Graph Theoretic Analysis of Multi-Agent System Structural Properties

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Contents

Summary

This dissertation aims to develop graph theoretical interpretations for properties of multi-agent systems, which usually stand for collections of individual agents with local interactions among the individuals. The interconnection topology has been proven to have a profound impact on the collective behavior of whole multi-agent system. In particular, we aim to reveal this kind of impact under external signals on system performance in terms of its controllability and disturbance rejection capability. Interaction link weight plays an important role in how interconnection topology affects multi-agent system behavior. Nonetheless, it is assumed that interaction links have no weight in most theoretical study, until recently. Consequently, in this dissertation, a weighted interconnection topology graph is adopted as the graphic representation of multi-agent system. What follows is that rather than the traditional controllability and disturbance rejection of multi-agent systems, we study these two problems of multi-agent system in a new structural sense.

In the controllability discussion, multi-agent systems with switching topologies are taken into consideration, which can be usually formulated as some kinds of hybrid system. Consequently, controllability of hybrid systems: switched linear system, representing time-dependent switching, and piecewise linear systems, representing state-dependent switching, is investigated first as a general case. More specifically, the structural controllability of switched linear systems is investigated first. Two kinds of graphic representations of switched linear systems are devised. Based on these topology graphs, graph theoretical necessary and sufficient conditions of the structural controllability for switched linear systems are presented, which show that the controllability purely bases on the graphic topologies among state and input vertices. Subsequently, as a special class of switched linear systems, the structural controllability of multi-agent systems under switching topologies is investigated. Graph-theoretic characterizations of the structural controllability are addressed and it turns out that the multi-agent system with switching topology is structurally controllable if and only if the union graph $\mathcal{G}$ of the underlying communication topologies is connected (single leader) or leader-follower connected (multi-leader). Besides, as predecessor research investigation for further study on multi-agent system with state-dependent switching topology, we consider the null controllability of piecewise linear system. An explicit and easily verifiable necessary and sufficient condition for a planar bimodal piecewise linear system to be null controllable is derived. What follows is a short discussion on how to adopt the results to the research process of controllability of state-dependent multi-agent systems.

The influence of interconnection topology on the disturbance rejection capability of multi-agent systems in a structural sense is also addressed. Multi-agent systems consisting of agents with non-homogeneous general linear dynamics are considered. With the aid of graph theory, criteria to determine the structural disturbance rejection capability of these systems are devised. These results show that using only the local disturbance rejection capability of each agent and the interconnection topology among local dynamics, the disturbance rejection capability of whole multi-agent system can be deduced. Besides, combination of disturbance rejection with controllability problem of multi-agent systems is introduced. We explicitly deduce the requirement on multi-agent interconnection topologies to guarantee the structural controllability and structural disturbance rejection capability simultaneously.

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Chapter 1

Introduction

Multi-agent systems, such as group of autonomous vehicles, power grid, sensor networks and so on, have brought great influence to our lives. However, due to the number of subsystems and the complexity of interactions among them, we still do not know how to control such large scale complex systems fully. Here we are specially interested in how multi-agent dynamics can be influenced by external signals and decisions in terms of controllability and disturbance rejection capability. In the following introduction part, we will introduce the background and motivation of this dissertation's research first. Followed by reviews on related research efforts in literature on multi-agent systems, such as consensus, controllability and disturbance rejection, together with review on structured systems and structural properties, which will be the basis for the whole dissertation's study. Finally, this chapter will summarize the organization and research contribution of this dissertation.

1.1 Multi-Agent Systems

1.1.1 Background and Motivation

Due to the latest advances in communication and computation, the distributed control and coordination of the networked dynamic agents has rapidly emerged as a hot multidisciplinary research area [1–3], which lies at the intersection of systems control theory, communication and mathematics. In addition, the advances of the research in multi-agent systems are strongly supported by their promising civilian and military applications, such as distributed plants (power grids, collaborative sensor arrays, sensor networks, transportation systems, distributed planning and scheduling, distributed supply chains), distributed computational systems (decentralized optimization, parallel processing, concurrent computing, cloud computing) and multi-robot systems (cooperative control of unmanned air vehicles(UAVs), autonomous underwater vehicles(AUVs), space exploration, air traffic control) [4, 5]. The behavior of these multi-agent systems has an important feature: all agents make their own local decisions while trying to coordinate the global goal with the other agents in the system, which is quite similar as the collective behavior of biological systems, such as ant colonies, bee flocking, and fish schooling. Usually, in these systems, each agent has very limited sensing, processing, and communication capabilities. However, a well coordinated group of these elementary agents can generate more remarkable capabilities and display highly complex group behaviors by following some simple rules which require only local intuitive interactions among the agents. This brings the fact that the group behavior is not a simple summation of the individual agent's behavior and can be greatly impacted by the communication protocols or interconnection topology among the agents, which poses several new challenges on control of such large scale complex systems that fall beyond the traditional methods. Hence, the cooperative control of multi-agent systems is still in its infancy and attracts more and more researchers' attention. Inspired by experience gained from biological systems, researchers have started focusing their attention on investigating how the group units make their whole group motions under control or get better performance just through limited and local interactions among them. In the next part we will review some research directions and methods in multi-agent systems.

1.1.2 Research Efforts in Literature

Lots of research have been done on multi-agent system in terms of its stability, controllability, observability, and performance. Due to the significance of local interactions among agents, there is a major on-going research effort in understanding how the interconnection topology influences the global behavior of multi-agent systems. On this topic, graph-theoretic approach has been widely utilized for encoding the local interactions and information flows in multi-agent systems. With the aid of algebraic graph theory [6], the interactions among agents and the information flow described by corresponding representative graphs can be translated into matrix representations, which can easily be incorporated into a dynamical system. In this graph-theoretic approach, a frequently adopted model is the Laplacian dynamics of multi-agent systems, which are built based on the Laplacian of representative graphs. This model has shown its significance in solving wide range of multi-agent related problems including consensus, social networks, flocking, formation control, and distributed computation [7–12]. In multi-agent consensus problem, the objective of multi-agent system is to make all agents agree upon certain quantities of interest, where such quantities might or might not be related to the motion of the individual agents (for example, the heading of a team of robots). In [7], the consensus problem was investigated under either fixed or switching interconnection topology with directed or undirected flow graphs. In [11], unmanned aerial vehicles (UAVs) formation control, which is concerned with whether a group of autonomous vehicles can follow a predefined trajectory while maintaining a desired spatial pattern, was studied using the the Laplacian of a formation graph and presented a Nyquist-like criterion. Besides this graph Laplacian, approaches like artificial potential functions [13–15], and navigation functions [16–19] have also been developed. In [13], using potential functions obtained naturally from the structural constraints of a desired formation, multiple autonomous vehicle systems distributed formation control problem was investigated. The navigation function method was adopted in [16] to deal with partially known environment for mobile robot motion planning.

Much more research investigation on the control and applications of multi-agent systems can be found in literature. A survey of recent research efforts, including formation control, cooperative tasking, spatio-temporal planning, and consensus, and possible future directions in cooperative control of multi-agent systems was introduced in [20]. Besides the aforementioned work, other directions of research efforts can be observed in literature, such as: parallel processing [21, 22], optimization based path planning [23–25], game theory based coordinations [26], geometrical swarming [27, 28], distributed learning [29], and observability of distributed sensor network [30].

As we can see, much of the prior work has concentrated on properties of stability (for example, consensus and formation control), observability (for example, observability of distributed sensor network), and performance (for example, optimization based path planning) of multi-agent systems. Our goal in this dissertation is to consider situations where multi-agent dynamics can be influenced by external signals and decisions. Consequently, this dissertation has particular interest in two new angles of properties of multi-agent systems: the controllability as well as another performance index in terms of the disturbance rejection capability. Section 1.1.3 will introduce the research efforts on controllability of multi-agent systems and in Section 1.1.4, some work on disturbance rejection of multi-agent systems will be addressed. Section 1.1.5 will give a short review of structural systems and structural properties, which will be the basis for the whole dissertation's study.

1.1.3 Controllability of Multi-Agent Systems

The controllability issue of multi-agent systems has recently attracted attentions. Actually, in control of multi-agent systems, it is desirable that people can drive the whole group of agents to any desirable configurations only based on local interactions between agents and possibly some limited commands to a few agents that serve as leaders. This can be straightforwardly transferred to the controllability problem, under which the multi-agent system would be considered as having the leader-follower framework: in this group of interconnected agents, some of the agents, referred to as the leaders, are influenced by an external control input, and the complement of the set of leaders in the system act as followers, who will abide by some agreement protocol. This multi-agent controllability problem was first proposed in [31], which formulated it as the classical controllability of a linear system and proposed a necessary and sufficient algebraic condition in terms of the eigenvectors of the graph Laplacian. Reference [31] focused on fixed topology situation with a particular member which acted as the single leader. Besides, an interesting finding was shown in [31] that increasing the connectivity of the interconnection topology graph will not necessarily do good to the controllability of corresponding multi-agent system. Subsequently, the problem was then developed in [32–41]. A notion of anchored systems was introduced in [34] and it was substantiated that symmetry with respect to the anchored vertices makes the system uncontrollable. This result was related to the symmetry and automorphism group of the interconnection topology graph. In [32], sufficient condition based on the null space of graph Laplacian for controllability of multi-agent systems was proposed. Furthermore, in [33], it was shown that a necessary and sufficient condition for controllability is not sharing any common eigenvalues between the Laplacian matrix of the follower set and the Laplacian matrix of the whole topology. To pursue a more intrinsic graph theoretical explanation of the controllability issue, in the same paper [33], the authors introduced the network equitable partitions and proposed a graph-theoretic necessary condition for the controllability of multi-agent systems. Following this new graphic characterization method, [35] subsequently investigated the graphic interpretation of controllability under multi-leader setting. In [41], the authors pushed the boundary further by introducing the notion of relaxed equitable partitions and provided a graph-theoretic interpretation for the controllability subspace when the multi-agent system is not completely controllable. The controllability of multi-agent system under switching topologies was studied in [37, 40], where some algebraic conditions for the controllability of multi-agent systems were introduced.

From the above literature review, it can be observed that so far the research progress using graph theory is quite limited and it remains elusive on getting a satisfactory graphic characterization of the controllability of multi-agent systems. Besides, the weights of communication links among agents have been demonstrated to have a great influence on the behavior of whole multi-agent group (see e.g., [42]). However, in the previous multi-agent controllability literature [31, 41], the communication weighting factor is usually ignored. One classical result under this no weighting assumption is that a multi-agent system with complete graphic communication topology is uncontrollable [31]. This is counter-intuitive since it means each agent can get direct information from each other but this leads to a bad global behavior as a team. This shows that too much information exchange may damage the controllability of multi-agent system. In contrast, if we set weights of unnecessary links to be zero and impose appropriate weights to other links so as to use the communication

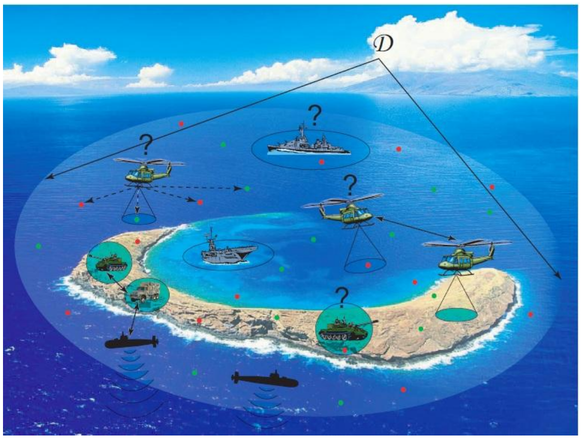

Fig. 1.1: Interconnections in a multi-agent system

information in a selective way, then it is possible to make the system controllable [43]. Motivated by the above observation, in this dissertation, the weighting factor is taken into account for multi-agent controllability problem. In particular, rather than the classical controllability of multi-agent systems, a new notion for the controllability of multi-agent systems, called structural controllability, which was proposed by us in [43], is investigated directly through the graph-theoretic approach for control systems. Besides, since fixed interconnection topologies may restrict their impacts on real applications, switching topologies will be adopted in investigation of multi-agent controllability in this dissertation. Take a real multi-agent system as example [44], which consists of helicopters, ships, tanks and submarines as depicted in Fig.1.1 . For the whole system, it's required to turn on/off some interconnection links to save energy and achieve the global goal with optimized communication energy usage. Under this situation, people can arbitrarily control the interconnections and this interconnection topology is called time-dependent switching topology of multi-agent systems. Under some other situations, the interconnections are influenced by factors that are out of control, such as distance and signal strength, which means the interconnection topology can not be fixed or arbitrarily controlled. Here it is assumed that the interconnections are fully determined by the agents' states and the corresponding multi-agent system interconnection topology is state-dependent switching topology. More details are provided in Chapter 3.

1.1.4 Disturbance Rejection

The problem of rejecting disturbances appears in a variety of applications including aircraft flight control systems [45, 46], active control systems of offshore structures affected by ocean wave forces [47], active noise control systems [48], rotating mechanical systems and vibration damping in industrial applications [49, 50], etc. As systems performing tasks in natural environments such as microsatellite clusters, formation flying of UAVs, automated highway systems and mobile robotics, the coordination of multi-agent systems also faces the challenge of external disturbance, which is a pervasive source of uncertainty in most applications. Hence, the control of such large scale complex systems must address the issue of disturbance rejection [51, 52].

Early research efforts on disturbance rejection problem can be traced back decades ago and rich literature can be found for the disturbance rejection for various control systems. In [53], under linear time-invariant systems, the authors discussed the problem of disturbance rejection by using state feedback, feed forward control and dynamic compensation in control u. A constructive solvability condition of disturbance rejection problem was introduced. Polynomial approach was adopted in [54] as a tool for analysis of the disturbance rejection problem of linear systems. Using an external polynomial model and the algebra of polynomials, solvability conditions were addressed together with a simple design procedure providing a stable dynamical solution. The authors in [55], investigated the disturbance rejection of nonlinear system. A sufficient condition was addressed to guarantee the existence of PI compensator of a given nonlinear plant to yield a stable closed-loop system with desired tracking and disturbance rejection performance. With the aid of neural networks, in [56] the state space of the disturbance-free plant was expanded to eliminate the effect of the disturbance. For some special cases, theoretical condition was introduced for complete rejection of the disturbance. In [57], under the disturbance-observer-based control (DOBC) framework, different observer designs were addressed for plants with different nonlinear dynamics for rejecting external disturbances. With the goal to find optimal disturbance rejection PID controller, the authors in [58] formulated this problem as a constrained optimization problem. Employing two genetic algorithms, a new method was developed for solving the constraint optimization problem. Inherited from traditional PID controller, active disturbance rejection control (ADRC) has been a work in progress [59–61]. With unknown system dynamics, [61] gave a detailed introduce of each component of ADRC as well as its structure and philosophy. Besides, the internal model principle is also adopted in disturbance rejection problem [62–64]. By using an adaptive observer, a compensator was designed to reject a biased sinusoidal disturbance in [63]. The authors of [64] proposed an internal model structure with adaptive frequency to cancel periodic disturbances.

Although the literature in disturbance rejection is rich, little attention has been paid to disturbance rejection of multi-agent systems, especially on the impact of interconnection topology among agents to disturbance rejection capability of whole multi-agent systems. In spite of this, some related research efforts can be observed in literature. Based on the Lyapunov function method, in [65], the problem of persistent disturbance rejection via state feedback for networked control systems was considered. The feedback gain to guarantee the disturbance rejection performance of the closed-loop system was derived with the aid of linear matrix inequalities. In [66], targeting analysis and growing analysis methods were adopted to the deadbeat disturbance rejection problem of multi-agent systems and a necessary condition for successful disturbance rejection was proposed. The authors in [67] built equivalence between the disturbance rejection problem of a multi-agent system and a set of independent systems whose dimensions are equal to that of a single agent. Besides, an interesting phenomenon was also observed is that the disturbance rejection capability of the whole multi-agent system coupled via feedback of merely relative measurements between agents will never be better than that of an isolated agent. In [51], the networked sensitivity transfer functions between any pair of agents for a given topology were developed for the convenience of disturbance rejection study of multi-agent systems. Disturbance rejection capability of uncertain multi-agent networks was investigated based on the proposed model reference adaptive control (MRAC) laws in [52].

Similar as the controllability problem, in this dissertation, under the disturbance rejection problem, we will consider multi-agent system described by weighted and directed interconnection topology, which most commonly emerges in complex system. Consequently, rather than the traditional disturbance rejection, the disturbance rejection in a structural sense will be discussed. Detailed motivation and discussions are addressed in Chapter 5.

Since both the controllability and disturbance rejection will be investigated in a structural sense, in the following part, we will give a short survey on structured system and what is going on in its structural properties study.

1.1.5 Structured System and Structural Properties

Motivated by the fact that the exact values of system parameters are usually difficult to obtain in practical applications due to uncertainties and noises, it is desirable to model physical systems into structured systems. A structured system is representative of a class of linear systems in the usual sense, whose system parameters are free parameters or fixed zeros. The structured systems viewpoint allows the determination of system properties to lie in the system structure and to remain invariant to changes in the parameter values. These so-called structural properties turn out to be true for almost all parameter values except for parameters in the zero set of some nontrivial polynomial with real coefficients in the system parameters.

The study of structured system was first introduced in [68], in which the structured system was associated with a directed graph whose vertices correspond to the input, state or output variables, and with an edge between two vertices if there is a free parameter relating the corresponding two variables in system dynamic equations and the structural controllability was investigated subsequently. The structural controllability study was further developed by [69] and alternatively investigated by [70, 71]. The authors in [72] further extended the structural controllability from linear system to interconnected linear systems. Following this framework, the structural controllability study for these composite structured systems was further derived using graph-theoretic method in [73–75]. In [75], criteria to determine the structural controllability of whole composite system using local structural controllability properties and the interconnection topology were developed. Lots of other control problems have been extensively investigated under this structured system framework. [76] addressed some basic issues and approaches related to structured property study and [77] provides a good survey of recent research efforts on structured systems. To describe the generic structure of transfer matrix at infinity, [78, 79] introduced disjoint input-output paths in the associated graph to deduce the infinite zero orders. The authors in [80–82] addressed how to determine the generic number of kinds of zeros for structured systems. Using graphic approaches to determine the state or feedback disturbance rejection problem was extensively addressed in [79, 83–86]. The authors in [83] substantiated that whenever the control inputs reach the outputs more quickly than the disturbances, the disturbance rejection is solvable by state feedback generically. In [85], necessary and sufficient conditions were derived for the generic solvability of the disturbance decoupling problem by measurement feedback for structured transfer matrix systems. In relation to the disturbance decoupling problem by measurement feedback, the sensor location and classification problem was introduced in [87]. [88] adopted graph-theoretic tools to show what kind of structured system can generically be decoupled into single-input, single-output systems by state feedback. This input-output decoupling problem was further investigated in [89]. Some investigations on decentralized control of structured systems were also addressed in [90, 91].

Based on the above literature review and motivation discussions, we are now ready to outline the detailed research problems attempted in this dissertation.

1.2 Contributions and Outline

The group behavior of multi-agent systems depends not only on the dynamics of each individual agent, but also on the local interactions among agents, i.e., the interconnection topology. In this dissertation, we aim to reveal the impact of interconnection topology on performance indexes of multi-agent systems under external signals (control signal and disturbance). In particular, controllability and disturbance rejection problems are addressed. The major novelties of this dissertation lie in the following several points:

- Rather than the traditional controllability and disturbance rejection, we consider a weighted interconnection topology of multi-agent systems, which quite commonly emerges in complex systems, and investigate the controllability and disturbance rejection in a structural sense, which are of more practical meaning, can overcome the inherently incomplete knowledge of the link weights and reduce the complexity of obtained justification algorithms. Besides, this kind of structural properties are true for almost all weight combinations except for some zero measure cases that occur when the system parameters satisfy certain accidental constraints.

- Graph theoretical interpretations of multi-agent system properties are addressed, which reveal the intrinsic relationship of interconnection topology and system behavior. This kind of graphic conditions make it convenient to verify system property just through the topology graph.
- Switching topologies are taken in to account under the multi-agent controllability problem, for which, to the best of our knowledge, there is almost no graph theory based study in literature.

Multi-agent systems with switching topologies are usually formulated as some kinds of hybrid system. Consequently, properties of hybrid systems: switched linear system and piecewise linear system, are investigated first as a general case. Then subsequent multi-agent properties study follows. The outline and contributions of this dissertation are addressed as follows.

First of all, the structural controllability of switched linear systems is investigated in Chapter 2. Two graphic representations of switched linear systems are presented. First, referring to the definition of structural controllability of linear system, we give the formal definition of structural controllability of switched linear systems. Subsequently, one representation graph named union graph is introduced. After addressing several graphic properties and their reflections in system matrix form, a sufficient condition for structural controllability of switched linear system is proposed. Furthermore, to obtain a elegant graphic interpretation of structural controllability, we devise another graph called colored union graph, in which edges from different subsystems (subgraphs) are labeled with different indexes (different colors). Based on this new proposed graph, two necessary and sufficient conditions for structural controllability are developed. Furthermore, the algorithm for verifying the graphic conditions is also presented together with the computation complexity estimation.

Following the discussion in Chapter 2, Chapter 3 formulates multi-agent system with switching topology as a special class of switched linear systems. Subsequently, the structural controllability of multi-agent systems is addressed and graphic interpretations of structural controllability under single/multi-leader under fixed/switching interconnection topology are proposed.

In Chapter 4, we consider the null controllability of piecewise linear system, which consists of two second order LTI systems separated by a line crossing through the origin. This can be treated as predecessor research effort for further study on multi-agent system with state-dependent switching topology. In the first part, the null controllability problem is addressed. First, the evolution directions from any non-origin state are studied from the geometric point of view, and it turns out that the directions usually span an open half space. Then, we derive an explicit and easily verifiable necessary and sufficient condition for a planar bimodal piecewise linear system to be null controllable. In the second part, a short survey of research efforts on state-dependent multi-agent systems together with possible application of the result obtained for piecewise linear system to state-dependent multi-agent system are presented.

Chapter 5 considers how the interconnection topology influences the structural disturbance rejection capability of multi-agent systems. The intrinsic interactions among states of agents are illustrated using a weighted and directed interconnection graph. Two kinds of systems models are considered: multi-agent systems with identical single integrator dynamics and multi-agent systems with non-homogeneous general linear dynamics. Criteria to determine the structural disturbance rejection capability of these systems using only the local disturbance rejection capability and the interconnection topology among local dynamics are devised. Besides, this chapter investigates the potential combination of obtained disturbance rejection results with the controllability problem of multi-agent systems. We explicitly show under what kind of interaction topologies, the whole multi-agent system is structurally controllable and meanwhile has structural disturbance rejection capability

In Chapter 6, we summarize the dissertation and discuss the results obtained. We also sketch the possible future steps to continue the work started with this dissertation.

Chapter 2

Structural Controllability of Switched Linear System

2.1 Introduction

In this chapter we will focus on providing solutions to the structural controllability problem of switched linear systems with the aid of graph theory. This work can be treated as a general case or predecessor research effort of the following study on structural controllability of multi-agent systems with switching topologies in Chapter 3. The structural controllability is a generalization of the traditional controllability concept for dynamical systems, and purely based on the graphic topologies among state and input vertices. The main underlying question here is to reveal how the graphic topologies of switched linear systems will influence or determine the structural controllability of switched linear systems and what are the graph theoretic necessary and sufficient conditions for the controllability. To fully describe the graphic topologies of switched linear systems, two kinds of graphic representations, union graph and colored union graph, will be addressed. After devising clear relations between several graphic properties and system matrices, graph theory based necessary and sufficient characterizations of the structural controllability for switched linear systems are presented. A brief literature review on controllability of switched linear systems and the motivation of the tackled problem in this chapter are given as follows.

As a special class of hybrid control systems, a switched linear system consists of several linear subsystems and a rule that orchestrates the switching among them. Switching between different subsystems or different controllers can greatly enrich the control strategies and may accomplish certain control objective which can not be achieved by conventional dynamical systems. For example, it provided an effective mechanism to cope with highly complex systems and/or systems with large uncertainties [92]. References [93] presented good examples that switched controllers could provide a performance improvement over a fixed controller. Besides, switched linear systems also have promising applications in control of mechanical systems, aircrafts, satellites and swarming robots. Driven by its importance in both theoretical research and practical applications, switched linear system has attracted considerable attention during the last decade [94–102].

Much work has been done on the controllability of switched linear systems. For example, the controllability and reachability for low-order switched linear systems have been presented in [96]. Under the assumption that the switching sequence is fixed, references [97] and [98] introduced some sufficient conditions and necessary conditions for controllability of switched linear systems. Complete geometric criteria for controllability and reachability were established in [99] and [101].

Up to now, all the previous work mentioned above has been based on the traditional controllability concept of switched linear systems. In this chapter, we propose a new notion for the controllability of switched linear system: structural controllability, which may be more reasonable in face of uncertainties. Actually, it is more often than not that most of system parameter values are difficult to identify and only known to certain approximations. On the other hand, we are usually pretty sure where zero elements are either by coordination or by the absence of physical connections among components of the system. Thus structural properties that are independent of a specific value of unknown parameters, e.g., the structural controllability studied here, are of particular interest. It is therefore assumed here that all the elements of matrices of switched linear systems are fixed zeros or free parameters. Furthermore, the switched linear system is said to be structurally controllable if one can find a set of values for the free parameters such that the corresponding switched linear system is controllable in the classical sense. For linear structured systems, generic properties including structural controllability have been studied extensively and it turns out that generic properties including structural controllability are true for almost all values of the parameters [68–71, 75–77, 103, 104]. That is also true for switched linear systems studied here and presents one of the reasons why this kind of structural controllability is of interest.

Graphic conditions can help to understand how the graphic topologies of dynamical systems influence the corresponding generic properties, here especially for the structural controllability. This would be helpful in many practical applications. For example, for multi-agent systems in Chapter 3, graphic interpretations for structural controllability help us to understand the necessary information exchange among agents to make the whole team well-behaved, e.g., controllable. Therefore, this motivates our pursuit on illuminating the structural controllability of switched linear systems from a graph theoretical point of view. In this chapter, we propose two graphic representations of switched linear systems and finally, it turns out that the structural controllability of switched linear systems only depends on the graphic topologies of the corresponding systems.

The organization of the rest of this chapter is as follows. In Section 2.2, we introduce some basic preliminaries and the problem formulation, followed by structural controllability study of switched linear systems in Section 2.3, where several graphic necessary and sufficient conditions for the structural controllability are devised. Numerical examples together with discussions on a more general case are also presented. Finally, some concluding remarks are drawn in Section 2.4.

2.2 Preliminaries and Problem Formulation

2.2.1 Graph Theory Preliminaries

First of all, the definition and example of a structured matrix are introduced as follows:

Definition 1. *P is said to be a structured matrix if its entries are either fixed zeros or independent free parameters. $\tilde{P}$ is called admissible (with respect to P) if it can be obtained by fixing the free parameters of P at some particular values. In addition P_{ij} is adopted to represent the element of P at row i and column j.*

Example 1. $P=\begin{bmatrix} 0 & \lambda_1 \\ \lambda_2 & \lambda_3 \end{bmatrix}$ *is a structured matrix, where λ_1, λ_2 and λ_3 are free parameters, and* $\tilde{P}=\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ *is admissible with respect to P.*

Following the above definition, now consider a linear control system:

$$\dot{x} = Ax(t) + Bu(t), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^r$. The matrices A and B are structured matrices, which means that their elements are either fixed zeros or free parameters. This structured system given

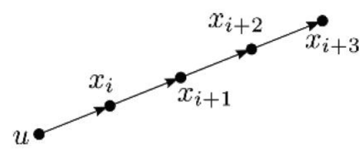

Fig. 2.1: Stem

by matrix pair (A, B) can be described by a directed graph [68]:

Definition 2. *The representation graph of structured system* (A, B) *is a directed graph* $\mathcal{G}$, *with vertex set* $\mathcal{V} = \mathcal{X} \cup \mathcal{U}$, *where* $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, *which is called state vertex set and* $\mathcal{U} = \{u_1, u_2, \ldots, u_r\}$, *which is called input vertex set, and edge set* $\mathcal{I} = \mathcal{I}_{UX} \cup \mathcal{I}_{XX}$, *where* $\mathcal{I}_{UX} = \{(u_i, x_j) | B_{ji} \neq 0, 1 \leq i \leq r, 1 \leq j \leq n\}$ *and* $\mathcal{I}_{XX} = \{(x_i, x_j) | A_{ji} \neq 0, 1 \leq i \leq n, 1 \leq j \leq n\}$ *are the oriented edges between inputs and states and between states defined by the interconnection matrices A and B above. This directed graph (for notational simplicity, we will use digraph to refer to directed graph)* $\mathcal{G}$ *is also called the graph of matrix pair* (A, B) *and denoted by* $\mathcal{G}(A, B)$.

Note that the total number of vertices in $\mathcal{G}(A, B)$ equals to the summation of dimension n of system states and dimension r of system control inputs. One important graphic definition in a digraph $\mathcal{G}$ is needed before we proceed forward:

Definition 3. *(Stem [68]) An alternating sequence of distinct vertices and oriented edges is called a directed path, in which the terminal node of any edge never coincide to its initial node or the initial or the terminal nodes of the former edges. A stem is a directed path in the state vertex set* $\mathcal{X}$, *that begins in the input vertex set* $\mathcal{U}$.

Two graphic properties 'accessibility' and 'dilation' were proposed by [68], which will serve as the basis of following discussion. We state them as follows:

Definition 4. *(Accessibility [68]) A vertex (other than the input vertices) is called nonaccessible if and only if there is no possibility of reaching this vertex through any stem of the*

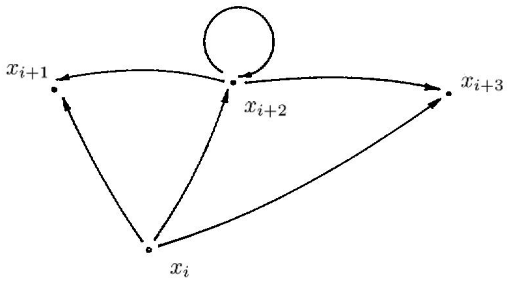

Fig. 2.2: Dilation

graph $\mathcal{G}$.

Definition 5. *(Dilation [68]) Consider one vertex set S formed by the vertices from the state vertices set $\mathcal{X}$ and determine another vertex set $T(S)$, which contains all the vertices v with the property that there exists an oriented edge from v to one vertex in S. Then the graph $\mathcal{G}$ contains a 'dilation' if and only if there exist at least a set S of k vertices in the vertex set of the graph such that there are no more than $k-1$ vertices in $T(S)$.*

Graphic illustrations for 'stem' and 'dilation' are shown in Fig.2.1 and Fig.2.2 respectively.

2.2.2 Switched Linear System, Controllability and Structural Controllability

In general, a switched linear system is composed of a family of subsystems and a rule that governs the switching among them, and is mathematically described by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ are the states, $u(t) \in \mathbb{R}^r$ are piecewise continuous input, $\sigma : [0, \infty) \rightarrow M \triangleq \{1, \ldots, m\}$, where time slot 0 is system initial time, is the switching signal. System (2.2) contains m subsystems (A_i, B_i), $i \in \{1, \ldots, m\}$ and $\sigma(t)= i$ implies that the ith subsystem (A_i, B_i) is active at time instance t.

In the sequel, the following definition of controllability of system (2.2) will be adopted [99]:

Definition 6. *Switched linear system (2.2) is said to be (completely) controllable if for any initial state x_0 and final state x_f, there exist a time instance $t_f > 0$, a switching signal $\sigma : [0, t_f) \to M$ and an input $u : [0, t_f) \to \mathbb{R}^r$ such that $x(0) = x_0$ and $x(t_f) = x_f$.*

For the controllability of switched linear systems, a well-known matrix rank condition was given in [100]:

Lemma 1. *([100]) If matrix:*

$$
\begin{aligned}
&[B_1, B_2, \ldots, B_m, A_1B_1, A_2B_1, \ldots, A_mB_1, A_1B_2, A_2B_2, \ldots, A_mB_2, \ldots, A_1B_m, A_2B_m,\\
&\ldots, A_mB_m, A_1^2B_1, A_2A_1B_1, \ldots, A_mA_1B_1, A_1A_2B_1, A_2^2B_1, \ldots, A_mA_2B_1, \ldots, A_1A_mB_m\\
&, A_2A_mB_m, \ldots, A_m^2B_m,\\
&, \ldots,\\
&A_1^{n-1}B_1, A_2A_1^{n-2}B_1, \ldots, A_mA_1^{n-2}B_1, A_1A_2A_1^{n-3}B_1, A_2^2A_1^{n-3}B_1, \ldots, A_mA_2A_1^{n-3}B_1 \ldots,\\
&A_1A_m^{n-2}B_m, A_2A_m^{n-2}B_m \ldots, A_m^{n-1}B_m]
\end{aligned}
\tag{2.3}
$$

has full row rank n, then switched linear system (2.2) is controllable, and vice versa.

Remark 1. *This matrix is called controllability matrix of switched linear system (2.2) and for simplicity, we will use $C(A_1, \ldots, A_m, B_1, \ldots, B_m)$ to represent it. If we use ImP to represent the range space of arbitrary matrix P, actually,*
$ImC(A_1, \ldots, A_m, B_1, \ldots, B_m)$ is the controllable subspace of switched linear system (2.2)(see [99] and [100]). The above lemma implies that system (2.2) is controllable if and only if

$ImC(A_1, \ldots, A_m, B_1, \ldots, B_m) = \mathcal{R}^n$. Besides, controllable subspace can be expressed as $\langle A_1, \ldots, A_m | B_1, \ldots, B_m \rangle$, which is the smallest subspace containing ImB_i, $i = 1, \ldots, m$ and invariant under the transformations $A_1, \ldots, A_m$ [102].

In view of structural controllability, system (2.2) will be treated as structured switched linear system defined as:

Definition 7. *For structured system (2.2), elements of all the matrices $(A_1, B_1, \ldots, A_m, B_m)$ are either fixed zero or free parameters and free parameters in different subsystems $(A_i, B_i), i \in M$ are independent. A numerically given matrices set $(\tilde{A}_1, \tilde{B}_1, \ldots, \tilde{A}_m, \tilde{B}_m)$ is called an admissible numerical realization (with respect to $(A_1, B_1, \ldots, A_m, B_m)$) if it can be obtained by fixing all free parameter entries of $(A_1, B_1, \ldots, A_m, B_m)$ at some particular values.*

Similar with the definition of structural controllability of linear system in [76], we have the following definition for structural controllability of switched linear system (2.2):

Definition 8. *Switched linear system (2.2) given by its structured matrices $(A_1, B_1, \ldots, A_m, B_m)$ is said to be structurally controllable if and only if there exists at least one admissible realization $(\tilde{A}_1, \tilde{B}_1, \ldots, \tilde{A}_m, \tilde{B}_m)$ such that the corresponding switched linear system is controllable in the usual numerical sense.*

Remark 2. *It turns out that once a structured system is controllable for one choice of system parameters, it is controllable for almost all system parameters, in which case the structured system then will be said to be structurally controllable [68, 77].*

Before proceeding further, we need to introduce the definition of g-rank of one matrix:

Definition 9. *The generic rank (g-rank) of a structured matrix P is defined to be the maximal rank that P achieves as a function of its free parameters.*

Then, we have the following algebraic condition for structural controllability of system (2.2):

Lemma 2. *Switched linear system (2.2) is structurally controllable if and only if*

$$g\text{-}rank\ C(A_1, \ldots, A_m, B_1, \ldots, B_m) = n.$$

2.3 Structural Controllability of Switched Linear Systems

2.3.1 Criteria Based on Union Graph

For switched linear system (2.2), digraph $\mathcal{G}_i(A_i, B_i)$ with vertex set $\mathcal{V}_i$ and edge set $\mathcal{I}_i$ can be adopted as the representation graph of subsystem (A_i, B_i), $i \in \{1, \ldots, m\}$.

As to the whole switched system, one kind of representation graph, which is called union graph, is described in the following definition:

Definition 10. *Switched linear system (2.2) can be represented by a union digraph $\mathcal{G}$ (sometimes named union graph without leading to confusion). Mathematically, $\mathcal{G}$ is defined as*

$$\mathcal{G}_1 \cup \mathcal{G}_2 \cup \ldots \cup \mathcal{G}_m = \{\mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_m; \mathcal{I}_1 \cup \mathcal{I}_2 \cup \ldots \cup \mathcal{I}_m\}$$

For union graph $\mathcal{G}$, the vertex set is the same as the vertex set of every subgraph $\mathcal{G}_i$. The edge set of $\mathcal{G}$ equals to the union of the edge sets of the subgraphs. Note that there are no multiple edges between any two vertices in $\mathcal{G}$.

Remark 3. *It turns out that union graph $\mathcal{G}$ is the representation graph of linear structured system: $(A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m)$. The reason is as follows: If the element* *at position $a_{ji}(b_{ji})$ in matrix $[A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m]$ is a free parameter, this implies that there exist some matrices $[A_p, B_p]$, $p = 1, \ldots, m$ such that the element at position $a_{ji}(b_{ji})$ is also a free parameter and in the corresponding subgraph $\mathcal{G}_p$, there is an edge from vertex i to vertex j. According to the definition of union graph, it follows that there is also an edge from vertex i to vertex j in union graph $\mathcal{G}$. If the element at position $a_{ji}(b_{ji})$ in $[A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + A_m]$ is zero, this implies that for every matrices $[A_p, B_p]$, $p = 1, \ldots, m$, the element at position $a_{ji}(b_{ji})$ is zero and in the corresponding subgraph $\mathcal{G}_p$, there is no edge from vertex i to vertex j. It follows that there is also no edge in union graph $\mathcal{G}$ from vertex i to vertex j.*

Before proceeding further, we need to introduce two definitions which were proposed in [68] for linear system (2.1) first:

Definition 11. *([68]) The matrix pair (A, B) is said to be reducible or of form I if there exists a permutation matrix P such that they can be written in the following form:*

$$PAP^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, PB = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \tag{2.4}$$

where $A_{11} \in \mathbb{R}^{p\times p}$, $A_{21} \in \mathbb{R}^{(n-p)\times p}$, $A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $B_{22} \in \mathbb{R}^{(n-p)\times r}$.

Remark 4. *Whenever the matrix pair (A, B) is of form I, the system is structurally uncontrollable ([68]) and meanwhile, the controllability matrix $C \triangleq \left[B, AB, \ldots, A^{n-1}B\right]$ will have at least one row which is identically zero for all parameter values [70]. If there is no such permutation matrix P, we say that the matrix pair (A, B) is irreducible.*

Definition 12. *([68]) The matrix pair (A, B) is said to be of form II if there exists a permutation matrix P such that they can be written in the following form:*

$$\left[PAP^{-1}, PB\right] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \tag{2.5}$$

where $P_2 \in \mathbb{R}^{(n-k)\times(n+r)}$, $P_1 \in \mathbb{R}^{k\times(n+r)}$ with no more than $k-1$ nonzero columns (all the other columns of P_1 have only fixed zero entries).

The following lemma, which will underpin the following analysis on switched linear systems, details the criteria for evaluating structural controllability of linear system (A, B) [68, 76]:

Lemma 3. *([68,76]) For linear structured system (2.1), the following statements are equivalent:*

a) the pair (A, B) is structurally controllable;

b) i)$[A, B]$ is irreducible or not of form I,
ii)$[A, B]$ has g-rank$[A, B] = n$ or is not of form II;

c) i)there is no nonaccessible vertex in $\mathcal{G}(A, B)$,
ii)there is no 'dilation' in $\mathcal{G}(A, B)$.

This lemma proposed interesting graphic conditions for structural controllability of linear systems and revealed that the structural controllability is totally determined by the underlying graph topology. However, for switched linear systems, to the best of our knowledge, proper graphic representations which can determine the structural controllability properties of switched linear systems are still lacking in the literature.

With the previous lemmas and definitions, we are in the position to present the first main result of the chapter, which is actually a graphic sufficient condition for structural controllability of switched linear systems:

Theorem 4. *Switched linear system (2.2) with graphic topologies $\mathcal{G}_i$, $i \in \{1, \dots, m\}$, is structurally controllable if its union graph $\mathcal{G}$ satisfies:*

i) there is no nonaccessible vertex in $\mathcal{G}$,

ii) there is no 'dilation' in $\mathcal{G}$.

Proof. Assume the two conditions in this theorem are satisfied. According to Remark 3 and Lemma 3, the corresponding linear system $(A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m)$ is structurally controllable. It follows that there exist some scalars for the free parameters in matrices $(A_i, B_i), i = 1, 2, \ldots, m$ such that controllability matrix

$$[B_1 + B_2 + \ldots + B_m, (A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m),$$
$$(A_1 + A_2 + \ldots + A_m)^2(B_1 + B_2 + \ldots + B_m), \ \ldots,$$
$$(A_1 + A_2 + \ldots + A_m)^{n-1}(B_1 + B_2 + \ldots + B_m)]$$

has full row rank n. Expanding the matrix, it follows that matrix

$$[B_1 + B_2 + \ldots + B_m, A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2$$
$$+ \ldots + A_mB_2 + \ldots + A_1B_m + A_2B_m \ldots + A_mB_m, \ \ldots,$$
$$A_1^{n-1}B_1 + A_2A_1^{n-2}B_1 + \ldots + A_m^{n-1}B_m]$$

has full rank n.

The following matrix can be got after adding some column vectors to the above matrix:

$$[B_1 + B_2 + \ldots + B_m, B_2, \ldots, B_m, A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2$$
$$+ \ldots + A_mB_2 + \ldots + A_1B_m + A_2B_m + \ldots + A_mB_m, A_2B_1, \ldots, A_mB_m, \ \ldots,$$
$$A_1^{n-1}B_1 + A_2A_1^{n-2}B_1 + \ldots + A_1A_m^{n-2}B_1 + \ldots + A_m^{n-1}B_m, A_2A_1^{n-2}B_1, \ldots,$$
$$A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m].$$

Since this matrix still has n linear independent column vectors, it follows that it has full row rank n. Next, subtracting $B_2, \ldots, B_m$ from $B_1 + B_2 + \ldots + B_m$; subtracting $A_2B_1, \ldots, A_mB_m$ from $A_1B_1 + A_2B_1 + \ldots + A_mB_1 + \ldots + A_1B_m + \ldots + A_mB_m$ and subtracting $A_2A_1^{n-2}B_1, \ldots,$ $A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m$ from $A_1^{n-1}B_1 + A_2A_1^{n-2}B_1 + \ldots + A_1A_m^{n-2}B_1 + \ldots + A_m^{n-1}B_m$, we can get the following matrix:

$$[B_1, B_2, \ldots, B_m, A_1B_1, A_2B_1, \ldots, A_mB_m, \ \ldots,$$
$$A_1^{n-1}B_1, A_2A_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_1, \ldots, A_m^{n-1}B_m],$$

which is the controllability matrix for switched linear systems (2.2). Since column fundamental transformation does not change the matrix rank, this matrix still has full row rank n. Hence, the switched linear system (2.2) is structurally controllable. □

Actually, from the proof, we can see that full rank of controllability matrix of linear system $(A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m)$ in Remark 3 implies the full rank of controllability matrix of system (2.2), which means that the structural controllability of this linear system implies structural controllability of system (2.2). It turns out that this criterion is not necessary for system (2.2) to be structurally controllable (see the example in subsection 3.4). This implies that the union graph does not contain enough information for determining structural controllability. This is because edges from different subsystems are not differentiated in union graph. In the following subsection, another graphic representation of switched linear systems is proposed, from which necessary and sufficient conditions for structural controllability arise.

2.3.2 Criteria Based on Colored Union Graph

Another graphic representation: *'colored union graph'* is defined as follows:

Definition 13. *Switched linear system (2.2) can be represented by a colored union digraph $\tilde{\mathcal{G}}(\tilde{\mathcal{V}}, \tilde{\mathcal{I}})$ (sometimes named colored union graph without leading to confusion), where vertex set $\tilde{\mathcal{V}} = \{\mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_m\}$ and edge set $\tilde{\mathcal{I}} = \{e | e \in \mathcal{I}_i, i = 1, 2, \ldots, m\}$, i.e., for $i \in \{1, \ldots, m\}$, to each edge e we associate index i in $\tilde{\mathcal{G}}$, if this edge is associated to the subsystem i (subgraph $\mathcal{G}_i$). Note that we associate several indexes (several different colors) to an edge e if it belongs to several subsystems.*

With this colored union graph, several graphic properties are introduced in the following lemmas.

Lemma 5. *There is no nonaccessible vertex in the colored union graph $\tilde{\mathcal{G}}$ of switched linear system (2.2) if and only if the matrix*

$$[A_1 + A_2 + \cdots + A_m, B_1 + B_2 + \cdots + B_m] \tag{2.6}$$

is irreducible or not of form I.

Proof. One vertex is accessible if and only if it can be reached by a stem. From Definitions 10 and 13, it follows that there is no nonaccessible vertex in the colored union graph if and only if there is no nonaccessible vertex in the union graph. Besides, from Remark 3, it is clear that the matrix representation of the union graph is $[A_1 + A_2 + \cdots + A_m, B_1 + B_2 + \cdots + B_m]$. According to Lemma 3, there is no nonaccessible vertex in the union graph if and only if matrix (2.6) is irreducible or not of form I. Consequently the equivalence between accessibility of colored union graph and irreducibility of matrix (2.6) gets proved. □

A new graphic property '*S-dilation*' in colored union graph needs to be introduced here:

Definition 14. *In colored union graph $\tilde{\mathcal{G}}$, which is composed of subgraphs $\mathcal{G}_i$, $i = 1, 2, \ldots, m$, consider one vertex set S formed by the vertices from the state vertex set $\mathcal{X}$ and determine another vertex set $T(S) = \{v | v \in T_i(S), i = 1, 2, \ldots, m\}$, where $T_i(S)$ is a vertex set in $\mathcal{G}_i$ which contains all the vertices w with the property that there exists an oriented edge from w to one vertex in S. Then $|T(S)| = \sum_{i=1}^{m} |T_i(S)|$. If $|T(S)| < |S|$, we say that there is a S-dilation in the colored union graph $\tilde{\mathcal{G}}$.*

Based on this new graphic property, the following lemma can be introduced:

Lemma 6. *There is S-dilation in the colored union graph $\tilde{\mathcal{G}}$ of switched linear system (2.2) if and only if matrix $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$ is of form II. It means that this matrix can be written into: $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$, where $P_1 \in \mathbb{R}^{p \times k}$ with no more than $p - 1$ nonzero columns (all the other columns of P_1 have only fixed zero entries).*

Proof. From [68] and [71] or Lemma 3, it is known that in linear systems, there is no 'dilation' in the corresponding graph if and only if the matrix pair $[A, B]$ can not be of form II or have g-rank $= n$. From the explanation of this result in [68] and Definition 12, P_1 in $[A, B]$ has p rows, which actually represents the p vertices of vertex set S (defined for dilation), and each nonzero element of each row of P_1 represents that there is one vertex pointing to the vertex presented by this row. Therefore, the number of nonzero columns in P_1 is the number of vertices pointing to some vertex in S, and actually equals to $|T(S)|$. Furthermore, by the definition of S*-dilation*, $|T(S)|$ is now the summation of $|T_i(S)|$, $i \in \{1, \ldots, m\}$, in every subgraph. Consequently, it follows that there is S*-dilation* in the colored union graph $\tilde{\mathcal{G}}$ if and only if matrix $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$ is of form II. □

Before going further to give another algebraic explanation of S*-dilation*, one definition and lemma proposed in [69] must be introduced first:

Definition 15. *([69]) A structured $n \times m'$ $(n \leq m')$ matrix A is of form (t) for some t, $1 \leq t \leq n$, if for some k in the range $m' - t < k \leq m'$, A contains a zero submatrix of order $(n + m' - t - k + 1) \times k$.*

Lemma 7. *([69]) g-rank of $A = t$*

i) for $t = n$ if and only if A is not of form (n);

ii) for $1 \leq t < n$ if and only if A is of form $(t + 1)$ but not of form (t).

From the above definition and lemma, another lemma is proposed here:

Lemma 8. *There is no S-dilation in the colored union graph $\tilde{\mathcal{G}}$ of switched linear system (2.2) if and only if the following matrix*

$$[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] \tag{2.7}$$

has g-rank n.

Proof. *Necessity:* If the matrix in (2.7) has g-rank $< n$, from Lemma 7, it follows that this matrix is of form (n). Then referring to Definition 15, the matrix in (2.7) must have a zero submatrix of order $(n + m' - t - k + 1) \times k$. Here, t can be chosen as n, then (2.7) has a zero submatrix of order $(m' - k + 1) \times k$. For this $(m' - k + 1)$ rows, there are only $(m' - k)$ nonzero columns. Consequently, the matrix in (2.7) is of form II and by Lemma 6, there is *S-dilation* in the colored union graph $\tilde{\mathcal{G}}$ of switched linear system (2.2).

Sufficiency: If there is *S-dilation* in the colored union graph $\tilde{\mathcal{G}}$, by Lemma 6, the matrix in (2.7) is of form II, then obviously P_1 in (2.7) can not have row rank equal to k and furthermore, the matrix in (2.7) can not have g-rank $= n$. □

With the above definitions and lemmas, a graphic necessary and sufficient condition for switched linear system to be structurally controllable can be proposed here:

Theorem 9. *Switched linear system (2.2) with graphic representations $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$, is structurally controllable if and only if its colored union graph $\tilde{\mathcal{G}}$ satisfies the following two conditions:*

i) there is no nonaccessible vertex in the colored union graph $\tilde{\mathcal{G}}$,

ii) there is no S-dilation in the colored union graph $\tilde{\mathcal{G}}$.

Proof. *Necessity: (i)* If there exist nonaccessible vertices in $\tilde{\mathcal{G}}$, by Lemma 5, the matrix $[A_1 + A_2 + \cdots + A_m, B_1 + B_2 + \cdots + B_m]$ is reducible or of form I. It follows that the controllability matrix

$$\begin{aligned}
&[B_1 + B_2 + \ldots + B_m, (A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m),\\
&(A_1 + A_2 + \ldots + A_m)^2(B_1 + B_2 + \ldots + B_m),\ \ldots,\\
&(A_1 + A_2 + \ldots + A_m)^{n-1}(B_1 + B_2 + \ldots + B_m)]
\end{aligned}$$

always has at least one row that is identically zero (Remark 4). It is clear that every component of the matrix, such as B_i, A_iB_j *and* $A_i^pA_j^qB_r$ has the same row always to be zero. As a result, the controllability matrix

$$[B_1, \ldots, B_m, A_1B_1, \ldots, A_mB_1, \ldots, A_mB_m, A_1^2B_1, \ldots, A_mA_1B_1, \ldots, A_1^2B_m, \ldots,$$
$$A_mA_1B_m, \ldots, A_1^{n-1}B_1, \ldots, A_mA_1^{n-2}B_1, \ldots, A_1A_m^{n-2}B_m, \ldots, A_m^{n-1}B_m]$$

always has one zero row and can not be of full rank n. Therefore, switched linear system (2.2) is not structurally controllable.

(ii) Suppose that switched linear system (2.2) is structurally controllable, i.e., the controllability matrix satisfies g-rank $C(A_1, \ldots, A_m, B_1, \ldots, B_m) = n$. Specifically, $Im[B_1, \ldots, B_m, A_1B_1, \ldots, A_mB_m, A_1^2B_1, \ldots, A_m^{n-1}B_m] = \mathbb{R}^n$. Since $\forall P \in \mathbb{R}^{n\times r}$, $Im(A_iP) \subseteq Im(A_i)$, we have that $Im[B_1, \ldots, B_m, A_1B_1, \ldots, A_mB_m, A_1^2B_1, \ldots, A_m^{n-1}B_m] \subseteq Im[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] \subseteq \mathbb{R}^n$. Thus condition g-rank $C(A_1, \ldots, A_m, B_1, \ldots, B_m) = n$ requires that $Im[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] = \mathbb{R}^n$ and therefore g-rank $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] = n$. However, if there is *S-dilation* in the colored union graph $\tilde{\mathcal{G}}$, by Lemma 6, g-rank $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m] < n$. Consequently, the switched linear system (2.2) is not structurally controllable.

Sufficiency: The general idea in the sufficiency proof is that we will assume that the two graphic conditions in the theorem hold. Then a contradiction will be found such that it is impossible that switched linear system (2.2) is structurally uncontrollable.

Before proceeding to switched linear system (2.2), firstly, consider a structured linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.8}$$

It is well known that system (2.8) is structurally controllable if and only if there exists a numerical realization $(\tilde{A}, \tilde{B})$, such that rank $(sI - \tilde{A}, \tilde{B}) = n, \forall s \in \mathbb{C}$. Otherwise, the PBH test [105] states that system (2.8) is uncontrollable if and only if for every numerical realization, there exists a row vector $q \neq 0$ such that $q\tilde{A} = s_0 q, s_0 \in \mathbb{C}$ and $q\tilde{B} = 0$, where rank $(s_0 I - \tilde{A}, \tilde{B}) < n$.

On one hand, if for every numerical realization rank $(sI - \tilde{A}, \tilde{B}) = n, \forall s \in \mathbb{C} \setminus \{0\}$, then the uncontrollability of system (2.8) implies necessarily that for every numerical realization there exists a vector $q \neq 0$ such that $q\tilde{A} = 0$ and $q\tilde{B} = 0$.

On the other hand, Lemma 14.1 of [76] states that, if in the digraph associated to (2.8), every state vertex is an end vertex of a stem (accessible), then g-rank $(sI - A, B) = n, \forall s \in \mathbb{C} \setminus \{0\}$, which means that for almost all numerical realization $(\tilde{A}, \tilde{B})$, rank $(sI - \tilde{A}, \tilde{B}) = n, \forall s \in \mathbb{C} \setminus \{0\}$.

Now considering switched linear system (2.2), assume that the two conditions in Theorem 9 are satisfied. Due to Lemma 14.1 of [76], as all the parameters of matrices $A_1, \ldots, A_m$, $B_1, \ldots, B_m$ are assumed to be free, the condition *(i)* of Theorem 9 implies that, for almost all vector values $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$, we have g-rank $(sI - (\bar{u}_1 A_1 + \ldots + \bar{u}_m A_m), (\bar{u}_1 B_1 + \ldots + \bar{u}_m B_m)) = n, \forall s \neq 0$. On the other hand, if switched linear system (2.2) is structurally uncontrollable, then for all constant values, $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$, linear systems defined by matrices $(\bar{A}, \bar{B})$ are also uncontrollable, where $\bar{A} = \sum_{i=1}^{m} \bar{u}_i A_i$ and $\bar{B} = \sum_{i=1}^{m} \bar{u}_i B_i$. We write the numerical realization of $(\bar{A}, \bar{B})$ as $(\tilde{\bar{A}}, \tilde{\bar{B}})$. This is due to the fact that for all constant values $\bar{u}$, $Im(C(\bar{A}, \bar{B}) \subseteq Im(C(A_1, \ldots, A_m, B_1, \ldots, B_m))$. Therefore, if the switched linear system is structurally uncontrollable, since for almost all $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$, g-rank $(sI - (\bar{u}_1 A_1 + \ldots + \bar{u}_m A_m), (\bar{u}_1 B_1 + \ldots + \bar{u}_m B_m)) = n, \forall s \neq 0$, we have that for every numerical realization matrix pair $(\tilde{\bar{A}}, \tilde{\bar{B}})$, there exists a nonzero vector q such that $q\tilde{\bar{A}} = 0$ and $q\tilde{\bar{B}} = 0$. Since this statement is true for almost all the values $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_m)$, we have that for almost all $n \cdot m$-tuple values $\bar{u}^j = (\bar{u}_1^j, \ldots, \bar{u}_m^j)$, $j = 1, \ldots, n \cdot m$, we can find nonzero vectors

q_j such that the following holds:

$$\begin{cases} \sum_{i=1}^{m} \bar{u}_i^j q_j \tilde{A}_i = 0, j = 1, \ldots, n \cdot m \\ \sum_{i=1}^{m} \bar{u}_i^j q_j \tilde{B}_i = 0. j = 1, \ldots, n \cdot m \end{cases} \tag{2.9}$$

Obviously, there can not exist more than n linear independent vectors q_j. Let us denote $q_1, q_2, \ldots, q_n$ the vectors such that $span\ (q_1, q_2, \ldots, q_{n \cdot m}) \subseteq$ $span\ (q_1, q_2, \ldots, q_n)$ (we can renumber the vectors if necessary). All the vectors $q_j, j = n+1, \ldots, n \cdot m$ are linear combinations of $q_1, q_2, \ldots, q_n$. Therefore, system (2.9) contains the following equations:

$$\begin{cases} \sum_{k=1}^{n} \sum_{i=1}^{m} a_{i,k}^j(\bar{u}) q_k \tilde{A}_i = 0 \quad j = 1, \ldots, n \cdot m \\ \sum_{k=1}^{n} \sum_{i=1}^{m} a_{i,k}^j(\bar{u}) q_k \tilde{B}_i = 0 \quad j = 1, \ldots, n \cdot m \end{cases} \tag{2.10}$$

where $a_{i,k}^j(\bar{u})$ are linear functions of $\bar{u}^j, j = 1, \ldots, n \cdot m$. Since system (2.9) is satisfied for almost all the values, we can find $\bar{u}^j, j = 1, \ldots, n \cdot m$ such that

$$det \begin{bmatrix} a_{1,1}^1(\bar{u}) & a_{1,2}^1(\bar{u}) & \ldots & a_{m,n}^1(\bar{u}) \\ a_{1,1}^2(\bar{u}) & a_{1,2}^2(\bar{u}) & \ldots & a_{m,n}^2(\bar{u}) \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,1}^{n \cdot m}(\bar{u}) & a_{1,2}^{n \cdot m}(\bar{u}) & \ldots & a_{m,n}^{n \cdot m}(\bar{u}) \end{bmatrix} \neq 0.$$

In this case, the only solution of (2.10) is $q_k \tilde{A}_1 = \ldots = q_k \tilde{A}_m = q_k \tilde{B}_1 = \cdots = q_k \tilde{B}_m = 0,\ k = 1, \ldots, n$. Obviously, if the switched linear system is structurally uncontrollable, then vector $q_k, k = 1, \ldots, n$ is nonzero. Consequently, switched linear system (2.2) is structurally uncontrollable only if for every numerical realization there exists at least one nonzero vector q such that $qA_1 = \ldots = qA_m = qB_1 = \cdots = qB_m = 0$. However, if condition *ii* of Theorem 9 is satisfied, then g-rank $[A_1, \ldots, A_m, B_1, \ldots, B_m] = n$ and therefore, for at least one numerical realization, there does not exist a vector $q \neq 0$ such that $qA_1 = \ldots = qA_m = qB_1 = \cdots = qB_m = 0$. Hence, the two conditions are sufficient to ensure the structural controllability of switched linear system (2.2). □

Actually, using the terminologies '*dilation*' and '*S-dilation*' as graphic criteria is not so numerically efficient. For example, to check the second condition of Theorem 9, we need to test for all possible vertex subsets to see whether there exist *S-dilation* in the colored union graph or not. Consequently, we will adopt another notion '*S-disjoint edges*' to form a more numerically efficient graphic interpretation of structural controllability.

Definition 16. *In the colored union graph $\tilde{\mathcal{G}}$, consider k edges $e_1 = (v_1, v_1'), e_2 = (v_2, v_2'), \ldots, e_k = (v_k, v_k')$. We define for $i = 1, \ldots, k$, S_i as the set of integers j such that $v_j = v_i$, i.e., $S_i = \{1 \leq j \leq k | v_j = v_i\}$. These k edges $e_1, e_2, \ldots, e_k$ are S-disjoint if the following two conditions are satisfied:*

i) edges $e_1, e_2, \ldots, e_k$ have distinct end vertices,

ii) for $i = 1, \ldots, k$, $S_i = \{i\}$ or there exist r distinct integers $i_1, i_2, \ldots, i_r$ such that $e_{j_1} \in \mathcal{I}_{i_1}, e_{j_2} \in \mathcal{I}_{i_2}, \ldots, e_{j_r} \in \mathcal{I}_{i_r}$, where $j_1, j_2, \ldots, j_r$ are all the elements of S_i.

Roughly speaking, k edges are *S-disjoint* if their end vertices are all distinct and if all the edges which have the same begin vertex can be associated to distinct indexes i. For this new graphic property, the following lemma can be given:

Lemma 10. *Considering switched linear system (2.2), there exist n S-disjoint edges in associated colored union graph $\tilde{\mathcal{G}}$ if and only if $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$ has g-rank $= n$.*

Proof. *Necessity:* If there exist n *S-disjoint* edges in $\tilde{\mathcal{G}}$, matrix $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$ contains at least n free parameters. Since the n *S-disjoint* edges have distinct end vertices, the corresponding n free parameters lie on n different rows. Besides, the n S-*disjoint* edges have distinct begin vertices or have same begin vertex that can be associated to distinct indexes i. This implies that these n free parameters lie on n different columns. keep these n free parameters and set all the other free parameters to be zero. We can see that matrix $[A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m]$ has following form: $\begin{bmatrix} 0 & \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ \lambda_n & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$, which has g-rank $= n$.

Sufficiency: From the Definition 12.3 and the following discussions of [76], for a structured matrix Q, g-rank Q = s-rank Q. where s-rank of Q is defined as the maximal number of free parameters that no two of which lie on the same row or column. If matrix $[A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_m]$ has g-rank $= n$, it follows that there exists n free parameters from n different rows, which implies that the corresponding n edges have different end vertices, from n different columns, which implies that these n edges start from different vertices or start from same vertices but can be associated to different indexes. Hence the condition that the matrix has g-rank $= n$ is sufficient to ensure the existence of n S-*disjoint* edges. □

With the above definition and lemma, another necessary and sufficient condition for structural controllability of system (2.2) can be proposed here:

Theorem 11. *Switched linear system (2.2) with graphic representations $\mathcal{G}_i$, $i \in \{1, \dots, m\}$, is structurally controllable if and only if its colored union graph $\tilde{\mathcal{G}}$ satisfies the following two conditions:*

i) there is no nonaccessible vertex in the colored union graph $\tilde{\mathcal{G}}$, *ii) there exist n S-disjoint edges in the colored union graph $\tilde{\mathcal{G}}$.*

Proof. Lemma 6 and Lemma 10 show that there exist n *S-disjoint* edges in the colored union graph $\tilde{\mathcal{G}}$ if and only if there is no *S-dilation* in $\tilde{\mathcal{G}}$. Then this theorem follows immediately. □

2.3.3 Computation Complexity of The Proposed Criteria

Compared with condition using '*S-dilation*', this condition using '*S-disjoint* edges' does not require to check all the vertex subsets, which is a more efficient criterion. The maximal number of '*S-disjoint* edges' can be calculated using bipartite graphs. For example, we can use the algorithm in [106], which allows to compute the cardinality of maximum matching into a bipartite graph. A bipartite graph is a graph whose vertices can be divided into two disjoint sets $\mathcal{U}$ and $\mathcal{W}$ such that every edge connects a vertex in $\mathcal{U}$ to one in $\mathcal{W}$. To build a bipartite graph in directed subgraph $\mathcal{G}_i(\mathcal{V}_i, \mathcal{I}_i)$, what we need to do is adding some vertices and making $\mathcal{U}_i = \{v \in \mathcal{V}_i | \exists (v, v') \in \mathcal{I}_i\}$, which implies that cardinality $|\mathcal{U}_i|$ equals to the number of nonzero columns in matrix $[A_i, B_i]$. Besides, $\mathcal{W}_i = \mathcal{X}_i$, i.e., the state vertex set. Then it follows that the maximum matching in this bipartite graph is the same as the maximal *S-disjoint* edge set in $\mathcal{G}_i(\mathcal{V}_i, \mathcal{I}_i)$. According to definition of *S-disjoint* edges, the beginning vertex from different subgraphs should be differentiated when building the bipartite graph for colored union graph $\tilde{\mathcal{G}}$. Therefore for the bipartite graph of $\tilde{\mathcal{G}}$, $\mathcal{U} = \{v | \exists (v, v') \in \mathcal{I}_i, i = 1, 2, \ldots, m\}$, which implies that cardinality $|\mathcal{U}|$ equals to the number of nonzero columns in matrix $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$. And $\mathcal{W} = \mathcal{X}$, i.e., the state vertex set. Similarly, the maximum matching in this bipartite graph is the same as the maximal *S-disjoint* edge set in colored union graph. Therefore the complexity order of algorithm using method in [106] is $O(\sqrt{p+n} \cdot q)$, where q is the number of edges

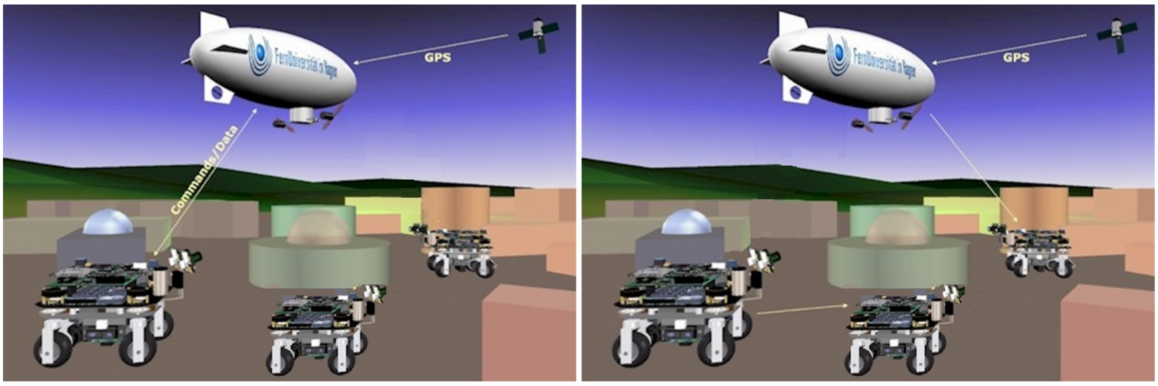

Fig. 2.3: Multi-agent system with switching topologies

in colored union graph, i.e., the number of free parameters in all system matrices, p is the number of nonzero columns in matrix $[A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m]$ and n is number of state variables. Compared with condition (ii) of Theorem 11, condition (i) of Theorem 11 is easier to check. We have to look for paths which connect each state vertex with one of the input vertex. This is a standard task of algorithmic graph theory. For example, depth-first search or breadth-first search algorithm for traversing a graph can be adopted and the complexity order is $O(|V| + |E|)$, where $|V|$ and $|E|$ are cardinalities of vertex set and edge set in union graph.

2.3.4 Numerical Examples

In the first example, we will consider a real group of control objects which can be modeled as switched linear system: A multi-agent systems with switching topologies as illustrated in Fig.2.3 [107].

This multi-agent system adopts a leader-flower structure, consisting of one airship, which works as the leader, and three ground robots, which work as followers. There are

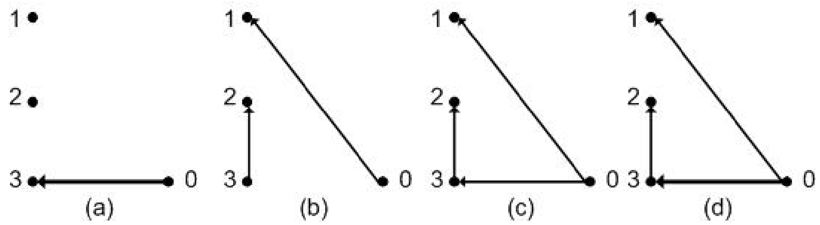

Fig. 2.4: Switched linear system with two subsystems

communications among the agents: the leader airship gives commands to some of follower ground robots and ground robots transmit position information to each other. All the commands and communication channels are under people's control, which means we can turn on/off communications as our wish to save energy. The structural controllability of this multi-agent system implies that all robots can reach any desired position configuration through proper choices of communication information weights among robots under command from the leader airship. Here we assume there are totally two kinds of communication topologies as shown in Fig.2.3. For the multi-agent system under specific communication topology, it can be modeled as a linear system. Consequently, for the whole multi-agent system with two switching topologies, it can be modeled as a switched linear system with two subsystems. This switched linear system is depicted by the graphic topologies in Fig. 2.4(a)-(b). In colored union graph $\tilde{\mathcal{G}}$ (Fig. 2.4(d)), thick lines represent edges from subgraph (a) and thin lines represent the edges from subgraph (b). Vertices 1,2 and 3 represent system states (ground robots positions) and vertex 0 represents system control input (command from airship). It turns out that the colored union graph $\tilde{\mathcal{G}}$ has no nonaccessible vertex and no S*-dilation*. Besides, the three edges are S*-disjoint* edges since they have different end vertices and one edge begins at vertex 3 and two edges begin at vertex 0 but they come from different subsystems. According to Theorem 9 or 11, the switched linear system is structurally controllable. On the other hand, the system matrices of each subsystem of

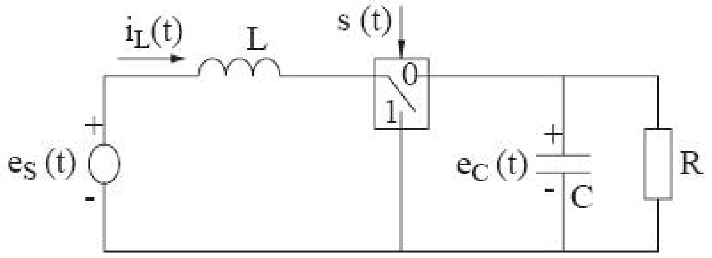

Fig. 2.5: The boost converter

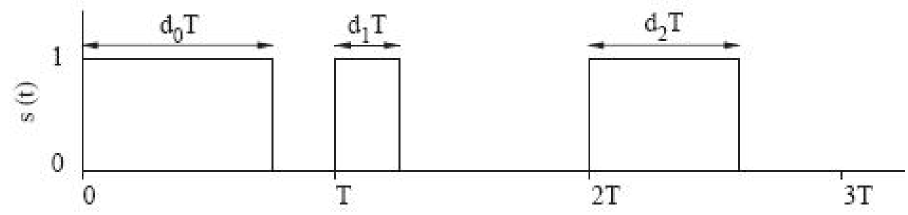

Fig. 2.6: Pulse-width modulation

corresponding subgraph are:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ \lambda_1 \end{bmatrix}; A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_2 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \lambda_3 \\ 0 \\ 0 \end{bmatrix}.$$

controllability matrix (2.3) can be calculated and can be shown to have *g-rank*=3. In addition, there exist a *dilation* in union graph Fig. 2.4(c), which shows that the condition in Theorem 4 is not necessary for structural controllability.

In the following example, we will consider another real control object with switched linear system model: A PWM-Driven Boost Converter [108] as illustrated in Fig.2.5.

In this electrical network, *L* is the inductance, *C* the capacitance, *R* the load resistance,

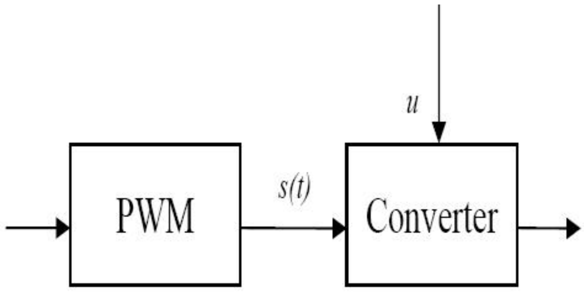

Fig. 2.7: PWM driven boost converter

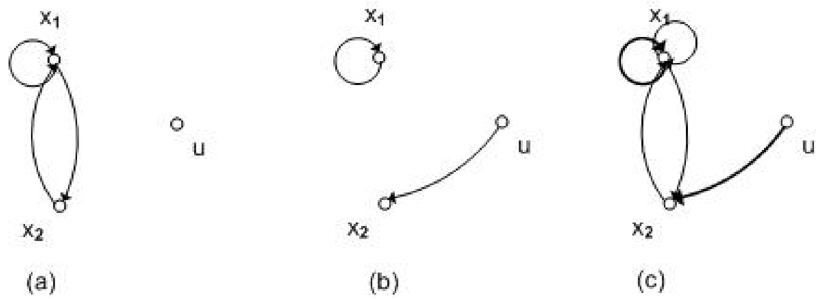

Fig. 2.8: Switched linear system with two subsystems

and $e_S(t)$ the source voltage. With this converter, the source voltage $e_S(t)$ can be transformed into a higher voltage $e_C(t)$ over the load R. The switch $s(t)$, which is supposed to have two states, namely, 0 and 1, is controlled by a PWM device. Then, we have $s(t) \in \{0, 1\}$ as shown in Fig.2.6. The schematic of the PWM-driven Boost converter is shown in Fig.2.7. By introducing the normalized variables $\tau = t/T$, $L_1 = L/T$, and $C_1 = C/T$, the dynamics for the Boost converter are described as follows:

$$\begin{aligned} \dot{e}_C(\tau) &= \tfrac{-1}{RC_1} e_C(\tau) + (1 - s(\tau))\tfrac{1}{C_1} i_L(\tau), \\ \dot{i}_L(\tau) &= -(1 - s(\tau))\tfrac{1}{L_1} e_C(\tau) + s(\tau))\tfrac{1}{L_1} e_S(\tau), \end{aligned} \tag{2.11}$$

Let $x_1 = e_C$, $x_2 = i_L$, $u = e_S$ $\sigma = s + 1$, then the system dynamics can be described as:

$$\dot{x} = A_\sigma x + B_\sigma u, \sigma \in \{1, 2\} \tag{2.12}$$

where:

$$A_1 = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{C_1} \\ -\frac{1}{L_1} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; A_2 = \begin{bmatrix} -\frac{1}{RC_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{1}{L_1} \end{bmatrix}.$$

Modeling this system using independent parameter and zero elements, we have that

$$A_1 = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; A_2 = \begin{bmatrix} \lambda_4 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \lambda_5 \end{bmatrix}.$$

The two subsystems are depicted by the graphic topologies in Fig. 2.8(a)-(b). In colored union graph $\tilde{\mathcal{G}}$ (Fig. 2.8(c)), thin lines represent edges from subgraph (a) and thick lines represent the edges from subgraph (b). It turns out that the colored union graph $\tilde{\mathcal{G}}$ has no nonaccessible vertex and no *S-dilation*. Besides, the edge starting from x_2 and ending at x_1 with index (a) together with the edge starting from u and ending at x_2 with index (b) consist of two *S-disjoint* edges since they have different starting and ending vertices. According to the results obtained above, this switched electrical network is structurally controllable and similarly the rank condition can be checked that it has full *g-rank* 2.

Form the above example, we can see that in some real applications there are some dependent parameters among subsystems (since under our independent case, the structural controllability holds for almost all values of the free parameters, the dependent case can be treated as a further extension but will not belittle the significance of results obtained above). For further investigation purpose, next we will use examples to illustrate that the dependence among system parameters will make some edges 'useless' or 'excessive' in judging the structural controllability. See the following switched linear system first

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}; A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \lambda_3 \\ \lambda_4 \end{bmatrix}.$$

According to Theorem 9 or 11, this system is structurally controllable. However, if dependent parameters are considered, see the following switched linear system (a linear system actually)

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}; A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}.$$

The dependence of all the parameters in matrix B_1 and B_2 makes this system not structurally controllable and the results in Theorem 9 or 11 not hold, even though it would be structurally controllable if the parameters in B_2 are replaced with λ_3 and λ_4 or simply remove λ_1 or λ_2 in the second subsystem.

2.4 Conclusions

In this chapter, structural controllability for switched linear systems has been investigated. Combining the knowledge in the literature of switched linear systems and graph theory, several graphic necessary and sufficient conditions for the structural controllability of switched linear systems have been proposed. These graphic interpretations provide us a better understanding on how the graphic topologies of switched linear systems will influence or determine the structural controllability of switched linear systems. This shows us a new perspective that we can design the switching algorithm to make the switched linear system structurally controllable conveniently just having to make sure some properties of the corresponding graph (union or colored union graph) are kept during the switching process. In this chapter, the parameters in different subsystem models are assumed to be independent. A more general assumption is that some free parameters remain the same among different subsystems switching, i.e., dependence among subsystems. It turns out that our necessary and sufficient condition derived here would be a necessary condition under this dependence assumption. Besides, our result can be treated as basic starting point for exploring the structural controllability of switched nonlinear systems: adopt Lie algebra or transfer function methods to get full characterizations for controllability of switched nonlinear system, then interpret each condition into graphic one and finally combine these conditions together to get graphic interpretations for structural controllability for switched nonlinear system. To obtain a full characterization for the dependent case or switched nonlinear case needs further investigation.

Next chapter will continue this research direction/method and investigate the structural controllability of multi-agent system with switching topologies. In particular we will study the influence of interconnection topology to group behavior of multi-agent systems.

Chapter 3

Structural Controllability of Multi-Agent System with Switching Topology

3.1 Introduction

In Chapter 2, graphic interpretations of structural controllability of general switched linear systems are addressed. In this chapter, we will follow the same methodology to investigate how the local interactions among agents influence the controllability of whole multi-agent systems. By controllability, we mean that people can drive the whole group of agents to any desirable configurations only based on local interactions between agents and possibly some limited commands to a few agents that serve as leaders. As we said in the very beginning, switching topologies are of more practical meaning than fixed topologies and the interaction link weight among agents has a profound impact on the collective behavior of multi-agent systems. Consequently, a weighted switching interconnection topology is chosen in this chapter to encode the local interactions and information flows in multi-agent systems. And the controllability of multi-agent systems will be studied in a structural sense, which holds for almost all values of interaction link weights. The main objective of the chapter is to get graph theoretic characterizations of the structural controllability for multi-agent systems. Graphic necessary and sufficient conditions of structural controllability under single/multi-leader under fixed/switching interconnection topology are proposed. It turns out that structural controllability of multi-agent system with switching topology is purely based on the interconnection topologies among agents. In what follows, we first give some basic preliminaries and the problem formulation in Section 3.2, and propose our graph based interpretation for the structural controllability under single leader case, in Section 3.3. In Section 3.4, graphic interpretation of structural controllability of multi-leader multi-agent system is addressed. Numerical examples are also presented as illustrations of obtained results in Section 3.5. Finally, some concluding remarks are drawn in the chapter.

3.2 Preliminaries and Problem Formulation

3.2.1 Graph Theory Preliminaries

A weighted graph is an appropriate representation for the communication or sensing links among agents because it can represent both existence and strength of these links among agents. The weighted graph $\mathcal{G}$ with N vertices consists of a vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{I} = \{e_1, e_2, \ldots, e_{N'}\}$, which is the interconnection links among the vertices. Each edge in the weighted graph represents a bidirectional communication or sensing media. Two vertices are known to be *neighbors* if $(i, j) \in \mathcal{I}$, and the number of neighbors for each vertex is its *valency*. An alternating sequence of distinct vertices and edges in the weighted graph is called a *path*. The weighted graph is said to be *connected* if there exists at least one path between any distinct vertices, and *complete* if all vertices are neighbors to each other.

The *adjacency matrix*, $\mathcal{A}$ is defined as

$$\mathcal{A}_{(i,j)} = \begin{cases} \mathcal{W}_{ij} & (i, j) \in \mathcal{I}, \\ 0 & otherwise, \end{cases} \tag{3.1}$$

where $\mathcal{W}_{ij} \neq 0$ stands for the weight of edge (j, i). Here, the adjacency matrix $\mathcal{A}$ is $\mid \mathcal{V} \mid \times \mid \mathcal{V} \mid$ and $\mid . \mid$ is the cardinality of a set.

The *Laplacian matrix* of a graph $\mathcal{G}$, denoted as $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ or $\mathcal{L}$ for simplicity, is defined as

$$\mathcal{L}_{(i,j)} = \begin{cases} \Sigma_{j\in N_i} \mathcal{W}_{ij} & i = j, \\ -\mathcal{W}_{ij} & i \neq j \; and \; (i, j) \in \mathcal{I}, \\ 0 & otherwise. \end{cases} \tag{3.2}$$

3.2.2 Multi-Agent Structural Controllability with Switching Topology

Specifically, controllability problem usually cares about how to control N agents based on the leader-follower framework. Take the case of single leader as example. Without loss of generality, assume that the N-th agent serves as the leader and takes commands and controls from outside operators directly, while the rest $N - 1$ agents are followers and take controls as the nearest neighbor law.

Mathematically, each agent's dynamics can be seen as a point mass and follow

$$\dot{x}_i = u_i. \tag{3.3}$$

The control strategy for driving all follower agents is

$$u_i = -\sum_{j\in\mathcal{N}_i} w_{ij}(x_i - x_j) + w_{ii}x_i, \tag{3.4}$$

where $\mathcal{N}_i$ is the neighbor set of the agent i (could contain i itself), and w_{ij} is weight of the edge from agent j to agent i. On the other hand, the leader's control signal is not influenced by the followers and needs to be designed, which can be represented by

$$\dot{x}_N = u_N. \tag{3.5}$$

In other words, the leader affects its nearby agents, but it does not get directly affected by the followers since it only accepts the control input from an outside operator. For simplicity, we will use z to stand for x_N in the sequel. It is known that the whole multi-agent system under fixed communication topology can be written as a linear system:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u_N \end{bmatrix}, \tag{3.6}$$

where $A \in \mathbb{R}^{(N-1)\times(N-1)}$ and $B \in \mathbb{R}^{(N-1)\times 1}$ are both sub-matrices of the corresponding graph Laplacian matrix $-\mathcal{L}$.

The communication network of dynamic agents with directed information flow under link failure and creation can be usually described by switching topology. Under m switching topologies, it is clear that the whole system equipped with m subsystems can be written in a compact form

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u_N \end{bmatrix}, \tag{3.7}$$

or, equivalently,

$$\begin{cases} \dot{x} &= A_i x + B_i z, \\ \dot{z} &= u_N, \end{cases} \tag{3.8}$$

where $i \in \{1, \ldots, m\}$. $A_i \in \mathbb{R}^{(N-1)\times(N-1)}$ and $B_i \in \mathbb{R}^{(N-1)\times 1}$ are both sub-matrices of the corresponding graph Laplacian matrix $-\mathcal{L}$. The matrix A_i reflects the interconnection among followers, and the column vector B_i represents the relation between followers and the leader under corresponding subsystems. Since the communication topologies among agents are time-varying, so the matrices A_i and B_i are also varying as a function of time. Therefore, the dynamical system described in (3.7) can be naturally modeled as a switched system (definition can be found latter).

Considering the structural controllability of multi-agent system, system matrices A_i and B_i, $i \in \{1, \ldots, m\}$ are structured matrices, which means that their elements are either fixed zeros or free parameters. Fixed zeros imply that there is no communication link between the corresponding agents and the free parameters stand for the weights of the communication links. Our main task here is to find out under what kinds of communication topologies, it is possible to make the group motions under control and steer the agents to the specific geometric positions or formation as a whole group. Now this controllability problem reduces to whether we can find a set of weights w_{ij} such that the multi-agent system (3.7) is controllable. Then the controllability problem of multi-agent system can now be formulated as the structural controllability problem of switched linear system (3.7):

Definition 17. *The multi-agent system (3.7) with switching topology, whose matrix elements are zeros or free parameters, is said to be structurally controllable if and only if there exist a set of communication weights w_{ij} that can make the system (3.7) controllable in the classical sense.*

3.3 Structural Controllability of Multi-Agent System with Single Leader

The multi-agent system with a single leader under switching topology has been modeled as switched linear system (3.7). Before proceeding to the structural controllability study, we first discuss the controllability of multi-agent system (3.7) when all the communication weights are fixed.

After simple calculation, the controllability matrix of switched linear system (3.7) can be shown to have the following form:

$$\begin{bmatrix} 0, & \ldots & ,0, & B_1, & \ldots, & B_m, & A_1B_1 & \ldots, & A_1A_m^{N-3}B_m, & \ldots, & A_m^{N-2}B_m \\ 1, & \ldots & ,1, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \end{bmatrix}$$

This implies that the controllability of the system (3.7) coincides with the controllability of the following system:

$$\dot{x} \;=\; A_i x + B_i z \qquad i \in \{1, \ldots, m\} \;. \tag{3.9}$$

Which is the extracted dynamics of the followers that correspond to the x component of the equation. Therefore,

Definition 18. *The multi-agent system (3.7) is said to be structurally controllable under leader z if system (3.9) is structurally controllable under control input z.*

For simplicity, we use (A_i, B_i) $i \in \{1, \ldots, m\}$ to represent switched linear system (3.9) in the sequel.

In (3.9), each subsystem (A_i, B_i) can be described by a directed graph [68]:

Definition 19. *The representation graph of structured system (A_i, B_i) is a directed graph*

$\mathcal{G}_i$, with vertex set $\mathcal{V}_i = \mathcal{X}_i \cup \mathcal{U}_i$, where $\mathcal{X}_i = \{x_1, x_2, \ldots, x_n\}$, which is called state vertex set and $\mathcal{U}_i = \{u_1, u_2, \ldots, u_r\}$, which is called input vertex set, and edge set $\mathcal{I}_i = \mathcal{I}_{U_iX_i} \cup \mathcal{I}_{X_iX_i}$, where $\mathcal{I}_{U_iX_i} = \{(u_p, x_q)|B_{qp} \neq 0, 1 \leq p \leq r, 1 \leq q \leq n\}$ and $\mathcal{I}_{X_iX_i} = \{(x_p, x_q)|A_{qp} \neq 0, 1 \leq p \leq n, 1 \leq q \leq n\}$ are the oriented edges between inputs and states and between states defined by the interconnection matrices A_i and B_i above. This directed graph (for notational simplicity, we will use digraph to refer to directed graph) $\mathcal{G}_i$ is also called the graph of matrix pair (A_i, B_i) and denoted by $\mathcal{G}_i(A_i, B_i)$.

For each subsystem, we have got a graph G_i with vertex set $\mathcal{V}_i$ and edge set $\mathcal{I}_i$ to represent the underlaying communication topologies. As to the whole switched system, the representing graph, which is called *union graph*, is defined as follows:

Definition 20. *The switched linear system (3.9) can be represented by a union digraph, defined as a flow structure $\mathcal{G}$. Mathematically, $\mathcal{G}$ is defined as*

$$\begin{aligned} \mathcal{G}_1 \cup \mathcal{G}_2 \cup \ldots \cup \mathcal{G}_m = \{&\mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_m; \\ &\mathcal{I}_1 \cup \mathcal{I}_2 \cup \ldots \cup \mathcal{I}_m\} \end{aligned} \tag{3.10}$$

Remark 5. *It turns out that union graph $\mathcal{G}$ is the representation of linear structured system: $(A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m)$.*

Remark 6. *In lots of literature about controllability of multi-agent systems [31–38, 40], the underlying communication topology among the agents is represented by undirected graph, which means that the communication among the agents is bidirectional. Here we still adopt this kind of communication topology. Then w_{ij} and w_{ji} are free parameters or zero simultaneously (in numerical realization, the values of w_{ij} and w_{ji} can be chosen to be different). Besides, one edge in undirected graph can be treated as two oriented edges.*

Consequently, even though all the analysis and proofs for structural controllability of multi-agent systems are based on the directed graph (the natural graphic representation of matrix pair (A_i, B_i) is digraph), the final result will be expressed in undirected graph form.

Before proceeding further, we need to introduce two definitions which were proposed in [68] for linear structured system $\dot{x} = Ax + Bu$ first:

Definition 21. *([68]) The matrix pair (A, B) is said to be reducible or of form I if there exist permutation matrix P such that they can be written in the following form:*

$$PAP^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, PB = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \tag{3.11}$$

where $A_{11} \in \mathbb{R}^{p\times p}$, $A_{21} \in \mathbb{R}^{(n-p)\times p}$, $A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $B_{22} \in \mathbb{R}^{(n-p)\times r}$.

Remark 7. *Whenever the matrix pair (A, B) is of form I, the system is structurally uncontrollable [68] and meanwhile, the controllability matrix $Q = \left[B, AB, \dots, A^{n-1}B\right]$ will have at least one row which is identically zero for all parameter values [70]. If there is no such permutation matrix P, we say that the matrix pair (A, B) is irreducible.*

Definition 22. *([68]) The matrix pair (A, B) is said to be of form II if there exist permutation matrix P such that they can be written in the following form:*

$$\left[PAP^{-1}, PB\right] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \tag{3.12}$$

where $P_2 \in \mathbb{R}^{(n-k)\times(n+r)}$, $P_1 \in \mathbb{R}^{k\times(n+r)}$ with no more than $k-1$ nonzero columns (all the other columns of P_1 have only fixed zero entries).

Here we need to recall a known result in literature for structural controllability of multi-agent system with fixed topology [43]:

Lemma 12. *([43]) The multi-agent system with fixed topology under the communication topology $\mathcal{G}$ is structurally controllable if and only if graph $\mathcal{G}$ is connected.*

This lemma proposed an interesting graphic condition for structural controllability in fixed topology situation and revealed that the controllability is totally determined by the communication topology. However, how about in the switching topology situation? According to Lemma 1, once we impose proper scalars for the parameters of the system matrix (A_i, B_i) to satisfy the full rank condition, the multi-agent system (3.9) is structurally controllable. However, this only proposed an algebraic condition. Do we still have very good graphic interpretation for the relationship between the structural controllability and switching interconnection topologies? The following theorem answers this question and gives a graphic necessary and sufficient condition for structural controllability under switching topologies.

Theorem 13. *The multi-agent system (3.9) with the communication topologies $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$ is structurally controllable if and only if the union graph $\mathcal{G}$ is connected.*

Proof. *Necessity:* Assume that the multi-agent switched system is structurally controllable, we want to prove that the union graph $\mathcal{G}$ is connected, which is equivalent to that the system has no isolated agents in the union graph $\mathcal{G}$ [43].

Suppose that the union graph $\mathcal{G}$ is disconnected and for simplicity, we will prove by contradiction for the case that there exits only one disconnected agent. The proof can be straightforwardly extended to more general cases with more than one disconnected agents. If there is one isolated agent in the union graph, there are two possible situations: the isolated agent is the leader or one of the followers. On one hand, if the isolated agent is the leader, it follows that $B_1 + B_2 + \ldots + B_m$ is identically a null vector. So every B_i is a null vector. Easily we can conclude that the controllability matrix for the switched system is never of full row rank $N - 1$, which means that the multi-agent system is not structurally controllable. On the other hand, if the isolated agent is one follower, we get that the matrix pair $(A_1 + A_2 + \ldots + A_m, B_1 + B_2 + \ldots + B_m)$ is reducible. By definition 21, the controllability matrix

$$
\begin{aligned}
&[B_1 + B_2 + \ldots + B_m,\\
&(A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m),\\
&,\cdots,\\
&(A_1 + A_2 + \ldots + A_m)^{N-2}(B_1 + B_2 + \ldots + B_m)]
\end{aligned}
$$

always has at least one row that is identically zero. Expanding the matrix yields

$$
\begin{aligned}
&[B_1 + B_2 + \ldots + B_m,\\
&A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2\\
&+ \ldots + A_mB_2 + \ldots + A_1B_m + A_2B_m \ldots + A_mB_m\\
&,\ldots,\\
&A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \ldots + A_m^{N-2}B_m].
\end{aligned}
$$

The zero row is identically zero for every parameter. This implies that every component in this matrix, such as B_i, A_iB_j *and* $A_i^pA_j^qB_r$, has the same row always to be zero. As a result, the controllability matrix

$$[B_1, \ldots, B_m, A_1B_1, \ldots, A_mB_1, \ldots, A_mB_m, A_1^2B_1, \ldots, A_mA_1B_1, \\ \ldots, A_1^2B_m, \ldots, A_mA_1B_m, \ldots, A_1^{n-1}B_1, \ldots, A_mA_1^{n-2}B_1, \ldots, \\ A_1A_m^{n-2}B_m, \ldots, A_m^{n-1}B_m]$$

always has one zero row. Therefore, the multi-agent system (3.9) is not structurally controllable. Until now, we have got the necessity proved.

Sufficiency: If the union graph $\mathcal{G}$ is connected, we want to prove that the multi-agent system (3.9) is structurally controllable.

According to Lemma 12, the connectedness of the union graph $\mathcal{G}$ implies that the corresponding system $(A_1 + A_2 + A_3 + \ldots + A_m, B_1 + B_2 + B_3 + \ldots + B_m)$ is structurally controllable. Then there exist some scalars for the parameters in system matrices that make the controllability matrix

$$[B_1 + B_2 + \ldots + B_m, \\ (A_1 + A_2 + \ldots + A_m)(B_1 + B_2 + \ldots + B_m), \\ , \ldots, \\ (A_1 + A_2 + \ldots + A_m)^{N-2}(B_1 + B_2 + \ldots + B_m)],$$

has full row rank $N-1$. Expanding the matrix, it follows that the matrix

$$
\begin{aligned}
&[B_1 + B_2 + \ldots + B_m,\\
&A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2\\
&+ \ldots + A_mB_2 + \ldots + A_1B_m + A_2B_m \ldots + A_mB_m\\
&, \ldots,\\
&A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \ldots + A_m^{N-2}B_m],
\end{aligned}
$$

has full rank $N-1$. Next, we add some column vectors to the above matrix and get

$$
\begin{aligned}
&[B_1 + B_2 + \ldots + B_m, B_2, \ldots, B_m,\\
&A_1B_1 + A_2B_1 + \ldots + A_mB_1 + A_1B_2 + A_2B_2 + \ldots + A_mB_2\\
&+ \ldots + A_1B_m + A_2B_m + \ldots + A_mB_m, A_2B_1, A_3B_1, \ldots, A_mB_m\\
&, \ldots,\\
&A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \ldots + A_m^{N-2}B_m, A_2A_1^{N-3}B_1, \ldots, A_m^{N-2}B_m].
\end{aligned}
$$

This matrix still have $N-1$ linear independent column vectors, so it has full row rank. Next, subtract $B_2, \ldots, B_m$ from $B_1 + B_2 + \ldots + B_m$; subtract $A_2B_1, \ldots, A_mB_m$ from $A_1B_1 + A_2B_1 + \ldots + A_mB_m$ and subtract $A_2A_1^{N-3}B_1, \ldots, A_m^{N-2}B_m$ from $A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \ldots + A_m^{N-2}B_m$. Since this column fundamental transformation will not change matrix rank, the matrix still has full row rank. Now the matrix becomes

$$
\begin{aligned}
&[B_1, \ldots, B_m, A_1B_1, \ldots, A_mB_1, \ldots, A_mB_m, A_1^2B_1, \ldots, A_mA_1B_1,\\
&\ldots, A_1^2B_m, \ldots, A_mA_1B_m, \ldots, A_1^{n-1}B_1, \ldots, A_mA_1^{n-2}B_1, \ldots,\\
&A_1A_m^{n-2}B_m, \ldots, A_m^{n-1}B_m]
\end{aligned}
$$

which is the controllability matrix of system (3.9) and has full row rank $N-1$. Therefore, the multi-agent system is structurally controllable. □

3.4 Structural Controllability of Multi-Agent System with Multi-Leader

In the above discussion, we assume the multi-agent system has totally N agents and the N-th agent serves as the leader and takes commands and controls from outside operators directly, while the rest $N-1$ agents are followers and take controls as the nearest neighbor law. In the following part, we will discuss the situation that several agents are chosen as the leaders of the whole multi-agent systems, which is actually an extension of single leader case.

Similar to the single leader case, the multi-agent system with multiple leaders is given by:

$$\begin{cases} \dot{x}_i = u_i, & i = 1, \dots, N \\ \dot{x}_{N+j} = u_{N+j}, & j = 1, \dots, n_l \end{cases} \tag{3.13}$$

where N and n_l represent the number of followers and leaders, respectively. x_i indicates the state of the ith agent, $i = 1, \dots, N + n_l$.

The control strategy u_i, $i = 1, \dots, N$ for driving all follower agents is the same as the single leader case. The leaders' control signal is still not influenced by the followers and we are allowed to pick u_{N+j}, $j = 1, \dots, n_l$ arbitrarily. For simplicity, we use vector x to stand for the followers' states and z to stand for the leaders' states.

Then the whole multi-agent system equipped with m communication topologies can be written in a compact form

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} , \tag{3.14}$$

or, equivalently,

$$\begin{cases} \dot{x} = A_i x + B_i z, \\ \dot{z} = u, \end{cases} \tag{3.15}$$

where $i \in \{1, \ldots, m\}$. $A_i \in \mathbb{R}^{N\times N}$ and $B_i \in \mathbb{R}^{N\times n_l}$ are both sub-matrices of the corresponding graph Laplacian matrix $-\mathcal{L}$.

The dynamics of the followers can be extracted as

$$\dot{x} = A_i x + B_i z, \quad i \in \{1, \ldots, m\}. \tag{3.16}$$

Remark 8. *Similar with the single leader case, the structural controllability of system (3.14) coincides with the structural controllability of system (3.16). And we say that the multi-agent system (3.13) with switching topology and multi-leader is structurally controllable if and only if the switched linear system (3.16) is structurally controllable with z as the control inputs.*

Before proceeding further, we first discuss the structural controllability of multi-agent systems with multi-leader under fixed topology with the following dynamics:

$$\dot{x} = Ax + Bz, \tag{3.17}$$

where $A \in \mathbb{R}^{N\times N}$ and $B \in \mathbb{R}^{N\times n_l}$ are both sub-matrices of the graph Laplacian matrix $-\mathcal{L}$.

In [36] and [38], a new graph topology: leader-follower connected topology was proposed:

Definition 23. *([36]) A follower subgraph $\mathcal{G}_f$ of the interconnection graph $\mathcal{G}$ is the subgraph induced by the follower set $\mathcal{V}_f$ (here is x). Similarly, a leader subgraph $\mathcal{G}_l$ is the subgraph induced by the leader set $\mathcal{V}_l$ (here is z).*

Denote by $\mathcal{G}_{c_1}, \ldots, \mathcal{G}_{c_\gamma}$, the connected components in the follower $\mathcal{G}_f$. The definition of leader-follower connected topology is as follows:

Definition 24. *([38]) The interconnection graph $\mathcal{G}$ of multi-agent system (3.17) is said to be leader-follower connected if for each connected component $\mathcal{G}_{c_i}$ of $\mathcal{G}_f$, there exists a leader in the leader subgraph $\mathcal{G}_l$, so that there is an edge between this leader and a node in $\mathcal{G}_{c_i}, i = 1, \ldots, \gamma$.*

Based on this new graph topology, we can derive the criterion for structural controllability for multi-agent system (3.17) under fixed topology:

Theorem 14. *The multi-agent system (3.17) with multi-leader and fixed topology under the communication topology $\mathcal{G}$ is structurally controllable if and only if graph $\mathcal{G}$ is leader-follower connected.*

Proof. *Necessity:* The idea of necessity proof is similar to the proof of Lemma 3 in [36]. We assume that there exists one connected component $\mathcal{G}_{c_p}$ not connected to the leader subgraph $\mathcal{G}_l$. Define A_i and B_i matrices as sub-matrices of A and B, the same as the F_i and R_i matrices in Lemma 3 of [36]. Following the analysis in Lemma 3 of [36], it can be easily got that the controllability matrix of multi-agent system (3.17) is:

$$C = \begin{bmatrix} B_1 & A_1 B_1 & A_1^2 B_1 & \cdots & A_1^{N-1} B_1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ B_\gamma & A_\gamma B_\gamma & A_\gamma^2 B_\gamma & \cdots & A_\gamma^{N-1} B_\gamma \end{bmatrix}. \tag{3.18}$$

Consequently, rank C = row rank $C < N$. The maximum rank of C is less than N, which implies that the corresponding multi-agent system (3.17) is not structurally controllable.

*Sufficiency:*We adopt the proof of Theorem 1 in [38] to help us prove the sufficiency. The communication graph $\mathcal{G}$ consists of several connected components $\mathcal{G}^{(i)}$, $i = 1, \ldots, \kappa$, which can be partitioned into two subgraphs: induced leader subgraph $\mathcal{G}_l^{(i)}$ and induced follower subgraph $\mathcal{G}_f^{(i)}$. For each connected components $\mathcal{G}^{(i)}$, $i = 1, \ldots, \kappa$, it can be modeled as a linear system with its system matrices being sub-matrices of A and B matrices. Following the analysis in Theorem 1 in [38], the following equation can be deduced:

$$rank\, C = rank\, C_1 + rank\, C_2 + \ldots + rank\, C_\kappa, \tag{3.19}$$

where C is the controllability matrix of multi-agent system (3.17) and C_i is the controllability matrix of connected component $\mathcal{G}^{(i)}$. The independence of these connected components guarantees the independence of free parameters in the corresponding matrices, which correspond to the communication weights of the links. Consequently, we have that

$$g\text{-}rank\, C = g\text{-}rank\, C_1 + g\text{-}rank\, C_2 + \ldots + g\text{-}rank\, C_\kappa$$

where g-rank of a structured matrix M is defined to be the maximal rank that M achieves as a function of its free parameters. Besides, if in some connected component $\mathcal{G}^{(i)}$, there is more than one leaders, we can split it into several connected components with single leader or choose one as leader and set all weights of the communication links between the followers and other leaders to be zero. After doing this, connected component $\mathcal{G}^{(i)}$ is a connected topology with single leader. According to Lemma 12, C_i has full *g-rank*, which equals to the number of follower agents in $\mathcal{G}^{(i)}$. Moreover, there is no common follower agent among the connected components. Consequently, *g-rank* C=N and multi-agent system (3.17) is structurally controllable. □

With the above definitions and theorems, we are in the position to present the graphic interpretation of structural controllability of multi-agent systems under switching topology with multi-leader:

Theorem 15. *The multi-agent system (3.14) or (3.16) with the communication topologies $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$ and multi-leader is structurally controllable if and only if the union graph $\mathcal{G}$ is leader-follower connected.*

Proof. As stated in Remark 5, the union graph $\mathcal{G}$ is the representation of the linear system: $(A_1 + A_2 + A_3 + \ldots + A_m, B_1 + B_2 + B_3 + \ldots + B_m)$. Therefore, the condition that the union graph $\mathcal{G}$ is leader-follower connected is equivalent to the condition that linear system $(A_1 + A_2 + A_3 + \ldots + A_m, B_1 + B_2 + B_3 + \ldots + B_m)$ is structurally controllable. Following the proof procedure in Theorem 13, this result can get proved. □

3.5 Numerical Examples

In the first example, we consider a real multi-agent system, as illustrated in Fig.3.1 [109], with fixed topology to show how to model multi-agent system with state space model and

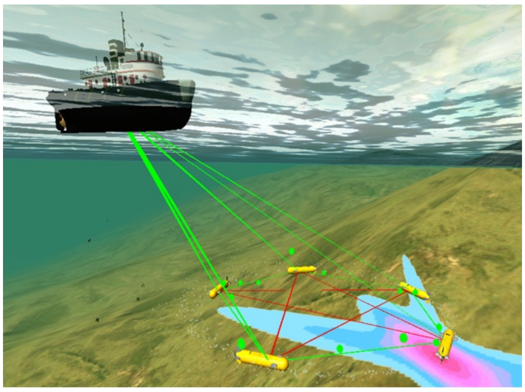

Fig. 3.1: Multi-agent system with full communications

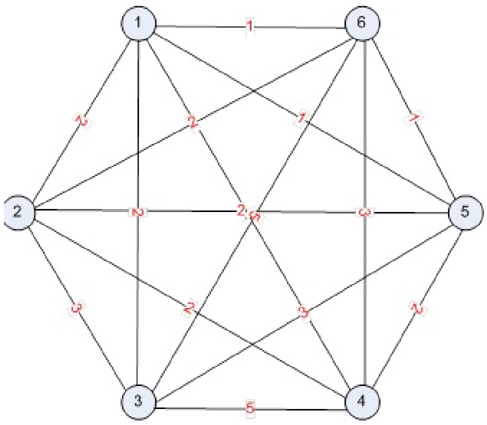

Fig. 3.2: Topology graph with weighted edges

also instance the role of communications link weights.

This multi-agent system adopts a leader-flower structure, consisting of one ship as the leader, and five submarines as followers. There are full communications among all the agents. The topology graph of this multi-agent system is shown in Fig.3.2: vertex 1 represents leader ship and vertices 2, 3, 4, 5, 6 represent the five follower submarines. The edges with numbers on them are used to represent weighted communication links among agents.

According to [31], this multi-agent system is not controllable if no weights are assigned to the communication links. After properly assigning weights as shown in Fig.3.2, the system matrices can be calculated as follows:

$$A = \begin{bmatrix} 8 & -2 & -2 & -2 & -1 \\ -2 & 11 & -3 & -2 & -2 \\ -2 & -3 & 18 & -5 & -3 \\ -2 & -2 & -5 & 14 & 2 \\ -1 & -2 & -3 & -2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ -2 \\ -5 \\ -3 \\ -1 \end{bmatrix}.$$

Checking controllability matrix, this multi-agent system can be proven to be controllable, which shows that proper choices of communication link weights can lead a good performance of whole system.

Next we will give two examples to illustrate the results under switching topologies and for simplicity, we take single leader case as examples.

We consider here a four-agent network with agent 0 as the leader and with switching topology described by the graphs in Fig.3.3(a)-(b) (the self-loops are not depicted, because it will not influence the connectivity). Overlaying the subgraphs together can get the union graph $\mathcal{G}$ of this example as shown in Fig.3.3(c). It turns out that the union graph of the

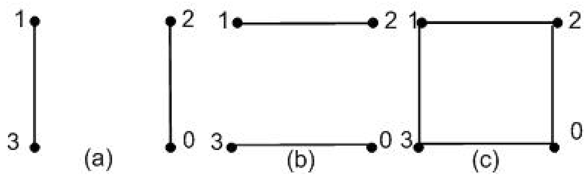

Fig. 3.3: Switched network with two subsystems

switched system is connected. By Theorem 13, it is clear that the multi-agent system is structurally controllable.

Next, the rank condition of this multi-agent system will be checked.

From Fig.3.3, calculating the Laplacian matrix for each subgraph topology, it can be obtained that the system matrices of each subsystem are (one thing we should mention here with the control strategy that each agent can use its own state information, the diagonal elements always have free parameters, so we can get the following form of sub-matrix of Laplacian matrix) :

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & \lambda_4 \\ 0 & \lambda_2 & 0 \\ \lambda_5 & 0 & \lambda_3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \lambda_6 \\ 0 \end{bmatrix}; A_2 = \begin{bmatrix} \lambda_7 & \lambda_{10} & 0 \\ \lambda_{11} & \lambda_8 & 0 \\ 0 & 0 & \lambda_9 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \lambda_{12} \end{bmatrix}.$$

According to Lemma 1, the controllability matrix for this switched linear system is: $[B_1, B_2, A_1B_1, A_2B_1, A_1B_2, A_2B_2, A_1^2B_1, A_2A_1B_1, A_1A_2B_1, A_2^2B_1, A_1^2B_2, A_2A_1B_2, A_1A_2B_2, A_2^2B_2]$. After simple calculation, we can find three column vectors in the controllability matrix:

$$\begin{bmatrix} 0 \\ \lambda_6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \lambda_{12} \end{bmatrix}, \begin{bmatrix} \lambda_4\lambda_{12} \\ 0 \\ \lambda_3\lambda_{12} \end{bmatrix}.$$

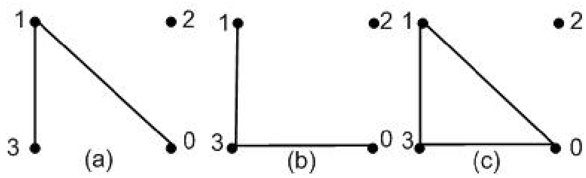

Fig. 3.4: Another switched network with two subsystems

Imposing all the parameters scalar 1, it follows that these three column vectors are linearly independent and this controllability matrix has full row rank. Therefore, the multi-agent system is structurally controllable.

In the second example, we still consider a four-agent network with agent 0 as the leader and with switching topology described by the graphs in Fig.3.4(a)-(b). Overlaying the subgraphs together can get the union graph $\mathcal{G}$ of this example shown in Fig.3.4(c). It turns out that the union graph of the switched system is disconnected, because agent 2 is isolated. According to Theorem 13, it is clear that the multi-agent system is not structurally controllable.

Similarly, the rank condition of this switched linear system needs to be checked to see whether it is structurally controllable or not.

From Fig.3.4, calculating the Laplacian matrix for each graphic topology, it is clear that the system matrices of each subsystem are :

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & \lambda_4 \\ 0 & \lambda_2 & 0 \\ \lambda_5 & 0 & \lambda_3 \end{bmatrix}, \ B_1 = \begin{bmatrix} \lambda_6 \\ 0 \\ 0 \end{bmatrix}; A_2 = \begin{bmatrix} \lambda_7 & 0 & \lambda_{10} \\ 0 & \lambda_8 & 0 \\ \lambda_{11} & 0 & \lambda_9 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 0 \\ \lambda_{12} \end{bmatrix}.$$

Computing the controllability matrix of this example yields the controllability matrix:

$$\begin{bmatrix} \lambda_6 & 0 & \lambda_1\lambda_6 & \dots & \lambda_7\lambda_{10}\lambda_{12} + \lambda_9\lambda_{10}\lambda_{12} \\ 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_{12} & \lambda_5\lambda_6 & \dots & \lambda_{10}\lambda_{11}\lambda_{12} + \lambda_9^2\lambda_{12} \end{bmatrix}.$$

This matrix has the second row always to be zero for all the parameter values, which makes the maximum rank of this matrix less than 3. Therefore, this multi-agent system is not structurally controllable.

3.6 Conclusions

In this chapter, the structural controllability problem of the multi-agent systems interconnected via a switching weighted topology has been considered. Based on known results in the literature of switched systems and graph theory, graphic necessary and sufficient conditions for the structural controllability of multi-agent systems under switching communication topologies were derived. It was shown that the multi-agent system is structurally controllable if and only if the union graph $\mathcal{G}$ is connected (single leader) or leader-follower connected (multi-leader). The graphic characterizations show a clear relationship between the controllability and interconnection topologies and give us a foundation to design the optimal control effect for the switched multi-agent system.

The switched linear system studied in Chapter 2 and the multi-agent system with switching topologies in this chapter can both be classified under time-dependent switching topology situation. Next question on the controllability topic would be how the state-dependent topology influences multi-agent system behavior. Consequently, in Chapter 4, the controllability of piecewise linear system will be addressed as a potential predecessor research attempt for the controllability of multi-agent system with state-dependent switching topology.

Chapter 4

Null controllability of Piecewise Linear System

4.1 Introduction

Time-dependent switching topology and state-dependent switching topology can be viewed as two main types of switching topologies. In Chapter 2 and Chapter 3, the impact of time-dependent switching topology on system performance has been addressed. In this chapter, we aim to investigate the null controllability of piecewise linear system, which can be a possible modeling of multi-agent system with state-dependent switching topology. In the first part of this chapter, the null controllability problem is addressed and we devise an explicit and easily verifiable necessary and sufficient condition for a planar bimodal piecewise linear system to be null controllable. In the second part, a short survey of research efforts on state-dependent multi-agent systems is presented and the relation between the result obtained for piecewise linear system and possible research directions of controllability of state-dependent multi-agent system is also introduced. A brief review on controllability of piecewise linear systems and the motivation of the tackled problem in this chapter are given as follows.

Piecewise linear systems refer to a subclass of hybrid systems that the whole state space is partitioned into polyhedral regions and a linear dynamics is active on each of these regions. A large class of nonlinear systems [110, 111] and lots of practical systems can be modeled as piecewise linear systems [112, 113]. For example, in [110], it was proven that piecewise linear systems can be used to analyze smooth nonlinear dynamics with arbitrary accuracy. In [113], the tunnel diode circuit was treated using framework of piecewise linear systems. Besides, piecewise linear systems can serve as an alternative system for the study of a particular hybrid system as indicated in [114], where equivalences among five classes of hybrid systems including piecewise linear systems were established. Due to their theoretical and practical importance, piecewise linear systems have drawn considerable attention these years [115–121].

[122] pointed out that observability and controllability properties of piecewise linear systems cannot be easily deduced from those of the component linear subsystems. Even if every subsystem is controllable, the whole piecewise linear system can not always be controllable. For example, consider the following bimodal piecewise linear system:

$$\dot{x}_1 = \begin{cases} x_2 & if \quad x_2 \geq 0 \\ \\ -x_2 & if \quad x_2 \leq 0 \end{cases}$$

$$\dot{x}_2 = u.$$

Each subsystem is controllable in the classical sense. However, the overall system is uncontrollable as the derivative of x_1 is always nonnegative. Conversely, even if some subsystem is uncontrollable, the whole piecewise linear system can still be controllable. For example, consider the following bimodal piecewise linear system:

$$\dot{x}_1 = \begin{cases} u_1 & if \quad x_2 \geq 0 \\ \\ 0 & if \quad x_2 \leq 0 \end{cases}$$

$$\dot{x}_2 = u_2.$$

The subsystem in $x_2 \geq 0$ is controllable and the subsystem in $x_2 \leq 0$ is uncontrollable because the derivative of x_1 is always 0. After simple observation, we can see that the whole system is controllable. Actually, due to the hybrid nature of piecewise linear systems, the controllability issues are far from being trivial as was pointed out in [123], where it was shown that even for simple classes of piecewise linear systems, the controllability problem turns out to be undecidable. Although it is difficult to obtain explicit conditions for controllability of general piecewise linear systems, it is still possible to get some explicit necessary and/or sufficient conditions for some special subclasses of piecewise linear systems. In [124], based on the geometric control theory, the authors investigated the controllability property of bimodal systems and a small-time local controllability condition was given. In [125–127], bimodal systems with continuous dynamics on the switching surface were considered. For example, in [125], the authors proposed a necessary and sufficient condition for the controllability of planar bimodal linear complementarity systems, which can be treated as a special class of piecewise linear systems. The controllability problem of conewise linear systems with dynamics continuous on the switching surface was studied in [128]. Some results about controllability of planar conewise linear systems were proposed in [129]. References [130] and [131] discussed the null controllability of discrete-time bimodal piecewise linear systems, in which some results that need to be checked case by case, were proposed.

In this chapter, the attention is paid to the continuous-time planar bimodal piecewise linear systems. In particular, the null controllability problem is investigated. First, the evolution directions from any non-origin state are studied from the geometric point of view, and it turns out that the directions usually span an open half space. After that, the whole state space is segmented into several spacial regions using the switching surface together with several new proposed dividing lines. Furthermore, using the classification discussion method according to the geometric position relation of system matrices, switching surface and the new proposed dividing lines, an explicit and easily verifiable geometric necessary and sufficient condition for the null controllability of planar bimodal piecewise linear systems is proposed.

The rest of this chapter is organized as follows. In Section 4.2, we introduce the class of systems to be studied, followed by null controllability study in Section 4.3, where one geometric necessary and sufficient condition, together with some necessary or sufficient conditions, for the null controllability is given. In Section 4.4, some examples are presented to illustrate the theoretical results. Possible application of the obtained results on the controllability of state-dependent multi-agent is addressed in Section 4.5. Finally, some concluding remarks are drawn in the chapter and some proofs are put into the appendix.

4.2 Problem Formulation

Consider the planar bimodal piecewise linear system with the following mathematical model:

$$\begin{cases} \dot{x}(t) &= A_1 x(t) + bu(t) \qquad c^T x \geq 0, \\ \\ \dot{x}(t) &= A_2 x(t) + bu(t) \qquad c^T x \leq 0, \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the control, $A_1,\ A_2$ and b ($b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$) are constant matrices with appropriate dimensions. c is a vector in $\mathbb{R}^2$. The whole state space is divided into two parts: $S_1 = \{x \in \mathbb{R}^2; c^T x \geq 0\}$ and $S_2 = \{x \in \mathbb{R}^2; c^T x \leq 0\}$, with one system active in each spacial part. Besides, on the switching surface $c^T x = 0$, each of the two subsystems is possible to be active.

In the sequel, we will adopt the following definition of a trajectory of system (4.1):

Definition 25. *An absolutely continuous function* $x(\cdot) : [0, T] \to R^2$ *is called (admissible) trajectory of system (4.1) if there exist a finite number of points* $0 = t_0 < t_1 < \cdots < t_p = T$ *and integers* $i_1, i_2, \ldots, i_p \in \{1, 2\}$ *such that for every* $k \in \{1, \ldots, p\}$*,*

i) $x(t) \in S_{i_k}$ *for all* $t \in [t_{k-1}, t_k]$*;*

ii) there exists a piecewise continuous function $u(\cdot)$ *such that* $\dot{x}(t) = A_{i_k} x(t) + bu(t)$ *for almost everywhere* $t \in [t_{k-1}, t_k]$.

What follows is the definition of null controllability of system (4.1):

Definition 26. *(Null controllability) A nonzero state x of system (4.1) is called controllable, if there exists a trajectory $x(\cdot)$ of (4.1) such that $x(0) = x$ and $x(t_f) = 0$ for some $t_f > 0$. System (4.1) is said to be null controllable if any nonzero state x is controllable.*

Our aim here is to find out under which condition, it is possible to drive any nonzero state in (4.1) to the origin with suitable choice of control input, namely that the continuous-time planar bimodal piecewise linear system (4.1) is null controllable.

4.3 Null Controllability

4.3.1 Evolution Directions

A question arises naturally when people study the trajectory of some system dynamics: which directions can the state evolve at specific point x_0, i.e., what are the directions of tangent vectors or derivatives of state x_0?

Before answering this question, we need to introduce some notations first: in system (4.1), the line parallel to vectors b and $-b$ and crossing zero is defined as $d^T x = 0$, where $d = \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}$. The line parallel to vectors b and $-b$ and crossing some point p is $d^T x = d^T p$. For notational convenience, this line will be represented by $d^T x(p)$ in the rest of this chapter. Let's use $\mathcal{X}_0$ to represent the set of evolution directions or derivative vectors at point x_0. It turns out that all the possible evolution directions of x_0 at a non-origin state x_0 usually span an open half space.

To answer the above question, what we need is to consider the coordinate centered at the point x_0 in a linear system. Then we can easily get the following lemma:

Lemma 16. *i) $\mathcal{X}_0 = \{x|d^T x = 0\}$ if Ax_0 is linearly dependent with b;*

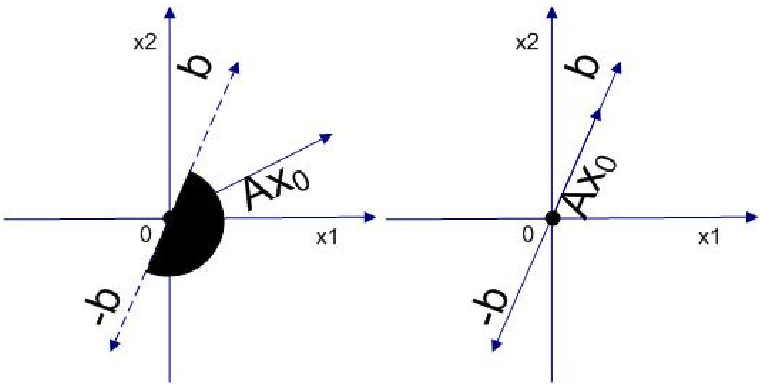

Fig. 4.1: Graphic illustration of Lemma 16

ii) $\mathcal{X}_0 = \{x|d^T x > 0\}$ *or* $\{x|d^T x < 0\}$ *if* Ax_0 *is linearly independent of* b*.*

Proof. For i, since Ax_0 is linearly dependent with b, $Ax_0 + bu, u \in R$ can be any vector that belongs to the line parallel to b and $-b$ and crossing 0, i.e., $d^T x(0)$. For ii, because Ax_0 is linearly independent of b, every vector f can be expressed as $f = \lambda_1 Ax_0 + \lambda_2 b$. If $d^T f > 0$, definitely, λ_1 is always positive (negative). Meanwhile, if another vector $f' = \lambda'_1 Ax_0 + \lambda'_2 b$ satisfies $d^T f' < 0$, definitely, λ'_1 is always negative (positive). Now consider arbitrary vectors f and f' satisfying $d^T f >$ and $d^T f' < 0$, respectively. Suppose $\lambda_1 > 0$ and $\lambda'_1 < 0$. Then we have $f/\lambda_1 = Ax_0 + \lambda_2/\lambda_1 b$ and $-f'/|\lambda'_1| = Ax_0 + \lambda'_2/\lambda'_1 b$. Consequently, $Ax_0+bu, u \in R$ can be and only be any vector that satisfies $d^T(Ax_0+bu) > 0$. Suppose $\lambda_1 < 0$ and $\lambda'_1 > 0$. $Ax_0 + bu, u \in R$ can be and only be any vector that satisfies $d^T(Ax_0 + bu) < 0$. Since Ax_0 is linearly independent of b, $Ax_0 \neq 0$. Therefore, $Ax_0 + bu, u \in R$ can not be any vector along the direction of b or $-b$. A graphic illustration is shown in Fig. 4.1. $\square$

Define another vector $E_i = [E_{i1}, E_{i2}] = d^T A_i = \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_i$, $i = \{1, 2\}$. The vector e_i

which belongs to $E_i x = 0$ is $e_i = \begin{bmatrix} -E_{i2} \\ E_{i1} \end{bmatrix}$ or $\begin{bmatrix} E_{i2} \\ -E_{i1} \end{bmatrix}$. e_1 and e_2 need to be chosen in the way that $c^T e_1 \geq 0, c^T e_2 \leq 0$, because subsystem 1 is only active when $c^T x \geq 0$ and subsystem 2 is only active when $c^T x \leq 0$. Now for each of the two subsystems, in the whole state space, we have:

Lemma 17. *i)* $\mathcal{X}_0 = \{x | d^T x = d^T x_0\}$ *if* $E_i x_0 = 0$*;*

ii) $\mathcal{X}_0 = \{x | d^T x > d^T x_0\}$*(or* $\{x | d^T x < d^T x_0\}$*) if* $E_i x_0 > 0$*. Meanwhile,* $\mathcal{X}_0 = \{x | d^T x < d^T x_0\}$*(or* $\{x | d^T x > d^T x_0\}$*) if* $E_i x_0 < 0$*.*

Proof. This lemma is a direct corollary of Lemma 16. □

4.3.2 Null Controllability

The following lemma presents a necessary condition for system (4.1) to be null controllable:

Lemma 18. *If both subsystems* (A_1, b) *and* (A_2, b) *are uncontrollable in the classical sense, the piecewise linear system (4.1) is not null controllable.*

Suppose that both subsystems are uncontrollable. For any subsystem (A_i, b), $i = 1$ or 2, the controllability matrix is $[b, A_i b]$. Since (A_i, b) is not controllable, the controllability matrix now has rank 1 and is of the form $[b]$. For a linear system, the range space of controllability matrix, i.e., λb here, is actually the reachability and controllability spaces, i.e., the largest set of states that can be driven to zero. This implies that any state that not belongs to $\lambda_i b$ is not controllable and can not be driven to zero under this linear dynamics. For the whole piecewise linear system (4.1), suppose $c^T b \neq 0$, i.e., neither of the controllability

Proof.

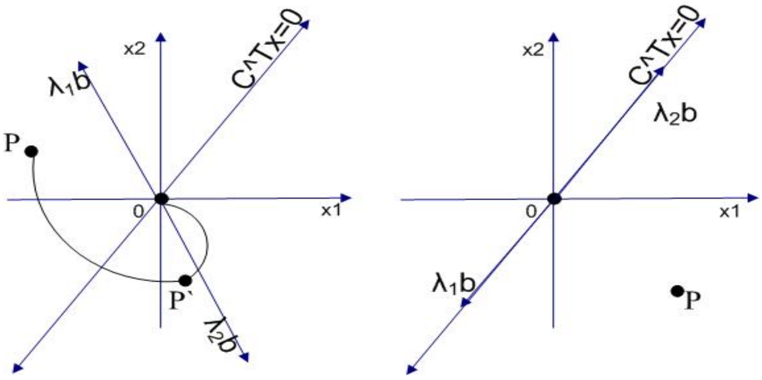

Fig. 4.2: Graphic illustration of Lemma 18

spaces of two subsystems coincides with line $c^T x = 0$ as depicted in left half of Fig.4.2. Consider an arbitrary state point p in $c^T x \geq 0$ but not in $\lambda_1 b$. Since (A_1, b) is not controllable, p can not reach zero in $c^T x \geq 0$. If there exists a trajectory starting from p, crossing the line $c^T x = 0$ and reaching zero as depicted in the figure, there must exist another state p' of this trajectory included in $c^T x \leq 0$ but not in $\lambda_2 b$. Besides, the trajectory starting from p' and reaching zero stays entirely in $c^T x \leq 0$. This conflicts with the assumption that system (A_2, b) is not controllable and its controllable space is limited in $\lambda_2 b$. Therefore, p can not be driven to zero and the piecewise linear system (4.1) is not null controllable under this case. If $c^T b = 0$ as depicted in right half of Fig.4.2, consider an arbitrary state p in $c^T x < 0$. The reachable set of subsystem (A_2, b) is now the line $c^T x = 0$ and there is no control input that can drive state p to zero or any point on $c^T x = 0$. Consequently, the piecewise linear system (4.1) is not null controllable. □

Remark 9. *the necessary condition in this lemma can be applied to a more general model as follows:*

$$\begin{cases} \dot{x}(t) &= A_1 x(t) + B_1 u(t) \qquad c^T x \geq 0, \\ \\ \dot{x}(t) &= A_2 x(t) + B_2 u(t) \qquad c^T x \leq 0. \end{cases} \tag{4.2}$$

Compared with system (4.1), this planar bimodal piecewise linear system has different B matrices in the two subsystems and the control input u may not be scalar now. Besides, it is also a necessary condition for system (4.2) to be completely controllable.

Lemma 19. *The linear system $\dot{x} = A_i x + bu$, i=1 or 2, is controllable if and only if $E_i b \neq 0$.*

Proof. If $b = 0$, the result follows directly. We assume that $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \neq 0$. If the system (A_i, b) is not controllable, the controllability matrix $[b, A_i b]$ has rank 1, which means that column vector $A_i b$ is linearly dependent with b, i.e., $A_i b = \lambda b$. Furthermore, $E_i = \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_i$ as defined. Therefore, $E_i b = \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_i b = \lambda \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0$. On the other hand, if $E_i b = 0$, we have $\begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T [b, A_i b] = [\begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T b, \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_i b] = [0, E_i b] = 0$. Since $b \neq 0$, $b_1 \neq 0$ or $b_2 \neq 0$. Therefore, the controllability matrix has rank less than 2 and the linear system (A_i, b) is uncontrollable. This completes the proof. □

Before proceeding further, we need to introduce the following definition for system (4.1):

Definition 27. *Define the convex cone formed by e_1 and e_2 as $\mathcal{V}$: specifically, $\mathcal{V}$ is defined as an open convex cone if $e_1 \neq \lambda e_2, \lambda > 0$ and we say a vector $v \in \mathcal{V}$ if there exist positive scalars λ_1 and λ_2 such that $v = \lambda_1 e_1 + \lambda_2 e_2$; When $e_1 = \lambda e_2, \lambda > 0$, we say a vector $v \in \mathcal{V}$ if there exists positive scalar λ_i such that $v = \lambda_i * e_i$. Moreover, the condition that state x is outside $\mathcal{V}$ means that vector $x \notin \mathcal{V}$ and vector $x \neq \lambda_1 e_1$ and $x \neq \lambda_2 e_2, \lambda_1 > 0, \lambda_2 > 0$.*

With the previous lemmas and definitions, we are in the position to present the main result of this chapter:

Theorem 20. *The bimodal piecewise linear system (4.1) is null controllable if and only if:*

i) there exist $i \in \{1, 2\}$, a scalar u, and a vector x outside $\mathcal{V}$ such that $A_i x + bu \in \mathcal{V}$;

ii) the corresponding subsystem (A_i, b) is controllable.

Remark 10. *If $c^T e_i \neq 0$, definitely we have an unique e_i. Otherwise, when $c^T e_i = 0$, both e_i and $-e_i$ satisfy the requirement that $c^T e_i \geqq 0$ ($c^T e_i \leqq 0$). Consequently, there are several convex cones formed by e_1 and e_2 (including $-e_i$). To satisfy conditions in Theorem 20, we should make sure for every cone, the two conditions should be satisfied.*

If the matrix c has the form $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, the following sufficient condition for system (4.1) to be null controllable can be given:

Corollary 1. *If the system matrices satisfy the following conditions:*

i) $\begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_1 \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} \neq 0$ *and* $\begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_2 \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} \neq 0$*, and*

ii) $b = \lambda_3 e_1 + \lambda_4 e_2$, for some λ_3, λ_4 that $\lambda_3 \lambda_4 > 0$,

then the bimodal piecewise linear system (4.1) is null controllable.

Proof. This is actually a special case of the main theorem. For detailed proof, please refer to the case $A.c$ in the appendix. □

Besides, we can get the following sufficient condition for system (4.1) to be not null controllable:

Corollary 2. *If the system matrices satisfy the following conditions:*

$$\textit{i)}\quad \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_1 \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} = 0 \textit{ or } \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_2 \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} = 0, \textit{ and}$$

$$\textit{ii)}\quad \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_1 b = 0 \textit{ or } \begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_2 b = 0,$$

then the bimodal piecewise linear system (4.1) is not null controllable.

Proof. This is actually a combination of several cases of the main theorem. For detailed proof, please refer to the cases *B.a*, *B.b* and *C.b* in the appendix. □

4.4 Numerical Examples

Example 2. *Consider the system dynamics described in the following equations:*

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \qquad -x_1 + x_2 \geq 0, \\ \\ \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \qquad -x_1 + x_2 \leq 0. \end{cases} \tag{4.3}$$

The system matrices are as follows:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, c = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

After simple calculation, it can be seen that: $d = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *or* $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *and furthermore, we can get the two dividing lines:*

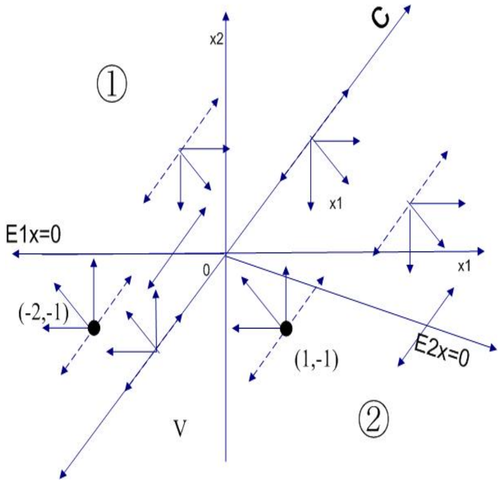

Fig. 4.3: Refinement of state space of system (4.3)

$$E_1 = d_1^T A_1 = [1 \quad -1]\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = [0 \quad 1],\ E_1 x = 0 \Leftrightarrow x_2 = 0;$$

$$E_2 = d_2^T A_2 = [-1 \quad 1]\begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} = [-1 \quad -2],\ E_2 x = 0 \Leftrightarrow x_1 + 2x_2 = 0.$$

The refinement of the whole state space according to the dividing lines is shown in Fig.4.3. We can easily see that for the cone $\mathcal{V}$, there exists some vector $A_1x + bu$, i.e., derivative vector of state x, $\in \mathcal{V}$, when x is in area ① outside $\mathcal{V}$ and also the subsystem (A_1, b) is controllable. According to Theorem 20, the system (4.3) is null controllable. Also, we can see that the conditions in corollary 1 are also satisfied, with $e_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\lambda_3 = -3$ and $\lambda_4 = -1$. Next, some simulation results are presented to illustrate the null controllability. Here, we choose one typical state, design suitable control input and drive it to zero.

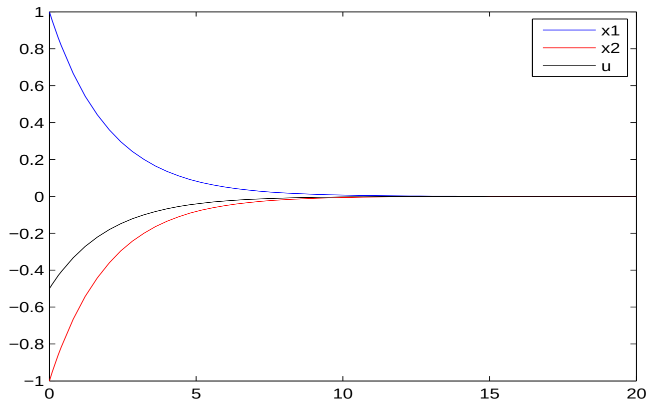

Fig. 4.4: Trajectory and control input of driving (1,-1) to 0 in system (4.3)

Starting from state $(1, -1)$*, designing suitable* u *can ensure the system trajectory be driven along a line trajectory to zero. The simulation result is shown in Fig.4.4.*

Example 3. *Consider another system dynamics described in the following equations:*

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) & -x_1 + x_2 \geq 0, \\ \\ \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), & -x_1 + x_2 \leq 0. \end{cases} \tag{4.4}$$

The system matrices are as follows:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, c = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Simple calculation yields that: $d = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *or* $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *and furthermore, we can get the two dividing lines:*

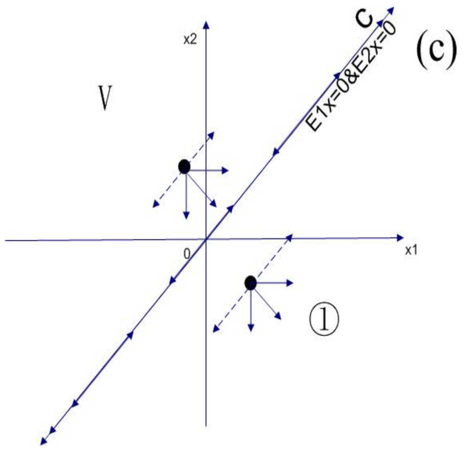

Fig. 4.5: Refinement of state space of system (4.4)

$$E_1 = d_1^T A_1 = [1 \quad -1]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = [-1 \quad 1],\ E_1 x = 0 \Leftrightarrow -x_1 + x_2 = 0;$$

$$E_2 = d_2^T A_2 = [-1 \quad 1]\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = [-1 \quad 1],\ E_2 x = 0 \Leftrightarrow -x_1 + x_2 = 0.$$

The refinement of the whole state space according to the dividing lines is depicted in Fig.4.5. It can be easily seen that for the cone $\mathcal{V}$, there is no vector $A_i x + bu$, i.e., derivative vector of state x, $\in \mathcal{V}$, where x is outside $\mathcal{V}$ (here is area ①). According to Theorem 20 or corollary 2, the system (4.3) is not null controllable. Actually, we can see that with such set of possible evolution directions as depicted in the figure for all states in area ①, arbitrary state in this area can not be driven to zero.

4.5 State Dependent Multi-Agent Systems

In this section, we will briefly discuss the possible application of above obtained result to the controllability study of multi-agent with state-dependent switching topologies.

In practice, the communication links between agents are unavoidably influenced by many factors that are out of control, such as distance, noise disturbance and signal strength. For these applications, it may become impossible to keep the communication topology fixed or be arbitrarily controlled by people for the whole period. Specifically, we consider a scenario where the existence of an interaction link between a pair of agents is determined fully by the agents' states. For example, a set of sensors collecting and processing information about a time-varying spatial field (e.g., to monitor temperature levels or chemical concentrations). Another real implementation example can a collection of mobile robots performing dynamic tasks which are divided regionally. Consequently, the interconnection topology of multi-agent system under this scenario inherently changes according to the spatial change of the states of agents. Namely, these kinds of multi-agent system are called multi-agent systems with state-dependent switching topologies.

Some research efforts using the concept of state-dependent interconnection topology can be observed, such as in [132], a global optimization problem was distributed to individual agents and the interactions among agents are assumed as a state-dependent communication model. In [133] an opinion consensus problem was addressed with the connectivity of communication topology is state-dependent. However, there is almost no research investigating the impact of state-dependent topology on the behavior of multi-agent systems. In [134], a distributed system described by discrete-event finite state space was introduced and the interaction topology was modeled into a state-dependent graph. Subsequently, a controllability concept for such state dependent graphs was devised together with some graphic interpretation for this controllability.

Consider the case that existence of an link between two agents only depends on the distance of two agents. Then the multi-agent system with state-dependent switching topology can be described as follows:

$$\begin{cases} \dot{x} &= A(x)x + B(x)z, \\ \dot{z} &= u_N, \end{cases} \tag{4.5}$$

where $A(x)$, $B(x)$ are sub-matrices of the corresponding graph Laplacian and dependent on system state x. Using i (j) and x_i (x_j) to represent agent i and its state (position here), the interconnection topology can be deduced in the following way:

$$\text{edge } (i, j) \text{ exist if and only if } \| x_i - x_j \| \leq d$$

The matrices $A(x)$, $B(x)$ can be derived accordingly.

In order to make this kind of modeling be really implemented, the system states need to be constrained to finite state set case or the system space needs to be divided to finitely many regions with each region corresponding to one interconnection topology.

For the fist situation, there are several possible problems needed to be solved before deducing the controllability of multi-agent system from the state-dependent topology graphs:

1. Solve graphic equations $g(x) = G$, where x is system state and G is one interconnection topology and deduce the state set corresponding to any specific interconnection topology.
2. Study the influence of states evolution and topologies evolution on each other.
3. Investigate the impact of state-dependent topologies on controllability of multi-agent systems.

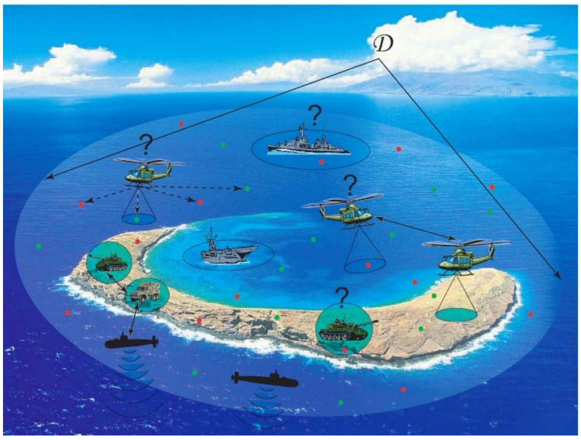

Fig. 4.6: Interconnections in a multi-agent system

For the second situation that system space can be divided to finitely many regions with each region corresponding to one interconnection topology, we can simplify the system model into a special class of piecewise linear system as we studied above. This situation is also of great practical meaning. As shown in Fig.4.6, this multi-agent system is conducting a surveillance task over an area, in which agents like helicopters, ships, tanks and submarines work in different roles and cooperate to finish the whole task. The information exchanges among agents, i.e., the interconnection links are unavoidably influenced by factors that are out of control, such as distance and signal strength, which can be assumed to be fully determined by the agents' states. This group of agents are required to monitor different some spacial regions. With moving to different regions, due to the circumstances change, role of each agent is changed and the cooperation relationships between agents are also changed, which requires totally different information exchange channels or interconnection topology. If two special regions with a spacial dividing line are considered under this scenario, multi-agent system with this kind of state-dependent switching topology can be described as follows:

$$\begin{cases} \dot{x}(t) &= A_1x(t) + B_1u(t) \qquad c^T x \geq 0, \\ \\ \dot{x}(t) &= A_2x(t) + B_2u(t) \qquad c^T x \leq 0, \end{cases} \tag{4.6}$$

where A_1, A_2, B_1, B_2 are sub-matrices of the corresponding graph Laplacian. This is actually one kind of piecewise linear system and other piece linear system model can be adopted according to the spacial region division.

However, as we said above, quite limited research progress has been achieved on the controllability of piecewise linear system. Consequently, a big gap lies here to get the graphic interpretation for controllability of multi-agent system with state-dependent switching topologies. Following this direction, several steps need to be finished in order to fully reveal the impact of state-dependent topologies on the controllability of multi-agents:

1. Solve the controllability of piecewise linear system.
2. Take the special structure of multi-agent system matrices into consideration and solve its controllability using algebraic method.
3. Interpret each algebraic condition into graphic expressions.
4. Combine all the graphic interpretations and devise condition on interconnection topologies to guarantee the controllability of multi-agent systems

4.6 Conclusions

In this chapter, we have investigated the null controllability of planar bimodal piecewise linear systems. An explicit and easily verifiable necessary and sufficient condition has been proposed in terms of the system parameters, followed by several necessary or sufficient conditions. The method of analyzing the evolution directions of system states brings us a deep insight of the relation between system trajectory and its controllability. Furthermore, we believe that using this kind of geometric analysis method, certain controllability of more general piecewise linear systems can also be considered, such as, high-order, multi-modal and complete controllability. In the last part of this chapter, the modeling of controllability of multi-agent system with state-dependent switching topology is addressed and the research effort on controllability of piecewise linear system is shown to light the way for solving the state-dependent multi-agent controllability problem.

In next chapter, the multi-agent behavior under another kind of external signal, external disturbance, will be discussed. Namely, the disturbance rejection problem will be investigated in structural way.

4.7 APPENDIX

4.7.1 Proof of Theorem 20

Proof. The basic idea here is to enumerate all the possible cases that there exists a vector $A_ix + bu \in \mathcal{V}$ when x is outside $\mathcal{V}$ and (A_i, b) is controllable, and then prove that every nonzero state can be driven to zero. Conversely, all the possible cases that at least one of the two conditions stated in the theorem can not be satisfied will be presented and proven

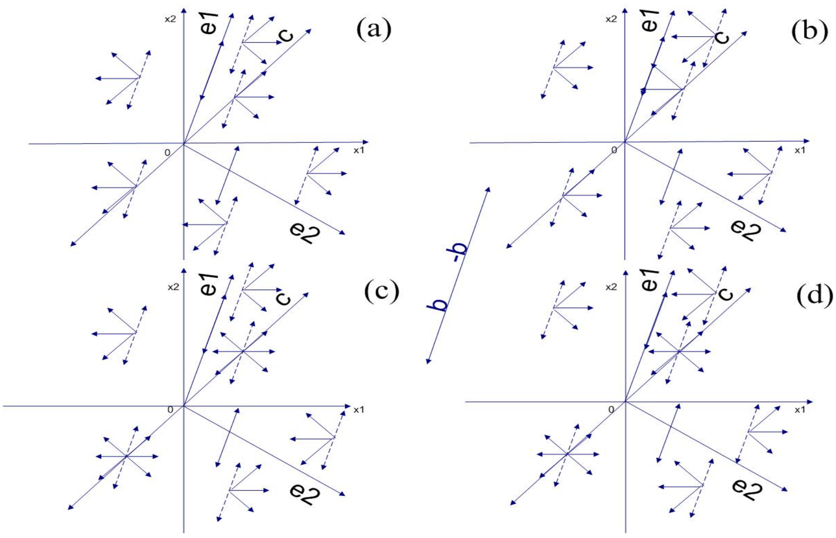

Fig. 4.7: Case A.a: $b = \lambda_1 e_1$ *or* $b = \lambda_2 e_2$

that there exists some nonzero state that can not be driven to zero. For simplicity, in all the following figures, let's use c, e_1 and e_2 to represent the lines $c^T x = 0$, $E_1 x = 0$ and $E_2 x = 0$, respectively. The derivative vectors or evolution directions of every state are depicted using the solid line with arrow. Besides, the dashed line with arrow represents the extreme derivative direction which can not be achieved, which actually is the direction of vectors b and $-b$.

case A:$c^T e_1 \neq 0, c^T e_2 \neq 0$:

As stated in Lemma 17, the evolving direction of one state p is actually along the line parallel to b and $-b$ and crossing p, i.e., $d^T x(p)$, or the right (left) open half plane divided by this line. Consequently, the geometric position relation of vector b and the cone $\mathcal{V}$ will clarify whether there exists vector $A_i x + bu$ that belongs to the cone $\mathcal{V}$:

case A.a:$b = \lambda_1 e_1$ *or* $b = \lambda_2 e_2$:

This is actually the case that the line parallel to b and $-b$ is parallel to or coincides with

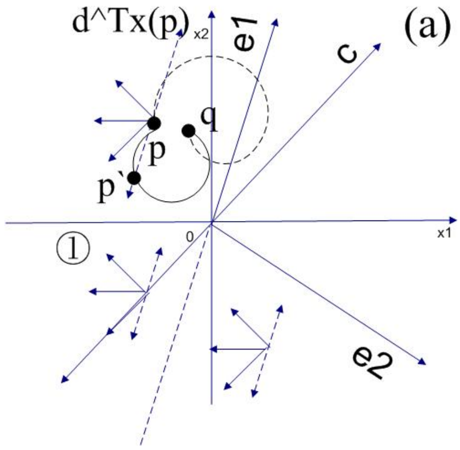

Fig. 4.8: Case (a) of $b = \lambda_1 e_1$ *or* $b = \lambda_2 e_2$

the boundary of $\mathcal{V}$. All the possible situations are shown in Fig.4.7.

A.a.a: consider the case depicted in Fig.4.8. It can be easily seen that there is no vector $A_i x + bu \in \mathcal{V}$ when x is outside $\mathcal{V}$. Furthermore, consider one point p in area ①. We will show that starting from p, the system can not reach any point in the right half space of line $d^T x(p)$. Suppose that there is a point q, who is reachable from p, in the right half space of line $d^T x(p)$ and outside the cone $\mathcal{V}$. If the system trajectory starting from p and reaching q crosses $\mathcal{V}$ as depicted using the dashed line, we can always find another point q' with the system trajectory from p to q' staying entirely outside the cone V. Thus, we can assume that the trajectory reaching q stays entirely outside the cone, which is represented by the solid line in the figure. Obviously, the trajectory must cross the line $d^T x(p)$ if it can reach the point q. We use p' to represent the crossing point. Consider another point that is infinitely close to p' in the right half space of the line $d^T x(p)$. The secant connecting these two points of this trajectory curve is the tangent which represents the derivative vector of p' when the two points are infinitely close. This implies that there is derivative vector of some point outside the cone $\mathcal{V}$ whose direction belongs to the right half plane of line $d^T x(p)$.

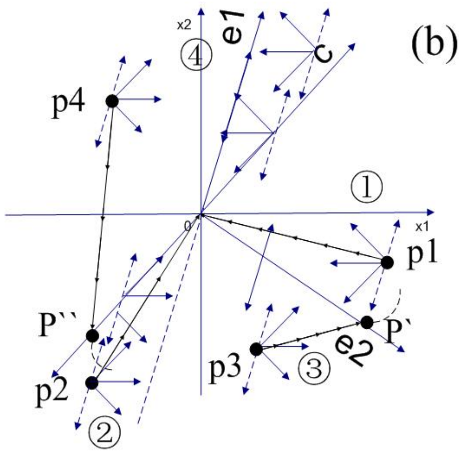

Fig. 4.9: Case (b) of $b = \lambda_1 e_1$ *or* $b = \lambda_2 e_2$

However, this is impossible since there is no such derivative vector of any point outside the cone $\mathcal{V}$ as shown in the figure. Consequently, p can not be driven to zero which implies that the system under this case is not null controllable.

A.a.b: consider the case depicted in Fig.4.9. After simple observation, it can be seen that there are four types of points (states) according to the geometric position, indicated by ①②③④, respectively in Fig.4.9 (the states belonging to the dividing lines and $c^T x = 0$ are not included. The analysis for these states is relatively easy, so we put it to the end of this case). For any point in area ② and ③, obviously, there exists some vector $A_2 x + bu \in \mathcal{V}$. Besides, subsystem (A_2, b) is controllable according to Lemma 19. For an arbitrary point p_1 in area ①, p_1 is connected with zero using the solid line. Since the line is entirely contained in cone $\mathcal{V}$, each point on this line can have its derivative vector along the direction of $-p_1$ with suitable choice of u. Therefore, we can design the control input u and make the system dynamics as:

$$\dot{x}(t) = A_2 x(t) + bu(t) = -\lambda(t) p_1, \;\; \lambda(t) > 0.$$

Solving this equation yields that the trajectory of this system is:

$$x(t) = -p_1 \int_0^t \lambda(t)dt + x_0.$$

The system (A_2, b) is controllable and vector b is not parallel with e_2, which implies that the derivative vector of system states with direction orthogonal to $E_2 x = 0$ can be chosen to be nonzero. Therefore, noticing that $\lambda(t)$ is always positive scalar, suitable t can be chosen and the integral can be made equal to 1 since $\lambda(t)$ will not converge to zero now. Clearly, x_0 equals to p_1 here. Hence, $x(T) = 0$ for some T, which implies that any state p_1 in area ① can be driven to zero. Using the similar analysis, it is easy to show that any point p_2 in area ② can be driven to zero and the possible trajectory is also depicted using solid line in the figure. Consider an arbitrary point p_3 in area ③. As shown above, there always exists some vector $A_i x + bu \in \mathcal{V}$ for any point in area ③. Choosing a derivative vector in the open cone $\mathcal{V}$, the corresponding line parallel to this derivative vector and crossing p_3 surely intersects $E_2 x = 0$, which is the boundary of $\mathcal{V}$, at some point p'. As the part of line between p_3 and p' is entirely outside $\mathcal{V}$, all the points in this part have the same possible evolution directions. Consequently, we can design the control u and make every state in this part have derivative direction of vector $p' - p_3$. The system dynamics now become:

$$\dot{x}(t) = A_2 x(t) + bu(t) = \lambda(t)(p' - p_3), \ \lambda(t) > 0.$$

Solving this equation, we can get the trajectory of this system as:

$$x(t) = (p' - p_3) \int_0^t \lambda(t)dt + x_0.$$

With similar reason as above, some suitable t can be chosen and the integral can be made equal to 1. Besides, x_0 equals to p_3 here. Thus, $x(T) = p'$ for some T. Furthermore, there are two reasons that the system dynamics can not stay on $E_2x = 0$ or go back to the outside of $\mathcal{V}$. One is that on $E_2x = 0$, we can choose the u to let the derivative direction still point to the inside of cone $\mathcal{V}$ (b or $-b$ direction). The other is, actually, the dynamical equations of system do not have description about second derivative of the state. Therefore, sudden change, (here inverse change) of evolution trajectory of system state is not possible. Due to these two reasons, the system dynamics will not stay at the point p' or go back. The system trajectory will reach some point in area ① and using the former control design strategy, the system trajectory can be driven to zero. Finally, any state p_3 in area ③ has been proven that it can be driven to zero and is controllable. Area ④ is defined as the area $c^T x > 0$ except the dividing line $E_1x = 0$. Consider an arbitrary point p_4 in area ④. According to Lemma 19, subsystem (A_1, b) is not controllable and its controllability space is limited in line $E_1x = 0$. Therefore, p_4 can not be driven to zero if its trajectory is only under linear dynamics $\dot{x} = A_1x + bu$. (Even though it seems that we can drive p_4 directly to zero along a line trajectory, it is actually not possible because the derivative vector orthogonal to $E_1x = 0$ and the parameter $\lambda(t)$ will converge to zero due to the geometric relation of b and $E_1x = 0$, which means the integral of $\lambda(t)$ can reach 1 only when t towards infinity). Fortunately, similar to the discussion about the states in area ③, we can design control u and let p_4 be driven to some point p'' and then into area ② (① for the states in right half of area ④) and finally to zero. For the points on $E_1x = 0$ or $E_2x = 0$, using similar control trajectory as discussed above, as shown in the figure, designing suitable control u can ensure the points on $E_1x = 0$ be driven to zero along the boundary of $\mathcal{V}$ and the points on $E_2x = 0$ be driven into area ① and then driven to zero. The states on $c^T x = 0$ can be treated as states in area ① or ② because we assume that any subsystem can be active on $c^T x = 0$. All the states here can be driven to zero and therefore, the system is null controllable in this case.

A.a.c: consider the case depicted in Fig.4.10. The analysis for this case is similar with

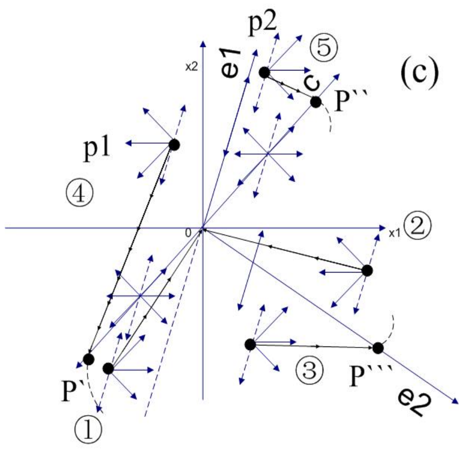

Fig. 4.10: Case (c) of $b = \lambda_1 e_1$ *or* $b = \lambda_2 e_2$

the above case and the system is null controllable. The corresponding trajectory for every state driven to zero is shown in the figure (proof details omitted due to length limit of chapter).

A.a.d: consider the case depicted in Fig.4.11. It can be easily seen that even though there exists some vector $A_1 x + bu \in \mathcal{V}$ when x is in area ②, the subsystem (A_1, b) is not controllable. The conditions in the theorem are not satisfied. Furthermore, consider an arbitrary point p in area ① in Fig.4.11 (notice that the long dashed line is the line parallel to b and $-b$ and crossing 0. Points in area ① are the points in the left open half plane of this line and in $c^T x < 0$). Using the same analysis as case A.a.(a), it can be shown that starting from p, the system trajectory can not reach any point in the right half space of line $d^T x(p)$ under linear dynamics $\dot{x} = A_2 x + bu$. The corresponding trajectory starting from p may enter area ②. Consider an arbitrary point p' in area ②. Similar to the analysis about the states in area ④ of case A.a.(b), it can be shown that starting from p', the system can not reach any point on the line $E_1 x = 0$ under linear dynamics $\dot{x} = A_1 x + bu$. The corresponding trajectory may go into area ①. If the trajectory reaches the dividing line of area ① and ②,

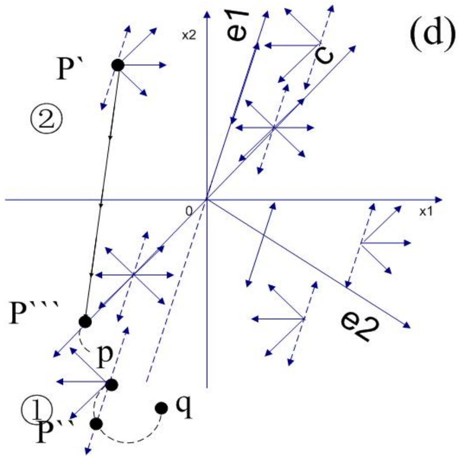

Fig. 4.11: Case (d) of $b = \lambda_1 e_1$ *or* $b = \lambda_2 e_2$

i.e., the left open segment of line $c^T x = 0$, any one of the two subsystems maybe active. However, no matter which system is active, the corresponding trajectory still can not reach zero according to the trajectory analysis for states in areas ① and ②. Consequently, any state p in area ① and any state p' in area ② can not be driven to zero point which implies that the system is not null controllable.

Remark 11. *All the proofs and graphic illustrations are based on* $b = \lambda_1 e_1$. *For the case that* $b = \lambda_2 e_2$*, the analysis method and result are similar. As a result, in this chapter, we will only give detailed analysis on* $b = \lambda_1 e_1$ *to stand for the analysis of the case that* $b = \lambda_1 e_1$ *or* $b = \lambda_2 e_2$ *if without leading to confusion.*

case A.b: b *and* $-b$ *are outside* $\mathcal{V}$:

All the possible situations are shown in Fig.4.12.

A.b.a: consider the case depicted in Fig.4.13. It can be easily seen that there is no vector $A_i x + bu \in \mathcal{V}$ when x is outside $\mathcal{V}$. Furthermore, consider one point p in area ① in Fig.4.13 (notice that the long dashed line is the line parallel to b and $-b$ and crossing

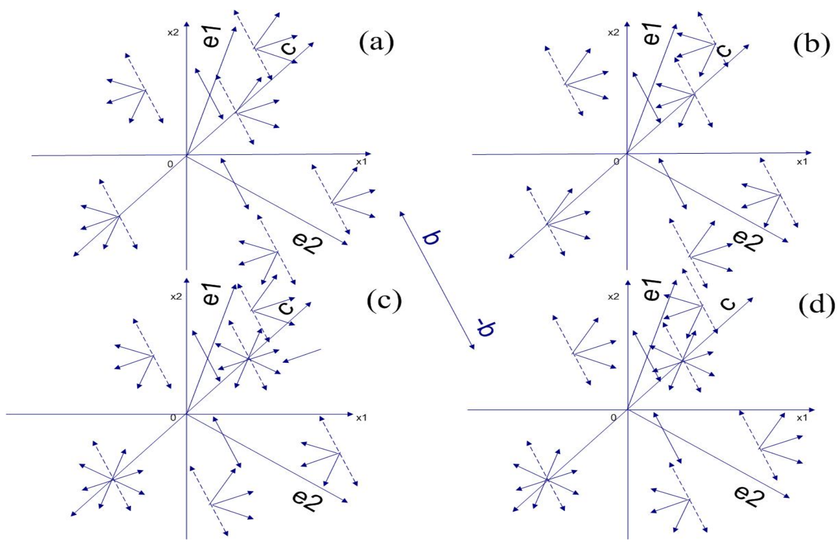

Fig. 4.12: Case A.b: $b\ and\ -b\ are\ outside\ \mathcal{V}$

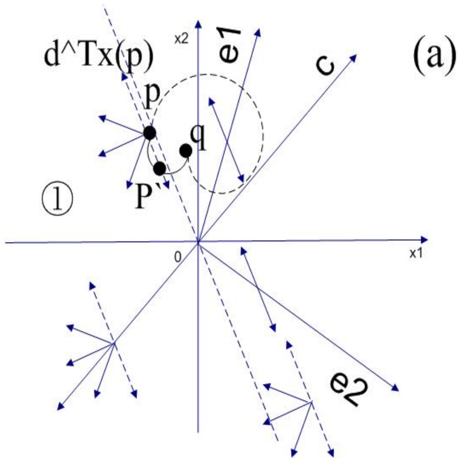

Fig. 4.13: Case (a) of $b\ and\ -b\ are\ outside\ \mathcal{V}$

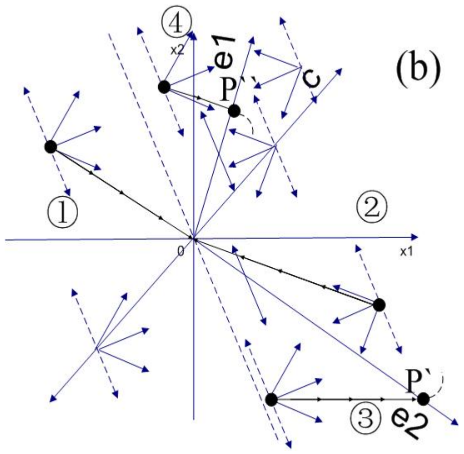

Fig. 4.14: Case (b) of $b\ and\ -b\ are\ outside\ \mathcal{V}$

0. Points in area ① are the points in the left open half plane of this line). Using the same analysis as case A.a.(a), it can be shown that starting from p, the system trajectory can not reach any point in the right half space of line $d^T x(p)$. Consequently, p can not be driven to zero which implies that the system under this case is not null controllable.

A.b.b: consider the case depicted in Fig.4.14. After simple observation, it is clear that there are four types of points according to the geometric position, indicated by ①②③④, respectively in Fig.4.14. For any point in area ①, ③ and ④, obviously, there exists some vector $A_i x + bu \in \mathcal{V}$ and (A_i, b) is controllable. Using the same analysis as case A.a.(b), it can be found that the points in ①② can be driven directly to zero along a line trajectory. Besides, the points in area ③(④) can be driven by a line trajectory to some point $p'(p'')$ and then into area ②(②) and finally driven to zero. The points on $E_1 x = 0$, $E_2 x = 0$ or $c^T x = 0$ can easily be shown to be controllable too. All the states here can be driven to zero and therefore, the system is null controllable in this case.

A.b.c and **A.b.d**: consider the cases depicted in Fig.4.15 and Fig.4.16. The analysis

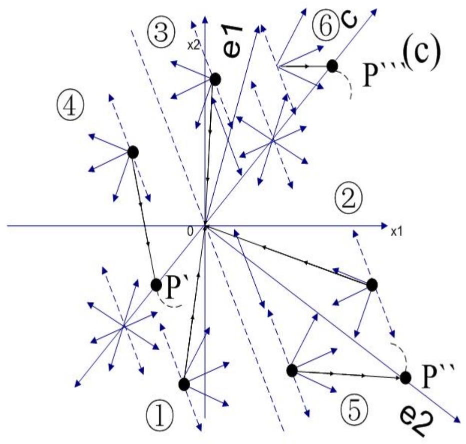

Fig. 4.15: Case (c) of b *and* $-b$ *are outside* $\mathcal{V}$

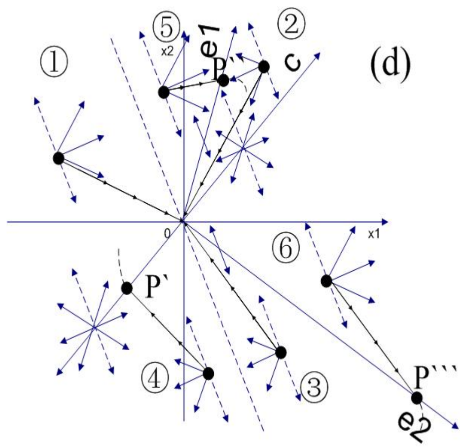

Fig. 4.16: Case (d) of b *and* $-b$ *are outside* $\mathcal{V}$

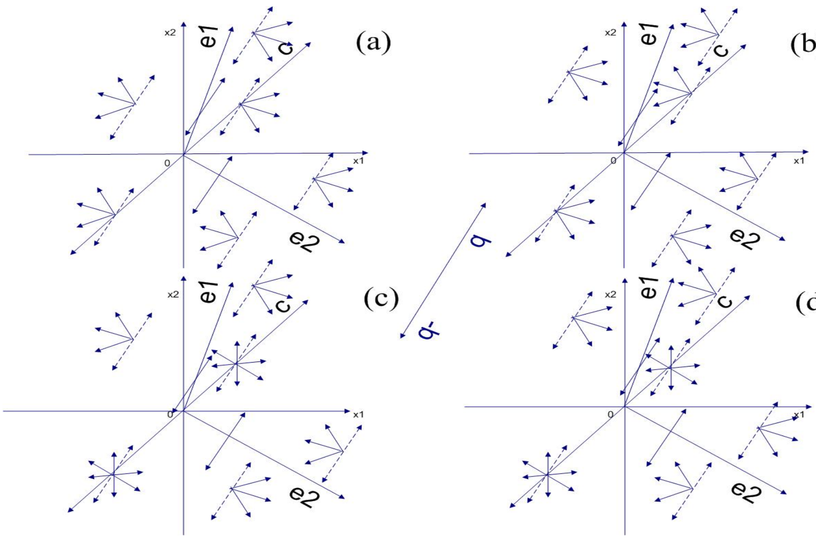

Fig. 4.17: Case A.c: b *or* $-b$ *is in* $\mathcal{V}$

for these cases are similar as the above case and the system is null controllable. The corresponding trajectory for every state driven to zero is shown in the figures (proof details omitted due to length limit of chapter).

case A.c:b *or* $-b$ *is in* $\mathcal{V}$:

All the possible situations are shown in Fig.4.17.

A.c.a, **A.c.b**, **A.c.c** and **A.c.d**: consider the cases depicted in Fig.4.18, Fig.4.19, Fig.4.20 and Fig.4.21). The analysis for these cases are similar as case A.b.(b). There exists some vector $A_i x + bu \in \mathcal{V}$, when x is outside $\mathcal{V}$ and (A_i, b) is controllable. The corresponding trajectory for every state driven to zero is shown in the figures. All the states can be driven to zero and therefore the system is null controllable in these cases.

Remark 12. *A special case contained in this case is that* e_1 *and* e_2 *are linearly dependent, which represents that line* $E_1 x = 0$ *coincides with* $E_2 x = 0$*. The proof for this special case is actually the same as the general case we presented above.*

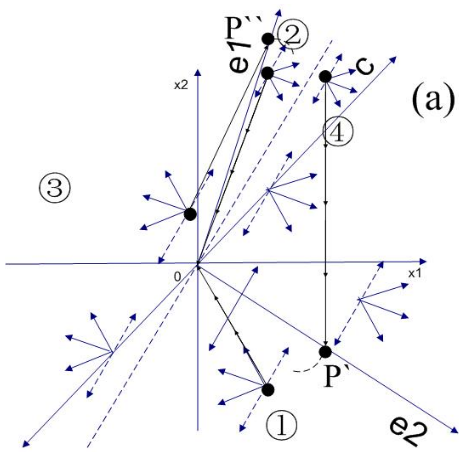

Fig. 4.18: Case (a) of b or $-b$ is in $\mathcal{V}$

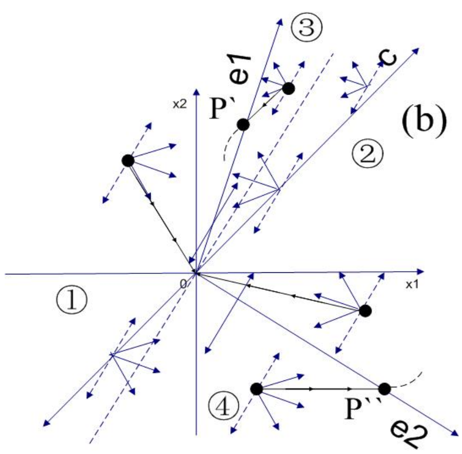

Fig. 4.19: Case (b) of b or $-b$ is in $\mathcal{V}$

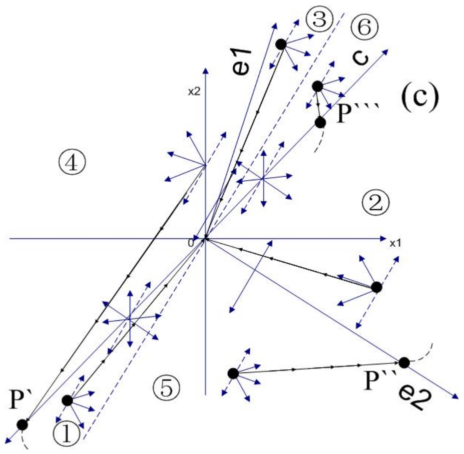

Fig. 4.20: Case (c) of b or $-b$ is in $\mathcal{V}$

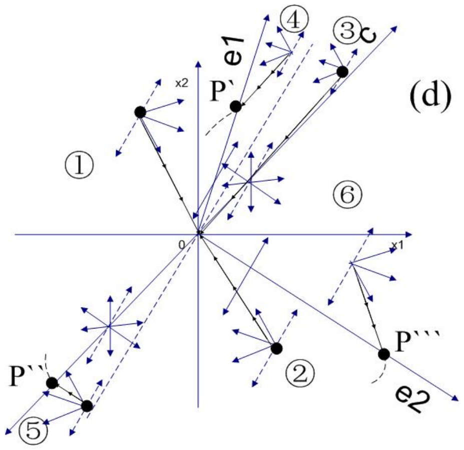

Fig. 4.21: Case (d) of b or $-b$ is in $\mathcal{V}$

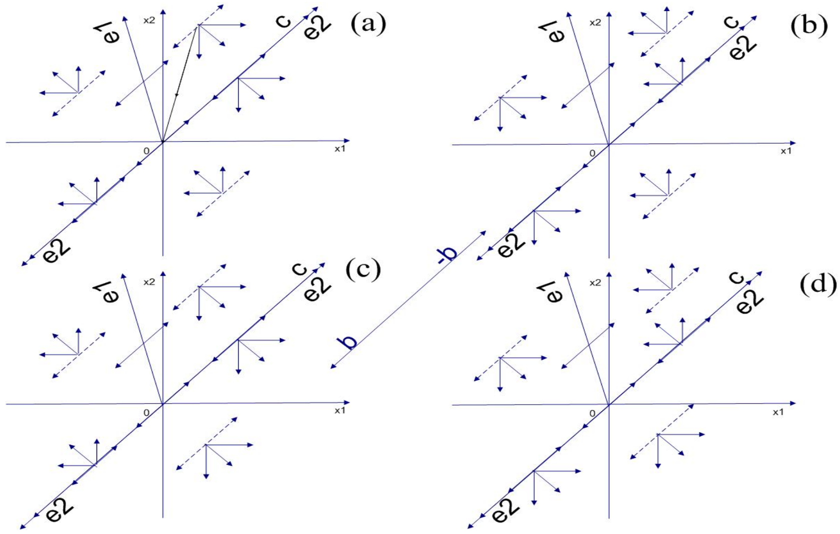

Fig. 4.22: Case B.a: $b = \lambda_2 e_2$

case B:$c^T e_1 \neq 0, c^T e_2 = 0$:

Remark 13. *We can also assume $c^T e_1 = 0, c^T e_2 \neq 0$. All the following proof would be the same, so we only prove this case with $c^T e_1 \neq 0, c^T e_2 = 0$.*

Remark 14. *As stated in Remark 10, from $c^T e_2 = 0$, we have e_2 and $-e_2$. It is necessary to consider simultaneously the convex cone formed by e_1 and e_2 and the convex cone e_1 and $-e_2$ when we verify the conditions stated in Theorem 20. For simplicity, we refer to the cone on the right side as cone $\mathcal{V}_1$ and the left one as $\mathcal{V}_2$.*

case B.a:$b = \lambda_2 e_2$:

This is actually the case that $d^T x(p)$ is parallel to or coincides with $c^T x = 0$. All the possible situations are shown in Fig.4.22.

B.a.a: consider the case depicted in Fig.4.23. Considering cone $\mathcal{V}_1$, for any point in area ③, obviously, there exists some vector $A_1 x + bu \in \mathcal{V}_1$ and (A_1, b) is controllable. However, for cone $\mathcal{V}_2$, even though for any point in area ①, there exists some vector

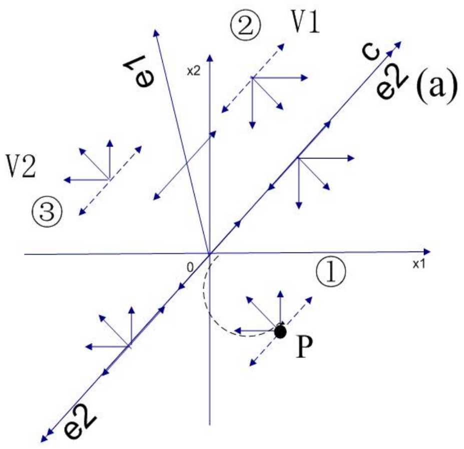

Fig. 4.23: Case (a) of $b = \lambda_2 e_2$

$A_2 x + bu \in \mathcal{V}_2$, (A_2, b) is uncontrollable. Therefore, the conditions in the theorem are not satisfied. For any state p in area ①, since subsystem (A_2, b) is uncontrollable, using the similar analysis about the states in area ④ of case A.a.(b), it can be shown that starting from p, the system can not reach any point on the line $E_2 x = 0$ (also $c^T x = 0$ here) under linear dynamics $\dot{x} = A_2 x + bu$. Consequently, state p can not be driven to zero and the system is not null controllable.

B.a.b, **B.a.c** and **B.a.d**: consider the cases depicted in Fig.4.24, Fig.4.25 and Fig.4.26. These cases are similar as the above case. In case B.a.(b), considering cone $\mathcal{V}_1$, for any point in area ①, there exists some vector $A_2 x + bu \in \mathcal{V}_1$, but (A_2, b) is uncontrollable. In case B.a.(c), for cone $\mathcal{V}_2$, the area outside $\mathcal{V}_2$ is now consisting of ①, ③ and their dividing line. Obviously, there is no point with a derivative vector, i.e., a vector $A_i x + bu$, that is in the open cone $\mathcal{V}_2$. In case B.a.(d), for cone $\mathcal{V}_1$, the area outside $\mathcal{V}_1$ is now consisting of ①, ② and their dividing line. Clearly, there is no point with a derivative vector, i.e., a vector $A_i x + bu$, that is in the open cone $\mathcal{V}_1$. The conditions in the theorem are not satisfied in all these cases. For the same reason as in the above case or case A.a.(a), one state p in

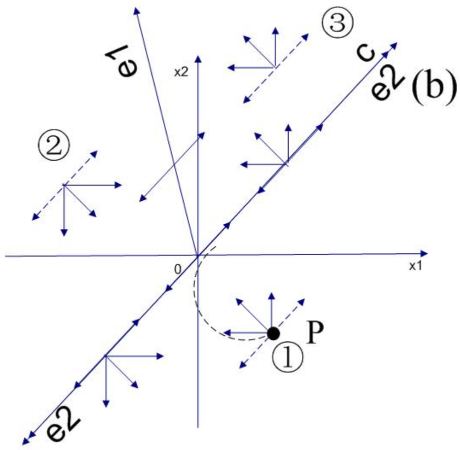

Fig. 4.24: Case (b) of $b = \lambda_2 e_2$

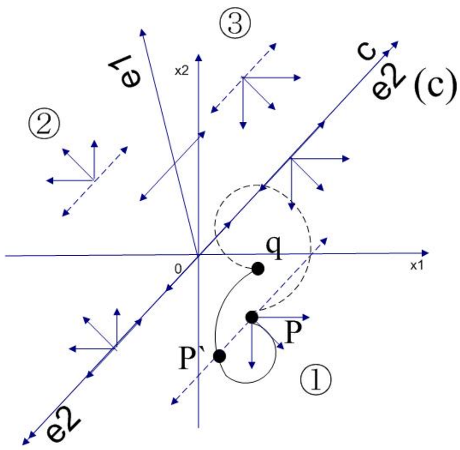

Fig. 4.25: Case (c) of $b = \lambda_2 e_2$

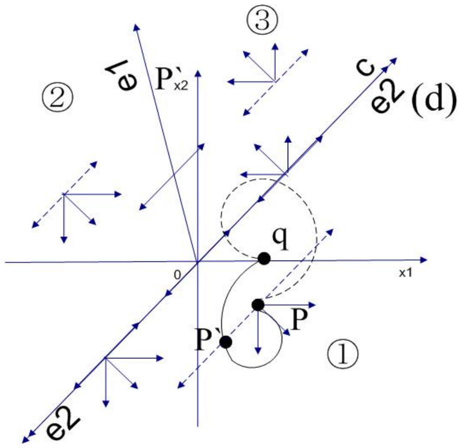

Fig. 4.26: Case (d) of $b = \lambda_2 e_2$

area ① can not reach zero. Therefore, the piecewise linear systems in these cases are not null controllable.

case B.b:$b = \lambda_1 e_1$:

This is actually the case that $d^T x(p)$ is parallel to or coincides with $E_1 x = 0$. All the possible situations are shown in Fig.4.27.

B.b.a: consider the case depicted in Fig.4.28. Considering cone $\mathcal{V}_1$, for any point in area ①, obviously, there exists some vector $A_i x + bu \in \mathcal{V}_1$. However, for cone $\mathcal{V}_2$, the area outside $\mathcal{V}_2$ is now consisting of ①, ③ and their dividing line. Obviously, there is no point with a derivative vector, i.e., a vector $A_i x + bu$, that is in the open cone $\mathcal{V}_2$. Furthermore, similar to case $A.a.(a)$, any point p in area ③ can not reach any point q in the left half space of line $d^T x(p)$ in area ③ and ①. Therefore, p can not be driven to zero which implies that the system is not null controllable.

B.b.b: consider the case depicted in Fig.4.29). This case is almost the same as the above case. The only difference is that under this case, it is that for cone $\mathcal{V}_1$ rather than cone $\mathcal{V}_2$, there is no desired vector $A_i x + bu \in \mathcal{V}_1$ (proof details omitted due to length limit

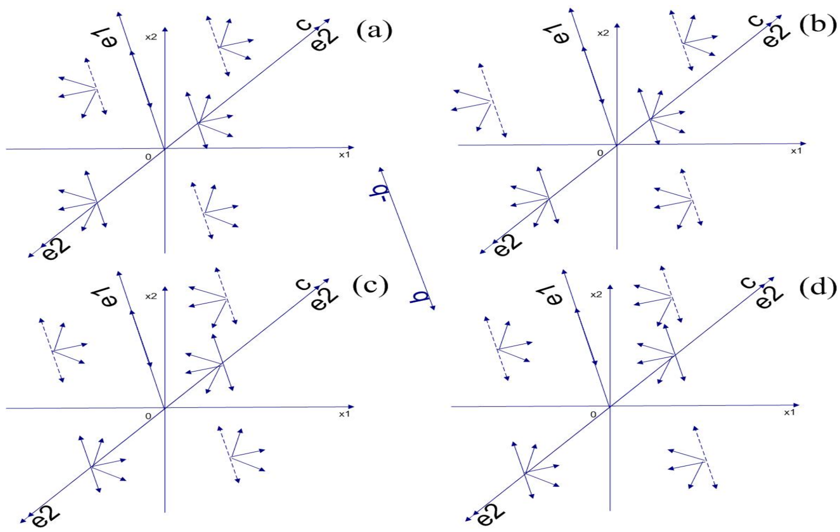

Fig. 4.27: Case B.b: $b = \lambda_1 e_1$

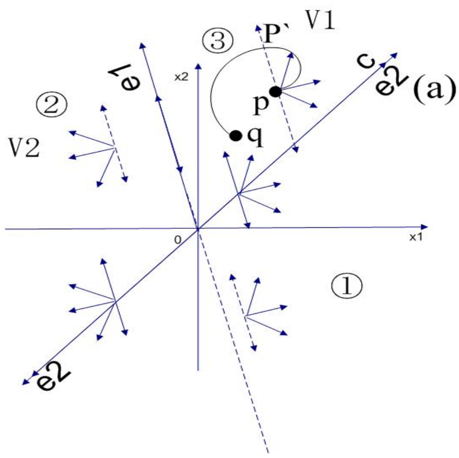

Fig. 4.28: Case (a) of $b = \lambda_1 e_1$

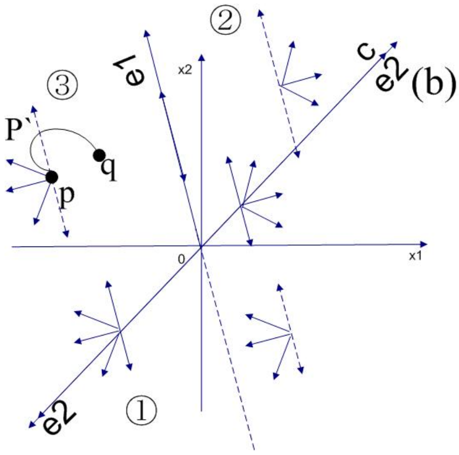

Fig. 4.29: Case (b) of $b = \lambda_1 e_1$

of chapter).

B.b.c: consider the case depicted in Fig.4.30. Considering cone $\mathcal{V}_1$, for any point in area ①, clearly, there exists some vector $A_2x+bu \in \mathcal{V}_1$ and (A_2, b) is controllable. However, considering cone $\mathcal{V}_2$, for any point in area ②, there exists some vector $A_1x + bu \in \mathcal{V}_1$, but (A_1, b) is uncontrollable. Therefore, the conditions in the theorem are not satisfied. Furthermore, similar to case $A.a.(d)$, although any state p in area ① can reach some states in area ② or dividing line of areas ① and ② and any state p' in area ② can reach some states in area ① or dividing line of areas ① and ②, no state in area ①, area ② and their dividing line can be driven to zero. Thus, the system under this case is not null controllable.

B.b.d: consider the case depicted in Fig.4.31. The analysis for this case is similar to the above case. Easily we can see for cone $\mathcal{V}_1$, even though for any point in area ②, there exists some vector $A_1x + bu \in \mathcal{V}_1$, (A_1, b) is uncontrollable. System (4.1) is not null controllable in this case (proof details omitted due to length limit of chapter).

case B.c: *b or $-b$ is in $\mathcal{V}_2$:*

Remark 15. *We can also assume that b or $-b$ is in $\mathcal{V}_1$. All the following proof would be*

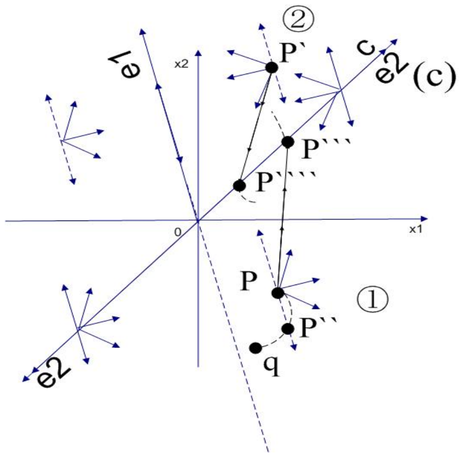

Fig. 4.30: Case (c) of $b = \lambda_1 e_1$

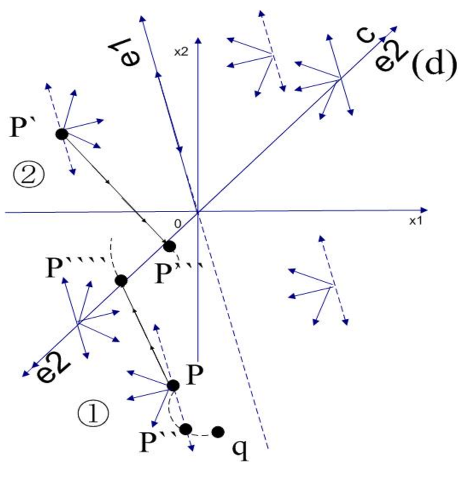

Fig. 4.31: Case (d) of $b = \lambda_1 e_1$

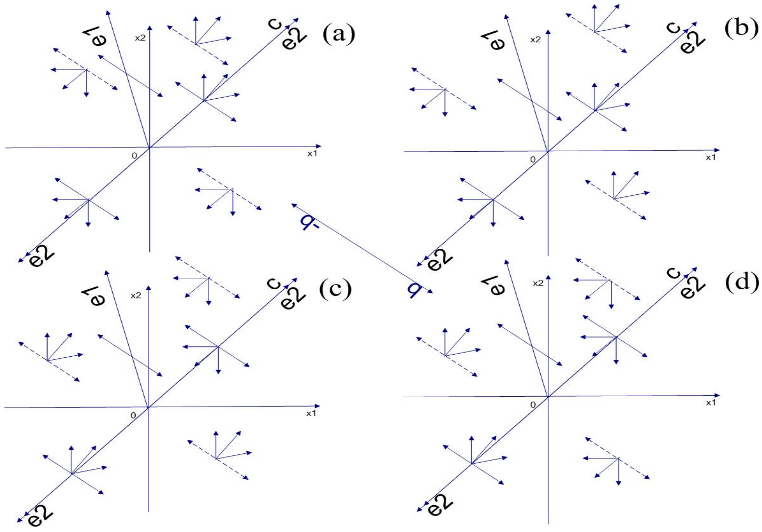

Fig. 4.32: Case B.c: $b\ or\ -b\ is\ in\ \mathcal{V}_2$

the same, so we only prove this case with b or $-b \in \mathcal{V}_2$.

All the possible situations are shown in Fig.4.32.

B.c.a: consider the case depicted in Fig.4.33. Considering cone $\mathcal{V}_2$, for any point in area ①, obviously, there exists some vector $A_2x + bu \in \mathcal{V}_2$ and (A_2, b) is controllable. However, for cone $\mathcal{V}_1$, the area outside $\mathcal{V}_1$ is now consisting of ①, ③ and their dividing line. Clearly, there is no point with a derivation vector, i.e., a vector $A_ix + bu$, that is in the open cone $\mathcal{V}_1$. Furthermore, similar to case $A.a.(a)$, some point p in area ① can not reach any point q in the right half space of line $d^T x(p)$ in area ① and ③. Consequently, p can not be driven to zero point which implies that the system under this case is not null controllable.

B.c.b, **B.c.c** and **B.c.d**: consider the cases depicted in Fig.4.34, Fig.4.35 and Fig.4.36. Easily we can see for both cone $\mathcal{V}_1$ and cone $\mathcal{V}_2$, there exists the desired vector and the corresponding subsystem is controllable. The system (4.1) is null controllable under these

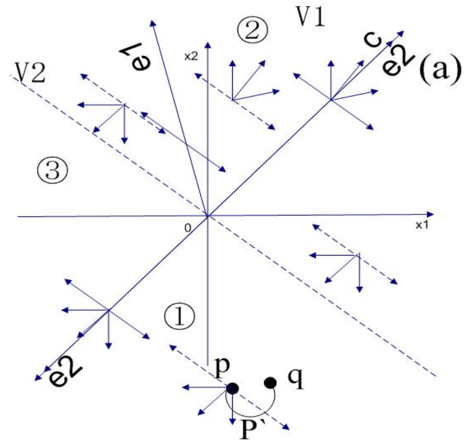

Fig. 4.33: Case (a) of $b\ or\ -b\ is\ in\ \mathcal{V}_2$

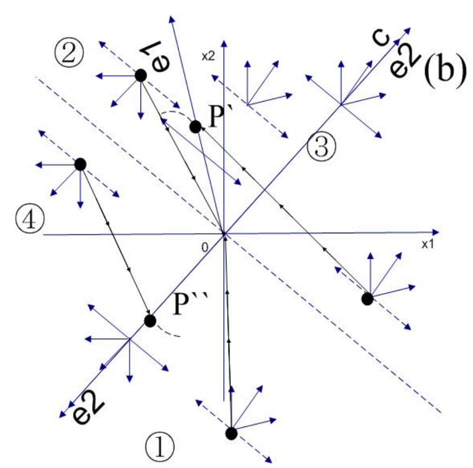

Fig. 4.34: Case (b) of $b\ or\ -b\ is\ in\ \mathcal{V}_2$

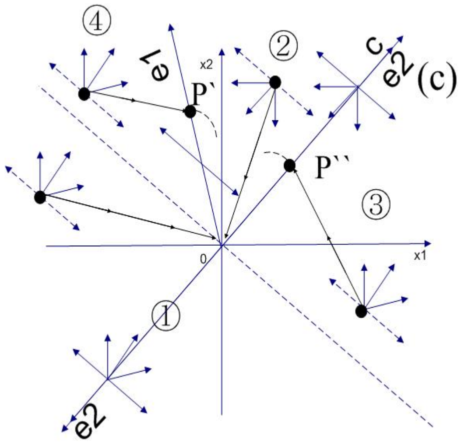

Fig. 4.35: Case (c) of $b\ or\ -b\ is\ in\ \mathcal{V}_2$

cases. The corresponding trajectory for every state driven to zero is shown in the figures (proof details omitted due to length limit of chapter).

case C:$c^T e_1 = c^T e_2 = 0$:

Remark 16. *As stated in Remark 10, from $c^T e_1 = c^T e_2 = 0$, it follows that e_1, $-e_1$, e_2 and $-e_2$ all satisfy the equation $c^T e_1 \geq 0$ and $c^T e_2 \leq 0$. Then we should consider simultaneously the convex cone formed by e_1 and e_2, the convex cone formed by$-e_1$ and e_2, convex cone formed by e_1 and $-e_2$ and convex cone formed by $-e_1$ and $-e_2$ when we verify the conditions stated in Theorem 20. According to definition 27, $\mathcal{V}$ is defined as the open convex cone if $e_1 \neq \lambda e_2, \lambda > 0$ and we say a vector $v \in \mathcal{V}$ if $v = \lambda_1 e_1 + \lambda_2 e_2, \lambda_1 > 0, \lambda_2 > 0$. When $e_1 = \lambda e_2, \lambda > 0$, we say a vector $v \in \mathcal{V}$ if $v = \lambda_i * e_i$, $\lambda_i > 0$. For simplicity, in the following proof, we denote the open cone opening up as cone $\mathcal{V}_1$ and the open cone opening down as $\mathcal{V}_2$. For $e_1 = \lambda e_2, \lambda > 0$, we refer to the right side cone as $\mathcal{V}_3$ and the left side cone as $\mathcal{V}_4$.*

case C.a:$b \neq \lambda_1 e_1$:

This is actually the case that $d^T x(p)$ is not parallel to or coincides with $c^T x = 0$. All the

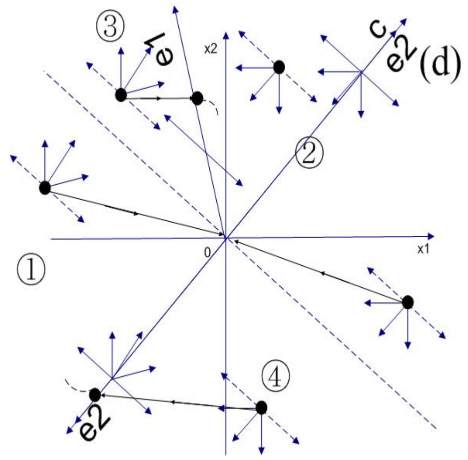

Fig. 4.36: Case (d) of $b\ or\ -b\ is\ in\ \mathcal{V}_2$

possible situations are shown in Fig.4.37.

C.a.a: consider the case depicted in Fig.4.38. There are four types of points according to the geometric position, indicated by ①②③④, respectively in Fig.4.38. First, considering cone $\mathcal{V}_1$ (the left open half plane of $c^T x = 0$ here), for any point in area ② and ③, clearly, there exists some vector $A_2 x + bu \in \mathcal{V}_1$ and (A_2, b) is controllable. Second, considering cone $\mathcal{V}_2$ (the right open half plane of $c^T x = 0$ here), for any point in area ① and ④, clearly, there exists some vector $A_1 x + bu \in \mathcal{V}_2$ and (A_1, b) is controllable. Third, considering cone $\mathcal{V}_3$ (the right half segment of line $c^T x = 0$), for any point in area ① and ④, clearly, there exists some vector $A_1 x + bu \in \mathcal{V}_3$ and (A_1, b) is controllable. Finally, considering cone $\mathcal{V}_4$ (the left half segment of line $c^T x = 0$), for any point in area ② and ③, clearly, there exists some vector $A_2 x + bu \in \mathcal{V}_4$ and (A_2, b) is controllable. The conditions in Theorem 20 are satisfied. Similarly, it can be shown that the points in ①② can be driven directly to zero along a line trajectory. Besides, the points in area ③ and ④ can be driven by a line trajectory to some point p' and p'' and then into area ① and ② respectively and finally driven to zero. The points on $c^T x = 0$ can be shown that they can be driven to area ① or ②

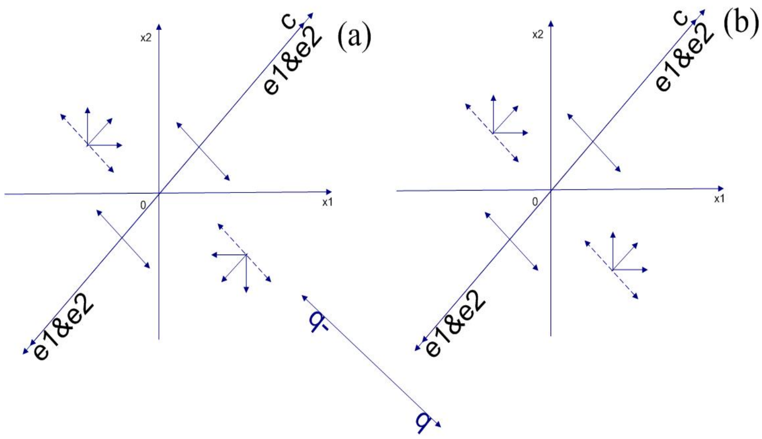

Fig. 4.37: Case C.a: $b \neq \lambda_1 e_1$

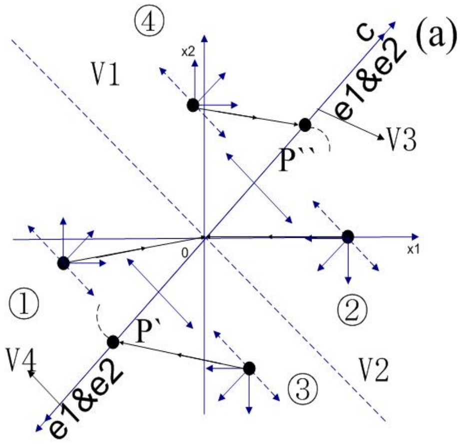

Fig. 4.38: Case (a) of $b \neq \lambda_1 e_1$

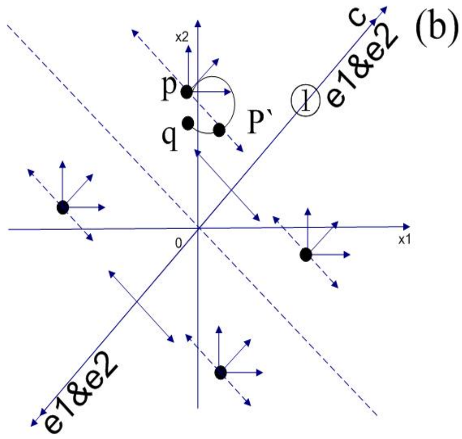

Fig. 4.39: Case (b) of $b \neq \lambda_1 e_1$

or ③ or ④. Consequently, all the states here can be driven to zero and therefore the system is null controllable in this case.

C.a.b: consider the case depicted in Fig.4.39. For cone $\mathcal{V}_4$, the area outside $\mathcal{V}_4$ is now consisting of all the areas except the left half of line $c^T x = 0$. Obviously, there is no state with a derivative vector, i.e., a vector $A_i x + bu$, that is in the cone $\mathcal{V}_4$. Furthermore, similar to case $A.a.(a)$, any point p at area ① can not reach any point q on the left half space of line $d^T x(p)$. Therefore, p can not be driven to zero which implies that the system under this case is not null controllable. All the possible situations are shown in Fig.4.40.

case C.b:$b = \lambda_1 e_1$:

This is actually the case that $d^T x(p)$ is parallel to or coincides with $c^T x = 0$. All the possible situations are shown in Fig.C.b.

C.b.a: consider the case depicted in Fig.4.41. For cone $\mathcal{V}_1$, although for any point in area ①, there exists some vector $A_2 x + bu \in \mathcal{V}_1$, the subsystem (A_2, b) is uncontrollable. Therefore, the conditions in the theorem are not satisfied. Furthermore, similar to the discussion analysis about the states in area ④ of case A.a.(b), it can be shown that starting

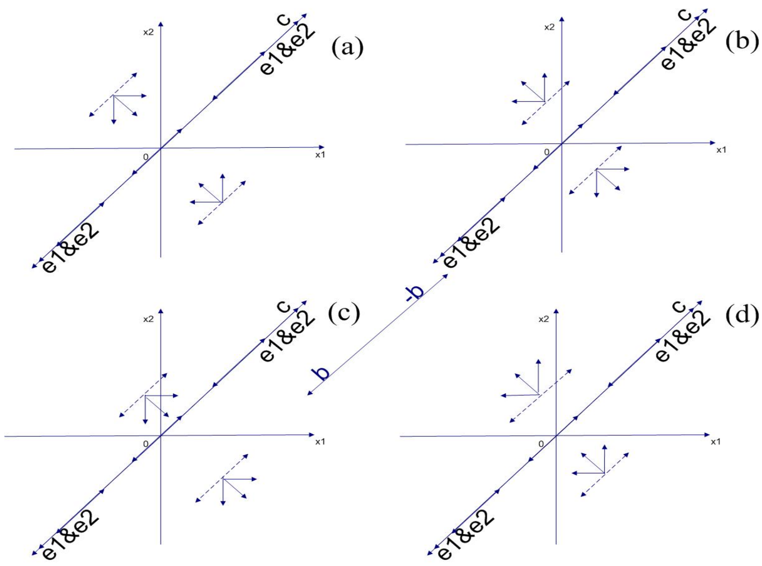

Fig. 4.40: Case C.b: $b = \lambda_1 e_1$

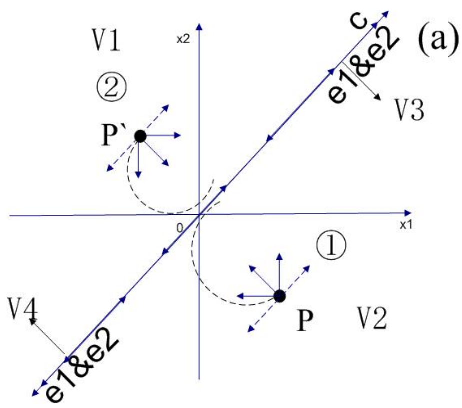

Fig. 4.41: Case (a) of $b = \lambda_1 e_1$

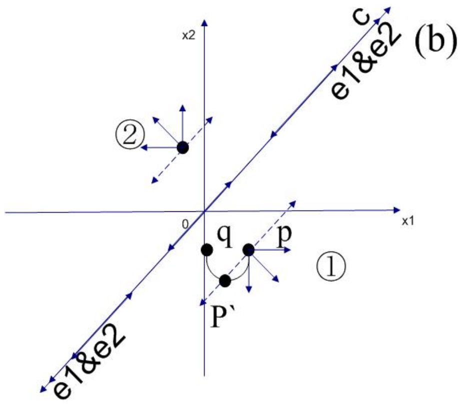

Fig. 4.42: Case (b) of $b = \lambda_1 e_1$

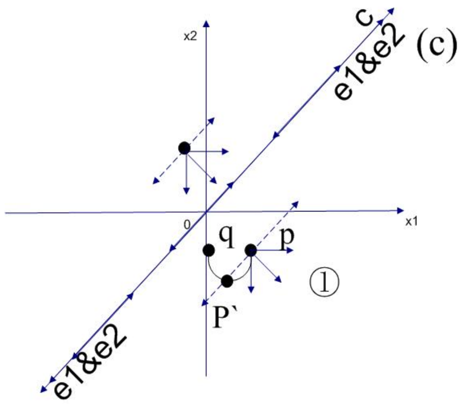

Fig. 4.43: Case (c) of $b = \lambda_1 e_1$

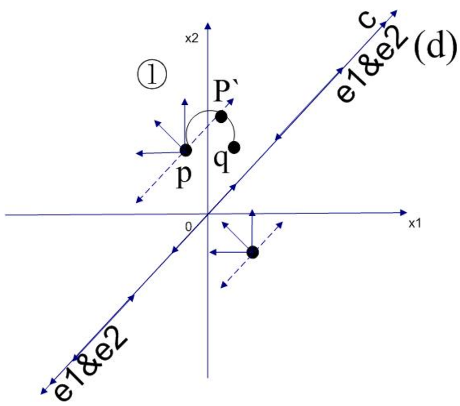

Fig. 4.44: Case (d) of $b = \lambda_1 e_1$

from arbitrary state p in area ①, the system can not reach any point on the line $E_2 x = 0$ (also $c^T x = 0$ here) under linear dynamics $\dot{x} = A_2 x + bu$. Hence, p can not be driven to zero which implies that the system under this case is not null controllable.

C.b.b, **C.b.c**, **C.b.d**: consider the cases depicted in Fig.4.42, Fig.4.43 and Fig.4.44. These cases are similar with the above case. In case C.b.(b), for cone $\mathcal{V}_1$, the area outside $\mathcal{V}_1$ is now area ①. Clearly, there is no state with a derivative vector, i.e., a vector $A_2 x + bu$, that is in the open cone $\mathcal{V}_1$. In case C.b.(c), for cone $\mathcal{V}_1$, the area outside $\mathcal{V}_1$ is now area ①. It is easy to see that there is no state with a derivative vector, i.e., a vector $A_2 x + bu$, that is in the open cone $\mathcal{V}_1$. In case C.b.(d), for cone $\mathcal{V}_2$, the area outside $\mathcal{V}_2$ is now area ①. It is clear that there is no state with a derivative vector, i.e., a vector $A_1 x + bu$, that is in the open cone $\mathcal{V}_2$. The conditions in the theorem are not satisfied in all these cases. For the same reason as in the above case or case A.a.(a), one state p in area ① can not reach zero. Consequently, the piecewise linear systems in these cases are not null controllable.

In conclusion, all the cases that there exists a vector $A_i x + bu \in \mathcal{V}$ when x is outside $\mathcal{V}$ and the corresponding subsystem (A_i, b) is controllable, are proven to be null controllable. Besides, all the possible cases that at least one of the two conditions can not be satisfied are proven that there always exists some nonzero state that can not be driven to zero and the system (4.1) is not null controllable. □

Chapter 5

Disturbance Rejection of Multi-agent System

5.1 Introduction

This chapter aims to investigate the impacts of local interactions between agents on the disturbance rejection property of the whole multi-agent system. The disturbance rejection property is important for multi-agent systems, since many applications of multi-agent systems usually require deployment of agents outdoors and possibly in unstructured environments. Hence external disturbances unavoidably influence the dynamical behavior of agents and the emergent behavior of the group. Furthermore, compared with traditional control objects, the disturbances may get exaggerated through information transferring along the network and bring much worse influence to the whole system behavior. Therefore, the control of such large scale complex systems must address the issue of disturbance rejection. Take the multi-agent system depicted in Fig.5.1 as example [135]. This group of UAVs and robots are moving towards a target area in a specific formation. Interactions

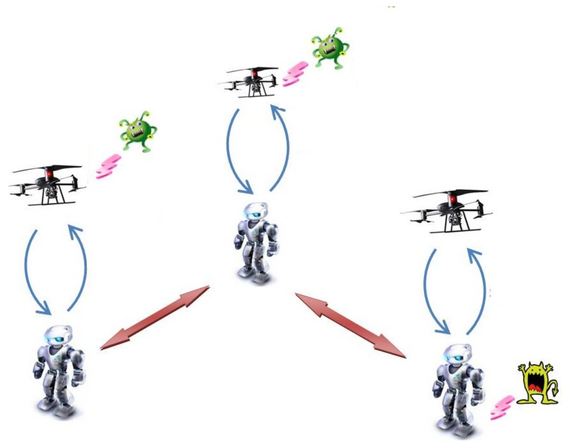

Fig. 5.1: Disturbed multi-agent system

among them couple their dynamics closely, which means a small unexpected change in the movement of one UAV or robot may greatly influence whole group's formation. Disturbances from surrounding environment, such as wind, external force and so on, may bring damage to whole system behavior. This motivates us to investigate under what kind of interconnections, this multi-agent system can dismiss the disturbance influence to the desired system behavior.

Different from the existing algebraic or geometric approaches to networked multi-agent systems, we model the multi-agent system as a structured system and study its disturbance rejection property from a structural sense. To be more specific, the interactions among agents are assumed to follow the nearest neighbor law while the weights are assumed to be adjustable. A multi-agent system is said to have structural disturbance rejection capability if it can reject disturbance for almost all such weighting parameters. In other words, we are interested in identifying interconnection topologies among agents that have the disturbance rejection capability. The advantage of the proposed method over the existing algebraic or geometric methods lies on the fact that the structural property does not depend on particular system parameters, the exact values of which are usually difficult to obtain in practical applications due to uncertainties and noises. In addition, the checking of structural property turns out to be of polynomial computational complexity and therefore can scale up for a network with large numbers of agents.

These considerations motivate us to investigate the disturbance rejection property of networked multi-agent systems from a structural sense. In particular, we consider agents with non-homogeneous linear structured dynamics (with single integrator dynamics as a special case) and the interconnection topology is modeled as a graph in which the states of each agent are represented by vertices, while the interaction links are the edges. With this graphic representation, the structural disturbance rejection problem is investigated directly through the graph-theoretic approach. For example, a special kind of graph structure called cacti was introduced in [68] to describe the structural controllability of linear system. Under this framework, the problem of disturbance rejection by measurement feedback has been solved using a graphic approach in [85, 136]. The main contribution of this chapter lies on the proposal of necessary and sufficient conditions for structural disturbance rejection capability of multi-agent systems. It turns out that the global disturbance rejection capability can be deduced by the local disturbance rejection capability of individual agents. These conditions can be easily verified from the associated graph, which is convenient and efficient. Besides, these results remove the necessity of requiring exact knowledge of the parameters of individual agent, focusing instead on the structure which is decided by positions of nonzero parameters and this makes it possible to conduct global network properties directly from the local properties of the interconnected agents. Furthermore, as a combination of obtained disturbance rejection results with other multi-agent control problems, we consider the the controllability of multi-agent systems with external disturbance. We explicitly show under what kind of interconnection topologies, the whole multi-agent system is structurally controllable and meanwhile has structural disturbance rejection capability.

The outline of this chapter is as follows: In Section 5.2, we introduce some basic preliminaries and problem formulation, followed by disturbance rejection study in Section 5.3, where a graphic necessary and sufficient condition for the disturbance rejection capability of networked multi-agent systems with non-homogeneous general linear structured dynamics is proposed. Besides, disturbance rejection capability of network with identical single integrator dynamics is also investigated as a special case. In Section 5.4, we combine the disturbance rejection results in controllability problem of multi-agent system. In Section 5.5, illustrative examples are presented to give the readers deeper understanding of our theoretical results. Finally, some concluding remarks are drawn in the chapter.

5.2 Preliminaries and Problem Formulation

5.2.1 Graph Theory Preliminaries

A directed graph is an appropriate representation for the interconnection topology among agents. The directed graph $\mathcal{G}$ with N vertices consists of a vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{I} = \{e_1, e_2, \ldots, e_{N'}\}$, which are the interaction links among the vertices. Each element $a = (i, j)$ of $\mathcal{I} \subset \mathcal{V}^2$ characterizes the relation between distinct pairs of vertices $i, j \in \mathcal{V}$. For an edge (i, j), we call i the tail and j the head. The in(out) degree of a vertex i is the number of edges with i as its head(tail). A directed path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. A directed path with no repeated vertices is called a simple directed path. A directed graph is called strongly connected if there is a directed path from each vertex in the graph to every other vertex. A directed graph is weakly connected if every vertex can be reached from every other but not necessarily following the directions of the edges. A simple cycle is a closed path that is self-avoiding (does not revisit vertices, other than the first). A acyclic directed graph is a directed graph without cycles.

5.2.2 Disturbance Rejection of Networked Multi-Agent Systems

In this chapter, we will consider a dynamic network composed of n agents with non-homogeneous general linear dynamics. In particular, the network is modeled as structured system and its structural properties are investigated. We are interested in how the interconnection topology will influence the disturbance rejection capability of the network. Namely, we would like to determine the disturbance rejection capability of the dynamical network composed of interconnected dynamical agent systems where each individual agent has the dynamics model:

$$\begin{cases} \dot{x}_i = A_i x_i + E_i \omega_i + \sum_j L_{ij} x_j \\ \qquad\quad y_i = C_i x_i \end{cases} \tag{5.1}$$

where $x_i \in R^{n_i}$ are the local states and y_i are the local output. $A_i \in R^{n_i \times n_i}$ and E_i represent the local dynamics and the local disturbance effectiveness respectively. The matrices $L_{ij} \in R^{n_i \times n_j}$ represent the network interconnections and determine the effects of the states of a given system on the evolution of neighboring systems. Since the self-influence is already captured by A_i, we set $L_{ii} = 0$ where 0 is the $n_i \times n_i$ matrix of zeros.

Here this dynamic networked multi-agent system is considered as a structured system, whose entries of the system matrices A_i, E_i, L_{ij} and C_i are either fixed zeros or independent parameters. If the system has totally p parameters, it can be parameterized by means of a parameter vector $q \in \mathcal{R}^p$. For such structured systems, one can study structural properties. A property is said to be structural if it is true for all values of the parameters except for those in a proper algebraic variety of the parameter space. Define the transfer function from external disturbance to output as $T_{\wedge}(s)$. Then for the structural disturbance rejection of networked multi-agent systems:

Definition 28. *The networked multi-agent system consisting of dynamic agents (5.1) has structural disturbance rejection capability if and if for almost all $q \in \mathcal{R}^p$, $T_{\wedge}(s) \equiv 0$.*

The aim of our work is to investigate the structural disturbance rejection capability of whole multi-agent systems from graphic point of view. Especially, we aim to reveal how to deduce the global disturbance rejection capability from the local disturbance rejection capability of individual agent and interconnection topology among them.

5.3 Structural Disturbance Rejection

5.3.1 Non-Homogeneous General Linear Dynamics Case

Before proceeding to the global structural disturbance rejection study, let us recall individual agent dynamics (5.1) first. We can see that it is possible to study the disturbance rejection capability of each agent dynamics with respect to both the local disturbance ω_i and the network interactions x_j. Since each agent has its own inherent disturbance rejection capability, by eliminating these network interactions x_j first, it sounds reasonable to analyze the disturbance rejection capability of each local system with respect to local disturbance ω_i and then determine a posteriori if the interaction structure will ensure a good disturbance rejection capability of whole networked multi-agent systems. This motivates the following definition about local disturbance rejection problem of each local individual agent dynamics:

Definition 29. *(local disturbance rejection) We perform a structural disturbance rejection* *analysis on each local agent dynamics:*

$$\begin{cases} \dot{x}_i = A_i x_i + E_i \omega_i + \sum_j L_{ij} x_j \\ y_i = C_i x_i \end{cases} \tag{5.2}$$

If the following linear structural system has structural disturbance rejection capability, we say that the agent system (5.2) has locally structural disturbance rejection capability:

$$\begin{cases} \dot{x}_i = A_i x_i + E_i \omega_i \\ y_i = C_i x_i \end{cases} \tag{5.3}$$

For each local agent, we have the following graphic representation:

Definition 30. *(local representation graph) For each agent dynamics (5.2), we describe its local interconnection topology using a directed graph $\mathcal{G}_i(\mathcal{V}_i, \mathcal{I}_i)$, where each vertex of vertex set $\mathcal{V}_i$ corresponds to one element of agent i state x_i or one element of local disturbance ω_i and each edge of edge set $\mathcal{I}_i$ corresponds to one independent parameter of structured matrices A_i or E_i.*

Combining all the local dynamics of individual agents, for the whole networked multi-agent systems, let $x = [x_1^T, x_2^T, \ldots, x_n^T]^T$, $y = [y_1^T, y_2^T, \ldots, y_n^T]^T$, $\omega = [\omega_1^T, \omega_2^T, \ldots, \omega_n^T]^T$, $C = [C_1^T, C_2^T, \ldots, C_n^T]^T$, $A = diag(A_1, A_2, \ldots, A_n)$, $L = \{L_{ij}\}$ and $E = diag(E_1, E_2, \ldots, E_n)$. Then the network dynamics can be described as follows:

$$\begin{cases} \dot{x} = A_\wedge x + L_\wedge x + E_\wedge \omega \\ y = C_\wedge x \end{cases} \tag{5.4}$$

where subscription $\wedge$ means the matrix is a structured matrix, i.e., $A_\wedge$, $L_\wedge$, $E_\wedge$ and $C_\wedge$ are all structured matrices.

Since each local dynamics is a general linear system, we consider the following linear structured system:

$$\begin{cases} \dot{x} = A_{\wedge}x + B_{\wedge}u \\ y = C_{\wedge}x + D_{\wedge}u \end{cases} \tag{5.5}$$

Suppose this structured system is parameterized by means of a parameter vector $q \in \mathcal{R}^p$, where p is number of parameters. Let $s-rank$ represent the structural rank, i.e. the number of infinite zeros of (5.5), $T_{\wedge}(s) = C_{\wedge}(sI - A_{\wedge})B_{\wedge} + D_{\wedge}$ represent transfer matrix. For each specific $q \in \mathcal{R}^p$, we call the following system as an admissible realization of (5.5):

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{5.6}$$

Transfer matrix of this system is $T(s) = C(sI - A)B + D$. Define $n - rank$ as *normal rank* of $T(s)$, i.e., the rank of $T(s)$ for almost all s.

Then for $s - rank$, we say:

Lemma 21. *([83]) $s - rank$ of linear structured system (5.5) is equal to the maximal number of input-output vertex disjoint paths.*

From [77], we have $s - rank$ equals $n - rank(T_{\wedge}(s))$ for almost all $q \in \mathcal{R}^p$. Then we have that

Lemma 22. *$T_{\wedge}(s)$=0 for all most all $q \in \mathcal{R}^p$ if and only if maximal number of input-output vertex disjoint paths is 0.*

One step forward, we can have the following lemma for the local structural disturbance rejection:

Lemma 23. *Local agent (5.1) has structural disturbance rejection capability if and only if maximal number of disturbance-output vertex disjoint paths is 0.*

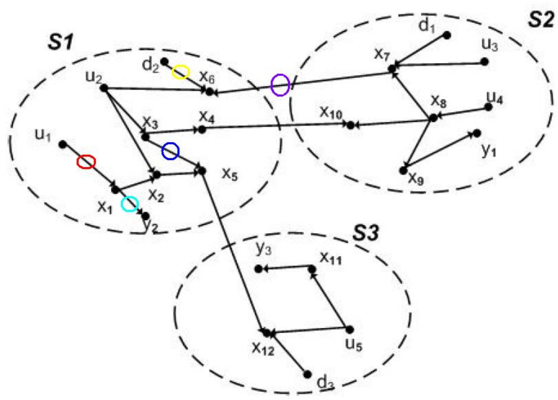

Fig. 5.2: Network and local representation graph

For the whole networked multi-agent systems, we also have the following graphic representation:

Definition 31. *(network representation graph) For the whole network dynamics (5.4), combine all the local graph $\mathcal{G}_i(\mathcal{V}_i, \mathcal{I}_i)$ together and build connection edges among agents. Then we have an additional edge set $\tilde{L}$, which corresponds to the independent parameters of L_{ij} and describes the interaction among agents. Then the network interconnection topology can be described using a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{I})$, where $\mathcal{V} = \{\mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_n\}$ and $\mathcal{I} = \{\mathcal{I}_1 \cup \mathcal{I}_2 \cup \ldots \cup \mathcal{I}_n \cup \tilde{L}\}$. The vertex set $\mathcal{V}$ contains all the vertices in each local graph and the edge set $\mathcal{I}$ contains all the edges in each local graph together with all the linking edges among the local graphs, which corresponds to the independent parameters in L_{ij}.*

A presentation topology graph is shown in Fig. 5.2. Each dash line circled subgraph is the local representation graph of each agent. The line with blue circle, green circle, yellow circle, and purple circle corresponds to one free parameter in $A_\wedge$, $C_\wedge$, $E_\wedge$, and $L_\wedge$ respectively. The line with red circle corresponds to possible external control input in $B_\wedge$ (for later use). In the interconnection topology graph $\mathcal{G}(\mathcal{V}, \mathcal{I})$ of network (5.4), we need to introduce some notations for vertices with special properties:

Definition 32. *Considering the network topology $\mathcal{G}(\mathcal{V}, \mathcal{I})$, we define the each state vertex of x_j with the corresponding element in L_{ij} nonzero as a virtual controller vertex for the agent system i and the corresponding state vertex of x_i, which is the head of edge starting from the virtual controller, is called virtual injector. In each local graph $\mathcal{G}_i(\mathcal{V}_i, \mathcal{I}_i)$, we define a state vertex of x_i as detectable if there is a directed path starting from this vertex and ending at one of the output vertices. Besides, we define a state vertex of x_i as disturbed if there is a directed path starting from one of the disturbance vertices and ending at this vertex.*

With all the above definitions and lemmas, we are ready to present the following result for the structural disturbance rejection capability of whole networked multi-agent systems:

Theorem 24. *The structured dynamic network of equation (5.4) has structural disturbance rejection capability if and only if*

C1) Each individual agent has locally structural disturbance rejection capability;

C2) For each directed path starting from a disturbed virtual controller, all the virtual injectors along this directed path are not detectable.

Proof. Since both L_{ii} and A_i stand for self-influence of states x_i, having $L_{ii} = 0$ avoids existence of duplicated parameters in $A_\wedge$ and $L_\wedge$. Consequently the independence of system matrices $A_\wedge$, $L_\wedge$, $E_\wedge$ and $C_\wedge$ is guaranteed. By Lemma 22, the whole networked multi-agent system (5.4) has structural disturbance rejection capability if and only if the maximal number of disturbance-output vertex disjoint paths is 0. One one hand, if C_1 does not hold, which means for some local graph, the maximal number of disturbance-output vertex disjoint paths is bigger than 0, this implies that in the whole network interconnection topology, the maximal number of disturbance-output vertex disjoint paths is bigger than 0. If C_2 does not hold, which means that for some virtual injector is reachable from a disturbed virtual controller, it is then concluded that the the maximal number of disturbance-output vertex disjoint paths is bigger than 0. On the other hand, if C_1 and C_2 holds, there is no possibility for a disturbance vertex to have a directed path to its own local graph's output vertices due to C_1 and no possibility for it to have a directed path to any output vertices in other local graphs since such a path among local graphs requires at least a disturbed virtual controller and a detectable virtual injector. Then the result follows. □

Remark 17. *This result highlights the importance of interconnection topology structures in determining the disturbance rejection capability of structured networked multi-agent systems. More specifically, the disturbance rejection capability of each agent and the interaction among the agents together decide the disturbance rejection capability of whole networked multi-agent systems. Although unavoidably we still need to know the whole topology, once the disturbance rejection capability of each agent is fixed, even when the interconnection topology changes, the only thing we need to do is to check the interaction of the virtual controller and virtual injector, which is only subset of the vertex set of whole multi-agent systems. Hence this result provides us graphic and convenient method for verifying the disturbance rejection capability of networked multi-agent systems. This makes it possible to conduct global network disturbance rejection capability directly from the local disturbance rejection capability of the interconnected agents.*

Checking condition $C2$ amounts to find paths which connect 'special vertices'. This is a standard task of algorithmic graph theory. For example, depth-first search or breadth-first search algorithm for traversing a graph can be adopted and the complexity order is $O(|V|+|E|)$, where $|V|$ and $|E|$ are cardinalities of vertex set and edge set in topology graph.

5.3.2 Single Integrator Case

In above general linear dynamics case, interactions among agents can not stand for the whole topology of multi-agent system since interactions exist among states of each local agent. Actually in multi-agent literature, individual agent is usually modeled as single integrator (mass point) dynamics, which can be treated as a special case of the above case here. This kind of modeling will bring convenience to reveal how the interactions among the agents determine the structural disturbance rejection capability of whole multi-agent system without considering the influence of local dynamics.

Specifically, considering a group of n agents, with dynamics of each agent being single integrator subject to external disturbance:

$$\begin{cases} \dot{x}_i = u_i + E_i\omega_i \\ y_i = C_i x_i \end{cases} \tag{5.7}$$

where x_i is the state, u_i is the control input, ω_i is the scalar external disturbance, y_i is the controlled output of agent i.

The interconnection topology of this networked multi-agent systems can be represented by a graph $\mathcal{G}(\mathcal{V}, \mathcal{I})$, where each element of vertex set $\mathcal{V}$ corresponds to an agent and each element of edge set $\mathcal{V}$ corresponds to an edge between two agents.

A coordination control law named the nearest neighbor control law is given by

$$u_i = -\sum\nolimits_{j\in\mathcal{N}_i} w_{ij}(x_i - x_j) + w_{ii}x_i, \tag{5.8}$$

where $\mathcal{N}_i$ is the neighbor set of the agent i.

Let $x = [x_1, x_2, \ldots, x_n]^T$, $y = [y_1, y_2, \ldots, y_n]^T$, $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T$, $C = [C_1, C_2, \ldots, C_n]^T$

and $E = diag(E_1, E_2, \ldots, E_n)$. Then by stacking the dynamics (5.7) of each agent, we can obtain that dynamics of whole networked multi-agent systems is:

$$\begin{cases} \dot{x} = \mathcal{L}x + E\omega \\ y = Cx \end{cases} \tag{5.9}$$

where $\mathcal{L}$ is the Laplacian matrix of interaction graph $\mathcal{G}$.

Similarly, we will investigate the disturbance rejection capability of above system in a structural sense. Thus we will consider the networked multi-agent systems of the type defined by Eq. (5.9) with parameterized entries and denoted by $\sum_\wedge$ as follows:

$$\begin{cases} \dot{x} = \mathcal{L}_\wedge x + E_\wedge\omega \\ y = C_\wedge x \end{cases} \tag{5.10}$$

This multi-agent system is called a structured multi-agent system if all the entries of system matrices $\mathcal{L}$ and J are either fixed zeros or independent parameters.

A directed graph $\tilde{\mathcal{G}}(\tilde{\mathcal{V}}, \tilde{\mathcal{I}})$ can be associated with structured system (5.10).

Definition 33. *The representation graph of structured system* (5.10) *is a directed graph $\tilde{\mathcal{G}}(\tilde{\mathcal{V}}, \tilde{\mathcal{I}})$, with vertex set $\tilde{\mathcal{V}} = \mathcal{X} \cup \mathcal{D} \cup \mathcal{Y}$, where $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$, which is called state vertices set, $\mathcal{D} = \{d_1, d_2, \ldots, d_n\}$, which is called disturbance vertices set and $\mathcal{Y} = \{y_1, y_2, \ldots, y_n\}$, which is called output vertices set, and edge set $\tilde{\mathcal{I}} = \mathcal{I}_{XX} \cup \mathcal{I}_{DX} \cup \mathcal{I}_{XY}$, where $\mathcal{I}_{XX} = \{(x_i, x_j) | \mathcal{L}_{\wedge ji} \neq 0\}$, $\mathcal{I}_{DX} = \{(e_i, x_j) | E_{\wedge ji} \neq 0\}$ and $\mathcal{I}_{XY} = \{(x_i, y_j) | C_{\wedge ji} \neq 0\}$ are the oriented edges between states, disturbance and output defined by the interaction matrices $\mathcal{L}_\wedge$, $E_\wedge$ and $C_\wedge$ above.*

Let V_1, V_2 be two nonempty subsets of the vertex set $\tilde{V}$. A simple path P is called a $V_1 - V_2$ path if its start vertex belongs to V_1 and its end vertex belongs to V_2. $V_1 - V_2$ paths are said to be disjoint if they have no common vertex.

Similarly, we are curious to look for a proper interconnection topology such that there exist appropriate coordinating law control $u_i = -\sum_{j\in\mathcal{N}_i} w_{ij}(x_i - x_j) + w_{ii}x_i$, such that the transfer matrix from disturbance to output is equal to zero, which means that the whole multi-agent system performance y is not affected by the external disturbance ω.

Specifically, for the networked multi-agent system (5.10), we have the following definition for the disturbance rejection problem:

Definition 34. *If under certain interconnection topology, there exists appropriate coordinating law control* $u_i = -\sum_{j\in\mathcal{N}_i} w_{ij}(x_i - x_j) + w_{ii}x_i$, $i \in \{1, 2, \ldots, n\}$, *such that the transfer matrix from disturbance to output is equal to zero, multi-agent system (5.10) is then said to have disturbance rejection capability.*

Then, we have the following result for the disturbance rejection problem for multi-agent system (5.10):

Theorem 25. *Multi-agent system (5.10) has structural disturbance rejection capability if and only if graphically, in the interconnection topology graph* $\tilde{\mathcal{G}}(\tilde{\mathcal{V}}, \tilde{\mathcal{I}})$, *maximal number of disturbance-output vertex disjoint paths is 0.*

Proof. For the closed loop dynamics of multi-agent system (5.10), we treat the disturbance as input, then the transfer function now is $C(sI - \mathcal{L})^{-1}E = 0$. Besides, by definition of coordination control law u, the independence of diagonal elements is guaranteed due to the existence of independent parameter w_{ii}. Then from Lemma (21) and (22), the result follows. $\square$

Remark 18. *Compared with the non-homogeneous general linear dynamics case, graphic* *interpretation of disturbance rejection capability of the single integrator case is much intuitive. This comes from the fact we mentioned at the beginning of this part that for single integrator case, the interactions among agents are actually the interactions among all the states of the whole network system. Consequently, there is no need to consider the impact of local dynamics of agents on the disturbance rejection capability of whole network, which makes the graphic interpretation more succinct.*

5.4 Structurally Controllable Multi-Agent System with Disturbance Rejection Capability

As a combination of obtained disturbance rejection results with other multi-agent control problem, we will consider the the controllability of multi-agent systems with external disturbance.

In control of multi-agent systems, it is desirable to apply proper control commands to agents and exchange information among the agents so that the whole group of agents can be positioned arbitrarily in the space. For example, group of unmanned air vehicles(UAVs) are required to fly in some specific formation in order to reduce the system cost, increase the robustness and efficiency. Following external commands for command center and exchanging position and speed information among all UAVs properly, this group of UAVs reach the required formation and finish the military task. Similar applications can be observed in lots of areas, such as space exploration, congestion control in communication networks, air traffic control and so on. This problem can be extracted as the controllability problem of multi-agent systems [31], during which the external disturbance would unavoidably affect the control process. The controllability of multi-agent system has also been studied in structural sense by our paper in groups [137, 138]. In what follows we will explicitly show under what kind of interconnection topologies, the whole multi-agent system is structurally controllable and meanwhile has structural disturbance rejection capability.

Adding control inputs to each individual agent in (5.1), we adopt the following mathematic description of each agent's dynamics model:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i + E_i \omega_i + \sum_j L_{ij} x_j \\ y_i = C_i x_i \end{cases} \tag{5.11}$$

where u_i are the local control input and B_i represents the local control effectiveness. $x_i, y_i, A_i, E_i, L_{ij}, C_i$ have the same meaning and setting as that in (5.1).

Similarly, with $x = [x_1^T, x_2^T, \ldots, x_n^T]^T$, $y = [y_1^T, y_2^T, \ldots, y_n^T]^T$, $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T$, $C = [C_1^T, C_2^T, \ldots, C_n^T]^T$, $A = diag(A_1, A_2, \ldots, A_n)$, $B = diag(B_1, B_2, \ldots, B_n)$, $L = \{L_{ij}\}$ and $E = diag(E_1, E_2, \ldots, E_n)$, the dynamics of whole multi-agent system can be described as:

$$\begin{cases} \dot{x} = A_\wedge x + L_\wedge x + B_\wedge u + E_\wedge \omega \\ y = C_\wedge x \end{cases} \tag{5.12}$$

here we assume $C_\wedge$ is nonsingular.

The representation graphs $\mathcal{G}_i(\mathcal{V}_i, \mathcal{I}_i)$ ($\mathcal{G}(\mathcal{V}, \mathcal{I})$) can be obtained by adding vertices standing for the control inputs to $\mathcal{V}_i(\mathcal{V})$ and adding edges standing for the parameters in control input matrix B_i to $\mathcal{I}_i(\mathcal{I})$. Similar as in Definition 33, the vertex set $\mathcal{V} = \mathcal{X} \cup \mathcal{U} \cup \mathcal{D} \cup \mathcal{Y}$ now, where $\mathcal{U}$ is *control input vertices set*.

One important graphic definition is needed before we proceed forward:

Definition 35. *(Stem [68]) A stem is an acyclic, directed path in the state vertex set $\mathcal{X}$, that begins in the input vertex set $\mathcal{U}$. The final state variable vertex in the stem is denoted as* *the terminal stem vertex.*

For this multi-agent system, we adopt the following definition of its controllability:

Definition 36. *We say multi-agent system (5.12) is controllable if for any initial state x_0 and final state x_f, there exist a time instance $t_f > 0$ and an input $u : [0, t_f)$ such that $x(0) = x_0$ and $x(t_f) = x_f$.*

In other words, we can drive the whole state of multi-agent system to any desirable state we want through external control input. Then the structural controllability of system (5.12) can be defined as:

Definition 37. *Multi-agent system (5.12), whose matrix elements are zeros or free parameters, is said to be structurally controllable if and only if there exist a set of parameter values that can make the system (5.12) controllable in the classical sense.*

To facilitate the following investigation, the output controllability of multi-agent system (5.12) needs to be introduced first:

Definition 38. *Multi-agent system (5.12) is output controllable if for any initial y_0 and final y_f, there exist a time instance $t_f > 0$ and an input $u : [0, t_f)$ such that $y(0) = y_0$ and $y(t_f) = y_f$.*

Similarly, we can have the definition for structural output controllability as follows:

Definition 39. *We say multi-agent system (5.12) is structurally output controllable if and only if there exist a set of system matrix parameter values that can make the system (5.12) output controllable in the classical sense.*

From (5.11), we can see that the controllability of each individual agent would be influenced by the interaction information from other agents. With the definition of virtual controllers in (32), for each agent i, we will treat these interaction information x_j in (5.11) as virtual control input although x_j may can not be arbitrarily chosen as real control input u_i. A subsequent definition for the local controllability of each agent named virtually structural controllability is adopted here [75]:

Definition 40. *If we treat $v_i = [x_1, x_2, \ldots, x_n]$ as virtual control input, $\hat{u}_i = [u_i, v_i]$ and $\hat{B}_i = [B_i, L_{ij}]$ where $L_{ij} = [L_{i1}, L_{i2}, \ldots, L_{in}]$ with $L_{ii} = 0$ are extended control input and extended control input matrix respectively. We say agent i is virtually structurally controllable if matrix pair $(A_i, \hat{B}_i)$ is structurally controllable.*

Based on this definition, the structural controllability of whole multi-agent systems has been proven to have close relation with local controllability of each agent [75]:

Lemma 26. *Multi-agent system (5.12) is structurally controllable if and only if*

C1) Each individual agent is virtually structurally controllable with respect to some local and virtual controllers and;

C2) Every virtual controller of (i) is connected to an unique terminal stem vertex.

With all the above lemmas and theorem, here we are in the position to reveal under what kind of interconnection topologies, the whole multi-agent system is structurally controllable and meanwhile has structural disturbance rejection capability.

Corollary 3. *Networked multi-agent system (5.12) is structurally controllable and simultaneously has structural disturbance rejection capability if and only if*

C1) Each individual agent is virtually structurally controllable with respect to some local and virtual controllers and;

C2) Each individual agent has locally structural disturbance rejection capability and;

C3) Every virtual controller of C1 is connected to an unique terminal stem vertex and;

C4) For each directed path starting from a disturbed virtual controller, all the virtual injectors along this directed path are not detectable.

Proof. One one hand, since the interactions among agents or 'state feedback' from others agents have been incorporated in matrix $L_\wedge$, the control input u is purely external control signal and will not influence transfer function from disturbance to output. On the other hand, the disturbance rejection capability of multi-agent system (5.12) excludes the influence of external disturbance to system output, which implies we can reduce the output controllability of original multi-agent system to the output controllability of system

$$\begin{cases} \dot{x} = A_\wedge x + L_\wedge x + B_\wedge u \\ \quad\quad y = C_\wedge x \end{cases} \tag{5.13}$$

Under the case the matrix $C_\wedge$ is square nonsingular, the output controllability matrix $[C_\wedge B_\wedge, C_\wedge(A_\wedge + L_\wedge)B_\wedge, \ldots, C_\wedge(A_\wedge + L_\wedge)^{N-1}B_\wedge]$ has the same row rank with that of state controllability matrix $[B_\wedge, (A_\wedge + L_\wedge)B_\wedge, \ldots, (A_\wedge + L_\wedge)^{N-1}B_\wedge]$, which implies the equivalence between these two controllability. Consequently, the theorem follows from combing the conditions under structural controllability and structural disturbance rejection capability. □

We can treat this as a 'separation rule' of controllability and disturbance rejection under this networked multi-agent systems. We note that in addition to using these results to evaluate the structural disturbance rejection of a multi-agent system it is also possible to apply these results to multi-agent cooperative control applications, such as controllability and optimal interconnection topologies design of networked multi-agent systems.

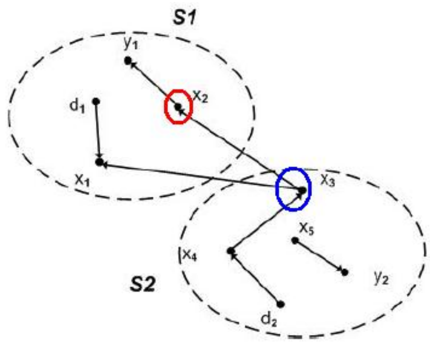

Fig. 5.3: Networked multi-agent system with two agents

5.5 Numerical Examples

The following simple examples are presented as illustrations of the obtained results.

For the first multi-agent system depicted in Fig. 5.3, we have totally two agents: S_1 and S_2 in the network. From our earlier results it is possible to study the structural disturbance rejection capability of whole networked multi-agent systems from the local disturbance rejection capability of each agent and the interaction among the agents. In Fig. 5.3 the digraphs of the two systems as well as the two proposed interactions ($x_3 \to x_1, x_3 \to x_2$) between the systems are displayed. Easily we can see that for each dynamic system S_1 and S_2, they both has structural disturbance rejection since maximal number of disturbance-output vertex disjoint paths is 0. However, we have one virtual controller x_3 (circled by blue line), two virtual injectors x_1 and x_2 (circled by red line), where x_3 is disturbed by d_2 and x_2 is detectable. Consequently, C_2 in Theorem 24 does not hold and the networked multi-agent system does not structurally have disturbance rejection capability.

Consider another multi-agent system consisting of three agents S_1 , S_2 and S_3 in Fig. 5.4. There are three interactions ($x_5 \to x_{12}, x_4 \to x_{10}$ *and* $x_7 \to x_6$) between the systems. With these three interactions, conditions of corollary 3 are satisfied and the individual cacti form a cactus cover for all the state variables in the network, thus ensuring that the network

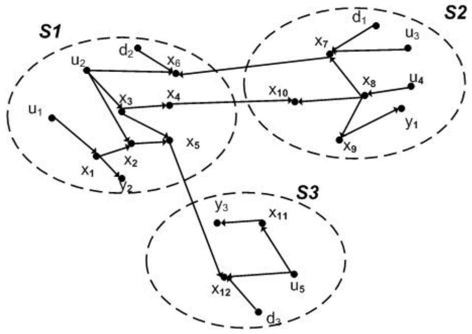

Fig. 5.4: Networked multi-agent systems with three agents

is structurally controllable. Besides, easily we can see that for each dynamic system, they all have structural disturbance rejection capability since maximal number of disturbance-output vertex disjoint paths is 0. Furthermore, we have three virtual controllers x_5, x_5 and x_7, three virtual injectors x_6 and x_{10} and x_{12}. We can see that for each directed path starting from a disturbed virtual controller, all the virtual injectors along this directed path are not detectable. Consequently, the networked multi-agent system is structurally controllable and has structural disturbance rejection capability. It should be noted that besides using these results to evaluate the disturbance rejection capability of a dynamical network, these results can also possibly be used to design systems and interconnection topologies that ensure the disturbance rejection capability of a dynamical network.

5.6 Conclusions and Future Work

In this chapter, the structural disturbance rejection capability of networked multi-agent system under arbitrary topologies is studied. Based on known results in the literature of linear structured system and graph theory, graphic necessary and sufficient conditions for the structural disturbance rejection capability of networked multi-agent systems were derived. The graphic characterizations show a clear relationship between the disturbance rejection capability of whole network and that of each agent and interconnection topologies. This gives us a convenient way to design a network of multi-agent systems which has a desired disturbance rejection capability using only the information of interconnection topologies. Subsequently, we investigate how the disturbance rejection results can be incorporated into the controllability problem of multi-agent systems. This shows possible combination of these results with multi-agent cooperative control problem, such as controllability and optimal interconnection topologies design of networked multi-agent systems and so on.

Some interesting remarks can be made on these results. First, it gives us a clear understanding on how the local disturbance rejection capability and the information exchanges among agents determine the disturbance rejection capability of whole group of agents. Second, the results developed in this chapter is based on the linear feedback law. If considering nonlinear feedback control law, the interactions among agents are different and the whole network of multi-agent systems should have a different interconnection topology. Consequently, the disturbance rejection under nonlinear feedback control laws needs further consideration. We will investigate this question in our future research.

Chapter 6

Conclusions

The profound impact of the interconnection topology on the collective behavior of whole multi-agent system has motivated us to develop graph theoretic interpretations of multi-agent system properties. Considering system dynamics under external signals, we explicitly described how the underlying interconnection topology affects properties of the overall system in terms of its controllability and disturbance rejection capability. Various graph theoretic interpretations of these two multi-agent properties were devised, which highlight the point that the collective behavior of multi-agent system in its controllability and disturbance rejection capability is purely based on the interconnection topologies among agents.

In particular, a weighted interconnection topology, which quite commonly emerges in large scale complex systems, was adopted as the graphic description of interactions among agents. Subsequently the controllability and disturbance rejection of multi-agent systems were addressed in a new structural sense. This kind of structural sense properties were shown to hold for almost all interaction link weight combinations and are of more practical meaning. The structural controllability and disturbance rejection capability help us to overcome our inherently incomplete knowledge of the link weights and from another angle, bring to light the effects of the interconnection topology on the controllability and disturbance rejection of multi-agent systems without worrying about the influence of weights factor. Besides, two kinds of switching topologies were adopted in the controllability discussion: time-dependent switching topology, modeled as switched linear system and state-dependent switching topology, modeled as piecewise linear system. With the aid of graph theory and geometric methods, controllability of general switched linear system and piecewise linear system were investigated as the predecessor research for the following controllability study of multi-agent system with time-dependent and state-dependent switching topologies.

More specifically, we will summarize and discuss each chapter's work of this dissertation. In addition, we also outline possible future research directions.

In Chapter 2, we have investigated structural controllability for switched linear systems. Two graphic representations of switched linear systems were devised. Subsequently, several graph-theoretic necessary and sufficient conditions for the structural controllability of switched linear systems have been introduced, which reveals the relationship between graphic topologies of switched linear system and its controllability. This brings light that we can design the switching algorithm to make the switched linear system structurally controllable conveniently just having to make sure some properties of the corresponding graph (union or colored union graph) are kept during the switching process.

A further extension of the work in this chapter can be observed from that the parameters in different subsystem models are assumed to be independent. However, in real practice, for example, in a multi-agent system, agent α uses information from agent β with some specific weight sometime and later even though whole interconnection topology changes, agent α still prefers to use information in totally the same way. This implies some free parameters may remain the same among different subsystems switching, i.e., dependence among subsystems. A possible consideration would be that the duplicated parameters should be indexed specially in the topology graph and their impact on system controllability should be a crucial point. Besides, since in real applications, lots of systems naturally have nonlinear dynamics and a time-varying composition relationship of system components, systems usually can be modeled as switched nonlinear systems, which can be treated as a natural extension of switched linear system models. Our result can be treated as basic starting point for exploring the structural controllability of switched nonlinear systems: adopt Lie algebra or transfer function methods to get full characterizations for controllability of switched nonlinear system, then interpret each condition into graphic one and finally combine these conditions together to get graphic interpretations for structural controllability for switched nonlinear system.

Chapter 3 represents a continuation of the work in Chapter 2. Multi-agent system interconnected via a switching weighted topology was modeled as a special class of switched linear system. Graph theoretic interpretations of its structural controllability were derived. It was shown that the multi-agent system is structurally controllable if and only if the union graph $\mathcal{G}$ is connected (single leader) or leader-follower connected (multi-leader). This work gives us a clear understanding on what are the necessary information exchanges among agents to make the group of agents behave in a desirable way.

One interesting research question arises from this scenario. In a real multi-agent system, high communication load will bring data drop or high energy consumption, which may greatly influence whole system's behavior. Based on developed results, this motivates us to reduce communication load by disabling certain links or making them on and off as long as the union graph is connected. Consequently, we will investigate how to find a minimum number of interaction links in our future work. Basically, as a multi-agent system with switching topologies, for each subgraph, we have several parameters for link weights.

The problem can be formulated as an optimized problem with the the total number of link weight parameters as the optimization index under the constraints on the link weight parameters such that the second smallest eigenvalue of union graph Laplacian matrix should be greater that 0 to guarantee the connectedness.

With time-dependent switching topology studied in Chapter 2 and Chapter 3, Chapter 4's work has proceeded to study state-dependent switching topology. As we can see in Section 4.5, multi-agent system could be modeled as a piecewise linear system under some application scenarios. Consequently, in this chapter, we have investigated the null controllability of planar bimodal piecewise linear systems. An explicit and easily verifiable necessary and sufficient condition has been proposed in terms of the system parameters, followed by several necessary or sufficient conditions. A subsequent short survey of existing research on state-dependent multi-agent systems was presented and we also devised modeling multi-agent system with state-dependent topologies into special class of piecewise linear systems.

The modeling of state-dependent multi-agent systems provokes several future steps in order to reveal the impact of state-dependent topology on controllability property:

1. Solve the controllability of piecewise linear system.
2. Take the special structure of multi-agent system matrices into consideration and solve its controllability using algebraic method.
3. Interpret each algebraic condition into graphic expressions.
4. Combine all the graphic interpretations and devise condition on interconnection topologies to guarantee the controllability of multi-agent systems

In Chapter 5, another system-theoretic property of the overall multi-agent system: the structural disturbance rejection capability multi-agent system under arbitrary interconnection topologies was studied. Graphic necessary and sufficient conditions for the structural disturbance rejection capability of multi-agent systems were derived. How to guarantee structural controllability and structural disturbance rejection capability of multi-agent systems simultaneously is also addressed graphically.

Possible further extension based on this chapter's work can be observed: the results developed in this chapter are based on the linear dynamics. Actually, the dynamics of agents and the coupling among the agents can usually be nonlinear. If considering this case, the interactions among agents would be different and the whole network of multi-agent systems should have a different interconnection topology. The first step can be developing algebraic conditions for disturbance rejection problem, using tools like transfer function, and subsequently next step is interpreting these conditions into graphic ones. We will investigate this question in our future research.

Bibliography

[1] J. Lawton and R. Beard, "Synchronized multiple spacecraft rotations," *Automatica*, vol. 38, no. 8, pp. 1359–1364, 2002.

[2] W. Dunbar and R. Murray, "Distributed receding horizon control for multi-vehicle formation stabilization," *Automatica*, vol. 42, no. 4, pp. 549–558, 2006.

[3] P. Bliman and G. Ferrari-Trecate, "Average consensus problems in networks of agents with delayed communications," *Automatica*, vol. 44, no. 8, pp. 1985–1995, 2008.

[4] C. Tomlin, G. Pappas, and S. Sastry, "Conflict resolution for air traffic management: A study in multiagent hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 509–521, 1998.

[5] J. How, E. King, and Y. Kuwata, "Flight demonstrations of cooperative control for uav teams," *AIAA 3rd"Unmanned Unlimite" Technical Conference, Workshop and Exhibit*, pp. 20–23, 2004.

[6] C. Godsil, G. Royle, and C. Godsil, *Algebraic graph theory*. Springer New York, 2001.

[7] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[8] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[9] F. Xiao, L. Wang, J. Chen, and Y. Gao, "Finite-time formation control for multi-agent systems," *Automatica*, vol. 45, no. 11, pp. 2605–2611, 2009.

[10] D. Bertsekas and J. Tsitsiklis, *Parallel and distributed computation*. Old Tappan, NJ (USA); Prentice Hall Inc., 1989.

[11] J. Fax and R. Murray, "Graph laplacians and stabilization of vehicle formations," *World Congress*, vol. 15, no. 1, pp. 88–88, 2002.

[12] D. Watts and S. Strogatz, "Collective dynamics of small-worldnetworks," *nature*, vol. 393, no. 6684, pp. 440–442, 1998.

[13] R. Olfati-Saber and R. Murray, "Distributed cooperative control of multiple vehicle formations using structural potential functions," *IFAC World Congress*, pp. 346–352, 2002.

[14] J. Reif and H. Wang, "Social potential fields: A distributed behavioral control for autonomous robots," *Robotics and Autonomous Systems*, vol. 27, no. 3, pp. 171–194, 1999.

[15] T. Laue and T. Röfer, "A behavior architecture for autonomous mobile robots based on potential fields," *RoboCup 2004: Robot Soccer World Cup VIII*, pp. 122–133, 2005.

[16] S. Loizou, H. Tanner, V. Kumar, and K. Kyriakopoulos, "Closed loop motion planning and control for mobile robots in uncertain environments," *42nd IEEE Conference on Decision and Control*, vol. 3, pp. 2926–2931, 2003.

[17] S. Loizou and K. Kyriakopoulos, "Multirobot navigation functions i," *Stochastic Hybrid Systems*, pp. 171–207, 2006.

[18] ——, "Multirobot navigation functions ii," *Stochastic Hybrid Systems*, pp. 171–207, 2006.

[19] H. Tanner and A. Kumar, "Towards decentralization of multi-robot navigation functions," *IEEE International Conference on Robotics and Automation*, pp. 4132–4137, 2005.

[20] R. Murray, "Recent research in cooperative control of multivehicle systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 129, p. 571, 2007.

[21] N. Carriero and D. Gelernter, "Linda in context," *Communications of the ACM*, vol. 32, no. 4, pp. 444–458, 1989.

[22] G. Almasi, "Overview of parallel processing," *(IBM Europe Institute, Course on Highly Parallel Processing, Davos, Switzerland, July 23-27, 1984) Parallel Computing,*, vol. 2, pp. 191–203, 1985.

[23] T. Schouwenaars, B. De Moor, E. Feron, and J. How, "Mixed integer programming for multi-vehicle path planning," *European Control Conference*, pp. 2603–2608, 2001.

[24] C. Reinl and O. von Stryk, "Optimal control of multi-vehicle-systems under communication constraints using mixed-integer linear programming," *1st international conference on Robot communication and coordination*, p. 3, 2007.

[25] Z. Cai and Z. Peng, "Cooperative coevolutionary adaptive genetic algorithm in path planning of cooperative multi-mobile robot systems," *Journal of Intelligent & Robotic Systems*, vol. 33, no. 1, pp. 61–71, 2002.

[26] E. Semsar-Kazerooni and K. Khorasani, "Multi-agent team cooperation: A game theory approach," *Automatica*, vol. 45, no. 10, pp. 2205–2213, 2009.

[27] M. Kloetzer and C. Belta, "Temporal logic planning and control of robotic swarms by hierarchical abstractions," *IEEE Transactions on Robotics*, vol. 23, no. 2, pp. 320–330, 2007.

[28] C. Cheah, S. Hou, and J. Slotine, "Region-based shape control for a swarm of robots," *Automatica*, vol. 45, no. 10, pp. 2406–2411, 2009.

[29] J. Choi, S. Oh, and R. Horowitz, "Distributed learning and cooperative control for multi-agent systems," *Automatica*, vol. 45, no. 12, pp. 2802–2814, 2009.

[30] M. Ji and M. Egerstedt, "Observability and estimation in distributed sensor networks," *46th IEEE Conference on Decision and Control*, pp. 4221–4226, 2007.

[31] H. Tanner, "On the controllability of nearest neighbor interconnections," *43rd IEEE Conference on Decision and Control*, vol. 3, pp. 2467–2472, 2004.

[32] M. Ji, A. Muhammad, and M. Egerstedt, "Leader-based multi-agent coordination: Controllability and optimal control," *American Control Conference*, pp. 1358–1363, 2006.

[33] M. Ji and M. Egersted, "A graph-theoretic characterization of controllability for multi-agent systems," *American Control Conference*, pp. 4588–4593, 2007.

[34] A. Rahmani and M. Mesbahi, "On the controlled agreement problem," *American Control Conferenc*, pp. 1376–1381, 2006.

[35] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 162–186, 2009.

[36] Z. Ji, H. Lin, and T. Lee, "A graph theory based characterization of controllability for multi-agent systems with fixed topology," *47th IEEE Conference on Decision and Control*, pp. 5262–5267, 2008.

[37] ——, "Controllability of multi-agent systems with switching topology," *IEEE Conference on Robotics, Automation and Mechatronics*, pp. 421–426, 2008.

[38] Z. Ji, Z. Wang, H. Lin, and Z. Wang, "Interconnection topologies for multi-agent coordination under leader-follower framework," *Automatica*, vol. 45, no. 12, pp. 2857–2863, 2009.

[39] B. Liu, G. Xie, T. Chu, and L. Wang, "Controllability of interconnected systems via switching networks with a leader," *IEEE International Conference on Systems, Man and Cybernetics*, vol. 5, pp. 3912–3916, 2006.

[40] B. Liu, T. Chu, L. Wang, and G. Xie, "Controllability of a leader–follower dynamic network with switching topology," *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1009–1013, 2008.

[41] S. Martini, M. Egerstedt, and A. Bicchi, "Controllability analysis of multi-agent systems using relaxed equitable partitions," *International Journal of Systems, Control and Communications*, vol. 2, no. 1, pp. 100–121, 2010.

[42] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.

[43] M. Zamani and H. Lin, “Structural controllability of multi-agent systems,” *American Control Conference*, pp. 5743–5748, 2009.

[44] Y. Wang, “Decision-making for search and classification using multiple autonomous vehicles over large-scale domains,” Ph.D. dissertation, Worcester Polytechnic Institute, 2011.

[45] S. Mulgund and R. Stengel, “Optimal nonlinear estimation for aircraft flight control in wind shear,” *Automatica*, vol. 32, no. 1, pp. 3–13, 1996.

[46] N. Seube, R. Moitie, and G. Leitmann, “Viability analysis of an aircraft flight domain for take-off in a windshear,” *Mathematical and computer modelling*, vol. 36, no. 6, pp. 633–641, 2002.

[47] W. Wang and G. Tang, “Feedback and feedforward optimal control for offshore jacket platforms,” *China Ocean Engineering*, vol. 18, no. 4, pp. 515–526, 2004.

[48] M. Bodson, J. Jensen, and S. Douglas, “Active noise control for periodic disturbances,” *IEEE Transactions on Control Systems Technology*, vol. 9, no. 1, pp. 200–205, 2001.

[49] Y. Matsuzaki, T. Ikeda, and C. Boller, “New technological development of passive and active vibration control: analysis and test,” *Smart materials and structures*, vol. 14, pp. 343–348, 2005.

[50] A. Den Hamer, G. Angelis, and N. Roozen, “Broad-band active vibration suppression using ppf focused on industrial application,” *IEEE/ASME Transactions on Mechatronics*, vol. 10, no. 2, pp. 146–153, 2005.

[51] S. Tonetti and R. Murray, "Limits on the network sensitivity function for homogeneous multi-agent systems on a graph," *American Control Conference*, pp. 3217–3222, 2010.

[52] J. Kim, V. Natarajan, S. Kelly, and J. Bentsman, "Disturbance rejection in robust pde-based mrac laws for uncertain heterogeneous multiagent networks under boundary reference," *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 3, pp. 484–495, 2010.

[53] S. Bhattacharyyta, "Disturbance rejection in linear systems," *International Journal of Systems Science*, vol. 5, no. 7, pp. 633–637, 1974.

[54] V. Kucera, "Disturbance rejection: a polynomial approach," *IEEE Transactions on Automatic Control*, vol. 28, no. 4, pp. 508–511, 1983.

[55] C. Desoer and C. Lin, "Tracking and disturbance rejection of mimo nonlinear systems with pi controller," *IEEE Transactions on Automatic Control*, vol. 30, no. 9, pp. 861–867, 1985.

[56] S. Mukhopadhyay and K. Narendra, "Disturbance rejection in nonlinear systems using neural networks," *IEEE Transactions on Neural Networks*, vol. 4, no. 1, pp. 63–72, 1993.

[57] L. Guo and W. Chen, "Disturbance attenuation and rejection for systems with nonlinearity via dobc approach," *International Journal of Robust and Nonlinear Control*, vol. 15, no. 3, pp. 109–125, 2005.

[58] R. Krohling and J. Rey, "Design of optimal disturbance rejection pid controllers using genetic algorithms," *IEEE Transactions on Evolutionary Computation*, vol. 5, no. 1, pp. 78–82, 2001.

[59] Y. Hou, Z. Gao, F. Jiang, and B. Boulter, "Active disturbance rejection control for web tension regulation," *40th IEEE Conference on Decision and Control*, vol. 5, pp. 4974–4979, 2001.

[60] Z. Gao, "Active disturbance rejection control: a paradigm shift in feedback control system design," *American Control Conference*, pp. 2339–2405, 2006.

[61] J. Han, "From pid to active disturbance rejection control," *IEEE transactions on Industrial Electronics*, vol. 56, no. 3, pp. 900–906, 2009.

[62] B. Francis and W. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, no. 5, pp. 457–465, 1976.

[63] R. Marino, G. Santosuosso, and P. Tomei, "Robust adaptive compensation of biased sinusoidal disturbances with unknown frequency," *Automatica*, vol. 39, no. 10, pp. 1755–1761, 2003.

[64] L. Brown and Q. Zhang, "Periodic disturbance cancellation with uncertain frequency," *Automatica*, vol. 40, no. 4, pp. 631–637, 2004.

[65] D. Yue, J. Lam, and Z. Wang, "Persistent disturbance rejection via state feedback for networked control systems," *Chaos, Solitons & Fractals*, vol. 40, no. 1, pp. 382–391, 2009.

[66] Y. Liu, "A targeting approach to disturbance rejection in multi-agent systems," Ph.D. dissertation.

[67] Z. Li, Z. Duan, and L. Huang, "H control of networked multi-agent systems," *Journal of Systems Science and Complexity*, vol. 22, no. 1, pp. 35–48, 2009.

[68] C. Lin, "Structural controllability," *IEEE Transactions on Automatic Control*, vol. 19, no. 3, pp. 201–208, 1974.

[69] R. Shields and J. Pearson, "Structural controllability of multiinput linear systems," *IEEE Transactions on Automatic Control*, vol. 21, no. 2, pp. 203–212, 1976.

[70] K. Glover and L. Silverman, "Characterization of structural controllability," *IEEE Transactions on Automatic Control*, vol. 21, no. 4, pp. 534–537, 1976.

[71] H. Mayeda, "On structural controllability theorem," *IEEE Transactions on Automatic Control*, vol. 26, no. 3, pp. 795–798, 1981.

[72] E. Davison, "Connectability and structural controllability of composite systems," *Automatica*, vol. 13, no. 2, pp. 109–123, 1977.

[73] C. Rech and R. Perret, "About structural controllability of interconnected dynamical systems," *Automatica*, vol. 27, no. 5, pp. 877–881, 1991.

[74] L. Kang, X. Yugeng, and Z. Zhang, "G-cactus and new results on structural controllability of composite systems," *International journal of systems science*, vol. 27, no. 12, pp. 1313–1326, 1996.

[75] L. Blackhall and D. Hill, "On the structural controllability of networks of linear systems," *Estimation and Control of Networked Systems*, pp. 245–250, 2010.

[76] K. Reinschke, "Multivariable control: A graph theoretic approach," 1988.

[77] J. Dion, C. Commault, and J. van der Woude, "Generic properties and control of linear structured systems: a survey," *Automatica*, vol. 39, no. 7, pp. 1125–1144, 2003.

[78] N. Suda, B. Wan, and I. Ueno, "The orders of infinite zeros of structured systems," *Transactions of the Society of Instrument Control Engineers*, vol. 25, pp. 346–348, 1989.

[79] J. Van der Woude, “On the structure at infinity of a structured system,” *Linear algebra and its applications*, vol. 148, pp. 145–169, 1991.

[80] E. Davison and S. Wang, “Properties and calculation of transmission zeros of linear multivariable systems,” *Automatica*, vol. 10, no. 6, pp. 643–658, 1974.

[81] V. Hovelaque, C. Commault, and J. Dion, “Zeros of structured linear systems,” *Automatica*, vol. 35, no. 10, pp. 1683–1688, 1999.

[82] J. van der Woude, “The generic number of invariant zeros of a structured linear system,” *SIAM Journal on Control and Optimization*, vol. 38, pp. 1–21, 1999.

[83] C. Commault, J. Dion, and A. Perez, “Disturbance rejection for structured systems,” *IEEE Transactions on Automatic Control*, vol. 36, no. 7, pp. 884–887, 1991.

[84] C. Commault, J. Dion, and M. Benahcene, “Output feedback disturbance decoupling graph interpretation for structured systems,” *Automatica*, vol. 29, no. 6, pp. 1463–1472, 1993.

[85] J. Van Der Woude, “Disturbance decoupling by measurement feedback for structured transfer matrix systems,” *Automatica*, vol. 32, no. 3, pp. 357–363, 1996.

[86] C. Commault, J. Dion, and T. Do, “The disturbance rejection by measurement feedback problem revisited,” *American Control Conference*, pp. 309–314, 2010.

[87] ——, “Sensor location and classification for disturbance rejection by measurement feedback,” *Automatica*, vol. 47, no. 12, pp. 2584–2594, 2011.

[88] A. Linnemann, “Decoupling of structured systems,” *Systems & Control Letters*, vol. 1, no. 2, pp. 79–86, 1981.

[89] J. Dion and C. Commault, "Feedback decoupling of structured systems," *IEEE Transactions on Automatic Control*, vol. 38, no. 7, pp. 1132–1135, 1993.

[90] A. Linnemann, "Fixed modes in parametrized systems," *International Journal of Control*, vol. 38, no. 2, pp. 319–335, 1983.

[91] Y. Hayakawa and D. Siljak, "On almost invariant subspaces of structural systems and decentralized control," *IEEE Transactions on Automatic Control*, vol. 33, no. 10, pp. 931–939, 1988.

[92] A. Leonessa, W. Haddad, and V. Chellaboina, "Nonlinear system stabilization via hierarchical switching control," *IEEE Transactions on Automatic Control*, vol. 46, no. 1, pp. 17–28, 2001.

[93] K. Narendra and J. Balakrishnan, "Adaptive control using multiple models," *IEEE Transactions on Automatic Control*, vol. 42, no. 2, pp. 171–187, 1997.

[94] H. Lin and P. Antsaklis, "Switching stabilizability for continuous-time uncertain switched linear systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 4, pp. 633–646, 2007.

[95] ——, "Stability and stabilizability of switched linear systems: a survey of recent results," *IEEE Transactions on Automatic control*, vol. 54, no. 2, pp. 308–322, 2009.

[96] K. Loparo, J. Aslanis, and O. Hajek, "Analysis of switched linear systems in the plane, part 2: global behavior of trajectories, controllability and attainability," *Journal of optimization theory and applications*, vol. 52, no. 3, pp. 395–427, 1987.

[97] J. Ezzine and A. Haddad, "Controllability and observability of hybrid systems," *International Journal of Control*, vol. 49, no. 6, pp. 2045–2055, 1989.

[98] F. Szigeti, "A differential-algebraic condition for controllability and observability of time varying linear systems," *31st IEEE Conference on Decision and Control*, pp. 3088–3090, 1992.

[99] Z. Sun, S. Ge, and T. Lee, "Controllability and reachability criteria for switched linear systems," *Automatica*, vol. 38, no. 5, pp. 775–786, 2002.

[100] S. Shuzhi and Z. Sun, *Switched linear systems: Control and design*. Springer Verlag, 2005.

[101] G. Xie and L. Wang, "Controllability and stabilizability of switched linear-systems," *Systems & Control Letters*, vol. 48, no. 2, pp. 135–155, 2003.

[102] Y. Qiao and D. Cheng, "On partitioned controllability of switched linear systems," *Automatica*, vol. 45, no. 1, pp. 225–229, 2009.

[103] J. Van der Woude, "A graph-theoretic characterization for the rank of the transfer matrix of a structured system," *Mathematics of Control, Signals, and Systems (MCSS)*, vol. 4, no. 1, pp. 33–40, 1991.

[104] K. Murota, *Systems analysis by graphs and matroids: structural solvability and controllability*. Springer-Verlag, 1987.

[105] T. Kailath, *Linear systems*. Prentice-Hall Englewood Cliffs, NJ, 1980.

[106] S. Micali and V. Vazirani, "An o (v— v— c— e—) algoithm for finding maximum matching in general graphs," *21st Annual Symposium on Foundations of Computer Science*, pp. 17–27, 1980.

[107] I. Masar, "The blimp project, prt," *University of Hagen*, 2007.

[108] W. De Koning, "Digital optimal reduced-order control of pulse-width-modulated switched linear systems," *Automatica*, vol. 39, no. 11, pp. 1997–2003, 2003.

[109] J. Dumon, P. Bellemain, S. Nicolas, N. Maciol, F. Martinez, and J. Caquas, "Masim, feednetback project," *INRIA, France*, 2011.

[110] A. Rantzer and M. Johansson, "Piecewise linear quadratic optimal control," *IEEE Transactions on Automatic Control*, vol. 45, no. 4, pp. 629–637, 2000.

[111] M. Johansson, *Piecewise linear control systems*. Springer Verlag, 2003, vol. 284.

[112] E. Sontag, "Nonlinear regulation: The piecewise linear approach," *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 346–358, 1981.

[113] H. Khalil and J. Grizzle, *Nonlinear systems*. Macmillan Publishing Company New York, 1992.

[114] W. Heemels, B. De Schutter, and A. Bemporad, "Equivalence of hybrid dynamical models," *Automatica*, vol. 37, no. 7, pp. 1085–1091, 2001.

[115] J. Imura and A. Van Der Schaft, "Characterization of well-posedness of piecewise-linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 9, pp. 1600–1619, 2000.

[116] J. Imura, "Well-posedness analysis of switch-driven piecewise affine systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 11, pp. 1926–1935, 2003.

[117] ——, "Optimal control of sampled-data piecewise affine systems," *Automatica*, vol. 40, no. 4, pp. 661–669, 2004.

[118] T. Hu and Z. Lin, "A complete stability analysis of planar linear systems under saturation," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 47, no. 4, pp. 498–512, 2000.

[119] ——, "A complete stability analysis of planar discrete-time linear systems under saturation," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 48, no. 6, pp. 710–725, 2001.

[120] G. Feng, "Stability analysis of piecewise discrete-time linear systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1108–1112, 2002.

[121] G. Ferrari-Trecate, F. Cuzzola, D. Mignone, and M. Morari, "Analysis of discrete-time piecewise affine and hybrid systems," *Automatica*, vol. 38, no. 12, pp. 2139–2146, 2002.

[122] A. Bemporad, G. Ferrari-Trecate, and M. Morari, "Observability and controllability of piecewise affine and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 10, pp. 1864–1876, 2000.

[123] V. Blondel and J. Tsitsiklis, "Complexity of stability and controllability of elementary hybrid systems," *Automatica*, vol. 35, pp. 479–490, 1999.

[124] V. Veliov and M. Krastanov, "Controllability of piecewise linear systems," *Systems & control letters*, vol. 7, no. 5, pp. 335–341, 1986.

[125] M. Camlibel, W. Heemels, and J. Schumacher, "Stability and controllability of planar bimodal linear complementarity systems," *42nd IEEE Conference on Decision and Control*, vol. 2, pp. 1651–1656, 2003.

[126] ——, "On the controllability of bimodal piecewise linear systems," *Hybrid Systems: Computation and Control*, pp. 571–614, 2004.

[127] W. Heemels and B. Brogliato, "The complementarity class of hybrid dynamical systems," *European journal of control*, vol. 9, no. 2-3, pp. 322–360, 2003.

[128] M. Camlibel, W. Heemels, and J. Schumacher, "Algebraic necessary and sufficient conditions for the controllability of conewise linear systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 3, pp. 762–774, 2008.

[129] A. Arapostathis and M. Broucke, "Stability and controllability of planar, conewise linear systems," *Systems & control letters*, vol. 56, no. 2, pp. 150–158, 2007.

[130] G. Xie, L. Wang, B. Xun, and L. Zhao, "Null controllability of planar piecewise linear systems," *IEEE International Conference on Systems, Man and Cybernetics*, vol. 1, pp. 154–159, 2003.

[131] J. Xu and L. Xie, "Null controllability of discrete-time planar bimodal piecewise linear systems," *International Journal of Control*, vol. 78, no. 18, pp. 1486–1496, 2005.

[132] I. Lobel, A. Ozdaglar, and D. Feijer, "Distributed multi-agent optimization with state-dependent communication," *Mathematical Programming*, pp. 1–30, 2010.

[133] V. Blondel, J. Hendrickx, and J. Tsitsiklis, "On krause's multi-agent consensus model with state-dependent connectivity," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2586–2597, 2009.

[134] M. Mesbahi, "On state-dependent dynamic graphs and their controllability properties," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 387–392, 2005.

[135] M. Zhu and S. Martinez, "Attack-resilient distributed formation control via online adaptation," *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, pp. 6624–6629, 2011.

[136] C. Commault, J. Dion, and V. Hovelaque, "A geometric approach for structured systems: application to disturbance decoupling," *Automatica*, vol. 33, no. 3, pp. 403–409, 1997.

[137] X. Liu, H. Lin, and B. Chen, "A graph-theoretic characterization of structural controllability for multi-agent system with switching topology," *48th IEEE Conference on Decision and Control*, pp. 7012–7017, 2009.

[138] S. Jafari, A. Ajorlou, and A. Aghdam, "Leader localization in multi-agent systems subject to failure: A graph-theoretic approach," *Automatica*, vol. 47, no. 8, pp. 1744–1750, 2011.

The contents of this dissertation are based on the following papers that have been published, accepted, or submitted to peer-reviewed journals and conferences.

Journal Papers:

1. X. Liu, H. Lin, and B. M. Chen,"Structural controllability of switched linear systems," Accepted provisionally by Automatica, 2012.
2. X. Liu, H. Lin, and B. M. Chen,"Graph-theoretic characterizations of structural controllability for multi-agent system with switching topology," To appear in International Journal of Control, 2013.
3. X. Liu, H. Lin, and B. M. Chen,"Null controllability of planar bimodal piecewise linear systems," International Journal of Control, vol. 88, no. 4, pp. 766-782, 2011.

Conference Papers:

1. X. Liu, H. Lin, and B. M. Chen,“Graphic interpretations of structural controllability for switched linear systems,” In Proc. of the 11th International Conference on Control, Automation, Robotics and Vision, Singapore Dec. 7-10, 2010.

2. X. Liu, H. Lin, and B. M. Chen,“A graph-theoretic characterization of structural controllability for multi-agent system with switching topology,” In Proc. of the 48th IEEE Conference on Decision and Control, Shanghai, Dec. 16-18, 2009.