CHARACTERIZATION AND COMPUTATION OF INVARIANT SETS FOR CONSTRAINED SWITCHED SYSTEMS

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DECLARATION

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

The thesis has also not been submitted for any degree in any university previously.

28/02/13 .....

Date

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S. Masood Dehghan B.

To

Dornoosh

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Summary

Standard results in the study of switched systems mostly consider unconstrained models with arbitrary switching functions. This thesis focuses on the stability of constrained switched systems when the switching function satisfies some minimal dwell-time requirement. Main contributions of the thesis include (i) a necessary and sufficient condition for the stability of switched systems when the switching function satisfies dwell-time requirement; (ii) an algorithm that computes the minimal common dwell-time needed for stability; (iii) a constructive procedure for computing the minimal mode-dependent dwell-times (mode-dependent dwell-times refers to the case where one dwell-time is used for each mode); (iv) a new characterization of robust invariant sets for dwell-time switched systems subject to disturbance inputs and constraints; and (v) algorithms that compute the minimal and maximal convex robust invariant sets under dwell-time considerations.

The above contributions are for the case where either a common dwell-time or mode-dependent dwell-times are imposed on the switched systems. They can be seen as the generalization of the special case where the system switches arbitrarily among the various modes. Finally, some of the above-mentioned theoretical results are applied to the problem of controlling the read/write head of a Hard Disk Drive (HDD) system. A mode switching control scheme with controller initialization is proposed that improves the performance of the HDD system compared to the conventional switching schemes.

Contents

List of Abbreviations

BMI	Bilinear Matrix Inequality
CADT	Constraint Admissible Dwell-Time
CADDT	Constraint Admissible Disturbance Dwell-Time
CQLF	Common Quadratic Lyapunov Function
DDT-invariance	Disturbance Dwell-Time Invariance
DOA	Domain of Attraction
DT-invariance	Dwell-Time Invariance
HDD	Hard Disk Drive
LDI	Linear Difference Inclusion
LMI	Linear Matrix Inequality
MLFs	Multiple Lyapunov Functions
MSC	Mode Switching Control
R/W head	Read/Write head
SNS	Saturated and Non-Saturated
VCM	Voice-Coil Motor

List of symbols

$A^\top$	Transpose of matrix A
$\mathbb{F}_\infty$	Minimal convex DT-invariant set
$\mathcal{I}_N$	Index set
$\mathbb{O}_\infty$	Maximal CADT-invariant set
$\mathbb{O}_\infty^\lambda$	Maximal CADT-contractive set
$\hat{P}$	One-step forward set
$\hat{P}_t$	t-step forward set
$\hat{Q}$	One-step backward set
$\hat{Q}_t$	t-step backward set
$\hat{\mathcal{Q}}_{sns}$	SNS-one-step backward set
$\hat{\mathcal{Q}}_{sns,\,t}$	SNS-t-step backward set
$\rho(A)$	Spectral radus of matrix A
σ	Switching signal
$\mathcal{S}_\tau$	Admissible switching signal
W	Disturbance set
X	Constraint set
$\mathbb{Z}^+$	Set of non-negative integers

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Chapter 1

Introduction and Review

The study of switched systems is motivated by their prevalence in numerous mechanical systems, power systems, biological systems, aircraft, traffic systems and others. For example control of read/write heads in hard disk drives requires precise positioning and rapid transitioning between tracks on a disk drive. To meet these objectives, commercial hard disk drives use switching control strategies [1–3] which combine a track-seeking controller and a track-following controller. The track-seeking controller rapidly steers the magnetic head to a neighborhood of the desired track, while the track-following controller regulates position and velocity, precisely and robustly, to enable read/write operations. This control strategy results in a switched system based on output feedback.

One main focus in the study of switched systems is to find conditions that ensure stability. This focus arises from several interesting phenomena. For example, when all the subsystems are exponentially stable, the switched system may have divergent trajectories for certain switching signals [4–6]. Another interesting example is that careful switching can stabilize a switched system with all individually unstable subsystems [5, 6]. In addition, there exists a large class of nonlinear systems which can be stabilized by switching control schemes, but cannot be stabilized by any continuous static state feedback control law [5, 7–9]. Given the wide applicability of switched

systems, the study of their stability and other analysis and design tools naturally arose.

1.1 Background

A switched system consists of a finite number of subsystems and some logical rules that govern the switching between these subsystems. The switching logic is specified in terms of a *switching signal* $\sigma(\cdot)$ that indicates the active mode of the system at any given time. In general, the active mode at time t not only depends on the time instant, but also on the current state $x(t)$ and/or previous active modes. Accordingly, switched systems are usually classified as *time-dependent* (switching depends on time t only), *state-dependent* (switching depends on state $x(t)$ as well), and with or without memory (switching also depends on the history of active modes) [5]. Switched systems can also be classified based on the dynamics of their subsystems, for example continuous-time or discrete-time, linear or nonlinear, etc. Of course, combinations of several types of switching is also possible.

A switched system with time-dependent switching is represented by

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \qquad \sigma : \mathbb{R}^+ \to \mathcal{I}_N \tag{1.1}$$

where $\mathcal{I}_N = \{1, 2, \cdots, N\}$ is a finite index set and f_i's are sufficiently regular (at least locally Lipschitz) functions. The switching signal $\sigma(\cdot)$ is a piecewise constant function that has a finite number of discontinuities, called *switching times*, on every bounded time interval. To avoid ambiguity at switching times, it is assumed that $\sigma(\cdot)$ is continuous from the right everywhere, i.e. $\sigma(t) = \lim_{h \to t^+} \sigma(h)$ for every $h \geq 0$. An example of such a switching signal for the case of $\mathcal{I}_N = \{1, 2\}$ is depicted in Figure 1.1.

In this thesis, we limit the scope of our study to the class of switched systems with linear modes and under time-dependent switching, for which a brief review of some of

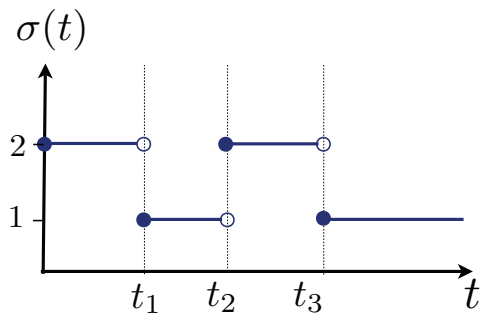


Figure 1.1: A time-dependent switching signal with switching times t_1, t_2, t_3.

the recent results are presented in this chapter. Throughout this thesis the following standard notations are used.

Notations: Given a matrix $A \in \mathbb{R}^{n\times n}$, $\rho(A)$ denotes its spectral radius. The floor function $\lfloor a \rfloor$ is the largest integer that is less than a. Symbol "$\top$" denotes the transpose of a matrix or a vector and $co\{\cdot\}$ denotes the convex-hull. Standard 2-norm is indicated by $\|\cdot\|$ while other p-norms are $\|\cdot\|_p, p = 1, \infty$. $\mathcal{B}(r) := \{x \in \mathbb{R}^n : \|x\| \leq r\}$ refers to the 2-norm ball of radius r. Positive definite (semi-definite) matrix, $P \in \mathbb{R}^{n\times n}$, is indicated by $P \succ 0 (\succeq 0)$ and I_n is the $n \times n$ identity matrix. Given a $P \succ 0$, $\mathcal{E}(P, c) := \{x : x^\top P x \leq c\}$ is the ellipsoidal set and $\mathcal{E}(P) := \{x : x^\top P x \leq 1\}$. A polyhedral set $S = \{x : Fx \leq \mathbf{1}\}$, where $F \in \mathbb{R}^{q\times n}$ is some matrix and the boldface $\mathbf{1} \in \mathbb{R}^q$ indicates the vector of all 1s. ∂S denotes the boundary of the set S. Suppose $\alpha > 0$ and $X, Y \subset \mathbb{R}^n$ are compact sets that contain 0 in their interiors. Then, the scaling of X is $\alpha X := \{\alpha x : x \in X\}$, image of X is $AX := \{y : y = Ax \text{ and } x \in X\}$ for some appropriate matrix A, the Minkowski sum is $X \oplus Y := \{z \in \mathbb{R}^n : z = x + y, x \in X, y \in Y\}$, the Pontryagin (or Minkowski) difference is $X \ominus Y := \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$ and $A(X \oplus Y) = AX \oplus AY$. The distance between $x \in \mathbb{R}^n$ and a set $Y \subset \mathbb{R}^n$ is $d(x, Y) := \inf_{y\in Y} \|x - y\|$.

1.2 Switched Linear Systems

Switched systems with all subsystems described by linear differential/difference equations are called *switched linear systems*, and have attracted most of the attention in the literature [5–7, 9, 10]. In particular, switched linear systems are represented by

$$\dot{x}(t) = A_i\, x(t), \qquad t \in \mathbb{R}^+,\ i \in \mathcal{I}_N \tag{1.2a}$$

$$\text{or,}\ \ x(t+1) = A_i\, x(t), \qquad t \in \mathbb{Z}^+,\ i \in \mathcal{I}_N \tag{1.2b}$$

where $x \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathcal{I}_N$ and the origin is an equilibrium point (maybe unstable) of the system.

In what follows, an overview of the recent results relevant to the stability of switched linear systems with time-dependent switching is presented. First, we focus on the stability of switched systems when switching among different modes is arbitrary, and highlight some necessary and sufficient stability conditions for arbitrary switching. Then, the stability problem is studied under restricted time-dependent switching. Finally, the gaps/challenges in this field that motivated the thesis are highlighted. It should be noted that since the literature on switched linear systems is so extensive, only the main ideas and drawbacks are presented here. Interested readers can find the detailed discussions of the results in the survey papers [8, 9, 11, 12] and the references therein.

1.2.1 Stability Analysis under Arbitrary Switching

One common question asked of a switched system is its stability conditions when there is no restriction on the switching signals. This issue is known as stability analysis under arbitrary switching and is of practical importance. For example, when multiple controllers are designed for a plant for performance enhancement, it is important that switching among these controllers does not cause instability. Clearly, this would not

be an issue if it is known a priori that system is stable under arbitrary switching.

The main tool for stability analysis of dynamical systems is the classical Lyapunov function [13, 14]. The main idea is to find a positive (norm-like) Lyapunov function $V(x(t)) > 0$ whose derivative is negative along the trajectories of the system (i.e. $\dot{V}(x(t)) < 0$)[1]. This would then implies that $x(t) \to 0$ as $t \to \infty$ and hence the origin of the system is asymptotically stable. Most of the recent works on the stability of switched linear systems is based on this method.

Consider a candidate Lyapunov function $V(x)$ that decreases along all trajectories of a switched linear system under arbitrary switching. Since the set of all arbitrary switching signals contains any constant switching signal $\sigma(t) = i$ for all $t \in \mathbb{R}$, it is concluded that such function $V(x)$ is also a Lyapunov function for each mode i of the system (1.2). Thus $V(x)$ has to be a "*common*" Lyapunov function for all the modes. It is well-known [5, 15–18] that if a common Lyapunov function exists for all the modes of a switched linear system, then the system is asymptotically stable under arbitrary switching. We now discuss different types of common Lyapunov functions proposed in the literature of switched linear systems.

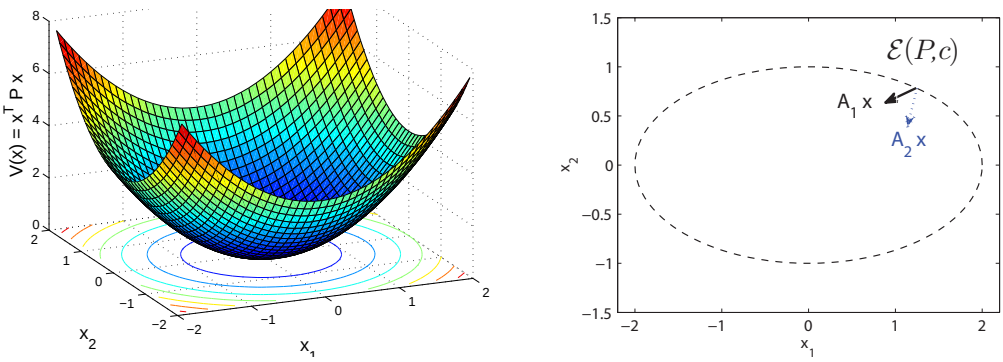
Common Quadratic Lyapunov functions: Recall that for a linear time-invariant (LTI) system $\dot{x}(t) = Ax(t)$ (respectively $x(t+1) = A\,x(t)$), the function $V(x) = x^\top P\,x$ is a quadratic Lyapunov function (QLF), if (i) P is symmetric and positive definite, and (ii) $A^\top P + PA$ (respectively $A^\top P\,A - P$) is negative definite. Similarly, for switched linear systems, the function $V(x) = x^\top P\,x$ is a common quadratic Lyapunov function (CQLF) if it is a QLF for each individual subsystem. More specifically, continuous-time switched system (1.2a) is asymptotically stable under arbitrary switching if there exists a $P \succ 0$ such that

$$PA_i + A_i^\top P \prec 0, \qquad \forall i \in \mathcal{I}_N. \tag{1.3}$$

[1]The discrete version of this condition requires $\Delta V(x(t)) := V(x(t+1)) - V(x(t)) < 0$.

Similarly, discrete-time switching system (1.2b) is asymptotically stable under arbitrary switching if there exists a $P \succ 0$ such that

$$A_i^\top P A_i - P \prec 0, \qquad \forall i \in \mathcal{I}_N. \tag{1.4}$$



(a) CQLF and its level-sets

(b) Contractive ellipsoid

Figure 1.2: Illustration of a common quadratic Lyapunov function in $\mathbb{R}^2$

The geometrical interpretation of the the above conditions is insightful. As it is shown Figure 1.2(a), the level-sets of a CQLF are the ellipsoids of the form $\mathcal{E}(P, c) = \{x : x^\top P x \leq c\}$. Condition (1.3) implies that for every point on the boundary of the ellipsoidal set $\mathcal{E}(P, c)$, the flow direction (i.e. $A_i x$, $i \in \{1, 2\}$) is pointing inwards to the set (see Figure 1.2(b)). This means when x is on the boundary of $\mathcal{E}(P, c)$, not only trajectory remains in the set but is also pushed inside with a guaranteed (boundary-crossing) speed. Since the subsystems are linear, by scaling the boundary of the set, we can see that the crossing speed implies the rate of convergence of the states to the origin. A set with such properties is called a *contractive* set.

The condition (1.3) or (1.4) is a linear matrix inequality (LMI) and can be solved using standard convex optimization routines (e.g. [19]). However, there are examples [5, 7] of switched systems that do not have a CQLF, but are exponentially stable under arbitrary switching. Hence, existence of CQLF is only a sufficient condition for stability and could be rather conservative.

Another approache resulting in CQLF, is the Lie algebraic method [4, 7, 20–22], which is based on the solvability of the Lie algebra generated by the subsystems' state matrices. It is shown that if the Lie algebra generated by the set of state matrices is solvable, then there exists a CQLF, and the switched linear system is stable under arbitrary switching.

Due to the conservatism of CQLFs, some attentions have been paid to a less conservative class of Lyapunov functions, called *switched quadratic Lyapunov functions* [23]. Basically, since every subsystem is stable, there exists a positive definite symmetric matrix $P_i \succ 0$ that solves the Lyapunov equation for each subsystem $i \in \mathcal{I}_N$. These matrices are then patched together based on the switching signals to construct a global Lyapunov function as $V(t, x(t)) = x(t)^\top P_{\sigma(t)} x(t)$. The stability condition under arbitrary switching is

$$A_i^\top P_j A_i - P_i \prec 0 \qquad \forall (i,j) \in \mathcal{I}_N \tag{1.5}$$

that grantees that $V(t+1, x(t+1)) < V(t, x(t))$ whenever the system switches from mode $i \in \mathcal{I}_N$ to mode $j \in \mathcal{I}_N$. Again, condition (1.5) is an LMI and can be solved efficiently. The geometrical interpretation of (1.5) is that the set $S := \bigcap_{i \in \mathcal{I}_N} \mathcal{E}(P_i, c)$ is a contractive, i.e. there exists a $\lambda \in (0,1)$ such that $A_i x \in \lambda S$ for every $x \in S$.

It is clear that when $P_i = P_j$ for all $i, j \in \mathcal{I}_N$, the switched quadratic Lyapunov function becomes the CQLF. Therefore, the stability criteria based on the switched quadratic Lyapunov function generalizes the CQLF approach and is less conservative. However, it is worth pointing out that the switched quadratic Lyapunov function method is still only a sufficient condition.

Polyhedral Lyapunov Functions: To obtain a condition that is both necessary and sufficient for stability of switched linear systems under arbitrary switching, a more complicated Lyapunov function than CQLF is required. This motivates the study of

non-quadratic Lyapunov functions.

The usage of non-quadratic functions has first appeared in the stability analysis of linear differential/difference inclusions (LDIs) of the form

$$\dot{x}(t) = A(t)\,x(t), \qquad A(t) \in co\{A_1, A_2, \cdots, A_N\} \tag{1.6a}$$

$$x(t+1) = A(t)\,x(t), \qquad A(t) \in co\{A_1, A_2, \cdots, A_N\} \tag{1.6b}$$

where $A(t)$ is constructed by a convex combination of A_i's. It is shown in [15] that stability of the above LDI systems, with infinite number of possible modes, is equivalent to the stability of the system when only the finite vertices $(A_i, i \in \mathcal{I}_N)$ are considered. This means stability of switched linear systems under arbitrary switching is equivalent to stability of LDI (1.6) and thus all the stability results of LDIs are also applicable to arbitrary switched systems.

For the LDIs and hence the arbitrary switched systems, it is known [15–17] that asymptotic stability is equivalent to existence of a *common* Lyapunov function (not necessarily quadratic) that is strictly convex and its directional derivative[2] along $A_i\,x$ is negative for all $i \in \mathcal{I}_N$. This statement also suggests that more complicated functions than quadratic functions should be used.

The first non-quadratic function described here is the class of Polyhedral Lyapunov Functions (PLFs), which are also known as piecewise linear Lyapunov functions. A PLF is defined by

$$V(x) = \max\{F_j\,x : j = 1, 2, \cdots, q\} \tag{1.7}$$

where $F_j \in \mathbb{R}^{1\times n}$ for $j = 1, \cdots, q$ and the linear functions $F_j\,x$ are called generators of

[2]When function $V(x)$ is not differentiable, directional derivative are used. The directional derivative of $V(x)$ in direction ζ is $\dot{V}(x;\zeta) := \lim_{h\to 0^+} \frac{V(x+h\zeta)-V(x)}{h}$.

the PLF [11]. The function $V(x)$ is induced by polyhedral set of the form

$$S = \{x : Fx \leq c\mathbf{1}\}, \quad c > 0 \tag{1.8}$$

where $F = [F_1^\top, F_2^\top, \cdots, F_q^\top]^\top \in \mathbb{R}^{q\times n}$ and $\mathbf{1} \in \mathbb{R}^q$ is a vector of all 1s. In other words, the polyhedral sets of (1.8) are the level-sets of the the PLF (1.7) as illustrated in Figure 1.3.

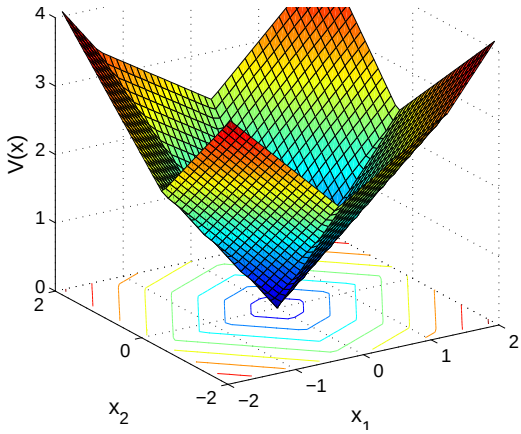


Figure 1.3: Polyhedral Lyapunov Function and its polyhedral level-sets in $\mathbb{R}^2$

It is clear that by increasing the number of generators (q) of a polyhedral set, the complexity of the PLF increases and hence it can be used as a non-conservative tool for stability analysis of switched systems. The following results, taken from [12, 24], summarizes a necessary and sufficient stability condition using PLFs.

Theorem 1.1 *Switched linear system $x(t+1) = A_i\, x(t)$, $i \in \mathcal{I}_N$ is asymptotically stable under arbitrary switching if and only if there exist $\lambda \in (0,1)$, $F \in \mathbb{R}^{q\times n}$, $q \geq n$ and non-negative matrices*[3] *$X_i \in \mathbb{R}^{q\times q}$ such that*

$$FA_i = X_i F, \qquad X_i\mathbf{1} \leq \lambda\mathbf{1}, \qquad \forall i \in \mathcal{I}_N \tag{1.9}$$

The above condition, simply implies that the polyhedral set $S = \{Fx \leq \mathbf{1}\}$ is con-

[3] A matrix is non-negative if all of its elements are non-negative.

tractive (with contraction rate λ). To see this, consider any $x(t)$ on the boundary of S. It follows from (1.9) that $Fx(t+1) = FA_i\,x(t) = X_iFx(t) \leq X_i\mathbf{1} \leq \lambda\mathbf{1}$ for any arbitrary $i \in \mathcal{I}_N$. This means $x(t+1) \in \lambda S$. Repeating this and noting that $\lambda < 1$ and S is bounded, $x(t) \to 0$ and asymptotic stability of the origin follows. Of course, the Lyapunov function induced by S is a PLF.

Theorem 1.2 *Switched linear system $\dot{x}(t) = A_ix(t)$ is asymptotically stable under arbitrary switching if and only if there exist $\beta > 0$, $F \in \mathbb{R}^{q\times n}$, $q \geq n$ and Metzler matrices*[4] *$Y_i \in \mathbb{R}^{q\times q}$ such that*

$$FA_i = Y_iF, \qquad Y_i\mathbf{1} \leq -\beta\mathbf{1}, \qquad \forall i \in \mathcal{I}_N \tag{1.10}$$

Condition (1.10) ensures that for any x on the boundary of $S = \{Fx \leq \mathbf{1}\}$, the directional derivative of $V(x) = \max_{j=1,\cdots,q}\{F_j\,x\}$ along the directions $A_i\,x$ is negative. To see this, consider any $x \in \partial S$. It follows that

$$\dot{V}(x; A_ix) = \max_j\{F_jA_ix\} = \max_j\{Y_{ij}Fx\} \leq \max_j\{Y_{ij}\mathbf{1}\} \leq -\beta \qquad \forall i \in \mathcal{I}_N$$

where Y_{ij} is the j-th row of matrix Y_i. This means for every $x \in \partial S$, the flow direction of A_ix is pointing inwards to the set S and hence S is a contractive set with convergence rate β (see Figure 1.4).

While the above theorems provide a necessary and sufficient set of stability conditions based on PLFs, it is generally difficult to specify "a priori" the number q of generators that are necessary for the construction of a common PLF. That is why several numerical algorithms have been developed for the construction of polyhedral Lyapunov functions. In [25], the authors propose an algorithm for difference inclusions which calculates a series of balanced polytopes converging to the level set of a common PLF after a finite number of steps. An alternative approach is given in [15, 16],

[4] A matrix is Metzler if all the off-diagonal elements are non-negative.

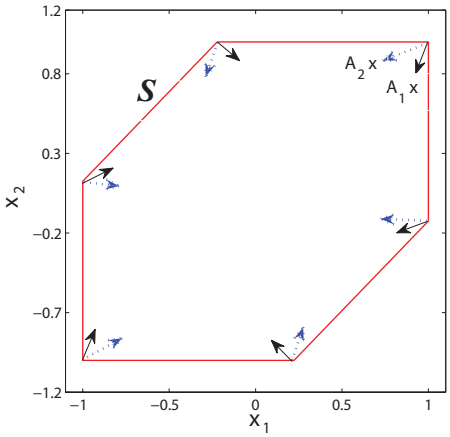


Figure 1.4: Illustration of a polyhedral contractive set S for a switched system with two modes: For all $x \in \partial S$, $A_1 x$ (solid line) and $A_2 x$ (dashed-line) points inward to the set S.

where linear programming based methods are developed for solving stability conditions. In [26], a numerical approach, called ray-griding, is suggested for the computation of PLFs based on a uniform partition of the state-space in terms of the ray directions. However, most of the above methods are applicable only to second-order or third-order systems. Blanchini and Miani [24, 27] proposed a method, based on recursive computation of backward sets of the system, that converges to a polyhedral contractive set. This method can be applied to high-dimensional systems and will be discussed in detail in the next chapter.

Composite Quadratic Lyapunov Functions: As stated earlier, piece-wise linear functions are universal tools for stability analysis, in the sense that existence of a common PLF is both necessary and sufficient for stability. It turns out that piece-wise quadratic functions can also be used as universal stability analysis tools. This is due to the fact that any polyhedral function can be arbitrarily approximated by a piece-wise quadratic function.

The usage of piece-wise quadratic Lyapunov functions has appeared recently in the analysis and design of LDIs [28–33]. One such function is the point-wise maximum of a family of quadratic functions, which is convex and homogeneous of degree two.

Since this functions is composed from a family of quadratic functions, it is also called a *composite quadratic function.*

Given s positive definite matrices $P_j \succ 0$, $j = 1, \cdots, s$, the max of quadratics is defined as $V_{max}(x) := \max\{x^\top P_j\, x : j = 1, \cdots, s\}$. and its level set is the intersection of the ellipsoids $\cap_{j=1}^{s} \mathcal{E}(P_j)$. A necessary and sufficient stability condition using composite quadratic functions is stated next.

Theorem 1.3 *[33] The switched system $\dot{x}(t) = A_i\, x(t)$, $i \in \mathcal{I}_N$ is asymptotically stable under arbitrary switching if and only if there exist an integer $s \geq N$, matrices $P_j \succ 0$ for $j = 1, \cdots, s$, and non-negative numbers $\eta_{ijk} \geq 0$, $i \in \mathcal{I}_N$, $j, k \in \{1, \cdots, s\}$ such that*

$$A_i^\top P_j + P_j A_i \prec \sum_{j \neq k} \eta_{ijk}\,(P_k - P_j) - \beta P_j \qquad \forall\, i, j, k \tag{1.11}$$

To understand this condition, we expand it for the case where $\mathcal{I}_N = \{1, 2\}$ and only two ellipsoids are used ($s = 2$). Then, (1.11) becomes

$$\begin{aligned}
A_1^\top P_1 + P_1 A_1 &\prec \eta_{112}\,(P_2 - P_1) - \beta P_1 && i = 1, j = 1, k = 2 \\
A_1^\top P_2 + P_2 A_1 &\prec \eta_{121}\,(P_1 - P_2) - \beta P_2 && i = 1, j = 2, k = 1 \\
A_2^\top P_1 + P_1 A_2 &\prec \eta_{212}\,(P_2 - P_1) - \beta P_1 && i = 2, j = 1, k = 2 \\
A_2^\top P_2 + P_2 A_2 &\prec \eta_{221}\,(P_1 - P_2) - \beta P_2 && i = 2, j = 2, k = 1
\end{aligned}$$

In what follows, we show that $S = \mathcal{E}(P_1) \cap \mathcal{E}(P_2)$ is a contractive set (with convergence rate β). For this purpose, consider any $x \in \partial S$. The directional derivative of $V_{max}(x)$ is given by

$$\dot{V}_{max}(x; A_i x) = \max\{x^\top (A_i^\top P_j + P_j A_i) x : j \in \{j : V_{max}(x) = V_j(x)\}, i \in \mathcal{I}_N\}$$

For every $x \in \partial S$ such that $x^\top P_1 x < x^\top P_2 x$, it follows that $V_{max}(x) = V_2(x)$ and hence

$\dot{V}_{max}(x; A_i x) = \max_i\{x^\top(A_i^\top P_2 + P_2 A_i)x\}$. From second and fourth inequality above it follows that $\dot{V}_{max}(x; A_i x) < \max_i\{\eta_{i21}\ x^\top(P_1 - P_2)x - \beta\ x^\top P_2 x\}$. Since $x^\top P_1 x < x^\top P_2 x$, $x^\top(P_1 - P_2)x < 0$ and hence $\dot{V}_{max}(x; A_i x) < -\beta\, x^\top P_2 x$. For the case when $x^\top P_2 x < x^\top P_1 x$, the same argument holds using the first and third inequalities and $\dot{V}_{max}(x; A_i x) < -\beta\, x^\top P_1 x$. Finally, when $x^\top P_2 x = x^\top P_1 x$, in all the four inequalities $x^\top(P_2 - P_1)x = 0$, and $\dot{V}_{max}(x; A_i x) < -\beta\ x^\top P_j x < 0$ for all $i \in \mathcal{I}_N$ and for all $j = 1, 2$. Thus $\dot{V}_{max}(x; A_i x) < -\beta\, V_{max}(x)$ for all $x \in \partial S$ and hence S is contractive (with convergence rate β).

Theorem 1.4 *[33] Switched system $x(t+1) = A_i x(t)$, $i \in \mathcal{I}_N$ is asymptotically stable under arbitrary switching if and only if there exist an integer $s \geq N$, $P_j \succ 0$, $j = 1, \cdots, s$ and non-negative numbers $\eta_{ijk} \geq 0$, $j, k \in \{1, \cdots, s\}$ and $\lambda \in (0, 1)$ such that $\sum_{j=1}^{s} \eta_{ijk} < 1$ and*

$$A_i^\top P_j\, A_i - \lambda \sum_{j \neq k} \eta_{ijk}\, P_k \prec 0 \qquad \forall\, i, j, k \tag{1.12}$$

Similar to the discussion above, condition (1.12) implies that the set $S = \bigcap_{j=1}^{s} \mathcal{E}(P)$ is contractive (with contraction rate λ).

The necessary and sufficient conditions of the above theorems is not surprising since any polyhedral function can be arbitrarily approximated by a piece-wise quadratic function provided that number of ellipsoids (s) is sufficiently large. As shown in [33], the number of ellipsoids (s) required for stability is equivalent to the number of piece-wise linear generators (q) of polyhedral Lyapunov functions appeared in Theorem 1.1 or Theorem 1.2. Similar to PLFs, the number of piece-wise quadratic functions (s) required is not known a priori. In addition, stability conditions (1.11) or (1.12) are bilinear matrix inequalities (BMIs) due to the existence of product of unknown variables, e.g. $\eta_{ijk} \times P_j$. Solving BMI problems obtained from composite quadratic functions, are much harder than the LMI conditions obtained using CQLFs. BMI problems are

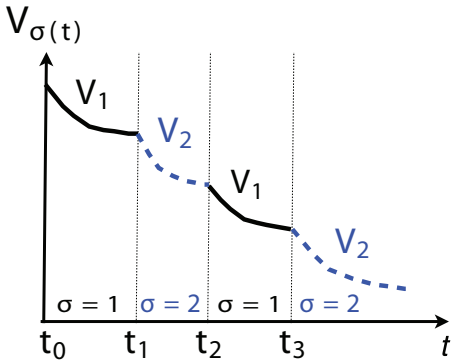
known to be NP-hard [19] and heuristic algorithms which involve approximation and local search should be used for solving them. In summary, finding non-conservative stability results, using piece-wise linear/quadratic functions, is computationally expensive.

1.2.2 Stability Analysis under Restricted Switching

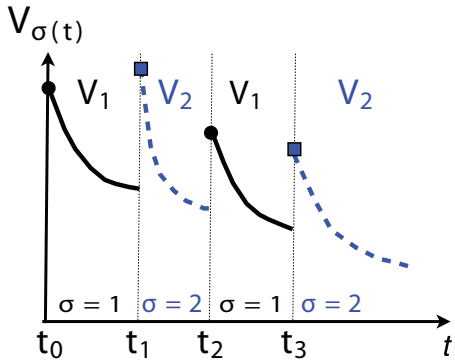
When a switched system is unstable under arbitrary switching, it is still possible to preserve stability of the origin by imposing some restrictions on the switching signal. The restrictions on switching may either be in time domain (e.g. dwell-time and average dwell-time) or in state space (e.g restrictions imposed by partitions of the state space). This section considers the stability problem of switched systems under restricted time-dependent switching and reviews the class of Multiple Lyapunov functions as their main stability analysis tool.

Multiple Lyapunov functions: The stability analysis under restricted switching is usually pursued in the framework of *Multiple Lyapunov Functions* (MLFs). The basic idea of MLFs is to concatenate several Lyapunov functions to construct a non-conventional Lyapunov function. The non-conventionality is in the sense that the MLF may not be monotonically decreasing along the state trajectories, may have discontinuities, and may only be piecewise differentiable. The reason for considering MLFs is that conventional Lyapunov functions may not exist for switched systems with restricted switching. For such cases, one may still construct a collection of Lyapunov-like functions, which only require local negativity of derivatives for certain subsystems/regions instead of global negativity [8].

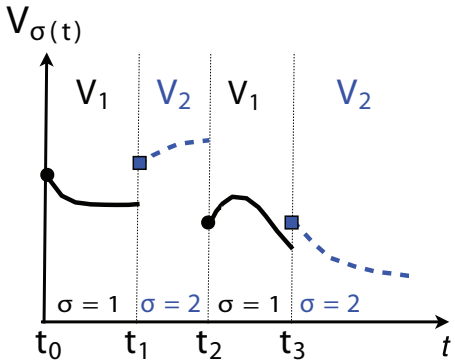
There are several versions of MLF results in the literature. A very intuitive MLF result [6] is illustrated in Fig. 1.5(a), for which the Lyapunov-like function is decreasing when the corresponding mode is active and its value does not increase at each switching



(a) V_σ at the switching instants form a decreasing sequence $V_{\sigma(t_i+1)}(x(t_{i+1})) < V_{\sigma(t_i)}(x(t_i))$.



(b) V_i at *entering times* form a decreasing sequence $V_1(t_2) < V_1(t_0), V_2(t_3) < V_2(t_1)$.



(c) MLFs increases during certain period.

Figure 1.5: Illustration of Multiple Lyapunov Functions

instant. Less conservative results can be obtained by relaxing the decreasing requirement at every switching time. For this purpose define $t_{i,k}$ to be the k-th time in which we switch into mode i of the system (i.e. $\sigma(t_{i,k}^-) \neq \sigma(t_{i,k}^+) = i$). The switched system is asymptotically stable provided that Lyapunov-like function values at every entering times to mode i, form a decreasing sequence i.e. $V_i(x(t_{i,k})) < V_i(x(t_{i,k-1}))$. The profile of typical Lyapunov functions associated with modes 1 and 2, for a switching sequence satisfying this condition, is depicted in Fig. 1.5(b). The following theorem, taken from [34], summarizes this result.

Theorem 1.5 *[34] Suppose that Lyapunov-like functions are associated for each mode*

i such that $V_i(x) > 0$ and $\dot{V}_i(x) < 0$ (respectively $\Delta V_i(x) < 0$). In addition suppose that $\sigma(t)$ is a switching sequence such that

$$V_i(x(t_{i,k})) < V_i(x(t_{i,k-1})) \qquad \textit{for all } i \in \mathcal{I}_N \tag{1.13}$$

where $t_{i,k}$ is the k-th time that vector field f_i is "switched in". Then the origin of the switched system $\dot{x}(t) = f_{\sigma(t)}x(t)$ (respectively $x(t+1) = f_{\sigma(t)}x(t)$) is asymptotically stable.

The above stability condition can be further relaxed, by letting the Lyapunov-like functions to increase their values in between switching times, provided that the increment is bounded by certain classes of functions [10,35]. This scenario is illustrated in Fig. 1.5(c).

While useful for stability analysis of both continuous-time and discrete-time switched systems, MLFs theorems have their drawbacks. First, applying MLFs theorems requires some information about the solutions of the system. Namely, the values of suitable Lyapunov functions at switching times must be known, which in general requires the knowledge of the state at these times. This is in contrast to the classical Lyapunov stability results, which do not require the knowledge of the state solutions. Second, extraction of the level-sets of MLFs is not obvious. Unlike common Lyapunov functions that their level-sets are convex and well-defined, the level-sets of MLFs have no clear structure. Finally, like most of the Lyapunov based methods, the stability results based on MLFs theorems are only sufficient conditions and may be rather conservative.

1.2.3 Stability under Time-Dependent Switching

It is well known that a switched system is stable if all individual subsystems are stable and the switching is sufficiently slow, so as to allow the transient effects to dissipate after

each switch [7, 36–38]. In this section we discuss how this property can be formulated and justified using multiple Lyapunov function techniques [9, 36–39].

The simplest way to specify slow switching is to introduce a number $\tau > 0$ and restrict the class of admissible switching signals with switching times $t_1, t_2, \cdots, t_k, \cdots$ to satisfy the inequality $t_{k+1} - t_k \geq \tau$ for all k. This number τ is usually called the *dwell-time* because σ "dwells" on each of its values for at least τ units of time. The set of all switching signals that satisfies the dwell-time restriction τ is denoted by $\mathcal{S}_\tau$.

Computation of dwell-time: When all subsystems are asymptotically stable, the lower bound on τ required for stability can be calculated using multiple Lyapunov functions. The following theorem, taken from [38], states a sufficient condition for stability of switched systems under dwell-time switching.

Theorem 1.6 *[38] Consider the switched system $\dot{x}(t) = f_{\sigma(t)}x(t)$ (respectively $x(t+1) = f_{\sigma(t)}x(t)$), where $f_i(0) = 0$, for all $i \in \mathcal{I}_N$. If there exist Lyapunov-like functions $V_i(x) > 0$ for each $i \in \mathcal{I}_N$, $\mu \geq 1$ and $\beta > 0$ (respectively $\lambda \in (0, 1)$) such that*

$$\dot{V}_i(x(t)) \leq -\beta V_i(x(t)) \quad (\textit{respectively } V_i(x(t+1)) \leq \lambda V_i(x(t)) \ ) \qquad \forall i \in \mathcal{I}_N \tag{1.14}$$

$$V_i(x(t)) < \mu V_j(x(t)) \qquad \forall (i, j) \in \mathcal{I}_N \times \mathcal{I}_N \tag{1.15}$$

Then, the switched system is asymptotically stable with any dwell-time $\tau \geq \frac{\ln \mu}{\beta}$ (respectively $\tau \geq -\frac{\ln \mu}{\ln \lambda}$).

The above conditions are direct usage of MLFs theorems. To see this, consider a switching signal with switching instants $t_1, t_2, \cdots, t_k, \cdots$. Then from (1.14), it follows

that

$$V_i(x(t_{k+1})) \leq e^{-\beta(t_{k+1}-t_k)} V_i(x(t_k)), \qquad \text{if } \sigma(t) = i \quad \forall t \in [t_k, t_{k+1})$$
$$\text{or} \;\; V_i(x(t_{k+1})) \leq \lambda^{(t_{k+1}-t_k)} V_i(x(t_k)), \qquad \text{if } \sigma(t) = i \quad \forall t \in [t_k, t_{k+1})$$

When a switching happens from mode i to mode j, from (1.15) it follows that

$$V_j(x(t_{k+1})) \leq \mu V_i(x(t_{k+1})) \leq \mu e^{-\beta(t_{k+1}-t_k)} V_i(x(t_k))$$
$$\text{or} \;\; V_j(x(t_{k+1})) \leq \mu V_i(x(t_{k+1})) \leq \mu \lambda^{(t_{k+1}-t_k)} V_i(x(t_k))$$

Assuming that $t_{k+1} - t_k \geq \tau$, it is clear that if $\mu e^{-\beta\tau} < 1$ (respectively $\mu\lambda^\tau < 1$), then $V_j(x(t_{k+1}) < V_i(x(t_k))$. Therefore, when $\tau > \frac{\ln \mu}{\beta}$ (respectively $\tau > -\frac{\ln \mu}{\ln \lambda}$), it follows that $V_i(x(t_{i,k+1}) < V_i(x(t_{i,k}))$. Asymptotic stability of the origin then follows from Theorem 1.5.

The conditions of Theorem 1.6 for switched linear systems is simplified to the existence of positive definite matrices $P_i \succ 0$ for each $i \in \mathcal{I}_N$ such that

$$A_i^\top P_i + P_i A_i \preceq -\beta P_i \quad (\text{respectively } A_i^\top P_i A_i \preceq \lambda P_i) \qquad \forall i \in \mathcal{I}_N \tag{1.16}$$
$$P_i \prec \mu P_j \qquad \forall (i,j) \in \mathcal{I}_N \times \mathcal{I}_N \tag{1.17}$$

Assuming that linear subsystems are all asymptotically stable, the first inequality is always feasible. A feasible value of parameter $\mu \geq 1$ that satisfies the second condition is $\mu = \max_{i,j}\{\frac{\lambda_{max}(P_i)}{\lambda_{min}(P_j)}\}$, where $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ denotes the maximum and minimum eigen-value of a matrix.

The above analysis justifies that when all the subsystems are asymptotically stable, there indeed exists a scalar τ such that the switched system is exponentially stable if the dwell time is larger than τ. An important and challenging problem under dwell-time switching, is to determine a minimal dwell time $\tau_{\min} > 0$ such that the origin

of the switched system is globally asymptotically stable under the time-dependent switching signal $\sigma(t) \in \mathcal{S}_{\tau_{\min}}$. Unfortunately, the conditions appeared in (1.16) - (1.17) are nonlinear in terms of variables β (or λ), μ and P_i's. Hence, finding the optimal values that results in the minimal τ is not easy. Even if optimal values are obtained, there is no guarantee that the resulting τ is the minimum since the conditions (1.16) - (1.17) are only sufficient stability conditions. Hence, several other approaches for computing an upper bound of $\tau_{\min}$ are proposed in the literature. Some of the notable results are reviewed next.

A simple method for computing an upper bound of $\tau_{\min}$ of switched linear systems is based on the exponential decay bounds on the transition matrices of the individual LTI subsystems. Due to the asymptotic stability of the linear subsystems, there exist positive constants α, β such that $\|e^{A_i(t-\bar{t})}\| \leq \alpha e^{-\beta(t-\bar{t})}$ for all $t \geq \bar{t} \geq 0$ and for all $i \in \mathcal{I}_N$. The constant β can be viewed as a common stability margin for all subsystems $A_i, i \in \mathcal{I}_N$. With $t_1, t_2, \cdots, t_k$ being the switching times in the interval (t_0, t), the solution of $\dot{x}(t) = A_{\sigma(t)}x(t)$ is

$$x(t) = e^{A_{\sigma(t_k)}(t-t_k)} e^{A_{\sigma(t_{k-1})}(t_k-t_{k-1})} \cdots e^{A_{\sigma(t_1)}(t_2-t_1)} e^{A_{\sigma(t_0)}(t_1-t_0)} x(t_0) \tag{1.18}$$

Assuming that $\sigma \in \mathcal{S}_\tau$, it follows that $t_{k+1} - t_k \geq \tau$ for all k and hence

$$e^{A_{\sigma(t_{k-1})}(t_k-t_{k-1})} \leq \alpha e^{-\beta(t_k-t_{k-1})} \leq \alpha e^{-\beta\tau}$$

To ensure asymptotic stability it is sufficient to have $\alpha e^{-\beta\tau} < 1$, which can be achieved with any $\tau \geq \frac{\ln \alpha}{\beta}$.

Similarly for discrete-time switched systems, there exist positive scalars α and λ with $\lambda \in (0, 1)$ such that $\|A_i^k\| < \alpha\lambda^k$ for all $k \in \mathbb{Z}^+$ and for all $i \in \mathcal{I}_N$. Again the constant λ can be viewed as a common stability margin for all subsystems $A_i, i \in \mathcal{I}_N$. With $t_1, t_2, \cdots, t_k$ being the switching times in the interval (t_0, t), the solu-

tion of discrete-time system at time t is $x(t) = A_{\sigma(t-1)}A_{\sigma(t-2)}\cdots A_{\sigma(1)}A_{\sigma(0)}x(t_0) = A_{\sigma(t_k)}^{t-t_k}A_{\sigma(t_{k-1})}^{t_k-t_{k-1}}\cdots A_{\sigma(t_0)}^{t_1-t_0}x(t_0)$. It follows that

$$\begin{aligned}\|x(t)\| &= \|A_{\sigma(t_k)}^{t-t_k}A_{\sigma(t_{k-1})}^{t_k-t_{k-1}}\cdots A_{\sigma(t_0)}^{t_1-t_0}x(t_0)\| \\ &< \alpha^{(t/\tau)}\lambda^t\|x(t_0)\| = \left(\alpha^{1/\tau}\lambda\right)^t\|x(t_0)\|\end{aligned}$$

The switched system is asymptotically stable with dwell-time τ provided that $\alpha^{1/\tau}\lambda < 1$ or equivalently $\tau > -\frac{\ln\alpha}{\ln\lambda}$.

The following theorem, taken from [36,38,40], provides an upper for minimal dwell-time required for stability of switched linear systems.

Theorem 1.7 *For each $i \in \mathcal{I}_N$, let $\mathcal{T}_i := \inf_{\alpha>0,\beta>0}\left\{\frac{\ln\alpha}{\beta} : \|e^{A_i t}\| \le \alpha e^{-\beta t}, \forall t \ge 0\right\}$ for continuous-time subsystems [38, 40] and let $\mathcal{T}_i := \inf_{\alpha>0,0<\lambda<1}\left\{-\frac{\ln\alpha}{\ln\lambda} : \|A_i{}^t\| \le \alpha\lambda^t, \forall t \ge 0\right\}$ for discrete-time subsystems [36]. Define $\mathcal{T} := \max_{i\in\mathcal{I}_N}\mathcal{T}_i$. Then, switched linear system $\dot{x}(t) = A_{\sigma(t)}x(t)$ (correspondingly $x(t+1) = A_{\sigma(t)}x(t)$) is asymptotically stable under dwell-time switching with any dwell-time $\tau \ge \mathcal{T}$.*

Since only $\|\cdot\|$ of states is considered, the minimal dwell-time obtained from the above result is rather conservative. Recently, Geromel and Colaneri [41, 42] propose a less conservative upper bound on the minimal dwell-time based on MLFs.

Theorem 1.8 *Assume that for some $\mathcal{T} > 0$, there exists a collection of positive definite matrices $P_i \succ 0, i \in \mathcal{I}_N$ of compatible dimensions such that*

$$A_i^\top P_i + P_i A_i \prec 0 \qquad \forall i \in \mathcal{I}_N \tag{1.19a}$$

$$e^{A_i^\top \mathcal{T}}\, P_j\, e^{A_i \mathcal{T}} - P_i \prec 0 \qquad \forall i \ne j, (i,j) \in \mathcal{I}_N \times \mathcal{I}_N \tag{1.19b}$$

Then, the switched system $\dot{x}(t) = A_{\sigma(t)}x(t)$ with dwell-time $\tau \ge \mathcal{T}$ is globally asymptotically stable [41].

Similarly, if for some $\mathcal{T} \geq 1$, there exists a collection of positive definite matrices $P_i \succ 0$, $i \in \mathcal{I}_N$ of compatible dimensions such that

$$A_i^\top P_i A_i - P_i \prec 0 \qquad \forall i \in \mathcal{I}_N \tag{1.20a}$$

$$A_i^{\mathcal{T}\top} P_j A_i^{\mathcal{T}} - P_i \prec 0 \qquad \forall i \neq j, (i,j) \in \mathcal{I}_N \times \mathcal{I}_N \tag{1.20b}$$

Then, the discrete-time switched system $x(t+1) = A_{\sigma(t)}x(t)$ with dwell-time $\tau \geq \mathcal{T}$ is globally asymptotically stable [42].

The upper bound of $\tau_{\min}$ obtained from these theorems is the minimum $\mathcal{T}$ such that conditions of (1.19) or (1.20) are satisfied. Again, these conditions are nonlinear with respect to the variables $\mathcal{T}$ and P_i's. However, for a fixed value of $\mathcal{T}$, optimization problems (1.19) and (1.20) are LMIs and can be solved using convex optimization algorithms. The minimum $\mathcal{T}$ that satisfy these LMIs can be found using a bisection search on $\mathcal{T}$.

Despite the various methods proposed in the literature, a constructive procedure for choosing the candidate Lyapunov functions that minimizes the dwell-time needed for stability is still lacking. As an alternative solution to this problem, several relaxations to the dwell-time concept are proposed. One is the use of average dwell-time [38] instead of strict dwell-time requirement at each switching instant. In the context of time-dependent switching, specifying a fixed dwell-time may be rather restrictive. If, after a switch occurs, there can be no more switches for the next τ units of time, then it is impossible to react to possible system failures during that time interval. Thus it is of interest to relax the concept of dwell-time, allowing the possibility of switching fast when it is necessary and then compensate for it by switching sufficiently slower later. The concept of average dwell time from [38] serves this purpose. Denote the number of switches in $\sigma(\cdot)$ in an interval (t, T) by $N_\sigma(t, T)$. We say that σ has an *average*

dwell-time τ_a, if there exist two positive numbers N_0 and τ_a such that

$$N_\sigma(t,T) \leq N_0 + \frac{T-t}{\tau_a} \tag{1.21}$$

The constant τ_a is called the *average dwell-time*, N_0 the *chatter bound* and $\mathcal{S}_{\tau_a,N_0}$ the set of all switching signals that satisfies the average dwell-time condition. Average dwell-time is less restrictive that the dwell-time condition. In fact, $N_0 = 1$ in (1.21) implies that σ cannot switch twice on any interval of length smaller than τ_a. Switching signals with this property are exactly the switching signals with dwell time τ_a. In general, if we discard the first N_0 switches, then the average time between consecutive switches is at least $\tau_a \geq \frac{T-t}{N_\sigma(t,T)-N_0}$. The constant N_0 affects the overshoot bound for Lyapunov stability but otherwise does not change stability properties of the switched system [7, 9].

In addition, it is possible to extend the results to the case where both stable and unstable subsystems coexist. When one considers unstable dynamics, slow switching (i.e., long enough dwell or average dwell time) is not sufficient for stability; it is also required to make sure that the switched system does not spend too much time in the unstable subsystems. The reason to consider unstable subsystems in switched systems is because there are cases where switching to unstable subsystems becomes unavoidable; e.g., when a failure occurs, or there are packet dropouts in communication. Sufficient conditions for the stability of the switched systems with unstable modes have appeared in [36]. Several other extensions and refinements on the dwell-time stability are also appeared in the literature [9, 36, 37, 41–46]. However, these results are conservative as they are based on MLFs which are sufficient conditions for stability. Necessary and sufficient conditions for stability of switched systems under dwell-time switching and procedures for computing minimal dwell-time need for stability are still lacking.

1.3 Motivation

As discussed in the previous section, despite the extensive work in the field of switched linear systems with time-dependent switching, there are some challenges that have not been studied thoroughly. Some of these issues, to be discussed in the thesis, are as follows:

(I1) **Necessary and sufficient condition for stability of switched linear systems under dwell-time switching:** While there has been much progress on stability analysis of switched systems under arbitrary switching [12,15,16,32,33], the work on dwell-time switching is much less. In literature, only sufficient stability conditions (based on MLFs) has been derived [36–38, 41, 44, 45, 47]. This motivates Chapter 2, in which a necessary and sufficient condition for stability under dwell-time switching is presented. The result is based on the polyhedral Dwell-Time contractive sets that can be seen as the generalization of polyhedral contractive sets appeared for arbitrary switching.

(I2) **Computation of the minimal dwell-time needed for stability:** Several approaches for computation of dwell-time [36, 41, 42, 46] and/or relaxation of the dwell-time needed for stability [38, 45, 47] are proposed in the literature. However, they all provide an upper bound on the minimal dwell-time needed for stability. Thus, a constructive procedure for computation of the $\tau_{\min}$ is still lacking [11,48]. This problem is addressed in Chapter 2 by providing an algorithm for the computation of the minimal dwell-time. In addition, relaxation of the dwell-time requirement is discussed in Chapter 3, by imposing a dwell-time for each mode of the system instead of one common dwell-time for all modes. A constructive procedure for computation of mode-dependent dwell-times is also discussed in this chapter.

(I3) **Effect of state and control constraints on switched systems:** Most of the real-world systems have physical constraints on states and/or control inputs. When constraints are present, one major focus of research is the characterization of invariant sets that are constraint admissible. A typical candidate for such sets is the level-set of the corresponding Lyapunov functions that is inside the constraint set [49]. While the level-sets of common Lyapunov functions used for arbitrary switched systems are convex and well-defined, the level-sets of MLFs have no clear structure. This motivates Chapters 2 and 4, which provide a new characterization for constraint admissible Dwell-Time invariant sets. These sets are constraint admissible at all times and invariant for every admissible switching that satisfies the dwell-time consideration. The case of control constraints is also covered in Chapter 5.

(I4) **Effect of disturbance on dwell-time switched systems:** Study of effect of disturbance on dynamical systems is crucial as it defines the robustness of the system with respect to the disturbances. This problem, which is only partially addressed in [46,50,51], is quite challenging for dwell-time switched systems since the effect of both exogenous disturbances and switching signals should be considered. Chapter 3 presents a complete characterization of robustly invariant sets and provides algorithms for computation of maximal and minimal robust invariant sets. These sets are important for obvious practical reasons. For example, the minimal robust invariant set characterizes the asymptotic behavior of the system due to the combined effect of switching and the exogenous disturbance input; while the maximal robust invariant set is used to ensure the satisfaction of physical state constraints, the violation of which can be detrimental in some applications.

1.4 Objectives and Scopes

The objective of this research is to develop tools for stability analysis and evaluation of effect of disturbances on discrete-time switched linear systems under dwell-time switching when they are subjected to constraints. The scope of the thesis will cover the following issues:

- Necessary and sufficient conditions for stability of switched systems under dwell-time switching.
- Algorithms for computation of the minimal common dwell-time needed for stability.
- Relaxation of the dwell-time requirement by imposing a dwell-time for each mode of the system instead of one single dwell-time for all modes and a constructive procedure for computation of a set of non-conservative (minimal under some conditions) mode-dependent dwell-times.
- Characterization and computation of constraint admissible invariant sets for dwell-time switched systems in the presence of constraints and disturbance inputs.
- Applying some of the above mentioned theoretical results, to the problem of controlling the read/write head of a Hard Disk Drive (HDD) system and showcase the performance improvement obtained using the proposed switching controller.

1.5 Thesis Organization

The rest of the thesis is organized as follows. Chapter 2 includes the characterization and computation of contractive sets for dwell-time switching systems. Based on this characterization, a necessary and sufficient condition for asymptotic stability and a

procedure for computation of the minimal dwell-time needed for asymptotic stability is provided. The relaxation of dwell-time requirement is described in Chapter 3 by introducing the mode-dependent dwell-times. Necessary and sufficient conditions for stability under such conditions and a constructive procedure of computing the minimal mode-dependent dwell-times are also discussed. Chapter 4 considers the characterization of robust invariant sets (robustness with respect to disturbance inputs) for dwell-time switching systems. Computation and convergence of the maximal and the minimal robust invariant sets are also discussed in this chapter. Chapter 5 describes the computation of domain of attraction of switched systems where the control input is subjected to saturation nonlinearity. The results of previous chapters are applied to the problem of controlling a HDD system in Chapter 6. A switching strategy with controller initialization that improves the performance of HDD is also proposed in this chapter. Finally, Chapter 7 summarizes the research contributions and describes the possible future works.

Chapter 2

Characterization and Computation of Contractive Sets

This chapter introduces the concepts of Dwell-Time invariant/contractive (DT-invariant/contractive) set and maximal constraint admissible DT-invariant set for discrete-time switching systems under dwell-time switching. Main contributions of this chapter include a characterization of DT-invariant/contractive sets; a numerical algorithm for the computation of the maximal constraint admissible DT-invariant set; a necessary and sufficient condition for asymptotic stability of the switching systems under dwell time switching and an algorithm for the computation of the minimal dwell-time needed for asymptotic stability.

2.1 Introduction

This chapter considers the following constrained discrete-time switched linear system:

$$x(t+1) = A_{\sigma(t)}\, x(t), \tag{2.1a}$$

$$x(t) \in X, \quad \forall t \in \mathbb{Z}^+ \tag{2.1b}$$

where $x(t) \in \mathbb{R}^n$ is the state variable and $\sigma(t) : \mathbb{Z}^+ \to \mathcal{I}_N := \{1, \cdots, N\}$ is a time-dependent switching signal that indicates the current active mode of the system among N possible modes in $\mathcal{A} := \{A_1, \cdots, A_N\}$. The constraint set $X \subset \mathbb{R}^n$ models physical state constraints imposed on the system, including those arising from the actuator via some appropriate state feedback if (2.1) is seen as a feedback system.

2.2 Preliminaries

This section begins with a review of the definitions of switching time, dwell time and admissible switching sequence/function. Suppose $t_0, t_1, \cdots, t_k, \cdots$ are the switching instants of (2.1) with $t_0 = 0$ and $t_k < t_{k+1}$. By definition, this means that $\sigma(t_k) \neq \sigma(t_{k+1})$ and $\sigma(t_k) = \sigma(t_k + 1) = \cdots = \sigma(t_{k+1} - 1)$ for all $k \in \mathbb{Z}^+$.

Definition 2.1 *An admissible switching sequence of system (2.1), $\mathcal{S}_\tau(t) = \{\sigma(t-1), \cdots, \sigma(0)\}$, with switching instants $t_0, t_1, \cdots, t_k, \cdots$ has a dwell time of τ means that $t_{k+1} - t_k \geq \tau$ for all $k \in \mathbb{Z}^+$. In addition, suppose t_{last} is the last switching time for an admissible sequence $\mathcal{S}_\tau(t)$, then $t - t_{last} \geq \tau$.*

Remark 2.1 *As defined, the dwell time condition corresponds to the minimal duration of stay in each mode required of the system. The last condition in Definition 2.1 requires further qualification. Suppose $\mathcal{A} = \{A_1, A_2\}$ and $\tau = 3$ then $\mathcal{S}_3(6) = \{1, 1, 1, 2, 2, 2\}$ is an admissible sequence. However, $\bar{\mathcal{S}}_3(6) = \{1, 1, 2, 2, 2, 2\}$ is not an admissible sequence because $t - t_{last} < 3$ and the dwell time consideration may be violated if $\sigma(6) = 2$. On the other hand, if $\sigma(6) = 1$ means $\bar{\mathcal{S}}_3(6)$ is a truncated subsequence of an admissible sequence. This is a key point that distinguishes systems under dwell time consideration and under arbitrary switching. Following the same reasoning, $\mathcal{S}_\tau(t)$ for $t < \tau$ is also not meaningful.*

Throughout this thesis, system (2.1) is assumed to satisfy the following assumptions:

(A1) The spectral radius of each individual subsystem $A_i, i \in \mathcal{I}_N$ is less than 1;

(A2) The constraint set X is a polytope represented by $X = \{x : Rx \leq \mathbf{1}\}$ for some appropriate matrix $R \in \mathbb{R}^{q \times n}$;

(A3) (A_i, R) is observable for at least one $A_i \in \mathcal{A}$;

(A4) A value of $\tau \geq 1$ has been identified such that the origin of the unconstrained switched system (2.1) with dwell time τ is asymptotically stable.

Assumption (A1) defines the family of systems considered in this work and is a reasonable requirement. The polyhedral assumption of (A2) is made to facilitate numerical computations of the invariant set of (2.1). Assumption (A3) ensures the compactness of the sets. It applies to only one $i \in \mathcal{I}_N$ since the invariance is applicable to *all* admissible sequences including one where $\sigma(k) = i$ for all $k \in Z^+$. Of course, if (A3) is not satisfied, system (2.1) can be reformulated to consider only the observable subsystem of A_i. Assumption (A4) follows from (A1) and poses minimal restriction as procedure for obtaining dwell-time that satisfies this condition is well-known [36, 42].

2.3 Main Results

This section begins with several definitions needed to precisely state the invariance condition for system with dwell time consideration. For notational convenience, $A_{\mathcal{S}_\tau(t)}$ refers to the product $\Pi_{r=0}^{t-1} A_{\sigma(r)}$ associated with the admissible sequence $\mathcal{S}_\tau(t) = \{\sigma(t-1), \cdots, \sigma(0)\}$.

Definition 2.2 *A set $\Omega \subset \mathbb{R}^n$ is said to be DT-invariant (Dwell-Time invariant) with respect to system (2.1a) with a dwell time τ if $x \in \Omega$ implies $A_{\mathcal{S}_\tau(t)} x \in \Omega$ for all admissible switching sequences $\mathcal{S}_\tau(t)$ and for all time t.*

While stating the requirement of DT-invariance for system (2.1), the above definition is of limited practical usefulness since $A_{\mathcal{S}_\tau(t)} x \in \Omega$ has to be satisfied by an infinite

number of admissible sequences for all time t. The next theorem shows how the infinite sequences can be avoided.

Theorem 2.1 *Suppose (A1) and (A4) are satisfied. A set $\Omega \subset \mathbb{R}^n$ is DT-invariant for system (2.1a) with dwell time τ if and only if for every $x \in \Omega$,*

$$A_i^t\, x \in \Omega, \qquad \text{for } t = \tau, \tau+1, \cdots, 2\tau-1, \quad \forall A_i \in \mathcal{A} \tag{2.2}$$

Proof:

(i) ($\Rightarrow$): The solution of (2.1) under an admissible switching function at time t is $x(t) = A_{\mathcal{S}_\tau(t)} x_0$ where

$$A_{\mathcal{S}_\tau(t)} = \cdots A_{i_\ell}^{k_\ell} \cdots A_{i_1}^{k_1} A_{i_0}^{k_0} \tag{2.3}$$

for some appropriate switching sequence $\mathcal{S}_\tau(t) = \{i_\ell, \cdots, i_\ell, i_{\ell-1}, \cdots, i_{\ell-1}, \cdots, i_0\}$ where $i_j \in \mathcal{I}_N$ and $k_j := t_{j+1} - t_j, j = 0, 1, \cdots, \ell$ being the corresponding duration times in each mode. Due to the dwell time requirement, each $k_j \geq \tau$. Without loss of generality, consider any of the A_i^k on the right hand side of (2.3). This term can be decomposed into a product of matrices involving A_i^τ and one matrix from $\{A_i^\tau, A_i^{\tau+1}, , \cdots, A_i^{2\tau-1}\}$. To see this, let $q = \lfloor \frac{k-\tau}{\tau} \rfloor$ with $\lfloor \cdot \rfloor$ being the floor function. Then,

$$A_i^k = (A_i^\tau)^q\, A_i^{k-q\tau} \tag{2.4}$$

Here, the superscript $k - q\tau$ of the last term corresponds to the remainder of $k - \tau$ when divided by τ and hence, assumes a value from $\{\tau, \tau+1, \cdots, 2\tau-1\}$. Consider the rightmost term of (2.4). From (2.2), it follows that $A_{i_0}^{k_0 - q_0\tau} x_0 \in \Omega$ for any $x_0 \in \Omega$. Similarly, $(A_{i_0}^\tau)^{q_0} A_{i_0}^{k_0 - q_0\tau} x_0 \in \Omega$ from (2.2). Repeating this process for the rest of the terms in (2.3) and for all admissible sequences completes the proof.

(ii) ($\Leftarrow$) Suppose there exists a $t \in \{\tau, \tau+1, \cdots 2\tau-1\}$ and some $A_i \in \mathcal{A}$ such that Ω is not invariant w.r.t. A_i^t. The sequence $\mathcal{S}_\tau(t) := \{i, i, \cdots, i\}$, which is an admissible

sequence, violates the DT-invariance of Ω. ■

An example that illustrates the proof is in order. Consider $\mathcal{A} = \{A_1, A_2\}$, $\tau = 3$ and $x(27) = A_{\mathcal{S}_\tau(27)}x_0 = A_1^8 A_2^9 A_1^{10} x_0$. Using the procedure described in the proof above, $x(27) = [A_1^3 A_1^5]\,[(A_2^3)^2 A_2^3]\,[(A_1^3)^2 A_1^4]\,x_0$. Since $A_i^t\,x \in \Omega$ for all $t = 3, 4, 5$ and for $i \in \{1, 2\}$, it follows that $A_{\mathcal{S}_\tau(27)}x_0 \in \Omega$ if $x_0 \in \Omega$.

An interesting and important connection can now be established between dwell time stability and stability under arbitrary switching for system (2.1a). While not needed for the rest of this section, this result is needed for the algorithm described in Section 2.4.

Theorem 2.2 *Consider an associated system of (2.1a) in the form*

$$\hat{x}(t+1) = \hat{A}\hat{x}(t), \qquad \hat{A} \in \{A_i^r : \textit{ for all } i \in \mathcal{I}_N \textit{ and } r = \tau, \cdots, 2\tau - 1\} \tag{2.5}$$

Then system (2.1a) is asymptotically stable with dwell time τ if and only if (2.5) is asymptotically stable under arbitrary switching.

Proof: (i) ($\Leftarrow$) We show that asymptotic stability of (2.5) implies asymptotic stability of (2.1a) with dwell time τ. It is well-known [15, 17] that (2.5) is asymptotically stable iff a polyhedral contractive set exists w.r.t. (2.5). This implies (2.5) is asymptotically stable iff a polyhedral set S and a $\lambda \in (0, 1)$ exist such that $\hat{A}x \in \lambda S$ for every $x \in S$ and for every $\hat{A} \in \{A_i^\tau, A_i^{\tau+1}, ..., A_i^{2\tau-1} \text{ for all } i \in \mathcal{I}_N\}$. Now consider an admissible switching sequence of the form (2.3) and $x(0) \in S$, it follows that

$$x(t) = A_{\mathcal{S}_\tau(t)}x(0) = \left(A_{s_n}^{k_n} \cdots A_{s_1}^{k_1} A_{s_0}^{k_0}\right) x(0) \in \lambda^{\bar{k}} S \tag{2.6}$$

where $\bar{k} := \lfloor k_0/\tau \rfloor + \lfloor k_1/\tau \rfloor + \cdots + \lfloor k_n/\tau \rfloor$. The rightmost condition of (2.6) follows from the fact that all $k_j \geq \tau$ and that $x(0) \in S$ implies $A_i^t\,x(0) \in \lambda S$ for all $i \in \mathcal{I}_N$ and for all $\tau \leq t \leq 2\tau - 1$. Since $\bar{k} \to \infty$ as $t \to \infty$, asymptotic stability of (2.1a) follows.

(ii) ($\Rightarrow$) We show that asymptotic stability of (2.1a) implies asymptotic stability of (2.5) with arbitrary switching. Proof of this part is by contradiction. Suppose that (2.1a) is asymptotically stable but (2.5) is not. This means there exist an arbitrary switching sequence w.r.t. (2.5) that is not converging to the origin. Clearly this switching sequence is an admissible switching sequence that satisfies the dwell time condition w.r.t. (2.1a) and hence it violates the asymptotic stability of (2.1a), which is a contradiction. ■

A consequence of Theorem 2.2 is that properties related to the stability of system (2.5) is also applicable to the stability of dwell-time switching systems.

Theorem 2.1 on DT-invariance for a set Ω requires that $x(t) \in \Omega$ for all t with $\tau \leq t \leq 2\tau - 1$ but no mention is made of the $x(t) \in X$ constraint stipulated in (2.1b). The next definition imposes this latter requirement for all time instants.

Definition 2.3 *A set Ω is said to be CADT-invariant (Constraint Admissible Dwell Time-invariant) with respect to system (2.1) with dwell time τ if it is DT-invariant and $x(t) \in X$ for all $t \in \mathbb{Z}^+$.*

Clearly, a necessary condition for constraint admissibility is that $\Omega \subseteq X$, but this is not sufficient. The following theorem states a necessary and sufficient condition for CADT-invariance of a set.

Theorem 2.3 *A DT-invariant set $\Omega \subset X$ is CADT-invariant for system (2.1) with dwell time τ, if and only if for any $x \in \Omega$,*

$$A_i^t x \in X, \text{ for all } i \in \mathcal{I}_N \text{ and for all } t = 1, \cdots, \tau - 1. \tag{2.7}$$

Proof: Consider the solution of switched system (2.1) under any admissible switching sequence of the form (2.3) with $x(0) \in \Omega$. Then, from DT-invariance of Ω and proof of *Theorem* 2.1, it follows that $x(t) \in \Omega \subset X$ for all $t \in [\tau, 2\tau - 1]$. If in

addition, $x(t) \in X$ for $t = 1, 2, \cdots, \tau - 1$, it is inferred that $x(t) = A_{\mathcal{S}_\tau(t)} x(0) \in X$ for all time $t \in \mathbb{Z}^+$ and thus Ω is CADT-invariant. The necessity can be easily shown with contradiction. ∎

2.3.1 Computation of polyhedral CADT-invariant sets

The results of Theorems 2.1 and 2.3 can be used to compute the maximal CADT-invariant set for (2.1). This set, denoted by $\mathbb{O}_\infty(\mathcal{A}, X, \tau)$, is the largest CADT-invariant set inside X in the sense that $x(t) = A_{\mathcal{S}_\tau(t)} x(0) \in \mathbb{O}_\infty(\mathcal{A}, X, \tau)$ if $x(0) \in \mathbb{O}_\infty(\mathcal{A}, X, \tau)$ for any admissible switching sequence $\mathcal{S}_\tau(t)$. For this purpose, let

$$\hat{Q}^i(\Omega) = \{x : A_i\, x \in \Omega\}$$

denote the one time backward set of Ω under subsystem A_i, $i \in \mathcal{I}_N$. It corresponds to the set of x that can be brought into Ω by system $x(t+1) = A_i\, x(t)$ in one time step. Similarly, repeating the above ℓ times lead to

$$\hat{Q}^i_\ell(\Omega) = \hat{Q}^i \cdots \hat{Q}^i(\Omega) = \{x : A^\ell x \in \Omega\} \tag{2.8}$$

and is referred to as the ℓ-step backward set of Ω with respect to mode $i \in \mathcal{I}_N$. For notational simplicity let $\mathbb{T} = \{\tau, \tau+1, \cdots, 2\tau - 1\}$ and define

$$Q^i(\Omega) := \bigcap_{\ell \in \mathbb{T}} \hat{Q}^i_\ell(\Omega) \tag{2.9}$$

as the intersection of $\hat{Q}^i_\ell(\Omega)$ over all $\ell = \tau, \tau+1, \cdots, 2\tau - 1$ for mode i. With this definition, the algorithm for computing the $\mathbb{O}_\infty(\mathcal{A}, X, \tau)$ set using Theorems 2.1 and 2.3 is now given. Hereafter, the dependence of $\mathbb{O}_\infty(\mathcal{A}, X, \tau)$ on $\mathcal{A}, X, \tau$ is dropped for notational convenience unless needed.

Step (1) of Algorithm 1 imposes the constraints according to Theorem 2.3. Sim-

Algorithm 1 Computation of maximal CADT-invariant set

`Input:` $\mathcal{A}$, X and τ.

1) Set $k = 0$ and let $\mathbb{O}_0 := X \bigcap_{i\in\mathcal{I}_N} \{ \bigcap_{\ell=\tau,\cdots,\tau-1} \hat{Q}^i_\ell(X)\}$.

2) Compute $Q^i(\mathbb{O}_k)$ for each mode $i \in \mathcal{I}_N$ and let $\mathbb{O}_{k+1} := \mathbb{O}_k \bigcap_{i\in\mathcal{I}_N} Q^i(\mathbb{O}_k)$.

3) If $\mathbb{O}_{k+1} \equiv \mathbb{O}_k$ set $\mathbb{O}_\infty = \mathbb{O}_k$ then stop, else set $k = k + 1$ and goto step (2).

ilarly, step (2) imposes the condition of Theorem 2.1. More exactly, each $Q^i(\mathbb{O}_k)$ of step (2) is $\bigcap_{\ell\in\mathbb{T}} \hat{Q}^i_\ell(\mathbb{O}_k)$ of (2.9) and is the intersection for ℓ-step backward set of each mode for $\ell = \tau, \cdots, 2\tau - 1$. By taking intersection of $Q^i(\mathbb{O}_k)$ over all $i \in \mathcal{I}_N$, step (2) captures all possible admissible sequences defined in Theorem 2.1. Obviously, the $\mathbb{O}_\infty$ obtained using the above algorithm depends on the choices of $\mathcal{A}$, X and τ. For notational convenience, such dependencies are not shown unless warranted.

Remark 2.2 *When $X = \{x : Rx \leq \mathbf{1}\}$ is a non-empty polytope as given under (A2), the associated computations of step (2) can be obtained noting that $\hat{Q}^i(X) = \{x : RA_i\, x \leq \mathbf{1}\}$, $\hat{Q}^i_\ell(X) = \{x : RA^\ell_i\, x \leq \mathbf{1}\}$.*

While not stated in Algorithm 1, fewer computations result if redundant inequalities are removed from $\mathbb{O}_{k+1}$ at the end of step (2). Properties of the $\mathbb{O}_\infty$ set obtained from the *Algorithm* 1 are stated next.

Theorem 2.4 *Suppose system (2.1) satisfies assumptions (A1)-(A4) and $\mathbb{O}_k$ is generated based on Algorithm 1. Then, (i) $\mathbb{O}_k \subset X$ and $\mathbb{O}_k \subseteq \mathbb{O}_{k-1}$ for all k. (ii) $\mathbb{O}_\infty := \lim_{k\to\infty} \mathbb{O}_k \subset \mathcal{X}$ exists, contains the origin and is finitely determined. (iii) $\mathbb{O}_\infty$ is the largest CADT-invariant set in the sense of Definition 2.3 and is the largest constraint-admissible domain of attraction under admissible switching sequences. (iv) When $\mathbb{O}_\infty$ is the largest CADT-invariant set for system (2.1) with constraint set X, $\beta\mathbb{O}_\infty$ is the corresponding set for system (2.1a) with constraint βX for any $\beta > 0$.*

Proof: (i) This result follows from step (2) of algorithm 1 that $\mathbb{O}_k \subseteq \mathbb{O}_{k-1}$ for all k. (ii) Suppose $\mathbb{O}_0 := \{x : \bar{R}_j\, x \leq 1 \text{ for all } j \in \mathcal{J}\}$. When $\mathbb{O}_k$ is incremented

to $\mathbb{O}_{k+1}$ in step (2) of Algorithm 1, additional inequalities are added to $\mathbb{O}_k$ in the form of $Q_t(\mathbb{O}_k)$ for $t = \tau, \cdots, 2\tau - 1$. For each $Q_t(\mathbb{O}_k)$, a total of N new inequalities are added. They are of the form $\bar{R}_j\, A_{i_1}^{t_1} A_{i_2}^{t_2} \cdots A_{i_{k+1}}^{t_{k+1}} x \leq 1$, for some $i_1, \cdots, i_k \in \mathcal{I}_N$, $t_1, \cdots, t_{k+1} \in \{\tau, \tau+1, \cdots, 2\tau-1\}$ and for all $i_{k+1} = 1, \cdots, N$, as discussed in *Remark* 2.2. This procedure of generating $\mathbb{O}_k$ captures all admissible sequences $\mathcal{S}_\tau(t)$ in the form of $A_{\mathcal{S}_\tau(t)} = A_{i_1}^{t_1} A_{i_2}^{t_2} \cdots A_{i_k}^{t_k}$ such that $t = t_1 + t_2 + \cdots + t_k$ and $k = \lfloor \frac{t}{\tau} \rfloor$. The main part of the proof is to show that after some sufficiently large step k, all these added inequalities are redundant to the $\mathbb{O}_k$ set.

It follows from Assumption (A4) that for every $\epsilon > 0$, there exist a $\hat{t} \in \mathbb{Z}^+$ such that $\left\| A_{\mathcal{S}_\tau(t)} \right\| < \epsilon$ for all $t \geq \hat{t}$. Choose $0 < \epsilon < \min\{\frac{1}{\|x\| \|\bar{R}_j\|} : j \in \mathcal{J}, x \in \mathbb{O}_0\}$. Then, for all $t_1 + \cdots + t_k \geq \hat{t}$ and every $j \in \mathcal{J}$,

$$\begin{aligned} \bar{R}_j (A_{i_1}^{t_1} \cdots A_{i_k}^{t_k}) x &= \bar{R}_j A_{\mathcal{S}_\tau(t)} x \leq \max_{\xi \in \mathcal{B}(\|x\|)} \bar{R}_j A_{\mathcal{S}_\tau(t)} \xi = \max_{\zeta \in \mathcal{B}(\|A_{\mathcal{S}_\tau(t)} x\|)} \bar{R}_j\, \zeta \\ &< \max_{\zeta \in \mathcal{B}(\epsilon \|x\|)} \bar{R}_j\, \zeta = \max_{\bar{\zeta} \in \mathcal{B}(\|x\|)} \epsilon\, \bar{R}_j\, \bar{\zeta} = \epsilon\, \|x\|\, \|\bar{R}_j\| < 1 \end{aligned}$$

where the last inequality follows from the choice of ϵ. Hence all inequalities added after $\hat{t}$-th iteration of algorithm 1 are redundant to the set $\mathbb{O}_{\hat{t}-1}$ and this shows finite termination of $\mathbb{O}_k$. The result of $0 \in \mathbb{O}_\infty$ follows from $0 \in \mathbb{O}_0$ and $0 \in Q_t(\mathbb{O}_k)$ for all $t \in \{\tau, \tau+1, \cdots, 2\tau-1\}$. (iii) When algorithm 1 terminates at some integer k^*, it is inferred that $\mathbb{O}_{k^*} = \mathbb{O}_{k^*+1}$. This and step (2) of the algorithm implies that $\mathbb{O}_{k^*}$ is t-invariant for all $\tau \leq t \leq 2\tau - 1$ w.r.t all $A_i \in \mathcal{A}$ and hence $\mathbb{O}_{k^*}$ is DT-invariant. Step (1) of algorithm 1 implies that $\mathbb{O}_{k^*}$ is constraint admissible for all of the first $\tau - 1$ steps. This and DT-invariance of $\mathbb{O}_{k^*}$ implies $\mathbb{O}_{k^*}$ is CADT-invariance. The proof of $\mathbb{O}_\infty$ being maximal is by contradiction. Suppose $\mathbb{O}_\infty$ is not maximal, therefore there exist a CADT-invariant set $\mathbb{O}^* \subseteq X$ such that $\mathbb{O}^* \nsubseteq \mathbb{O}_\infty$. Since $\mathbb{O}^*$ must be constraint admissible for any switching sequence that is less than τ, $\mathbb{O}^* \subset \mathbb{O}_0$. Let $x \in \mathbb{O}^*$. As $\mathbb{O}^*$ is CADT-invariant, $A_i^t x \in \mathbb{O}^* \subset \mathbb{O}_0$ for all $t = \tau, \cdots, 2\tau - 1$ and for all $i \in \mathcal{I}_N$.

This implies that $x \in Q_t(\mathbb{O}_0)$ for all $\tau \leq t \leq 2\tau - 1$, or, $x \in \mathbb{O}_1$. Hence, $\mathbb{O}^* \subseteq \mathbb{O}_1$. Repeating the above argument shows that $\mathbb{O}^* \subseteq \mathbb{O}_k$ for all k and $\mathbb{O}^* \subseteq \lim_{k\to\infty} \mathbb{O}_k = \mathbb{O}_\infty$ which violates $\mathbb{O}^* \nsubseteq \mathbb{O}_\infty$. That $\mathbb{O}_\infty$ is the largest domain of attraction follows from it being a CADT-invariant and assumption (A4). (v) Since $\hat{Q}_\ell(\beta\Omega, A_i) = \beta\hat{Q}_\ell(\Omega, A_i)$, it follows that $Q_\ell(\beta\Omega) = \bigcap_i \hat{Q}_\ell(\beta\Omega, A_i) = \bigcap_i \beta\hat{Q}_\ell(\Omega, A_i) = \beta Q_\ell(\Omega)$. Using this result in algorithm 1 yields $\mathbb{O}_\infty(\beta X) = \beta\mathbb{O}_\infty(X)$. ■

Remark 2.3 *It is important to highlight the precise meaning of result (iii) of the preceding Theorem. As mentioned in Remark 2.1 and Definition 2.1, a sequence that violates the $t - t_{last} \geq \tau$ condition is not admissible, yet it may be a truncated subsequence of an admissible sequence. As Algorithm 1 is for system (2.1) under all admissible sequences, the presence of such inadmissible sequences results in $\mathbb{O}_\infty$ being CADT-invariant and not positive invariant in the conventional sense. This means that $x(0) \in \mathbb{O}_\infty$ implies $x(\tau) \in \mathbb{O}_\infty$ and $x(t) \in X$ for all t. There is no requirement that $x(t) \in \mathbb{O}_\infty$ when $t = 1, \cdots, \tau - 1$. A set with such property is also known as a constraint admissible returnable set. Figure 2.1 shows the $\mathbb{O}_\infty$ set based on an example with $X = \{x \in \mathbb{R}^2 : ||x||_\infty \leq 1\}$, $\mathcal{I}_N = \{1, 2\}$, $A_1 = \begin{bmatrix} 0.7 & 1 \\ 0 & 0.2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.8 & 0 \\ 0.4 & 0.6 \end{bmatrix}$ and $\tau = 2$. Trajectories under admissible sequences of two initial states $(\pm(0.846, 0.408))$ within $\mathbb{O}_\infty$ are shown. Clearly, $x(1) \notin \mathbb{O}_\infty$ but $x(2)$ is.*

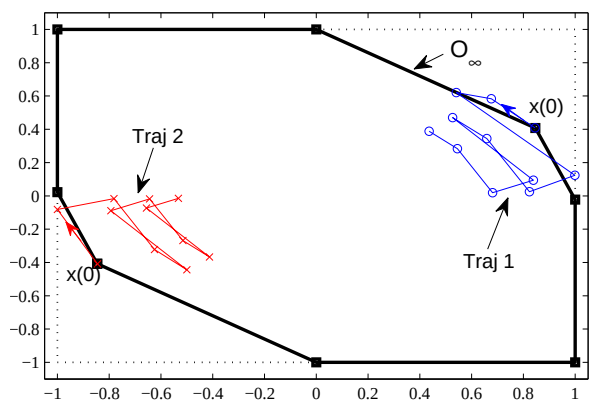


Figure 2.1: Maximal CADT set with sample trajectories from $x(0) = \pm(0.846, 0.408)$

2.3.2 Computation of piece-wise quadratic CADT-invariant sets

The results of Theorems 2.1 and 2.3 can also be extended to obtain a CADT-invariant set defined by the intersection of ellipsoidal sets. The next theorem shows the basic results needed.

Theorem 2.5 *Suppose system (2.1) satisfies assumptions (A1)-(A4) with dwell time τ. If there exist $P_i \succ 0$ for $i = 1, \cdots, N$ such that*

$$\left(A_i^k\right)^\top P_{jk} \left(A_i^k\right) - P_{ik} \prec 0, \quad \forall (i,j) \in \mathcal{I}_N \times \mathcal{I}_N, \forall k \in \{\tau, \tau+1, \cdots, 2\tau-1\} \tag{2.10}$$

Then, (i) $\Psi := \bigcap_{i \in \mathcal{I}_N, k \in \mathbb{T}} \mathcal{E}(P_{ik})$ where $\mathcal{E}(P) = \{x : x^\top P x \leq 1\}$ is a DT-invariant set for system (2.1). (ii) Let $\mathbb{O}_0 := X \bigcap_{i \in \mathcal{I}_N, \ell = \tau, \cdots, \tau-1} \hat{Q}_\ell^i(X)$. There exists an $\bar{\alpha} > 0$, such that $\alpha\Psi$ is CADT-invariant for all $\alpha \leq \bar{\alpha}$.

Proof: (i) For a positive definite $P \succ 0$, define $\|x\|_P := \{x : x^\top P x \leq 1\}$ and let $x \in \Psi$. By (2.10) and the fact that $P_{ik} \succ 0$, it follows that $\|A_i^k x\|_{P_{jk}}^2 < \|x\|_{P_{ik}}^2 \leq 1$ for all $(i,j) \in \mathcal{I}_N \times \mathcal{I}_N$ and for all $k \in \mathbb{T}$. This means that $A_i^k x \in \Psi$, for all $i \in \mathcal{I}_N$ and for all $k \in \mathbb{T}$, which shows the DT-invariance of Ψ. (ii) Suppose $\mathbb{O}_0$ is represented as a polyhedral of the form $\mathbb{O}_0 := \{x : a_j^\top x \leq 1 \text{ for all } j \in \mathcal{J}\}$. Consider the optimization problem $\alpha_j := \max_{x,\gamma}\{\gamma : x^\top P_{ik} x \leq \gamma^2, a_j^\top x \leq 1\}$ for the jth inequality of $\mathbb{O}_0$. The solution of this problem can be shown to be $(a_j^\top P_{ik}^{-1} a_j)^{-0.5}$. Hence, $\alpha_j \Psi$ is the largest scaled Ψ set that is contained in the half-space of $\{x : a_j^\top x \leq 1\}$. Repeating this procedure over all inequalities of $\mathbb{O}_0$ yields $\bar{\alpha}\Psi$ being the largest scaled Ψ set within $\mathbb{O}_0$. This, together with definition of $\mathbb{O}_0$, show that $\bar{\alpha}\Psi$ is CADT-invariance. That $\alpha\Psi$ is also CADT-invariance for any $\alpha < \bar{\alpha}$ follows from $\alpha\Psi \subseteq \bar{\alpha}\Psi \subseteq \mathbb{O}_0$. ∎

Part (i) of the above theorem can be seen as the equivalence of Theorem 2.1 but with Ω replaced by Ψ. Like Theorem 2.1, part (i) does not impose the $x(t) \in X$

condition. Instead, constraint satisfaction is imposes via the $\mathbb{O}_0$ set in a similar fashion as Theorem 2.3 and step (1) of Algorithm 1. Closed-form expression of $\bar{\alpha}$ also exists under (A2). More exactly, when $\mathbb{O}_0$ is expressed as $\mathbb{O}_0 := \{x : a_j^\top x \leq 1 \text{ for all } j \in \mathcal{J}\}$ for some appropriate row vectors $a_j, j \in \mathcal{J}$, the value of $\bar{\alpha}$ of $\bar{\alpha}\Psi$ is obtained by finding the largest α such that $\alpha\mathcal{E}(P_{ik}) \subseteq \mathbb{O}_0$ for all $i \in \mathcal{I}_N$ and for all $k \in \mathbb{T}$. This is done by considering the largest $\alpha\mathcal{E}(P_{ik})$ contained in each half space $\{x : a_j^\top x \leq 1\}$. In addition, it is easy to show that $\alpha_j = \sqrt{\left(a_j^\top P_{ik}^{-1} a_j\right)^{-1}} := \max_{\alpha,y}\{\alpha : y^\top P_{ik}\, y \leq \alpha^2, a_j^\top y \leq 1\}$ and $\bar{\alpha}$ can be determined. Figure 2.2 shows the corresponding $\bar{\alpha}\Psi$ set for the same problem given in Remark 2.3. Clearly, $\bar{\alpha}\Psi \subset \mathbb{O}_\infty$ since $\mathbb{O}_\infty$ is the largest DT-invariant set.

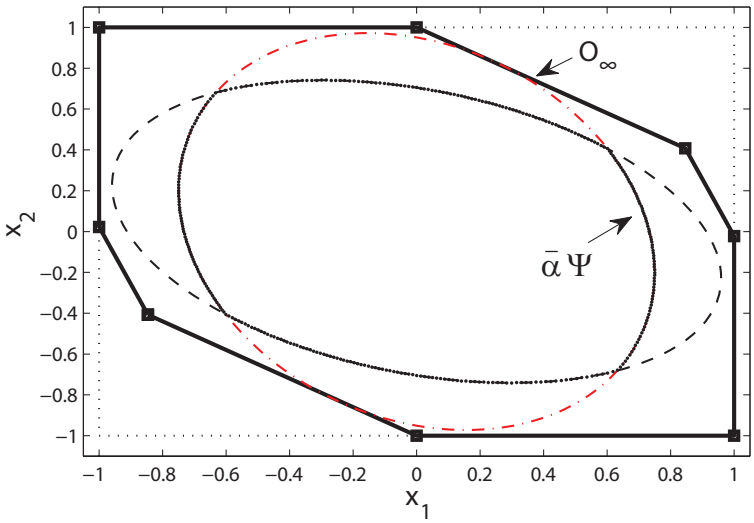


Figure 2.2: Comparison of maximal polyhedral and piece-wise quadratic CADT-invariant sets.

2.4 Computation of the minimal dwell time

An algorithm that finds the minimal dwell time which ensures stability of the origin of system (2.1) can be obtained based on Algorithm 1. This is motivated by the observation that an empty $\mathbb{O}_\infty$ set results if the τ used in Algorithm 1 does not satisfy (A4). Since τ is a scalar, a bisection search with Algorithm 1 as a sub-routine, can be used to find the minimal τ needed for stability. Such an approach, however, suffers

from two drawbacks.

- (I1) The implication of violation of assumption (A4): when a given τ is not known *a priori* to satisfy (A4), there is no guarantee that the origin is asymptotically stable even when Algorithm 1 terminates successfully. Only Lyapunov stability can be ascertained.

- (I2) The implication of the characterization of the $\mathbb{O}_\infty$ set in Algorithm 1. If Algorithm 1 fails for a given τ when $\mathbb{O}_\infty$ is polyhedral, does there exists a different characterization of $\mathbb{O}_\infty$ (quadratic, piece-wise quadratic or otherwise) for which the origin is asymptotically stable?

These issues are now addressed. As assumption (A4) no longer holds in this section, a new definition is needed, which is motivated from the definition of standard contractive sets [17].

Definition 2.4 *A set $\Omega \subset \mathbb{R}^n$ containing the origin is said to be DT-contractive (with contraction λ) w.r.t. (2.1), if there exists a $\lambda \in (0,1)$ such that $x \in \Omega$ implies $A_{\mathcal{S}_\tau(t)}x \in \lambda\Omega$ for all admissible switching sequences $\mathcal{S}_\tau(t)$ and for all time t.*

Again, the above definition is of limited applicability since all admissible sequences are needed. The adaption of DT-contractive set to a result similar to Theorem 2.1 is therefore desirable and can be easily achieved.

Corollary 2.1 *A non-empty set $\Omega \subset \mathbb{R}^n$ is DT-contractive, with contraction $\lambda \in (0,1)$, if and only if $A_i^t\,\Omega \subseteq \lambda\Omega$ for all $i \in \mathcal{I}_N$ and for all $\tau \leq t \leq 2\tau - 1$.*

With this, a necessary and sufficient condition for stability of (2.1) with dwell time τ, is now given.

Theorem 2.6 *Suppose (A1)-(A3) are satisfied. The origin of system (2.1) is asymptotically stable under admissible switching with dwell time τ if and only if system (2.1) admits a polyhedral DT-contractive set, that contains the origin, for some $\lambda \in (0,1)$.*

Proof: ($\Leftarrow$) Suppose a polyhedral DT-contractive set, S, with contractive factor $\lambda \in (0,1)$ exists for system (2.1a). Consider an admissible switching sequence of the form (2.3) and $x(0) \in S$, it follows that

$$x(t) = A_{\mathcal{S}_\tau(t)} x(0) = \left(A_{s_n}^{k_n} \cdots A_{s_1}^{k_1} A_{s_0}^{k_0}\right) x(0) \in \lambda^{\bar{k}} S \tag{2.11}$$

where $\bar{k} := \lfloor k_0/\tau \rfloor + \lfloor k_1/\tau \rfloor + \cdots + \lfloor k_n/\tau \rfloor$. The rightmost condition of (2.11) follows from the fact that all $k_j \geq \tau$ and that $x(0) \in S$ implies $A_i^t x(0) \in \lambda S$ for all $i \in \mathcal{I}_N$ and for all $\tau \leq t \leq 2\tau - 1$. Since $\bar{k} \to \infty$ as $t \to \infty$, the asymptotic stability of system (2.1a) follows.

($\Rightarrow$) In view of theorem 2.2, the origin of (2.1) is asymptotically stable under dwell time switching iff (2.5) is asymptotically stable under arbitrary switching. In addition, (2.5) is asymptotically stable iff there exist a polyhedral contractive set w.r.t. (2.5) [15,17]. This implies that (2.5) is asymptotically stable iff there exist a $\lambda \in (0,1)$ and a polyhedral set S such that $\hat{A}S \subseteq \lambda S$ for every $\hat{A} \in \{A_i^\tau, A_i^{\tau+1}, ..., A_i^{2\tau-1} \text{ for all } i \in \mathcal{I}_N\}$. This implies, from definition 2.4, that S is DT-contractive w.r.t. (2.1) and the result follows. ■

A polyhedral DT-contractive set can be computed by a slight modification to Algorithm 1 by incorporating a choice of $\lambda \in (0,1)$. Computation of polyhedral CADT-contractive set is described in Algorithm 1a.

Algorithm 1a Computation of polyhedral CADT-contractive set

`Input:` $\mathcal{A}$, X, λ and τ.

1) Set $k = 0$ and let $\mathbb{O}_0^\lambda := X \bigcap_{i \in \mathcal{I}_N, \ell = \tau, \cdots, \tau-1} \hat{Q}_\ell^i(X)$.
2) Compute $Q^i(\lambda \mathbb{O}_k^\lambda)$ for each $i \in \mathcal{I}_N$ and let $\mathbb{O}_{k+1}^\lambda := \mathbb{O}_k^\lambda \bigcap_{i \in \mathcal{I}_N} Q^i(\lambda \mathbb{O}_k^\lambda)$.
3) If $\mathbb{O}_{k+1}^\lambda \equiv \mathbb{O}_k^\lambda$ set $\mathbb{O}_\infty^\lambda = \mathbb{O}_k^\lambda$ then stop, else set $k = k+1$ and goto step (2).

It is worthy to note that step (1) above ensures constraint satisfaction according to Theorem 2.3 and, hence, does not require the consideration of λ.

Theorem 2.7 *Suppose the origin of system (2.1) is asymptotically stable under dwell time switching with dwell time τ. Algorithm 1a with dwell time τ yields a non-empty $\mathbb{O}_\infty^{\bar\lambda}$ for some $\bar\lambda \in (0,1)$. In addition, Algorithm 1a with dwell time τ will yield a non-empty $\mathbb{O}_\infty^{\lambda}$ for any $\lambda \in [\bar\lambda, 1]$.*

Proof: From the result of *Theorem* 2.6, asymptotic stability of (2.1) implies the existence of a polyhedral DT-contractive set, S, which contains the origin and a $\bar\lambda \in (0,1)$ such that $A_i^t S \subseteq \bar\lambda S$ for all $i \in \mathcal{I}_N$ and for all $t = \tau, \tau+1, \cdots, 2\tau-1$. Let $\|\cdot\|_S$ be the norm induced by S. DT-contractivity of S implies that

$$\|A_i^t\|_S < \bar\lambda < 1, \qquad \text{for all } i \in \mathcal{I}_N \text{ and for all } t = \tau, \tau+1, \cdots, 2\tau-1 \tag{2.12}$$

The rest of the proof follows similar development as in Theorem 2.4 and hence will be brief. Suppose Algorithm 1a is invoked with some $\lambda < \bar\lambda$ and let $\mathbb{O}_0^\lambda := \{x : \tilde R_j\, x \leq 1 \text{ for all } j \in \mathcal{J}\}$. Additional inequalities are added to $\mathbb{O}_k^\lambda$, when $\mathbb{O}_k^\lambda$ is incremented to $\mathbb{O}_{k+1}^\lambda$ in step (2) of Algorithm 1a. These additional inequalities are of the form $\tilde R_j\, A_{i_1}^{t_1} A_{i_2}^{t_2} \cdots A_{i_{k+1}}^{t_{k+1}} x \leq \lambda^{k+1}$, where $i_1, i_2, \cdots, i_{k+1} \in \mathcal{I}_N$ and $t_1, t_2, \cdots, t_{k+1} \in \mathbb{T}$. For each added inequalities at the k-th iteration of algorithm 1a, for some positive real numbers δ_1 and δ_2, we have

$$\begin{aligned}
\tilde R_j (A_{i_1}^{t_1} \cdots A_{i_k}^{t_k}) x = \tilde R_j A_{\mathcal{S}_\tau(t)} x \;&\leq \max_{\xi \in \mathcal{B}\left(\|A_{\mathcal{S}_\tau(t)} x\|\right)} \tilde R_j\, \xi \leq \max_{\zeta \in \mathcal{B}\left(\|A_{\mathcal{S}_\tau(t)} x\|_S\right)} \delta_1 \tilde R_j\, \zeta \\
&< \max_{\zeta \in \mathcal{B}(\bar\lambda^k \|x\|_S)} \delta_1 \tilde R_j\, \zeta = \max_{\bar\zeta \in \mathcal{B}(\|x\|_S)} \bar\lambda^k\, \delta_1 \tilde R_j\, \bar\zeta \\
&\leq \max_{\bar\xi \in \mathcal{B}(\|x\|)} \bar\lambda^k\, \delta_1 \delta_2\, \tilde R_j\, \bar\xi \leq \bar\lambda^k\, \delta_1 \delta_2\, \|x\|\, \|\tilde R_j\| \\
&< \lambda^k
\end{aligned}$$

The last inequality holds for a sufficiently large k due to the fact that $\lambda < \bar\lambda$. This means that for some sufficiently large k, all new inequalities are redundant to $\mathbb{O}_{k-1}^\lambda$ and the iteration converges, yielding $\mathbb{O}_\infty^\lambda$. That the above argument holds for all $\lambda \in (\bar\lambda, 1]$

completes the proof. ■

Together, Theorems 2.6 and 2.7 address issues (I1) and (I2). Successful termination of Algorithm 1a means that $x(t) \to 0$ for any $x(0) \in \mathbb{O}_\infty^\lambda$ and hence issue (I1) is resolved. While the use of a polyhedral set is both necessary and sufficient for determining the asymptotic stability by Theorem 2.6, Theorem 2.7 also shows that there is a range of λ, $[\bar{\lambda}, 1)$, that is admissible for Algorithm 1a. In practice, it is prudent to chose λ close to 1, say $\lambda = 0.999$.

With the above observations, the next algorithm outlines the steps for finding the minimal dwell-time needed for stability. It is based on a bisection search on τ starting with an initial τ_0 that satisfies (A4).

Algorithm 2 Computation of minimum dwell time

`Input:` $\mathcal{A}, X, \tau_0$
Initialization: Let $\overline{\tau} = \tau_0$ and $\underline{\tau} = 1$.
`while` $\overline{\tau} > \underline{\tau} + 1$

1) Let $\tau = \lfloor (\overline{\tau} + \underline{\tau})/2 \rfloor$ and invoke Algorithm 1 using $\mathcal{A}$, X and τ.
2) If $\mathbb{O}_\infty = \emptyset$, then $\underline{\tau} = \tau$, else $\overline{\tau} = \tau$.

`end while`
Let $\bar{\tau} := \tau$.

3) Invoke Algorithm 1a using $\mathcal{A}$, X, $\bar{\tau}$ and $\lambda = 0.999$.
4) If $\mathbb{O}_\infty^\lambda \neq \emptyset$, then $\tau_{\min} = \bar{\tau}$ and terminate, else $\bar{\tau} = \bar{\tau} + 1$. Goto step (3)

The "while" loop in Algorithm 2 compute $\mathbb{O}_\infty$ based on Algorithm 1. Following the discussion of (I1) above, the second part of algorithm 2 is needed to ensure that all $x \in \mathbb{O}_\infty$ converges to the origin. Clearly, if only DT-invariance is needed but not asymptotic stability of the origin, this second part can be omitted.

Remark 2.4 *The $\mathbb{O}_\infty^\lambda$ obtained from Algorithm 1a can be interpreted as a "generalized" Lyapunov function for switching system (2.1). Since $\mathbb{O}_\infty^\lambda$ is a polytope and contains the origin, it induces a norm $\|x\|_{\mathbb{O}_\infty^\lambda} := \min\{\mu \geq 0 : x \in \mu\mathbb{O}_\infty^\lambda\}$ (or the Minkowski*

distance function of $\mathbb{O}_\infty^\lambda$). Let $V(x(t)) := \|x(t)\|_{\mathbb{O}_\infty^\lambda}$. Unlike conventional Lyapunov functions, $V(x(t))$ does not decreases at every step, but decreases at every τ time steps or at every switching instant. Contractivity of $\mathbb{O}_\infty^\lambda$ ensures that $V(x(t_{k+1})) \leq \lambda V(x(t_k))$ where t_k and t_{k+1} are consecutive switching instants. Hence, the sequence of $V(x(t_k))$ with respect to index k is a decreasing sequence that converges to zero. This also means that $V(t)$ may increase in between switching instants, see example in Section 2.5.

Remark 2.5 (Systems with parametric uncertainty)
Consider the switched system with parametric uncertainty where A_i matrices are uncertain and represented in the form of

$$A_i \in co\{\bar{A}_{i,1}, \bar{A}_{i,2}, \cdots, \bar{A}_{i,M_i}\} \qquad \textit{for each} \quad i \in \mathcal{I}_N.$$

Accordingly, the robust [1] *backward sets are defined as*

$$\hat{Q}^i(\Omega) = \{x : A_i\, x \in \Omega\} = \big\{x : \bar{A}_{i,r}\, x \in \Omega, \forall r \in \{1, \cdots, M_i\}\big\} = \bigcap_{r=1}^{M_i} \big\{x : \bar{A}_{i,r}\, x \in \Omega\big\}$$

$$\hat{Q}^i_\ell(\Omega) = \hat{Q}^i \cdots \hat{Q}^i(\Omega)$$

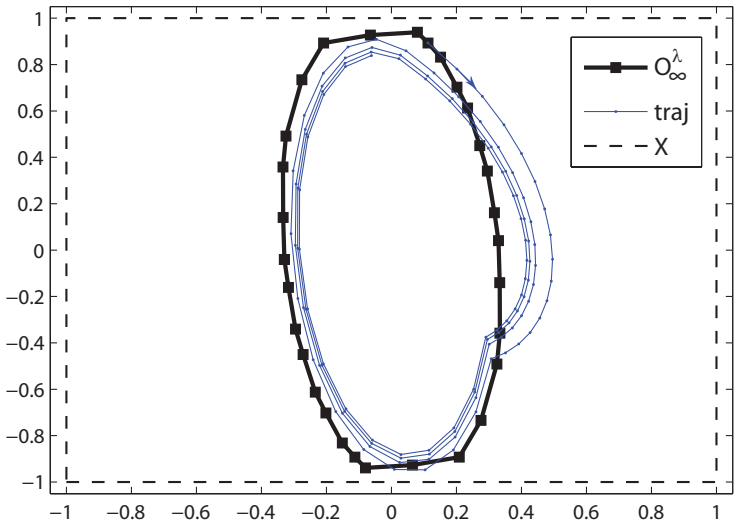
With this modification, Algorithm 1 or 1a can be used for computation of polyhedral CADT-invariant/contractive sets of switched systems with parametric uncertainty. Results of Theorem 2.4, 2.6 and 2.7 are also valid for uncertain switched systems.

2.5 Numerical Examples

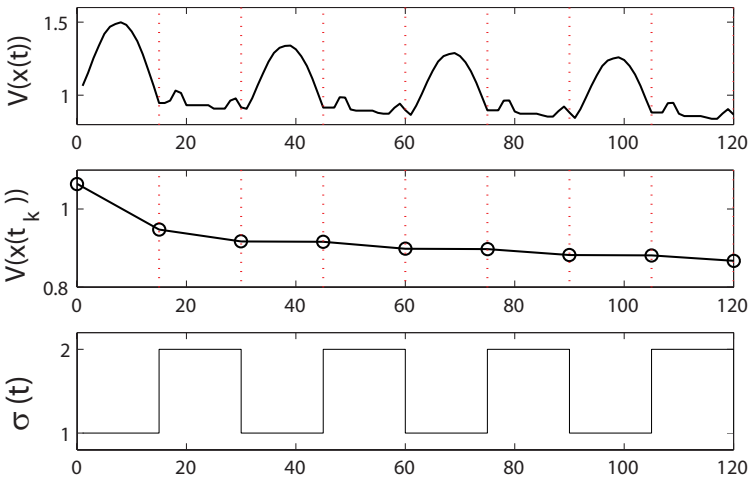
The numerical example is on a switching system with $\mathcal{A} = \{A_1, A_2\}$, $A_1 = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0.1 \\ -0.9 & 0.9 \end{bmatrix}$ with state constraints $X = \{x \in \mathbb{R}^2 : ||x||_\infty \leq 1\}$. The intention is to determine the minimum dwell time and the maximal constraint admissible domain of attraction under dwell time switching for this system. It is worthy to note that existing

[1] The robustness considered here is with respect to parametric uncertainty.

techniques [12, 24] meant for systems under arbitrary switching is not applicable for this example.



(a) State trajectory in $x_1 - x_2$ space; $x(0) = (0.1124, 0.8929)$



(b) Lyapunov function $V(x(t)) = \|x(t)\|_{O_\infty^\lambda}$

Figure 2.3: Illustration of CADT-contractive set $\mathbb{O}_\infty^\lambda$ for $\tau_{\min} = 15$.

Using the approach of [36], an upper bound on $\tau_{\min}$, $\tau^Z := 233$ is obtained for this example. Using the piece-wise quadratic Lyapunov function of Theorem 2.5 discussed in Section 2.3, it is observed that the smallest τ for which (2.10) admits a solution is at $\tau^{LMI} := 16$. Algorithm 2, however, yields a minimum dwell time of $\tau_{\min} = 15$. This is due to the fact that the conditions in the literature are all sufficient conditions for

stability. Figure 2.3(a) shows the $\mathbb{O}_\infty^\lambda$ (with $\lambda = 0.999$) set and a state trajectory under a periodic switching sequence where $t_{s_{k+1}} - t_{s_k} = 15$ for all k and $\sigma(0) = 1$. That the state moves out of $\mathbb{O}_\infty^\lambda$ is clear but it comes back in no more than 15 steps. Moreover, $x(t) \in X$ at all times. The "generalized" Lyapunov function of $V(x(t)) = \|x(t)\|_{\mathbb{O}_\infty^\lambda}$ for this trajectory is shown in Figure 2.3(b). Again, $V(t)$ is not monotonically decreasing with respect to t but is monotonically decreasing with k and $V(t) \to 0$ as $t \to \infty$.

The example above shows that there is a significant improvement on stability conditions in terms of dwell time calculations when compared to the results available in the literature to date; see [36]. Moreover, constraint admissible domain of attraction of dwell time switching systems is obtained, which is appeared to be the first of its kind.

2.6 Summary

Definitions and characterization of DT-contractive and CADT-contractive sets for discrete-time switched systems under dwell-time switching are given in this chapter. It is shown that existence of a polyhedral DT-contractive set is both necessary and sufficient for asymptotic stability of switched systems under dwell-time switching. A numerical algorithm for computation of the maximal CADT-invariant/contractive set is also provided. Using this algorithm as a sub-algorithm, a procedure for the computations of the minimal dwell time needed for stability of the switched system is obtained.

Chapter 3

Computations of Mode Dependent Dwell Times

3.1 Introduction

The same switched system considered in last chapter is considered here:

$$x(t+1) = A_{\sigma(t)}\, x(t), \tag{3.1a}$$

$$x(t) \in X, \quad \forall t \in \mathbb{Z}^+ \tag{3.1b}$$

where $\sigma(t) : \mathbb{Z}^+ \to \mathcal{I}_N := \{1, \cdots, N\}$ is a time-dependent switching signal and $X \subset \mathbb{R}^n$ is the constraint set.

Results of Chapter 2 provide necessary and sufficient conditions for asymptotic stability of (3.1) and algorithms for determination of the minimal dwell time. As mentioned in Chapter 1, several relaxations to the dwell-time approach have also appeared. One is the use of average dwell-time [52] instead of strict dwell-time requirement at each switching instant. However, average dwell-time requirement may result in the state moving far away from the origin and violating physical constraints. Such a behavior is not likely to happen under strict dwell-time requirement so long as the initial

state is within reasonable bound from the origin. Another relaxation is to impose a dwell-time for each mode of the system instead of one single dwell time for all modes of the system [45, 47].

This chapter is concerned with the determination of the mode-dependent dwell times and is an extension of the work presented in Chapter 2. Algorithms presented in Chapter 2, together with several other results introduced hereafter, are used to compute the mode-dependent dwell times for (3.1).

3.2 Preliminaries

The assumptions on the problem data are the same as those appeared in Chapter 2. Also recall that a set $\Omega \subset \mathbb{R}^n$ is DT-contractive for system (3.1a) with dwell time τ, if and only if there exists a $\lambda \in (0, 1)$ such that for every $x \in \Omega$,

$$A_i^t x \in \lambda\Omega \quad \text{for all } t \in \mathbb{T} = \{\tau, \tau+1, \cdots, 2\tau-1\} \text{ and for all } i \in \mathcal{I}_N. \tag{3.2}$$

This result was used in Algorithm 1a to compute the maximal constraint admissible DT-contractive set, $\mathbb{O}_\infty^\lambda$, of system (3.1) for a particular choice of $\lambda \in (0, 1)$. The input to the algorithm were $\mathcal{A} := \{A_1, \cdots, A_N\}$, X, λ and τ while the output of the algorithm was $\mathbb{O}_\infty^\lambda$. When $\mathbb{O}_\infty^\lambda \neq \emptyset$, then the given τ is known to be equal or greater than the minimal dwell time, τ_{min}, for stability. On the other hand, $\mathbb{O}_\infty^\lambda = \emptyset$ implies that $\tau < \tau_{min}$. With this fact, the bisection Algorithm 2 on variable τ was used to compute τ_{min}.

3.3 Main Results

This section and hereafter consider the case where one dwell time is associated with each mode of system (3.1a). Let these dwell times be collective denoted by $\Gamma := \{\tau_1, \cdots, \tau_N\}$

with τ_i being the dwell time for mode i. In addition, τ_c is the common dwell time when all the dwell times are the same. Clearly, an admissible switching sequence, $\mathcal{S}_\Gamma(t)$, with switching instants $t_0, t_1, \cdots, t_k, \cdots$ means that $t_{k+1} - t_k \geq \tau_i$ when $\sigma(t_k) = i$ for all $k \in \mathbb{Z}^+$. Similarly, let

$$\mathbb{T}_i := \{\tau_i, \tau_i + 1, \cdots, 2\tau_i - 1\} \text{ for any } i \in \mathcal{I}_N, \text{ and} \tag{3.3}$$

$$\mathbb{T}_\Gamma := \cup_{i \in \mathcal{I}_N} \mathbb{T}_i. \tag{3.4}$$

Theorem 3.1 *Suppose (A1) is satisfied with Γ and $\mathbb{T}_\Gamma$ as defined above. A set $\Omega \subset \mathbb{R}^n$ is DT-contractive with contraction factor $\lambda \in (0,1)$ for system (3.1a) with dwell times Γ, if and only if for every $x \in \Omega$,*

$$A_i^t x \in \lambda\Omega \quad \textit{for all } t \in \mathbb{T}_i \textit{ and for all } i \in \mathcal{I}_N. \tag{3.5}$$

This result is a direct extension of Theorem 2.1 and its proof follows similar arguments. The key to the application of Theorem 3.1 lies in the computation of the set Ω. Algorithm 1a computes the maximal constraint admissible DT-contractive set for the case of a single dwell time for system (3.1). That algorithm is extended here for the case where there are mode-dependent dwell times. Let $\hat{Q}^i(\Omega) := \{x : A_i x \in \Omega\}$ be the set of x that can be brought into Ω by system A_i in one time step. Repeating the above ℓ times lead to $\hat{Q}^i_\ell(\Omega) = \hat{Q}^i \cdots \hat{Q}^i(\Omega) = \{x : A_i^\ell x \in \Omega\}$ and is referred to as the ℓ-step backward set of Ω under system A_i. Define

$$Q^i(\Omega) := \bigcap_{\ell \in \mathbb{T}_i} \hat{Q}^i_\ell(\Omega) \tag{3.6}$$

as the intersection of $\hat{Q}^i_\ell(\Omega)$ for $\ell = \tau_i, \cdots, 2\tau_i - 1$ for A_i. With this definition, the algorithm for computing the DT-contractive set for a particular choice of $\lambda \in (0,1)$, denoted by $\mathbb{O}^\lambda_\infty$, is now given.

Algorithm 3 Computation of polyhedral DT-contractive set with Mode dependent dwell times

`Input:` $\mathcal{I}_N$, Γ, $\mathcal{X}$ and λ.
`Output:` $\mathbb{O}_\infty^\lambda$.

(i) Set $k = 0$ and let $\mathbb{O}_0^\lambda := X$.

(ii) Compute $Q^i(\lambda\mathbb{O}_k^\lambda)$ for every $i \in \mathcal{I}_N$ and let $\mathbb{O}_{k+1}^\lambda := \mathbb{O}_k^\lambda \bigcap_{i\in\mathcal{I}_N} Q^i(\lambda\mathbb{O}_k^\lambda)$.

(iii) If $\mathbb{O}_{k+1}^\lambda \equiv \mathbb{O}_k^\lambda$ set $\mathbb{O}_\infty^\lambda = \mathbb{O}_k^\lambda$ then stop, else set $k = k + 1$ and goto step (ii).

Step (ii) imposes the condition of Theorem 3.1 with $Q^i(\Omega) = \cap_{\ell\in\mathbb{T}_i}\hat{Q}_\ell^i(\Omega)$ as given by (3.6). For mode i, the Q operator is applied for $\tau_i, \cdots, 2\tau_i - 1$ to obtain the set $Q^i(\Omega)$ such that points starting from it will return to $\lambda Q^i(\Omega)$ after τ_i to $2\tau_i - 1$ steps. Obviously, the $\mathbb{O}_\infty^\lambda$ obtained using the above algorithm depends on the choices of $\mathcal{A}$, X, λ and τ. For notational convenience, such dependencies are not shown unless warranted.

Remark 3.1 *In the subsequent reference to Algorithm 1, the input variables, X and λ will not be mentioned unless warranted. The X set is defined by (3.1b) and $\lambda := 1 - \epsilon$ for some small $\epsilon > 0$ in our numerical experiments involving Algorithm 1.*

Remark 3.2 *If a constraint admissible DT-contractive set is needed instead of DT-contractive, step (i) of Algorithm 1 can be replaced by $\mathbb{O}_0^\lambda := X \cap_{i\in\mathcal{I}_N,\ell=1,2,\cdots,\tau_i-1} \hat{Q}_\ell^i(X)$. Algorithm 1 uses DT-contractive set for the purpose of determining stability of (3.1a) under dwell time switching, as indicated in Theorem 2.2.*

The following result provides a necessary and sufficient condition for asymptotic stability of (3.1). The proof follows similar reasoning given in Chapter 2 and is therefore omitted.

Theorem 3.2 *Suppose (A1)-(A3) are satisfied with Γ as defined above. System (3.1) is asymptotically stable under Γ if and only if Algorithm 1 yields a non-empty DT-contractive set with a contraction factor $\lambda \in (0, 1)$.*

Using the procedure given in Algorithm 2, a minimal common dwell time, τ_c, for system (3.1a) can be computed. This choice of τ_c is both necessary and sufficient for stability of (3.1a) under any admissible dwell-time switching sequence that satisfy the common dwell time requirement. In addition, an associated non-empty set Ω_c and a $\lambda_c \in (0, 1)$ are available such that Ω_c is DT-contractive.

Given this setting, the procedure hereafter describes a procedure that determines Γ with an associated DT-contractive set Ω in the sense of Theorem 3.1. Clearly, the trivial choice of $\tau_1 = \cdots = \tau_N = \tau_c$ is admissible. Hence, a more meaningful search is to restrict Γ to the set

$$\Upsilon_c := \{\Gamma : \tau_i \leq \tau_c \text{ for all } i \in \mathcal{I}_N\}. \tag{3.7}$$

The above definition on Υ_c is useful to define the meaning of optimal dwell time.

Definition 3.1 *Given system (3.1a) and an optimal common dwell time τ_c such that Υ_c is defined by (3.7). The system has an optimal mode-dependent dwell time $\Gamma \in \Upsilon$ if Γ is a stabilizing dwell time having the smallest value of $\sum_{i=1}^{N} \tau_i$.*

Several useful lemmas are first stated to facilitate the approach towards searching for the stabilizing mode-dependent dwell time.

Lemma 3.1 *Suppose $\Gamma_a = \{\tau_1, \cdots, \tau_k, \cdots, \tau_N\}$ is a stabilizing dwell time for system (3.1a), so is $\Gamma_b = \{\tau_1, \cdots, \tau_k + 1, \cdots, \tau_N\}$.*

Proof: Consider the case when $i = k$, Theorem 3.2 implies that there exists an Ω such that $A_k^t x \in \lambda\Omega$ for all $t \in \{\tau_k, \cdots, 2\tau_k - 1\}$. This also implies that $A_k^{2\tau_k} x \in \lambda\Omega$ and $A_k^{2\tau_k+1} x \in \lambda\Omega$ for any $x \in \Omega$ since $A_k^{2\tau_k} x = A_k^{\tau_k} A_k^{\tau_k} x \in \lambda^2\Omega \subset \lambda\Omega$ and $A_k^{2\tau_k+1} x = A_k^{\tau_k} A_k^{\tau_k+1} x \in \lambda\Omega$. These two inclusions imply that $A_k^t x \in \lambda\Omega$ for all $t \in \{\tau_k + 1, \cdots, 2\tau_k - 1, 2\tau_k, 2\tau_k + 1\}$ which proves the assertion. ∎

Corollary 3.1 *From Lemma 3.1, it follows that any* $\hat{\Gamma} = \{\hat{\tau}_1, \cdots, \hat{\tau}_N\}$ *such that* $\hat{\tau}_i \geq \tau_i$ *for all* $i \in \mathcal{I}_N$ *is a stabilizing mode-dependent dwell time if* $\Gamma = \{\tau_1, \cdots, \tau_N\}$ *is one.*

Lemma 3.2 *Suppose a common minimal dwell time* τ_c *is known for system (3.1a) and* Υ_c *is that defined by (3.7). Let* $\Gamma \subset \Upsilon_c$ *be a stabilizing dwell time such that system (3.1a) is stable. Then, at least one* $\tau_i \in \Gamma$ *is equal to* τ_c.

Proof: Suppose the assertion is not true. This means that there exists a $\hat{\Gamma} = \{\hat{\tau}_1, \cdots, \hat{\tau}_N\}$ with $\hat{\tau}_i < \tau_c$ for all $i \in \mathcal{I}_N$ such that system (3.1a) is stable under dwell time switching. Let $\hat{\tau}_{max} = \max_{i \in \mathcal{I}_N} \hat{\tau}_i$. By Corollary 3.1, it follows that $\{\hat{\tau}_{max}, \cdots, \hat{\tau}_{max}\}$ is a stabilizing mode-dependent dwell time. Since $\hat{\tau}_{max} < \tau_c$, this contradicts τ_c being the minimal common dwell time. ∎

This search of a stabilizing dwell time is facilitated by a bisection algorithm with Algorithm 1 as a subalgorithm and Lemma 3.2. The bisection search algorithm is now described. Besides standard inputs of Algorithm 1, it requires $\underline{\Gamma}$ and $\overline{\Gamma}$ to be non-stabilizing and stabilizing dwell times respectively.

Algorithm 4 Bisection Algorithm for mode-dependent dwell-times

`Input:` $\mathcal{I}$, $\{A_i : i \in \mathcal{I}\}$, $\mathcal{X}$, λ, $\underline{\Gamma}$, and $\overline{\Gamma}$.
`Output:` A stabilizing dwell time Γ.

(i) Let $\Gamma = \lfloor \frac{\underline{\Gamma}+\overline{\Gamma}}{2} \rfloor$ (implemented elementwise). If $\Gamma = \underline{\Gamma}$, set $\Gamma = \overline{\Gamma}$ and terminate.

(ii) Invoke Algorithm 1 with inputs $\mathcal{I}$ and Γ.

(iii) If $\mathbb{O}_\infty^\lambda = \emptyset$, set $\underline{\Gamma} = \Gamma$. Otherwise, set $\overline{\Gamma} = \Gamma$. Goto step (i).

The results of Lemma 3.2 and the bisection algorithm is used for the search of stabilizing mode-dependent dwell times. The basic idea is first described in the exemplary case of a two-mode system.

3.3.1 System with two modes

Given system (3.1a) with $\mathcal{I}_N = \{1, 2\}$ and the common minimal dwell time, τ_c, for this system. Let $\underline{\Gamma} = \{\tau_c, 0\}$ and $\overline{\Gamma} = \{\tau_c, \tau_c\}$. Invoke the bisection algorithm using $\underline{\Gamma}$, $\overline{\Gamma}$, $\mathcal{I} = \mathcal{I}_N$, $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{X}$. Let the solution to be Γ_1. Repeat the above but with $\underline{\Gamma} = \{0, \tau_c\}$ and $\overline{\Gamma} = \{\tau_c, \tau_c\}$ and let the solution be Γ_2. The optimal dwell time is one with the smallest total dwell time and is given by $\min\{\tau_1 + \tau_2 | (\tau_1, \tau_2) \in \Gamma_i, i = 1, 2\}$.

The setting of $\underline{\Gamma} = \{\tau_c, 0\}$ (or $\underline{\Gamma} = \{0, \tau_c\}$) in the above is a notation to denote that $\underline{\Gamma}$ is an inadmissible set for the proper working of the bisection Algorithm (note that step (i) sets $\Gamma = \lfloor \frac{\underline{\Gamma}+\overline{\Gamma}}{2} \rfloor$ elementwise). It should not be interpreted as one of the two modes is dropped from consideration. Also, see Remark 3.5 in the event that $\Gamma_1 \neq \Gamma_2$ but having the same value of $\tau_1 + \tau_2$.

Lemma 3.3 *The mode-dependent dwell time obtained using the above procedure is optimal (in the sense of Definition 3.1) for system (3.1a) with $\mathcal{I}_N = \{1, 2\}$.*

Proof: The result is based on the fact that Algorithm 1 is both necessary and sufficient for Γ being a stabilizing dwell time, the result of Lemma 3.2 and that $\tau_1 + \tau_2$ is the smaller of Γ_1 and Γ_2. ∎

3.3.2 System with more than two modes

In the general case where $N > 2$, two procedures are needed prior to the computation of the dwell times. The first is the common dwell time, τ_c, for the N-mode system. The second is the dwell times of all K-mode combinations for $K = 2, 3, \cdots, N-1$. These dwell times for the K-mode system are generated incrementally starting from $K = 2$, one step at a time. For every value of K, there are $C_K^N := \frac{N!}{K!(N-K)!}$ distinct choices to pick K modes from the N-mode system. Let the collection of these distinct

C_K^N index sets and a specific choice of one of these sets be denoted respectively by

$$\mathfrak{C}^K := \{\mathcal{I} = \{i_1, \cdots, i_K\} : i_j < i_{j+1}\ j \in \mathbb{Z}_1^{K-1} \text{ and } i_j \in \mathcal{I}_N\ \forall j \in \mathbb{Z}_1^K\}, \tag{3.8}$$

$$\Gamma_{\mathcal{I}} := \{\tau_{i_1}, \cdots, \tau_{i_K}\} \text{ for some } \mathcal{I} \in \mathfrak{C}^K. \tag{3.9}$$

Suppose $\Gamma_{\mathcal{J}}$ for all $\mathcal{J} \in \mathfrak{C}^K$ are known. Without loss of generality, these $\mathcal{J}$ are denoted respectively as $\mathcal{J}_1, \cdots, \mathcal{J}_r$ where $r = C_K^N$. Let $\mathcal{I} \in \mathfrak{C}^{K+1}$ be a specific choice of the $(K+1)$-mode system and $\Gamma_c = \{\tau_c, \cdots, \tau_c\}$ where τ_c is the single common dwell time for the $(K+1)$-mode system corresponding to $\mathcal{I}$. Note that $|\mathcal{I}| = K+1$ while $|\mathcal{J}_\ell| = K$ for all $\ell = 1, \cdots, r$. The procedure that computes the $\Gamma_{\mathcal{I}}$ is now described.

Algorithm 5 Computation of $\Gamma_{\mathcal{I}}$

`Input:` $\{\Gamma_{\mathcal{J}} : \mathcal{J} = \mathcal{J}_1, \cdots, \mathcal{J}_r\}$, τ_c, $\Gamma_c = \{\tau_c, \cdots, \tau_c\}$ and $\mathcal{I} = \{i_1, \cdots, i_{K+1}\}$.
`Output:` A stabilizing dwell time $\Gamma_{\mathcal{I}}$ for the corresponding $(K+1)$-mode system.

(0) Set $j = 1$.

(i) Let $\mathcal{I}_j = \mathcal{I} \backslash i_j$ and let $\mathcal{J}^*$ denote the $\mathcal{J}$ index set that has the same elements as $\mathcal{I}_j$.

(ii) Let $\Gamma_{\mathcal{I}+} = \{\tau_{i_1}, \cdots, \tau_{i_{K+1}}\}$ where $\tau_{i_j} = \tau_c$ and the rest of the τ_i are set equal to those from $\Gamma_{\mathcal{J}^*}$. Let $\Gamma_{\mathcal{I}-} = \Gamma_{\mathcal{I}+}$ except that $\tau_{i_j} = 0$. Invoke Algorithm 3 with $\mathcal{I}$ and $\Gamma_{\mathcal{I}+}$ and yielding output set $\mathbb{O}_\infty$.

(iii) If $\mathbb{O}_\infty = \emptyset$, set $\overline{\Gamma} = \Gamma_c$ and $\underline{\Gamma} = \Gamma_{\mathcal{I}+}$. Else, set $\overline{\Gamma} = \Gamma_{\mathcal{I}+}$ and $\underline{\Gamma} = \Gamma_{\mathcal{I}-}$.

(iv) Invoke the bisection algorithm 4 with $\underline{\Gamma}$ and $\overline{\Gamma}$. Let $\Gamma_{\mathcal{I}}^j$ be the solution of the bisection algorithm. If $j < K+1$, set $j = j+1$ and goto step (i).

(v) Let $k = \arg_j \min\{\tau_1 + \cdots + \tau_{K+1} : \Gamma_{\mathcal{I}}^j, j = 1, \cdots, K+1\}$ and set $\Gamma_{\mathcal{I}} = \Gamma_{\mathcal{I}}^k$. Stop.

The choice of $\Gamma_{\mathcal{I}+}$ in step (ii) is motivated from Lemma 3.2. The rest of τ are set equal to those from $\Gamma_{\mathcal{J}^*}$ because they are necessary conditions for stability for the $(K+1)$-mode system. Similarly, the choice of $\tau_{i_j} = \tau_c$ in step (ii) is a result of Lemma 3.2. Lemma 3.2 has a further implication that is not captured in the algorithm above.

Remark 3.3 *In step (iii) under the case where $\mathbb{O}_\infty \neq \emptyset$, the bisection algorithm is invoked with $\overline{\Gamma} = \Gamma_{\mathcal{I}+}$ and $\underline{\Gamma} = \Gamma_{\mathcal{I}-}$. This step is fine if there are more than one τ in*

$\Gamma_{\mathcal{I}+}$ that are equal to τ_c. If τ_{i_j} is the only τ having a value of τ_c, then the bisection algorithm can be avoided by setting $\Gamma_{\mathcal{I}}^j = \Gamma_{\mathcal{I}+}$ as a result of Lemma 3.2.

Lemma 3.4 *Suppose for some value of K, $\Gamma_{\mathcal{I}}$ are the optimal dwell times for all $\mathcal{I} \in \mathfrak{C}^K$. (This is true for the $K = 2$ case following Lemma 3.3). Consider a particular choice of j in step (i) of Algorithm 5. Suppose the output set $\mathbb{O}_\infty$ of step (ii) is non-empty, then the corresponding $\Gamma_{\mathcal{I}}^j$ so obtained in step (iv) is the optimal dwell time in the sense of Definition 3.1.*

Proof: Note that when $\mathbb{O}_\infty$ of step (ii) is non-empty, $\Gamma_{\mathcal{I}}^j$ is obtained from the bisection algorithm with $\Gamma_{\mathcal{I}+} = \{\tau_{i_1}, \cdots, \tau_c, \cdots, \tau_{i_{K+1}}\}$ and $\Gamma_{\mathcal{I}-} = \{\tau_{i_1}, \cdots, 0, \cdots, \tau_{i_{K+1}}\}$. Hence, $\Gamma_{\mathcal{I}}^j$ takes the form of $\{\tau_{i_1}, \cdots, \tau_{i_j}, \cdots, \tau_{i_{K+1}}\}$. Now suppose the assertion is not true and there exists a $\hat{\tau}_\ell < \tau_\ell$ for some ℓ satisfying $1 \leq \ell \leq K+1$ such that $\Gamma := \{\tau_{i_1}, \cdots, \hat{\tau}_\ell, \cdots, \tau_{i_{K+1}}\}$ is a stabilizing dwell time. If $\ell = j$, this leads to a contradiction since τ_ℓ is already the optimal under the bisection algorithm. If $\ell \neq j$, this also leads to a contradiction since $\Gamma \backslash \{\hat{\tau}_\ell\}$ is the optimal dwell time for the K-mode system. ∎

Remark 3.4 *The result of Lemma 3.4 is not as limited as it appears. In step (v) of algorithm 5, the mode-dependent dwell times for a particular choice of $\mathcal{I}$ is identified by the index k that achieves the minimal value of $\tau_1 + \cdots + \tau_{K+1}$. Hence, if the case of index k (and not for all $j = 1, \cdots, K+1$) satisfies the conditions stipulated in Lemma 3.4, the mode-dependent dwell times for $\mathcal{I}$ is optimal. In many of the numerical examples tested (including the two examples of Section 3.4), the output Γ is obtained under such a situation and is, therefore, the optimal mode-dependent dwell time.*

Remark 3.5 *While never experienced in all our numerical examples, it is possible that there are more than one j that attains the minimal value of $\tau_1 + \cdots + \tau_{K+1}$ in step (v) of Algorithm 5. In such an event, the Γ corresponding to all the non-unique minima*

should be stored under the choice of $\mathcal{I}$. Correspondingly, steps (i)-(iv) of Algorithm 5 will have to be run more than once to obtain the $\Gamma_{\mathcal{I}}$. Specifically, suppose there are two $\Gamma_{\mathcal{J}_\ell}$, $\Gamma_{\mathcal{J}_\ell^1}$ and $\Gamma_{\mathcal{J}_\ell^2}$, for a specific choice of $\mathcal{J}$. Step (i)-(iv) has to be done twice, one each of the $\Gamma_{\mathcal{J}_\ell^i}$. Step (v) should also include the two values of $\Gamma_{\mathcal{I}}^j$ in determining the optimal k.

3.4 Numerical Examples

The algorithms described in the prior sections are illustrated using two numerical examples and the results are also compared with the results of a recent work in the literature [47]. The first example considered has the following details: $X = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$ and $\mathcal{I}_N = \{1, 2, 3, 4\}$ with $A_1 = [1, 0.1; -0.2, 0.9]$, $A_2 = [1, 0.1; -0.9, 0.8]$, $A_3 = [0.95, 0.09; -0.94, 0.86]$ and $A_4 = [0.99, -0.04; 0.4, 0.95]$. The corresponding spectral radii are 0.959, 0.943, 0.950, 0.978 respectively.

index j of Alg. 3			$\{1,2\}$ $\tau_c = 7$	$\{1,3\}$ $\tau_c = 8$	$\{1,4\}$ $\tau_c = 15$	$\{2,3\}$ $\tau_c = 1$	$\{2,4\}$ $\tau_c = 5$	$\{3,4\}$ $\tau_c = 1$
	$\underline{\Gamma}$	$\overline{\Gamma}$	Γ_{output}	Γ_{output}	Γ_{output}	Γ_{output}	Γ_{output}	Γ_{output}
1	τ_c 0	τ_c τ_c	7 7	8 8	**15 1**	**1 1**	**5 1**	**1 1**
2	0 τ_c	τ_c τ_c	**1 7**	**1 8**	14 15	1 1	5 5	1 1

Table 3.1: Intermediate mode-dependent dwell times for all 2-mode subsystems of Example I.

Table 3.1 shows the intermediate mode-dependent dwell times computed for all 2-mode subsystems as described in section 3.3.1. The minimal dwell times are indicated in bold font. They are used to compute the mode-dependent dwell time for the 3-mode subsystems shown in Table 3.2 according to the procedure described in Algorithm 5 incorporating the features mentioned in Remarks 3.3 and 3.4.

index	$\{1,2,3\}$, $\tau_c = 8$			$\{1,2,4\}$, $\tau_c = 15$			$\{1,3,4\}$, $\tau_c = 15$			$\{2,3,4\}$, $\tau_c = 5$		
j	$\underline{\Gamma}$	$\overline{\Gamma}$	Γ_{output}	$\underline{\Gamma}$	$\overline{\Gamma}$	Γ_{output}	$\underline{\Gamma}$	$\overline{\Gamma}$	Γ_{output}	$\underline{\Gamma}$	$\overline{\Gamma}$	Γ_{output}
1	8 1 1	8 8 8	8 7 8	15 5 1	–	15 5 1	15 1 1	–	15 1 1	5 1 1	–	5 1 1
2	1 0 8	1 8 8	**1 7 8**	15 0 1	15 15 1	**15 5 1**	15 0 1	15 15 1	**15 1 1**	5 0 1	5 5 1	**5 1 1**
3	1 7 8	–	1 7 8	1 7 15	15 15 15	14 7 15	1 8 15	15 15 15	14 8 15	1 1 5	5 5 5	5 1 5

Table 3.2: Intermediate mode-dependent dwell times for all 3-mode subsystems of Example I.

index	$\{1,2,3,4\}$, $\tau_c = 15$		
j	$\underline{\Gamma}$	$\overline{\Gamma}$	Γ_{output}
1	15 5 1 1	15 15 15 15	15 5 3 1
2	15 0 1 1	15 15 1 1	**15 6 1 1**
3	15 5 0 1	15 5 15 1	15 5 3 1
4	1 7 8 15	15 15 15 15	15 7 8 15

Table 3.3: Intermediate mode-dependent dwell times for the 4-mode system of Example I.

The result of $K = 3$ is shown in Table 3.2[1]. Each of the 3 rows corresponds to a particular choice of j for $j = 1, 2, 3$ described in step (i) of Algorithm 5. Also, the $\underline{\Gamma}$, $\overline{\Gamma}$ are those set according to step (iii) of the same Algorithm. The results of Table 3.2 are used to compute the results of Table 3.3. Hence, the mode-dependent dwell times obtained form algorithm 5 is $\Gamma = [\tau_1, \tau_2, \tau_3, \tau_4] = [15, 6, 1, 1]$ and is optimal because it satisfies the conditions of Remark 3.4.

As a comparison, the procedure for mode-dependent dwell time is computed using the MLFs approach of [47]. They show that if there exist $P_i \succ 0$, $0 < \lambda_i < 1$, and $\mu_i \geq 1$ for each $i \in \mathcal{I}_N$ such that

$$A_i^T P_i A_i \preceq \lambda_i P_i \quad \forall i \in \mathcal{I}_N \tag{3.10}$$

$$P_i \preceq \mu_j P_j \quad \forall (i,j) \in \mathcal{I}_N \times \mathcal{I}_N \tag{3.11}$$

[1]The symbol "–" in the Tables 3.2 indicates that the bisection Algorithm is not invoked as a result of Remark 3.3.

Then, the system is asymptotically stable under dwell times $\tau_i \geq -\frac{\ln \mu_i}{\ln \lambda_i}$ for all $i \in \mathcal{I}_N$.

Clearly, τ_i can be minimized by minimizing $\mu_i \geq 1$ and $\lambda_i \in (0, 1)$ simultaneously. Unfortunately, (3.10)-(3.11) is a Bilinear Matrix Inequality (BMI) in λ_i, μ_i and P_i and the optimal solution is not easily determinable. If λ_i and μ_i are fixed, then P_i is easily solvable since it is a Linear Matrix Inequality. The same is true when P_i for all $i \in \mathcal{I}_N$ are fixed and solving for λ_i and μ_i. The procedure proceeds by alternatively fixing these two groups of variables. The mode-dependent dwell times found using their method are $\Gamma^Z = [\tau_1, \tau_2, \tau_3, \tau_4] = [43, 16, 14, 8]$.

Details of the second example are $X = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$, $\mathcal{I}_N = \{1, 2, 3, 4, 5\}$, $A_1 = [1, 0.1; -0.2, 0.9]$, $A_2 = [1, 0.1; -0.9, 0.9]$, $A_3 = [0.95, 0.09; -0.94, 0.8]$, $A_4 = [1, -0.04; 0.4, 0.95]$ and $A_5 = [0.8, 0.5; 0\ , 0.5]$ and their respective spectral radii are 0.959, 0.995, 0.919, 0.983 and 0.8. The mode-dependent dwell times obtained from algorithm 3 is found to be $\Gamma = [16, 8, 1, 16, 7]$ or $\sum \tau_i = 48$ and is known to be optimal (from Remark 3.4) while the result of [47] yields to $\Gamma^Z = [35, 157, 13, 31, 9]$ with $\sum \tau_i = 245$.

3.5 Summary

This chapter proposes an algorithmic approach to the determination of mode-dependent dwell times of a system switching among N linear subsystems. The approach builds up progressively by computing the mode-dependent dwell times of the K-mode subsystems for $K = 2, \cdots, N$. The K-mode dwell times provide necessary conditions for the stabilizing dwell times for the $(K + 1)$-mode subsystems and, under appropriate conditions, sufficient conditions for the optimal mode-dependent dwell times. In the numerical examples considered, where some of N modes having spectral radii close to 1, the approach yields the optimal mode-dependent dwell times that are significantly smaller than the results of a recent work in the literature.

Chapter 4

Computation of Disturbance Invariant Sets

4.1 Introduction

This chapter considers the following constrained discrete-time switched linear system with additive disturbance:

$$x(t+1) = A_{\sigma(t)}x(t) + w(t) \tag{4.1a}$$

$$x(t) \in X, w(t) \in W, \quad \forall t \in \mathbb{Z}^+ \tag{4.1b}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^n$ are the state and disturbance variables respectively, $W \subset \mathbb{R}^n$ is the disturbance set, $\sigma(t) : \mathbb{Z}^+ \to \mathcal{I}_N := \{1, \cdots, N\}$ is a time-dependent switching that satisfies some dwell time conditions and $X \subset \mathbb{R}^n$ is the constraint set.

Most of the literature of the switched systems [5, 12, 23, 36–38, 53] is concerned with stability condition when disturbance inputs are absent. A few of them also consider the presence of constraints and/or disturbances [24, 54–56] when switching is arbitrary. This chapter is concerned with the characterization and computation of

suitably defined disturbance-invariant sets (also known as robust[1]-invariant sets) for system (4.1) when $\sigma(\cdot)$ is an admissible switching function that respects the dwell-time consideration. Since only dwell-time switching is allowed, the invariance condition is termed Disturbance Dwell-Time invariance (DDT-invariance). Other contributions of this chapter include algorithms for the computations of the maximal and the minimal convex DDT-invariant sets for system (4.1). In the limiting case where the dwell-time is one sample period, $\sigma(\cdot)$ becomes an arbitrary switching function, and the corresponding invariant sets and their computations have appeared in the literature, see for example, [24] and [55, 56]. Hence, this work can also be seen as a generalization of those obtained for arbitrary switching systems. Note that the results presented in this chapter are for the case when a common dwell-time is considered. Extension of the results to mode-dependent dwell-time is obvious and hence will not be considered here. Also for the convenience of following the main ideas, the proofs of theorems are deferred to the end of the chapter.

4.2 Preliminaries

Recall that $\mathcal{S}_\tau$ is the class of admissible switching signals that satisfies the dwell-time consideration. Let $\{i\}^\ell := \{i, i, \cdots, i\}$ be a sequence of ℓ elements of i with $i \in \mathcal{I}_N$ and $\mathcal{W}_\ell$ be the set of sequences $\{w(\cdot)\}$ of length ℓ with every $w(\cdot) \in W$. Then, a switching sequence can equivalently be represented by $\mathcal{S}_\tau(t) = \{\{i_m\}^{k_m}, \cdots, \{i_1\}^{k_1}, \{i_0\}^{k_0}\}$ for some appropriate $i_j \in \mathcal{I}_N$ for all $j = 0, \cdots, m$ and $\sum_{j=0}^{m} k_j = t$.

Assumptions on problem data are **(A1)**-**(A4)** of Chapter 2. In addition, the disturbance set of system (4.1) satisfies **(A5)** W is a polytope and contains 0 in its interior. The polyhedral assumption of (A5) is made to facilitate numerical computations described in this chapter and it is not needed for the theoretical development of Section 4.3.

[1]Robustness is with respect to additive disturbance

4.3 Main Results

This section begins with definitions of Disturbance-Dwell-Time-invariant (DDT-invariant) set and Constraint Admissible Disturbance Dwell-Time-invariant (CADDT-invariant) set for system (4.1a) with admissible input sequences.

Definition 4.1 *A set $\Omega \subset \mathbb{R}^n$ is said to be DDT-invariant w.r.t. (4.1a) with dwell-time τ, if $x(0) \in \Omega$ implies $x(t) \in \Omega$ for every admissible sequence $S_\tau(t)$ and for every allowable disturbance sequence $\{w(0), \cdots, w(t-1)\} \in \mathcal{W}_t$.*

Definition 4.2 *A set $\Omega \subset \mathbb{R}^n$ is said to be CADDT-invariant w.r.t. (4.1a) with dwell-time τ, if it is DDT-invariant and $x(t) \in X$ for all $t \in \mathbb{Z}^+$.*

The definition of DDT-invariance is closely related to the definition of an admissible sequence. Aa an example, suppose that Ω is DDT-invariant and $x(0) \in \Omega$. With the admissible sequence $\mathcal{S}_3(6) = \{1, 1, 1, 2, 2, 2\}$ it follows that $x(6) \in \Omega$ but sequence $\bar{\mathcal{S}}_3(6) = \{1, 1, 2, 2, 2, 2\}$ may not result in $x(6) \in \Omega$. Similarly, $x(7) \in \Omega$ if $\bar{\mathcal{S}}_3(7) = \{1, 1, 1, 2, 2, 2, 2\}$ is obtained from $\bar{\mathcal{S}}_3(6)$ with $\sigma(6) = 1$.

While stating the requirements of DDT-invariance and CADDT-invariance, the above definitions are of limited practical usefulness since the reachable set of system (4.1) for all admissible switching input and disturbance sequences of length t have to be considered. Clearly, such an approach is not computationally tractable. This difficulty can be circumvented using the characterization of Dwell-Time invariance of Chapter 2. For the disturbance-free system it was shown that any admissible sequence of the form

$$\mathcal{S}_\tau(t) = \{\{i_m\}^{k_m}, \cdots, \{i_1\}^{k_1}, \{i_0\}^{k_0}\} \tag{4.2}$$

with $i_j \in \mathcal{I}_N$, $k_j \geq \tau$ for all $j = 0, \cdots, m$ and $\sum_{j=0}^{m} k_j = t$ can be written as a unique ordering of a finite number of subsequences as

$$\mathcal{S}_\tau(t) = \{\{i_m\}^{q_m\tau}, \{i_m\}^{r_m}, \cdots, \{i_1\}^{q_1\tau}, \{i_1\}^{r_1}, \{i_0\}^{q_0\tau}, \{i_0\}^{r_0}\} \tag{4.3}$$

where, for all $j = 0, \cdots, m$, $q_j = \lfloor \frac{k_j - \tau}{\tau} \rfloor$ is the remainder of $k_j - \tau$ when divided by τ and $r_j \in \mathbb{T}$ with

$$\mathbb{T} := \{\tau, \tau + 1, \cdots, 2\tau - 1\}. \tag{4.4}$$

Motivated by this result, a parameterization of all admissible sequences can be obtained using an alternative representation of (4.2). This takes the form of

$$\mathcal{S}_\tau(t) = \{\ \{j_{p-1}\}^{\ell_{p-1}}, \cdots, \{j_1\}^{\ell_1}, \{j_0\}^{\ell_0}\} \tag{4.5}$$

for some appropriate integers $\ell_0, \ell_1, \cdots \ell_{p-1}$ with $\sum_{i=0}^{p-1} \ell_i = t$ where each $\ell_i \in \mathbb{T}$, $j_i \in \mathcal{I}_N$ for $i = 0, \cdots, p-1$. This form shows that an admissible sequence is a concatenation of p-stage subsequences (as opposed to a m-mode subsequences of (4.2)): the first stage is in mode j_0 for ℓ_0 steps, the second in mode j_1 for ℓ_1 steps and so on with the possibility that $j_i = j_{i+1}$. For example, $\mathcal{S}_3(10) = \{\{1\}^4, \{2\}^6\}$ with $\mathcal{I}_N = \{1, 2\}$ can be represented as $\{\{1\}^4, \{2\}^3, \{2\}^3\}$ in the format of (4.5). Such a representation facilitates the representation of all admissible sequences up till time t. For this purpose, several operations are introduced. They are slight modifications of well-known one-step forward (backward) operator for standard linear system.

Given a set $\Omega \subset \mathbb{R}^n$, let $\hat{P}(\Omega, A, W) := \{Ax + w : x \in \Omega, w \in W\} = A\Omega \oplus W$ be the set of reachable states in one time step from Ω with respect to system $x(t+1) = Ax(t) + w(t)$ driven by disturbance $w(\cdot) \in W$. Repeating this operation ℓ times lead

to the ℓ-step reachable set of Ω given by

$$\begin{aligned}\hat{P}_\ell(\Omega, A, W) &= \{A^\ell x + A^{\ell-1}w + \cdots + Aw + w : x \in \Omega, w \in W\} \\ &= A^\ell\Omega \oplus A^{\ell-1}W \oplus \cdots \oplus AW \oplus W \end{aligned} \tag{4.6}$$

In the case of (4.5), mode j_i can be any index of $\mathcal{I}_N$ and ℓ_i is any element in $\mathbb{T}$. This motivates the definition of

$$P(\Omega, W) := \bigcup_{\ell\in\mathbb{T}} \Big\{ \bigcup_{i\in\mathcal{I}_N} \hat{P}_\ell(\Omega, A_i, W)\Big\} = \bigcup_{\ell\in\mathbb{T}, i\in\mathcal{I}_N} \hat{P}_\ell(\Omega, A_i, W) \tag{4.7}$$

and it characterizes the reachable set of one stage based on the representation given by (4.5). This means that $x(\ell_0) \in P(\{0\}, W)$ and

$$x(\ell_1 + \ell_0) \in \bigcup_{\ell\in\mathbb{T}, i\in\mathcal{I}_N} \hat{P}_\ell(P\big(\{0\}, W), A_i, W\big) = P\big(P(\{0\}, W), W\big) = P_2(\{0\}, W).$$

This continues till the p-th stage where

$$x(\ell_{p-1} + \cdots + \ell_0) \in \bigcup_{\ell\in\mathbb{T}, i\in\mathcal{I}_N} \hat{P}_\ell\Big(P_{p-1}(\{0\}, W), A_i, W\Big) = P_p(\{0\}, W). \tag{4.8}$$

Another interpretation of the above is that the family of *all* admissible sequences up to time $p(2\tau - 1)$ is

$$\bigcup_{\ell_0\in\mathbb{T},\cdots,\ell_{p-1}\in\mathbb{T}} \left(\bigcup_{j_0\in\mathcal{I}_N,\cdots,j_{p-1}\in\mathcal{I}_N} \{ \{j_{p-1}\}^{\ell_{p-1}}, \cdots, \{j_1\}^{\ell_1}, \{j_0\}^{\ell_0}\} \right) \tag{4.9}$$

The above analysis is based on the forward operation of $\hat{P}(\cdot, \cdot, \cdot)$. Another operation needed in the sequel is that given by the one-step backward operator. Formally, this one-step and ℓ-step backward sets of a given non-empty $\Omega \subset \mathbb{R}^n$ w.r.t. system $x(t+1) = Ax(t) + w(t)$ are known respectively to be $\hat{Q}(\Omega, A, W) = \{x : Ax + w \in \Omega, w \in W\} =$

$\{x : Ax \in (\Omega \ominus W)\}$ and

$$\begin{aligned}\hat{Q}_\ell(\Omega, A, W) = \hat{Q} \cdots \hat{Q}(\Omega, A, W) &= \{x : A^\ell x + \cdots + Aw + w \in \Omega, w \in W\} \\ &= \left\{x : A^\ell x \in \left(\Omega \ominus W \ominus \cdots \ominus A^{\ell-1}W\right)\right\} \end{aligned} \tag{4.10}$$

In the characterization of (4.5), the first stage consists of ℓ_0 time steps where ℓ_0 can be any element in $\mathbb{T}$ while j_0 can be any element of $\mathcal{I}_N$. Hence, the set of state that can be brought into Ω for one stage of an admissible sequence is

$$Q(\Omega, W) := \bigcap_{\ell \in \mathbb{T}} \Big\{ \bigcap_{i \in \mathcal{I}_N} \hat{Q}_\ell(\Omega, A_i, W) \Big\} = \bigcap_{\ell \in \mathbb{T}, i \in \mathcal{I}_N} \hat{Q}_\ell(\Omega, A_i, W) \tag{4.11}$$

and it is the backward set for one stage in an admissible sequence.

Theorem 4.1 *Suppose (A1), (A4) and (A5) are satisfied and a non-empty set Ω is given. Let $P(\cdot,\cdot)$ and $Q(\cdot,\cdot)$ be as defined by (4.7) and (4.11) respectively. The following statements are equivalent:*
(i) A set $\Omega \subset \mathbb{R}^n$ is DDT-invariant for system (4.1a);
(ii) $P(\Omega, W) \subseteq \Omega$;
(iii) $\Omega \subseteq Q(\Omega, W)$.

Theorem 4.1 shows that $x(0) \in \Omega$ implies $x(t) \in \Omega$ for all $t \in \mathbb{T}$. However, no mention is made of the constraints $x(t) \in X$ for all t as stipulated in (4.1b). Clearly, the constraint admissibility requires more conditions than $\Omega \subseteq X$. Imposing $x(t) \in X$ for $t = 0, 1, \cdots, \tau - 1$ ensures that Ω is CADDT-invariant. This result is therefore obvious and stated in the following corollary without a proof.

Corollary 4.1 *Suppose (A1), (A4) and (A5) are satisfied and a non-empty set Ω is given. Let $\hat{P}_\ell(\cdot,\cdot,\cdot)$ and $\hat{Q}_\ell(\cdot,\cdot,\cdot)$ be as defined by (4.6) and (4.10) respectively. A DDT-invariant set $\Omega \subseteq X$ is CADDT-invariant for system (4.1) with dwell-time τ, if*

and only if (i) $\hat{P}_\ell(\Omega, A_i, W) \subseteq X$ *for all* $\ell = 0, 1, \cdots, \tau - 1$ *and for all* $i \in \mathcal{I}_N$ *or (ii)* $\Omega \subseteq \hat{Q}_\ell(X, A_i, W)$ *for all* $\ell = 0, 1, \cdots, \tau - 1$ *and for all* $i \in \mathcal{I}_N$.

4.4 Minimal DDT-invariant set and its computation

The solution of (4.1a) is

$$\begin{aligned} x(t) = {} & A_{\sigma(t-1)} A_{\sigma(t-2)} \cdots A_{\sigma(1)} A_{\sigma(0)} x(0) + A_{\sigma(t-1)} \cdots A_{\sigma(1)} w(0) + \\ & A_{\sigma(t-1)} \cdots A_{\sigma(2)} w(1) + \cdots + A_{\sigma(t-1)} w(t-2) + w(t-1). \end{aligned} \tag{4.12}$$

The first term on the righthand side of (4.12) approaches zero as t approaches infinity for any admissible switching sequence under (A4). The sum of the rest of the terms on the righthand side of (4.12) characterizes the asymptotic behavior of switching system (4.1a) in the presence of disturbance sequences. Let $F_t(\mathcal{A}, W, \tau)$ be the set of states that can be reached in t steps from the origin for all admissible sequences with dwell-time τ and all disturbance sequences of length t. Using (4.12) with $x(0) = 0$, it follows that

$$F_t(\mathcal{A}, W, \tau) := \bigcup_{\sigma(\cdot) \in \mathcal{S}_\tau(t)} \left(A_{\sigma(t-1)} A_{\sigma(t-2)} \cdots A_{\sigma(1)} W \oplus \cdots \oplus A_{\sigma(t-1)} W \oplus W \right) \tag{4.13}$$

with $F_0(\mathcal{A}, W, \tau) := \{0\}$. For notational simplicity, the dependence of F_t and other derived sets on $(\mathcal{A}, W, \tau)$ will be dropped unless warranted by context. The limiting condition of (4.13), existence of which is shown in Theorem 4.2, becomes

$$F_\infty = \lim_{t \to \infty} F_t \tag{4.14}$$

Hence, F_∞ characterizes the asymptotical behavior of (4.1) and, typically, a small F_∞ set is desirable. The computation of F_t based on the elapsed time t for system (4.1) is difficult because it has no clear structure. A more useful representation is that given by (4.5) which characterizes the reachable states by stages instead of time. Let $\mathcal{F}_k$ be the set of reachable states at the k-th stage. Using the same reasoning in Section 4.3 leading to equation (4.9), define

$$\mathcal{F}_k := P(\mathcal{F}_{k-1}, W) = \bigcup_{\ell \in \mathbb{T}, i \in \mathcal{I}_N} \hat{P}_\ell(\mathcal{F}_{k-1}, A_i, W) \tag{4.15}$$

with $\mathcal{F}_0 = \{0\}$. Since the above union operation is taken over all $\ell \in \mathbb{T}$ and all $i \in \mathcal{I}_N$, (4.15) captures all admissible sequences of length $k(2\tau - 1)$ and hence, $\mathcal{F}_k = F_{k(2\tau-1)}$ for all $k \in \mathbb{Z}^+$. Taking the limit as $k \to \infty$, $F_\infty = \lim_{t\to\infty} F_t = \lim_{k\to\infty} F_{k(2\tau-1)} = \lim_{k\to\infty} \mathcal{F}_k = \mathcal{F}_\infty$.

The union operation of (4.15) remains problematic computationally as it does not preserve convexity. This problem can be circumvented by computing a convex outer-bound of $\mathcal{F}_k$, denoted by $\mathbb{F}_k$ in the form of $\mathbb{F}_k := co\{\mathcal{F}_k\}$. Similarly, $\mathbb{F}_\infty := \lim_{k\to\infty} \mathbb{F}_k$. Conceptually, the procedure of computing $\mathbb{F}_\infty$ is to first compute $\mathcal{F}_k$ based on (4.15) at every stage k and then compute its convex hull, starting from $k = 0$. The exact algorithmic computation of $\mathbb{F}_\infty$ is given below.

Algorithm 6 Computation of $\mathbb{F}_\infty$

`Input:` $\mathcal{A}$, X, W and τ.

(a) `Set` $k = 0$, $\ell = \tau$, $i = 1$, $\mathbb{F}_0 = \{0\}$, $\mathbb{F}_1 = \{0\}$.

(b) `While` $\ell \leq 2\tau - 1$,
  `While` $i \leq N$,
    $\mathbb{F}_{k+1} = co\{\, \mathbb{F}_{k+1}, \hat{P}_\ell\,(\mathbb{F}_k, A_i, W)\,\}$
  `next` i
`next` ℓ

(c) `If` $\mathbb{F}_{k+1} \equiv \mathbb{F}_k$, `set` $\mathbb{F}_\infty = \mathbb{F}_k$ `and stop, else set` $k = k+1$ `and goto step` (b).

Clearly, step **(b)** of Algorithm 6 computes

$$\mathbb{F}_{k+1} = co\{P(\mathbb{F}_k, W)\} = co\{\hat{P}_\ell\,(\mathbb{F}_k, A_i, W) : \ell \in \mathbb{T} \,\text{and}\, i \in \mathcal{I}_N\}. \tag{4.16}$$

This step can be computed when W is a polytope under (A5). Properties of the $\mathbb{F}_\infty$ set obtained from Algorithm 6 are stated next.

Theorem 4.2 *Suppose system (4.1) satisfies assumptions (A1), (A4), (A5) and $\mathbb{F}_k$ is generated based on Algorithm 6. The following properties hold:*

(i) $\mathbb{F}_k \equiv co\{\mathcal{F}_k\}$ *for all* k.

(ii) $0 \in \mathbb{F}_k$ *and* $\mathbb{F}_k \subseteq \mathbb{F}_{k+1}$ *for all* k.

(iii) $\mathbb{F}_k \supseteq \mathcal{F}_k = F_{k(2\tau-1)}$ *for all* k.

(iv) $\mathcal{F}_\infty := \lim_{k\to\infty} \mathcal{F}_k$ *exists and it is bounded.*

(v) $\mathbb{F}_\infty := \lim_{k\to\infty} \mathbb{F}_k$ *exists and the set sequence* $\{\mathbb{F}_k : k \in \mathbb{Z}^+\}$ *of Algorithm 6 converges to* $\mathbb{F}_\infty$.

(vi) $\mathbb{F}_\infty = co\{F_\infty\}$.

(vii) $\mathbb{F}_\infty$ *is DDT-invariant.*

(viii) $\mathbb{F}_\infty$ *is the minimal convex DDT-invariant set.*

(ix) The state of system (4.1a) starting from any $x(0)$ *converges to* F_∞ *for every admissible sequence in the sense that* $d(x(t), F_\infty) \to 0$ *as* $t \to \infty$.

4.5 Maximal Constraint Admissible DDT-invariant set

This section deals with the characterization and computation of the maximal constraint admissible DDT-invariant set, $\mathbb{O}_\infty(\mathcal{A}, X, W, \tau)$, for system (4.1). This set defines the largest region starting from which system (4.1) remains constraint admissible for all admissible sequences. A necessary assumption for the existence of such a set is that **(A6)** $F_\infty \subset X$ and is CADDT-invariant. This is a reasonable assumption requiring that the effect of disturbance be small and not to exceed the constraint set.

Let $\mathbb{O}_1(\mathcal{A}, X, W, \tau)$ be the set of states that can be brought into constraint set X in one stage for system (4.1) under an appropriate $\mathcal{S}_\tau(t)$. This means that $x(t) \in X$ if $x(0) \in \mathbb{O}_1(\mathcal{A}, X, W, \tau)$ for all appropriate t for one stage. Using (4.5), $x(0)$ belongs to the set $\mathbb{O}_1(\mathcal{A}, X, W, \tau) := Q(X, W) = \bigcap_{\ell \in \mathbb{T}, i \in \mathcal{I}_N} \hat{Q}_\ell(X, A_i, W)$ since (j_0, ℓ_0) of (4.5) can be any element of $\mathbb{T} \times \mathcal{I}_N$ and $x(t) \in X$ has to be satisfied for all such sequences. Using the above recursively leads to

$$\begin{aligned} \mathbb{O}_k(\mathcal{A}, X, W, \tau) = & \quad Q(\mathbb{O}_{k-1}(\mathcal{A}, X, W, \tau), W) \\ = & \bigcap_{\ell \in \mathbb{T}, i \in \mathcal{I}_N} \hat{Q}_\ell(\mathbb{O}_{k-1}(\mathcal{A}, X, W, \tau), A_i, W) \end{aligned} \tag{4.17}$$

with $\mathbb{O}_0 := X \bigcap_{i \in \mathcal{I}_N, \ell = 1, 2, \cdots \tau - 1} \hat{Q}_\ell(X, A_i, W)$. The detailed algorithmic computation of $\mathbb{O}_\infty$ is given in Algorithm 7. Hereafter, the dependence of $\mathbb{O}_k$ on $(\mathcal{A}, X, W, \tau)$ is dropped for notational convenience unless needed.

Algorithm 7 Computation of maximal CADDT-invariant set

`Input:` $\mathcal{A}$, X, W and τ.

(a) `Set` $k = 0$, $\ell = 1$, $i = 1$, $\mathbb{O}_0 = X$.

(b) `While` $\ell \leq \tau - 1$,
  `While` $i \leq N$,
    $\mathbb{O}_0 = \mathbb{O}_0 \cap \hat{Q}_\ell(X, A_i, W)$
  `next` i
`next` ℓ

(c) `Set` $\ell = \tau$, $i = 1$, $\mathbb{O}_{k+1} = \mathbb{O}_k$.

(d) `While` $\ell \leq 2\tau - 1$,
  `While` $i \leq N$,
    $\mathbb{O}_{k+1} = \mathbb{O}_{k+1} \cap \hat{Q}_\ell(\mathbb{O}_k, A_i, W)$
  `next` i
`next` ℓ

(e) `If` $\mathbb{O}_{k+1} \equiv \mathbb{O}_k$, `set` $\mathbb{O}_\infty = \mathbb{O}_k$ `then stop, else set` $k = k + 1$ `and goto step` (c).

Step **(b)** of Algorithm 7 imposes the constraints for the first $\tau - 1$ steps to ensure the constraint admissibility of $\mathbb{O}_\infty$ according to Corollary 4.1. Similarly, step **(d)** imposes (4.17) and captures all possible admissible switching sequences.

When X and W are polytopes under assumptions (A2) and (A5), so is $\mathbb{O}_k$. The corresponding numerical operations for each step of Algorithm 7 are also straight forward, including the computation of $\hat{Q}(X, A, W)$ (see [57]). More exactly, $\hat{Q}(X, A, W) = \{x : R(Ax + w) \leq \mathbf{1}, \forall w \in W\} = \{x : RAx \leq \mathbf{1} - \max_{w \in W} Rw\}$. Hence,

$$\hat{Q}_\ell(X, A_i, W) = \left\{x : RA_i^\ell x \leq \mathbf{1} - \max_{w \in W} Rw - \max_{w \in W} RA_i w - \cdots - \max_{w \in W} RA_i^{\ell-1} w\right\}. \tag{4.18}$$

If $\left(1 - \max_{w \in W} R_j A_i w - \cdots - \max_{w \in W} R_j A_i^{r-1} w\right)$ of (4.18) is negative for any of its rows, Algorithm 7 terminates with $\mathbb{O}_\infty = \emptyset$. While not stated in Algorithm 7, fewer computations results if redundant inequalities are removed from $\mathbb{O}_{k+1}$ at the end of step (2). Properties of the $\mathbb{O}_\infty$ obtained from Algorithm 7 are stated next.

Theorem 4.3 *Suppose system (4.1) satisfies assumptions (A1)-(A5) and $\mathbb{O}_k$ is generated based on Algorithm 7, such that $\mathbb{O}_k \neq \emptyset$ for all k. Then, the following results are known:*

(i) $\mathbb{O}_k \subset X$ and $\mathbb{O}_{k+1} \subseteq \mathbb{O}_k$ for all k.

(ii) $\mathbb{O}_\infty := \lim_{k\to\infty} \mathbb{O}_k$ exists, contains the origin in its interior and is finitely determined.

(iii) $\mathbb{O}_\infty$ is the largest CADDT-invariant set contained in X.

(iv) For every $x(0) \in \mathbb{O}_\infty$, $x(t)$ converges to F_∞ for every admissible switching sequence in the sense that $d(x(t), F_\infty) \to 0$ as $t \to \infty$.

Remark 4.1 *Suppose system (4.1) satisfies assumptions (A1)-(A5) and $\mathbb{O}_\infty \neq \emptyset$. Then, minimality of F_∞ implies that $F_\infty \subset \mathbb{O}_\infty \subset X$. Conversely, (A6) implies the existence of at least one CADDT-invariant set in X. Thus, $\mathbb{O}_\infty \neq \emptyset$ if and only if (A6) is satisfied.*

Remark 4.2 *If $x(0) \in \mathbb{O}_\infty(\mathcal{A}, X, W, \tau)$ then $x(0) \in \mathbb{O}_\infty(\mathcal{A}, X, W, \bar{\tau})$ for any $\bar{\tau} \geq \tau$. This follows because any admissible sequence with dwell-time $\bar{\tau}$ is also an admissible sequence with dwell time τ. In addition, the trajectory starting from $x(0)$ with any admissible sequences with dwell-time $\bar{\tau}$ is also constraint admissible because $x(0) \in \mathbb{O}_\infty(\tau)$. Hence, $\mathbb{O}_\infty(\mathcal{A}, X, W, \bar{\tau}) \supseteq \mathbb{O}_\infty(\mathcal{A}, X, W, \tau)$. A similar argument but considering the union instead of intersection leads to $\mathbb{F}_\infty(\mathcal{A}, W, \bar{\tau}) \subseteq \mathbb{F}_\infty(\mathcal{A}, W, \tau)$.*

Remark 4.3 *Following (4.10), (4.11) and Algorithm 7, the following properties can be easily verified:*

(i) $\mathbb{O}_\infty(\mathcal{A}, X, W, \tau) \subseteq \mathbb{O}_\infty(\mathcal{A}, X, \bar{W}, \tau)$ for any $\bar{W} \subseteq W$;

(ii) $\mathbb{O}_\infty(\mathcal{A}, X, W, \tau) \subseteq \mathbb{O}_\infty(\bar{\mathcal{A}}, X, W, \tau)$ for any $\bar{\mathcal{A}} \subseteq \mathcal{A}$;

(iii) $\mathbb{O}_\infty(\mathcal{A}, X, W, \tau) \subseteq \mathbb{O}_\infty(\mathcal{A}, \bar{X}, W, \tau)$ for any $\bar{X} \supseteq X$;

(iv) $\mathbb{O}_\infty(\mathcal{A}, \alpha X, \alpha W, \tau) = \alpha \mathbb{O}_\infty(\mathcal{A}, X, W, \tau)$ for any $\alpha > 0$. The last property follows from $\hat{Q}(\alpha X, A, \alpha W) = \{x : R(Ax + \alpha w) \leq \alpha \mathbf{1}, \forall w \in W\} = \{x : RAx \leq \alpha(\mathbf{1} - \max_{w\in W} Rw)\} = \alpha \hat{Q}(X, A, W)$.

Remark 4.4 *Following* Remark *4.2,* $\mathbb{O}_\infty(\mathcal{A}, X, W, \tau) \subseteq \mathbb{O}_\infty(\mathcal{A}, X, W, \tau = \infty) = \cap_{i\in\mathcal{I}_N}\mathbb{O}_\infty(A_i, X, W)$ *and* $\mathbb{F}_\infty(\mathcal{A}, W, \tau) \supseteq \mathbb{F}_\infty(\mathcal{A}, W, \tau = \infty) = \cup_{i\in\mathcal{I}_N}\mathbb{F}_\infty(A_i, W)$ *where* $\mathbb{O}_\infty(A_i, X, W)$ *and* $\mathbb{F}_\infty(A_i, W)$ *are the maximal and minimal disturbance invariant sets for the standard linear system* $x(t+1) = A_i x(t) + w(t)$ *with constraint* $x(t) \in X$.

Remark 4.5 *When* A_i *matrices are uncertain and parameterized in the form of*

$$A_i \in co\{\bar{A}_{i,1}, \bar{A}_{i,2}, \cdots, \bar{A}_{i,M_i}\} \text{ for each } i \in \mathcal{I}_N.$$

Then, the robust[2] *forward/backward sets are modified as*

$$\begin{aligned}\hat{P}(\Omega, A_i, W) &= \left\{\bar{A}_{i,r}\, x + w \in \Omega, x \in \Omega, w \in W, r \in \{1, \cdots, M_i\}\right\} \\ &= \bigcup_{r=1}^{M_i} \left(\bar{A}_{i,r}\, \Omega \oplus W\right)\end{aligned}$$

$$\hat{P}_\ell(\Omega, A_i, W) = \hat{P} \cdots \hat{P}(\Omega, A_i, W)$$

$$\begin{aligned}\hat{Q}(\Omega, A_i, W) &= \left\{x : \bar{A}_{i,r}\, x + w \in \Omega, w \in W, r \in \{1, \cdots, M_i\}\right\} \\ &= \bigcap_{r=1}^{M_i} \left\{x : \bar{A}_{i,r}\, x \in \Omega \ominus W\right\}\end{aligned}$$

$$\hat{Q}_\ell(\Omega, A_i, W) = \hat{Q} \cdots \hat{Q}(\Omega, A_i, W)$$

With these modifications, Algorithm 6/7 can be used for computation of minimal/maximal CADT-invariant sets of switched systems with parametric uncertainty. Results of Theorems 4.1, 4.2 and 4.3 are also valid for uncertain system.

[2]Robustness considered here is with respect to parametric uncertainty.

4.6 Numerical Examples

The first example is on a system with $\mathcal{A} = \{A_1, A_2\}$, $A_1 = \begin{bmatrix} 0.1321 & 0.2494 \\ -2.4940 & -0.1173 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.9885 & 0.4406 \\ -0.0441 & 0.7682 \end{bmatrix}$. The constraint and disturbance sets are $X = \{x \in \mathbb{R}^2 : ||x||_\infty \leq 1\}$ and $W = \{w \in \mathbb{R}^2 : ||w||_\infty \leq 0.001\}$ respectively. It can be verified that the disturbance-free system is asymptotically stable with any dwell-time $\tau \geq 6$. Equivalently, this means that the system is unstable under arbitrary switching and existing computational techniques [24,55,56] for arbitrary switched systems cannot be used. With $\tau = 6$, both the minimal and maximal CADDT-invariant sets are computed for this system and are shown in Figure 4.1. A typical state trajectory of this example starting from $x(0) = (0.3969, 0.0769)$ is also shown. The input sequence used is periodic with $\sigma(0) = 2$, $t_{s_{k+1}} - t_{s_k} = 6$ for all $k \geq 0$ while the disturbance sequence is generated from a random uniform distribution over W.

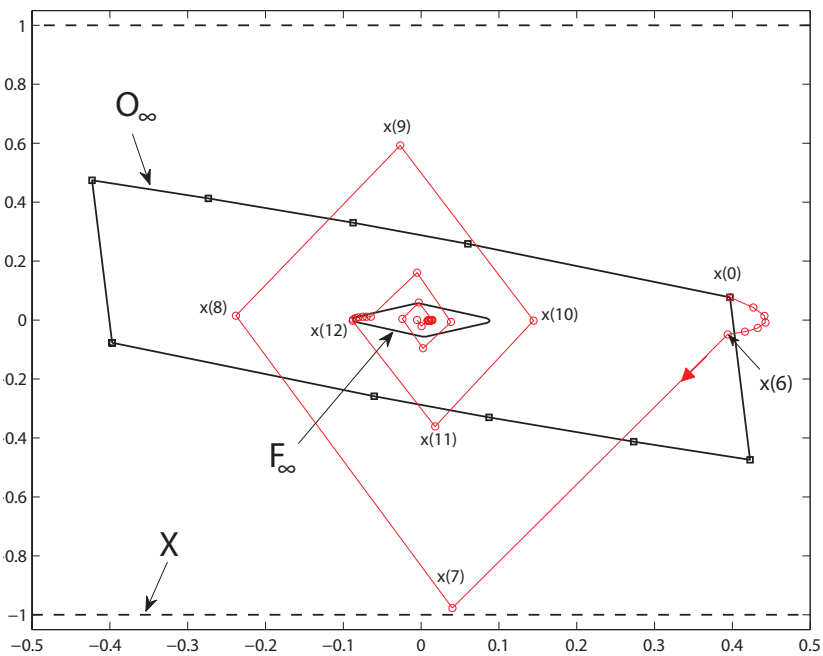


Figure 4.1: Illustration of maximal and minimal CADDT-invariant sets

It is interesting to note that $x(t) \notin \mathbb{O}_\infty$ for $t = 1, \cdots, 5$ but $x(6) \in \mathbb{O}_\infty$. Subsequently, $x(t)$ leave $\mathbb{O}_\infty$ momentarily at $t = 7, 9$ and 11. Hence, $\mathbb{O}_\infty$ is not *positively invariant* but CADDT-invariant, the behavior of which is described in *Defintion* 2.1.

The trajectories have two properties that are clearly different from those of standard invariant set: $x(t) \in X$ but not necessarily in $\mathbb{O}_\infty$ for all $t \geq 0$ and $x(t+\tau) \in \mathbb{O}_\infty$ if $x(t) \in \mathbb{O}_\infty$ for all $t \geq 0$. Finally, the state trajectories converge to F_∞, as claimed in property (iv) of *Theorem* 4.3.

Additional runs on this example are done to illustrate the points mentioned in *Remarks* 4.2 and 4.4. Figure 4.2(a) shows $\mathbb{F}_\infty$ and $\mathbb{O}_\infty$ for $\tau = 6$ and 10 respectively. Clearly, $\mathbb{O}_\infty(\tau = 10) \supseteq \mathbb{O}_\infty(\tau = 6)$ and $\mathbb{F}_\infty(\tau = 10) \subseteq \mathbb{F}_\infty(\tau = 6)$ as discussed in *Remark* 4.2. Figures 4.2(b) and 4.2(c) show the fact that $\mathbb{O}_\infty(\mathcal{A}, X, W, \tau) \subseteq \cap_{i \in \mathcal{I}_N} \mathbb{O}_\infty(A_i, X, W)$ and $\cup_{i \in \mathcal{I}_N} \mathbb{F}_\infty(A_i, W) \subseteq \mathbb{F}_\infty(\mathcal{A}, W, \tau)$ as claimed in *Remark* 4.4.

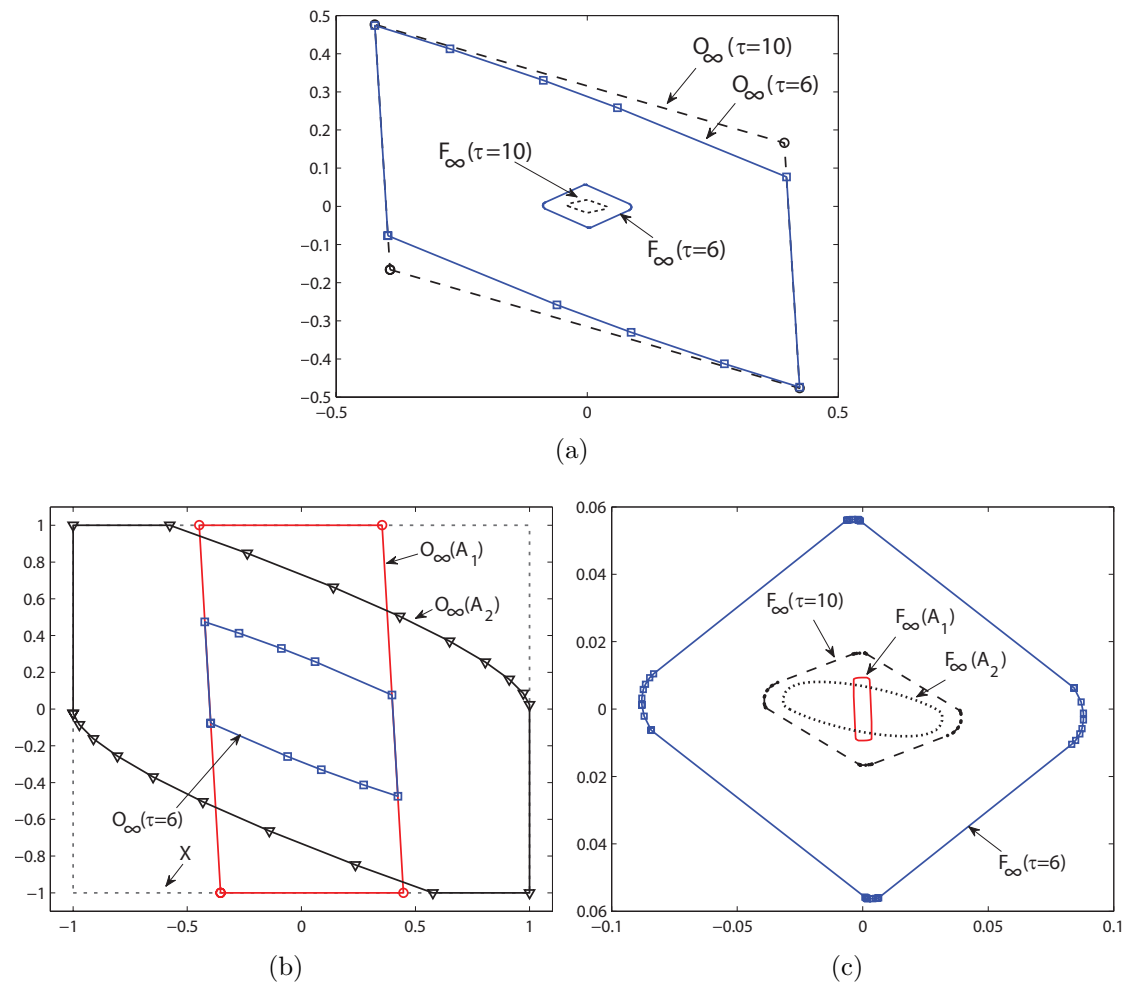


Figure 4.2: Illustration of (a) minimal/maximal CADDT-invariant sets for $\tau = 6, \tau = 10$, (b) maximal invariant set of linear subsystems, (c) minimal invariant set of linear subsystems.

					$\mathbb{O}_\infty$			$\mathbb{F}_\infty$		
Ex.	n	N	τ	$\rho_{\max}$	time	#	k_{it}	time	#	k_{it}
Ia	2	2	6	0.472	8.10	10	3	472.41	38	88
Ib	2	2	10	0.287	17.39	4	3	390.38	44	19
II	4	4	3	0.753	28.25	50	7	18148	1184	68
III	11	2	5	0.828	10415	5478	12	∞	-	-

Table 4.1: Computational results

Computational results for the examples considered in this paper are presented in Table 4.1. These results include the dimension of the problem (n), number of modes (N), a measure of the spectral radius of the system ($\rho_{\max} := \max\{\rho(A_i^\tau) : i \in \mathcal{I}_N\}$), the computational (wall-clock) time[3] in seconds, the number of inequalities (#) that represents $\mathbb{O}_\infty/\mathbb{F}_\infty$ set and the iteration (k_{it}) at which algorithms converge. Examples Ia and Ib use the system considered earlier but with dwell-time 6 and 10 respectively. Example II is a dynamical model of longitudinal flight of F8 aircraft [59] discretized with a sampling period of 0.1 second. In this example, $\mathcal{A} := \{A_1, A_2, A_3, A_4\}$ with $A_i := A + BK_i$ where K_i's are LQR controller gains obtained for $Q_1 = Q_2 = Q_3 = 0.5Q_4 = I_4$, $R_1 = 0.5R_3 = R_4 = I_2$ and $R_2 = diag([1, 0.1])$. The constraints arise from the control inputs $\{u \in \mathbb{R}^2 : ||u||_\infty \leq \frac{25\pi}{180}\}$. Example III is an 11-dimensional model of a UAV helicopter taken from [60] discretized with a sampling period of 0.02 second. Similarly, it has $A_i := A + BK_i$ where K_i's are LQR controllers obtained with $Q_1 = Q_2 = I_{11}$, $R_1 = diag([10, 0.1, 0.1, 1])$ and $R_2 = I_4$. The constraints for this example are $\{x \in \mathbb{R}^{11} : ||x||_\infty \leq 10\}$ and those arise from the control inputs $\{u \in \mathbb{R}^4 : ||u||_\infty \leq \pi/4\}$. In all the above examples, $W = \{w \in \mathbb{R}^n : \|w\|_\infty \leq 0.001\}$.

Characteristics of the complexity of Algorithm 7 are similar to that used for computing the maximal invariant set for standard linear system [57]. For example, much of the computational load is on the verification of $\mathbb{O}_{k+1} \equiv \mathbb{O}_k$; computational load increases when the dimension of the problem increases. Of course, the complexity also

[3]All the algorithms of this paper are implemented in Matlab 7 using MPT toolbox solvers [58] and the computations are performed on a dual-core CPU with 3.2 GHz processor.

increases when $\rho_{\max}$ approaches 1, N increases, or τ increases.

The computational effort for $\mathbb{F}_\infty$ is much higher than that for $\mathbb{O}_\infty$, due to the Minkowski sum and the convex-hull operations of (4.6) and (4.16) needed by Algorithm 6. Of course, the complexity also increases with the dimension of the system, see result of example II. This complexity issue is likely to remain unless significant improvement is made to the Minkowski sum and convex hull operations. For the time being, it may be desirable to develop efficient outer-approximation algorithms for $\mathbb{F}_\infty$ like those given by [61] and [55].

4.7 Summary

Definitions of a DDT-invariant set and a CADDT-invariant set are given for constrained discrete-time switching systems under input sequence that respects dwell-time consideration. Using a characterization of all admissible sequences, numerical algorithms for the computation of the minimal and maximal convex CADDT-invariant sets are provided. Examples of maximal CADDT-invariant sets are provided including a system of moderately high dimension. The minimal convex CADDT-invariant set requires the convex hull operation and can be computationally intensive for system with large dimension.

Proofs

Proof of Theorem 4.1:

$(i) \Rightarrow (ii)$: This proof is by contradiction. Suppose Ω is DDT-invariant but (ii) is not satisfied. This means there exists an $\ell \in \{\tau, \tau+1, \cdots, 2\tau-1\}$ and $i \in \mathcal{I}_N$ such that $\hat{P}_\ell(\Omega, A_i, W) \nsubseteq \Omega$. However, $\{i\}^\ell = \{i, i, \cdots, i\}$ is an admissible switching sequence and for any $x(0) \in \Omega$ it follows that $x(t) \in \hat{P}_\ell(\Omega, A_i, W) \nsubseteq \Omega$. This implies $x(t) \notin \Omega$ for some admissible switching sequence, which contradicts the DDT-invariance of Ω. $(ii) \Rightarrow (iii)$: With (4.7), condition (ii) holds means $\hat{P}_\ell(\Omega, A_i, W) \subseteq \Omega$ for all $\ell \in \mathbb{T}$ and for all $i \in \mathcal{I}_N$. Applying $\hat{Q}_t(\cdot)$ operator on both sides of the above inclusion yields

$$\hat{Q}_\ell\Big(\hat{P}_\ell(\Omega, A_i, W), A_i, W\Big) \subseteq \hat{Q}_\ell(\Omega, A_i, W), \quad \forall \ell \in \mathbb{T}, \forall i \in \mathcal{I}_N \tag{4.19}$$

because $\hat{Q}_t(\Omega_1, A, W) \subseteq \hat{Q}_t(\Omega_2, A, W)$ for any Ω_1 and Ω_2 such that $\Omega_1 \subseteq \Omega_2$. The left-hand side of (4.19) is $\hat{Q}_\ell\big(\hat{P}_\ell(\Omega, A_i, W), A_i, W\big) = \Omega$ and taking the intersection of $\hat{Q}_\ell(\Omega, A_i, W)$ over all $\ell \in \mathbb{T}$ and $i \in \mathcal{I}_N$ leads to $\Omega \subseteq Q(\Omega, W)$. $(iii) \Rightarrow (i)$: Let $x(0) \in \Omega$, this implies $x(0) \in Q(\Omega, W)$ by (iii). Consider all admissible sequence, $\mathcal{S}_\tau(t)$ of the form (4.5). It follows that $x(\ell_0) = A_{j_0}^{\ell_0}\,\Omega + A_{j_0}^{\ell_0-1} w_0 + \cdots + A_{j_0} w_{\ell_0-2} + w_{\ell_0-1} \in \Omega$ for any $j_0 \in \mathcal{I}_N$ and any $\ell_0 \in \mathbb{T}$. Repeating this for all stages until the last stage of $\ell_p - 1$ shows that $x(t) \in \Omega$. This shows that Ω is DDT-invariant. ∎

Proof of Theorem 4.2:

(i) The proof is induction. For notational simplicity, let $D(i,t) := A_i^{t-1}W \oplus A_i^{t-2}W \oplus \cdots \oplus A_iW \oplus W$. With $\mathbb{F}_0 = \mathcal{F}_0 = \{0\}$, it follows from (4.16) that $\mathbb{F}_1 = co_{i\in\mathcal{I}_N,\ell\in\mathbb{T}}\big\{D(i,\ell)\big)\big\} = co\big\{\bigcup_{i\in\mathcal{I}_N,\ell\in\mathbb{T}} D(i,\ell)\big\} = co\big\{\mathcal{F}_1\big\}$. Assume that $\mathbb{F}_k = co\{\mathcal{F}_k\}$, the proof is complete if $\mathbb{F}_{k+1} = co\{\mathcal{F}_{k+1}\}$. To show this, note that

$$\mathcal{F}_{k+1} = \bigcup_{i\in\mathcal{I}_N,\ell\in\mathbb{T}} \Big(A_i^\ell \mathcal{F}_k \oplus D(i,\ell)\Big) \tag{4.20}$$

from (4.15). $\mathcal{F}_k$ is also the reachable set at the k-th stage of all admissible sequences characterized by (4.5). Since $j_i \in \mathcal{I}_N, \ell_i \in \mathbb{T}$ in (4.5) and there are N elements in $\mathcal{I}_N$ and τ elements in $\mathbb{T}$, there are altogether $(N\tau)^k$ admissible sequences at the k-th stage. Let the reachable set for each of these sequences be $\Psi_1, \Psi_2, \cdots, \Psi_{(N\tau)^k}$. Then, $\mathcal{F}_k = \bigcup_{s=1}^{(N\tau)^k} \Psi_s$. This and (4.20) imply that $\mathcal{F}_{k+1}$ is the union of $(N\tau)^{k+1}$ sets:

$$\begin{aligned}\mathcal{F}_{k+1} &= \bigcup_{i\in\mathcal{I}_N,\ell\in\mathbb{T}} \Big(A_i^\ell \mathcal{F}_k \oplus D(i,\ell)\Big) = \bigcup_{i\in\mathcal{I}_N,\ell\in\mathbb{T}} \Big(A_i^\ell (\bigcup_{s=1}^{(N\tau)^k} \Psi_s) \oplus D(i,\ell)\Big) \\ &= \bigcup_{\substack{s=1,\cdots,(N\tau)^k \\ i\in\mathcal{I}_N,\ell\in\mathbb{T}}} \Big(A_i^\ell \Psi_s \oplus D(i,\ell)\Big) := \bigcup_{r=1}^{(N\tau)^{k+1}} \Phi_r \end{aligned} \tag{4.21}$$

From (4.16), it follows that

$$\mathbb{F}_{k+1} = co_{\substack{i\in\mathcal{I}_N, \\ \ell\in\mathbb{T}}}\Big\{\big(A_i^\ell \mathbb{F}_k \oplus D(i,\ell)\big)\Big\} = \Big\{\sum_{\substack{j=1,\cdots,(N\tau) \\ i\in\mathcal{I}_N,\ell\in\mathbb{T}}} \alpha_j\big(A_i^\ell \mathbb{F}_k \oplus D(i,\ell)\big) : \alpha_j \geq 0, \sum_{j=1}^{N\tau} \alpha_j = 1\Big\}.$$

Since $\mathbb{F}_k = co\{\mathcal{F}_k\} = \big\{ \sum_{s=1}^{(N\tau)^k} \gamma_s \Psi_s : \gamma_s \geq 0, \Sigma_{s=1}^{(N\tau)^k} \gamma_s = 1 \big\}$,

$$\begin{aligned}
\mathbb{F}_{k+1} &= \left\{ \sum_{\substack{j=1,\cdots,(N\tau)\\ i\in\mathcal{I}_N, \ell\in\mathbb{T}}} \alpha_j \Big[A_i^\ell \Big(\sum_{s=1}^{(N\tau)^k} \gamma_s \Psi_s \Big) \oplus D(i,\ell) \Big] : \alpha_j \geq 0, \gamma_s \geq 0, \sum_{j=1}^{N\tau} \alpha_j = 1, \sum_{s=1}^{(N\tau)^k} \gamma_s = 1 \right\} \\
&= \left\{ \sum_{\substack{j=1,\cdots,(N\tau), i\in\mathcal{I}_N, \ell\in\mathbb{T}\\ s=1,\ldots,(N\tau)^k}} \alpha_j \gamma_s \Big[A_i^\ell \Psi_s \oplus D(i,\ell) \Big] : \alpha_j \geq 0, \gamma_s \geq 0, \sum_{j=1}^{N\tau} \alpha_j = 1, \sum_{s=1}^{(N\tau)^k} \gamma_s = 1 \right\} \\
&= \left\{ \sum_{r=1}^{(N\tau)^{k+1}} \beta_r \Phi_r : \beta_r \geq 0, \sum_{r=1}^{(N\tau)^{k+1}} \beta_r = 1 \right\} \equiv co\{\mathcal{F}_{k+1}\}.
\end{aligned}$$

(ii) Since $0 \in W$, $0 \in P(\mathcal{F}_{k-1}, W)$ for all k from (4.15) and (4.6). This implies $0 \in \mathcal{F}_k \subseteq \mathbb{F}_k$ for all k from (4.16). Moreover, $\mathbb{F}_k \subseteq P(\mathbb{F}_k, W) \subseteq co\{P(\mathbb{F}_k, W)\} = \mathbb{F}_{k+1}$ and hence $\mathbb{F}_k \subseteq \mathbb{F}_{k+1}$ for all k.

(iii) Since step (b) of Algorithm 6 captures all admissible switching sequences of length $k(2\tau - 1)$, it is inferred that $\mathcal{F}_k = F_t$ with $t = k(2\tau - 1)$ for all $k \in \mathbb{Z}^+$. This and result (i) imply $F_{k(2\tau-1)} = \mathcal{F}_k \subseteq co\{\mathcal{F}_k\} = \mathbb{F}_k$ for all $k \in \mathbb{Z}^+$.

(v) Let $\widetilde{W} := co_{i\in\mathcal{I}_N}\{D(i,\tau), D(i,\tau+1), \cdots, D(i, 2\tau-1)\}$. Compactness of $\widetilde{W}$ and (A1) imply [57] the existence of a $\lambda \in (0,1)$, a constant μ and an appropriate norm ball $\mathcal{B}(\eta)$ such that $\widetilde{W} \subseteq \mu\mathcal{B}(\eta)$ and $A_i^\ell \widetilde{W} \subseteq \mu\lambda\mathcal{B}(\eta)$ for all $\ell \in \mathbb{T}$ and for all $i \in \mathcal{I}_N$.

From (4.16), $\mathbb{F}_{k+1} = co_{i\in\mathcal{I}_N}\Big\{ \big(A_i^\tau \mathbb{F}_k \oplus D(i,\tau)\big), \cdots, \big(A_i^{2\tau-1}\mathbb{F}_k \oplus D(i, 2\tau-1)\big) \Big\} \subseteq co_{i\in\mathcal{I}_N}\Big\{ \big(A_i^\tau \mathbb{F}_k \oplus \widetilde{W}\big), \cdots, \big(A_i^{2\tau-1}\mathbb{F}_k \oplus \widetilde{W}\big) \Big\}$.

Expanding $\mathbb{F}_k$ recursively till $\mathbb{F}_0$, the above becomes

$$\begin{aligned}
\mathbb{F}_k &\subseteq co_{\substack{i_1,i_2,\cdots,i_k\in\mathcal{I}_N\\ \ell_1,\ell_2,\cdots,\ell_k\in\mathbb{T}}} \Big\{ A_{i_k}^{\ell_k} A_{i_{k-1}}^{\ell_{k-1}} \cdots A_{i_1}^{\ell_1} \mathbb{F}_0 \oplus A_{i_k}^{\ell_k} A_{i_{k-1}}^{\ell_{k-1}} \cdots A_{i_2}^{\ell_2} \widetilde{W} \oplus \cdots \oplus A_{i_k}^{\ell_k} \widetilde{W} \oplus \widetilde{W} \Big\} \\
&\subseteq \mu\big(\lambda^{k-1} + \cdots + \lambda + 1\big)\mathcal{B}(\eta).
\end{aligned}$$

The last inclusion follows from $\widetilde{W} \subseteq \mu\mathcal{B}(\eta)$ and $A_i^\ell \widetilde{W} \subseteq \mu\lambda\mathcal{B}(\eta)$ stated earlier and $\mathbb{F}_0 = \{0\}$.

Recall that the family of compact sets in $\mathbb{R}^n$ forms a complete metric space [62] with the Hausdorff metric $\mathcal{H}(X,Y) := \max\big\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\big\}$. Thus, the inclusion $\mathbb{F}_k \subseteq \mu\big(\lambda^{k-1}+\cdots+\lambda+1\big)\mathcal{B}(\eta)$ implies that the Hausdorff distance of $\mathbb{F}_{k+1}$ and $\mathbb{F}_k$ is bounded by $\mu\lambda^k$. Hence, the set sequence $\{\mathbb{F}_k : k \in \mathbb{Z}^+\}$ is Cauchy and has a limit-set, $\mathbb{F}_\infty$, such that $\mathcal{H}(\mathbb{F}_\infty, \mathbb{F}_k) \to 0$ as $k \to \infty$.

(iv) This proof is similar to that of part (v) except for the replacement of $\mathbb{F}_k$ by $\mathcal{F}_k$ and the convex hull operator by the union operator.

(vi) Result (iv) and (v) show that the limit sets $\mathcal{F}_\infty$ and $\mathbb{F}_\infty$ exist. This and results (i), (iii) imply that $\mathbb{F}_\infty = co\,\{\mathcal{F}_\infty\} = co\,\{F_\infty\}$.

(vii) Algorithm 6 terminates when $\mathbb{F}_{k^*+1} = \mathbb{F}_{k^*}$ for some k^* and hence $\mathbb{F}_\infty = \mathbb{F}_{k^*}$. This means that $\hat{P}_\ell(\mathbb{F}_\infty, A_i, , W) \subseteq \mathbb{F}_\infty$ for all $\ell \in \mathbb{T}$ and for all $i \in \mathcal{I}_N$. DDT-invariance of $\mathbb{F}_\infty$ then follows from part (ii) of Theorem 4.1.

(viii) By definition, F_∞ is the minimal DDT-invariant set. Its convex-hull, $\mathbb{F}_\infty$, is the minimal convex DDT-invariant set.

(ix) Considering the evolution of $x(t)$ of system in (4.12), the first term approaches 0 as $t \to \infty$ and the sum of the rest of the terms correspond to a point in F_∞ for any admissible switching sequence $\mathcal{S}_\tau(t)$. This means that $d(x(t), F_\infty) \to 0$ as $t \to \infty$. ■

Proof of Theorem 4.3:

(i) This result follows directly from step (d) of Algorithm 7 and $\mathbb{O}_0 \subseteq X$.

(ii) Suppose $\mathbb{O}_0 := \{x : \bar{R}_j\, x \leq 1 \text{ for all } j \in \mathcal{J}\}$. When $\mathbb{O}_k$ is incremented to $\mathbb{O}_{k+1}$ in step (d) of Algorithm 7, additional inequalities are added to $\mathbb{O}_k$ in the form of $\hat{Q}_\ell(\mathbb{O}_k, A_i, W)$ for $\ell = \tau, \cdots, 2\tau - 1$ and for $i = 1, \cdots, N$. For each $\bar{R}_j$, $j \in \mathcal{J}$, these inequalities are of the form

$$\begin{aligned}\bar{R}_j A_{i_1}^{\ell_1} A_{i_2}^{\ell_2} \cdots A_{i_k}^{\ell_k} A_{i_{k+1}}^{\ell_{k+1}} x \leq 1 - \theta(\bar{R}_j, \ell_1, i_1) - \theta(\bar{R}_j A_{i_1}^{\ell_1}, \ell_2, i_2) \\ - \cdots - \theta(\bar{R}_j A_{i_1}^{\ell_1} \cdots A_{i_k}^{\ell_k}, \ell_{k+1}, i_{k+1})\end{aligned} \tag{4.22}$$

where $\theta(\bar{R}_j, \ell, i) := \max_{w \in W} \bar{R}_j w + \max_{w \in W} \bar{R}_j A_i w + \cdots + \max_{w \in W} \bar{R}_j A_i^{\ell-1} w$ as discussed in (4.18), $i_1, \cdots, i_{k+1} \in \mathcal{I}_N$ and $\ell_1, \cdots, \ell_{k+1} \in \mathbb{T}$. Since $\mathbb{O}_k \neq \emptyset$ for all k, the righthand side of (4.22) is greater than 0. The remainder part of this proof shows that these added inequalities are redundant to $\mathbb{O}_k$ after some sufficiently large k. If $\theta(\bar{R}_j, \ell, i) = \bar{\varepsilon}$, then $\theta(\bar{R}_j A_k, \ell, i) \leq \beta_k \lambda_k \bar{\varepsilon}$ for some $\beta_k > 0$ and $\lambda_k \in (0, 1)$ and any $k \in \mathcal{I}_N$. Hence, define $\varepsilon := \max_{\ell \in \mathbb{T}, i \in \mathcal{I}_N} \theta(\bar{R}_j, \ell, i)$. This means that there exist a $\lambda \in (0, 1)$ and a $\beta > 0$ such that

$$\begin{aligned}
&\max_{\substack{i_1, i_2 \in \mathcal{I}_N \\ \ell_1, \ell_2 \in \mathbb{T}}} \theta(\bar{R}_j A_{i_1}^{\ell_1}, \ell_2, i_2) \leq \beta \varepsilon \lambda, \\
&\max_{\substack{i_1, i_2, i_3 \in \mathcal{I}_N \\ \ell_1, \ell_2, \ell_3 \in \mathbb{T}}} \theta(\bar{R}_j A_{i_1}^{\ell_1} A_{i_2}^{\ell_2}, \ell_3, i_3) \leq \beta \varepsilon \lambda^2, \\
&\quad \vdots \\
&\max_{\substack{i_1, \ldots, i_{k+1} \in \mathcal{I}_N \\ \ell_1, \ldots, \ell_{k+1} \in \mathbb{T}}} \theta(\bar{R}_j A_{i_1}^{\ell_1} A_{i_2}^{\ell_2} \cdots A_{i_k}^{\ell_k}, \ell_{k+1}, i_{k+1}) \leq \beta \varepsilon \lambda^k
\end{aligned}$$

With this, the righthand side of (4.22) is

$$\begin{aligned} RHS &= 1 - \left[\theta(\bar{R}_j, \ell_1, i_1) + \theta(\bar{R}_j A_{i_1}^{\ell_1}, \ell_2, i_2) + \cdots + \theta(\bar{R}_j A_{i_1}^{\ell_1} A_{i_2}^{\ell_2} \cdots A_{i_k}^{\ell_k}, \ell_{k+1}, i_{k+1})\right] \\ &\geq 1 - \left[\, \varepsilon \; + \; \beta\varepsilon\lambda \; + \; \cdots \; + \; \beta\varepsilon\lambda^k\right] = 1 - \varepsilon - \beta\varepsilon\lambda \frac{1-\lambda^{k-1}}{1-\lambda} \qquad (4.23) \\ &\geq 1 - \varepsilon - \beta\varepsilon\lambda \frac{1}{1-\lambda} \qquad (4.24) \end{aligned}$$

In addition, the lefthand side of (4.22) satisfy

$$LHS = \bar{R}_j (A_{i_1}^{\ell_1} \cdots A_{i_k}^{\ell_k} A_{i_{k+1}}^{\ell_{k+1}}) x \leq \lambda^{k+1} \, \alpha \, \|x\| \, \|\bar{R}_j\| \qquad (4.25)$$

for some positive real number α. Since $0 < \lambda < 1$, there exist a sufficiently large k such that $\max\left\{\lambda^{k+1}\alpha\|x\| \; \|\bar{R}_j\| : j \in \mathcal{J}, x \in \mathbb{O}_0\right\} < (1 - \varepsilon - \beta\varepsilon\lambda\frac{1}{1-\lambda})$. Then, (4.24) and (4.25) imply that

$$\begin{aligned} LHS = \bar{R}_j (A_{i_1}^{\ell_1} \cdots A_{i_{k+1}}^{\ell_{k+1}}) x &\leq \lambda^{k+1}\alpha\|x\| \, \|\bar{R}_j\| < (1 - \varepsilon - \beta\varepsilon\lambda \frac{1}{1-\lambda}) \\ &< (1 - \varepsilon - \beta\varepsilon\lambda \frac{1-\lambda^{k-1}}{1-\lambda}) \leq RHS \end{aligned}$$

Hence, all new inequalities of (4.22) added at iteration $(k+1)$ of algorithm 7 are redundant to $\mathbb{O}_k$ and $\mathbb{O}_k = \mathbb{O}_{k+1}$ and $\mathbb{O}_\infty$ is finitely determined. The result of $0 \in \mathbb{O}_\infty$ follows from $0 \in \mathbb{O}_0$ and $0 \in \hat{Q}_\ell(\mathbb{O}_k, A_i, W)$ for all $\ell \in \mathbb{T}$ and for all $i \in \mathcal{I}_N$.

(iii) Algorithm 7 terminates when $\mathbb{O}_{k^*+1} = \mathbb{O}_{k^*} = \mathbb{O}_\infty$ for some k^* and hence $\mathbb{O}_\infty \subseteq \hat{Q}_\ell(\mathbb{O}_\infty, A_i, W)$ for all $\ell \in \mathbb{T}$ and for all $i \in \mathcal{I}_N$. DDT-invariance of $\mathbb{O}_\infty$ then follows from part (iii) of Theorem 4.1. Moreover, step (b) of algorithm 2 ensures that $\mathbb{O}_\infty$ is constraint admissible for all of the first $\tau - 1$ steps. This and DDT-invariance of $\mathbb{O}_\infty$ together, imply CADDT-invariance of $\mathbb{O}_\infty$. The proof of $\mathbb{O}_\infty$ being maximal is by contradiction. Suppose $\mathbb{O}_\infty$ is not maximal and there exist a CADDT-invariant set $\mathbb{O}^* \subseteq X$ such that $\mathbb{O}^* \nsubseteq \mathbb{O}_\infty$. Since $\mathbb{O}^*$ must be constraint admissible for any switching sequence that is less than τ, $\mathbb{O}^* \subset \mathbb{O}_0$. Let $x \in \mathbb{O}^*$. As $\mathbb{O}^*$ is DDT-

invariant, $x(t) \in \mathbb{O}^* \subset \mathbb{O}_0$ for all $\ell = \tau, \cdots, 2\tau - 1$ and for all $i \in \mathcal{I}_N$. This implies that $x \in \hat{Q}_\ell(\mathbb{O}_0, A_i, W)$ for all $\ell \in \mathbb{T}$ and for all $i \in \mathcal{I}_N$, or, $x \in \mathbb{O}_1$. Therefore, $\mathbb{O}^* \subseteq \mathbb{O}_1$. Repeating the above argument shows that $\mathbb{O}^* \subseteq \mathbb{O}_k$ for all k and hence $\mathbb{O}^* \subseteq \lim_{k\to\infty} \mathbb{O}_k = \mathbb{O}_\infty$ which violates $\mathbb{O}^* \nsubseteq \mathbb{O}_\infty$.

(iv) The result follows from (A6) that $F_\infty \subseteq \mathbb{F}_\infty \subset \mathbb{O}_\infty$ and property (ix) of Theorem 4.2. ■

Chapter 5

Domain of Attraction of Saturated Switched Systems

This chapter proposes two approaches to compute domain of attraction (DOA) of switched systems under dwell-time switching and in the presence of saturation nonlinearity. The first approach uses linear difference inclusions (LDI) to represent the saturation nonlinearity. Accordingly, we derive sufficient conditions, in terms of linear matrix inequalities (LMIs), for asymptotic stability of the switched system that simultaneously enlarge the DOA. The second approach generalizes the concept of Saturated and Non-Saturated (SNS) invariance [63] and SNS-domain of attraction for switched systems under dwell-time switching. An algorithm for computing the maximal SNS domain of attraction is also provided. In addition, it is shown that any DOA obtained from LDI approach is a subset of SNS-domain of attraction and hence SNS approach is less conservative but requires additional computations.

5.1 Introduction

Consider the discrete-time switched systems with saturation nonlinearity in the form of

$$\begin{cases} x(t+1) = A_{\sigma(t)}\, x(t) + B_{\sigma(t)} sat(u(t)) \\ \quad u(t) = K_{\sigma(t)}\, x(t) \end{cases} \tag{5.1}$$

where $\sigma(t) : \mathbb{Z}^+ \to \mathcal{I}_N := \{1, \cdots, N\}$ is a time-dependent switching signal that satisfies the common dwell-time restriction. Symbol $sat(\cdot)$ is the standard vector-valued saturation function, i.e., $sat(u) = [sat(u_1), \cdots, sat(u_m)]^T$, with $sat(u_j) = sgn(u_j) \min\{1, |u_j|\}$. Without loss of generality, the saturation limit is normalized to one, by appropriately scaling the B_σ and K_σ matrices.

Most of the literature of switched systems is concerned with conditions that ensure stability of the unsaturated system (5.1) when switching signal $\sigma(\cdot)$ is arbitrary [7, 12, 23], or when it satisfies some dwell-time restrictions [36, 37, 46]. The earlier chapters of this thesis extended those results to include the case where constraints are present. A special case of constraint is the saturation limit of the control. Such a constraint is common in many applications and may cause instability and/or performance degradation of the system. Consequently, estimation of the DOA of (5.1) is important and has received the attention of many researchers (see, e.g., [64–68]).

While several approaches have been proposed to handle saturation nonlinearity, two of them appear promising. The first approach is the polytopic representation of saturation nonlinearity (see, e.g., [69–71]). Accordingly, the saturation nonlinearity can be represented as a linear differential/difference inclusion (LDI). Such a representation allows one to use conventional tools for linear systems to deal with the saturated systems [70]. The second approach is based on the concept of SNS-invariance introduced by Alamo et al. for a single linear system [63]. They also show its role in enlarging the DOA.

Although the above mentioned approaches has been used for switched systems under arbitrary switching, (see, e.g., [64–67] for LDI approach and [68] for SNS approach), the extension of these methods for dwell-time switched systems has not been done due to the complex structure of switching sequences with dwell-time restriction. To the best of our knowledge there are very few results on such systems [72, 73]. These results are somewhat conservative and a comparison with these results is presented in section 5.6.

The following additional notations are used in this chapter. Given an integer $m \geq 1$, define $\mathcal{V}_m := \{S : S \subseteq \{1, ..., m\}\}$ as the set of all subsets of $\{1, ..., m\}$. Hence, there 2^m elements in $\mathcal{V}_m$ including the empty set, $\{\emptyset\}$. Also $S^c = \{j \in \{1, ..., m\} : j \notin S\}$ is the complement of S in $\mathcal{V}_m$. Given a vector $x \in \mathbb{R}^n$, x_j is the j-th element of x and $|x|$ is the absolute operator that is applied element-wise. Given a matrix $Y \in \mathbb{R}^{m \times n}$, $Y^{i\bullet}$ is the i-th row and $Y^{\bullet j}$ is the j-th column of Y.

5.2 Preliminaries

Recall from Chapter 4 that a set Ω is DT-contractive if the solution of the system after t-steps for $t = \tau, \cdots, 2\tau - 1$ is inside Ω. Characterization of DT-contractive sets of system (5.1) follow this result, but requires additional operators and hence, notations. Consider the i-th mode of (5.1),

$$x(t+1) = A_i\, x(t) + B_i\, sat(K_i x(t)) \tag{5.2}$$

Then the successor state of x, $F_i(x)$, is

$$F_i(x) = A_i\, x + B_i\, sat(K_i x).$$

Repeating the above leads to

$$\begin{aligned} F_i^2(x) &= F_i(F_i(x)) = A_i F_i(x) + B_i\, sat(K_i F_i(x)) \\ &\vdots \\ F_i^t(x) &= F_i(F_i^{t-1}(x)) = F_i(F_i(\cdots F_i(x))) \end{aligned} \tag{5.3}$$

where $F_i^t(x)$ is the state evolution of (5.1) after t-steps with $x(0) = x$ and $\mathcal{S}_\tau(t) = \{i, i, \cdots, i\}$. Then, a set Ω is DT-contractive (with contraction $\lambda \in (0,1)$) w.r.t. system (5.1) if and only if for every $x \in \Omega$, $F_i^t(x) \in \lambda\Omega$ for all $i \in \mathcal{I}_N$ and for all $t \in \mathbb{T} = \{\tau, \tau+1, \cdots, 2\tau-1\}$.

5.3 LDI approach

In this section, we generalize the LDI approach for estimating DOA of switched system (5.1) under dwell-time switching. LDI approach uses auxiliary terms and exploits their convex hull to represent the saturation function as summarized in the following lemmas:

Lemma 5.1 *[70, 71] Single-input case ($u \in \mathbb{R}$): For all $u, v \in \mathbb{R}$ such that $|v| \leq 1$,*

$$sat(u) \in co\,\{u, v\}$$

Lemma 5.2 *[70] Multi-input case ($u \in \mathbb{R}^m$): For any $S \in \mathcal{V}_m$, define D_S to be the $m \times m$ diagonal matrix with diagonal elements $D_S(j,j)$, whose value is 1 if $j \in S$ and 0 otherwise. Also define $D_{S^c} = I_m - D_S$. Then, for all $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^m$ such that $|v_j| \leq 1$ for all $j \in \{1, \cdots, m\}$:*

$$sat(u) \in co\,\{D_{S^c} u + D_S\, v : \forall S \in \mathcal{V}_m\} \tag{5.4}$$

To illustrate the main idea of the LDI approach, consider any $u \in \mathbb{R}^2$ as an example. According to Lemma 5.2, for any $v \in \mathbb{R}^2$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $|v_1| \leq 1, |v_2| \leq 1$, it follows that

$$sat\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) \in co\left\{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\}.$$

In other words, the above lemma states that $sat(u)$ can be expressed as a convex hull of vectors formed by choosing some rows (those belonging to S) from v and the rest (those belonging to S^c) from u. Using (5.4) and assuming that $u = K_i x$ and v is replaced by some linear function $H_i x$, it follows that

$$sat(K_i\, x) \in co\,\{D_{S^c} K_i\, x + D_S\, H_i\, x : \forall S \in \mathcal{V}_m\} \tag{5.5}$$

for all $x \in \mathcal{L}(H_i) := \{x : \|H_i x\|_\infty \leq 1\}$. Define

$$E_{i,H_i}(x, S) := \Big(A_i + \sum_{j \in S^c} B_i^{\bullet j} K_i^{j\bullet}\Big)x + \Big(\sum_{j \in S} B_i^{\bullet j} H_i^{j\bullet}\Big)x \tag{5.6}$$

and it follows from (5.5) and (5.6) that for every $x \in \mathcal{L}(H_i)$,

$$F_i(x) = A_i\, x + B_i\, sat(K_i\, x) \in co\{E_{i,H_i}(x, S) : \forall S \in \mathcal{V}_m\}. \tag{5.7}$$

It is known that saturated system (5.1) is asymptotically stable under arbitrary switching if a common quadratic Lyapunov function $V(x) = x^\top P\, x$ exists such that

$$V(F_i(x)) < V(x), \qquad \forall i \in \mathcal{I}_N \tag{5.8}$$

Since $V(x)$ is a convex function and $F_i(x)$ is a convex combination of $E_{i,H_i}(x, S)$ according to (5.7), condition (5.8) is satisfied if $V(E_{i,H_i}(x, S)) < V(x)$ for each $i \in \mathcal{I}_N$ and for each $S \in \mathcal{V}_m$. The following theorem, taken from [64], summarizes this result.

Theorem 5.1 *[64] If there exist $P \succ 0$ and matrices H_i for each $i \in \mathcal{I}_N$ such that*

$$[E_{i,H_i}(x,S)]^\top P[E_{i,H_i}(x,S)] - x^\top P\, x < 0, \qquad \forall i \in \mathcal{I}_N, \forall S \in \mathcal{V}_m \tag{5.9}$$

$$\mathcal{E}(P) \subseteq \mathcal{L}(H_i), \qquad \forall i \in \mathcal{I}_N \tag{5.10}$$

Then, (i) the origin of the (5.1) is asymptotically stable under arbitrary switching, (ii) $\mathcal{E}(P) := \{x : x^\top P\, x \leq 1\}$ is the estimate of DOA of the system.

The DOA, which is inside $\mathcal{L}(H_i)$, can be enlarged by maximizing the size of the ellipsoid $\mathcal{E}(P)$. The key point here is to choose the auxiliary variables H_i's so that the size of DOA is maximized. Since $E_{i,H_i}(x,S)$ of (5.9) is a linear function of the variable H_i, the above constraints can be converted into LMIs and solved with convex optimization routines.

If the switched system is unstable under arbitrary switching, condition (5.9) becomes infeasible. This is because the strictly decreasing requirement of the Lyapunov function is too restrictive. This requirement can be relaxed by requiring the Lyapunov function to be decreasing after t-steps for $t = \tau, \tau+1, \cdots, 2\tau-1$.

While the stability condition is reasonable, the LDI representation of $F_i^t(x)$ is difficult as $F_i^t(x)$ consists of t nested saturation functions. The rest of this section describes the LDI representation of $F_i^t(x)$ by introducing t auxiliary variables $H_{i,1}, \cdots, H_{i,t}$. Each of these variables are introduced for LDI representation of one of the nested saturations.

Consider $F_i^2(x)$ and suppose that $H_{i,1}$ and $H_{i,2}$ are associated for LDI representation of $sat(K_i x)$ and $sat(K_i F_i(x))$, respectively. Define

$$E_{i,H_{i,2}}\big(F_i(x),S\big) := \Big(A_i + \sum_{j\in S^c} B_i^{\bullet j} K_i^{j\bullet}\Big) F_i(x) + \Big(\sum_{j\in S} B_i^{\bullet j} H_{i,2}^{j\bullet}\Big) x \tag{5.11}$$

Then, from (5.6) and (5.11), it follows that

$$F_i(x) \in co\{E_{i,H_{i,1}}(x, S_1) : \forall S_1 \in \mathcal{V}_m\}, \qquad \forall x \in \mathcal{L}(H_{i,1}) \tag{5.12}$$

$$F_i^2(x) = F_i(F_i(x)) \in co\{E_{i,H_{i,2}}(F_i(x), S_2) : \forall S_2 \in \mathcal{V}_m\}, \qquad \forall x \in \mathcal{L}(H_{i,2}) \tag{5.13}$$

Since $F_i^2(x)$ is represented by the convex-hull of $E_{i,H_{i,2}}(F_i(x), S_2)$, and $F_i(x)$ is by itself a convex-hull of $E_{i,H_{i,1}}(x, S_1)$, it is straightforward[1] to expand $F_i^2(x)$ as

$$F_i^2(x) \in co\{E_{i,H_{i,2}}\big(E_{i,H_{i,1}}(x, S_1), S_2\big) : \forall S_1, S_2 \in \mathcal{V}_m\}, \qquad \forall x \in \mathcal{L}(H_{i,1}) \cap \mathcal{L}(H_{i,2}). \tag{5.14}$$

An example that illustrates this is given next. Consider a single-input system where $m = 1$ and hence $\mathcal{V}_m = \{\{\emptyset\}, \{1\}\}$. From (5.14), it follows that $E_{i,H_{i,2}}\big(E_{i,H_{i,1}}(x, S_1), S_2\big)$ takes one of the following four expressions, depending on the value of $S_1, S_2 \in \mathcal{V}_m$.

$$S_1 = \{\emptyset\}, S_2 = \{\emptyset\} : E_{i,H_{i,2}}\big(E_{i,H_{i,1}}(x, \{\emptyset\}), \{\emptyset\}\big) = (A_i + B_iK_i)^2 x \tag{5.15}$$

$$S_1 = \{1\}, S_2 = \{\emptyset\} : E_{i,H_{i,2}}\big(E_{i,H_{i,1}}(x, \{1\}), \{\emptyset\}\big) = (A_i + B_iK_i)A_i x + (A_i + B_iK_i)B_i\, H_{i,1} x \tag{5.16}$$

$$S_1 = \{\emptyset\}, S_2 = \{1\} : E_{i,H_{i,2}}\big(E_{i,H_{i,1}}(x, \{\emptyset\}), \{1\}\big) = A_i(A_i + B_iK_i)x + (B_i)\, H_{i,2} x \tag{5.17}$$

$$S_1 = \{1\}, S_2 = \{1\} : E_{i,H_{i,2}}\big(E_{i,H_{i,1}}(x, \{1\}), \{1\}\big) = A_i^2 x + (A_iB_i)\, H_{i,1} x + (B_i)\, H_{i,2} x \tag{5.18}$$

It is important to note that each one of the above expressions is an affine function of $H_{i,1}$, $H_{i,2}$. Therefore, $F_i^2(x)$ which is the convex-hull of them, is also an affine function of $H_{i,1}$ and $H_{i,2}$.

[1] When $\alpha \in co\{\alpha_i : i = 1, \cdots, n_\alpha\}$ and $\beta \in co\{\beta_j : j = 1, \cdots, n_\beta\}$, then $\gamma = \alpha + \beta \in co\{\alpha_i + \beta_j : i \in \{1, \cdots, n_\alpha\}, j \in \{1, \cdots, n_\alpha\}\}$.

Similar to the above procedure, by associating auxiliary matrices $H_{i,1}$, $H_{i,2}$, $\cdots$, $H_{i,t}$ to each one of the nested saturation functions appeared in $F_i^t(x)$, it follows that

$$F_i^t(x) \in co\big\{E_{i,H_{i,t}}(E_{i,H_{i,t-1}} \cdots (E_{i,H_{i,1}}(x, S_1), \cdots), S_{t-1}), S_t) : \forall S_1, S_2, \cdots, S_t \in \mathcal{V}_m\big\},$$
$$\forall x \in \mathcal{L}(H_{i,1}) \cap \cdots \cap \mathcal{L}(H_{i,t}). \quad (5.19)$$

In order to simplify the notations of $F_i^t(x)$, let

$$\begin{aligned}
&E_i\,(x, S_1) := E_{i,H_{i,1}}(x, S_1)\\
&E_i^2(x, S_1, S_2) := E_{i,H_{i,2}}\big(E_{i,H_{i,1}}(x, S_1), S_2\big)\\
&\quad\vdots\\
&E_i^t(x, S_1, \cdots, S_t) := E_{i,H_{i,t}}(E_{i,H_{i,t-1}} \cdots (E_{i,H_{i,1}}(x, S_1), \cdots), S_{t-1}), S_t)
\end{aligned} \quad (5.20)$$

With these notations, the following theorem states a sufficient condition for stability of (5.1) under dwell-time switching and provides an estimate of its DOA.

Theorem 5.2 *If there exist a $P \succ 0$ and matrices $H_{i,1}, H_{i,2}, \cdots, H_{i,2\tau-1} \in \mathbb{R}^{m\times n}$ for all $i \in \mathcal{I}_N$ such that*

$$\big[E_i^\ell(x, S_1, \cdots, S_\ell)\big]^\top P\big[E_i^\ell(x, S_1, \cdots, S_t)\big] - x^\top P x < 0,$$
$$\forall i \in \mathcal{I}_N, S_1, \cdots, S_\ell \in \mathcal{V}_m, \ell \in \mathbb{T} \quad (5.21)$$

$$\mathcal{E}(P) \subseteq \mathcal{L}(H_{i,t}), \qquad \forall i \in \mathcal{I}_N, t \in \{1, \cdots, 2\tau - 1\} \quad (5.22)$$

Then, (i) the origin of the saturated system (5.1) with dwell-time τ is locally asymptotically stable; (ii) there exists an $\bar{\alpha} \in (0, 1)$ such that $\alpha\mathcal{E}(P)$ is DT-contractive for all $\alpha \le \bar{\alpha}$ and $\alpha\mathcal{E}(P)$ is inside the DOA of (5.1).

Remark 5.1 *Setting $\tau = 1$ in the above theorem, retrieves the stability conditions for saturated systems under arbitrary switching (see e.g. Theorem 5.1 or results appeared*

in [64, 66, 67]). Hence, the LDI approach presented here can be seen as a generalization of those obtained for arbitrary switched systems.

5.3.1 Enlarging the DOA using LDI approach

The estimate of DOA of system (5.1) obtained from Theorem 5.2 is the ellipsoidal set, $\mathcal{E}(P)$. To enlarge this set, one must chose $P \succ 0$ and auxiliary matrices $H_{i,1}, \cdots, H_{i,2\tau-1}$ such that the size of $\mathcal{E}(P)$ is maximized. This can be done by solving the following constrained optimization problem.

$$\begin{aligned} &\max_{P, H_{i,1}, \cdots, H_{i,2\tau-1}} \text{volume } \mathcal{E}(P) \\ &\qquad s.t. \text{ (5.21) and (5.22).} \end{aligned} \tag{5.23}$$

In what follows, we describe how to transform (5.23), into an optimization problem with LMI constraints that can be efficiently solved with convex optimization routines.

The key point is that $E_i^t(x, S_1, \cdots, S_t)$ of (5.21) is an affine function of variable $H_{i,1}$, $\cdots$, $H_{i,t}$ for each $S_1, S_2, \cdots, S_t \in \mathcal{V}_m$ (See equations (5.15)-(5.18) for the affine expression of $E_i^2(x, S_1, S_2)$ in terms of $H_{i,1}$ and $H_{i,2}$). Thus it can be rewritten as

$$E_i^t(x, S_1, \cdots, S_t) = (\Theta_{i,0} + \Theta_{i,1}H_{i,1} + \cdots + \Theta_{i,t}H_{i,t})x \tag{5.24}$$

where $\Theta_{i,\cdot}$'s are known functions of A_i, B_i, K_i only. In order to transform (5.21) into an LMI constraint, pre- and post-multiply it by P^{-1}. Using (5.24), it follows that

$$x^\top \Big[P^{-1}(\Theta_{i,0} + \Theta_{i,1}H_{i,1} + \cdots + \Theta_{i,t}H_{i,t})^\top P(\Theta_{i,0} + \Theta_{i,1}H_{i,1} + \cdots + \Theta_{i,t}H_{i,t})P^{-1} - P^{-1}\Big]x < 0.$$

Let $Z = P^{-1}$ and $Y_{i,t} = H_{i,t}P^{-1}$. Then, the above equation is converted into

$$\big(\Theta_{i,0}Z + \Theta_{i,1}Y_{i,1} + \cdots + \Theta_{i,t}Y_{i,t}\big)^\top Z^{-1}\big(\Theta_{i,0}Z + \Theta_{i,1}Y_{i,1} + \cdots + \Theta_{i,t}Y_{i,t}\big) - Z \prec 0$$

Utilizing the property of Schur complement [19], the above is equivalent to

$$\begin{bmatrix} Z & (\Theta_{i,0}Z + \Theta_{i,1}Y_{i,1} + \cdots + \Theta_{i,t}Y_{i,t})^\top \\ * & Z \end{bmatrix} \succ 0 \tag{5.25}$$

where $*$ denotes the transpose of the off-diagonal term. Equation (5.25) is an LMI in terms of the variables $Z, Y_{i,t}$'s.

Constraint (5.22) can also be converted into an LMI using the fact that $\mathcal{E}(P) \subseteq \{x : a^\top x \leq 1\}$ is equivalent to $a^\top P^{-1} a \leq 1$ [70]. Again, using the Schur complement it follows that

$$a^\top P^{-1} a \leq 1 \Leftrightarrow \begin{bmatrix} 1 & a^\top P^{-1} \\ * & P^{-1} \end{bmatrix} \succeq 0.$$

With this, constraint (5.22) is equivalent to

$$\begin{bmatrix} 1 & Y_{i,t}^{j\bullet} \\ * & Z \end{bmatrix} \succeq 0\,, \qquad \forall j \in \{1, \cdots, m\}, \forall i \in \mathcal{I}_N, \;\; \forall t \in \{1, \cdots, 2\tau - 1\} \tag{5.26}$$

where $Y_{i,t}^{j\bullet}$ is the j-th row of $Y_{i,t}$.

Finally, $Trace(P^{-1})$ can be used as a measure of size of the ellipsoid $\mathcal{E}(P)$ and thus the estimate of the DOA of (5.1) can be enlarged by solving the following constrained LMI optimization problem with $Z \succ 0, Y_{i,1}, \cdots, Y_{i,2\tau-1}$ for $i = 1, 2, \cdots, N$ as variables:

$$\begin{aligned} \max_{Z, Y_{i,1}, \cdots, Y_{i,2\tau-1}} \quad & Trace(Z) \\ \text{s.t. LMIs in (5.25) and (5.26).} \end{aligned} \tag{5.27}$$

The auxiliary matrices $H_{i,t}$ can then be obtained from $H_{i,t} = Y_{i,t}P$ where $P = Z^{-1}$.

Remark 5.2 *Any feasible solution of optimization problem (5.27) with dwell-time τ, is also a feasible solution for optimization problem (5.27) with any $\bar{\tau} \geq \tau$. This follows*

because for any $x \in \mathcal{E}(P)$ *and any* $\bar{\tau} \geq \tau$, $F_i^{\bar{\tau}}(x) \in \mathcal{E}(P)$ *if* $F_i^{\tau}(x) \in \mathcal{E}(P)$.

5.4 SNS approach

In this section, we generalize the SNS-approach of [63] to switched systems under dwell-time switching. To proceed, some set operations are required.

Recall from Chapter 4 that the one-step backward set of Ω w.r.t. mode i is $\mathcal{Q}^i(\Omega) := \{x : F_i(x) = A_i x + B_i sat(K_i x) \in \Omega\}$. Using this, the one-step set of the switched system (5.1) is

$$\mathcal{Q}(\Omega) = \bigcap_{i \in \mathcal{I}_N} \mathcal{Q}^i(\Omega) = \{x : F_i(x) \in \Omega, \forall i \in \mathcal{I}_N\} \tag{5.28}$$

In general, $\mathcal{Q}(\Omega)$ is not necessarily convex when Ω is. An example that illustrate non-convexity of $\mathcal{Q}(\Omega)$ is in order.

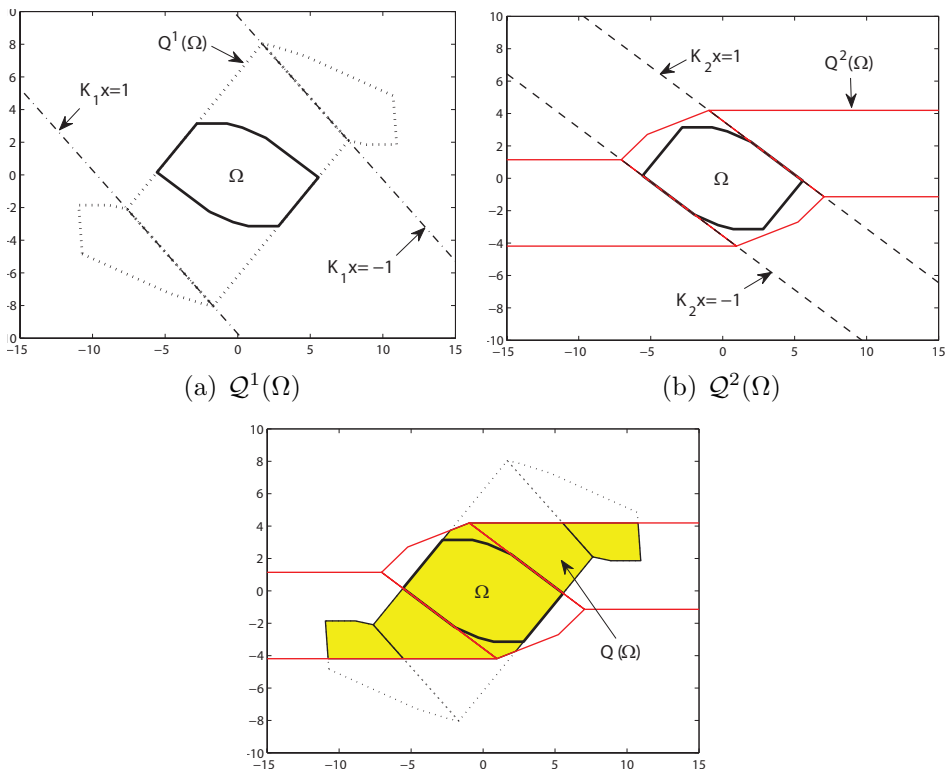
Example 1: Consider a switched system with $\mathcal{I}_N = \{1, 2\}$, $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B_1 = [10, 5]^\top$, $K_1 = [-0.1,\ 0.1]$, $A_2 = \begin{bmatrix} 0 & -1 \\ 0.0001 & 1 \end{bmatrix}$, $B_2 = [0.5,\ -2]^\top$ and $K_2 = [0.02,\ 0.03]$. Figure 5.1 shows $\mathcal{Q}^1(\Omega)$ (dotted lines), $\mathcal{Q}^2(\Omega)$ (solid lines) and $\mathcal{Q}(\Omega)$ (in shade).

The non-convex nature of $\mathcal{Q}(\Omega)$ makes its use for computations of DOA of (5.1) undesirable. To circumvent the non-convexity problem we make use of the SNS-operator introduced in [63] for single saturated system and generalize it for switched systems. The key idea, which is illustrated in Fig. 5.2, is to use a piece-wise linear function as an upper-bound of the saturation function:

$$a\ sat(u) \leq \max\{au, -|a|\}\ , \qquad \forall a, u \in \mathbb{R} \tag{5.29}$$

The way in which (5.29) can be used, is discussed in the following example.

Example 2: Consider a saturated system $x(t+1) = Ax(t) + B\,sat\big(u(t)\big)$, where $u \in \mathbb{R}^2$, $u = Kx$, $B \in \mathbb{R}^{n \times 2}$ and $K \in \mathbb{R}^{2 \times n}$. Also let $B^{\bullet j}, K^{j\bullet}$ to be the j-th column of



(a) $\mathcal{Q}^1(\Omega)$ (b) $\mathcal{Q}^2(\Omega)$

(c) $\mathcal{Q}(\Omega) = \mathcal{Q}^1(\Omega) \cap \mathcal{Q}^2(\Omega)$

Figure 5.1: Illustration of non-convex one-step sets

B and the j-th row of K, respectively. Then,

$$
\begin{aligned}
x(t+1) = A\,x(t) + B sat\big(K\,x(t)\big) &= A\,x(t) + \big[B^{\bullet 1}\ B^{\bullet 2}\big] \begin{bmatrix} sat(K^{1\bullet}x) \\ sat(K^{2\bullet}x) \end{bmatrix} \\
&= A\,x(t) + B^{\bullet 1} sat\big(K^{1\bullet}x(t)\big) + B^{\bullet 2} sat\big(K^{2\bullet}x(t)\big) \\
&\leq A\,x(t) + \max\left\{B^{\bullet 1}K^{1\bullet}x(t), -|B^{\bullet 1}|\right\} + \max\left\{B^{\bullet 2}K^{2\bullet}x(t), -|B^{\bullet 2}|\right\} \\
&\leq \max\Big\{\big(A+BK\big)x(t)\ , \\
&\qquad \big(A + B^{\bullet 2}K^{2\bullet}\big)x(t) - |B^{\bullet 1}|\ , \\
&\qquad \big(A + B^{\bullet 1}K^{1\bullet}\big)x(t) - |B^{\bullet 2}|\ , \\
&\qquad A\,x(t) - |B^{\bullet 1}| - |B^{\bullet 2}|\Big\}
\end{aligned}
$$

which is an upper bound of state-evolution of the saturated system after one step. Note

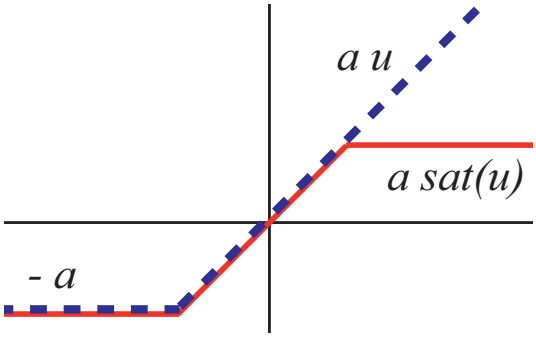


Figure 5.2: Piece-wise linear upper bound of saturation function $a\,sat(u)$ for $a > 0$.

that in the above expression, there are four terms that define the max function. The first term corresponds to the unsaturated system, the second term is associated with the case where the first input ($u_1 = K^{1\bullet}x$) is saturated and $u_2 = K^{2\bullet}x$ is not saturated. The third term is when u_2 is saturated but not u_1 and the last term corresponds to the case where both inputs are saturated. This motivates the SNS-invariance, which is invariance for all possible modes of saturation. To obtain a the formal definition of SNS-invariant set, denote

$$G(x,S) := A\,x + \sum_{j\in S^c} B^{\bullet j}K^{j\bullet}x + \sum_{j\in S} B^{\bullet j}sat\big(K^{j\bullet}x\big) \tag{5.30}$$

as the successor state of x when only the inputs from $S \in \mathcal{V}_m$ are saturated but not the rest. With this, the definition of SNS-invariance is now given.

Definition 5.1 *A set Ω is said to be SNS-invariant if for every $x \in \Omega$, $G(x,S) \in \Omega$ for all $S \in \mathcal{V}_m$.*

With this definition, the SNS-one-step set is defined as

$$\hat{Q}_{sns}(\Omega) = \{x : G(x,S) \in \Omega, \forall S \in \mathcal{V}_m\}. \tag{5.31}$$

To understand this operator, consider system of Example 2. The SNS-one-step set of

$\Omega = \{x : R\,x \leq \mathbf{1}\}$ for this example is

$$\begin{aligned}
\hat{\mathcal{Q}}_{sns}(\Omega) = \quad & \{x : R(A + B^{\bullet 1}K^{1\bullet} + B^{\bullet 2}K^{2\bullet})x \leq \mathbf{1}\} && : G(x, \{\emptyset\}) \in \Omega \\
& \cap \{x : R(A + B^{\bullet 2}K^{2\bullet})x + RB^{\bullet 1}sat(K^{1\bullet}x) \leq \mathbf{1}\} && : G(x, \{1\}) \in \Omega \\
& \cap \{x : R(A + B^{\bullet 1}K^{1\bullet})x + RB^{\bullet 2}sat(K^{2\bullet}x) \leq \mathbf{1}\} && : G(x, \{2\}) \in \Omega \\
& \cap \{x : R(A)x + RB^{\bullet 1}sat(K^{1\bullet}x) + RB^{\bullet 2}sat(K^{2\bullet}x) \leq \mathbf{1}\} && : G(x, \{\{1\}, \{2\}\}) \in \Omega
\end{aligned}$$

While the $\hat{\mathcal{Q}}_{sns}(\Omega)$ defined in (5.31) is not easily computable due to the existence of saturation functions, equation (5.29) can be used to obtain an equivalent representation of (5.31) as follows:

$$\hat{\mathcal{Q}}_{sns}(\Omega) = \bigcap_{S \in \mathcal{V}_m} \Big\{ x : R\big(A + \sum_{j \in S^c} B^{\bullet j}K^{j\bullet}\big)x - \sum_{j \in S} \big|RB^{\bullet j}\big| \leq \mathbf{1} \Big\} \tag{5.32}$$

Equation (5.32) shows that $\hat{\mathcal{Q}}_{sns}(\Omega)$ can be computed from the intersection of 2^m polyhedral sets and hence is convex. $\hat{\mathcal{Q}}_{sns}(\Omega)$ for the system of Example 2 where $m = 2$ becomes

$$\begin{aligned}
\hat{\mathcal{Q}}_{sns}(\Omega) = \quad & \{x : R(A + B^{\bullet 1}K^{1\bullet} + B^{\bullet 2}K^{2\bullet})x \leq \mathbf{1}\} && : S = \{\emptyset\} \\
& \cap \{x : R(A + B^{\bullet 2}K^{2\bullet})\,x - |RB^{\bullet 1}| \leq \mathbf{1}\} && : S = \{1\} \\
& \cap \{x : R(A + B^{\bullet 1}K^{1\bullet})\,x - |RB^{\bullet 2}| \leq \mathbf{1}\} && : S = \{2\} \\
& \cap \{x : R(A)\,x - |RB^{\bullet 1}| - |RB^{\bullet 2}| \;\leq\; \mathbf{1}\} && : S = \{\{1\}, \{2\}\}
\end{aligned} \tag{5.33}$$

Then, for any $x \in \hat{\mathcal{Q}}_{sns}(\Omega)$,

$$\begin{aligned}
R\big(Ax + Bsat(Kx)\big) \leq \max \Big\{ & R\big(A + BK\big)x \;,\; R\big(A + B^{\bullet 2}K^{2\bullet}\big)x - |RB^{\bullet 1}| \;, \\
& R\big(A + B^{\bullet 1}K^{1\bullet}\big)x - |RB^{\bullet 2}| \;,\; RA\,x - |RB^{\bullet 1}| - |RB^{\bullet 2}| \Big\}
\end{aligned} \tag{5.34}$$

Each of the four terms in the max function of (5.34) corresponds to one of the terms appeared (5.33) and is less than $\mathbf{1}$, thus $R\big(Ax + Bsat(Kx)\big) \leq \mathbf{1}$. This means $Ax + Bsat(Kx) \in \Omega$. Moreover, $\hat{\mathcal{Q}}_{sns}(\Omega)$ is convex since it is obtained as the intersection of finite number of polyhedral sets.

The $\hat{\mathcal{Q}}_{sns}(\Omega)$ of (5.32), is defined for a (non-switched) saturated system. The SNS-one-step set for mode i and for switched system (5.1) are respectively defined as:

$$\mathcal{Q}^i_{sns}(\Omega) = \bigcap_{S \in \mathcal{V}_m} \Big\{ x : R\big(A_i + \sum_{j \in S^c} B_i^{\bullet j} K_i^{j\bullet}\big)x - \sum_{j \in S} \big|RB_i^{\bullet j}\big| \leq \mathbf{1} \Big\} \tag{5.35}$$

$$\mathcal{Q}_{sns}(\Omega) = \bigcap_{i \in \mathcal{I}_N} \mathcal{Q}^i_{sns}(\Omega) \tag{5.36}$$

From (5.35), (5.36), it follows that

$$x \in \mathcal{Q}_{sns}(\Omega) \Rightarrow F_i(x) \in \Omega, \ \forall i \in \mathcal{I}_N \Rightarrow x \in \mathcal{Q}(\Omega).$$

Therefore, $\mathcal{Q}_{sns}(\Omega)$ is a convex inner approximation of non-convex one-step set $\mathcal{Q}(\Omega)$ of system (5.1). Figure 5.3 illustrates these sets for switched system of Example 1.

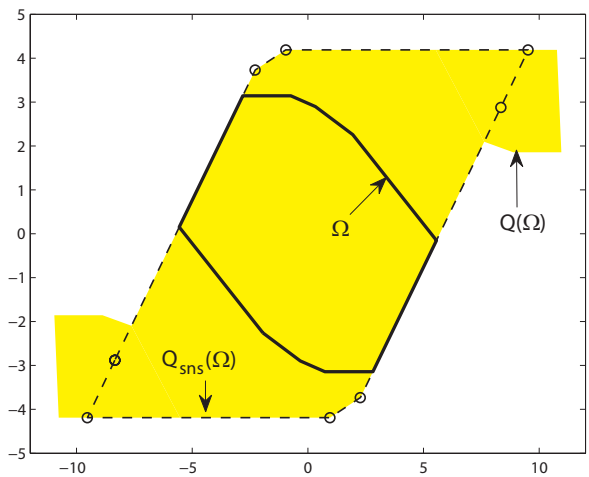


Figure 5.3: Illustration of the convex SNS-one-step set.

With the help of the convex SNS-one-step operator, a simple procedure for computation of DOA of (5.1) is to start from an initial polyhedral DOA and recursively use

the SNS-one-step operator to enlarge it. Fig 5.4 shows the resulting DOA of Example 1 when the SNS operator is used.

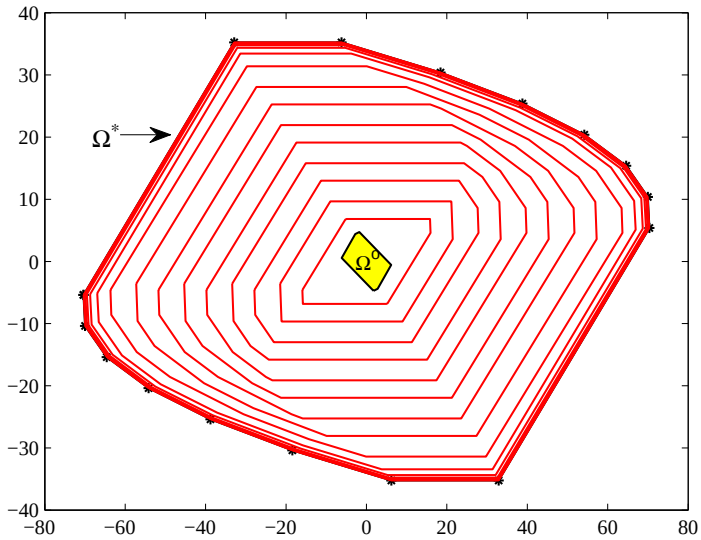


Figure 5.4: Enlarging the domain of attraction by recursively computing the one-step sets.

The above procedure, however, can only be used for switched systems that are stable under arbitrary switching. When the system is not stable under arbitrary switching the above recursive procedure results in an empty set as it requires the states of the system to contract at every step. When a dwell-time τ is imposed for stability, this condition can be relaxed by requiring the states of the system to contract only after t-steps for $t = \tau, \tau+1, \cdots, 2\tau-1$. This means that $F_i^t(x)$ may not be in Ω for $t = 1, 2, \cdots, \tau-1$.

Define $\mathcal{Q}_t^i(\Omega) := \mathcal{Q}^i \cdots \mathcal{Q}^i(\Omega) = \{x : F_i^t(x) \in \Omega\}$ as the t-step set of Ω w.r.t. mode i. The relaxation above requires computation of

$$\mathbb{Q}(\Omega) := \bigcap_{i \in \mathcal{I}_N} \{ \bigcap_{t \in \mathbb{T}} \mathcal{Q}_t^i(\Omega) \}. \tag{5.37}$$

As described earlier $\mathcal{Q}^i(\Omega)$ is non-convex (see e.g. Fig. 5.3) and so is $\mathcal{Q}_t^i(\Omega)$ and $\mathbb{Q}(\Omega)$. Again the non-convexity issue can be resolved if a convex (inner) approximation of $\mathcal{Q}_t^i(\Omega)$ can be obtained. The SNS-t-step set, defined below, serves this purpose.

Recall that $x \in \mathcal{Q}^i_{sns}(\Omega)$ implies $F_i(x) \in \Omega$. Similarly, $x \in \mathcal{Q}^i_{sns}(\mathcal{Q}^i_{sns}(\Omega))$ implies that $F_i^2(x) = F_i(F_i(x)) \in \Omega$. Repeating this for t times, results in the SNS-t-step set of Ω for mode i, given by

$$\mathcal{Q}^i_{sns,t}(\Omega) := \mathcal{Q}^i_{sns} \cdots \mathcal{Q}^i_{sns}(\Omega) \tag{5.38}$$

$\mathcal{Q}^i_{sns,t}(\Omega)$ is convex and $\mathcal{Q}^i_{sns,t}(\Omega) \subseteq \mathcal{Q}^i_t(\Omega)$. With this, a convex inner approximation of $\mathbb{Q}(\Omega)$ of (5.37) is

$$\mathbb{Q}_{sns}(\Omega) := \bigcap_{i \in \mathcal{I}_N} \left\{ \bigcap_{t \in \mathbb{T}} \mathcal{Q}^i_{sns,t}(\Omega) \right\} = \bigcap_{i \in \mathcal{I}_N} \mathcal{Q}^i_{sns,\tau}(\Omega) \tag{5.39}$$

Note that the second equality follows from the fact that $\mathcal{Q}^i_{sns,t}(\Omega) \subseteq \mathcal{Q}^i_{sns,t+1}(\Omega)$ and thus the inner bracket of (5.39) is $\mathcal{Q}^i_{sns,\tau}(\Omega) \cap \mathcal{Q}^i_{sns,\tau+1}(\Omega) \cap \cdots \cap \mathcal{Q}^i_{sns,2\tau-1}(\Omega) = \mathcal{Q}^i_{sns,\tau}(\Omega)$.

In summary, $\mathbb{Q}_{sns}(\Omega)$ has the following properties: (i) $x \in \mathbb{Q}_{sns}(\Omega) \Rightarrow F_i^t(x) \in \Omega$, $\forall i \in \mathcal{I}_N$, $\forall t \in \mathbb{T} \Rightarrow x \in \mathbb{Q}(\Omega)$. (ii) $\mathbb{Q}_{sns}(\Omega)$ is convex and can be computed efficiently when Ω is polyhedral.

The following algorithm summarizes the procedure for enlarging DOA of (5.1) under dwell-time switching.

Algorithm 8 Enlarging DOA for saturated switched systems.

(i) Set $k = 0$ and let Ω^0 to be an initial DOA that is DT-contractive.

(ii) Let $\Omega^{k+1} := \mathbb{Q}_{sns}(\lambda\Omega^k)$.

(iii) If $\Omega^{k+1} \equiv \Omega^k$ then stop, else set $k = k + 1$ and goto step (ii).

Algorithm 8 generates a sequence of DOAs of (5.1) and requires an initial DT-contractive set Ω^0. A proper choice of Ω^0 is the maximal DT-contractive set $\mathbb{O}^\lambda_\infty(X, \tau)$ of unsaturated system $x(t+1) = (A_\sigma + B_\sigma K_\sigma)x(t)$, with $X = \mathcal{L}$. The $\mathbb{O}^\lambda_\infty(X, \tau)$ can be obtained using Algorithm 1a presented in Chapter 2.

Note that a contraction factor of $\lambda = 1-\varepsilon$, $\varepsilon > 0$ is added in Algorithm 8 to impose a strict contraction to the origin after every τ-steps. Properties of the sets obtained from Algorithm 8 are stated next.

Theorem 5.3 *Let $\mathcal{L} := \cap_{i\in\mathcal{I}_N}\mathcal{L}(K_i) = \cap_{i\in\mathcal{I}_N}\{x : \|K_i\, x\|_\infty \leq 1\}$. Suppose that the origin of the unsaturated system $x(t+1) = (A_\sigma + B_\sigma K_\sigma)x(t)$, $\sigma \in \mathcal{S}_\tau$ is asymptotically stable with dwell-time τ. Suppose also that $\Omega^0 \subseteq \mathcal{L}$ is a DT-contractive polyhedral set that contains the origin in its interior and Ω^k's are computed based on Algorithm 8. Then,*

(i) each Ω^k is a convex polyhedron;

(ii) $\Omega^k \subseteq \Omega^{k+1}$ for all $k \geq 0$;

(iii) each Ω^k is DT-invariant w.r.t. system (5.1);

(iv) $x \in \Omega^k \Rightarrow F_i^t(x) \in \lambda\Omega^{k-1}$, $\forall t \in \mathbb{T}, \forall i \in \mathcal{I}_N$;

(v) each Ω^k is a DOA of (5.1);

Remark 5.3 *Any DOA obtained from Algorithm 8 with dwell-time τ is also a DOA of system (5.1) with dwell-time $\bar{\tau} \geq \tau$. This follows because $\mathbb{Q}_{sns,\bar{\tau}}(\Omega) \supseteq \mathbb{Q}_{sns,\tau}(\Omega)$ and $\mathbb{O}_\infty^\lambda(X,\bar{\tau}) \supseteq \mathbb{O}_\infty^\lambda(X,\tau)$.*

Remark 5.4 *Algorithm 8 with $\tau = 1$ can be used for computation of DOA of switched systems under arbitrary switched systems.*

5.5 Comparison of SNS and LDI approaches

This section shows that any DT-contractive set obtained from LDI approach is contained inside the DOA obtained from Algorithm 8 and hence, SNS approach is less conservative than the LDI approach.

To show this, an additional property of the $\mathcal{Q}_{sns}^i(\cdot)$ and $\mathbb{Q}_{sns}(\cdot)$ operators must be

highlighted. Similar to (5.30), for mode i of the system define

$$G_i(x,S) := A_i\,x + \sum_{j\in S^c} B_i^{\bullet j} K_i^{j\bullet} x + \sum_{j\in S} B_i^{\bullet j} sat\big(K_i^{j\bullet} x\big) \tag{5.40}$$

as the successor state of x when only the inputs from $S \in \mathcal{V}_m$ are saturated but not the rest. Repeating this, leads to

$$\begin{aligned} G_i^2(x,S_1,S_2) &= G_i(G_i(x,S_1),S_2) \\ &\vdots \\ G_i^t(x,S_1,S_2,\cdots,S_t) &= G_i(G_i\cdots G_i(x,S_1)\cdots),S_t) \end{aligned} \tag{5.41}$$

From equivalence of (5.31) and (5.32), it follows that

$$\begin{aligned} \mathcal{Q}_{sns}^i(\Omega) &= \{x : G_i(x,S) \in \Omega, \forall S \in \mathcal{V}_m\} \\ &= \big\{x : R\big(A_i + \sum_{j\in S^c} B_i^{\bullet j} K_i^{j\bullet}\big)x - \big|\sum_{j\in S} RB_i^{\bullet j}\big| \le \mathbf{1}, \forall S \in \mathcal{V}_m\big\} \end{aligned}$$

This means,

$$x \in \mathcal{Q}_{sns}^i(\Omega) \iff G_i(x,S) \in \Omega, \forall S \in \mathcal{V}_m. \tag{5.42}$$

Repeating this t times, implies that

$$x \in \mathcal{Q}_{sns,t}^i(\Omega) \iff G_i^t(x,S_1,\cdots,S_t) \in \Omega, \forall S_1,\cdots,S_t \in \mathcal{V}_m.$$

and hence

$$\begin{aligned} x \in \mathbb{Q}_{sns}(\Omega) \iff G_i^t(x,S_1,\cdots,S_t) \in \Omega, \qquad & \forall S_1,\cdots,S_t \in \mathcal{V}_m \\ & \forall t \in \mathbb{T}, \forall i \in \mathcal{I}_N \end{aligned} \tag{5.43}$$

An associated definition is now given.

Definition 5.2 *A set Ω is said to be* DT-SNS-invariant (*or* DT-SNS-contractive)*, if for every $x \in \Omega$, $G_i^t(x, S_1, \cdots, S_t) \in \Omega$ (or $\lambda\Omega$) for all $S_1, \cdots, S_t \in \mathcal{V}_m$, for all $t \in \mathbb{T} = \{\tau, \cdots, 2\tau - 1\}$ and for all $i \in \mathcal{I}_N$.*

From (5.43) and Theorem 4.1 of Chapter 4, it follows that $\mathbb{Q}_{sns}(\Omega)$ is DT-SNS-contractive if and only if Ω is DT-SNS-contractive, i.e. $\Omega \subseteq \mathbb{Q}_{sns}(\Omega)$.

This property of $\mathbb{Q}_{sns}(\cdot)$ implies that, if we start from a DT-SNS-contractive set and use Algorithm 8, the sequence of sets obtained are all DT-SNS-contractive.

Denote Ψ as a DOA obtained form LDI approach and recall that computation of Ψ requires auxiliary matrices $H_{i,1}, \cdots, H_{i,2\tau-1}$ such that $E_i^\ell(x, S_1, \cdots, S_\ell) \in \lambda\Psi$ for all $\ell \in \mathbb{T}$ and for all $S_1, \cdots, S_\ell \in \mathcal{V}_m$ for some $\lambda \in (0, 1)$. The following lemma shows the relationship between $E_i^t(x, S_1, S_2, \cdots, S_t)$ of LDI approach and $G_i^t(x, S_1, S_2, \cdots, S_t)$ of SNS approach.

Lemma 5.3 *Let $E_i^t(x, S)$ and $E_i^t(x, S_1, S_2, \cdots, S_t)$ be defined as (5.6) and (5.20), respectively. Then,*

$$G_i(x, S) \in co\{E_i^t(x, \bar{S}) : \forall \bar{S} \in \mathcal{V}_m\} \tag{5.44}$$

$$G_i^t(x, S_1, S_2, \cdots, S_t) \in co\{E_i^t(x, \bar{S}_1, \bar{S}_2, \cdots, \bar{S}_t) : \forall \bar{S}_1, \cdots, \bar{S}_t \in \mathcal{V}_m\} \tag{5.45}$$

From Lemma 5.3 and DT-contractivity of Ψ, it follows that for every $x \in \Psi$,

$$G_i^t(x, S_1, S_2, \cdots, S_t) \in co\{E_i^t(x, \bar{S}_1, \bar{S}_2, \cdots, \bar{S}_t)\} \in \lambda\Psi \tag{5.46}$$

Thus, Ψ is also SNS-DT-contractive. From (5.43) and Algorithm 8, it follows that $\Psi \subset \Omega^*$. The following theorem, summarizes this key result.

Theorem 5.4 *Suppose that Ψ is a DT-contractive set obtained from LDI approach for some auxiliary matrices $H_{i,1}, \cdots, H_{i,2\tau-1}$, for all $i \in \mathcal{I}_N$ and $\Psi \subseteq \bigcap_{i \in \mathcal{I}_N, t \in \{1, \cdots, 2\tau-1\}} \mathcal{L}(H_{i,t})$.*

Then, (i) Ψ is a SNS-DT-contractive set, (ii) When sequence of Ω^k is generated from Algorithm 8, there exist a $k > 0$ such that $\Psi \subset \Omega^k$.

5.6 Numerical Examples

The example considered is a saturated switched system, taken from [73], with $\mathcal{I}_N = \{1, 2\}$, $A_1 = \left[\begin{smallmatrix} -0.7 & 1.0 \\ -0.5 & -1.2 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} 0.26 & -1.0 \\ 1.7 & -1.5 \end{smallmatrix}\right]$, $B_1 = [1,\ 0]^T$, $B_2 = [0,\ -1]^T$, $K_1 = [1.1759,\ 0.1089]$, $K_2 = [1.5114,\ -0.7765]$.

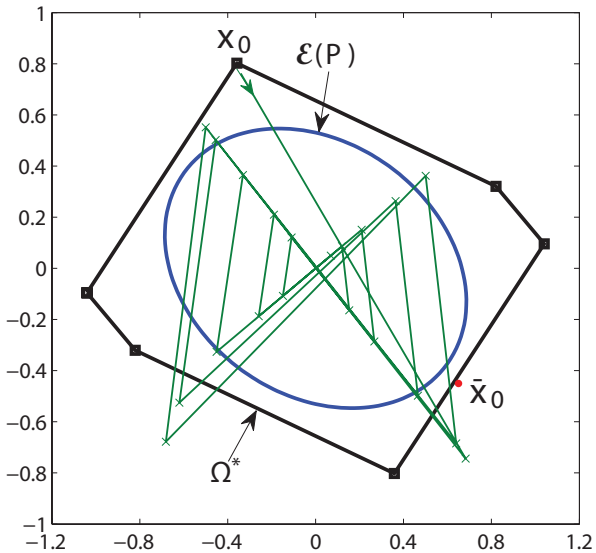


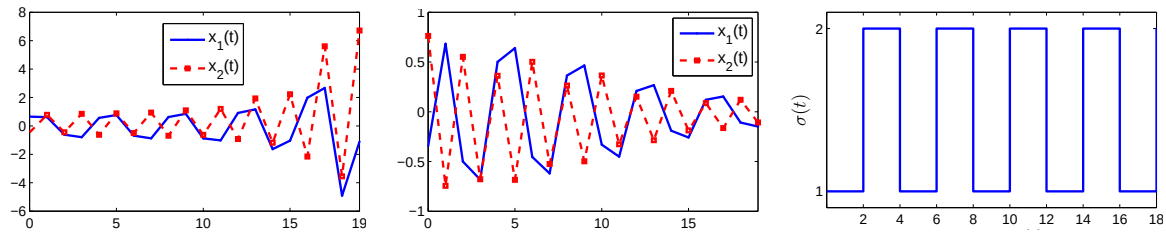
Figure 5.5: Comparison of DOAs for $\tau = 2$: SNS approach, Ω^* and LDI approach, $\mathcal{E}(P)$.

It can be shown that this system is asymptotically stable with any dwell-time $\tau \geq 2$. The intention is to compute the DOA of the system using the LDI approach of Section 5.3 and SNS approach of Section 5.4 for different values of dwell-time $\tau \geq 2$ and compare the results.

The solution of the optimization problem (5.27) with $\tau = 2$ are $P = \left[\begin{smallmatrix} 2.2549 & 0.6805 \\ * & 3.5523 \end{smallmatrix}\right]$, $H_{1,1} = [1.2215, 0.7638]$, $H_{1,2} = [0.5291, 1.4426]$, $H_{1,3} = [-0.4630, -1.0011]$, $H_{2,1} = [1.1322, -0.8601]$, $H_{2,2} = [-0.0894, -0.7529]$, $H_{2,3} = [-0.3201, 0.3926]$. Figure 5.5 compares the DOA obtained from LDI approach (ellipsoid $\mathcal{E}(P)$) and the SNS approach

(Ω^*). Note that $\mathcal{E}(P) \subset \Omega^*$ as claimed in Theorem 5.4.

Figure 5.5 also shows a sample trajectory of the system starting from x_0 on the boundary of Ω^* under the periodic switching sequence shown in Fig. 5.6(c). The solution of the system under the same input sequence starting from $\bar{x}_0$ outside Ω^* (but very close to the boundary of Ω^*) diverges (see Fig. 5.6(a)).



(a) Diverging response from $\bar{x}_0 = (0.653, -0.468) \notin \Omega^*$

(b) Converging response from $x_0 = (-0.358, 0.802) \in \Omega^*$

(c) input switching signal

Figure 5.6: State response of the system under periodic switching with dwell time $\tau = 2$.

5.6.1 Comparison with other methods

As a comparison, the authors of [73] use an LDI-based method together with MLFs to obtain as estimate of DOA of (5.1). They show that if there exist $\lambda \in (0, 1)$, $\mu \geq 1$, $P_i \succ 0$ and H_i for each $i \in \mathcal{I}_N$ such that

$$[(E_{i,H_i}(x, S))]^\top P_i \, [(E_{i,H_i}(x, S))] \leq \lambda \, x^\top P_i \, x \qquad \forall i \in \mathcal{I}_N, \forall S \in \mathcal{V}_m \tag{5.47a}$$

$$P_i \preceq \mu \, P_j \qquad \forall (i, j) \in \mathcal{I}_N \times \mathcal{I}_N \tag{5.47b}$$

$$\mathcal{E}(P_i) \subseteq \mathcal{L}(H_i) \qquad \forall i \in \mathcal{I}_N \tag{5.47c}$$

Then, the saturated system (5.1) is asymptotically stable with dwell time $\tau \geq -\frac{\ln \mu}{\ln(\lambda)}$. The above condition can be easily converted into LMIs and solved for P_i's such that the size of $\mathcal{E}(P_i)$ are maximized. Since MLFs are used, the estimate of DOA is then the largest norm-2 ball $\mathcal{B}_r = \{x : \|x\| \leq r\} \subseteq \cap_{i \in \mathcal{I}_N} \mathcal{E}(P_i)$ such that if $x(0) \in \mathcal{B}_r$ then $x(t) \in \mathcal{E}(P_i)$ for all $t \in \mathbb{Z}^+$. An admissible choice of r that guarantees this condition is

$r = \min_{i \in \mathcal{I}_N} \sqrt{\frac{1}{\lambda_{max}(P_i)}}$.

For the example considered in this section, the smallest dwell-time τ that results in a feasible solution for optimization problem (5.47) is $\tau = 5$. The resulting DOA ($\mathcal{B}_r$) is compared with the DOA obtained from our LDI approach $\mathcal{E}(P)$ and SNS approach (Ω^*) in Fig. 5.7.

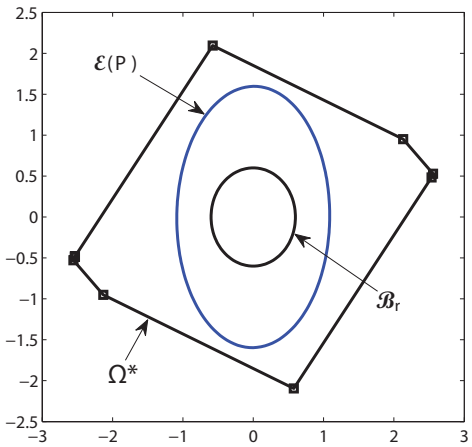


Figure 5.7: Comparison of Ω^* with $\mathcal{E}(P)$ and $\mathcal{B}_r$ for $\tau = 5$.

Computational results for $\tau = 2$ and $\tau = 5$ are also presented in Table 5.1. These results include the size of DOA obtained from different approaches, the iteration at which Algorithm 8 converges (k^*) and the number of inequalities (#) that characterizes the Ω^* set. From Table 5.1, it can be seen that our LDI approach is less conservative, in terms of both minimal dwell-time needed for stability and the size of DOA, than the LDI method of [73]. Moreover, LDI results are subset of SNS domain of attraction, i.e. $\mathcal{B}_r \subset \mathcal{E}(P) \subset \Omega^*$.

τ	$Area(\mathcal{B}(r))$	$Area(\mathcal{E}(P))$	$Area(\Omega^*)$	k^*	#
2	–	1.143	1.896	29	6
5	1.131	6.443	12.011	17	8

Table 5.1: Computational results of saturated switched system

5.7 Summary

In this chapter, two approaches are proposed for to the computation of DOA of switched systems under dwell-time switching and in the presence of saturation nonlinearity. In the LDI approach sufficient conditions, in terms of linear matrix inequalities, for asymptotic stability of the switched system is derived. The second approach is based on the convex SNS-operator defined for switched systems. An algorithm that starts from an initial DOA, inside the linear region of controllers, and recursively enlarges the DOA is also provided. It is also shown that any estimate of domain of attraction using LDI approach is a subset of SNS approach and thus SNS approach is less conservative.

Proofs:

Proof of Theorem 5.2:

Let $Y_0 := \cap_{i\in\mathcal{I}_N, t=1,\cdots,2\tau-1}\mathcal{L}(H_{i,t})$. Then, (5.22) implies that $\mathcal{E}(P) \subseteq Y_0$ and there exists a $\bar{\alpha} < 1$ such that for every $x \in \bar{\alpha}\mathcal{E}(P)$, $F_i^t(x) \subseteq Y_0$ for all $t = 1, \cdots, \tau - 1$ and for all $i \in \mathcal{I}_N$. This and equations (5.21) and (5.22), imply that there exist $\lambda \in (0, 1)$ such that for every $x \in \alpha\mathcal{E}(P)$ with $\alpha \leq \bar{\alpha}$,

$$E_i^t(x, S_1, \cdots, S_t) \in \lambda\alpha\mathcal{E}(P), \;\; \forall S_1, \cdots, S_t \in \mathcal{V}_m, \forall t \in \mathbb{T}.$$

Since $F_i^t(x)$ is a convex combination of $E_i^t(x, S_1, \cdots, S_t)$, it follows that for every $x \in \alpha\mathcal{E}(P)$, $\alpha \leq \bar{\alpha}$ $F_i^t(x) \in \lambda\alpha\mathcal{E}(P)$ and hence $\alpha\mathcal{E}(P)$ is DT-contractive. Asymptotic stability of origin of system (5.1) then follows from Theorem 2.6. ■

Proof of Theorem 5.3:

Results of (i,ii,iv) follow directly from the definition of $\mathbb{Q}_{sns}$ operator. (iii) DT-invariance of Ω^0 implies that $\Omega^1 = \mathbb{Q}_{sns}(\lambda\Omega^0) \supset \Omega^0$. DT-invariance of Ω^1 then follows from property (iii) of Theorem 4.1. Repeating this recursively completes the proof. (v) From result (iv) it follows that for every $x \in \Omega^k$, there exist a $t \in \mathbb{Z}^+$ such that $x(t) \in \Omega^0$ for every admissible switching sequence. Since Ω^0 is a DOA, it follows that $\lim_{t\to\infty} x(t) \to 0$. ■

Proof of Theorem 5.4:

(i) For every $x \in \Psi$, it follows from (5.46) that $G_i^t(x, S_1, \cdots, S_t) \in \lambda\Psi$ for every $S_1, \cdots, S_t \in \mathcal{V}_m$ and for all $t \in \mathbb{T}$. Thus Ψ is a SNS-DT-contractive set. (ii) From DT-contractivity of Ψ, it follows that for every $x(0) \in \Psi$, there exists a t such that $x(t) \in \Omega^0$. This, (5.43) and recursive procedure of Algorithm 8 imply that $x(0) \in \Omega^{\bar{k}}$ for some $\bar{k}$. ■

Chapter 6

Switching Controllers for Hard Disk Drives

Hard disk drives (HDDs) are used as data-storage devices in computers and other data-processing systems. Figure 6.1 illustrates a typical hard disk system, in which data are arranged in concentric circles called tracks. The data are read from or written on the rotating disks using the read/write (R/W) head, which is actuated by the voice-coil motor (VCM). The VCM is controlled by a servo controller, which has two main responsibilities: *track-seeking* and *track-following*. In track-seeking, the controller moves the R/W head from one track to another as fast as possible, while in track-following it keeps the head on the specific track so that data can be read/written. In general, these two tasks cannot be achieved with only one controller. A common practice in HDD industry is therefore to design separate controllers for each task, and then use a Mode Switching Controller (MSC) that switches between them [3, 74–79].

MSCs activate the track-seeking controller for quickly moving the R/W head to the vicinity of the desired track; and switch to the track-following controller for keeping the head on that specific track. The main challenges in MSCs are to guarantee the stability and the switching conditions under which the performance is optimal [3, 76–80].

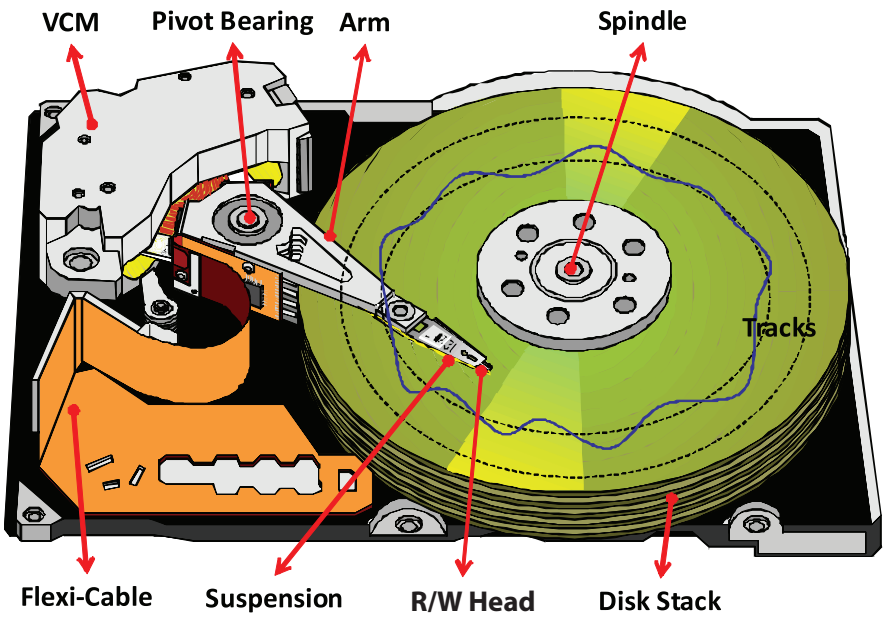


Figure 6.1: A conventional hard disk drive

Assuming that the track-seeking and track-following controllers are given for a HDD, this chapter explores the application of our developed methods in providing switching conditions between these controllers such that the stability of the system is guaranteed. Performance of the overall system is also improved by appropriate initialization of the states of controller at the instance of mode switching.

6.1 Dynamical Model of HDD

It is well known [79, 80] in the research community of the HDD servo systems, that VCM actuators have a characteristic of a double integrator cascaded with some high-frequency resonance modes. The transfer function of the VCM actuator is

$$G_{VCM}(s) = \frac{y(s)}{u(s)} = \frac{k}{s^2} \prod_{i=1}^{N} G_{res,i}(s) \tag{6.1}$$

where u is the actuator input (in volts), y is the position of R/W head (in tracks), $v = \dot{y}$ is the velocity of the R/W head, k is the gain of the actuator[1], and $G_{res,i}(s)$ for $i = 1, \cdots, n_r$ are the transfer functions of the significant resonance modes.

Throughout this chapter, the following 10-th order model of the Maxtor-51536U3 HDD, taken from [3, 79], is used:

$$G_{VCM}(s) = \frac{6.4013 \times 10^7}{s^2} \prod_{i=1}^{4} G_{res,i}(s) \ ,$$

$$G_{res,1}(s) = \frac{0.912s^2 + 457.4s + 1.433 \times 10^8}{s^2 + 359.2s + 1.433 \times 10^8}, \quad G_{res,2}(s) = \frac{0.759s^2 + 962.6s + 2.491 \times 10^8}{s^2 + 789.1s + 2.49 \times 10^8}$$
$$G_{res,3}(s) = \frac{9.917 \times 10^8}{s^2 + 1575s + 9.917 \times 10^8} \ , \quad G_{res,4}(s) = \frac{2.731 \times 10^9}{s^2 + 2613s + 2.731 \times 10^9}$$

In order to minimize the effect of the high-frequency resonance modes, it is a common practice to use notch filters to cancel the unwanted responses as much as possible. For the above VCM model, the following notch filter is used to cancel the first three resonance modes (see frequency response of VCM with the notch filter in Fig. 6.2).

$$G_{notch}(s) = \left(\frac{s^2 + 238.8s + 1.425 \times 10^8}{s^2 + 2388s + 1.425 \times 10^8}\right) \times \left(\frac{s^2 + 314.2s + 2.467 \times 10^8}{s^2 + 3142 + 2.467 \times 10^8}\right) \times \left(\frac{s^2 + 628.3s + 9.87 \times 10^8}{s^2 + 12570 + 9.87 \times 10^8}\right)$$

[1] $k = k_y k_v$, where k_y is the position measurement gain and $k_v = k_t/m$, with k_t being the current-force conversion coefficient and m being the mass of the VCM actuator.

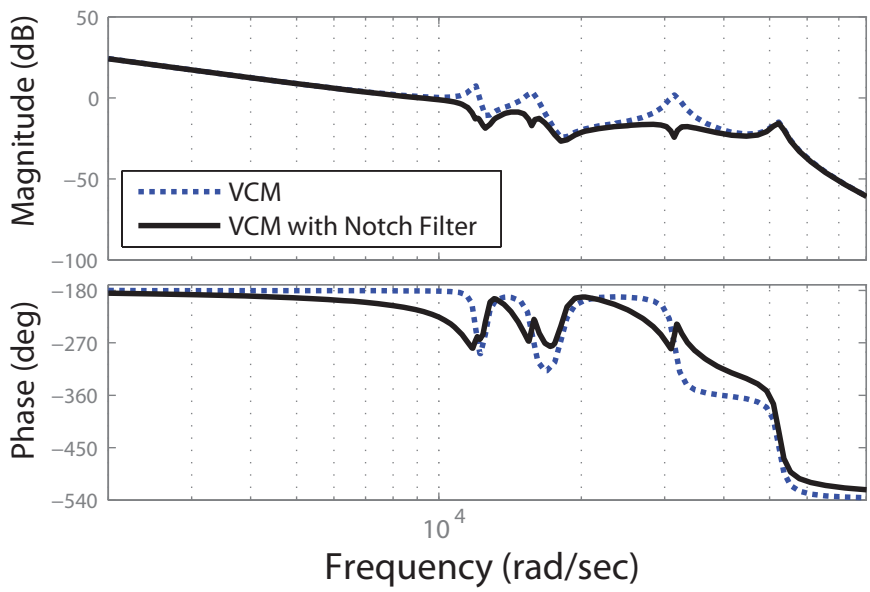


Figure 6.2: Frequency responses of the VCM with/without the notch filter

6.1.1 Model of the plant and controllers

Assuming that the high frequency resonance modes are compensated by the notch filters, the simplified second-order model for the VCM actuator will be used hereafter as the plant model for the purpose of controller design. In the implementation, the overall control action would be $\tilde{u} = G_{notch}(s) \times u$, with u being the controller output.

The second-order state space model of the plant is

$$\text{Plant}: \begin{cases} x_p(t+1) = A_p x_p(t) + B_p\, u(t) + E_p\, w(t) \\ y(t) = C_p\, x(t) \end{cases} \tag{6.2}$$

where $x_p = \begin{bmatrix} y \\ v \end{bmatrix}$, $A_p = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}$, $C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $B_p = \begin{bmatrix} kT_s^2/2 \\ kT_s \end{bmatrix}$, $k = 6.4013 \times 10^7$ and $T_s = 0.05$ milli-seconds is the sampling period. The disturbance w is an unknown input that models the effect of friction and other nonlinearities. It is assumed that $u \in U = \{u : |u| \leq 3\}$, $w \in W = \{w : |w| \leq 3 \times 10^{-3}\}$, and $E_p = B_p$. Also, only variable y is measurable but not v.

The track-following controller (denoted by mode 2) is taken from [79] and satisfies certain frequency domain specifications like gain margin greater 6 dB, phase margin greater than and 30°, and maximum peaks of the sensitivity and complementary sensitivity functions less than 6 dB. The track-seeking controller (denoted by mode 1) is designed such that it has a fast response for seek lengths up to 200 tracks. These controllers are of the form (6.3) and their details are given in Table 6.1.

$$\text{Controller}\,(i): \begin{cases} x_c(t+1) = A_{c_i}\, x_c(t) + B_{y_i}\, y(t) + B_{r_i}\, r(t) \\ u_c(t) = C_{c_i}\, x_c(t) + D_{y_i}\, y(t) + D_{r_i}\, r(t) \end{cases} \tag{6.3}$$

with x_c being the controller states, y being the measured position, and r being the reference track number.

Mode (i)	A_{c_i}	$B_{r_i} \times 10^5$	$B_{y_i} \times 10^5$	C_{c_i}	D_{y_i}	D_{r_i}
1	$\begin{bmatrix} -0.809 & -0.371 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	[2.975, -5.002]	1	-1
2	$\begin{bmatrix} 1 & 0 \\ -0.041 & 0.644 \end{bmatrix}$	$\begin{bmatrix} -5 \\ 417 \end{bmatrix}$	$\begin{bmatrix} 5 \\ -1843 \end{bmatrix}$	[-1.604, -7.524]	0.161	-0.462

Table 6.1: Details of the track-seeking (mode 1) and track-following (mode 2) controllers.

Combining (6.2) and (6.3), the closed-loop equation for each mode i is

$$x(t+1) = A_{cl,i}\, x(t) + B_{cl,i}\, r(t) + E_{cl}\, w(t) \tag{6.4}$$

where $x = [x_p^\top,\, x_c^\top]^\top$ is the closed-loop states and

$$A_{cl,i} = \begin{bmatrix} A_p + B_p D_{y,i} C_p & B_p C_{c,i} \\ B_{y,i} C_p & A_{c,i} \end{bmatrix}, \quad B_{cl,i} = \begin{bmatrix} B_p D_{r,i} \\ B_{r,i} \end{bmatrix}, \quad E_{cl} = \begin{bmatrix} E_p \\ 0 \end{bmatrix}$$

Assuming that r_d is the desired track number, the equilibrium state of (6.4) is $x_e =$

$(I - A_{cl,i})^{-1} B_{cl,i}\, r_d$. With the change of variables $\bar{x} := \begin{bmatrix} \bar{x}_p \\ \bar{x}_c \end{bmatrix} = x - x_e$, the closed-loop system (6.4) is transformed into

$$\bar{x}(t+1) = A_{cl,i}\, \bar{x}(t) + E_{cl}\, w(t) \tag{6.5}$$

where $\bar{x} = 0$ is now the equilibrium solution.

6.2 Stability analysis of MSC

In this section, we briefly describe the conventional method for stability analysis of MSC. Then, we apply our results presented in Chapters 2 and 3 to find time-dependent switching conditions that ensures stability of MSC.

The simple switching strategy that ensures stability of conventional MSC is illustrated in Fig. 6.3. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are respectively the DOAs of track-seeking and track-following controllers such that $\mathcal{D}_2 \subset \mathcal{D}_1$. MSC uses the track-seeking controller whenever $\bar{x} \in \mathcal{D}_1$, $\bar{x} \notin \mathcal{D}_2$ and switches to track-following controller only if $\bar{x} \in \mathcal{D}_2$ (see Fig. 6.3). Note that since $\mathcal{D}_1$ is a domain of attraction and $\mathcal{D}_1 \supset \mathcal{D}_2$, for any initial condition $\bar{x}(0) \in \mathcal{D}_1$ there exists a time $t^* \in \mathbb{Z}^+$ such that $\bar{x}(t^*) \in \mathcal{D}_2$. Once $\bar{x}(t^*)$ is inside $\mathcal{D}_2$, MSC switches to track-following mode and the convergence of $\bar{x}$ to the origin is guaranteed since $\mathcal{D}_2$ is a contractive DOA of the track-following controller. A sample trajectory of the system with such a switching strategy is also shown in Fig. 6.3. It starts from $\bar{x}(0)$ outside $\mathcal{D}_2$ under track-seeking controller ($i = 1$) and then switches to track-following controller ($i = 2$) when $\bar{x} \in \mathcal{D}_2$.

The switching condition above is state-dependent and easy to use when only two controllers are involved. In the case of multiple controllers, the state-dependent switching condition become complicated as multiple nesting conditions has to be considered.

On the other hand, system (6.5) can be modeled as a switched system with time-

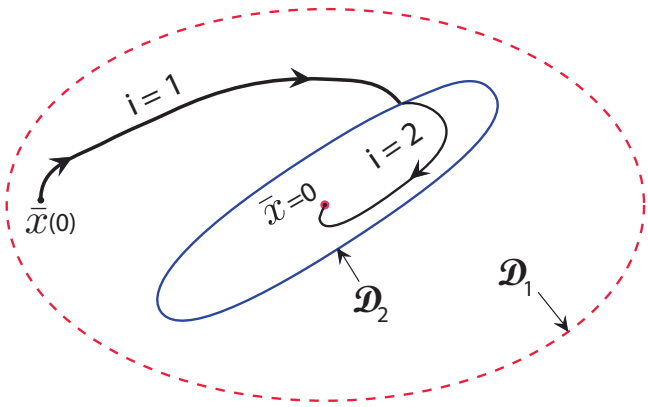


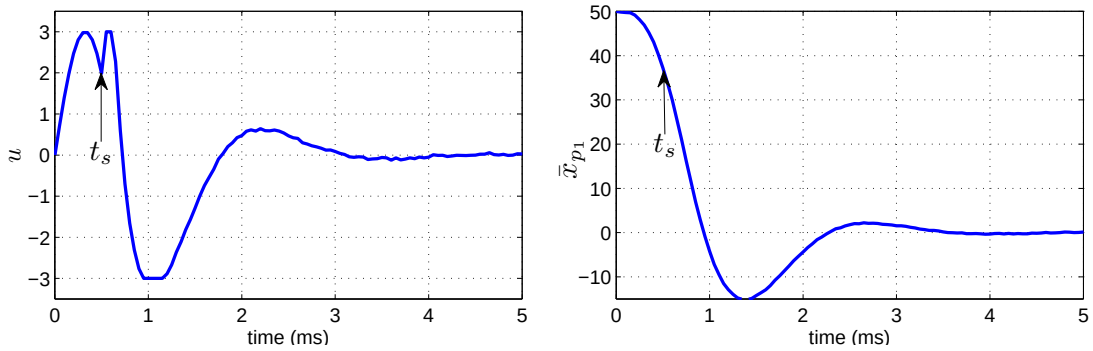
Figure 6.3: Illustration of conventional MSC strategy: when $\bar{x} \in \mathcal{D}_1$, $\bar{x} \notin \mathcal{D}_2$ MSC uses track-seeking and switch to track-following when $\bar{x} \in \mathcal{D}_2$.

dependent switching. Stability of such system can be verified with the results provided in Chapters 2 and 3. Using Algorithm 2, the minimal common dwell-time of $\tau_c = 16$ is obtained for this system. It is also easy to find the minimal mode-dependent dwell-times $\tau_1 = 11$, $\tau_2 = 16$ that guarantees the asymptotic stability of (6.5). This means that if the track-seeking controller is in operation for at least $\tau_1 \geq 11 \times T_s = 0.55$ msec, then it is safe to switch to the track-following controller provided that it is in operation for at least $\tau_2 \geq 16 \times T_s = 0.8$ msec. Unlike the state-dependent switching condition, our time-dependent switching conditions can be easily applied to the case where multiple controller are used.

6.2.1 Performance of MSC

Stability of MSC is guaranteed with any switching sequence satisfying the above dwell-time conditions. However, as it will be shown in the following example, the performance of the system is highly dependent on the instance of switching and also on the saturation limit of the actuator.

Figure 6.4 shows the response of MSC for the reference signal of $r_d = 50$ tracks, when switching to track-following controller happens at $t_s = 0.55$ msec. Although the system remains stable, $\bar{x}_{p_1}$ has a large overshoot (see Fig. 6.4(b)) and the performance

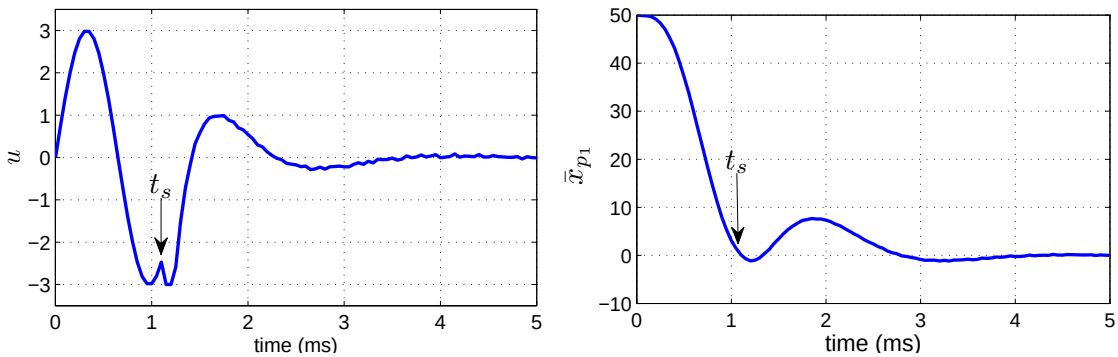


(a) Input response $u(t)$ (b) State response $\bar{x}_1(t)$

Figure 6.4: Response of the MSC for $r_d = 50\mu m$ with switching at $t_s = 0.55$ msec.

of the switched system is not satisfactory. This is mainly due to (i) the abrupt change in the control effort at the instant of switching and (ii) saturation of track-following controller (see Fig. 6.4(a)).

To minimize the effect of saturation of track-following controller, the conventional MSC techniques choose $\mathcal{D}_2$ as the largest DOA inside the linear region of track-following controller. Fig. 6.5 shows the input/output response of the state-dependent MSC for $r_d = 50$ tracks with this $\mathcal{D}_2$. While the overshoot is reduced, the response is not satisfactory yet and the abrupt change of control action is still present (see Fig. 6.5(a)).



(a) Input response $u(t)$ (b) state response $\bar{x}_1(t)$

Figure 6.5: Response of the conventional MSC: switching at $t_s = 1.2$ msec, where $x(t_s) \in \mathcal{D}_2$.

To further improve the performance of MSC, one common approach is to properly initialize the states of track-following controller at the instance of switching. This

approach, known as initial value compensation (IVC) [74–78], is used to obtain smooth and fast transient responses. The basic idea is to minimize a cost function, $J(\bar{x})$, of position error and control effort at the moment of switching. Recall that states of the plant $\bar{x}_p$ are measurable[2] and $\bar{x} = \begin{bmatrix} \bar{x}_p \\ \bar{x}_c \end{bmatrix}$, the problem is then to find the initial values of the controller at the switching instance, $\bar{x}_c(t_s)$, that minimizes J:

$$\begin{aligned} &\min_{\bar{x}_c} \; J(\bar{x}) \\ &\text{subject to} \;\; \bar{x} \in \mathcal{D}_2 \end{aligned} \tag{6.6}$$

where $\mathcal{D}_2$ is a DOA of track-following controller. In the conventional MSC $\mathcal{D}_2$ is chosen as the largest contractive ellipsoid that is inside the linear region of track-following controller.

The transient response can be improved if MSC can switch earlier to the track-following controller with proper initialization. This can be achieved by finding a larger domain of attraction of track-following controller that considers the saturation effect into account. The following section describes how algorithms presented in Chapter 5 can be used for this purpose.

6.2.2 Computation of DOA of saturated controller

In the VCM model of (6.2), the control effort u is constrained by $u \in U = \{u : |u| \leq 3\}$. In order to use the results of Chapter 5, which require the saturation limit of 1, we normalize the plant model (6.2) as follows:

$$\begin{cases} x_p(t+1) = A_p\, x_p(t) + (3B_p)\, sat\big(\frac{u(t)}{3}\big) + E\, w(t) \\ y(t) = C_p\, x(t) \end{cases} \tag{6.7}$$

[2]position y is directly measured and velocity v is estimated using an state-observer.

For simplicity, we set $\tilde{B}_p = 3B_p$, $\tilde{C}_{ci} = C_{ci}/3$, $\tilde{D}_{yi} = D_{yi}/3$ and $\tilde{D}_{ri} = D_{ri}/3$. Combining (6.3) and (6.7), the saturated closed-loop system is given by:

$$\bar{x}(t+1) = \tilde{A}_i\, \bar{x}(t) + \tilde{B}\, sat(\tilde{K}_i\, \bar{x}(t)) + \tilde{E}w(t) \tag{6.8}$$

where $\bar{x} = \begin{bmatrix} \bar{x}_p \\ \bar{x}_c \end{bmatrix} = x - x_e$ and

$$\tilde{A}_i = \begin{bmatrix} A_p & 0 \\ B_{y_i}C_p & A_{c_i} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_p \\ 0 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} \tilde{D}_{y_i}C_p & \tilde{C}_{ci} \end{bmatrix}, \quad \tilde{E} = E_{cl}$$

Since disturbance $w \in W = \{w : |w| \leq 3 \times 10^{-3}\}$ is present in (6.8), we need to modify Algorithm 8 such that it considers the effect of disturbances. This can be easily done by replacing step 2 of Algorithm 8 with $\Omega^{k+1} = \mathbb{Q}_{sns}(\Omega^k \ominus EW)$, where $\mathbb{Q}_{sns}(\Omega \ominus EW)$ computes the robust one-step set described in Chapter 4. We can now compute the DOA of the track-following controller under saturated control, using modified version of Algorithm 8 with $\mathcal{I}_N = \{2\}$ and $\tau = 1$, as only mode 2 is considered.

The DOA of the track-following controller obtained from Algorithm 8 is denoted by X_2 and has a polyhedral characterization in the form of:

$$X_2 = \{\bar{x} : H_p\, \bar{x}_p + H_c\, \bar{x}_c \leq \mathbf{1}\}. \tag{6.9}$$

Switching condition and Controller initialization:

Once the characterization of X_2 set is obtained, a switching strategy similar to conventional MSC can be used. Accordingly, we start with the track-seeking controller and switch to track-following controller at the moment where $\bar{x}_p \in \text{Proj}(X_2) := \{\bar{x}_p : \exists \bar{x}_c \text{ s.t. } [\bar{x}^\top, \bar{x}_c^\top]^\top \in X_2\}$. Similarly, the track-following controller is initialized (at

the moment of switching) by solving the following optimization problem:

$$\begin{aligned} \bar{x}_c(t_s) &:= \arg\min_{\bar{x}_c} \; J(\bar{x}(t)) \\ s.t. \quad & H_c\,\bar{x}_c(t) \leq \mathbf{1} - H_p\,\bar{x}_p(t) \end{aligned} \tag{6.10}$$

The following algorithm summarizes the MSC strategy:

Algorithm 9 MSC with initialization of track-following controller.

(a) Let $t = 0$.

(b) Solve optimization problem (6.10) with $\bar{x}_p(t)$ and $\bar{x}_c(t)$.

(c) **if** there is no feasible solution **then**

- Set $i = 1$ (track-seeking controller), compute u with $i = 1$ from (6.3) and apply it to the plant.
- $t = t + 1$ and goto step (b).

else

- Switch to mode 2 (track-following controller), with $\bar{x}_c(t_s)$ being the minimizer of (6.10).

(d) Compute u from (6.3) with $i = 2$ and apply it to the plant.

(e) $t = t + 1$ and goto (d).

Note that the optimization problem (6.10) is solved until the states of the plant enter $\text{Proj}(X_2)$. At this moment MSC switches to track-following mode with its states initialized using (6.10). From then onwards no optimization is required and the controller is updated according to (6.3). It should be mentioned that computation of X_2 is expensive and it is done off-line. When the characterization of X_2 is obtained, (6.10) is a convex optimization problem with linear inequality constraints in $\mathbb{R}^2$, and hence can be solved efficiently in real-time.

In summary, the purposed MSC method differs from the conventional MSC by considering the saturation effect into the account and obtaining a larger DOA of the

track-following controller.

6.2.3 Simulation Results

In this section, we compare the performance of our method, with the one of the MSC with initialization proposed in [78].

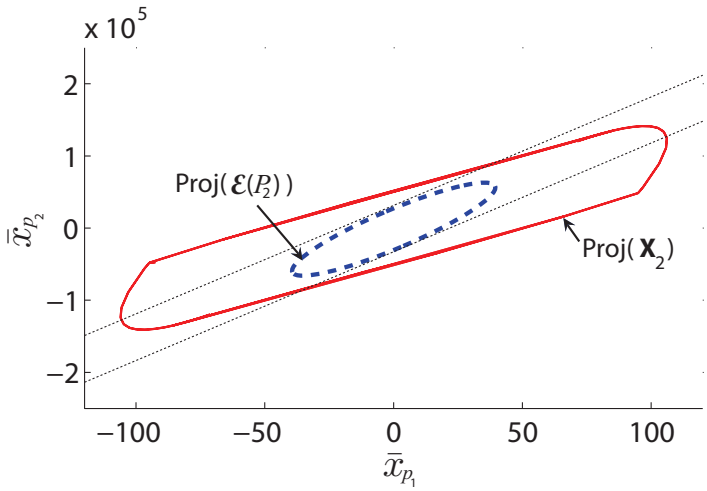


Figure 6.6: Comparison of the DOAs of track-following controller.

The cost function is the same for both methods and it is of the form

$$J(\bar{x}) = \|\tilde{K}_2\,\bar{x} - \,u_1\,\| \tag{6.11}$$

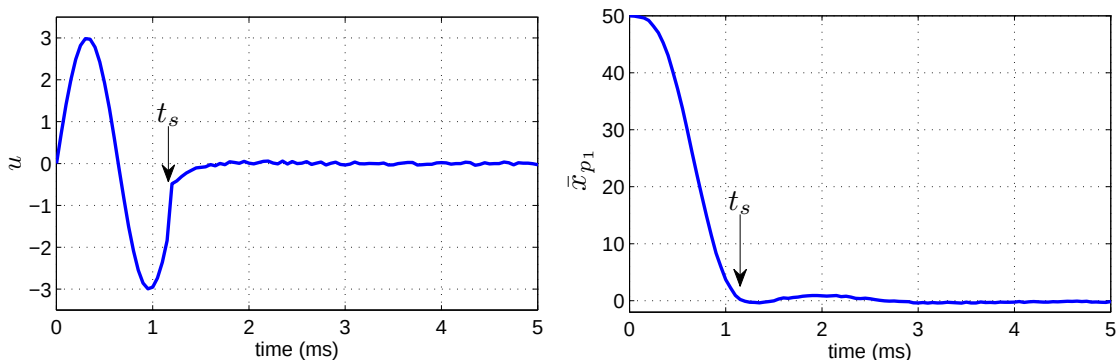
that minimizes the difference between the control effort of the track-seeking controller (u_1) and track-following controller ($\tilde{K}_2\,\bar{x}$) at the moment of switching. The difference between the two methods is the choice of DOA of the track-following controller. For our method, the polyhedral set X_2 obtained from Algorithm 8 is used as $\mathcal{D}_2$ while for

the method of [78], $\mathcal{D}_2$ is chosen as the largest ellipsoid $\mathcal{E}(P_2)$ such that

$$
\begin{aligned}
&P_2 \succ 0 \\
&(A_{cl,2}\,\bar{x} + E_{cl}\,w)^\top P_2 (A_{cl,2}\,\bar{x} + E_{cl}\,w) - P_2 \prec 0, \qquad \forall w \in W, \forall \bar{x} \in \mathcal{E}(P_2) \\
&\mathcal{E}(P_2) \subseteq \mathcal{L}(\tilde{K}_2 \bar{x})
\end{aligned}
$$

Figure 6.6 shows the projection of the X_2 into the $\bar{x}_{p_1}$ - $\bar{x}_{p_2}$ space and compare its size with the maximal ellipsoidal set, denoted by $\mathcal{E}(P_2)$, used in the conventional MSC. Note that $\mathcal{E}(P_2)$ is inside the linear region of track-following controller illustrated with dotted lines in Fig. 6.6.

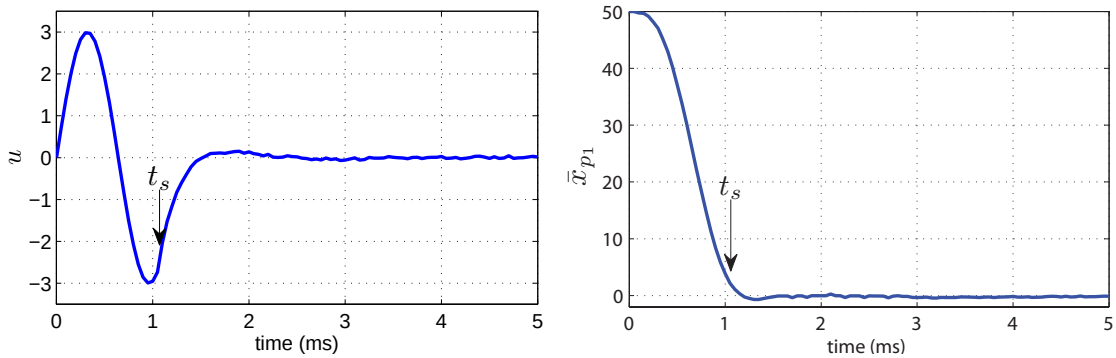
The input/state response obtained from method of [78] and our method are shown in Figs. 6.7 and 6.8, respectively. It can be seen that the larger X_2 set allows MSC to switch faster to the track-following controller and thus has a better performance in terms of (i) faster settling time and (ii) reduced response overshoot.



(a) Input response $u(t)$ (b) state response $\bar{x}_1(t)$

Figure 6.7: Response of the conventional MSC with switching at $t_s = 1.2$ ms and initial states of $\bar{x}_c(t_s) = [-0.031\,,\,-1.943]^\top$.

To further illustrate the superiority of our proposed method, the settling time improvement for different values of r are shown in Fig. 6.9. It can be seen that the gain in the settling time is more profound in the larger seek lengths.



(a) Input response $u(t)$ (b) state response $\bar{x}_1(t)$

Figure 6.8: Response of the proposed MSC with switching at $t_s = 1.05$ ms and initial states of $\bar{x}_c(t_s) = [-0.012\,,\ -1.546]^\top$.

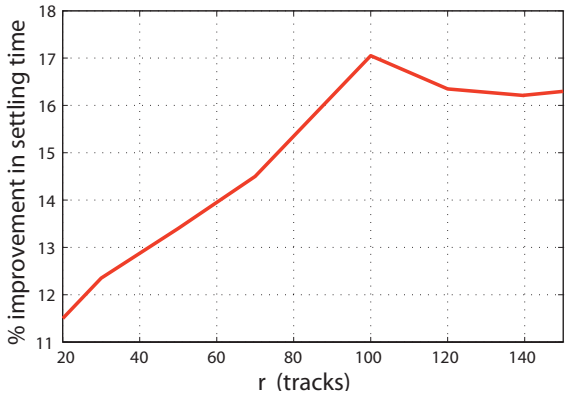


Figure 6.9: Improvement of settling time for different values of seek length r.

6.3 Summary

In this chapter, a mode switching controller is proposed for controlling the R/W head of HDDs. The proposed method uses an optimization problem to find the optimal instant of switching, together with a proper initialization of the second controller. It was shown that this design can enhance the performance of the currently used MSCs.

Chapter 7

Conclusions and Future Works

This chapter concludes the thesis with the summary of the main contributions and provides possible future research directions.

7.1 Contributions

(a) **Necessary and sufficient stability conditions:**

Characterization of all admissible switching sequences that satisfy the dwell-time consideration is provided. Based on this characterization, polyhedral DT-contractive sets are introduced and it was shown in Theorem 2.6 that existence of such sets is both necessary and sufficient for asymptotic stability of switched linear systems under dwell-time switching. This condition is a generalization of the results appeared in the literature of arbitrary switched systems.

(b) **Computation of the minimal dwell-time:**

An algorithm is proposed for the computation of the minimal dwell-time needed for stability. In addition, relaxation of the dwell-time requirement is considered by imposing a dwell-time for each mode of the system instead of one common dwell-time for all modes. A constructive procedure for computation of mode-

dependent dwell-times, which under suitable conditions results in the minimal mode-dependent dwell-times, is provided.

(c) **Characterization of constraint admissible DT-invariant sets:**

A new characterization of constraint admissible sets is proposed for dwell-time switched systems. These sets are constraint admissible at all times and DT-invariant for every admissible switching sequence. An algorithm that computes the maximal constraint admissible DT-invariant set is also presented.

(d) **Characterization of robust invariant sets:**

Similar to part (c) but in the case where disturbance is present, characterization of robust invariant sets is presented. Algorithms for computation of maximal and minimal robust invariant sets are also proposed.

(e) **Domain of attraction of saturated systems:**

Two approaches are proposed for computation of domain of attraction of switched systems in the presence of saturation nonlinearity. These results are useful as they can enlarge the domain of attraction beyond the linear region of controllers.

(f) **Application:**

A mode switching controller is proposed for the control of read/write head of a hard disk drive system that switches between the track-seeking and track-following modes. In addition, a procedure for initialization of the track-following controller is proposed that minimizes the jerk in the control signal and improves the performance of the mode switching controller.

7.2 Future Works

This thesis opens up some interesting directions for further investigation:

(a) **Stability condition when unstable modes are present:**

There are situations where switching to unstable modes becomes unavoidable; (e.g. when failure of sensors/components occurs in a servo system or there are packet dropouts in communication networks). When unstable dynamics are present, slow switching (i.e., long enough dwell time) is not sufficient for stability; additional requirement is that switched system does not spend too much time in the unstable modes [8, 36]. One way to tackle this problem is to extend the results of Chapter 2:

Suppose $\mathcal{A} = \big\{A_i : i \in \mathcal{I}_N = \{1, \cdots, N\}\big\}$ is the set of stable modes and $\bar{\mathcal{A}} = \big\{\bar{A}_j : j \in \mathcal{I}_{\bar{N}} = \{1, \cdots, \bar{N}\}\big\}$ is the set of unstable modes. Suppose a minimum dwell-time τ for stable modes and a maximum duration of stay on unstable modes $\bar{\tau}$ is considered. Then, switched system is asymptotically stable iff there exists a $\lambda \in (0, 1)$ and a bounded polyhedral set Ω such that

$$\big(A_i^k\big)\big(\bar{A}_{j_1}^{\bar{k}_1} \cdots \bar{A}_{j_{\bar{N}}}^{\bar{k}_{\bar{N}}}\big)\,\Omega \subseteq \lambda\,\Omega, \qquad \forall i \in \mathcal{I}_N, \forall j_1, \cdots, j_{\bar{N}} \in \mathcal{I}_{\bar{N}}, \forall k \in \{\tau, \cdots, 2\tau - 1\}$$
$$\forall \bar{k}_1, \cdots, \bar{k}_{\bar{N}} \in \{0, 1, \cdots, \bar{\tau}\}, \sum_{j=1}^{j=\bar{N}} \bar{k}_j \leq \bar{\tau}$$

This condition, basically considers all switching signals in which the dwell-time in the stable modes is greater than τ and the total duration of stay in the unstable modes is less than $\bar{\tau}$.

The above condition implies that system remains stable if the total amount of divergence from Ω due to unstable modes is compensated by long enough dwell-time on stable modes. Note that when $\bar{\tau}$ is fixed, a bisection search can be used to find the minimal τ of stable modes that ensures stability. Similarly, when τ is fixed it is possible to find the maximal duration of stay, $\bar{\tau}$, in unstable modes

that ensures stability. While the condition is both necessary and sufficient, the total number of constraints to be considered grows rapidly with increasing of $\bar{N}$ or $\bar{\tau}$, making the computation of such sets intractable. Further research should be carried out to obtain reasonable and practical relaxations of this condition.

(b) **Stability of switched systems with state jumps:**

Currently, the thesis does not consider the systems with state jumps (also know as *Impulsive Systems*). Effect of state jumps is usually specified with a reset-map $R_{i,j}$ that defines the new vale of the states when the system switches from mode i to j, i.e. $x^+ = R_{i,j}x$, with $R_{i,i} = I$. Extending the results to switched impulsive systems is of practical importance as it is customary for systems with multi-controllers to reset the states of the controllers at switching instants to improve the transient response. If the reset map is given a priori, a necessary and sufficient condition for stability is existence of compact set Ω and a $\lambda \in (0, 1)$ such that:

$$R_{i,j}\, A_i^k\, \Omega \subseteq \lambda\Omega \qquad\qquad \forall i, j, \forall k \in \{\tau, \cdots, 2\tau - 1\}$$

Of course, the above problem becomes challenging if the reset matrices $R_{i,j}$'s are to be designed.

(c) **Stability of switched systems with time delay:**

In practical applications, due to the transmission delay or the mode identifying delay, there may exist a time delay in the state or the control input. Extending the results to switched systems with time delay should be investigated. Obtaining the necessary and sufficient conditions are challenging as the effect of time delay and mode switching should be considered simultaneously.

(d) **Performance of time-dependent switched systems:**

Performance of a switched system in the presence of disturbances is of practical

importance. Consider the following switched system

$$x(t+1) = A_{\sigma(t)}x(t) + B\,w(t) \tag{7.1a}$$

$$z(t) \;=\; C_{\sigma(t)}\,x(t) + E\,w(t) \tag{7.1b}$$

where z is the output of the system and $w \in W$ is the disturbance input. A measure of performance of system (7.1) is to find a $\gamma > 0$ such that $\sum_{t=0}^{\infty} \|z(t)\| \leq \gamma \sum_{t=0}^{\infty} \|w(t)\|$. This problem is commonly referred to as the $\mathcal{L}_2$-gain problem and it determines the maximum output energy that can be excited with a given input energy. The $\mathcal{L}_2$-gain problem is quite challenging for dwell-time switched systems, as the effect of both exogenous inputs and switching signals should be considered. Finding the minimum $\mathcal{L}_2$-gain (γ_{min}) is still an open problem [48].

The results presented in Chapter 4 suggest a possible way to tackle this problem. Specifically, the minimal robust DT-invariant set ($\mathbb{F}_\infty$) can be used to obtain the limit-set of all trajectories of (7.1), denoted by X_∞.

$$X_\infty := \bigcup_{i \in \mathcal{I}_N} \hat{P}_{\tau-1}(\mathbb{F}_\infty, A_i, BW).$$

where $\hat{P}_t(\cdot)$ is the forward operator after t-steps defined in Chapter 4. Since, $z(t) = C_{\sigma(t)}x(t) + Ew(t)$, it follows that $Z_\infty := \bigcup_{i \in \mathcal{I}_N} C_i X_\infty \oplus EW$ is the limit-set of z, i.e. $\lim_{t\to\infty} z(t) \to Z_\infty$. Once the characterization of Z_∞ is obtained, $\gamma := \max_{z \in Z_\infty} \|z\|$ is the upper bound of the minimal $\mathcal{L}_2$-gain of (7.1).

While the above procedure provides a solution to the $\mathcal{L}_2$-gain problem, efficient methods for computation/approximation of $\mathbb{F}_\infty, X_\infty, Z_\infty$ remain challenging and need to be addressed.

(e) **Extension of the results to continuous-time systems:**

The stability condition provided in Chapter 2 has a direct counterpart for con-

tinuous systems; namely system $\dot{x}(t) = A_{\sigma(t)}x(t)$ is asymptotically stable with dwell-time τ if there exists a $\lambda \in (0, 1)$ and a bounded polyhedral set Ω such that

$$e^{A_i t}\,\Omega \subseteq \lambda\Omega, \qquad \forall t \in [\tau, 2\tau], \forall i \in \mathcal{I}_N \tag{7.2}$$

However, verification of the above condition is not easy as it requires consideration of infinite number of inclusions. Since each mode is stable it might be possible to resolve this issue by considering some upper bounds on the solutions of each mode.

Unlike the discrete counterpart, (7.2) is only a sufficient condition and it is not clear whether it is also necessary for asymptotic stability or not. Hence, establishing a necessary and sufficient stability condition for continuous-time systems remains a challenging problem.

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- **M. Dehghan**, C-J. Ong, "Discrete-time switching linear system with constraints: Characterization and computation of invariant sets under dwell-time consideration," ***Automatica***, Vol. 48, No. 5, pp. 964–969, 2012.
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