_1$ model, I is a Σ_1 cut in $\mathcal{M}$, $a \in \mathcal{M}$ is greater than all numbers in I, and $\{\Phi_e\}_{e\in\mathcal{M}}$ is a recursive enumeration of all r.e. Turing functionals of $\mathcal{M}$. The following two lemmas are straightforward.

Lemma 2.6. *For every nonempty $\mathcal{M}$-finite set X,*

$$X \subseteq I \leftrightarrow \max X \in I,$$
$$X \subseteq \overline{I} \leftrightarrow \min X \in \overline{I},$$

where $\max X$ *(*$\min X$*, respectively) is the maximum (minimum) element in X.*

Lemma 2.7. *For any set $G \subseteq \mathcal{M}$, $I \leq_T G$ if and only if $I \leq_p G$.*

For every $e, s \in \mathcal{M}$, we define

$$\Psi_e^I[s] = \{\langle X, z, n\rangle : \exists P \subseteq I_s \, \exists N \subseteq \overline{I}_s \, (\langle X, z, P, N\rangle \in \Phi_e^I[s] \wedge n = \min(N \cup \{a\}))\}. \tag{2.3}$$

That is, we only consider the minimum element of the negative condition of the computation. Ψ_e^I also satisfies *local downward closure property* with respect to $\Psi_e^I[s]$. If $G = \Phi_e^I$, then for any $\mathcal{M}$-finite set X,

$$X \subseteq G \leftrightarrow \exists n \in \overline{I} \, (\langle X, 1, n\rangle \in \Psi_e^I),$$
$$X \subseteq \overline{G} \leftrightarrow \exists n \in \overline{I} \, (\langle X, 0, n\rangle \in \Psi_e^I).$$

Therefore, we also say that $G \leq_T I$ *via* Ψ_e or $G = \Psi_e^I$. $\{\Psi_e^I\}_{e\in\mathcal{M}}$ can be seen as a *recursive enumeration of all r.e. Turing functionals with oracle I.*

2.2 Second Order Arithmetic

In this section, We recall some useful facts about second order arithmetic. The reader may consult [39, 44] for details.

2.2.1 Language and analytic hierarchy

The language of second order arithmetic is a two sorted language. 0 and 1 are constant symbols. Variables include number variables m, n, k ranging over ω (the set of natural numbers), and set variables X, Y, Z ranging over $\mathcal{P}(\omega)$ (the power set of ω). Quantifiers in front of number variables are number quantifiers, and those in front of set variables are set quantifiers. Functions include $+$ (plus) and $\cdot$ (times).

Atomic formulas are $t = s$, $t < s$ and $X(n) = k$, where t and s are number theoretic terms, n is a number variable and $k = 0$ or 1. Analytical formulas are built up from atomic formulas, propositional connectives and quantifiers as usual. A formula without any quantifier is Σ_0. We say a formula φ is *arithmetic* , if φ only contains number quantifiers. An arithmetic formula φ is Σ^0_n (or Σ_n for short), if there is a Π_{n-1} formula ψ such that φ is in the form of $\exists n\psi(n)$. A formula is Π^0_n (or Π_n for short), if its negation is Σ_n. A formula is Δ_n if it is both Σ_n and Π_n.

Now we define the analytic hierarchy. Let $\Sigma^1_0 = \Pi^1_0$ denote arithmetic formulas. As in the arithmetic hierarchy, a formula φ is Σ^1_n if there is a Π^1_{n-1} formula ψ such that φ is in the form of $\exists X\psi(X)$. Similarly, we define Π^1_n and Δ^1_n formulas.

An arithmetical formula $R(X, m)$ is *recursive* if there is an index e such that Φ^X_e is total and for all X and m,

$$R(X, m) \leftrightarrow \Phi^X_e(m) = 1.$$

Proposition 2.2.1 (Kleene, 1955). *Every analytical formula $\varphi(X, m)$ can be put in one of the following forms:*

$$\begin{array}{lll} A(X,m) & \exists Y\,\forall n\, R(X,Y,m,n) & \exists Z\,\forall Y\,\exists n\, R(X,Y,Z,m,n)\ldots \\ & \forall Y\,\exists n\, R(X,Y,m,n) & \forall Z\,\exists Y\,\forall n\, R(X,Y,Z,m,n)\ldots \end{array}$$

where A is arithmetic and R is recursive.

2.2.2 Hyperarithemtic theory

Kleene' s $\mathcal{O}$ and hyperarithmetic sets

The well ordering $<_\mathcal{O}$ over ω is the smallest subset of ω^2 such that

(i) $1 <_\mathcal{O} 2$.

(ii) $\forall n\,(n$ is in the field of $<_\mathcal{O} \to\ n <_\mathcal{O} 2^n)$.

(iii) $\forall e\,(\Phi_e$ is a total function $\wedge \forall n\,(\Phi_e(n) <_\mathcal{O} \Phi_e(n+1)) \to \forall n\,(\Phi_e(n) <_\mathcal{O} 3 \cdot 5^e))$.

(iv) $\forall i\,\forall j\,\forall k\,(i <_\mathcal{O} j\ \wedge\ j <_\mathcal{O} k\ \to\ i <_\mathcal{O} k)$.

Kleene' s $\mathcal{O}$ is the field of $<_{\mathcal{O}}$. The function $|\quad|_{\mathcal{O}} : \mathcal{O} \to Ord$, where Ord is the class of ordinals, is defined by induction as follows.

$$|1|_{\mathcal{O}} = 0$$
$$|2^n|_{\mathcal{O}} = |n|_{\mathcal{O}} + 1, \quad n \in \mathcal{O}$$
$$|3 \cdot 5^e|_{\mathcal{O}} = \sup_n |\Phi_e(n)|_{\mathcal{O}}, \quad 3 \cdot 5^e \in \mathcal{O}$$

Let $\omega_1^{CK} = \sup\{|n|_{\mathcal{O}} : n \in \mathcal{O}\}$.

Now we iterate the Turing jump through $\mathcal{O}$.

$$H_1 = \emptyset$$
$$H_{2^n} = (H_n)' \quad n \in \mathcal{O}$$
$$H_{3 \cdot 5^e} = \{(n, m) : m \in H_{\Phi_e(n)}, 3 \cdot 5^e \in \mathcal{O}\}$$

A set A is *hyperarithmetic* if $A \leq_T H_n$ for some $n \in \mathcal{O}$.

Theorem 2.8 (Kleene, [39]). *Hyperarithemtic sets are exactly the Δ^1_1 sets.*

Π^1_1 completeness

A Π^1_1 set $A \subset \omega$ is Π^1_1 *complete* if every Π^1_1 set is many-one reducible to A.

Lemma 2.9. *There is a Π^1_1 set that is not Σ^1_1. Thus, if A is Π^1_1 complete, then A is not Σ^1_1.*

Suppose T is a tree. A function $f : T \to Ord$ is *order preserving* if for all $\sigma, \tau \in T$, $\sigma \subsetneq \tau$ implies that $f(\sigma) < f(\tau)$. We say T is *well founded* if $[T] = \{p \in \omega^\omega : \forall n\, (p \upharpoonright n \in T)\}$ is empty.

Lemma 2.10 ([39]). *Suppose $T \subseteq \omega^{<\omega}$ is a recursive tree. Then T is well founded if and only if there is an order preserving function $f : T \to \omega_1^{CK}$.*

Let $\{T_e\}_{e<\omega}$ be a recursive list of all (partial) recursive functions from $\omega^{<\omega}$ to $\{0, 1\}$. For every $\sigma \in \omega^{<\omega}$, we say $T_{e,s}(\sigma) = j$, if $T_e(\sigma)$ converges within s steps and is equal to j. Call T_e *describes* a well founded tree if T_e is a total function and $\{\sigma \in \omega^{<\omega} : T_e(\sigma) = 1\}$ is a well founded tree. Define

$$\mathrm{WF} = \{e < \omega : T_e \text{ describes a well founded tree}\}.$$

Proposition 2.2.2 ([39]). WF *and* $\mathcal{O}$ *are* Π^1_1 *complete. Therefore,* WF *and* $\mathcal{O}$ *are not* Σ^1_1.

Lemma 2.11. *Suppose* $<^*$ *is a* Σ^1_1 *well ordering over* ω. *Then the order type of* $<^*$ *is less than* ω_1^{CK}.

Proof. For the sake of a contradiction, we assume $<^*$ is a Σ^1_1 well ordering of order type at least ω_1^{CK}. Then

$$\begin{aligned} n \in \mathrm{WF} \leftrightarrow\ & (T_n \text{ is total}) \wedge \{\sigma \in \omega^{<\omega} : T_n(\sigma) = 1\} \text{ is a tree} \wedge \\ & \exists f\, (f : \omega^{<\omega} \to \omega \wedge \forall \sigma \forall \tau\, (\sigma \subsetneq \tau \wedge T_n(\tau) = 1 \to f(\sigma) <^* f(\tau))). \end{aligned}$$

That is a contradiction since WF is not Σ^1_1. $\square$

Proposition 2.2.3 ([39]). *Given any linear ordering* R *over* ω, $\mathsf{WO}(R)$, *which states that* R *is a well ordering, is* Π^1_1 *not* Σ^1_1.

Inductive definitions

Suppose $A \subseteq \omega$ and $\Gamma(A) = \{\Gamma_n(A)\}_{n<\omega}$ is a sequence of functions from 2^ω to 2^ω. We define the arithmetic or hyperarithmetic complexity of Γ to be that of the predicate $m \in \Gamma_n(X)$. Γ is *monotonic* if $\Gamma_n(A) \supseteq \Gamma_n(B)$ for all $A \supseteq B$ and every n.

For each $n < \omega$ and ordinal α, define $\Gamma_n^\alpha(A)$ as follows:

$$\begin{aligned} &\Gamma_n^0(A) = A \\ &\Gamma_n^{\alpha+1}(A) = \Gamma_n^\alpha(A) \cup \Gamma_n(\Gamma^\alpha(A)) \\ &\Gamma_n^\lambda(A) = \textstyle\bigcup_{\alpha<\lambda} \Gamma_n^\alpha(A) \quad \lambda \text{ is a limit ordinal.} \end{aligned}$$

Since ω is countable, there is a least countable ordinal $\alpha(\Gamma_n, A)$, such that for all $\alpha \geq \alpha(\Gamma_n, A)$, $\Gamma_n^\alpha(A) = \Gamma_n^{\alpha(\Gamma_n,A)}(A) = \Gamma_n^\infty(A)$. Let the *rank* of $\Gamma_n(A)$ be $|\Gamma_n(A)| = \alpha(\Gamma_n, A)$ and the *rank* of $\Gamma(A)$ be $|\Gamma(A)| =$ the least $\alpha \geq |\Gamma_n(A)|$ for all n. Then $|\Gamma(A)| < \aleph_1$.

Proposition 2.2.4 was originally proved by Spector [48] in 1955. His result applies to any Γ that is Π^1_1. Here we give a different but simpler proof for hyperarithmetic Γ. And that suffices for our discussion in Chapter 5.

Proposition 2.2.4 (Spector). *Suppose Γ is monotonic and hyperarithmetic, and A is hyperarithmetic. Then $|\Gamma(A)| \leq \omega_1^{CK}$. If moreover, $\Gamma_n^\infty(A) = \omega$ for all n, then $|\Gamma(A)| < \omega_1^{CK}$.*

Proof. Note that we only need to prove the proposition for $A = \emptyset$ instead of the general case. For an arbitrary hyperarithmetic A, we consider $\Gamma^*(\emptyset) = \{\Gamma_n^*(\emptyset)\}_{n<\omega}$ defined by $m \in \Gamma_n^*(X)$ if and only if $m \in \Gamma^*(X)$ or $m \in A$. Then Γ^* preserves hyperarithmetic and monotonic properties, and $|\Gamma^*| = |\Gamma(A)|$, whenever $|\Gamma| \geq \omega$.

We may further assume $\forall m\, \Gamma_m(A) = \Gamma_0(A)$ (then $|\Gamma(A)| = |\Gamma_0(A)|$) for the following reason. Let $A^{**} = \{(n, x) : x \in A\}$. Then A^{**} is also hyperarithmetic. For all m and X, define

$$\Gamma_m^{**}(X) = \{(n, x) \in \omega^2 : x \in \Gamma_n(X^{[n]})\},$$

where $X^{[n]} = \{x : (n, x) \in X\}$ is the n^{th} column of X. Then $\Gamma^{**} = \{\Gamma_n^{**}\}_{n<\omega}$ preserves hyperarithmetic and monotonic properties, and $|\Gamma^{**}(A^{**})| = |\Gamma_0^{**}(A^{**})| = |\Gamma(A)|$.

For the rest of this proof, we always assume $A = \emptyset$ and $\forall n\, \Gamma_n(A) = \Gamma_0(A)$ with $\Gamma_0(\emptyset) \neq \emptyset$. We denote $\Gamma_0(A)$ by Γ.

For every $n \in \Gamma^\infty$, we define its rank by $\text{rank}(n) =$ the least α such that $n \in \Gamma^{\alpha+1}$. Now we define a liner order $<^*$ over Γ^∞:

$$m <^* n \leftrightarrow (\text{rank}(m) < \text{rank}(n)) \vee (\text{rank}(m) = \text{rank}(n) \wedge m < n).$$

Then $<^*$ is a well ordering over Γ^∞ and the order type of $<^*$ is at least $|\Gamma|$.

We claim that for all $k \in \Gamma^\infty$,

$$\begin{aligned}(m, n \in \Gamma^\infty \wedge \text{rank}(m), \text{rank}(n) < \text{rank}(k) \wedge m <^* n) \leftrightarrow \\ \exists R\, \exists X\, [R \in \text{LO}_0 \wedge X^{[0]} = \emptyset \wedge \forall i \in F(R)\, [i > 0 \rightarrow \\ (X^{[i]} = \textstyle\bigcup_{jRi} \Gamma(X^{[j]}) \wedge k \notin \bigcup_{jRi} X^{[j]})] \\ \wedge [\exists i \in F(R)\, (n \notin X^{[i]} \wedge m \in X^{[i]}) \\ \vee \forall i \in F(R)\, ((m \in X^{[i]} \leftrightarrow n \in X^{[i]}) \wedge m < n)]], \end{aligned} \tag{2.1}$$

where LO_0 is the collection of all the linear ordering over ω such that 0 is the least element, and $F(R) = \{n < \omega : \exists m < \omega\, (mRn \vee nRm)\}$ is the field of R.

If (2.1) is true, then any initial segment of $<^*$ is a Σ^1_1 well ordering. By Lemma 2.11, $|\Gamma| \leq \omega_1^{CK}$. Suppose $\Gamma^\infty = \omega$ and $|\Gamma|$ is a limit ordinal. Then $m <^* n$ if and only if there exists k such that $\mathrm{rank}(m), \mathrm{rank}(n) < \mathrm{rank}(k)$ and $m <^* n$. By (2.1), $<^*$ is hyperarithmetic and so $|\Gamma| < \omega_1^{CK}$.

It remains to prove our claim (2.1). The direction from left to right is obvious and we only check the direction from right to left by showing R is a well ordering. Suppose R and X satisfy the matrix of the right hand side. Then for every $i \in F(R)$, let $\alpha(i)$ =the least α such that $\Gamma^{\alpha+1} \not\subseteq \bigcup_{jRi} X^{[j]}$ ($\alpha(i)$ exists since $k \in \Gamma^\infty \setminus \bigcup_{jRi} X^{[j]}$). Now we will show that

$$jRi \to \alpha(j) < \alpha(i).$$

For all j, $\bigcup_{lRj} X^{[l]} \supseteq \bigcup_{\alpha<\alpha(j)} \Gamma^{\alpha+1} = \Gamma^{\alpha(j)}$. Therefore, for all jRi, $\bigcup_{jRi} X^{[j]} \supseteq \Gamma^{\alpha(j)+1}$. Then $\alpha(j) < \alpha(i)$. Thus, R is a well ordering. $\square$

Recall in Chapter 1, we defined the Cantor-Bendixson derivative and rank as follows. Let $T \subseteq 2^{<\omega}$ be a tree and $[T] = \{x \in 2^\omega : \forall n\,(x \upharpoonright n \in T)\}$. The Cantor-Bendixson derivative of T, denoted as $\mathrm{CB}(T)$, is

$$\mathrm{CB}(T) = \{\sigma \in 2^{<\omega} : \exists x, y \in [T]\,(x \neq y \text{ extend } \sigma)\}.$$

We may iteratively apply the Cantor-Bendixson derivative through the ordinals, i.e. let $T_0 = T$ and for every $\alpha > 0$, let $T_\alpha = \bigcap_{\beta<\alpha} \mathrm{CB}(T_\beta)$. The least ordinal α such that $T_\alpha = \bigcap_\beta T_\beta$ is the Cantor-Bendixson rank of T (denoted $|T|_{\mathrm{CB}}$).

Consider the Cantor-Bendixson derivative to be on the complement of a tree T. Then this operation is monotone and hyperarithmetic. Thus, by Proposition 2.2.4, $|T|_{\mathrm{CB}} \leq \omega_1^{CK}$. If $\bigcap_\alpha T_\alpha = \emptyset$, then $|T|_{\mathrm{CB}} < \omega_1^{CK}$.

2.2.3 Reverse mathematics

The axioms of second order arithmetic are the following.

(i) Basic axioms: P^-.

(ii) Induction axiom: $(0 \in X \wedge \forall n\,(n \in X \to n+1 \in X)) \to \forall n\,(n \in X)$.

(iii) Comprehension scheme: $\exists X\,\forall n\,(n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is an analytic formula (possibly with parameters).

Subsystems of second order arithmetic included RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0 in a strictly increasing logical strength order. In this thesis, we focus on the last three principles. Each of ACA_0, ATR_0 and Π^1_1-CA_0 includes basic axioms and induction axiom. ACA_0 and Π^1_1-CA_0 contain the comprehension schema restricted to arithmetic formulas and Π^1_1 formulas respectively. ATR_0 is ACA_0 plus the following principle

$$\forall R\,(\mathsf{WO}(R) \to \exists X\, H_\theta(R,X)),$$

where θ is arithmetical and $H_\theta(R,X)$ is a formula which says that R is a linear order, and for all i in the field of R, $X^{[i]} = \{n : \theta(n, \bigcup_{jRi}(\{j\} \times X^{[j]}))\}$.

Proposition 2.2.5 (Simpson, [44])**.** *The following are equivalent over ACA_0.*

1. Π^1_1 *comprehension.*

2. *For any sequence of trees $\{T_k\}_{k<\omega}$, $T_k \subseteq \omega^{<\omega}$, there exists a set X such that $\forall k(k \in X \leftrightarrow T_k$ has a path).*

Recall the Cantor-Bedixson kernel we defined in Chapter 1 (see also Section 2.2.2).

Proposition 2.2.6 (Simpson, [44])**.** *Over ACA_0,*

(1) ATR_0 implies that for any tree $T \subseteq \omega^\omega$ either $\mathrm{Ker}_{\mathrm{CB}}(T) = \emptyset$ or T contains a nonempty perfect subtree.

(2) Π^1_1 comprehension holds if and only if $\mathrm{Ker}_{\mathrm{CB}}(T)$ exists for every $T \subseteq \omega^\omega$.

2.3 α-Recursion

We recall some basic definitions and results in α-recursion theory. A detailed introduction to the subject can be found in [2, 31, 32, 39].

2.3.1 Admissible ordinals

The language of α-recursion theory is the language of Zermelo-Fraenkel set theory (ZF). Formulas and Levy hierarchy of formulas are defined as usual. Given a formula

φ, we write $\mu x\, \varphi(x)$ to denote the least ordinal x such that $\varphi(x)$ holds, and $[x, y]$ ($[x, y)$ respectively) to denote $\{z : x \leq z \leq y\}$ ($\{z : x \leq z < y\}$ respectively). An ordinal α is said to be Σ_1 *admissible* if L_α satisfies Σ_1 replacement.

Suppose α is a Σ_1 admissible ordinal. A set is *α-r.e.*, if it is Σ_1 definable over L_α. If the set is Δ_1 definable over L_α, then it is *α-recursive*. A set is α-finite if it is in L_α. A set is *regular* if its intersection with any α-finite set is still α-finite. For each nonempty α-finite set $C \subset \alpha$, define $\sup C = \mu y\, \forall x \in C(x < y)$, $\max^* C = \mu y\, \forall x \in C(x \leq y)$, $\min C = \mu x\, (x \in C)$. Given a non-empty α-finite set C, the least element $\min C$ always exists, however there may not be the maximal element $\max C$ in C. If $\max C$ exists in C, then $\max^* C = \max C$; if there is no maximal element in C, then by their definitions $\max^* C = \sup C$.

Suppose $\beta < \delta \leq \alpha$. β is said to be *δ-stable*, if $L_\beta \prec_1 L_\delta$. β is said to be an *α-cardinal* if there is no α-finite one-to-one correspondence between β and any $\gamma < \beta$. Every α-cardinal greater than ω is α-stable.

Each α-finite set has an α-cardinality. The α-cardinality of an α-finite set C is denoted by $|C|_\alpha$.

Recall that there exists a one-one, α-recursive (total) function f that maps α onto L_α. That is, α-finite sets can be effectively coded as ordinals. Thus, there is no harm in identifying α-finite sets with ordinals below α, and identifying subsets of L_α with subsets of α. From now on, by an α-r.e. set without specification, we always mean an α-r.e. subset of α. Also, f yields a recursive bijection from α^2 to α. Fix such a bijection, and denote it by $\langle \cdot, \cdot \rangle$.

It is straightforward to verify that there is a Gödel numbering of α-r.e. sets, which we denote as $\{W_e\}_{e<\alpha}$. For an arbitrary numbering $\{A_e\}_{e<\alpha}$ and any stage $\eta < \alpha$, the set $A_{e,\eta}$ is defined to be the collection of elements which are less than η and are enumerated into A_e by stage η. In other words, suppose $x \in A_e$ if and only if $\exists y\, \varphi(e, x, y)$, where φ is Σ_0, then $A_{e,\eta} = \{x < \eta : \exists y < \eta\, \varphi(e, x, y)\}$.

2.3.2 Σ_n projectum and cofinality

Let $n \geq 1$. The *Σ_n projectum of* α, denoted by $\sigma np\,(\alpha)$, is defined to be the least ordinal β such that there is a Σ_n (partial) function from β onto α.

Theorem 2.12 (Jensen, [21]). *$\sigma np\,(\alpha)$ is the least β such that some Σ_n (over L_α)*

subset of β is not α-finite. Thus, if $I \subset \alpha$ is an α-finite set such that $|I|_\alpha < \sigma np\,(\alpha)$, then each Σ_n subset of I is α-finite.

The Σ_n *cofinality of* $\delta \leq \alpha$, denoted by $\sigma ncf\,(\delta)$, is defined to be

$$\mu\gamma\,\exists f\left[\,f : \gamma \xrightarrow{\text{one-one}} \delta, (\text{total on } \gamma),\ \text{is } \Sigma_n \text{ over } L_\alpha \text{ and } f \text{ is cofinal (in } \delta)\right].$$

It is obvious that $\sigma np\,(\alpha)$ and $\sigma ncf(\alpha)$ are α-cardinals.

2.3.3 Tameness

The notion of tameness was introduced by Lerman [30]. It has many applications, especially in constructions involving Σ_2 functions.

Let $f : \beta \to \alpha$ for some $\beta \leq \alpha$. Then f is said to be *tame* Σ_2 if it is total and there exists an α-recursive f' such that

$$\forall\gamma < \beta\,\exists\tau\,\forall x < \gamma\,\forall\eta > \tau\,(f'(\eta, x) = f(x)).$$

Such an f' is said to *tamely generate* f. The tameness of f refers to the way f' approximates f on proper initial segments of $\mathrm{dom}(f)$. A Σ_2 function need not be tame Σ_2.

The *tame* Σ_2 *projectum of* α, denoted by $t\sigma 2p\,(\alpha)$, is defined to be

$$\mu\beta\,\exists f\left[\,f : \beta \xrightarrow[\text{onto}]{\text{one-one}} \alpha, (\text{total on } \beta), \text{ is tame } \Sigma_2\right].$$

A set is *tame* Σ_2 if its characteristic function is tame Σ_2. Analogous to $\sigma 2p\,(\alpha)$, we have

Lemma 2.13 (Simpson, [2, 31])**.** *$t\sigma 2p\,(\alpha)$ is the least β such that not every tame Σ_2 subset of β is α-finite.*

Lemma 2.14 ([31])**.** *For all $\delta \leq \alpha$, there exists a strictly increasing tame Σ_2 cofinal function $f : \sigma 2cf(\delta) \to \delta$. Every Σ_2 function from $\vartheta \leq \sigma 2cf(\alpha)$ to α is tame.*

Corollary 2.15 ([2, 31, 39])**.** (1) $\omega \leq \sigma 2cf(\alpha) \leq t\sigma 2p\,(\alpha) \leq \sigma 1p\,(\alpha) \leq \alpha$,

(2) $\sigma 2cf(\sigma 1p\,(\alpha)) = \sigma 2cf(t\sigma 2p\,(\alpha)) = \sigma 2cf(\alpha)$.

Corollary 2.16 (Local Σ_2 Replacement)**.** *Let $a < \sigma 2cf(\alpha)$ and $R \subseteq \alpha \times \alpha$ be a Σ_2 relation. Then*

$$\forall x < a\, \exists y\, R(x, y) \to \exists z\, \forall x < a\, \exists y < z\, R(x, y). \tag{2.1}$$

Moreover,

$$\forall x < a\, \exists y < \sigma 2cf(\alpha)\, R(x, y) \to \exists z < \sigma 2cf(\alpha)\, \forall x < a\, \exists y < z\, R(x, y). \tag{2.2}$$

Proof. (2.1) is immediate from the definition of $\sigma 2cf(\alpha)$. By Lemma 2.14, it is straightforward to get (2.2) from (2.1). We omit the details here. $\square$

Chapter 3

Degree Structures Without Σ_1 Induction

3.1 Proper D-r.e. Degree and Σ_1 Induction

Cooper [12] proved the existence of a proper d-r.e. degree in the standard model ω, using a $\mathbf{0}'$-priority construction. As we see in Section 3.1.1, his proof remains valid under the weaker assumption of Σ_1 induction. The remain problem is therefore the converse: is Σ_1 induction necessary for the existence of a proper d-r.e. degree? In Section 3.1.2, we give a negative answer to this question.

3.1.1 $I\Sigma_1$ implies the existence of a proper d-r.e. degree

Theorem 3.1 (Kontostathis [26]). *Let $\mathcal{M} \models P^- + I\Sigma_1$. Then there exists a d-r.e. set D such that $D \not\equiv_T W$ for any r.e. set W.*

Proof. Suppose $\mathcal{M} \models P^- + I\Sigma_1$. Let $\{W_e\}_{e\in\mathcal{M}}$, $\{\Phi_e\}_{e\in\mathcal{M}}$, and functions $\langle\cdot,\ldots,\cdot\rangle$ be as above. The objective in the construction is to meet, for all $e, i, j \in \mathcal{M}$, the requirement

$$R_{\langle e,i,j\rangle} : D \neq \Phi_i^{W_e} \text{ or } W_e \neq \Phi_j^D.$$

In $\mathcal{M}$, we can perform Cooper's construction by Σ_1 induction. Moreover, by Σ_1 induction again, each requirement $R_{\langle e,i,j\rangle}$ is injured at most $3^{\langle e,i,j\rangle} - 1$ times. To show $R_{\langle e,i,j\rangle}$ is satisfied, we consider

$$A_{\langle e,i,j\rangle} = \{\langle e', i', j', k\rangle : \langle e', i', j'\rangle < \langle e, i, j\rangle \wedge R_{\langle e',i',j'\rangle} \text{ receives attention at least } k \text{ times}\}$$

is bounded and Σ_1, therefore is $\mathcal{M}$-finite by Lemma 2.2. Thus the range of the recursive function $f : A_{\langle e,i,j\rangle} \to \mathcal{M}$ defined by

$$f(\langle e', i', j', k\rangle) = \mu s\, (R_{\langle e',i',j'\rangle} \text{ receives attention at least } k \text{ times by stage } s)$$

is bounded. Suppose $\text{ran}(f) \subseteq [0, s)$. Then after stage s, $R_{\langle e,i,j\rangle}$ is never be injured and receives attention at most twice and $R_{\langle e,i,j\rangle}$ is satisfied eventually. $\square$

Remark. Let $D = A \setminus B$, where A and B are r.e. such that $B \subseteq A$. $I\Sigma_1$ implies that A and B are regular. Then by $I\Sigma_1$,

$$X \subseteq D \leftrightarrow X \subseteq A \wedge X \subseteq \overline{B}, \tag{3.1}$$

$$X \subseteq \overline{D} \leftrightarrow \exists X_1 \subseteq \overline{A}\, \exists X_2 \subseteq B\, (X = X_1 \cup X_2), \tag{3.2}$$

for every $\mathcal{M}$-finite set X. Hence, $P^- + I\Sigma_1$ is sufficient to show that every d-r.e. set is Turing reducible to $\emptyset'$ and that there is a proper d-r.e. degree below $\mathbf{0'}$ by Theorem 3.1.

Suppose $\mathcal{M}$ is a $B\Sigma_1$ model. Then (3.1) remains valid but (3.2) fails: if $X \subseteq \overline{D}$ and A is not regular, then $X_2 = X \cap A$ (a subset of B) may not be $\mathcal{M}$-finite. For this reason, D may not be reducible to $\emptyset'$. The observation here will be important for our construction of a proper d-r.e. degree (not below $\mathbf{0'}$) in a $B\Sigma_1$ model.

3.1.2 $B\Sigma_1$ implies the existence of a proper d-r.e. degree

Theorem 3.2. *If $\mathcal{M} \models P^- + B\Sigma_1 + \text{Exp}$, then there is a proper d-r.e. degree in $\mathcal{M}$.*

By Theorem 3.1, we only need to show the existence of a proper d-r.e. degree in any $B\Sigma_1$ model $\mathcal{M}$. Suppose $I \subseteq \mathcal{M}$ is a Σ_1 cut, a is an upper bound of all numbers in I, and $f : I \to \mathcal{M}$ is a Δ_1 strictly increasing cofinal function.

The difficulty of applying the $\mathbf{0'}$-priority method in a $B\Sigma_1$ model is as follows. Fix a requirement R_e and suppose each requirement $R_{e'}$, $e' < e$ is injured only $\mathcal{M}$-finitely many times. Then the set $A_e = \{\langle e', n\rangle : e' < e \wedge R_{e'}$ requires attention at least n times$\}$ is r.e. Without $I\Sigma_1$, the enumeration of A_e may not terminate at any stage s and there may not be any opportunity to satisfy R_e.

Proof of Theorem 3.2. We will construct a d-r.e. set D such that $D \not\leq_T \emptyset'$ in stages along the cut I, without the use of a priority argument. (At any stage $i \in I$, we compute $f(i)$ many steps.) For every $e \in \mathcal{M}$, the *requirement* is

$$Q_e : D \neq \Phi_e^{\emptyset'}.$$

The strategy of meeting a requirement Q_e is to attach a *witness* $X_e = [ea, (e+1)a)$ to Q_e and to look for a stage $i > 0$ such that

$$\Phi_e^{\emptyset'}[f(i-1)] \restriction X_e = \emptyset.$$

If no such stage exists, Q_e is automatically satisfied with witness X_e. If i exists, then we enumerate $ea + i$ into D at stage i. Now consider whether there is a stage $j > i$ such that

$$\Phi_e^{\emptyset'}[f(j-1)] \restriction X_e = \{ea + i\}.$$

If there is no such stage j, then Q_e is satisfied, as $\Phi_e^{\emptyset'} \restriction X_e \neq \{ea + i\} = D \restriction X_e$. If j exists, then we extract $ea + i$ from D at stage j, look for a stage $k > j$ such that

$$\Phi_e^{\emptyset'}[f(k-1)] \restriction X_e = \emptyset,$$

and repeat the strategy over again.

Notice that different requirements here do *not conflict* with one another and this strategy allows us to accommodate all requirements simultaneously.

According to the strategy, the function $\langle i, e\rangle \mapsto \langle f(i), D_{f(i)} \restriction X_e\rangle$ is recursive, where $e \in \mathcal{M}$ and $i \in I$. Thus by $B\Sigma_1$ which is equivalent to $I\Delta_1$ according to Theorem 2.1, we can easily prove the following results:

(i) For every $x \in \mathcal{M}$, x is enumerated into D at most once and is extracted from D at most once,

(ii) For every $e \in \mathcal{M}$ and $i \in I$, there is at most one element in $D_{f(i)} \restriction X_e$.

Therefore, D is d-r.e. and $D \restriction X_e$ contains at most one element.

For the sake of a contradiction, suppose Q_e is not satisfied.

Case 1. $D \upharpoonright X_e = \Phi_e^{\emptyset'} \upharpoonright X_e = \emptyset$. Suppose $i \in I$ is a stage such that there is a computation $\langle X_e, 0, P, N\rangle \in \Phi_e^{\emptyset'}[f(i-1)]$, where P has been enumerated as a subset of $\emptyset'$ by stage $f(i-1)$ and $N \subseteq \overline{\emptyset'}$. Let $j \geq i$ be the first stage by which the element in $D_{f(i)} \upharpoonright X_e$, if any, is extracted from D (such a stage exists for $D \upharpoonright X_e = \emptyset$). Then at stage $j+1$, the element $ea+j+1$ is enumerated into D and will never be extracted, contradicting the assumption that $D \upharpoonright X_e = \emptyset$.

Case 2. $D \upharpoonright X_e = \Phi_e^{\emptyset'} \upharpoonright X_e = \{ea+i\}$. Then $D \upharpoonright X_e = D_{f(i)} \upharpoonright X_e = \Phi_e^{\emptyset'}[f(j)] \upharpoonright X_e$ for some $j > i$. Thus $ea+i$ is extracted from D at stage j. Again, that is a contradiction. $\square$

The cut I plays a significant role in the proof of Theorem 3.2. It exploits the recursive cofinal function f and *compresses* time and space to achieve the diagonalization against $\Phi_e^{\emptyset'}$ for every e. Notice that the set D constructed in the proof of Theorem 3.2 is unbounded. With the aid of I, we can actually further compress the space and construct a bounded d-r.e. set $D \not\leq_T \emptyset'$.

3.1.3 Bounded sets

Let $\mathcal{M}, I, a, f$ be as in Section 3.1.2. Suppose $D = A \setminus B$, where A and B are bounded r.e. sets and $B \subseteq A$. Let b be an upper bound of all elements in A. We may further assume that $A_{f(0)} = B_{f(0)} = \emptyset$ and for all $i \in I$, $B_{f(i+1)} \subseteq A_{f(i)}$ (this is to ensure that, along the time axis I, none appears in B before it is enumerated in A). Since the set

$$H = \{(x,i) : x < b,\, i \in I,\, x \in (A_{f(i)} \setminus A_{f(i-1)}) \cup (B_{f(i)} \setminus B_{f(i-1)})\}, \tag{3.3}$$

which records the stages of enumeration, is Δ_1 on $[0,b) \times I$, H is coded by Lemma 2.4. Suppose $\hat{H} \subseteq [0,b) \times [0,a)$ is a code of H satisfying for every $x < b$, there are exactly two i's such that $(x,i) \in \hat{H}$. Define $i_x = \min\{i < a : (x,i) \in \hat{H}\}$ and $j_x = \max\{i < a : (x,i) \in \hat{H}\}$. Then for every $x < b$,

$$j_x \in I \to x \in \overline{D}, \quad (i_x \in I \wedge j_x \in \overline{I}) \to x \in D, \quad i_x \in \overline{I} \to x \in \overline{D}.$$

Fix $e \in \mathcal{M}$. To ensure $D \neq \Phi_e^{\emptyset'}$, we need to implement a diagonalization strategy as in Theorem 3.2. Given any r.e. set R, we say x that *escapes from computation* Φ_e^R *at stage* s, if $x \in B$ and for every computation of the form $\langle \{x\}, 0, P, N\rangle$ in

$\Phi_e^R[s]$, $N \cap R \neq \emptyset$. Note that if $X \subseteq \overline{D}$ is $\mathcal{M}$-finite such that for every stage s, there is an $x \in X$ such that x escapes from computation Φ_e^R at stage s, then $\Phi_e^R \upharpoonright X \neq D \upharpoonright X = \emptyset$ by the local downward closure property of Φ_e^R. This idea leads to the following Lemma.

Lemma 3.3. *If R is r.e. and $\Phi_e^R = D$, then for some stage s, there is no $x < b$ with $f(i_x) \geq s$ such that x escapes from computation Φ_e^R at stage s.*

Proof. We prove this by contradiction. Suppose $\Phi_e^R = D$ and for every $i \in I$, there is an $x < b$ with $i_x \geq i$ such that x escapes from computation Φ_e^R at stage $f(i)$. Then the function $\alpha : I \to I^2$ defined by $i \mapsto (i_x, j_x)$ where $x < b$ is the first enumerated number satisfying $i_x \geq i$, $j_x \in I$ and x escapes from computation Φ_e^R at stage $f(i)$. The function α is total on I by assumption.

Since α is recursive, α is coded on I^3 by a function $\hat{\alpha} : [0, a) \to [0, a)^2$. We denote the first coordinate of $\hat{\alpha}(i)$ by $\hat{\alpha}_1(i)$ and the second by $\hat{\alpha}_2(i)$. We may further assume that for every $i < a$, $\hat{\alpha}_2(i) > \hat{\alpha}_1(i) \geq i$.

Now let $X = \{x < b : (i_x, j_x) \in \operatorname{ran}(\hat{\alpha})\}$. Then for every $x \in X$,

Case 1. There is an $i \in I$ such that $\hat{\alpha}(i) = (i_x, j_x)$. Then $\alpha(i) = (i_x, j_x)$. Thus, $i_x, j_x \in I$.

Case 2. There is an $i \in \overline{I}$ such that $\hat{\alpha}(i) = (i_x, j_x)$. Then $i_x, j_x \in \overline{I}$ since $i_x \geq i$.

Therefore, $X \subseteq \overline{D}$. Moreover, by the definition of α, for every stage $f(i)$, there is an $x \in X$ such that x escapes from computation Φ_e^R at stage $f(i)$, and so $\Phi_e^R \upharpoonright X \neq \emptyset$, contradicting the assumption that $\Phi_e^R = D$. □

Corollary 3.4. *If $\Phi_e^{\emptyset'} = D$, then for some stage s, there is no $x < b$ such that x escapes from computation $\Phi_e^{\emptyset'}$ at stage s and $f(i_x) \geq s$.*

The above definition can be generalized to computations $\{\Psi_e\}_{e \in \mathcal{M}}$. We say an element x *escapes from computation Ψ_e^I at stage s*, if $x \in B$ and $n \in I$ for all $\langle \{x\}, 0, n \rangle$ in $\Psi_e^I[s]$.

Corollary 3.5. *If $\Psi_e^I = D$, then for some $i \in I$, there is no $x < b$ such that x escapes from computation Ψ_e^I at stage $f(i)$ and $i_x \geq i$.*

Lemma 3.6. *$D \leq_T \emptyset'$ if and only if $D \leq_T I$.*

Proof. We only need to show the " only if" part. Since every $\mathcal{M}$-finite set X,

$$X \subseteq D \leftrightarrow (X \subseteq [0,b) \wedge X \subseteq D),$$
$$X \subseteq \overline{D} \leftrightarrow (X \cap [0,b) \subseteq \overline{D}),$$

we only need to consider subsets of $[0,b)$ in the computation of D.

Suppose $D = \Phi_e^{\emptyset'}$. Define G to be a set that codes the approximation of $\Phi_e^{\emptyset'}$ as follows:

$$G = \{(X,i,j,z) : X \subseteq [0,b) \wedge i < j \in I \wedge (z = 0 \vee z = 1) \wedge \\ \exists P \, \exists N \, (\langle X, z, P, N\rangle \in \Phi_e^{\emptyset'}[f(i)] \wedge N \subseteq \overline{\emptyset'_{f(j)}})\}.$$

That is, $(X,i,j,z) \in G$ if and only if the computation $\Phi_e^{\emptyset'}(X)[f(i)] = z$ is still valid at stage $f(j)$. Since G is Δ_1 on $[0,2^b) \times I \times I \times [0,2)$, by Lemma 2.4, G is coded by $\hat{G} \subseteq [0,2^b) \times [0,a) \times [0,a) \times [0,2)$.

Suppose $X \subseteq [0,b)$. If $X \subseteq D$, then there is a quadruple $\langle X, 1, P, N\rangle \in \Phi_e^{\emptyset'}[f(i)]$ such that $N \subseteq \overline{\emptyset'}$. Thus, for every $j > i$, $j \in I$, we have $N \subseteq \overline{\emptyset'_{f(j)}}$ and $(X,i,j,1) \in \hat{G}$. Since I is not $\mathcal{M}$-finite,

$$u = \sup\{j < a : \forall j' \, (i < j' < j \rightarrow (X,i,j',1) \in \hat{G})\} \in \overline{I}. \tag{3.4}$$

Therefore,

$$\exists i \in I \, \exists u \in \overline{I} \, \forall j \, (i < j < u \rightarrow (X,i,j,1) \in \hat{G}). \tag{3.5}$$

Conversely, if (3.5) holds, then there is some $\langle X, 1, P, N\rangle \in \Phi_e^{\emptyset'}[f(i)]$ such that $N \subseteq \overline{\emptyset'}$. Thus, for every $\mathcal{M}$-finite set $X \subseteq [0,b)$,

$$X \subseteq D \leftrightarrow \exists i \in I \, \exists u \in \overline{I} \, \forall j \, (i < j < u \rightarrow (X,i,j,1) \in \hat{G}),$$

and similarly,

$$X \subseteq \overline{D} \leftrightarrow \exists i \in I \, \exists u \in \overline{I} \, \forall j \, (i < j < u \rightarrow (X,i,j,0) \in \hat{G}). \qquad \square$$

To construct a bounded d-r.e. set $D \not\leq_T \emptyset'$, by Lemma 3.6, it is enough to ensure that $D \not\leq_T I$: For each i and e, if there is an $x \in D$ with $i_x \geq i$ such that x escapes from Ψ_e^I at stage $f(i)$, then $D \neq \Psi_e^I$ by Corollary 3.5.

Lemma 3.7. *For every $e, s \in \mathcal{M}$, if $\Psi_e^I = D$, then the set*

$$J_{e,s} = \{j \in I : \exists x < b\, \exists i \in I\, \exists n \in \overline{I}\, (\langle \{x\}, 0, n\rangle \in \Psi_e^I[s] \wedge f(i) \leq s \wedge i = i_x \wedge j = j_x)\}$$

is bounded in I. Moreover, there is a recursive function $\beta : \mathcal{M}^2 \to \mathcal{M}$ such that whenever $\Psi_e^I = D$, $\beta(e,s) \in I$ is an upper bound of all elements in $J_{e,s}$.

Proof. Fix s and e. Let $i^* \in I$ be the largest i such that $f(i^*) \leq s$. Then

$$\begin{aligned} J_{e,s} &= \{j \in I : \exists x < b\, \exists i \leq i^*\, \exists n \in \overline{I}\, (\langle \{x\}, 0, n\rangle \in \Psi_e^I[s] \ \wedge\ i = i_x \ \wedge\ j = j_x)\} \\ &\subseteq \{j < a : \exists x < b\, \exists i \leq i^*\, \exists n > j\, (\langle \{x\}, 0, n\rangle \in \Psi_e^I[s] \ \wedge\ i = i_x \ \wedge\ j = j_x)\}. \end{aligned} \tag{3.6}$$

We denote the set in the second line of (3.6) by $\tilde{J}_{e,s}$, which is $\mathcal{M}$-finite. Let $\beta(e,s) = \sup \tilde{J}_{e,s}$. We only need to show that if $\Psi_e^I = D$, then $\tilde{J}_{e,s} \subset I$.

Suppose $\Psi_e^I = D$ and $\langle \{x\}, 0, n\rangle \in \Psi_e^I$, where $x < b$, $i_x \leq i^*$ and $j_x < n$. Since $i_x \in I$, $x \in A$. If $n \in \overline{I}$, then $x \in \overline{D}$, and so $x \in B$ with $j_x \in I$. If $n \in I$, then $j_x \in I$ since $j_x < n$. In any case $j_x \in I$, so $\tilde{J}_{e,s} \subset I$. □

Proof of Theorem 3.2 (bounded set D). I, a, f are defined as above. Let

$$\begin{aligned} \hat{H} = \{(x,i) \in [0, 2^a) \times [0, a) :\ & \text{If the } \mathcal{M}\text{-finite set } X \text{ represented by the binary} \\ & \text{expansion of } x \text{ has at least two elements, then } i = \min X \text{ or} \\ & i = \max X, \text{and if } X \text{ has less than two elements, then } i = 0 \text{ or } i = 1\}. \end{aligned}$$

That is, for every pair $(i,j) \in [0,a) \times [0,a)$ with $i < j$, we have some $x < 2^a$ such that $(i,j) = (i_x, j_x)$, where

$$i_x = \min\{i : (x,i) \in \hat{H}\}, \quad j_x = \max\{i : (x,i) \in \hat{H}\}.$$

Then $\hat{H}$ codes the enumeration of a d-r.e. set D:

$$D = \{x < 2^a : i_x \in I, j_x \in \overline{I}\}.$$

Let β be defined as in Lemma 3.7.

Now we claim that $D \not\leq_T I$, so $D \not\leq_T \emptyset'$ by Lemma 3.6. For the sake of contradiction, suppose $D = \Psi_e^I$. For each $i \in I$, let $x(i)$ be the least such that $(i_{x(i)}, j_{x(i)}) = (i, \beta(e, f(i)) + 1)$. By Lemma 3.7, $x(i)$ escapes from Ψ_e^I at stage $f(i)$. Then by Lemma 3.5, $D \neq \Psi_e^I$. That is a contradiction. □

3.1.4 $B\Sigma_1 + \neg I\Sigma_1$ implies d-r.e. degrees below $\mathbf{0}'$ are r.e.

In Sections 3.1.2 and 3.1.3, it was shown that in a $B\Sigma_1$ model, there is a proper d-r.e. degree not below $\mathbf{0}'$. In this section, we prove that in any $B\Sigma_1$ model, it is impossible to find a proper d-r.e. degree below $\mathbf{0}'$.

Let $\mathcal{M}, f, I, a$ be as in Section 3.1.2 and 3.1.3. Let $D = A \setminus B$, where A and B are r.e. (may not be bounded) and $B \subseteq A$. As before, we may assume that $A_{f(0)} = B_{f(0)} = \emptyset$ and for all $i \in I$, $B_{f(i+1)} \subseteq A_{f(i)}$. If D is recursive, then clearly $\deg(D)$ is r.e. For the rest of this section, we always assume that $D = \Phi_e^{\emptyset'}$ is not recursive. The object is to construct an r.e. set $W \equiv_T D$.

3.1.5 Regular sets

We first consider the case when D is regular. The idea of considering regular and non-regular sets can be traced back to Chong and Mourad [5].

Lemma 3.8. *If $D = \Phi_e^{\emptyset'}$ is regular, then $D \leq_T I$.*

Proof. Suppose D is regular. The method here is similar to that in Lemma 3.6. For each $k \in I$ and stage s, we say (E_0, E_1) is a *partition of* $[0, f(k))$ *at stage* s, if

(i) $E_0 \cup E_1 = [0, f(k))$, $E_0 \cap E_1 = \emptyset$, and

(ii) There are $\mathcal{M}$-finite sets P_0, P_1, N_0, N_1 such that $\langle E_0, 0, P_0, N_0\rangle, \langle E_1, 1, P_1, N_1\rangle \in \Phi_e^{\emptyset'}[f(i)]$.

Note that at each stage $f(i)$, there is at most one partition of $[0, f(k))$. Let

$$G = \{(i,j,k) \in I^3 : i \leq j \wedge \text{There are partitions of } [0, f(k)) \text{ at stage } f(i) \text{ and } f(j) \wedge \text{The two partitions are equal}\}.$$

By Lemma 2.4, G is coded on I^3. Suppose $E_{0,k} = \overline{D} \cap [0, f(k))$ and $E_{1,k} = D \cap [0, f(k))$, then there is some $i \in I$ such that

$$\exists P_0, P_1 \subseteq \emptyset' \, \exists N_0, N_1 \subseteq \overline{\emptyset'} \, (\langle E_{0,k}, 0, P_0, N_0\rangle \in \Phi_e^{\emptyset'}[f(i)] \wedge \langle E_{1,k}, 1, P_1, N_1\rangle \in \Phi_e^{\emptyset'}[f(i)]).$$

That is, at stage $f(i)$, Φ_e *correctly computes a partition of* $[0, f(k))$. Then for any stage $s \geq f(i)$, the partition of $[0, f(k))$ at stage s must be $(E_{0,k}, E_{1,k})$, so $(i,j,k) \in \hat{G}$

for all $j \geq i$. Thus for any $k \in I$,

$$E_{0,k} = \overline{D} \cap [0, f(k)) \wedge E_{1,k} = D \cap [0, f(k)) \rightarrow \exists i \in I\, \exists i' \in \overline{I}\, \forall j\, ((E_{0,k}, E_{1,k}) \text{ is a partition of } [0, f(k)) \text{ at stage } f(i) \wedge (i \leq j \leq i' \rightarrow (i, j, k) \in \hat{G})). \quad (3.7)$$

Now suppose the conclusion in (3.7) holds for (E_0, E_1), and we prove the hypothesis in (3.7). Thus, the hypothesis and conclusion in (3.7) are actually equivalent and $D \leq_T I$.

For the sake of a contradiction, we assume that $\overline{D} \cap [0, f(k)) \neq E_0$, $D \cap [0, f(k)) \neq E_1$, $i \in I$, (E_0, E_1) is a partition of $[0, f(k))$ at stage $f(i)$, $i' \in \overline{I}$, and $\forall j\, (i \leq j \leq i' \rightarrow (i, j, k) \in \hat{G})$. Let $\tilde{j} > i$ such that $\tilde{j} \in I$ and Φ_e correctly computes a partition of $[0, f(k))$ at stage $f(\tilde{j})$. Then $(i, \tilde{j}, k) \notin \hat{G}$. That is a contradiction. $\square$

Lemma 3.9. *If $D = \Phi_e^{\emptyset'}$ is regular and non-recursive, then $D \geq_T I$.*

Proof. Consider the following set:

$$G_0 = \{(i, j, k) \in I^3 : i < j < k \wedge \exists x \in A_{f(i)}\, (x \in B_{f(k)} \setminus B_{f(j)})\},$$

i.e. $f(k)$ is a stage that we find $A_{f(i)} \setminus B_{f(j)}$ is not a subset of D. By Lemma 2.4, G_0 can by coded on I^3 by a set $\hat{G}_0 \subseteq [0, a)^3$ such that for every $(i, j, k) \in \hat{G}_0$, $i < j < k$ and for each $i < j \in I$, there is some $k < a$ such that $(i, j, k) \in \hat{G}_0$. (If $(i, j, k) \in I^2 \times [0, a)$ and $A_{f(i)} \setminus B_{f(j)} \subseteq D$, then $k \in \overline{I}$.)

Suppose the set $C_0 = \{k < a : \exists i, j \in I\, (A_{f(i)} \setminus B_{f(j)} \subseteq D \wedge (i, j, k) \in \hat{G}_0)\} \subseteq \overline{I}$ is *unbounded in* $\overline{I}$, i.e. no $i' \in \overline{I}$ is a lower bound of numbers in C_0, then

$$i' \in \overline{I} \leftrightarrow \exists k < i'\, (k \in C_0)$$

It follows that $\overline{I}$ is r.e. in D. Hence $I \leq_p D$. By Lemma 2.7, $I \leq_T D$.

Now suppose C_0 is bounded in $\overline{I}$. Then there is an $i' \in \overline{I}$ such that

$$\forall i, j \in I\, (A_{f(i)} \setminus B_{f(j)} \subseteq D \leftrightarrow \exists k > i'\, ((i, j, k) \in \hat{G}_0)).$$

Furthermore, since D is regular, for every $i \in I$ there is $j \in I$ such that $A_{f(i)} \setminus B_{f(j)} \subseteq D$. Thus, $D = \{x : \exists i, j \in I\, \exists k > i'\, ((i, j, k) \in \hat{G}_0 \wedge x \in A_{f(i)} \setminus B_{f(j)})\}$ is r.e. In that case, by modifying the enumeration of D, we may assume that $B = \emptyset$. Let

$$G_1 = \{(i, j, k) \in I^3 : i < j \wedge \exists x < f(k)\, (x \in A_{f(j)} \setminus A_{f(i)})\},$$

i.e. $f(j)$ is a stage that we find $A_{f(i)} \upharpoonright [0, f(k))$ is not $D \upharpoonright [0, f(k))$. Again G_1 is coded by $\hat{G}_1 \subseteq [0, a)$ in I^3. We also assume that for all $i < k \in I$, there is some j such that $(i, j, k) \in \hat{G}_1$. (If $A_{f(i)} \upharpoonright [0, f(k)) = D \upharpoonright [0, f(k))$ and $(i, j, k) \in \hat{G}_1$, then $j \in \overline{I}$.)

Suppose $C_1 = \{j : \exists i, k \in I\, (A_{f(i)} \upharpoonright [0, f(k)) = D \upharpoonright [0, f(k)) \wedge (i, j, k) \in \hat{G}_1)\}$ is unbounded in $\overline{I}$. Then for all i'',

$$i'' \in \overline{I} \leftrightarrow \exists i \in I\, \exists k \in I\, \exists j < i''\, (A_{f(i)} \upharpoonright [0, f(k) = D \upharpoonright [0, f(k)) \wedge (i, j, k) \in \hat{G}_1).$$

Hence, $I \leq_T D$.

If C_1 is bounded in $\overline{I}$, then for some $i'' \in \overline{I}$,

$$\forall i, k \in I\, (A_{f(i)} \upharpoonright [0, f(k)) = D \upharpoonright [0, f(k)) \leftrightarrow \exists k > i''\, (i, j, k) \in \hat{G}_1). \tag{3.8}$$

Again, since D is regular, for all $k \in I$, there is some $i \in I$ such that $A_{f(i)} \upharpoonright [0, f(k)) = D \upharpoonright [0, f(k))$. Then (3.8) implies that D is recursive. That is a contradiction. □

Corollary 3.10. *If $D = \Phi_e^{\emptyset'}$ is regular and non-recursive, then $D \equiv_T I$.*

Non-regular sets

Similar to Lemma 3.9, we have

Lemma 3.11. *If D is non-regular, then $D \geq_T I$.*

Proof. Suppose $d \in \mathcal{M}$ and $D \upharpoonright [0, d)$ is not $\mathcal{M}$-finite. As in Section 3.1.3, let

$$H = \{(x, i) : x < d \wedge i \in I \wedge (x \in A_{f(i)} \setminus A_{f(i-1)} \vee x \in B_{f(i)} \setminus B_{f(i-1)})\},$$

and let $\hat{H} \subseteq [0, d) \times [0, a)$ be a code of H in $[0, d) \times I$ such that for every $x < d$, there are exactly two i's with $(x, i) \in \hat{H}$. Define $i_x = \min\{i < a : (x, i) \in \hat{H}\}$ and $j_x = \max\{i < a : (x, i) \in \hat{H}\}$.

For every $x < d$, if $x \in D$, then $i_x \in I$ and $j_x \in \overline{I}$. Suppose such j_x's are *unbounded in* $\overline{I}$, i.e.

$$\forall i' \in \overline{I}\, \exists x < d\, (x \in D \wedge j_x < i'). \tag{3.9}$$

Then for any i',

$$i' \in \overline{I} \leftrightarrow \exists x < d\, (x \in D \wedge j_x < i').$$

Thus, $I \leq_T D$.

If (3.9) fails, then let $i' \in \overline{I}$ be such that $\forall x < d\,(x \in D \rightarrow j_x > i')$. Then we consider the x's in $\overline{D}$ such that $j_x > i'$. For such an x, $i_x \in \overline{I}$. If

$$\forall i'' \in \overline{I}\, \exists x < d\,(x \in \overline{D} \wedge j_x > i' \wedge i_x < i''). \tag{3.10}$$

Then for every i'',

$$i'' \in \overline{I} \leftrightarrow \exists x < d\,(x \in \overline{D} \wedge j_x > i' \wedge i_x < i'').$$

Thus, $I \leq_T D$.

Suppose (3.10) fails again, and let $i'' \in \overline{I}$ be such that $\forall x < d\,(x \in \overline{D} \wedge j_x > i' \rightarrow i_x > i'')$. Then for all $x < d$,

(i) If $j_x > i'$, then $x \in D$ if and only if $i_x < i''$.

(ii) If $j_x \leq i'$, then $x \in \overline{D}$.

Thus, $D \upharpoonright [0, d)$ is Δ_1. According to Lemma 2.4, $D \upharpoonright [0, d)$ is $\mathcal{M}$-finite, contradicting our assumption. □

Given $k \in I$, we say $r_k \in I$ is a *separating point of* $[0, f(k))$ if

$$\forall x \in ([0, f(k)) \setminus A_{f(r_k)})\,(x \in B \rightarrow \exists P \subseteq \emptyset'\, \exists N \subseteq \overline{\emptyset'}\,(\langle \{x\}, 0, P, N\rangle \in \Phi_e^{\emptyset'}[f(r_k)])).$$

By Corollary 3.4, a separating point of $[0, f(k))$ exists for every $k \in I$. Moreover, the predicate "r is not a separating point of $[0, f(k))$", whose variables are r and k, is Σ_1, so it is reducible to I by Lemma 3.12.

Lemma 3.12 (Chong and Yang [8]). *Every bounded r.e. set is reducible to I.*

If $B = \emptyset$, then clearly D is of r.e. degree. For the general case, intuitively, we modify the enumeration of D to be more " effective": we enumerate x into D only if we see that x is enumerated into A at some stage s and all the computations of the form $\langle \{x\}, 0, P, N\rangle \in \Phi_e^{\emptyset'}[s]$ are fake by $N \cap \emptyset' \neq \emptyset$. That is, we define

$$A^* = \{x : \exists s\, \exists t > s\,[x \in A_s \setminus B_t \wedge \forall P, N\,(\langle \{x\}, 0, P, N\rangle \in \Phi_e^{\emptyset'}[s] \rightarrow N \cap \emptyset'_t \neq \emptyset)]\}$$
$$B^* = A^* \cap B.$$

Then $D = A^* \setminus B^*$. We show that $D \equiv_T A^* \oplus B^* \oplus I$. By Lemma 3.11, $D \geq_T I$. Thus, we only need to show:

Lemma 3.13. *If D is non-regular, then $D \oplus I \equiv_T A^* \oplus B^* \oplus I$.*

Proof. Fix any $k \in I$. By Lemma 3.12, we can get a separating point r (may not be unique) of $[0, f(k))$ recursively in I. Then the interval $[0, f(k))$ is separated into two parts: $[0, f(k)) \setminus A_{f(r)}$ and $[0, f(k)) \cap A_{f(r)}$. By definition of separating point,

$$\forall x \in [0, f(k)) \setminus A_{f(r)} \, (x \in B \rightarrow \exists P \subseteq \emptyset' \, \exists N \subseteq \overline{\emptyset'} \, (\langle \{x\}, 0, P, N \rangle \in \Phi_e^{\emptyset'}[f(r)])).$$

Therefore,

$$\forall x \in [0, f(k)) \setminus A_{f(r)} \, (x \in B \rightarrow x \notin A^*),$$

i.e.

$$D \restriction ([0, f(k)) \setminus A_{f(r)}) = A^* \restriction ([0, f(k)) \setminus A_{f(r)}), \quad B^* \restriction ([0, f(k)) \setminus A_{f(r)}) = \emptyset. \tag{3.11}$$

In addition, we claim that

$$A^* \restriction ([0, f(k)) \cap A_{f(r)}) \text{ is recursive.}$$

For every $x \in [0, f(k)) \cap A_{f(r)}$, since x is enumerated into A at some stage s, there exists a stage $t > s$ such that either

Case 1. x is enumerated into B at stage t, or

Case 2. All computations of the form $\langle \{x\}, 0, P, N \rangle$ in $\Phi_e^{\emptyset'}[s]$ are fake, i.e. $N \cap \emptyset'_t \neq \emptyset$.

Thus, $x \in A^*$ if and only if Case 2 occurs first, which can be determined recursively. Therefore, to determine whether X is a subset of $D \restriction ([0, f(k)) \cap A_{f(r)})$ or a subset of $\overline{D} \restriction ([0, f(k)) \cap A_{f(r)})$, we only need to take $B^* \restriction ([0, f(k)) \cap A_{f(r)})$ as an oracle, and vice versa. This property and (3.11) combine to produce $D \oplus I \equiv_T A^* \oplus B^* \oplus I$. □

Theorem 3.14. *In a $B\Sigma_1$ model, every d-r.e. degree below $\mathbf{0'}$ is r.e.*

We therefore have:

Corollary 3.15. *Assume $P^- + B\Sigma_1 + \mathrm{Exp}$. Then*

(1) There is a proper d-r.e. degree;

(2) $I\Sigma_1$ is equivalent to the existence of a proper d-r.e. degree below $\mathbf{0'}$.

3.2 Degrees Below $\mathbf{0}'$ in a Saturated Model

As shown in Section 3.1, any proper d-r.e. degree in a $B\Sigma_1$ model is not below $\mathbf{0}'$. In this section, we expand our investigation with an analysis of the degrees below $\mathbf{0}'$ for $B\Sigma_1$ models. The main result is:

Theorem 3.16. $P^- + B\Sigma_1 + \mathrm{Exp} \nvdash$ *There is a non-r.e. degree below* $\mathbf{0'}$.

To show this theorem, we consider a $B\Sigma_1$ model $\mathcal{M}$ with the following properties:

(i) $\omega \subset \mathcal{M}$ is a Σ_1 cut of $\mathcal{M}$.

(ii) Every subset of ω is coded (on ω) in the model $\mathcal{M}$.

Such a model $\mathcal{M}$ is called a *saturated* $B\Sigma_1$ (or *saturated*, for short) model. In [46], Slaman and Woodin showed that a saturated $B\Sigma_1$ model exists. Let I denote ω, $a \in \mathcal{M}$ be such that $I \subset [0, a)$, and $f : I \to \mathcal{M}$ be a strictly increasing cofinal Δ_1 function with $f(0) = 0$. We may further assume that $\langle \cdot, \cdot \rangle \restriction I^2$ maps onto I.

The proof that every d-r.e. set $D = A \setminus B$ reducible to $\emptyset'$ is of r.e. degree in Section 3.1.4 could be simplified if $\mathcal{M}$ is saturated: Suppose $D = \Phi_e^{\emptyset'}$ and

$$G = \{\langle k, r \rangle : r \text{ is the least separating point of } [0, f(k+1))\}.$$

Then G is coded by $\hat{G} \subseteq [0, a)$. For each $k < a$, let r_k be the least r such that $\langle k, r \rangle \in \hat{G}$ and $[0, f(k))$ can be recursively separated into two parts:

- $P_0 = \{x : \exists k \in I\, (x \in [f(k), f(k+1)) \,\wedge\, x \notin A_{f(r_k)})\}$ and
- $P_1 = \{x : \exists k \in I\, (x \in [f(k), f(k+1)) \,\wedge\, x \in A_{f(r_k)})\}$.

For any $x \in [f(k), f(k+1))$,

- If $x \in A_{f(r_k)}$, then $x \in D$ if and only if $x \notin B$.
- If $x \notin A_{f(r_k)}$, then by the definition of separating point, $x \in D$ if and only if all computations of the form $\langle \{x\}, 0, P, N \rangle$ in $\Phi_e^{\emptyset'}[f(r_k)]$ are fake, (i.e. $N \cap \emptyset' \neq \emptyset$) and $x \in A$.

Thus, $D \upharpoonright P_0$ is Σ_1, $D \upharpoonright P_1$ is Π_1 and D is of r.e. degree. Clearly, the key to this proof is the separating points.

Now suppose $V = \Phi_e^{\emptyset'}$, which may not be d-r.e. We generalize the notion of separating points as follows: Let $k \in I$ and

$$H_k = \{(x,i) : x \in [f(k), f(k+1)) \wedge i \in I \wedge \Phi_e^{\emptyset'}(x)[f(i)] = 1\}.$$

That is, H_k records the approximation of $\Phi_e^{\emptyset'} \upharpoonright [f(k), f(k+1))$. Since H_k is recursive on $[f(k), f(k+1)) \times I$, it is coded by some $\hat{H}_k \subseteq [f(k), f(k+1)) \times [0, a)$. For each k, we fix a code $\hat{H}_k$. For any $i < a$ and $x \in [f(k), f(k+1))$, we define

$$V_i(x) = \begin{cases} 1 & \text{if } (x,i) \in \hat{H}_k \\ 0 & \text{otherwise} \end{cases}$$

and so $V(x) = \lim_{i \in I} V_i(x)$.

Suppose $i \in I$. Then

(i) x is said to be *i-honest*, if for any $j \in I$ greater than i, $V_j(x) = V_i(x)$; otherwise, x is an *i-liar.*

(ii) x is *found to be an i-liar by stage j*, if x is an i-liar, $j \in I$, $j > i$ and

$$\exists k \leq j\, (k > i \wedge V_k(x) \neq V_i(x));$$

(iii) x is called an *i-white liar*, if x is an i-liar and $V(x) = V_i(x)$;

(iv) x is an *i-malicious liar*, if x is an i-liar and $V(x) \neq V_i(x)$.

We observe that *white liars* correspond to *escaping elements* in Section 3.1.3. Similar to Lemma 3.3, we have

Lemma 3.17. *For any $i, k \in I$, there is a $j > i$ such that all i-white liars in $[f(k), f(k+1))$ are found by stage j.*

Proof. For the sake of a contradiction, we suppose $i, k \in I$ and for each $j > i$ in I, there is an i-white liar not founded by stage j, and without loss of generality, we assume all such i-white liars are not in V. Then consider the function $\delta : I \backslash [0, i] \to I$, $j \mapsto \langle n_j, z_0^j, z_1^j, \ldots, z_{n_j - 1}^j \rangle$, where

(i) z^j is the first i-white liar not in V that is found at a least stage $j' > j$ but is not found by stage j, and

(ii) $z_0^j < z_1^j < \ldots < z_{n_j-1}^j$ is a list of all stages $l \in I$ such that $V_l(z^j) \neq V_{l-1}(z^j)$.

According to the saturation of $\mathcal{M}$, δ is coded on I^2 by an $\mathcal{M}$-finite partial function $\hat{\delta} : [i+1, a) \to [0, a)$ with the following properties:

(i) $\mathrm{dom}(\hat{\delta}) \supset \mathrm{dom}(\delta)$, and

(ii) For each $j \in \mathrm{dom}(\hat{\delta})$, $\hat{\delta}(j) = \langle n_j, z_0^j, z_1^j, \ldots, z_{n_j-1}^j \rangle$ for some $n_j, z_0^j, \ldots, z_{n_j-1}^j$ such that

  (a) $z_0^j < z_1^j < \ldots, < z_{n_j-1}^j < a-1$, and

  (b) $\forall m\, (z_m^j > i \leftrightarrow z_m^j > j)$.

Then for any $j \in \mathrm{dom}(\hat{\delta})$, we may recursively find an $x \in [f(k), f(k+1))$ with $V_i(x) = 0$ such that $z_0^j, \ldots, z_{n_j-1}^j$ are the first n_j many l's satisfying $V_l(x) \neq V_{l-1}(x)$ and the $(n_j+1)^{\text{th}}$ l is the largest possible according to $\hat{H}_k$. This x is said to be *corresponding to* j. Notice that if x corresponds to a $j \in I$, then x is also an i-white liar not found by j, and if x corresponds to a $j \in \overline{I}$, then x is i-honest.

Now let

$$X = \{x \in [f(k), f(k+1)) : \exists j \in \mathrm{dom}(\hat{\delta})\, (x \text{ is corresponding to } j)\}.$$

Since each $x \in X$ is either i-honest or an i-white liar, $X \subseteq \overline{V}$. But then there is some $j > i$ such that $\Phi_e^{\emptyset'}[f(j)] \upharpoonright X = \emptyset$. According to the local downward closure property of $\Phi_e^{\emptyset'}$, all i-liars in X are found by stage j. That is a contradiction. □

Suppose all i-white liars in $[f(k), f(k+1))$ are found by stage j, $x \in [f(k), f(k+1))$ and the approximation $V_l(x)$ does not "change its mind" between i and j, i.e. $\forall l \in [i, j]\, (V_i(x) = V_l(x))$. Then x cannot be an i-white liar. Thus, for all such x's,

$$V(x) = V_i(x) \leftrightarrow \neg \exists j \in I\, (j > i \wedge V_j(x) \neq V_i(x)). \tag{3.1}$$

Conversely, for any $x \in [f(k), f(k+1))$, there are $i < j \in I$ such that x is i-honest and all i-white liars in $[f(k), f(k+1))$ are found by stage j. Then the approximation $V_l(x)$ does not " change its mind" between i and j for all $j > i$.

Lemma 3.18. *There exists $u \in I$ with property $\rho(k,u)$:*

For any $x \in [f(k), f(k+1))$, there are $i < j < u$ such that all i-white liars in $[f(k), f(k+1))$ are found by stage j and $V_i(x) = V_l(x)$ for all $l \in [i,j]$.

Proof. Let $T_k = \{\langle i,j\rangle \in I^2 : i < j \wedge \text{All } i\text{-white liars in } [f(k), f(k+1)) \text{ are found by stage } j\}$. Since $\mathcal{M}$ is saturated, T_k is coded by $\hat{T}_k \subseteq [0,a)$ so that for all $\langle i,j\rangle \in \hat{T}_k$, $i < j$. Now consider the function $\epsilon : [f(k), f(k+1)) \to \hat{T}_k$, $x \mapsto \mu\langle i,j\rangle(\langle i,j\rangle \in \hat{T}_k \wedge \forall l \in [i,j]\, V_i(x) = V_l(x))$. For every x in $[f(k), f(k+1))$, since $\langle\cdot,\cdot\rangle$ maps I^2 onto I and there is a pair $\langle i,j\rangle \in T_k$ such that $\forall l \in [i,j]\,(V_i(x) = V_l(x))$, we have $\epsilon(x) \in I$. By $B\Sigma_1$, $\text{ran}(\epsilon)$ is bounded in I. Let $u \in I$ be an upper bound of all elements in $\text{ran}(\epsilon)$ and it is straightforward to verify that $\rho(k,u)$ holds. $\square$

For each $k \in I$, let u_k be the least u satisfying $\rho(k,u)$ and

$$F = \{\langle k,i,j\rangle \in I^3 : i < j < u_k \wedge j \text{ is the least such that}$$
$$\text{all } i\text{-white liars in } [f(k), f(k+1)) \text{ are found by stage } j\}.$$

Suppose $\hat{F} \subseteq [0,a)$ is a code of F such that for all $\langle k,i,j\rangle \in \hat{F}$ with $k \in I$, $i < j < u_k$ and $\langle k,i,j\rangle \in F$.

We recursively separate $\mathcal{M}$ into countably many parts $\{E_{k,i}\}_{k\in I, i<u_k}$:

$$E_{k,i} = \{x \in [f(k), f(k{+}1)) \setminus \textstyle\bigcup_{i'<i} E_{k,i'} : \exists j\,(\langle k,i,j\rangle \in \hat{F} \wedge \forall l \in [i,j]\,(V_i(x) = V_l(x)))\}.$$

For every $k \in I$ with $\langle k,i,j\rangle \in \hat{F}$ and every $x \in E_{k,i}$, x cannot be an i-white liar since all i-white liars in $[f(k), f(k+1))$ are found by stage j. Thus (3.1) holds for all $x \in E_{k,i}$. Define the r.e. set A on each $E_{k,i}$ by

$$x \in A \restriction E_{k,i} \leftrightarrow \exists j \in I\,(j > i \wedge V_j(x) \neq V_i(x)).$$

By (3.1) again, $A \equiv_T V$ and $\deg(V)$ is r.e.

Remark. Note that the argument in this section only requires that every arithmetically definable (in the sense of $\mathcal{M}$) subset of ω is coded on ω. Thus, for a countable model of $P^- + B\Sigma_1 + \neg I\Sigma_1 +$ Exp, if ω is a Σ_1 cut and every arithmetically definable subset of ω is coded, then all degrees below $\mathbf{0}'$ are r.e.

In general, for a given model of $P^- + B\Sigma_1 + \neg I\Sigma_1 +$ Exp, it is still unknown whether there is a non-r.e. degree below $\mathbf{0}'$ due to the complexity of coding. In the above argument, note that in the proof of Lemma 3.17, δ is not Δ_1 on I^2. More importantly, the complexity of F defined above is far beyond Δ_1 on I^3.

Chapter 4

Friedberg Numbering In Reverse Recursion Theory And α-Recursion Theory

4.1 Weak Fragments of PA

The known constructions of a Friedberg numbering ([15, 29]) make strong use of existence of the least index for each r.e. set, in order to construct a Friedberg numbering for ω. This is equivalent to proving the theorem in the theory $P^- + I\Sigma_2$, as we discuss below. We will prove in this section that over the base theory $P^- + B\Sigma_2$, $I\Sigma_2$ is both sufficient and necessary for the existence of such a numbering.*

4.1.1 Towards Friedberg numbering in fragments of PA

Let $\{W_e\}$ be a Gödel numbering in a model of $P^- + I\Sigma_0 + \text{Exp}$. Note that the statement "$W_i = W_e$" is Π_2. Therefore, $L\Pi_2$ suffices to show that every r.e. set has a least index in $\{W_e\}$. By Theorem 2.1, $L\Pi_2 \Leftrightarrow I\Sigma_2$. In fact, the induction needed to carry out the construction of a Friedberg numbering for ω is just $I\Sigma_2$. Thus,

Lemma 4.1 ($P^- + I\Sigma_2$). *There exists a Friedberg numbering.*

*Without $B\Sigma_2$, a Friedberg numbering may exist. Let $\mathcal{M}$ be a model of $P^- + I\Sigma_1$ such that (i) $B\Sigma_2$ fails and (ii) there is a Σ_2 one-one projection from ω onto $\mathcal{M}$. (The existence of such a model $\mathcal{M}$ was shown by Groszek and Slaman [17].) In $\mathcal{M}$, the construction of a Friedberg numbering can be carried out by exploiting the existence of the Σ_2 projection.

Now we consider the case that $I\Sigma_2$ fails. From now on in this section, $\mathcal{M}$ is a $B\Sigma_2$ model and $I \subset \mathcal{M}$ is a Σ_2 cut. Let $\{A_e\}_{e\in\mathcal{M}}$ be a one-one numbering of r.e. sets in $\mathcal{M}$. Our purpose is to construct an r.e. set X such that $X \neq A_e$, for all $e \in \mathcal{M}$. Hence, $\{A_e\}_{e\in\mathcal{M}}$ is not a Friedberg numbering.

By Lemma 2.3, let $f : I \to \mathcal{M}$ be a nondecreasing Δ_2 cofinal function with $f(0) = 0$. That makes it possible to establish a partition of $\mathcal{M}$, $\{[f(i), f(i+1)) : i \in I\}$. The interval $[f(i), f(i+1))$ is said to be the i^{th} *block* (or *block* i) of $\mathcal{M}$. Then X is constructed by diagonalizing against A_e's in each block.

For any $a \in \mathcal{M}$,

$$\forall d, e < a\, \exists x\, (d \neq e \;\to\; A_d(x) \neq A_e(x)).$$

Since $\{A_e\}_{e\in\mathcal{M}}$ is a one-one numbering. It follows from $B\Sigma_2$ that there is a $b \in \mathcal{M}$ such that

$$\forall d, e < a\, \exists x < b\, (d \neq e \;\to\; A_d(x) \neq A_e(x)). \tag{4.1}$$

Here, b is said to be a *bound of differences relative to* $[0, a)$. (4.1) implies that there is at most one $e < a$ such that

$$A_e \restriction [0, b) = X \restriction [0, b). \tag{4.2}$$

Therefore, diagonalizing against $\{A_e\}_{e<a}$ amounts to diagonalizing against the sole A_e satisfying (4.2), if any, by one witness greater than or equal to b. In short, to diagonalize against one block it suffices to diagonalize against one special r.e. set.

Let us recall the definition of limit as follows. Suppose $h : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a total function. Then

$$\lim_s h(s, a) = n,$$

if either $n \in \mathcal{M}$ and there exists t such that

$$\forall s > t\, (h(s, a) = n),$$

or else $n = \infty$ and

$$\forall m\, \exists t\, \forall s > t\, (h(s, a) > m).$$

Since $f : I \to \mathcal{M}$ is Δ_2, (4.1) yields a Δ_2 function $g : I \to \mathcal{M}$ such that $g(i)$ is a bound of differences relative to $[0, f(i))$ for each $i \in I$. A careful examination of the proof of the Limit Lemma [47] in standard model ω shows that the proof of the Limit

Lemma only requires $P^- + B\Sigma_1$ and the regularity of Δ_2 sets. Then in the $B\Sigma_2$ model $\mathcal{M}$, the Limit Lemma implies that f and g have recursive approximations. A more precise statement of this situation is that f and g may chosen to have nondecreasing recursive approximations, as proved in Lemma 4.2. Based on those approximations, it will be shown later that X can be constructed in an effective manner.

Lemma 4.2. *Let $\mathcal{M}$ be a $B\Sigma_2$ model and $I \subset \mathcal{M}$ be a Σ_2 cut. Then there exist (total) recursive functions $f', g' : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ such that*

(i) $\lambda s(f'(s,i))$, $\lambda i(f'(s,i))$, $\lambda s(g'(s,i))$ *and* $\lambda i(g'(s,i))$ *are nondecreasing;*

(ii) functions f and g given by $f(i) = \lim_s f'(s,i)$, $g(i) = \lim_s g'(s,i)$ are well defined and less than ∞ on I and equal ∞ on $\mathcal{M} \setminus I$;

(iii) $f : I \to \mathcal{M}$ is cofinal;

(iv) $\forall i, j, s, t\, (i \neq j \to g'(s,i) \neq g'(t,j))$, *i.e.* $\mathrm{ran}(\lambda s\, g'(s,i)) \cap \mathrm{ran}(\lambda s\, g'(s,j)) = \emptyset$ *for any $i \neq j$;*

(v) $\forall i \in I\, \forall d, e < f(i)\, \exists x < g(i)\, (d \neq e \to A_d(x) \neq A_e(x))$, *i.e. $g(i)$ is a bound of differences relative to* $[0, f(i))$.

Proof. Functions f' and f satisfying (i)-(iii) may be defined from the Σ_2 definition of I (See [3, 4]). We omit the details and directly define g' satisfying (i), (ii) and (v). Then (i), (ii), (iv) and (v) will be satisfied for g'', defined by $g''(s,i) = \langle i, g'(s,i)\rangle$ for any $s, i \in \mathcal{M}$ ($\langle \cdot, \cdot \rangle$ is a recursive code of pairs).

Now define g' by induction on s as follows.

$$
\begin{aligned}
g'(0,i) &= i \\
g'(s+1,i) &= \begin{cases} g'(s,i) & \text{if } g'(s,i) > f'(s,i) \text{ and} \\ & \forall d, e < f'(s,i)\, \exists x < g'(s,i)\, (d \neq e \to \\ & \qquad\qquad\qquad A_{d,s}(x) \neq A_{e,s}(x)), \\ g'(s,i) + 1 & \text{otherwise.} \end{cases}
\end{aligned}
$$

By $I\Sigma_1$, g' is total recursive and $\lambda s(g'(s,i))$ is nondecreasing.

To see that $\lambda i(g'(s,i))$ is nondecreasing, it suffices to show

$$\forall i\,(g'(s,i+1) \geq g'(s,i)). \tag{4.3}$$

by induction on s and $I\Pi_1$. The induction is straightforward and we omit the details here.

Observe that a recursive set either has a maximum element or is unbounded in $\mathcal{M}$ by $L\Pi_1$. Then it follows immediately from the nondecreasing property of $\lambda s(g'(s,i))$ that

$$\lim_s g'(s,i) < \infty \leftrightarrow \{g'(s,i) : s \in \mathcal{M}\} \text{ is bounded.}$$

Then it is easy to check that (ii) and (v) hold by $B\Sigma_2$ and the properties of f and the defintion of g. □

4.1.2 Nonexistence of Friedberg numbering

Let $\mathcal{M}$, I and $\{A_e\}_{e\in\mathcal{M}}$ be as in Section 4.1.1. In this section it will be shown that there exists an r.e. set $X \notin \{A_e\}_{e\in\mathcal{M}}$. The method here converts the diagonalization strategy in Section 4.1.1 to an effective one so as to obtain an r.e. counterexample X.

Theorem 4.3. *There is no Friedberg numbering in a $B\Sigma_2$ model.*

Proof. Again, $\mathcal{M}$, I and $\{A_e\}_{e\in\mathcal{M}}$ are as in Section 4.1.1. Let f, f', g, g' be as in Lemma 4.2. The construction below defines X such that

(i) $X \subseteq \operatorname{ran}(g')$;

(ii) $\forall i \notin I\, \forall s\, (g'(s,i) \in X)$

(iii) $\forall i \in I\, \forall s\, (g'(s,i) < g(i) \to g'(s,i) \in X)$;

(iv) $\forall i \in I\, [(g(i) \notin X) \leftrightarrow \exists c < f(i)\, (A_c \upharpoonright [0,g(i)) = X \upharpoonright [0,g(i)) \wedge g(i) \in A_c)]$.

According to Lemma 4.2, $g(i)$ is a bound of differences relative to $[0,f(i))$. Then at most one $c < f(i)$ satisfies

$$A_c \upharpoonright [0,g(i)) = X \upharpoonright [0,g(i)).$$

Thus, (iv) implies $X \neq A_c$, for any $c < f(i), i \in I$.

Since $\lambda s(g'(s,i))$ is nondecreasing, (ii) and (iii) are satisfied easily via the approximation g'. However, that approximation strategy fails for Clause (iv). This is because in the matrix of (iv), the right hand side of "$\leftrightarrow$" is Δ_2 and so we cannot recursively determine its truth value. More precisely, at stage s, it is tempting to (perhaps mistakenly) enumerate $g'(s,i)$ if

$$\neg\exists c < f'(s,i)\,(A_{c,s} \restriction [0, g'(s,i)) = X_s \restriction [0, g'(s,i)) \;\wedge\; g'(s,i) \in A_{c,s}). \tag{4.4}$$

By (4.4), guessing whether $g'(s,i)$ should be enumerated into X could be wrong even if $g'(s,i) = g(i)$ and $f'(s,i) = f(i)$. We may find a $c < f(i)$ at a later stage satisfying $A_c \restriction [0, g(i)) = X \restriction [0, g(i))$ and $g(i) \in A_c$ in the sense of that stage. But once $g(i) = g'(s,i)$ is mistakenly enumerated into X, $g(i)$ cannot be removed from X.

The problem can be solved with the aid of Lemma 2.5.

A non-effective construction of X is carried out inductively on I, with the intention of finding a set G such that

$$G(i) = 0 \leftrightarrow \exists c < f(i)\,(A_c \restriction [0, g(i)) = X \restriction [0, g(i)) \wedge g(i) \in A_c). \tag{4.5}$$

Define

$$X_0 = \textstyle\bigcup_{n \in \mathcal{M}} \{g'(s,n) : g'(s,n) < g(n)\}.$$

$$G(0) = \begin{cases} 0 & \text{if } \exists c < f(0)\,(A_c \restriction [0, g(0)) = X_0 \restriction [0, g(0)) \wedge g(0) \in A_c), \\ 1 & \text{otherwise.} \end{cases}$$

$$X_{i+1} = \begin{cases} X_i & \text{if } G(i) = 0, \\ X_i \cup \{g(i)\} & \text{if } G(i) = 1. \end{cases}$$

$$G(i+1) = \begin{cases} 0 & \text{if } \exists c < f(i+1)\,(A_c \restriction [0, g(i+1)) = X_{i+1} \restriction [0, g(i+1)) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \wedge\, g(i+1) \in A_c), \\ 1 & \text{otherwise.} \end{cases}$$

for all $i \in I$. Here, $g(n) = \infty$, if $n \notin I$, by Lemma 4.2.

Let

$$X = \textstyle\bigcup_{i \in I} X_i.$$

It is immediate from Lemma 2.5 that X_i and $G(i)$ are well defined on I, G is Δ_2 on I and coded on I. Suppose $\hat{G}$ is a code of G on I. Then

$$\begin{aligned} X &= X_0 \cup (\textstyle\bigcup_{i\in I}\{g(i) : G(i) = 1\}) \\ &= \textstyle\bigcup_{n\in\mathcal{M}}\{g'(s,n) : \exists t > s\,(g'(s,n) < g'(t,n)) \lor \hat{G}(n) = 1\}. \end{aligned}$$

and X is r.e.

By Lemma 4.2, g is strictly increasing on I. Thus,

$$X \upharpoonright [0, g(i+1)) = X_{i+1} \upharpoonright [0, g(i+1)).$$

(4.5) and Clause (iv) are satisfied according to the construction. □

Theorem 4.3 and Lemma 4.1 combine to yield

Corollary 4.4 ($P^- + B\Sigma_2$)**.** *$I\Sigma_2$ is equivalent to the existence of a Friedberg numbering.*

Remark. A numbering $\{B_e\}_{e\in\mathcal{M}}$ is *acceptable* (K*-acceptable*, respectively) if for any other numbering $\{D_e\}_{e\in\mathcal{M}}$ there is a recursive ($\emptyset'$-recursive, respectively) function f such that $D_e = B_{f(e)}$ for all e. Clearly, Gödel numbering is acceptable. In classical recursion theory, a Friedberg numbering is an example of non-acceptable universal numbering and non-K-acceptable universal numbering.

In a $B\Sigma_2$ model $\mathcal{M}$, no Friedberg numbering exists, but a non-K-acceptable universal numbering, thus a non-acceptable universal numbering still exists.

For instance, suppose $\langle\cdot,\cdot\rangle : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ is a recursive injection, and let

$$B_e = \begin{cases} \mathcal{M} & \text{if } e = 0, \\ W_i \setminus \{j\} & \text{if } e > 0 \text{ and } \exists i,j < e\,(\langle i,j\rangle = e), \\ W_e \setminus \{0\} & \text{if } e > 0 \text{ and } \neg\,\exists i,j < e\,(\langle i,j\rangle = e). \end{cases} \tag{4.6}$$

where $\{W_e\}_{e\in\mathcal{M}}$ is a Gödel numbering. Then $\{B_e\}_{e\in\mathcal{M}}$ is a universal numbering and

$$B_e = \mathcal{M} \Leftrightarrow e = 0.$$

Thus, there is no K-recursive function $g : \mathcal{M} \to \mathcal{M}$ satisfying

$$W_e = B_{g(e)}.$$

A *K_e-numbering* $\{C_e\}$ is a universal numbering for which the grammar equivalence problem $\{(e,d) : C_e = C_d\}$ is $\emptyset'$-recursive (See [20]). A Friedberg numbering is a K_e-numberings. In a $B\Sigma_2$ model, no Friedberg numbering exists, and also no K_e-numbering exists. The reason is as follows. Suppose $\mathcal{M}$ is a $B\Sigma_2$ model and $\{C_e\}_{e\in\mathcal{M}}$ is a K_e-numbering. Then for each e in $\mathcal{M}$,

$$\{d < e : C_d = C_e\}$$

is a Δ_2 set, and has a least element. It follows that the least index exists for every r.e. set in the numbering $\{C_e\}_{e\in\mathcal{M}}$, which is not the case for $\{W_e\}_{e\in\mathcal{M}}$. By [29], a Friedberg numbering can be constructed using a priority-free method via the numbering $\{C_e\}_{e\in\mathcal{M}}$, a contradiction. Hence,

Corollary 4.5 ($P^- + B\Sigma_2$)**.** *$I\Sigma_2$ is equivalent to the existence of a K_e-numbering.*

4.2 Σ_1 Admissible Ordinals

In this section, we investigate the problem of the existence of a Friedberg numbering in the context of admissible ordinals.

4.2.1 Towards Friedberg numbering in α-recursion

Assume $\{W_e\}_{e<\alpha}$ is a Gödel numbering. We attempt to lift the construction of a Friedberg numbering from ω to α. No difficulty arises when L_α satisfies Σ_2 replacement. The proof remains valid because Σ_2 replacement suffices to show

$$\begin{aligned}&(e \text{ is the least index for } W_e) \Leftrightarrow \\ &\qquad \exists b\, \exists \eta\, \forall d < e\, (W_d \restriction b = W_{d,\eta} \restriction b \neq W_{e,\eta} \restriction b = W_e \restriction b). \quad (4.1)\end{aligned}$$

Therefore the least index can be approximated effectively.

If L_α does not satisfy Σ_2 replacement, then the approach above for constructing a Friedberg numbering fails by noticing that Σ_2 replacement is also necessary for (4.1) to hold. In this situation, the straightforward adaptation of the argument in $B\Sigma_2$ models is not applicable neither: Suppose $\{A_e\}_{e<\alpha}$ is a one-one numbering, then it is not always true that for an arbitrary $\beta < \alpha$,

$$\exists b\, \forall d, e < \beta\, (d \neq e \rightarrow \exists x < b\, (A_d(x) \neq A_e(x))). \quad (4.2)$$

In this chapter, we introduce three strategies which will either yield a successful construction of a Friedberg numbering for suitable α's or allow a diagonalization argument to be implemented showing the nonexistence of such a numbering.

The intuition is that the shorter the list of α-r.e. sets is, the more likely (4.1) and (4.2) can be made to hold. The first strategy attempts to rearrange the order of α-r.e. sets so as to produce a short, necessarily non-recursive, list of these sets. A further idea is to force every proper initial segment of the list to be correctly approximated from some stage onwards, for the sake of computing the least indices and upper bounds of differences correctly in the limit. To achieve this, we arrange for the list of the α-r.e. sets to have length $t\sigma 2p(\alpha)$. More precisely, α-r.e. sets are listed by a tame Σ_2 projection $g : t\sigma 2p(\alpha) \xrightarrow[\text{onto}]{\text{one-one}} \alpha$. Thus, for an arbitrary numbering $\{A_e\}_{e<\alpha}$, the set A_d respectively is listed before A_e, if $g^{-1}(d) < g^{-1}(e)$.

The second strategy is to exploit the key property of $\sigma 2cf(\alpha)$, i.e. Corollary 2.16. According to Corollary 2.16, it is possible to apply Σ_2 replacement on lengths less than $\sigma 2cf(\alpha)$. The first two strategies combine to suggest the possibility that a Friedberg numbering exists when $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$. In particular, if L_α satisfies Σ_2 replacement, then $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha) = \alpha$.

If $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, then Lemma 2.13 implies that coding a tame Σ_2 subset of $\sigma 2cf(\alpha)$ is possible. The problem left to adapt the proof in $B\Sigma_2$ models to L_α is to give an effective method of searching for an upper bound b in (4.2). However, such an upper bound may not exist.

The third strategy is aimed at devising diagonalization method to show the nonexistence of a Friedberg numbering in the situation that $\sigma 2cf(\alpha) < t\sigma 2p\,(\alpha)$. This is done by analyzing α-finite sets together with a property we call *pseudostability.* Pseudostable ordinals will be used to get suitable upper bounds for witnesses that differentiae two α-r.e. sets in a given α-finite collection for the purpose of a diagonalization. (See Section 4.2.3 and 4.2.4).

Let $C, I \subset \alpha$ be α-finite. If $|I|_\alpha < \sigma 1p\,(\alpha)$, then for any simultaneous enumeration of α-r.e. sets $\{A_e\}_{e\in I}$, the set

$$I_C = \{e \in I : A_e \supsetneq C\}$$

is α-finite, by Theorem 2.12. Thus $\exists \eta\, \forall e \in I\,(e \in I_C \leftrightarrow A_{e,\eta} \supsetneq C\,)$ by Σ_1 replacement. Therefore, any set $X \supsetneq C$ such that $X \restriction \eta = C$ would not be in $\{A_e\}_{e\in I}$

(recall that $A_{e,\eta} \subseteq [0,\eta)$ for every $e, \eta < \alpha$).

Note that the recursive search for η strongly relies on the parameter I_C. That would be a problem if C varies, as the parameters of I_C may not be recovered effectively. Nevertheless, there are special cases when the parameter I_C can be omitted (i.e. the ordinal η can be derived directly from C). For example, when

(i) C is never in the list of r.e. sets $\{A_e\}_{e\in I}$ as C changes,

(ii) a final segment of C is an interval of ordinals with $\sup C = \eta$ being an α-stable ordinal, and

(iii) roughly speaking, $\sup C$ is large enough,

then we have

$$\forall e \in I\,(A_e \supsetneq C \leftrightarrow A_e \supseteq C \leftrightarrow A_{e,\sup C} \supseteq C).$$

The only problem with the use of α-stable ordinals is that α-stable ordinals need not be cofinal in α. Therefore, the notion of *pseudostablility*, a weak form of α-stability, is introduced. As will be seen in Section 4.2.3 and 4.2.4, pseudostable ordinals are cofinal in α and enjoy the properties required for our construction.

4.2.2 When $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$

The main result of this section is

Theorem 4.6. *If $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$, then there exists a Friedberg numbering.*

The strategy here is to adapt Kummer's construction [29] by introducing a shorter list of all α-r.e. sets on $t\sigma 2p\,(\alpha)$ and applying local Σ_2 replacement (Corollary 2.16) on $\sigma 2cf(\alpha)$.

Let $\hat{\alpha} = t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$, $f : \hat{\alpha} \xrightarrow[\text{cofinal}]{\text{strictly increasing}} \alpha$ and $g : \hat{\alpha} \xrightarrow[\text{onto}]{\text{one-one}} \alpha$ be tame Σ_2, and $f', g' : \alpha \times \hat{\alpha} \to \alpha$ be recursive functions that tamely generate f, g such that for all $\eta < \hat{\alpha}$,

(i) $\lambda x\,(f'(\eta, x))$ and $\lambda x\,(g'(\eta, x))$ are one-one,

(ii) $\lambda x\,(f'(\eta, x))$ is strictly increasing, and $\lambda x\,(g'(\eta, x))$ is strictly increasing for $x > \eta$, if any.

For simplicity, f_η, g_η will be used to denote functions $\lambda x\,(f'(\eta,x))$, $\lambda x\,(g'(\eta,x))$ respectively.

Lemma 4.7. *Suppose $\{W_e\}_{e<\alpha}$ is a Gödel numbering. Then there are numberings $\{P_e\}_{e<\alpha}$ and $\{Q_e\}_{e<\alpha}$ such that*

(i) $\{P_e\}_{e<\alpha} \cap \{Q_e\}_{e<\alpha} = \emptyset$*;*

(ii) $\{P_e\}_{e<\alpha} \cup \{Q_e\}_{e<\alpha} = \{W_e\}_{e<\alpha}$*;*

(iii) $P_e \neq P_d$ *whenever* $e \neq d$*;*

(iv) $\{e < \alpha : P_e \supseteq C\}$ *is cofinal in* α*, for every* α*-finite set* C*.*

Proof. Let

$$P_e = [0, e),$$
$$Q_e = \begin{cases} \alpha & \text{if } e = 0, \\ W_{e'} \cup \{e''\} \setminus \{e'''\} & \text{if } e = \langle\langle e', e''\rangle, e'''\rangle \text{ and } e'' > e''', \\ W_e \cup \{e\} \setminus \{0\} & \text{otherwise.} \end{cases}$$

Then (i)-(iv) are immediate from the definitions of P_e and Q_e. □

Requirements and strategy

Fix numberings $\{P_e\}_{e<\alpha}$ and $\{Q_e\}_{e<\alpha}$ as in Lemma 4.7. For any $e < \alpha$, e is said to be *the least index for* $\{e' : Q_{e'} = Q_e\}$ *via* g, if

$$\exists i < \hat{\alpha}\, \forall j < i\, (g(i) = e \wedge Q_{g(i)} \neq Q_{g(j)}).$$

We denote the characteristic function of this predicate by $\mathsf{L}_{Q,g}(e)$ (or $\mathsf{L}(e)$ for short).

A Friedberg numbering $\{A_e\}_{e<\alpha}$ will be constructed and the requirements are as follows.

$$\text{Requirement } e: \quad R_{P,e} : \exists! \rho\, (P_e = A_\rho),$$
$$R_{Q,e} : \exists! \rho\, (Q_e = A_\rho).$$

The strategy for satisfying requirement e consists of the following:

(i) assign a unique *follower* $\rho = F^*_P(e)$ to P_e with the objective of making $A_{F^*_P(e)}$ equal to P_e;

(ii) assign a unique follower $\rho = F^*_Q(e)$ to Q_e, whenever $\mathsf{L}(e) = 1$, with the objective of making $A_{F^*_Q(e)}$ equal to Q_e; and

(iii) for every $\rho < \alpha$, assign ρ to a unique set from $\{P_e\}_{e<\alpha} \cup \{Q_e\}_{e<\alpha, \mathsf{L}(e)=1}$, such that ρ is the follower of the corresponding set.

More precise definitions of F^*_P and F^*_Q will be given in the part of construction and the part of verification.

The strategy works effectively, except for the fact that "$\mathsf{L}(e) = 1$" is not a recursive predicate. Nevertheless, it will soon be seen that "$\mathsf{L}(e) = 1$" has an effective approximation $\mathsf{L}'(\eta, e)$ (See Lemma 4.9). For the moment assume that Lemma 4.9 holds, i.e.

$$\mathsf{L}(e) = 1 \leftrightarrow \lim_{\eta \to \alpha} \mathsf{L}'(\eta, e) = 1,$$

where L' is α-recursive. Then at each stage η, the construction will proceed as follows.

Step One assign a follower to Q_e, if $e < \eta$, Q_e has no follower and $\mathsf{L}'(\eta, e) = 1$; *release* the follower of Q_e (which was assigned before stage η, if any), whenever $e \geq \eta$ or $\mathsf{L}'(\eta, e) = 0$;

Step Two assign a follower to P_e, if $e < \eta$ and P_e has no follower;

Step Three for all $\rho \in [0, \eta) \cup \{\rho : \rho \text{ is relased at step one}\}$, if ρ has not been assigned to any set by the end of step two, then assign ρ to some P_d such that P_d has not been assigned to any follower and $P_d \supseteq \bigcup_{\delta<\eta} A_{\rho,\delta}$.

Step Four if $F_P(\eta, e)$ is a follower of P_e and $F_Q(\eta, e')$ is a follower of $Q_{e'}$ by the end of step three, then let $A_{F_P(\eta,e),\eta} = P_{e,\eta}$ and $A_{F_Q(\eta,e'),\eta} = Q_{e',\eta}$.

This strategy succeeds, because

(i) each P_e has a follower and never releases its follower,

(ii) eventually Q_e has a permanent follower after some stage if and only if $\mathsf{L}(e) = 1$, and

(iii) each ρ, as a follower, is released at most once, after which it will be a permanent follower of a P set or a Q set.

More details will be given in the part of verification.

To approximate $\mathsf{L}(e)$, the notion of the *greatest common length* of $Q_{g(i)}$ and $Q_{g(j)}$, $\forall j < i$ will be introduced. To define this notion, we first prove Lemma 4.8. Lemma 4.8 claims that for an $i < \hat{\alpha}$, the statement that $W_{g(i)}$ is not equal to $W_{g(j)}$ for any $j < i$ is equivalent to the existence of an upper bound $b < \hat{\alpha}$, such that the least difference of $W_{g(i)}$ with any $W_{g(j)}$ for $j < i$, after the mapping g^{-1}, lies below b and is seen by stage $f(b)$.

Lemma 4.8. *If $i < \hat{\alpha}$, then*

$$\forall j < i\, (Q_{g(i)} \neq Q_{g(j)}) \leftrightarrow \exists b < \hat{\alpha} \forall j < i\, \exists x < b$$
$$(Q_{g(i),f(b)}(g(x)) = Q_{g(i)}(g(x)) \neq Q_{g(j)}(g(x)) = Q_{g(j),f(b)}(g(x))). \quad (4.3)$$

Proof. We only prove the direction from left to right.

Suppose $\forall j < i\, (Q_{g(i)} \neq Q_{g(j)})$. Then

$$\forall j < i\, \exists x < \hat{\alpha}\, \exists \gamma < \hat{\alpha}\, (Q_{g(i),f(\gamma)}(g(x)) = Q_{g(i)}(g(x)) \neq Q_{g(j)}(g(x)) = Q_{g(j),f(\gamma)}(g(x))).$$

Since the matrix of the above formula is Σ_2, Lemma 2.16 provides a $b < \hat{\alpha}$ such that the right hand side of (4.3) holds. □

In the proof of Theorem 4.6, Lemma 4.8 is the only place where the function f is involved. In its proof, Lemma 4.8 essentially applies the condition $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$. By Lemma 4.8, the greatest common length is measured within $\hat{\alpha}$ through the map g. One advantage of this measure has to do with the regularity. That is, $(g^{-1} \upharpoonright W) \cap \delta$ is α-finite for any α-r.e. set W and $\delta < \hat{\alpha}$, since $\hat{\alpha} = t\sigma 2p\,(\alpha) \leq \sigma 1p\,(\alpha)$ and g is tame Σ_2 (the tame Σ_2 property of g ensures that $g_\eta \upharpoonright \delta = g \upharpoonright \delta$, so $(g^{-1} \upharpoonright W) \cap \delta = (g_\eta^{-1} \upharpoonright W) \cap \delta$ for all sufficiently large η; and $\delta < \hat{\alpha} \leq \sigma 1p\,(\alpha)$ ensures that any α-r.e. subset of $g \upharpoonright \delta$ is α-finite). An arbitrary α-r.e. set W, however, need not be regular.

Suppose $e, \eta < \alpha$. The *greatest common length with respect to e through g at* *stage* η is defined as

$$c_g(\eta, e) = \begin{cases} \max^*\{b < \min\{\hat{\alpha}, \eta\} : \exists j < g_\eta^{-1}(e)(Q_{e,\eta} \upharpoonright \operatorname{ran}(g_\eta \upharpoonright b) = \\ \qquad\qquad\qquad\qquad Q_{g_\eta(j),\eta} \upharpoonright \operatorname{ran}(g_\eta \upharpoonright b))\} \\ \qquad\qquad\qquad\qquad \text{if } e < \eta \text{ and } e \in \operatorname{ran}(g_\eta \upharpoonright \min\{\hat{\alpha}, \eta\}), \\ 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Note that c_g is an α-recursive function.

The index e is said to be *the least index for* $\{e' : Q_{e'} = Q_e\}$ *via* g *at stage* η, if

$$\exists \delta < \eta \, \forall \rho \, (\delta \leq \rho \leq \eta \rightarrow c_g(\rho, e) = c_g(\eta, e) < \hat{\alpha}),$$

and the characteristic function of this relation is denoted by $\mathsf{L}'_{Q,g}(\eta, e)$ (or $\mathsf{L}'(\eta, e)$ for short). Notice that $\mathsf{L}'_{Q,g}(\eta, e)$ (or $\mathsf{L}'(\eta, e)$) is α-recursive.

Lemma 4.9. $\mathsf{L}(e) = 1 \leftrightarrow \lim_{\eta \to \alpha} \mathsf{L}'(\eta, e) = 1$.

Proof. Let $i = g^{-1}(e) < \hat{\alpha}$.

Suppose $\mathsf{L}(e) = 1$. Then $\forall j < i \, (Q_{g(i)} \neq Q_{g(j)})$. As in Lemma 4.8, there is a $b_0 < \hat{\alpha}$ such that

$$\forall j < i \, \exists x < b_0 \, (Q_{e,f(b_0)}(g(x)) = Q_e(g(x)) \neq Q_{g(j)}(g(x)) = Q_{g(j),f(b_0)}(g(x))).$$

Thus,

$$\forall j < i \, \forall \eta > f(b_0) \, (Q_{e,\eta} \upharpoonright \operatorname{ran}(g \upharpoonright b_0)) \neq Q_{g(j),\eta} \upharpoonright (\operatorname{ran}(g \upharpoonright b_0))).$$

Let η_0 be a stage such that

$$\forall \eta > \eta_0 \, (g_\eta \upharpoonright (\max\{i, b_0\} + 1) = g \upharpoonright (\max\{i, b_0\} + 1)).$$

Also, it follow easily from $t\sigma 2p\,(\alpha) \leq \sigma 1p\,(\alpha)$ and Theorem 2.12 that there is an η_1 such that

$$\forall j \leq i \, (Q_{g(j)} \upharpoonright \operatorname{ran}(g \upharpoonright b_0) = Q_{g(j),\eta_1} \upharpoonright \operatorname{ran}(g \upharpoonright b_0)).$$

Then for any $\eta > \max\{\eta_0, \eta_1, f(b_0)\}$,

$$\begin{aligned} c_g(\eta, e) &= \max^*\{b < \min\{\hat{\alpha}, \eta\} : \exists j < i \, (Q_{e,\eta} \upharpoonright \operatorname{ran}(g_\eta \upharpoonright b) = Q_{g_\eta(j),\eta} \upharpoonright \operatorname{ran}(g_\eta \upharpoonright b))\} \\ &= \max^*\{b < b_0 : \exists j < i \, (Q_e \upharpoonright \operatorname{ran}(g \upharpoonright b) = Q_{g(j)} \upharpoonright \operatorname{ran}(g \upharpoonright b))\} \end{aligned}$$

is a constant less than $\hat{\alpha}$, and so $\lim_{\eta\to\alpha} \mathsf{L}'(\eta, e) = 1$.

Now assume δ is a stage such that $\forall\eta > \delta\,(\mathsf{L}'(\eta, e) = 1)$. Then $\forall\eta > \delta\,(c_g(\eta, e) = c_g(\delta, e) < \hat{\alpha})$. For the sake of contradiction, suppose $j < i$ and $Q_{g(j)} = Q_{g(i)} = Q_e$.

Similar to the existence of η_0 and η_1 above, there is a stage $\eta_2 > c_g(\delta, e) + 1$ such that

$$\begin{aligned}\forall\eta > \eta_2\,[g_\eta \restriction (\max\{i, c_g(\delta, e)\} + 1) &= g \restriction (\max\{i, c_g(\delta, e)\} + 1)\\ \wedge\ Q_{g(j)} \restriction \mathrm{ran}(g \restriction (c_g(\delta, e) + 1)) &= Q_{g(j),\eta} \restriction \mathrm{ran}(g \restriction (c_g(\delta, e) + 1))\\ \wedge\ Q_e \restriction \mathrm{ran}(g \restriction (c_g(\delta, e) + 1)) &= Q_{e,\eta} \restriction \mathrm{ran}(g \restriction (c_g(\delta, e) + 1))].\end{aligned}$$

Thus, $c_g(\eta, e) \geq c_g(\delta, e) + 1$ for each $\eta > \eta_2$, a contradiction. $\square$

Construction

At each stage η, the construction below is carried out in four steps as described earlier. Two α-recursive functions $F_P(\eta, e)$ and $F_Q(\eta, e)$ are defined to denote the follower of P_e at stage η and the follower of Q_e at stage η respectively. During the construction, ρ is said to be *unused* if ρ has not been in the range of F_P and F_Q defined so far.

The construction proceeds as follows.

At stage η. **Step One.** For each $e < \alpha$,

Case 1.1: $e \geq \eta$ or $\mathsf{L}'(\eta, e) = 0$. Set $F_Q(\eta, e) = -1$.

Case 1.2: Case 1.1 fails and either η is a limit ordinal such that $\lim_{\gamma\to\eta} F_Q(\gamma, e) \neq -1$ exists or $\eta = \eta' + 1$ is a successor ordinal such that $F_Q(\eta', e) \geq 0$. Then let

$$F_Q(\eta, e) = \begin{cases} \lim_{\gamma\to\eta} F_Q(\gamma, e) & \text{if } \eta \text{ is a limit ordinal,}\\ F_Q(\eta', e) & \text{if } \eta = \eta' + 1.\end{cases}$$

Case 1.3: Case 1.1 and Case 1.2 fail. Let $e_0 < e_1 < \ldots < e_\zeta < \ldots$ be a list of all e's of Case 1.3 and $\rho_0 < \rho_1 < \ldots < \rho_\zeta < \ldots$ be a list of all unused ρ. Let $F_Q(\eta, e_\zeta) = \rho_\zeta$ for each e_ζ.

Step Two. For any $e < \alpha$,

Case 2.1. Either η is a limit ordinal such that $\lim_{\gamma\to\eta} F_P(\gamma, e) \neq -1$ exists or

$\eta = \eta' + 1$ is a successor ordinal such that $F_P(\eta', e) \geq 0$. Then set

$$F_P(\eta, e) = \begin{cases} \lim_{\gamma \to \eta} F_P(\gamma, e) & \text{if } \eta \text{ is a limit ordinal,} \\ F_P(\eta', e) & \text{if } \eta = \eta' + 1. \end{cases}$$

Case 2.2. Case 2.1 fails and $e < \eta$. Similar to Case 1.3, define $F_P(\eta, e)$ to be ρ'_ζ, whenever e is the ζ^{th} ordinal in Case 2.2 and ρ'_ζ is the ζ^{th} unused ρ by the end of step one.

Case 2.3. Case 2.1 and Case 2.2 fail. $F_P(\eta, e)$ will be defined in step three.

Step Three. Let $\rho''_0 < \rho''_1 < \ldots < \rho''_\zeta < \ldots$ be a list of ρ's such that

(i) The ordinal ρ is not $F_P(\eta, e)$ and not $F_Q(\eta, e')$ for any defined $F_P(\eta, e)$ and $F_Q(\eta, e')$, and

(ii) Either $\rho < \eta$ or $\rho \in \{F_P(\delta, d) : d < \alpha, \delta < \eta\} \cup \{F_Q(\delta, d) : d < \alpha, \delta < \eta\} \setminus \{-1\}$.

Now recursively define

$$e''_\zeta = \text{ the first enumerated } e > \sup_{\zeta' < \zeta} e''_{\zeta'} \text{ such that } P_e \supseteq \textstyle\bigcup_{\delta < \eta} A_{\rho''_\zeta, \delta}$$
$$\text{and that } F_P(\eta, e) \text{ is undefined by the end of step two.}$$

Define $F_P(\eta, e''_\zeta)$ to be ρ''_ζ.

Finally, for $F_P(\eta, e)$ still undefined, let $F_P(\eta, e) = -1$.

Step Four. For any $\rho < \alpha$, if $\rho = F_P(\eta, e)$, then let $A_{\rho,\eta} = (\bigcup_{\zeta<\eta} A_{\rho,\zeta}) \cup P_{e,\eta}$; if $\rho = F_Q(\eta, e)$, then let $A_{\rho,\eta} = (\bigcup_{\zeta<\eta} A_{\rho,\zeta}) \cup Q_{e,\eta}$. Otherwise, let $A_{\rho,\eta} = \bigcup_{\zeta<\eta} A_{\rho,\zeta}$.

Verification

Clause (iii) of the next lemma implies that the above construction is α-recursive.

Lemma 4.10. *Assume $\eta < \alpha$.*

(i) For all $e < \eta$, $F_P(\eta, e) \geq 0$ and ($F_Q(\eta, e) \geq 0 \leftrightarrow \mathsf{L}'(\eta, e) = 1$);

(ii) $\eta \subseteq \mathrm{ran}(F_P \upharpoonright (\{\eta\} \times \alpha)) \cup \mathrm{ran}(F_Q \upharpoonright (\{\eta\} \times \alpha))$, i.e. each $\rho < \eta$ becomes a follower of some set from stage η onwards;

(iii) $\{e : F_P(\eta, e) \neq -1\}$, $\{e : F_Q(\eta, e) \neq -1\}$, $\mathrm{ran}(F_P \restriction (\{\eta\} \times \alpha)) \setminus \{-1\}$ *and* $\mathrm{ran}(F_Q \restriction (\{\eta\} \times \alpha)) \setminus \{-1\}$ *are α-finite;*

(iv) $\forall e, e' \, (F_P(\eta, e), F_Q(\eta, e') \geq 0 \rightarrow F_P(\eta, e) \neq F_Q(\eta, e'))$ *and* $\forall e, e' \, ((F_P(\eta, e) = F_P(\eta, e') \geq 0) \vee (F_Q(\eta, e) = F_Q(\eta, e') \geq 0) \rightarrow e = e')$*. In other words, at stage η, the assignment of followers is one-one;*

(v) $\forall e \, (F_P(\eta, e) \geq 0 \rightarrow \forall \delta > \eta \, F_P(\delta, e) = F_P(\eta, e))$*, i.e. P_e never releases its follower for any e;*

(vi) $\forall e \, (\eta > e \wedge \forall \delta \geq \eta \, (\mathsf{L}'(\delta, e) = 1) \rightarrow \forall \delta > \eta \, (F_Q(\delta, e) = F_Q(\eta, e)))$*, i.e. Q_e never release its follower after stage η if e is thought to be the least index via g from stage η onwards;*

(vii) *$A_{\rho,\eta}$ is equal to $P_{e,\eta}$ if $F_P(\eta, e) = \rho$, and is equal to $Q_{e,\eta}$ if $F_Q(\eta, e) = \rho$.*

Proof. By induction on η and δ (δ is as in Clause (v)-(vi)). $\square$

Define $F_P^*, F_Q^* : \alpha \rightarrow \alpha \cup \{-1\}$ by

$$F_P^*(e) = \lim_{\eta \to \alpha} F_P(\eta, e), \quad F_Q^*(e) = \lim_{\eta \to \alpha} F_Q(\eta, e).$$

That is, $F_P^*(e)$ is the permanent follower of P_e; and $F_Q^*(e)$, if defined, is the permanent follower of Q_e.

Part (i), (v) and (vi) of Lemma 4.10 together imply that

$$\forall e \, (F_P^*(e) \downarrow \neq -1), \quad \forall e \, (\mathsf{L}(e) = 1 \rightarrow F_Q^*(e) \downarrow \neq -1).$$

For $e < \alpha$ such that $\mathsf{L}(e) = 0$, Lemma 4.9 implies that there are cofinally many stages η satisfying $\mathsf{L}'(\eta, e) = 0$, and so there are cofinally many stages η such that $F_Q(\eta, e) = -1$. Thus,

$$\forall e \, (\mathsf{L}(e) = 0 \rightarrow F_Q^*(e) \uparrow \vee F_Q^*(e) = -1).$$

By (iv), the assignment of permanent followers is one-one, i.e.

$$\begin{aligned} \forall e, e' \, [(\mathsf{L}(e') = 1 \rightarrow F_P^*(e) \neq F_Q^*(e')) \wedge (e \neq e' \rightarrow F_P^*(e) \neq F_P^*(e')) \\ \wedge \, (e \neq e' \wedge \mathsf{L}(e) = \mathsf{L}(e') = 1 \rightarrow F_Q^*(e) \neq F_Q^*(e'))]. \quad (4.4) \end{aligned}$$

According to (vii),

$$\forall e\,(P_e = A_{F^*_P(e)}),\ \text{and}\ \forall e\,(\mathsf{L}(e) = 1 \to Q_e = A_{F^*_Q(e)}).$$

Consequently, $\{A_e\}_{e<\alpha}$ is a universal numbering of all α-r.e. sets. To show that $\{A_e\}_{e<\alpha}$ is a Friedberg numbering, it is only necessary to show that $\{A_e\}_{e<\alpha}$ is one-one. Observe that by (4.4), $\{A_e\}_{e<\alpha}$ being one-one is immediate once $\alpha \subseteq \mathrm{ran}(F^*_P) \cup \mathrm{ran}(F^*_Q)$, i.e. each ρ is a permanent follower for someone, has been proved.

Let $\rho < \alpha$. By (v), if $\rho = F_P(\eta, e)$ for some η, e, then $\rho = F^*_P(e) \in \mathrm{ran}(F^*_P)$. Now suppose $\rho \neq F_P(\eta, e)$ for all η and e. Then at stage $\rho + 1$, according to (ii), $\rho = F_Q(\rho + 1, e')$. Moreover, $\forall \eta > \rho + 1\,(\rho = F_Q(\eta, e'))$. Otherwise, at the least stage $\eta > \rho + 1$ with $\rho \neq F_Q(\eta, e')$, it is defined in step three that $F_P(\eta, e'') = \rho$ for some e'', yielding a contradiction. Since $\forall \eta > \rho\,(\rho = F_Q(\eta, e'))$, we immediately get $\mathsf{L}(e') = 1$ and $\rho = F^*_Q(e')$.

4.2.3 Pseudostability

Through out this section of pseudostability, we make the assumption that $\sigma 1p\,(\alpha) > \omega$, which is only necessary for Lemma 4.20. In this section, we introduce the notion of pseudostability and generalize some properties of α-stable ordinals to pseudostable ordinals. In Section 4.2.4, pseudostability will be used to show the nonexistence of a Friedberg numbering when $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, which is stronger than $\sigma 1p\,(\alpha) > \omega$ (since $\sigma 2cf(\alpha) \geq \omega$). Under the assumption that $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, all proofs in this section remain the same.

Suppose $\{A_e\}_{e<\alpha}$ is an arbitrary numbering. As noticed in Section 4.2.1, α-stable ordinals are used to obtain, roughly speaking, an upper bound of the least differences between a given α-finite set C and α-finitely many α-r.e. sets of the numbering. That idea succeeds mainly because of the following property: for any ζ and α-finite set C, if $\delta < \sigma 1p\,(\alpha)$ and β is a large enough α-stable ordinal, then

$$\forall e < \delta\,(A_e \supseteq C \cup [\zeta, \beta) \leftrightarrow A_{e,\beta} \supseteq C \cup [\zeta, \beta)).$$

Pseudostable ordinals are defined mainly by this property.

Lemma 4.11. *Suppose $\{A_e\}_{e<\alpha}$ is a numbering. Then there exists an α-recursive function $h : \alpha^4 \to \alpha$, such that: for any $\gamma < \alpha$, α-finite set $C \subset \alpha$, and α-finite (partial) function $p : \alpha \xrightarrow{\text{one-one}} \alpha$ satisfying $|dom(p)|_\alpha < \sigma 1p\,(\alpha)$, we have* *(i) For each $\eta < \alpha$, $h(\eta, \gamma, C, p) \leq \eta$ is defined.*

(ii) The sequence $\{h(\eta, \gamma, C, p)\}_{\eta<\alpha}$ is nondecreasing.

(iii) There is a $\beta < \alpha$ such that

$$\beta = \lim_{\eta \to \alpha} h(\eta, \gamma, C, p) = h(\beta, \gamma, C, p) > \max\{\gamma, C, \sup C, p\}, ^\dagger$$

and

$$\forall e \in \mathrm{ran}(p)\, (A_e \supseteq C \cup [\sup C, \beta) \;\leftrightarrow\; A_{e,\beta} \supseteq C \cup [\sup C, \beta)). \tag{4.5}$$

The rest of this section is devoted to the proof of Lemma 4.11.

An ordinal $\beta < \alpha$ is said to be *pseudostable relative to the numbering* $\{A_e\}_{e<\alpha}$, if $\beta = \lim_{\eta \to \alpha} h(\eta, \gamma, C, p)$ for some h, γ, C, p satisfying all the requirements in Lemma 4.11. Immediately from the definition, for any C, p as in Lemma 4.11, pseudostable ordinals $\{\lim_{\eta \to \alpha} h(\eta, \gamma, C, p) : \gamma < \alpha\}$ are cofinal in α.

In the construction given in Section 4.2.4, Lemma 4.11 is applied as follows: the function p is an initial segment of the graph of a tame Σ_2 projection from $t\sigma 2p\,(\alpha)$ to α, γ is a stage such that all approximations related to the initial segment have reached their final limit, and C is an initial segment of the set to be constructed.

The method of proof of Lemma 4.11 consists of a Skolem hull argument below α with respect to the property (4.5) and, roughly speaking, coding the approximation of the Skolem hull construction into the enumeration of an α-r.e. subset with α-cardinality less than $\sigma 1p\,(\alpha)$. By Theorem 2.12, the α-r.e. set is α-finite. Thus, its enumeration terminates before α. Consequently, the Skolem hull is also below α.

Skolem hull argument

From now on, γ, C, p are as in Lemma 4.11 and fixed. For each $n < \omega$, define the Skolem function

$$\begin{aligned} z_0(\gamma, C, p) &= \max\{\gamma, C, \sup C, p\} + 1, \\ z_{n+1}(\gamma, C, p) &= \mu z \geq z_n(\gamma, C, p) (\forall e \in \mathrm{ran}(p)\, (A_e \supseteq C \cup [\sup C, z_n(\gamma, C, p)) \;\rightarrow \\ &\qquad\qquad A_{e,z} \supseteq C \cup [\sup C, z_n(\gamma, C, p))). \end{aligned}$$

†The canonical enumeration of sets in L_α allows a canonical effective coding of α-finite sets (See [39]). Here, $\beta > C$ and $\beta > p$ mean that β is greater than the code of C and p respectively.

To simply the notation, we suppress the parameters of $z_n(\gamma, C, p)$ unless the possibility of confusion arises. Note that $\{z_n : n < \omega\}$ may not be stable or pseudostable ordinals. In fact, later we will see that $\max^*_{n<\omega} z_n$ is a pseduostable ordinal.

Lemma 4.12. $\{z_n : n < \omega\} \subseteq \alpha$.

Proof. Since p is one-one, $|\operatorname{ran}(p)|_\alpha = |\operatorname{dom}(p)|_\alpha < \sigma 1p\,(\alpha)$. Thus, any α-r.e. subset of $\operatorname{ran}(p)$ is α-finite, by Theorem 2.12.

By induction on n, if $z_n < \alpha$, the set $\{e \in \operatorname{ran}(p) : A_e \supseteq C \cup [\sup C, z_n)\}$ is α-finite. Hence $z_{n+1} < \alpha$ by Σ_1 replacement. It follows that $\{z_n : n < \omega\} \subseteq \alpha$. □

Lemma 4.13. $\forall n \forall \eta \geq z_{n+1} \forall e \in \operatorname{ran}(p)\,(A_e \supseteq C \cup [\sup C, z_n) \leftrightarrow A_{e,\eta} \supseteq C \cup [\sup C, z_n))$.

Proof. By the definition of z_{n+1} and the fact that $\{A_e\}_{e<\alpha}$ are α-r.e. sets. □

Let $\beta(\gamma, C, p) = \max^*_{n<\omega} z_n(\gamma, C, p)$. Again, we suppress parameters of $\beta(\gamma, C, p)$ for simplicity.

Lemma 4.14. $\forall e \in \operatorname{ran}(p)\,(A_e \supseteq C \cup [\sup C, \beta) \leftrightarrow A_{e,\beta} \supseteq C \cup [\sup C, \beta))$.

Proof. For any $e \in \operatorname{ran}(p)$,

$$\begin{aligned}
&A_e \supseteq C \cup [\sup C, \beta) \\
\leftrightarrow\ &\forall n < \omega\,(A_e \supseteq C \cup [\sup C, z_n)) \\
\leftrightarrow\ &\forall n < \omega\,(A_{e,\beta} \supseteq C \cup [\sup C, z_n)) \quad \text{by Lemma 4.13} \\
\leftrightarrow\ &A_{e,\beta} \supseteq C \cup [\sup C, \beta). \square
\end{aligned}$$

It will be shown later that $\beta < \alpha$. For the moment assume that this is true. To prove Lemma 4.11, it remains to define h by the approximation of $\{z_n\}_{n<\omega}$, so that $\beta = \lim_{\eta \to \alpha} h(\eta, \gamma, C, p)$.

At stage η, define the approximation of $\{z_n\}_{n<\omega}$ by induction on $n < \omega$ as follows:

$$z_{0,\eta} = \min\{z_0, \eta\},$$

$$\begin{aligned}
z_{n+1,\eta} = \max\{&\max^*_{\eta'<\eta} z_{n+1,\eta'}, \mu z \leq \eta[(z \geq z_{n,\eta}) \wedge \\
&\forall e \in \operatorname{ran}(p)\,(A_{e,\eta} \supseteq C \cup [\sup C, z_{n,\eta}) \to A_{e,z} \supseteq C \cup [\sup C, z_{n,\eta}))]\}.
\end{aligned}$$

In the definition of $z_{n+1,\eta}$, "$\max^*_{\eta'<\eta} z_{n+1,\eta'}$" ensures that $z_{n+1,\eta}$ is nondecreasing with respect to η, and "$A_{e,\eta} \supseteq C \cup [\sup C, z_{n,\eta}) \to A_{e,z} \supseteq C \cup [\sup C, z_{n,\eta})$" is a Skolem hull construction.

Lemma 4.15. *Suppose $n < \omega$. Then*

(i) $\{z_{n,\eta}\}_{\eta<\alpha}$ *is a nondecreasing sequence;*

(ii) $\forall \eta\, (z_{n,\eta} \leq \min\{z_n, \eta\})$*;*

(iii) $\forall \eta \geq z_n\, (z_{n,\eta} = z_n)$*.*

Proof. Clause (i) is immediate from the definition of $z_{n,\eta}$. Also from the definition of $z_{n,\eta}$, an induction on η shows $\forall \eta\, \forall n\, (z_{n,\eta} \leq \eta)$. Hence $\forall \eta < z_n\, \forall n\, (z_{n,\eta} \leq \min\{z_n, \eta\})$. Therefore, to prove (ii), only (iii) needs to be shown.

Clause (iii) is proved by induction on n and η. We omit the details. $\square$

For any $\eta < \alpha$, define

$$h(\eta, \gamma, C, p) = \max{}^*_{n<\omega} z_{n,\eta}.$$

By Lemma 4.15,

$$\forall \eta \geq \beta\, (h(\eta, \gamma, C, p) = \beta).$$

It is easy to check (i)-(iii) of Lemma 4.11. To complete the proof of Lemma 4.11, it remains only to verify that $\max^*_{n<\omega} z_n < \alpha$, i.e. $\beta < \alpha$. The following lemma deals with a special case and is straightforward to verify.

Lemma 4.16. *If $z_{n+1} = z_n$, then $\forall m < \omega\, (m > n \to z_m = z_n)$.*

Lemma 4.16 suggests that if $z_n = z_{n+1}$, for some $n < \omega$, then $\beta = \max^*_{m<\omega} z_m = z_n < \alpha$. Thus, to show $\beta < \alpha$ in general, we only need to check the case when $\{z_n\}_{n<\omega}$ is strictly increasing. That case will be addressed in the coding part below.

Coding

Let γ, C, p be given and $\{z_n\}_{n<\omega}$ be defined as in previous part of Skolem hull argument. In this part, we always assume that $\{z_n\}_{n<\omega}$ is strictly increasing. Then it is immediate from the definition of z_n that

$$\forall n < \omega\, \exists\, e \in \operatorname{ran}(p)\, (A_e \supseteq C \cup [\sup C, z_n)). \tag{4.6}$$

With the above formula in mind, it is straightforward to code the approximation of β by enumerating (n,e) such that, (n,e) is enumerated at stage η if $A_{e,\eta} \supseteq C \cup [\sup C, z_{n,\eta})$. It is tempting to assume (mistakenly) that $z_{n+1,\eta} = z_{n+1}$ if and only if the (n,e)'s have completed their enumeration at stage η. Nevertheless, in that event, the enumeration of (n,e)'s may terminate before the enumeration of some (m,e'), $m < n$, due to the approximation of z_n and z_m, $m < n$. The trick to cover this possibility is to incorporate the enumeration of the (m,e')'s, for all $m < n$, in the enumeration of the (n,e)'s: Suppose at stage η, $A_{e,\eta} \supseteq C \cup [\sup C, z_{n,\eta})$. Then $(n, e_0, e_1, \ldots, e_n)$ is enumerated if $(n-1, e_0, e_1, \ldots, e_{n-1})$ is enumerated by stage η. Then for $n > 0$, the enumeration of the $(n, e_0, e_1, \ldots, e_n)$'s does not terminate whenever some $(n-1, e'_0, e'_1, \ldots, e'_{n-1})$ is yet to be enumerated.

More precisely, define an α-r.e. set $D \subseteq \bigcup_{n<\omega}(\{n\} \times \mathrm{ran}(p)^{n+1})$ as follows, where

$$\bigcup_{n<\omega}(\{n\} \times \mathrm{ran}(p)^{n+1}) = \{(n, e_0, e_1, \ldots, e_n) : n < \omega, e_0, e_1, \ldots, e_n \in \mathrm{ran}(p)\}.$$

Suppose $\eta < z_0$. Then let $D_\eta = \emptyset$.

At stage $\eta \geq z_0$, the enumeration of D_η is carried out in ω steps. Let

$$D_{\eta,0} = (\bigcup_{\eta'<\eta} D_{\eta'}) \cup \{(0,e) : e \in \mathrm{ran}(p) \wedge A_{e,\eta} \supseteq C \cup [\sup C, z_0)\},$$

and if $n > 0$,

$$D_{\eta,n} = (\bigcup_{m<n} D_{\eta,m}) \cup \{(n, e_0, e_1, \ldots, e_n) : e_0, e_1, \ldots, e_n \in \mathrm{ran}(p) \wedge \\ A_{e_n,\eta} \supseteq C \cup [\sup C, z_{n,\eta}) \wedge (n-1, e_0, e_1, \ldots, e_{n-1}) \in D_{\eta,n-1}\}.$$

$$D_\eta = \bigcup_{n<\omega} D_{\eta,n}.$$

Then let $D = \bigcup_{\eta<\alpha} D_\eta$.

Lemma 4.17. *If $n > 0$, $(n-1, e_0, e_1, \ldots, e_{n-1}) \in D$ and $A_{e_n} \supseteq C \cup [\sup C, z_n)$, then $(n, e_0, e_1, \ldots, e_n) \in D$.*

Proof. Let $\eta > z_n$ be large enough such that $(n-1, e_0, e_1, \ldots, e_{n-1}) \in D_\eta$, and $C \cup [\sup C, z_n) \subseteq A_{e_n,\eta}$. Since $\eta > z_n$, we have $z_{n,\eta} = z_n$. Thus, $(n, e_0, e_1, \ldots, e_n) \in D_\eta$. □

Lemma 4.18. *For any $n < \omega$ and $\eta < \alpha$,*

$$\eta \geq z_{n+1} \leftrightarrow D_\eta \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}) = D \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}). \tag{4.7}$$

Proof. The lemma is proved by induction on n.

Let $n = 0$. By the definition of D_η, for all $e \in \mathrm{ran}(p)$ and $\eta < \alpha$,

$$(0, e) \in D_\eta \leftrightarrow (\eta \geq z_0 \wedge A_{e,\eta} \supseteq C \cup [\sup C, z_0)).$$

Thus, for any $e \in \mathrm{ran}(p)$,

$$(0, e) \in D \leftrightarrow A_e \supseteq C \cup [\sup C, z_0),$$

According to (4.6), $D \restriction (\{0\} \times \mathrm{ran}(p)) \neq \emptyset$. Therefore,

$$D \restriction (\{0\} \times \mathrm{ran}(p)) = D_\eta \restriction (\{0\} \times \mathrm{ran}(p)) \rightarrow \eta \geq z_1.$$

The other direction of (4.7) for $n = 0$ is immediate from the definition of z_1.

With the intention of showing (4.7) when $n > 0$, assume that (4.7) is true for $0, \ldots, n-1$. Pick any $\eta < \alpha$. We consider three cases.

Case 1. $\eta < z_n$. Since (4.7) is true for $n-1$, let $(n-1, e_0, e_1, \ldots, e_{n-1}) \in D \setminus D_\eta$. Let $e_n \in \mathrm{dom}(p)$ be any index such that $A_{e_n} \supseteq C \cup [\sup C, z_n)$. Then $(n, e_0, e_1, \ldots, e_n) \in D \setminus D_\eta$. Hence $D_\eta \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}) \neq D \restriction (\{n\} \times \mathrm{ran}(p)^{n+1})$.

Case 2. $z_n \leq \eta < z_{n+1}$. Then $z_{n,\eta} = z_n$ and by the definition of z_{n+1}, there is some $e_n \in \mathrm{ran}(p)$ such that $A_{e_n} \supseteq C \cup [\sup C, z_n)$ but $A_{e_n,\eta} \not\supseteq C \cup [\sup C, z_n)$. Let $x \in C \cup [\sup C, z_n) \setminus A_{e_n,\eta}$. Since $z_n > \sup C$, we have $x < z_n$.

Subcase 2.1. there exists $(n-1, e_0, e_1, \ldots, e_{n-1}) \in D \setminus \bigcup_{\eta' < z_n} D_{\eta'}$. Then

(i) Since $(n-1, e_0, e_1, \ldots, e_{n-1}) \notin \bigcup_{\eta' < z_n} D_{\eta'}$, $(n, e_0, e_1, \ldots, e_n) \notin \bigcup_{\eta' < z_n} D_{\eta'}$;

(ii) For any δ such that $z_n \leq \delta \leq \eta$, we have $(n, e_0, e_1, \ldots, e_n) \notin D_\delta \setminus \bigcup_{\delta' < \delta} D_{\delta'}$, as $A_{e_n,\delta} \not\supseteq C \cup [\sup C, z_n)$ and $z_n = z_{n,\delta}$.

Thus, $(n, e_0, e_1, \ldots, e_n) \in D \setminus D_\eta$. Hence $D_\eta \restriction (\{n\} \times \mathrm{ran}(p)^{n+1}) \neq D \restriction (\{n\} \times \mathrm{ran}(p)^{n+1})$.

Subcase 2.2. Subcase 2.1 fails. Then we claim that $\max^*_{\eta' < z_n} z_{n,\eta'} = z_n$. It will be proved in a moment. For now assume the claim and let $\eta' < z_n$ be such that $z_{n,\eta'} > x$. Since $\eta' < z_n$, there is $(n-1, e_0, e_1, \ldots, e_{n-1}) \in D \setminus D_{\eta'}$. Therefore,

(i) If $\delta \leq \eta'$, then $(n, e_0, e_1, \ldots, e_n) \notin D_\delta$ since $(n-1, e_0, e_1, \ldots, e_{n-1}) \notin D_\delta$;

(ii) If $\eta' < \delta \leq \eta$, then $z_{n,\delta} > x$ and $x \in C \cup [\sup C, z_{n,\delta}) \setminus A_{e_n,\delta}$. Therefore, $A_{e_n,\delta} \not\supseteq C \cup [\sup C, z_{n,\delta})$ and $(n, e_0, e_1, \ldots, e_n) \notin D_\delta \setminus \bigcup_{\delta'<\delta} D_{\delta'}$.

Thus, $(n, e_0, e_1, \ldots, e_n) \in D \setminus D_\eta$. Hence $D_\eta \upharpoonright (\{n\} \times \mathrm{ran}(p)^{n+1}) \neq D \upharpoonright (\{n\} \times \mathrm{ran}(p)^{n+1})$.

Case 3. $\eta \geq z_{n+1}$. One can see immediately that $z_{n,\eta} = z_n = z_{n,z_{n+1}}$. Suppose $A_{e_n,\eta} \supseteq C \cup [\sup C, z_{n,\eta})$ and $(n-1, e_0, e_1, \ldots, e_{n-1}) \in D_{\eta,n-1}$. Then, by the definition of z_{n+1}, $A_{e_n,z_{n+1}} \supseteq C \cup [\sup C, z_{n,z_{n+1}})$ and by (4.7) for $n-1$, $(n-1, e_0, e_1, \ldots, e_{n-1}) \in D_{z_{n+1}}$. Thus, $(n, e_0, e_1, \ldots, e_n) \in D_{z_{n+1}}$. Hence $D_\eta \upharpoonright (\{n\} \times \mathrm{ran}(p)^{n+1}) = D_{z_{n+1}} \upharpoonright (\{n\} \times \mathrm{ran}(p)^{n+1}) = D \upharpoonright (\{n\} \times \mathrm{ran}(p)^{n+1})$.

Finally, in Subcase 2.2, to see $\max^*_{\eta'<z_n} z_{n,\eta'} = z_n$, assume for a contradiction that $M = \max\{z_{n-1}, \max^*_{\eta'<z_n} z_{n,\eta'}\} < z_n$. Then there exists $(n-1, e^*_0, e^*_1, \ldots, e^*_{n-1}) \in D \setminus D_M$. Let δ be the first stage that $(n-1, e^*_0, e^*_1, \ldots, e^*_{n-1})$ is enumerated into D. Then by (4.7) for $n-1$ and the assumption of Subcase 2.2, we conclude $M < \delta < z_n$. Now

(a) If $n = 1$, then $z_0 \leq M < \delta < z_1$ and $(0, e^*_0) \in D_\delta \setminus D_M$. Since $(0, e^*_0) \in D_\delta$, by the definition of D_δ, $A_{e^*_0,\delta} \supseteq C \cup [\sup C, z_0)$. Then by the definition of $z_{1,\delta}$, $A_{e^*_0,z_{1,\delta}} \supseteq C \cup [\sup C, z_0) = C \cup [\sup C, z_{0,\delta})$. Therefore, $A_{e^*_0,M} \supseteq C \cup [\sup C, z_0)$ and $(0, e^*_0) \in D_M$, a contradiction.

(b) If $n \geq 2$, then $z_{n-1} \leq M < \delta < z_n$ and $(n-1, e^*_0, e^*_1, \ldots, e^*_{n-1}) \in D_\delta \setminus \bigcup_{\delta'<\delta} D_{\delta'}$. By definition of D_δ, $A_{e^*_{n-1},\delta} \supseteq C \cup [\sup C, z_{n-1,\delta}) = C \cup [\sup C, z_{n-1}) = C \cup [\sup C, z_{n-1,M})$ and $(n-2, e^*_0, e^*_1, \ldots, e^*_{n-2}) \in D_\delta$. Similar to the proof in (a), we have
$$A_{e^*_{n-1},M} \supseteq C \cup [\sup C, z_{n-1,M}).$$
And by (4.7) for $n-2$, $(n-2, e^*_0, e^*_1, \ldots, e^*_{n-2}) \in D_M$. Thus, $(n-1, e^*_0, e^*_1, \ldots, e^*_{n-1})$ is in D_M, again a contradiction. $\square$

Corollary 4.19. *For any $\eta < \alpha$,*
$$\eta \geq \beta \leftrightarrow D_\eta = D.$$

The next task to show that D is α-finite.

Lemma 4.20. *Every α-r.e. subset of $\bigcup_{n<\omega}(\{n\} \times \mathrm{ran}(p)^{n+1})$ is α-finite.*

Proof. Let $\kappa = \max\{|\mathrm{dom}(p)|_\alpha, \omega\}$. Since $|\mathrm{dom}(p)|_\alpha, \omega < \sigma 1p\,(\alpha)$, we have $\kappa < \sigma 1p\,(\alpha)$. Since p is one-one, it follows immediately that $|\,\mathrm{ran}(p)|_\alpha \leq \kappa$. Therefore, $|\{n\} \times \mathrm{ran}(p)^{n+1}|_\alpha \leq \kappa$ for all $n < \omega$. Furthermore, the α-finite bijections from $\{n\} \times \mathrm{ran}(p)^{n+1}$ to κ may be defined uniformly for all $n < \omega$. Hence $|\bigcup_{n<\omega}(\{n\} \times \mathrm{ran}(p)^{n+1})|_\alpha \leq \kappa < \sigma 1p\,(\alpha)$ and the lemma follows by Theorem 2.12. □

Lemma 4.19 and 4.20 combine to imply that D is α-finite. Hence

Lemma 4.21. $\max^*_{n<\omega} z_n < \alpha$, *i.e.* $\beta < \alpha$.

Observe at this point that Lemma 4.11 holds whenever $\sigma 1p\,(\alpha) > \omega$. Since no restriction on the numbering is required, if $\sigma 1p\,(\alpha) > \omega$, then Lemma 4.11 is applicable for any type of numberings. In particular, Lemma 4.11 is also true for a Gödel numbering when $\sigma 1p\,(\alpha) > \omega$. Notice that a Gödel numbering exists in L_α for all Σ_1 admissible ordinal α. Thus, in general, the nonexistence of a Friedberg numbering when $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$ (See Section 4.2.4) is not due to the existence of pseudostable ordinals.

4.2.4 When $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$

In this section, we prove

Theorem 4.22. *If $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, then there is no Friedberg numbering of α-r.e. sets.*

Since $t\sigma 2p\,(\alpha) > \sigma 2cf(\alpha)$, by Clause (1) of Corollary 2.15 , $\omega < \sigma 1p\,(\alpha)$ and $\sigma 2cf(\alpha) < \alpha$. Therefore, in this situation, the notion of pseudostability is applicable and Σ_2 replacement fails.

Let $\{A_e\}_{e<\alpha}$ be a one-one numbering, and let h be an α-recursive function satisfying Lemma 4.11. The objective is to construct an α-r.e. set X, so that $X \notin \{A_e\}_{e<\alpha}$. Thus, $\{A_e\}_{e<\alpha}$ is not a Friedberg numbering.

Fix the terminology as follows. Let

$$g : t\sigma 2p\,(\alpha) \xrightarrow[\text{onto}]{\text{one-one}} \alpha$$

be a tame Σ_2 projection, and according to Lemma 2.14 and Clause (2) of Corollary 2.15, let

$$f : \sigma 2cf(\alpha) \to t\sigma 2p\,(\alpha)$$

be a strictly increasing tame Σ_2 cofinal function so that $f(0) = 0$. Moreover, assume $f' : \alpha \times \sigma 2cf(\alpha) \to t\sigma 2p(\alpha)$, $g' : \alpha \times t\sigma 2p(\alpha) \to \alpha$ tamely generate f and g respectively. As in Section 4.2.2, f_η, g_η will be used to denote functions $\lambda x\,(f'(\eta, x))$ and $\lambda x\,(g'(\eta, x))$. Moreover, we assume that for all $\eta < \alpha$, f_η is nondecreasing and $\mathrm{ran}(f_\eta), \mathrm{ran}(g_\eta) \subseteq [0, \eta]$.

Strategy

As in Section 4.2.2, g makes it possible to arrange the indices of $\{A_e\}_{e<\alpha}$ on $t\sigma 2p(\alpha)$. The function f partitions $t\sigma 2p(\alpha)$ into $\sigma 2cf(\alpha)$ many blocks: $\{[f(i), f(i+1)) : i < \sigma 2cf(\alpha)\}$. $[f(i), f(i+1)))$ is said to be *the i^{th} block* (or *block i*) *of* $t\sigma 2p(\alpha)$. By *α-r.e. sets in the i^{th} block* (or *α-r.e. sets in block i*), we mean the α-r.e. sets are from the collection $\{A_e : g(e) \in [f(i), f(i+1))\}$. Since the numbering $\{A_e\}_{e<\alpha}$ is one-one, each α-r.e. set is in at most one block. The set X is constructed by diagonalizing against α-r.e. sets in each block.

Suppose $i < \sigma 2cf(\alpha)$, $\gamma < \alpha$, $C \subset \alpha$ is an α-finite set, and $\beta = \beta(\gamma, C, g \restriction f(i))$ is the pseudostable ordinal obtained in Lemma 4.11 when $p = g \restriction f(i)$, i.e.

$$\beta = \lim_{\eta \to \alpha} h(\eta, \gamma, C, g \restriction f(i)),$$

and $X \restriction \beta = C \cup [\sup C, \beta)$. Then it follows from Lemma 4.11 that

$$\forall e \in \mathrm{ran}(g \restriction f(i))\,(A_e \supseteq X \restriction \beta \leftrightarrow A_{e,\beta} \supseteq X \restriction \beta). \tag{4.8}$$

Since $\{A_e\}_{e<\alpha}$ is a one-one numbering, there is at most one e in the range of $g \restriction f(i)$ such that $A_e = X \restriction \beta$. Therefore, by (4.8), the set $\{e \in \mathrm{ran}(g \restriction f(i)) : A_e \supsetneq X \restriction \beta\}$ is α-finite. According to Σ_1 replacement, let $u \geq \beta$ be such that

$$\forall e \in \mathrm{ran}(g \restriction f(i))\,(A_e \supsetneq X \restriction \beta \leftrightarrow A_{e,u} \supsetneq X \restriction \beta). \tag{4.9}$$

Now suppose e is in the range of $g \restriction f(i)$, then

(i) if $A_e \not\supseteq X \restriction \beta$, then there is a least $w < \beta$ such that $A_e(w) \neq X(w)$;

(ii) if $A_e \supseteq X \restriction \beta$, then either $A_e = X \restriction \beta$ or $A_{e,u} \supsetneq X \restriction \beta$.

Thus, to diagonalize against A_e in block j for all $j < i$ (i.e. $e \in \mathrm{ran}(g \restriction f(i))$), it suffices to define $X \restriction (u+1) = C \cup [\sup C, \beta) \cup \{u\}$. In our construction, X is defined by iterating this strategy through $i < \sigma 2cf(\alpha)$.

This strategy may be converted to an effective one, largely because f, g and h are effectively and tamely approximated. The only difficulty concerns obtaining a nice recursive approximation of u in (4.9) (notice that the intention is to make $X \upharpoonright [\beta, u) = \emptyset$). A recursive approximation of u requires information regarding $I_{\beta,i} = \{e \in \operatorname{ran}(g \upharpoonright f(i)) : A_e \supsetneq X \upharpoonright \beta\}$. Lemma 2.13 and Lemma 4.11 provide a way around this difficulty. Notice that a correct guess of the set $I'_{\beta,i} = \{e \in \operatorname{ran}(g \upharpoonright f(i)) : A_e \supseteq X \upharpoonright \beta\}$ is obtained from stage β onwards. Thus, only a coding of the existence of an A_e which is equal to $X \upharpoonright \beta$, where e is in the range of $g \upharpoonright f(i)$, is needed to determine $I_{\beta,i}$: if such an A_e exists, then $I_{\beta,i}$ is obtained by enumerating all $e \in I'_{\beta,i}$ such that $A_e \supsetneq X \upharpoonright \beta$ until only one index in $I'_{\beta,i}$ remains to be enumerated; if no such A_e exists, then $I_{\beta,i} = I'_{\beta,i}$. As will be seen in a moment, the coding is tame Σ_2 and hence, by Lemma 2.13, is α-finite.

The above strategy is an analogue of that in $B\Sigma_2$ models. The difference between the two constructions mainly arises from the upper bound established in the constructions. In $B\Sigma_2$ models, it is an upper bound of the least differences between any pair of r.e. sets in some blocks; in L_α, since Σ_2 replacement fails, the upper bound is only for the least differences between X and the α-r.e. sets in some α-finite part of the numbering.

Construction

X is first constructed recursively in $\emptyset'$ by induction through $\sigma 2cf(\alpha)$ with the intention of coding the existence of A_e such that A_e is equal to $X \upharpoonright \beta_i$, where e is in the range of $g \upharpoonright f(i)$, $i < \sigma 2cf(\alpha)$, and β_i is a pseudostable ordinal specified below.

Let $i < \sigma 2cf(\alpha)$. Suppose for all $j < i$, the values of $\gamma_j, \beta_j, u_j, X[j]$ and $G(j)$ have been defined. For i, the values of $\gamma_i, \beta_i, u_i, X[i]$ and $G(i)$ are defined as follows.

> Stage γ_i is defined to be a stage such that the approximation of f below $i+1$ and the approximation of g below $f(i)+1$ have reached their limits from stage γ_i onwards:
>
> $$\gamma_i = \max\{\mu\zeta\,(\forall \zeta' \geq \zeta\,(f_{\zeta'} \upharpoonright (i+1) = f \upharpoonright (i+1))), \\ \mu\zeta\,(\forall \zeta' \geq \zeta\,(g_{\zeta'} \upharpoonright (f(i)+1) = g \upharpoonright (f(i)+1))\}.$$
>
> Let C_i be $\bigcup_{j<i} X[j]$. If C_i is α-finite, then let β_i be the pseudostable

ordinal obtained in Lemma 4.11 when $\gamma = \gamma_i$, $C = C_i$, and $p = g \upharpoonright f(i)$, i.e.

$$\beta_i = \lim_{\zeta \to \alpha} h(\zeta, \gamma_i, C_i, g \upharpoonright f(i)).$$

If C_i is not α-finite, then β_i, together with ordinals defined below u_i, X_i and $G(i)$, is undefined. It will follow from Lemma 4.23 that C_i is α-finite for all $i < \sigma 2cf(\alpha)$.

The pseudostable ordinal β_i together with an upper bound u_i defined below will be applied to diagonalize A_e in block j for all $j < i$. Intuitively, the upper bound u_i is a stage at which all α-r.e. sets with indices in the range of $g \upharpoonright f(i)$ containing $C_i \cup [\sup C_i, \beta_i)$ as a proper subset have been enumerated. More precisely, we define

$$\begin{aligned} u_i = \mu u \geq \beta_i\, [\forall e \in \operatorname{ran}(g \upharpoonright f(i))\, (A_e \supsetneq C_i \cup [\sup C_i, \beta_i) \\ \to A_{e,u} \supsetneq C_i \cup [\sup C_i, \beta_i))]. \end{aligned}$$

$X[i]$ is defined to be an end extension of C_i using β_i and u_i as parameters with the intention of diagonalizing A_e in block j for all $j < i$:

$$X[i] = C_i \cup [\sup C_i, \beta_i) \cup \{u_i\}.$$

X succeeds in diagonalizing A_e in a block j for all $j < i$ if X is an end extension of $X[i]$, for the reason shown in the part of the strategy.

$G(i)$ is defined below to provide the desired code of the existence of A_e in a block $j < i$ such that A_e is identical with $C_i \cup [\sup C_i,\ \beta_i)$, i.e.

$$G(i) = \begin{cases} 1 & \text{if } \exists e \in \operatorname{ran}(g \upharpoonright f(i))\, (A_e = C_i \cup [\sup C_i, \beta_i)), \\ 0 & \text{otherwise.} \end{cases}$$

$G(i)$ will be a parameter of the recursive approximation of u_i as shown in the section we described the strategy. We review the idea briefly in the following.

For the rest of this paragraph we only consider A_e's such that e is in the range of $g \upharpoonright f(i)$. Also for simplicity, let Υ_i denote the α-finite set $C_i \cup [\sup C_i, \beta_i)$. Since β_i is pseudostable, whether A_e contains Υ_i as a subset is determined at stage β_i. If $G(i) = 0$, then all A_e containing Υ_i

as a subset will contain Υ_i as a proper subset. Therefore when $G(i) = 0$, to determine u_i, one only needs to wait until each A_e containing Υ_i as a subset at stage β_i has enumerated an element not in Υ_i. If $G(i) = 1$, then all but one A_e containing Υ_i as a subset would contain Υ_i as a proper subset. Thus when $G(i) = 1$, to determine u_i, one only needs to wait until all but one A_e containing Υ_i as a subset at stage β_i has enumerated an element not in Υ_i.

Lemma 4.23. *The function $q : i \mapsto (\gamma_i, \beta_i, u_i, X[i], G(i))$ is tame Σ_2 and has domain $\sigma 2cf(\alpha)$.*

Proof. Suppose $\delta = \mathrm{dom}(q) \leq \sigma 2cf(\alpha)$. Notice that

(i) f, g are tame Σ_2;

(ii) For every $i < \delta$ and $\zeta \geq \beta_i$, $h(\zeta, \gamma_i, C_i, g \restriction f(i)) = \beta_i$, where $C_i = \bigcup_{j<i} X[j]$, i.e. the approximation to β_i reaches its limit at stage β_i and does not change thereafter;

(iii) For every $i < \delta$, by Lemma 4.11 and definitions of β_i and u_i,

$$\begin{aligned} &G(i) = 0 \leftrightarrow \\ &\quad \forall e \in \mathrm{ran}(g \restriction f(i))(A_{e,\beta_i} \supseteq C_i \cup [\sup(C_i, \beta_i) \to A_{e,u_i} \supsetneq C_i \cup [\sup C_i, \beta_i)). \end{aligned}$$

Now it is straightforward to verify that the function q is Σ_2.

Moreover, q can be viewed as a (partial) function on $\sigma 2cf(\alpha)$. Since Lemma 2.14 implies that q is tame Σ_2, we have $q \restriction a$ is α-finite, whenever $a \leq t\sigma 2p(\alpha)$.

For the sake of contradiction, assume $\delta < \sigma 2cf(\alpha)$. Then $q \restriction \delta$ and C_δ are α-finite. This implies that γ_δ and β_δ are defined. Since $f(\delta) < t\sigma 2p(\alpha) \leq \sigma 1p(\alpha)$ and g is a tame Σ_2 one-one function, by Theorem 2.12, each α-r.e. subset of $\mathrm{ran}(g \restriction f(\delta))$ is α-finite. Hence

$$\{e \in \mathrm{ran}(g \restriction f(\delta)) : A_e \supsetneq C_\delta \cup [\sup C_\delta, \beta_\delta)\}$$

is α-finite. Thus, u_δ is well defined by Σ_1 replacement, and so are $X[\delta]$ and $G(\delta)$, a contradiction. $\square$

Lemma 4.24. *$G : \sigma 2cf(\alpha) \to \{0,1\}$ is α-finite.*

Proof. By Lemma 4.23, $\{i < \sigma 2cf(\alpha) : G(i) = 1\}$ is tame Σ_2. Since $\sigma 2cf(\alpha) < t\sigma 2p\,(\alpha)$, according to Lemma 2.13, G is α-finite. $\square$

Let

$$X = \bigcup_{i<\sigma 2cf(\alpha)} X[i].$$

Lemma 4.25. *$X \notin \{A_e\}_{e<\alpha}$.*

Proof. Assume $X \in \{A_e\}_{e<\alpha}$ for a contradiction. Since f is cofinal and g is onto, there is $i < \sigma 2cf(\alpha)$ and $e \in \mathrm{ran}(g \restriction f(i))$ such that $X = A_e$. Let Υ_i denote the α-finite set $(\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i)$ for simplicity.

Observe that for every $i' > i$, $X[i']$ is an end extension of $X[i]$. Hence

$$\Upsilon_i = X \restriction u_i = A_e \restriction u_i.$$

Since $X \supsetneq \Upsilon_i$, it follows from the definition of u_i that $A_{e,u_i} \supsetneq \Upsilon_i$. But notice that $A_{e,u_i} \subseteq A_e \restriction u_i = \Upsilon_i$, a contradiction. $\square$

Verifying that X is α-r.e.

Lemma 4.25 states that X is not in the numbering $\{A_e\}_{e<\alpha}$. To see that $\{A_e\}_{e<\alpha}$ is not a universal numbering, we only need to show that X is α-r.e. We will effectively reconstruct the set X as an α-r.e. set using the α-finite code G as a parameter.

Again, let C_i, Υ_i denote $\bigcup_{j<i} X[j]$ and $(\bigcup_{j<i} X[j]) \cup [\sup(\bigcup_{j<i} X[j]), \beta_i)$ respectively for simplicity. Note that by Lemma 4.11, for any $e \in \mathrm{ran}(g \restriction f(i))$, A_e contains Υ_i if and only if Υ_i is enumerated into A_e by stage β_i. Moreover, at most one $e \in \mathrm{ran}(g \restriction f(i))$ satisfies $A_e = \Upsilon_i$. And by the definition of G, such an e exists if and only if $G(i) = 1$. These observations yield an alternative definition of u_i with parameter G:

$$\begin{aligned} u_i = \mu u \geq \beta_i \,[G(i) = 0 \;\to \forall e \in \mathrm{ran}(g \restriction f(i))\,(A_{e,\beta_i} \supseteq \Upsilon_i \to A_{e,u} \supsetneq \Upsilon_i) \\ \wedge\; G(i) = 1 \;\to\; \forall^{\geq 1} e \in \mathrm{ran}(g \restriction f(i))(A_{e,\beta_i} \supseteq \Upsilon_i \;\to\; A_{e,u} \supsetneq \Upsilon_i)]. \end{aligned} \tag{4.10}$$

Here, by "$\forall^{\geq 1} e \in C$", where C is any α-finite set, we mean "$\exists e_0 \in C\, \forall e \in C \setminus \{e_0\}$". Definition (4.10) implies that u_i is α-recursively defined by β_i, $g \restriction f(i)$ and C_i.

At each stage $\eta < \alpha$, the approximation of $\{X[i]\}_{i<\sigma 2cf(\alpha)}$ inductively for $i < \sigma 2cf(\alpha)$ is given as follows.

Stage $\gamma_{i,\eta}$ is defined to be a stage not exceeding η such that the approximation of f below $i+1$ and approximation of g below $f_\eta(i)+1$ have attained their values at stage η and do not change thereafter until stage η:

$$\begin{aligned}\gamma_{i,\eta} = \max\{&\mu\zeta \leq \eta\,(\forall \zeta' \in [\zeta,\eta]\,(f_{\zeta'} \upharpoonright (i+1) = f_\eta \upharpoonright (i+1))), \\ &\mu\zeta \leq \eta\,(\forall \zeta' \in [\zeta,\eta]\,(g_{\zeta'} \upharpoonright (f_\eta(i)+1) = g_\eta \upharpoonright (f_\eta(i)+1)))\}.\end{aligned} \tag{4.11}$$

Let $C_{i,\eta}$ be $\bigcup_{j<i} X_\eta[j]$. Pseudostable ordinal β_i is approximated via the function h, i.e.

$$\beta_{i,\eta} = h(\eta, \gamma_{i,\eta}, C_{i,\eta}, g_\eta \upharpoonright f_\eta(i)).$$

An upper bound $u_{i,\eta} < \eta$ is defined by substituting $\beta_{i,\eta}, g_\eta, f_\eta, X_\eta[j]$ for $\beta_i, g, f, X[j]$ respectively and restricting u to the set $[\sup C_{i,\eta}, \eta)$ in (4.10): Let $\Upsilon_{i,\eta}$ denote $C_{i,\eta} \cup [\sup C_{i,\eta}, \beta_{i,\eta})$. Then

$$\begin{aligned}u_{i,\eta} = \mu u \geq \beta_{i,\eta}\,[\,&(u \geq \sup C_{i,\eta}\ \wedge\ (u < \eta) \wedge \\ &G(i) = 0\ \rightarrow \forall e \in \mathrm{ran}(g_\eta \upharpoonright f_\eta(i))\,(A_{e,\beta_{i,\eta}} \supseteq \Upsilon_{i,\eta} \rightarrow A_{e,u} \supsetneq \Upsilon_{i,\eta}) \wedge \\ &G(i) = 1\ \rightarrow \forall^{\geq 1} e \in \mathrm{ran}(g_\eta \upharpoonright f_\eta(i))(A_{e,\beta_{i,\eta}} \supseteq \Upsilon_{i,\eta}\ \rightarrow\ A_{e,u} \supsetneq \Upsilon_{i,\eta})].\end{aligned}$$

For some η, $u_{i,\eta}$ may be undefined.

Now define

$$X_\eta[i] = \begin{cases} \Upsilon_{i,\eta} \cup \{u_{i,\eta}\} & \text{if for each } j \leq i, u_{j,\eta} \text{ is defined,} \\ \emptyset & \text{otherwise.}\end{cases}$$

Lemma 4.26. *Suppose $i < \sigma 2cf(\alpha)$. Then*

$$\lim_{\eta\to\alpha}(\gamma_{i,\eta}, \beta_{i,\eta}, u_{i,\eta}, X_\eta[i]) = (\gamma_{i,\eta_0}, \beta_{i,\eta_1}, u_{i,\eta_2}, X_{\eta_2}[i]) = (\gamma_i, \beta_i, u_i, X[i]),$$

for any $\eta_0 \geq \gamma_i$, $\eta_1 \geq \beta_i$, and $\eta_2 > u_i$.

Proof. All equations are proved simultaneously by induction on i. Suppose Lemma 4.26 is proved for each $j < i$. Let $\eta < \alpha$ be a stage. We have

(i) If $\eta \geq \gamma_i$, then the definition of γ_i implies

$$f_\eta \restriction (i+1) = f \restriction (i+1), \quad g_\eta \restriction (f(i)+1) = g \restriction (f(i)+1).$$

Thus, according to (4.11), $\gamma_{i,\eta} = \gamma_i$.

(ii) Let $\eta \geq \beta_i$. Then by their definitions and Lemma 4.11, $\beta_i > \max\{\gamma_i, \sup_{j<i} u_j\}$. According to the inductive hyphothesis and (1) of this proof, $(\gamma_{i,\eta}, C_{i,\eta}, g_\eta \restriction f_\eta(i)) = (\gamma_i, C_i, g \restriction f(i))$. By Lemma 4.11 again, $\beta_{i,\eta} = \beta_i$.

(iii) Now suppose $\eta > u_i$. Since $u_i \geq \beta_i$, by (2) of the present proof, $(\beta_{i,\eta}, \gamma_{i,\eta}, C_{i,\eta}, g_\eta \restriction f_\eta(i)) = (\beta_i, \gamma_i, C_i, g \restriction f(i))$. So $u_{i,\eta}$ is defined and equal to u_i. Combine this with the inductive hypothesis, then we have $\forall j \leq i\, (u_{j,\eta} \downarrow = u_j)$. Hence $X_\eta[i] = X[i]$. $\square$

Lemma 4.27. *$X_\eta[i] \subseteq X$ for all $\eta < \alpha$, $i < \sigma 2cf(\alpha)$.*

Proof. Fix a stage η. Lemma 4.11 implies that for every $i < \sigma 2cf(\alpha)$, we have

$$\beta_i > \max\{\operatorname{ran}(g \restriction (f(i)+1)), \sup_{j<i} u_j\} \text{ and } u_i \geq \beta_i.$$

Thus, $\{\beta_i\}_{i<\sigma 2cf(\alpha)}$ and $\{u_i\}_{i<\sigma 2cf(\alpha)}$ are strictly increasing and cofinal in α.

Let $i^* < \sigma 2cf(\alpha)$ be the least i that $u_i \geq \eta$. Then for every $j < i^*$, $u_j < \eta$, and Lemma 4.26 implies that $X_\eta[j] = X[j] \subseteq X$.

Suppose $i \geq i^*$. $X_\eta[i] \subseteq X$ is trivially true if $X_\eta[i] = \emptyset$. Now assume $X_\eta[i] \neq \emptyset$. Then by its definition, $X_\eta[i] \subseteq \eta$ and $X_\eta[i] \restriction \sup C_{i^*} = C_{i^*}$, where $C_{i^*} = \bigcup_{j<i^*} X[j]$.

Case 1. $\eta \leq \beta_{i^*}$. By the definition of $\{X_\eta[j]\}_{j \leq i}$, we have

$$X_\eta[i] \subseteq C_{i^*} \cup [\sup C_{i^*}, \eta) \subseteq X[i^*].$$

Therefore, $X_\eta[i] \subseteq X$.

Case 2. $\beta_{i*} < \eta \leq u_{i^*}$. By Lemma 4.26, $(\gamma_{i^*,\eta}, \beta_{i^*,\eta}, C_{i^*,\eta}) = (\gamma_{i^*}, \beta_{i^*}, C_{i^*})$. Then $\Upsilon_{i^*,\eta} = \Upsilon_{i^*}$. Since $\gamma_{i^*} = \gamma_{i^*,\eta}$, $f_\eta \restriction (i+1) = f \restriction (i+1)$ and $g_\eta \restriction f(i) = g \restriction f(i)$. Therefore, in the formula of the definition of $u_{i^*,\eta}$, every subscript η can be omitted. Note that $\eta \leq u_{i^*}$, so $u_{i^*,\eta}$ is undefined. Therefore $X_\eta[i] = \emptyset$, a contradiction. $\square$

Note that the sequence $\{(\gamma_{i,\eta}, \beta_{i,\eta}, u_{i,\eta}, X_\eta[i])\}_{i<\sigma 2cf(\alpha), \eta<\alpha}$ is α-recursive. Thus, Lemma 4.26 and Lemma 4.27 combine to produce the following corollary.

Corollary 4.28. *X is α-r.e. Hence $\{A_e\}_{e<\alpha}$ is not a Friedberg numbering.*

Since $\{A_e\}_{e<\alpha}$ is an arbitrary one-one numbering, Corollary 4.28 implies that there is no Friedberg numbering when $t\sigma 2p(\alpha) > \sigma 2cf(\alpha)$, proving Theorem 4.22.

The code G plays a significant role in the proof of Theorem 4.22. It exploits the property that the numbering $\{A_e\}_{e<\alpha}$ is one-one and makes the use of pseudostable ordinals to achieve the diagonalization against $\{A_e\}_{e<\alpha}$. Pseudostability is applicable for any numbering. Yet, if the numbering is not one-one in some blocks and the number of such repetitions is cofinal in $t\sigma 2p(\alpha)$ as the number of blocks increases, then a code such as G may not exist. Therefore, the diagonalization construction may not be applicable for other types of numberings.

Remark. As in Section 4.1.2, Gödel numbering is a K-acceptable numbering and (4.6) is still a valid example of non-K-acceptable numbering in L_α, for all Σ_1 admissible α, by simply replacing the notations appropriately.

Now we consider K_e-numbering. If $t\sigma 2p(\alpha) = \sigma 2cf(\alpha)$, then the Friedberg numbering constructed in Section 4.2.1 is a natural example of a K_e-numbering (ref. Section 4.1.2 and 4.2.1). If $t\sigma 2p(\alpha) > \sigma 2cf(\alpha)$, then there is no K_e-numbering as the situation of $B\Sigma_2$ models, but for a different reason: For the sake of contradiction assume $\{C_e\}_{e<\alpha}$ is a K_e-numbering. Then $\{e' < e : C_{e'} = C_e\}$ is Δ_2 for every $e < \alpha$. Therefore the least indices of $\{C_e\}_{e<\alpha}$ have an α-recursive approximation. Then the straightforward adaptation of the proof in [29] provides a Friedberg numbering in L_α. Hence, we have

Corollary 4.29. *If $t\sigma 2p(\alpha) > \sigma 2cf(\alpha)$, there is no K_e-numbering in L_α.*

Corollary 4.30. *The following are equivalent:*

1. *$t\sigma 2p(\alpha) = \sigma 2cf(\alpha)$;*
2. *There is a Friedberg numbering in L_α;*
3. *There is a K_e-numbering in L_α.*

4.3 Friedberg Numbering of N-r.e. Sets

A sequence $\{D_e\}_{e\in\mathcal{M}}$ of n-r.e. sets is a *Friedberg numbering of n-r.e. sets* if

(i) For any n-r.e. set $X \subseteq \omega$, there is a unique e such that $D_e = X$, and

(ii) $\{(x,e) : x \in D_e\}$ is also n-r.e.

Notice that the discussion in Section 4.1 and 4.2.2 is applicable to n-r.e. sets for every $n \geq 1$. Then we get the following results.

Corollary 4.31. *Suppose $n \geq 1$, $\mathcal{M}$ is a $B\Sigma_2$ model and $\{D_e\}_{e\in\mathcal{M}}$ is a sequence of n-r.e. sets without repetition such that $\{(x,e) : x \in D_e\}$ is also n-r.e. Then there is an r.e. set X such that $X \neq D_e$ for all $e \in \mathcal{M}$.*

Corollary 4.32. *For every $n \geq 1$, the following are equivalent over $P^- + B\Sigma_2$:*

1. *Σ_2 induction.*

2. *There exists a Friedberg numbering of n-r.e. sets.*

Corollary 4.33. *If α is admissible and $t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$, then there is a Friedberg numbering of n-r.e. sets in L_α for any $n \geq 1$.*

Now suppose α is admissible, $\{D_e = A_e \setminus B_e\}_{e<\alpha}$ is a sequence of d-r.e. sets without repetition such that $\{(x,e) : x \in A_e\}$, $\{(x,e) : x \in B_e\}$ are r.e. Then for every $i < \sigma 2cf(\alpha)$, we modify the definition of γ_i as follows:

$$
\begin{aligned}
\gamma_i = \max\{&\mu\zeta\,(\forall\zeta' \geq \zeta\,(f_{\zeta'} \restriction (i+1) = f \restriction (i+1))),\\
&\qquad\qquad \mu\zeta\,(\forall\zeta' \geq \zeta\,(g_{\zeta'} \restriction (f(i)+1) = g \restriction (f(i)+1))\\
&\qquad\qquad\qquad \forall e < f(i)\,(B_e \supsetneq (\sup\nolimits_{j<i} u_j \setminus C_i) \;\rightarrow\\
&\qquad\qquad\qquad\qquad\qquad \exists x < \zeta\,(x \in B_{e,\zeta} \setminus (\sup\nolimits_{j<i} u_j) \,\vee\, x \in C_i))\}.
\end{aligned}
$$

We modify $\gamma_{i,\eta}$ accordingly and other definitions are the same as in Section 4.2.4. Then we get the following results.

Corollary 4.34. *There is an r.e. set X such that $X \neq D_e$ for all $e < \alpha$.*

Corollary 4.35. *For every admissible ordinal α, the following are equivalent:*

1. *$t\sigma 2p\,(\alpha) = \sigma 2cf(\alpha)$;*

2. *There exists a Friedberg numbering of d-r.e. sets in L_α.*

Chapter 5

Recursive Aspects Of An Everywhere Differentiable Function

In this chapter, we investigate properties of everywhere differentiable functions in the aspect of hyperarithmetic theory and reverse mathematics. The descriptive set theoretic aspect of this topic was done by Kechris and Woodin [24]. This chapter can be viewed as an effective version of their results.

To describe intervals, real numbers, subsets of real numbers and functions in our context, we need to "translate" them into the language of second order arithmetic. Section 5.2 is devoted to this work. In Section 5.3 and 5.4, we study everywhere differentiable functions in the context of hyperarithemtic theory. In Section 5.3, $D = \{e < \omega : \Phi_e$ describes an everywhere differentiable function on $[0, 1]\}$ is proved to be a Π^1_1 complete set. In Section 5.4, to every continuous function on $[0, 1]$, the Kechris-Woodin rank is assigned in terms of Cantor-Bendixson type analysis, from which the Kechris-Woodin kernel is defined. In Section 5.4, the existence of Kechris-Woodin kernel is shown to be equivalent to Π^1_1-CA_0 in reverse mathematics.

5.1 Convention and Notations

In this chapter, our study involves natural numbers, integers, rational numbers, real numbers, arithmetic operations of these numbers, sequences of these numbers, their subsets and functions. For convenience, e, i, j, k, l, m, n are reserved for natural numbers; u, v, w are reserved for integers; a, b, c, d, p, q, r, s, ε are rational numbers; x, y, z are variables over the set of real numbers; and f, g, h are intended to denote functions on the closed interval $[0, 1]$. All these notations allow superscripts or subscripts. To avoid confusion, in this chapter, we will use $(\_,\_)$ to denote an open interval and $\langle \_,\_ \rangle$ ($\langle \_,\_,\dots,\_ \rangle$, respectively) to denote a pair (a finite sequence, respectively) of numbers.

In addition, let $\mathbf{C}[0, 1]$ be the collection of all continuous functions on $[0, 1]$ and $\mathbf{D}$ be the collection of everywhere differentiable functions on $[0, 1]$. (At 0 and 1, we consider the right hand side derivative and the left hand side derivative respectively.)

5.2 Second Order Arithmetic Descriptions

In this section, we set up a system to code numbers, sequences, subsets and functions in second order arithmetic. In general, we will add an $\hat{}$ to denote the codes. For instance, we denote the set of codes of rational numbers by $\hat{Q}$.

Let $\pi : \omega^2 \to \omega$ be a recursive bijection such that $\pi\langle 0, 0\rangle = 0$. If $\pi\langle i, j\rangle = k$, then we say k *codes* the ordered pair $\langle i, j\rangle$. We define

$$\hat{Z} = \{\pi\langle i, 0\rangle : i < \omega\} \cup \{\pi\langle 0, j\rangle : j < \omega\}.$$

The code $\pi\langle i, j\rangle \in \hat{Z}$ is to denote the integer $i - j$, in particular $\pi\langle 0, 0\rangle = 0$ denotes 0. The absolute value function, order relation and arithmetic operations over $\hat{Z}$ are denoted by $|\pi\langle i, j\rangle|_{\mathbb{Z}}$, $<_{\mathbb{Z}}$, $+_{\mathbb{Z}}$, $-_{\mathbb{Z}}$, $\cdot_{\mathbb{Z}}$, $\div_{\mathbb{Z}}$, respectively. Then,

$$|\pi\langle i, 0\rangle|_{\mathbb{Z}} = \pi\langle i, 0\rangle, \quad |\pi\langle 0, j\rangle|_{\mathbb{Z}} = \pi\langle j, 0\rangle$$

$$\pi\langle i, j\rangle <_{\mathbb{Z}} \pi\langle i', j'\rangle \leftrightarrow i + j' < i' + j, \text{ where } \pi\langle i, j\rangle, \pi\langle i', j'\rangle \in \hat{Z}$$

$$\pi\langle i, 0\rangle +_{\mathbb{Z}} \pi\langle i', 0\rangle = \pi\langle i + i', 0\rangle, \quad \pi\langle 0, j\rangle +_{\mathbb{Z}} \pi\langle 0, j'\rangle = \pi\langle 0, j + j'\rangle$$

$$\pi\langle i, 0\rangle +_{\mathbb{Z}} \pi\langle 0, j\rangle = \begin{cases} \pi\langle i - j, 0\rangle & \text{if } i \geq j, \\ \pi\langle 0, j - i\rangle & \text{otherwise} \end{cases}$$

$$\pi\langle i, j\rangle -_{\mathbb{Z}} \pi\langle i', j'\rangle = \pi\langle i'', j''\rangle \leftrightarrow \pi\langle i', j'\rangle +_{\mathbb{Z}} \pi\langle i'', j''\rangle = \pi\langle i, j\rangle$$

$$\pi\langle i, 0\rangle \cdot_{\mathbb{Z}} \pi\langle i', 0\rangle = \pi\langle ii', 0\rangle, \quad \pi\langle 0, j\rangle \cdot_{\mathbb{Z}} \pi\langle 0, j'\rangle = \pi\langle jj', 0\rangle$$

$$\pi\langle i, 0\rangle \cdot_{\mathbb{Z}} \pi\langle 0, j\rangle = \pi\langle 0, ij\rangle$$

$$\pi\langle i, j\rangle \div_{\mathbb{Z}} \pi\langle i', j'\rangle = \pi\langle i'', j''\rangle \leftrightarrow \neg(i' = j' = 0) \wedge \pi\langle i', j'\rangle \cdot_{\mathbb{Z}} \pi\langle i'', j''\rangle = \pi\langle i, j\rangle$$

Suppose $u(m)$ and $u(n)$ are the integers coded by m and n respectively and $u(n) > 0$. Let $\gcd_{\mathbb{Z}}(m,n)$ be $\pi\langle \gcd(u(m), u(n)), 0\rangle$, the code of the greatest common divisor of $u(m)$ and $u(n)$. Now we introduce the field $\mathbb{Q}$. We define

$$\hat{Q} = \{\pi\langle m, n\rangle : m, n \in \hat{Z} \wedge n >_{\mathbb{Z}} 0 \wedge \gcd_{\mathbb{Z}}(m,n) = \pi\langle 1, 0\rangle\}$$

Here $\pi\langle m, n\rangle \in \hat{Q}$ codes the rational number $\frac{u(m)}{u(n)}$, where $u(m)$ and $u(n)$ are the integers coded by m and n respectively. In $\hat{Q}$, we use $\pi\langle 0, 1\rangle$ to denote 0. Then,

$$|\pi\langle m, n\rangle|_{\mathbb{Q}} = \pi\langle |m|_{\mathbb{Z}}, n\rangle$$

$$\pi\langle m, n\rangle <_{\mathbb{Q}} \pi\langle m', n'\rangle \leftrightarrow m \cdot_{\mathbb{Z}} n' <_{\mathbb{Z}} m' \cdot_{\mathbb{Z}} n$$

$$\pi\langle m, n\rangle +_{\mathbb{Q}} \pi\langle m', n'\rangle = \pi\langle m'', n''\rangle, \text{ where} \begin{cases} m'' = (m \cdot_{\mathbb{Z}} n' +_{\mathbb{Z}} m \cdot_{\mathbb{Z}} n') \div_{\mathbb{Z}} d, \\ n'' = (n \cdot_{\mathbb{Z}} n') \div_{\mathbb{Z}} d, \text{ and} \\ d = \gcd_{\mathbb{Z}}(m \cdot_{\mathbb{Z}} n' +_{\mathbb{Z}} m \cdot_{\mathbb{Z}} n', n \cdot_{\mathbb{Z}} n') \end{cases}$$

$$\pi\langle m, n\rangle -_{\mathbb{Q}} \pi\langle m', n'\rangle = \pi\langle i'', j''\rangle \leftrightarrow \pi\langle m', n'\rangle +_{\mathbb{Q}} \pi\langle m'', n''\rangle = \pi\langle m, n\rangle$$

$$\pi\langle m, n\rangle \cdot_{\mathbb{Q}} \pi\langle m', n'\rangle = \pi\langle m \cdot_{\mathbb{Z}} m', n \cdot_{\mathbb{Z}} n'\rangle$$

$$\pi\langle m, n\rangle \div_{\mathbb{Q}} \pi\langle m', n'\rangle = \pi\langle m'', n''\rangle \leftrightarrow \neg(m' = 0) \wedge \pi\langle m', n'\rangle \cdot_{\mathbb{Q}} \pi\langle m'', n''\rangle = \pi\langle m, n\rangle$$

In the sequel, if $l \in \hat{Q}$, then let $a(l)$ be the rational number coded by l.

This coding of $\mathbb{Q}$ provides a way to describe open intervals with rational end points. We say $\langle l, l'\rangle$ *describes* an open interval $(a(l), a(l'))$ if $l, l' \in \hat{Q}$, and $l <_{\mathbb{Q}} l'$. $A_0 \subset \omega^2$ *describes* an open set if

(i) Each $\langle l, l'\rangle \in A_0$ describes an open interval $(a(l), a(l'))$.

(ii) If $\langle m, m'\rangle \in A_0$ and $\langle l, l'\rangle$ describes an open interval $(a(l), a(l')) \subseteq (a(m), a(m'))$, then $\langle l, l'\rangle \in A_0$, i.e. A_0 is closed under subsets.

(iii) If for every $k < \omega$, $\langle m_k, m'_k\rangle \in A_0$, and $\langle l, l'\rangle$ describes an open interval $(a(l), a(l')) = \bigcup_k (a(m_k), a(m'_k))$, then $\langle l, l'\rangle \in A_0$, i.e. A_0 is closed under countable union.

The open set described by A_0 is $\bigcup_{\langle l,l'\rangle \in A_0} (a(l), a(l'))$. We may also say its complement, $\overline{\bigcup_{\langle l,l'\rangle \in A_0} (a(l), a(l'))}$, is described by A_0. If $A \subset \omega$, then A *describes* an open set if $\{\langle l, l'\rangle : \pi\langle l, l'\rangle \in A\}$ describes an open set.

Lemma 5.1. *$\hat{Z}$, $\hat{Q}$ are recursive subsets of ω. Their absolute value functions, order relations and arithmetic operations defined above are recursive. Moreover, "$A \subset \omega$ describes an open set" is an arithmetic property of A.*

Proof. We only need to show that "$A_0 \subset \omega^2$ describes an open set" is arithmetic. In its definition, Clause (i) and (ii) are clearly arithmetic. So we only need to check Clause (iii). Notice that (iii) is equivalent to the following statement:

If for every $m, m' \in \hat{Q}$ with $l <_{\mathbb{Q}} m <_{\mathbb{Q}} m' <_{\mathbb{Q}} l'$, there is a finite sequence $\{\langle m_k, m_k' \rangle\}_{k<n} \subset A_0$ such that $\{(a(m_k), a(m_k')) : k < n\}$ is an open cover of the closed interval $[a(m), a(m')]$, then $\langle l, l' \rangle \in A_0$.

$\{(a(m_k), a(m_k')) : k < n\}$ is an open cover of the closed interval $[a(m), a(m')]$ if and only if there is a permutation of these open intervals, say $\{(a(m_k), a(m_k')) : k < n\}$ itself, such that for all $k < n - 1$, $m_{k+1} <_{\mathbb{Q}} m_k' <_{\mathbb{Q}} m_{k+1}'$, $m_0 <_{\mathbb{Q}} m <_{\mathbb{Q}} m_0'$ and $m_{n-1} <_{\mathbb{Q}} m' <_{\mathbb{Q}} m_{n-1}'$. □

Remark 5.2. From now on, we we expand the second order language to include the language for the field of rational numbers. In this expanded language, the order relation and arithmetic operations without subscribes are the usual ones over rationals. And the complexity of a formula is determined by coding rational numbers into natural numbers. Using Lemma 5.1, we may treat rational numbers as natural numbers in a formula without changing its arithmetic or analytic complexity. In the following, we will further expand the language to the field of reals with its arithmetic operations and we may treat real numbers as a sequence of rationals thus as a subset of ω.

We now introduce $\mathbb{R}$ by a sequence of rational numbers. Let

$$\begin{aligned}\hat{R} = \{f : f : \omega \to \mathbb{Q} \text{ is a function} \wedge \\ \forall \epsilon \in \mathbb{Q}^+ \, \exists i < \omega \, \forall m, n < \omega \, (m, n > i \to |f(m) - f(n)| < \epsilon)\}.\end{aligned}$$

For any $f \in \hat{R}$, $\{f(n)\}_{n<\omega}$ is a Cauchy sequence and has a limit. The intuition is to denote the real number $\lim_n f(n)$ by f. In particular, we use 0, where $0(n) = 0$ for all n, to denote the real number 0. We define the order relation and arithmetic operations over $\hat{R}$ as those over $\mathbb{R}$.

$$|f|_{\mathbb{R}}(n) = |f(n)|, \text{ for all } n < \omega$$

$$f =_{\mathbb{R}} g \leftrightarrow \forall \epsilon \in \mathbb{Q}^+ \, \exists i < \omega \, \forall m, n < \omega \, (m, n > i \to |g(m) - f(n)| < \epsilon)$$

$$f <_{\mathbb{R}} g \leftrightarrow \exists \epsilon \in \mathbb{Q}^+ \, \exists i < \omega \, \forall m, n < \omega \, (m, n > i \to g(m) - f(n) > \epsilon)$$

$$(f +_{\mathbb{R}} g)(n) = f(n) + g(n), \text{ for all } n < \omega$$

$$f -_{\mathbb{R}} g = h \leftrightarrow g +_{\mathbb{R}} h = f$$

$$(f \cdot_{\mathbb{R}} g)(n) = f(n) \cdot g(n), \text{ for all } n < \omega$$

$$f \div_{\mathbb{R}} g = h \leftrightarrow \forall n < \omega \, (g(n) \neq 0) \wedge \neg(g =_{\mathbb{R}} 0) \wedge g \cdot_{\mathbb{R}} h = f$$

Lemma 5.3. *"$f \in \hat{R}$" and the arithmetic operations over $\hat{R}$ are arithmetic.*

For continuous functions over $\mathbb{R}$, we may consider using functions over $\hat{R}$ as their codes. However, that idea leads to the third order definition of functions on $\hat{R}$. By the continuity, we may narrow down the complexity of descriptions of continuous functions. Intuitively, we may code all quadruple of rational numbers $\langle a, b, r, s\rangle$ such that the function maps real numbers in the open interval (a, b) into (r, s). From this view point, we say $\hat{f} \subset \mathbb{Q}^4$ *describes* a continuous function on $[p, q]$, where $p < q$ are rational numbers, if

(i) Any quadruple $\langle a, b, r, s\rangle \in \hat{f}$ satisfies $a < b$, $r < s$ and $(a, b) \cap [0, 1] \neq \emptyset$.

(ii) (Consistency) If $\langle a, b, r, s\rangle, \langle a', b', r', s'\rangle \in \hat{f}$ and $(a, b) \cap (a', b') \neq \emptyset$, then $(r, s) \cap (r', s') \neq \emptyset$.

(iii) (Preciseness) For every $\varepsilon \in \mathbb{Q}^+$, there is a sequence $\{\langle a_i, b_i, r_i, s_i\rangle\}_{i<n}$ in $\hat{f}$ such that for all $i < n$, $s_i - r_i < \varepsilon$, and $\{(a_i, b_i)\}_{i<n}$ is an open cover of the interval $[p, q]$.

Then let

$$\hat{C}[0, 1] = \{\hat{f} \subset \mathbb{Q}^4 : \hat{f} \text{ describes a continuous function on } [0, 1]\}$$

Lemma 5.4. *"$\hat{f} \in \hat{C}[0, 1]$" is an arithmetic property of $\hat{f}$.*

Lemma 5.5. *If $\hat{f}$ describes a continuous function on $[0, 1]$, then there is a (unique) continuous function $f : [0, 1] \to \mathbb{R}$ such that*

$$\forall \langle a, b, r, s\rangle \in \hat{f} \, (\mathrm{ran}(f \upharpoonright (a, b)) \subseteq [r, s]). \tag{5.1}$$

We say $\hat{f}$ describes f.

Proof. For every $x \in [0,1]$, pick a sequence $\{\langle a_i, b_i, r_i, s_i\rangle\}_{i<\omega}$ in $\hat{f}$ such that $x \in \bigcap_i (a_i, b_i)$ and $\lim_i (s_i - r_i) = 0$, and we call $\{\langle a_i, b_i, r_i, s_i\rangle\}_{i<\omega}$ an x-approximation sequence. Then define

$$f(x) = \lim_i r_i.$$

Firstly, the function f is well-defined. Note that for every $x \in [0,1]$ there is an x-approximation sequence $\{\langle a_i, b_i, r_i, s_i\rangle\}_{i<\omega}$. And since $x \in \bigcap_i (a_i, b_i)$, for every i, j, we have $(a_i, b_i) \cap (a_j, b_j) \neq \emptyset$ and so $(r_i, s_i) \cap (r_j, s_j) \neq \emptyset$. Then $\lim_{i,j\to\infty} |r_i - r_j| \leq \lim_{i,j\to\infty} (s_i - r_i) + (s_j - r_j) = 0$. Thus, $\lim_i r_i$ exists.

To see the definition does not depend on the choice of the x-approximation sequence, suppose $\{\langle a_i, b_i, r_i, s_i\rangle\}_{i<\omega}, \{\langle a'_i, b'_i, r'_i, s'_i\rangle\}_{i<\omega}$ are x-approximation sequences in $\hat{f}$. Then $\{\langle a''_i, b''_i, r''_i, s''_i\rangle\}_{i<\omega}$ defined by

$$\langle a''_i, b''_i, r''_i, s''_i\rangle = \begin{cases} \langle a_i, b_i, r_i, s_i\rangle & \text{if } i \text{ is even} \\ \langle a'_i, b'_i, r'_i, s'_i\rangle & \text{if } i \text{ is odd.} \end{cases}$$

is also an x-approximation sequence. By the argument in the previous paragraph, $\lim_i r''_i$ exists. Hence, $\lim_i r_i = \lim_i r'_i$.

Secondly, we show that $\forall \langle a, b, r, s\rangle \in \hat{f}\,(\operatorname{ran}(f \restriction (a,b)) \subseteq [r,s])$. Pick any $x \in [0,1] \cap (a,b)$ and x-approximation sequence $\{\langle a_i, b_i, r_i, s_i\rangle\}_{i<\omega}$ in $\hat{f}$ with $\langle a_0, b_0, r_0, s_0\rangle = \langle a, b, r, s\rangle$. Then $f(x) = \lim_i r_i = \lim_i (r_i + s_i)/2$. Note that for all $j > i$,

$$|(r_j + s_j)/2 - (r_i + s_i)/2| \leq (s_j - r_j)/2 + (s_i - r_i)/2.$$

Let $j \to \infty$, we have $|f(x) - (r_i + s_i)/2| \leq (s_i - r_i)/2$. In particular, $|f(x) - (r+s)/2| \leq (s-r)/2$, and so $f(x) \in [r,s]$.

Moreover, for each $\varepsilon \in \mathbb{Q}^+$, there is a sequence $\{\langle a_i, b_i, r_i, s_i\rangle\}_{i<n}$ in $\hat{f}$ such that for all $i < n$, $s_i - r_i < \varepsilon$, and $\{(a_i, b_i)\}_{i<n}$ is an open cover of $[0,1]$. Since $\operatorname{ran}(f \restriction (a_i, b_i)) \subseteq [r_i, s_i]$, f is continuous on $[0,1]$.

Thirdly, f is unique. Suppose f and g are continuous functions on $[0,1]$ satisfying (5.1), and suppose $f(x) \neq g(x)$ for some $x \in [0,1]$. Let $\varepsilon \in \mathbb{Q}^+$ such that $|f(x) - g(x)| > \varepsilon$, and $\langle a, b, r, s\rangle$ in $\hat{f}$ such that $x \in (a,b)$, $s - r < \varepsilon$. Then $f(x), g(x) \in [r,s]$ and so $|f(x) - g(x)| < \varepsilon$. We get a contradiction. □

Lemma 5.6. *If $f : [0,1] \to \mathbb{R}$ is a continuous function, then there is an $\hat{f} \in \hat{C}[0,1]$ that describes f.*

Proof. Consider

$$\hat{f} = \{\langle a,b,r,s\rangle \in \mathbb{Q}^4 : a < b, r < s,\ (a,b) \cap [0,1] \neq \emptyset,\ \mathrm{ran}(f \restriction (a,b)) \subseteq (r,s)\}.$$

It is straightforward to check that $\hat{f}$ is in $\hat{C}[0,1]$. $\square$

For every continuous function f on $[0,1]$, let $f^{scan}_{p,q}((q-p)x+p) = (q-p)f(x)$ for all $x \in [0,1]$, and $f^{scan}_{p,q}$ is called the *scanned copy of f on* $[p,q]$.

Lemma 5.7. *Suppose $\hat{f}$ describes the continuous function f on $[0,1]$. Then*

(i) For any rational number $l > 0$, $l\hat{f} = \{\langle a,b,lr,ls\rangle : \langle a,b,r,s\rangle \in \hat{f}\}$ describes lf.

(ii) For any rational numbers $p < q$, $\hat{f}^{scan}_{p,q} = \{\langle (q-p)a+p, (q-p)b+p, (q-p)r, (q-p)s\rangle : \langle a,b,r,s\rangle \in \hat{f}\}$ describes $f^{scan}_{p,q}$.

If $\hat{f}$ describes f, then we say $\hat{f}$ *describes* an everywhere differentiable function on $[0,1]$ if f is everywhere differentiable on $[0,1]$, i.e. $f \in \mathbf{D}$. We say a total function Φ_e *describes* an everywhere differentiable function on $[0,1]$, if $\Phi_e = A$ for some $A \subset \omega$ and

$$\{\langle a(i), a(j), a(k), a(l)\rangle : \pi\langle i, \pi\langle j, \pi\langle k,l\rangle\rangle\rangle \in A,\ \text{where } a(i), a(j), a(k), a(l) \text{ are the rational numbers coded by } i,j,k,l \text{ respectively.}\}$$

describes an everywhere differentiable function on $[0,1]$.

Lemma 5.8. *$D = \{e < \omega : \Phi_e$ describes an everywhere differentiable function on $[0,1]\}$ is Π^1_1.*

5.3 Π^1_1 Completeness of D

Recall that in Section 2.2, we have seen that WF is Π^1_1 complete. This section is devoted to the proof of following theorem. Its proof idea is from Mazurkiewicz [33] (see also [24]).

Theorem 5.9. *D is Π^1_1 complete.*

For this purpose, we construct a recursive function $\chi : \omega \to \omega$ such that

$$e \in \mathrm{WF} \leftrightarrow \chi(e) \in D.$$

For the convenience of further discussion, we first "convert" every partial recursive function $T_e : \omega^{<\omega} \to \{0, 1\}$ to a total recursive function $T_{g(e)}$ such that $\{\sigma : T_{g(e)}(\sigma) = 1\}$ is a tree and $e \in \mathrm{WF} \leftrightarrow g(e) \in \mathrm{WF}$.

Fix a recursive bijection $\ulcorner \cdot \urcorner : \omega^{<\omega} \to \omega$. For every $e < \omega$, define $T_{g(e)}(\sigma) = 1$ if and only if

(i) σ is the empty string, $\langle 0\rangle$, $\langle 1\rangle$ or $\langle 2\rangle$; or

(ii) (Code whether T_e is a total function) $\sigma = \langle 0n\rangle\hat{}\langle\underbrace{0\ldots0}_{m \text{ times}}\rangle$ and $T_{e,m}(n)\uparrow$; or

(iii) (Code whether T_e describes a tree) $\sigma = \langle 1n\rangle\hat{}\langle\underbrace{0\ldots0}_{m \text{ times}}\rangle$ and for some τ and τ' with $\ulcorner\tau\urcorner, \ulcorner\tau'\urcorner < n$, τ is an initial segment of τ' but $T_{e,n}(\tau)\downarrow= 0$ and $T_{e,n}(\tau')\downarrow= 1$ (i.e. at stage n, we find T_e does not describe a tree); or

(iv) (Code whether T_e describes a well founded tree) $\sigma = \langle 2\rangle\hat{}\tau$ and for every k such that $2k + 1 < |\tau|$, $T_{e,\tau(2k)}(\langle\tau(1), \tau(3), \ldots, \tau(2k + 1)\rangle) = 1$ (i.e. an even digit $\tau(2k)$ codes the steps after which the string of odd digits up to $\tau(2k+1)$ is computed to be in the tree T_e).

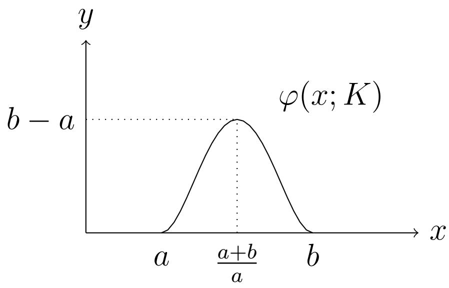
Otherwise, define $T_{g(e)}(\sigma) = 0$.

Note that $T_{g(e)}$ is always total recursive, $\{\sigma : T_{g(e)}(\sigma) = 1\}$ is a tree, and $g(e) \in \mathrm{WF} \leftrightarrow e \in \mathrm{WF}$.

Now we construct continous functions on $[0, 1]$. For any closed interval $K = [a, b] \subseteq [0, 1]$, $a, b \in \mathbb{Q}$, let

$$\varphi(x; K) = \begin{cases} \frac{16(x-a)^2(x-b)^2}{(b-a)^3}, & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

The graph of $\varphi(x; K)$ is as follows.



In the following, we construct two sequences $\{J_\sigma = (a_\sigma, b_\sigma)\}_{\sigma\in\omega^{<\omega}}$ and $\{K_\sigma = [c_\sigma, d_\sigma]\}_{\sigma\in\omega^{<\omega}}$ such that:

(i) $0 \le a_\sigma < c_\sigma < d_\sigma < b_\sigma \le 1$ are rational numbers.

(ii) K_σ is concentric with J_σ and the length of K_σ is less than $\frac{1}{2^{\ulcorner\sigma\urcorner}}$ of that of J_σ, i.e. $d_\sigma - c_\sigma < \frac{1}{2^{\ulcorner\sigma\urcorner}}(b_\sigma - a_\sigma)$.

(iii) If σ is a proper initial segment of τ, then $J_\tau \subset K^{(L)}_\sigma$. Here $K^{(L)}_\sigma = [c_\sigma, (c_\sigma + d_\sigma)/2]$ and $K^{(R)}_\sigma = [(c_\sigma + d_\sigma)/2, d_\sigma]$.

(iv) If σ and τ are incompatible (i.e. there exists $n < \min\{|\sigma|, |\tau|\}$ such that $\sigma(n) \ne \tau(n)$), then $J_\sigma \cap J_\tau = \emptyset$.

Lemma 5.10. *For every pair $\sigma \ne \tau$ in $\omega^{<\omega}$, $K^{(R)}_\sigma \cap K^{(R)}_\tau = \emptyset$.*

Proof. If σ and τ are incompatible, then $J_\sigma \cap J_\tau = \emptyset$. Therefore, $K^{(R)}_\sigma \cap K^{(R)}_\tau = \emptyset$.

If σ is a proper initial segment of τ, since $K^{(R)}_\tau \subset J_\tau \subset K^{(L)}_\sigma$, $K^{(R)}_\sigma \cap K^{(R)}_\tau = \emptyset$. $\square$

Lemma 5.11. *There are recursive sequences (in the sense of Remark 5.2) $\{a_\sigma\}_{\sigma\in\omega^{<\omega}}$, $\{b_\sigma\}_{\sigma\in\omega^{<\omega}}$, $\{c_\sigma\}_{\sigma\in\omega^{<\omega}}$ and $\{d_\sigma\}_{\sigma\in\omega^{<\omega}}$ satisfying Clause (i)-(iv).*

Proof. If σ is the empty string, then let $a_\sigma = 0, b_\sigma = 1, c_\sigma = 1/4, d_\sigma = 3/4$.

Inductively, suppose we have defined J_σ and K_σ. Then for every $n < \omega$, let

$$\begin{aligned}
a_{\sigma\hat{}\langle n\rangle} &= \frac{n}{2(n+1)} d_\sigma + \frac{n+2}{2(n+1)} c_\sigma \\
b_{\sigma\hat{}\langle n\rangle} &= \frac{n+1}{2(n+2)} d_\sigma + \frac{n+3}{2(n+2)} c_\sigma \\
c_{\sigma\hat{}\langle n\rangle} &= \frac{a_{\sigma\hat{}\langle n\rangle} + b_{\sigma\hat{}\langle n\rangle}}{2} - \frac{b_{\sigma\hat{}\langle n\rangle} - a_{\sigma\hat{}\langle n\rangle}}{2^{\ulcorner\sigma\urcorner+3}} \\
d_{\sigma\hat{}\langle n\rangle} &= \frac{a_{\sigma\hat{}\langle n\rangle} + b_{\sigma\hat{}\langle n\rangle}}{2} + \frac{b_{\sigma\hat{}\langle n\rangle} - a_{\sigma\hat{}\langle n\rangle}}{2^{\ulcorner\sigma\urcorner+3}}
\end{aligned}$$

It is straightforward to check that Clause (i)-(iv) are satisfied. $\square$

We fix sequences $\{J_\sigma\}_{\sigma\in\omega^{<\omega}}$ and $\{K_\sigma\}_{\sigma\in\omega^{<\omega}}$ satisfying Lemma 5.11. Define

$$F_T(x) = \sum_{\sigma\in T} \varphi(x; K_\sigma^{(R)}),$$

for any tree $T \subseteq \omega^{<\omega}$. Note that F_T is continuous since every $\varphi(x; K_\sigma^{(R)})$ is continuous and

$$\max_{x\in[0,1]} \varphi(x; K_\sigma^{(R)}) < 2^{-\ulcorner\sigma\urcorner+1}. \tag{5.2}$$

By Lemma 5.10, for every $x \in [0,1]$, there is at most one σ such that $\varphi(x; K_\sigma^{(R)}) \neq 0$ and for any σ, $F_T((c_\sigma + d_\sigma)/2) = F_T(d_\sigma) = 0$.

Pick any $\langle a, b, r, s\rangle \in \mathbb{Q}$ such that $a < b$, $r < s$ and $(a, b) \cap [0, 1] \neq \emptyset$. To determine whether $\mathrm{ran}(F_T \restriction (a, b)) \subseteq (r, s)$, we consider the following three cases.

Case 1. $s \leq 0$. Then $\mathrm{ran}(F_T \restriction (a, b)) \not\subseteq (r, s)$, since $F_T(x) \geq 0$ for all $x \in [0, 1]$.

Case 2. $r < 0 < s$. Then

$$\mathrm{ran}(F_T \restriction (a, b)) \subseteq (r, s) \leftrightarrow \max_{a<x<b} F_T(x) < s.$$

Let n_s be the maximal n such that $2^{-n+1} \geq s$. Then by (5.2),

$$\max_{a<x<b} F_T(x) < s \leftrightarrow \max_{\substack{a<x<b,\\ \sigma\in T;\ulcorner\sigma\urcorner\leq n_s}} \varphi(x; K_\sigma^{(R)}) < s.$$

Case 3. $r \geq 0$. Then

$$\mathrm{ran}(F_T \restriction (a, b)) \subseteq (r, s) \leftrightarrow \exists\sigma \in T\,((a, b) \subset K_\sigma^{(R)} \wedge \\ \mathrm{ran}(\varphi(x; K_\sigma^{(R)}) \restriction (a, b)) \subseteq (r, s)).$$

Let n_{b-a} be the maximal n such that $2^{-n+1} \geq b - a$. Then for all $\sigma \in T$ such that $\ulcorner\sigma\urcorner > n_{b-a}$, $1/2(d_\sigma - c_\sigma) < b - a$ and so $(a, b) \not\subseteq K^{(R)}$. Therefore,

$$\mathrm{ran}(F_T \restriction (a, b)) \subseteq (r, s) \leftrightarrow \exists\sigma \in T\,(\ulcorner\sigma\urcorner \leq n_{b-a} \wedge \\ (a, b) \subset K_\sigma^{(R)} \wedge \mathrm{ran}(\varphi(x; K_\sigma^{(R)}) \restriction (a, b)) \subseteq (r, s)).$$

Hence, $\mathrm{ran}(F_T \restriction (a, b)) \subseteq (r, s)$ is uniformly recursive in T. Let

$$\hat{F}_T = \{\langle a, b, r, s\rangle \in \mathbb{Q}^4 : a < b, r < s, (a, b) \cap [0, 1] \neq \emptyset, \mathrm{ran}(F_T \restriction (a, b)) \subseteq (r, s)\}$$

and

$$\Phi_{\chi(e)} = \{\pi\langle i, \pi\langle j, \pi\langle k, l\rangle\rangle\rangle : \langle a(i), a(j), a(k), a(l)\rangle \in \hat{F}_{T_{g(e)}}\},$$

where $a(i)$, $a(j)$, $a(k)$, $a(l)$ are the rational numbers coded by i, j, k, l respectively and π is as defined in Section 5.2. Then $\Phi_{\chi(e)}$ describes $F_{T_{g(e)}}$.

To see that $e \in \mathrm{WF}$ if and only if $\chi(e) \in D$, it suffices to prove the following lemma.

Lemma 5.12 ([33]). *T is a well founded tree if and only if F_T is everywhere differentiable on* $[0, 1]$.

Proof. For very $\alpha \in \omega^\omega$, let $x_\alpha = \bigcap_{n<\omega} K_{\alpha\restriction n}$ and $G_T = \{x_\alpha : \alpha \in [T]\}$. Then it suffices to show that

$$x \in G_T \leftrightarrow F'_T(x) \text{ does not exist.}$$

Firstly, if $x \in G_T$, then $x \in K^{(L)}_{\alpha\restriction n}$ for all n. By Lemma 5.10, $F_T(x) = 0$. Let $\eta_n = 1/4(d_{\alpha\restriction n} - c_{\alpha\restriction n})$, (i.e. half of the length of $K^{(R)}_{\alpha\restriction n}$), and $\xi_n = 1/4\, c_{\alpha\restriction n} + 3/4\, d_{\alpha\restriction n}$, (i.e. the middle point of $K^{(R)}_{\alpha\restriction n}$). Then $\lim_n \eta_n = 0, \lim_n \xi_n = x$. Note that

$$\frac{F_T(\xi_n + \eta_n) - F_T(x)}{\xi_n + \eta_n - x} = 0,$$

since $\xi_n + \eta_n$ is the right end point of $K^{(R)}_{\alpha\restriction n}$ and

$$\frac{F_T(\xi_n) - F_T(x)}{\xi_n - x} = \frac{2\eta_n}{\xi_n - x} \geq \frac{2\eta_n}{3\eta_n} = \frac{2}{3}.$$

Hence $F'_T(x)$ does not exist.

Secondly, suppose $x \notin G_T$. Note that $G_T = \bigcap_n \bigcup_{\sigma\in T;|\sigma|=n} J_\sigma$. So there is some n, such that $x \notin J_\sigma$ for any $\sigma \in T$ of length at least n. Moreover, for any $n' < n$, there is at most one $\tau \in T$ of length n' such that $x \in J_\tau$. Therefore, $\{\sigma \in T : x \in J_\sigma\}$ is finite. Let $N \geq 2$ be large enough such that for all $\sigma \in T$ if $\ulcorner\sigma\urcorner \geq N$, then $x \notin J_\sigma$. Then for any $y \in [0, 1]$, if $y \in K^{(R)}_\sigma$, $\ulcorner\sigma\urcorner \geq N$,

$$\left|\frac{\varphi(y; K^{(R)}_\sigma) - \varphi(x; K^{(R)}_\sigma)}{y - x}\right| \leq \frac{1/2(d_\sigma - c_\sigma)}{1/2(1 - 1/2^{\ulcorner\sigma\urcorner})(b_\sigma - a_\sigma)} < \frac{1}{2^{\ulcorner\sigma\urcorner - 1}}$$

For $n \geq N$, let

$$F_{T,n} = \sum_{\sigma\in T;\ulcorner\sigma\urcorner\leq n} \varphi(x; K^{(R)}_\sigma).$$

Then for all $n \geq N$, $y \in [0,1] \setminus \{x\}$,

$$\left|\frac{F_T(y) - F_T(x)}{y - x} - \frac{F_{T,n}(y) - F_{T,n}(x)}{y - x}\right| \leq \sum_{\sigma \in T; \ulcorner \sigma \urcorner > n} \left|\frac{\varphi(y; K_\sigma^{(R)}) - \varphi(x; K_\sigma^{(R)})}{x - y}\right| \leq \frac{1}{2^{n-1}}.$$

Let y approach x, then

$$\left|\liminf_{y \to x} \frac{F_T(y) - F_T(x)}{y - x} - \limsup_{y \to x} \frac{F_T(y) - F_T(x)}{y - x}\right| \leq \frac{1}{2^{n-2}}$$

for any $n \geq N$. Thus, $F_T'(x)$ exists. □

5.4 Effective Ranks of Continuous Functions

In this section, we study the Kechris-Woodin derivative defined in [24] and assign a Kechris-Woodin rank to each continuous function. Then we show the correspondence between recursive ordinals and recursively described continuous (including everywhere differentiable) functions on $[0, 1]$.

For each function $f \in \mathbf{C}[0, 1]$, $\varepsilon \in \mathbb{Q}^+$ and closed set $P \subseteq [0, 1]$, let the *Kechris-Woodin derivative* of P with respect to f and ε be

$$\begin{aligned} P'_{\varepsilon,f} = P'_\varepsilon = \{x \in P : \ & \text{For every open neighborhood } U \text{ of } x \text{ there are} \\ & \text{rational points } p < q, r < s \text{ in } U \cap [0,1] \text{ with} \\ & [p,q] \cap [r,s] \cap P \neq \emptyset \text{ and } |\Delta_f(p,q) - \Delta_f(r,s)| > \varepsilon\}, \end{aligned}$$

where

$$\Delta_f(x, y) = \frac{f(x) - f(y)}{x - y}, \quad x, y \in [0,1], \quad x \neq y.$$

We may iterate the Kechris-Woodin derivative along ordinals as follows.

$$\begin{aligned} & P^0_{\varepsilon,f} = P^0_\varepsilon = [0,1] \\ & P^{\alpha+1}_{\varepsilon,f} = P^{\alpha+1}_\varepsilon = (P^\alpha_\varepsilon)'_\varepsilon \\ & P^\lambda_{\varepsilon,f} = P^\lambda_\varepsilon = \textstyle\bigcap_{\alpha<\lambda} P^\alpha_\varepsilon, \quad \lambda \text{ is a limit oridinal} \end{aligned}$$

By its definition, we have the following properties of the Kechris-Woodin derivative.

Proposition 5.4.1. *For all ordinals $\alpha \leq \beta$, $P^{\alpha}_{\varepsilon} \supseteq P^{\beta}_{\varepsilon}$ and for any positive rational numbers $\varepsilon \leq \varepsilon'$, $P^{\alpha}_{\varepsilon} \supseteq P^{\alpha}_{\varepsilon'}$.*

Thus, there is a least $\alpha_f(\varepsilon) = \alpha(\varepsilon) < \aleph_1$ such that for all $\alpha \geq \alpha(\varepsilon)$, $P^{\alpha}_{\varepsilon} = P^{\alpha(\varepsilon)}_{\varepsilon} = P^{\infty}_{\varepsilon}$. Let the *Kechris-Woodin rank* of f,

$$|f|_{\mathrm{KW}} = \mu\alpha\{\alpha : \alpha \geq \alpha_f(\varepsilon),\ \text{for all } \varepsilon \in \mathbb{Q}^+\},$$

and the *Kechris-Woodin kernel* of f,

$$\mathrm{Ker}_{\mathrm{KW}}(f) = \bigcup_{\varepsilon \in \mathbb{Q}^+} \bigcap_{\alpha} P^{\alpha}_{\varepsilon} = \bigcup_{\varepsilon \in \mathbb{Q}^+} P^{\infty}_{\varepsilon}.$$

Lemma 5.13 ([24]). *f is everywhere differentiable if and only if* $\mathrm{Ker}_{\mathrm{KW}}(f) = \emptyset$.

Proof. Suppose for some $x \in [0,1]$, $f'(x)$ does not exist, then for some $\varepsilon \in \mathbb{Q}^+$ we have the following property:

For every open neighborhood U of x, there are rational numbers $r, s, p, q \in U \cap [0,1]$ such that $r \leq x \leq s$, $p \leq x \leq q$, $|\Delta_f(r,s) - \Delta_f(p,q)| > \varepsilon$.

By induction, $x \in P^{\alpha}_{\varepsilon}$ for all α and $\mathrm{Ker}_{\mathrm{KW}}(f) \neq \emptyset$.

Now we consider the case that f is everywhere differentiable. Then we claim that for every ε and any nonempty closed set $P \subseteq [0,1]$, $P'_{\varepsilon} \neq P$, therefore $\mathrm{Ker}_{\mathrm{KW}}(f) = \emptyset$. For the sake of a contradiction, we assume that $P \subseteq [0,1]$ is closed and $P'_{\varepsilon} = P$. Then by the definition of P'_{ε} we have for every $n < \omega$, the set

$$\begin{aligned} E_n = \{x \in P : \exists p, q, r, s \in \mathbb{Q} \cap [0,1]\, (p < x < q \ \wedge\ r < x < s \wedge \\ q - p, s - r < 1/n \ \wedge\ |\Delta_f(p,q) - \Delta_f(r,s)| > \varepsilon)\} \\ \bigcup \{x \in P : \exists q, s \in \mathbb{Q} \cap [0,1]\, (x < q < 1/n \ \wedge\ x < s < 1/n \wedge \\ |\Delta_f(0,q) - \Delta_f(0,s)| > \varepsilon)\} \\ \bigcup \{x \in P : \exists p, r \in \mathbb{Q} \cap [0,1]\, (1 - 1/n < p < x \wedge \\ 1 - 1/n < r < x \ \wedge\ |\Delta_f(p,1) - \Delta_f(r,1)| > \varepsilon)\} \end{aligned}$$

is open dense in P. By the Baire Category Theorem, $\bigcap_n E_n \neq \emptyset$. Let $x \in \bigcap_n E_n$. Then $f'(x)$ does not exist. That is a contradiction. □

Proposition 5.4.2. *Suppose f is everywhere differentiable on* $[0,1]$. *Then f' is continuous if and only if* $|f|_{\mathrm{KW}} = 1$.

Proof. Assume that f' is continuous. Then for every $\varepsilon \in \mathbb{Q}^+$ and $x \in [0,1]$, there is an open neighborhood U of x, such that for any $y \in U \cap [0,1]$, $|f'(y) - f'(x)| < \varepsilon/2$. Therefore, if $p < q, r < s$ in $U \cap [0,1]$, then $|\Delta_f(p,q) - \Delta_f(r,s)| = |f'(y) - f'(z)| < \varepsilon$ for some $y, z \in U \cap [0,1]$. Hence $P^1_\varepsilon = \emptyset$.

Now suppose f' is not continuous at x. Then there is an $\varepsilon \in \mathbb{Q}^+$ such that for all open neighborhood U of x there is a $y \in U \cap [0,1]$ with $|f'(y) - f'(x)| > \varepsilon$. Therefore, there are rational numbers $p < q, r < s$ in $U \cap [0,1]$ with $[p,q] \cap [r,s] \neq \emptyset$ and $|\Delta_f(p,q) - \Delta_f(r,s)| > \varepsilon$. Thus, $P^1_\varepsilon \neq \emptyset$. $\square$

Now we consider sets of natural numbers that describe the Kechris-Woodin derivatives. For any ordinal α, let

$$\hat{P}^\alpha_{\varepsilon,f} = \hat{P}^\alpha_\varepsilon = \{\pi\langle l, l'\rangle : l, l' < \omega, a(l) < a(l'), P^\alpha_{\varepsilon,f} \cap (a(l), a(l')) = \emptyset\},$$

where $a(l)$, $a(l')$ are rational numbers coded by l and l' respectively and π is defined as in Section 5.3. Then $\hat{P}^\alpha_\varepsilon$ describes the closed set P^α_ε. Let

$$\begin{aligned}\hat{\mathrm{Ker}}_{\mathrm{KW}}(f) = \{\pi\langle e, \pi\langle l, l'\rangle\rangle : e, l, l' \in \omega, a(l) < a(l'), a(e) > 0,& \\ \pi\langle l, l'\rangle \in \hat{P}^\alpha_{a(e)} \text{ for some } \alpha\}.&\end{aligned}$$

Then we say $\hat{\mathrm{Ker}}_{\mathrm{KW}}(f)$ *describes* $\mathrm{Ker}_{\mathrm{KW}}(f)$.

Proposition 5.4.3. *For any ordinal α,*

$$\hat{P}^{\alpha+1}_{\varepsilon,f} = \Gamma_{\varepsilon,f}(\hat{P}^\alpha_\varepsilon),$$

where for any $A \subseteq \{\pi\langle l, l'\rangle : l, l' < \omega, a(l) < a(l')\}$,

$$\begin{aligned}\Gamma_{\varepsilon,f}(A) = \{\pi\langle l, l'\rangle : l, l' < \omega, a(l) < a(l'), \textit{and for all rational points } p < q,& \\ r < s \textit{ in } (a(l), a(l')) \cap [0,1], \ \textit{if } [p,q] \cap [r,s] \cap \overline{\textstyle\bigcup_{\pi\langle l,l'\rangle \in A} (a(l), a(l'))} \neq \emptyset,& \\ \textit{then } |\Delta_f(p,q) - \Delta_f(r,s)| \leq \varepsilon\}.&\end{aligned}$$

In particular, if $\lambda > 0$ is a limit ordinal, then

$$\hat{P}^{\lambda+1}_\varepsilon = \Gamma_{\varepsilon,f}(\textstyle\bigcup_{\alpha<\lambda} \hat{P}^\alpha_\varepsilon).$$

Proof. By the definition of Kechris-Woodin derivative. $\square$

Suppose $\hat{f} \subset \omega$ describes f. Then in Proposition 5.4.3 $\Gamma_{\varepsilon,f}$ is a monotonic operator arithmetic in $\hat{f}$. Therefore, by Proposition 2.2.4, we have Theorem 5.14.

Theorem 5.14. *If f has a recursive description, then $|f|_{\mathrm{KW}} \leq \omega_1^{CK}$. Moreover, if f is everywhere differentiable on $[0,1]$, then $|f|_{\mathrm{KW}} < \omega_1^{CK}$.*

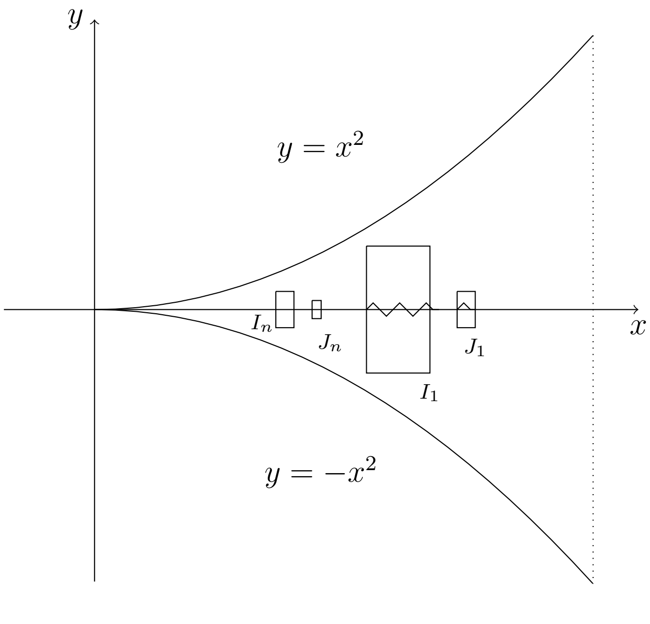
The rest of this section is devoted to the proof of Theorem 5.15. Part of the proof idea can be found in [24].

An everywhere differentiable nonnegative function f on $[0,1]$ is *nice*, if $\max_{x\in[0,1]} f(x) < 1$, $\max_{x\in[0,1]} f'(x) < 1$ and $f(0) = f(1) = f'(0) = f'(1) = 0$.

Theorem 5.15. *There is a recursive function $F : \omega \to \omega$ such that if $e \in \mathcal{O}$ and $e \neq 1$, then $\Phi_{F(e)}$ describes a nice function $f_{|e|_{\mathcal{O}}}$ on $[0,1]$ such that $|f_{|e|_{\mathcal{O}}}|_{\mathrm{KW}} = |e|_{\mathcal{O}}$ and if $|e|_{\mathcal{O}}$ is a successor, then $|f_{|e|_{\mathcal{O}}}|_{\mathrm{KW}} = \alpha_{f_{|e|_{\mathcal{O}}}}(1/9)$.*

Recall $\varphi(x;K)$ in Section 5.3. Let $f_1(x) = 1/64\varphi(x;[1/4,3/4])$. Then f_1 is nice and by Proposition 5.4.2, $|f_1|_{\mathrm{KW}} = \alpha_{f_1}(1/9) = 1$. Let e_0 be an index such that Φ_{e_0} describes f_1 (e_0 exists by the argument in Section 5.3) and define $F(n,2) = \Phi(n,0) = e_0$. Next, we will construct f_α inductively and show there is an effective induction of their recursive descriptions with parameter n. By the Recursion Theorem, the parameter n could be fixed.

Consider the following picture.



Lemma 5.16. *There are recursive sequences of rational numbers* $\{a_n\}_{n<\omega}$, $\{b_n\}_{n<\omega}$, $\{c_n\}_{n<\omega}$, $\{d_n\}_{n<\omega}$ *such that* $\lim_n a_n = 0$ *and for all* $n < \omega$,

(i) If $n > 0$, *then* $0 < a_n < b_n < c_n < d_n < a_{n-1} \leq a_0 < b_0 < c_0 < d_0 < 1$. *We denote* $[a_n, b_n]$ *by* I_n *and* $[c_n, d_n]$ *by* J_n.

(ii) The squares with basis I_n *or* J_n *are between the graphs of* $y = x^2$ *and* $y = -x^2$, *i.e.* $d_n - c_n < (c_n)^2$ *and* $b_n - a_n < (a_n)^2$.

(iii) Let the middle point of J_n *be* ξ_n. *Then* $\frac{d_n - c_n}{\xi_n - a_n} = 1/4$.

Here "recursive" is defined in the sense of Remark 5.2.

Proof. For every $n < \omega$ let

$$\begin{aligned}
a_n &= \frac{1}{n+2}, \\
s_n &= a_n + (a_n)^2, \\
b_n &= \frac{1}{4}(s_n - a_n) + a_n, \\
c_n &= \frac{7}{16}(s_n - a_n) + a_n, \\
d_n &= \frac{9}{16}(s_n - a_n) + a_n.
\end{aligned}$$

s_n is applied here for the sake of Clause (ii). It is straightforward to check that Clause (i) to (iii) are satisfied. $\square$

For the rest of this section, we fix sequences $\{I_n = [a_n, b_n]\}_{n<\omega}$ and $\{J_n = [c_n, d_n]\}_{n<\omega}$ as in Lemma 5.16. A function f_α is constructed so that $f_\alpha \restriction I_n$ is a scanned copy of some $f_{\alpha'}$, $\alpha' < \alpha$, $f_\alpha \restriction J_n$ depends on the property of α, and f_α equals 0 on the complement of $\bigcup_n I_n \cup \bigcup_n J_n$.

We consider the following three cases.

Case 1. α is a limit ordinal and $\{\beta_n\}_{n<\omega}$ is a strictly increasing sequence of ordinals such that $\lim_n \beta_n = \alpha$. Define

$$f_\alpha(x) = \begin{cases} (f_{\beta_n}/(n+1))^{scan}_{a_n, b_n}(x) & \text{if } x \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then f_α is nice on [0,1]. Moreover, for every $n<\omega$, $\max_{x\in I_n}((f_{\beta_n}/(n+1))^{scan}_{a_n,b_n})'(x) = \max_{x\in[0,1]} f'_{\beta_n}(x)/(n+1) < 1/(n+1)$. Then for any $\varepsilon \in \mathbb{Q}^+$ and closed set $P \subseteq [0,1]$, we have $P'_{\varepsilon,f_\alpha} \subseteq \bigcup_{2/(n+1)>\varepsilon} I_n$. Thus, $|f_\alpha|_{\mathrm{KW}} = \sup_n |f_{\beta_n}/(n+1)|_{\mathrm{KW}} = \alpha$.

Case 2. $\alpha = \lambda+1$, where λ is a limit and $\{\beta_n\}_{n<\omega}$ is a strictly increasing sequence of successor ordinals such that $\lim_n \beta_n{=}\lambda$. Define

$$f_\alpha(x) = \begin{cases} (f_{\beta_n})^{scan}_{a_n,b_n}(x) & \text{if } x\in I_n,\\ 0 & \text{otherwise.}\end{cases}$$

Then f_α is nice on [0,1]. For any $\varepsilon\in\mathbb{Q}^+$, and closed set $P\subseteq[0,1]$, $P'_{\varepsilon,f_\alpha} \subseteq \bigcup_n I_n \cup \{0\}$. Thus, $|f_\alpha|_{\mathrm{KW}} \geq \sup_n |f_{\beta_n}|_{\mathrm{KW}} = \lambda$. On the other hand, for all $\varepsilon\in\mathbb{Q}^+$, $P^\lambda_{\varepsilon,f_\alpha} \subseteq \{0\}$. To see $|f_\alpha|_{\mathrm{KW}} = \alpha_{f_\alpha}(1/9) = \lambda+1$, it suffices to show that $0 \in P^\lambda_{1/9,f_\alpha}$. Since $|f_{\beta_n}|_{\mathrm{KW}} = \alpha_{f_{\beta_n}}(1/9) = \beta_n$, $P^\beta_{1/9,f_{\beta_n}} \upharpoonright I_n \neq \emptyset$ for every $\beta<\lambda$ and $\beta_n>\beta$. Therefore, $0\in P^\beta_{1/9,f_\alpha}$ for all $\beta<\lambda$. Thus, $0\in P^\lambda_{1/9,f_\alpha}$.

Case 3. $\alpha=\beta+1$, where β is a successor ordinal. Then let

$$f_\alpha(x) = \begin{cases} (f_\beta)^{scan}_{a_n,b_n}(x) & \text{if } x\in I_n,\\ (\varphi(x;[1/4,3/4]))^{scan}_{c_n,d_n}(x) & \text{if } x\in J_n,\\ 0 & \text{otherwise.}\end{cases}$$

Then f_α is nice on [0,1]. As in Case 2, for any $\varepsilon\in\mathbb{Q}^+$, and closed set $P\subseteq[0,1]$, $P'_{\varepsilon,f_\alpha} \subseteq \bigcup_n I_n\cup\{0\}$. Thus, $P^\alpha_{\varepsilon,f_\alpha}\subseteq\{0\}$.To see $|f_\alpha|_{\mathrm{KW}} = \alpha_{f_\alpha}(1/9) = \beta+1$, we only need to show that $0\in P^\beta_{1/9,f_\alpha}$. Suppose $\beta=\gamma+1$. Then for any n, $P^\gamma_{1/9,f_\alpha}\upharpoonright I_n\neq\emptyset$. Let ξ_n be the middle point of J_n. Note that

$$\Delta_{f_\alpha}(\xi_n,a_n) - \Delta_{f_\alpha}(b_n,a_n) = \frac{1/2(d_n-c_n)}{\xi_n-a_n} = \frac{1}{8},$$

and $[a_n,\xi_n]\cap[a_n,b_n] = I_n$. Thus, $0\in P^\beta_{1/9,f_\alpha}$.

Now we consider recursive descriptions of $\{f_\alpha\}_{\alpha<\omega_1^{CK}}$.

Suppose $\{g_n\}_{n<\omega}$ and $\{h_n\}_{n<\omega}$ are sequences of nice functions on $[0,1]$, and $\{\Psi_{n;g}\}_{n<\omega}$, $\{\Psi_{n;h}\}_{n<\omega}$ are uniformly recursive sequences of total recursive functions such that for all n, $\Psi_{n;g}$ describes g_n and $\Psi_{n;h}$ describes h_n. Then define

$$a_{-1} = 1,$$
$$C_{n;g} = \{\pi\langle i,\pi\langle j,\pi\langle l,k\rangle\rangle\rangle : a_n - a(i), a(j)-b_n < \min\{a_n-d_{n+1}, c_n-b_n\}\},$$
$$C_{n;h} = \{\pi\langle i,\pi\langle j,\pi\langle l,k\rangle\rangle\rangle : c_n - a(i), d(j)-b_n < \min\{c_n-b_n, a_{n-1}-d_n\}\},$$

$$\Psi_g = (\bigcup_n (\Psi_{n;g})^{scan}_{a_n,b_n} \cap C_{n;g}) \cup \{\pi\langle i, \pi\langle j, \pi\langle k,l\rangle\rangle\rangle : a(k) < 0 < a(l) \wedge$$
$$a(i) < a(j) \wedge [a(j) \leq 0 \vee a(i) \geq b_0$$
$$\vee \exists n\, (a(i) = b_n, b(i) = a_{n-1})]\}, \quad (5.1)$$

and

$$\Psi_{g,h} = (\bigcup_n (\Psi_{n;g})^{scan}_{a_n,b_n} \cap C_{n;g}) \cup (\bigcup_n (\Psi_{n;h})^{scan}_{c_n,d_n} \cap C_{n;h}) \cup \{\pi\langle i, \pi\langle j, \pi\langle k,l\rangle\rangle\rangle :$$
$$a(k) < 0 < a(l) \wedge a(i) < a(j) \wedge [a(j) \leq 0 \vee a(i) \geq d_0$$
$$\vee \exists n\, [(a(i) = b_n, b(i) = c_n) \vee (a(i) = d_n, b(i) = a_{n-1})]]\}, \quad (5.2)$$

where $a : \omega \to \mathbb{Q}$, $\pi : \omega^2 \to \omega$ and $(\quad)^{scan}_{a,b}$ are as defined in Section 5.2 in the sense of Remark 5.2.

Lemma 5.17. Ψ_g *describes* g *and* $\Psi_{g,h}$ *describes* $g \oplus h$, *where*

$$g(x) = \begin{cases} g_n(x) & \text{if } x \in I_n, \\ 0 & \text{otherwise.} \end{cases} \qquad g \oplus h(x) = \begin{cases} g_n(x) & \text{if } x \in I_n, \\ h_n(x) & \text{if } x \in J_n, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.17 is straightforward and we skip its proof.

Now we are back to the proof of Theorem 5.15.

Let $|e|_{\mathcal{O}} = \alpha > 1$. Suppose for all $1_{\mathcal{O}} < e' <_{\mathcal{O}} e$ we have defined $F(n,e')$, such that $\forall e' <_{\mathcal{O}} e\, [\forall e'' <_{\mathcal{O}} e'\, (\Phi_{F(n,e'')}$ describes $f_{|e''|_{\mathcal{O}}}) \to \Phi_{F(n,e')}$ describes $f_{|e'|_{\mathcal{O}}}]$. Now we define $F(n,e)$.

Case 1. $e = 3 \cdot 5^m$. Then α is a limit. Let $g_k = f_{|\Phi_m(k)|_{\mathcal{O}}}/(k+1)$, $\Psi_{k;g} = (1/(k+1))((\Phi_{F(n,\Phi_m(k))})^{scan}_{a_k,b_k})$ for all $k < \omega$, and $\Phi_{F(n,e)} = \Psi_g$ be defined as in (5.1). Then $\Phi_{F(n,e)}$ describes f_α.

Case 2. $e = 2^{3 \cdot 5^m}$. So $\alpha = \lambda + 1$, where λ is a limit ordinal. Then let $g_k = f_{|2^{\Phi_m(k)}|_{\mathcal{O}}}$, $\Psi_{k;g} = (\Phi_{F(n,2^{\Phi_m(k)})})^{scan}_{a_k,b_k}$ for all $k < \omega$, and $\Phi_{F(n,e)} = \Psi_g$ be defined as in (5.1). $\Phi_{F(n,e)}$ describes f_α.

Case 3. $e = 2^m$, where $m \neq 3 \cdot 5^{m'}$ for any $m' < m$. Then $\alpha = \beta + 1$, where β is a successor ordinal. Then let $g_k = f_{|m|_{\mathcal{O}}}$, $h_k = \varphi(\_; [1/4, 3/4])$, $\Psi_{k;g} = (\Phi_{F(n,m)})^{scan}_{a_k,b_k}$, $\Psi_{k;h}$ be a recursive description of $\varphi(\_; [1/4, 3/4])$. Define $\Phi_{F(n,e)} = \Psi_{g,h}$ as in (5.2). Then $\Phi_{F(n,e)}$ describes f_α.

By the Recursion theorem, there is a fixed parameter n_0, such that $\Phi_{F(e)} = \Phi_{F(n_0,e)}$ describes $f_{|e|_\mathcal{O}}$ for all $e \in \mathcal{O} \setminus \{1\}$, where $F(e) = F(n_0, e)$. The proof of Theorem 5.15 is complete.

5.5 Kechris-Woodin Kernel and Π^1_1-CA$_0$

In this section, we discuss the existence of Kechris-Woodin kernel from the view point of reverse mathematics. In a model $\mathcal{M}$ of second order arithmetic, we say a continuous function f (a closed set, $\mathrm{Ker}_{\mathrm{KW}}(f)$) exists, if f (the closed set, $\mathrm{Ker}_{\mathrm{KW}}(f)$, respectively) has a description in the second order part of $\mathcal{M}$.

Theorem 5.18 (ATR$_0$)**.** *For every continuous function f on $[0,1]$, either* $\mathrm{Ker}_{\mathrm{KW}}(f) = \emptyset$ *or there is a nonempty closed set $P \subseteq Ker_{\mathrm{KW}}(f) \neq \emptyset$.*

Proof. Suppose $\mathcal{M}$ is a model of ATR$_0$ and f in $\mathcal{M}$ is a continuous function on $[0,1]$. And suppose R is a well ordering in $\mathcal{M}$ and 0 is the R-least element. Then there is a sequence of sets $\{A_n\}_{n\in F(R)}$ such that $A_0 = \emptyset$ and

$$\forall n > 0\,(n \in F(R) \rightarrow A_n = \textstyle\bigcup_{iRn}\Gamma(A_i)),$$

where $\Gamma(A_i) = \bigcup_n \{\pi\langle n,k\rangle : a(n) > 0, k \in \Gamma_{a(n),f}(A_i^{[n]})\}$, $a(n)$ is the rational number coded by n and $\Gamma_{a(n),f}$ is defined as in Proposition 5.4.3.

Since "R is an well ordering" is not Σ^1_1 property, there is a linear order but not well ordering R (in the sense of $\mathcal{M}$) such that 0 is the R-least element, and there exists a sequence of sets $A = \{A_n\}_{n\in F(R)}$ satisfying the requirements in last paragraph and R has no (A,R)-recursive infinite R-decreasing sequence.

For any $n \in F(R)$, we say n is *nonstandard* if there is an infinite R-decreasing sequence (in the sense of the universe) in $\{i \in F(R) : iRn\}$.

Pick any nonstandard n. We claim that $A_n \supseteq \hat{\mathrm{Ker}}_{\mathrm{KW}}(f)$, where $\hat{\mathrm{Ker}}_{\mathrm{KW}}(f)$ describes $\mathrm{Ker}_{\mathrm{KW}}(f)$. Suppose not, then there is an $m \in \hat{\mathrm{Ker}}_{\mathrm{KW}}(f) \setminus A_n$. By the proof of (2.1) (let $k = m$ in (2.1)), $\{jRi : iRn\}$ is a well ordering (in the sense of the universe), deriving a contradiction. Now if $A_n = \hat{\mathrm{Ker}}_{\mathrm{KW}}(f)$ or $\mathrm{Ker}_{\mathrm{KW}}(f) = \emptyset$, then we are done. Otherwise, suppose $k \in A_n \setminus \hat{\mathrm{Ker}}_{\mathrm{KW}}(f)$. Then there is an R-least element $n' \in F(R)$ such that $k \in A_{n'}$ (If n' does not exist, then there is an infinite (A,R)-recursive infinite R-decreasing sequence). Clearly, n' is nonstandard. Then we have a nonstandard n'' such that $n''Rn'$. Since $k \notin A_{n''}$ and $A_{n''} \supseteq \hat{\mathrm{Ker}}_{\mathrm{KW}}(f)$, $A_{n''}$ describes a nonempty closed subset of $\mathrm{Ker}_{\mathrm{KW}}(f)$. $\square$

Theorem 5.19. *The existence of Kechris-Woodin kernel is equivalent to Π^1_1-CA_0 over ACA_0*

By Proposition 2.2.5, it suffices to show Lemma 5.20.

Lemma 5.20. *The following are equivalent over ACA_0.*

1. *For any sequence of trees $\{T_k\}_{k<\omega}$, $T_k \subseteq \omega^{<\omega}$, there exists a set X such that $\forall k(k \in X \leftrightarrow T_k$ has a path).*
2. *For every f continuous on $[0,1]$, $\mathrm{Ker}_{\mathrm{KW}}(f)$ exists.*

Proof. $(1 \to 2)$. Suppose f continuous in $[0,1]$. And $A = \{\langle r,s\rangle : 0 \le r < s \le 1\}$. We construct a sequence of trees $T_{\varepsilon,f,(a,b)} \subseteq A^{<\omega}$ arithmetically in a description of f, where $\varepsilon \in \mathbb{Q}$ and $a < b$ are rational numbers.

$$\begin{aligned}T_{\varepsilon,f,(a,b)} = \{\emptyset\} &\cup \{\langle r,s\rangle \in A : a < r < s < b\}\\ &\cup \{\langle\langle r_0,s_0\rangle,\ldots,\langle r_n,s_n\rangle\rangle \in A^{n+1} : \varepsilon \in \mathbb{Q} \wedge n < \omega \wedge \forall i \le n\,(s_i - r_i < \tfrac{1}{2^i})\\ &\qquad \wedge \textstyle\bigcap_{i=0}^{n}[r_i,s_i] \neq \emptyset \wedge \forall i < n\,(|\Delta_f(r_{i+1},s_{i+1}) - \Delta_f(r_i,s_i)| > \varepsilon)\}.\end{aligned}$$

To show $1 \to 2$, it suffices to prove that $T_{\varepsilon,f,(a,b)}$ is well founded if and only if $(a,b) \cap P^{\infty}_{\varepsilon,f} = \emptyset$.

Suppose $T_{\varepsilon,f,(a,b)}$ is not well founded. Let $\langle\langle r_0,s_0\rangle,\langle r_1,s_1\rangle,\ldots,\langle r_n,s_n\rangle,\ldots\rangle$ be an infinite path in $T_{\varepsilon,f,(a,b)}$. Then $\bigcap_n [r_n,s_n] \neq \emptyset$. Let $x_0 \in \bigcap_n [r_n,s_n] \subset (a,b)$. By induction, $x_0 \in P^{\alpha}_{\varepsilon,f}$ for all α. Thus, $(a,b) \cap P^{\infty}_{\varepsilon,f} \neq \emptyset$.

Now assume $P = (a,b) \cap P^{\infty}_{\varepsilon,f} \neq \emptyset$. As in the proof of Lemma 5.13, define

$$\begin{aligned}E_n = \{x \in P : \exists p,q,r,s \in \mathbb{Q}\cap[0,1]\,(p < x < q \,\wedge\, r < x < s \,\wedge\\ q-p, s-r < 1/n \,\wedge\, |\Delta_f(p,q) - \Delta_f(r,s)| > \varepsilon)\}\\ \bigcup\{x \in P : \exists q,s \in \mathbb{Q}\cap[0,1]\,(x < q < 1/n \,\wedge\, x < s < 1/n \,\wedge\\ |\Delta_f(0,q) - \Delta_f(0,s)| > \varepsilon)\}\\ \bigcup\{x \in P : \exists p,r \in \mathbb{Q}\cap[0,1]\,(1-1/n < p < x \,\wedge\\ 1-1/n < r < x \,\wedge\, |\Delta_f(p,1) - \Delta_f(r,1)| > \varepsilon)\},\end{aligned}$$

and E_n is open and dense in P since $(P^{\infty}_{\varepsilon,f})'_{\varepsilon,f} = P^{\infty}_{\varepsilon,f}$. By the Baire Category Theorem, $\bigcap_n E_n \neq \emptyset$. Let $x_1 \in \bigcap_n E_n$. Then by the definition of $T_{\varepsilon,f,(a,b)}$, there is an infinite path $\langle\langle r_0, s_0\rangle, \langle r_1, s_1\rangle, \ldots, \langle r_n, s_n\rangle, \ldots\rangle$ such that for all n, $r_n < x_1 < s_n$. Then $T_{\varepsilon,f,(a,b)}$ is not well founded.

$(2 \to 1)$. Suppose $\{T_n\}_{n<\omega}$ is a sequence of trees. Let F_{T_n} be defined as in Section 5.3 and let

$$\begin{aligned} p_n &= \frac{1}{4}(\frac{1}{n+1} - \frac{1}{n+2}) + \frac{1}{n+2}, \\ q_n &= \frac{3}{4}(\frac{1}{n+1} - \frac{1}{n+2}) + \frac{1}{n+2}, \quad \forall n < \omega. \\ F(x) &= \begin{cases} (F_{T_n})^{scan}_{p_n,q_n}(x) & \text{if } p_n \leq x \leq q_n \text{ for some } n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Then for n,

$$\begin{aligned} & T_n \text{ is well founded} \\ \leftrightarrow\ & F_{T_n} \text{ is everywhere differentiable on } [0,1] \\ \leftrightarrow\ & \forall \varepsilon \in \mathbb{Q}^+ \, ((\frac{1}{n+2}, \frac{1}{n+1}) \cap P^{\infty}_{\varepsilon,F} = \emptyset).\Box \end{aligned}$$

Chapter 6

Open problems

We conclude this thesis with four open problems.

1. In any $B\Sigma_1$ model, a proper d-r.e. degree exists. Does a proper 3-r.e. degree exist? In general, is there a proper n-r.e. degree in $B\Sigma_1$ models, for $n \geq 3$? If R is r.e. and Q is a subset of $\overline{R}$, then for all stage s, $Q \subseteq \overline{R_s}$. However, if R is 2-r.e. for $n \geq 2$ and $Q \subseteq \overline{R}$, then Q may not be a subset of $\overline{R_s}$ for any s and the computation Φ_e^R may not be correctly approximated. This raises a main difficulty to diagonalize Φ_e^R.

2. Is there a $B\Sigma_1$ model with a non-r.e. degree below $\mathbf{0'}$? We have seen a $B\Sigma_1$ model in which every degree below $\mathbf{0'}$ is r.e. A careful analysis of the proof shows if $B\Sigma_1$ model has a Σ_1 cut on which every Π_2 subset is coded, then in the model, all degrees below $\mathbf{0'}$ are r.e. It is tempting to conjecture that there is a characterization of the existence of non-r.e. degrees below $\mathbf{0'}$ in terms of the existence of codes in the model.

3. It is shown that there is no Friedberg numbering in a $B\Sigma_2$ model. Also, we have seen that in the projection model which satisfies $I\Sigma_1$ but not $B\Sigma_2$, a Friedberg numbering exists. In general, does a Friedberg numbering exist when $B\Sigma_2$ fails?

4. In α-recursion theory, for every admissible ordinal α, $t\sigma 2p(\alpha) = \sigma 2cf(\alpha)$ if and only if there is a Friedberg numbering of r.e. sets in L_α, if and only if there is a Friedberg numbering of d-r.e. sets in L_α. Yet, for $n \geq 3$, we only know that

$t\sigma 2p(\alpha) = \sigma 2cf(\alpha)$ implies there is a Friedberg numbering of n-r.e. sets. The other direction is still open.

For instance, let $\alpha = \aleph_\omega^L$ and $\{D_e\}_{e<\alpha}$ be an effective list of 3-r.e. sets without repetition. To construct an r.e. set X not in the list, we may still diagonalize $\{D_e\}_{e<\alpha}$ block by block. For one block of 3-r.e. sets, whether one set contains $[0, \aleph_n^L)$ as a subset may be determined by stage $\aleph_n^L$ if n is large enough. But if one 3-r.e. set containing $[0, \aleph_n^L)$ enumerates one more element, then that element may further change its mind two more times. A way to deal with this difficulty is to let n be large enough such that we have found all indices in the block, for which the enumeration of D_e is r.e. or d-r.e. This method works if the enumeration of D_e is d-r.e. for all e (see Section 4.3). But it is not sufficient for 3-r.e. sets. By this method, $X \upharpoonright \aleph_n^L = [0, \aleph_n^L)$ cannot "filtrate" any set D_e with a 3-r.e. enumeration. It only filtrates every D_e with a d-r.e. enumeration. If we relax the requirement of X, i.e. let X be a d-r.e. or 3-r.e. set, then it seems that the above problem is solved: we may extract one element from $[0, \aleph_n^L)$, for every 3-r.e. set in the block with a 3-r.e. enumeration. Yet, in practice, we need to recursively approximate $\aleph_n^L$, during which we may mistakenly exhaust all the chances to change one's mind before getting the true value of $\aleph_n^L$. Perhaps there is a Friedberg numbering of 3-r.e. sets in $L_{\aleph_\omega^L}$. In any case, some new technic is needed here.

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