CHARACTERIZATION AND TRANSFORMATION OF COUNTRYMAN LINES AND $\mathbb{R}$-EMBEDDABLE COHERENT TREES IN ZFC

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DECLARATION

I hereby declare that the thesis is my original work and it has been written by me in its entirety.
I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

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January 2013

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Contents

Summary

Chapter 1 will review the background of linear orders and tree orders.

Chapter 2 will review and generate the results on transformation under MA_{ω_1} mainly coming from [1]. In particular, we prove that, under MA_{ω_1}, coherent and Countryman are two equivalent conditions.

Chapter 3 will investigate the transformation from a $\mathbb{R}$-embeddable coherent tree to a Countryman line. We first show that every Countryman line has a $\mathbb{R}$-embeddable partition tree and explain the necessity of $\mathbb{R}$-embeddability. We then show that for any countable linear order O that cannot be embedded into $\mathbb{Z}$, it is consistent to have a $\mathbb{R}$-embeddable coherent tree $T \subset O^{<\omega_1}$ which contains no Countryman suborder (with its lexicographical order). This gives a negative answer to the question whether there is a transformation from $\mathbb{R}$-embeddable coherent trees to Countryman lines. In chapter 3, we will also give an equivalent formulation for a $\mathbb{R}$-embeddable coherent tree T to contain a Countryman suborder $X - T_X$ is a subtree of some $\mathbb{R}$-embeddable coherent $T' \subset \mathbb{Z}^{<\omega_1}$. As for the problem of containing Countryman suborder, a condition under which an $\mathbb{R}$-embeddable coherent tree is Countryman (with its lexicographical order) is also discussed in chapter 3.

Chapter 4 will discuss several properties related to the transformation from Countryman to coherent. As a particular case, we show how a small part of a partition tree of some Countryman line affects a large part of the partition tree. We also show how different Countryman types affect the minimal size of basis for Countryman lines. We expect that the properties found in this chapter may lead to interesting applications.

Chapter 1

Introduction

The notion of a linear order is a useful concept and plays an important role in mathematics. The theory of linear orders was first systematically studied by Cantor (Cantor, 1895) when he proved that $\mathbb{Q}$ is the unique countable dense linear order without endpoint. Since then, the class of countable linear orders has been well studied. There are some very pleasant and deep properties of countable lines: Laver's theorem (Laver, 1971) – the class of countable scattered orderings is well-quasi-ordered under embeddability; the class of countable orders admits a 2-element basis – every countable line contains a subline of type ω or ω^*; Universal object (Cantor, 1895) – every countable line can be embedded into $\mathbb{Q}$; etc. So the next natural step is to attempt to develop a similar structure theory for linear orders of the next cardinality – $\aleph_1$.

As we can see, well ordering ω_1, its reverse ω_1^* and uncountable subset of real numbers are uncountable lines, but there are more. Kurepa (Kurepa, 1935) first systematically studied trees and their connection to lines, using lexicographical order to get a line from a tree and partition tree to get a tree from a line. Then Aronszajn (Aronszajn, 1935) constructed an uncountable tree (which is today called Aronszajn tree) and its lexicographical order contains no suborder isomorphic to

ω_1, ω_1^* or uncountable set of reals. It has been also observed (see, for example, S. Todorcevic, 1984) that every Aronszajn line – uncountable line with no suborders of type ω_1, ω_1^* or uncountable set of reals – is isomorphic to a lexicographically ordered Aronszajn tree.

The structure of ω_1 and ω_1^* is very clear. For uncountable sets of real numbers, Baumgartner showed in (Baumgartner, 1973) that PFA implies that any two $\aleph_1$ dense subsets of reals are isomorphic. Hence, under PFA, a strong structure theory for sets reals of size $\aleph_1$ follows: well-quasi-ordered, one element basis, universal, etc. However, similar problem about Aronszajn line was not clear until recently.

This chapter will historically introduce a subclass of Aronszajn linear orders, called Countryman lines and the corresponding class of coherent trees (their lexicographical order – coherent line – too). Moreover, we also present some results characterizing the Countryman line using coherent trees.

1.1 Countryman lines and coherent trees

Even after the method of partition trees and lexicographical orders was introduced, the structure of Aronszajn line remained still not clear enough to solve problems like, for example, the basis for Aronszajn lines. In 1970, R. Countryman asked whether there is an uncountable linear order whose cartesian square is a countable union of chains. Such a linear order is today known as *Countryman line* and it is noticed by Countryman himself that every Countryman line is an Aronszajn line. Then Shelah in [9] constructed the first example of a Countryman line and pointed out that a Countryman line and its reverse have no uncountable isomorphic suborder which means that any basis for Countryman lines and hence Aronszajn lines must have at least 2 elements. Shelah then conjectured that it is consistent that Countryman lines serve as a basis for Aronszajn lines and moreover that it is consistent to have a 2-element basis for Countryman lines – a Countryman line C and its reverse C^*.

Shelah's conjecture remained out of reach until S. Todorcevic in [5] introduced his method of minimal walks on ordinals and used it to produce a number of concrete trees and lines that are both Countryman (with their lexicographical orders) and coherent (any 2 elements are different at only finite many places as functions). His paper [4] makes this connection even more explicit by building a deep structure theory of Aronszajn and coherent trees valid under PFA. Then, building on this, J. Moore in [6] proved that PFA gives a positive answers to both of the two conjectures of Shelah. In his theorem, the two Countryman lines that form the basis for the class of Aronszajn lines are also coherent and in fact the coherence plays an important role in his proof.

Besides the solution to the basis problem for Aronszajn lines, Countryman lines and coherent trees are useful in other problems. For example [8], J. Moore has used Countryman lines to construct (under PFA) a universal Aronszajn line – every Aronszajn line can be embedded into it (an analogue to the role of $\mathbb{Q}$ in the class of countable lines). In [21], Martinez-Ranero proved (under PFA) the analogue of Laver's theorem in this context, stating that the class of Aronszajn lines is well-quasi-ordered under embeddability. Countryman property and coherence property have other important usages too, e.g., partition problems solved in [5], L space problems solved in [7], etc.

All above mentioned papers (and almost all papers about Countryman lines or coherent trees that I know) use Countryman (or coherent) lines that are both Countryman and coherent. Therefore, a nature question arises: are they the same? The question actually has two sides:

1. is every Countryman line coherent?

2. is every coherent line[1] Countryman?

The answer to each question should give us a better description of both Countryman lines and coherent trees and lines, and moreover, it should direct us towards a new direction for further research and application of Countryman lines and coherent trees.

1.2 Transformations under MA_{ω_1}

As we know that every partition tree of a Countryman line is an Aronszajn tree. However, Aronszajn is not enough to describe the Countryman property, since for example, Countryman line is preserved by forcing which preserves ω_1 while Aronszajn may be not. Here we choose the coherent property to characterize Countryman and investigating the deep connection between them. One motivation is that there are some earlier results on transformation between Countryman lines and coherent trees.

First, S. Todorcevic in [5] showed that trees with properties ω-ranging (i.e., every element of the tree is a function from some ordinal to ω), coherent (i.e. the difference of two functions is finite) and finite-to-one (i.e. the pre-image of any element is finite) can be transformed into Countryman lines (i.e., the tree with its canonical lexicographical order is Countryman). However, these properties are very strong and it is not likely to be invertible. Then J. T. Moore in [6] showed under PFA that every Aronszajn tree contains an uncountable subset that is a Countryman line with its lexicographical order. His proof uses a part of the Proper Forcing Axiom whose consistency needs some large cardinal assumption. Recently, S. Todorcevic has done more research on coherent trees and given a clearer description of the transformation under MA_{ω_1} which is equiconsistent with ZFC.

[1] The coherent line mentioned here should be $\mathbb{R}$-embeddable.

In [1], he proved that every special coherent Aronszajn tree is a Countryman line with its lexicographical order. Together with a well known fact that under MA_{ω_1}, every Aronszajn tree is special, we can observe the transformation from the coherent tree to Countryman line, i.e., under MA_{ω_1}, every coherent tree with its canonical lexicographical order is a Countryman line. This result is highly useful for providing important information on transformation from coherent tree to Countryman line. There is also an unpublished S. Todorcevic's result whose idea can also be found in [1]: every Countryman line has a Lipschitz partition tree. Lipschitz mentioned above is a property for trees introduced by S. Todorcevic and in [1] he proved that under MA_{ω_1}, every Lipschitz tree has a lexicographical order which is coherent, i.e., for any Lipschitz tree, there is a coherent tree which is tree isomorphic to it. The above mentioned two results give an important contribution on transformation from Countryman lines into coherent trees, i.e., under MA_{ω_1}, every Countryman line is isomorphic to an uncountable subset of some coherent tree with its lexicographical order. This gives a clear description between coherent trees and Countryman lines under MA_{ω_1}, and gives a natural guessing that this may be true in ZFC. However, the above results are still limited to some forcing axiom which is independent of ZFC and neither transformation is known in ZFC. One major difficulty in studying the transformation between linear order and tree order is that the transformation needs the lexicographical order which means tree isomorphism cannot describe the linear structure and what we need is lexicographically isomorphism(i.e., preserves both tree order and linear order) which is not well studied.

1.3 Objectives

In view of the previous review, the following gaps still exist in the study on transformation between Countryman lines and $\mathbb{R}$-embeddable coherent trees:

1. Whether $\mathbb{R}$-embeddable is necessary.

2. The transformation between Countryman lines and $\mathbb{R}$-embeddable coherent trees under ZFC.

The transformation exists by assuming some additional forcing axiom, and this can decide the consistency of the transformation. But the transformation without any additional axiom is still unclear. So to see the whole picture of the transformation between Countryman lines and $\mathbb{R}$-embeddable coherent trees, we still need to find whether it is a consequence of ZFC or its negation is consistent.

The main aims of this study were:

1. to summarize and give a complete description of the transformation between Countryman lines and $\mathbb{R}$-embeddable coherent trees under MA_{ω_1}.

2. to investigate whether $\mathbb{R}$-embeddable is necessary.

3. to investigate the transformation between Countryman lines and $\mathbb{R}$-embeddable coherent trees in ZFC.

4. to investigate several properties of Countryman line and coherent tree for better understanding and further research.

Chapter 2 proves the existence of the transformation under MA_{ω_1} which completes aim 1. Chapter 3.1 proves the necessity of $\mathbb{R}$-embeddability which complete aim 2. Chapter 3.2 constructs different models that provides different relations between Countryman property and coherent property and some model contains a coherent line which cannot be transformed to a Countryman line. This completes one side of aim 3. Chapter 4 investigates the size of basis for Countryman lines. All chapter are related to aim 4.

The results of this study may contribute to a better understanding of relations between Countryman lines and coherent trees. Also, result on connection between $\mathbb{R}$-embeddable and ranging type for Countryman may give a new view on different type of Countryman lines, i.e., how simple can a countable linear order O be such that the Countryman line can be partitioned into a $\mathbb{R}$-embeddable O-ranging tree. It is understood that besides $\mathbb{R}$-embeddable coherent trees, there are some other kinds of interesting coherent trees, for example, Souslin coherent trees, which can be transformed into a Souslin line and there might be some deeper connection between these tree orders and linear orders. But since they cannot be transformed into Countryman lines, they are beyond the scope of this study.

1.4 Preliminaries

Definition 1. An uncountable linear ordering L is *Countryman* (or say *a Countryman line*) if its square is a countable union of chains under product order, i.e., there is a partition

$$c : L^2 \to \omega$$

such that for any $(a, b), (a', b')$, if $c(a, b) = c(a', b')$, then $a < a' \to b \leq b'$.

Definition 2. (1) A partial order $(T, <_T)$ is a *tree order* if the set of predecessors of each element is a well order, i.e., for any x in T, $pred_T(x) = \{y \in T : y <_T x\}$ is a well order. In this case, call T (or $(T, <_T)$) a tree.
(2) If $(T, <_T)$ is a tree order, x is in T, the *height of x in T* (written as $ht_T(x)$ or $ht(x)$ if there is no confusing) is the order type of $pred_T(x)$, i.e., the ordinal α such that $(\alpha, \in)$ is isomorphic to $(pred_T(x), <_T)$. The *height of T* is $ht(T) = sup\{ht(x) + 1 : x \in T\}$. The *$\alpha$-th level of T* is $T_\alpha = \{x \in T : ht(x) = \alpha\}$. Also some notation: $T \restriction_\alpha = \bigcup_{\beta<\alpha} T_\beta$, $T^t = \{x \in T : t \leq_T x\}$ and for $\alpha \leq ht(x)$, $x \restriction_\alpha$ is the $y \in T_\alpha$ such that $y \leq_T x$.
(3) If $(T, <_T)$ is a tree, we say T *has unique limits* if for any $x, y \in T$ such that

$ht(x) = ht(y)$ is a limit ordinal, $x = y$ iff $pred_T(x) = pred_T(y)$.[2]

(4) If $(T, <_T)$ is a tree, we say $x, y \in T$ are *incomparable* (written as $x \perp y$) if $x \nleq_T y \wedge y \nleq_T x$, otherwise they are *comparable* (written as $x \not\perp y$); for $X \subset T$, we say X is a *chain* of T if any two elements of X are comparable; for $Y \subset T$, we say Y is an *antichain* of T if any two elements of Y are incomparable; b is a *branch* of T if b is a maximal chain.

(5) For a tree T which has unique limits and two elements x, y in T, $\Delta_T(x, y) = max\{\alpha \leq ht(x), ht(y) : x \restriction_\alpha = y \restriction_\alpha\}$, simply use $\Delta(x, y)$ if there is no confusing.

(6) A tree $(T, <_T)$ which has unique limits is an *Aronszajn tree* if $ht(T) = \omega_1$, T_α is countable for any $\alpha < \omega_1$ and T has no uncountable branch.[3]

(7) For any Aronszajn tree T and $X \subset T$, say X is *special* if X is a countable union of chains.[4]

(8) For any tree $(T, <_T)$ a *lexicographical order* $<_{lex}$ (or written as $<_l$ or $<_{lT}$) of T is a linear order such that for any $x, y \in T$, $x <_{lex} y$ iff

(a) $x <_T y$ or

(b) x, y are incomparable and $x \restriction_{\Delta(x,y)+1} <_{lex} y \restriction_{\Delta(x,y)+1}$.

Notation: (1) Recall that every Aronszajn tree in this thesis has unique limits.

(2) Every Aronszajn tree in this thesis is one-rooted.[5] So when we mention a non-one-rooted Aronszajn tree T, we assume T has already been changed into T' where T'_0 is a singleton, $T'_{n+1} = T_n$ for $n < \omega$ and $T'_\alpha = T_\alpha$ for $\alpha \geq \omega$.

Definition 3. (1) For any two partial orders $(A, <_A)$ and $(B, <_B)$, we say A *is B-embeddable* (or say A *can be embedded into* B) if there is a mapping $\pi : A \to B$ such that

$\forall x, y \in A\ x <_A y \to \pi(x) <_B \pi(y)$.

[2] This definition comes from [15].

[3] Although in some paper Aronszajn tree is also defined for trees which doesnot have unique limits, in this thesis, Aronszajn trees are restricted to trees which has unique limits.

[4] When X equals T, this definition agrees with the usual definition of "T is special".

[5] T is one-rooted if T_0 is a singleton.

(2)Two trees $(T_1, <_{T_1})$, $(T_2, <_{T_2})$ are *tree isomorphic* (also called isomorphic) if there is a bijection $\pi : T_1 \to T_2$ that preserves the tree order, i.e., $x <_{T_1} y$ implies $\pi(x) <_{T_2} \pi(y)$ for any x, y in T_1; two trees with lexicographical orders $(T_1, <_{T_1}, <_{lex1})$, $(T_2, <_{T_2}, <_{lex2})$ are *lexicographically isomorphic* if there is a bijection from T_1 to T_2 that preserves both tree order and lexicographical order.

Remark: $<_T$ denotes the tree order and $<_{lex}$ (or $<_l$ or $<_{lT}$) denotes the lexicographical order as long as there is no confusing.

Definition 4. (1) For any linear order O, we say an Aronszajn tree T is *O-ranging* if T is a subset of $O^{<\omega_1}$.
(2) For any linear order O, a lexicographically ordered tree $(T, <_T, <_l)$ is *O-branching* if for any $t \in T$, $(succ_T(t), <_l)$ can be embedded into O, where $succ_T(t) = \{s : ht(s) = ht(t) + 1 \wedge t <_T s\}$

Remark: O-ranging tree is also O-branching.

As in this thesis we do not need to differ two lexicographically isomorphic trees, from now on we assume every lexicographical ordered Aronszajn tree is a subset of $\mathbb{Q}^{<\omega_1}$ and the lexicographical order is the canonical lexicographical order.[6]

Definition 5. (1) A lexicographically ordered Aronszajn tree T is *coherent*[7] if T $D_{xy} = \{\alpha < dom(x), dom(y) : x(\alpha) \neq y(\alpha)\}$ is finite for any $x, y \in T$. Assume $dom(x) \leq dom(y)$, we use $x =^* y \restriction_{dom(x)}$ to denote above property.
(2) A line is *coherent* if it can be embedded into the lexicographical order of a coherent tree.

Definition 6. (1) For an Aronszajn tree $(T, <_T)$, we say a subset T' is a *downward closure subtree* of T if there is a $X \subset T$ such that $T' = T_X$ where $T_X = \{t \in T :$

[6]For $s \in \mathbb{Q}^\alpha, t \in \mathbb{Q}^\beta$, the canonical lexicographical order between them is $s <_l t$ if (1) $s \subset t$ and $s \neq t$ or (2) $s(\Delta(s,t)) < t(\Delta(s,t))$ and $\Delta(s,t) < min\{\alpha, \beta\}$ where $\Delta(s,t) = min\{\xi : s(\xi) \neq t(\xi)\}$.

[7]Since we don't differ lexicographically isomorphic trees, an Aronszajn tree is coherent if it is lexicographically isomorphic to some coherent tree.

$\exists x \in X\ t \leq_T x\}$ and the tree order of T' agrees with T i.e., $<_{T'} = <_T \cap (T')^2$ (and the lexicographical order agrees too if T is lexicographically ordered)[8].

(2) For an Aronszajn tree $(T, <_T)$, we say T'' is a *club restriction subtree* of T if there is a club C such that $T'' = T \restriction_C$ where $T \restriction_C = \{t \in T : ht_T(t) \in C\}$ and the tree order (and lexicographical order if exists) agrees with T.[9]

Definition 7. For any Aronszajn tree T and club C, for any $s, t \in T$, $\Delta_C(s,t)$ is the α such that $C(\alpha) \leq \Delta(s,t) < C(\alpha+1)$ where $C(\alpha)$ is the α-th element of C.

Note for $s, t \in T \restriction_C$, $\Delta_C(s,t) = \Delta_{T\restriction_C}(s,t)$.

Definition 8. A *partition tree* of a linear order L is a lexicographically ordered tree T which contains a $X \subset T$ such that $T = T_X$ and L order isomorphic to X with the lexicographical order on T.

We don't differ two isomorphic linear order. So from now on, we just assume X as mentioned in above definition is L as long as there is no confusion.

Definition 9. Two Aronszajn trees T_1 and T_2 are *tree (or lexicographically) isomorphic when restrict to a club* (or say *tree (or lexicographically) isomorphic on a club set of levels*) if there is a club C such that $T_1 \restriction_C$ is tree (or lexicographically) isomorphic to $T_2 \restriction_C$; T_1 and T_2 are *near* each other if there are a club C and $X_i \in [T_i]^{\omega_1}$ for $i = 1, 2$ such that $(T_1)_{X_1} \restriction_C$ is tree isomorphic to $(T_2)_{X_2} \restriction_C$.

Definition 10. An Aronszajn tree T is a *Lipschitz tree* if for any $X \in [T]^{\omega_1}$,[10] for any mapping $\pi : X \to T$ such that $ht(x) = ht(\pi(x))$ for any x in X (i.e., π is level preserving), there is $X' \in [X]^{\omega_1}$ such that

For any x, y in X', $\Delta(x,y) \leq \Delta(\pi(x), \pi(y))$.

[8]Note a downward closure subtree of an Aronszajn tree still has unique limits and hence an Aronszajn tree.

[9]Note $ht_{T\restriction_C}(t) = \alpha$ iff $ht_T(t) = C(\alpha)$ for any $t \in T \restriction_C$.

[10]For any set A, for any cardinal κ, $[A]^\kappa = \{B \subset A : |B| = \kappa\}$.

Remark: It is equivalent to replace "$\Delta(x,y) \leq \Delta(\pi(x),\pi(y))$" in above definition by "$\Delta(x,y) = \Delta(\pi(x),\pi(y))$": π^{-1} is also level preserving, so apply Lipschitz property to $\pi^{-1} : \pi''X' \to T$ will get an uncountable subset X'' of X' (and hence of X) such that for any x, y in X'', $\Delta(x,y) = \Delta(\pi(x),\pi(y))$.

Definition 11. For any function $a : [\omega_1]^2 \to \mathbb{Q}$, the tree induced by a is $T(a) = \{a_\beta \restriction_\alpha : \alpha \leq \beta < \omega_1\}$ where $a_\beta(\xi) = a(\xi,\beta)$[11] for any $\xi < \beta$ and the tree order is extension as functions, the lexicographical order is the canonical lexicographical order.

[11] For convenience, we will use $a(\xi,\beta)$ to denote $a(\{\xi,\beta\})$ for any $\xi < \beta$.

Chapter 2

Countryman lines and coherent trees and their connections

A coherent line can canonically induce or be induced from a coherent tree. So both tree order and linear order – lexicographical order – are considered in this thesis. This section will provide some basic connection between tree order and linear order. Some results on transformation between Countryman line and coherent line under MA_{ω_1} will also be presented in this section.

2.1 Partition trees and lexicographical orders

Let's first present some facts.

Fact 2.1. *Every Aronszajn tree with lexicographical order is lexicographically isomorphic to some subset of* $\mathbb{Q}^{<\omega_1}$,[1] *where the tree order of* $\mathbb{Q}^{<\omega_1}$ *is extension as functions and the lexicographical order of* $\mathbb{Q}^{<\omega_1}$ *is the canonical lexicographical order.*

[1] $\mathbb{Q}^{<\omega_1} = \bigcup_{\alpha<\omega_1} \mathbb{Q}^{\alpha}$ and $\mathbb{Q}^{\alpha}=\{f$: f is a function from α to $\mathbb{Q}\}$.

Fact 2.2. *A club restriction subtree of a tree which has unique limits still has unique limits and a club restriction subtree of a coherent tree is still coherent.*

Let's recall that in this thesis, "partition tree" is used to transform a linear order into a tree and "lexicographical order" is used to transform a tree order into a line. See [13] or [2] for more on partition tree and lexicographical order. It is easy to see that for a tree with only tree order there are different ways to define lexicographical order on it. But in this section we will see that the partition tree of a linear order is kind of "unique" up to take club restriction subtree.[2]

Remark: While in some paper only the tree order of a partition tree is considered, in this thesis, a partition tree of a linear order is always lexicographically ordered.

One standard way to get a partition tree from a linear order L can be find in [13] or [2]. Let's recall the procedure:

(1) $T_0 = \{L\}$;

(2) $T_{\alpha+1} = \cup\{\{I_0, I_1\}$: there is a $I \in T_\alpha$ such that $I = I_0 \cup I_1, I_0 \cap I_1 = \phi$ and $I_0 \neq \phi, I_1 \neq \phi\}$;

(3) $T_\alpha = \{\cap b : b \subset \bigcup_{\beta<\alpha} T_\beta, \forall\beta < \alpha\ b \cap T_\beta \neq \phi$ and $\cap b \neq \phi\}$ for limit ordinal α.

The partition tree will be $T = \cup\{T_\alpha : \alpha < ht(T)\}$ where $ht(T) = min\{\alpha : T_\alpha = \phi\}$.

Now we turn to prove the "uniqueness" of the partition of a linear order.

Definition 12. A sequence of models $\langle N_\alpha : \alpha < \xi\rangle$ is an *elementary chain* of length ξ if there is a large enough cardinal κ, such that:

(1) $\forall\alpha < \xi\ (N_\alpha, \in) \prec (H(\kappa), \in)$;[3]

(2) for any $\alpha < \beta < \xi$, $(N_\alpha, \in) \prec (N_\beta, \in)$ and $N_\alpha \in N_\beta$.

And call it *continuous elementary chain* if it has the following additional property:

(3) If α is an infinite limit ordinal, then $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$.

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[2] i.e., any two partition trees of a linear order are isomorphic when restricted to a club level (going to club restriction subtrees).

[3] $H(\kappa) = \{x \in V_\kappa : |tc(x)| < \kappa\}$ where $tc(x)$ is the transitive closure of x. And we are also allowed to write $N_\alpha \prec H(\kappa)$ instead of writing $(N_\alpha, \in) \prec (H(\kappa), \in)$.

Remark: In this thesis, $\kappa = \omega_2$ if κ in the above definition is omitted.

Fact 2.3. *If $\langle N_\alpha : \alpha < \omega_1 \rangle$ is a continuous elementary chain and each N_α is countable, then*
(1) $N_\alpha \cap \omega_1 = sup(N_\alpha \cap \omega_1)$ is a countable ordinal and $C = \{N_\alpha \cap \omega_1 : \alpha < \omega_1\}$ is a club.
(2) If $A \subset \omega_1$, $N_\alpha \cap \omega_1 \in A$ and $A \in N_\alpha$, then A is stationary.

Let's call $C = \{N_\alpha \cap \omega_1 : \alpha < \omega_1\}$ the club induced from the continuous elementary chain $\langle N_\alpha : \alpha < \omega_1 \rangle$. This club will be frequently used, especially when we are taking a club restriction subtree. Since in this thesis most properties related to a club are closed under taking a subclub, for such a property, if there is a club which satisfies the property, then there is a club in N_0 which satisfies the property, and then C satisfies the property since C is a subclub of any club in N_0.
The following theorem shows that for Aronszajn trees, the lexicographical order can determine the tree order:

Proposition 2.4. *If $(S, <_S, <_{lS}), (T, <_T, <_{lT})$ are two lexicographically ordered Aronszajn trees, X, Y are uncountable subsets of S and T respectively and $\pi : (X, <_{lS}) \to (Y, <_{lT})$ is an isomorphism, then there is a club C such that $S_X \restriction_C$ is tree and lexicographical isomorphic to $T_Y \restriction_C$. Moreover, there is a lexicographically isomorphism $f : S_X \restriction_C \to T_Y \restriction_C$ such that f agrees with π, i.e., $\pi''(S^s \cap X) = T^{f(s)} \cap Y$ for any $s \in S_X \restriction_C$.*

Proof. Going to downward closure subtrees, we can assume $S = S_X$ and $T = T_Y$. Let $\langle N_\alpha : \alpha < \omega_1 \rangle$ be a continuous elementary chain, N_0 contains all relevant objects and each N_α is countable and $C = \{N_\alpha \cap \omega_1 : \alpha < \omega_1\}$. Then we are going to show that C is such a club we need.

First, we make some notation: for any $s \in S$, $t \in T$

$A_s = A \cap S^s = \{x \in X : s \leq_S x\}$, $B_t = B \cap T^t$.

It is easy to see that A_s, B_t are intervals of X and Y repectively.

Define $f : S \restriction_C \to T \restriction_C$ by: for any $\alpha \in C$, for any $s \in S_\alpha$, $f(s)$ is the $t \in T_\alpha$ such that $B_t = \pi'' A_s$. The following claim will show that the mapping is well-defined:

Claim 1: For any $\alpha \in C$ for any $s \in S_\alpha$ there is a $t \in T_\alpha$ such that $B_t = \pi'' A_s$.

proof of claim 1: Fix $\alpha \in C$, $s \in S_\alpha$ and ξ such that $\alpha = N_\xi \cap \omega_1$. Pick $x \in A_s$ (x exists since we have assumed $S = S_X$). Then $x \notin N_\xi$ and hence $\pi(x) \notin N_\xi$ and hence $ht(\pi(x)) \geq \alpha$. Let $t \in T_\alpha$ and $t \leq_T \pi(x)$. It is suffice to show $B_t = \pi'' A_s$.

Subclaim 1.1 $\pi'' A_s \subset B_t$.

proof of subclaim 1.1: Suppose otherwise, there is some $x' \in X \cap A_s$ such that $\pi(x') \notin B_t$ and WLOG assume $x <_{lS} x'$. Repeat previous procedure we can find some $t' \in T_\alpha$ such that $\pi(x') \in B_{t'}$ and hence $t <_{lS} t'$. As $t, t' \in T_\alpha$ and $t \neq t'$, we know $\Delta(t, t') < \alpha$. Then a standard argument (e.g., see Corollary 2.6) will show that there is a $t'' \in N_\xi \cap T$ such that t, t', t'' are pairwise incomparable and $t <_{lT} t'' <_{lT} t'$. Now by elementarity of N_ξ, pick $y'' \in B_{t''} \cap N_\xi$ and let $x'' = \pi^{-1}(y'')$ (hence $ht(x'') < \alpha$). Then $t <_{lT} y'' <_{lT} t'$ and hence $x <_{lS} x'' <_{lS} x'$. As A_s is an interval, $x'' \in A_s$ and hence $ht(x'') \geq \alpha$. A contradiction. This finishes the proof of subclaim 1.1.

Subclaim 1.2 $\pi'' A_s \supset B_t$.

proof of subclaim 1.2: Just notice π^{-1} is a isomorphism from B to A. Then repeat the previous proof we can get $\pi^{-1''} B_t \subset A_s$. This finishes the proof of subclaim 1.2 and hence the proof of claim 1.

Claim 2: f is a tree isomorphism and hence a lexicographical isomorphism.

proof of claim 2: It is easy to see that f is injective, and f is surjective since π is surjective. To show f preserves the tree order, pick arbitrary $s, s' \in S \restriction_C$ such that $s <_S s'$. Pick any $x \in A_{s'} (\subset A_s)$. Then $\pi(x) \in B_{f(s)}$ and $\pi(x) \in B_{f(s')}$. Then $f(s) <_T \pi(x)$ and $f(s') <_T \pi(x)$. So $f(s) <_T f(s')$ since f is level preserving. So f is a tree isomorphism. And the following fact is suffice to show the lexicographical isomorphism:

if $s <_{lS} s'$ and s is incomparable with s' for some $s, s' \in S \restriction_C$, then $A_s <_{lS} A_{s'}$[4] and hence $B_{f(s)} <_{lT} B_{f(s')}$ and hence $f(s) <_{lT} f(s')$.

This finishes the proof of claim 2 and hence proof of the proposition. □

Remark: Then the "uniqueness" of the partition tree of a linear order easily follows: two partition trees of an Aronszajn line are lexicographically isomorphic on a club level. In particular, one partition tree is special iff the other partition tree is special.

This theorem also suggests that under some condition, tree orders can also tell the difference of linear orders–there lexicographical orders– and this gives a way to get different linear order types from different tree order types (for different tree order types, readers are referred to [11]):

Corollary 2.5. *If T_0 and T_1 are partition trees of L_0, L_1 respectively and they are not tree isomorphic when restrict to a club,*[5] *then L_0 is not isomorphic to L_1. If moreover T_0 and T_1 are not near each other, then L_0 and L_1 contain no uncountable isomorphic suborder.*

Above corollary can be applied with special property: if T_0 mentioned above is special while T_1 is non-special, then L_0 and L_1 are not isomorphic. Moreover, L_1 cannot be embedded into L_0, and if T_1 contains no special subtree, then L_0 cannot be embedded into L_1 either.

Most time we are interested in uncountable subset instead of the whole tree itself. The reason is although we can transform the whole tree into a line, we may not able to transform a line into a whole tree:

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[4]i.e. for any $x \in A_s$ and $x' \in A_{s'}$, $x <_{lS} x'$.

[5]i.e., for any club C, $T_0 \restriction_C$ is not isomorphic to $T_1 \restriction_C$.

Corollary 2.6. *For any lexicographically ordered Aronszajn tree* $(T, <_T, <_{lT})$*, there is an uncountable subset* X *that cannot be partitioned into a whole tree, i.e., for any lexicographically ordered tree* $(S, <_S, <_{lS})$*,* $(X, <_{lT})$ *is not isomorphic to* $(S, <_{lS})$*.*

Proof. By going to an uncountable subtree ($\{t \in T : T^t$ is uncountable.$\}$ will work), we can assume for any $t \in T$, T^t is uncountable.

Let $\langle N_\alpha : \alpha < \omega_1 \rangle$ and $C = \{N_\alpha \cap \omega_1 : \alpha < \omega_1\}$ be as before. Let C_1 be all nonaccumulate points of C. Then it is suffice to prove that $X = T \restriction_{C_1}$ cannot be partitioned into a whole tree.

Suppose otherwise, $(S, <_S, <_{lS})$ is an lexicographically ordered tree and $\pi : (X, <_{lT}) \to (S, <_{lS})$ is an isomorphism. Let $f : T \restriction_D \to S \restriction_D$ be the lexicographical isomorphism guaranteed by Proposition 2.4 where D is some club (note here $T = T_X$) and we can assume $D \subset C'$ (C' is the set of all accumulate points in C) and hence $D \cap C_1 = \phi$. Pick a $t \in T \restriction_D$, by Proposition 2.4, $\pi'' T^t \cap X = S^{f(s)}$. Note $f(s)$ is the $<_{lS}$-least element of $S^{f(s)}$. So $T^t \cap X$ has a $<_{lT}$-least element too which contradicts the following fact:

Claim: $T^t \cap X$ contains no least element.

proof of claim: Suppose otherwise, u is the least element. Note $u \neq t$ since $t \notin X$. Let $ht(u) = N_\eta \cap \omega_1$. Define in N_η:

$A = \{z \in T : t <_T z$ and z is the $<_{lT}$-least element in $T^t \cap T_{ht(z)}\}$.

Then A is uncountable since $u \in A$. It is easy to see that A is a chain which contradicts that T is an Aronszajn tree. This finishes the proof of the claim and hence the proof of the corollary. □

Remark: Although some Aronszajn line cannot be partitioned into a whole tree, it may contain some Aronszajn subline that can be partitioned into a whole tree. Recall that it is shown in [6], that under PFA, every Aronszajn line contains either C or C^* where C is the lexicographical order of arbitrary coherent tree. Later (see remark after Lemma 3.8) we will also give an example that in some model of ZFC there is an Aronszajn line which contains no Aronszajn subline that can be partitioned into a whole tree.

Recall that a linear order is coherent if it can be embedded into the lexicographical order of some coherent tree. Now it should be clear that "almost" every partition tree will work:

Fact 2.7. *If L is a coherent line and $(T, <_T, <_{lT})$ is its partition tree, then there is a club C such that $T\restriction_C$ is coherent. Moreover, we can extend $T\restriction_C$ to a coherent tree $(T', <_{T'}, <_{lT'})$ such that L and even $(T, <_{lT})$ can be embedded into $(T', <_{lT'})$.*

Proof. Let S be a coherent tree such that L can be embedded into S. It is easy to see that S_L is still coherent. Then by Proposition 2.4, there is a club C such that $S_L\restriction_C$ lexicographically isomorphic to $T\restriction_C$. So $T\restriction_C$ is coherent.

For the moreover part, let's just define $T' \supset T\restriction_C \cup L$ by put every point in $T\restriction_{\omega_1\setminus C} \cap L$ as an endpoint:

(1) If $\alpha = \beta + 1 < \omega_1$ is a successor ordinal, first, T''_α is (lexicographically) order isomorphic to $T\restriction_{(C(\beta),C(\beta+1)]}$ via f'_α such that:

(i) tree order is preserved;

(ii) T''_α is a subset of $\{s^\frown q : s \in T'_\beta \text{ and } q \in O\}$ for some countable linear order O;

(iii) $t(\beta) = t'(\beta)$ implies $f'_\alpha(t)(\beta) = f'_\alpha(t')(\beta)$ for any $t, t' \in T_{C(\alpha)}$.

Then embed O into $\mathbb{Q}$ and get T'_α isomorphic to T''_α which is a subset of $\{s^\frown q : s \in T'_\beta \text{ and } q \in \mathbb{Q}\}$.

(2) If $\alpha < \omega_1$ is a limit ordinal, then $T'_\alpha = \{s :$ there exists some $t \in T_{C(\alpha)}$ such that for all $\beta < \alpha$, $f_\beta(s\restriction_\beta) <_T t\}$, i.e., the sequences induced from T.

It is easy to see that T' is the desired tree we want. $\square$

The proof of above fact actually shows the following:

Corollary 2.8. *(1) An Aronszajn line L is coherent iff there is a partition tree T of L and a club C such that $T\restriction_C$ is coherent.*
(2) For any lexicographically ordered Aronszajn tree $(T, <_T, <_l)$, for any club C, $(T, <_l)$ is a coherent line iff $(T\restriction_C, <_l)$ is a coherent line.

2.2 Some results under MA_{ω_1}

A transformation from coherent trees to Countryman lines under MA_{ω_1} is guaranteed by the following theorem which only needs a weaker condition than MA_{ω_1}:

Theorem 2.9 ([1])**.** *Every special coherent tree is Countryman with its lexicographical order.*[6] *So in particular, MA_{ω_1} implies that every coherent tree is Countryman with its lexicographical order.*

The in particular part is actually using the following well-known fact (the proof can also be found in [10]):

Fact 2.10 ([14])**.** *MA_{ω_1} implies that every Aronszajn tree is special.*

S. Todorcevic actually proved a stronger result which is unpublished, I will refer it here with his permission:

Theorem 2.11 (Todorcevic)**.** *If T is an $\mathbb{R}$-embeddable coherent tree and T is ω-ranging, then T is Countryman with its lexicographical order.*

Proof. This follows from Corollary 3.15 below. □

Note every special Aronszajn tree can be canonically extended to a binary special tree (i.e., a special Aronszajn tree is a club restriction subtree of some binary special tree). So above theorem can imply Theorem 2.9.

The following fact will be frequently used:

[6] i.e., the linear ordering for the tree is its lexicographical order.

Fact 2.12. *(1) For an Aronszajn tree T, the following are equivalent:*

(a) T is special.

(b) There is a club C such that $T\restriction_C$ is special.[7]

(c) T is $\mathbb{Q}$-embeddable.

(2) For an Aronszajn tree T, the following are equivalent:

(a) T is $\mathbb{R}$-embeddable.

(b) There is a club C such that $T\restriction_{C+1}$ is special.[8]

(c) For any nonstationary set X, $T\restriction_X$ is special.

Fodor's Lemma is well-known and frequently used:

Lemma 2.13 (Fodor's Lemma [19])**.** *Every regressive function on a stationary set is constant on a stationary subset.*

To get the transformation from Countryman lines to coherent trees, I will use another unpublished works of S.Todorcevic with his permission:

Theorem 2.14 (Todorcevic)**.** *If T is an Aronszajn tree $X \subset T$ and X is Countryman with its lexicographical order, then there is a club C such that the club restriction subtree $T_X\restriction_C$ is Lipschitz.*

See appendix for a proof.

The following theorem is a slight generalization of [1] Lemma 4.2.7 and the proof is similar too:

Theorem 2.15 (MA_{ω_1})**.** *If T is a Lipschitz tree, $X \subset T$ and X is Countryman, then T_X is lexicographically isomorphic to a coherent tree. Moreover, X is Countryman can be replaced by the following weaker property:*

for any $n < \omega$, for any $\mathscr{A}$ consists of uncountable pairwise disjoint subsets of X^n,

[7] Recall that $T\restriction_X = \cup_{\alpha\in X} T_\alpha$ for any $X \subset \omega_1$.

[8] $C + 1 = \{\alpha + 1 : \alpha \in C\}$.

there are $a, b \in \mathscr{A}$ s.t. for any $i < n$, $a_i <_{lex} b_i$, where a_i, b_i are i-th elements of a, b.

Proof. WLOG, assume $T = T_X$. Define a poset
$\mathcal{P} = \{p : p$ is a finite partial function from T to $\mathbb{Q}^{[\omega_1]^{<\omega}}$ such that:
(1) for any $t \in dom(p)$, $p(t)$ is a finite partial function from $ht(t)$ to $\mathbb{Q}$;
(2) for any two elements $s <_{lex} t$ in $dom(p)$, $\Delta(s, t) \in dom(p(s)) \cap dom(p(t))$ and for any $\xi \in dom(p(s)) \cap dom(p(t))$, $p(s)(\xi) = p(t)(\xi)$ if $\xi < \Delta(s, t)$, $p(s)(\xi) < p(t)(\xi)$ if $\xi = \Delta(s, t)$. $\}$
And $p < q$ (p is stronger than q) if $p \neq q$ and
(a) $dom(p) \supset dom(q)$ and for any $t \in dom(q)$, $p(t)$ extends $q(t)$ as a function;
(b) for any s, t in $dom(q)$ and for any $\xi \in (dom(p(s)) \cap dom(p(t))) \setminus (dom(q(s)) \cup dom(q(t)))$, $p(s)(\xi) = p(t)(\xi)$.
First, we need to show that $\mathcal{P}$ is c.c.c.
Fix $\{p_\alpha : \alpha < \omega_1\} \subset \mathcal{P}$. By Fodor's Lemma, we can find a stationary subset Γ_1 and a countable ordinal α_0 such that:
(1) for any $\alpha \in \Gamma_1$, for any $s, s' \in dom(p_\alpha)$, $(ht(s) < \alpha_0) \vee (ht(s) \geq \alpha)$, $(\Delta(s, s') < \alpha_0) \vee (\Delta(s, s') \geq \alpha)$ and $dom(p_\alpha(s)) \cap \alpha < \alpha_0$;
(2) p_α ($\alpha \in \Gamma_1$) is constant below α_0 level, i.e., p_α's ($\alpha \in \Gamma_1$) have the same size m and for any $i < m$, for any $\alpha, \beta \in \Gamma_1$, $s \restriction_{\alpha_0} = t \restriction_{\alpha_0}$ and $p_\alpha(s) \restriction_{\alpha_0} = p_\beta(t) \restriction_{\alpha_0}$ where s, t are $<_{lex}$-ith element of $dom(p_\alpha), dom(p_\beta)$ respectively.

Define $a_\alpha = \{s \restriction_\alpha : s \in dom(p_\alpha) \wedge ht(s) \geq \alpha\}$. Now we can find an uncountable subset Γ of Γ_1 such that a_α's has the same size n and for all $\alpha \neq \beta$ in Γ:
(i) $a_\alpha(i)$ and $a_\beta(j)$ are incomparable for all $i, j < n$;
(ii)$\Delta(a_\alpha(i), a_\beta(i)) = \Delta(a_\alpha(j), a_\beta(j))$ for all $i, j < n$.
By the fact of X is Countryman or use the property mentioned in the theorem, we can find $\gamma \neq \delta$ in Γ such that $a_\gamma(i) <_{lex} a_\delta(i)$ for all $i < n$. Now we can find a $p \in \mathcal{P}$ that is stronger than p_γ and p_δ: $dom(p) = dom(p_\gamma) \cup dom(p_\delta)$ and for

$s \in dom(p)$

(1) if $ht(s) < \gamma$ then $ht(s) < \alpha_0$ and $p_\gamma(s) = p_\delta(s)$, define $p(s) = p_\gamma(s)$;

(2) if $ht(s) \geq \gamma$ and $s \in dom(p_\gamma)$, define $p(s) = p_\gamma(s) \cup \{(\Delta(p_\gamma(0), p_\delta(0)), 0)\}$;

(3) if $ht(s) \geq \gamma$ and $s \in dom(p_\delta)$, define $p(s) = p_\delta(s) \cup \{(\Delta(p_\gamma(0), p_\delta(0)), 1)\}$.

Then it is easy to check that p is a member of $\mathcal{P}$ and p is stronger than p_γ and p_δ. This shows that $\mathcal{P}$ is c.c.c.

Now note that $D_{t,\xi} = \{p \in \mathcal{P} : t \in dom(p) \wedge \xi \in dom(p(t))\}$ is dense for each $t \in T$ and $\xi < ht(t)$. By MA_{ω_1}, assume G is a filter that intersects each $D_{t,\xi}$ for all $t \in T$ and $\xi < ht(t)$. Then $dom(\cup G)$ is T, $rang(\cup G)$ is a subset of $\mathbb{Q}^{<\omega_1}$. By definition of $\mathcal{P}$ we can see $\cup G$ is a lexicographical isomorphism between $rang(\cup G)$ and $dom(\cup G)$. By definition (b) of forcing extension, it is easy to see $rang(\cup G)$ is coherent. This finishes the proof. □

Corollary 2.16 (MA_{ω_1}). *Every coherent tree is Countryman with respect to its lexicographical order and every Countryman line has a partition tree which is coherent. In particular, a line is Countryman iff it is coherent.*

Proof. "A coherent tree is Countryman" follows from Theorem 2.9. Assume L is a Countryman line and T is its partition tree. Then by Theorem 2.14 and Theorem 2.15, there is a club C such that $T \restriction_C$ is coherent. Then by Corollary 2.8, L is coherent. □

Under MA_{ω_1}, coherence can be slightly generalized:

Definition 13. For any $\alpha < \omega_1$, an uncountable subset $A \subset [\mathbb{Q}]^{<\omega_1}$ is *α-coherent* if for any $s, t \in A$, $\{\xi < dom(s), dom(t) : s(\xi) \neq t(\xi)\}$ has order type less than α. And an Aronszajn tree is *α-coherent* if it is lexicographically isomorphic to some α-coherent uncountable subset of $[\mathbb{Q}]^{<\omega_1}$.

Note the usual coherent means ω-coherent. Now we will list under MA_{ω_1} some equivalent statement for Countryman.

Corollary 2.17 (MA_{ω_1}). *For any Aronszajn line L, the following are equivalent:*

(1) L is Countryman.

(2) L is coherent.

(3) For any partition tree T of L, there is a club C such that $T \restriction_C$ is Countryman (or coherent).

(4) L is α-coherent for some $\alpha < \omega_1$, i.e., it has an α-coherent partition tree T.

(5) L contains a Countryman dense subset.

(6) For any $0 < \alpha < \omega_1$, for any $L' \subset L^\alpha$ where the order for L^α is the lexicographical order, if L' is an Aronszajn line, then L' is Countryman.

Proof. Corollary 2.16 gives the equivalence between (1) and (2). Corollary 2.8 gives the equivalence between (2) and (3). (1) $\to$ (4) is trivial, let's prove (4) $\to$ (1). Fix an infinite $\alpha < \omega_1$ and a α-coherent partition tree T. Let's just assume T itself is a α-coherent subset of $[\mathbb{Q}]^{<\omega_1}$ and $<_l$ is its lexicographical order. We will prove that T is Lipschitz and has the property mentioned in Theorem 2.15 and hence T is coherent (and Countryman).

Claim 1: T is Lipschitz.

proof of claim 1: Let $f : X \to T$ be a level preserving map for some $X \in [T]^{\omega_1}$. Define for any $x \in X$, $D_x = \{\beta < ht(x) : x(\beta) \neq f(x)(\beta)\} \cup \{\alpha\}$. Then D_x has order type $\leq \alpha + 1$. Find least $\xi \leq \alpha$ such that $\{D_x(\xi) : x \in X\}$ is unbounded where $D_x(\xi)$ is the ξ-th element of D_x if exists and undefined otherwise. Find an uncountable subset $Y \subset X$ and a $\delta < \omega_1$ such that:

(1) δ bounds $<\xi$-th elements of D_x for any $x \in Y$, i.e., $D_x \cap D_x(\xi) \subset \delta$ for any $x \in Y$;

(2) $\{x \restriction_{D_x(\xi)} : x \in Y\}$ is an antichain;

(3) for any x, y in Y, $x \restriction_\delta = y \restriction_\delta$ and $f(x) \restriction_\delta = f(y) \restriction_\delta$.

Then for any x, y in Y, $\delta \leq \Delta(x, y) < min\{D_x(\xi), D_y(\xi)\}$ by (2) and (3). And hence $\Delta(f(x), f(y)) = \Delta(x, y)$ by (3) and definition of D_x, D_y. This finishes the proof of claim 1.

Claim 2: For any $n < \omega$, for any uncountable subset $\mathscr{A} \subset T^n$ consists of pairwise disjoint subsets, there are a, b in $\mathscr{A}$ such that $a_i <_l b_i$ for any $i < n$ where a_i, b_i are i-th elements of a, b respectively.

proof of claim 2: Without loss of generality, we can assume that for any $a \in \mathscr{A}$, $ht(a_0) \leq ht(a_i)$ for any $i < n$. Let $X_0 = \{a_0 : a \in \mathscr{A}\}$, define $f_0 : X \to T$ by $f_0(a_0) = a_1 \restriction_{ht(a_0)}$. Then repeat the proof of claim 1, we can find uncountable $X_1 \subset X_0$ (corresponding to the Y in proof of claim 1) such that for any $a_0, b_0 \in X_1$, $a_0 <_l b_0$ iff $f_0(a_0) <_l f_0(b_0)$ iff $a_1 <_l b_1$.

Repeat above argument $n-1$ times we can find $X_0 \supset X_1 \supset ... \supset X_{n-1}$ such that for any $i < n$, for any $a_0.b_0 \in X_i$, $a_0 <_l b_0$ iff $a_i <_l b_i$. Then pick $a, b \in \mathscr{A}$ such that $a_0, b_0 \in X_{n-1}$ and $a_0 <_l b_0$, and hence $a_i <_l b_i$ for any $i < n$. This finishes the proof of claim 2.

Then by Theorem 2.15, T is coherent. This shows (4) $\to$ (1).

(1) $\to$ (5) is trivial, let's prove (5) $\to$ (1). Fix a dense subset L' which is Countryman, and let S be a partition tree of L. Then $S_{L'}$ is a partition tree of L'. By Fact 2.7, there is a club C such that $S_{L'} \restriction_C$ is coherent. Note $S \setminus S_{L'}$ consists only endpoints since L' is dense in L, and hence $S \restriction_C \setminus S_{L'} \restriction_C$ consists only endpoints. So $S \restriction_C$ is $\omega + 1$-coherent and hence coherent. Then L is coherent by Corollary 2.8. This shows (5) $\to$ (1).

(6) $\to$ (1) is trivial, let's prove (1) $\to$ (6).

Pick a partition tree T of L such that T is a coherent subset of $[\mathbb{Q} \cap (0,1)]^{<\omega_1}$ and the lexicographical order is the canonical lexicographical order. Without loss of generality, assume L itself is a subset of $[\mathbb{Q} \cap (0,1)]^{<\omega_1}$. Fix $0 < \alpha < \omega_1$ and $L' \subset L^\alpha$ such that L' is an Aronszajn line. For any $l \in L^\alpha$, fix a countable sequence $t_l = l(0)^\frown 0^\frown l(1)^\frown 0^\frown ...^\frown l(\xi)^\frown 0....$ Then it is easy to see that $\{t_l : l \in L'\}$ is isomorphic to L'. Let S be the downward closure of $\{t_l : l \in L'\}$ and the tree order of S is extensions as functions and the lexicographical order of S is the canonical lexicographical order. It is suffice to show that S is coherent. Fix a club C such that for any $\beta \in C$, for any $\gamma < \beta$, $\gamma\alpha < \beta$. Use the construction in Fact 2.2, we can assume $S' = S \upharpoonright_C$ is a subset of $[\mathbb{Q}]^{<\omega_1}$ such that for any $s, t \in S'$, for any $\xi < ht_{S'}(s), ht_{S'}(t)$, $s(\xi) = t(\xi)$ if $s \upharpoonright_{[C(\xi),C(\xi+1))} = t \upharpoonright_{[C(\xi),C(\xi+1))}$, i.e., for each $\xi < \omega_1$, we embed $\{t \upharpoonright_{[C(\xi),C(\xi+1))} : t \in T \upharpoonright_C\}$ into $\mathbb{Q}$.

Now we are going to show that S' is $(\alpha+\omega)^2$-coherent by:

Claim 3: For any $s \in S$, for any $t \in T_{ht_S(s)}$, $D_{st} = \{\xi \in C$: there is some $\eta \in [C(\xi), C(\xi+1))$ such that $s(\eta) \neq t(\eta)\}$ has order type less than $\alpha + \omega$.

proof of claim 3: Let $s = \langle s(i) : i < \alpha\rangle$ and note each $s(i) \in T$. For each $i < \alpha$, let $\beta_i = \sum_{j<i}(ht(s(j)) + 1)$. Then $s \upharpoonright_{[\beta_i,\beta_{i+1})} = s(i)$. Note by definition of C, for any ξ such that $C(\xi) \in (\beta_i, \beta_{i+1})$, $C(\xi) = \beta_i + C(\xi)$ and hence $s \upharpoonright_{[C(\xi),\beta_{i+1})} = s(i) \upharpoonright_{[C(\xi),\beta_{i+1})}$. This shows that $[\beta_i, \beta_{i+1}) \cap D_{st}$ is finite and hence finishes the proof of the claim.

Then for any $s, t \in S$, $D_{st}, s(i), \beta_i$ are as above, $D_{st} \cap [\beta_i, \beta_{i+1})$ has order type less than $\alpha+\omega$. Then D_{st} has order type less than $(\alpha+\omega)\alpha$. So S' is $(\alpha+\omega)^2$-coherent. Then S' and hence S is Countryman. This finishes the proof of the corollary. $\square$

Remark: Note it is not hard to use $\diamondsuit$ (diamond principle)[9] to construct an Aronszajn tree which contains a dense Countryman and coherent subset but is neither Countryman nor coherent itself. So MA_{ω_1} is necessary for above corollary.

[9]See [18] or [10] for $\diamondsuit$.

Chapter 3

From $\mathbb{R}$-embeddable coherent trees to Countryman lines

3.1 $\mathbb{R}$-embeddablity

This section will give the reason we choose $\mathbb{R}$-embeddable coherent trees other than arbitrary or special coherent trees.

Definition 14. Assume L is a Countryman line and $L^2 = \bigcup_{n<\omega} C_n$ where each C_n is a chain. Say this partition $L^2 = \bigcup_{n<\omega} C_n$ has *maximal property* if:
for all $n < m$ in ω, for all (a, b) in C_m, $C_n \cup \{(a, b)\}$ is not a chain, i.e. for some $(c, d) \in C_n$, either $a < c \wedge b > d$ or $a > c \wedge b < d$.

Definition 15. Let Λ denote the set of countable limit ordinals, i.e., $\Lambda = \{\lambda < \omega_1 : \lambda \text{ is a limit ordinal.}\}$.

Lemma 3.1. *For any Countryman line L, there is a partition with maximal property.*

Proof. Let $L^2 = \bigcup_{n<\omega} C'_n$ be arbitrary partition of L^2 into countably many chains. Let $L^2 = \{l_\alpha : \alpha < \omega_1\}$. Define C_n by induction on n. If C_m is defined for any

$m < n$, then define $C_n = \bigcup_{\alpha<\omega_1} D_\alpha$ where D_α is defined by induction on α:
$D_0 = C'_n \setminus (\bigcup_{m<n} C_m)$;
$D_\alpha = \bigcup_{\beta<\alpha} D_\beta$, if α is a limit ordinal;
$D_{\alpha+1} = D_\alpha \cup \{l_\alpha\}$ if $l_\alpha \notin \bigcup_{m<n} C_m$ and $D_\alpha \cup \{l_\alpha\}$ is a chain, and $D_{\alpha+1} = D_\alpha$ otherwise.
It is easy to check that each D_α is a chain and hence C_n is a chain. It is also easy to see that $L^2 = \bigcup_{n<\omega} C_n$ is a partition of L^2 into countably many chains. Note that the construction of D_α at successor stages will guarantee that this partition has the maximal property. □

Now we are ready to show that any partition tree of a Countryman line is ℝ-embeddable if we are allowed to take club restriction subtree.

Theorem 3.2. *If T is a partition tree of some Countryman line X, then T is ℝ-embeddable when restricted to a club level, i.e., $T \restriction_C$ is ℝ-embeddable for some club C. Hence, every Countryman line has an ℝ-embeddable partition tree.*

Proof. Let $\langle N_\alpha : \alpha < \omega_1 \rangle$ be a continuous elementary chain, N_0 contains all relevant objects and each N_α is countable and $C = \{N_\alpha \cap \omega_1 : \alpha < \omega_1\}$. Then we just need to prove that $T \restriction_C$ is ℝ-embeddable.
First, as X is Countryman, assume $X^2 = \bigcup_{n<\omega} C_n$ is a partition with maximal property. Define $C : X^2 \to \omega$ by $C(x, y) = n$ iff $(x, y) \in C_n$.
To get a embedding into ℝ, the following property will be needed:
Claim 1: For any $\alpha \in C$, for any $s \in T \restriction_\alpha \cap X$, for any $x, y \in X$, if $\Delta(x, y) \geq \alpha$, then $C(s, x) = C(s, y)$ and $C(x, s) = C(y, s)$.
proof of claim 1: Assume otherwise, fix $\alpha \in C$ and $s, x, y \in X$ be such that $\Delta(x, y) \geq \alpha$ and $C(s, x) \neq C(s, y)$ (the proof for $C(x, s) = C(y, s)$ is similar.). WLOG, assume $C(s, x) < C(s, y)$. Assume $x \restriction_\alpha = y \restriction_\alpha = t$ and $\alpha = N_\beta \cap \omega_1$. By elementarity, $A = \{z : C(s, z) = C(s, x)\}$ is uncountable since $A \in N_\beta$ while $x \in A$

is not in N_β. Then $N_\beta \models$ "A is uncountable". Now we need the following:

Subclaim 1.1:there are z, w in $A \cap N_\beta$ that are incomparable with t such that $z <_{lex} t <_{lex} w$.

proof of subcliam 1.1: Suppose otherwise, assume there is no such z (similar for no such w), i.e., for any $r \in A \cap N_\beta$, $r \geq_{lex} t$. Then for any γ such that $ht(s) < \gamma < ht(t)$, $t \restriction_\gamma$ is the least element in $(T_A)_\gamma$ under the order $<_{lex}$, i.e., $min((T_A)_\gamma) = t \restriction_\gamma$. Then

$N_\beta \models min((T_A)_\gamma)$ exists for any $\gamma > ht(s)$ and $\{min((T_A)_\gamma) : \gamma > ht(s)\}$ is an uncountable chain.

Then by elementarity, T contains an uncountable chain too. This contradicts the fact that T is an Aronszajn tree. This finishes the proof of subclaim 1.1.

Now fix z, w guaranteed by subclaim 1.1. By maximal property of the partition, assume $(a, b) \in C_{C(s,x)}$ such that $a <_{lex} s \wedge b >_{lex} y$ or $a >_{lex} s \wedge b <_{lex} y$. Then we have $a <_{lex} s \wedge b >_{lex} z$ or $a >_{lex} s \wedge b <_{lex} w$. This is a contradiction since $C(a, b) = C(s, z) = C(s, w)$. This finishes the proof of claim 1.

Now define $f : T \restriction_C \to [\omega]^\omega$ by $f(t) = \{C(s, x) : s \in T \restriction_{ht(t)}$ and $x \in T^t \cap X\}$ for any $t \in T \restriction_C$. It follows from claim 1 that we can fix a x and $f(t)$ won't change and hence $t <_T t'$ in $T \restriction_C$ implies $f(t) \subset f(t')$. To prove f is an embedding, we need to prove $t <_T t'$ in $T \restriction_C$ implies $f(t) \neq f(t')$. It is enough to prove the following:

Claim 2: For any $\alpha \in C$, for any $s \neq s' \in T \restriction_\alpha \cap X$, for any $x \in X \setminus T \restriction_\alpha$ $C(s, x) \neq C(s', x)$ and $C(x, s) \neq C(x, s')$.

proof of claim 2: Suppose otherwise, $C(s, x) = C(s', x) = m$ (the proof for $C(x, s) \neq C(x, s')$ is similar). Define $B = \{a \in T : C(s, a) = C(s', a) = m\}$. Then $B \in N_\beta$ where $\alpha = N_\beta \cap \omega_1$ and B is uncountable by elementarity and the fact that x is in B. Now pick $r <_{lex} p$ in $B \cap N_\beta$ and WLOG assume $s <_{lex} s'$. Then $C(s, p) = C(s', r) = m$ and $s <_{lex} s'$ while $p >_{lex} r$. Contradict the fact that C_m is a chain. This finishes the proof of claim 2.

For the hence part, see Fact 2.7 for extending $T \restriction_C$ by adding elements in $T \setminus T \restriction_C$ as endpoints. This finishes the proof of the theorem. □

The following explains the reason we restrict ourselves to $\mathbb{R}$-embeddable trees:

Corollary 3.3. *If a Countryman line has a coherent partition tree, it has a $\mathbb{R}$-embeddable coherent partition tree too.*

Proof. Fix a Countryman line X and its coherent partition tree $(T, <_T, <_l)$. Let $T'' = T \restriction_C$ be $\mathbb{R}$-embeddable for some club C. Use method described in Fact 2.7 to extend T'' to a coherent tree T' with $X \setminus T''$ as endpoints. Then T' is $\mathbb{R}$-embeddable by Fact 2.12. □

On the other hand, $\mathbb{R}$-embeddable is the best we can expect, i.e., we cannot expect the partition tree of a Countryman line to be special. The following example can be found in [1] (see Lemma 2.2.2 Lemma 2.2.4 and Lemma 2.2.17 in [1] for a proof)[1]:

Example 3.4. *([1]) In ZFC, there is always a finite-to-one*[2] *(in particularly it is $\mathbb{R}$-embeddable) coherent tree $T \subset \omega^{<\omega_1}$ (i.e. T is ω-ranging) that is Countryman*[3]*; adding a Cohen real will add a finite-to-one (and also $\mathbb{R}$-embeddable) coherent tree $T \subset \omega^{<\omega_1}$ that is Countryman and contains no stationary antichain*[4]*. So in particularly, it is consistent to have a coherent tree that is Countryman while it contains no special subtree.*

Also there are some coherent tree that is Countryman while it is not $\mathbb{R}$-embeddable. And so take a club restriction subtree is necessary.

Example 3.5. *It is consistent to have a coherent tree that is Countryman and not $\mathbb{R}$-embeddable:*

[1] A similar construction will be given in latter proof.

[2] finite-to-one as functions, i.e., the preimage of any element is finite

[3] $T(\rho_1)$ constructed in [1] is such a tree.

[4] A stationary antichain is an antichain X such that $ht(X) = \{ht(x) : x \in X\}$ is stationary.

firstly, start from a $\mathbb{R}$-embeddable coherent tree $T \subset \mathbb{Q}^{<\omega_1}$ that is Countryman and non-special (e.g. previous example);
secondly, define a 1-shift $T^{(1)}$ of T as following:
For $s \in T$, $s^{(1)} \in \mathbb{Q}^{ht(s)+1}$, $s^{(1)}(\beta+1) = s(\beta)$ for $\beta < ht(s)$ and $s^{(1)}(\alpha) = 0$ for limit ordinal α;
finally, $T^{(1)}$– the downward closure of $\{s^{(1)} : s \in T\}$– is what we need.

To prove that the example has the required properties, first, it is easy to see that $T^{(1)}$ is coherent.
Then, as all successor levels of $T^{(1)}$ is lexicographically isomorphic to T, $(T^{(1)} \restriction_{\Lambda+1}, <_{lT^{(1)}})$[5] can be embedded into $(T, <_{lT})$. Note $(T^{(1)} \restriction_{\Lambda}, <_{lT^{(1)}})$ can also be embedded into $(T, <_{lT})$. A partition of $(T, <_{lT})$ into countably many chains can easily induce a partition of $(T^{(1)}, <_{lT^{(1)}})$ into countably many chains (see also Proposition 3.16). Then $T^{(1)}$ is Countryman.
At last, if $T^{(1)}$ is $\mathbb{R}$-embeddable, then $T^{(1)}$ is special by Fact 2.12 and the fact that $T^{(1)} \restriction_{\Lambda}$ tree isomorphic to $T^{(1)} \restriction_{\Lambda+1}$.

3.2 An $\mathbb{R}$-embeddable coherent tree may be not Countryman

As mentioned before, S.Todorcevic showed that every ω-ranging $\mathbb{R}$-embeddable coherent tree is Countryman with its lexicographical order. In this section, we will see that it will not be true if we replace ω-ranging by $\mathbb{Q}$-ranging (or even just ω^*-ranging). But if we just want an uncountable subset of a $\mathbb{R}$-embeddable coherent tree to be Countryman, $\mathbb{Z}$-ranging is enough and it is the best we can expect. This section will also give an example with stronger negation (i.e., $\mathbb{R}$-embeddable coherent trees that contain no Countryman suborder with their lexicographical

[5] Λ is the set of limit ordinals below ω_1 and recall that $\Lambda + 1 = \{\lambda + 1 : \lambda \in \Lambda\}$.

orders). So whether there is a transformation from $\mathbb{R}$-embeddable coherent tree to Countryman line depends on the ranging.

Theorem 3.6. *Assume O is a countable linear order.*

(1) If O can't be embedded into ω, then it is consistent to have a $\mathbb{R}$-embeddable O-ranging coherent tree T such that $(T, <_{lT})$ is not Countryman.

(2) If O can't be embedded into $\mathbb{Z}$, then it is consistent to have a $\mathbb{R}$-embeddable O-ranging coherent tree T $(\subset O^{<\omega_1})$ such that $(T, <_{lT})$ contains no Countryman suborder.

Before we prove the theorem, we need a few lemmas. First, note if $O \subset O'$, then an O-ranging tree is also an O'-rang tree. So we just need to deal with several linear order O's:

Fact 3.7. *(1) If O can't be embedded into ω, then O contains a subset of type ω^* or $\omega + 1$.*

(2) If O can't be embedded into $\mathbb{Z}$, then O contains a subset of type $(\omega + 1)^*$ or $\omega + 1$.

The following lemma proves case ω^* for (1) of Theorem 3.6:

Lemma 3.8. *If $T \subset (\omega^*)^{<\omega_1}$ is a finite-to-one coherent tree with no stationary antichain, then T contains no stationary Countryman suborder, i.e., for any $X \in [T]^{\omega_1}$ such that $ht(X)$ is stationary, X is not Countryman.*

Proof. First, let's denote ω^* by $\omega^* = \{-n : n \in \omega\}$ with order $-n > -(n+1)$ for any $n \in \omega$.

Suppose otherwise, X is a stationary Countryman suborder (assume $|X \cap T_\alpha| \leq 1$ for any $\alpha < \omega_1$) and $X^2 = \bigcup_{n<\omega} C_n$ is a partition that witnesses the Countryman property. Define $X_0 = \{x \in X : \exists t_x \in X \ x <_T t_x\}$. Since $X \setminus X_0$ is an antichain and T contains no stationary antichain, $ht(X_0)$ is stationary. For any $x \in X_0$, fix a $t_x \in X$ such that $x <_T t_x$. Define a function $f : X_0 \to \omega$ by $f(y) = i$ iff $(y, t_y) \in C_i$. Then we can find $X_1 \subset X_0$ and $m, n \in \omega$ such that $ht(X_1)$ is stationary, $f''X_1 = \{m\}$ and $t_x(ht(x)) = -n$ for any $x \in X_1$.

Claim: There are $x \in X_1$ and $\{x_n \in X_1 : n < \omega\}$ such that $x_0 <_T x_1 <_T ... <_T x_k <_T x_{k+1} <_T ... <_T x$.

proof of Claim: Suppose otherwise, for any $x \in X_1$, $\{y \in X_1 : y <_T x\}$ is finite. Assume By going into a stationary subset assume $ht(X_1)$ only consists of limit ordinals. Then define a regressive function $h : ht(X_1) \to \omega_1$ by

$h(\alpha) = max\{ht(y) : y \in X_1 \wedge y <_T x_\alpha\}$ where x_α is the element in $T_\alpha \cap X_1$.

Going to a subset Z of X_1 such that $ht(Z)$ is stationary and h is constant on $ht(Z)$. As T has no stationary antichain, there are z_1, z_2 in Z such that $z_1 <_T z_2$ and hence $h(ht(z_2)) \geq ht(z_1)$. But $h(ht(z_2)) = h(ht(z_1)) < ht(z_1)$, a contradiction. This finishes the proof of the claim.

Now fix x and $\{x_n : n < \omega\}$ guaranteed by the claim. As T is finite-to-one, we can find a $k < \omega$ such that $x(ht(x_k)) < -n$. Then we have $x_k <_l x$ since $x_k <_T x$ and $t_{x_k} >_l t_x$ since $t_{x_k} \restriction_{ht(x_k)} = x_k = x \restriction_{ht(x_k)} = t_x \restriction_{ht(x_k)}$ and $t_{x_k}(ht(x_k)) = -n > x(ht(x_k)) = t_x(ht(x_k))$. This contradict the fact that both (x_k, t_{x_k}) and (x, t_x) are in C_m. $\square$

Remark: As we said before, that it is consistent to have an Aronszajn line which contains no Aronszajn subline that can be partitioned into a whole tree. One such example is $L = T \restriction_X$ where T is the tree mentioned in the above lemma and X is any uncountable nonstationary subset of ω_1. If $L' \in [L]^{\omega_1}$ can be partitioned into a whole tree $(S, <_S, <_{lS})$ (i.e., $(L', <_{lT})$ is order isomorphic to $(S, <_{lS})$ where $<_{lT}$ is the lexicographical order of T), then we will get a contradiction since on one hand, $(S, <_{lS})$ is Countryman as by Theorem 3.14, $(L, <_{lT})$ and hence $(L', <_{lT})$ is, while on the other hand, by Proposition 2.4, $S \restriction_C$ is lexicographically isomorphic to $T_{L'} \restriction_C$ for some club C, and so $S \restriction_C$ is not Countryman since according to previous lemma, $T_{L'} \upharpoonright_C$ is not. Note above argument actually shows that for any $L' \in [L]^{\omega_1}$, for any partition tree S of L', $ht_S(L')$ is nonstationary.

Definition 16. For any countable linear orders O, O' such that $O \subset O'$, for any coherent tree $T \subset O^{<\omega_1}$, the *completion of the coherent tree T for O'* (or say *the completion of T* if $O = O'$) is $T^* = \{t \in O'^{<\omega_1}$: there is some $s \in T_{dom(t)}$ such that $\{\alpha \in dom(t) : t(\alpha) \neq s(\alpha)\}$ is finite.$\}$.
A coherent tree T is *complete* if its completion is T itself.

Remark: Above lemma will also induce a $\mathbb{R}$-embeddable $\omega + 1$ (or $(\omega + 1)^*$)-ranging coherent tree which is not Countryman. If there is a T as mentioned in the previous proposition and T^* is its completion for $(\omega+1)^*$, then T^* contains an antichain (with respect to the tree order) which is not Countryman (with respect to the lexicographical order). Hence, there is a $\mathbb{R}$-embeddable $(\omega + 1)$-ranging coherent tree with this property too.

Now, we will construct a $\omega + 1$-ranging $\mathbb{R}$-embeddable coherent tree which contains no uncountable Countryman suborder. First, let's describe some sufficient condition for this.

Definition 17. An Aronszajn tree $T \subset (\omega + 1)^{<\omega_1}$ has *property (*)* if:
(1) for any $X \in [T]^{\omega_1}$, for any club C, there is a $x \in T_X \upharpoonright_C$ such that $x^\frown\omega \in T_X$;
(2) for any $Y \in [T]^{\omega_1}$ such that $ht(Y)$ is stationary, for any $n < \omega$, there are $n < m < \omega$ and t_1, t_2 in Y such that $t_1^\frown m <_T t_2$

Lemma 3.9. *If $T \subset (\omega + 1)^{<\omega_1}$ is an $\mathbb{R}$-embeddable Aronszajn tree has property (*) and contains no stationary antichain, then T contains no Countryman suborder with its canonical lexicographical order.*

Proof. Suppose otherwise, $X \in [T]^{\omega_1}$ and $(X, <_l)$ is Countryman where $<_l$ is the lexicographical order. Let $X^2 = \bigcup_{n<\omega} C_n$ be a partition witness that X is Countryman. Let $\langle N_\alpha : \alpha < \omega_1\rangle$ be a continuous elementary chain, N_0 contains all relevant objects and each N_α is countable and $C = \{N_\alpha \cap \omega_1 : \alpha < \omega_1\}$. Let $A : T \restriction_\Lambda \to \omega$ be a partition of $T \restriction$ into countably many chains.

Let x witness (1) of property (*) for X and C. Note there is some $i < \omega$ such that $x^\frown i \in T_X$ (otherwise $Z = \{t \in T_X : t^\frown \omega \in T_X$ and x has only one immediate successor in $T_X\}$ has stationary height by Fact 2.3. Let $Z_n = \{t \in Z : A(t^\frown \omega) = n\}$. Then for some n, Z_n has stationary height. This contradict that T contains no stationary antichain.). Fix such an i. Define $Y = \{t \in T_X : t^\frown i \in T_X \wedge t^\frown \omega \in T_X\}$. Then $Y \in [T]^{\omega_1}$ and $ht(Y)$ is stationary by Fact 2.3. Define $Y_n = \{t \in Y :$ there are u, v in X such that $(u, v) \in C_n$, $t^\frown i <_T u$ and $t^\frown \omega <_T v\}$. Fix a n such that $ht(Y_n)$ is stationary.

Apply (2) of property (*) to Y_n and i, we get $m \in (i, \omega)$ and t_1, t_2 in Y_n such that $t_1^\frown m <_T t_2$. By definition of Y_n, we have $(u_j, v_j) \in C_n$ such that $t_j^\frown i <_T u_j$ and $t_j^\frown \omega <_T v_j$ for $j = 1, 2$. Now we have $u_1 <_l u_2$ and $v_1 >_l v_2$ (note $t_1^\frown i <_l t_1^\frown m <_l t_1^\frown \omega$ and hence $u_1 <_l u_2 <_l v_2 <_l v_1$) which contradict the fact that C_n is a chain for product order. □

Now we will construct a coherent tree with properties mentioned in above lemma and it will be suffice for Theorem 3.6. The construction will use the method of minimal walk introduced by S. Todorcevic in [5].

Definition 18. (1) A *C-sequence* is a sequence $\langle C_\alpha : \alpha < \omega_1\rangle$ such that

(i) $C_{\alpha+1} = \{\alpha\}$;

(ii) if α is a limit ordinal, then C_α is a cofinal subset of α of order type ω.

(2) A *step* from a countable ordinal β towards a smaller ordinal α is the minimal point of C_β that is $\geq \alpha$. The cardinality of the set $C_\beta \cap \alpha$, or better to say the order-type of this set, is the *weight* of the step.

(3) For a C-sequence, a *walk* (or a *minimal walk*) from a countable ordinal β to a smaller ordinal α is the sequence $\beta = \beta_0 > \beta_1 > ... > \beta_n = \alpha$ such that for each

$i < n$, the ordinal β_{i+1} is the step from β_i towards α.

(4) For a C-sequence, the full code of the walk is the function $\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$, defined recursively by $\rho_0(\alpha, \beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha, min(C_\beta \setminus \alpha))$ with boundary value $\rho_0(\alpha, \alpha) = \phi$[6]

(5) For a C-sequence, The *maximal weight* of the walk is the function $\rho_1 : [\omega_1]^2 \to \omega$, defined recursively by $\rho_1(\alpha, \beta) = max\{|C_\beta \cap \alpha|, \rho_1(\alpha, min(C_\beta \setminus \alpha))\}$ with boundary value $\rho_1(\alpha, \alpha) = 0$.

(6) For a C-sequence, the number of steps of the minimal walk is the function $\rho_2 : [\omega_1]^2 \to \omega$, defined recursively by $\rho_2(\alpha, \beta) = \rho_2(\alpha, min(C_\beta \setminus \alpha)) + 1$ with boundary value $\rho_2(\alpha, \alpha) = 0$

Here are some basic fact about the ρ-functions from [1]:

Lemma 3.10. *([1]) (1) For any $\alpha < \beta < \omega_1$, if α is a limit ordinal, there is an ordinal $\xi < \alpha$ such that for any $\eta \in (\xi, \alpha)$[7], α is in the walk from β to ξ.*

(2) ρ_1 is finite to one and hence $\mathbb{R}$-embeddable.

(3) ρ_1 is coherent.

(4) For any $\alpha < \beta$, $\rho_2(\alpha, \beta) > 0$.

(5) For any $\alpha \leq \beta$, if α is a limit ordinal, then there is a $\xi_{\alpha\beta} < \alpha$ such that for any $\gamma \in [\xi_{\alpha\beta}, \alpha)$, $\rho_2(\gamma, \beta) = \rho_2(\alpha, \beta) + \rho_2(\gamma, \alpha)$.

(6) For any $\alpha < \beta$, if $\rho_{2\alpha} \not<_{T(\rho_2)} \rho_{2\beta}$, then $\Delta(\rho_{2\alpha}, \rho_{2\beta})$ is a successor ordinal.

Now we can start our construction. Note our construction is a little different from that in [1].

Construction: First in ground model V, fix a partition of ω_1 into countably many stationary sets: $\omega_1 = \bigcup_{n<\omega} S_n$ and an uncountable almost disjoint family

[6]Note firstly we write $\rho_0(\{\alpha, \beta\})$ as $\rho_0(\alpha, \beta)$ where the smaller appears first; secondly, although $\{\alpha\}$ is not in the domain, $\rho_0(\alpha, \alpha)$ is defined for convenience.

[7]Here $(*, *)$ denotes the interval of ordinals.

$\{A_\alpha : \alpha \in \Lambda\}$, i.e., for any $\alpha < \beta$ in Λ, $A_\alpha \subset \omega$ is infinite and $A_\alpha \cap A_\beta$ is finite. For any infinite $\alpha \in \Lambda$, fix a surjection $\pi_\alpha : A_\alpha \to \alpha$ such that every $\xi < \alpha$ has infinitely many preimage.

Then force with Cohen forcing $Fn(\omega, 2) = \{p : p$ is a finite partial function from ω to $2\}$ where the order is extension as functions. Let G be a generic filter and $r = \cup\{p : p \in G\}$. We don't distinguish a subset of ω and its characteristic function so we can assume that $r \subset \omega$.

Now our C-sequence will be defined as $(C_{\alpha+1} = \{\alpha\})$:

for infinite $\alpha \in \Lambda$, define $C_\alpha(n)$ – the nth element of C_α: $a_0 = min(A_\alpha \cap r)$, $C_\alpha(0) = \pi_\alpha(a_0)$; $a_{n+1} = min\{i \in A_\alpha \cap r : i > a_n$ and $\pi_\alpha(i) > C_\alpha(n)\}$, $C_\alpha(n+1) = \pi_\alpha(a_{n+1})$.

The final coherent function $a : [\omega_1]^2 \to \omega + 1$ and hence the coherent tree $T(a) = \{a_\beta \restriction_\alpha : \alpha < \beta < \omega_1\}$ will be defined by: for any $\alpha < \beta < \omega_1$,

$$a(\alpha, \beta) = \begin{cases} \omega & : \alpha \in S_{\rho_1(\alpha,\beta)} \\ \rho_1(\alpha, \beta) & : otherwise \end{cases}$$

Remark: For any $p \in Fn(\omega, 2)$, if $dom(p) = n$ for some n, then for any infinite $\alpha \in \Lambda$, p computes C_α correctly, i.e., if replace r in the computation of C_α by p and $C_\alpha(i)$ is computed to be ξ for some $i < \omega$, $\xi < \alpha$, then $p \Vdash C_\alpha(i) = \xi$, and vise-verse. Moreover, if p computes $C_\alpha(i)$, then p computes $C_\alpha(j)$ for any $j < i$.

First, we need to check that the coherent function and tree are well-defined:

Lemma 3.11. *C-sequence, ρ_1 and a are as constructed above.*
(1) For any infinite $\alpha \in \Lambda$, C_α has order type ω. So ρ_1 and hence a is well-defined.
(2) For any $\alpha < \beta < \gamma < \omega_1$, $\rho_1(\alpha, \beta) = \rho_1(\alpha, \gamma)$ iff $a(\alpha, \beta) = a(\alpha, \gamma)$. So $T(a)$ is coherent and tree isomorphic to $T(\rho_1)$. Hence $T(a)$ is $\mathbb{R}$-embeddable.

Proof. (1) It is suffice to prove that for any $\alpha \in \Lambda$, C_α is unbounded in α and for any $\xi < \alpha$, $C_\alpha \cap \xi$ is finite. Now fix a $\alpha \in \Lambda$.

Claim: For any $\xi < \alpha$, $C_\alpha \cap \xi$ is finite and $C_\alpha \not\subset \xi$.

Proof of claim: Fix a $\xi < \alpha$. For any $p \in Fn(\omega, 2)$, pick a $n < \omega$ such that $dom(p) \subset n$ and $\pi_\alpha(n) > \xi$. Then extend p to q such that $q(n) = 1$ and $q(i) = 0$ for $i \in dom(q) \setminus (dom(p) \cup \{n\})$. It is easy to check that $q \Vdash |C_\alpha \cap \xi| \leq n \wedge C_\alpha \not\subset \xi$. So a density argument shows that $C_\alpha \cap \xi$ is finite and $C_\alpha \not\subset \xi$: $D = \{p : p \Vdash C_\alpha \cap \xi$ is finite.$\}$ and $E = \{p : p \Vdash C_\alpha \not\subset \xi\}$ are dense.

(2) Trivial. Note for the hence part we can take the $\mathbb{R}$-embedding to be $f : T(\rho_1) \to \omega^\omega$:

$f(s)(n) = |\{\xi < ht(s) : s(\xi) = n\}|$. □

The following fact will be needed in following proof:

Fact 3.12. *If $\mathcal{P}$ is a countable forcing poset, $X \subset V$ is a set in $V[G]$ where G is a generic filter over $\mathcal{P}$, then there is a sequence $\langle Y_n : n < \omega \rangle$ in $V[G]$ such that $Y_n \in V$ for any $n < \omega$ and $X = \bigcup_{n<\omega} Y_n$. In particular, if $X \in V[G]$ and $X \subset V$ is an uncountable (or stationary) subset of ω_1, there is an uncountable (or stationary) $Y \in V$ such that $Y \subset X$. Hence, a club in $V[G]$ contains a subset that is a club in V and a stationary set in V is stationary in $V[G]$.*

Now we come to the key lemma:

Lemma 3.13. *(1) $T(\rho_1)$ and hence $T(a)$ contains no stationary antichain.*
(2) $T(a)$ has property (*).

Proof. (1) Fix $X \subset T(\rho_1)$ a subset of stationary height. By coherence and going to a stationary subset we can assume $X = \{\rho_{1\alpha} : \alpha \in \Gamma\}$ for some stationary $\Gamma \subset \Lambda$. By previous fact and going to a stationary subset we can assume $\Gamma \in V$.

Using density argument, it is suffice to prove the following:

Claim 1: For any $p \in Fn(\omega, 2)$, there are some $q \leq p$ and $\alpha < \beta$ in Γ such that

$q \Vdash \rho_{1\alpha} <_{T(\rho_1)} \rho_{1\beta}$.

Proof of claim 1: Fix a $p \in Fn(\omega, 2)$. Extend p if necessary, we can assume $dom(p) = n$ for some $n < \omega$. Note only a finite part (of size $\leq n$) of C_α is computed by p for any $\alpha \in \Lambda$. Find a stationary subset $\Gamma_1 \subset \Gamma$ and an ordinal $\alpha_0 < \omega_1$ such that (let C^p_α be the finite set computed by p)

(i) $C^p_\alpha \subset \alpha_0$ for any $\alpha \in \Gamma_1$;

(ii) $C^p_\alpha = C^p_\beta$ for any $\alpha, \beta \in \Gamma_1$.

Now pick $\alpha < \beta$ in Γ_1 and $m > n$ such that $\pi_\beta(m) = \alpha$ (such m exists since preimage of α is infinite). Extend p to $q \subset (m+1) \times 2$ such that $q \restriction_{[n,m)} = 0$ and $q(m) = 1$. Then $C^q_\beta = C^p_\beta \cup \{\alpha\} = C^p_\alpha \cup \{\alpha\}$. It follows from the definition of ρ_1 that $q \Vdash \rho_{1\alpha} = \rho_{1\beta} \restriction_\alpha$, i.e., $q \Vdash \rho_{1\alpha} <_{T(\rho_1)} \rho_{1\beta}$.

(2) Let's first prove (1) of property (*).

In $V[G]$, fix $X \in [T(a)]^{\omega_1}$ and club C. By previous fact and going to a subclub we can assume $C \in V$ and $C \subset \Lambda$. Pick any $\alpha \in \bigcap_{n<\omega} acc(S_n \cap C)$[8]. Let's first prove the following:

Claim 2: $\{\xi \in C \cap \alpha : a(\xi, \alpha) = \omega\}$ is unbounded below α.

proof of claim 2: Fix a $\eta < \alpha$, we want to find some $\xi \in [\eta, \alpha)$ such that $a(\xi, \alpha) = \omega$. For any $p \in Fn(\omega, 2)$, extend p if necessary we can assume $dom(p) = n$ for some $n < \omega$. Pick $\eta' > \eta, max(C^p_\alpha)$ and $\eta' < \alpha$. $\alpha \in acc(S_n \cap C)$, so we can find a $\xi \in S_n \cap C \setminus (\eta' + n)$ below α. Note $|C^p_\alpha| \leq n$. Pick an increasing sequence $\langle m_i < \omega : i < n + 1 - |C^p_\alpha| \rangle$ such that $\pi_\alpha(m_i) = \eta' + i$ for $i < n - |C^p_\alpha|$ and $\pi_\alpha(m_{n-|C^p_\alpha|}) = \xi$ (we can do this since each ordinal has infinite preimage).

Now extend p to $q \subset (m_{n-|C^p_\alpha|} + 1) \times 2$ such that $q \restriction_n = p$, $q(m_i) = 1$ for $i < n + 1 - |C^p_\alpha|$ and $q(j) = 0$ for the rest $j \leq m_{n-|C^p_\alpha|}$. Then $C^q_\alpha = C^p_\alpha \cup \{\eta' + i : i < n - |C^p_\alpha|\} \cup \{\xi\}$. Hence $q \Vdash \rho_1(\xi, \alpha) = n$. So $q \Vdash a(\xi, \alpha) = \omega$. This finishes the proof of claim 2.

[8] For any uncountable $X \subset \omega_1$, $acc(X) = \{\alpha < \omega_1 : sup(X \cap \alpha) = \alpha\}$. Note $acc(X)$ is always a club.

Now (1) of property (*) follows from the coherence of a and claim 2.

Then we prove (2) of property (*).

By coherence of a, it is suffice to prove for $Y = \{a_\alpha : \alpha \in \Gamma\}$ for some stationary Γ. Now fix a $n < \omega$ and such a Y together with the stationary Γ. Going to a stationary subset we can assume $\Gamma \in V$ and $\Gamma \subset \bigcap_{n<\omega} acc(S_n) \cap \Lambda$.

For any $p \in Fn(\omega, 2)$, extend p if necessary we can assume $dom(p) = k$ for some $k < \omega$. Find a stationary subset $\Gamma_1 \subset \Gamma$ and an ordinal $\alpha_0 < \omega_1$ such that

(i) $C^p_\alpha \subset \alpha_0$ for any $\alpha \in \Gamma_1$;

(ii) $C^p_\alpha = C^p_\beta$ for any $\alpha, \beta \in \Gamma_1$.

Fix $\alpha < \beta$ in Γ_1 such that $\alpha \in S_{n'}$ and $m > n, k, n'$. Pick an increasing sequence $\langle l_i < \omega : i < m - |C^p_\alpha|\rangle$ such that $l_0 > n + max(A_\alpha \cap A_\beta)$ and $\pi_\alpha(l_i) = max(C^p_\alpha) + i$ for any $i < m - |C^p_\alpha|$. Then pick an increasing sequence $\langle l'_j < \omega : j < m + 1 - |C^p_\beta|\rangle$ such that $l'_0 > l_{m-1-|C^p_\alpha|}$ and $\pi_\beta(l'_j) = max(C^p_\alpha) + j$ for any $j < m - |C^p_\beta|$ and $\pi_\beta(l_{m-|C^p_\beta|}) = \alpha$.

Now extend p to $q \subset (l'_{m-|C^p_\alpha|} + 1) \times 2$ such that $q \restriction_n = p$, $q(l_i) = 1$, $q(l'_j) = 1$ for above mentioned l_i's and l'_j's and $q(i) = 0$ for the rest $i \leq l'_{m-|C^p_\alpha|}$. Then $C^q_\alpha = C^p_\alpha \cup \{max(C^p_\alpha) + i : i < m - |C^p_\alpha|\rangle\}$ and $C^q_\beta = C^p_\beta \cup \{max(C^p_\alpha) + i : i < m - |C^p_\alpha|\rangle\} \cup \{\alpha\}$. Recall $C^p_\alpha = C^p_\beta$. So $q \Vdash$ "$\rho_{1\alpha} <_{T(\rho_1)} \rho_{1\beta}$ and $\rho_1(\alpha, \beta) = m$". Note $\alpha \notin S_m$. So $a_\alpha <_{T(a)} a_\beta$ and $a(\alpha, \beta) = m$. This finishes the proof of (2) of property (*) and hence the proof of the lemma. □

Now Theorem 3.6 follows:

Proof of Theorem 3.6: (1) By Fact 3.7, we just need to prove for $O = \omega^*$ or $O = \omega + 1$. The case $O = \omega^*$ follows from Lemma 3.8 and note then $T(\rho_1)$ we just constructed is a finite to one coherent tree with no stationary antichain (so the tree for Lemma 3.8 can be induced from $T(\rho_1)$ from a bijection between ω and ω^*). The case $O = \omega + 1$ follows from Lemma 3.9, Lemma 3.11 and Lemma 3.13.

(2)By Fact 3.7, we just need to prove for $O = \omega + 1$ or $O = (\omega + 1)^*$. The case $O = \omega + 1$ follows from Lemma 3.9, Lemma 3.11 and Lemma 3.13. For the case $O = (\omega + 1)^*$. Let $T \subset (\omega + 1)^{<\omega_1}$ be an $\mathbb{R}$-embeddable coherent tree with no Countryman suborder. Here we don't distinguish $(\omega + 1)^*$ and $-(\omega + 1) = \{-\alpha : \alpha < \omega + 1\}$. Denote $-T = \{-t : t \in T\}$ where $(-t)(\xi) = -(t(\xi))$ for any $\xi < ht(t)$. Then $-T$ is an $\mathbb{R}$-embeddable coherent tree. If $-T$ contains a Countryman suborder, then $-T$ contains an antichain $-X$ whose lexicographical is Countryman. Then its reverse order $(X, <_{lT})$ is Countryman. A contradiction. Hence $-T$ is a witness for case $O = (\omega+1)^*$. This finishes the proof of the theorem.

3.3 An equivalent condition for coherence being Countryman

Previous section shows that non-ω-ranging (non-$\mathbb{Z}$-ranging) $\mathbb{R}$-embeddable coherent tree may be not Countryman (or may contain no Countryman suborder) with its lexicographical order. This gives a negative answer to the question "whether $\mathbb{R}$-embeddable coherence implies Countryman". However, as we said before, there are some positive answer: every $\mathbb{R}$-embeddable ω-ranging coherent tree is Countryman (S. Todorcevic). So to make the situation on transforming $\mathbb{R}$-embeddable coherence to Countryman clear, we need to answer the following two questions:

1. what is the sufficient (or equivalent) condition for a $\mathbb{R}$-embeddable coherent tree to be Countryman (with its lexicographical order)?

2. what is the sufficient (or equivalent) condition for a $\mathbb{R}$-embeddable coherent tree to contain a Countryman suborder?

This section will give some sufficient conditions for $\mathbb{R}$-embeddable coherence to be Countryman and contain a Countryman suborder respectively and then prove that these conditions are actually the "equivalent" conditions. As the investigating goes on, we can see that what really matters is not when a $\mathbb{R}$-embeddable coherent tree contains a Countryman suborder, but when the nonstationary levels of a $\mathbb{R}$-embeddable coherent tree is Countryman. So the real question is:

3. for an uncountable subset $X \subset \omega_1$, what is the equivalent condition for a $\mathbb{R}$-embeddable coherent tree to have $(T \restriction_X, <_{lT})$ to be Countryman?

Let's start from a sufficient condition for a $\mathbb{R}$-embeddable coherent tree to contain a Countryman suborder. By previous section, the condition shouldn't be weaker than $\mathbb{Z}$-ranging. However, we do have a "weaker" (looking) condition:

Theorem 3.14. *If $(T, <_T, <_l)$ is a $\mathbb{R}$-embeddable $\mathbb{Z}$-branching coherent tree and X is an uncountable nonstationary subset of ω_1, then $(T \restriction_X, <_l)$ is Countryman.*

Proof. WLOG, we can assume T is $\mathbb{Q}$-ranging and X contains all successor ordinals. By Fact 2.12, let $T \restriction_X = \bigcup_{n<\omega} A_n$ where each A_n is an antichain. Define function $A : T \restriction_X \to \omega$ by $A(x) = n$ iff $x \in A_n$ (assume A_n's are pairwise disjoint). Now for any $(a, b) \in X^2$, define $D_{ab} = \{\alpha < ht(a), ht(b) : a(\alpha) \neq b(\alpha)\}$ (D_{ab} is finite by coherence), $A_a^b : D_{ab} \to [\omega]^{<\omega}$ where $A_a^b(\alpha) = \{A(a \restriction_\alpha^\frown r) : \alpha \in D_{ab}$, $(a(\alpha) < r < b(\alpha))$ or $(b(\alpha) < r < a(\alpha))$ and $a \restriction_\alpha^\frown r \in T$ and $b \restriction_\alpha^\frown r \in T\} \cup \{A(a \restriction_{\alpha+1}\}$ and A_b^a is similarly defined. Then define a structure for (a, b): $S(a, b) = \langle D_{ab}, A(a), A(b), a, b, A_a^b, A_b^a\rangle$ where a serves as a function with domain D_{ab} (similarly for b). To prove each structure is finite, it is enough to show the following:

Claim 1: $A_a^b(\alpha)$ (or similarly $A_b^a(\alpha)$)is finite for any (a, b) and any $\alpha \in D_{ab}$.

proof of claim 1: Suppose otherwise, and WLOG assume $a(\alpha) < b(\alpha)$, then $\{r : a(\alpha) < r < b(\alpha) \wedge a \restriction_\alpha^\frown r \in T \wedge b \restriction_\alpha^\frown r \in T\}$ is infinite. Pick an infinite increasing subset (or infinite decreasing subset if no increasing one). Then together with $b(\alpha)$ we can find a subset of $(succ_T(b \restriction_\alpha), <_l)$ with order type $\omega + 1$. Contradict the fact that T is $\mathbb{Z}$-branching. This finishes the proof of claim 1.

Say two structures $S(a, b), S(c, d)$ are isomorphic if $A(a) = A(c)$, $A(b) = A(d)$

and there is an order isomorphism $\pi : D_{ab} \to D_{cd}$ such that $a(\alpha) = c(\pi(\alpha))$, $b(\alpha) = d(\pi(\alpha))$, $A_a(\alpha) = A_c(\pi(\alpha))$, $A_b(\alpha) = A_d(\pi(\alpha))$ for any $\alpha \in D_{ab}$.

Then define $C(\langle D, m, n, f_1, f_2, f_3, f_4 \rangle) = \{(a, b) \in X^2 : \langle D, m, n, f_1, f_2, f_3, f_4 \rangle$ structure isomorphic to $S(a, b)\}$ for each $D \in [\omega]^{<\omega}$, $m, n \in \omega$, f_1, f_2 functions from D to $\mathbb{Q}$, f_3, f_4 functions from D to $[\omega]^{<\omega}$. And the partition for X^2 will be $X^2 = \cup\{C(\langle D, m, n, f_1, f_2, f_3, f_4\rangle) : D \in [\omega]^{<\omega}, m, n \in \omega, f_1, f_2$ are functions from D to $\mathbb{Q}$ and f_3, f_4 are functions from D to $[\omega]^{<\omega}.\}$. There are only countably many $C(\langle D, m, n, f_1, f_2, f_3, f_4\rangle)$'s, so it is suffice to prove that each $C(\langle D, m, n, f_1, f_2, f_3, f_4\rangle)$ is a chain for product order.

Assume, towards a contradiction, (a, b), (c, d) are in the same part of the partition, i.e., $S(a, b)$ structure isomorphic to $S(c, d)$, and $a <_l c$ while $b >_l d$. a, c are incomparable for the tree order as $A(a) = A(c)$, so $\Delta(a, c) < ht(a), ht(c)$.

Claim 2: $D_{ab} \cap \Delta(a, c) = D_{cd} \cap \Delta(a, c)$.

proof of claim 2: Suppose otherwise, let $\alpha = min(D_{ab} \triangle D_{cd}) < \Delta(a, c)$. WLOG, assume $\alpha \in D_{ab} \setminus D_{cd}$ and $\alpha = D_{ab}(n) < D_{cd}(n)$ (nth element of D_{ab} and D_{cd} correspondingly). Then $A(c \restriction_{\alpha+1}) = A(a \restriction_{\alpha+1}) \in A_a(\alpha)$ since $\alpha < \Delta(a, c)$, and $A_a(D_{ab}(n)) = A_c(D_{cd}(n))$ by structure isomorphism. Then we have $A(c \restriction_{\alpha+1}) \in A_c(D_{cd}(n))$ and so $A(c \restriction_{\alpha+1}) = A((c \restriction_{D_{cd}(n)})^\frown r)$ for some r. Note $c \restriction_{\alpha+1} <_T (c \restriction_{D_{cd}(n)})^\frown r$. A contradiction. This finishes the proof of claim 2.

Then for any $\alpha < \Delta(a, c)$, if $\alpha \notin D_{ab}$, $b(\alpha) = a(\alpha) = c(\alpha) = d(\alpha)$ and if $\alpha \in D_{ab}$, $b(\alpha) = d(\alpha)$ by the above claim and the structure isomorphism. So $\Delta(b, d) \geq \Delta(a, c)$. Similarly, we have $\Delta(a, c) \geq \Delta(b, d)$. So $\Delta(a, c) = \Delta(b, d)$.

Denote $\xi = \Delta(a, c)$. We have $a(\xi) < c(\xi)$ as $a <_l c$ and $b(\xi) > d(\xi)$ as $b >_l d$. There are several cases:

Case 1: $\xi \in D_{ab} \cap D_{cd}$.

Then $a(\xi) = c(\xi)$ by claim 2 and the structure isomorphism. Contradiction.

Case 2: $\xi \notin D_{ab} \cup D_{cd}$.

Then $a(\xi) = b(\xi)$ and $c(\xi) = d(\xi)$. Contradict that $a(\xi) < c(\xi)$ and $b(\xi) > d(\xi)$.

Case 3: $\xi \in D_{ab} \setminus D_{cd}$.

Then $c(\xi) = d(\xi)$ and so $a(\xi) < c(\xi) = d(\xi) < b(\xi)$. Then $A(a \restriction_\xi^\frown c(\xi)) \in A_a(\xi)$ and so $A(c \restriction_{\xi+1}) \in A_a(D_{ab}(n))$ where n is the number such that $\xi = D_{ab}(n)$. By structure isomorphism, $A(c \restriction_{\xi+1}) \in A_c(D_{cd}(n))$, i.e., there is some r such that $(c \restriction_{D_{cd}(n)})^\frown r \in T$ and $A(c \restriction_{\xi+1}) = A((c \restriction_{D_{cd}(n)})^\frown r)$. But $c \restriction_{\xi+1} <_T (c \restriction_{D_{cd}(n)})^\frown r$. A contradiction.

Case 4: $\xi \in D_{cd} \setminus D_{ab}$.

Then $d(\xi) < a(\xi) = b(\xi) < c(\xi)$. Repeat the proof of case 3 we can get a contradiction.

In any case, there is a contradiction. This finishes the proof of the theorem. □

For any ω-branching coherent tree T (assume T is $\mathbb{Q} \cap (0,1)$-ranging), let T' be the downward closure of $\{t^\frown 0 : t \in T\}$, then T' is still ω-branching ($\mathbb{Q} \cap [0,1)$-ranging) and coherent, moreover, $(T, <_{lT})$ can be embedded into $(T' \restriction_{\Lambda+1}, <_{lT'})$[9] via $\pi(t) = t^\frown 0$ for $t \in T$. Then Theorem 2.11 is easily deduced from Theorem 3.14:

Corollary 3.15. *If T is a $\mathbb{R}$-embeddable ω-branching coherent tree, then T is Countryman with its lexicographical order.*

By Theorem 3.6, we see that Theorem 3.14 is the best we can expect. And although above theorem shows for the whole nonstationary levels, by Proposition 3.16, it shows no more than a subset that captures the tree (i.e., a subset whose downward closure subtree is the whole tree). Let's now see for a $|mathbbR$-embeddable coherent tree T the difference between containing an uncountable Countryman suborder and having $T \restriction_A$ is Countryman for any nonstationary $A \subset \omega_1$.

Proposition 3.16. *(Countryman is closed under countable self union) (1) If $(L, <_L)$ is a Countryman line, $(A, <_A)$ is an uncountable linear order and $A = \bigcup_{n<\omega} A_n$ is a*

[9] Recall that $\Lambda + 1 = \{\alpha + 1 < \omega_1 : \alpha \text{ is a limit ordinal.}\}$.

partition such that $(A_n, <_A)$ can be embedded into $(L, <_L)$ for any n, then $(A, <_A)$ is Countryman.

(2) For any partition tree T of some Countryman line, if $X \in [T]^{\omega_1}$ is special, then X is Countryman. In particular, if T is $\mathbb{R}$-embeddable, then $T \upharpoonright_A$ is Countryman for any uncountable nonstationary $A \subset \omega_1$.

(3) T is an $\mathbb{R}$-embeddable coherent tree containing an uncountable Countryman suborder, iff T is contained in the completion of some $\mathbb{R}$-embeddable coherent tree T' such that $T' \upharpoonright_A$ is Countryman for any nonstationary $A \subset \omega_1$

Proof. (1) Let $L^2 = \bigcup_{n<\omega} C_n$ be a partition witnesses that L is Countryman. Let $\pi_n : A_n \to L$ be an embedding.

Now we can partition A^2 into countably many chains: for any m, n, k in ω, define $D(m, n, k) = \{(a, b) : a \in A_m \wedge b \in A_n \wedge (\pi_m(a), \pi_n(b)) \in C_k\}$. It is easy to see that $A^2 = \bigcup_{m,n,k<\omega} D(m, n, k)$ is a countable partition. So it suffice to show that $D(m, n, k)$ is a chain (for product order) for any $m, n, k < \omega$.

Fix $m, n, k < \omega$ and pick $(a, b), (c, d) \in D(m, n, k)$. Assume $a <_A c$ and we need to show $b \leq_A d$. First, $\pi_m(a) <_L \pi_m(c)$ since π_m is an embedding. Second, $\pi_n(b) \leq_L \pi_n(d)$ since $(\pi_m(a), \pi_n(b)) \in C_k$ and $(\pi_m(c), \pi_n(d)) \in C_k$. Last, $b \leq_A d$ since π_n is an embedding.

(2) follows from (1).

(3) Note the "if" part follows from Fact 2.12 and Corollary 4.7. For the "only if" part. Assume $X \in [T]^{\omega_1}$ is Countryman. Just take $T' = T_X$. Then T' has the desired property by (2) and T is contained in the completion follows from the fact that T is coherent. □

Note by remark after definition 16, an $\mathbb{R}$-embeddable coherent tree T contains a Countryman suborder doesn't imply $T \upharpoonright_{\Lambda+1}$ is Countryman. So unlike $\mathbb{R}$-embeddability, property that every (uncountable) nonstationary set of levels is

Countryman is not closed under taking completion. So it is necessary that T is a contained in a completion of some $\mathbb{R}$-embeddable coherent tree with this property. As $\mathbb{R}$-embeddable coherent tree that contains a Countryman suborder can be viewed as a (subtree of) completion of $\mathbb{R}$-embeddable coherent tree with property that every (uncountable) nonstationary set of levels is Countryman. So the question becomes that when does a $\mathbb{R}$-embeddable coherent tree has this property?

Theorem 3.17. *(1) If $(T, <_T, <_{lT})$ is a $\mathbb{R}$-embeddable coherent tree, then the following are equivalent:*
(a) $(T, <_{lT})$ is Countryman.
(b) T is a subtree of some $\mathbb{R}$-embeddable ω-ranging coherent tree T'.
(2) If $(T, <_T, <_{lT})$ is an $\mathbb{R}$-embeddable coherent tree, then the following are equivalent:
(c) $(T \restriction_A, <_{lT})$ is Countryman for any uncountable nonstationary $A \subset \omega_1$.
(d) T is a subtree of some $\mathbb{R}$-embeddable $\mathbb{Z}$-ranging coherent tree T'.
(e) $(T \restriction_{\Lambda+1}, <_{lT})$ is Countryman.

Before we start the proof, we should note that above mentioned "T is a subtree of some T'" is necessary since we can easily get a $\mathbb{Q}$-branching tree by taking a club restriction subtree.

First, note only levels in $\Lambda + 1$ matters:

Fact 3.18. *If T is an $\mathbb{R}$-embeddable Aronszajn tree and T is ω-ranging (or $\mathbb{Z}$-ranging) at levels $C+1$ for some club C, i.e., for any $s \in T\restriction_C$, for any immediate successor t, $t = s^\frown n$ for some $n < \omega$ (or $n \in \mathbb{Z}$), then T is a subtree of some $\mathbb{R}$-embeddable ω-ranging (or $\mathbb{Z}$-ranging) Aronszajn tree T'. Moreover, T' is coherent (or Countryman or both) if T is.*

Proof. Without loss of generality, just prove for ω-ranging and assume $T \subset (\omega^{<\omega})^{<\omega_1}$ and is ω-ranging at levels $C+1$, i.e., $t(\alpha) \in \omega$ for any $t \in T$ for any $\alpha \in C$. Embedd

T to $\omega^{<\omega_1}$ by

for any $t \in T$, $f(t)$ has height (length) $\omega * ht(t)$ and $f(t) \restriction_{[\omega\alpha,\omega\alpha+\omega)} = t(\alpha)$[10] for $\alpha < ht(t)$.

Then it is easy to see that T lexicographically isomorphic to $(f''T) \restriction_\Lambda$, i.e., T can be viewed as a subtree of $f''T$. We only need to prove $f''T$ is $\mathbb{R}$-embeddable. Recall that T is ω-ranging at levels $C+1$ for some club C. Go to an uncountable subclub C' such that for any $\alpha \in C'$, $\omega\alpha = \alpha$. Then $T \restriction_{C'+1}$ is isomorphic to $(f''T) \restriction_{C'+\omega}$ which is isomorphic to $(f''T) \restriction_{C'+1}$. So $(f''T) \restriction_{C'+1}$ is special and by Fact 2.12 and hence $f''T$ is $\mathbb{R}$-embeddable. For the moreover part, note the construction will never destroy the coherence. And the Countryman part follows from Proposition 3.16. □

Let's prove the theorem starting from an easy case:

Lemma 3.19. *(1) If $(T, <_T, <_{lT})$ is an $\mathbb{R}$-embeddable Aronszajn tree such that $(T, <_{lT})$ is Countryman, then T is a subtree of some $\mathbb{R}$-embeddable ω-ranging Aronszajn tree T' such that $(T', <_{lT'})$ is Countryman.*
(2) If $(T, <_T, <_{lT})$ is an $\mathbb{R}$-embeddable Aronszajn tree such that $(T \restriction_{\Lambda+1}, <_{lT})$ is Countryman, then T is a subtree of some $\mathbb{R}$-embeddable $\mathbb{Z}$-ranging Aronszajn tree T' such that $(T' \restriction \Lambda+1, <_{lT'})$ is Countryman.

Proof. (1) Assume $T \subset \mathbb{Q}^{<\omega_1}$ and by Fact 2.12 $A : T \restriction_{\Lambda+1} \to \omega$ is a partition into antichains. Let $c : T^2 \to \omega$ be a partition witnesses Countryman. For any $s \in T$, let $B_s = \{q \in \mathbb{Q} : s^\frown q \in T\}$ and fix a cofinal sequence C_s in B_s with order type 1 or ω. Let $C_s(n)$ denote nth element of C_s. Now we want to embed T to some T' satisfying the assumption of previous fact with the club $C = \Lambda$. For any $t \in T$, define $f(t)$ to be a sequence of length $\leq ht(t)+1$ such that for any $\alpha \leq ht(t)$

(1) if $\alpha = \beta + 1$, then $f(t)(\alpha) = t(\beta)$;

[10] Here we don't distinguish a finite sequence σ and its extension with eventually 0's – $\sigma^\frown \vec{0}$.

(2) if $\alpha \in \Lambda$ and $\alpha < ht(t)$, then $f(t)(\alpha) = n$ if $t(\alpha) \in (C_s(n-1), C_s(n)]$ where $s = t\restriction_\alpha$;

(3) if $\alpha \in \Lambda$ and $\alpha = ht(t)$, then $f(t)(\alpha)$ is undefined and $f(t)$ has length $ht(t)$.

To show that $f''T$ (let's don't distinguish $f''T$ and its downward closure) is $\mathbb{R}$-embeddable, by Fact 2.12, just need to define a partition of $(f''T)\restriction_{\Lambda+1}$ into countably many antichains. Define $g : (f''T)\restriction_{\Lambda+1} \to \omega^2$ by for any $s \in (f''T)\restriction_\Lambda$, let $s' \in T_{ht(s)}$ be such that $f(s') = s$ and $g(s^\frown n) = (A(s'^\frown C_{s'}(n), c(s', s'^\frown C_{s'}(n)))$.

Assume $s^\frown m <_{f''T} t^\frown n$ and s', t' in T are corresponding preimages. If $t'(ht(s')) = C_{s'}(m)$, then $g(s^\frown m) \neq g(t^\frown n)$ by first coordinate. If $t'(ht(s')) < C_{s'}(m)$, then $g(s^\frown m) \neq g(t^\frown n)$ by second coordinate. This finishes the proof of (1).

(2) For any $s \in T$, B_s is defined as before and C_s is a sequence in B_s that is both cofinal and anti-cofinal with order type 1 or 2 or ω or $-\omega$ or $\mathbb{Z}$ depending on whether B_s has minimal or maximal element. Define f similarly except for $\alpha \in \Lambda$ and $\alpha < ht(t)$, for $-n < 0$, $f(t)(\alpha) = -n$ if $t(\alpha) \in [C_{t\restriction_\alpha}(-n), C_{t\restriction_\alpha}(-n+1))$. And define g similarly except the second of $g(s^\frown n)$ is $c(s'^\frown C_{s'}(0), s'^\frown C_{s'}(n)$ for any $s \in T\restriction_\Lambda$ and $n \in \mathbb{Z}$. □

Above lemma suggests that Countryman should have some connection to ω-ranging (or $\mathbb{Z}$-ranging for not whole Countryman). However, above construction will probably destroy the coherent property since partition for immediate successors of each $s \in T$ is considered separately, i.e., no interaction between $s \neq s'$ in T_α for some α is considered.

Before we go to the main lemma, we need some definition:

Definition 19. (1) If T is a coherent tree, the *canonical completion* of T is $T^* = \{t : \exists t' \in T_{ht(t)} t =^* t'$ and $\forall \alpha < ht(t) t(\alpha) \in A_\alpha\}$ where $A_\alpha = \{q : s^\frown q \in T_{\alpha+1}$ for some $s\}$.

(2) If T is a coherent tree, the *one step canonical completion* of T is $T^{(1)} = \{t \in$

$T^* : \exists t' \in T_{ht(t)} \;\; D_{tt'} = \{\alpha < ht(t) : t(\alpha) \neq t'(\alpha)\}$ has size at most 1.}.

Now we come to our key lemma which takes care of the whole level instead of each $s \in T$ at the same time:

Lemma 3.20. *(1) If T is an $\mathbb{R}$-embeddable coherent tree and $(T, <_{lT})$ is Countryman, then its canonical completion T^* is also $\mathbb{R}$-embeddable coherent and $(T^*, <_{lT^*})$ is Countryman.*
(2) If T is an $\mathbb{R}$-embeddable coherent tree and $(T \restriction_{\Lambda^c}, <_{lT})$ is Countryman where Λ^c is the complement of Λ, then its canonical completion T^* is also $\mathbb{R}$-embeddable coherent and $(T^* \restriction_{\Lambda^c}, <_{lT^*})$ is Countryman.

Proof. (1) Note $\mathbb{R}$-embeddability follows from Corollary 4.7. Assume $T \subset (\mathbb{Q} \cap (0,1))^{<\omega_1}$. Then T can be lexicographically embedded into Λ^c levels of T' = downward closure of $\{t^\frown 0 : t \in T\}$. Moreover, T's canonical completion can be embedded into Λ^c levels of canonical completion of T'. So (1) follows from (2) and let's just prove (2)

(2) Assume $T \subset \mathbb{Q}^{<\omega_1}$. Let $c : (T \restriction_{\Lambda^c})^2 \to \omega$ witnesses that $(T \restriction_{\Lambda^c}, <_{lT})$ is Countryman.

Note $T^* = \bigcup_{n<\omega} T^{(n)}$ where $T^{(n+1)} = (T^{(n)})^{(1)}$. In fact, we just need to prove the following:

claim: $(T^{(1)} \restriction_{\Lambda^c}, <_{lT^{(1)}})$ is Countryman where $T^{(1)}$ is the one step canonical completion.

Let's first assume that claim is true and prove the lemma. First, $(T^{(n+1)} \restriction_{\Lambda^c}, <_{lT^{(n+1)}})$ is Countryman (just treat $T^{(n)}$ as a new "T"). Let $c_n : T^{(n)} \restriction_{\Lambda^c} \to \omega$ be a partition witnesses Countryman. Then $c^* : T^* \restriction_{\Lambda^c} \to \omega^3$ is defined by

$c^*(s,t) = (m, n, c_i(s,t))$ where m (or respectively n) is least j (k) such that $s \in T^{(j)}$ ($t \in T^{(k)}$) and $i = max\{m, n\}$.

It is easy to check that c^* witnesses that $T^* \restriction_{\Lambda^c}$ is Countryman.

Proof of claim: Let $A : T\restriction_{\Lambda^c} \to \omega$ be a partition into antichains. For any $s \in T^{(1)} \setminus T$, fix $s' \in T_{ht(s)}$ such that $D_{ss'} = \{\alpha < ht(s) : s(\alpha) \neq s'(\alpha)\}$ has size 1, fix α_s to be the ordinal in $D_{ss'}$ and fix $s'' \in T_{\alpha_s+1}$ such that $s''(\alpha_s) = s(\alpha_s)$. And if $s \in T$, then denote $s' = s'' = s$. For any $x, y \in T^{(1)}$, let

$D_{xy} = \{\alpha < min\{ht(x), ht(y)\} : x(\alpha) \neq y(\alpha)\}$.

Now for any $s, t \in T^{(1)}$,

$E_{st} = D_{st} \cup D_{ss'} \cup D_{ss''} \cup D_{tt'} \cup D_{tt''}$.

Now use a finite structure to characterize s, t:

$S(s,t) = \langle E_{st}, s, t, s', t', s'', t'', A', c, \pi\rangle$ where s, t, s', t', s'', t'' are functions (may be partial) on E_{st}, A' is defined for $x \in \{s, t, s', t', s'', t''\}$ and $\alpha \in E_{st}$ by $A'(x, \alpha) = A(x\restriction_{\alpha+1})$ and $\pi(x) = 0$ if $x \in T$, $\pi(x) = 1$ if $x \in T^{(1)} \setminus T$.

Now we partition $T^{(1)}$ into isomorphic structures. And we just need to show that for any $s, t, u, v \in T^{(1)}$, if $S(s,t)$ is isomorphic to $S(u,v)$ and $s <_{lT^{(1)}} u$, then $t \leq_{lT^{(1)}} v$.

Fix $s, t, u, v \in T^{(1)}$ such that $S(s,t)$ is isomorphic to $S(u,v)$ and $s <_{lT^{(1)}} u$. We will prove the case that $s, t, u, v \in T^{(1)} \setminus T$, and we will see that the proof works for the rest cases.

First, repeat the argument in Theorem 3.14, we can get that $\Delta(s,u) = \Delta(t,v) = \Delta(s',u') = \Delta(t',v')$ and $E_{st} \cap \Delta(s,u) = E_{uv} \cap \Delta(s,u)$. Let $\xi = \Delta(s,u)$. Note if $\alpha_s \geq \xi$ (or $\alpha_t \geq \xi$), then $\Delta(s'',u'') = \xi$ (or $\Delta(t'',v'') = \xi$).

If $t = v$, then it is trivial. Now assume $t \neq v$. Now we will discuss by cases.

Case 1: $\xi \in E_{st} \cap E_{uv}$

Assume $\xi = E_{st}(n) = E_{uv}(n)$, then by structure isomorphism, $s(\xi) = u(\xi)$. A contradiction. So this case never happens.

Case 2: $\xi \notin E_{st} \cup E_{uv}$.

Then $t(\xi) = s(\xi) < u(\xi) = v(\xi)$. Hence $t <_{lT^{(1)}} v$.

Case 3: $\xi \in E_{st} \setminus E_{uv}$.

First note that if $\alpha_s \neq \xi$, then $s(\xi) = s'(\xi)$ and if $\alpha = \xi$, then $s(\xi) = s''(\xi)$. Similarly for t, u, v. Now we will discuss several subcases.

subcase 3.1: $\alpha_s \neq \xi$ and $\alpha_t \neq \xi$.

Then $s' \restriction_\xi = u' \restriction_\xi$ (recall $\Delta(s', u') = \xi$) and $s'(\xi) = s(\xi) < u(\xi) = u'(\xi)$. So $s' <_{lT} u'$. By $c(s', t') = c(u', v')$, $t' \leq_{lT} v'$. Note also $t' \restriction_\xi = v' \restriction_\xi$ and $\xi = \Delta(t', v')$. So $t' <_{lT} v'$ and $t'(\xi) < v'(\xi)$. So $t(\xi) < v(\xi)$ and hence $t <_{lT^{(1)}} v$.

subcase 3.2: $\alpha_s \neq \xi$ and $\alpha_t = \xi$.

Then $s' <_{lT} u'$ and hence $t'' \leq_{lT} v''$ $(c(s', t'') = c(u', v''))$. So $t''(\xi) < v''(\xi)$ $(\Delta(t'', v'') = \xi)$. Note also $t(\xi) = t''(\xi)$ since $\xi = \alpha_t$ and $v(\xi) = v''(\xi)$ since $\xi \notin E_{uv}$. So $t(\xi) < v(\xi)$ and hence $t <_{lT^{(1)}} v$.

subcase 3.3: $\alpha_s = \xi$ and $\alpha_t \neq \xi$.

Similar as subcase 3.2 using s'', u'' and t', v'.

subcase 3.4: $\alpha_s = \xi$ and $\alpha_t = \xi$.

Similar as before using s'', u'' and t'', v''.

Case 4: $\xi \in E_{uv} \setminus E_{st}$

Similar as case 3.

Now we have proved that in any possible case, $t <_{lT^{(1)}} v$. This finishes the proof of the claim and hence the proof of lemma. □

Remark: Canonical completion is necessary here. For example, if $T \subset \omega^{<\omega_1}$ is a finite to one coherent tree. Then $(T, <_{lT})$ is Countryman. But if T contains no stationary antichain, then T^* – its completion for $\omega + 1$ – is not Countryman and actually, $(T^* \restriction_{\Lambda+1}, <_{lT^*})$ is not Countryman.

Now we are ready to prove Theorem 3.17:

Proof of Theorem 3.17: (1) $(b) \rightarrow (a)$ follows from Theorem 3.14 or Corollary 3.15.

$(a) \rightarrow (b)$. Assume T^* is the canonical completion of T. Then T^* is $\mathbb{R}$-embeddable coherent and Countryman by Lemma 3.20. Let $A_\alpha = \{q : \exists s \in T^* s^\frown q \in T^*_{\alpha+1}\}$. Let C_α be a cofinal sequence in A_α of order type 1 or ω. Then repeat the proof in Lemma 3.19 using A_α (C_α) to replace B_s (C_s) for $s \in T^*_\alpha$. It is easy to check that the resulting tree is moreover coherent.

(2) $(c) \to (d)$ is similar as $(a) \to (b)$.

$(d) \to (c)$ follows from Theorem 3.14.

$(d) \to (e)$ follows from Theorem 3.14.

$(e) \to (d)$. Assume $T \subset \mathbb{Q}^{<\omega_1}$ and $\{q_n : n < \omega\}$ is an enumeration of $\mathbb{Q}$. Fix $A : T\restriction_{\Lambda^c} \to \omega$ be a partition into chains. For any $s \in T\restriction_{\Lambda+1}$, let $\{s_n : n < \omega\}$ lists $T^s \cap \bigcup_{n<\omega} T_{ht(s)+n}$. $X_{nm} = \{t \in T\restriction_{\Lambda^c} : t = s_n \text{ for some } s \in T_{\Lambda+1} \text{ and } A(s) = m.\}$. It is easy to see that each $(X_{nm}, <_{lT})$ can be embedded into $(T\restriction_{\Lambda+1}, <_{lT})$. So $T\restriction_{\Lambda^c}$ is Countryman by Proposition 3.16. Hence (d) holds (similar as $(a) \to (b)$). This finishes the proof of Theorem 3.17.

Theorem 3.17 has very interesting applications. For example, every coherent tree – unless it is special – comes from (i.e., is a subset of) a coherent tree that is not Countryman:

Corollary 3.21. *If T is an $\mathbb{R}$-embeddable coherent tree and T is not special, then it has a super tree T' which is $\mathbb{R}$-embeddable and coherent and $(T'\restriction_{\Lambda+1}, <_{lT'})$ is not Countryman.*

Proof. Going to a super tree we can assume T is $\mathbb{Z}$-ranging (if no such super tree, then $T' = T$ works). Let T' be a completion of T for $\{-\omega, \omega\} \cup \mathbb{Z}$. Then we can get that $(T'\restriction_{\Lambda+1}, <_{lT'})$ is not Countryman. Otherwise, going to a super tree T'', by Proposition 2.4 and Theorem 3.17 (actually the proof of Theorem 3.17), $T''_{T'\restriction_{\Lambda+1}}$ is $\mathbb{Z}$-ranging and has maximal and minimal immediate successor on a club (i.e., for some club C, for any $s \in T''_{T'\restriction_{\Lambda+1}}$ with height in C, s has maximal and minimal immediate successor in $T''_{T'\restriction_{\Lambda+1}}$). So $T''_{T'\restriction_{\Lambda+1}}$ is finitely branching and hence special.

Hence T is special. A contradiction. $\square$

Chapter 4

From Countryman lines to $\mathbb{R}$-embeddable coherent trees

4.1 Special trees

Previously, we code the difference of two elements in a coherent tree and by using coherence we can guarantee that there are at most countably many codes. And this is how we get the Countryman property from the coherent property by assuming some additional assumption. It is then natural to use the coherence to countably code some other structure, such as special property:

Theorem 4.1. *If T is a coherent tree, X is a subset of T such that $ht(X)$ is a club and X is special, then T is special.*

Proof. Assume $ht(X) = \omega_1$ (otherwise, replace T by $T' = T \restriction_{ht(X)}$ which is coherent by Fact 2.2 and T' is special iff T is special by Fact 2.12). By going to a subset assume $|X \cap T_\alpha| = 1$ for any $\alpha < \omega_1$. And assume $T \subset \omega^{<\omega_1}$ since we can have T tree isomorphic to some coherent $T' \subset \omega^{<\omega_1}$ and T is special iff T' is special. Let $X = \bigcup_{1 \leq n < \omega} A_n$ witness that X is special and define $A : X \to \omega \setminus \{0\}$ by $A(x) = n$

iff $x \in A_n$.

Define $f : T \to V_\omega$[1] by induction on levels of T:

If $t \in T_0$, then $f(t) = \{0\}$.

If f is defined on T_β for any $\beta < \alpha$. For any $t \in T_\alpha$, let $\{x\} = X \cap T_\alpha$, $D_t = \{\gamma < \alpha : t(\gamma) \neq x(\gamma)\}$ and assume $\gamma_0, ..., \gamma_n$ is an increasing enumeration of D_t. $f(t) = \{A(x)\} \cup \{(i, x(\gamma_i), t(\gamma_i), f(t \restriction_{\gamma_i})) : i \leq n\}$.

Since V_ω is countable, it is suffice to prove that each preimage is an antichain. Suppose otherwise, there are $s <_T t$ such that $f(s) = f(t)$ and $(ht(s), ht(t))$ is least (with respect to the product order). Let $\{x\} = X \cap T_{ht(s)}$ and $\{y\} = X \cap T_{ht(t)}$. $f(s) = f(t)$ implies $A(x) = A(y)$ and hence x is incomparable with y. Let $\delta = \Delta(x, y)$. It is easy to see that $D_s \cap \delta = D_t \cap \delta$. And δ is in D_s or D_t since $x(\delta) \neq y(\delta)$ while $s(\delta) = t(\delta)$.

We will prove for the case that $\delta \in D_s$ and the other case is similar. Assume $\delta = D_s(i)$. Then $\delta \notin D_t$ since otherwise $\delta = D_t(i)$ and then $x(\delta) = y(\delta)$ by $f(s) = f(t)$ which contradicts the definition of δ. Then $D_s(i) < D_t(i)$ and hence $s \restriction_{D_s(i)} <_T t \restriction_{D_t(i)}$ and $f(s \restriction_{D_s(i)}) = f(t \restriction_{D_t(i)})$ by $f(s) = f(t)$. This contradicts the assumption that $(ht(s), ht(t))$ is the least one with such property. This finishes the proof of the theorem. □

If we can find a Countryman line without the above mentioned property, then it will contain no coherent suborder. However, it was found later that the Countryman line has a even stronger property:

Theorem 4.2. *For any Countryman line L and its partition tree T, there is a club C such that, for any uncountable subset $X \subset T$, if X is special, then $T \restriction_{ht(X) \cap C}$ is special. In particular, if X is special and $ht(X)$ is a club, then T is special.*

Let's fix a partition $c : L^2 \to \omega$ of L^2 into countably many chains.

[1] Here V is the universe and note V_ω is countable.

The following lemma will be needed:

Lemma 4.3. *There is a club C such that for any $x, y, z, w \in L$, if $c(x,y) = c(z,w)$, $x \perp z$ and $y \perp w$, then $\Delta_C(x,z) = \Delta_C(y,w)$.*

Proof. Let $\langle N_\alpha : \alpha < \omega_1 \rangle$ be a continuous elementary chain, N_0 contains all relevant objects and each N_α is countable and $C = \{N_\alpha \cap \omega_1 : \alpha < \omega_1\}$.
Suppose this C doesn't work. Assume $c(x,y) = c(z,w) = n$. WLOG, assume $x <_{lT} z$ and $\Delta_C(x,z) < \Delta_C(y,w)$. Then there is some $\beta = N_\alpha \cap \omega_1$ such that $\Delta(x,z) < \beta \le \Delta(y,w)$. Fix $\xi = \Delta(x,z) + 1 < \beta$. Define
$D = \{t >_T y \restriction_\xi$: there are $y', w' >_T t$, $x' >_T x \restriction_\xi$ and $z' >_T z \restriction_\xi$ such that $c(x',y') = c(z',w') = n\}$.
It is easy to see that $D \in N_\alpha$ and $y \restriction_\beta \in D$. So D is uncountable. Pick a $t \in D$ such that $t \perp y \restriction_\beta$. We can assume $t <_{lT} y \restriction_\beta$. According to definition of D, fix $w' >_T t$ and $z' >_T z \restriction_\xi$ such that $c(z',w') = n$. Then $c(x,y) = c(z',w')$. But $x <_{lT} z'$ and $y >_{lT} w'$. A contradiction. This finishes the proof of the lemma. □

Proof of Theorem 4.2: Let C be a club guaranteed by above lemma. Let $A : X \to \omega$ be a partition of X to countably many antichains. For any $t \in T \restriction_{ht(X)\cap C}$, fix $l_t \in L$ such that $t \le_T l_t$ (l_t exists because T is the partition tree of L) and fix $x_t \in X$ such that $ht(t) = ht(x_t)$.
Now we can define a partition of $T \restriction_{ht(X)\cap C}$ into countably many antichains:
$T \restriction_{ht(X)\cap C} = \cup\{B(m,n) : m < \omega \text{ and } n < \omega\}$ where $B(m,n) = \{t \in T \restriction_{ht(X)\cap C}$: $A(x_t) = m$ and $c(l_{x_t}, l_t) = n\}$.
It is suffice to prove that each $B(m,n)$ is an antichain.
Suppose towards a contradiction that there are $s, t \in B(m,n)$ such that $s <_T t$. Note, $x_s \perp x_t$ and hence $\Delta(l_{x_s}, l_{x_t}) < ht(s) \in C$. By previous lemma, $\Delta(l_s, l_t) < ht(s)$ since $c(l_{x_s}, l_s) = c(l_{x_t}, l_t)$. This contradicts the fact that $s <_T t$. This finishes the proof of the theorem.

Also above two theorems can be used to construct a type of Aronszajn trees whose lexicographical order contains no coherent or Countryman suborder.
Above theorem can also be applied to coherent trees:

Corollary 4.4. *If T is coherent and $\mathbb{R}$-embeddable (or at least there is an uncountable $\Gamma \subset \omega_1$ such that $T \restriction_\Gamma$ is special), then there is a club C such that for any uncountable subset $X \subset T$, if X is special, then $T \restriction_{ht(X) \cap C}$ is special.*

Proof. First prove the case when T is $\mathbb{R}$-embeddable. Then T is tree isomorphic to some ω-ranging $\mathbb{R}$-embeddable coherent tree T'. Then T' is Countryman by Theorem 2.11 or Corollary 3.15. Now the corollary follows from the Theorem 4.2. Then prove the case when $T \restriction_\Gamma$ is special for some uncountable $\Gamma \subset \omega_1$. Then $T \restriction_{\overline{\Gamma}}$ is $\mathbb{R}$-embeddable where $\overline{\Gamma}$ is the closure of Γ, i.e., $\overline{\Gamma} = \{\alpha < \omega_1 : \alpha \in \Gamma$ or $sup(\Gamma \cap \alpha) = \alpha\}$. Treat $T \restriction_{\overline{\Gamma}}$ as a new tree and repeat above argument we can get the conclusion. □

Remark: We will see that the $\mathbb{R}$-embeddability required in above corollary is necessary: start with a coherent Souslin tree[2] $T \subset \omega^{<\omega_1}$, let T^* be the completion of T for $\mathbb{Z}$. Then T^* contains an antichain of arbitrary nonstationary height, but T^* contains no uncountable special levels.

Above remark tells that a coherent tree which contains nonstationary antichain (i.e., an antichain whose height is nonstationary) may contain no uncountable special levels, and Theorem 4.1 tells that a coherent tree which contains a special subset with club height is special. There is still a gap that whether a coherent tree T which contains a stationary co-stationary antichain (or equivalently special subset) contains uncountable special levels (i.e., an uncountable subset $\Gamma \subset \omega_1$

[2]The existence of a coherent Souslin tree follows from $\diamondsuit$ (just a slightly generalization of the proof of existence of a Souslin tree in [16] or [15]) or see another construction in [5].

such that $T\restriction_\Gamma$ is special).

The answer is no. To prove this, let's first introduce the following:

Definition 20. (1) For a stationary set $S\subset\omega_1$, a $\diamondsuit_S$-sequence (or $\diamondsuit(S)$-sequence) is a sequence $\langle S_\alpha:\alpha\in S\rangle$ such that $S_\alpha\subset\alpha$ and for any $X\subset\omega_1$, $\{\alpha\in S: X\cap\alpha = S_\alpha\}$ is stationary.
(2) For a stationary set S, $\diamondsuit_S$ asserts that there is a $\diamondsuit_S$ sequence.

The following can be found in [18]:

Fact 4.5 ([18]). *If $V=L$, then $\diamondsuit_S$ holds for any stationary S.*

Now we can give the proof:

Proposition 4.6. *S is a stationary and co-stationary subset of ω_1. Assume $\diamondsuit_S$, then there is a coherent tree T and a subset $X\subset T$ such that, X is an antichain, $ht(X)=\Lambda\setminus S$ and T is not special when restrict to an uncountable set of levels, i.e., for any $A\in[\omega_1]^{\omega_1}$, $T\restriction_A$ is not special.*

Proof. Assume S consists of only limit ordinals. Fix a $\diamondsuit_S$ sequence $\langle A_\alpha:\alpha\in S\rangle$. Now construct a coherent T and an antichain $X\subset T$ with properties:
(1) $ht(X)=\Lambda\setminus S$ and $|X\cap T_\alpha|\leq 1$ for any $\alpha<\omega_1$;
(2) for any $\alpha<\beta<\omega_1$, there is a $s\in\omega^{\beta-\alpha}$ [3] such that for any $t\in T_\alpha\setminus X_\alpha$, $t^\frown s\in T_\beta\setminus X_\beta$, where $X_\xi = X\cap T\restriction_{\xi+1}$;
(3) for any $\alpha<\omega_1$, X_α is an antichain;
(4) for any $\alpha<\omega_1$, T_α is coherent, i.e., $\{\xi<\alpha: s(\xi)\neq t(\xi)\}$ is finite for any $s,t\in T_\alpha$.
Let $\varphi(\alpha,\beta,s)$ denote the statement:
for any $t\in T_\alpha\setminus X_\alpha$, $t^\frown s\in T_\beta\setminus X_\beta$.
Now let's start the construction:

[3] $\beta-\alpha$ is the ordinal ξ such that $\alpha+\xi=\beta$.

$T_0 = \{\phi\}, X_0 = \phi$.

Now assume for any $\beta < \alpha$, T_β, X_β has been defined and property (1),(2) are preserved.

Case 1: $\alpha = \beta + 1$ for some β.

$T_\alpha = \{t^\frown n : t \in T_\beta \setminus X_\beta \text{ and } n \in \omega\}$, $X_\alpha = X_\beta$. It is easy to see that property (1),(3),(4) are preserved. To see (2), fix $\eta < \gamma \leq \alpha$. Case $\gamma < \alpha$ follows from induction and we can assume $\gamma = \alpha$. By induction, we can find s such that $\varphi(\eta, \beta, s)$ holds. Then $\varphi(\eta, \gamma, s^\frown n)$ holds for any $n < \omega$.

Case 2: $\alpha \in \Lambda \setminus S$.

Fix $\langle \alpha_n : n < \omega \rangle$ increasing and cofinal in α. Fix $x_0 \in T_{\alpha_0} \setminus X_{\alpha_0}$.

For each $n < \omega$, pick $a_n \neq b_n$ such that $\varphi(\alpha_n, \alpha_{n+1}, a_n)$ and $\varphi(\alpha_n, \alpha_{n+1}, b_n)$ (note by construction of successor steps, there are infinitely many s such that $\varphi(\alpha_n, \alpha_{n+1}, s)$ holds). $T_\alpha = \{s^\frown b_n^\frown \langle a_i : n < i < \omega \rangle : s \in T_{\alpha_n} \setminus X_{\alpha_n} \text{ and } n < \omega\} \cup \{x_0^\frown \langle a_i : i < \omega \rangle\}$. $X_\alpha = \bigcup_{\beta < \alpha} X_\beta \cup \{x_0^\frown \langle a_i : i < \omega \rangle\}$.

It is easy to see that property (1) holds. For (2), note for any $\xi < \alpha$, $\xi < \alpha_n$ for some n, then there is some a such that $\varphi(\xi, \alpha_n, a)$, then $\varphi(\xi, \alpha, a^\frown b_n^\frown \langle a_i : n < i < \omega \rangle)$. For (3), if there is some $x \in X_\alpha$ such that $x <_T x_0^\frown \langle a_i : i < \omega \rangle$, then $x <_T x_0^\frown \langle a_i : i < n \rangle$ for some $n < \omega$. This contradict the choice of a_i. For (4), if $s, t \in T_\alpha$, then by definition of T_α, $\{\xi < \alpha : s(\xi) \neq t(\xi)\} = \{\xi < \alpha_n : s \restriction_{\alpha_n} (\xi) \neq t \restriction_{\alpha_n} (\xi)\}$ for some $n < \omega$. Hence (4) follows by induction hypothesis.

Case 3: $\alpha \in S$.

Fix $\langle \alpha_n : n < \omega \rangle$ increasing and cofinal in α. Fix $x_0 \in T_{\alpha_0}$.

Pick a_n by induction on n. If a_m is defined for any $m < n$, pick a_n to be any $a \in \omega^{\alpha_{n+1} - \alpha_n}$ such that:

(i) $\varphi(\alpha_n, \alpha_{n+1}, a)$;

(ii) if A_α code a $f : T \restriction_{\Gamma \cap \alpha} \to \omega$ partition into antichains where Γ is an unbounded subset of α (note we can assume at every stage $\xi > 0$ we have fixed a bijection between $T_\xi \to [\omega\xi, \omega(\xi+1))$ and f is a subset of $\omega\alpha \times \omega$), then $d(a) = min\{f(t)+1 : t \in T\restriction_{\Gamma\cap(\alpha_n,\alpha_{n+1}]}$ and $t \leq_T x_0^\frown\langle a_i : i < n\rangle^\frown a\}$ is the least we can achieve. (Here we assume $min(\phi) = 0$.)

Then $T_\alpha = \{s^\frown\langle a_i : n \leq i < \omega\rangle : s \in T_{\alpha_n} \setminus X_{\alpha_n}$ and $n < \omega\}$, $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.

It is easy to see that property (1),(3) holds. For (2), note for any $\xi < \alpha$, $\xi < \alpha_n$ for some n, then there is some a such that $\varphi(\xi, \alpha_n, a)$, then $\varphi(\xi, \alpha, a^\frown\langle a_i : n \leq i < \omega\rangle)$.

For (4), if $s, t \in T_\alpha$, then by definition of T_α, $\{\xi < \alpha : s(\xi) \neq t(\xi)\} = \{\xi < \alpha_n : s\restriction_{\alpha_n}(\xi) \neq t\restriction_{\alpha_n}(\xi)\}$ for some $n < \omega$. Hence (4) follows by induction hypothesis.

This finishes the construction and $T = \bigcup_{\alpha<\omega_1} T_\alpha$, $X = \bigcup_{\alpha<\omega_1} X_\alpha$.

It is easy to see that T is coherent and $X \subset T$ is an antichain and $ht(X) = \Lambda \setminus S$.

So we are left to show that for any $\Gamma \in [\omega_1]^{\omega_1}$, $T\restriction_\Gamma$ is not special.

Suppose otherwise, there is a $\Gamma \in [\omega_1]^{\omega_1}$ and $f : T\restriction_\Gamma \to \omega$ is a partition into antichains. Then fix a continuous elementary chain $\langle N_\alpha : \alpha < \omega\rangle$ such that N_0 contains all relevant objects and let $C = \{sup(N_\alpha \cap \omega_1) : \alpha < \omega_1\}$. Fix a $\alpha \in S \cap C'$ such that A_α codes $f\restriction_{T\restriction_{\Gamma\cap\alpha}}$, where C' is the set of accumulate points of C (note $sup(N_\alpha \cap \omega_1) = \alpha$).

Let $\langle \alpha_n : n < \omega\rangle$ and x_0 be the fixed elements in the construction and a_n be the chosen ones in the construction for each $n < \omega$. Let $x = x_0^\frown\langle a_n : n < \omega\rangle$. Let $\beta = min(\Gamma \setminus \alpha)$. Pick $y \in T^x \cap T_\beta$ such that $\varphi(\alpha, \beta, y\restriction_{[\alpha,\beta)})$. Let $f(y) = m$. There is a $n_0 < \omega$ such that for any $n > n_0$, $d(a_n)$ defined in the construction is either 0 or greater than $m + 1$. Since $\alpha \in C'$, we can pick $\delta < \alpha$ such that $sup(N_\delta \cap \omega_1) > \alpha_{n_0+1}$. Let $n_1 < \omega$ be such that $\alpha_{n_1} < sup(N_\delta \cap \omega_1) \leq \alpha_{n_1+1}$.

Then $H(\omega_2) \models \beta > \alpha_{n_1}$ and there is some $y \in T_\beta$ such that $f(y) = m$ and $\varphi(\alpha_{n_1}, \beta, y\restriction_{[\alpha_{n_1},\beta)})$.

By elementarity, $N_\delta \models$ there is a $\beta' > \alpha_{n_1}$ and there is some $y' \in T_{\beta'}$ such that $f(y') = m$ and $\varphi(\alpha_{n_1}, \beta', y'\restriction_{[\alpha_{n_1},\beta')})$.

Pick such β' and y' in N_δ and hence $\beta < \sup(N_\delta \cap \omega_1) \leq \alpha_{n_1+1}$. Pick a c such that

$\varphi(\beta', \alpha_{n_1+1}, c)$. Then $\varphi(\alpha_{n_1}, \alpha_{n_1+1}, (y' \upharpoonright_{[\alpha_{n_1}, \beta')})^\frown c)$ and $d((y' \upharpoonright_{[\alpha_{n_1}, \beta')})^\frown c) \leq m+1 < d(a_{n_1})$ (note $d(a_{n_1})$ cannot be 0 since $\Gamma \cap (\alpha_{n_1}, \alpha_{n_1+1}] \neq 0$). Contradict the choice of a_{n_1}. □

Here is a summary for speciality of a coherent tree under different conditions:

Corollary 4.7. *Suppose T is a coherent tree and $X \subset T$ is an uncountable subset such that X is special.*
(1) If $ht(X)$ contains a club, then T is special.
(2) If $ht(X)$ is stationary and $T \upharpoonright_\Gamma$ is special for some uncountable set $\Gamma \subset \omega_1$, then $T \upharpoonright_{ht(X)\cap C}$ is special for some club C.
(3) It is consistent that $ht(X)$ is stationary and T contains no special uncountable set of levels, i.e., $T \upharpoonright_\Gamma$ is not special for any uncountable $\Gamma \subset \omega_1$.
(4) For any uncountable set $\Gamma \subset \omega_1$, $T \upharpoonright_\Gamma$ is special iff for some downward closed subtree T', $T' \upharpoonright_\Gamma$ is special. In particular, if $T \upharpoonright_\Gamma$ is special, $T^* \upharpoonright_\Gamma$ is special where T^* is the completion of T.

Proof. (1) follows from Theorem 4.1, (2) follows from Corollary 4.1 and (3) follows from Proposition 4.6. Let's prove (4). Only the "if" part need a proof. Fix an uncountable $\Gamma \subset \omega_1$ and assume $T' \upharpoonright_\Gamma$ is special where T' is a downward closed subtree of T.
Fix a partition $A : T' \upharpoonright_\Gamma \rightarrow \omega$ such that each $A^{-1''}\{n\}$ is an antichain.
For each $\alpha < \omega_1$, fix an injection $\pi_\alpha : \alpha + 1 \rightarrow \omega$.
Define $D_{st} = \{\xi < ht(s), ht(t) : s(\xi) \neq t(\xi)\}$ for any $s, t \in T$. Note if T is coherent, then $T \subset O^{<\omega_1}$ for some countable linear order O. Hence $s(\xi)$ is defined.
Define $f : T \rightarrow \omega$ by: $f(t) = min\{|D_{st}| : s \in T'_{ht(t)}\}$.[4]
Define $g : T \rightarrow \omega_1$ by: $g(t) = min\{\xi \leq ht(t) : f(t \upharpoonright_\xi) = f(t)\}$.
Now we can define a partition of $T \upharpoonright_\Gamma$ into antichains. For any $t \in T \upharpoonright_\Gamma$,

[4]Note for $s \in T'$, $ht_T(s) = ht_{T'}(s)$. So we donot distinguish ht_T and $ht_{T'}$.

(1) if $t \in T'$, then let $F(t) = A(t)$;

(2) if $t \notin T'$, then $f(t) > 0$. Fix $t' \in T'_{ht(t)}$ such that $|D_{tt'}| = f(t)$. Enumerate $D_{tt'}$ in increasing order: $\xi_0 < \xi_1 < ... < \xi_n$. Note by definition of $g(t)$, $\xi_n \leq g(t)$. Then let $F(t) = A(t')^\frown\langle\langle t(\xi_i), t'(\xi_i), \pi_{g(t)}(\xi_i)\rangle : i \leq n\rangle$.

According to above definition of F, the range of F is countable. It is suffice to prove that each preimage is an antichain.

Suppose otherwise, there are $s, t \in T \upharpoonright_\Gamma$ such that $F(s) = F(t)$ and $s <_T t$. If $s \in T'$, then $t \in T'$ by length of $F(t)$. Then $A(s) = A(t)$. A contradiction.

Now assume $s \notin T'$. Note $f(s) = \frac{|F(s)|-1}{3} = \frac{|F(t)|-1}{3} = f(t)$. Then $s <_T t$ implies $g(s) = g(t)$. As $F(s) = F(t)$ and $\pi_{g(s)} = \pi_{g(t)}$, we can get that $D_{ss'} = D_{tt'}$ where s', t' are chosen to define $F(s), F(t)$ respectively. Hence $s'(\xi) = t'(\xi)$ for any $\xi \in D_{ss'} = D_{tt'}$. Note also for any $\xi \in ht(s) \setminus D_{ss'}$, $s'(\xi) = s(\xi) = t(\xi) = t'(\xi)$. This shows that $s' <_{T'} t'$. This contradicts that $A(s') = A(t')$ and $s', t' \in T' \upharpoonright_\Gamma$. This finishes the proof of (4) and hence the proof of the corollary. □

4.2 Basis for Countryman lines

Since there are some kinds of Countryman lines that are coherent, if there is some Countryman line which is not coherent (or contains no uncountable coherent subset), then it must be different from those that are both Countryman and coherent. So it should be helpful to know different Countryman types, especially if we try to find some Countryman line which contains no coherent subline. One interesting problem about different Countryman types is the basis problem. This section will define an equivalence relation on Countryman lines, investigate the size of the equivalence class and relate it to size of basis.

Definition 21. For two Countryman lines L_1, L_2, say they are *equivalent* (written as $L_1 \sim L_2$) if they have something in common, i.e., there is an uncountable line

L which can be embedded into both L_1 and L_2.[5]

We need the following to prove that " $\sim$ " is an equivalence relation.

Lemma 4.8. *If L_1, L_2 are both Countryman, then the following are equivalent:*
(1) $L_1 \sim L_2$.
(2) There is a countable partition of L_1 — $L_1 = \bigcup_{n<\omega} X_n$ — such that each $(X_n, <_{L_1})$ can be embedded into $(L_2, <_{L_2})$.
(3) $L_1 + L_2$ is Countryman where $L_1 + L_2$ is the lexicographical order of $(\{0\} \times L_1) \cup (\{1\} \times L_2)$.
(4) For any uncountable partial injection $f : L_1 \to L_2$, there is a $X \in [dom(f)]^{\omega_1}$ such that X is isomorphic to $f''X$ via f.

Proof. (1) $\to$ (2). Assume $c : L_1^2 \to \omega$ witnesses that L_1 is Countryman. Assume L can be embedded into both L_1 and L_2 and WLOG assume L is a subline of L_1. Fix a bijection $j : L_1 \to L$. Define for each $n < \omega$
$X_n = \{x \in L_1 : c(x, j(x)) = n\}$.
Then each X_n is isomorphic to $f''X_n \subset L$ and hence can be embedded into L_2.
(2) $\to$ (3) follows from Proposition 3.16.
(3) $\to$ (4). Assume $c : (L_1 + L_2)^2 \to \omega$ witnesses that $L_1 + L_2$ is Countryman. Let $X_n = \{x \in L_1 : c(x, f(x)) = n\}$. Then pick X to be some uncountable X_n.
(4) $\to$ (1). Trivial. □

Equivalence follows easily:

Lemma 4.9. *$\sim$ is an equivalence relation.*

Proof. Only transitivity needs a proof. Now assume L_i $(i < 3)$ are Countryman and $L_0 \sim L_1$, $L_1 \sim L_2$. Then by previous lemma, $L_0 + L_1$ and $L_1 + L_2$ are Countryman

[5]Or say they *near* each other as Shelah mentioned in [9].

and hence $L_0 + L_1 + L_1 + L_2$ are Countryman (since $L_0 + L_1 \sim L_1 + L_2$). Then $L_0 + L_2$ is Countryman. Hence $L_0 \sim L_2$ by previous lemma. $\square$

Fact 4.10. *The size of equivalence ($\sim$) class on Countryman is greater than or equal to the size of basis for Countryman.*

In most time, they have the same size. And we are actually going to investigate the size of the equivalence class.

The filter defined on coherent trees introduced in [1] will be used to investigate the equivalence relation:

Definition 22. For an Aronszajn tree T,
$\mathcal{U}(T) = \{A \subset \omega_1 : A \supset \Delta(X)$ for some uncountable $X \subset T\}$.
$\mathcal{I}(T)$ is its dual ideal, i.e., $\mathcal{I}(T) = \{X : X^c \in \mathcal{U}(T)\}$.

The following fact comes from [1]:

Fact 4.11. *[[1]] If T is Lipschitz (or coherent with no Suslin subtree), then $\mathcal{U}(T)$ is a filter.*

Following gives the connection between filter $\mathcal{U}(T)$ and non-equivalent Countryman lines.

Definition 23. Assume $T \subset \mathbb{Q}^{<\omega_1}$ is an Aronszajn tree and $X \subset \omega_1$. $T(X)$ is defined to be the Aronszajn tree $T(X) = \{t_X : t \in T\}$ where for $\alpha < ht(t)$

$$t_X(\alpha) = \begin{cases} t(\alpha) & : \alpha \notin X \\ -t(\alpha) & : \alpha \in X \end{cases}$$

i.e., preserve the tree order and lexicographical order on levels in X^c while change the lexicographical order on levels in X to its reverse. We will simply use $<_{T_X}$ (or $<_T$ since tree order is not changed) and $<_{lX}$ to denote the tree order and lexicographical order of $T(X)$ respectively.

Lemma 4.12. *Assume T is a Lipschitz tree such that $(T(X)\restriction_{\Lambda^c}, <_{lX})$ is Countryman for any $X \subset \omega_1$. Then for any $X, Y \subset \omega_1$, $(T(X)\restriction_{\Lambda^c}, <_{lX}) \sim (T(Y)\restriction_{\Lambda^c}, <_{lY})$ iff $X\Delta Y \in \mathcal{I}(T)$.*

Proof. Fix $X, Y \subset \omega_1$.

$(\Rightarrow)$. Fix bijection π between them mapping t_X to t_Y. By (4) of Lemma 4.8, there is some uncountable $A \subset T$ such that π is an isomorphism between $(\{t_X : t \in A\}, <_{lX})$ and $(\{t_Y : t \in A\}, <_{lY})$. It is easy to check that $\Delta(A) \cap (X\Delta Y) = \phi$. So $X\Delta Y \in \mathcal{I}(T)$.

$(\Leftarrow)$. Pick an uncountable $A \subset T$ such that $\Delta(A) \cap (X\Delta Y) = \phi$. Then $(\{t_X : t \in A\}, <_{lX})$ is isomorphic to $(\{t_Y : t \in A\}, <_{lY})$. Then $(T(X)\restriction_{\Lambda^c}, <_{lX}) \sim (T(Y)\restriction_{\Lambda^c}, <_{lY})$. □

Assumption in above lemma is easily satisfied for coherent trees that are Countryman (on nonstationary levels) by Theorem 3.17 (going to a super tree if necessary). Surprisingly, we will see the assumption is easily satisfied by just Countryman too.

Lemma 4.13. *If T is a partition tree of some Countryman line L, then there is some club C such that for any $X \subset \omega_1$, $(T\restriction_C (X)\restriction_{\Lambda^c}, <_{lX})$ is Countryman.*

Proof. Pick a club C_1 such that $T\restriction_C$ is $\mathbb{R}$-embeddable. By Proposition 3.16, $(T\restriction_{C_1}\restriction_{\Lambda^c}, <_{lT})$ is Countryman. Let $c : T\restriction_{C_1}\restriction_{\Lambda^c} \to \omega$ witness this. Pick a subclub C with property mentioned in Lemma 4.3. Now we are going to prove that this C works. Let $A : T\restriction_C\restriction_{\Lambda^c} \to \omega$ witness the $\mathbb{R}$-embeddability.

Fix any $X \subset \omega_1$. Define $c_X : T\restriction_C (X)\restriction_{\Lambda^c} \to \omega^3$ by for any $s_X, t_X \in T\restriction_C (X)\restriction_{\Lambda^c}$ $c_X(s_X, t_X) = (A(s), A(t), c(s,t))$.

Now assume $c_X(s_X, t_X) = c_X(s'_X, t'_X)$ and $s_X <_{lX} s'_X$. $t = t'$ is trivial so assume $t \neq t'$. Then $s \perp s'$ since $A(s) = A(t)$ and similarly $t \perp t'$. By Lemma 4.3, $\Delta_C(s, s') = \Delta_C(t, t')$. So either $\Delta_C(s, s') \in X$ hence $s >_{lT} s'$, $t \geq_{lT} t'$ and hence

$t_X \leq_{lX} t'_X$ or $\Delta_C(s,s') \notin X$ hence $s <_{lT} s'$, $t \leq_{lT} t'$ and hence $t_X \leq_{lX} t'_X$. This finishes the proof of the lemma. □

The following fact can be used to compute the size of equivalence class of Countryman lines (see also in [10]):

Fact 4.14. *If $\mathcal{I}$ is a σ-complete proper ideal on ω_1, then there are ω_1 many disjoint sets in $\mathcal{I}^+$ $(= \mathcal{P}(\omega_1) \setminus \mathcal{I})$. In particular, $|\mathcal{P}(\omega_1)/\mathcal{I}| = 2^{\omega_1}$.*

Proof. Use Ulam's matrix. □

Corollary 4.15. *Assume there is a nonspecial Countryman line[6]. Then there are 2^{ω_1} many pairwise non-equivalent Countryman lines. In particular, the basis for Countryman lines has size at least 2^{ω_1}.*

Proof. Fix a non-special Countryman line L and its partition tree T. By previous lemma, going to a club restriction subtree we can assume $(T(X) \restriction_{\Lambda^c}, <_{lX})$ is Countryman for any $X \subset \omega_1$.

Claim 1: For any $X \subset \omega_1$, if $X \in \mathcal{I}(T)$, then $T \restriction_X$ is special.

proof of claim 1: Fix $X \in \mathcal{I}(T)$ and uncountable $A \subset T$ such that $\Delta(A) \cap X = \phi$. Then T_A never split at levels in X and hence the partition of $T_A \restriction_{X+1}$ induces a partition of $T_A \restriction_X$. So $T_A \restriction_X$ is special and by Theorem 4.2 $T \restriction_X$ is special.

Now let $\mathcal{I} = \{X \subset \omega_1 : T \restriction_X \text{ is special.}\}$ be the ideal consists of sets of special levels. $\mathcal{I}$ is proper since T is nonspecial. By Lemma 4.12 and claim 1, we have that the size of equivalence classes $\geq |\mathcal{P}(\omega_1)/\mathcal{I}(T)| \geq |\mathcal{P}(\omega_1)/\mathcal{I}|$. Note $\mathcal{I}$ is σ-complete. Then the corollary follows from above fact. □

We just introduced one type of non-equivalence – $T(X) \nsim T(Y)$ for some specific T and X, Y such that $X \Delta Y \notin \mathcal{I}(T)$. Note their partition trees may be tree isomorphic. Now we are going to introduce a new type of non-equivalence.

[6]Nonspecial means some (equivalently every) partition tree is not special.

Definition 24. For two Lipschitz trees T, T'

(1) say $T \equiv T'$ if there is an uncountable partial tree isomorphism, i.e., an uncountable partial level preserving map $\pi : T \to T'$ such that for any $x, y \in dom(\pi)$, $\Delta(x, y) = \Delta(\pi(x), \pi(y))$ [7].

(2) say $T \equiv_C T'$ if for some club C, $T \restriction_C \equiv T' \restriction_C$.

(3) say $T < T'$ if there is an uncountable partial level preserving map $\pi : T \to T'$ such that for any $x, y \in dom(\pi)$, $\Delta(x, y) < \Delta(\pi(x), \pi(y))$.

(4) say $T <_C T'$ if for any club C, $T \restriction_C < T' \restriction_C$.

Here are some properties of $\equiv, \equiv_C, <$ and $<_C$:

Fact 4.16. *Assume T_i $(i < 2)$ are two Lipschitz trees.*

(1) $\equiv, \equiv_C$ are equivalence relations and $<, <_C$ are partial orderings.

(2) If $T_0 < T_1$ and $T_1 \equiv T_2$, then $T_0 < T_2$. Similarly for $>$. If $T_0 <_C T_1$ and $T_1 \equiv_C T_2$, then $T_0 <_C T_2$. Similarly for $>_C$.

(3) If T_i $(i < 2)$ are partition trees for Countryman lines L_i $(i < 2)$ such that $L_0 \sim L_2$, then $T_0 \equiv_C T_1$.

(4) If $T_0 <_C T_1$, then for any $A_i \in [T]^{\omega_1}$ $(i < 2)$, $(A_0, <_{lT_0}) \nsim (A_1, <_{lT_1})$ (if they are Countryman).

(5) If $T_0 <_C T_1$ and $\mathbb{P}$ is a c.c.c. poset, then $V^{\mathcal{P}} \models T_0 <_C T_1$.

(6) If $T_0 < T_1$ and $\pi : T_0 \to T_1$ is any uncountable level preserving partial mapping, then there is an uncountable $A \subset dom(\pi)$ such that $\pi \restriction_A$ witnesses $T_0 < T_1$.

Proof. (1) For "$\equiv$", only transitivity needs a proof. Fix $T_0 \equiv T_1$ via π and $T_1 \equiv T_2$ via σ. For any $s \in dom(\pi)$, fix a $s' \in (T_1)_{dom(\sigma)}$ such that $ht(s) = ht(s')$ and $s'' \in (T_2)_{rang(\sigma)}$ such that $s'' = \sigma(x) \restriction_{ht(s')}$ for some $x \in dom(\sigma) \cap T^{s'}$. Then $f : rang(\pi) \to T_1$ defined by $f(\pi(s)) = s'$ is a level preserving map on T_1. By Lipschitz, there is an uncountable $A \subset dom(\pi)$ such that f is tree isomorphic on

[7] π will induce a tree isomorphism between $T_{dom(\pi)}$ and $T'_{rang(\pi)}$.

$\pi''A$. Going to an uncountable subset we can assume that $A, \pi''A, A' = \{s' : s \in A\}$ and $A'' = \{s'' : s \in A\}$ are antichains. Then the map sends every $s \in A$ to s'' is an uncountable partial tree isomorphism. So $T_0 \equiv T_2$ and hence $\equiv$ is an equivalence relation.

For " $\equiv_C$ ", just note that if $C' \subset C$ are clubs, then $T \restriction_C \equiv T' \restriction_C$ implies $T \restriction_{C'} \equiv T' \restriction_{C'}$.

For " $<$ ", the proof of transitivity is the same as $\equiv$. The irreflexivity follows from Lipschitz.

For " $<_C$ ", just use the definition and the fact that $<$ is a partial ordering.

(2) The construction for (1) works for (2) too.

(3) Follows from Proposition 2.4.

(4) Follows from (2) and (3).

(5) Note that if $C' \subset C$ are clubs, then $T \restriction_{C'} < T' \restriction_{C'}$ implies $T \restriction_C < T' \restriction_C$. Then (5) follows from the fact that every club in a c.c.c. forcing extension contains a subclub in the ground model.

(6) Let $\sigma : T_0 \to T_1$ witnesses $T_0 < T_1$. For any $s \in dom(\pi)$, fix $s' \in (T_0)_{dom(\sigma)}$. Going to an uncountable subset $A' \subset dom(\pi)$ and by Lipschitz, we can assume for any $s_0, s_1 \in A'$, $\Delta(s_0, s_1) = \Delta(s'_0, s'_1)$. For any $t = \pi(s)$, let $t' = \sigma(s')$. Going to an uncountable subset $A \subset A'$ and by Lipschitz, we can assume for any $t_0, t_1 \in \pi''A$, $\Delta(t_0, t_1) = \Delta(t'_0, t'_1)$. Then for any $s_0, s_1 \in A$, $\Delta(s_0, s_1) = \Delta(s'_0, s'_1) < \Delta(\sigma(s'_0), \sigma(s'_1)) = \Delta(\pi(s_0), \pi(s_1))$. □

Note $\equiv, \equiv_C, <$ and $<_C$ are relations concern tree orders, they have nothing to do with the lexicographical order. So (4) of above fact actually says that some tree property may in some sense decide linear order property.

Definition 25. $gen(\mathcal{F}_{club}) = min\{|F| : F \text{ generates } \mathcal{F}_{club}\}$ where F generates $\mathcal{F}_{club}$ means that every club (on ω_1) contains a subclub in F.

Now we can get non-equivalent Countryman lines by iterated forcing a set of coherent trees linearly ordered by $<_C$.

Lemma 4.17. *Assume κ, λ are two cardinals (may be finite). If $\langle S_i : i < \kappa\rangle$ is $<_C$-increasing sequence of Lipschitz trees, $\langle T_j : j < \lambda\rangle$ is $<_C$-decreasing sequence of Lipschitz trees and $S_i <_C T_j$ for any $i < \kappa, j < \lambda$, then there is a c.c.c. poset $\mathcal{P}$ of size $max\{\kappa, \lambda, gen(\mathcal{F}_{club})\}$ which forces an $\mathbb{R}$-embeddable coherent tree T such that $S_i <_C T <_C T_j$ for any $i < \kappa, j < \lambda$.*

Proof. First fix a poset which forces an $\mathbb{R}$-embeddable coherent tree (see also 3.17): $\mathcal{P}_0 = \{p : p : [\omega_1]^2 \to \omega$ is a finite partial function such that

(1) $p(\cdot, \beta)$[8] is one to one, i.e., if $(\alpha, \beta) \neq (\alpha', \beta) \in dom(p)$, then $p(\alpha, \beta) \neq p(\alpha', \beta)$,[9]

(2) for any $\alpha \in dom_0(p) = \{\xi : \exists\delta\ (\xi, \delta) \in dom(p)\}$, for any $\beta \in dom_1(p) = \{\delta : \exists\xi\ (\xi, \delta) \in dom(p)\}$, if $\alpha < \beta$, then $(\alpha, \beta) \in dom(p)$.$\}$.

The order is that for $p, q \in \mathcal{P}_0$, $p \leq_0 q$ iff

(1) $q \subset p$ and

(2) for any $\beta, \gamma \in dom_1(q)$, for any $\alpha \in dom_0(p) \setminus dom_0(q)$, $p(\alpha, \beta) = p(\alpha, \gamma)$.

Then fix a set F of clubs which generates the club filter and has size $gen(\mathcal{F}_{club})$.

Now for any $i < \kappa.j < \lambda$, for any $\alpha < \omega_1$, fix $s^i_\alpha \in (S_i)_\alpha$ and $t^j_\alpha \in (T_j)_\alpha$. For any club $C \in F$, by (6) of previous fact, choose $X_{ijC} \in [\omega_1]^{\omega_1}$ such that for any $\alpha < \beta$ in X_{ijC}, $\Delta_{C'}(s^i_\alpha, s^i_\beta) < \Delta_{C'}(t^j_\alpha, t^j_\beta)$ where $C' = \{\alpha \in C : sup(C \cap \alpha) = \alpha\}$.

The required forcing will be as following:

$\mathcal{P} = \{p = (p^0, A_p, B_p, C_p, \langle X^p_{ijC} : i \in A_p, j \in B_p, C \in C_p\rangle) : p^0 \in \mathcal{P}_0, A_p \in [\kappa]^{<\omega}, B_p \in [\lambda]^{<\omega}, C_p \in [F]^{<\omega}$ and $X^p_{ijC} \in [X_{ijC}]^{<\omega}$ satisfying that

(1) $X^p_{ijC} \subset dom_1(p^0)$ and

(2) for any $\beta, \gamma \in X^p_{ijC}$, $\Delta_C(s^i_\beta, s^i_\gamma) < \Delta_C(p^0(\cdot, \beta), p^0(\cdot, \gamma)) < \Delta_C(t^j_\beta, t^j_\gamma)$. $\}$.

[8] Recall that for unordered pair $\{\alpha, \beta\}$, we may always write it ordered as (α, β) for $\alpha < \beta$.

[9] Here one to one is to make sure the final tree is $\mathbb{R}$-embeddable. Another way is to use Theorem 4.1 to make the final tree special.

Note condition (2) implies that there is some $\alpha < \beta, \gamma$ such that $p^0(\alpha, \beta) \neq p^0(\alpha, \gamma)$.

The order is that for $p, q \in \mathcal{P}$, $p \leq q$ if $p^0 \leq_0 q^0$ and $A_q \subset A_p, B_q \subset B_p, C_q \subset C_p$ and $X^q_{ijC} \subset X^p_{ijC}$ for any $i \in A_q, j \in B_q, C \in C_q$.

To prove that this $\mathcal{P}$ works. First, it is easy to see that $|\mathcal{P}| = max\{\kappa, \lambda, gen(\mathcal{F}_{club})\}$.

Second, let's prove that $\mathcal{P}$ is c.c.c.. Fix $\{p_\alpha : \alpha < \omega_1\} \subset \mathcal{P}$. Going to a stationary subset $\Gamma \subset \omega_1$ we can assume that

(1) $dom(p^0_\alpha), dom_0(p^0_\alpha), dom_1(p^0_\alpha)(\alpha \in \Gamma)$ form Δ-systems respectively, $p^0_\alpha(\alpha \in \Gamma)$ are constant when restrict to the root and for $\alpha < \beta$ in Γ, they don't overlap, i.e., $dom_0(p^0_\alpha), dom_1(p^0_\alpha) < \beta \leq dom_0(p^0_\beta)\setminus$ root of dom_0, $dom_1(p^0_\beta)\setminus$ root of dom_1,

(2) $A_{p_\alpha}, B_{p_\alpha}, C_{p_\alpha}(\alpha \in \Gamma)$ form Δ-systems respectively,

(3) for each i, j, C in the corresponding root, $X^{p_\alpha}_{ijC}(\alpha \in \Gamma)$ forms a Δ-system with the same size and they don't overlap.

Now letA_0, B_0, C_0 be the roots of $A_{p_\alpha}, B_{p_\alpha}, C_{p_\alpha}$ respectively. $D = \bigcap C_0$ and $D' = \{\alpha \in D : sup(D \cap \alpha) = \alpha\}$.

Now fix some i, j, C in the root, assume $X^{p_\alpha}_{ijC} \setminus \alpha = \{\alpha_0, ..., \alpha_{n-1}\}$. Going to a stationary subset $\Gamma'_1 \subset \Gamma$, we can find a $\tau < \omega_1$ such that

(4) $dom_0(p^0_\alpha) \cap [\tau, \alpha) = \phi, dom_1(p^0_\alpha) \cap [\tau, \alpha) = \phi$,

(5) $X^{p_\alpha}_{ijC} \cap \alpha < \tau$ and are constant (for $\alpha \in \Gamma'_1$),

(6) the splitting points of $\{s^i_{\alpha_k} \restriction_\alpha : k < n\}$ are below τ and $\{s^i_{\alpha_k} \restriction_\tau : k < n\}$ are constant (for $\alpha \in \Gamma'_1$) (so we can assume $\{s^i_{\alpha_k} \restriction_\alpha : k < n\} = \{s_\alpha(k) : k < n'\}$ for some $n' \leq n$),

(7) the splitting points of $\{t^j_{\alpha_k} \restriction_\alpha : k < n\}$ are below τ and $\{t^j_{\alpha_k} \restriction_\tau : k < n\}$ are constant (for $\alpha \in \Gamma'_1$) (so we can assume $\{t^j_{\alpha_k} \restriction_\alpha : k < n\} = \{t_\alpha(k) : k < n''\}$ for some $n'' \leq n$).

Repeat above procedure for each i, j, C in the root (which is finite) and we get a stationary Γ_1.

Then going to an uncountable subset $\Gamma_2 \subset \Gamma_1$ and by Lipschitz, we can assume that for each i, j, C in the root, $\{s_\alpha(k) : \alpha \in \Gamma_2\}$ $\{t_\alpha(k') : \alpha \in \Gamma_2\}$ are antichains for each $k < n'$, $k' < n''$ and

(8) for any $k_0 < k_1 < n'$, for any $\alpha < \beta$ in Γ_2, $\Delta(s_\alpha(k_0), s_\beta(k_0)) = \Delta(s_\alpha(k_1), s_\beta(k_1))$,

(9) for any $k_0 < k_1 < n''$, for any $\alpha < \beta$ in Γ_2, $\Delta(t_\alpha(k_0), t_\beta(k_0)) = \Delta(t_\alpha(k_1), t_\beta(k_1))$,

(10) for each $k < n$, the mapping sends $s^i_{\alpha_k}$ to $t^j_{\alpha_k}$ (with domain $\{s^i_{\alpha_k} : \alpha \in \Gamma_2\}$) witnesses $S_i \restriction_{D'} < T_j \restriction_{D'}$.

Now pick $\alpha < \beta$ in Γ_2 and we want to find some $p < p_\alpha, p_\beta$. First let $A_p = A_{p_\alpha} \cup A_{p_\beta}$, $B_p = B_{p_\alpha} \cup B_{p_\beta}$ and $C_p = C_{p_\alpha} \cup C_{p_\beta}$. Then extend p^0_α, p^0_β to q^0 such that $dom_i(q^0)$ ($i < 2$) is their union and $q^0(\xi, \delta) = n_\xi$ where (ξ, δ) is new added and n_ξ's are different (for different ξ's) natural numbers not appeared in $rang(p^0_\alpha), rang(p^0_\beta)$. For i, j, C,

(a) if $\langle i, j, C\rangle \in A_{p_\alpha} \times B_{p_\alpha} \times C_{p_\alpha} \setminus A_{p_\beta} \times B_{p_\beta} \times C_{p_\beta}$, then $X^p_{ijC} = X^{p_\alpha}_{ijC}$,

(b) if $\langle i, j, C\rangle \in A_{p_\beta} \times B_{p_\beta} \times C_{p_\beta} \setminus A_{p_\alpha} \times B_{p_\alpha} \times C_{p_\alpha}$, then $X^p_{ijC} = X^{p_\beta}_{ijC}$,

(c) if $\langle i, j, C\rangle \notin A_{p_\alpha} \times B_{p_\alpha} \times C_{p_\alpha} \cup A_{p_\beta} \times B_{p_\beta} \times C_{p_\beta}$, then $X^p_{ijC} = \phi$,

(d) if $\langle i, j, C\rangle \in A_{p_\alpha} \times B_{p_\alpha} \times C_{p_\alpha} \cap A_{p_\beta} \times B_{p_\beta} \times C_{p_\beta}$ (i.e., in the root), then $X^p_{ijC} = X^{p_\alpha}_{ijC} \cup X^{p_\beta}_{ijC}$ and extend q^0 to p^0: let $\delta = \Delta_{C'}(t_\alpha(0), t_\beta(0)) > \Delta_{C'}(s_\alpha(0), s_\beta(0)) = \delta'$ and choose $\xi_C \in (\Delta(s_\alpha(0), s_\beta(0)), C(\Delta_C(t_\alpha(0), t_\beta(0)))) \cap C$; define $p^0(\xi_C, \eta) = m_0$ for $\eta \in dom_1(p^0_\alpha) \setminus dom_1(p^0_\beta)$ and $p^0(\xi_C, \eta) = m_1$ for $\eta \in dom_1(p^0_\beta) \setminus dom_1(p^0_\alpha)$ where m_0, m_1 are two different natural numbers that never appeared before.

We first need that $p \in \mathcal{P}$. We have from the construction that $p^0 \in \mathcal{P}_0$, $A_p \in [\kappa]^{<\omega}, B_p \in [\lambda]^{<\omega}, C_p \in [F]^{<\omega}$, $X^p_{ijC} \in [\omega_1]^{<\omega}$ and $X^p_{ijC} \subset dom_1(p^0)$. Now we want to prove (2) of definition $\mathcal{P}$ is satisfied. Fix i, j, C and $\theta, \gamma \in X^p_{ijC}$. If i, j, C falls in case (a)-(c), then it is trivial. Now assume i, j, C are as in case (d). If β, γ are both in $dom_1(p^0_\alpha)$ (or $dom_1(p^0_\beta)$), then it is trivial.

To make notation simple, assume $\theta = \alpha_k$, $\gamma = \beta_l$ and $\gamma' = \alpha_l$ for some $k, l < n$. Now we will discuss by cases.

Case 1: $\Delta(t^j_\theta, t^j_{\gamma'}) < \tau$.

$\Delta_C(s^i_\theta, s^i_\gamma) = \Delta_C(s^i_\theta, s^i_{\gamma'}) < \Delta_C(p(\cdot,\theta), p(\cdot,\gamma')) = \Delta_C(p(\cdot,\theta), p(\cdot,\gamma)) < \Delta_C(t^j_\theta, t^j_{\gamma'}) = \Delta_C(t^j_\theta, t^j_\gamma)$.

Case 2: $\Delta(p(\cdot,\theta), p(\cdot,\gamma')) < \tau \leq \Delta(t^j_\theta, t^j_{\gamma'})$.

Then $\Delta(s^i_\theta, s^i_\gamma) = \Delta(s^i_\theta, s^i_{\gamma'})$, $\Delta(p(\cdot,\theta), p(\cdot,\gamma')) = \Delta(p(\cdot,\theta), p(\cdot,\gamma)) \leq \xi_C$ and $\Delta(t^j_\theta, t^j_\gamma) = \Delta(t^j_{\gamma'}, t^j_\gamma) > \xi_C$. Hence (2) of definition $\mathcal{P}$ follows.

Case 3: $\Delta(s^i_\theta, s^i_{\gamma'}) < \tau \leq \Delta(p(\cdot,\theta), p(\cdot,\gamma'))$.

Then $\Delta(s^i_\theta, s^i_\gamma) = \Delta(s^i_\theta, s^i_{\gamma'}) < \tau$, $\Delta(p(\cdot,\theta), p(\cdot,\gamma')) = \Delta(p(\cdot,\theta), p(\cdot,\gamma)) = \xi_C$ and $\Delta(t^j_\theta, t^j_\gamma) = C'(\delta)$. Hence (2) of definition $\mathcal{P}$ follows.

Case 4: $\Delta(s^i_\theta, s^i_{\gamma'}) \geq \tau$.

Then $\Delta(s^i_\theta, s^i_\gamma) = C'(\delta')$, $\Delta(p(\cdot,\theta), p(\cdot,\gamma)) = \xi_C$ and $\Delta(t^j_\theta, t^j_\gamma) = C'(\delta)$. Hence (2) of definition $\mathcal{P}$ follows.

Hence p is a condition in $\mathcal{P}$. And it is easy to see from the construction that $p < p_\alpha, p_\beta$. So $\mathcal{P}$ is c.c.c..

Finally we need to prove that this $\mathcal{P}$ forces a coherent tree we want. Assume G is a generic filter. First note for any $\alpha < \beta < \omega_1$, $D_{\alpha\beta} = \{p : (\alpha,\beta) \in dom(p^0)\}$ is dense and hence $a = \cup\{p^0 : p \in G\}$ is a one to one coherent map (see also Theorem 2.15) which induces the desired $\mathbb{R}$-embeddable coherent tree $T = T(a)$. For each $i < \kappa, j < \lambda$, to prove $S_i <_C T <_C T_j$, we just need for each $C \in F$, $X'_{ijC} = \cup\{X^p_{ijC} : p \in G\}$ is uncountable. It is suffice to prove that for any $\alpha < \omega_1$, $E_{ijC\alpha} = \{p : i \in A_p, j \in B_p, C \in C_p$ and $X^p_{ijC} \not\subset \alpha\}$ is dense. Given $p \in \mathcal{P}$ and extend p if necessary we can assume $i \in A_p, j \in B_p, C \in C_p$. Pick some β large enough in X_{ijC}. Extend p to q such that

(1) $X^q_{ijC} = X^p_{ijC} \cup \{\beta\}$ and the rest are the same as p,

(2) for each $\xi \in X^p_{ijC}$, fix $\delta_\xi \in C$ such that $\Delta(s^i_\xi, s^i_\beta) < \delta_\xi < C(\Delta_C(t^j_\xi, t^j_\beta))$. Define $dom_0(q^0) = dom_0(p^0) \cup \{\delta_\xi : \xi \in X^p_{ijC}\}$ and $dom_1(q^0) = dom_1(p^0) \cup \{\beta\}$. Define $q^0(\delta_\xi, \beta) = n_\xi, q^0(\delta_\xi, \xi') = n'_\xi$ for $\xi \in X^p_{ijC}, \xi' \in dom_1(p^0)$ where n_ξ, n'_ξ are different (and different for different ξ's) natural numbers that never appeared

before. $q^0(\gamma, \beta)$ is arbitrarily defined to satisfy the definition of $\mathcal{P}$ for $\gamma \in dom_0(p^0)$. It is easy to check that $q < p$ and $q \in E_{ijC\alpha}$. So $E_{ijC\alpha}$ is dense. This finishes the proof of the lemma. □

Using the fact that $<_C$ is preserved by c.c.c. forcing, we can add many different Countryman lines by iterating (with finite support) above forcing poset. Note $MA_{gen(\mathcal{F}_{club})}$ can be achieved by finitely support iterating c.c.c. forcing with sufficient length since c.c.c. forcing won't change $gen(\mathcal{F}_{club})$.

Corollary 4.18. *(1) Assume $MA_{gen(\mathcal{F}_{club})}$. Every basis for Countryman lines has size 2^{ω_1}.*[10]

(2) Assume $MA_{gen(\mathcal{F}_{club})}$. There are 2^{ω_1} many Aronszajn trees such that any two of them contain no uncountable isomorphic subtree when restrict to a club.

Remark: It has been proved in [11] that it is consistent with MA_κ that any two Aronszajn trees contain isomorphic subtree (restrict to a club). Above corollary provides a new view of this problem.

Now we are going to investigate the consistency of "least size of basis for Countryman lines is not 2 or 2^{ω_1}". First let's fix some notation.

Definition 26. (1)$\mathcal{K} = \{\mathcal{P} : \mathcal{P}$ is a poset of size ω_1 and $\mathcal{P}$ has property (K) which is guaranteed by a set in $H(\omega_2)\}$[11].

(2) $MA_{\mathcal{K}}(\omega_1)$ is the statement that for any poset $\mathcal{P} \in \mathcal{K}$, for any $\mathcal{D}$ connection of ω_1 many dense subsets of $\mathcal{P}$, there is a filter G which meets each dense set $D \in \mathcal{D}$.

Lemma 4.19. *Assume $MA_{\mathcal{K}}(\omega_1)$ and for any two $\mathbb{R}$-embeddable coherent tree tree S, T, $S \equiv_C T$. Then $\{(T(X), <_X) : X \subset \omega_1\}$ is a basis for Countryman lines where T is arbitrary $\mathbb{R}$-embeddable coherent tree. In particular, the least size of basis for Countryman lines is $|\mathcal{P}(\omega_1)/\mathcal{I}(T)|$.*

[10] This is related to a claim of [1].

[11] X guarantees that $\mathcal{P}$ has property (K) if any model that contains $\mathcal{P}$ and X satisfies that $\mathcal{P}$ has property (K).

Proof. According to the proof of Theorem 2.15, for any Countryman line L and its Lipschitz partition tree S, the poset which forces S to be coherent is in $\mathcal{K}$ (note the property (K) is guaranteed by the Countryman property – the partition of L^2 into countably many chains – which has size ω_1). So every Countryman line is coherent.

Note every $\mathbb{R}$-embeddable tree is special since the poset which forces such tree to be special is in $\mathcal{K}$.

Now given any Countryman line L and its $\mathbb{R}$-embeddable coherent partition tree S. Pick a club C such that $S\restriction_C \equiv T\restriction_C$. By $\equiv$, let A, B be uncountable subset of $S\restriction_C, T\restriction_C$ respectively such that $(S\restriction_C)_A$ is tree isomorphic to $(T\restriction_C)_B$ via π. Let $S' = (S\restriction_C)_A, T' = (T\restriction_C)_B$. It is suffice to prove that $(S', <_{lS'}) \sim (T'(X), <_{lX})$ for some $X \subset \omega_1$.

To find the X, let's first prove this:

Claim 1: S' contains an uncountable binary subtree.

proof of claim 1: Let $\mathcal{P} = \{p \in [S']^{<\omega} : S'_p \text{ is binary.}\}$ and the order is $p \leq q$ iff $p \supset q$. Let's check the property (K). Fix $\{p_\alpha : \alpha < \omega_1\} \subset \mathcal{P}$. Going to an uncountable subset we can assume p_α's form a Δ-system with root r and all have same size n. By coherence, we can find a stationary $\Gamma \subset \omega_1$ and $\alpha_0 < \omega_1$ such that

(1) for any $\alpha \in \Gamma$, for any $s, t \in p_\alpha \setminus r$, $s\restriction_{[\alpha_0,\alpha)} = t\restriction_{[\alpha_0,\alpha)}$;

(2) for any $\alpha, \beta \in \Gamma$, $S'_{p_\alpha}\restriction_{\alpha_0} = S'_{p_\beta}\restriction_{\alpha_0}$.

Going to an uncountable subset $\Gamma_1 \subset \Gamma$ such that $\cup\{(p_\alpha \setminus r)\restriction_\alpha : \alpha \in \Gamma_1\}$ is an antichain. So any two elements in $\{p_\alpha : \alpha \in \Gamma_1\}$ are compatible. Then $\mathcal{P}$ has property (K) and note this is witnessed by coherence and $\mathbb{R}$-embeddability which can be coded in $H(\omega_2)$. So $\mathcal{P} \in \mathcal{K}$. Note $\mathcal{P}$ is c.c.c., so there is a $p \in \mathcal{P}$ which forces the generic filter to be uncountable. Then $\mathcal{P}(\leq p) = \{q \in \mathcal{P} : q \leq p\}$ will force an uncountable binary subtree (note $\mathcal{P}(\leq p) \in \mathcal{K}$). A filter which meets each $D_\alpha = \{p : p \not\subset T\restriction_\alpha\}$ will give an uncountable subset whose downward closure is binary. This finishes the proof of claim 1.

Now let S'' be an uncountable binary subtree of S' and $T'' = \pi''S''$. Note for any $s \in S''$, π either preserves the lexicographical order of immediate successor of s or maps the lexicographical order of immediate successor of s reversely. This is the reason we choose a binary subtree. Let $A = \{s \in S'' : s$ has two immediate successor $s_0 <_{lS} s_1$ in S'' such that $\pi(s_0) >_{lT} \pi(s_1).\}$. Now we need to find a subtree such that splitting points on the same level behaves the same way, i.e., π preserves (or reversely maps) the lexicographical order of immediate successor on the entire level.

Define a poset $\mathcal{Q} = \{p \in [S'']^{<\omega}$: for any $s_i \in p$ $(i < 4)$, if $\Delta(s_0, s_1) = \Delta(s_2, s_3)$, then $s_0 \wedge s_1 \in A$ iff $s_2 \wedge s_3 \in A.\}$. Recall that $s \wedge t = s \restriction_{\Delta(s,t)}$.

Claim 2: $\mathcal{Q} \in \mathcal{K}$.

proof of claim 2: First it is easy to see that $|\mathcal{Q}| = \omega_1$. Now fix $\{p_\alpha : \alpha < \omega_1\} \subset \mathcal{Q}$. So $(S'', <_{lS})$ is Countryman. Fix $c_0 : S''^2 \to \omega$ a partition into chains and similarly $c_1 : T''^2 \to \omega$ a partition into chains. Going to an uncountable subset we can assume p_α's form a Δ-system with root r and all have same size n. By coherence, we can find a stationary $\Gamma \subset \omega_1$ and $\alpha_0 < \omega_1$ such that

(1) for any $\alpha \in \Gamma$, for any $s, t \in p_\alpha \backslash r$, $s \restriction_{[\alpha_0,\alpha)} = t \restriction_{[\alpha_0,\alpha)}$ and $\pi(s) \restriction_{[\alpha_0,\alpha)} = \pi(t) \restriction_{[\alpha_0,\alpha)}$;

(2) for any $\alpha, \beta \in \Gamma$, $S''_{p_\alpha} \restriction_{\alpha_0} = S''_{p_\beta} \restriction_{\alpha_0}$ and $T''_{\pi(p_\alpha)} \restriction_{\alpha_0} = T''_{\pi(p_\beta)} \restriction_{\alpha_0}$,

(3) for any $\alpha, \beta \in \Gamma$, $c_0 \restriction_{p_\alpha^2} = c_0 \restriction_{p_\beta^2}$ and $c_1 \restriction_{\pi(p_\alpha)^2} = c_1 \restriction_{\pi(p_\beta)^2}$[12],

(4) they don't overlap, i.e., for any $\alpha < \beta$ in Γ, for any $s \in p_\alpha$, $t \in p_\beta \setminus r$, $ht(s) < \beta \leq ht(t)$.

For any $\alpha < \beta$ in Γ, we want to show $p_\alpha \cup p_\beta \in \mathcal{Q}$ and hence p_α is compatible with p_β. Pick $s_i \in p_\alpha \cup p_\beta$ $(i < 4)$ such that $\Delta(s_0, s_1) = \Delta(s_2, s_3)$. The only nontrivial case is $\Delta(s_0, s_1) = \Delta(p_\alpha(0), p_\beta(0)) = \xi \in [\alpha_0, \alpha)$ where $p_\alpha(0)$ is the $<_{lS}$-least element of $p_\alpha \setminus r$. WLOG, assume $s_0, s_2 \in p_\alpha$ and $s_1, s_3 \in p_\beta$. Let $s_0', s_2' \in p_\beta$

[12]Here "=" actually means isomorphic.

be the corresponding elements of s_0, s_2 in p_α, i.e., $s'_0 = p_\beta(n)$ iff $s_0 = p_\alpha(n)$. So $s_1 \upharpoonright_\beta = s'_0 \upharpoonright_\beta$ and $s_3 \upharpoonright_\beta = s'_2 \upharpoonright_\beta$. Then $s_0 \wedge s_1 = s_0 \wedge s'_0$, $s_2 \wedge s_3 = s_2 \wedge s'_2$. By (3), $s_0 <_{lS} s'_0$ iff $s_2 <_{lS} s'_2$ and $\pi(s_0) <_{lT} \pi(s'_0)$ iff $\pi(s_2) <_{lT} \pi(s'_2)$. WLOG, assume $s_0 <_{lS} s'_0$. Then either $\pi(s_0) <_{lT} \pi(s'_0)$ or $\pi(s_0) >_{lT} \pi(s'_0)$. Hence either $s_0 \wedge s'_0 \notin A$ and $s_2 \wedge s'_2 \notin A$ or $s_0 \wedge s'_0 \in A$ and $s_2 \wedge s'_2 \in A$. So $p_\alpha \cup p_\beta \in \mathcal{Q}$. This proves that $\mathcal{Q}$ has property (K). Note property (K) is guaranteed by coherence and Countryman which can be expressed by a set of size ω_1. This finishes the proof of claim 2.

Like in claim 1, we can find an uncountable subset $B \subset S''$ such that for any $s_i \in B$ $(i < 4)$, if $\Delta(s_0, s_1) = \Delta(s_2, s_3)$, then $s_0 \wedge s_1 \in A$ iff $s_2 \wedge s_3 \in A$. Let $A' = \{s \in S''_B : s$ has two immediate successor $s_0 <_{lS} s_1$ in S''_B such that $\pi(s_0) >_{lT} \pi(s_1).\}$. Let $X = \{\alpha < \omega_1 : \exists s \in A' \ ht_{S''}(s) = \alpha\}$. So S'' is lexicographically isomorphic to $T''_{\pi'' B}(X)$ via π. Hence $S \sim S''_B \sim T''_{\pi'' B}(X) \sim T(X')$ where $X' = \cup\{[C(\alpha), C(\alpha+1)) : \alpha \in X\}$. So $\{T(Y) : Y \subset \omega_1\}$ is a basis for Countryman line. The in particular part follows from Lemma 4.12. □

Now we are going to force a model which satisfies assumption of previous lemma and $|\mathcal{P}(\omega_1)/\mathcal{I}(T)| = 4$.

First, let's start from a model which satisfies GCH and contains a special complete coherent tree T such that there is $X_0 \subset \omega_1$, $X_0 \notin \mathcal{I}(T)$ and $X_1 = X^c \notin \mathcal{I}(T)$ where $X^c = \omega_1 \setminus X$ (for example, we can choose L). We then fix a well ordering $<$ on $H(\omega_2)$. Then fix a surjection $f : \omega_2 \to \omega_2 \times \omega_2 \times 3$ such that each preimage has size ω_2.

Then let's introduce some notation:

Definition 27. (1) For a special complete coherent tree S, $\mathcal{P}_S = \{p = (c_p, f_p) : c_p \in [\omega_1]^{<\omega}$, f_p is a finite level preserving partial mapping from S to T such that for any $s, t \in dom(f_p)$, for any $\alpha \in c_p$, $\Delta(s,t) < \alpha$ iff $\Delta(f_p(s), f_p(t)) < \alpha\}$.
$p \leq q$ iff $c_p \supset c_q$ and $f_p \supset f_q$.

(2) For an uncountable $X \subset \omega_1$, $\mathcal{P}_X = \{p \in [T]^{<\omega} : \Delta(p) = \{\Delta(s,t) : s \perp t$ in $p\} \subset X\}$.
$p \leq q$ iff $p \supset q$.

Our goal is to get the assumption in Lemma 4.19 and
$\mathcal{P}(\omega_1)/\mathcal{I}(T) = \{\phi/\mathcal{I}(T), X_0/\mathcal{I}(T), X_1/\mathcal{I}(T), \omega_1/\mathcal{I}(T)\}$.
The assumption in lemma 4.19 can be achieved by iterating $\mathcal{P}_S$ for every possible complete special coherent tree S and every possible poset $\mathcal{P}$ with property (K). And by iterating $\mathcal{P}_X$ for $X \cap X_i \notin \mathcal{I}(T)$ $(i < 2)$ we can get $\mathcal{P}(\omega_1)/\mathcal{I}(T) \subset \{\phi/\mathcal{I}(T), X_0/\mathcal{I}(T), X_1/\mathcal{I}(T), \omega_1/\mathcal{I}(T)\}$. So the only problem is to preserve $X_i \notin \mathcal{I}(T)$ $(i < 2)$. Let's first see the two step iteration case.

Lemma 4.20. *For any special complete coherent tree S, $\mathcal{P}_S$ is proper. Moreover, for large enough regular θ, for countable $\mathcal{M} \prec H(\theta)$, $p \in \mathcal{P}_S$ is $(M, \mathcal{P}_S)$-generic iff $\mathcal{M} \cap \omega_1 \in c_p$.*

Proof. Fix large regular θ and countable $\mathcal{M} \prec H(\theta)$. Assume $\delta = \mathcal{M} \cap \omega_1$. Fix $A : S \to \omega$ witnesses that S is special and $A' : T \to \omega$ witnesses that T is special.
We just need to prove the moreover part since for any $p \in \mathcal{P}_S$, $(c_p \cap \{\delta\}, f_p) \in \mathcal{P}_S$.
For the moreover part, fix $p \in \mathcal{P}_S$. First, assume $\delta \in c_p$. Fix $D \in \mathcal{M}$ dense in $\mathcal{P}_S$. It is suffice to assume $p \in D$ and find some $q \in D \cap \mathcal{M}$ compatible with p. Let $\alpha < \delta$ be such that
(1) for any $s \in dom(f_p)$, $ht(s) < \alpha$ or $ht(s) \geq \delta$;
(2) $c_p \cap \delta < \alpha$;
(3) for any $s, t \in dom(f_p)$, $s \restriction_{[\alpha,\delta)} = t \restriction_{[\alpha,\delta)}$ and $f_p(s) \restriction_{[\alpha,\delta)} = f_p(t) \restriction_{[\alpha,\delta)}$.
Assume $dom(f_p) \setminus S \restriction_\delta = \{s_i : i < n\}$. By elementarity, find a $q \in \mathcal{M} \cap D$ such that there is $\delta' > \alpha$ and
(i) p and q agree below α, i.e., $c_p \cap \alpha = c_q \cap \alpha$, $S_{dom(f_p)} \restriction_\alpha = S_{dom(f_q)} \restriction_\alpha$ and $T_{rang(f_p)} \restriction_\alpha = T_{rang(f_p)} \restriction_\alpha$;

(ii) (1)(2)(3) mentioned above holds for q when replace p, δ by q, δ';

(iii) $dom(f_q)$ has the same tree structure as $dom(f_p)$ and $dom(f_q) \setminus S \restriction_\delta = \{s'_i : i < n\}$, $A(s_i \restriction_\delta) = A(s'_i \restriction'_\delta)$ for any $i < n$ and $A'(f_p(s_i) \restriction_\delta) = A'(f_q(s'_i) \restriction'_\delta)$ for any $i < n$.

Note for any $i, j < n$, either $\Delta(s_i, s'_j) < \alpha$ when $\Delta(s_i, s'_j) = \Delta(s_i, s_j)$ and $\Delta(f_p(s_i), f_q(s'_j)) = \Delta(f_p(s_i), f_p(s_j))$ or $\Delta(s_i, s'_j) \in [\alpha, \delta')$. Hence, it is easy to check that $(c_p \cup c_q, f_p \cup f_q) \in \mathcal{P}_S$.

Second, assume p is $(\mathcal{M}, \mathcal{P}_S)$-generic. Suppose towards a contradiction that $\delta \notin c_p$. We can fix a extension $p' < p$ such that there are $s, t \in dom(f_{p'})$ such that $\Delta(s, t) < \delta \leq \Delta(f_{p'}(s), f_{p'}(t))$. Then $D = \{q \in \mathcal{P}_S : c_q \not\subset \Delta(s, t)\} \in \mathcal{M}$ is dense. It is easy to check that for any $q \in D \cap \mathcal{M}$, p' is incompatible with q. So p is not $(\mathcal{M}, \mathcal{P}_S)$-generic. A contradiction. □

Lemma 4.21. *For any uncountable $X \subset \omega_1$, $\mathcal{P}_X$ either is c.c.c. or collapses ω_1. Moreover,*

(1) $\mathcal{P}_X$ is c.c.c. iff $X \notin \mathcal{I}(T)$;

(2) $\mathcal{P}_X$ collapses ω_1 iff $X \in \mathcal{I}(T)$.

Proof. It is suffice to prove that $X \notin \mathcal{I}(T) \to \mathcal{P}_X$ is c.c.c. and $X \in \mathcal{I}(T) \to \mathcal{P}_X$ collapses ω_1.

First, let's assume $X \in \mathcal{I}(T)$. Let G be a generic filter and $A = \cup G$. Then A is unbounded below ω_1^V since $D_\alpha = \{p \in \mathcal{P}_X : p \not\subset T \restriction_\alpha\}$ is dense for any $\alpha < \omega_1^V$. And $\Delta(A) \subset X \in \mathcal{I}(T)$. So $\mathcal{I}(T)$ is not a filter in the generic extension. Hence ω_1 is collapsed.

Second, assume $X \notin \mathcal{I}(T)$. Let $\{p_\alpha : \alpha < \omega_1\}$ be a subset of $\mathcal{P}_X$. We can find a stationary set $\Gamma \subset \omega_1$ and a countable ordinal α_0 such that

(1) for any $\alpha \in \Gamma$, for any $s \in p_\alpha$, either $ht(s) < \alpha_0$ or $ht(s) \geq \alpha$;

(2) for any $\alpha \in \Gamma$, for any $s, t \in p_\alpha \setminus T \restriction_{\alpha_o}$, $s \restriction_{[\alpha_0, \alpha)} = t \restriction_{[\alpha_0, \alpha)}$;

(3) for any $\alpha < \beta$ in Γ, $T_{p_\alpha} \restriction_{\alpha_0} = T_{p_\beta} \restriction_{\alpha_0}$.

Let $A = \{p_\alpha(0) \restriction_\alpha : \alpha \in \Gamma\}$ where $p_\alpha(0)$ is the $<_{lT}$-least $s \in p_\alpha$ such that $ht(s) \geq \alpha$. Since $X \notin \mathcal{I}(T)$, we can find $\alpha < \beta$ in Γ such that $p_\alpha(0) \restriction_\alpha \perp p_\beta(0) \restriction_\beta$ and $\Delta(p_\alpha(0), p_\beta(0)) \in X$. It is then easy to check that $p_\alpha \cup p_\beta \in \mathcal{P}_X$ and stronger than both p_α and p_β. □

Corollary 4.22. *If $\mathcal{P}$ is a forcing notion such that $\mathcal{P} \times \mathcal{P}_{X_i}$ $(i < 2)$ preserves ω_1, then $\Vdash_{\mathcal{P}} X_i \notin \mathcal{I}(T)$ $(i < 2)$. In particular, for any $\mathcal{P}$ having property (K) or $\mathcal{P} = \mathcal{P}_S$ for some special complete coherent tree S or $\mathcal{P} = \mathcal{P}_X$ for some $X \subset \omega_1$ such that $X \cap X_i \notin \mathcal{I}(T)$ $(i < 2)$, $\Vdash_{\mathcal{P}} X_i \notin \mathcal{I}(T)$ $(i < 2)$.*

Proof. Let's just prove for $i = 0$ and $i = 1$ is similar.

Suppose otherwise, $\nVdash_{\mathcal{P}} X_0 \notin \mathcal{I}(T)$. Then there is a $p \in \mathcal{P}$ such that $p \Vdash_{\mathcal{P}} X_0 \in \mathcal{I}(T)$. Then by previous lemma, $\mathcal{P}(\leq p) * \mathcal{P}_{X_0}$ and hence $\mathcal{P}(\leq p) \times \mathcal{P}_{X_0}$ will collapse ω_1. Contradict the fact that $\mathcal{P} \times \mathcal{P}_{X_0}$ preserves ω_1.

For the moreover part, if $\mathcal{P}$ has property (K), then $\mathcal{P} \times \mathcal{P}_{X_0}$ is c.c.c. and hence preserves ω_1. If $\mathcal{P} = \mathcal{P}_S$ for some special complete coherent tree S. Then S is still special complete and coherent in $V^{\mathcal{P}_{X_0}}$. So $V^{\mathcal{P}_{X_0}} \models \mathcal{P}_S$ is proper. Hence $\mathcal{P}_{X_0} * \mathcal{P}_S$ is proper. So the product forcing $\mathcal{P}_S \times \mathcal{P}_{X_0}$ is proper and hence preserves ω_1. If $\mathcal{P} = \mathcal{P}_X$ for some $X \cap X_0 \notin \mathcal{I}(T)$. Then according to the proof of previous lemma, $\mathcal{P}_X \times \mathcal{P}_{X_0}$ is c.c.c.. □

Definition 28. Let $\{N_i : i < m\}$ be a finite set of countable subsets of $H(\omega_2)$. We will say that $\{N_i : i < m\}$ is a *symmetric system* if

(1) For every $i < m$, $(N_i, \in, <)$ is a countable elementary substructure of $(H(\omega_2), \in, <)$[13].

(2) Given distinct $i, i' < m$, if $\delta_{N_i} = \delta_{N_{i'}}$ (recall $\delta_{N_i} = N_i \cap \omega_1$), then there is a (unique) isomorphism $\Psi_{N_i N_{i'}} : N_i \to N_{i'}$. Furthermore, we require that $\Psi_{N_i N_{i'}}$ is

[13]Recall $<$ is a well-ordering fixed in the ground model.

the identity on $N_i \cap N_{i'}$.

(3) For all $i, j < m$, if $\delta_{N_i} < \delta_{N_j}$, then there is some $j' < m$ such that $\delta_{N_j} = \delta_{N_{j'}}$ and $N_i \in N_{j'}$.

(4) For all $i, j, j' < m$, if $N_i \in N_j$ and $\delta_{N_j} = \delta_{N_{j'}}$, then there is some $i' < m$ such that $\Psi_{N_j N_{j'}}(N_i) = N_{i'}$.

We will now inductively define our forcing $\mathcal{P}_\alpha$ using the method introduced in [20].

$\mathcal{P}_0 = \{p = (\phi, \{(N_i, 0) : i < m\}) : m < \omega$ and $\{N_i : i < m\}$ is a symmetric system.$\}$.

For $p, q \in \mathcal{P}_0$, $p \leq_0 q$ iff $\{N_i^p : i < m^p\} \supset \{N_i^q : i < m^q\}$.

Not just $\mathcal{P}_0$, for any $\alpha \leq \omega_2$, for any condition $p \in \mathcal{P}_\alpha$, $p = (p_0, \{(N_i, \beta_i) : i < m\})$ where p_0 is a sequence of length α, $\beta_i \leq \alpha$ and $\{N_i : i < m\}$ is a symmetric system.

Definition 29. For any condition $p = (p_0, \{(N_i, \beta_i) : i < m\})$, $\Delta_p = \{(N_i, \beta_i) : i < m\}$.

For $\alpha + 1 < \omega_2$, denote $f(\alpha) = (\alpha_0, \alpha_1, \alpha_2)$ (recall that $f : \omega_2 \to \omega_2 \times \omega_2 \times 3$ is fixed at first in the ground model). Let's define a $\mathcal{P}_\alpha$-name for a forcing notion $\dot{\mathcal{Q}}_\alpha$.

(1) $\alpha_2 = 0$. Let $(\dot{\mathcal{P}}, \dot{<})$ be the α_1-th[14] forcing with property (K) in $V^{\mathcal{P}_{\alpha_0}}$ such that $\Vdash_{\mathcal{P}_{\alpha_0}} \dot{\mathcal{P}} \subset \omega_1$. Then $\dot{\mathcal{Q}}_\alpha$ is either the trivial forcing $\{1\}$ or $\dot{\mathcal{P}}$ and
$||\dot{\mathcal{Q}}_\alpha = \dot{\mathcal{P}}||_\alpha = ||\dot{\mathcal{P}}$ has property (K)$||_\alpha$.

(2) $\alpha_2 = 1$. Let $\dot{X}$ be the α_1-th subset of ω_1 in $V^{\mathcal{P}_{\alpha_0}}$. Then $\dot{\mathcal{Q}}_\alpha$ is either the trivial forcing $\{1\}$ or $\mathcal{P}_{\dot{X}}$ and
$||\dot{\mathcal{Q}}_\alpha = \mathcal{P}_{\dot{X}}||_\alpha = ||\dot{X} \cap X_i \notin \mathcal{I}(T)\ (i < 2)||_\alpha$.

(3) $\alpha_2 = 2$. Let $\dot{S}$ be the α_1-th special complete coherent tree in $V^{\mathcal{P}_{\alpha_0}}$. Then $\dot{\mathcal{Q}}_\alpha$ is either the trivial forcing $\{1\}$ or $\mathcal{P}_{\dot{S}}$ and

[14] Here we require $\Vdash_{\mathcal{P}_{\alpha_0}} 2^{\omega_1} = \omega_2$ and a canonical ordering of $\mathcal{P}(\omega_1)$ of type ω_2. We will prove later that $\Vdash_{\mathcal{P}_\xi} 2^{\omega_1} = \omega_2$ for any $\xi \leq \omega_2$. And the canonical ordering can be given by the $<$-least $\mathcal{P}_{\alpha_0}$ name for a bijection between 2^{ω_1} and ω_2 where $<$ is the well ordering of $H(\omega_4)$ fixed at the beginning.

$||\dot{\mathcal{Q}}_\alpha = \mathcal{P}_{\dot{S}}||_\alpha = ||\dot{S}$ is a special complete coherent tree $||_\alpha$.

Then $\mathcal{P}_{\alpha+1} = \{p = (p_0, \{(N_i, \beta_i) : i < m\}) : m < \omega, p_0$ is a sequence of length $\alpha+1$ and $\{N_i : i < m\}$ is a symmetric system such that

(1) $\beta_i \leq (\alpha+1) \cap sup(N_i \cap \omega_2)$ for $i < m$;

(2) $q \restriction_\alpha = (p_0 \restriction_\alpha, \{(N_i, min\{\beta_i, \alpha\}) : i < m\}) \in \mathcal{P}_\alpha$;

(3) $p_0(\alpha)$ is either 1 – the weakest condition in $\dot{\mathcal{Q}}_\alpha$ or $p \restriction_\alpha \Vdash_{\mathcal{P}_\alpha} p_0(\alpha)$ is $(N_i[\dot{G}], \dot{\mathcal{Q}}_\alpha)$-generic for any N_i such that $\alpha \in N_i$ and $(N_i, \alpha+1) \in \Delta_p$.$\}$

For $p, q \in \mathcal{P}_{\alpha+1}$, $p \leq_{\alpha+1} q$ iff

(1) $p \restriction_\alpha \leq_\alpha q \restriction_\alpha$;

(2) $p \restriction_\alpha \Vdash_{\mathcal{P}_\alpha} p_0(\alpha) \leq q_0(\alpha)$;

(3) for any $(N_i, \beta_i) \in \Delta_q$, there is a $\beta' \geq \beta_i$ such that $(N_i, \beta') \in \Delta_p$.

We now come to the case $\alpha \leq \omega_2$ is a limit ordinal.

$\mathcal{P}_\alpha = \{p = (p_0, \{(N_i, \beta_i) : i < m\}) : m < \omega, p_0$ is a sequence of length α and $\{N_i : i < m\}$ is a symmetric system such that

(1) $\beta_i \leq (\alpha+1) \cap sup(N_i \cap \omega_2)$ for $i < m$;

(2) for any $\gamma < \alpha$, $q \restriction_\gamma = (p_0 \restriction_\gamma, \{(N_i, min\{\beta_i, \gamma\}) : i < m\}) \in \mathcal{P}_\gamma$.

For $p, q \in \mathcal{P}_\alpha$, $p \leq_\alpha q$ iff $p \restriction_\gamma \leq q \restriction_\gamma$ for any $\gamma < \alpha$.

Now we are going to prove that $\mathcal{P}_{\omega_2}$ forces the assumption of Lemma 4.19 and $|\mathcal{P}(\omega_1)/\mathcal{I}(T)| = 4$. First, we will use some facts from [20].

Definition 30. (1) For a countable ordinal $\alpha > 0$, a forcing poset $\mathcal{P}$ is *α-proper* iff for sufficiently large regular cardinal λ and every continuous elementary chain $\langle M_i : i < \alpha \rangle$ of countable elementary submodel of $H(\lambda)$ containing $\mathcal{P}$, the following holds: every $p \in \mathcal{P} \cap M_0$ has an extension $q \leq p$ that is $(M_i, \mathcal{P})$-generic for every $i < \alpha$.

(2) Say $\mathcal{P}$ is *finitely proper* iff $\mathcal{P}$ is n-proper for every $n < \omega$.

Remark: In [20], Aspero-Mota iterated a different notation – V-finitely proper.

Note every finitely proper poset is V-finitely proper so the same procedure works. In this thesis we just need to iterate finitely proper posets. And by Lemma 4.20, for any special complete coherent tree S, $\mathcal{P}_S$ is finitely proper. Hence, by corollary 4.22, $\mathcal{P}_S \times \mathcal{P}_{X_i}$ $(i < 2)$ is finitely proper. Hence $\dot{\mathcal{Q}} \times \mathcal{P}_{X_i}$ $(i < 2)$ is always finitely proper since c.c.c. posets are finitely proper.

Lemma 4.23 ([20]). *Assume $\mathcal{P}_{\omega_2}$ is a defined as above with $\dot{\mathcal{Q}}_\alpha$ possibly replace by some finitely proper poset. Then the following holds.*
(1) If $\mathcal{N}$ is a symmetric system and $N \in \mathcal{N}$, then $N \cap \mathcal{N}$ is a symmetric system and for any symmetric system $\mathcal{W} \in N$ such that $\mathcal{W} \supset N \cap \mathcal{N}$, $\mathcal{N} \cup \{\Psi_{NN'}(W) : W \in \mathcal{W}, N' \in \mathcal{N}, \delta_{N'} = \delta_N\}$ is a symmetric system extending both $\mathcal{N}$ and $\mathcal{W}$.
(2) $\mathcal{P}_{\omega_2}$ is $\aleph_2$-c.c..
(3) For any $\alpha \leq \omega_2$, $\mathcal{P}_\alpha$ is proper. Moreover, if $N^* \prec H(\theta)$ is a countable elementary submodel for some large regular θ and $(N^* \cap H(\omega_2), \alpha) \in \Delta_p$ (or $(N^* \cap H(\omega_2), sup(N^* \cap \omega_2)) \in \Delta_p$ for the case $\alpha = \omega_2$), then p is $(N^*, \mathcal{P}_\alpha)$-generic.
(4) $\Vdash_{\mathcal{P}_{\omega_2}} 2^{\omega_1} = \omega_2$.
(5) For any $\alpha < \omega_2$, $\mathcal{P}_{\omega_2}$ forces $\dot{G}^+_\alpha = \{p_0(\alpha) : p \in \dot{G}_{\omega_2}\}$ generates a $V^{\mathcal{P}_\alpha}$-generic filter over $\dot{\mathcal{Q}}_\alpha$.

Then $\mathcal{P}_{\omega_2}$ preserves cardinals and every subset of ω_1 in $V^{\mathcal{P}_{\omega_2}}$ appears in $V^{\mathcal{P}_\gamma}$ for some $\gamma < \omega_2$ (since it has ω_1 elements and by ω_2-c.c. each element is determined by ω_1 many conditions).

Corollary 4.24. *(1) $V^{\mathcal{P}_{\omega_2}} \models$ "$MA_{\mathcal{K}}(\omega_1)$ and for any $\mathbb{R}$-embeddable coherent tree S, $S \equiv_C T$".*
(2) If $X_i \notin \mathcal{I}(T)$ in $V^{\mathcal{P}_{\omega_2}}$ $(i < 2)$, then

$$V^{\mathcal{P}_{\omega_2}} \models \mathcal{P}(\omega_1)/\mathcal{I}(T) = \{\phi/\mathcal{I}(T), X_0/\mathcal{I}(T), X_1/\mathcal{I}(T), \omega_1/\mathcal{I}(T)\}$$

.

Proof. (1) If in $V^{\mathcal{P}_{\omega_2}}$ $\dot{\mathcal{P}} \in \mathcal{K}$ and $\dot{\mathcal{D}}$ is a collection of ω_1 dense sets., WLOG, assume $\dot{\mathcal{P}} \subset \omega_1$. Assume $\dot{X} \in H^{V^{\mathcal{P}_{\omega_2}}}(\omega_2)$ witnesses that $\dot{\mathcal{P}}$ has property (K). Then $\dot{\mathcal{P}}, \dot{\mathcal{D}}\dot{X}$ appear in some early stage $\gamma < \omega_2$, i.e., $\dot{\mathcal{P}}, \dot{\mathcal{D}}\dot{X} \in V^{\mathcal{P}_\gamma}$. Assume $\dot{\mathcal{P}}$ is the δ-th such forcing with property (K). Let α be large enough such that $f(\alpha) = (\gamma, \delta, 0)$. Then $\dot{\mathcal{P}}$ has property (K) in $V^{\mathcal{P}_\alpha}$ since $\dot{X} \in V^{\mathcal{P}_\alpha}$. Hence $\dot{\mathcal{Q}}_\alpha = \dot{\mathcal{P}}$. Hence at the final model $V^{\mathcal{P}_{\omega_2}}$ there is a filter meets each dense set in $\dot{\mathcal{D}}$.

If $\dot{S}$ is an $\mathbb{R}$-embeddable coherent tree in $V^{\mathcal{P}_{\omega_2}}$. Then the forcing which forces $\dot{S}$ to be special is in $\mathcal{K}$. So $\dot{S}$ is special. Let $\dot{S}'$ be its completion. Then repeat above argument, we can get that $\dot{S}' \equiv_C T$. Hence $\dot{S} \equiv_C T$.

(2) For any $\dot{X} \subset \omega_1$ in $V^{\mathcal{P}_{\omega_2}}$, we have 4 cases:

Case 1: $\dot{X} \cap X_0 \in \mathcal{I}(T)$ and $\dot{X} \cap X_1 \in \mathcal{I}(T)$.

Then $\dot{X}/\mathcal{I}(T) = \phi/\mathcal{I}(T)$.

Case 2: $\dot{X} \cap X_0 \notin \mathcal{I}(T)$ and $\dot{(X)} \cap X_1 \notin \mathcal{I}(T)$.

Then $\dot{\mathcal{Q}}_\alpha = \mathcal{P}_{\dot{X}}$ for some $\alpha < \omega_2$. And hence $\dot{(X)} \in \mathcal{U}(T)$. Hence $\dot{X}/\mathcal{I}(T) = \omega_1/\mathcal{I}(T)$.

Case 3: $\dot{X} \cap X_0 \in \mathcal{I}(T)$ and $\dot{(X)} \cap X_1 \notin \mathcal{I}(T)$.

Then as in case 2, $\dot{X} \cup X_0 \in \mathcal{U}(T)$. Then $\dot{X}/\mathcal{I}(T) = X_1\mathcal{I}(T)$.

Case 4: $\dot{X} \cap X_0 \notin \mathcal{I}(T)$ and $\dot{(X)} \cap X_1 \in \mathcal{I}(T)$.

As in case 3, $\dot{X}/\mathcal{I}(T) = X_0/\mathcal{I}(T)$. This shows that in $V^{\mathcal{P}_{\omega_2}}$, $\mathcal{P}(\omega_1)/\mathcal{I}(T) = \{\phi/\mathcal{I}(T), X_0/\mathcal{I}(T), X_1/\mathcal{I}(T), \omega_1/\mathcal{I}(T)\}$. □

Now we are left to prove that $\phi/\mathcal{I}(T), X_0/\mathcal{I}(T), X_1/\mathcal{I}(T), \omega_1/\mathcal{I}(T)$ are pairwise different in $V^{\mathcal{P}_{\omega_2}}$. And we just need to prove that $X_i \notin \mathcal{I}(T)$ $(i < 2)$ in $V^{\mathcal{P}_{\omega_2}}$.

Lemma 4.25. *$X_i \notin \mathcal{I}(T)$ $(i < 2)$ in $V^{\mathcal{P}_{\omega_2}}$.*

Proof. By symmetry, we just need to show $X_0 \notin \mathcal{I}(T)$. By Corollary 4.22, we just need to show that $\mathcal{P}_{\omega_2} \times \mathcal{P}_{X_0}$ preserves ω_1. We will prove by induction that for each $\alpha \leq \omega_2$, $\mathcal{P}_\alpha \times \mathcal{P}_{X_0}$ is proper.

$\alpha = 0$. Fix a large enough regular cardinal θ and a countable $N^* \prec H(\theta)$. Let, $N = N^* \cap H(\omega_2)$. It is suffice to prove that for any $(p, q) \in \mathcal{P}_0 \times \mathcal{P}_{X_0}$ such that $(N, 0) \in \Delta_p$, (p, q) is $(N^*, \mathcal{P}_0 \times \mathcal{P}_{X_0})$-generic. Now let's fix such a (p, q) and a dense subset $D \in N^*$ of $\mathcal{P}_0 \times \mathcal{P}_{X_0}$. Going to a stronger condition we can assume $(p, q) \in D$ and we just need to find a $(p', q') \in D \cap N^*$ compatible with (p, q).

Let $D' = \{(p', q') \in D : \Delta_{p'} \supset \Delta_p \cap N\}$. Then $D' \in N^*$ (since $\Delta_p \cap N \in N^*$) and $(p, q) \in D'$. Let $E = \{q' : \exists p' \ (p', q') \in D'\} \in N^*$. Fix a maximal antichain $E' \subset E$ and $E' \in N^*$. Note E' is countable since $\mathcal{P}_{X_0}$ is c.c.c.. Hence $E' \subset N^*$. Pick a $q' \in E'$ such that $q \not\perp q'$. By elementarity, pick a $p' \in N^*$ such that $(p', q') \in D'$. Hence, $(p', q') \in N^*$ and $(p', q') \not\perp (p, q)$ by Lemma 4.23.

$\alpha = \bar{\alpha} + 1$. First, let's prove that $\mathcal{P}_\alpha \times \mathcal{P}_{X_0}$ is isomorphic to $\mathcal{P}_{\bar{\alpha}} * (\dot{\mathcal{Q}} \times \mathcal{P}_{X_0})$: (1) $((p, \dot{q}), r) \to (p, (\dot{q}, r))$ is a bijection from $\mathcal{P}_\alpha \times \mathcal{P}_{X_0}$ to $\mathcal{P}_{\bar{\alpha}} * (\dot{\mathcal{Q}}_{\bar{\alpha}} \times \mathcal{P}_{X_0})$ since $p \Vdash_{\mathcal{P}_{\bar{\alpha}}} \dot{q}$ is $(N[\dot{G}_{\bar{\alpha}}], \dot{\mathcal{Q}}_{\bar{\alpha}})$-generic iff $p \Vdash_{\mathcal{P}_{\bar{\alpha}}} (\dot{q}, r)$ is $(N[\dot{G}_{\bar{\alpha}}], \dot{\mathcal{Q}}_{\bar{\alpha}} \times \mathcal{P}_{X_0})$-generic; (2) the order for the poset is also preserved.

Then by Lemma 4.23, $\mathcal{P}_{\bar{\alpha}} * (\dot{\mathcal{Q}} \times \mathcal{P}_{X_0}$ and hence $\mathcal{P}_\alpha \times \mathcal{P}_{X_0}$ is proper.

α is an infinite limit ordinal. Fix a countable $N^* \prec H(\theta)$ for some large regular θ and $p \in N^* \cap \mathcal{P}_\alpha$. Let $N = N^* \cap H(\omega_2)$. It follows from the construction of $\mathcal{P}_\alpha$ and Lemma 4.20, Lemma 4.21 that there is a condition $p' <_\alpha p$ such that $(N, \alpha) \in \Delta_{p'}$ (here we assume $\alpha < \omega_2$, since if $\mathcal{P}_{\omega_2} \times \mathcal{P}_{X_0}$ is not proper, then in $V^{\mathcal{P}_{\omega_2}}$, there is some $A \in [T]^{\omega_1}$ such that $\Delta(A) \cap X_0 = \phi$. Then such A appears at some previous stage, $A \in V^{\mathcal{P}_\gamma}$ for some $\gamma < \omega_2$. Then $\mathcal{P}_\gamma \times \mathcal{P}_{X_0}$ collapses ω_1.). Now we just need to show (p', r) is $(N, \mathcal{P}_\alpha \times \mathcal{P}_{X_0})$-generic for any $r \in \mathcal{P}_{X_0}$.

Fix $D \in N^*$ a dense open subset of $\mathcal{P}_\alpha \times \mathcal{P}_{X_0}$. Extending p' we can assume $p' \in D$ and we just need to find some $(q, r') \in D \cap N$ compatible with (p', r).

Suppose first $cf(\alpha) = \omega$. Then we can find some $\sigma \in N^* \cap \alpha$ above $supp(p') = \{\xi < \alpha : p'(\xi)$ is not the weakest element.$\}$. Let $G_\sigma \times G$ be $\mathcal{P}_\sigma \times \mathcal{P}_{X_0}$-generic and $(p' \restriction_\sigma, r) \in G_\sigma \times G$. By Lemma 4.23, $p' \restriction_\sigma$ is $(N^*, \mathcal{P}_\sigma)$-generic. Then by induction hypothesis, $(p' \restriction_\sigma, r)$ is $(N^*, \mathcal{P}_\sigma \times \mathcal{P}_{X_0})$-generic. So

$$N^*[G_\sigma \times G] \models \exists (q, r') \in D \ \ (q \restriction_\sigma, r') \in G_\sigma \times G \wedge supp(q) \subset \sigma.$$

Pick such a $(q, r') \in N^*[G_\sigma \times G]$. $(q, r') \in N^*$ since $(q, r') \in D$. Then $q \not\perp p'$ since $q \restriction_\sigma \not\perp p' \restriction_\sigma$ (both in G_σ) and their supports are below σ. So $(p', r) \not\perp (q, r')$.

Now suppose $cf(\alpha) > \omega$. Fix a $\sigma \in N \cap \alpha$ such that

(i) for any $(N', \xi) \in \Delta_{p'}$ such that $\Delta_{N'} < \Delta_N$, $sup(N' \cap N \cap \alpha) < \sigma$ [15];

(ii) σ bounds $supp(p') \cap sup(N \cap \alpha)$.

Like in case $cf(\alpha) = \omega$, we can find $(q, r') \in D$ such that $(q \restriction_\sigma, r') \not\perp (p' \restriction_\sigma, r)$. According to the proof of Lemma 4.23, $q \not\perp p'$. Hence $(q, r') \not\perp (p', r)$.

This finishes the proof for limit α and hence the proof of the lemma. □

Now we have the following:

Corollary 4.26. *(1) It is consistent to have the least size of the basis for Countryman lines to be 4.*

(2) For any $n < \omega$, it is consistent to have the least size of the basis for Countryman lines to be 2^{n+1}.

Proof. (1) follows from Lemma 4.19, Corollary 4.24 and Lemma 4.25.

(2) The proof for (1) works for (2) too. Just replace the partition $\omega_1 = \bigcup_{i<2} X_i$ in construction for (1) by a partition $\omega_1 = \bigcup_{i<n+1} X_i$. And replace 2 by $n+1$ in appropriate place in the proof. □

4.3 Some applications

Now we are ready to get the following application:

[15] Note there are finitely many such N' and for each N', there is a $\bar{N}$ such that $\delta_{\bar{N}} = \delta_N$ and $N' \in \bar{N}$, then $sup(N' \cap N \cap \alpha) \leq sup(\Psi_{\bar{N}N}(N') \cap \alpha) \in N \cap \alpha$.

Theorem 4.27. *(1) $T(\rho_2)$ is lexicographically isomorphic to a coherent tree.*
(2) $T(\rho_2)$ is special. In particular, $T(\rho_2)$ is Countryman with respect to its lexicographical order.

Proof. (1) Define a function $a : [\omega_1]^2 \to \mathbb{Z}$ by: for any $\alpha < \beta$
(i) $a(\alpha, \beta) = 0$ if α is a limit ordinal;
(ii) $a(\alpha, \beta) = \rho_2(\alpha, \beta) - \rho_2(\alpha - 1, \beta)$ if α is a successor ordinal.
Notation: $<_l$ denotes the canonical lexicographical order for sequences and hence $<_l$ is the lexicographical order for both $T(\rho_2)$ and $T(a)$.
We want to prove that $T(a)$ is the tree we want. First, we show that it is coherent:
Claim 1: $T(a)$ is coherent.
proof of claim 1: Suppose otherwise, there are $\alpha < \beta$ such that $D_{\alpha\beta} = \{\eta < \alpha : a(\eta, \alpha) \neq a(\eta, \beta)\}$ is infinite. Note $D_{\alpha\beta}$ consists of only successor ordinals. Let $\gamma \leq \alpha$ be the least δ such that $D_{\alpha\beta} \cap \delta$ is infinite and it is easy to see that γ is a limit ordinal. Let $\xi_{\gamma\alpha}, \xi_{\gamma\beta}$ be as guaranteed by Lemma 3.10 and let $\xi = max\{\xi_{\gamma\alpha}, \xi_{\gamma\beta}\}$. Then $\xi < \gamma$ and hence $D_{\alpha\beta} \cap [\xi, \gamma)$ is infinite. Now pick $\eta \in D_{\alpha\beta} \cap [\xi, \gamma)$ and we have $a(\eta, \alpha) \neq a(\eta, \beta)$. But on the other hand, by Lemma 3.10,
$a(\eta, \alpha) = \rho_2(\eta, \alpha) - \rho_2(\eta - 1, \alpha) = (\rho_2(\gamma, \alpha) + \rho_2(\eta, \gamma)) - (\rho_2(\gamma, \alpha) + \rho_2(\eta - 1, \gamma)) = \rho_2(\eta, \gamma) - \rho_2(\eta - 1, \gamma) = a(\eta, \gamma)$.
And similarly, $a(\eta, \beta) = a(\eta, \gamma)$. Hence $a(\eta, \alpha) = a(\eta, \beta)$. A contradiction. This finishes the proof of claim 1.
Now define a mapping $\pi : T(\rho_2) \to T(a)$ by $\pi(\rho_{2\beta} \restriction_\alpha) = a_\beta \restriction_\alpha$. Then we just need to show:
Claim 2: π is a lexicographical isomorphism.
Proof of claim 2: First we need to show that π is well defined, i.e., for any $\alpha \leq \beta < \gamma$, $\rho_{2\beta} \restriction_\alpha = \rho_{2\gamma} \restriction_\alpha$ implies that $\pi(\rho_{2\beta} \restriction_\alpha) = \pi(\rho_{2\gamma} \restriction_\alpha)$, and this is easily followed by the definition of a and π. Hence, it is followed that π preserves the tree order.
Then we need to show that π is a bijection, it is enough to show that it is an

injection. Pick any two distinct elements in $T(\rho_2)$: $\rho_{2\beta} \restriction_\alpha \neq \rho_{2\gamma} \restriction_\alpha$. Let $\delta = \Delta(\rho_{2\beta} \restriction_\alpha, \rho_{2\gamma} \restriction_\alpha)$. Then δ is a successor ordinal by Lemma 3.10. And hence $a(\delta,\beta) = \rho_2(\delta,\beta) - \rho_2(\delta-1,\beta) \neq \rho_2(\delta,\gamma) - \rho_2(\delta-1,\gamma) = a(\delta,\gamma)$. So π is an injection.

At last, we are left to show π preserves the lexicographical order. Pick any $\rho_{2\beta} \restriction_\alpha <_l \rho_{2\gamma} \restriction_\alpha$, we need to show $a_\beta \restriction_\alpha = a_\gamma \restriction_\alpha$. If $\rho_{2\beta} \restriction_\alpha <_{T(\rho_2)} \rho_{2\gamma} \restriction_\alpha$, then we are done since π preserves the tree order. Now assume $\tau = \Delta(\rho_{2\beta} \restriction_\alpha, \rho_{2\gamma} \restriction_\alpha) < \alpha$ and $\rho_2(\tau,\beta) < \rho_2(\tau,\gamma)$. Note τ is a successor ordinal by Lemma 3.10. So $a(\tau,\beta) = \rho_2(\tau,\beta) - \rho_2(\tau-1,\beta) < \rho_2(\tau,\gamma) - \rho_2(\tau-1,\gamma)$. And also note $a_\beta \restriction_\tau = a_\gamma \restriction_\tau$ since we have proven that π is well defined. Now we have $a_\beta \restriction_\alpha <_l a_\gamma \restriction_\alpha$ and so π preserves the lexicographical order. This finishes the proof of claim 2.

Now we have that $T(\rho_2)$ is lexicographically isomorphic to $T(a)$ which is coherent.

(2) According to Theorem 4.1, it is suffice to prove that $\{\rho_{2\alpha} \restriction_\alpha : \alpha < \omega_1$ is an infinite limit ordinal$\}$ is an antichain. Pick two arbitrary infinite limit ordinals $\alpha < \beta < \omega_1$. Find a $\xi_{\alpha\beta} < \alpha$ guaranteed by Lemma 3.10 and pick $\gamma \in [\xi_{\alpha\beta}, \alpha)$. We have $\rho_2(\gamma,\beta) = \rho_2(\alpha,\beta) + \rho_2(\gamma,\alpha) > \rho_2(\gamma,\alpha)$ and so $\rho_{2\alpha} \restriction_\alpha$ is incomparable with $\rho_{2\beta} \restriction_\beta$. This finishes the proof of the theorem. □

Remark: Above proof can be generalized to prove that $T(\rho_0)$ is lexicographically isomorphic to a coherent tree. See also [21] for a proof that $T(\rho_0)$ is tree isomorphic to a coherent tree.

Appendix **A**

Appendix

Proof of Theorem 2.14: We will use Lemma 4.3 to prove this theorem and use notation in Theorem 4.2. Let $T' = T \upharpoonright_C$ and we are going to show that T' is Lipschitz. For any level preserving map $\pi : X \to T'$ where $X \in [T']^{\omega_1}$, let $X_n = \{x \in X : C(l_x, l_{\pi(x)}) = n\}$. Pick a $n < \omega$ such that X_n is uncountable. It is suffice to prove that for any $x, y \in X_n$, $\Delta_{T'}(x,y) = \Delta_{T'}(\pi(x), \pi(y))$ (here $\Delta_{T'}(s,t)$ is the first difference of s and t with respect to T'). Suppose otherwise, assume there are $x, y \in X_n$ such that $\Delta_{T'}(x,y) < \Delta_{T'}(\pi(x), \pi(y))$ (similar for the case $\Delta_{T'}(x,y) > \Delta_{T'}(\pi(x), \pi(y))$). Let $\alpha = \Delta_{T'}(\pi(x), \pi(y))$. Then as points in T, $\Delta(x,y) < C(\alpha)$ and $\Delta_T(\pi(x), \pi(y)) \geq C(\alpha)$ where $C(\alpha)$ is the α-th element of the club C. Then $C(l_x, l_{\pi(x)}) = C(l_y, l_{\pi(y)}) = n$ which contradicts Lemma 4.3 since $l_x \upharpoonright_{C(\alpha)} \neq l_y \upharpoonright_{C(\alpha)}$ while $l_{\pi(x)} \upharpoonright_{C(\alpha)} = l_{\pi(y)} \upharpoonright_{C(\alpha)}$. This finishes the proof of the theorem.

Proof of Fact 2.1: Let $(T, <_T, <_{lT})$ be arbitrary lexicographically ordered Aronszajn tree. Define $S \subset \mathbb{Q}^{<\omega_1}$ and a lexicographical isomorphism $\pi : T \to S$ by induction on the height of S:

$S_0 = \{\phi\}$ and $\pi'' T_0 = S_0$.

If S_α is defined and $\pi \upharpoonright_{T\upharpoonright_{\alpha+1}}$ is defined and lexicographically isomorphic, let

$f_\alpha : T_{\alpha+1} \to \mathbb{Q}$ be an embedding preserving $<_{lT}$, then $S_{\alpha+1} = \{s^\frown f_\alpha(t) : s \in S_\alpha,$ $t \in T_{\alpha+1}$ and $\pi(t \restriction_\alpha) = s\}$ and $\pi(t) = \pi(t \restriction_\alpha)^\frown f_\alpha(t)$ for any $t \in T_{\alpha+1}$. It is easy to see that $\pi \restriction_{T\restriction_{\alpha+2}}$ is also lexicographically isomorphic.

If α is a limit ordinal and for any $\beta < \alpha$, S_β is defined and $\pi \restriction_{T\restriction_\beta}$ is defined and lexicographically isomorphic, then $S_\alpha = \{\cup\{\pi(t \restriction_\beta) : \beta < \alpha\} : t \in T_\alpha\}$ and $\pi(t) = \cup\{\pi(t \restriction_\beta) : \beta < \alpha\}$ for any $t \in T_\alpha$. It is easy to check that $\pi \restriction_{T\restriction_{\alpha+1}}$ is also lexicographically isomorphic.

Proof of Fact 2.2: If T has unique limits and C is a club in $ht(T)$. To see $T \restriction_C$ has unique limits, just note that any element which has limit height in $T \restriction_C$ has limit height in T too and the predecessors in $T \restriction_C$ is cofinal in predecessors in T.

Assume $(T, <_T, <_l)$ is coherent and $C \subset \omega_1$ is a club. Assume $0 \in C$. $T \restriction_C = \{t \in T : ht_T(t) \in C\}$ and the tree order and the lexicographical order of $T \restriction_C$ are the restriction of $<_T$ and $<_l$ on $T \restriction_C$. Now we just need to define a coherent tree S and a lexicographical isomorphism $\pi : T \restriction_C \to S$. Still do induction on α:

If $\alpha = 0$, $S_0 = \{\phi\}$ and $\pi''(T \restriction_C)_0 = S_0$.

If S_α is defined and $\pi \restriction_{(T\restriction_C)\restriction_{\alpha+1}}$ is defined and lexicographically isomorphic, let $A_\alpha = \{t \restriction_{[C(\alpha),C(\alpha+1))} : t \in T_{C(\alpha+1)}\}$, and since A_α is countable, we can find $f_\alpha : A_\alpha \to \mathbb{Q}$ to be any embedding preserving lexicographical order, then $S_{\alpha+1} = \{s^\frown f_\alpha(t) : s \in S_\alpha,\ t \in (T \restriction_C)_{\alpha+1}$ and $\pi(t \restriction_{C(\alpha)}) = s\}$ and $\pi(t) = \pi(t \restriction_{C(\alpha)})^\frown f_\alpha(t)$ for any $t \in (T \restriction_C)_{\alpha+1}$. Note that $f_\alpha(t) = f_\alpha(t')$ iff $t \restriction_{[C(\alpha),C(\alpha+1))} = t' \restriction_{[C(\alpha),C(\alpha+1))}$. It is easy to see that $\pi \restriction_{(T\restriction_C)\restriction_{\alpha+2}}$ is also an lexicographical isomorphism.

If α is a limit ordinal and for any $\beta < \alpha$, S_β is defined and $\pi \restriction_{T\restriction_\beta}$ is defined and lexicographically isomorphic, then $S_\alpha = \{\cup\{\pi(t \restriction_\beta) : \beta < \alpha\} : t \in (T \restriction_C)_\alpha\}$ and $\pi(t) = \cup\{\pi(t \restriction_\beta) : \beta < \alpha\}$ for any $t \in (T \restriction_C)_\alpha$. It is easy to check that $\pi \restriction_{(T\restriction_C)\restriction_{\alpha+1}}$ is also lexicographically isomorphic.

Then we just need to show that S is coherent. Pick $x, y \in S$. Recall that in the construction of successor levels, we have $s(\alpha) \neq t(\alpha)$ iff $\pi^{-1}(s) \restriction_{[C(\alpha),C(\alpha+1))} \neq$

$\pi^{-1}(t) \restriction_{[C(\alpha),C(\alpha+1))}$. Then by the fact that T is coherent, we can get that $\{\alpha < ht(s), ht(t) : s(\alpha) \neq t(\alpha)\}$ is finite. And hence S is coherent.

Proof of Fact 2.3: (1) If $\beta \in N_\alpha \cap \omega_1$, then $N \models \beta$ is countable. Let $f : \omega \to \beta$ be a surjection in N_α. Then $\beta \subset N_\alpha$ since $f \in N_\alpha$ and $\omega \subset N_\alpha$.

C is unbounded since $N_\alpha \cap \omega_1 \geq \alpha$ and C is closed since $N_\alpha = \cup_{\beta<\alpha} N_\beta$ for limit α.

(2) By elementary, we just need to prove that $N_\alpha \models A$ is stationary. Pick any club $D \in N_\alpha$ and then $H(\omega_2) \models D$ is a club. Then D is unbounded in $N_\alpha \cap \omega_1$ and hence $N_\alpha \cap \omega_1 \in D$. So $H(\omega_2) \models D \cap A \neq \phi$. By elementary again, $N_\alpha \models D \cap A \neq \phi$. Then $N_\alpha \models A$ is stationary.

Proof of Fact 2.12: (1) (a) implies (b) and (c) implies (a) are trivial. So we just prove (b) implies (c). Assume $T \restriction_C = \cup_{n<\omega} A_n$ be a partition of $T \restriction_C$ into countably many antichains. Define a map $f_0 : T \restriction_C \to \mathbb{Q}$ by induction on n:

$f_0'' A_0 = \{0\}$.

If $f_0 \restriction_{\cup_{m \leq n} A_m}$ is defined and the range is finite, then we define $f_0 \restriction_{A_{n+1}}$. For any $t \in A_{n+1}$,

if t is maximal (with respect to the tree order) in $\cup_{m\leq n+1} A_n$, $f_0 = max(rang(f_0 \restriction_{\cup_{m\leq n} A_m})) + 1$;

if t is minimal in $\cup_{m\leq n+1} A_n$, $f_0 = min(rang(f_0 \restriction_{\cup_{m\leq n} A_m})) - 1$;

if t is neither maximal nor minimal, $f_0(t) = (min(a + b)/2$ where $a = max\{f_0(s) : s \in \cup_{m\leq n} A_m$ and $s <_T t\}$ and $b = min\{f_0(s) : s \in \cup_{m\leq n} A_m$ and $t <_T s\}$.

Now $f_0 \restriction_{\cup_{m\leq n+1} A_m}$ is defined and the range is still finite.

It is easy to see that $f_0 : T \restriction_C \to \mathbb{Q}$ defined above is an embedding. For any $\alpha < \omega_1$, fix $\pi_\alpha : T \restriction_{[C(\alpha),C(\alpha+1))} \to \mathbb{Q}$ to be any embedding for the tree order. Define $f : T \to \mathbb{Q} \times \mathbb{Q}$ (the order for $\mathbb{Q} \times \mathbb{Q}$ is the lexicographical order) by:

for any $t \in T$, assume $ht(t) \in [C(\alpha), C(\alpha + 1))$ and $t' \in T_{C(\alpha)}$ such that $t' \leq_T t$, then $f(t) = (f_0(t'), \pi_\alpha(t))$. It is easy to see that f is an embedding. And hence T is $\mathbb{Q}$-embeddable.

$(2)(a) \to (b)$. Let $f : T \to \mathbb{R}$ be an embedding. Just pick C to be ω_1. Then $C+1$ is the set of successor ordinals below ω_1. Define $a : T \restriction_{C+1} \to \mathbb{Q}$ by: for any $t \in T$ such that $ht(t) = \alpha+1$, define $a(t)$ to be any rational in the interval $(f(t \restriction_\alpha), f(t))$. It is easy to see that for any $q \in \mathbb{Q}$, $a^{-1''}\{q\}$ is an antichain. So $T \restriction_{C+1}$ is special.
$(b) \to (c)$. Let C be any club and $T \restriction_{C+1} = \cup_{n<\omega} A_n$ be a partition of $T \restriction_{C+1}$ into countably many antichains. Let X be any nonstationary set. Going to a subclub of C we can assme that $C \cap X = \phi$. Let $g_\alpha : T \restriction_{[C(\alpha), C(\alpha+1))} \to \omega$ be an injection. Define a partition $T \restriction_{\omega_1 \setminus C} = \cup_{m,n<\omega} B(m,n)$ by: for any $t \in T \restriction_{\omega_1 \setminus C}$, if $C(\alpha) < ht(t) < C(\alpha+1)$ and $t \restriction_{C(\alpha)+1} \in A_m$, then $t \in B(m, g_\alpha(t))$. It is easy to check that each $B(m,n)$ is an antichain. So $T \restriction_{\omega_1 \setminus C}$ and hence $T \restriction_X$ is special.
$(c) \to (a)$. Assume X is the set of successor ordinals below ω_1. Repeat the proof in (1), we can get that $T \restriction_X$ is $\mathbb{Q}$-embeddable. Let $f_0 : T \restriction_X \to \mathbb{Q}$ be an embedding. Extend f_0 to $f : T \to \mathbb{R}$ by: $f(t) = sup\{f_0(s) : s <_T t \text{ and } s \in T \restriction_X\}$ for any $t \in T$ such that $ht(t)$ is a limit ordinal.

Proof of Fodor's Lemma: Suppose otherwise, f is regressive on a stationary set S and for any $\alpha < \omega_1$, $f^{-1}\{\alpha\}$ is nonstationary. Let C_α be a club that disjoint from $f^{-1}\{\alpha\}$. It is straightforward to check that $C = \triangle_{\alpha<\omega_1} C_\alpha = \{\beta : \text{for any } \alpha < \beta, \beta \in C_\alpha\}$ is a club and disjoint from S. Contradict that S is stationary.

Proof of Fact 3.7: (1) Note O is infinite. If O contains no minimal element, then O contains a subset of tyoe ω^* and we are done. Now assume m is the minimal element of O.

If for any $x \in O$, the interval (m,x) is finite, then O can be embedded into ω. So we can find some $x \in O$ such that the interval (m,x) is infinite. Then interval (m,x) contains a subset of ω or ω^* and hence O contains a subset of type $\omega+1$ or ω^*.

(2) If for any $m < x$ in O, the interval (m,x) is finite, then O can be embedded into $\mathbb{Z}$. So we can find $m < x$ in O such that the interval (m,x) is infinite. Then

the interval (m, x) contains a subset of ω or ω^* and hence O contains a subset of type $\omega + 1$ or $(\omega + 1)^*$.

Proof of Lemma 3.10: (1) Let $\beta_0 > ... > \beta_n$ be the walk from β to α. Let $\xi = max \bigcup_{i<n} (C_{\beta_i} \cap \alpha)$. It follows from the definition of minimal walk that this ξ works.

(2) Suppose otherwise, there are $\beta < \omega_1$, $n < \omega$ and a sequence $\{\alpha_i : i < \omega\}$ below β such that $\rho_1(\alpha_i, \beta) = n$. Then $\alpha = sup\{\alpha_i : i < \omega\} \leq \beta$. Pick a $i < \omega$ such that $|C_\alpha \cap \alpha_i| > n$ and $\alpha_i > \xi$ if $\alpha < \beta$ where ξ is guaranteed by (1). Then α is in the walk from β to α_i. Hence $\rho_1(\alpha_i, \beta) \geq \rho_1(\alpha_i, \alpha) \geq i > n$. A contradiction.

(3) It is suffice to prove that for any $\alpha < \beta\omega_1$, $\rho_{1\alpha} = \rho_{1\beta} \upharpoonright_\alpha$. We will do induction on α. Take ξ a witness for (1). Enlarge ξ if necessary we can assume that $|C_\alpha \cap \xi| > \rho_1(\alpha, \beta)$. For any $\eta \in (\xi, \alpha)$, first note it follows from the definition of ρ_1 that $\rho_1(\eta, \beta) = max\{\rho_1(\alpha, \beta), \rho_1(\eta, \alpha)\}$; then we get that $\rho_1(\eta, \beta) = \rho_1(\eta, \alpha)$. So we get $\rho_{1\alpha} \upharpoonright_{(\xi,\alpha)} = \rho_1 1\beta \upharpoonright_{(\xi,\alpha)}$. Then $\rho_{1\alpha} =^* \rho_{1\beta} \upharpoonright_\alpha$ since by induction $\rho_{1\alpha} \upharpoonright_\xi =^* \rho_{1\xi} =^* \rho_{1\beta} \upharpoonright_\xi$.

(4) is trivial. (5) follows from (1) and the definition of ρ_2. (6) follows from (5).

Proof of Fact 3.12: In $V[G]$, assmue $G = \{g_n : n < \omega\}$. Let $\dot{X} \in V$ be a name for X. Then for each $n < \omega$, define in V: $Y_n = \{x : g_n \Vdash x \in \dot{X}\}$. $\langle Y_n : n < \omega \rangle$ is the sequence required.

If $X \subset \omega_1$ is uncountable (or stationary), then there is a $n < \omega$ such that Y_n is uncountable (or stationary) in $V[G]$. So $Y = Y_n$ is what we need.

If $C \subset \omega_1$ is a club in $V[G]$, then by previous conclusion, we can find uncountable $Y \in V$ such that $Y \subset C$. Then the closure of Y is a club in V and contained in C. And if $S \subset \omega_1$ is stationary, then S intersects every club in V. By previous conclusion, S intersects every club in $V[G]$. So S is stationary in $V[G]$.

Proof of Fact 4.11: Given uncountable $X, Y \subset T$, it is suffice to find uncountable $Z \subset T$ such that $\Delta(X) \cap \Delta(Y) \supset \Delta(Z)$. For any $\alpha < \omega_1$, pick $x_\alpha \in X, y_\alpha \in Y$ of

height $\geq \alpha$. Define level preserving map π which maps each $x_\alpha \restriction_\alpha$ to $y_\alpha \restriction_\alpha$. By Lipschitz (use Fodor's Lemma for coherent tree with no Suslin subtree), going to an uncountable subset $\Gamma \subset \omega_1$, we can assume π is tree isomorphism on $\{x_\alpha \restriction_\alpha : \alpha \in \Gamma\}$. Going to an uncountable we can assume $\{x_\alpha \restriction_\alpha : \alpha \in \Gamma\}$ and $\{y_\alpha \restriction_\alpha : \alpha \in \Gamma\}$ are antichains. So $Z = \{x_\alpha \restriction_\alpha : \alpha \in \Gamma\}$ works.
Note for any uncountable $A \subset T$, we can find an uncountable subset which is an antichain (For any $\alpha < \omega_1$, pick $a_\alpha, b_\alpha \in A$ such that $\alpha \leq \Delta(a_\alpha, b_\alpha) < ht(a_\alpha) \leq ht(b_\alpha)$. Define π' mapping each a_α to $b_\alpha \restriction_\alpha$. Then an uncountable subset of A such that π' is tree isomorphism guaranteed by Lipschitz is an antichain.

Sketch proof of Lemma 4.23: (1) Follows straightforwardly from the definition. See [20] for more details.

(2) Prove inductively on $\alpha \leq \omega_2$ that every $\{p_\xi : i < \omega_2\} \subset \mathcal{P}_\alpha$ contains a pairwise compatible subset of size ω_2. Recall GCH is true in the ground model. Now fix $\{p_\xi : i < \omega_2\} \subset \mathcal{P}_\alpha$ and $p_\xi = (p_0^\xi, \{(N_i^\xi, \beta_i^\xi) : i < m\})$ for some $m < \omega$ (WLOG, assume m is independent of ξ). For each $\xi < \omega_2$, going to a stronger condition, we can assume for any $\tau < \xi$, $p_0^\xi(\tau)$ is either the weakest element or can be viewed as a countable ordinal ξ_τ (since $\dot{\mathcal{Q}}$ is forced to be of size $\leq \omega_1$, ξ_τ can be picked to be the η such that $p_0^\xi(\tau)$ is forced to be the η-th element of $\dot{\mathcal{Q}}$). We may also assume the transitive collapses of N_i^ξ are the same for different ξ and $\bigcup_{i<m} N_i^\xi$ form a Δ-system with root X. Moreover, $\bigcup_{i<m}(N_i^\xi \setminus X) \cap \omega_2$ don't overlap and $\langle \bigcup_{i<m} N_i^\xi, X, \langle N_i^\xi : i < m\rangle\rangle$ are all isomorphic.
For $\alpha = 0$, it is straightforward to check that any two p_α and p_β are compatible.

For $\alpha = \bar{\alpha} + 1$, going to a subset, we can assume $\xi_\tau = \eta$ for some fix $\eta < \omega_1$. Then by induction we can find a subset $\Gamma \in [\omega_2]^{\omega_2}$ such that $\{p_\xi \restriction_{\bar{\alpha}} : \xi \in \Gamma\}$ are pairwise compatible. Then $\{p_\xi : \xi \in \Gamma\}$ are pairwise compatible.

For α infinite limit, if $\alpha < \omega_2$, then find a subset $\Gamma \in [\omega_2]^{\omega_2}$ and a $\sigma < \omega_2$ such that $supp(p_\xi) \subset \sigma$ for all $\xi \in \Gamma$. Then it follows from the induction hypothesis. Now assume $\alpha = \omega_2$. Going to a subset, we can assume $\{supp(p_\xi) : \xi < \omega_2\}$ forms a Δ-system with root s. Now pick $\sigma < \omega_2$ that bounds X and s. Then by induction, we can find a subset $\Gamma \in [\omega_2]^{\omega_2}$ such that $\{p_\xi \restriction_\sigma : \xi \in \Gamma\}$ are pairwise compatible. It is straightforward to check that $\{p_\xi : \xi \in \Gamma\}$ are pairwise compatible.

(3) By Lemma 4.20 and Lemma 4.21, we just need to prove the moreover part. We will prove by induction on α. Fix countable $N^* \prec H(\lambda)$ for some large regular λ. Let $N = N^* \cap H(\omega_2)$. Fix $D \in N^*$ a dense open subset of $\mathcal{P}_\alpha$. Fix $p \in \mathcal{P}_\alpha$ such that $(N, \alpha) \in \Delta_p$ if $\alpha < \omega_2$ and $(N, sup(N \cap \omega_2)) \in \Delta_p$ if $\alpha = \omega_2$. Extend p if necessary, we assume $p \in D$. We need to find some $q \in D \cap N^*$ compatible with p.

For $\alpha = 0$, it is the same as usual forcing with model as side condition (use (1) above).

For $\alpha = \bar{\alpha} + 1$, fix $G_{\bar{\alpha}}$ a generic filter for $\mathcal{P}_{\bar{\alpha}}$ and $p \restriction_{\bar{\alpha}} \in G_{\bar{\alpha}}$. Let
$E = \{r \in \dot{\mathcal{Q}}_{\bar{\alpha}}/G_{\bar{\alpha}} : r = q_0(\bar{\alpha})/G_{\bar{\alpha}}$ for some $q \in D$ such that $q \restriction_{\bar{\alpha}} \in G_{\bar{\alpha}}$ or no $q \in D$ such that $q \restriction_{\bar{\alpha}} \in G_{\bar{\alpha}}$ and $q_0(\bar{\alpha})/G_{\bar{\alpha}} \leq r\}$.

It is easy to see that E is dense. By induction hypothesis, $p \restriction_{\bar{\alpha}}$ is $(N^*, G_{\bar{\alpha}})$-generic. Then by definition of $\mathcal{P}_\alpha$, $p_0(\bar{\alpha})/G_{\bar{\alpha}}$ is $(N[G_{\bar{\alpha}}], \dot{\mathcal{Q}}_{\bar{\alpha}})$-generic. So there is some $r \in E \cap N^*[G_{\bar{\alpha}}]$ such that $r \not\perp p_0(\bar{\alpha})/G_{\bar{\alpha}}$. Note the second case in definition of E won't happen for r. Let $q \in D \cap N^*[G_{\bar{\alpha}}]$ such that $r = q_0(\bar{\alpha})/G_{\bar{\alpha}}$. Then $q \in N^*$ since $D \in V$. To show $q \not\perp p$, fix $p' \in G_{\bar{\alpha}}$ extends both $p \restriction_{\bar{\alpha}}$ and $q \restriction_{\bar{\alpha}}$ and $p' \Vdash r \not\perp p_0(\bar{\alpha})/G_{\bar{\alpha}}$. Then $(p'_0{}^\frown r', \Delta_{p'} \cup \Delta_p)$ extends both p and q where r' is forced by $p' \restriction_{\bar{\alpha}}$ to be stronger than r and $p_0(\bar{\alpha})$.

For α infinite limit, suppose first $cf(\alpha) = \omega$. Then we can find some $\sigma \in N^* \cap \alpha$ above $supp(p) = \{\xi < \alpha : p(\xi)$ is not the weakest element.$\}$. Let G_σ be $\mathcal{P}_\sigma$-generic and $p \restriction_\sigma \in G_\sigma$. Then by induction hypothesis, $p \restriction_\sigma$ is $(N^*, \mathcal{P}_\sigma)$-generic. So

$$N^*[G_\sigma] \models \exists q \in D q \restriction_\sigma \in G_\sigma \wedge supp(q) \subset \sigma.$$

Pick such a $q \in N^*[G_\sigma]$. Then $q \in N^*$ since $q \in D$. Then $q \not\perp p$ since $q \restriction_\sigma \not\perp p \restriction_\sigma$ (both in G_σ) and their supports are below σ.

Now suppose $cf(\alpha) > \omega$. Fix a $\sigma \in N \cap \alpha$ such that

(i) for any $(N', \xi) \in \Delta_p$ such that $\Delta_{N'} < \Delta_N$, $sup(N' \cap N \cap \alpha) < \sigma$;

(ii) σ bounds $supp(p) \cap sup(N \cap \alpha)$.

Like in case $cf(\alpha) = \omega$, we can find $q \in D$ such that $q \restriction_\sigma \not\perp p \restriction_\sigma$. Let $r \in \mathcal{P}_\sigma$ extending both $q \restriction_\sigma$ and $p \restriction_\sigma$. Let $r' = (r'_0, \Delta_r \cup \Delta_p \cup \Delta_q) \in \mathcal{P}_\alpha$ where $supp(r') = supp(r) \cup supp(q) \cup supp(p)$, $r' \restriction_\sigma = r \restriction_\sigma$, for $\tau \in supp(q) \setminus \sigma$, $r' \restriction_\tau \Vdash_{\mathcal{P}_\tau}$ "$r'_0(\tau) \leq q_0(\tau)$ and is $(N'[\dot{G}_\tau], \dot{\mathcal{Q}}_\tau)$-generic for N' such that $\delta_{N'} \geq \delta_N$ and $(N', \xi) \in \Delta_p$ for some ξ" (note $\dot{\mathcal{Q}}_\tau$ is either c.c.c. or has form $\mathcal{P}_S$, this $r'_0(\tau)$ can be chosen by Lemma 4.20) and for $\tau \in supp(p) \setminus \sigma$, $r'_0(\tau) = p_0(\tau)$.

Let's now check that $r' \in \mathcal{P}_\alpha$. First note $\{M : \exists \xi (M, \xi) \in \Delta_{r'}\} = \{M : \exists \xi (M, \xi) \in \Delta_r\}$ and hence is a symmetric system. Then we just need to check the generic condition, i.e., for any $\tau \in supp(r')$, for any $(M, \xi) \in \Delta_{r'}$, if $\tau \in M$ and $\tau \leq \xi$, then $r' \restriction_\tau \Vdash_{\mathcal{P}_\tau}$ "$r'_0(\tau)$ is $(M[\dot{G}_\tau], \dot{\mathcal{Q}}_\tau)$-generic". For $\tau < \sigma$, this is true since $r' \restriction_\sigma = r \restriction_\sigma$. For $\tau \in [\sigma, sup(N \cap \alpha))$, $\tau \in supp(q)$. If $(M, \xi) \in \Delta_q$, then the generic condition is true since $r'_0(\tau) \leq q_0(\tau)$. If $(M, \xi) \in \Delta_p$, then $\delta_M \geq \delta_N$ by choice of σ (note $\tau \in M \cap N \cap \alpha$) and hence the generic condition is true by choice of $r'_0(\tau)$. For $\tau \geq sup(N \cap \alpha)$, $(M, \xi) \in \Delta_p$ since $q \in N$ and $\xi \notin N$. Then the generic condition is true since $r'_0(\tau) = p_0(\tau)$.

It is easy to see that r' extends both p and q. This finishes the $cf(\alpha) > \omega$ case since $q \in D \cap N^*$.

(4) Recall GCH is true in the ground model. So $|\mathcal{P}_{\omega_2}| = \omega_2$. Together with ω_2-c.c., we have $(2^{\omega_1})^{V^{\mathcal{P}_{\omega_2}}} \leq (|\mathcal{P}_{\omega_2}|^{\omega_1})^V = (\omega_2)^V$. So $V^{\mathcal{P}_{\omega_2}} \models 2^{\omega_1} = \omega_2$.

(5) It is suffice to prove that $\dot{G}^+$ is forced to meet every dense open subset of $\dot{\mathcal{Q}}_\alpha$ in $V^{\mathcal{P}_\alpha}$. Fix $\dot{D}$ a dense open subset of $\dot{\mathcal{Q}}_\alpha$ in $V^{\mathcal{P}_\alpha}$. Fix $p \in \mathcal{P}_{\omega_2}$. By extending p we can assume $p_0(\alpha)$ is not the weakest condition (hence forced to be generic). Since

$\dot{D}$ is dense, we can find a name $\dot{r}$ such that $p\restriction_\alpha \Vdash_{\mathcal{P}_\alpha}$ "$\dot{r} \leq_{\dot{\mathcal{Q}}_\alpha} p_0(\alpha)$ and $\dot{r} \in \dot{D}$". Then p' is obtained from p by replacing $p_0(\alpha)$ by $\dot{r}$. And $p' \Vdash_{\mathcal{P}_{\omega_2}} \dot{G}^+ \cap \dot{D} \neq \phi$.

Bibliography

[1] Stevo Todorcevic, Walks on Ordinals and their Characteristics. Progress in Mathematics, 263. Birkhäuser Verlag, Basel, 2007. vi+324 pp. ISBN: 978-3-7643-8528-6.

[2] Stevo Todorcevic. Trees and linearly ordered sets. In K. Kunen and J.E. Vaughan, editors, Handbook of Set-theoretic topology, pages 235C293. North- Holland, 1984.

[3] Stevo Todorcevic. Coherent sequences. In Matthew Foreman and Akihiro Kanamori, editors, handbook of set theory, pages 215-296, Springer, 2010.

[4] Stevo Todorcevic. Lipschitz maps on trees. Journal of the Inst. of Math. Jussieu, 6:527C556, 2007.

[5] Stevo Todorcevic. Partitioning pairs of countable ordinals. Acta. Math. 159 (1987), no. 3-4, 261-294.

[6] Justin T. Moore. A five element basis for the uncountable linear orders. Annals of Math., 163:669C688, 2006.

[7] Justin T. Moore. A solution to the L space problem. J. Amer. Math. Soc. 19:717C736, 2006.

[8] Justin T. Moore. A universal Aronszajn line. Math. Res. Lett. 16 (2009), no. 1, 121C131.

[9] Saharon Shelah, Decomposing uncountable squares to countably many chains, J. Combinatorial Theory, Ser. A 21 (1976), 110C114.

[10] Thomas Jech. Set theory. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[11] Uri Abraham and Saharon Shelah. Isomorphism types of Aronszajn trees. Israel journal of mathematics, Vol. 50, Nos. 1-2, 1985.

[12] Georg Cantor. Beitrage zur Begrundung der transfiniten Mengenlehre. Math. Ann. 46 (1895), 481-512; 49(1897), 207-246.

[13] Duro Kurepa. Ensembles Ordonnés et Ramifiés. Publ. Math. Univ. Belgrade, 4, 1-138, 1935.

[14] J. Baumgartner, J. Malitz, and W. Reinhardt. Embedding Trees in the Rationals. Proc. Nat. Acad. Sci. U.S.A. 67 (1970), 1748C1753.

[15] Keith J. Devlin. Constructibility. Springer, Berlin, 1984.

[16] Keith J. Devlin and Håvard Johnsbråten. The Souslin problem. Lecture Notes in Mathematics, Vol. 405. Springer-Verlag, Berlin, 1974.

[17] Richard Laver. On Fraïśse's order type conjecture, Annals of Math. 93 (1971), 89-111.

[18] Ronald B. Jensen. The fine structure of the constructible hierarchy. Ann. Math. Logic 4 (1972), 229C308.

[19] Geza Fodor. Eine Bemerkung zur Theorie der regressiven Funktionen. Acta Sci. Math. Szeged 17 (1956), 139C142. MR 18,551d.

[20] David Aspero and Miguel Angel Mota. Forcing consequences of PFA together with the continuum large. Submitted.

[21] Carlos Martinez-Ranero. Contributions towards a fine structure theory of Aronszajn orderings. PhD thesis, University of Toronto, 2010.

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abstract This thesis investigates the transformation between $\mathbb{R}$-embeddable coherent trees and Countryman lines. I will show that it is consistent to have a $\mathbb{R}$-embeddable coherent tree whose lexicographical order is not Countryman and moreover it has no Countryman suborder. I will also give an equivalence relation for an $\mathbb{R}$-embeddable coherent tree to be Countryman. Some properties about basis for Countryman lines which is related to transform Countryman lines to coherent trees will be given in this thesis too.

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