MULTIPERIOD PORTFOLIO OPTIMIZATION WITH TRANSACTION COSTS

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Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety.
I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Fu Yinghui

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Summary

In this thesis, we study a multiperiod mean-variance portfolio optimization problem in the presence of proportional transaction costs. Many existing studies have shown that transaction costs can significantly affect investors' behaviour. However, even under simple assumptions, closed-form solutions are not easy to obtain when transaction costs are considered. As a result, they are often ignored in multiperiod portfolio analysis, which leads to suboptimal solutions. To tackle this complex problem, this thesis studies a market consisting of one risk-free and one risky asset. Whenever there is a trade after the initial asset allocation, the investor incurs a linear transaction cost. The single-period and the two-period cases are investigated before we extend the results to a longer horizon. For single-period and two-period problems, we derive the closed-form expressions of the optimal thresholds for investors to re-allocate their resources. These thresholds divide the action space into three regions. In every region, one investment strategy is recommended out of three options, namely, buy, sell and hold. Some important properties of the analytical solutions to the single-period and two-period models are identified, which shed light on solving investment problems involving more time periods. When more time periods are considered, it becomes intractable since the quadratic structure of the model cannot be retained due to the incorporation of transaction costs. Therefore, based on the features of the optimal solutions identified in single-period and two-period analyses, we develop an approximation method to obtain near optimal solutions. The approximation can work efficiently and effectively under mild assumptions. A series of numerical experiments are conducted to show that the proposed method can significantly improve the investment performance compared to the case when transaction costs are ignored. The recursive property of the proposed approximation method also makes it efficient to solve the multiperiod problem over a long planning horizon.

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Chapter 1

Introduction

Portfolio optimization is a class of studies aiming at optimizing the allocation of an investor's wealth among a basket of assets available on the market according to the investor's preference. The way for modern portfolio selection theory has been paved by Markowitz (1952) using mean and variance as the measures of reward and risk of the portfolio respectively. Such a mean-variance analysis is the first effective approach to treat the trade-off between reward and risk quantitatively. For decades, the mean-variance approach has received great attention, and intensive research has been done in this area. Steinbach (2001) gives an extensive literature review of this approach.

Portfolio optimization can be studied by single-period models and long-term models. To distinguish from single-period models, long-term portfolio optimization models defined in this thesis allow for interim rebalancing of the portfolio. The long-term portfolio selection can be further classified into two categories, namely discrete-time multiperiod models and continuous-time models.

1.1 Mean-Variance Theory

The classic mean-variance model considers a one-period investment problem. The objective is either to maximize the expected final wealth while keeping the variance within a certain level or to minimize the variance while ensuring that the expected final wealth meets a desired level. With such an objective, an investor constructs a portfolio by selecting a number of assets and allocating a portion of his wealth to each asset selected at the beginning of the period, based on observation of the market and anticipation of the performance of financial assets. After the initial investment decision has been made, it is assumed that there is no further adjustment allowed during the investment period.

1.1.1 Return and Risk

Suppose there are N assets in the market to choose from to construct a portfolio. Let $\boldsymbol{x}$ be the portfolio vector which indicates the amount of wealth allocated in each asset, $\mathring{\boldsymbol{R}} = (\mathring{R}_1, \mathring{R}_2, \ldots, \mathring{R}_N)'$ be the random return rate vector with expectation $\bar{\boldsymbol{R}} = (\,\mathrm{E}(\mathring{R}_1),\, \mathrm{E}(\mathring{R}_2), \ldots,\, \mathrm{E}(\mathring{R}_N))'$, and $\boldsymbol{C}$ be the covariance matrix where

$$\boldsymbol{C} = \begin{pmatrix} \mathrm{Var}(\mathring{R}_1) & \cdots & \mathrm{Cov}(\mathring{R}_1, \mathring{R}_N) \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}(\mathring{R}_N, \mathring{R}_1) & \cdots & \mathrm{Var}(\mathring{R}_N) \end{pmatrix}$$

Each portfolio is associated with two performance indicators, return and risk. The return is represented by the expected final wealth $\bar{\boldsymbol{R}}\boldsymbol{x}$, and the risk is indicated by the variance $\boldsymbol{x}'\boldsymbol{C}\boldsymbol{x}$. Under the assumption of quadratic utility functions, mean-variance analysis provides the exact optimal strategies. Within a certain range of returns, quadratic functions provide good approximation to general concave utility functions (Markowitz, 1959, Chap. 13).

1.1.2 Efficient Portfolios

In mean-variance theory, every portfolio is associated with two indicators , i.e., expected return (mean) and variance. For a portfolio y_1, if there is a portfolio y_2 that has the same mean and variance as y_1, then the two portfolios are called equal. If there exists a portfolio y_3 that outperforms y_1 by having higher mean and no higher variance, or lower variance and no lower mean than y_1, then y_1 is called inefficient (Markowitz et al., 2000). All the feasible portfolios which cannot be outperformed by others are called efficient. Figure 1.1 shows a curve that contains all the efficient portfolios. This curve is called the efficient frontier. All the portfolios under the curve are considered inefficient.

1.1.3 Mean-Variance Formulation

In a classic one-period mean-variance problem, an investor observes the financial market and makes prediction on the performance of the assets, and then selects a number of assets from the market and allocates a certain proportion of his wealth to each of them to construct a portfolio. Suppose the investor with an original total wealth of w_0 wants to allocate his wealth to the N assets

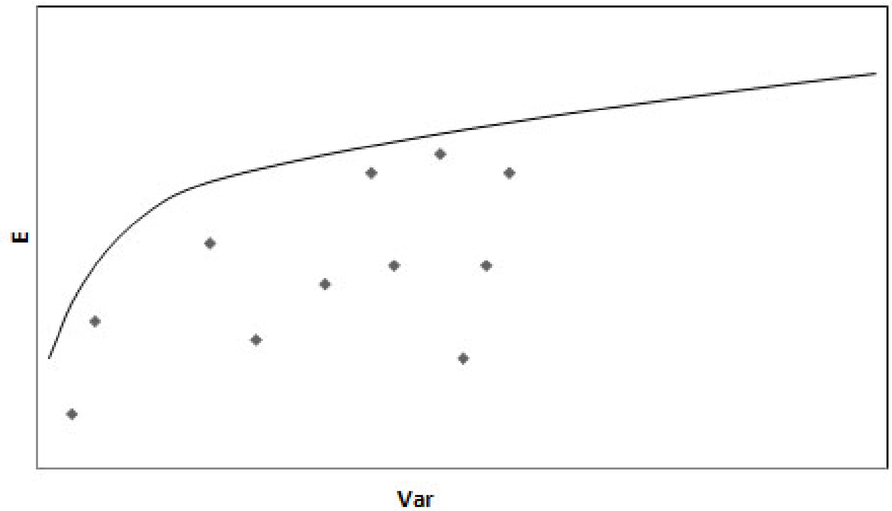


Figure 1.1: Efficient frontier

in the market. As discussed in Section 1.1.1, the portfolio is evaluated by its reward and risk, represented by $\bar{\boldsymbol{R}}\boldsymbol{x}$ and $\boldsymbol{x}'\boldsymbol{C}\boldsymbol{x}$ respectively. The investor can either choose to maximize his reward within a certain risk level σ or minimize the risk of the portfolio given a desired reward μ . The two formulations are shown as follows:

$$\begin{aligned} \max\ & \bar{\boldsymbol{R}}\boldsymbol{x} \\ s.t.\ & \boldsymbol{x}'\boldsymbol{C}\boldsymbol{x} \leq \sigma \\ & \mathbf{1}'\boldsymbol{x} = w_0 \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &\min \boldsymbol{x}'\boldsymbol{C}\boldsymbol{x} \\ &s.t.\ \bar{\boldsymbol{R}}\boldsymbol{x} \geq \mu \\ &\quad \boldsymbol{1}'\boldsymbol{x} = w_0 \\ &\quad \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{1.2}$$

where $\boldsymbol{1}$ is a vector of all ones, and $\boldsymbol{0}$ is a vector of all zeros. There is another commonly used formulation equivalent to (1.1) and (1.2). It maximizes the expected value of a concave quadratic utility. The formulation is given below.

$$\begin{aligned} &\max \bar{\boldsymbol{R}}\boldsymbol{x} - \lambda \boldsymbol{x}'\boldsymbol{C}\boldsymbol{x} \\ &s.t.\ \boldsymbol{1}'\boldsymbol{x} = w_0 \\ &\quad \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{1.3}$$

where $\lambda > 0$ is a parameter reflecting the investor's risk tolerance. A higher value of λ indicates a stronger aversion towards investment risk. By changing the value of λ, different risk attitudes can be addressed by the mean-variance model. The Markowitz efficient frontier can be generated by solving (1.3) parametrically in terms of λ. A risk-averse investor will be expecting his portfolio on the left side of the efficient frontier as shown in Figure 1.1, while a risk-seeker will select a portfolio on the right side of the efficient frontier. Throughout this thesis our analysis is based on a similar mean-variance framework but with transaction costs.

1.2 Transaction Costs

In the portfolio studies, perfect liquidity of the market is often a basic assumption. Such a market is characterized by the absence of transaction costs. Ignorance of transaction costs is unrealistic and may result in overly active trading strategies. Unnecessary transactions will reduce the profit of the investment. Therefore, one of the main goals of our research is to find investment strategies with better performance in the situation when transaction costs are incorporated.

In asset investment, transaction costs are those fees triggered by trading activities in asset investment including brokerage fees, big-ask spreads and other forms of costs. In real practice, transaction costs are often proportional to the trade amount, such as fixed-rate commission or bid-ask spreads. Other arrangements also exist such as lump-sum charge or transaction fee brackets. In our study, transaction fees are assumed to be charged at constant rates, i.e. linear, at all times.

1.3 Thesis Contribution

This thesis studies discrete-time multiperiod mean-variance portfolio optimization models incorporating proportional transaction costs. We start the analysis with a simple single-period problem considering two assets. Closed-form results are obtained. The results allow us to gain important management insights on the optimal investment strategy and shed light on the solution to multiperiod investment problems. Then the analysis is extended to solve two-period problems assuming that the rate of return follows a uniform distribution and a discrete distribution respectively. By investigating the two-period models, we grasp the key complication involved in the multiperiod case. To improve the performance for investment over more time periods, we develop an approximation method to solve the problem. Numerical experiments show that the proposed approximation method provides close-to-optimal solutions under certain assumptions. The main contributions of this thesis include:

1. We solve a single-period problem analytically considering one risk-free asset and one risky asset under proportional transaction-cost assumption. In addition to the results obtained in the existing studies, we provide theoretical insights on the optimal investment strategies for investors with different risk attitudes by conducting sensitivity analysis on the parameter λ.

2. We solve a two-period problem considering two assets with no borrowing and short selling under the assumption that the return rate of the risky asset follows a uniform distribution. Such problem has never been solved before in the existing literature. The problem becomes very complicated due to the fact that the value function of the first period becomes cubic. The solution obtained is interpreted as the optimal investment decisions. An analysis of how transaction cost rate would affect the investment strategy, the non-transaction region and the mean-variance efficient frontier is also performed. The closed-form expressions of the thresholds where the optimal investment action is changed are obtained as well.

3. We develop an efficient approximation method to provide close-to-optimal solution to the multiperiod portfolio optimization problem with proportional transaction costs. Numerical experiments show that the approximate method provides almost identical efficient frontiers to the true efficient frontiers in various situations under certain distribution assumptions of the random return rate.

   Compared to the existing approximation methods, our model makes more realistic assumptions and imposes less constraints for application. Additionally, our methods can be applied to solve investment problems with more periods.

   Compared to solving the model exactly in the static manner, the approximation method requires significantly less computational effort. Besides, the solution to multiperiod mean-variance problem is often unattainable for most random distributions of return rates. In these cases, our approximation method can still be used to recommend investment strategy with satisfactory performance.

The theoretical contribution of this thesis is that it provides an approach to tackle a typical class of problem in dynamic programming whose true value functions are continuous. The solution to such problem is not attainable using the standard backward dynamic programming method. Therefore, we try to find a good approximate value function to replace the true value function as in the approach of Approximate Dynamic Programming. Interested readers can refer to Powell (2007) for more details on Approximate Dynamic Programming.

1.4 Thesis Overview

This thesis is organized as follows.

The next chapter gives an extensive review of the research work on both single-period and long-term portfolio optimization, ignoring and considering transaction cost influence.

In Chapter 3, we describe the mean-variance model of portfolio optimization in the presence of transactions costs and embed the mean-variance model in an equivalent quadratic model.

In Chapter 4, we first provide the solution to a single-period problem with transaction costs. Then, the special case where there is no transaction cost is also discussed.

In Chapter 5, we consider a two-period investment problem and formulate the problem as a stochastic dynamic program. Analytical solution for the two-period problem is obtained. Numerical experiments were conducted to illustrate the method and mean-variance efficient frontiers were plotted under different transaction fee schedules.

Chapter 6 describes an approximation method to get near-optimal solutions. The performance of the approximation is discussed over extensive numerical experiments.

Lastly, we draw conclusions and suggest future research directions in Chapter 7.

Chapter 2

Literature Review

This chapter reviews literature in portfolio optimization, especially, focusing on studies under the mean-variance framework. Firstly, different choices of risk measure are discussed. Then, portfolio optimization without and with transaction costs are reviewed. For each assumption, we classify the research work into single-period and long-term studies. Research gaps are identified at the end of this chapter.

2.1 Measures of Risk

The mean-variance theory uses mean as the indicator of the portfolio return and variance as the portfolio risk. There exist other common measures of risk in addition to variance. One class of measures favored by the academic world is downside risk measures. Compared to variance, measures that minimize the downside risk seem more plausible since in most cases only capital loss is undesirable for investors. However, usually the optimal solutions to the models with downside risk control can only be found through numerical algorithms (Nawrocki, 1999; Estrada, 2007a). Common quantitative indicators of downside risk include downside volatility (Roy, 1952), semivariance (Estrada, 2007b; Huang, 2008b), Value-at-Risk (Huang, 2008a), probability of exceeding certain loss level (Campbell et al., 2001), and worst case (Gülpınar and Rustem, 2007). Among these measures, VaR and CVaR are widely used in the finance field. Semivariance is also common in research studies due to its long history and developed numerical solving techniques (Nawrocki, 1999). Another popular definition of risk is the downside volatility, firstly proposed by Roy (1952) for a single-period problem. In his work, risk is defined to be the probability of the occurrence of an investment "disaster". Later, this approach has been extended to the long-term portfolio optimizations (Li, 1998; Chiu and Li, 2009; Karatzas et al., 1987; Chiu et al., 2012). Markowitz (1959) analyzes the pros and cons of different measures of risk.

In this thesis, we follow the mean-variance framework and use variance as the measure of risk. The reason is threefold. Firstly, the mean-variance theory is well-established. Closed-form solutions have been found. Therefore, we can take advantage of the existing theories and methods to solve the problem. Secondly, in the case when the underlying distribution of returns is symmetric, it is equivalent to use either variance or downside risk measures. Lastly, it has been shown that mean-variance also provides a good approximation for various utility functions and empirical distributions of returns (Markowitz et al., 2000). Therefore, it can be applied to a wider range of problems.

2.2 Portfolio Optimization without Transaction Costs

The modern portfolio theory is built on the groundwork of Markowitz's mean-variance analysis. Since then, the mean-variance theory has gained great attention from both the academic world and the business world. Numerous papers have been published in this field.

The classical Markowitz mean-variance model analyzes a single-period problem. Later on, Merton (1971) extends Markowitz's mean-variance analysis to solve continuous-time investment problems. However, there is greater difficulty in applying the mean-variance method to the discrete-time multiperiod scenario. Not until the recent paper by Li and Ng (2000) has the discrete-time multiperiod mean-variance portfolio optimization gained much success. The mean-variance theory also inspires Treynor (1961a,b), Sharpe (1970), Lintner (1965) and Mossin (1966) to develop the capital asset pricing model (CAPM) that incorporates risk into the asset pricing (Frencha, 2003).

2.2.1 Single-Period Portfolio Optimization without Transaction Costs

The paper of the Nobel Prize winner Harry Markowitz (1952) has built the foundation of modern portfolio theory. His work in 1950s has initiated intensive studies in academia. The classic model has been extended in various directions. Tobin (1958) uses the mean-variance approach to study the investment with liquidation. That is, the portfolio to be managed contains a risk-free cash account. Investors can lend the risk-free asset at a positive interest rate r^0, or borrow cash at the same rate. Black (1972) forbids borrowing and lending but allows unlimited short selling in risky assets, i.e. the non-negativity constraints were removed from the classic model and thus the only constraint is $\mathbf{1'}\boldsymbol{x} = 1$. Lin and Liu (2008) consider minimum transaction lots for the one-period portfolio selection problem and use genetic algorithm to solve the model. Feasible solutions can be found efficiently, and near optimality can be achieved in some cases. Bonami and Lejeune (2009) extend the classical model by adding a probabilistic constraint over the asset returns. Markowitz et al. (2000) examines the case when upper bounds are imposed on the holdings of each asset.

2.2.2 Long-Term Portfolio Optimization without Transaction Costs

The mean-variance framework has been widely used to solve single-period problems. However, this method has gained relatively less success when applied to long-term investment analysis. This is partly due to the aggregated uncertainty when long-term investment is considered (Li and Ng, 2000). Difficulties of extending the single-period mean-variance model to a multiperiod scenario have been reported by Chen et al. (1971), Li and Ng (2000) and Steinbach (2001).

Discrete-Time Multiperiod Portfolio Optimization without Transaction Costs

Long-term problems include two categories, i.e., discrete-time models and continuous-time models. In discrete-time multiperiod models, portfolio rebalancing can be conducted at the beginning of each period. Hakansson (1971) models the discrete-time multiperiod investment problem with the objective of maximizing the average return rather than optimizing the trade-off between mean and variance of the portfolio. Gülpınar and Rustem (2007) measure risk by the downside volatility instead of variance. Li et al. (1998) and Li and Ng (2000) embed the multiperiod mean-variance model into an auxiliary quadratic model. The solution to the auxiliary model was then used to derive the solution to the original mean-variance model. Zhu et al. (2004) later extend the study by imposing the bankruptcy control, i.e., the total wealth of the investor should be non-negative for all time periods. Çakmak and Özekici (2006) adopt the same embedding method as Li and Ng's to solve an investment problem with the assumption that the market moves according to a Markov chain. Çelikyurt and Özekici (2007) further generalize the study of Çakmak and Özekici (2006) by considering the problems under various assumptions including safety-first approach, coefficient of variation and quadratic utility functions. Wei and Ye (2007) then follow their work and modify the model by adding the bankruptcy constraint. However, in all these studies, no transaction costs are considered in the formulation. The drawbacks of neglecting transaction costs will be addressed in Section 2.3.1.

Continuous-Time Portfolio Optimization without Transaction Costs

The continuous-time portfolio optimization problems assume that the portfolio can be rebalanced at any time. Instead of using mean-variance objectives, most continuous-time models maximize the expected constant relative risk aversion (CRRA) form of utility of consumption over a time horizon or the investor's wealth at the end of the investment horizon. Samuelson (1969) discusses lifetime portfolio optimization under deterministic asset return rates. Merton (1971), known as the pioneering work in continuous-time investment planning, develops the optimal investment policies under the assumptions that the utility functions belong to the HARA (hyperbolic absolute risk-aversion) family and the asset price process follow a geometric Brownian motion. Cox and Huang (1989) later improve the model by imposing non-negativity constraints on the investor's consumption amounts and the final wealth. The optimal consumption and investment strategies are derived by a technique using martingales. These studies adopt an objective function maximizing the cumulative discounted utility of consumption or final wealth. Since then, there have been enormous amounts of literature studying continuous-time approaches for portfolio selection problems.

The continuous-time problem with mean-variance objectives has received less attention and success as compared to Merton's model. Closed-form solution is only achievable with various constraints. Efficient solving methods have been developed to tackle the problem and various properties of the solutions have been identified through insightful analyses. Zhou and Li (2000) adopt the method used by Li and Ng (2000) for solving discrete-time problems and embed the continuous-time mean-variance model into auxiliary linear-quadratic models. Though the analytical solution is still unattainable, the closed-form of the efficient frontier is derived. Later, their model under the linear-quadratic framework is extended for various assumptions. Li et al. (2002) study the model that forbids short selling of assets. Lim and Zhou (2002) assume random coefficients including interest rates, appreciation rates, and volatility coefficients. Bielecki et al. (2005) use the same framework with an additional constraint to prohibit bankruptcy. However, none of these studies have found the closed-form of the optimal solutions.

As discussed by Merton (1971), the main advantage of the continuous-time approach is that it makes use of the established research in stochastic processes and reduces the number of parameters in the model. Nevertheless, discrete-time models are still important in practice. They are easier to understand and implement. Besides, it may not be optimal to have infinitesimal trading and we can shorten the period of discrete-time models to approximate the performance of the continuous-time models.

2.3 Portfolio Optimization with Transaction Costs

Transaction costs are fees triggered by trading activities. Common transaction cost schemes include linear fees that are proportional to the trading amount, fixed lump-sum charges for each transaction and mixture of fixed and proportional fees. The bid-ask spread can also be viewed as a type of transaction costs incurred by the trader (Demsetz, 1968). Arnott and Wagner (1990) treat transaction costs as the commission charged for the transaction together with the impact of the executed trade over the market. This section reviews portfolio optimization studies considering transaction costs.

2.3.1 Impact of Transaction Costs

Most existing studies on portfolio optimization ignore the impact of transaction costs on investment decisions. No consideration of transaction costs may result in investment policies characterized by extremely heavy trading. It is reported that investors' behaviors can be significantly affected when taking into account the transaction costs. Constantinides (1979) shows that transaction costs lead to less frequent trading. Pelsser and Vorst (1996) prove that transaction fees as modest as 0.5% of the trading amount can make the optimal strategy of considering no transaction costs inferior to simple stop-loss and lock-in strategies. Yoshimoto (1996) shows by numerical analysis that discarding the impact of transaction costs leads to suboptimal solutions. Atkinson and Mokkhavesa (2003) consider overseas investments and concluded that markets with low transaction costs attract more equity investments. Fleten and Lindset (2008) study the strategies for insurance companies to hedge multiperiod guarantee in the framework of stochastic programming. It has been shown that the case with proportional hedging costs results in less active rebalancing compared to the case ignoring trading fees. Feng et al. (2011) examine investment with a strategy adhering to constant fraction of wealth allocated to each asset in the presence of transaction costs and concluded that transaction costs lead to less frequent rebalancing. Therefore, it is a known fact that ignoring transaction costs affects the performance of a portfolio. However, incorporating such transaction costs brings challenges to obtaining closed-form solutions. In this section, we review portfolio selection models with transaction costs and their corresponding solving methods for both single-period and multiperiod settings.

2.3.2 Single-Period Portfolio Optimization with Transaction Costs

Single-period portfolio optimization with transaction costs is first studied by Constantinides (1979). In this paper, a non-transaction region has been defined qualitatively. Perold (1984) later incorporates concave piecewise linear transaction costs into a one-period mean-variance portfolio revision model. This methodology is later adopted by Konno and Wijayanayake (2001) to analyze a mean-absolute-deviation model. Yoshimoto (1996) obtains numerical solutions to a mean-variance portfolio revision model using a nonlinear programming algorithm. Li et al. (2000) use linear approximation to efficiently solve the quadratic model assuming constant transaction fee rates. Best and Hlouskova (2005) develop an efficient algorithm that solves a large-scale single-period investment problem with proportional transaction costs. Lobo et al. (2007) obtain approximate results for a one-period rebalancing problem with concave transaction costs. Dan (2008) solves the single period mean-variance problem with proportional transaction costs. The solution obtained agrees with the solution derived in Chapter 4 of this thesis. Kozhan and Schmid

(2009) study an investment problem with transaction costs where the investment risk is determined based on subjective beliefs instead of using any probability distribution. Jana et al. (2009) incorporate proportional transaction costs into a mean-variance model with an additional diversification criterion. Zhang et al. (2010) consider transaction costs in a one-period rebalancing model treating the random return rates as fuzzy numbers. Although single-period models involving multiple assets or various transaction costs have been developed, their results cannot be easily extended to multiperiod scenarios as the complexity of the problem grows rapidly with increasing number of investment periods.

2.3.3 Long-Term Portfolio Optimization with Transaction Costs

Recently, more attention has been placed on long-term portfolio optimization in the presence of transaction costs. Transaction costs usually have larger impact on the performance of a multiperiod investment, because, in such an investment, the investor may adjust the portfolio one or more times during the investment horizon. In this section, we review studies on both discrete-time models and the continuous-time models.

Discrete-Time Multiperiod Portfolio Optimization with Transaction Costs

There is a limited number of existing studies focusing on the discrete-time mean-variance portfolio optimization with transaction costs. The main challenge to solve such problems lies in the fact that for each period, there exists three action spaces, namely, buy, sell and hold. Therefore, a general expression of the value function cannot be obtained easily. This creates difficulty for solving the problem recursively. Mulvey and Vladimirou (1989) use the stochastic network framework to consider financial planning problems. Proportional transaction costs are incorporated by assigning multipliers to scale the trading amounts. Dantzig and Infanger (1993) study a multiperiod linear model where the nonlinear utility function is approximated by a piecewise linear function. Gennotte and Jung (1994) adapt the model of Dumas and Luciano (1991) for continuous-time analysis to solve a discrete-time multiperiod problem. It maximizes a power utility function but no closed-form solution is obtained. Following the work of Gennotte and Jung, Schroder (1995) further considers fixed costs and identifies some properties of the optimal solution through numerical studies. Boyle and Lin (1997) adopt a similar method but it maximizes the indirect utility function, which is the maximum value of the power utility function used by Gennotte and Jung. In this way, the dimension of the problem is reduced, and thus, analytical solutions are provided. Chryssikou (1998) models the multiperiod portfolio optimization problems with transaction costs under the mean-variance framework and obtains near-optimal solutions. However, the transaction costs are assumed to be quadratic functions of the trading amounts, which means that the unit transaction costs increase as the trading amounts go up. Such cost schemes are rarely seen in practice. Bertsimas and Pachamanova (2008) solve the corresponding problem by maximizing the worst-case return to get a robust solution instead of providing optimal solutions according to the investor's risk tolerance. This robust design offers a safe investment option but it is conservative and restricts the risk-taking investors from seeking higher returns.

Continuous-Time Portfolio Optimization with Transaction Costs

In terms of continuous-time problems, most existing models focus on maximizing the discounted utility of consumption or bequest instead of studying the trade-off between reward and risk as in the mean-variance models. Magill and Constantinides (1976) are the first to incorporate linear transaction costs into Merton's continuous-time portfolio optimization model. They gain qualitative insights on the change of trading behaviors when transaction is no longer costless. Davis and Norman (1990) obtain closed-form solutions to a problem that maximizes the cumulative utility of consumption on an infinite horizon under proportional transaction costs. Dumas and Luciano (1991) solve a similar model with an objective to increase the final utility. Shreve and Soner (1994) generalize the model of Davis and Norman by removing some of the restrictions and identify the properties of the value function and the non-transaction region. Liu (2004) further incorporates fixed transaction costs into the continuous-time model, but no closed-form transaction thresholds are identified. Muthuraman and Kumar (2006) extend the model of Liu (2004) by imposing restrictions on borrowing. Numerical results to a continuous-time mean-variance problem have been obtained by Dai et al. (2010) assuming two assets and proportional transaction costs. In their work, the return rate of the risky asset is assumed to follow a standard one-dimensional Brownian motion, i.e., the price of the risky asset has a constant volatility. On the contrary, our model puts a much milder restriction on the return rates. The method introduced in this thesis can be applied to any symmetric distributions, which is a basic assumption for mean-variance analysis.

2.4 Research Gaps

Since the first introduction of the mean-variance theory by Markowitz in 1950s, intensive research has been done in this field in the past decades. The mean-variance studies can be categorized into single-period, discrete-time multi-period and continuous-time models. For the single-period and continuous-time scenarios, the studies are well established. Closed-form solutions have been found for the single-period models. The continuous-time studies pioneered by Merton have also gained much success in obtaining solutions to problems maximizing the utility of consumption or bequest. On the contrary, the discrete-time mean-variance problems have attracted less attention due to the great difficulties in decomposing the variance term. Nonetheless, discrete-time models are an important class of problems because of their common presence in real life.

Another challenge that this thesis tackles is to incorporate the transaction costs. Compared to the counter-parting portfolio optimization without transaction costs, investment with transaction costs is greatly under-studied. This is because that considering transaction costs often leads to intractable models. However, as many studies have shown, ignoring such costs may lower the investment performance to a considerable extent.

In order to address the research gaps that have been discussed in the previous paragraphs, this thesis studies a discrete-time multiperiod mean-variance portfolio optimization problem in the presence of linear transaction costs. To our best knowledge, no closed-form solution for this type of problem has been obtained.

Chapter 3

Problem Formulation with Transaction Costs

3.1 Problem Definition

We consider a multiperiod portfolio optimization problem using mean-variance framework. Suppose the investor wants to maximize his mean-variance objective function over an investment horizon which is divided into T time periods. The time variable t is set to be 0 at the beginning of the entire investment horizon, and t equals $1, 2, ..., T$ at the end of each period. At $t = 0$, a portfolio y will be constructed by selecting assets from a market containing two assets, and later be rebalanced at $t = 1, 2, ..., T - 1$. Asset 0 is a risk-free asset with a fixed return rate r^0 through all the time periods. This asset can be a cash deposit or a government bond. Such investment is considered as a safe investment with no risk, therefore the variance of the return of this asset equals to zero. Asset 1 is a risky asset that yields a random return in every time period. Examples of such asset include stocks, mutual funds and other variable-return securities. The return rate of the risky asset is denoted by a random variable $\mathring{R}_t^1$ from time t to time $t+1$. The return rate of the risky asset in each period is assumed to be independent from each other. Further let h_t^i denote the holding of asset i at time t, and $w_t = h_t^0 + h_t^1$ represent the investor's wealth at time t. Observing h_t^i, the investor chooses transaction amount to rebalance the portfolio. The amount of risky asset bought and sold at time t are denoted by x_t^{1+} and x_t^{1-}. All the money which is not invested in the risky asset goes to the risk-free account. Once a transaction occurs on the risky asset, the investor has to pay a proportional transaction fee which is deducted from the risk-free account at a rate of α for buying and β for selling. Thus the buying and selling costs are αx_t^{1+} and βx_t^{1-} respectively.

3.2 Conditions on x_t^{1+} and x_t^{1-} :

Both x_t^{1+} and x_t^{1-} should be non-negative. It then follows that at least one of the two decision variables at each period should be equal to zero if it is an optimal action, for otherwise we can always find a better action which can reduce the transaction cost and thus increase the reward. The proof to this theorem will be given in Section 3.6.

Furthermore, if borrowing and short selling behaviors are prohibited, then

$$x_t^{1+} \leq \frac{h_t^0}{1+\alpha} \tag{3.1a}$$

$$x_t^{1-} \leq h_t^i. \tag{3.1b}$$

This set of constraints indicates that all the capital is financed from internal sources. At the same time, the investor can only sell the amount of asset up to the maximum amount on hand.

3.3 Recursive Rebalancing Equations

At the beginning of each period, the investor can change his portfolio by trading the risky asset. And the transaction costs incurred will be deducted from the risk-free asset. The recursive rebalancing equations are

$$h_{t+1}^0 = r^0\big(h_t^0 - (1+\alpha)x_t^{1+} + (1-\beta)x_t^{1-}\big) \tag{3.2}$$

for the risk-free asset, and

$$h_{t+1}^1 = \mathring{R}_t^1 \left(h_t^1 + x_t^{1+} - x_t^{1-}\right) \tag{3.3}$$

for the risky assets respectively.

Here we introduce a new set of post-decision state variables $\hat{h}_t^0$ and $\hat{h}_t^1$ to represent the asset levels immediately after the transactions have been performed:

$$\widehat{h}_t^0 = h_t^0 - (1+\alpha)x_t^{1+} + (1-\beta)x_t^{1-} \tag{3.4}$$

$$\widehat{h}_t^1 = h_t^1 + x_t^{1+} - x_t^{1-} \tag{3.5}$$

Combining (3.2), (3.3), (3.4) and (3.5) yields the rebalancing equations in terms of $\hat{h}_t^0$ and $\hat{h}_t^i$:

$$h_{t+1}^0 = r^0 \widehat{h}_t^0, \tag{3.6}$$

$$h_{t+1}^1 = \mathring{R}_t^1 \widehat{h}_t^1, \tag{3.7}$$

The recursive rebalancing equations reflect how the wealth is accumulated through each time period.

3.4 Multiperiod Mean-Variance Formulation

Having defined the recursive dynamics in the previous section, the single-period mean-variance formulation (1.3) can be extended to the multiperiod case by incorporating the rebalancing equations.

The multiperiod mean-variance portfolio optimization problem can be formulated as

$$\text{PB}[\mathring{\lambda}]: \qquad \max \mathrm{E}\left(w_T\right) - \lambda \operatorname{Var}\left(w_T\right)$$
$$s.t.\ w_T = h_T^0 + (1-\beta)\, h_T^1$$
$$h_{t+1}^0 = r^0\big(h_t^0 - (1+\alpha)x_t^{1+} + (1-\beta)x_t^{1-}\big)$$
$$h_{t+1}^1 = \mathring{R}_t^1\left(h_t^1 + x_t^{1+} - x_t^{1-}\right)$$
$$x_t^{1+}, x_t^{1-} \geq 0, \tag{3.8}$$

for $t = 0, 1, \ldots, T-1$. The final wealth w_T is the sum of the risk-free asset and the risky asset after liquidation. For notational simplicity, denote

$$R_t^1 = \mathring{R}_t^1 \qquad \text{for} \quad t = 0, 1, \ldots T-2,$$
$$R_{T-1}^1 = \mathring{R}_{T-1}^1\,(1-\beta)\,. \tag{3.9}$$

Note that if liquidation is not required at the end of the investment horizon, then $R_{T-1}^1 = \mathring{R}_{T-1}^1$. Now PB$[\mathring{\lambda}]$ can be rewritten as

$$\text{PB}[\lambda]: \qquad \max \mathrm{E}\left(w_T\right) - \lambda \operatorname{Var}\left(w_T\right)$$
$$s.t.\ w_T = h_T^0 + h_T^1$$
$$h_{t+1}^0 = r^0\big(h_t^0 - (1+\alpha)x_t^{i+} + (1-\beta)x_t^{1-}\big)$$
$$h_{t+1}^1 = R_t^1\left(h_t^1 + x_t^{1+} - x_t^{1-}\right)$$
$$x_t^{1+}, x_t^{1-} \geq 0. \tag{3.10}$$

Since the variance term cannot be decomposed with respect to time stages,

$PB[\lambda]$ cannot be solved directly using dynamic programming. In the following sections, we obtain the optimal solutions by solving an auxiliary model.

3.5 Auxiliary Model

In the multiperiod mean-variance model, the objective at each time period is to maximize

$$\mathrm{E}\left(w_T\right) - \lambda \, \mathrm{Var}\left(w_T\right) \tag{3.11}$$

Since the variance term is non-decomposable with respect to time stages, the multiperiod mean-variance model cannot be solved directly using dynamic programming. Under certain assumptions, the mean-variance model can be embedded in a quadratic model to be solved by dynamic programming (Li and Ng, 2000). The objective function of the quadratic model takes the form

$$\delta \, \mathrm{E}\left(w_T\right) - \lambda \, \mathrm{E}\left(w_T\right)^2 . \tag{3.12}$$

As proven in Li and Ng (2000), under same constraints, the optimal solution to a model optimizing (3.11) is same as the optimal solution to a model optimizing (3.12), given that

$$\delta = 1 + 2\lambda \, \mathrm{E}(w_T)|_{\pi^*} \tag{3.13}$$

where π^* is the optimal investment policy. Conclusively, the auxiliary quadratic model PB[δ, λ] is defined as follows:

$$
\begin{aligned}
\text{PB}[\delta, \lambda]: \qquad & \max \delta \, \mathrm{E}(w_T) - \lambda \, \mathrm{E}(w_T)^2 \\
& s.t. \; w_T = h_T^0 + h_T^1 \\
& \quad h_{t+1}^0 = r^0 \left(h_t^0 - (1+\alpha) x_t^{1+} + (1-\beta) x_t^{1-}\right) \\
& \quad h_{t+1}^1 = R_t^1 \left(h_t^1 + x_t^{1+} - x_t^{1-}\right) \\
& \quad x_t^{1+} \leq \frac{h_t^0}{1+\alpha} \\
& \quad x_t^{1-} \leq h_t^1 \\
& \quad x_t^{1+}, x_t^{1-} \geq 0,
\end{aligned}
\tag{3.14}
$$

for $t = 0, 1, \ldots, T-1$. For notational convenience, we use $\mathrm{E}(\cdot)^2$ to represent $\mathrm{E}[(\cdot)^2]$ and $\mathrm{E}^2(\cdot)$ to represent $[\,\mathrm{E}(\cdot)]^2$ throughout this thesis.

3.6 Property of the Solution

This section provides an important property of the optimal solution to make it simpler to solve PB[δ, λ]. The property of the optimal solutions is given in Proposition 3.1.

Proposition 3.1. *Under the optimal investment policy, we have the following condition:*

$$
x_t^{1+} \cdot x_t^{1-} = 0 \qquad \textit{for } t = 0, 1, ..., T-1. \tag{3.15}
$$

Proof. Let U be the objective function in PB[λ], and $\boldsymbol{x_t} = (x_t^{1+}, x_t^{1-})$, for $t =$

$0, 1, ...T-1$. Assume that $\dot{\boldsymbol{\pi}} = (\dot{\boldsymbol{x}}_{\boldsymbol{0}}, ..., \dot{\boldsymbol{x}}_{\boldsymbol{s}}, ..., \dot{\boldsymbol{x}}_{\boldsymbol{T-1}})$ is a feasible investment policy for PB[λ], where $\dot{\boldsymbol{x}}_{\boldsymbol{s}}$ contains two strictly positive elements $\dot{x}_s^{1+}$ and $\dot{x}_s^{1-}$, $\forall i$. Suppose there is another feasible policy $\ddot{\boldsymbol{\pi}}$ which is identical to $\dot{\boldsymbol{\pi}}$ in all the periods except period s where

$$\begin{cases} \ddot{x}_s^{1+} = \dot{x}_s^{1+} - \dot{x}_s^{1-} \\ \ddot{x}_s^{1-} = 0 & \text{if } \dot{x}_s^{1+} \geq \dot{x}_s^{1-}, \end{cases}$$

or

$$\begin{cases} \ddot{x}_s^{1+} = 0 \\ \ddot{x}_s^{1-} = \dot{x}_s^{1-} - \dot{x}_s^{1+} & \text{if } \dot{x}_s^{1+} < \dot{x}_s^{1-}. \end{cases}$$

The two policies make the same adjustment on the risky asset while $\dot{\boldsymbol{\pi}}$ induces more transaction costs charged from the risk-free account. If $\dot{x}_s^{1+} \geq \dot{x}_s^{1-}$, then

$$U|_{\dot{\pi}} = U|_{\ddot{\pi}} - (r^0)^{T-s}(\alpha + \beta)\dot{x}_s^{1-} < U|_{\ddot{\pi}}.$$

This means that a policy can always be improved for PB[λ] if there are both purchasing and selling actions at the same time. In other words, a necessary condition for an optimal solution is

$$x_t^{1+} \cdot x_t^{1-} = 0 \qquad \text{for } t = 0, 1, ..., T-1.$$

Similarly, we have the same conclusion for the case of $\dot{x}_s^{1+} \leq \dot{x}_s^{1-}$. As PB[$\lambda$] and PB[$\delta, \lambda$] have the same optimal solution, this property also applies to the latter. □

Chapter 4

The Single-Period Problem

In order to gain some first insights, we start with solving a single-period investment problem considering two assets, asset 0 as the risk-free asset and asset 1 as the risky asset. The single-period problem can be viewed as a special case of the multiperiod problem.

Define

$$\xi_0^\alpha = R_0^1 - r^0(1+\alpha) \tag{4.1}$$

$$\xi_0^\beta = R_0^1 - r^0(1-\beta) \tag{4.2}$$

ξ_0^α and ξ_0^β can be interpreted as the excess return. We assume $\mathrm{E}(R_0^1) \geq r^0(1+\alpha)$ to ensure that the investor expects a higher return by taking up extra risk. It can be easily verified that $\mathrm{E}(\xi_0^\alpha) < \mathrm{E}(\xi_0^\beta)$. The objective function in

PB$[\delta, \lambda]$ can be rewritten in terms of x_0^{1+} and x_0^{1-}:

$$\begin{aligned} f(x_0^{1+}, x_0^{1-}) =& \delta \mathrm{E}(h_1^0 + h_1^1) - \lambda \mathrm{E}(h_1^0 + h_1^1)^2. \\ =& \delta \, \mathrm{E} \left\{ r^0 h_0^0 + R_0^1 h_0^1 + \xi_0^\alpha x_0^{1+} - \xi_0^\beta x_0^{1-} \right\} \\ & - \lambda \, \mathrm{E} \left\{ r^0 h_0^0 + R_0^1 h_0^1 + \xi_0^\alpha x_0^{1+} - \xi_0^\beta x_0^{1-} \right\}^2 \end{aligned} \tag{4.3}$$

The one-period model with transaction costs can be written as

$$\begin{aligned} \max \delta \, \mathrm{E} & \left\{ r^0 h_0^0 + R_0^1 h_0^1 + \xi_0^\alpha x_0^{1+} - \xi_0^\beta x_0^{1-} \right\} \\ & - \lambda \, \mathrm{E} \left\{ r^0 h_0^0 + R_0^1 h_0^1 + \xi_0^\alpha x_0^{1+} - \xi_0^\beta x_0^{1-} \right\}^2 \\ s.t. \, & x_0^{1+} \le \frac{h_0^0}{1+\alpha} \\ & x_0^{1-} \le h_0^1 \\ & x_0^{1+}, x_0^{1-} \ge 0. \end{aligned} \tag{4.4}$$

4.1 Analytical Solution to the Single-Period Problem

Since it has been proved in Proposition 3.1 that at least one of the decision variables equals zero, it is easy to tell that the objective function in (4.4) is a concave objective function of both x_0^{1+} and x_0^{1-}. Therefore, we can find the optimal solution by simply applying KKT conditions. The KKT conditions for the quadratic program (QP) (4.4) are

$$x_0^{1+} \le \frac{h_0^0}{1+\alpha} \tag{4.5}$$

$$x_0^{1-} \le h_0^1 \tag{4.6}$$

$$x_0^{1+} \ge 0 \tag{4.7}$$

$$x_0^{1-} \ge 0 \tag{4.8}$$

$$\begin{aligned} y_0^0 \ge{} & \delta\,\mathrm{E}(\xi_0^\alpha) - 2\lambda \left[r^0\,\mathrm{E}(\xi_\alpha)\left(h_0^0 - (1+\alpha)x_0^{1+} + (1-\beta)x_0^{1-}\right)\right. \\ & \left. + \mathrm{E}(R_0^1\xi_0^\alpha)\left(h_0^1 + x_0^{1+} - x_0^{1-}\right)\right] \end{aligned} \tag{4.9}$$

$$\begin{aligned} y_0^1 \ge{} & -\delta\,\mathrm{E}(\xi_0^\beta) + 2\lambda \left[r^0\,\mathrm{E}(\xi_0^\beta)\left(h_0^0 - (1+\alpha)x_0^{1+} + (1-\beta)x_0^{1-}\right)\right. \\ & \left. + \mathrm{E}(R_0^1\xi_0^\beta)\left(h_0^1 + x_0^{1+} - x_0^{1-}\right)\right] \end{aligned} \tag{4.10}$$

$$\begin{aligned} x_0^{1+} & \left\{ y_0^0 - \delta\,\mathrm{E}(\xi_0^\alpha) + 2\lambda \left[r^0\,\mathrm{E}(\xi_0^\alpha)\left(h_0^0 - (1+\alpha)x_0^{1+} + (1-\beta)x_0^{1-}\right)\right.\right. \\ & \left.\left. + \mathrm{E}(R_0^1\xi_0^\alpha)\left(h_0^1 + x_0^{1+} - x_0^{1-}\right)\right]\right\} = 0 \end{aligned} \tag{4.11}$$

$$\begin{aligned} x_0^{1-} & \left\{ y_0^1 + \delta\,\mathrm{E}(\xi_0^\beta) - 2\lambda \left[r^0\,\mathrm{E}(\xi_0^\beta)\left(h_0^0 - (1+\alpha)x_0^{1+} + (1-\beta)x_0^{1-}\right)\right.\right. \\ & \left.\left. + \mathrm{E}(R_0^1\xi_0^\beta)\left(h_0^1 + x_0^{1+} - x_0^{1-}\right)\right]\right\} = 0 \end{aligned} \tag{4.12}$$

$$y_0^0 \left(\frac{h_0^0}{1+\alpha} - x_0^{1+} \right) = 0 \tag{4.13}$$

$$y_0^1 \left(h_0^1 - x_0^{1-} \right) = 0 \tag{4.14}$$

where y_0^0 and y_0^1 are the Lagrange multipliers.

We further denote

$$d_0^\alpha = \frac{\mathrm{E}^2(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)^2} \tag{4.15}$$

$$d_0^\beta = \frac{\mathrm{E}^2(\xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)^2} \tag{4.16}$$

$$q_0^\alpha = \frac{\mathrm{E}\left(R_0^1\xi_0^\alpha\right)}{\mathrm{E}(\xi_0^\alpha)} \tag{4.17}$$

$$q_0^\beta = \frac{\mathrm{E}(R_0^1\xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)} \tag{4.18}$$

Since $\mathrm{E}(\xi_0^\alpha) < \mathrm{E}(\xi_0^\beta)$, it can be derived that $d_0^\alpha < d_0^\beta$ and $q_0^\alpha > q_0^\beta$.

Having proved that at least one of the two decision variables equals zero, we will discuss the three cases where (1) only $x_0^{1-} = 0$; (2) only $x_0^{1+} = 0$ and (3) both are zero. The optimal strategy is found after solving the KKT conditions. The results are summarized in Table 4.1. It shows that for different initial asset holding levels which are represented by the values of h_0^0 and h_0^1, five investment strategies are recommended respectively. A correspondent piecewise value function representing the optimal value of the objective is associated with each optimal investment strategy. Detailed derivation of the optimal solution and the value function can be found in Appendix A and Appendix B respectively.

As shown in Table 4.1, $g_0^{(1)}$, $g_0^{(2)}$, $g_0^{(3)}$ and $g_0^{(4)}$ are the four thresholds for change in investment strategy. They divide the action space into five regions.

Table 4.1: Solution to the one-period problem

	Range of $\frac{\delta}{2\lambda}$	Solution & value function $V_0(h_0^0, h_0^1)$
1	$\left[g_0^{(1)}, \infty\right)$	$x_0^{1+} = \frac{h_0^0}{1+\alpha}, \quad x_0^{1-} = 0$ $V_0^{(1)} = \delta\,\mathrm{E}(R_0^1)\left(\frac{h_0^0}{1+\alpha} + h_0^1\right) - \lambda\,\mathrm{E}(R_0^1)^2\left(\frac{h_0^0}{1+\alpha} + h_0^1\right)^2$
2	$\left[g_0^{(2)}, g_0^{(1)}\right)$	$x_0^{1+} = \frac{\mathrm{E}(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)^2}\left(\frac{\delta}{2\lambda} - g_0^{(2)}\right), \quad x_0^{1-} = 0$ $V_0^{(2)} = \delta\,\mathrm{E}(r^0h_0^0 + R_0^1h_0^1) - \lambda\,\mathrm{E}(r^0h_0^0 + R_0^1h_0^1)^2 + \lambda d_0^\alpha\left(\frac{\delta}{2\lambda} - g_0^{(2)}\right)^2$
3	$\left[g_0^{(3)}, g_0^{(2)}\right)$	$x_0^{1+} = 0, \quad x_0^{1-} = 0$ $V_0^{(3)} = \delta\,\mathrm{E}(r^0h_0^0 + R_0^1h_0^1) - \lambda\,\mathrm{E}(r^0h_0^0 + R_0^1h_0^1)^2$
4	$\left[g_0^{(4)}, g_0^{(3)}\right)$	$x_0^{1+} = 0, \quad x_0^{1-} = \frac{\mathrm{E}(\xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)^2}\left(g_0^{(2)} - \frac{\delta}{2\lambda}\right)$ $V_0^{(4)} = \delta\,\mathrm{E}(r^0h_0^0 + R_0^1h_0^1) - \lambda\,\mathrm{E}(r^0h_0^0 + R_0^1h_0^1)^2 + \lambda d_0^\beta\left(\frac{\delta}{2\lambda} - g_0^{(3)}\right)^2$
5	$\left[0, g_0^{(4)}\right)$	$x_0^{1+} = 0, \quad x_0^{1-} = h_0^1$ $V_0^{(5)} = \delta r^0\left(h_0^0 + (1-\beta)h_0^1\right) - \lambda(r^0)^2\left(h_0^0 + (1-\beta)h_0^1\right)^2$

The four thresholds are defined as follows:

$$g_0^{(1)} = \frac{q_0^\alpha}{1+\alpha}h_0^0 + q_0^\alpha h_0^1 \tag{4.19}$$

$$g_0^{(2)} = r^0h_0^0 + q_0^\alpha h_0^1 \tag{4.20}$$

$$g_0^{(3)} = r^0h_0^0 + q_0^\beta h_0^1 \tag{4.21}$$

$$g_0^{(4)} = r^0h_0^0 + r^0(1-\beta)h_0^1 \tag{4.22}$$

According to the interval that $\frac{\delta}{2\lambda}$ falls in, five different investment strategies are recommended respectively.

In the first two intervals, when $g_0^{(2)} \leq \frac{\delta}{2\lambda}$, the optimal strategy is to increase the holding level of the risky asset using the fund deposited in the risk-free account. The threshold $g_0^{(2)}$ is written as

$$g_0^{(2)} = r^0 h_0^0 + q_0^\alpha h_0^1 = \left(r^0 h^0 + \mathrm{E}(R_0^1) h_0^1\right) + \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)} h_0^1. \tag{4.23}$$

It can be interpreted in such a way: $(r^0 h^0 + \mathrm{E}(R_0^1) h_0^1)$ represents the expected return of the current portfolio before rebalancing, and the latter part $\frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)} h_0^1$ is the ratio of the variance of the random return rate over the excess return, also known as the dispersion index (Cox and Lewis, 1966), times the amount of risky asset on hand. This could be interpreted as the risk borne by the current portfolio. Therefore, $g_0^{(2)} \leq \frac{\delta}{2\lambda}$ means that the expected return of the current portfolio plus the risk of purchasing more risky asset is below the desired level. Thus the investment decision is to purchase more risky asset to increase the return and risk simultaneously.

In the second interval, where $\frac{\delta}{2\lambda} \in \left[g_0^{(2)}, g_0^{(1)}\right)$, it is recommended to buy more of the risky asset to rebalance the holdings to the target levels $\widehat{h}_{T-1}^{0\alpha}$ and $\widehat{h}_{T-1}^{1\alpha}$ with the following expressions:

$$\begin{aligned}
\widehat{h}_0^{0\alpha} &= h_0^0 - (1+\alpha) x_0^{1+} \\
&= -d_0^\alpha \left((1+\alpha)\frac{\delta}{2\lambda} - q_0^\alpha h_0^0 - q_0^\alpha (1+\alpha) h_0^1 \right) && (4.24) \\
\widehat{h}_0^{1\alpha} &= h_0^1 + x_0^{1+} \\
&= d_0^\alpha \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - r^0 (1+\alpha) h_0^1 \right) && (4.25)
\end{aligned}$$

In the first interval where $\frac{\delta}{2\lambda} \geq g_0^{(1)}$, the buying amount has hit the upper limit $\frac{h_0^0}{1+\alpha}$, the maximum amount of fund that can be financed internally. Therefore, the optimal solution is to buy as much as possible, i.e., to use up all the fund deposited in the risk-free asset to buy the risky asset.

Symmetrically, for the last two intervals, the optimal solution is to sell the risky asset since the current portfolio exposes the investor to too much risk. It is indicated by $g_0^{(3)} > \frac{\delta}{2\lambda}$, where

$$g_0^{(3)} = r^0 h_0^0 + q_0^\beta h_0^1 = \left(r^0 h^0 + \mathrm{E}(R_0^1) h_0^1\right) + \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\beta)} h_0^1. \tag{4.26}$$

Similar to $g_0^{(2)}$, it can be interpreted as the expected return of the current portfolio plus the risk of holding the current risky asset level. When $\frac{\delta}{2\lambda}$ is less than $g_0^{(3)}$, the investment strategy is to shift the fund in the risky asset to the risk-free account. The post-decision holding levels are

$$\begin{aligned} \widehat{h}_0^{0\beta} &= h_0^0 + (1-\beta) x_0^{1-} \\ &= -d_0^\beta \left((1-\beta)\frac{\delta}{2\lambda} - q_0^\beta h_0^0 - q_0^\beta (1-\beta) h_0^1 \right), \end{aligned} \tag{4.27}$$

$$\begin{aligned} \widehat{h}_0^{1\beta} &= h_0^1 - x_0^{1-} \\ &= d_0^\beta \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - r^0 (1-\beta) h_0^1 \right). \end{aligned} \tag{4.28}$$

Lastly, between the buying and selling intervals, there exists a non-transaction region (when $g_0^{(3)} \leq \frac{\delta}{2\lambda} < g_0^{(2)}$). The investor is recommended to take no action since the increase of the investor's expected utility cannot compensate the transaction costs incurred from rebalancing the portfolio.

The optimal value function that we obtained is found to be smooth on the entire domain. (see Appendix C)

4.1.1 Numerical Example

This section presents a numerical example to illustrate the results obtained for the single-period problem. Suppose an investor with 10 million dollars deposited in the risk-free asset is to make a yearly investment plan. The investor makes investment and constructs his portfolio at the beginning of the year with the goal to maximize his mean-variance utility defined by (4.3) at the end of the year. It is assumed that there is no further adjustment to the portfolio during the year. The allocation is based on one risky asset and one risk-free asset. The risk-free asset has a return rate of 1.04, while the return rate of the risky asset follows a uniform distribution $U(0.95, 1.3)$ for both periods. λ takes the value of 0.5 to represent the investor's risk profile. The buying and selling transaction costs are both 1.5%.

As it is assumed that there is no balance in the risky asset at the beginning of the investment horizon, x_0^{1-} should equal to zero naturally. This statement is further supported by studying the value of the breakpoints defined by (4.19) to (4.22):

$$g_0^{(1)} = 12.5329 \tag{4.29}$$

$$g_0^{(2)} = 10.4 \tag{4.30}$$

$$g_0^{(3)} = 10.4 \tag{4.31}$$

$$g_0^{(4)} = 10.4 \tag{4.32}$$

Since $g_0^{(2)} = g_0^{(3)} = g_0^{(4)} = 10.4$, the third, fourth and fifth intervals become empty. Thus x_0^{1-} can only take the value of zero in all the situations.

Now we study the value of $\frac{\delta}{2\lambda}$ for the two intervals. Since $h_0^1 = 0$, then $g_0^{(1)} = \frac{q_0^\alpha}{1+\alpha} h_0^0 = \frac{\mathrm{E}(R_0^1)+\frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)}}{1+\alpha} h_0^0$. From (3.13), we obtain the expressions for the two intervals.

For interval 1,

$$\begin{aligned}
\frac{\delta}{2\lambda}^{(1)} &= \frac{1}{2\lambda} + \frac{\mathrm{E}(R_0^1)}{1+\alpha} h_0^0 \\
&= \frac{1}{2\lambda} + \frac{q_0^\alpha - \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)}}{1+\alpha} h_0^0 \\
&= g_0^{(1)} + \left(\frac{1}{2\lambda} - \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 \right). \qquad (4.33)
\end{aligned}$$

For interval 2,

$$\begin{aligned}
\frac{\delta}{2\lambda}^{(2)} &= \frac{1}{2\lambda} + r^0 h_0^0 + \frac{\mathrm{E}^2(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)^2} \left(\frac{\delta}{2\lambda} - r^0 h_0^0 \right) \\
&= \frac{1}{2\lambda} + r^0 h_0^0 + \left(1 - \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)^2} \right) \left(\frac{\delta}{2\lambda} - r^0 h_0^0 \right) \\
\Rightarrow \frac{\delta}{2\lambda}^{(2)} &= \frac{\mathrm{E}(\xi_0^\alpha)^2}{\mathrm{Var}(R_0^1)} \cdot \frac{1}{2\lambda} + r^0 h_0^0 \\
&= \frac{\mathrm{E}(\xi_0^\alpha)^2}{\mathrm{Var}(R_0^1)} \cdot \frac{1}{2\lambda} + g_0^{(1)} - \frac{\mathrm{E}(R_0^1) + \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)}}{1+\alpha} h_0^0 + r^0 h_0^0 \\
&= \frac{\mathrm{E}(\xi_0^\alpha)^2}{\mathrm{Var}(R_0^1)} \cdot \frac{1}{2\lambda} + g_0^{(1)} - \frac{\mathrm{E}(\xi_0^\alpha)^2}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 \\
&= g_0^{(1)} + \frac{\mathrm{E}(\xi_0^\alpha)^2}{\mathrm{Var}(R_0^1)} \left(\frac{1}{2\lambda} - \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 \right). \qquad (4.34)
\end{aligned}$$

From (4.33) and (4.34), it can be concluded that if $\frac{1}{2\lambda} - \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 \geq 0$, then the value of $\frac{\delta}{2\lambda}$ will fall into the first interval. Otherwise, it will fall into the second one. Since

$$\frac{1}{2\lambda} - \frac{\text{Var}(R_0^1)}{\text{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 = -0.4492 < 0, \tag{4.35}$$

$\frac{\delta}{2\lambda}$ falls into the second interval. Therefore, the optimal solution for this single-period problem is

$$x_0^{1+*} = 6.7984 \tag{4.36}$$

$$x_0^{1-*} = 0. \tag{4.37}$$

The expected final wealth is

$$\text{E}(w_1) = r^0 h_0^0 + \text{E}(\xi_0^\alpha) x_0^{1+*} = 10.8718. \tag{4.38}$$

The variance of the final wealth is

$$\text{Var}(w_1) = \text{Var}(R_0^1) x_0^{1+*} = 0.4718. \tag{4.39}$$

4.1.2 Sensitivity Analysis on λ

The example given above has $\frac{1}{2\lambda} - \frac{\text{Var}(R_0^1)}{\text{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 < 0$. This section studies the change of optimal investment strategy with the change in value of λ.

When $\frac{1}{2\lambda} - \frac{\text{Var}(R_0^1)}{\text{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 \geq 0$, i.e., $0 < \lambda \leq 0.3450$, $\frac{\delta}{2\lambda}$ falls into the first interval $[g_0^{(1)}, +\infty)$. Thus, we have

$$x_0^{1+*} = \frac{h_0^0}{1+\alpha} = 9.8522 \tag{4.40}$$

$$x_0^{1-*} = 0. \tag{4.41}$$

The value of optimal x_0^{1+*} is the maximum amount of the risky asset that can be purchased due to the no-borrowing constraint.

When $\frac{\mathrm{Var}(R_0^1)(g^{(2)}-g^{(1)})}{\mathrm{E}(\xi_0^\alpha)^2} \leq \frac{1}{2\lambda} - \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 < 0$, i.e., $\lambda > 0.3450$, $\frac{\delta}{2\lambda}$ falls into the second interval $[g_0^{(2)}, g_0^{(1)})$. The optimal solution is

$$\begin{aligned} x_0^{1+*} &= \frac{\mathrm{E}(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)^2}\left(\frac{\delta}{2\lambda} - g_0^{(2)}\right) \\ &= \frac{\mathrm{E}(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)^2}\left(g_0^{(1)} + \frac{\mathrm{E}(\xi_0^\alpha)^2}{\mathrm{Var}(R_0^1)}\left(\frac{1}{2\lambda} - \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)}h_0^0\right) - g_0^{(2)}\right) \\ &= 4.6191\left(2.1329 + 1.4718\left(\frac{1}{2\lambda} - 1.4492\right)\right) \\ &= 6.7984\left(\frac{1}{2\lambda}\right) \end{aligned} \tag{4.42}$$

$$x_0^{1-*} = 0. \tag{4.43}$$

From Equation 4.42, we can conclude that the optimal amount of risky asset to be purchased decreases as the value of λ increases. Since a higher value of λ indicates a greater level of risk aversion, thus a lower purchase amount is expected in real-life situations. Another interesting observation of Equation 4.42 is that x_0^{1+*} will always be positive as long as λ stays positive. It means that as long as the investor has the slightest intention to take extra risk for higher return, he will purchase some amount of risky asset given that the expected return of the risky asset is higher than the risk-free asset. As the value of λ decreases, the investor becomes more risk-taking. Therefore, he is willing to invest more in the risky asset until the purchase amount hits the limit.

4.2 Special Case: No Transaction Costs

In this section, we consider a special case with no transaction costs. Short selling and borrowing are still prohibited. The model with no transaction costs is equivalent to setting α and β to zero in (4.4). Therefore, the optimal solution to (4.4) can be applied to the no-transaction-cost formulation with some minor changes. First of all, two threshold $g_0^{(2)}$ and $g_0^{(3)}$ now equals to each other and consequently the third interval in Table 4.1 disappears. The following updated notations need to be introduced:

$$\xi_0 = R_0^1 - r^0 \tag{4.44}$$

$$q_0 = \frac{\mathrm{E}(R_0^1 \xi_0)}{\mathrm{E}(\xi_0)} \tag{4.45}$$

$$d_0 = \frac{\mathrm{E}^2(\xi_0)}{\mathrm{E}(\xi_0)^2} \tag{4.46}$$

$$g_0^{(1')} = q_0 h_0^0 + q_0 h_0^1 \tag{4.47}$$

$$g_0^{(2,3)} = r^0 h_0^0 + q_0 h_0^1 \tag{4.48}$$

$$g_0^{(4')} = r^0 h_0^0 + r^0 h_0^1 \tag{4.49}$$

The solution and optimal objective are summarized in Table 4.2.

In Table 4.2, the investment strategy for the second and third intervals

Table 4.2: Solution to the one-period problem with no transaction costs

	range of $\frac{\delta}{2\lambda}$	solution & value function $V_0(h_0^0, h_0^1)$
1	$\left[g_0^{(1')}, \infty\right)$	$x_0^{1+} = h_0^0, \quad x_0^{1-} = 0$ $V_0^{(1)} = \delta\,\mathrm{E}(R_0^1)\,(h_0^0 + h_0^1) - \lambda\,\mathrm{E}(R_0^1)^2\,(h_0^0 + h_0^1)^2$
2	$\left[g_0^{(2,3)}, g_0^{(1')}\right)$	$x_0^{1+} = \frac{\mathrm{E}(\xi_0)}{\mathrm{E}(\xi_0)^2}\left(\frac{\delta}{2\lambda} - g_0^{(2,3)}\right), \quad x_0^{1-} = 0$ $V_0^{(2)} = \delta\,\mathrm{E}(r^0 h_0^0 + R_0^1 h_0^1) - \lambda\,\mathrm{E}(r^0 h_0^0 + R_0^1 h_0^1)^2 + \lambda d_0\left(\frac{\delta}{2\lambda} - g_0^{(2,3)}\right)^2$
3	$\left[g_0^{(4')}, g_0^{(2,3)}\right)$	$x_0^{1+} = 0, \quad x_0^{1-} = \frac{\mathrm{E}(\xi_0)}{\mathrm{E}(\xi_0)^2}\left(g_0^{(2,3)} - \frac{\delta}{2\lambda}\right)$ $V_0^{(4)} = \delta\,\mathrm{E}(r^0 h_0^0 + R_0^1 h_0^1) - \lambda\,\mathrm{E}(r^0 h_0^0 + R_0^1 h_0^1)^2 + \lambda d_0\left(\frac{\delta}{2\lambda} - g_0^{(2,3)}\right)^2$
4	$\left[0, g_0^{(4')}\right)$	$x_0^{1+} = 0, \quad x_0^{1-} = h_0^1$ $V_0^{(5)} = \delta r^0\,(h_0^0 + h_0^1) - \lambda (r^0)^2\,(h_0^0 + h_0^1)^2$

result in the same expression for asset holding levels after rebalancing. The new holding levels for the two assets are

$$\begin{aligned}
\widehat{h}_0^0 = h_0^0 - x_0^{1+} + x_0^{1-} &= h_0^0 - \frac{\mathrm{E}(\xi_0)}{\mathrm{E}(\xi_0)^2}\left(\frac{\delta}{2\lambda} - g_0^{(2,3)}\right) \\
&= h_0^1 - \frac{\mathrm{E}(\xi_0)}{\mathrm{E}(\xi_0)^2}\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1 \xi_0)}{\mathrm{E}(\xi_0)} h_0^1\right) \\
&= \frac{\mathrm{E}(\xi_0)}{\mathrm{E}(\xi_0)^2}\left(q_0 h_0^0 - q_0 h_0^1 - \frac{\delta}{2\lambda}\right),
\end{aligned} \tag{4.50}$$

and

$$\begin{aligned}
\widehat{h}_0^1 = h_0^1 + x_0^{1+} - x_0^{1-} =& h_0^1 + \frac{\mathrm{E}(\xi_0)}{\mathrm{E}(\xi_0)^2}\left(\frac{\delta}{2\lambda} - g_0^{(2,3)}\right) \\
=& h_0^1 + \frac{\mathrm{E}(\xi_0)}{\mathrm{E}(\xi_0)^2}\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1 \xi_0)}{\mathrm{E}(\xi_0)} h_0^1\right) \\
=& \frac{\mathrm{E}(\xi_0)}{\mathrm{E}(\xi_0)^2}\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - r^0 h_0^1\right). \qquad (4.51)
\end{aligned}$$

That is the reason why the two intervals have the same value functions. Note that no borrowing or short selling is allowed in our formulation. Interval 1 indicates that when the target buying amount of risky asset has exceeded the amount of risk-free asset on hand, the investment strategy is to transfer all the fund invested in the risk-free asset to the risky account. Symmetrically, interval 4 shows the situation when the target selling amount has hit the upper limit, the strategy is just to sell as much as possible. When borrowing and short selling are allowed, the first and the fourth intervals in Table 4.2 will no longer exist. These results exactly agree with the solution obtained by Li and Ng (2000).

Chapter 5

The Two-Period Problem

In this chapter, we extend the results obtained in the previous chapter to a multiperiod setting by investigating a two-period portfolio optimization. The two-period problem is modeled as a dynamic program. We still assume a market consisting of two assets, one risk-free asset and one risky asset. Borrowing and short selling are allowed. Two models assuming two types of distribution of the random return rates for the risky asset are studied.

5.1 Uniformly Distributed Return Rate

5.1.1 Analytical Solution to the Second Period

Assume that the solution to the first period has been obtained. For the second period, the value function of dynamic programming is $V_1(x_1^{1+}, x_1^{1-} | h_1^0, h_1^1) = \delta\,\mathrm{E}(h_2^0 + h_2^1) - \lambda\,\mathrm{E}(h_2^0 + h_2^1)^2$. The one-period optimization problem is modeled as follows:

$$\begin{aligned}
\max\ & V_1(x_1^{1+}, x_1^{1-} | h_1^0, h_1^1) \\
s.t.\ & h_2^0 = r^0\big(h_1^0 - (1+\alpha)x_1^{1+} + (1-\beta)x_1^{1-}\big) \\
& h_2^1 = R_1^1(h_1^1 + x_1^{1+} - x_1^{1-}) \\
& x_1^{1+}, x_1^{1-} \geq 0. \qquad (5.1)
\end{aligned}$$

The results obtained in the previous chapter for single-period problem can be adapted to derive the solution to the second period in the two-period problem. The optimal solution for the second period allowing borrowing and short selling is concluded in Table 5.1. It can be seen that g_1^α and g_1^β are two thresholds for switching the investment strategy. They divide the action space into three regions. According to the range that $\frac{\delta}{2\lambda}$ falls into, three investment strategies are recommended respectively.

Table 5.1: Solution to the last period of PB$[\delta, \lambda]$

range of $\frac{\delta}{2\lambda}$	solution & value function $V_1(h_1^0, h_1^1)$
$[g_1^\alpha, +\infty)$	$x_1^{1+} = b_1^\alpha\left(\frac{\delta}{2\lambda} - g_1^\alpha\right), \quad x_1^{1-} = 0$ $V_1^{(1)} = \delta\,\mathrm{E}(r^0h_1^0 + R_1^1h_1^1) - \lambda\,\mathrm{E}(r^0h_1^0 + R_1^1h_1^1)^2 + \lambda\, d_1^\alpha\left(\frac{\delta}{2\lambda} - g_1^\alpha\right)^2$
$\left[g_1^\beta, g_1^\alpha\right)$	$x_1^{1+} = 0, \quad x_1^{1-} = 0$ $V_1^{(2)} = \delta\,\mathrm{E}(r^0h_1^0 + R_1^1h_1^1) - \lambda\,\mathrm{E}(r^0h_1^0 + R_1^1h_1^1)^2$
$\left[0, g_1^\beta\right)$	$x_1^{1+} = 0, \quad x_1^{1-} = b_1^\beta\left(g_1^\beta - \frac{\delta}{2\lambda}\right)$ $V_1^{(3)} = \delta\,\mathrm{E}(r^0h_1^0 + R_1^1h_1^1) - \lambda\,\mathrm{E}(r^0h_1^0 + R_1^1h_1^1)^2 + \lambda\, d_1^\beta\left(\frac{\delta}{2\lambda} - g_1^\beta\right)^2$

The following notations are used in Table 5.1

$$\xi_1^\alpha = R_1^1 - r^0(1+\alpha) \tag{5.2}$$

$$\xi_1^\beta = R_1^1 - r^0(1-\beta) \tag{5.3}$$

$$d_1^\alpha = \frac{\mathrm{E}^2(\xi_1^\alpha)}{\mathrm{E}(\xi_1^\alpha)^2} \tag{5.4}$$

$$d_1^\beta = \frac{\mathrm{E}^2(\xi_1^\beta)}{\mathrm{E}(\xi_1^\beta)^2} \tag{5.5}$$

$$q_1^\alpha = \frac{\mathrm{E}\left(R_1^1\xi_1^\alpha\right)}{\mathrm{E}(\xi_1^\alpha)} \tag{5.6}$$

$$q_1^\beta = \frac{\mathrm{E}(R_1^1\xi_1^\beta)}{\mathrm{E}(\xi_1^\beta)} \tag{5.7}$$

$$g_1^\alpha = r^0 h_1^0 + q_1^\alpha h_1^1 \tag{5.8}$$

$$g_1^\beta = r^0 h_1^0 + q_1^\beta h_1^1. \tag{5.9}$$

Note that the intervals in Table 5.1 are simply the second, third and fourth intervals of Table 4.1. Therefore, the optimal value function is also smooth on the entire domain.

5.1.2 Analytical Solution to the First Period

At the beginning of period 1, we are faced with a two-period problem ahead (period 1 and period 2). Solving the two-period problem is the same as finding the optimal actions to take in the next two time periods, i.e., looking for the optimal values of x_0^{1+}, x_0^{1-}, x_1^{1+} and x_1^{1-}. Given the initial asset level of h_0^0 and h_0^1 at the beginning of period 1, we have to decide how much risky asset to buy or sell (x_0^{1+} and x_0^{1-}), and the action taken will affect the asset levels of the next period, h_1^0 and h_1^1. Once h_1^0 and h_1^1 are known, we can find the best actions at period 2 from Table 5.1 and the final reward is immediately known. Thus, we can obtain the optimal set of actions to maximize the final reward.

We apply backward dynamic programming for solving. Combining (3.6), (3.6), (5.8) and (5.9), the two thresholds g_1^α and g_1^β can be rewritten as

$$g_1^\alpha = (r^0)^2 \widehat{h}_0^0 + q_1^\alpha R_0^1 \widehat{h}_0^1 \tag{5.10}$$

$$g_1^\beta = (r^0)^2 \widehat{h}_0^0 + q_1^\beta R_0^1 \widehat{h}_0^1 \tag{5.11}$$

From equations (5.10) and (5.11), we can derive the two thresholds for R_0^1 :

$$B_0^\alpha = \frac{\frac{\delta}{2\lambda} - (r^0)^2 \widehat{h}_0^0}{q_1^\alpha \widehat{h}_0^1} \tag{5.12}$$

$$B_0^\beta = \frac{\frac{\delta}{2\lambda} - (r^0)^2 \widehat{h}_0^0}{q_1^\beta \widehat{h}_0^1} \tag{5.13}$$

Then the value function for the first period can be written as

$$V_0 = \max \int_{\underline{r_u}}^{B_0^\alpha} p(r_0^1) V_1^{(1)} \, \mathrm{d}r_0^1 + \int_{B_0^\alpha}^{B_0^\beta} p(r_0^1) V_1^{(2)} \, \mathrm{d}r_0^1 + \int_{B_0^\beta}^{\overline{r_u}} p(r_0^1) V_1^{(3)} \, \mathrm{d}r_0^1 \tag{5.14}$$

where $\underline{r_u}$ and $\overline{r_u}$ are the lower bound and the upper bound of the random return rate of the risky asset.

In the multiperiod setting, considering sophisticated distributions adds much difficulty for solving. Therefore, existing studies often choose simple return rate assumptions, e.g., Gennotte and Jung (1994) assume a distribution with two possible outcomes, Bertsimas and Pachamanova (2008) use worst-case returns with correlations for robust optimization. Instead of considering only a finite set of realizations of the return rate, we use the uniform distribution assumption for the first period in our paper for its simplicity and frequent usage in financial analysis (McFarland, 1988; Shalit and Yitzhaki, 2002; Kuan et al., 2009; Wagner, 2010). The only information needed to apply a uniform distribution is the upper bound and lower bound of the return rate. With the uniform distribution assumption, a closed-form solution to the two-period problem can be obtained. It is also worth noting that our analysis for the second period is not affected by the choice of probability distribution of the return rate for the first period, i.e., there will always exist three solution intervals as long as the transaction costs are present.

We assume $R_0^1 \sim U(\underline{r_u}, \overline{r_u})$, where $\underline{r_u}$ and $\overline{r_u}$ satisfy

$$\overline{r_u} > r^0 \frac{1+\alpha}{1-\beta} \tag{5.15}$$

$$\underline{r_u} < r^0 \tag{5.16}$$

to ensure that investing in one asset is not strictly dominated by the other. Additionally, all the wealth is assumed to be deposited in the risk-free account at the beginning of the first period, i.e., $h_0^0 = w_0$ and $h_0^1 = 0$. The investor decides the amount of wealth to be shifted from the risk-free asset to the risky asset. Therefore, $x_0^{1+} \geq 0$ and $x_0^{1-} = 0$. The resulting value function for the first period is

$$
\begin{aligned}
V_0 =&\delta\,\mathrm{E}\left[(r^0)^2\widehat{h}_0^0 + R_1^1R_0^1\widehat{h}_0^1\right] - \lambda\,\mathrm{E}\left[(r^0)^2\widehat{h}_0^0 + R_1^1R_0^1\left(\widehat{h}_0^1\right)\right]^2 - \frac{\lambda}{3(\overline{r_u}-\underline{r_u})\widehat{h}_0^1} \\
&\times\left\{\frac{d_1^\beta}{q_1^\beta}\left[\frac{\delta}{2\lambda} - (r^0)^2\widehat{h}_0^0 - q_1^\beta\overline{r_u}\,\widehat{h}_0^1\right]^3 - \frac{d_1^\alpha}{q_1^\alpha}\left[\frac{\delta}{2\lambda} - (r^0)^2\widehat{h}_0^0 - q_1^\alpha\underline{r_u}\,\widehat{h}_0^1\right]^3\right\}.
\end{aligned}
$$

The derivation details can be found in Appendix D.

In order to find the global optima to the problem, the concavity of V_0 needs to be established. As such, we further define a random variable

$$
\xi_0^\alpha = R_1^1R_0^1 - (r^0)^2(1+\alpha) \tag{5.17}
$$

and two constants

$$
\overline{L}_0 = q_1^\beta\overline{r_u} - (r^0)^2(1+\alpha) \tag{5.18}
$$

$$
\underline{L}_0 = q_1^\alpha\underline{r_u} - (r^0)^2(1+\alpha) \tag{5.19}
$$

Then we have the following results:

Lemma 5.1. $\mathrm{E}\,(\xi_0^\alpha)^2 > \frac{\frac{d_1^\beta}{q_1^\beta}\left(\overline{L}_0\right)^3 - \frac{d_1^\alpha}{q_1^\alpha}\left(\underline{L}_0\right)^3}{3(\overline{r_u}-\underline{r_u})}$.

Proof.

$$
\begin{aligned}
&\mathrm{E}\left(\xi_0^\alpha\right)^2 - \frac{\frac{d_1^\beta}{q_1^\beta}\left(\overline{L_0}\right)^3 - \frac{d_1^\alpha}{q_1^\alpha}\left(\underline{L_0}\right)^3}{3(\overline{r_u}-\underline{r_u})} \\
=&\mathrm{E}(R_1^1)^2\frac{\overline{r_u}^3-\underline{r_u}^3}{3(\overline{r_u}-\underline{r_u})} - \mathrm{E}(R_1^1)(\underline{r_u}+\overline{r_u})(r^0)^2(1+\alpha) + \left[(r^0)^2(1+\alpha)\right]^2 \\
&- \frac{1}{3(\overline{r_u}-\underline{r_u})}\left[\left(d_1^\beta(q_1^\beta)^2\overline{r_u}^3 - d_1^\alpha(q_1^\alpha)^2\underline{r_u}^3\right) - 3\left(d_1^\beta q_1^\beta\overline{r_u}^2 - d_1^\alpha q_1^\alpha\underline{r_u}^2\right)(r^0)^2(1+\alpha)\right. \\
&\quad \left. +3\left(d_1^\beta\overline{r_u} - d_1^\alpha\underline{r_u}\right)\left((r^0)^2(1+\alpha)\right)^2 - \left(\frac{d_1^\beta}{q_1^\beta} - \frac{d_1^\alpha}{q_1^\alpha}\right)\left((r^0)^2(1+\alpha)\right)^3\right] \\
=&\frac{\mathrm{Var}(R_1^1)}{3(\overline{r_u}-\underline{r_u})r^0}\left[\frac{\left(r^0(1-\beta)\overline{r_u}-(r^0)^2(1+\alpha)\right)^3}{\mathrm{E}(\xi_1^\beta)(1-\beta)} - \frac{\left(\underline{r_u}-r^0\right)^3}{\mathrm{E}(\xi_1^\alpha)(1+\alpha)}\right] \\
&+ \frac{\mathrm{Var}(R_0^1)}{r^0(1+\alpha)(1-\beta)\,\mathrm{E}(R_1^1\xi_1^\alpha)\,\mathrm{E}(R_1^1\xi_1^\beta)}\left[\left(\alpha\,\mathrm{E}(\xi_1^\alpha)+\beta\,\mathrm{E}(\xi_1^\beta)\right)\mathrm{E}(R_1^1-r^0)\right. \\
&\quad \left. + (\alpha+\beta)\,\mathrm{Var}(R_1^1) + r^0(\alpha+\beta)\,\mathrm{E}\left(R_1^1 - r^0(1+\alpha\beta)\right)\right] > 0
\end{aligned}
$$

Thus, Lemma 5.1 is proved. □

Lemma 5.2. *Under the optimal investment policy π^*, $\frac{\delta}{2\lambda} > (r^0)^2\hat{h}_0^0 + (r^0)^2(1+\alpha)\widehat{h}_0^1$.*

Proof. As PB(λ) is equivalent to PB(δ,λ) if and only if $\delta = 1 + 2\lambda\,\mathrm{E}(w_T)|_{\pi^*}$, thus under the optimal investment policy,

$$
\begin{aligned}
\frac{\delta}{2\lambda} =& \frac{1}{2\lambda} + \mathrm{E}(w_T)|_{\pi^*} \\
\geq& \frac{1}{2\lambda} + (r^0)^2\widehat{h}_0^0 + \mathrm{E}(R_1^1)\,\mathrm{E}(R_1^1)\widehat{h}_0^1 \\
>& (r^0)^2\widehat{h}_0^0 + (r^0)^2(1+\alpha)\widehat{h}_0^1.
\end{aligned} \tag{5.20}
$$

□

Lemmas 5.1 and 5.2 lay a preliminary foundation for proving the concavity of V_0. It will be shown in Theorem 5.1 that V_0 is strictly concave with respect to x_0^{1+} and the KKT conditions can be applied for solving the problem.

Theorem 5.1. *The KKT conditions are sufficient for finding the optimal solution to the two-period problem that has the objective function given by* (5.17)*.*

Proof. From Lemma 5.1 and 5.2, we can derive

$$\frac{\mathrm{d}^2 V_0(x_0^{1+})}{\mathrm{d}(x_0^{1+})^2} = -2\lambda\left[\mathrm{E}\left(\xi_0^{\alpha}\right)^2 - \frac{\frac{d_1^{\beta}}{q_1^{\beta}}\left(\overline{L}_0\right)^3 - \frac{d_1^{\alpha}}{q_1^{\alpha}}\left(\underline{L}_0\right)^3}{3(\overline{r_u} - \underline{r_u})}\right] - \left(\frac{d_1^{\beta}}{q_1^{\beta}} - \frac{d_1^{\alpha}}{q_1^{\alpha}}\right)\left(\frac{\frac{\delta}{2\lambda} - (r^0)^2\hat{h}_0^0 - (r^0)^2(1+\alpha)\widehat{h}_0^1}{\widehat{h}_0^1}\right)^3 < 0$$

Hence, the objective function is a concave function of x_0^{1+}, and the KKT conditions are sufficient for finding the optimal solution. □

Theorem 5.1 allows the optimal solution to be obtained by solving the KKT conditions as listed below.

(i) $\frac{\mathrm{d}V_0(x_0^{1+})}{\mathrm{d}x_0^{1+}} - y = 0$

(ii) $x_0^{1+} \geq 0$

(iii) $y \geq 0$

(iv) $x_0^{1+} y = 0$

where y is the Lagrange multiplier.

By solving the KKT conditions, the optimal solution is found and given by Corollary 5.30.

Corollary 5.1. *The optimal solution for the first period is*

$$x_0^{1+*} = \frac{\frac{\gamma}{2} - (r^0)^2 w_0}{u_0} \tag{5.21}$$

$$x_0^{1-*} = 0 \tag{5.22}$$

where $\gamma = \delta/\lambda$.

Proof. Let $y = 0$. Then condition (i) becomes

$$\begin{aligned}
0 =& \frac{\mathrm{d}V_0(x_0^{1+})}{\mathrm{d}x_0^{1+}} \\
=& \delta\,\mathrm{E}\left(\xi_0^\alpha\right) - 2\lambda\,\mathrm{E}\left[\xi_0^\alpha\left((r^0)^2 h_0^0 + R_1^1 R_0^1 h_0^1 + \xi_0^\alpha x_0^{1+}\right)\right] \\
&+ \frac{\lambda}{3(\overline{r_u} - \underline{r_u})\left(x_0^{1+}\right)^2}\left\{3\frac{d_1^\beta}{q_1^\beta}\left[\frac{\delta}{2\lambda} - (r^0)^2 w_0 - \overline{L_0} x_0^{1+}\right]^2 \overline{L_0} x_0^{1+}\right. \\
&-3\frac{d_1^\alpha}{q_1^\alpha}\left[\frac{\delta}{2\lambda} - (r^0)^2 h_0^0 - \underline{L_0} x_0^{1+}\right]^2 \underline{L_0} x_0^{1+} + \frac{d_1^\beta}{q_1^\beta}\left[\frac{\delta}{2\lambda} - (r^0)^2 w_0 - \overline{L_0} x_0^{1+}\right]^3 \\
&\left. - \frac{d_1^\alpha}{q_1^\alpha}\left[\frac{\delta}{2\lambda} - (r^0)^2 w_0 - \underline{L_0} x_0^{1+}\right]^3\right\}.
\end{aligned} \tag{5.23}$$

Multiplying both sides by $\frac{3(\overline{r_u}-\underline{r_u})}{\lambda x_0^{1+}}$ yields

$$\begin{aligned}
0 =& \left(\frac{d_1^\beta}{q_1^\beta} - \frac{d_1^\alpha}{q_1^\alpha}\right)\left(\frac{\frac{\delta}{2\lambda} - (r^0)^2 w_0}{x_0^{1+}}\right)^3 \\
&+ 3\left(2(\overline{r_u} - \underline{r_u})\,\mathrm{E}(\xi_0^\alpha) - \frac{d_1^\beta}{q_1^\beta}\left(\overline{L_0}\right)^2 + \frac{d_1^\alpha}{q_1^\alpha}\left(\underline{L_0}\right)^2\right)\left(\frac{\frac{\delta}{2\lambda} - (r^0)^2 w_0}{x_0^{1+}}\right) \\
&- 6(\overline{r_u} - \underline{r_u})\,\mathrm{E}(\xi_0^\alpha)^2 + 2\frac{d_1^\beta}{q_1^\beta}\left(\overline{L_0}\right)^3 - 2\frac{d_1^\alpha}{q_1^\alpha}\left(\underline{L_0}\right)^3. \qquad (5.24)
\end{aligned}$$

Let

$$u = \frac{\frac{\delta}{2\lambda} - (r^0)^2 w_0}{x_0^{1+}} \qquad (5.25)$$

$$c_1 = \frac{d_1^\beta}{q_1^\beta} - \frac{d_1^\alpha}{q_1^\alpha} \qquad (5.26)$$

$$c_2 = 3\left(2(\overline{r_u} - \underline{r_u})\,\mathrm{E}(\xi_0^\alpha) - \frac{d_1^\beta}{q_1^\beta}\left(\overline{L_0}\right)^2 + \frac{d_1^\alpha}{q_1^\alpha}\left(\underline{L_0}\right)^2\right) \qquad (5.27)$$

$$c_3 = -6(\overline{r_u} - \underline{r_u})\,\mathrm{E}(\xi_0^\alpha)^2 + 2\frac{d_1^\beta}{q_1^\beta}\left(\overline{L_0}\right)^3 - 2\frac{d_1^\alpha}{q_1^\alpha}(L^a)^3. \qquad (5.28)$$

Then (5.24) can be written as

$$c_1 u^3 + c_2 u + c_3 = 0 \qquad (5.29)$$

As proven in Lemma 5.1, $\frac{c_3}{c_1}$ is negative, which implies that (5.29) has at least one positive root. Additionally, V_0 is strictly concave and so there can at most be one optimal solution. Denoting u_0 as the positive root of (5.29), x_0^{1+*} can be expressed as

$$x_0^{1+*} = \frac{\frac{\gamma}{2} - (r^0)^2 w_0}{u_0}, \tag{5.30}$$

where $\gamma = \delta/\lambda$. □

With the optimal policy for the first period provided in Corollary 5.1, we can further derive the mean-variance frontier of our model. The derivation details are as follows:

The targeted holdings of assets under such a policy are given by

$$\begin{aligned}
\widehat{h}_0^0 =& \frac{-(1+\alpha)\frac{\gamma}{2} + q_0^\alpha w_0}{q_0^\alpha - (r^0)^2(1+\alpha)} \\
\widehat{h}_0^1 =& \frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q_0^\alpha - (r^0)^2(1+\alpha)}
\end{aligned}$$

where $q_0^\alpha = u_0 + (r^0)^2(1+\alpha)$. The expected final wealth is (see Appendix E for derivation details)

$$\begin{aligned}
\mathrm{E}(w_2(\gamma)) =& \int_{\underline{r_u}}^{B_{T-2}^\alpha} p(r_0^1) w_1^{(1)} \, \mathrm{d}r_0^1 + \int_{B_{T-2}^\alpha}^{B_{T-2}^\beta} p(r_0^1) w_1^{(2)} \, \mathrm{d}r_0^1 + \int_{B_{T-2}^\beta}^{\overline{r_u}} p(r_0^1) w_1^{(3)} \, \mathrm{d}r_0^1 \\
=& (r^0)^2 w_0 + \zeta \left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q_0^\alpha - (r^0)^2(1+\alpha)} \right),
\end{aligned} \tag{5.31}$$

where

$$\zeta = \mathrm{E}(\xi_0^\alpha) + \frac{1}{2(\overline{r_u} - \underline{r_u})} \frac{d_1^\alpha}{q_1^\alpha} \left(q_0^\alpha - q_1^\alpha \underline{r_u} \right)^2 - \frac{1}{2(\overline{r_u} - \underline{r_u})} \frac{d_1^\beta}{q_1^\beta} \left(q_0^\alpha - q_1^\beta \overline{r_u} \right)^2 . \tag{5.32}$$

Further, we have

$$
\begin{aligned}
&\mathrm{E}\left(w_2(\gamma)\right)^2 \\
&= \int_{\underline{r_u}}^{B^{\alpha}_{T-2}} p(r^1_0)\big(w^{(1)}_1\big)^2 \,\mathrm{d}r^1_0 + \int_{B^{\alpha}_{T-2}}^{B^{\beta}_{T-2}} p(r^1_0)\big(w^{(2)}_1\big)^2 \,\mathrm{d}r^1_0 + \int_{B^{\beta}_{T-2}}^{\overline{r_u}} p(r^1_0)\big(w^{(3)}_1\big)^2 \,\mathrm{d}r^1_0 \\
&= \nu\left(\frac{\frac{\gamma}{2}-(r^0)^2w_0}{q^{\alpha}_0-(r^0)^2(1+\alpha)}\right)^2 + 2\zeta(r^0)^2w_0\left(\frac{\frac{\gamma}{2}-(r^0)^2w_0}{q^{\alpha}_0-(r^0)^2(1+\alpha)}\right) + \left((r^0)^2w_0\right)^2
\end{aligned}
\tag{5.33}
$$

where

$$
\begin{aligned}
\nu =& \mathrm{E}(\xi^{\alpha}_0)^2 + \frac{1}{3(\overline{r_u}-\underline{r_u})}\left[\frac{d^{\beta}_1}{q^{\beta}_1}\left(q^{\alpha}_0-q^{\beta}_1\overline{r_u}\right)^3 - \frac{d^{\alpha}_1}{q^{\alpha}_1}\left(q^{\alpha}_0-q^{\alpha}_1\underline{r_u}\right)^3\right] \\
&- \frac{1}{\overline{r_u}-\underline{r_u}}\left[\frac{d^{\beta}_1}{q^{\beta}_1}\left(q^{\alpha}_0-q^{\beta}_1\overline{r_u}\right)^2 - \frac{d^{\alpha}_1}{q^{\alpha}_1}\left(q^{\alpha}_0-q^{\alpha}_1\underline{r_u}\right)^2\right].
\end{aligned}
\tag{5.34}
$$

Combining (5.31) and (5.33) yields

$$
\mathrm{Var}(w_2(\gamma)) = \mathrm{E}\left(w_2(\gamma)\right)^2 - \mathrm{E}^2\left(w_2(\gamma)\right) = \left(\nu-\zeta^2\right)\left(\frac{\frac{\gamma}{2}-(r^0)^2w_0}{u^*_0}\right)^2. \tag{5.35}
$$

Thus let

$$
\begin{aligned}
U =& \mathrm{E}\left(w_2(\gamma)\right) - \lambda\,\mathrm{Var}(w_2(\gamma)) \\
=&(r^0)^2w_0 + \zeta\left(\frac{\frac{\gamma}{2}-(r^0)^2w_0}{u^*_0}\right) - \lambda\left(\nu-\zeta^2\right)\left(\frac{\frac{\gamma}{2}-(r^0)^2w_0}{u^*_0}\right)^2
\end{aligned}
\tag{5.36}
$$

Differentiating (5.36) with respect to γ yields

$$\frac{\mathrm{d}U}{\mathrm{d}\gamma} = \frac{\zeta}{u_0^*} - 2\lambda \frac{\nu - \zeta^2}{u_0^*} \left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{u_0^*} \right) \tag{5.37}$$

Solving $\frac{\partial \widetilde{U}}{\partial \gamma} = 0$ yields

$$\gamma^* = \frac{\zeta u_0^*}{\lambda(\nu - \zeta^2)} + 2(r^0)^2 w_0 \tag{5.38}$$

Substitute (5.38) into (5.31) and (5.35), we have

$$\mathrm{E}\left(w_2(\gamma^*)\right) = (r^0)^2 w_0 + \frac{\zeta^2}{2\lambda(\nu - \zeta^2)}, \tag{5.39}$$

$$\mathrm{Var}(w_2(\gamma^*)) = \frac{\zeta^2}{4\lambda^2(\nu - \zeta^2)}. \tag{5.40}$$

The mean-variance efficient frontier is

$$\mathrm{Var}(w_2) = \frac{\nu - \zeta^2}{\zeta^2} \left(\mathrm{E}(w_2) - (r^0)^2 w_0 \right)^2. \tag{5.41}$$

$\mathrm{Var}(w_2)$ is a quadratic function of $\mathrm{E}(w_2)$ in (5.41), indicating a risk-averse utility function. Additionally, when $\mathrm{E}(w_2)$ is equal to $(r^0)^2 w_0$, $\mathrm{Var}(w_2) = 0$. It means that if all the wealth has been allocated to the risk-free asset in the first and second periods, there is no uncertainty in this investment strategy.

5.1.3 Numerical Example

We modify the single-period investment example discussed in Section 4.1.1 into a two-period problem by allowing a rebalancing time point at the end of the sixth month. All the rest of the assumptions remain the same. Then the optimal strategy for the first period is

$$x_0^{1+*} = 9.4849 \qquad \text{and} \qquad x_0^{1-*} = 0, \tag{5.42}$$

while at the second period, the optimal strategy is

$$\begin{cases} x_1^{1+*} = 4.1478(12.4871 - 12.2989R_0^1), x_1^{1-*} = 0, & \text{if } 0.95 \le R_0^1 < 1.0153 \\ x_1^{1+*} = 0, x_1^{1-*} = 0, & \text{if } 1.0153 \le R_0^1 < 1.0735 \\ x_1^{1+*} = 0, x_1^{1-*} = 4.9500(11.6324R_0^1 - 12.4871), & \text{if } 1.0735 \le R_0^1 \le 1.3 \end{cases} \tag{5.43}$$

The optimal portfolio yields an expected final wealth of 11.8904 million dollars with a variance of 1.0744 square million dollars.

In (5.43), 1.0153 and 1.0735 are the two thresholds where the optimal action changes. When R_0^1 is below 1.0153, a relatively low level of holdings of the risky asset will occur at the end of the first period, and therefore the optimal solution is to buy more risky asset. On the other hand, if R_0^1 exceeds 1.0735, the investor should shift the wealth from the risky asset to the risk-free asset in order to reduce the risk. When R_0^1 is between 1.0153 and 1.0735, we have $x_1^{1+*} = x_1^{1-*} = 0$, which means that no adjustment should be made to the current portfolio. This is the non-transaction region.

In conclusion, at the end of the first period, the realization of R_0^1 can be observed. It will decide the asset levels at the beginning of the second period (h_1^0 and h_1^1). As shown in Table 5.1, based on the observation of h_1^0 and h_1^1, the investor can easily find the best investment strategy for the second period.

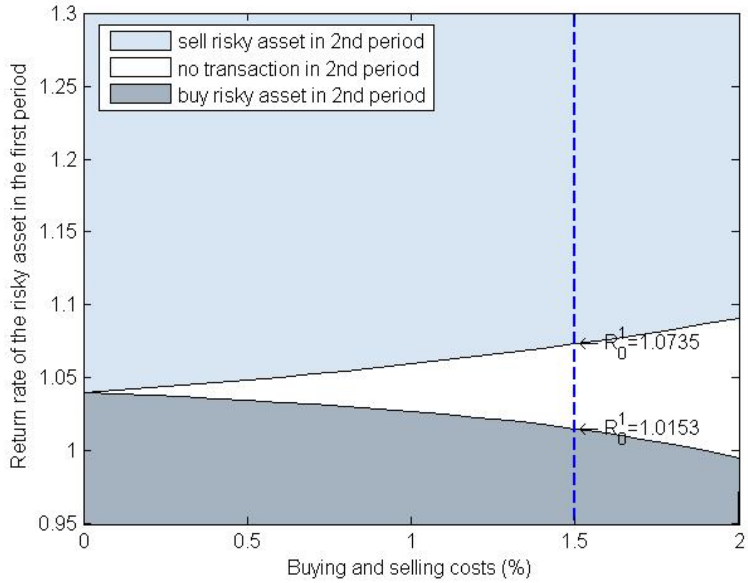


Figure 5.1: Optimal investment strategy in the second period

In Figure 5.1, we plotted values of the two thresholds for R_0^1 by changing the value of transaction costs (α and β), while keeping other data the same as those in the example in Section 4.1.1.

Figure 5.2 shows the efficient frontiers under three transaction fee schedules. The three frontiers meet at one end when $\mathrm{Var}(w_2) = 0$ and $\mathrm{E}(w_2) = 10.8160$. As the transaction fee increases, the expectation of the investors' final wealth decreases for the same exposure to investment risk. The efficient frontier with no transaction costs is an upper bound of efficient frontiers with transaction fees. If transaction costs are ignored, an investor will be faced with a lower reward than expected from a portfolio constructed according to the investor's risk tolerance or exposed to a higher risk for the investor's target return.

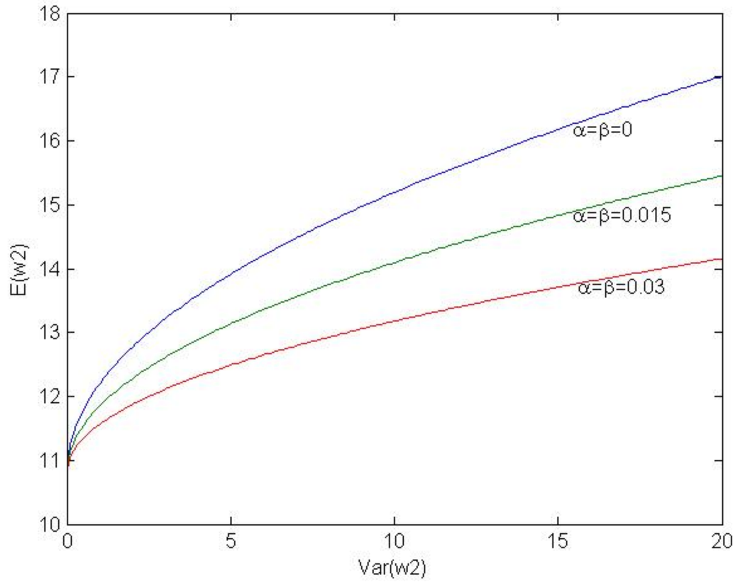


Figure 5.2: Markowitz efficient frontiers with proportional transaction costs.

5.2 Discretely Distributed Return Rate

Section 5.1 shows that the value function has a non-quadratic structure assuming that the random return rate follows a simple uniform distribution. Therefore, the results for the two-period problem assuming a continuous distribution has little potential to be extended to a multiperiod case. In order to maintain the quadratic structure, we explore a model under the assumption that R_0^1 follows a discrete distribution and use a scenario tree to represent its realizations. Note that any continuous distribution can be approximated by a discrete distribution. In other words, the realizations of a random variable can be represented by a scenario tree with sufficiently many scenarios. For simplicity, we illustrate the method by assuming that there are only two possible random outcomes. The representing two-branch scenario tree is shown in Figure 5.3.

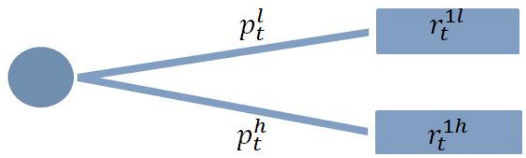


Figure 5.3: A two-branch scenario tree.

In Figure 5.3, r_t^{1l} and r_t^{1h} are the two outcomes of R_t^1 which represent respectively the lower return and the higher return of the risky asset for period t, and p_t^l and p_t^h are the corresponding probabilities for the two outcomes which satisfy $p_t^l + p_t^h = 1$.

We start the analysis from the beginning of the second period (the end of the first period), i.e. $t = 1$. For the second period, the analytical solution is obtained in Section 5.1.1 and summarized in Table 5.1. Three different investment strategies are recommended and three corresponding optimal objectives are derived.

Now let us move backward to $t = 0$. For the first period, we assume that the asset holding levels h_0^0 and h_0^1 are given at the beginning the period. Since $h_1^1 = R_0^1(h_0^1 + x_0^{1+} - x_0^{1-})$, the realization of R_0^1 and the decisions taken in period $T-2$ (x_0^{1+} and x_0^{1-}) determine the value of h_1^1 , and thus determine the value of the two thresholds for the next period. As we assume a two-branch realization of R_0^1, once the investor takes an action, i.e. chooses a set of x_0^{1+} and x_0^{1-}, there is a probability p_0^l that h_1^1 equals $r_0^{1l}(h_0^1 + x_0^{1+} - x_0^{1-})$, and the other probability p_0^h that h_1^1 equals $r_0^{1h}(h_0^1 + x_0^{1+} - x_0^{1-})$. The two realizations of

h_1^1, denoted by h_1^{1l} and h_1^{1h}, may fall into any of the three intervals in Table 5.1 with the corresponding objective functions V_1^l and V_1^h. The value function for $T-2$ is

$$\begin{aligned} & V_0(h_0^0, h_0^1) \\ & = \max_{x_0^{1+}, x_0^{1-} \geq 0} \left\{ \mathrm{E}\left(V_1(h_1^0, h_1^{1l}, h_1^{1h})\right) \right\} \\ & = \max_{x_0^{1+}, x_0^{1-} \geq 0} \left\{ p_0^l V_1^l(h_1^0, h_1^{1l}) + p_0^h V_1^h(h_1^0, h_1^{1h}) \right\}, \end{aligned} \tag{5.44}$$

where $V_1^l(h_1^0, h_1^{1l})$ and $V_1^h(h_1^0, h_1^{1h})$ is one of the three value functions listed in Table 5.1, depending on which interval $\frac{\delta}{2\lambda}$ falls into. In this way, the stochastic model is transferred to a deterministic model.

We manage to formulate the problem as a dynamic program, though a general form to express the optimal solution is still not available. However, we are able to understand some of the complication involved in the solving process.

The two thresholds for the random return rate, B_0^α and B_0^β, defined in (5.12) and (5.13), divide the space into three regions. r_0^{1l} and r_0^{1h} can fall into any of the three regions as long as $r_0^{1l} < r_0^{1h}$ is satisfied. Therefore, there are totally six possible region combinations of two return rate realizations. Consequently, there will be six possible forms of the objective functions. As the number of time periods and return rate realizations increase, the problem becomes intractable. Moreover, the two thresholds contain the decision variables, i.e. the values of the thresholds change according to the values of decision variables. This makes it even harder to formulate the objective function. Therefore, a closed-form solution is very difficult to be obtained.

Chapter 6

An Approximation Method for Solving Multiperiod Portfolio Optimization

As shown in Chapter 5, even for the two-period investment problem considering two assets under the simple assumption that the random return rate follows a uniform distribution or a discrete distribution, the problem becomes intractable. In order to solve the multiperiod problem, we introduce an approximation method in this chapter.

We use dynamic programming approach to solve PB$[\delta, \lambda]$. A series of value functions $V_t(h_t^0, h_t^1)$ is defined to represent the utility to hold certain amounts of the two assets (h_t^0 and h_t^1) at time t and it is written as

$$V_t(h_t^0, h_t^1) = \max_{x_t^{1+}, x_t^{1-} \geq 0} \left\{ \mathrm{E}\left(V_{t+1}(h_{t+1}^0, h_{t+1}^1)\right) \right\} \tag{6.1}$$

with the boundary condition

$$V_T(h_T^0, h_T^1) = \delta\left(h_T^0 + h_T^1\right) - \lambda\left(h_T^0 + h_T^1\right)^2. \tag{6.2}$$

Thus we have formulated the multiperiod portfolio optimization problem as a dynamic program.

6.1 Notations

For easy reference, before introducing the approximation method, we make the following notations which are used in the rest of the thesis. Their expressions will be derived later in this chapter.

m_t : the minimizer of the 2-norm distance between the value function and the actual value function. Its expression is given in Appendix F

$$\xi_t^\alpha = \prod_{i=t}^{T-1} R_i^1 - \left(r^0\right)^{(T-t)}(1+\alpha) \tag{6.3}$$

$$\xi_t^\beta = \prod_{i=t}^{T-1} R_i^1 - \left(r^0\right)^{(T-t)}(1-\beta) \tag{6.4}$$

$$b_t^\alpha = \frac{\mathrm{E}(\xi_t^\alpha) - \frac{1}{2}\sum_{i=t+1}^{T-1} m_i \,\mathrm{E}\left(Q_{t,i}^{\alpha,\alpha} + Q_{t,i}^{\beta,\alpha}\right)}{\mathrm{E}(\xi_t^\alpha)^2 - \sum_{i=t+1}^{T-1} m_i \,\mathrm{E}\left(Q_{t,i}^{\alpha,\alpha} Q_{t,i}^{\beta,\alpha}\right)} \tag{6.5}$$

$$b_t^\beta = \frac{\mathrm{E}(\xi_t^\beta) - \frac{1}{2}\sum_{i=t+1}^{T-1} m_i \,\mathrm{E}\left(Q_{t,i}^{\alpha,\beta} + Q_{t,i}^{\beta,\beta}\right)}{\mathrm{E}(\xi_t^\beta)^2 - \sum_{i=t+1}^{T-1} m_i \,\mathrm{E}\left(Q_{t,i}^{\alpha,\beta} Q_{t,i}^{\beta,\beta}\right)} \tag{6.6}$$

$$d_t^\alpha = \frac{\left(\mathrm{E}(\xi_t^\alpha) - \frac{1}{2}\sum_{i=t+1}^{T-1} m_i \,\mathrm{E}\left(Q_{t,i}^{\alpha,\alpha} + Q_{t,i}^{\beta,\alpha}\right)\right)^2}{\mathrm{E}(\xi_t^\alpha)^2 - \sum_{i=t+1}^{T-1} m_i \,\mathrm{E}\left(Q_{t,i}^{\alpha,\alpha} Q_{t,i}^{\beta,\alpha}\right)} \tag{6.7}$$

$$d_t^\beta = \frac{\left(\mathrm{E}(\xi_t^\beta) - \frac{1}{2}\sum_{i=t+1}^{T-1} m_i \,\mathrm{E}\left(Q_{t,i}^{\alpha,\beta} + Q_{t,i}^{\beta,\beta}\right)\right)^2}{\mathrm{E}(\xi_t^\beta)^2 - \sum_{i=t+1}^{T-1} m_i \,\mathrm{E}\left(Q_{t,i}^{\alpha,\beta} Q_{t,i}^{\beta,\beta}\right)} \tag{6.8}$$

$$q_t^\alpha = \frac{1}{b_t^\alpha} + (r^0)^{(T-t)}(1+\alpha) \tag{6.9}$$

$$q_t^\beta = \frac{1}{b_t^\beta} + (r^0)^{(T-t)}(1-\beta) \tag{6.10}$$

$$g_t^\alpha = (r^0)^{(T-t)} h_t^0 + q_t^\alpha h_t^1 \tag{6.11}$$

$$g_t^\beta = (r^0)^{(T-t)} h_t^0 + q_t^\beta h_t^1 \tag{6.12}$$

for $t = 0, 1, \ldots, T-1$. We further define

$$G_{t,s}^\alpha = (r^0)^{(T-t)} h_t^0 + q_s^\alpha \prod_{j=t}^{s-1} R_j^1 h_t^1 \tag{6.13}$$

$$G_{t,s}^\beta = (r^0)^{(T-t)} h_t^0 + q_s^\beta \prod_{j=t}^{s-1} R_j^1 h_t^1 \tag{6.14}$$

$$Q_{t,s}^{\alpha,\alpha} = q_s^\alpha \prod_{j=t}^{s-1} R_j^1 - (r^0)^{(T-t)}(1+\alpha) \tag{6.15}$$

$$Q_{t,s}^{\alpha,\beta} = q_s^\alpha \prod_{j=t}^{s-1} R_j^1 - (r^0)^{(T-t)}(1-\beta) \tag{6.16}$$

$$Q_{t,s}^{\beta,\alpha} = q_s^\beta \prod_{j=t}^{s-1} R_j^1 - (r^0)^{(T-t)}(1+\alpha) \tag{6.17}$$

$$Q_{t,s}^{\beta,\beta} = q_s^\beta \prod_{j=t}^{s-1} R_j^1 - (r^0)^{(T-t)}(1-\beta) \tag{6.18}$$

for $t = 0, 1, \ldots, T-2$ and $s = t+1, t+2, \ldots, T-1$.

6.2 The Approximation Steps

In this section, we introduce an approximation method for obtaining the value functions which works even when α or β is non-zero. It includes the following steps:

1. Find the approximate value function for the last period.

2. Solve the second last period using the approximate value function.

3. Derive approximate solutions and value functions recursively for the earlier periods.

4. Find the approximate value for δ.

In the following sections, we will develop the approximation method according to the steps listed above. A series of numerical experiments will be conducted to show the performance of the method.

6.3 Approximate Value Function for the Last Stage

The solution and the value function for the last period of a multiperiod problem can be easily obtained by adapting the results to the second period of a two-period problem provided in Table 5.1. The only adaption needed is to change the period-indicating subscription from "1" to "$T-1$". In Table 5.1, the value function contains three parts. The only difference among the three parts is the last part. Thus the exact value function for the last period V_{T-1} can be written as:

$$V_{T-1} = \delta\, \mathrm{E}\left(r^0 h^0_{T-1} + R^1_{T-1} h^1_{T-1}\right) - \lambda\, \mathrm{E}\left(r^0 h^0_{T-1} + R^1_{T-1} h^1_{T-1}\right)^2 + \hat{V}_{T-1} \quad (6.19)$$

where

$$\hat{V}_{T-1} = \begin{cases} \lambda d^{\alpha}_{T-1}\left(\rho - g^{\alpha}_{T-1}\right)^2 & \text{when} \quad \rho \geq g^{\alpha}_{T-1} \\ 0 & \text{when} \quad g^{\beta}_{T-1} \leq \rho < g^{\alpha}_{T-1} \\ \lambda d^{\beta}_{T-1}\left(\rho - g^{\beta}_{T-1}\right)^2 & \text{when} \quad 0 \leq \rho < g^{\beta}_{T-1} \end{cases} \quad (6.20)$$

As such a problem cannot be solved recursively, we construct an approximate quadratic value function which has only one expression by interpolation. Let

$$\rho = \frac{\delta}{2\lambda}. \quad (6.21)$$

The breakpoints $\rho = g^{\alpha}_{T-1}$ and $\rho = g^{\beta}_{T-1}$ are chosen as the two data points for interpolation as these are the two thresholds where the investment decision changes. Therefore, the approximate function would be of the following form:

$$\begin{aligned}\widetilde{V}_{T-1} =& \delta \, \mathrm{E}\left(r^0 h^0_{T-1} + R^1_{T-1} h^1_{T-1}\right) - \lambda \, \mathrm{E}\left(r^0 h^0_{T-1} + R^1_{T-1} h^1_{T-1}\right)^2 \\ &+ \lambda m_{T-1} \left(\rho - g^{\alpha}_{T-1}\right) \left(\rho - g^{\beta}_{T-1}\right)\end{aligned} \tag{6.22}$$

The coefficient m_{T-1} assigned to the term $(\rho - g^{\alpha}_{T-1})(\rho - g^{\beta}_{T-1})$ minimizes the cumulative 2-norm distance between $\widetilde{V}_{T-1}$ and V_{T-1}. To obtain the explicit expression for such a cumulative distance, we impose the following conditions:

$$\begin{cases} d^{\alpha}_{T-1} \left(\rho - g^{\alpha}_{T-1}\right) \leq \frac{h^0_{T-1}}{1+\alpha} \\ d^{\beta}_{T-1} \left(g^{\beta}_{T-1} - \rho\right) \leq h^1_{T-1} \end{cases}$$

These conditions set upper bounds for the optimal transaction amount for the last period. It indicates that the investment is still self-financing, i.e., borrowing and short selling are not considered. Thus, m_{T-1} can be derived by solving the below quadratic function, in which the subscript $T-1$ is omitted to simplify notations.

$$\begin{aligned}&\left(\widetilde{V} - V\right)^2 \\ &= \int_0^{\frac{\rho}{r^0(1-\beta)}} \left\{ \int_{\frac{\rho - q^{\alpha} h^1}{q^{\alpha}/(1+\alpha)}}^{\frac{\rho - q^{\alpha} h^1}{r^0}} \left[m \left(\rho - g^{\alpha}\right)\left(\rho - g^{\beta}\right) - d^{\alpha}\left(\rho - g^{\alpha}\right)^2\right]^2 \, \mathrm{d}h^0 \right. \\ &\quad + \int_{\frac{\rho - q^{\alpha} h^1}{r^0}}^{\frac{\rho - q^{\beta} h^1}{r^0}} \left[m \left(\rho - g^{\alpha}\right)\left(\rho - g^{\beta}\right)\right]^2 \mathrm{d}h^0 \\ &\quad \left. + \int_{\frac{\rho - q^{\beta} h^1}{r^0}}^{\frac{\rho - r^0(1-\beta) h^1}{r^0}} \left[m \left(\rho - g^{\alpha}\right)\left(\rho - g^{\beta}\right) - d^{\beta}\left(\rho - g^{\beta}\right)^2\right]^2 \mathrm{d}h^0 \right\} \mathrm{d}h^1\end{aligned} \tag{6.23}$$

It is easy to see the strict concavity of (6.23) with respect to m_{T-1}, and so there exists a single optimal m_{T-1} minimizing the cumulative 2-norm distance between $\widetilde{V}_{T-1}$ and V_{T-1}. The expression of the optimal m_{T-1} is given in Appendix F. When α and β are equal to zero, $m_{T-1} = d^{\alpha}_{T-1} = d^{\beta}_{T-1}$.

6.4 Approximate Solution to the Second Last Stage

After deriving the approximate value function for the last period, we go backwards and find the approximate solution to the second last period. Since there is only one value function instead of three for the last period there is no need to determine the asset level at the beginning of the last period in order to find out which value function to use. Using the notations defined in Section 6.1, we can write the approximate model for the second last period as

$$
\begin{aligned}
&\mathrm{APB}_{T-2}[\delta,\lambda]: \\
&V_{T-2} = \max_{x^{1+}_{T-2},x^{1-}_{T-2}\geq 0} \mathrm{E}\left(\widetilde{V}_{T-1}\left(h^0_{T-1},h^1_{T-1}\right)\right) \\
&\quad = \max_{x^{1+}_{T-2},x^{1-}_{T-2}\geq 0} \delta\,\mathrm{E}\left(\left(r^0\right)^2 h^0_{T-2} + R^1_{T-1}R^1_{T-2}h^1_{T-2} + \xi^{\alpha}_{T-2}x^{1+}_{T-2} - \xi^{\beta}_{T-2}x^{1-}_{T-2}\right) \\
&\qquad\qquad -\lambda\,\mathrm{E}\left(\left(r^0\right)^2 h^0_{T-2} + R^1_{T-1}R^1_{T-2}h^1_{T-2} + \xi^{\alpha}_{T-2}x^{1+}_{T-2} - \xi^{\beta}_{T-2}x^{1-}_{T-2}\right)^2 \\
&\qquad\qquad +\lambda m_{T-1}\,\mathrm{E}\left[\left(\rho - r^0 h^0_{T-2} - q^{\alpha}_{T-1}h^1_{T-2} - Q^{\alpha,\alpha}_{T-2,T-1}x^{1+}_{T-2} + Q^{\alpha,\beta}_{T-2,T-1}x^{1-}_{T-2}\right)\right. \\
&\qquad\qquad\qquad \left.\cdot\left(\rho - r^0 h^0_{T-2} - q^{\beta}_{T-1}h^1_{T-2} - Q^{\beta,\alpha}_{T-2,T-1}x^{1+}_{T-2} + Q^{\beta,\beta}_{T-2,T-1}x^{1-}_{T-2}\right)\right]
\end{aligned}
$$

The solution to $\mathrm{APB}_{T-2}[\delta, \lambda]$ is summarized in Table 6.1. Comparing Table 6.1 with Table 5.1, the following similarities between the solutions to the last two periods were observed:

1. For both periods, there are three optimal investment strategies depending on the relationship between ρ and the two thresholds.

2. The expressions for all the parameters in both periods are similar in form.

3. The optimal solution and value function for $t = T - 2$ retain the same structure as that for $t = T - 1$.

From Table 6.1, the value function is again a piecewise function containing three intervals. To continue to analyze the earlier periods, the piecewise value function for $\mathrm{APB}_{T-2}[\delta, \lambda]$ has to be approximated by a general function $\widetilde{V}_{T-2}$ using the same method that has been applied to the last period. The corresponding approximate value function is

$$\begin{aligned}\widetilde{V}_{T-2} =&\delta \,\mathrm{E}\left(\left(r^0\right)^2 h^0_{T-2} + R^1_{T-1} R^1_{T-2} h^1_{T-2}\right) - \lambda \,\mathrm{E}\left(\left(r^0\right)^2 h^0_{T-2} + R^1_{T-1} R^1_{T-2} h^1_{T-2}\right)^2 \\ &+ \lambda \, m_{T-1} \,\mathrm{E}\left[\left(\rho - G^{\alpha}_{T-2}\right)\left(\rho - G^{\beta}_{T-2}\right)\right] + \lambda \, m_{T-2} \left(\rho - g^{\alpha}_{T-2}\right)\left(\rho - g^{\beta}_{T-2}\right)\end{aligned} \tag{6.24}$$

Thus, $\widetilde{V}_{T-2}$ also shows a similar structure as $\widetilde{V}_{T-1}$. In the next section, the results obtained for the last two periods will be extended to an approximation method that can be applied to all time periods.

Table 6.1: Solution to $APB_{T-2}[\delta, \lambda]$

Range of ρ	Solution & value function $V_{T-2}(h^0_{T-2}, h^1_{T-2})$
$\left[g^{\alpha}_{T-2}, +\infty\right)$	$x^{1+}_{T-2} = b^{\alpha}_{T-2}\left(\rho - g^{\alpha}_{T-2}\right), \quad x^{1-}_{T-2} = 0$ $V^{(1)}_{T-2} = \delta\,\mathrm{E}\Big((r^0)^2 h^0_{T-2} + R^1_{T-1}R^1_{T-2}h^1_{T-2}\Big) - \lambda\,\mathrm{E}\Big((r^0)^2 h^0_{T-2} + R^1_{T-1}R^1_{T-2}h^1_{T-2}\Big)^2 + \lambda\, m_{T-1}\,\mathrm{E}\left[\left(\rho - G^{\alpha}_{T-2,T-1}\right)\left(\rho - G^{\beta}_{T-2,T-1}\right)\right] + \lambda\, d^{\alpha}_{T-2}\left(\rho - g^{\alpha}_{T-2}\right)^2$
$\left[g^{\beta}_{T-2}, g^{\alpha}_{T-2}\right)$	$x^{1+}_{T-2} = 0, \quad x^{1-}_{T-2} = 0$ $V^{(2)}_{T-2} = \delta\,\mathrm{E}\Big((r^0)^2 h^0_{T-2} + R^1_{T-1}R^1_{T-2}h^1_{T-2}\Big) - \lambda\,\mathrm{E}\Big((r^0)^2 h^0_{T-2} + R^1_{T-1}R^1_{T-2}h^1_{T-2}\Big)^2 + \lambda\, m_{T-1}\,\mathrm{E}\left[\left(\rho - G^{\alpha}_{T-2,T-1}\right)\left(\rho - G^{\beta}_{T-2,T-1}\right)\right]$
$\left[0, g^{\beta}_{T-2}\right)$	$x^{1+}_{T-2} = 0, \quad x^{1-}_{T-2} = b^{\beta}_{T-2}\left(g^{\beta}_{T-2} - \rho\right)$ $V^{(3)}_{T-2} = \delta\,\mathrm{E}\Big((r^0)^2 h^0_{T-2} + R^1_{T-1}R^1_{T-2}h^1_{T-2}\Big) - \lambda\,\mathrm{E}\Big((r^0)^2 h^0_{T-2} + R^1_{T-1}R^1_{T-2}h^1_{T-2}\Big)^2 + \lambda\, m_{T-1}\,\mathrm{E}\left[\left(\rho - G^{\alpha}_{T-2,T-1}\right)\left(\rho - G^{\beta}_{T-2,T-1}\right)\right] + \lambda\, d^{\beta}_{T-2}\left(\rho - g^{\beta}_{T-2}\right)^2$

6.5 Approximation for the Earlier Periods

The similarities shared by the solutions for the last two periods lead us to make the following proposition:

Proposition 6.1. *Using the approximation method applied to the last two periods, the solution to the approximate problem* $APB_t[\delta, \lambda]$ *is*

$$\begin{cases} x_t^{1+} = b_t^{\alpha}\left(\rho - g_t^{\alpha}\right), x_t^{1-} = 0 & \text{if} \quad \rho \geq g_t^{\alpha} \\ x_t^{1+} = 0, x_t^{1-} = 0 & \text{if} \quad g_t^{\beta} \leq \rho < g_t^{\alpha} \\ x_t^{1+} = 0, x_t^{1-} = b_t^{\beta}\left(g_t^{\beta} - \rho\right) & \text{if} \quad \rho < g_t^{\beta} \end{cases}$$

The corresponding approximate value function is

$$\begin{aligned} \widetilde{V}_t(h_t^0, h_t^1) = & \delta \, \mathrm{E}\left((r^0)^{(T-t)} h_t^0 + \prod_{i=t}^{T-1} R_i^1 h_t^1 \right) - \lambda \, \mathrm{E}\left((r^0)^{(T-t)} h_t^0 + \prod_{i=t}^{T-1} R_i^1 h_t^1 \right)^2 \\ & + \lambda \sum_{i=t+1}^{T-1} m_i \, \mathrm{E}\left[\left(\rho - G_{t,i}^{\alpha}\right) \left(\rho - G_{t,i}^{\beta}\right) \right] + \lambda \, m_t \left(\rho - g_t^{\alpha}\right) \left(\rho - g_t^{\beta}\right) \end{aligned} \tag{6.25}$$

for $t = 0, 1, \ldots, T-1$.

Proof. We will use mathematical induction to prove this proposition.

Proposition 6.1 is already shown to be true for $t = T-1$ and $t = T-2$ in sections 6.3 and 6.4 respectively.

Assume that Proposition 6.1 holds for t where $t \in \{2, \ldots, T-1\}$. Then

$$
\begin{aligned}
V_{t-1} =& \max\Big\{ \mathrm{E}(\widetilde{V}_t) \Big\} \\
=& \max\Bigg\{ \delta\, \mathrm{E}\Bigg((r^0)^{(T-t+1)} h_t^0 + \prod_{i=t-1}^{T-1} R_i^1 h_t^1 + \xi_{t-1}^{\alpha} x_{t-1}^{1+} - \xi_{t-1}^{\beta} x_{t-1}^{1-} \Bigg) \\
& - \lambda\, \mathrm{E}\Bigg((r^0)^{(T-t+1)} h_t^0 + \prod_{i=t-1}^{T-1} R_i^1 h_t^1 + \xi_{t-1}^{\alpha} x_{t-1}^{1+} - \xi_{t-1}^{\beta} x_{t-1}^{1-} \Bigg)^2 \\
& + \lambda\, m_{T-1}\, \mathrm{E}\Bigg[\Big(\rho - G_{t-1,T-1}^{\alpha} - Q_{t-1,T-1}^{\alpha,\alpha} x_{t-1}^{1+} - Q_{t-1,T-1}^{\alpha,\beta} x_{t-1}^{1-} \Big) \\
& \qquad \cdot \Big(\rho - G_{t-1,T-1}^{\beta} - Q_{t-1,T-1}^{\beta,\alpha} x_{t-1}^{1+} - Q_{t-1,T-1}^{\beta,\beta} x_{t-1}^{1-} \Big) \Bigg] \\
& + \lambda\, m_{T-2}\, \mathrm{E}\Bigg[\Big(\rho - G_{t-1,T-2}^{\alpha} - Q_{t-1,T-2}^{\alpha,\alpha} x_{t-1}^{1+} - Q_{t-1,T-2}^{\alpha,\beta} x_{t-1}^{1-} \Big) \\
& \qquad \cdot \Big(\rho - G_{t-1,T-2}^{\beta} - Q_{t-1,T-1}^{\beta,\alpha} x_{t-1}^{1+} - Q_{t-1,T-2}^{\beta,\beta} x_{t-1}^{1-} \Big) \Bigg] \\
& \ldots \\
& + \lambda\, m_t\, \mathrm{E}\Bigg[\Big(\rho - G_{t-1,t}^{\alpha} - Q_{t-1,t}^{\alpha,\alpha}\, x_{t-1}^{1+} - Q_{t-1,t}^{\alpha,\beta}\, x_{t-1}^{1-} \Big) \\
& \qquad \cdot \Big(\rho - G_{t-1,t}^{\beta} - Q_{t-1,t}^{\beta,\alpha}\, x_{t-1}^{1+} - Q_{t-1,t}^{\beta,\beta}\, x_{t-1}^{1-} \Big) \Bigg] \Bigg\}. \qquad (6.26)
\end{aligned}
$$

Since the approximate objective function for $t-1$ is concave with respect to x_t^{1+} and x_t^{1-}, we can obtain the following solutions by solving the KKT conditions:

1. When $\frac{\delta}{2\lambda} \geq g^{\alpha}_{t-1}$, the optimal solution is

$$
\begin{aligned}
x^{1+}_{t-1} =& \left(\frac{\mathrm{E}(\xi^{\alpha}_{t-1}) - \dfrac{1}{2}\displaystyle\sum_{j=t}^{T-1} m_i \,\mathrm{E}\left(Q^{\alpha,\alpha}_{t-1,i} + Q^{\beta,\alpha}_{t-1,i}\right)}{\mathrm{E}(\xi^{\alpha}_{t-1})^2 - \displaystyle\sum_{i=t}^{T-1} m_i \,\mathrm{E}\left(Q^{\alpha,\alpha}_{t-1,i} Q^{\beta,\alpha}_{t-1,i}\right)} \right) \\
& \cdot \left(\frac{\delta}{2\lambda} - (r^0)^{(T-t+1)} h^0_{t-1} - (r^0)^{(T-t+1)}(1+\alpha) h^1_{t-1} \right) - h^1_{t-1} \\
=& \, b^{\alpha}_{t-1} \left(\frac{\delta}{2\lambda} - g^{\alpha}_{t-1} \right) \qquad (6.27) \\
x^{1-}_{t-1} =& \, 0. \qquad (6.28)
\end{aligned}
$$

The value function is

$$
\begin{aligned}
V_{t-1} =& \delta \,\mathrm{E}\left((r^0)^{(T-t+1)} h^0_{T-1} + \prod_{i=t-1}^{T-1} R^1_i h^1_t \right) \\
& - \lambda \,\mathrm{E}\left((r^0)^{(T-t+1)} h^0_{T-1} + \prod_{i=t-1}^{T-1} R^1_i h^1_t \right)^2 \\
& + \lambda \, m_{T-1} \,\mathrm{E}\left[\left(\frac{\delta}{2\lambda} - G^{\alpha}_{t-1,T-1} \right) \left(\frac{\delta}{2\lambda} - G^{\beta}_{t-1,T-1} \right) \right] \\
& + \lambda \, m_{T-2} \,\mathrm{E}\left[\left(\frac{\delta}{2\lambda} - G^{\alpha}_{t-1,T-2} \right) \left(\frac{\delta}{2\lambda} - G^{\beta}_{t-1,T-2} \right) \right] \\
& \quad \cdots \\
& + \lambda \, m_t \,\mathrm{E}\left[\left(\frac{\delta}{2\lambda} - G^{\alpha}_{t-1,t} \right) \left(\frac{\delta}{2\lambda} - G^{\beta}_{t-1,t} \right) \right] \\
& + \lambda \, d^{\alpha}_{t-1} \left(\frac{\delta}{2\lambda} - g^{\alpha}_{t-1} \right)^2 \qquad (6.29)
\end{aligned}
$$

2. When $g^{\beta}_{t-1} \le \frac{\delta}{2\lambda} < g^{\alpha}_{t-1}$, the optimal solution is

$$x^{1+}_{t-1} = 0 \tag{6.30}$$

$$x^{1-}_{t-1} = 0. \tag{6.31}$$

The value function is

$$\begin{aligned} V_{t-1} =& \delta \operatorname{E}\left(\left(r^0\right)^{(T-t+1)} h^0_{T-1} + \prod_{i=t-1}^{T-1} R^1_i h^1_t \right) \\ & - \lambda \operatorname{E}\left(\left(r^0\right)^{(T-t+1)} h^0_{T-1} + \prod_{i=t-1}^{T-1} R^1_i h^1_t \right)^2 \\ & + \lambda\, m_{T-1} \operatorname{E}\left[\left(\frac{\delta}{2\lambda} - G^{\alpha}_{t-1,T-1} \right) \left(\frac{\delta}{2\lambda} - G^{\beta}_{t-1,T-1} \right) \right] \\ & + \lambda\, m_{T-2} \operatorname{E}\left[\left(\frac{\delta}{2\lambda} - G^{\alpha}_{t-1,T-2} \right) \left(\frac{\delta}{2\lambda} - G^{\beta}_{t-1,T-2} \right) \right] \\ & \ldots \\ & + \lambda\, m_t \operatorname{E}\left[\left(\frac{\delta}{2\lambda} - G^{\alpha}_{t-1,t} \right) \left(\frac{\delta}{2\lambda} - G^{\beta}_{t-1,t} \right) \right] \end{aligned} \tag{6.32}$$

3. When $0 \leq \frac{\delta}{2\lambda} < g_{t-1}^{\beta}$, the optimal solution is

$$x_{t-1}^{1+} = 0 \tag{6.33}$$

$$\begin{aligned} x_{t-1}^{1-} &= h_{t-1}^{1} - \left(\frac{\mathrm{E}(\xi_{t-1}^{\beta}) - \frac{1}{2} \sum_{j=t}^{T-1} m_i \,\mathrm{E}\left(Q_{t-1,i}^{\alpha,\beta} + Q_{t-1,i}^{\beta,\beta}\right)}{\mathrm{E}(\xi_{t-1}^{\beta})^2 - \sum_{i=t}^{T-1} m_i \,\mathrm{E}\left(Q_{t-1,i}^{\alpha,\beta} Q_{t-1,i}^{\beta,\beta}\right)} \right) \\ &\quad \cdot \left(\frac{\delta}{2\lambda} - (r^0)^{(T-t+1)} h_{t-1}^{0} - (r^0)^{(T-t+1)} (1-\beta) h_{t-1}^{1} \right) \\ &= b_{t-1}^{\beta} \left(g_{t-1}^{\beta} - \frac{\delta}{2\lambda} \right). \end{aligned} \tag{6.34}$$

The value function is

$$\begin{aligned} V_{t-1} =& \delta \,\mathrm{E}\left((r^0)^{(T-t+1)} h_{T-1}^{0} + \prod_{i=t-1}^{T-1} R_i^1 h_t^1 \right) \\ & - \lambda \,\mathrm{E}\left((r^0)^{(T-t+1)} h_{T-1}^{0} + \prod_{i=t-1}^{T-1} R_i^1 h_t^1 \right)^2 \\ & + \lambda \, m_{T-1} \,\mathrm{E}\left[\left(\frac{\delta}{2\lambda} - G_{t-1,T-1}^{\alpha} \right) \left(\frac{\delta}{2\lambda} - G_{t-1,T-1}^{\beta} \right) \right] \\ & + \lambda \, m_{T-2} \,\mathrm{E}\left[\left(\frac{\delta}{2\lambda} - G_{t-1,T-2}^{\alpha} \right) \left(\frac{\delta}{2\lambda} - G_{t-1,T-2}^{\beta} \right) \right] \\ & \quad \cdots \\ & + \lambda \, m_{t} \,\mathrm{E}\left[\left(\frac{\delta}{2\lambda} - G_{t-1,t}^{\alpha} \right) \left(\frac{\delta}{2\lambda} - G_{t-1,t}^{\beta} \right) \right] \\ & + \lambda \, d_{t-1}^{\beta} \left(\frac{\delta}{2\lambda} - g_{t-1}^{\beta} \right)^2 \end{aligned} \tag{6.35}$$

From (6.29), (6.32), and (6.35), the value function can be concluded as

$$
\begin{aligned}
V_{t-1} = {} & \delta \, \mathrm{E} \left((r^0)^{(T-t+1)} h_t^0 + \prod_{i=t-1}^{T-1} R_i^1 h_t^1 \right) - \lambda \, \mathrm{E} \left((r^0)^{(T-t+1)} h_t^0 + \prod_{i=t-1}^{T-1} R_i^1 h_t^1 \right)^2 \\
& + \lambda \, m_{T-1} \, \mathrm{E} \left[\left(\rho - G_{t-1,T-1}^{\alpha} \right) \left(\rho - G_{t-1,T-1}^{\beta} \right) \right] \\
& + \lambda \, m_{T-2} \, \mathrm{E} \left[\left(\rho - G_{t-1,T-2}^{\alpha} \right) \left(\rho - G_{t-1,T-2}^{\beta} \right) \right] \\
& \quad \cdots \\
& + \lambda \, m_t \, \mathrm{E} \left[\left(\rho - G_{t-1,t}^{\alpha} \right) \left(\rho - G_{t-1,t}^{\beta} \right) \right] \\
& + \hat{V}_{t-1} \qquad (6.36)
\end{aligned}
$$

where

$$
\hat{V}_{t-1} = \begin{cases} \lambda \, d_{t-1}^{\alpha} \left(\rho - g_{t-1}^{\alpha} \right)^2 & \text{when} \quad \rho \geq g_{t-1}^{\alpha} \\ 0 & \text{when} \quad g_{t-1}^{\beta} \leq \rho < g_{t-1}^{\alpha} \\ \lambda \, d_{t-1}^{\beta} \left(\rho - g_{t-1}^{\beta} \right)^2 & \text{when} \quad 0 \leq \rho < g_{t-1}^{\beta} \end{cases} \qquad (6.37)
$$

Using the earlier approximation method for the piecewise value function V_{t-1}, the approximate value function for $t-1$ is

$$
\begin{aligned}
\widetilde{V}_{t-1} = {} & \delta \, \mathrm{E} \left((r^0)^{(T-t+1)} h_t^0 + \prod_{i=t-1}^{T-1} R_i^1 h_t^1 \right) - \lambda \, \mathrm{E} \left((r^0)^{(T-t+1)} h_t^0 + \prod_{i=t-1}^{T-1} R_i^1 h_t^1 \right)^2 \\
& + \lambda \sum_{i=t}^{T-1} m_i \, \mathrm{E} \left[\left(\rho - G_{t-1,i}^{\alpha} \right) \left(\rho - G_{t-1,i}^{\beta} \right) \right] \\
& + \lambda \, m_{t-1} \left(\rho - g_{t-1}^{\alpha} \right) \left(\rho - g_{t-1}^{\beta} \right) \qquad (6.38)
\end{aligned}
$$

Thus we have shown that if Proposition 6.1 is true for t where $t \in \{2, \ldots, T-1\}$, it also holds for $t-1$. Therefore, it can be concluded that Proposition 6.1 is true for all time periods. $\square$

6.6 The Value of δ

In the previous section, we have derived the general expression of the approximate solution for every time period. However, the value of δ remains to be determined. Equation (3.13) indicates that the value of δ is related to the value of $\mathrm{E}(w_T^*)$, where w_T^* is the final wealth at the end of the entire horizon under the optimal investment policy. Now let us study the value of $\mathrm{E}(w_T^*)$ in our approximate model.

Starting with the last period, it is being treated as a one-period problem. With the optimal solution provided in Table 5.1, We can calculate the corresponding expected final wealth at time $T-1$, which is denoted by $\mathrm{E}_{T-1}(w_T)$. The results are listed in Table 6.2.

Table 6.2: Expected final wealth for the last period

Range of ρ	Expected final wealth $\mathrm{E}_{T-1}(w_T)$
$\left[g_{T-1}^{\alpha}, +\infty\right)$	$\mathrm{E}_{T-1}(w_T)^{(1)} = r^0 h_{T-1}^0 + \mathrm{E}(R_{T-1}^1) h_{T-1}^1 + d_{T-1}^{\alpha}\left(\rho - g_{T-1}^{\alpha}\right)$
$\left[g_{T-1}^{\beta}, g_{T-1}^{\alpha}\right)$	$\mathrm{E}_{T-1}(w_T)^{(2)} = r^0 h_{T-1}^0 + \mathrm{E}(R_{T-1}^1) h_{T-1}^1$
$\left[0, g_{T-1}^{\beta}\right)$	$\mathrm{E}_{T-1}(w_T)^{(3)} = r^0 h_{T-1}^0 + \mathrm{E}(R_{T-1}^1) h_{T-1}^1 + d_{T-1}^{\beta}\left(\rho - g_{T-1}^{\beta}\right)$

Similar to the value function for the last period, the function of the expected final wealth also contains three intervals as shown in Table 6.2. For portfolio selection with more time periods, the closed-form of $\mathrm{E}_t(w_T)$ cannot be obtained. Therefore, we will use an approximation that replaces the actual function with three intervals by a general linear function for the expected wealth in order to obtain an approximate $\mathrm{E}_t(\widetilde{w}_T)$ recursively. $\mathrm{E}_t(\widetilde{w}_T)$ represents the approximate expected final wealth at t. For the last period, we define

$$\mathrm{E}_{T-1}(\widetilde{w}_T) = r^0 h^0_{T-1} + \mathrm{E}(R^1_{T-1}) h^1_{T-1} + m_{T-1}\left(\rho - \frac{1}{2}\left(g^{\alpha}_{T-1} + g^{\beta}_{t-1}\right)\right) \quad (6.39)$$

The first part of $\mathrm{E}_{T-1}(\widetilde{w}_T)$, $r^0 h^0_{T-1} + \mathrm{E}(R^1_{T-1}) h^1_{T-1}$, comes from the exact expected final wealth function as it is a common expression shared by the three intervals. The second part of $\mathrm{E}_{T-1}(\widetilde{w}_T)$ contains $\rho - \frac{1}{2}\left(g^{\alpha}_{t-1} + g^{\beta}_{t-1}\right)$, which is the arithmetic mean of $(\rho - g^{\alpha}_{T-1})$ in $\mathrm{E}_{T-1}(w_T)^{(1)}$ and $(\rho - g^{\beta}_{T-1})$ in $\mathrm{E}_{T-1}(w_T)^{(3)}$, and m_{T-1} is the corresponding coefficient.

Replacing h^0_{T-1} and h^1_{T-1} in (6.39) with the recursive rebalancing equations (3.2) and (3.3), we obtain

$$\begin{aligned}\mathrm{E}_{T-2}(\widetilde{w}_T) = &(r^0)^2\left(h^0_{T-2} - (1+\alpha)x^{1+}_{T-2} + (1-\beta)x^{1-}_{T-2}\right)\\ &+ \mathrm{E}(R^1_{T-1})\,\mathrm{E}(R^1_{T-2})\left(h^1_{T-2} + x^{1+}_{T-2} - x^{1-}_{T-2}\right)\\ &+ m_{T-1}\left(\rho - (r^0)^2\left(h^0_{T-2} - (1+\alpha)x^{1+}_{T-2} + (1-\beta)x^{1-}_{T-2}\right)\right.\\ &\left.-\frac{q^{\alpha}_{T-1} + q^{\beta}_{T-1}}{2}\,\mathrm{E}(R^1_{T-2})\left(h^1_{T-2} + x^{1+}_{T-2} - x^{1-}_{T-2}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= (r^0)^2 h^0_{T-2} + \mathrm{E}(R^1_{T-1})\,\mathrm{E}(R^1_{T-2}) h^1_{T-2} \\
&\quad + m_{T-1}\left(\rho - (r^0)^2 h^0_{T-2} - \frac{q^\alpha_{T-1} + q^\beta_{T-1}}{2}\,\mathrm{E}(R^1_{T-2}) h^1_{T-2}\right) \\
&\quad + \left(\mathrm{E}(\xi^\alpha_{T-2}) - \frac{1}{2}\,\mathrm{E}\left(Q^{\alpha,\alpha}_{T-2,T-1} + Q^{\beta,\alpha}_{T-2,T-1}\right)\right) x^{1+}_{T-2} \\
&\quad - \left(\mathrm{E}(\xi^\beta_{T-2}) - \frac{1}{2}\,\mathrm{E}\left(Q^{\alpha,\beta}_{T-2,T-1} + Q^{\beta,\beta}_{T-2,T-1}\right)\right) x^{1-}_{T-2} \qquad (6.40)
\end{aligned}
$$

After substituting the optimal solution for $\mathrm{APB}_{T-2}(\delta, \lambda)$ presented in Table 6.1, the approximate expected final wealth for $T-2$ is as shown in Table 6.3.

Table 6.3: Approximate expected final wealth for the second last period

Range of ρ	Solution & value function $\mathrm{E}(w_{T-2}(h^0_{T-2}, h^1_{T-2})$
$\left[g^\alpha_{T-2}, +\infty\right)$	$\mathrm{E}_{T-2}(\widetilde{w}_T)^{(1)} = (r^0)^2 h^0_{T-2} + \mathrm{E}(R^1_{T-1})\,\mathrm{E}(R^1_{T-2}) h^1_{T-2}$ $+m_{T-1}\,\mathrm{E}\left(\rho - \frac{1}{2}\left(G^\alpha_{T-2,T-1} + G^\beta_{T-2,T-1}\right)\right)$ $+d^\alpha_{T-2}\left(\rho - g^\alpha_{T-2}\right)$
$\left[g^\beta_{T-2}, g^\alpha_{T-2}\right)$	$\mathrm{E}_{T-2}(\widetilde{w}_T)^{(2)} = (r^0)^2 h^0_{T-2} + \mathrm{E}(R^1_{T-1})\,\mathrm{E}(R^1_{T-2}) h^1_{T-2}$ $+m_{T-1}\,\mathrm{E}\left(\rho - \frac{1}{2}\left(G^\alpha_{T-2,T-1} + G^\beta_{T-2,T-1}\right)\right)$
$\left[0, g^\beta_{T-2}\right)$	$\mathrm{E}_{T-2}(\widetilde{w}_T)^{(3)} = (r^0)^2 h^0_{T-2} + \mathrm{E}(R^1_{T-1})\,\mathrm{E}(R^1_{T-2}) h^1_{T-2}$ $+m_{T-1}\,\mathrm{E}\left(\rho - \frac{1}{2}\left(G^\alpha_{T-2,T-1} + G^\beta_{T-2,T-1}\right)\right)$ $+d^\beta_{T-2}\left(\rho - g^\beta_{T-2}\right)$

Using the same method applied to the last period to approximate the piecewise function of the expected final wealth then yields

$$\begin{aligned}
\mathrm{E}_{T-2}(\widetilde{w}_T) =& (r^0)^2 h^0_{T-2} + \mathrm{E}(R^1_{T-1})\,\mathrm{E}(R^1_{T-2}) h^1_{T-2} \\
&+ m_{T-1}\,\mathrm{E}\left(\rho - \frac{1}{2}\left(G^{\alpha}_{T-2,T-1} + G^{\beta}_{T-2,T-1}\right)\right) \\
&+ m_{T-2}\left(\rho - \frac{1}{2}\left(g^{\alpha}_{T-2} + g^{\beta}_{T-2}\right)\right) \qquad (6.41)
\end{aligned}$$

The expression retains the same structure from the last period. Based on the observed structure of $\mathrm{E}_t(\widetilde{w}_T)$, the following proposition can be obtained:

Proposition 6.2. *The approximate expected final wealth at time t is*

$$\begin{aligned}
\mathrm{E}_t(\widetilde{w}_T) =& (r^0)^{(T-t)} h^0_t + \prod_{i=t}^{T-1} \mathrm{E}(R^1_i) h^1_t \\
&+ \sum_{i=t+1}^{T-1} m_i \left(\rho - \frac{1}{2}\,\mathrm{E}(G^{\alpha}_{t,i} + G^{\beta}_{t,i})\right) + m_t\left(\rho - \frac{1}{2}\left(g^{\alpha}_t + g^{\beta}_t\right)\right)
\end{aligned} \qquad (6.42)$$

Proof. We will again use mathematical induction to prove Proposition 6.2.

Proposition 6.2 is already shown to be true for $t = T - 1$ and $t = T - 2$.

Assume that Proposition 6.2 holds for t, where $t \in \{1, 2, \ldots, T-1\}$.

Substituting (3.2) and (3.3) into (6.42) yields

$$
\begin{aligned}
\mathrm{E}_t(\widetilde{w}_T) = {} & (r^0)^{(T-t+1)} \left(h_{t-1}^0 - (1+\alpha)x_{t-1}^{1+} + (1-\beta)x_{t-1}^{1-}\right) \\
& + \prod_{i=t-1}^{T-1} \mathrm{E}(R_i^1) \left(h_{t-1}^1 + x_{t-1}^{1+} - x_{t-1}^{1-}\right) \\
& + \sum_{i=t+1}^{T-1} m_i \Bigg(\rho - (r^0)^{(T-t+1)} \left(h_{t-1}^0 - (1+\alpha)x_{t-1}^{1+} + (1-\beta)x_{t-1}^{1-}\right) \\
& \qquad - \frac{q_i^\alpha + q_i^\beta}{2} \prod_{j=t-1}^{i-1} \mathrm{E}(R_j^1) \left(h_{t-1}^1 + x_{t-1}^{1+} - x_{t-1}^{1-}\right)\Bigg) \\
& + m_t \Bigg(\rho - (r^0)^{(T-t+1)} \left(h_{t-1}^0 - (1+\alpha)x_{t-1}^{1+} + (1-\beta)x_{t-1}^{1-}\right) \\
& \qquad - \frac{q_t^\alpha + q_t^\beta}{2} \mathrm{E}(R_{t-1}^1) \left(h_{t-1}^1 + x_{t-1}^{1+} - x_{t-1}^{1-}\right)\Bigg) \\
= {} & (r^0)^{(T-t+1)} h_{t-1}^0 + \prod_{i=t-1}^{T-1} \mathrm{E}(R_i^1) h_{t-1}^1 \\
& + \sum_{i=t}^{T-1} m_i \left(\rho - \frac{1}{2}\mathrm{E}(G_{t-1,i}^\alpha + G_{t-1,i}^\beta)\right) \\
& + \left(\mathrm{E}(\xi_{t-1}^\alpha) - \frac{1}{2}\sum_{i=t}^{T-1} m_i \,\mathrm{E}\left(Q_{t-1,i}^{\alpha,\alpha} + Q_{t-1,i}^{\beta,\alpha}\right)\right) x_{t-1}^{1+} \\
& - \left(\mathrm{E}(\xi_{t-1}^\beta) - \frac{1}{2}\sum_{i=t}^{T-1} m_i \,\mathrm{E}\left(Q_{t-1,i}^{\alpha,\beta} + Q_{t-1,i}^{\beta,\beta}\right)\right) x_{t-1}^{1-}
\end{aligned}
\tag{6.43}
$$

By further substituting the solution for $t-1$, we can then obtain the three intervals for $\mathrm{E}_{t-1}(\widetilde{w}_T)^{(1)}$, which are shown in Table 6.4. Using a general function

Table 6.4: Approximate expected final wealth for $t-1$

Range of ρ	Solution & value function $\mathrm{E}(w_{t-1}(h^0_{t-1}, h^1_{t-1})$
$\left[g^{\alpha}_{t-1}, +\infty\right)$	$\mathrm{E}_{t-1}(\widetilde{w}_T)^{(1)} = (r^0)^{(T-t+1)} h^0_{t-1} + \prod_{i=t-1}^{T-1} \mathrm{E}(R^1_i) h^1_{t-1}$ $+\sum_{i=t}^{T-1} m_i \left(\rho - \frac{1}{2}\mathrm{E}\left(G^{\alpha}_{t-1,i} + G^{\beta}_{t-1,i}\right)\right)$ $+d^{\alpha}_{t-1}\left(\rho - g^{\alpha}_{t-1}\right)$
$\left[g^{\beta}_{t-1}, g^{\alpha}_{t-1}\right)$	$\mathrm{E}_{t-1}(\widetilde{w}_T)^{(2)} = (r^0)^{(T-t+1)} h^0_{t-1} + \prod_{i=t-1}^{T-1} \mathrm{E}(R^1_i) h^1_{t-1}$ $+\sum_{i=t}^{T-1} m_i \left(\rho - \frac{1}{2}\mathrm{E}\left(G^{\alpha}_{t-1,i} + G^{\beta}_{t-1,i}\right)\right)$
$\left[0, g^{\beta}_{t-1}\right)$	$\mathrm{E}_{t-1}(\widetilde{w}_T)^{(3)} = (r^0)^{(T-t+1)} h^0_{t-1} + \mathrm{E}(R^1_{T-1})\,\mathrm{E}(R^1_{t-1}) h^1_{t-1}$ $+\sum_{i=t}^{T-1} m_i \left(\rho - \frac{1}{2}\mathrm{E}\left(G^{\alpha}_{t-1,i} + G^{\beta}_{t-1,i}\right)\right)$ $+d^{\beta}_{t-1}\left(\rho - g^{\beta}_{t-1}\right)$

to approximate the piece-wise function of $\mathrm{E}_{t-1}(\widetilde{w}_T)^{(1)}$ yields

$$\mathrm{E}_{t-1}(\widetilde{w}_T) = (r^0)^{(T-t+1)} h^0_{t-1} + \prod_{i=t-1}^{T-1} \mathrm{E}(R^1_i) h^1_{t-1} + \sum_{i=t}^{T-1} m_i \left(\rho - \frac{1}{2}\mathrm{E}(G^{\alpha}_{t-1,i} + G^{\beta}_{t-1,i})\right) + m_{t-1}\left(\rho - \frac{1}{2}\left(g^{\alpha}_{t-1} + g^{\beta}_{t-1}\right)\right) \tag{6.44}$$

Thus, Proposition 6.2 also holds for $t-1$ and this concludes the proof. □

Combining (6.21), (3.13) and (6.42) yields the approximate δ:

$$\widetilde{\delta} = \frac{1 + 2\lambda\left((r^0)^T h^0_0 + \prod_{i=0}^{T-1} \mathrm{E}(R^1_i) h^1_0\right) - \lambda \sum_{i=1}^{T-1} m_i \,\mathrm{E}(G^{\alpha}_{0,i} + G^{\beta}_{0,i}) - \lambda(g^{\alpha}_0 + g^{\beta}_0)}{1 - \sum_{i=0}^{T-1} m_i} \tag{6.45}$$

This completes the description of our proposed approximation method for solving the multiperiod mean-variance portfolio optimization problem. The main advantage of such an approximation is that it retains the structure of the optimal solutions and value functions at each period. This allows the problem to be solved recursively.

6.7 Numerical Experiments

In this section, results of numerical experiments are presented to show the performance of the proposed approximation method. We use a four-month investment problem to compare the investment policy obtained by the approximation method with the actual optimal policy. Here, one period is set to be one month. At the beginning of each month, the portfolio will be reviewed and rebalanced to maximize the final mean-variance utility. The rate of return of the risky asset is assumed to follow a binary distribution. At each period, the return rate of the risky asset has two possible outcomes r_u and r_d. r_u represents the return rate of the risky asset in the scenario that its price goes up and the r_d indicates the corresponding return rate when its price goes down. Figure 6.1 illustrates such a process. At time t, there are 2^t possible scenarios in total. A node in the scenario tree can be located by its time and scenario. For instance, node (2,4) indicates the fourth scenario at $t = 2$. Assuming that the random return rate is symmetrically distributed, the probabilities of the realization of the two outcomes are both equal to 50%. For the risk-free asset, we assume that it has a constant monthly return denoted by r_0.

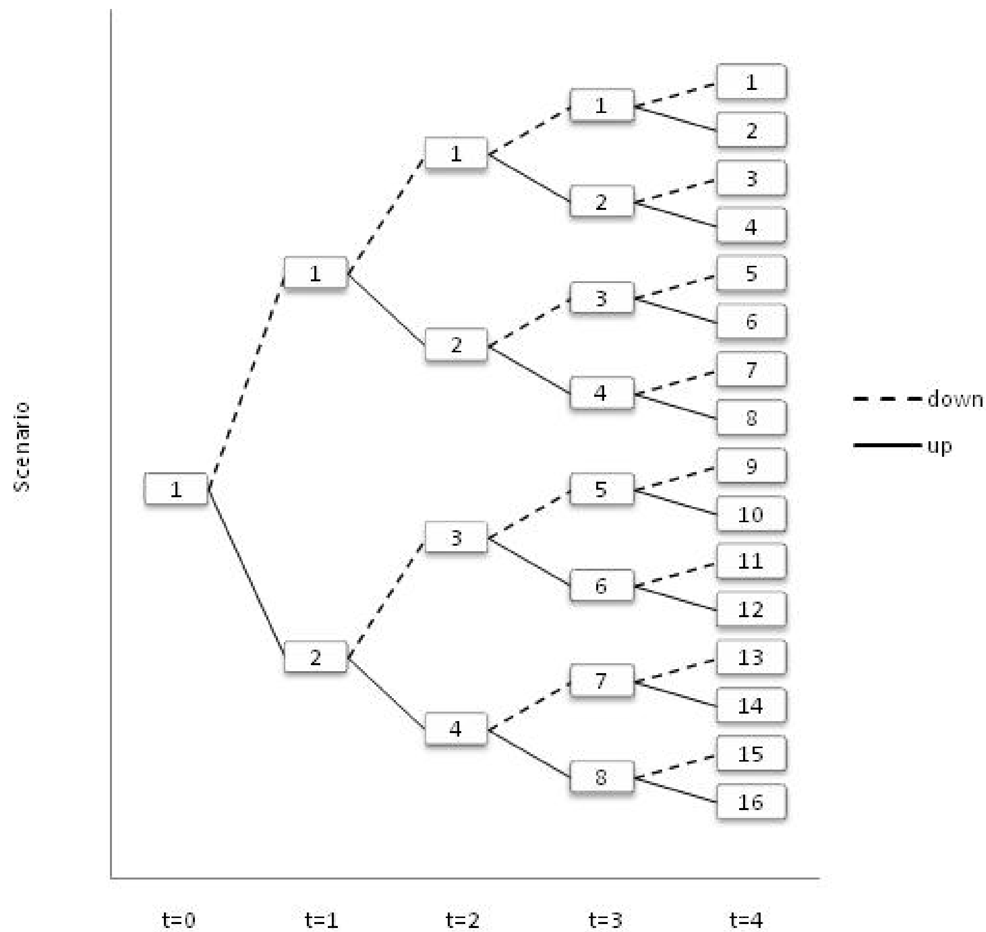


Figure 6.1: Four-period two-branch scenario tree

We will use the mean and standard deviation of the S&P500 annualized returns from year 1928 to 2010 as the inputs of the risky asset, and the Treasury Bill as the risk-free asset. The annual inputs are

Average annual return of Treasury Bill:	$r_{[Tbill]}$	$= 1.0370$
Average annual return of S&P500:	$r_{[SP500]}$	$= 1.1165$
Average std of the annual return of S&P500:	$\sigma_{[SP500]}$	$= 0.2069$

Thus, the monthly returns and standard deviation are

$$r_0 = 1.0030 \tag{6.46}$$

$$\mathrm{E}(R_i) = 1.0092 \tag{6.47}$$

$$\sigma = 0.0597 \tag{6.48}$$

$$r_u = 1.0690 \tag{6.49}$$

$$r_d = 0.9495 \tag{6.50}$$

The initial holdings of the risk-free asset and the risky asset are assumed to be $h_0^0 = 10$ and $h_0^1 = 0$ respectively. In order to evaluate the performance of the proposed approximation method for investment under different conditions, experiments were performed under different assumptions of transaction fee rates, i.e., 0%, 0.3%, 0.5% and 1% transaction costs. In these four situations, the approximation method is compared with the method ignoring transaction costs, with the optimal results as the benchmark. Later, the proposed approximation method is also compared with the optimal method in terms of the solving efficiency.

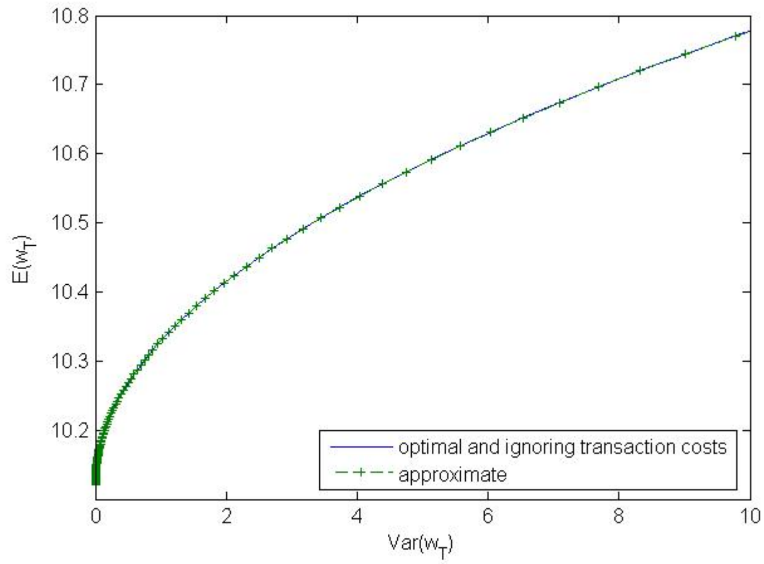


Figure 6.2: Efficient frontier at $t = 0$ for $\alpha = \beta = 0\%$

6.7.1 Investment with no transaction cost

We start with the special case when there is no transaction cost. Figure 6.2 plots the efficient frontiers at the beginning of the investment horizon. For this special case, the method ignoring transaction costs provides the optimal solution. The two frontiers in Figure 6.2 are obtained by (1) solving the multiperiod mean-variance model ignoring transaction costs (it is also the optimal frontier) and (2) using the proposed approximation method. As shown in Figure 6.2, the three methods generate identical efficient frontiers.

It is shown that our approximation method solves the multiperiod portfolio optimization problem exactly.

Proposition 6.3. *Under the assumption that there is no transaction cost, the approximation method provides the exact solutions.*

Proof. When there is no transaction cost, i.e., $\alpha = \beta = 0$, the value function for the last period of the multiperiod portfolio optimization problem is

$$\begin{aligned}\mathring{V}_{T-1} = \delta\,\mathrm{E}\left(r^0 h^0_{T-1} + R^1_{T-1} h^1_{T-1}\right) - \lambda\,\mathrm{E}\left(r^0 h^0_{T-1} + R^1_{T-1} h^1_{T-1}\right)^2 \\ +\lambda \mathring{d}_{T-1}\left[\rho - r^0 h^0_{T-1} - \mathring{q}_{T-1} h^1_{T-1}\right]^2\end{aligned} \tag{6.51}$$

where $\mathring{d}_{T-1} = \frac{\mathrm{E}^2(R^1_{T-1}-r^0)}{\mathrm{E}(R^1_{T-1}-r^0)^2}$ and $\mathring{q}_{T-1} = \left(\frac{\mathrm{E}^2(R^1_{T-1}-r^0)}{\mathrm{E}(R^1_{T-1}-r^0)} + r^0\right)$. By solving recursively, the value function for period t is

$$\begin{aligned}\mathring{V}_t =& \delta\,\mathrm{E}\left(\left(r^0\right)^{(T-t)} h^0_t + \prod_{i=t}^{T-1} R^1_i h^1_t\right) - \lambda\,\mathrm{E}\left(\left(r^0\right)^{(T-t)} h^0_t + \prod_{i=t}^{T-1} R^1_i h^1_t\right)^2 \\ &+ \lambda \sum_{i=t+1}^{T-1} \mathring{d}_i\,\mathrm{E}\left(\rho - \mathring{G}_{t,i}\right)^2 + \lambda\,\mathring{d}_t\left(\rho - \mathring{g}_t\right)^2\end{aligned} \tag{6.52}$$

where $\mathring{G}_{t,i} = \left(r^0\right)^{(T-t)} h^0_t + \mathring{q}_s \prod_{j=t}^{s-1} R^1_j h^1_t$ and $\mathring{g}_t = \left(r^0\right)^{(T-t)} h^0_t + \mathring{q}_t h^1_t$.

In the case of no transaction cost, the proposed approximation method generates $\mathring{m}_t = \mathring{d}_t$ (see Appendix F). Therefore, the three models are equivalent when there is no transaction cost. □

6.7.2 Investment with Transaction Costs

Consider the case in which the investors have to pay transaction fees whenever there is purchasing or selling of the risky asset. The results obtained by the proposed approximation method are compared with the optimal results and the results obtained by ignoring the transaction costs. In addition, the mean-variance efficient frontiers for the three cases are plotted as well.

Figure 6.3 shows the efficient frontiers at 0.3% transaction fee rate. In this case, the transaction fees are low and therefore, efficient frontiers for the three methods, i.e., the optimal solution, the approximate solution and the solution ignoring transaction costs, are close to each other. The extra costs incurred due to the active rebalancing are negligible, and so ignoring transaction costs has little impact on the investment performance for this case.

In Figure 6.4 where the transaction fee rate increases to 0.5%, it can be seen that the proposed approximation method generates an almost optimal efficient frontier, while the difference between the efficient frontiers for the optimal solution and that of ignoring transaction costs becomes larger. This is because the increase in transaction costs results in less frequent trading activities in the optimal solution. However, if transaction costs are ignored, trading of assets occurs at every time period. This results in more undesired transactions and thus lower expected return for the same risk level. On the other hand, the proposed approximation method has taken into account of the transaction costs. For every time period, the approximate solution has a non-transaction region where the investor is recommended not to make any rebalancing action on the existing portfolio. This improves the overall performance of the investment.

When the transaction fee rate further increases to 1%, Figure 6.5 shows the proposed approximation method still provides almost the same efficient frontier as the optimal one, and the portfolio ignoring transaction costs is noticeably inferior to that of the first two investment policies. The reason is that for high transaction costs, the two methods both recommend "buy-and-hold" investment strategy, i.e., to buy the risky asset at the beginning of the entire investment horizon to a desired level and hold the same portfolio

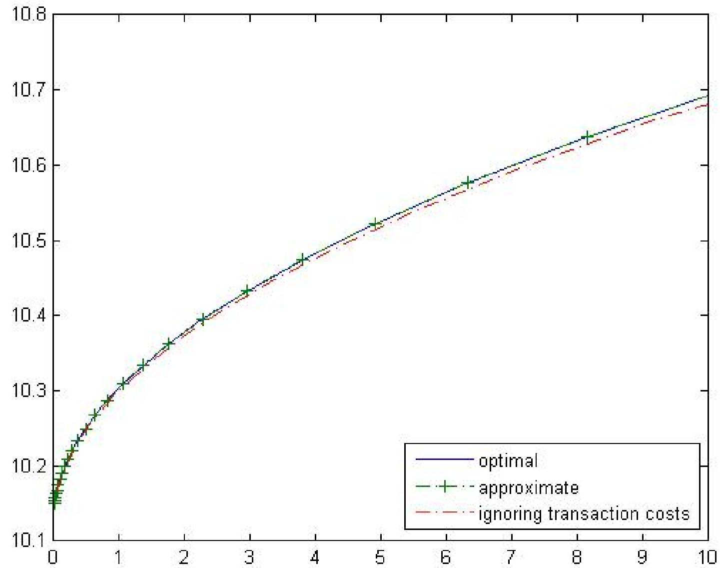


Figure 6.3: Efficient frontier at $t = 0$ for $\alpha = \beta = 0.3\%$

without any further trade until the end of the last period. In contrast, the solution obtained when the transaction costs are ignored is still characterized by heavy trading.

In Figures 6.2 to 6.5, under the four different transaction fee rate assumptions, all the curves start at the same point of $\mathrm{Var}(w_T) = 0$, where the investor is completely risk averse, i.e., the maximum tolerable variance for the investment is 0. In this case, the investor allocates all his wealth in the risk-free asset. After four months, the final wealth will be accumulated to

$$w_4 = (h^0)\big(r^0\big)^4 = 10 \times 1.0370^{(4/12)} = 10.1218 \tag{6.53}$$

Comparing Figures 6.2 to 6.5, we can also conclude that the increase in the

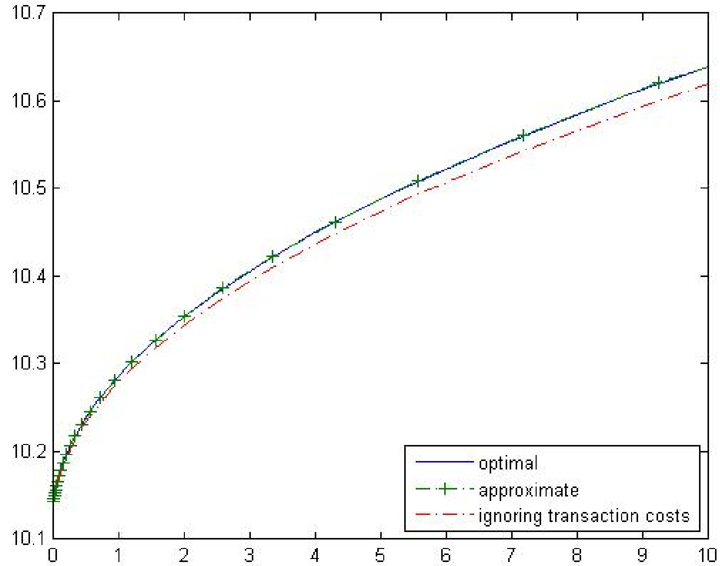


Figure 6.4: Efficient frontier at $t = 0$ for $\alpha = \beta = 0.5\%$

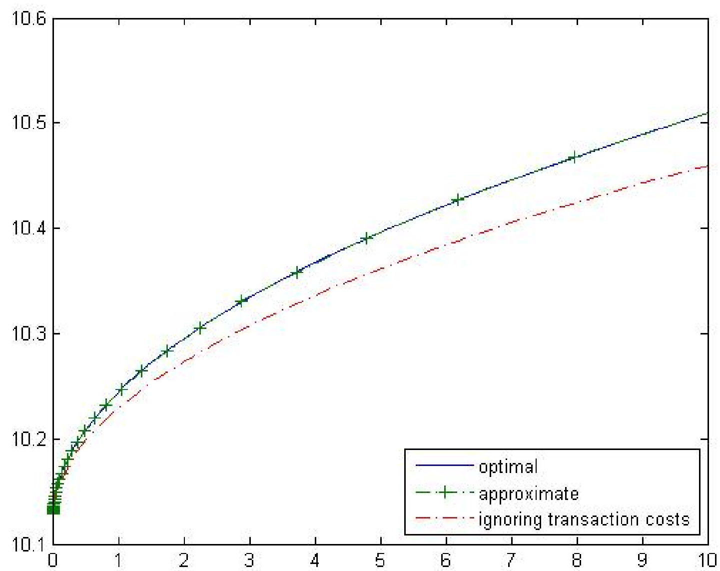


Figure 6.5: Efficient frontier at $t = 0$ for $\alpha = \beta = 1\%$

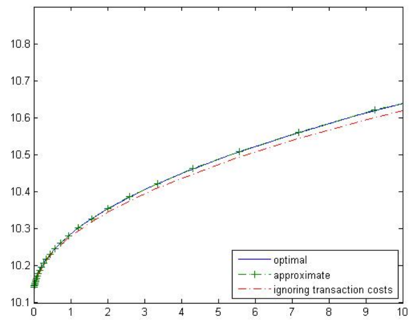
transaction costs reduces the expected final wealth for the same tolerable risk level. However, the approximation method mitigates the loss resulting from the transaction fees when a solution ignoring transaction costs is used.

6.7.3 Sensitivity Analysis

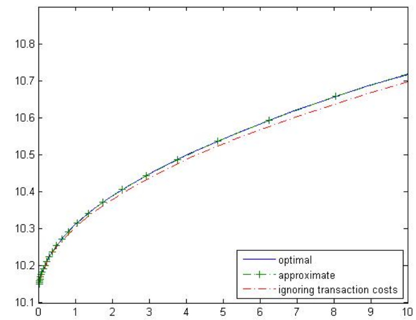
In this section we study how sensitive the performance of the approximation methods is to changes in the expected value and standard deviation of the return rate of the risky asset, assuming other parameters remain same and 0.5% of transaction fee rate.

Figure 6.6 plot the mean-variance efficient frontiers for the yearly expected returns of SP500 to be 1.1165, 1.265, 1.1365 and 1.1465 respectively. As it is shown in the figures, the overall expected return of the portfolio grows for a given risk level as the expected return of SP500 increases. The performance of the approximation method remains stable with the expected annual return of SP500 changing from 1.1165 to 1.1465. It provides an almost identical efficient frontier as the optimal solution. The approximation method outperforms the method ignoring transaction cost in all the four situations.

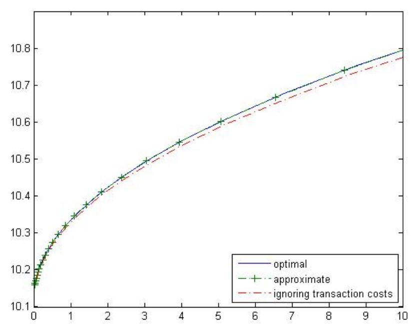
Figure 6.7 show the mean-variance efficient frontiers for the yearly standard deviation of the returns of SP500 to be 0.2069, 0.2269, 0.2469 and 0.2669 respectively. The overall expected return of the portfolio decreases for a given risk level as volatility of the market increases. The approximation method continues to provide very good performance with the standard deviation increases from 0.2069 to 0.2669.



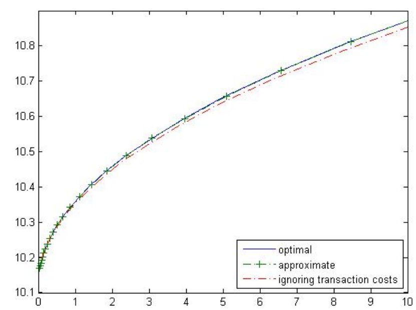
(a) Efficient frontier at $t = 0$ for $r_{[SP500]} = 1.1165$



(b) Efficient frontier at $t = 0$ for $r_{[SP500]} = 1.1265$



(c) Efficient frontier at $t = 0$ for $r_{[SP500]} = 1.1365$

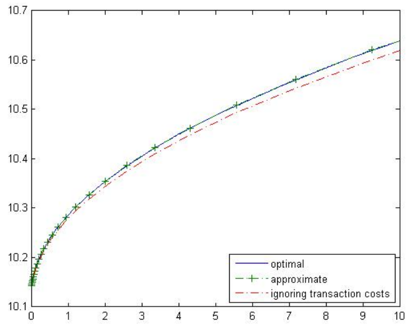


(d) Efficient frontier at $t = 0$ for $r_{[SP500]} = 1.1465$

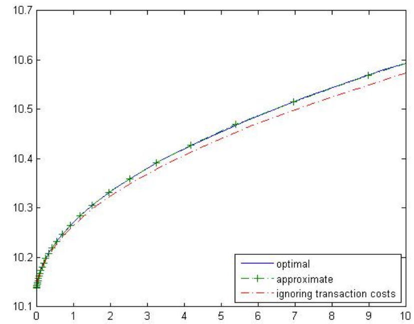
Figure 6.6: Efficient frontiers by changing the value of expected return of SP500

6.7.4 Investment for More Time Periods

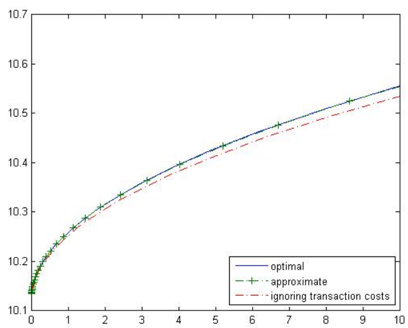
When there are more time periods, the number of scenarios grows exponentially, and it becomes computationally expensive to solve the multiperiod problem exactly. Figure 6.8 compares the calculation times of using the exact method and the approximation method given that $\lambda = 0.035$ and the transaction fee rates are 1%. It can be seen that the computing time using the exact method grows drastically as the number of time periods increases while



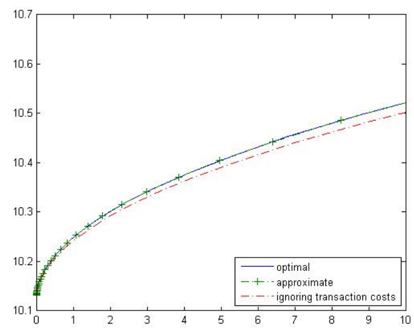
(a) Efficient frontier at $t = 0$ for $\sigma_{[SP500]} = 0.2069$



(b) Efficient frontier at $t = 0$ for $\sigma_{[SP500]} = 0.2269$



(c) Efficient frontier at $t = 0$ for$\sigma_{[SP500]} = 0.2469$



(d) Efficient frontier at $t = 0$ for $\sigma_{[SP500]} = 0.2669$

Figure 6.7: Efficient frontiers by changing the value of expected return of SP500

the computing time using the approximation method rises in a much slower pace. For a 7-period problem, the total solving time of using the exact method reaches as high as 787.26 seconds on a computer with Intel Core 2 Duo CPU and 4GB RAM. If the problem considers more than 7 periods, the computer fails to provide an optimal solution. In contrast, the proposed approximation method makes it possible to solve much more periods as it significantly reduces the computational effort needed for solving. Figure 6.9 plots the efficient fron-

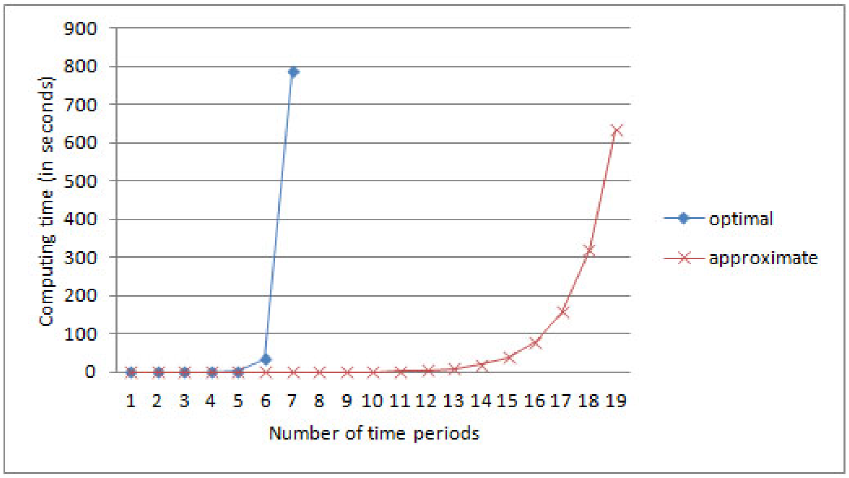


Figure 6.8: Computing times using the exact method and the approximate method

tiers for problems with an investment horizon consisting of 2 to 12 months. As shown in Figure 6.9, the expected wealth accumulated increases as the length of the investment horizon increases.

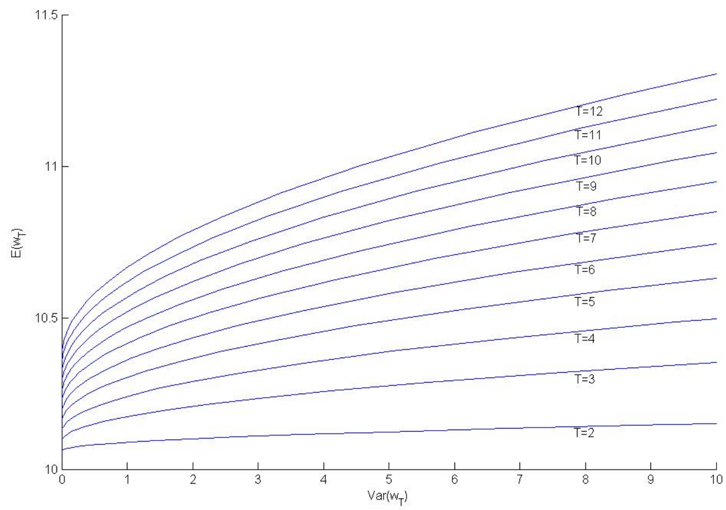


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Figure 6.9: Efficient frontiers at $t = 0$ for multiperiod problems using approximation method

Chapter 7

Conclusion and Future Research Directions

This thesis studies a multiperiod mean-variance investment problem incorporating proportional transaction costs. Since the variance term is non-decomposable in terms of dynamic programming, we get the solution to the mean-variance model by solving a quadratic model. Under some assumptions, the two models are proved to be equivalent. Another difficulty to solve the multiperiod problem is the incorporation of transaction costs. Most existing studies neglect such costs for simplicity. However, our results show that ignorance of transaction fees can lead to suboptimal solutions and thus significantly affect the portfolio performance. In order to find an investment policy that improves the overall investment performance, this thesis incorporates linear transaction costs into a multiperiod mean-variance portfolio optimizations problem.

7.1 Conclusion and Contribution

In this thesis, closed-form solutions for the single-period and two-period problems are obtained. The optimal solutions obtained are translated into investment decisions. As it has been shown, the results for the two-period problem agree with the multiperiod mean-variance portfolio studies when transactions are costless. When the transaction costs are in presence, there will be a non-transaction region where the investor take no action since the enhanced utility by rebalancing cannot cover the transaction costs incurred. The explicit expressions of the upper bound and lower bound of this non-transaction region are given. They are also referred to as the thresholds where the optimal investment decision switches.

For investment problems containing more periods, the quadratic structure of the value function cannot be retained when transaction costs are considered. Therefore, it becomes challenging to solve the multiperiod optimization problem by dynamic programming. Solving the model in a static manner is also difficult as it becomes considerably expensive since the scenarios of the random returns increases exponentially as the number of time periods rises.

In order to obtain near optimal solutions in an efficient manner, we develop an approximation method which works for mild assumptions of the return rates of the risky asset. The approximation overcomes the difficulty of preserving the quadratic structure of the value function for the multiperiod model so that dynamic programming can be used to find the solutions. By mathematical induction, we have proved that the approximate solution and the value function retain the same structure. Therefore, the model can be solved recursively. Such an approximation provides the exact solution in the special case when there is no transaction cost. In the case when transaction costs are applicable, a series of numerical experiments show that the approximation provides close-to-optimal solutions for a four-period problem assuming that the return rate has two possible outcomes for each period. The approximation method provides almost optimal results for various values of parameters. The development of such an approximation thus enables the investor to obtain near-optimal solutions in an efficient manner.

7.2 Future Research Directions

One straight continuation of the present work is to develop an approximation method using a similar approach discussed in this paper to handle a portfolio containing more than one risky asset. The model defined in Chapter 3 can be easily extended to address a N-asset problem. For solving, we need to consider correlation among the return rates of the risky assets. Different assumptions can be made on the correlation. The future study can examine different types of correlation and develop approximation methods for solving under each assumption.

Another interesting research direction is to consider different transaction cost schemes. In this thesis, we assumed the transaction costs to be linear. Other common cost schemes in real practice include fixed lump-sum, piece-wise linear transaction fees, combination of fixed and linear fees, and transaction fee brackets. Assuming nonlinear transaction costs may result in a different structure of the value functions from this thesis. Therefore, new approximation methods may need to be developed to solve such problems. However, similar techniques can still be adopted to solve the problem recursively with the objective of retaining the structure of the value function.

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Appendices

Appendix A

Optimal Solution for the Single-Period Problem

The optimal investment decision for the single-period problem with no borrowing and short selling is represented by x_0^{1+} and x_0^{1-}, which stand for the amount of the risky asset to buy and sell respectively. By solving the KKT conditions, the optimal solution can be obtained. The derivation details are as follows:

A.1 $x_0^{1+} > 0$ and $x_0^{1-} = 0$

From (4.13), $y_0^0 = 0$ or $\frac{h_0^0}{1+\alpha} - x_0^{1+} = 0$.

A.1.1 When $y_0^0 = 0$

From (4.11),

$$
\begin{aligned}
& \delta \, \mathrm{E}(\xi_0^\alpha) - 2\lambda \left[r^0 \, \mathrm{E}(\xi_0^\alpha) \left(h_0^0 - (1+\alpha) x_0^{1+} \right) + \mathrm{E}(R_0^1 \xi_0^\alpha) \left(h_0^1 + x_0^{1+} \right) \right] = 0 \\
\Rightarrow \quad & \delta \, \mathrm{E}(\xi_0^\alpha) - 2\lambda \left[r^0 \, \mathrm{E}(\xi_0^\alpha) h_0^0 + \mathrm{E}(R_0^1 \xi_0^\alpha) h_0^1 + \mathrm{E}(\xi_0^\alpha)^2 x_0^{1+} \right] = 0 \\
\Rightarrow \quad & x_0^{1+} = \frac{\frac{\delta}{2\lambda} \mathrm{E}(\xi_0^\alpha) - r^0 \, \mathrm{E}(\xi_0^\alpha) h_0^0 - \mathrm{E}(R_0^1 \xi_0^\alpha) h_0^1}{\mathrm{E}(\xi_0^\alpha)^2} \\
\Rightarrow \quad & x_0^{1+} = \frac{\mathrm{E}(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)^2} \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1 \right) \qquad \text{(A.1)}
\end{aligned}
$$

and $x_0^{1-} = 0$. To ensure that $0 \le x_0^{1+} < \frac{h_0^0}{1+\alpha}$, the following condition should be satisfied:

$$
r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1 \le \frac{\delta}{2\lambda} < \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1 \qquad \text{(A.2)}
$$

A.1.2 When $\frac{h_0^0}{1+\alpha} - x_0^{1+} = 0$

$$
x_0^{1+} = \frac{h_0^0}{1+\alpha} \quad \text{and} \quad x_0^{1-} = 0. \qquad \text{(A.3)}
$$

This case happens when the ideal x_0^{1+} exceeds its upper limit $\frac{h_0^0}{1+\alpha}$, i.e.

$$
\frac{\delta}{2\lambda} \ge \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1 \qquad \text{(A.4)}
$$

A.2 $x_0^{1+} = 0$ and $x_0^{1-} > 0$

From (4.14), $y_0^1 = 0$ or $h_0^1 - x_0^{1-} = 0$.

A.2.1 When $y_0^1 = 0$

From (4.12),

$$
\begin{aligned}
& \delta\, \mathrm{E}(\xi_0^\beta) - 2\lambda \left[r^0\, \mathrm{E}(\xi_0^\beta) \left(h_0^0 + (1-\beta) x_0^{1-} \right) + \mathrm{E}(R_0^1 \xi_0^\beta) \left(h_0^1 - x_0^{1-} \right) \right] = 0 \\
\Rightarrow \quad & \delta\, \mathrm{E}(\xi_0^\beta) - 2\lambda \left[r^0\, \mathrm{E}(\xi_0^\beta) h_0^0 + \mathrm{E}(R_0^1 \xi_0^\beta) h_0^1 - \mathrm{E}(\xi_0^\beta)^2 x_0^{1+} \right] = 0 \\
\Rightarrow \quad & x_0^{1-} = - \frac{\frac{\delta}{2\lambda}\, \mathrm{E}(\xi_0^\beta) - r^0\, \mathrm{E}(\xi_0^\beta) h_0^0 - \mathrm{E}(R_0^1 \xi_0^\beta) h_0^1}{\mathrm{E}(\xi_0^\beta)^2} \\
\Rightarrow \quad & x_0^{1-} = \frac{\mathrm{E}(\xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)^2} \left(r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)} h_0^1 - \frac{\delta}{2\lambda} \right) \qquad \text{(A.5)}
\end{aligned}
$$

and $x_0^{1+} = 0$.

To eusure that x_0^{1-} is within the correct range, we have

$$
r^0 h_0^0 + r^0 (1-\beta) h_0^1 \le \frac{\delta}{2\lambda} < r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)} h_0^1 \qquad \text{(A.6)}
$$

A.2.2 When $h_0^1 - x_0^{1-} = 0$

$$
x_0^{1+} = 0 \quad \text{and} \quad x_0^{1-} = h_0^1, \qquad \text{(A.7)}
$$

when

$$
\frac{\delta}{2\lambda} < r^0 h_0^0 + r^0 (1-\beta) h_0^1. \qquad \text{(A.8)}
$$

A.3 $x_0^{1+} = 0$ and $x_0^{1-} = 0$

$$x_0^{1+} = 0 \quad \text{and} \quad x_0^{1-} = 0, \tag{A.9}$$

when

$$r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)} h_0^1 \le \frac{\delta}{2\lambda} < r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1. \tag{A.10}$$

Appendix B

Value Functions for the Single-Period Problem

In this section, the value functions for the five intervals in terms of $\frac{\delta}{2\lambda}$ will be presented.

1. When $\frac{\delta}{2\lambda} \geq \frac{\mathrm{E}(R_0^1\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)}h_0^0 + \frac{\mathrm{E}(R_0^1\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)}h_0^1$

   - The optimal solution is

$$x_0^{1+} = \frac{h_0^0}{1+\alpha} \tag{B.1}$$

$$x_0^{1-} = 0. \tag{B.2}$$

   We substitute the optimal solution into the objective function (4.3):

   - The value function is

$$V_0(h_0^0, h_0^1) = \delta\mathrm{E}(R_0^1)\left(\frac{h_0^0}{1+\alpha} + h_0^1\right) - \lambda\mathrm{E}(R_0^1)^2\left(\frac{h_0^0}{1+\alpha} + h_0^1\right)^2 \tag{B.3}$$

2. When $r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1 \leq \frac{\delta}{2\lambda} < \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)} h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1$

- The optimal solution is

$$x_0^{1+} = \frac{\mathrm{E}(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)^2} \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1 \right) \tag{B.4}$$

$$x_0^{1-} = 0 \tag{B.5}$$

- The value function is

$$\begin{aligned} V_0(h_0^0, h_0^1) =& \delta\, \mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big) - \lambda\, \mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big)^2 \\ &+ \lambda \frac{\mathrm{E}^2(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)^2} \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1 \right)^2 \end{aligned} \tag{B.6}$$

3. When $r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)} h_0^1 \leq \frac{\delta}{2\lambda} < r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} h_0^1$

- The optimal solution is

$$x_0^{1+} = 0 \tag{B.7}$$

$$x_0^{1-} = 0 \tag{B.8}$$

- The value function is

$$V_0(h_0^0, h_0^1) = \delta\, \mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big) - \lambda\, \mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big)^2 \tag{B.9}$$

4. When $r^0 h_0^0 + r^0(1-\beta) h_0^1 \leq \frac{\delta}{2\lambda} < r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)} h_0^1$

- The optimal solution is

$$x_0^{1+} = 0 \tag{B.10}$$

$$x_0^{1-} = \frac{\mathrm{E}(\xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)^2}\left(r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)} h_0^1 - \frac{\delta}{2\lambda}\right) \tag{B.11}$$

- The value function is

$$\begin{aligned} V_0(h_0^0, h_0^1) =& \delta\,\mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big) - \lambda\,\mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big)^2 \\ &+ \lambda \frac{\mathrm{E}^2(\xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)^2}\left(r^0 h_0^0 + \frac{\mathrm{E}(R_0^1 \xi_0^\beta)}{\mathrm{E}(\xi_0^\beta)} h_0^1 - \frac{\delta}{2\lambda}\right)^2 \end{aligned} \tag{B.12}$$

5. When $\frac{\delta}{2\lambda} < r^0 h_0^0 + r^0(1-\beta) h_0^1$

- The optimal solution is

$$x_0^{1+} = 0 \tag{B.13}$$

$$x_0^{1-} = h_0^1 \tag{B.14}$$

- The value function is

$$V_0(h_0^0, h_0^1) = \delta r^0 \left(h_0^0 + (1-\beta) h_0^1\right) - \lambda (r^0)^2 \left(h_0^0 + (1-\beta) h_0^1\right)^2 \tag{B.15}$$

Appendix C

Piecewise Continuity of the Value Functions

In this section, the piecewise continuity of value functions will be investigated. Firstly the partial derivatives for the five value functions will be derived. Then we will examine the continuity at the four thresholds by matching the left and right derivative of the two neighbouring value functions at each threshold.

1. When $\frac{\delta}{2\lambda} \geq g_0^{(1)}$,

$$V_0^{(1)}(h_0^0, h_0^1) = \delta\,\mathrm{E}(R_0^1)\left(\frac{h_0^0}{1+\alpha} + h_0^1\right) - \lambda\,\mathrm{E}(R_0^1)^2\left(\frac{h_0^0}{1+\alpha} + h_0^1\right)^2 \quad \text{(C.1)}$$

The partial derivative of V_0 with respect to h_0^0 is

$$\frac{\partial V_0^{(1)}(h_0^0, h_0^1)}{\partial h_0^0} = \frac{2\lambda\,\mathrm{E}(R_0^1)}{1+\alpha}\left[\frac{\delta}{2\lambda} - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)}\left(\frac{h_0^0}{1+\alpha} + h_0^1\right)\right], \quad \text{(C.2)}$$

and the partial derivative of V_0 with respect to h_0^1 is

$$\frac{\partial V_0^{(1)}(h_0^0, h_0^1)}{\partial h_0^1} = 2\lambda \, \mathrm{E}(R_0^1) \left[\frac{\delta}{2\lambda} - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} \left(\frac{h_0^0}{1+\alpha} + h_0^1 \right) \right] \tag{C.3}$$

2. When $g_0^{(2)} \leq \frac{\delta}{2\lambda} < g_0^{(1)}$,

$$\begin{aligned} V_0^{(2)} =& \delta \, \mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big) - \lambda \, \mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big)^2 + d_0^\alpha \lambda \bigg(\frac{\delta}{2\lambda} - g_0^{(2)} \bigg)^2 \\ =& \delta \, \mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big) - \lambda \, \mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big)^2 + d_0^\alpha \lambda \bigg(\frac{\delta}{2\lambda} - r^0 h_0^0 - q_0^\alpha h_0^1 \bigg)^2 \end{aligned} \tag{C.4}$$

The partial derivatives are

$$\begin{aligned} & \frac{\partial V_0^{(2)}(h_0^0, h_0^1)}{\partial h_0^0} \\ &= 2\lambda r^0 \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \mathrm{E}(R_0^1) h_0^1 \right) - d_0^\alpha \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - q_0^\alpha h_0^1 \right) \right] \\ &= 2\lambda r^0 \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \mathrm{E}(R_0^1) h_0^1 \right) - d_0^\alpha \left(\frac{\delta}{2\lambda} - g_0^{(2)} \right) \right] \end{aligned} \tag{C.5}$$

and

$$\begin{aligned} & \frac{\partial V_0^{(2)}(h_0^0, h_0^1)}{\partial h_0^1} \\ &= 2\lambda \, \mathrm{E}(R_0^1) \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1 \right) - \frac{d_0^\alpha q_0^\alpha}{\mathrm{E}(R_0^1)} \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - q_0^\alpha h_0^1 \right) \right] \\ &= 2\lambda \, \mathrm{E}(R_0^1) \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1 \right) - \frac{d_0^\alpha q_0^\alpha}{\mathrm{E}(R_0^1)} \left(\frac{\delta}{2\lambda} - g_0^{(2)} \right) \right] \end{aligned} \tag{C.6}$$

3. When $g_0^{(3)} \leq \frac{\delta}{2\lambda} < g_0^{(2)}$,

$$V_0^{(3)} = \delta\,\mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big) - \lambda\,\mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big)^2 \tag{C.7}$$

The partial derivatives are

$$\frac{\partial V_0^{(3)}(h_0^0, h_0^1)}{\partial h_0^0} = 2\lambda r^0 \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \ \mathrm{E}(R_0^1) h_0^1\right) \tag{C.8}$$

$$\frac{\partial V_0^{(3)}(h_0^0, h_0^1)}{\partial h_0^1} = 2\lambda\,\mathrm{E}(R_0^1) \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1\right) \tag{C.9}$$

4. When $g_0^{(4)} \leq \frac{\delta}{2\lambda} < g_0^{(3)}$,

$$V_0^{(4)} = \delta\,\mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big) - \lambda\,\mathrm{E}\big(r^0 h_0^0 + R_0^1 h_0^1\big)^2 + d_0^\beta \lambda \left(\frac{\delta}{2\lambda} - g_0^{(3)}\right)^2 \tag{C.10}$$

The partial derivatives are

$$\begin{aligned}
&\frac{\partial V_0^{(4)}(h_0^0, h_0^1)}{\partial h_0^0} \\
&\quad = 2\lambda r^0 \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \ \mathrm{E}(R_0^1) h_0^1\right) - d_0^\beta \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - q_0^\beta h_0^1\right)\right] \\
&\quad = 2\lambda r^0 \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \ \mathrm{E}(R_0^1) h_0^1\right) - d_0^\beta \left(\frac{\delta}{2\lambda} - g_0^{(3)}\right)\right]
\end{aligned} \tag{C.11}$$

and

$$\frac{\partial V_0^{(4)}(h_0^0, h_0^1)}{\partial h_0^1}$$
$$= 2\lambda \operatorname{E}(R_0^1) \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\operatorname{E}(R_0^1)^2}{\operatorname{E}(R_0^1)} h_0^1 \right) - \frac{d_0^\beta q_0^\beta}{\operatorname{E}(R_0^1)} \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - q_0^\beta h_0^1 \right) \right]$$
$$= 2\lambda \operatorname{E}(R_0^1) \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\operatorname{E}(R_0^1)^2}{\operatorname{E}(R_0^1)} h_0^1 \right) - \frac{d_0^\beta q_0^\beta}{\operatorname{E}(R_0^1)} \left(\frac{\delta}{2\lambda} - g_0^{(3)} \right) \right] \tag{C.12}$$

5. When $0 \leq \frac{\delta}{2\lambda} < g_0^{(4)}$,

$$V_0^{(5)} = \delta r^0 \left(h_0^0 + (1-\beta) h_0^1 \right) - \lambda (r^0)^2 \left(h_0^0 + (1-\beta) h_0^1 \right)^2 \tag{C.13}$$

The partial derivatives are

$$\frac{\partial V_0^{(5)}(h_0^0, h_0^1)}{\partial h_0^0}$$
$$= \delta r^0 - 2\lambda (r^0)^2 \left(h_0^0 + (1-\beta) h_0^1 \right)$$
$$= 2\lambda r^0 \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - r^0 (1-\beta) h_0^1 \right) \tag{C.14}$$

$$\frac{\partial V_0^{(5)}(h_0^0, h_0^1)}{\partial h_0^1}$$
$$= \delta r^0 (1-\beta) - 2\lambda (r^0)^2 (1-\beta) \left(h_0^0 + (1-\beta) h_0^1 \right)$$
$$= 2\lambda r^0 (1-\beta) \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - r^0 (1-\beta) h_0^1 \right) \tag{C.15}$$

Having all the five sets of differentiation equations derived, the next step is to study the continuity at the four thresholds. The left and right derivatives of the neighbouring functions at each threshold will be compared.

1. $\frac{\delta}{2\lambda} = g_0^{(1)}$

   $V_0^{(1)}$ and $V_0^{(2)}$ are connected at $\frac{\delta}{2\lambda} = g_0^{(1)}$. We will match the left and right partial derivatives with respect to h_0^0 and h_0^1. The right derivative of $V_0^{(1)}$ with regards to h_0^0 at threshold $\frac{\delta}{2\lambda} = g_0^{(1)}$ is

$$\begin{aligned}
&\left.\frac{\partial_+ V_0^{(1)}}{\partial h_0^0}\right|_{\frac{\delta}{2\lambda}=g_0^{(1)}} \\
&\quad= \frac{1}{1+\alpha}\left[\delta \mathrm{E}(R_0^1) - 2\lambda \mathrm{E}(R_0^1)^2\left(\frac{h_0^0}{1+\alpha} + h_0^1\right)\right]\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(1)}} \\
&\quad= \frac{2\lambda}{1+\alpha}\left(\mathrm{E}(R_0^1) q_0^\alpha - \mathrm{E}(R_0^1)^2\right)\left(\frac{h_0^0}{1+\alpha} + h_0^1\right) \\
&\quad= \frac{2\lambda}{1+\alpha}\,\frac{\mathrm{E}^2(R_0^1)\,\mathrm{E}(\xi_0^\alpha) + \mathrm{E}(R_0^1)\,\mathrm{Var}(R_0^1) - \mathrm{E}^2(R_0^1)\,\mathrm{E}(\xi_0^\alpha) - \mathrm{Var}(R_0^1)\,\mathrm{E}(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)} \\
&\qquad\cdot\left(\frac{h_0^0}{1+\alpha} + h_0^1\right) \\
&\quad= \frac{2\lambda r^0\,\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)}\left(\frac{h_0^0}{1+\alpha} + h_0^1\right),
\end{aligned} \tag{C.16}$$

and the left derivative at this threshold is

$$
\begin{aligned}
&\left.\frac{\partial_- V_0^{(2)}}{\partial h_0^0}\right|_{\frac{\delta}{2\lambda}=g_0^{(1)}} \\
&\quad = 2\lambda r^0 \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right) - d_0^\alpha \left(\frac{\delta}{2\lambda} - g_0^{(2)}\right)\right]\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(1)}} \\
&\quad =2\lambda r^0 \left[q_0^\alpha \left(\frac{h_0^0}{1+\alpha} + h_0^1\right) - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right. \\
&\qquad\quad \left. -d_0^\alpha \left(q_0^\alpha \left(\frac{h_0^0}{1+\alpha} + h_0^1\right) - r^0 h_0^0 - q_0^\alpha h_0^1\right)\right] \\
&\quad =2\lambda r^0 \left[\left(\frac{q_0^\alpha}{1+\alpha} - r^0\right) h_0^0 + \left(q_0^\alpha - \mathrm{E}(R_0^1)\right) h_0^1\right. \\
&\qquad\quad \left. -d_0^\alpha \left(\frac{q_0^\alpha}{1+\alpha} - r^0\right) h_0^0\right] \\
&\quad =2\lambda r^0 \left[\left(\frac{\mathrm{E}(\xi_0^\alpha)^2}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)}\right)\left(1 - \frac{\mathrm{E}^2(\xi_0^\alpha)}{\mathrm{E}(\xi_0^\alpha)^2}\right) h_0^0 + \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)} h_0^1\right] \\
&\quad =\frac{2\lambda r^0\,\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)} \left(\frac{h_0^0}{1+\alpha} + h_0^1\right) \qquad \text{(C.17)}
\end{aligned}
$$

Therefore, the left and right derivatives of the two neighbouring functions with respect to h_0^0 agree at breakpoint $\frac{\delta}{2\lambda} = g_0^{(1)}$. Now, let us move on to the case with respect to h_0^1. Since

$$
\begin{aligned}
&\left.\frac{\partial_+ V_0^{(1)}}{\partial h_0^1}\right|_{\frac{\delta}{2\lambda}=g_0^{(1)}} \\
&\quad = \delta \mathrm{E}(R_0^1) - 2\lambda \mathrm{E}(R_0^1)^2 \left(\frac{h_0^0}{1+\alpha} + h_0^1\right)\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(1)}} \\
&\quad =\frac{2\lambda r^0\,(1+\alpha)\,\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)} \left(\frac{h_0^0}{1+\alpha} + h_0^1\right), \qquad \text{(C.18)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left.\frac{\partial_- V_0^{(2)}}{\partial h_0^1}\right|_{\frac{\delta}{2\lambda}=g_0^{(1)}} \\
&= 2\lambda\,\mathrm{E}(R_0^1)\left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1\right) - \frac{d_0^\alpha q_0^\alpha}{\mathrm{E}(R_0^1)}\left(\frac{\delta}{2\lambda} - g_0^{(2)}\right)\right]\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(1)}} \\
&=2\lambda\,\mathrm{E}(R_0^1)\left[q_0^\alpha\left(\frac{h_0^0}{1+\alpha} + h_0^1\right) - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1\right. \\
&\qquad \left. - \frac{\mathrm{E}(R_0^1\xi_0^\alpha)\,\mathrm{E}(\xi_0^\alpha)}{\mathrm{E}(R_0^1)\,\mathrm{E}(\xi_0^\alpha)^2}\left(q_0^\alpha\left(\frac{h_0^0}{1+\alpha} + h_0^1\right) - r^0 h_0^0 - q_0^\alpha h_0^1\right)\right] \\
&=2\lambda\,\mathrm{E}(R_0^1)\left[\left(1 - \frac{\mathrm{E}(R_0^1\xi_0^\alpha)\,\mathrm{E}(\xi_0^\alpha)}{\mathrm{E}(R_0^1)\,\mathrm{E}(\xi_0^\alpha)^2}\right)\left(\frac{q_0^\alpha}{1+\alpha} - r^0\right) h_0^0\right. \\
&\qquad \left. + \left(q_0^\alpha - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)}\right) h_0^1\right] \\
&=2\lambda\,\mathrm{E}(R_0^1)\left[\left(\frac{\mathrm{Var}(R_0^1) r^0 (1+\alpha)}{\mathrm{E}(R_0^1)\,\mathrm{E}(\xi_0^\alpha)^2}\right)\left(\frac{\mathrm{E}(\xi_0^\alpha)^2}{\mathrm{E}(\xi_0^\alpha)(1+\alpha)}\right) h_0^0\right. \\
&\qquad \left. + \mathrm{Var}(R_0^1)\frac{r^0(1+\alpha)}{\mathrm{E}(R_0^1)} h_0^1\right] \\
&=\frac{2\lambda r^0\,(1+\alpha)\,\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)}\left(\frac{h_0^0}{1+\alpha} + h_0^1\right),
\end{aligned}
\tag{C.19}
$$

Thus, the value function is shown to be continuous with respect to h_0^0 and h_0^1 at threshold $\frac{\delta}{2\lambda} = g_0^{(1)}$.

2. $\frac{\delta}{2\lambda} = g_0^{(2)}$

   $V_0^{(2)}$ and $V_0^{(3)}$ are connected at this threshold.

With respect to h_0^0,

$$\left.\frac{\partial_+ V_0^{(2)}}{\partial h_0^0}\right|_{\frac{\delta}{2\lambda}=g_0^{(2)}}$$

$$\begin{aligned}
&=2\lambda r^0 \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right) - d_0^\alpha\left(\frac{\delta}{2\lambda} - g_0^{(2)}\right)\right]\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(2)}} \\
&=2\lambda r^0 \left(g_0^{(2)} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right) \\
&=2\lambda r^0 \left(q_0^\alpha - \mathrm{E}(R_0^1)\right) h_0^1 \\
&=2\lambda r^0 \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)} h_0^1
\end{aligned} \tag{C.20}$$

and

$$\left.\frac{\partial_- V_0^{(3)}}{\partial h_0^0}\right|_{\frac{\delta}{2\lambda}=g_0^{(2)}}$$

$$\begin{aligned}
&=2\lambda r^0 \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right)\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(2)}} \\
&=2\lambda r^0 \left(g_0^{(2)} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right) \\
&=2\lambda r^0 \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)} h_0^1.
\end{aligned} \tag{C.21}$$

For h_0^1,

$$
\begin{aligned}
&\left.\frac{\partial_+ V_0^{(2)}}{\partial h_0^1}\right|_{\frac{\delta}{2\lambda}=g_0^{(2)}} \\
&\quad = 2\lambda \,\mathrm{E}(R_0^1)\left.\left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1\right) - \frac{d_0^\alpha q_0^\alpha}{\mathrm{E}(R_0^1)}\left(\frac{\delta}{2\lambda} - g_0^{(2)}\right)\right]\right|_{\frac{\delta}{2\lambda}=g_0^{(2)}} \\
&\quad = 2\lambda \,\mathrm{E}(R_0^1)\left(g_0^{(2)} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1\right) \\
&\quad = 2\lambda \,\mathrm{E}(R_0^1)\left(q_0^\alpha - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)}\right) h_0^1 \\
&\quad = 2\lambda r^0 (1+\alpha)\frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)} h_0^1, \qquad \text{(C.22)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left.\frac{\partial_- V_0^{(3)}}{\partial h_0^1}\right|_{\frac{\delta}{2\lambda}=g_0^{(2)}} \\
&\quad = 2\lambda \,\mathrm{E}(R_0^1)\left.\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1\right)\right|_{\frac{\delta}{2\lambda}=g_0^{(2)}} \\
&\quad = 2\lambda \,\mathrm{E}(R_0^1)\left(g_0^{(2)} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1\right) \\
&\quad = 2\lambda r^0 (1+\alpha)\frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\alpha)} h_0^1. \qquad \text{(C.23)}
\end{aligned}
$$

Therefore, the value function is continuous with respect to h_0^0 and h_0^1 at threshold $\frac{\delta}{2\lambda} = g_0^{(2)}$.

3. $\frac{\delta}{2\lambda} = g_0^{(3)}$

$V_0^{(3)}$ and $V_0^{(4)}$ are connected at this threshold.

For h_0^0,

$$\begin{aligned}
&\left.\frac{\partial_+ V_0^{(3)}}{\partial h_0^0}\right|_{\frac{\delta}{2\lambda}=g_0^{(3)}} \\
&\quad= 2\lambda r^0 \left.\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right)\right|_{\frac{\delta}{2\lambda}=g_0^{(3)}} \\
&\quad= 2\lambda r^0 \left(g_0^{(3)} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right) \\
&\quad= 2\lambda r^0 \left(q_0^{\beta} - \mathrm{E}(R_0^1)\right) h_0^1 \\
&\quad= 2\lambda r^0 \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^{\beta})} h_0^1
\end{aligned} \tag{C.24}$$

and

$$\begin{aligned}
&\left.\frac{\partial_- V_0^{(4)}}{\partial h_0^0}\right|_{\frac{\delta}{2\lambda}=g_0^{(3)}} \\
&\quad= 2\lambda r^0 \left.\left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right) - d_0^{\beta}\left(\frac{\delta}{2\lambda} - g_0^{(3)}\right)\right]\right|_{\frac{\delta}{2\lambda}=g_0^{(3)}} \\
&\quad= 2\lambda r^0 \left(g_0^{(3)} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right) \\
&\quad= 2\lambda r^0 \frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^{\beta})} h_0^1.
\end{aligned} \tag{C.25}$$

For h_0^1,

$$\left.\frac{\partial_+ V_0^{(3)}}{\partial h_0^1}\right|_{\frac{\delta}{2\lambda}=g_0^{(3)}}$$
$$=2\lambda\,\mathrm{E}(R_0^1)\left(\frac{\delta}{2\lambda}-r^0h_0^0-\frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)}h_0^1\right)\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(3)}}$$
$$=2\lambda\,\mathrm{E}(R_0^1)\left(g_0^{(3)}-r^0h_0^0-\frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)}h_0^1\right)$$
$$=2\lambda\,\mathrm{E}(R_0^1)\left(q_0^\beta-\frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)}\right)h_0^1$$
$$=2\lambda r^0(1-\beta)\frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\beta)}h_0^1, \tag{C.26}$$

and

$$\left.\frac{\partial_- V_0^{(4)}}{\partial h_0^1}\right|_{\frac{\delta}{2\lambda}=g_0^{(3)}}$$
$$=2\lambda\,\mathrm{E}(R_0^1)\left[\left(\frac{\delta}{2\lambda}-r^0h_0^0-\frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)}h_0^1\right)-\frac{d_0^\beta q_0^\beta}{\mathrm{E}(R_0^1)}\left(\frac{\delta}{2\lambda}-g_0^{(3)}\right)\right]\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(3)}}$$
$$=2\lambda r^0\left(g_0^{(3)}-r^0h_0^0-\frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)}h_0^1\right)$$
$$=2\lambda r^0(1-\beta)\frac{\mathrm{Var}(R_0^1)}{\mathrm{E}(\xi_0^\beta)}h_0^1. \tag{C.27}$$

Therefore, the value function is continuous with respect to h_0^0 and h_0^1 at threshold $\frac{\delta}{2\lambda}=g_0^{(3)}$.

4. $\frac{\delta}{2\lambda} = g_0^{(4)}$

For h_0^0,

$$\left.\frac{\partial_+ V_0^{(4)}}{\partial h_0^0}\right|_{\frac{\delta}{2\lambda}=g_0^{(4)}}$$

$$= 2\lambda r^0 \left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \mathrm{E}(R_0^1)h_0^1\right) - d_0^\beta \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - q_0^\beta h_0^1\right)\right]\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(4)}}$$

$$= 2\lambda r^0 \left[\left(r^0(1-\beta) - \mathrm{E}(R_0^1)\right) h_0^1 - d_0^\beta \left(r^0(1-\beta) - q_0^\beta\right) h_0^1\right]$$

$$= 2\lambda r^0 \left[- \mathrm{E}(\xi_0^\beta)h_0^1 - +d_0^\beta \frac{\mathrm{E}(\xi_0^\beta)^2}{\mathrm{E}(\xi_0^\beta)} h_0^1\right]$$

$$= 0 \tag{C.28}$$

and

$$\left.\frac{\partial_- V_0^{(5)}}{\partial h_0^0}\right|_{\frac{\delta}{2\lambda}=g_0^{(4)}}$$

$$= 2\lambda r^0 \left(\frac{\delta}{2\lambda} - r^0 h_0^0 - r^0(1-\beta)h_0^1\right)\Bigg|_{\frac{\delta}{2\lambda}=g_0^{(4)}}$$

$$= 0 \tag{C.29}$$

For h_0^1,

$$
\begin{aligned}
&\left.\frac{\partial_+ V_0^{(4)}}{\partial h_0^1}\right|_{\frac{\delta}{2\lambda}=g_0^{(4)}} \\
&= 2\lambda\,\mathrm{E}(R_0^1)\left.\left[\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)} h_0^1\right) - \frac{d_0^\beta q_0^\beta}{\mathrm{E}(R_0^1)}\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - q_0^\beta h_0^1\right)\right]\right|_{\frac{\delta}{2\lambda}=g_0^{(4)}} \\
&= 2\lambda\,\mathrm{E}(R_0^1)\left[\left(r^0(1-\beta) - \frac{\mathrm{E}(R_0^1)^2}{\mathrm{E}(R_0^1)}\right) h_0^1 - \frac{d_0^\beta q_0^\beta}{\mathrm{E}(R_0^1)}\left(r^0(1-\beta) - q_0^\beta\right) h_0^1\right] \\
&= 2\lambda\,\mathrm{E}(R_0^1)\left[-\frac{\mathrm{E}(R_0^1\xi_0^\beta)}{\mathrm{E}(R_0^1)} h_0^1 + \frac{\mathrm{E}(R_0^1\xi_0^\beta)}{\mathrm{E}(R_0^1)} h_0^1\right] \\
&= 0 \qquad \text{(C.30)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left.\frac{\partial_- V_0^{(5)}}{\partial h_0^1}\right|_{\frac{\delta}{2\lambda}=g_0^{(4)}} \\
&= 2\lambda r^0(1-\beta)\left.\left(\frac{\delta}{2\lambda} - r^0 h_0^0 - r^0(1-\beta) h_0^1\right)\right|_{\frac{\delta}{2\lambda}=g_0^{(4)}} \\
&= 0 \qquad \text{(C.31)}
\end{aligned}
$$

Therefore, the value function is continuous with respect to h_0^0 and h_0^1 at threshold $\frac{\delta}{2\lambda} = g_0^{(4)}$ as well.

In conclusion, since the value function of the one-period problem is a piecewise function containing five pieces of smooth functions and it is continuous on each thresholds, the value function is a continuous function on the entire domain.

Appendix D

Value Function for the First Period

We apply the backward DP algorithm. After we have obtained the piece-wise value function for the latter period, we can derive the value function for the first period:

$$
\begin{aligned}
&V_0(\widehat{h}_0^0, \widehat{h}_0^1) \\
&\quad = \max \int_{\underline{r_u}}^{B_0^\alpha} \frac{1}{\overline{r_u} - \underline{r_u}} \left[\delta \, \mathrm{E} \left((r^0)^2 \widehat{h}_0^0 + R_1^1 r_0^1 \widehat{h}_0^1 \right) - \lambda \, \mathrm{E} \left((r^0)^2 \widehat{h}_0^0 + R_1^1 r_0^1 \widehat{h}_0^1 \right)^2 \right. \\
&\qquad\qquad \left. + d_1^\alpha \lambda \left(\frac{\delta}{2\lambda} - (r^0)^2 \widehat{h}_0^0 - q_1^\alpha r_0^1 \widehat{h}_0^1 \right)^2 \right] \mathrm{d}r_0^1 \\
&\qquad\quad + \int_{B_0^\alpha}^{B_0^\beta} \frac{1}{\overline{r_u} - \underline{r_u}} \left[\delta \, \mathrm{E} \left((r^0)^2 \widehat{h}_0^0 + R_1^1 r_0^1 \widehat{h}_0^1 \right) - \lambda \, \mathrm{E} \left((r^0)^2 \widehat{h}_0^0 + R_1^1 r_0^1 \widehat{h}_0^1 \right)^2 \right] \mathrm{d}r_0^1
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{B_0^\beta}^{\overline{r_u}} \frac{1}{\overline{r_u}-\underline{r_u}} \left[\delta \operatorname{E}\left((r^0)^2\widehat{h}_0^0 + R_1^1 r_0^1 \widehat{h}_0^1\right) - \lambda \operatorname{E}\left((r^0)^2\widehat{h}_0^0 + R_1^1 r_0^1 \widehat{h}_0^1\right)^2 \right.\\
&\qquad \left. + d_1^\beta \lambda \left(\frac{\delta}{2\lambda} - (r^0)^2\widehat{h}_0^0 - q_1^\beta r_0^1 \widehat{h}_0^1\right)^2\right] \mathrm{d}r_0^1 \\
=&\max \int_{\underline{r_u}}^{\overline{r_u}} \frac{1}{\overline{r_u}-\underline{r_u}} \left[\delta \operatorname{E}\left((r^0)^2\widehat{h}_0^0 + R_1^1 r_0^1 \widehat{h}_0^1\right) - \lambda \operatorname{E}\left((r^0)^2\widehat{h}_0^0 + R_1^1 r_0^1 \widehat{h}_0^1\right)^2\right] \mathrm{d}r_0^1 \\
&+\int_{\underline{r_u}}^{B_0^\alpha} \frac{1}{\overline{r_u}-\underline{r_u}} d_1^\alpha \lambda \left(\frac{\delta}{2\lambda} - (r^0)^2\widehat{h}_0^0 - q_1^\alpha r_0^1 \widehat{h}_0^1\right)^2 \mathrm{d}r_0^1 \\
&+\int_{B_0^\beta}^{\overline{r_u}} \frac{1}{\overline{r_u}-\underline{r_u}} d_1^\beta \lambda \left(\frac{\delta}{2\lambda} - (r^0)^2\widehat{h}_0^0 - q_1^\beta r_0^1 \widehat{h}_0^1\right)^2 \mathrm{d}r_0^1 \qquad \text{(D.1)}
\end{aligned}
$$

Since

$$
\left.\left(\frac{\delta}{2\lambda} - (r^0)^2\widehat{h}_0^0 - q_1^\alpha r_0^1 \widehat{h}_0^1\right)^3\right|_{r_{T-2}^1 = B_{T-2}^\alpha} = 0 \qquad \text{(D.2)}
$$

$$
\left.\left(\frac{\delta}{2\lambda} - (r^0)^2\widehat{h}_0^0 - q_1^\beta r_0^1 \widehat{h}_0^1\right)^3\right|_{r_{T-2}^1 = B_{T-2}^\beta} = 0, \qquad \text{(D.3)}
$$

the value function can be simplified to

$$
\begin{aligned}
&V_0(\widehat{h}_0^0, \widehat{h}_0^1) \\
&=\max -\frac{1}{\overline{r_u}-\underline{r_u}} \left\{\delta\left[(r^0)^2\widehat{h}_0^0\underline{r_u} + \frac{1}{2}\operatorname{E}(R_1^1)\widehat{h}_0^1\underline{r_u}^2\right]\right. \\
&\qquad -\lambda\left[\left((r^0)^2\widehat{h}_0^0\right)^2\underline{r_u} + (r^0)^2\operatorname{E}(R_1^1)\widehat{h}_0^0\widehat{h}_0^1\underline{r_u}^2 + \frac{1}{3}\operatorname{E}(R_1^1)^2\left(\widehat{h}_0^1\right)^2\underline{r_u}^3\right] \\
&\qquad \left. -\frac{1}{3}d_1^\alpha\lambda\left(q_1^\alpha\widehat{h}_0^1\right)^{-1}\left[\frac{\delta}{2\lambda} - (r^0)^2\widehat{h}_0^0 - q_1^\alpha\widehat{h}_0^1\underline{r_u}\right]^3\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\overline{r_u}-\underline{r_u}}\left\{\delta\left[(r^0)^2\widehat{h}_0^0\overline{r_u}+\frac{1}{2}\,\mathrm{E}(R_1^1)\widehat{h}_0^1\overline{r_u}^2\right]\right.\\
&\qquad-\lambda\left[\left((r^0)^2\widehat{h}_0^0\right)^2\overline{r_u}+(r^0)^2\,\mathrm{E}(R_1^1)\widehat{h}_0^0\widehat{h}_0^1\overline{r_u}^2+\frac{1}{3}\,\mathrm{E}(R_1^1)^2\left(\widehat{h}_0^1\right)^2\overline{r_u}^3\right]\\
&\qquad\left.-\frac{1}{3}d_1^\beta\lambda\left(q_1^\beta\widehat{h}_0^1\right)^{-1}\left[\frac{\delta}{2\lambda}-(r^0)^2\widehat{h}_0^0-q_1^\beta\widehat{h}_0^1\overline{r_u}\right]^3\right\}\\
=&\max\delta\left[(r^0)^2\widehat{h}_0^0+\,\mathrm{E}(R_1^1)\widehat{h}_0^1\frac{\underline{r_u}+\overline{r_u}}{2}\right]-\lambda\left[\left((r^0)^2\widehat{h}_0^0\right)^2\right.\\
&\qquad\left.+(r^0)^2\,\mathrm{E}(R_1^1)\widehat{h}_0^0\widehat{h}_0^1(\underline{r_u}+\overline{r_u})+\frac{1}{3}\,\mathrm{E}(R_1^1)^2\left(\widehat{h}_0^1\right)^2(\underline{r_u}^2+\underline{r_u}\overline{r_u}+\overline{r_u}^2)\right]\\
&-\frac{\lambda}{\overline{r_u}-\underline{r_u}}\left\{\frac{1}{3}\frac{\left(\frac{\delta}{2\lambda}-(r^0)^2\widehat{h}_0^0\right)^3}{\widehat{h}_0^1}\left(\frac{d_1^\beta}{q_1^\beta}-\frac{d_1^\alpha}{q_1^\alpha}\right)\right.\\
&\qquad-\left(\frac{\delta}{2\lambda}-(r^0)^2\widehat{h}_0^0\right)^2\left(d_1^\beta\overline{r_u}-d_1^\alpha\underline{r_u}\right)\\
&\qquad+\left(\frac{\delta}{2\lambda}-(r^0)^2\widehat{h}_0^0\right)\widehat{h}_0^1\left(d_1^\beta q_1^\beta\overline{r_u}^2-d_1^\alpha q_1^\alpha\underline{r_u}^2\right)\\
&\qquad\left.-\frac{1}{3}\left(\widehat{h}_0^1\right)^2\left(d_1^\beta(q_1^\beta)^2\overline{r_u}^3-d_1^\beta(q_1^\alpha)^2\underline{r_u}^3\right)\right\}
\end{aligned}
\tag{D.4}
$$

We assumed that $R_0^1 \sim U(\underline{r_u}, \overline{r_u})$, which implies

$$
\mathrm{E}(R_0^1) = \frac{\underline{r_u}+\overline{r_u}}{2} \tag{D.5}
$$

$$
\mathrm{E}(R_0^1)^2 = \frac{\underline{r_u}^2+\underline{r_u}\overline{r_u}+\overline{r_u}^2}{3}. \tag{D.6}
$$

Therefore, the value function can be further simplified to:

$$
\begin{aligned}
&V_0(\widehat{h}_0^0, \widehat{h}_0^1) \\
&= \max \delta \, \mathrm{E}\left[(r^0)^2 \widehat{h}_0^0 + R_1^1 R_0^1 \widehat{h}_0^1\right] - \lambda \, \mathrm{E}\left[(r^0)^2 \widehat{h}_0^0 + R_{T-1}^1 R_0^1 \left(\widehat{h}_0^1\right)\right]^2 \\
&\quad - \frac{\lambda}{\overline{r_u} - \underline{r_u}} \left\{ \frac{1}{3} \frac{\left(\frac{\delta}{2\lambda} - (r^0)^2 \widehat{h}_0^0\right)^3}{\widehat{h}_0^1} \left(\frac{d_1^\beta}{q_1^\beta} - \frac{d_1^\alpha}{q_1^\alpha}\right) - \left(\frac{\delta}{2\lambda} - (r^0)^2 \widehat{h}_0^0\right)^2 \left(d_1^\beta \overline{r_u} - d_1^\alpha \underline{r_u}\right) \right. \\
&\qquad + \left(\frac{\delta}{2\lambda} - (r^0)^2 \widehat{h}_0^0\right) \widehat{h}_0^1 \left(d_1^\beta q_1^\beta \overline{r_u}^2 - d_1^\alpha q_1^\alpha \underline{r_u}^2\right) \\
&\qquad \left. - \frac{1}{3} \left(\widehat{h}_0^1\right)^2 \left(d_1^\beta (q_1^\beta)^2 \overline{r_u}^3 - d_1^\beta (q_1^\alpha)^2 \underline{r_u}^3\right) \right\} \qquad \text{(D.7)} \\
&= \max \delta \, \mathrm{E}\left[(r^0)^2 \widehat{h}_0^0 + R_{T-1}^1 R_0^1 \widehat{h}_0^1\right] - \lambda \, \mathrm{E}\left[(r^0)^2 \widehat{h}_0^0 + R_{T-1}^1 R_0^1 \left(\widehat{h}_0^1\right)\right]^2 \\
&\quad - \frac{\lambda}{3(\overline{r_u} - \underline{r_u}) \widehat{h}_0^1} \left\{ \frac{d_1^\beta}{q_1^\beta} \left[\frac{\delta}{2\lambda} - (r^0)^2 \widehat{h}_0^0 - q_1^\beta \overline{r_u} \widehat{h}_0^1\right]^3 \right. \\
&\qquad \left. - \frac{d_1^\alpha}{q_1^\alpha} \left[\frac{\delta}{2\lambda} - (r^0)^2 \widehat{h}_0^0 - q_1^\alpha \underline{r_u} \widehat{h}_0^1\right]^3 \right\} \qquad \text{(D.8)}
\end{aligned}
$$

Appendix E

Expressions for $\mathrm{E}(w_2(\gamma))$ and $\mathrm{E}(w_2(\gamma))^2$

The expectation of w_2 is

$$\begin{aligned}
&\mathrm{E}(w_2(\gamma)) \\
&\quad = \int_{\underline{r_u}}^{B^\alpha_{T-2}} p(r^1_0) w^{(1)}_1 \,\mathrm{d}r^1_0 + \int_{B^\alpha_{T-2}}^{B^\beta_{T-2}} p(r^1_0) w^{(2)}_1 \,\mathrm{d}r^1_0 + \int_{B^\beta_{T-2}}^{\overline{r_u}} p(r^1_0) w^{(3)}_1 \,\mathrm{d}r^1_0 \\
&\quad = (r^0)^2 w_0 + \mathrm{E}(\xi^\alpha_0) \left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q^\alpha_0 - (r^0)^2 (1+\alpha)} \right) \\
&\qquad + \frac{1}{2(\overline{r_u} - \underline{r_u})} \frac{d^\alpha_1}{q^\alpha_1} \left(q^\alpha_0 - q^\alpha_1 \underline{r_u} \right)^2 \left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q^\alpha_0 - (r^0)^2 (1+\alpha)} \right) \\
&\qquad - \frac{1}{2(\overline{r_u} - \underline{r_u})} \frac{d^\beta_1}{q^\beta_1} \left(q^\alpha_0 - q^\beta_1 \overline{r_u} \right)^2 \left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q^\alpha_0 - (r^0)^2 (1+\alpha)} \right) \\
&\quad = (r^0)^2 w_0 + \rho \left(\frac{\frac{\gamma}{2} - (r^0)^2 h^0_0}{q^\alpha_0 - (r^0)^2 (1+\alpha)} \right),
\end{aligned} \tag{E.1}$$

where

$$\rho = \mathrm{E}(\xi_0^\alpha) + \frac{1}{2(\overline{r_u} - \underline{r_u})}\frac{d_1^\alpha}{q_1^\alpha}\left(q_0^\alpha - q_1^\alpha \underline{r_u}\right)^2 - \frac{1}{2(\overline{r_u} - \underline{r_u})}\frac{d_1^\beta}{q_1^\beta}\left(q_0^\alpha - q_1^\beta \overline{r_u}\right)^2.$$

The expectation of $(w_2)^2$ is

$$\begin{aligned}
&\mathrm{E}(w_2(\xi))^2 \\
&\quad= \int_{\underline{r_u}}^{B_{T-2}^\alpha} p(r_0^1)\left(w_1^{(1)}\right)^2 \mathrm{d}r_0^1 + \int_{B_{T-2}^\alpha}^{B_{T-2}^\beta} p(r_0^1)\left(w_1^{(2)}\right)^2 \mathrm{d}r_0^1 + \int_{B_{T-2}^\beta}^{\overline{r_u}} p(r_0^1)\left(w_1^{(3)}\right)^2 \mathrm{d}r_0^1 \\
&\quad= \mathrm{E}\left[(r^0)^2\widehat{h}_0^0 + R_1^1 R_0^1 \widehat{h}_0^1\right]^2 + \frac{1}{(\overline{r_u} - \underline{r_u})\widehat{h}_0^1}\left\{-\frac{1}{3}\frac{d_1^\alpha}{q_1^\alpha}\left(\frac{\gamma}{2} - (r^0)^2\widehat{h}_0^0 - q_1^\alpha \underline{r_u}\widehat{h}_0^1\right)^3\right. \\
&\qquad + \frac{1}{3}\frac{d_1^\beta}{q_1^\beta}\left(\frac{\gamma}{2} - (r^0)^2\widehat{h}_0^0 - q_1^\beta \overline{r_u}\widehat{h}_0^1\right)^3 + \frac{d_1^\alpha}{q_1^\alpha}\left(\frac{\gamma}{2} - (r^0)^2\widehat{h}_0^0 - q_1^\alpha \underline{r_u}\widehat{h}_0^1\right)^2 \frac{\gamma}{2} \\
&\qquad \left. - \frac{d_1^\beta}{q_1^\beta}\left(\frac{\gamma}{2} - (r^0)^2\widehat{h}_0^0 - q_1^\beta \overline{r_u}\widehat{h}_0^1\right)^2 \frac{\gamma}{2}\right\} \\
&\quad= \left((r^0)^2 w_0\right)^2 + 2(r^0)^2 w_0\, \mathrm{E}(\xi_0^\alpha)\left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q_0^\alpha - (r^0)^2(1+\alpha)}\right) \\
&\qquad + \mathrm{E}(\xi_0^\alpha)^2\left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q_0^\alpha - (r^0)^2(1+\alpha)}\right)^2 \\
&\qquad + \frac{1}{\overline{r_u} - \underline{r_u}}\left\{\frac{1}{3}\left[\frac{d_1^\beta}{q_1^\beta}\left(q_0^\alpha - q_1^\beta \overline{r_u}\right)^3 - \frac{d_1^\alpha}{q_1^\alpha}\left(q_0^\alpha - q_1^\alpha \underline{r_u}\right)^3\right]\left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q_0^\alpha - (r^0)^2(1+\alpha)}\right)^2\right. \\
&\qquad \left. - \left[\frac{d_1^\beta}{q_1^\beta}\left(q_0^\alpha - q_1^\beta \overline{r_u}\right)^2 - \frac{d_1^\alpha}{q_1^\alpha}\left(q_0^\alpha - q_1^\alpha \underline{r_u}\right)^2\right]\left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q_0^\alpha - (r^0)^2(1+\alpha)}\right)\frac{\gamma}{2}\right\} \\
&\quad= \nu\left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q_0^\alpha - (r^0)^2(1+\alpha)}\right)^2 + 2\rho(r^0)^2 w_0\left(\frac{\frac{\gamma}{2} - (r^0)^2 w_0}{q_0^\alpha - (r^0)^2(1+\alpha)}\right) + \left((r^0)^2 w_0\right)^2,
\end{aligned} \tag{E.2}$$

where

$$\nu = \mathrm{E}(\xi_0^\alpha)^2 + \frac{1}{3(\overline{r_u} - \underline{r_u})} \left[\frac{d_1^\beta}{q_1^\beta} \left(q_0^\alpha - q_1^\beta \overline{r_u} \right)^3 - \frac{d_1^\alpha}{q_1^\alpha} \left(q_0^\alpha - q_1^\alpha \underline{r_u} \right)^3 \right]$$

$$- \frac{1}{\overline{r_u} - \underline{r_u}} \left[\frac{d_1^\beta}{q_1^\beta} \left(q_0^\alpha - q_1^\beta \overline{r_u} \right)^2 - \frac{d_1^\alpha}{q_1^\alpha} \left(q_0^\alpha - q_1^\alpha \underline{r_u} \right)^2 \right]. \tag{E.3}$$

Appendix F

The Value of m_t

The optimal m_t is a minimizer of function $\left(\tilde{V}_t - V_t\right)^2$:

$$\begin{aligned}
&\left(\tilde{V}_t - V_t\right)^2 \\
&\quad = \int_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}} \left\{ \int_{\frac{\rho - q_t^\alpha h_t^1}{q_t^\alpha/(1+\alpha)}}^{\frac{\rho - q_t^\alpha h_t^1}{(r^0)^{(T-t)}}} \left[m_t \left(\rho - g_t^\alpha\right) \left(\rho - g_t^\beta\right) - d_t^\alpha \left(\rho - g_t^\alpha\right)^2 \right]^2 \mathrm{d}h_t^0 \right. \\
&\qquad + \int_{\frac{\rho - q_t^\alpha h_t^1}{(r^0)^{(T-t)}}}^{\frac{\rho - q_t^\beta h_t^1}{r^0}} \left[m_t \left(\rho - g_t^\alpha\right) \left(\rho - g_t^\beta\right) \right]^2 \mathrm{d}h_t^0 \\
&\qquad \left. + \int_{\frac{\rho - q_t^\beta h_t^1}{r^0}}^{\frac{\rho - (r^0)^{(T-t)}(1-\beta) h_t^1}{(r^0)^{(T-t)}}} \left[m_t \left(\rho - g_t^\alpha\right) \left(\rho - g_t^\beta\right) - d_t^\beta \left(\rho - g_t^\beta\right)^2 \right]^2 \mathrm{d}h_t^0 \right\} \mathrm{d}h_t^1 \qquad \text{(F.1)}
\end{aligned}$$

As $\left(\tilde{V}_t - V_t\right)^2$ is obviously a strictly convex function of m_t, the optimal m_t is

$$m_t = -\frac{c_2}{2c_1} \qquad \text{(F.2)}$$

where c_1 and c_2 are the coefficients of the square term and linear term respectively. The expressions of c_1 and c_2 are as follows.

$$
\begin{aligned}
c_1 &= \int_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}} \left\{ \int_{\frac{\rho - q_t^\alpha h_t^1}{q_t^\alpha/(1+\alpha)}}^{\frac{\rho-(r^0)^{(T-t)}(1-\beta)h_t^1}{(r^0)^{(T-t)}}} \left[\left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right) \right.\right. \\
&\qquad \left.\left. \cdot \left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\beta h_t^1\right)\right]^2 \, \mathrm{d}h_t^0 \right\} \mathrm{d}h_t^1 \\
&= \int_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}} \left\{ \int_{\frac{\rho - q_t^\alpha h_t^1}{q_t^\alpha/(1+\alpha)}}^{\frac{\rho-(r^0)^{(T-t)}(1-\beta)h_t^1}{(r^0)^{(T-t)}}} \left[\left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^4 \right.\right. \\
&\qquad - 2\left(q_t^\beta - q_t^\alpha\right) h_t^1 \left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^3 \\
&\qquad \left.\left. + \left(q_t^\beta - q_t^\alpha\right)^2 (h^1)^2 \left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^2 \right] \mathrm{d}h_t^0 \right\} \mathrm{d}h_t^1 \\
&= \int_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}} \left\{ \left[\frac{\left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^5}{-5(r^0)^{(T-t)}} \right.\right. \\
&\qquad - 2\left(q_t^\beta - q_t^\alpha\right) h_t^1 \frac{\left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^4}{-4(r^0)^{(T-t)}} \\
&\qquad \left.\left. + \left(q_t^\beta - q_t^\alpha\right)^2 (h_t^1)^2 \frac{\left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^3}{-3(r^0)^{(T-t)}} \right] \Bigg|_{\frac{\rho - q_t^\alpha h_t^1}{q_t^\alpha/(1+\alpha)}}^{\frac{\rho-(r^0)^{(T-t)}(1-\beta)h_t^1}{(r^0)^{(T-t)}}} \right\} \mathrm{d}h_t^1
\end{aligned}
\tag{F.3}
$$

$$
\begin{aligned}
&= \int_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}} \Bigg\{ \Bigg[\frac{\left(q_t^\alpha - (r^0)^{(T-t)}(1-\beta)\right)^5}{5(r^0)^{(T-t)}} (h_t^1)^5 \\
&\qquad + \frac{\left(q_t^\beta - q_t^\alpha\right)\left(q_t^\alpha - (r^0)^{(T-t)}(1-\beta)\right)^4}{2(r^0)^{(T-t)}} (h_t^1)^5 \\
&\qquad + \frac{\left(q_t^\beta - q_t^\alpha\right)^2\left(q_t^\alpha - (r^0)^{(T-t)}(1-\beta)\right)^3}{3(r^0)^{(T-t)}} (h_t^1)^5 \Bigg] \\
&\qquad - \Bigg[-\frac{\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{5(r^0)^{(T-t)}} \left(\rho - q_t^\alpha h_t^1\right)^5 \\
&\qquad + \frac{\left(q_t^\beta - q_t^\alpha\right)\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{2(r^0)^{(T-t)}} \left(\rho - q_t^\alpha h_t^1\right)^4 h_t^1 \\
&\qquad - \frac{\left(q_t^\beta - q_t^\alpha\right)^2\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{3(r^0)^{(T-t)}} \left(\rho - q_t^\alpha h_t^1\right)^3 (h_t^1)^2 \Bigg] \Bigg\} \, \mathrm{d}h_t^1 \\
&= \Bigg[\frac{\left(q_t^\alpha - (r^0)^{(T-t)}(1-\beta)\right)^5}{30(r^0)^{(T-t)}} (h_t^1)^6 + \frac{\left(q_t^\beta - q_t^\alpha\right)\left(q_t^\alpha - (r^0)^{(T-t)}(1-\beta)\right)^4}{12(r^0)^{(T-t)}} (h_t^1)^6 \\
&\quad + \frac{\left(q_t^\beta - q_t^\alpha\right)^2\left(q_t^\alpha - (r^0)^{(T-t)}(1-\beta)\right)^3}{18(r^0)^{(T-t)}} (h_t^1)^6 - \frac{\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{30(r^0)^{(T-t)} q_t^\alpha} \left(\rho - q_t^\alpha h_t^1\right)^6 \\
&\quad + \frac{\left(q_t^\beta - q_t^\alpha\right)\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{2(r^0)^{(T-t)}(q_t^\alpha)^2} \frac{\rho + 5q_t^\alpha h_t^1}{30} \left(\rho - q_t^\alpha h_t^1\right)^5 \\
&\quad - \frac{\left(q_t^\beta - q_t^\alpha\right)^2\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{3(r^0)^{(T-t)}(q_t^\alpha)^3} \Bigg(\frac{\left(\rho - q_t^\alpha h_t^1\right)^6}{6} - 2\rho \frac{\left(\rho - q_t^\alpha h_t^1\right)^5}{5} \\
&\qquad + (\rho)^2 \frac{\left(\rho - q_t^\alpha h_t^1\right)^4}{4} \Bigg) \Bigg] \Bigg|_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}} \qquad \text{(F.4)}
\end{aligned}
$$

$$
\begin{aligned}
= \Bigg[\Bigg(& \frac{\left(q_t^\alpha - (r^0)^{(T-t)}(1-\beta)\right)^5}{30(r^0)^{(T-t)}} + \frac{\left(q_t^\beta - q_t^\alpha\right)\left(q_t^\alpha - (r^0)^{(T-t)}(1-\beta)\right)^4}{12(r^0)^{(T-t)}} \\
& + \frac{\left(q_t^\beta - q_t^\alpha\right)^2\left(q_t^\alpha - (r^0)^{(T-t)}(1-\beta)\right)^3}{18(r^0)^{(T-t)}} \Bigg) \left(\frac{1}{(r^0)^{(T-t)}(1-\beta)}\right)^6 \\
& - \frac{\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{30(r^0)^{(T-t)} q_t^\alpha} \left(1 - \frac{q_t^\alpha}{(r^0)^{(T-t)}(1-\beta)}\right)^6 \\
& - \frac{\left(q_t^\beta - q_t^\alpha\right)\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{2(r^0)^{(T-t)}(q_t^\alpha)^2} \left(\frac{\left(1 - \frac{q_t^\alpha}{(r^0)^{(T-t)}(1-\beta)}\right)^6}{6} - \frac{\left(1 - \frac{q_t^\alpha}{(r^0)^{(T-t)}(1-\beta)}\right)^5}{5} \right) \\
& - \frac{\left(q_t^\beta - q_t^\alpha\right)^2\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{3(r^0)^{(T-t)}(q_t^\alpha)^3} \Bigg(\frac{\left(1 - \frac{q_t^\alpha}{(r^0)^{(T-t)}(1-\beta)}\right)^6}{6} - \frac{2\left(1 - \frac{q_t^\alpha}{(r^0)^{(T-t)}(1-\beta)}\right)^5}{5} \\
& \qquad + \frac{\left(1 - \frac{q_t^\alpha}{(r^0)^{(T-t)}(1-\beta)}\right)^4}{4} \Bigg) \\
& + \frac{\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{30(r^0)^{(T-t)} q_t^\alpha} - \left(q_t^\beta - q_t^\alpha\right) \frac{\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{60(r^0)^{(T-t)}\left(q_t^\alpha\right)^2} \\
& + \left(q_t^\beta - q_t^\alpha\right)^2 \frac{\left(1 - \frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{180(r^0)^{(T-t)}\left(q_t^\alpha\right)^3} \Bigg] (\rho)^6 \qquad \text{(F.5)}
\end{aligned}
$$

and

$$
\begin{aligned}
c_2 =& \int_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}} \left\{ \int_{\frac{\rho-q_t^\alpha h_t^1}{q_t^\alpha/(1+\alpha)}}^{\frac{\rho-q_t^\alpha h_t^1}{(r^0)^{(T-t)}}} \left[-2d_t^\alpha \left(\rho - g_t^\alpha\right)^3 \left(\rho - g_t^\beta\right)\right] \mathrm{d}h_t^0 \right. \\
&+ \left. \int_{\frac{\rho-q_t^\beta h_t^1}{r^0}}^{\frac{\rho-(r^0)^{(T-t)}(1-\beta)h_t^1}{(r^0)^{(T-t)}}} \left[-2d_t^\beta \left(\rho - g_t^\alpha\right) \left(\rho - g_t^\beta\right)^3\right] \mathrm{d}h_t^0 \right\} \mathrm{d}h_t^1 \\
=& \int_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}} \left\{ \int_{\frac{\rho-q_t^\alpha h_t^1}{q_t^\alpha/(1+\alpha)}}^{\frac{\rho-q_t^\alpha h_t^1}{(r^0)^{(T-t)}}} -2d_t^\alpha \left[\left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^4 \right. \right. \\
&- \left. \left(q_t^\beta - q_t^\alpha\right) h_t^1 \left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^3 \right] \mathrm{d}h_t^0 \\
&+ \int_{\frac{\rho-q_t^\beta h_t^1}{r^0}}^{\frac{\rho-(r^0)^{(T-t)}(1-\beta)h_t^1}{(r^0)^{(T-t)}}} -2d_t^\beta \left[\left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\beta h_t^1\right)^4 \right. \\
&+ \left. \left. \left(q_t^\beta - q_t^\alpha\right) h_t^1 \left(\rho - (r^0)^{(T-t)} h_t^0 - q_t^\beta h_t^1\right)^3 \right] \mathrm{d}h_t^0 \right\} \mathrm{d}h_0^1 \\
=& 2\int_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}} \left\{ -d_t^\alpha \left(-\frac{\left(\frac{\delta}{2\lambda} - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^5}{5(r^0)^{(T-t)}} \right. \right. \\
&+ \left. \left(q_t^\beta - q_t^\alpha\right) h_t^1 \frac{\left(\frac{\delta}{2\lambda} - (r^0)^{(T-t)} h_t^0 - q_t^\alpha h_t^1\right)^4}{4(r^0)^{(T-t)}} \right) \Bigg|_{\frac{\rho-q_t^\alpha h_t^1}{q_t^\alpha/(1+\alpha)}}^{\frac{\rho-q_t^\alpha h_t^1}{(r^0)^{(T-t)}}} \\
&- d_t^\beta \left(-\frac{\left(\frac{\delta}{2\lambda} - (r^0)^{(T-t)} h_t^0 - q_t^\beta h_t^1\right)^5}{5(r^0)^{(T-t)}} \right. \\
&- \left. \left. \left(q_t^\beta - q_t^\alpha\right) h_t^1 \frac{\left(\frac{\delta}{2\lambda} - (r^0)^{(T-t)} h_t^0 - q_t^\beta h_t^1\right)^4}{4(r^0)^{(T-t)}} \right) \Bigg|_{\frac{\rho-q_t^\beta h_t^1}{r^0}}^{\frac{\rho-(r^0)^{(T-t)}(1-\beta)h_t^1}{(r^0)^{(T-t)}}} \right\} \mathrm{d}h_t^1
\end{aligned}
$$

$$
\begin{aligned}
=&2\int_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}}\left\{d_t^\alpha\left(-\frac{\left(1-\frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{5(r^0)^{(T-t)}}\left(\rho-q_t^\alpha h_t^1\right)^5\right.\right.\\
&\qquad\left.+\left(q_t^\beta-q_t^\alpha\right)\frac{\left(1-\frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{4(r^0)^{(T-t)}}\left(\rho-q_t^\alpha h_t^1\right)^4 h_t^1\right)\\
&\quad-d_t^\beta\left(\frac{\left(q_t^\beta-(r^0)^{(T-t)}(1-\beta)\right)^5}{5(r^0)^{(T-t)}}(h_t^1)^5\right.\\
&\qquad\left.\left.-\left(q_t^\beta-q_t^\alpha\right)\frac{\left(q_t^\beta-(r^0)^{(T-t)}(1-\beta)\right)^4}{4(r^0)^{(T-t)}}(h_t^1)^5\right)\right\}\,\mathrm{d}h_t^1\\
=&\left[2d_t^\alpha\left(\frac{\left(1-\frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{30(r^0)^{(T-t)}q_t^\alpha}\left(\rho-q_t^\alpha h_t^1\right)^6\right.\right.\\
&\quad\left.+\left(q_t^\beta-q_t^\alpha\right)\frac{\left(1-\frac{(r^0)^{(T-t)}}{q_t^\alpha/(1+\alpha)}\right)}{4(r^0)^{(T-t)}(q_t^\alpha)^2}\left(\frac{\left(\rho-q_t^\alpha h_t^1\right)^6}{6}-\rho\frac{\left(\rho-q_t^\alpha h_t^1\right)^5}{5}\right)\right)\\
&\quad-2d_t^\beta\left(\frac{\left(q_t^\beta-(r^0)^{(T-t)}(1-\beta)\right)^5}{30(r^0)^{(T-t)}}\left(h_t^1\right)^6\right.\\
&\quad\left.\left.-\left(q_t^\beta-q_t^\alpha\right)\frac{\left(q_t^\beta-(r^0)^{(T-t)}(1-\beta)\right)^4}{24(r^0)^{(T-t)}}(h_t^1)^6\right)\right]\Bigg|_0^{\frac{\rho}{(r^0)^{(T-t)}(1-\beta)}}
\end{aligned}
$$

$$
\begin{aligned}
=\Bigg\{ & 2d_t^{\alpha}\left[\frac{\left(1-\frac{(r^0)^{(T-t)}}{q_t^{\alpha}/(1+\alpha)}\right)}{30(r^0)^{(T-t)}q_t^{\alpha}}\left(1-\frac{q_t^{\alpha}}{(r^0)^{(T-t)}(1-\beta)}\right)^6\right.\\
&\left.+\left(q_t^{\beta}-q_t^{\alpha}\right)\frac{\left(1-\frac{(r^0)^{(T-t)}}{q_t^{\alpha}/(1+\alpha)}\right)}{4(r^0)^{(T-t)}(q_t^{\alpha})^2}\left(\frac{\left(1-\frac{q_t^{\alpha}}{(r^0)^{(T-t)}(1-\beta)}\right)^6}{6}-\frac{\left(1-\frac{q_t^{\alpha}}{(r^0)^{(T-t)}(1-\beta)}\right)^5}{5}\right)\right]\\
&-2d_t^{\alpha}\left[\frac{\left(1-\frac{(r^0)^{(T-t)}}{q_t^{\alpha}/(1+\alpha)}\right)}{30(r^0)^{(T-t)}q_t^{\alpha}}-\left(q_t^{\beta}-q_t^{\alpha}\right)\frac{\left(1-\frac{(r^0)^{(T-t)}}{q_t^{\alpha}/(1+\alpha)}\right)}{120(r^0)^{(T-t)}(q_t^{\alpha})^2}\right]\\
&-2d_t^{\beta}\left[\frac{\left(q_t^{\beta}-(r^0)^{(T-t)}(1-\beta)\right)^5}{30(r^0)^{(T-t)}}\left(\frac{1}{(r^0)^{(T-t)}(1-\beta)}\right)^6\right.\\
&\left.-\left(q_t^{\beta}-q_t^{\alpha}\right)\frac{\left(q_t^{\beta}-(r^0)^{(T-t)}(1-\beta)\right)^4}{24(r^0)^{(T-t)}}\left(\frac{1}{(r^0)^{(T-t)}(1-\beta)}\right)^6\right]\Bigg\}(\rho)^6 \quad \text{(F.6)}
\end{aligned}
$$