BUNDLING RETAILING UNDER STOCHASTIC MARKET

WANG QIANG

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WANG QIANG

(B.Eng., Shanghai Jiao Tong University of China)

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DECLARATION

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

WANG QIANG

17 Aug, 2012

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Summary

This research studies bundling retailing problem under stochastic market. Compared to the conventional bundling study, the incorporation of inventory issue becomes significant when demands are uncertain. First, a two-stage stochastic programming model involving both pricing and ordering decisions has been built for two-product mixed bundling strategy under stochastic market. Reservation price model is adopted to demonstrate the relationship between prices and demands. Two different policies, namely non-sharing policy and sharing policy are proposed and compared. In the latter model, concavity in order quantities has been proved. The algorithm with Downhill Simplex Method has been proposed to search for the prices. Considerable numerical analysis was carried out to examine the effects of relevant factors, such as cost structure and demand variation, on the performance of mixed bundling. These results can serve as guidelines for practitioners who face particular market conditions. Second, based on the first work, many assumptions are relaxed and more realistic conditions are considered, including Multinomial Model Logit Model for customer choice behavior, joint reservation price distribution and different types of product pairs. Sample Average Approximation with IPA (Infinitesimal Perturbation Analysis) gradient estimator is used to solve this model, with extensive numerical studies for various parameters. The results provide more managerial insights regarding several important factors like correlation coefficient between the reservation prices for the individual products and degree of contingency. Third, a special type of bundling is analyzed using dynamic pricing. We defined the bundling consisting of an advertising component and a main component in terms of

pricing effect. Closed-form results have been obtained and comparisons with some heuristics have been conducted.

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NOMENCLATIVE

MNL: Multinomial Logit Model

SAA: Sample Average Approximation

IPA: Infinitesimal Perturbation Analysis

Chapter 1 Introduction

1.1 Background

Bundling, first introduced into the field of Economics by Stigler (1963), has been widely studied especially in the literature of marketing and economics. Known as selling several products as one combined product, bundling was originally invented as a marketing strategy to extract more consumer surplus, followed by copious works in delineating the rationales of bundling, providing great guidelines for real business when bundling strategy can be used. An excellent review can be found in Stremersch and Tellis (2002).

Bundling has become a pervasive business phenomenon in a broad range of industries in mainly two forms: pure bundling and mixed bundling. Pure bundling only permits sales of bundles, while mixed bundling allows selling separate products and bundles at the same time. Tremendous bundling cases can be found in business practices: computer options (integrated computer or just components like hard disk) in high-tech industry; car options (whether add some extra services like decoration based on preliminary purchase) in auto mobile industry; consumable goods (whether offer packages of shampoo and conditioner) in retailing industry; season tickets, traveling packages, food menus, subscription of multiple programs in service industry; information goods like software and music in on-line selling industry, etc. In some sense, quantity discount sales can also be considered as a special case of mixed bundling where the components are same.

Bundling can be further divided into two classes: price bundling and product bundling, with regard to the bundling process. The distinction between the two classes was vague until first clearly addressed by Stremersch and Tellis (2002). Price bundling is more like a marketing practice by selling several products together without any physical integration, which means no separate inventory is needed for bundles; while product bundling involves physical integration among the individual products to form bundles, which usually requires additional manufacturing process, adding values by the bundling process. Separate inventory should be kept for the bundle in product bundling so as to fulfill the order from the customers because the bundling process takes time. Examples like bundling selling of computer options can be viewed as product bundling. Eppen et al. (1991) state managers should consider bundles in product bundling as new products, highlighting its strategic meaning for companies.

Many advantages of bundling have been revealed under various situations. Nalebuff (2003) presents a good summary about motivations for bundling. Generally, companies find more opportunities to promote new products and bring great convenience by one-stop shopping which increases customer service level by implementing bundling. In the demand side, bundling is a tool of price discrimination, more efficiently helping capture heterogeneity of customers, thus increasing sales and total profitability. In addition, it may also incur cost savings and improve quality when bundling several products together, especially when bundling process facilitates production and brings additional integrated functions that separate products lack. More specifically, monopolists can use bundling to preserve or expand their market power, setting barriers for entry; while in present of competition, sellers can achieve competitive advantages via bundling.

When exploring the suitability of bundling, the most common factors under investigation include reservation prices correlation, heterogeneity of customer valuations, relationship between separate products (complementary or substitutable), customer reservation price distributions, etc. Through these factors, plentiful useful conclusions like conditions under which bundling is more profitable and which form of bundling is better are elicited, e.g. negatively correlated reservation prices make bundling more efficient (Stigler 1963, Adams and Yellen 1976), and mixed bundling usually dominates pure bundling without cost savings (Schmalensee 1984). With a correct recognition of practical situations, these findings can be served as guidelines when managers consider implementing bundling strategy.

1.2 Motivation of the study

We notice that most works in the literature of bundling are conducted from the perspective of sellers who directly sell products to the customers, and the issue of how to design the bundling strategy, including bundle form and pricing for all alternatives, often is the focus. The direction of introducing competition by studying multiple sellers is also explored to some extent (Carbajo et al., 1990; Martin 1999; etc.). However, the work of studying bundling within a supply chain just begins to draw researchers' attention. As well known, optimization solely within the marketing department or the operational department will lead to overall sub-optimality (MacDonald and Rasmussen, 2010). Joint analysis of marketing and other issues like inventory decisions is of great interest of market players. In the bundling literature, inventory decision is out of consideration because the market size is usually assumed to be fixed. So in our research, we intend to study mixed bundling retailing for perishable products which usually face uncertain market. The market size of a product can be measured by the population of the customers who have the potential to buy it. For perishable products, the market size is generally uncertain which is hard to predict because of their short life circles. The bundling here refers to price bundling. In addition, when bundling involves no cost savings, mixed bundling can be viewed as a mixture of unbundling and pure bundling in some sense, outperforming either unbundling or pure bundling. Thus, we choose to focus on mixed bundling. The main objective is to design the optimal mixed bundling strategy for the retailer, including pricing and ordering decisions. This work is under a newsevendor sentting. Besides, we also conduct a preliminary study on dynamic pricing for bundling strategy.

The existing literature lacks study of bundling from supply chain perspective, The most related paper to our work is McCardle et al. (2007), which considers pricing and ordering issues simultaneously only for unbundling and pure bundling strategies. In essence, it is a single product problem. Our mixed bundling model is much more complicated as there are multiple alternatives that need to be analyzed jointly. Besides, the market size in their work is assumed uniformly distributed, while our model considers a more general situation. Another related paper is Ernst and Kouvelis (1999). They optimize stock levels for two individual products and packaged product with a certain substitution pattern among them. It is a product bundling problem as separate inventory needs to be ordered for the packaged product. But the joint demand function for the products is given, pricing issue not being considered. In contrast, Bulut et al. (2005) use numerical method to determine optimal prices in a two-product mixed bundling given initial inventories of individual products.

1.3 Objectives and scope

Bundling has been widely studied under various settings from the perspective of pricing because it helps to effectively charge prices to different types of customers so as to gain more profit. By noticing the importance of the issue of inventory due to high market uncertainty and enhanced requirement for supply chain coordination for the retailers who need to make decisions on inventory, some researchers began to consider the element of inventory in the context of bundling, which has been shown to bring significant advantages when inventory is not an issue. The problem of joint pricing and inventory decisions under mixed bundling has not been formally explored, and it could be really complex because of its specific multi-product structure. It is worthy to carry out this study to fill the gap.

The objective of this work is to jointly determine prices and inventories for the three alternatives under mixed bundling strategy by constructing a mathematical model and thereafter to find solutions using numerical methods. In addition, we also intend to study dynamic pricing for bundling under certain assumptions. Detailed aims included:

- To determine the inventory decisions under mixed bundling, i.e., ordering decision before the selling season starts and allocation decision after demands are realized
- To incorporate pricing as decision variables to jointly tackle pricing and inventory

- To gain comprehensive management insights through numerical experiments, examining factors like demand correlation between reservation prices, degree of contingency between the products, etc.
- To conduct comparisons between mixed bundling strategy and no bundling strategy.
- To investigate the effects of dynamic pricing on bundling compared to the conventional fixed price algorithms.

The results of this thesis should demonstrate the rationale of mixed bundling under stochastic market and provide optimal solutions for the model. It should be able to give managerial implications as guidelines for practitioners who may adopt bundling strategy in uncertain environment. All the analysis in this study can be further extended to other more complicated circumstances, e.g. duopoly and oligopoly.

The research on this topic is limited to monopoly situation, where only one retailer is under consideration. It would be interesting to investigate application of game theory on both pricing and inventory under mixed bundling, but it is beyond scope of this thesis.

1.4 Research results and managerial insights

1) Mixed bundling under stochastic market is still outperforming unbundling, attributed by pricing effect and inventory pooling effect. The relative magnitude of the two effects depends on the parameters like cost structure and coefficient variation of the market size distribution. By considering both the pricing and inventory factors, the performance gap is almost doubled in terms of total expected profit compared to the case only considering the inventory factor under most circumstances in the numerical experiments.

2) To be practical, we consider joint reservation price distribution, discrete choice of Multinomial Logit Model and different degree of contingency between the individual products in Chapter 4. The factors have significant impact on the performance of mixed bundling strategy. Therefore, it is important to study these factors profoundly before adopting bundling strategies.

3) Dynamic bundling pricing can further improve the profitability over static pricing. For bundling practices in industries like restaurants, more expected profit can be achieved if the bundling strategy and dynamic pricing technique can be combined into consideration. Inventory level and lapsed time decide the bundling pricing.

1.5 Contributions

This research makes following major contributions:

1) Theoretically, it is the first work to jointly study the pricing and inventory problem for bundling under stochastic market. Based on the reservation price consumer choice model, a two-stage stochastic model is built and extensively analyzed. We derive meaningful managerial insights by conducting various numerical experiments, providing useful guidance of practical implementation.

2) The work is further extended by considering more realistic problems, i.e., using the Multinomial Logit Model instead of the reservation price model for consumer choice behavior, considering joint distribution for the reservation prices of the individual products, assuming different degree of contingency between the individual products for each customer. Due to the complexity of the extended model, simulation optimization technique (Sample Average Approximation with

IPA gradient estimation) is employed. What's more, we exploit the model structure to enable the use of IPA method.

3) We also develop a model for dynamic pricing of a bundling practice and derive closed-form results under several different demand functions.

1.6 Thesis structure

This thesis contains six chapters. Chapter 2 gives a comprehensive literature review about this study, including bundling, component commonality and dynamic pricing. Bundling literature is further classified into product bundling and price bundling, with the latter as the main focus.

Chapter 3 presents the basic mathematic model for the joint pricing and inventory problem of mixed bundling under stochastic market. In this base model, we assume the customers would not choose secondary option if their favorite is not available. The reservation price model is used to describe the customer purchasing behavior. Regarding the inventory part, two policies (non-sharing policy and sharing policy) are examined and compared. An algorithm with Downhill Simplex Method is proposed to numerically solve the model. At the end, we discuss the impact on the model of inventory decisions when considering substitutions for customer behavior.

In chapter 4, a more comprehensive study is conducted based on the model in chapter 3. Several major extensions are considered. First of all, instead of the reservation price model, we use the more realistic Multinomial Logit Model to model the customer choice behavior. Secondly, a joint distribution function is adopted for the reservation price distribution for the individual products. Thirdly, we consider the full range of degree of contingency for the products, namely substitutable, independent and complementary products. Due to the added complexity, we turn to simulation method for numerical results, i.e., Sample Average Approximation with IPA gradient Approximation.

Chapter 5 studies the dynamic pricing problem for a special type of bundling with one product as main component and the other product as advertising component. Customer arrival is assumed to follow Poisson distribution, of which the parameter is affected by the price of the advertising component. After the customers arrive, the purchasing decision is determined by the price of and the reservation price for the main component. Three different demand functions are considered, i.e. linear demand function, power function and sigmoid function.

Chapter 6 summarizes the work of this thesis and discusses several directions for further research.

Chapter 2 Literature Review

2.1 Bundling

The literature about bundling is relatively rich in fields of marketing and economics where bundling is treated as a price discrimination device to extract more consumer surplus or win advantage over competitors.

2.1.1 Product bundling

Research on product bundling has not been formally conducted, though similar results may be expected as that of price bundling, especially in demand side: the way of customers choosing products. Porter (1985) mentioned some qualitative savings via product bundling, e.g. manufacturing set-up cost. Eppen et al. (1991) stated that bundles under product bundling should be treated as new products instead of a marketing tool only. Under mixed bundling strategy with fixed price structure, Ernst and Kouvelis (1999) built a model to theoretically determine and numerically search for the optimal stocking levels for two individual products and packaged product while substitution exists between them. However, they do not explicitly show how customers make purchasing decisions between these products, simply assuming a joint demand distribution for them and suggesting a fraction of unmet demand can be substituted by the other products (no substitution between the two separate products).When intermediaries exist between a monopolist and customers, Gal-Or (2004) examines profitability of product bundling by negotiations between the monopolist and its intermediaries.

2.2.2 Price bundling

Price bundling attracts major attention in the literature, which can be mainly classified into six categories as follows:

The main stream is to examine profitability of bundling for a monopolist. Through numerical examples, Adams and Yellen (1976) indicated that mixed bundling was optimal for two independently valued products especially when their reservation prices are negatively correlated. Still based on additivity assumption, Schmalansee (1984) finds bundling benefits even if reservation prices are positively correlated when using joint normal distribution as demand function. There is no dominating strategy between unbundling and pure bundling, the comparison depending on parameters like unit profit and standard deviation of demand. But mixed bundling combines advantages of both. The finding was further generalized to general distribution function by McAfee, McMillan and Whinston (1989). When products are complementary or substitutable, each form of bundling could be optimal as degree of contingency and other parameters vary (Lewbel 1985, Venkatesh and Kamakura 2003). McAfee et al. (1989) examine bundling in a two-product model under general reservation price distribution. Through graphical illustration, Salinger (1995) investigates effects of cost saving (manufacturing-related cost) on performance of bundling by comparing bundle demand and aggregate demand of separate products. Based on the model in Schmalansee (1984), Olderog and Skiera (2000) provide a comprehensive sensitivity analysis for three bundling strategies by simulation. While maximal expected profit is the common objective in most studies, Scott and Highfill (2001) add market share as another objective in the context of bundling.

Considerable studies about bundling in market where there are competitors have been carried out. Carbajo et al. (1990) explore incentives to bundle in imperfect competition. Some researchers advocate for bundling or tying as a leveraging tool that can expand market power into another market or block entry of potential entrants (Matutes and Regibeau 1992, Martin 1999, Choi 2003, Peitz 2006, Spector 2006), though some opposite views exist (Seidmann, 1991). A basic form is to examine possibility and profitability of bundling between competitors in duopoly market, where an equilibrium bundling strategy should exist, though no unanimous conclusions are presented (Economides 1993, Anderson and Leruth 1993, Kopalle et al. 1999). Shy (1996) showed that firms prefer to tie products together under oligopoly to differentiate themselves and the resulting Bertrand competition can increase profits through tying. Following Shy's work, Chen (1997) showed that at least one firm would prefer to tie their products and both firms would earn positive profits while social welfare is reduced. Vanboug (2005) further extended the discussion to tying with two bundles and showed that in equilibrium one firm would choose to pure tie the products while others would practice mixed tying. Extended issues like bundling in oligopoly or of asymmetric players and profit sharing between competitors who bundle their products together have also been addressed (Gans and King 2006, Ginsburgh and Zang 2007, Ghosh and Balachander 2007).

Another category is about design of bundle, finding optimal bundle prices. Hanson and Martin (1990), a cornerstone work, formulate a mixed integer linear program to construct optimal bundles among a number of components and search the according optimal bundle prices. Considering two criteria, available time and reservation price, in customer decision, Venkatesh and Mahajan (1993) propose a probabilistic approach to optimally price performance tickets in different bundling strategies. Instead of pursuing maximal profit, Ansari et al. (1996) study pricing of bundle for the sake of maximized usage from view of nonprofit organizations. A real case (pay TV) can be found in Crampes and Hollander (2004).

With the development of information industry, bundling has become pervasive in the selling of information goods, an obvious characteristic of which differing from other normal commodities is its low marginal cost. Bakos and Brynjolfsson (1999) state well prediction of customer evaluation as advantage of bundling large number of information goods and that pure bundling is usually optimal when inventory is not a constraint. They restudy the issue in the environment of competition in another paper (Bakos and Brynjolfsson, 2000). Specifically, Geng et al. (2005) examine optimal bundling strategies of information goods whose value decrease with time. Like bundle design in traditional industries, similar analysis is also conducted for information goods, e.g. Hitt and Chen (2005) where a customized bundling strategy proved better than unbundling and pure bundling under some conditions is proposed.

Some authors focus on explicitly demonstrating how customers measure the bundle consisting of several components. Simonin and Ruth (1995) indicate that component brands have significant influence on customers' reservation price for the bundle via a qualitative experiment-alike method when introducing a new product in the bundle. A detailed discussion on bundle valuation can be found in Fishburn and Pekec (2002). Johnson et al. (1999) carry out a study to show that how price and discount information should be presented to customers for the bundle. Based on utility theory, Jedidi et al

(2003) build and test a model for joint distribution of reservation prices for components and bundles, capturing heterogeneity in the distribution.

Last but not least, recently some researcher start to consider supply chain related issues in price bundling, e.g. product inventory, no longer solely based on marketing or sales view. This is reasonable and necessary because any option (bundling) taken at downstream of a supply chain (between sellers and customers) usually would influence performance and decisions at upstream (say sellers and suppliers). For the sake of global utility and supply chain coordination, it is worthwhile to study bundling from a broader view along the supply chain. Given initial inventories for individual products, Bulut et al. (2008) investigate price bundling strategies by modeling customer arrive as a Poisson process for both single period and multi-period cases. For the first time, McCardle et al. (2007) study price bundling (unbundling and pure bundling) for both basic and fashion products in retailing merchandising, determining order quantities and prices simultaneously.

2.2 Literature for joint pricing and newsvendor problem

In the literature of joint pricing and newsvendor problem, multi-product based case has not yet been studied. By assuming demand is price dependent but randomness of demand is price independent, Mills (1959) and Karlin and Carr (1962) design demand function as additive case and multiplicative case respectively. The resulting optimal price has an up bound in the former paper, while inversely optimal price has a low bound in the latter one. Young (1978) propose a general model that combines additive and multiplicative cases, and give some optimality conditions by examining PF_2 distributions as well as log-normal distribution. Petruzzi and Dada (1999) reviews and extends this problem, providing more general optimality conditions that the distribution should has non-decreasing hazard rate. This problem has also been extended to multiple periods (Ernst 1970, Zabel 1972, Thowsen 1975 and Petruzzi and Dada 1999).

2.3 Literature for component commonality

In the component commonality literature, assemble-to-order system attracts much attention. ATO model was first studied based on one common component and single period (Baker et al. 1986, Gerchak and Henig 1986 and Gerchak et al. 1988), and later developed into more than one common component or multiple periods (Eynan and Tosenblatt 1996, Rudi 1998, Hillier 1999a and Cheung 2002). Some researchers also investigate component commonality in assemble-to-stock (AOS) system (Eppen and Schrage 1981, Grotzinger 1993 and Bollapragada et al. 1998). Chew et al. (2006) put forward the issue of component-mismatch and explore this effect under equal-fractile allocation policy.

2.4 Literature for dynamic pricing on multiple products

Dynamic pricing has been extensively studied since its inception for the airline industry, in which it is often called revenue management. Before the departure, the airline has a limited number of seats to sell over a certain period of time. During the selling period, the airline can adjust the airfare depending on the time to departure and available number of seats. On one hand, the airline tends to offer a promotion when there are still a large number of seats unsold while departure time is approaching. On the other hand, the airline has incentives to reserve some seats under anticipation of potential customers who are willing to pay a higher price. In essential, the contradiction of whether to sell a seat is a comparison between instant revenue and expected marginal revenue. This results in dynamic pricing for the airfare, and with this dynamic balancing between the two contradictory considerations the expected revenue over the selling period can be maximized. The general results are that the price decreases in the remaining inventory level and increases in time to departure.

An excellent review can be found in McGill and van Ryzin (1999), Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003), and Chiang et al. (2007). The most basic dynamic pricing problem is for single-product single-period with standard demand assumption like constant Poisson customer arrival. Gallego and van Ryzin (1994) built the foundation for dynamic pricing policies under Poisson demand. Many variations have been developed based on the general dynamic pricing problem. Considering the implementation difficulty for continuously changing price in practice, some researchers restrict the price to a set of discrete values (Chatwin, 2000) and/or only allow price change on a few predefined time points (Feng and Gallego, 1995, 1996). They discussed the finite price changes, markup and markdown (a special case of one opportunity for price change), and timing of price changes. Responding to the dynamic pricing scheme adopted by the retailer, some strategic customers may hold back their purchases for the period when the price is expected to drop. In addition, retailers have motivation to gather demand information in advance before making any decisions. Therefore, reservation systems have been employed to better control customer demand. Resulted problems include overbooking and cancellation, which have been examined to some extent (Rothstein, 1971, 1985; Chatwin, 1996, 1998, 1999). Another variation is regarding the randomness of demand, which consists of customer arrival and customer reservation price. Wen Zhao and Yu-Sheng Zheng (2000) explored the impact of customers' reservation price changes and non-homogeneous customer arrivals on the performance of optimal dynamic pricing policies. The numerical revenue improvement could be very significant. Kyle Y. Lin (2005) further studied the real-time demand learning problem as the demand information becomes more accurate while time goes on. Advertising is one important factor that could affect demand besides price. MacDonald and Rasmussen (2010) incorporated advertising effect into the classic dynamic pricing problem, assuming advertising affecting the customer arrival intensity by power model. They identified the advertising mechanism works in a different way than the pricing mechanism in controlling expected sales and thus net revenue, though their effects are same: pricing changes the probability of purchasing while advertising affects customer arrival. Extending single-product dynamic pricing to multi-products is another important direction (Gallego and van Ryzin, 1997; Bitran and Caldentey, 2003; Maglaras and Meissner, 2006). Talluri and van Ryzin (2004) decomposed multiproduct problem into single product problems and neglected the cross-effect in demands.

Some researcher began to investigate the dynamic pricing in the context of bundling, in one way or another. Guler et al. (2009) attempted to study product bundling in the framework of revenue management. It is a deterministic bundling pricing problem for two perishable products. The bundle and price decisions are made at the beginning of the selling season and kept unchanged over the time horizon. Along with the prices, the number of bundles to be formed at the beginning of selling season is optimized. They also investigated the effect of product bundling cost. MacDonald and Rasmussen (2010) developed a model for dynamic pricing and advertising and used the two mechanisms to control the sales and revenue. Closed form results were derived from a system of ordinary differential equations. Bulut et al. (2009) first attempted to discuss dynamic pricing under bundling strategies. They formulated the multi-period mixed bundling pricing problem using a dynamic programming approach. To solve the model, they exhaustively searched through the whole price space to find the optimal prices for each period. In their numerical study, they fixed the individual product prices and exclusively examined the bundle price.

Chapter 3 Mixed Bundling Retailing under Stochastic Market

We study mixed bundling strategy of two fashion products from a retailer's view in this model. The retailer orders two individual products from his suppliers and sells them to the customers separately or jointly, total in three forms. Therefore, the customers in the market have three alternatives to choose. From the view of the retailer, the three alternatives face the same market, which is uncertain. So his aim is to simultaneously determine selling prices for these three alternatives and order quantities for the two individual products in order to maximize total expected profit.

In the conventional pricing and newsvendor model literature, price and demand are assumed to follow a certain relationship in deterministic case. Noise is added additively or multiplicatively for stochastic case. Petruzzi and Data (1999) give a good review for this problem in single-product situation. To some extent, our problem is an extension to multi-product case with a special structure (mixed bundling). Besides, we also use consumer behavior knowledge to unveil the relationship of price and demand, as presented in the model part. We believe this is a good method to capture the demand information. In other words, we combine the techniques in marketing area and inventory area to solve our multi-product based joint pricing and ordering problem.

In another aspect, the sharing policy makes use of the fact that three alternatives share two components in the mixed bundling, so our research is also related to literature of component commonality. In this field, the main problem under investigation is how to determine optimal stocking level for each component under certain constraints like limited budget and pre-specified service level, in either single period or multiple periods. Our sharing policy can be viewed as a specially structured component commonality problem: three alternatives with two components. It is actually an assemble-to-order model because no separate inventory is needed for the bundle in price bundling. The difference is that the alternative prices are endogenous in our model, and these prices determine products' demands, satisfying certain conditions in mixed bundling as shown later, while prices and demands are often exogenous in component commonality literature.

3.1 Model preliminaries

3.1.1 Problem description

The problem is how to jointly make pricing and inventory decisions for the products that are ordered separately from suppliers but sold to customers under mixed bundling strategy from the perspective of one retailer who monopolizes a regional market. Only two individual products are considered for bundling. This problem is an extension of conventional study on bundling strategy under deterministic market, where prices are only decision variables and the inventory issue is out of consideration.

The market size is uncertain, assumed following a distribution. The reservation price model (Schmalansee, 1984) will still be used to model customer choice as that under deterministic market, except that the resulted demands for the products are stochastic rather than deterministic. When bundling involves no cost savings, mixed bundling can be viewed as a mixture of no bundling and pure bundling in some sense, outperforming either no bundling or pure bundling. Thus, we choose to focus on mixed bundling.

Two types of products are ordered from suppliers and offered to the customers in three alternatives: namely, product 1, product 2 and the bundle.

In the demand side, we utilize reservation price model to demonstrate the relationship between market shares and prices of the three alternatives. While in the inventory side, since each alternative contains either of or both product 1 and product 2, the retailer need to first make ordering decision before the selling season starts and then allocate products to fulfill the demands for these three alternatives

3.1.2 Assumptions

Several preliminary assumptions are made as below:

- The bundle consists of one unit of product 1 and product 2.
- Reservation price for the bundle is the sum of the reservation prices for product 1 and product 2, which means the two products are neither complementary nor substitutable.
- Single period is considered. Salvage value and penalty cost are assumed to be zero for the sake of brevity of formulas. The results of our model can be easily extended if these factors are nonzero.
- Each customer purchases at most one unit of product 1 and product 2.
- A crucial assumption for the model we propose in this study is that the customers in the market would stick to their most preferred alternative, i.e., the alternative which gives them highest consumer surplus. If their favorite is unavailable, they would exit

without buying anything instead of switching to secondary alternative which may be available and generates positive surplus.

3.1.3 Notations

The following notations are used throughout the study.

i --- Alternative index, $i = 1, 2, or\ b$, b stands for the bundle;

R_i --- Customer's reservation price for alternative i, and we have $R_b = R_1 + R_2$ from the assumptions;

M --- Random market size; $f_M(m)$ is pdf of market size distribution, with mean μ_M and $sd\ \sigma_M$.

D_i --- Realized demand for alternative i; Demand vector $\overline{D} = (D_1, D_2, D_b)$; $f_{D_i}(x)$ is pdf of demand distribution of alternative i, with mean μ_{D_i} and $sd\ \sigma_{D_i}$.

c_i --- Unit cost of alternative i; $c_b = c_1 + c_2$

α_i --- Market share of alternative i at a realized market size;

Decision variables:

p_i --- Price of alternative i; Price vector $\overline{p} = (p_1, p_2, p_b)$

Q_i --- Order quantity for product i, $i = \{1, 2\}$; Quantity vector $\overline{Q} = (Q_1, Q_2)$

3.2 The general model for joint pricing and ordering decisions

The decision process is divided into two stages. In the first stage, the retailer determines the prices for the three alternatives under mixed bundling and ordering quantities for the two components at the time when facing uncertain market size. In the second stage, demand for each alternative becomes known after market size is realized. Based on the realized demands and available inventories ordered in the first stage, the retailer makes inventory allocation decision to satisfy customer demands. Assume the retailer is risk neutral, so the objective is to maximize the total expected profit. The objective function is as below:

$$\underset{\overline{p},\overline{Q}}{Max}\,\Pi = E_{\overline{D}}[p_1 q_1(\overline{Q},\overline{D}) + p_2 q_2(\overline{Q},\overline{D}) + p_b q_b(\overline{Q},\overline{D}) - c_1 Q_1 - c_2 Q_2] \qquad 3\text{-}1$$

Subject to the constraint where $\max\{p_1, p_2\} < p_b < p_1 + p_2$.

Where $q_i(\overline{p},\overline{Q})$ is the allocated inventory for demand of alternative i in the second stage, which depends on the ordering decision made in the first stage and demand vector $\overline{D}$. The relationship between demands and prices will be delineated in the next section.

To solve this multivariate problem, we first find the optimal ordering decisions by assuming that the prices are given and fixed, and then incorporate the pricing decision into the optimization problem. Before that, we employ the reservation price model in bundling literature to model demands in mixed bundling under stochastic market.

3.2.1 The reservation price model

Reservation price model is mostly widely used in the literature of bundling, delineating the relationship between market share and price with consumer behavior knowledge. It is a good method especially for multi-product situation. In reservation price model, price is the only factor when customers make purchasing decisions. Four options are available under mixed bundling: purchasing product 1, product 2, the bundle and nothing. The customer will choose only one option which yields largest consumer surplus, which is the difference between reservation price and option price. The market shares are easy to identify, shown in Figure3-1.

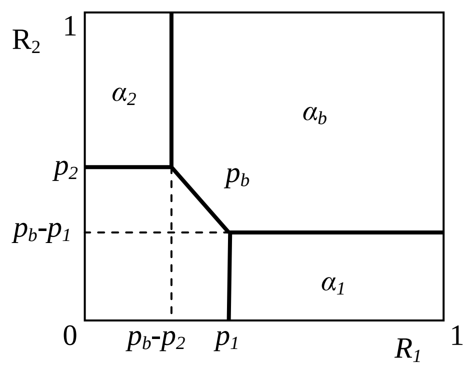


Figure 3-1: Consumer choice under mixed bundling

In this study, without loss of generality, we assume that the reservation prices for the two individual products are independent and follow standard uniform distribution, i.e., $R_i \sim U(0,1)$. The probability density function is $g_{R_i}(r_i)=1$. Thus, market share for each alternative can be derived, unveiling the relationship between alternatives' market shares and prices: $\alpha_1=(1-p_1)(p_b-p_1)$, $\alpha_2=(1-p_2)(p_b-p_2)$ and $\alpha_b=(1+p_1-p_b)(1+p_2-p_b)-\frac{1}{2}(p_1+p_2-p_b)^2$.

The demand for each alternative will depend on its market share α_i, we will use two different types of demand distributions in this study. Firstly, we assume the whole market size is M, which follows a certain distribution, and so the demand for each alternative will be $D_i = M\alpha_i$, depending on the realized market size. In this case, the demands for the three alternatives will be perfectly positively correlated. In the conventional study of bundling under deterministic market, one common market is usually presumed. Therefore, the numerical study will be mainly conducted under this demand type (perfect positive correlation) for a fair comparison with that in deterministic market, so as to discover the bundling performance under stochastic market. For the second type, we assume the demands follow a general multivariate distribution $f_{D_1,D_2,D_b}(x_1, x_2, x_b)$, where mean demand for each alternative is proportionate to its market share. This type is general in the sense that an arbitrary demand correlation matrix could be adopted instead of perfect positive correlation in the first type. That is, the first type is a special case of the second demand type. We separately consider these two demand types because the model we proposed can be applied to general demand situation while more detailed results can be derived under perfect positive demand.

The inventory decisions will be developed based on this general demand model in next subsection.

3.3 Determination of prices and order quantities

We propose two policies to handle this multi-product pricing and ordering problem. One is non-sharing policy, determining order quantity for each alternative separately. This policy is straightforward and easy to understand. The other one is sharing policy, pooling inventories together to fulfill the demands for individual products and the bundle. The underlying idea for the two policies is the same, which is substituting optimal order quantity in terms of prices into the expected profit function, reducing decision variables from five to three, i.e., the three prices.

Given a price vector (p_1, p_2, p_b), each alternative faces a demand distribution with mean and standard deviation proportionate to that of the market size distribution, as their market share solely depends on the price vector. Demands for the three alternatives are perfectly positively correlated once prices are fixed in this mixed bundling model.

$f_{D_i}(x_i) = \frac{1}{\alpha_i} f_M(\frac{x_i}{\alpha_i})$, and $\mu_{Di} = \alpha_i \mu_M$, $\sigma_{Di} = \alpha_i \sigma_M$

3.3.1 Non-sharing policy

The randomness of the market size results in a stochastic demand for each alternative for a possible price vector (p_1, p_2, p_b). We can find the optimal order quantity for each alternative separately as solving a newsvendor problem, and then the optimization problem is only with respect to the prices.

The total expected profit function is:

$$\begin{aligned} E\Pi(Q_i, p_i) &= \sum_i (p_i \min\{Q_i, x_i\} - c_i Q_i) \\ &= \sum_i [p_i (\int_0^{Q_i} x f_{D_i}(x)dx + Q_i \int_{Q_i}^{\infty} f_{D_i}(x)dx) - c_i Q_i] \end{aligned} \qquad 3\text{-}2$$

Let $Q_i = \alpha_i L_i$, then we have

$$E\Pi(p_i, L_i) = \sum_i \alpha_i (p_i \int_0^{L_i} m f_M(m)dm + p_i L_i \int_{L_i}^{\infty} f_M(m)dm - c_i L_i) \qquad 3\text{-}3$$

As in the classic newsvendor model,

$$\frac{\partial E\Pi(p_i, L_i)}{\partial L_i} = \alpha_i (p_i - c_i - p_i F_M(L_i)) \text{ and } \frac{\partial^2 E\Pi(p_i, L_i)}{\partial {L_i}^2} = -\alpha_i p_i f_M(L_i) < 0$$

For a given price vector (p_1, p_2, p_b), $F_M(L_i^*) = \dfrac{p_i - c_i}{p_i}$ and hence $Q_i^* = \alpha_i L_i^*$. So, the optimal order quantities Q_1 and Q_2 for the two components are exclusive: $Q_1 = Q_1^* + Q_b^*$ and $Q_2 = Q_2^* + Q_b^*$.

Substitute the result of L_i^* into (3-3), the optimization problem can be expressed as:

$$\begin{aligned} &\max E\Pi(p_i) = \sum_i \alpha_i p_i \int_0^{L_i^*} m f_M(m) dm \\ &s.t. \quad \max(p_1, p_2) < p_b < p_1 + p_2 \end{aligned} \qquad \text{3-4}$$

To find the optimal price vector (p_1^*, p_2^*, p_b^*), we use multidimensional gradient search method as no closed form of results exists. However, the function above might not be concave. But alternatively we can enumerate all possible price solutions to measure the accuracy of results from the searching method.

3.3.2 The sharing policy

As discussed in previous section, demands for the three alternatives follow the joint demand distribution after prices are decided. At the first stage, the retailer makes the order decision for the two individual products: product 1 and product 2. At the second stage when demands are realized, the retailer decides how to allocate the inventories, based on the existing inventories ordered at the first stage. It is comparable to the component commonality problem of two-stage assembly system, but has its unique features because of the special price structure $\max(p_1, p_2) < p_b < p_1 + p_2$ under mixed bundling.

The two-stage stochastic programming model:

main problem $\quad \underset{\bar{Q}}{Max}\, \Pi' = E_{\bar{D}}[\, p_1 q_1(\bar{Q}, \bar{D}) + p_2 q_2(\bar{Q}, \bar{D}) + p_b q_b(\bar{Q}, \bar{D}) - c_1 Q_1 - c_2 Q_2 \,]$

sub-problem $\quad \underset{\bar{q}}{Max}\, \Pi'' = p_1 q_1 + p_2 q_2 + p_b q_b$

$$
\begin{aligned}
s.t. \quad & q_1 + q_b \le Q_1 \\
& q_2 + q_b \le Q_2 \\
& q_1 \le D_1 \\
& q_2 \le D_2 \\
& q_b \le D_b \\
& q_1, q_2, q_b \ge 0
\end{aligned}
$$

q_i is the fulfilled part of demand for alternative i. They should not exceed the realized demands, and products used in individual and bundle selling are all from the existing inventories, as shown in the constraints. $q_i(\bar{Q}, \bar{D})$ is the solution to the model of second stage, under predetermined order quantities $\bar{Q}$ and one realization of demands $\bar{D}$. The procedure to solve the model is to probe the best allocation rule for a given quantity vector (Q_1, Q_2) by solving the sub-problem first, thereafter substitute the solution back into the main problem to find the optimal order quantities Q_1^* and Q_2^* that maximize the total expected profit over the stochastic demands $\bar{D}$. In a valid mixed bundling strategy, the condition $\max(p_1, p_2) < p_b < p_1 + p_2$ holds, which indicates that selling one unit of bundle is more profitable than only selling one unit of either product 1 or product 2, but less profitable than selling them both separately. Therefore, we can have following best allocation rule:

1) If the inventory of product 1 or product 2 exceeds sum of the demands for itself and the bundle, the maximum profit can be achieved by first fulfilling the demand for the bundle as many as possible, then using the remaining inventories to satisfy the demands for product 1 and product 2.

2) If both inventories of product 1 and product 2 are fewer than sum of the demands for itself and the bundle respectively, the individual product with larger difference between its inventory and demand has the highest priority, followed by the bundle and the other individual product at the last.

Consider the possible demand realizations, as shown in Figure 3-2. Without loss of generality, we assume $p_1 < p_2 < p_b < p_1 + p_2$, from which we have $\alpha_1 > \alpha_2$.

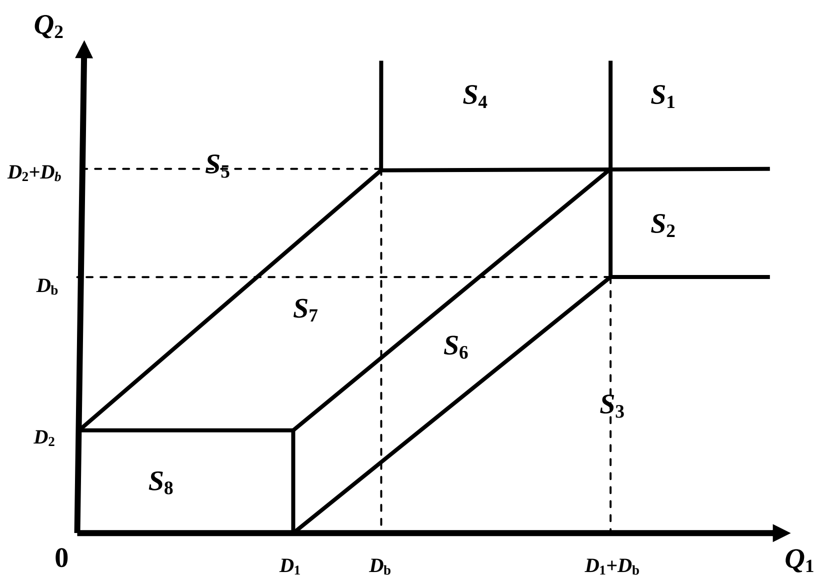


Figure 3-2: Inventory allocation cases

For a certain (Q_1, Q_2), allocation of the inventories has eight cases at most. Case S_1 and case S_8 represent two extreme situations: realized demand $\overline{D}$ is too small such that all demands for the three alternatives can be fulfilled in case S_1, while $\overline{D}$ is too large so that all inventories are allocated to individual product demands in case S_8, where mixed bundling turns out to be no bundling. Case S_2 represents the situation when demands for alternative 1 and alternative *b* are fully satisfied with the remaining inventory of product 2 for the demand of alternative 2. Case S_3 is an extension of case S_2 where all inventory of product 2 is used for the demand of alternative *b*. In case S_6, demand for alternative 1 is first satisfied, followed by the bundle and alternative 2 at last. Case S_4, S_5 and S_7 are the symmetrical situation of case S_2, S_3 and S_6 respectively, as product 1 and product 2 in mixed bundling retailing are structurally symmetric. For each case, the profit function is linear in Q_i, e.g. the according profit in case S_2 is $p_1D_1 + p_2(Q_2 - D_b) + p_bD_b - c_1Q_1 - c_2Q_2$. On the boundaries, adjacent cases give same allocation results. The optimal allocation rule can be summarized in Table 3-1.

Table 3-1: Optimal allocation rule

Cases	q_1	q_2	q_b	Conditions
S_1	D_1	D_2	D_b	$Q_1 \geq D_1+D_b$, $Q_2 \geq D_2+D_b$
S_2	D_1	Q_2-D_b	D_b	$Q_1 \geq D_1+D_b$, $D_b \leq Q_2 \leq D_2+D_b$
S_3	D_1	0	Q_2	$Q_1-Q_2 \geq D_1$, $Q_2 \leq D_b$
S_4	Q_1-D_b	D_2	D_b	$D_b \leq Q_1 \leq D_1+D_b$, $Q_2 \geq D_2+D_b$
S_5	0	D_2	D_b	$Q_1 \leq D_b$, $Q_1-Q_2 \leq -D_2$
S_6	D_1	$Q_2-Q_1+D_1$	Q_1-D_1	$D_1 \leq Q_1 \leq D_1+D_b$, $D_1-D_2 \leq Q_1-Q_2 \leq D_1$

S_7	Q_1-Q_2+D_2	D_2	Q_2-D_2	$D_2 \le Q_2 \le D_2+D_b$, $-D_2 \le Q_1-Q_2 \le D_1-D_2$
S_8	Q_1	Q_2	0	$Q_1 \le D_1$, $Q_2 \le D_2$

The total expected profit function is:

$$
\begin{aligned}
E\Pi(Q_1,Q_2) = &\sum_{i=1}^{8} E\Pi_i(Q_1,Q_2) \\
= &\int_{x_1,x_2,x_b \in S_1} (p_1x_1 + p_2x_2 + p_bx_b) f_{D_1,D_2,D_b}(x_1,x_2,x_b)dx_1dx_2dx_b \\
&+ \int_{x_1,x_2,x_b \in S_2} [p_1x_1 + p_2(Q_2 - x_b) + p_bx_b] f_{D_1,D_2,D_b}(x_1,x_2,x_b)dx_1dx_2dx_b \\
&+ \int_{x_1,x_2,x_b \in S_3} (p_1x_1 + p_bQ_2) f_{D_1,D_2,D_b}(x_1,x_2,x_b)dx_1dx_2dx_b \\
&+ \int_{x_1,x_2,x_b \in S_4} [p_1(Q_1 - x_b) + p_2x_2 + p_bx_b] f_{D_1,D_2,D_b}(x_1,x_2,x_b)dx_1dx_2dx_b \\
&+ \int_{x_1,x_2,x_b \in S_5} (p_2x_2 + p_bQ_1) f_{D_1,D_2,D_b}(x_1,x_2,x_b)dx_1dx_2dx_b \\
&+ \int_{x_1,x_2,x_b \in S_6} [p_1x_1 + p_2(Q_2 - Q_1 + x_1) + p_b(Q_1 - x_1)] f_{D_1,D_2,D_b}(x_1,x_2,x_b)dx_1dx_2dx_b \\
&+ \int_{x_1,x_2,x_b \in S_7} [p_1(Q_1 - Q_2 + x_2) + p_2x_2 + p_b(Q_2 - x_2)] f_{D_1,D_2,D_b}(x_1,x_2,x_b)dx_1dx_2dx_b \\
&+ \int_{x_1,x_2,x_b \in S_8} (p_1Q_1 + p_2Q_2) f_{D_1,D_2,D_b}(x_1,x_2,x_b)dx_1dx_2dx_b - c_1Q_1 - c_2Q_2 \\
= &\sum_{i=1}^{8} \int_{x_1,x_2,x_b \in S_i} (p_1x_1 + p_2x_2 + p_bx_b) f_{D_1,D_2,D_b}(x_1,x_2,x_b)dx_1dx_2dx_b
\end{aligned}
$$

Where x_1, x_2 and x_b is substituted by q_1, q_2 and q_b in Table 3-1 respectively for each case.

Proposition 1 $E\Pi(Q_1,Q_2)$ is concave in (Q_1,Q_2).

Proof: See Appendix A.

Proposition 2 The optimal order quantities satisfy the following two first-order equations in general.

$$\frac{\partial E\Pi(Q_1,Q_2)}{\partial Q_1} = p_1(P(S_4)+P(S_7)+P(S_8)) + p_b P(S_5) + (p_b - p_2)P(S_6) - c_1 = 0 \qquad 3\text{-}5$$

$$\frac{\partial E\Pi(Q_1,Q_2)}{\partial Q_2} = p_2(P(S_2)+P(S_6)+P(S_8)) + p_b P(S_3) + (p_b - p_1)P(S_7) - c_2 = 0 \qquad 3\text{-}6$$

$P(S_i)$ is the probability of case S_i given a certain order decision (Q_1, Q_2), $i = 1, 2, ..., 8$.

Proof is omitted as the partial derivation is straightforward.

3.4 A specific application: when the three alternatives' demands are perfectly positively correlated

Specifically, demands for the three alternatives are of perfect positive correlation when uncertainty is with respect to the common market size M. That is, $D_i = M\alpha_i$. So, D_i follows same distribution as M with parameter relationship: $\mu_{D_i} = \alpha_i \mu_M$ and $\sigma_{D_i} = \alpha_i \sigma_M$. Here, the market size distribution is assumed to be continuous and differentiable. Then we can express the results in terms of market size M.

From Fig. 3-2 we can see that all allocation cases are possible for one order decision under a general joint demand distribution. Now in this specific model, demands for the three alternatives depend on the realized market size and they are perfectly positively correlated. Therefore, the allocation pattern depicted in Figure 3-2 keeps unchanged when the market size varies, except for the magnitude of the demands (D_1, D_2 and D_b). That is, not all eight allocation cases are possible for an order decision (Q_1, Q_2). We need to identify the cases that could happen for a given order decision and their corresponding boundaries. We first express the conditions of each case in Table 3-1 in the form of market size M, as presented in Table 3-2.

Table 3-2: Optimal allocation rule in form of market size M

Cases	Conditions	In form of M
S_1	$Q_1 \geq D_1+D_b$, $Q_2 \geq D_2+D_b$	$M \leq \min\{Q_1/(\alpha_1+\alpha_b), Q_2/(\alpha_2+\alpha_b)\}$
S_2	$Q_1 \geq D_1+D_b$, $D_b \leq Q_2 \leq D_2+D_b$	$Q_2/(\alpha_2+\alpha_b) \leq M \leq \min\{Q_1/(\alpha_1+\alpha_b), Q_2/\alpha_b\}$
S_3	$Q_1-Q_2 \geq D_1$, $Q_2 \leq D_b$	$Q_2/\alpha_b \leq M \leq (Q_1-Q_2)/\alpha_1$
S_4	$D_b \leq Q_1 \leq D_1+D_b$, $Q_2 \geq D_2+D_b$	$Q_1/(\alpha_1+\alpha_b) \leq M \leq \min\{Q_2/(\alpha_2+\alpha_b), Q_1/\alpha_b\}$
S_5	$Q_1 \leq D_b$, $Q_1-Q_2 \leq -D_2$	$Q_1/\alpha_b \leq M \leq (Q_2-Q_1)/\alpha_2$
S_6	$D_1 \leq Q_1 \leq D_1+D_b$, $D_1-D_2 \leq Q_1-Q_2 \leq D_1$	$\max\{Q_1/(\alpha_1+\alpha_b), (Q_1-Q_2)/\alpha_1\} \leq M \leq \min\{Q_1/\alpha_1, (Q_1-Q_2)/(\alpha_1-\alpha_2)\}$
S_7	$D_2 \leq Q_2 \leq D_2+D_b$, $-D_2 \leq Q_1-Q_2 \leq D_1-D_2$	$\max\{Q_2/(\alpha_2+\alpha_b), (Q_2-Q_1)/\alpha_2, (Q_1-Q_2)/(\alpha_1-\alpha_2)\} \leq M \leq Q_2/\alpha_2$
S_8	$Q_1 \leq D_1$, $Q_2 \leq D_2$	$M \geq \max\{Q_1/\alpha_1, Q_2/\alpha_2\}$

The total expected profit function becomes:

$$
\begin{aligned}
E\Pi(Q_1,Q_2) &= \sum_{i=1}^{8} E\Pi_i(Q_1,Q_2) \\
&= \int_{m\in S_1} (p_1 m\alpha_1 + p_2 m\alpha_2 + p_b m\alpha_b)\, f_M(m)dm + \int_{m\in S_2} [p_1 m\alpha_1 + p_2(Q_2 - m\alpha_b) + p_b m\alpha_b]\, f_M(m)dm \\
&+ \int_{m\in S_3} (p_1 m\alpha_1 + p_b Q_2)\, f_M(m)dm + \int_{m\in S_4} [p_1(Q_1 - m\alpha_b) + p_2 m\alpha_2 + p_b m\alpha_b]\, f_M(m)dm \\
&+ \int_{m\in S_5} (p_2 m\alpha_2 + p_b Q_1)\, f_M(m)dm + \int_{m\in S_6} [p_1 m\alpha_1 + p_2(Q_2 - Q_1 + m\alpha_1) + p_b(Q_1 - m\alpha_1)]\, f_M(m)dm \\
&+ \int_{m\in S_7} [p_1(Q_1 - Q_2 + m\alpha_2) + p_2 m\alpha_2 + p_b(Q_2 - m\alpha_2)]\, f_M(m)dm + \int_{m\in S_8} (p_1 Q_1 + p_2 Q_2)\, f_M(m)dm - c_1 Q_1 - c_2 Q_2
\end{aligned}
$$

We can compute the optimal order quantities based on first-order optimality conditions due to the property of concavity derived in the general model. But we can see from Figure 3-1 that different order decisions at the first stage may result in different combination of allocation cases during the second stage. Hence we need to give more indepth analysis of the model.

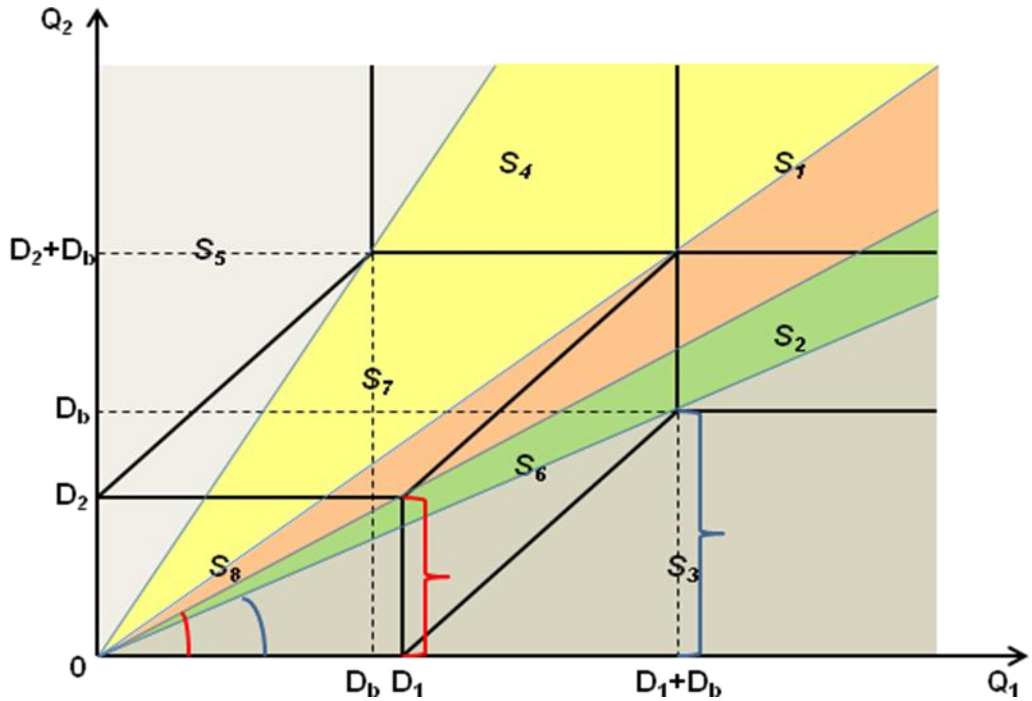


Figure 3-3: Dissected allocation scenarios when $\alpha_1 > \alpha_2$ and $\alpha_2 / \alpha_1 \geq \alpha_b / (\alpha_1 + \alpha_b)$

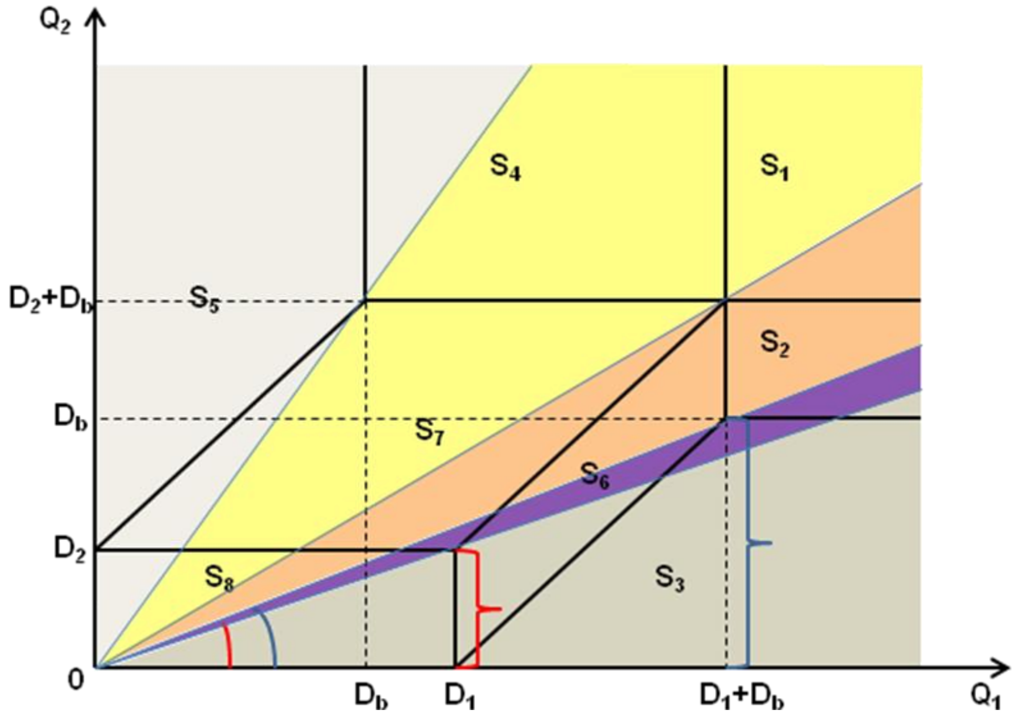


Figure 3-4: Dissected allocation scenarios when $\alpha_1 > \alpha_2$ and $\alpha_2 / \alpha_1 < \alpha_b / (\alpha_1 + \alpha_b)$

Here we only examine the situation when $\alpha_1 > \alpha_2$ due to symmetry. From Figure 3-3 and Figure 3-4 (found in Appendix A), we find that the resulted allocation cases depend on the magnitude of the ratio of order quantities. Table 3-3 enumerates the situation of Figure 3-3, where $\alpha_2 / \alpha_1 \geq \alpha_b / (\alpha_1 + \alpha_b)$.

Table 3-3: Specific allocation scenarios when $\alpha_1 > \alpha_2$ and $\alpha_2 / \alpha_1 \geq \alpha_b / (\alpha_1 + \alpha_b)$

scenario	The range of $t=Q_2/Q_1$	Cases	Boundary for each case
1	$t \leq \alpha_b/(\alpha_1+\alpha_b)$	S_1	$M \leq Q_2/(\alpha_2+\alpha_b)$
		S_2	$Q_2/(\alpha_2+\alpha_b) \leq M \leq Q_2/\alpha_b$
		S_3	$Q_2/\alpha_b \leq M \leq (Q_1-Q_2)/\alpha_1$
		S_6	$(Q_1-Q_2)/\alpha_1 \leq M \leq Q_1/\alpha_1$

		S_8	$M \geq Q_1/\alpha_1$
2	$\alpha_b/(\alpha_1+\alpha_b) \leq t \leq \alpha_2/\alpha_1$	S_1	$M \leq Q_2/(\alpha_2+\alpha_b)$
		S_2	$Q_2/(\alpha_2+\alpha_b) \leq M \leq Q_1/(\alpha_1+\alpha_b)$
		S_6	$Q_1/(\alpha_1+\alpha_b) \leq M \leq Q_1/\alpha_1$
		S_8	$M \geq Q_1/\alpha_1$
3	$\alpha_2/\alpha_1 \leq t \leq (\alpha_2+\alpha_b)/(\alpha_1+\alpha_b)$	S_1	$M \leq Q_2/(\alpha_2+\alpha_b)$
		S_2	$Q_2/(\alpha_2+\alpha_b) \leq M \leq Q_1/(\alpha_1+\alpha_b)$
		S_6	$Q_1/(\alpha_1+\alpha_b) \leq M \leq (Q_1-Q_2)/(\alpha_1-\alpha_2)$
		S_7	$(Q_1-Q_2)/(\alpha_1-\alpha_2) \leq M \leq Q_2/\alpha_2$
		S_8	$M \geq Q_2/\alpha_2$
4	$(\alpha_2+\alpha_b)/(\alpha_1+\alpha_b) \leq t \leq (\alpha_2+\alpha_b)/\alpha_b$	S_1	$M \leq Q_1/(\alpha_1+\alpha_b)$
		S_4	$Q_1/(\alpha_1+\alpha_b) \leq M \leq Q_2/(\alpha_2+\alpha_b)$
		S_7	$Q_2/(\alpha_2+\alpha_b) \leq M \leq Q_2/\alpha_2$
		S_8	$M \geq Q_2/\alpha_2$
5	$t \geq (\alpha_2+\alpha_b)/\alpha_b$	S_1	$M \leq Q_1/(\alpha_1+\alpha_b)$
		S_4	$Q_1/(\alpha_1+\alpha_b) \leq M \leq Q_1/\alpha_b$
		S_5	$Q_1/\alpha_b \leq M \leq (Q_2-Q_1)/\alpha_2$
		S_7	$(Q_2-Q_1)/\alpha_2 \leq M \leq Q_2/\alpha_2$
		S_8	$M \geq Q_2/\alpha_2$

The difference between Figure 3-3 and Figure 3-4 is that scenario 2 is replaced by scenario 6, consisting of case S_1, S_2, S_3, S_6, S_7 and S_8, while the other scenarios are same.

Table 3-4: Scenario 6 in Figure 3-3

Scenario	The range of $t=Q_2/Q_1$	Cases	Boundary for each case
6	$\alpha_2/\alpha_1 \le t \le \alpha_b/(\alpha_1+\alpha_b)$	S_1	$M \le Q_2/(\alpha_2+\alpha_b)$
		S_2	$Q_2/(\alpha_2+\alpha_b) \le M \le Q_2/\alpha_b$
		S_3	$Q_2/\alpha_b \le M \le (Q_1-Q_2)/\alpha_1$
		S_6	$(Q_1-Q_2)/\alpha_1 \le M \le (Q_1-Q_2)/(\alpha_1-\alpha_2)$
		S_7	$(Q_1-Q_2)/(\alpha_1-\alpha_2) \le M \le Q_2/\alpha_2$
		S_8	$M \ge Q_2/\alpha_2$

Similarly, symmetric results can be derived if $\alpha_1 < \alpha_2$. Specifically, resulted scenarios could only be scenario 1, scenario 2, scenario 4 and scenario 5 when $\alpha_1 = \alpha_2$.

For the brevity of presentation, we just give an example of derivation under scenario 1 as a representative, while the other scenarios can be analyzed similarly. Under scenario 1, specifically, the total expected profit function is:

$$
\begin{aligned}
E\Pi(Q_1,Q_2) = & \int_0^{\frac{Q_2}{\alpha_2+\alpha_b}} (p_1 m\alpha_1 + p_2 m\alpha_2 + p_b m\alpha_b) f_M(m)dm + \int_{\frac{Q_2}{\alpha_2+\alpha_b}}^{\frac{Q_2}{\alpha_b}} [p_1 m\alpha_1 + p_2(Q_2 - m\alpha_b) + p_b m\alpha_b] f_M(m)dm \\
& + \int_{\frac{Q_2}{\alpha_b}}^{\frac{Q_1-Q_2}{\alpha_1}} (p_1 m\alpha_1 + p_b Q_2) f_M(m)dm + \int_{\frac{Q_1-Q_2}{\alpha_1}}^{\frac{Q_1}{\alpha_1}} [p_1 m\alpha_1 + p_2(Q_2 - Q_1 + m\alpha_1) + p_b(Q_1 - m\alpha_1)] f_M(m)dm \\
& + \int_{\frac{Q_1}{\alpha_1}}^{\infty} (p_1 Q_1 + p_2 Q_2) f_M(m)dm - c_1 Q_1 - c_2 Q_2
\end{aligned}
$$

As this expected profit function is concave in (Q_1, Q_2), we can derive the following first-order conditions.

$$
\begin{aligned}
\frac{\partial E\Pi(Q_1,Q_2)}{\partial Q_1} &= (p_1\frac{Q_1-Q_2}{\alpha_1}\alpha_1+p_bQ_2)\,f_M(\frac{Q_1-Q_2}{\alpha_1})\frac{1}{\alpha_1}+(p_b-p_2)\int_{\frac{Q_1-Q_2}{\alpha_1}}^{\frac{Q_1}{\alpha_1}} f_M(m)dm \\
&\quad +(p_1Q_1+p_2Q_2)\,f_M(\frac{Q_1}{\alpha_1})\cdot\frac{1}{\alpha_1}-[p_1\frac{Q_1-Q_2}{\alpha_1}\alpha_1+p_2(Q_2-Q_1+\frac{Q_1-Q_2}{\alpha_1}\alpha_1)+p_b(Q_1-\frac{Q_1-Q_2}{\alpha_1}\alpha_1)]\,f_M(\frac{Q_1-Q_2}{\alpha_1})\frac{1}{\alpha_1} \\
&\quad +p_1\int_{\frac{Q_1}{\alpha_1}}^{\infty} f_M(m)dm-(p_1Q_1+p_2Q_2)\,f_M(\frac{Q_1}{\alpha_1})\frac{1}{\alpha_1}-c_1 \\
&=(p_b-p_2)\int_{\frac{Q_1-Q_2}{\alpha_1}}^{\frac{Q_1}{\alpha_1}} f_M(m)dm+p_1\int_{\frac{Q_1}{\alpha_1}}^{\infty} f_M(m)dm-c_1 \\
&=(p_b-p_2)P(S_6)+p_1P(S_8)-c_1 \\
&=0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E\Pi(Q_1,Q_2)}{\partial Q_2} &= (p_1\alpha_1+p_2\alpha_2+p_b\alpha_b)\frac{Q_2}{\alpha_2+\alpha_b}\,f_M(\frac{Q_2}{\alpha_2+\alpha_b})\frac{1}{\alpha_2+\alpha_b}+p_2\int_{\frac{Q_2}{\alpha_2+\alpha_b}}^{\frac{Q_2}{\alpha_b}} f_M(m)dm \\
&\quad +(p_1\frac{Q_2}{\alpha_b}\alpha_1+p_bQ_2)\,f_M(\frac{Q_2}{\alpha_b})\frac{1}{\alpha_b}-(p_1\frac{Q_2}{\alpha_2+\alpha_b}\alpha_1+p_2(Q_2-\frac{Q_2}{\alpha_2+\alpha_b}\alpha_b)+p_b\frac{Q_2}{\alpha_2+\alpha_b}\alpha_b)\,f_M(\frac{Q_2}{\alpha_2+\alpha_b})\frac{1}{\alpha_2+\alpha_b} \\
&\quad +p_b\int_{\frac{Q_2}{\alpha_b}}^{\frac{Q_1-Q_2}{\alpha_1}} f_M(m)dm+(p_1(Q_1-Q_2)+p_bQ_2)\,f_M(\frac{Q_1-Q_2}{\alpha_1})\frac{-1}{\alpha_1}-(p_1\frac{Q_2}{\alpha_b}\alpha_1+p_bQ_2)\,f_M(\frac{Q_2}{\alpha_b})\frac{1}{\alpha_b} \\
&\quad +p_2\int_{\frac{Q_1-Q_2}{\alpha_1}}^{\frac{Q_1}{\alpha_1}} f_M(m)dm-(p_1(Q_1-Q_2)+p_bQ_2)\,f_M(\frac{Q_1-Q_2}{\alpha_1})\frac{-1}{\alpha_1}+p_2\int_{\frac{Q_1}{\alpha_1}}^{\infty} f_M(m)dm-c_2 \\
&= p_2\int_{\frac{Q_2}{\alpha_2+\alpha_b}}^{\frac{Q_2}{\alpha_b}} f_M(m)dm+p_b\int_{\frac{Q_2}{\alpha_b}}^{\frac{Q_1-Q_2}{\alpha_1}} f_M(m)dm+p_2\int_{\frac{Q_1-Q_2}{\alpha_1}}^{\frac{Q_1}{\alpha_1}} f_M(m)dm+p_2\int_{\frac{Q_1}{\alpha_1}}^{\infty} f_M(m)dm-c_2 \\
&= p_2(P(S_2)+P(S_6)+P(S_8))+p_bP(S_3)-c_2 \\
&=0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E\Pi(Q_1,Q_2)}{\partial Q_1} &= (p_b-p_2)\int_{\frac{Q_1-Q_2}{\alpha_1}}^{\frac{Q_1}{\alpha_1}} f_M(m)dm+p_1\int_{\frac{Q_1}{\alpha_1}}^{\infty} f_M(m)dm-c_1=0 \\
\frac{\partial E\Pi(Q_1,Q_2)}{\partial Q_2} &= p_2\int_{\frac{Q_2}{\alpha_2+\alpha_b}}^{\frac{Q_2}{\alpha_b}} f_M(m)dm+p_b\int_{\frac{Q_2}{\alpha_b}}^{\frac{Q_1-Q_2}{\alpha_1}} f_M(m)dm+p_2\int_{\frac{Q_1-Q_2}{\alpha_1}}^{\frac{Q_1}{\alpha_1}} f_M(m)dm+p_2\int_{\frac{Q_1}{\alpha_1}}^{\infty} f_M(m)dm-c_2=0
\end{aligned}
$$

That is $p_2(P(S_2)+P(S_6)+P(S_8))+p_bP(S_3)-c_2=0$ and $(p_b-p_2)P(S_6)+p_1P(S_8)-c_1=0$, a special case of the general version in Proposition 2. Similarly, the corresponding first-order conditions can be obtained under other scenarios.

The solution to the first-order optimality conditions under each scenario is not valid unless it satisfies the according conditions, i.e. Q_2/Q_1 is within the region that causes the scenario to happen. But due to the global concavity of the model proved above, at least one scenario would yield the solution which is truly the global optimal result.

When demands are independent, the optimal order quantities will be calculated from the general first-order optimality conditions because all the eight allocation cases are possible for one order decision.

3.5 Search for best prices

We proved the total expected profit function is concave in order quantities $\overline{Q}$ when prices $\overline{p}$ are fixed. That is to say, the optimal order quantities can be efficiently found for any arbitrage price vector. Therefore, we can reduce the problem dimensions from five to three, i.e., the three prices variables. However, due to the quadratic form of prices in market share expressions α_i, we cannot prove the concavity in prices. Theoretically, there are multiple local maximum points. We propose an algorithm using Downhill Simplex Method (Nelder and Mead, 1965), see below. The algorithm is based on the special case of perfect positive demand correlation. As different initial price vectors may end in different local maximums, we also tried different initial price vectors and chose the best from the obtained solutions. To further verify the optimal solution, we enumerated possible prices values (satisfying the price constraint) with a step size of 0.02. The verification shows that the optimal solution from our algorithm is close to the global best result. In the next section, we will conduct detailed numerical experiments.

The Algorithm

Step 0: Initialization:

0.1 Set k=0, select four initial price vectors p_1, p_2, p_3, p_4 and let $\Theta_k = \{p_1, p_2, p_3, p_4\}$.

Step 1: Compute expected profit:

1.1 for each price vector in set Θ_k, calculate market share α_i, and then determine the optimal ordering quantity Q_1 and Q_2.

1.2 Compute the expected profit respectively for each price vector j, $E\Pi_j, j = 1,2,3,4$.

Step 2: Update price vector, if $(E\Pi_{[4]} - E\Pi_{[1]})^2 < \varepsilon$ (ε is a predefined precision):

2.1 Rank the price vectors in Θ_k in the non-decreasing order of the profit values. Let $p_{[k]}$ denote the kth ranked price vector.

2.2 Let $p_0 = (p_{[2]} + p_{[3]} + p_{[4]})/3$.

2.3 conduct reflection: $p_r = p_0 + \lambda_0(p_0 - p_{[1]})$, and evaluate the expected profit $E\Pi_r$.

2.4 Case 1($E\Pi_r < E\Pi_{[2]}$), conduct contraction: $p_c = p_0 + \lambda_c(p_0 - p_{[1]})$ and compute expected profit $E\Pi_c$, If $E\Pi_c > E\Pi_{[1]}$, $\Theta_k = \{p_c, p_{[2]}, p_{[3]}, p_{[4]}\}$, otherwise conduct reduction $p_i' = p_{[4]} + \lambda_r(p_{[i]} - p_{[4]}), i = 1, 2, 3$, $\Theta_k = \{p'_1, p'_2, p'_3, p_{[4]}\}$.

Case 2 ($E\Pi_{[2]} \le E\Pi_r \le E\Pi_{[4]}$), $\Theta_k = \{p_r, p_{[2]}, p_{[3]}, p_{[4]}\}$.

Case 3 ($E\Pi_r > E\Pi_{[4]}$), conduct expansion: $p_e = p_0 + \lambda_e(p_0 - p_{[1]})$ and compute expected profit $E\Pi_e$, If $E\Pi_e > E\Pi_r$, $\Theta_k = \{p_e, p_{[2]}, p_{[3]}, p_{[4]}\}$, otherwise, $\Theta_k = \{p_r, p_{[2]}, p_{[3]}, p_{[4]}\}$.

Step 3: Set k=k+1, if k<K (maximum iteration number), repeat step 2.

Downhill Simplex Method is an optimization algorithm that requires no derivatives but function values. It generates next point by actions including reflection, expansion, contraction and reduction. The worst point is reflected to a new point (p_r) by the centroid of other points (p_0). If the reflected point is worse than the second worst point (Case 1), a contracted point (p_c) is conducted for the worst point, which leads to all other points being contracted towards the best point if the contracted point is no better than the worst point. If the reflected point is better than the best point (Case 3), an expanded point is conducted to find a better point along this direction. λ_0 , λ_c , λ_r and λ_e are parameters for procedures of reflection, contraction, reduction and expansion respectively, which usually take values of 1, -1/2, 1/2 and 2.

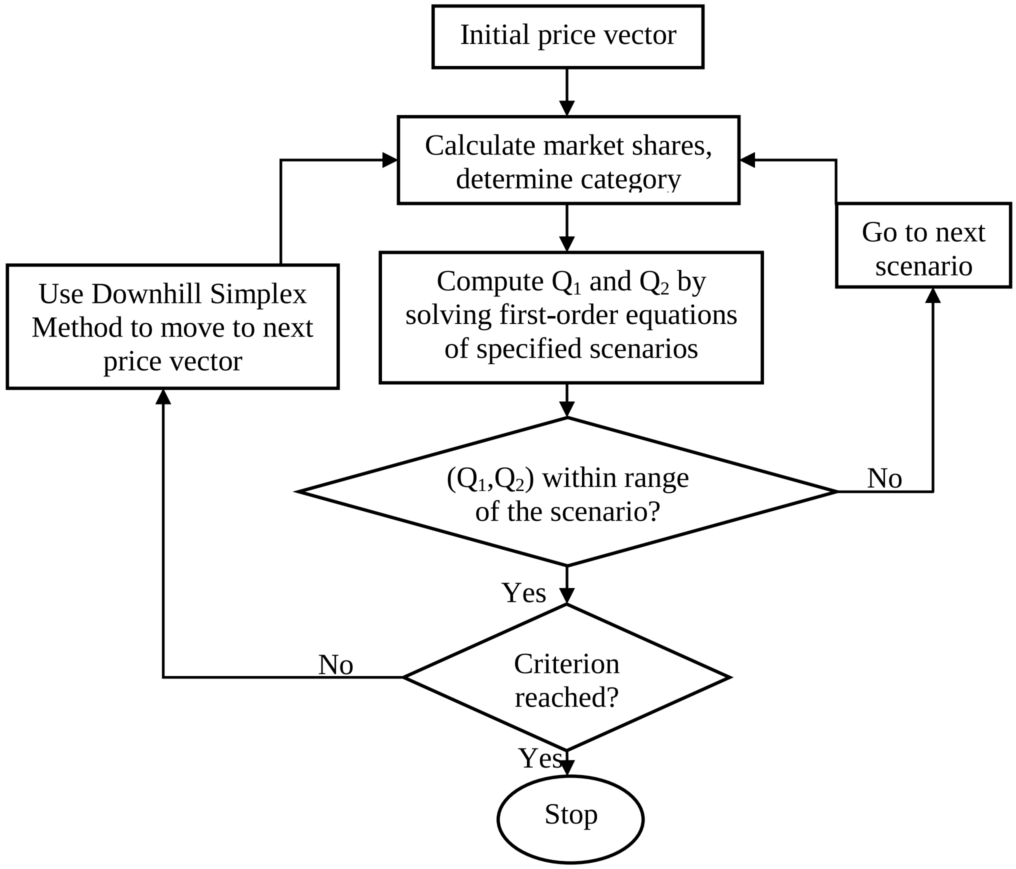


Figure 3-5: Flowchart of the algorithm

3.6 Computational study

Imperfect knowledge about the retailing environment highlights the importance of sensitivity analysis with respect to key parameters, like coefficient of variation of the market size distribution. The mean of the market size is relatively easy to estimate, which can be approximated by the population of the specific group who are potential targeted customers characterized by age, income level and so on, in the region that the retailer covers. But the variance of the market size is hard to estimate, especially for perishable products or new products. The retailer would also be interested to know how the unit costs of the products he orders affect the optimal pricing and inventory decisions because the mixed bundling strategy studied in this model can be applied to any two products which satisfy the assumptions made about them above. In this section, we conduct sensitivity analysis with respect to these two factors, showing their impact on mixed bundling strategy under both non-sharing and sharing policies. In addition, we will compare these two policies in terms of important indexes such as overall profitability, alternatives' prices, etc. Bundle discount, an important index often displayed to the customers in bundle sales, will also be closely tracked in the sensitivity study, and it indicates the attractiveness of the bundle, calculated by formula (3-7).

$$Dis = \frac{p_1 + p_2 - p_b}{p_1 + p_2} * 100\% \qquad 3\text{-}7$$

We introduce the issue of inventory into the conventional mixed bundling strategy by adding uncertainty to demands (market size). Consequently, one natural question is what benefit of this extension is and whether it is worth the extra effort of making market prediction and thereafter taking the inventory issue into picture. The second question of interest is to quantitatively measure the performance of mixed bundling strategy over no bundling strategy and assess the contributions of pooling inventories and price bundling respectively. We will mainly consider two important factors: product cost and coefficient of variation of the market size distribution. All the numerical experiments mentioned above are based on the special case where the demands for the three alternatives are perfectly positively correlated.

In the experiment, we assume the market size follows a normal distribution with mean of 500 and vary the coefficient of variation from 0.1 to 0.4 to represent market uncertainty. The possibility of negative realization of market size is small and thus can be neglected.

For the factor of product cost, we investigate three different cost levels: low (0.1), mid (0.2) and high (0.3), defined as the average of unit cost of the two individual products ($ac = (c_1 + c_2)/2$). Cost structure is represented by cost ratio c_2 / c_1, which varies from 1 to 6 in the numerical experiments.

3.6.1 The base case

The base case is set at $cv = 0.2$ and mid cost level with symmetric cost structure: $M \sim N(500, 100^2)$ and $c_1 = c_2 = 0.2$. Using the algorithm presented in last section, we are able to obtain the optimal prices and order quantities for this two-product mixed bundling problem. Table 3-5 gives the results for the basic case under no bundling strategy and mixed bundling strategy. The second column is the optimal prices for the alternatives in light with the selling strategies. The third and fourth column record the optimal ordering quantity for product 1 and product 2 respectively, while the last two columns report the associated expected profit and total ordering cost.

Comparing their performance shown in table 3-5, mixed bundling generates 6.8% more expected profit at the cost of 9.9% higher ordering capital. The reason can be found in Figure 3-6, which shows that under stochastic market mixed bundling strategy is able to gain more profit through selling more bundles by increasing prices for the individual products and offering a discount for the bundle, though demands for the two individual products would decrease and more customers buy nothing. Increased aggregated sales improve the total expected profit. These results are consistent with that under deterministic market, i.e., more products can be sold because of increased aggregated demand under mixed bundling strategy, regardless of market condition.

Table 3-5: Comparison for the basic case

Strategies	(p_1^*, p_2^*, p_b^*)	Q_1^*	Q_2^*	$profit$	$total\ cost$
No bundling	(0.61, 0.61, -)	210	210	142.7	84
Mixed bundling	(0.69, 0.69, 1.11)	231	231	152.4	92.4

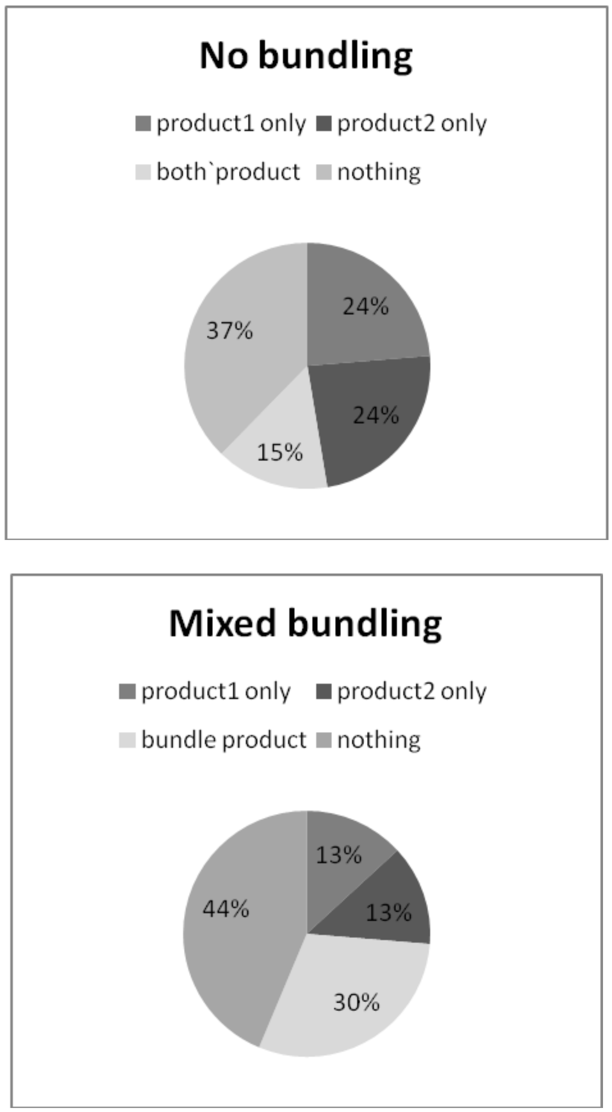


Figure 3-6: Market share for each strategy

3.6.2 The algorithm efficiency study

In the algorithm, the Downhill Simplex method was used for price vector updating without gradient information. In order to verify the results from our algorithm, we exhaustively searched the price space for optimal results by enumeration (0.02 of step size). We compared the results and efficiencies using the base case example. All numerical experiments were conducted using MATLAB 2010b in a 2.67GHz Dell desktop. The comparison results are shown in Table 3-6.

Table 3-6: Algorithms comparison

Algorithms	(p_1^*, p_2^*, p_b^*)	Q_1^*	Q_2^*	*profit*	**CPU Time (sec)**
Downhill Simplex Method	(0.69, 0.69, 1.11)	231	231	152.4	25
Exhaustive Search Method	(0.68, 0.68, 1.12)	232	231	152.5	14562

We can see that the Downhill Simplex Method consumes much less time than the Exhaustive Search Method, and the comparison verified that the results obtained from our algorithm are close to the global optimal.

3.6.3 The prices

In this section, we aim to discover some insightful results with respect to optimal prices for both individual products and the bundle under mixed bundling strategy and also conduct comparison to optimal prices under no bundling. Table 3-7 summarizes the simulation results for optimal prices under different cost ratios and different coefficients of variation of market size distribution based on mid cost level. Dis is used to measure the attractiveness of the bundle to individual products under mixed bundling, defined as $Dis = (p_1^* + p_2^* - p_b^*)/(p_1^* + p_2^*) \times 100\%$. Δp_i^* is the price increase for individual product of mixed bundling in comparison to no bundling. The numerical results show that smaller discount of the bundle should be offered as the market becomes more uncertain. With respect to cost ratio, highest discount can be achieved when cost structure is symmetrical while low limit of discount approached when one individual product's cost reduces to near zero. As for the individual price increase when comparing mixed bundling with no bundling, it decreases in both coefficient of variation of market size and cost ratio. These findings can be served as guidelines in price issues for practical implementation of mixed bundling strategy under stochastic market.

Table 3-7: Optimal prices under stochastic market

c_2 / c_1	cv	No bundling	Mixed bundling	Dis	Δp_1^*	Δp_2^*
1	0.1	(0.61, 0.61)	(0.68, 0.68, 1.10)	20.0%	12.3%	12.3%
	0.2	(0.61, 0.61)	(0.69, 0.69, 1.11)	19.4%	11.9%	11.9%
	0.3	(0.62, 0.62)	(0.69, 0.69, 1.12)	18.8%	11.1%	11.1%
	0.4	(0.63, 0.63)	(0.70, 0.70, 1.14)	18.3%	10.5%	10.5%
4	0.1	(0.55, 0.67)	(0.60, 0.76, 1.10)	19.3%	10.8%	13.5%
	0.2	(0.55, 0.68)	(0.60, 0.76, 1.12)	18.2%	9.8%	12.6%
	0.3	(0.55, 0.68)	(0.60, 0.77, 1.13)	17.0%	8.7%	11.8%
	0.4	(0.56, 0.69)	(0.60, 0.77, 1.15)	16.0%	7.7%	11.1%

3.6.4 Significance of consideration of inventory issue

When the market is deterministic, only prices are considered as decision variables since the quantities of the individual products can be implicitly calculated based on the deterministic market size and their market shares derived from the obtained optimal prices. However, when the market becomes stochastic, it is necessary to jointly determine prices and order quantities in order to achieve the maximal expected profit. It would be of great interest to retail managers to investigate how much benefit that correctly predicting potential market can bring when the market involves uncertainty. To conduct such investigation, we measure the expected profit loss for various conditions, which is the difference between the result from our joint optimization model and the expected profit when optimal prices and according order quantities obtained under deterministic market are used in the corresponding stochastic market.

Figure 3-7 shows the profit loss for both symmetric ($c_2 / c_1 = 1$) and highly asymmetric ($c_2 / c_1 = 4$) cost structure at mid cost level and different cv. We can clearly see from the illustration that the expected profit loss enlarges at higher market uncertainty and higher cost ratio. This is because it is implicitly assumed that cost structure is symmetric and market cv is zero under deterministic market. At the point where $cv = 0.4$ and $c_2 / c_1 = 4$, if the retail manager does not see the latent uncertainty in market but instead treat the market as deterministic, the expected profit loss could be as large as 5%. This amount of profit usually will induce retail managers to make decision in favor of investing in market prediction effort and incorporating inventory issue into implementation of mixed bundling strategy.

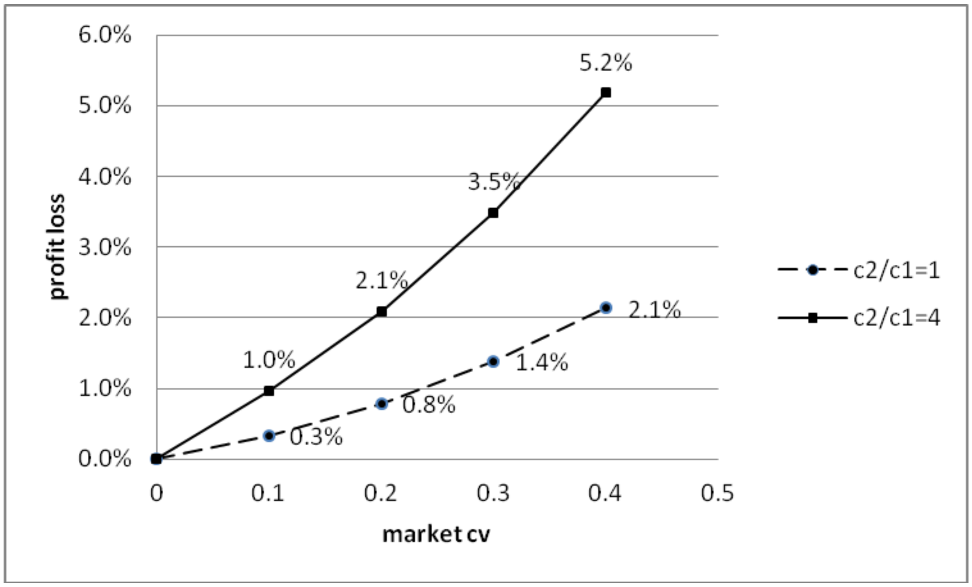


Figure 3-7: Profit loss at mid cost level ($ac = 0.2$)

3.6.5 Effect of bundling pricing and inventory pooling to overall mixed bundling performance

With inventory issue considered under mixed bundling strategy, we can identify two effects that contribute to the advantage of mixed bundling strategy compared to no bundling strategy. One is called bundling pricing effect, which expands aggregate demand by offering the option of bundle besides individual products and setting proper prices for them. The other one is inventory pooling effect, which is caused by the pooled inventories that are used to hedge against uncertain demand. In our main model, these two effects are integrated because pricing and ordering decisions are determined simultaneously. We need to isolate the two effects so as to assess the contribution of each effect to the performance of mixed bundling strategy under stochastic market. In order to do that, we compute results for an extra case in addition to the no bundling and mixed bundling cases that we have already examined above. In this extra case, prices are same as the optimal prices obtained under mixed bundling case. But ordering quantities for the three alternatives are calculated separately from each other, which can be viewed as news vendor problem with three independent products. We name this extra case as intermediate case. So difference between no bundling case and intermediate case reflects the bundling pricing effect, while difference between intermediate case and mixed bundling case represents the inventory pooling effect.

Figure 3-8 plots mixed bundling performance against no bundling strategy at various parameter settings. To better understand how mixed bundling outperforms no bundling under stochastic market, we also draw the contributions of bundling pricing effect and inventory pooling effect, as shown by the gray region and dark region in the graphs respectively. Comparing the four subfigures in Figure 3-8, we find that bundling pricing effect decreases in market uncertainty, cost level and cost ratio. The monotonous negative impact of market uncertainty on bundling pricing effect is easy to explain as pricing is always less effective when demand is of larger variance. When cost level is lifted up, the profit margin space for all alternatives is narrowed so that the ability of bundling pricing is further limited, causing its effect to contract. High cost ratio has negative impact on bundling pricing effect, indicating it is more beneficial to adopt mixed bundling strategy for cost-similar individual products when market uncertainty is under consideration. For inventory pooling effect, it is seen that high uncertainty makes inventory pooling more important as the inventories are more likely to serve different demands. The impact of cost level in inventory pooling effect depends on the capital reserved in inventories. In Figure 3-8, as cost level rise from 0.1 to 0.3, total investment in inventories increases, so the according inventory pooling effect becomes more significant. Note that if cost level further rises, the total investment in inventories will decrease in the end due to reduced order quantities. The effect of cost ratio on inventory pooling is nearly neglectable. In our numerical experiments, the magnitude of inventory pooling effect is less than that of bundling pricing effect in most cases, sometimes even can be ignored. But if cost level is properly set and market variance is large enough, inventory pooling effect can be dominant, as the point of market *cv* at 0.4 in Figure 3-8(d), where it is almost twice of the bundling pricing effect.

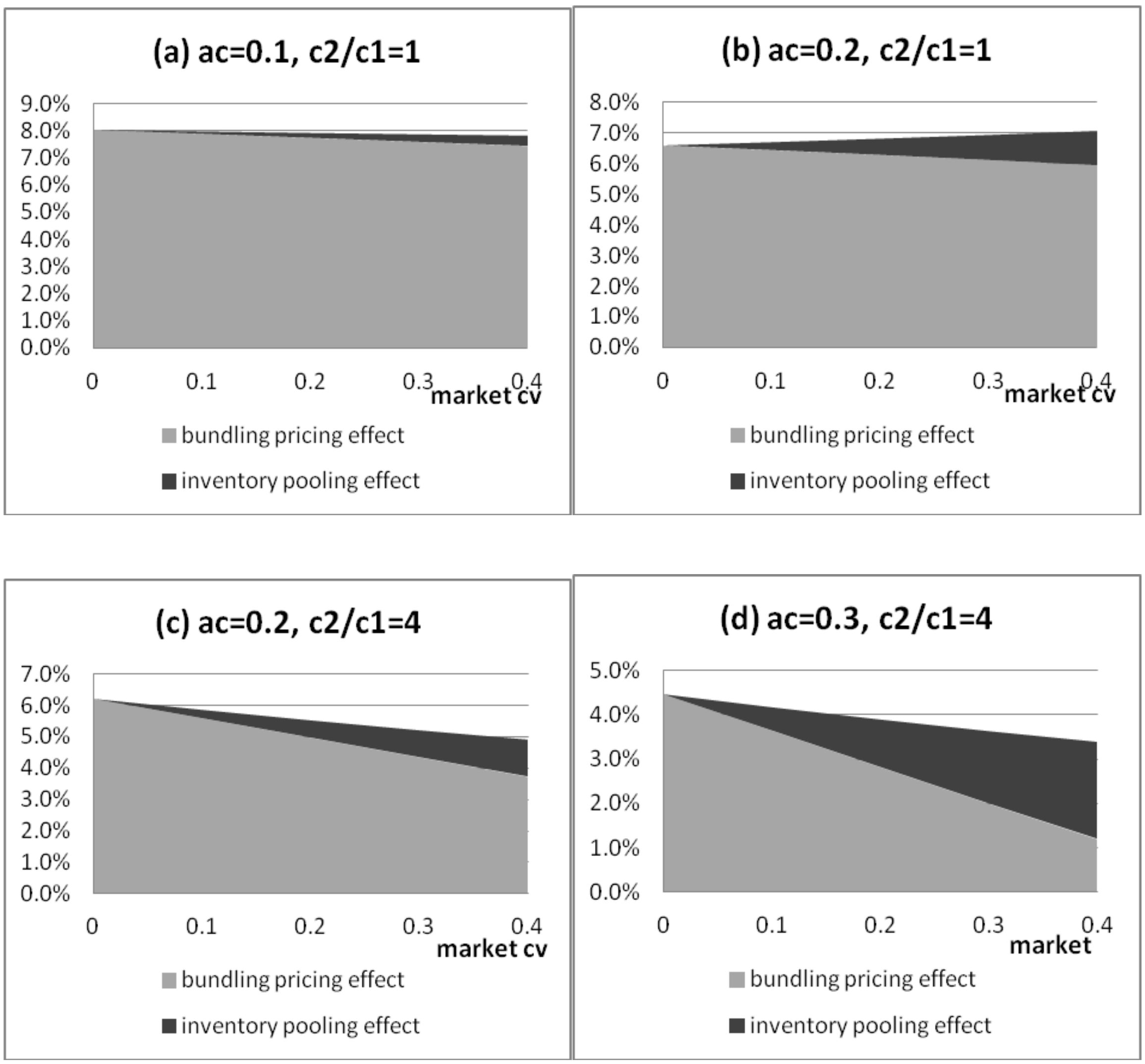


Figure 3-8: Contributions to mixed bundling performance

By synthesizing bundling pricing effect and inventory pooling effect, we obtain the performance gap between mixed bundling strategy and no bundling strategy, which is the criterion for deciding which selling strategy to choose. From the discussion on these two effects above, we can see that the performance gap is enlarged when cost ratio increases, suggesting more potential expected profit gain through mixed bundling when individual products are of symmetrical cost structure. The impact of cost level or market uncertainty is not monotonous, depending on values of other factors. For instant, market uncertainty helps to expand performance gap when cost level is of a proper value and cost ratio is small, where its positive impact on inventory pooling effect is more than offsetting its negative impact on bundling pricing effect (see Figure 3-8(b), where the performance gap goes upward as market cv increases), while it reduces the gap at other values of cost level and cost ratio (see Figure 3-8(a), 3-8(c), 3-8 (d), where the performance gap goes downward as market cv increases).

3.7 Two extensions

3.7.1 Impact of demand correlation

The numerical study is solely based on the case of perfect positive demand correlation. To study the impact of demand correlation, we simply run some simulations for the case of independent demands, and compare the results with that from perfect correlation that we have already obtained. Though this way of studying demand correlation is not thorough, we believe that comparison of these two special cases can shed some light on how demand correlation works under mixed bundling of stochastic market. Each alternative is assumed to follow a normal distribution $N(500\times\alpha_i,(100\times\alpha_i)^2)$ in the base case of independent demands, where α_i is the according market share for alternative i.

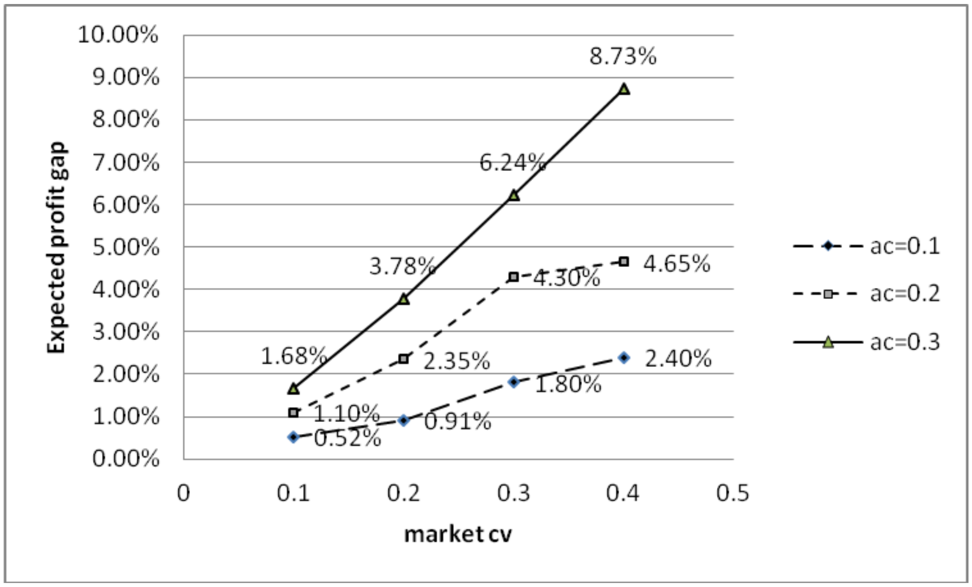


Figure 3-9: Independent demands vs. perfectly positively correlated demands

Figure 3-9 demonstrates the impact of demand correlation by comparing the performance of profitability of these two special cases when the cost structure is symmetric. It shows that expected profit from independent demand case always outperforms that from perfect correlation case and their gap monotonously enlarges as market size becomes more uncertain. This can be explained by the fact that inventory pooling effect is more significant due to smaller aggregated demand variance when demands are of less correlation. Though prices can be adjusted to a better solution, the simulation results show that obtained optimal prices for these two cases are almost same and nearly all the profit increment is attributed by the inventory pooling effect. Thus, we can conclude that the special case of perfect positive demand correlation provides a lower bound for mixed bundling performance under stochastic market since the actual demand correlation is always no greater than perfect correlation.

3.7.2 Determination of order quantities with substitutions

When demands of customers who originally prefer alternative i are not satisfied, the customers can turn to existing alternative as long as the consumer surplus is positive. All the customers do not stick to their originally preferred alternative anymore if their original demand is not met. They can switch to buy secondary alternative which still gives non-negative consumer surplus to them should the inventories allow.

Assumption: selling to the original demand is more profitable than selling to the secondary demand because of substitution cost. Therefore, the retailer would not deliberately force the customers from their preference to the secondary choice. So when allocating existing inventories for three demands after market size realized, the retailer behaves exactly like that in the model without substitution. The substitution could happen only when the customers who originally choose the bundle are not all satisfied. They may turn to the remaining individual product 1 or product 2. The reverse is not true as unfilled demand from customers who originally prefer individual products cannot turn to the bundle because at least one component of the bundle is stock out, i.e., either product 1 or product 2. The eight cases without substitutions still hold in the situation with substitution but some modifications need conducted in the cases where unmet demand for bundles may turn to remaining individual products. These cases include case S_3, where customers whose original demand for the bundle is unmet may choose to buy remaining product 1. Symmetrically, substitution occurs from the bundle to product 2 in case S_5.

Based on the reservation price model, the probability of customers who originally prefer the bundle turn to individual product 1 is:

$$R_{b1} = \frac{(1-p_1)(1+p_1-p_b)}{\alpha_b}$$

$$R_{b2} = \frac{(1-p_2)(1+p_2-p_b)}{\alpha_b}$$

So modified profit under case S_3 is:

$$E_3 = p_1 M\alpha_1 + p_b Q_2 + (p_1 - d)\min\{R_{b1}(M\alpha_b - Q_2), Q_1 - M\alpha_1 - Q_2\} - c_1 Q_1 - c_2 Q_2$$

Symmetrically, modified profit under case S_5 is:

$$E_5 = p_2 M\alpha_2 + p_b Q_1 + (p_2 - d)\min\{R_{b2}(M\alpha_b - Q_1), Q_2 - M\alpha_2 - Q_1\} - c_1 Q_1 - c_2 Q_2$$

Where d is the substitution cost, assumed to be constant and same for both cases. Should the assumption hold, d satisfies: $p_b \geq R_{b1}p_1 + R_{b2}p_2 - d$. Since $0 < R_{b1} < 1$ and $0 < R_{b2} < 1$, d is further confined as: $p_b \geq p_1 + p_2 - d$.

We divide case S_3 into two sub-cases, case S_{31} and S_{32}, whose profit functions are:

$$E_{31} = p_1 M\alpha_1 + p_b Q_2 + (p_1 - d)R_{b1}(M\alpha_b - Q_2) - c_1 Q_1 - c_2 Q_2$$

$$E_{32} = p_1 M\alpha_1 + p_b Q_2 + (p_1 - d)(Q_1 - M\alpha_1 - Q_2) - c_1 Q_1 - c_2 Q_2$$

Similarly analysis applies to case S_5. These modifications can be illustrated in the following figure:

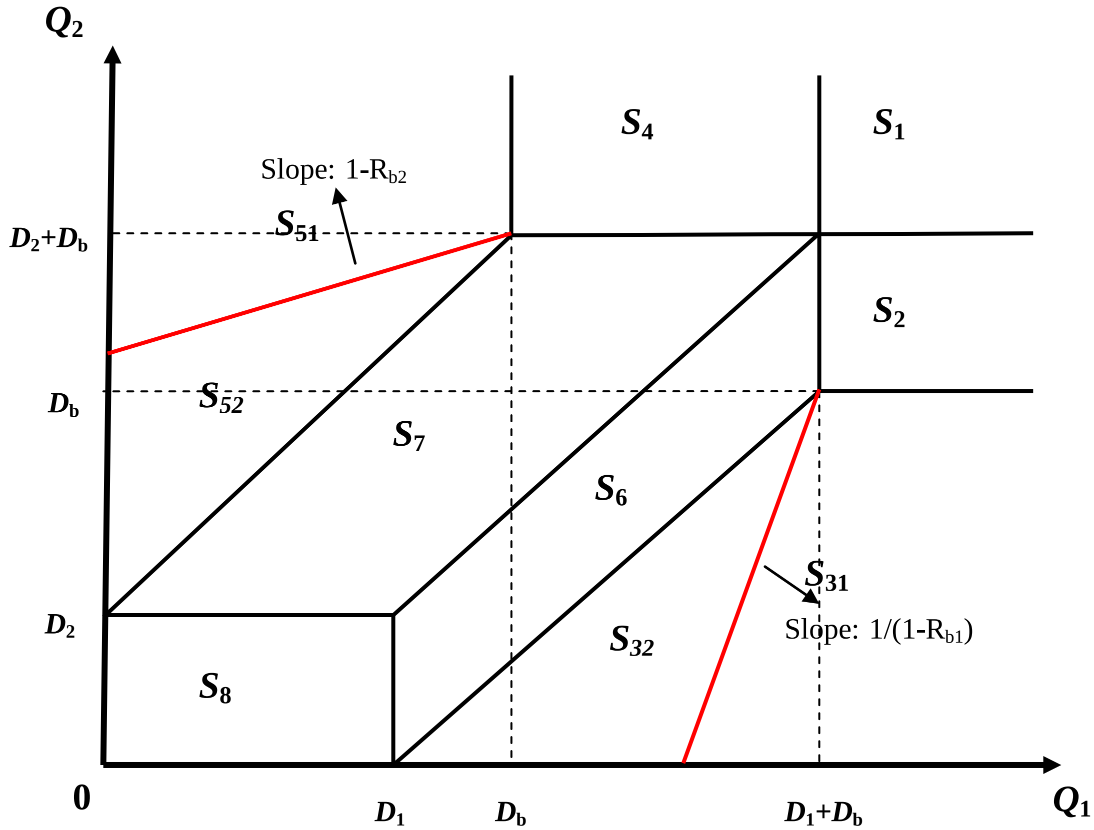


Figure 3-10: Inventory allocation with substitutions

The expected total profit function is:

$$
\begin{aligned}
E\Pi(Q_1,Q_2) &= \sum_{i=1}^{8} E\Pi_i(Q_1,Q_2) \\
&= \int_{m\in S_1} (p_1 m\alpha_1 + p_2 m\alpha_2 + p_b m\alpha_b) f_M(m)dm + \int_{m\in S_2} [p_1 m\alpha_1 + p_2(Q_2 - m\alpha_b) + p_b m\alpha_b] f_M(m)dm \\
&+ \int_{m\in S_{31}} (p_1 m\alpha_1 + p_b Q_2 + (p_1 - d)R_{b1}(M\alpha_b - Q_2)) f_M(m)dm + \int_{m\in S_{32}} (p_1 m\alpha_1 + p_b Q_2 + (p_1 - d)(Q_1 - M\alpha_1 - Q_2)) f_M(m)dm \\
&+ \int_{m\in S_4} [p_1(Q_1 - m\alpha_b) + p_2 m\alpha_2 + p_b m\alpha_b] f_M(m)dm \\
&+ \int_{m\in S_{51}} (p_2 m\alpha_2 + p_b Q_1 + (p_2 - d)R_{b2}(M\alpha_b - Q_1)) f_M(m)dm + \int_{m\in S_{52}} (p_2 m\alpha_2 + p_b Q_1 + (p_2 - d)(Q_2 - M\alpha_2 - Q_1)) f_M(m)dm \\
&+ \int_{m\in S_6} [p_1 m\alpha_1 + p_2(Q_2 - Q_1 + m\alpha_1) + p_b(Q_1 - m\alpha_1)] f_M(m)dm \\
&+ \int_{m\in S_7} [p_1(Q_1 - Q_2 + m\alpha_2) + p_2 m\alpha_2 + p_b(Q_2 - m\alpha_2)] f_M(m)dm + \int_{m\in S_8} (p_1 Q_1 + p_2 Q_2) f_M(m)dm - c_1 Q_1 - c_2 Q_2
\end{aligned}
$$

This function is still concave in (Q_1, Q_2). Similar analytical results and algorithm can be derived as that in the model without substitution except that two more cases need to be considered. As remaining individual products may also be purchased by the customers who originally prefer the bundle, we expect that more quantities for both products should be ordered at the first stage compared to the results under model without substitution.

3.8 Chapter conclusion

In this chapter, we study the problem of joint pricing and inventory decisions for mixed bundling under stochastic market. The reservation price model was used for modeling of demands and a two-stage stochastic model was proposed for inventory decision. Global concavity in inventory decision was proved at given prices. We used a numerical search method for joint pricing and inventory optimization. The numerical experiments showed that under stochastic market mixed bundling still outperforms no bundling as it does in stable market. We showed the significance of considering inventory issue under mixed bundling strategy when the market is stochastic. Besides, we quantitatively measure the sources of outperformance of mixed bundling over no bundling: bundling pricing effect and inventory pooling effect. At last, by numerically comparing two special cases where demands of the three alternatives under mixed bundling are independent and perfectly positively correlated, we showed that mixed bundling would perform even better if real demands are of less correlation under stochastic market. Further study can be carried out by considering substitution, where customers may switch to other alternatives as long as they are available and yield positive consumer surplus when their most preferred choice is out of stock.

Chapter 4 A More General Model for Mixed Bundling under Stochastic Market

Based on the previous chapter, which studied joint pricing and inventory problem for mixed bundling strategy under stochastic market, we conduct several extensions to cope with more general market conditions in this work. These extensions include a more realistic customer choice model (MNL), a joint reservation price distribution for the two individual products, and the degree of contingency for each customer demonstrating product relationship when forming the bundle (substitutable, independent and supplementary). Because of the high complexity of the problem, we use simulation optimization for optimal solutions, namely Sample Average Approximation with IPA gradient estimation. We solve a linear optimization model and its dual problem to calculate one component in the gradient decomposition chain. Finally, we carry out various numerical simulations to evaluate mixed bundling performance. Sensitivity analysis is also conducted for the newly added factors from the extensions.

4.1 Problem description

The problem framework is still same as the one in our previous work: a monopolistic retailer makes joint pricing and ordering decisions for two products that will be sold to customers in mixed bundling strategy (three alternatives as two individual products and the bundle) before demand is known. Demands for the three alternatives will be derived from the to-be-realized market size and according customer choices. After market size becomes known, the retailer allocates the products to fulfill the three types of demands. The objective is to maximize the total expected revenue.

In the previous work, we were able to derive closed-form gradients with respect to prices because we assumed independent uniform reservation price distributions and the simple deterministic reservation price model for customers' choice issue. In this study, we intend to relax these two assumptions to make the model more realistic and use simulation method for solution.

Generally, each customer has his own reservation price vector (R_1, R_2, R_b). To describe the reservation price distribution, we choose the general joint distribution $f(r_1, r_2)$, with mean and variance as $\overline{\mu} = (\mu_1, \mu_2)$ and $Var = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$, where ρ denotes when correlation coefficient between the two individual reservation prices. The reservation price for the bundle R_b is derived based on the two individual reservation prices R_1 and R_2. Previously, we made the additivity assumption for this issue, which means the two individual products are independent. In this work, we assume the relationship between the bundle and the two individual products is perceived differently for each customer. Under this extension, the two individual products can be substitutable, independent or complementary, and the degree of contingency (Venkatesh and Kamakura 2003) is random. That is, we have $R_b = \kappa(R_1 + R_2)$, where κ can be less than 1 (substitutable), equals 1 (independent) or greater than 1 (complementary). When the individual products are substitutable or complementary, each customer may have different κ value to indicate different degree of contingency. This can be simply implemented in simulation by randomly generating one κ (within the boundary) for each customer.

In the literature of bundling, the reservation price model is widely used for customer behavior. In this model, customers simply choose the alternative which yields largest consumer surplus, i.e., the difference between reservation price and alternative price. This model assumes price is the only factor that affects customers' purchasing decision and associated utility function is deterministic. However, some unobservable factors may have impact on choice making, such as mood. An error term should be considered in the utility function to reflect such influences. McFadden (1974) initiated the conditional logit analysis for discrete choice problems by assuming the error term in utility function following a specific statistical distribution. For more than two alternatives and under some other technical assumptions, it is known as Multinomial Logit Model (MNL). In a set of *n* alternative, the purchasing probability for *alternative i* is $prob_i = e^{U_i} / \sum_{i \in n} e^{U_i}$, where U_i is the known part of utility function. This model has been widely accepted for discrete choice systems. In this study, for the first time we will use Multinomial Logit Model (MNL) to model customer behavior under mixed bundling structure. Compared to the reservation price mode, this model is more realistic in modeling customer's purchasing behavior by considering uncertainty in their utility function. Under the mixed bundling structure, a customer with reservation price vector (R_1, R_2, R_b) faces four alternatives: buying product 1, buying product 2, buying the bundle, and buying nothing. The deterministic part in the utility function is consumer surplus, whose value for alternative i is $S_i = R_i - p_i, i = 1, 2, b$, and $S_0 = 0$. Adapting the assumptions for MNL model, the purchasing probability for each alternative is calculated as:

$$\pi_i = \frac{\exp(\lambda S_i)}{\sum_i \exp(\lambda S_i) + 1} \qquad \text{4-1}$$

We can see that under MNL model the alternative with larger consumer surplus has a higher probability of being chosen. Even if the according surplus is negative, the alternative still has a chance being bought.

4.2 Notations

Below are the notations used throughout the study.

i --- Alternative index, $i = 1, 2, or\ b$, b stands for the bundle;

N --- Number of samples drawn (market size realizations), $n = 1, 2, ..., N$

M_n --- Number of customers in nth sample, $j = 1, 2, ..., M_n$

R_{nji} --- Customer's reservation price of *jth* customer in *nth* sample for alternative i;

S_{nji} --- Consumer surplus of *jth* customer in *nth* sample for alternative *i*;

$f(r_1, r_2)$ --- pdf of the joint reservation price distribution with mean and variance as $\overline{\mu} = (\mu_1, \mu_2)$ and $Var = [\begin{matrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{matrix}]$, where ρ is the correlation coefficient;

κ --- Degree of contingency between the two individual products, $R_{njb} = \kappa(R_{nj1} + R_{nj2})$. κ could be greater or less than 1 or equal to 1. For the former two cases, different customer can have different κ value.

M --- Random market size; $f_M(m)$ is pdf of market size distribution, with mean μ_M and sd σ_M .

D_{ni} --- Aggregated demand in *nth* sample for alternative i ; Demand vector $\overline{D} = (D_1, D_2, D_b)$;

π_{nji} --- purchasing probability of *jth* customer in *nth* sample for alternative *i*;

c_i --- Unit cost of alternative i ; $c_b = c_1 + c_2$

Decision variables:

p_i --- Price of alternative i ; Price vector $\overline{p} = (p_1, p_2, p_b)$

Q_i --- Order quantity for product i , $i = \{1, 2\}$; Quantity vector $\overline{Q} = (Q_1, Q_2)$

4.3 The mathematical model

$$First-stage \quad \underset{(\overline{p},\overline{Q})}{Max}\, \Pi = E_{\overline{D}}[\, p_1 q_1(\overline{Q}, \overline{D}) + p_2 q_2(\overline{Q}, \overline{D}) + p_b q_b(\overline{Q}, \overline{D}) - c_1 Q_1 - c_2 Q_2\,]$$

Subject to the constraint $\max\{p_1, p_2\} < p_b < p_1 + p_2$

After the demand realized in the second stage, the retailer decides how to allocate the inventory to maximize profit.

$$
\begin{aligned}
second-stage \quad & \underset{\bar{q}}{Max}\, \Pi' = p_1 q_1 + p_2 q_2 + p_b q_b \\
s.t. \quad & q_1 + q_b \le Q_1 \\
& q_2 + q_b \le Q_2 \\
& q_1 \le D_1 \\
& q_2 \le D_2 \\
& q_b \le D_b \\
& q_1, q_2, q_b \ge 0
\end{aligned}
$$

$\bar{q} = (q_1, q_2, q_b)$ is subsidiary variable of inventory allocation amount for the demands.

This stochastic two-stage model is difficult to solve analytically as the demands $\bar{D}$ are complicatedly related with price variables $\bar{p}$ due to the heterogeneous reservation prices and the probabilistic MNL model. Even with known prices $\bar{p}$ and order quantities $\bar{Q}$, the objective function does not have an analytical expression. Therefore, we use the Sample Average Approximation (SAA) (Shapiro, 2003) method to approximate the stochastic problem for numerical results. N fixed samples are drawn to simulate the demand realizations. By the law of large numbers, as long as the sample size is large enough, we can approximate original optimization objective with the sample average value. For fixed N samples, the original stochastic problem turns into the following deterministic optimization model:

$$\begin{aligned} &\underset{\bar{p},\bar{Q}}{Max}\,\Omega = \frac{1}{N}\sum_{n=1}^{N}(p_1 q_{n1} + p_2 q_{n2} + p_b q_{nb}) - (c_1 Q_1 + c_2 Q_2) \\ &s.t. \qquad q_{n1} + q_{nb} \le Q_1 \\ &\qquad\qquad q_{n2} + q_{nb} \le Q_2 \\ &\qquad\qquad\quad q_{n1} \le D_{n1} \\ &\qquad\qquad\quad q_{n2} \le D_{n2} \\ &\qquad\qquad\quad q_{nb} \le D_{nb} \\ &\qquad\quad q_{n1}, q_{n2}, q_{nb} \ge 0 \end{aligned} \tag{4-2}$$

In order to use gradient-based searching algorithms, we first need to figure out how to calculate gradients with respect to decision variables. We only need to calculate gradients with respect to the prices $\bar{p}$ because the optimal order quantities $\bar{Q}^*$ can be directly figured out by solving above large scale linear programming for given prices. Based on the chain rule, we have the following guide for gradient derivation:

$$\frac{\partial\Omega}{\partial p_i} = \frac{1}{N}\left(\sum_{n=1}^{N} q_{ni} + \sum_{n=1}^{N}\frac{\partial\Omega_n}{\partial D_{n1}}\frac{\partial D_{n1}}{\partial p_i} + \sum_{n=1}^{N}\frac{\partial\Omega_n}{\partial D_{n2}}\frac{\partial D_{n2}}{\partial p_i} + \sum_{n=1}^{N}\frac{\partial\Omega_n}{\partial D_{nb}}\frac{\partial D_{nb}}{\partial p_i}\right) \tag{4-3}$$

This is the IPA (Infinitesimal Perturbation Analysis) gradient estimators of the original stochastic optimization problem. In the below sections, we will show that the IPA gradients are unbiased estimators of true gradients in the model. Before that, we first explain the formation of demands during simulation procedure.

4.3.1 The demands

In the nth sample, total M_n customers are generated, where $M_n, n = 1,...,N$ follows market size distribution $f_M(m)$. The reservation price vector $(R_{nj1}, R_{nj2}, R_{njb})$ for customer $j, j = 1,...,M_n$ in nth sample is generated based on the reservation price distribution $f(r_1, r_2)$ and the bundle reservation price equation $R_{njb} = \kappa_{nj}(R_{nj1} + R_{nj2})$. For the jth customer, the consumer surplus for alternative i is $S_{nji} = R_{nji} - p_i$. Then according to the MNL model, the purchasing probability for this alternative is $\pi_{nji} = \frac{\exp(\lambda S_{nji})}{\sum_i \exp(\lambda S_{nji}) + +1}$. We approximate the total demands D_{ni} by aggregating the according purchasing probability across the M_n customers for this sample:

$$D_{ni} \approx \sum_{j=1}^{M_n} \pi_{nji} = \frac{\exp(\lambda S_{nji})}{\sum_i \exp(\lambda S_{nji}) + +1} \qquad 4\text{-}4$$

4.3.2 IPA method

Infinitesimal Perturbation Analysis was first introduced by Y.C. Ho in 1979 for analyzing a production line problem. It provides gradient estimate based on one single simulation run. In order for the IPA to work, the system must satisfy the conditions stated in the theorem by L'Ecuyer (1995) as below:

$E[f(\theta, \xi)], \theta \in \Upsilon,\ P(\xi \in H) = 1$

(i) for all $z \in \mathrm{H}$, $f(\cdot, z)$ is continuous everywhere in Υ

(ii) for all $z \in \mathrm{H}$, $f(\cdot, z)$ is differentiable everywhere in $\Upsilon \setminus D(z)$, where $D(z)$ is at most countable

(iii) $f(\theta, \xi)$ is almost surely differentiable at $\theta = \theta_0$

(iv) $f'(\theta, \xi)$ is uniformly dominated by as integrable function of ξ, i.e.,

$$\sup_{\theta \in \Upsilon \setminus D(\xi)} | f'(\theta, \xi) | \leq \phi(\xi),\ \mathrm{E}[\phi(\xi)] < \infty$$

Then $f(\cdot) = E[f(\cdot,\xi)]$ is differentiable at $\theta = \theta_0$, and $f'(\theta_0) = E[f'(\theta_0,\xi)]$

Our objective function is continuous and differentiable in the decision space, satisfying all the conditions for the exchange of derivative and integral. Therefore, we can use the exact gradients in equation 4-3 in gradient-based simulation optimization algorithms as they are unbiased estimators of true gradients:

$$g(\overline{p}) = \frac{\partial \Pi}{\partial \overline{p}} = \nabla E[\pi(\overline{p},\overline{D})] = E[\nabla \pi(\overline{p},\overline{D})]$$

Next, we show in detail how to calculate the gradients during simulation process.

In the traditional stochastic approximation problems, usually N i.i.d. replications are run to obtain an unbiased estimator for the gradient:

$$\hat{g}(\theta) = \frac{1}{N}\sum_{i=1}^{N} \nabla f(\theta,\xi_i) \qquad 4\text{-}5$$

However in our problem, another two decision variables (Q_1, Q_2) need to be determined based on the choice of price vector $\overline{p}$ and realized demands $\overline{D}$, adding complexity to the use of IPA method. As mentioned before, the deterministic optimization problem becomes a LP model for given prices:

$$\begin{aligned}
\underset{\overline{q_n},\overline{Q}}{Max}\,\Omega = \frac{1}{N}\sum_{n=1}^{N}(p_1 q_{n1} + p_2 q_{n2} + p_b q_{nb}) - (c_1 Q_1 + c_2 Q_2)& \\
s.t. \quad q_{n1} + q_{nb} \le Q_1& \\
q_{n2} + q_{nb} \le Q_2& \\
q_{n1} \le D_{n1}& \\
q_{n2} \le D_{n2}& \\
q_{nb} \le D_{nb}& \\
q_{n1}, q_{n2}, q_{nb} \ge 0&
\end{aligned} \qquad 4\text{-}6$$

Solving above LP gives the optimal order quantities Q_1^* and Q_2^* under given prices and fixed samples. Besides, we also know the values for q_{ni} from the LP solutions. As $\partial D_{ni} / \partial p_i$ is obtainable from equation 4-4, we only need to calculate $\partial \Omega_n / \partial D_{ni}$ for gradient estimators $\partial \Omega / \partial p_i$ based on equation 4-2. For this term, we study the dual problem of the LP, assuming the dual variables are $y_{ni}, i = 1,2,3,4,5$:

$$
\begin{aligned}
& \underset{y_{n1},\ldots,y_{n5}}{MinW} = \frac{1}{N}\sum_{n=1}^{N}(Q_1^* y_{n1} + Q_2^* y_{n2} + D_{n1}y_{n3} + D_{n2}y_{n4} + D_{n3}y_{n5}) - (c_1 Q_1^* + c_2 Q_2^*) \\
& s.t. \qquad y_{n1} + y_{n3} \geq p_1 \\
& \qquad\qquad y_{n2} + y_{n4} \geq p_2 \\
& \qquad y_{n1} + y_{n2} + y_{n5} \geq p_b \\
& y_{n1}, y_{n2}, y_{n3}, y_{n4}, y_{n5} \geq 0
\end{aligned}
\tag{4-7}
$$

The dual variables are the sensitivity of the primal objective with respect to the RHS in the constraints. That is:

$$
\frac{\partial \Omega_n}{\partial D_{n1}} = y_{n3}, \frac{\partial \Omega_n}{\partial D_{n2}} = y_{n4}, \frac{\partial \Omega_n}{\partial D_{n3}} = y_{n5} \tag{4-8}
$$

From equation 4-6, we have:

$$
\frac{\partial D_{ni}}{\partial p_i} = \sum_{j=1}^{M_n} \frac{\partial \pi_{nji}}{\partial S_{nji}} \frac{\partial S_{nji}}{\partial p_i} = -\sum_{j=1}^{M_n} \frac{\partial \pi_{nji}}{\partial S_{nji}} = -\lambda \sum_{j=1}^{M_n} \frac{\exp(S_{nji})[\sum_i \exp(S_{nji}) + 1 - \exp(S_{nji})]}{[\sum_i \exp(S_{nji}) + 1]^2}
$$

$$
\frac{\partial D_{ni'}}{\partial p_i}\Big|_{i' \neq i} = \sum_{j=1}^{M_n} \frac{\partial \pi_{nji'}}{\partial S_{nji}} \frac{\partial S_{nji}}{\partial p_i} = -\lambda \sum_{j=1}^{M_n} \frac{\partial \pi_{nji'}}{\partial S_{nji}} = \lambda \sum_{j=1}^{M_n} \frac{\exp(S_{nji}) \cdot \exp(S_{nji'})}{[\sum_i \exp(S_{nji}) + 1]^2}
$$

Then the gradients in equation 4-5 can be expressed in complete form as below:

$$\frac{\partial\Omega}{\partial p_1}=\sum_{n=1}^{N}q_{n1}+\sum_{n=1}^{N}[-y_{n3}\lambda\sum_{j=1}^{M_n}\frac{\exp(S_{nj1})[\sum_i\exp(S_{nji})+1-\exp(S_{nj1})]}{[\sum_i\exp(S_{nji})+1]^2}]$$
$$+\sum_{n=1}^{N}[y_{n4}\lambda\sum_{j=1}^{M_n}\frac{\exp(S_{nj1})\cdot\exp(S_{nj2})}{[\sum_i\exp(S_{nji})+1]^2}]+\sum_{n=1}^{N}[y_{n5}\lambda\sum_{j=1}^{M_n}\frac{\exp(S_{nj1})\cdot\exp(S_{njb})}{[\sum_i\exp(S_{nji})+1]^2}] \quad 4\text{-}9$$

$$\frac{\partial\Omega}{\partial p_2}=\sum_{n=1}^{N}q_{n2}+\sum_{n=1}^{N}[-y_{n4}\lambda\sum_{j=1}^{M_n}\frac{\exp(S_{nj2})[\sum_i\exp(S_{nji})+1-\exp(S_{nj2})]}{[\sum_i\exp(S_{nji})+1]^2}]$$
$$+\sum_{n=1}^{N}[y_{n3}\lambda\sum_{j=1}^{M_n}\frac{\exp(S_{nj2})\cdot\exp(S_{nj1})}{[\sum_i\exp(S_{nji})+1]^2}]+\sum_{n=1}^{N}[y_{n5}\lambda\sum_{j=1}^{M_n}\frac{\exp(S_{nj2})\cdot\exp(S_{njb})}{[\sum_i\exp(S_{nji})+1]^2}] \quad 4\text{-}10$$

$$\frac{\partial\Omega}{\partial p_b}=\sum_{n=1}^{N}q_{nb}+\sum_{n=1}^{N}[-y_{n5}\lambda\sum_{j=1}^{M_n}\frac{\exp(S_{njb})[\sum_i\exp(S_{nji})+1-\exp(S_{njb})]}{[\sum_i\exp(S_{nji})+1]^2}]$$
$$+\sum_{n=1}^{N}[y_{n3}\lambda\sum_{j=1}^{M_n}\frac{\exp(S_{njb})\cdot\exp(S_{nj1})}{[\sum_i\exp(S_{nji})+1]^2}]+\sum_{n=1}^{N}[y_{n4}\lambda\sum_{j=1}^{M_n}\frac{\exp(S_{njb})\cdot\exp(S_{nj2})}{[\sum_i\exp(S_{nji})+1]^2}] \quad 4\text{-}11$$

4.4 The simulation procedure

a. Set an initial price vector (p_1, p_2, p_b), which satisfies the price constraint

b. Run N replications in terms of market size realization

c. In each replication, generate M_n customers and generate reservation price for each customer.

d. Compute purchasing probability π_{nji} and aggregated demands D_{ni}

e. Compute gradient estimator $g(\hat{\bar{p}})$

f. $p_i^{k+1}=p_i^k+\frac{1}{k}g(\hat{\bar{p}})^k$, where k means the kth simulation batch of N replications. Same N samples are used across the simulation steps. g_k

g. Exist the loop when $\| \frac{1}{k} g(\hat{p})^k \| < \varepsilon$.

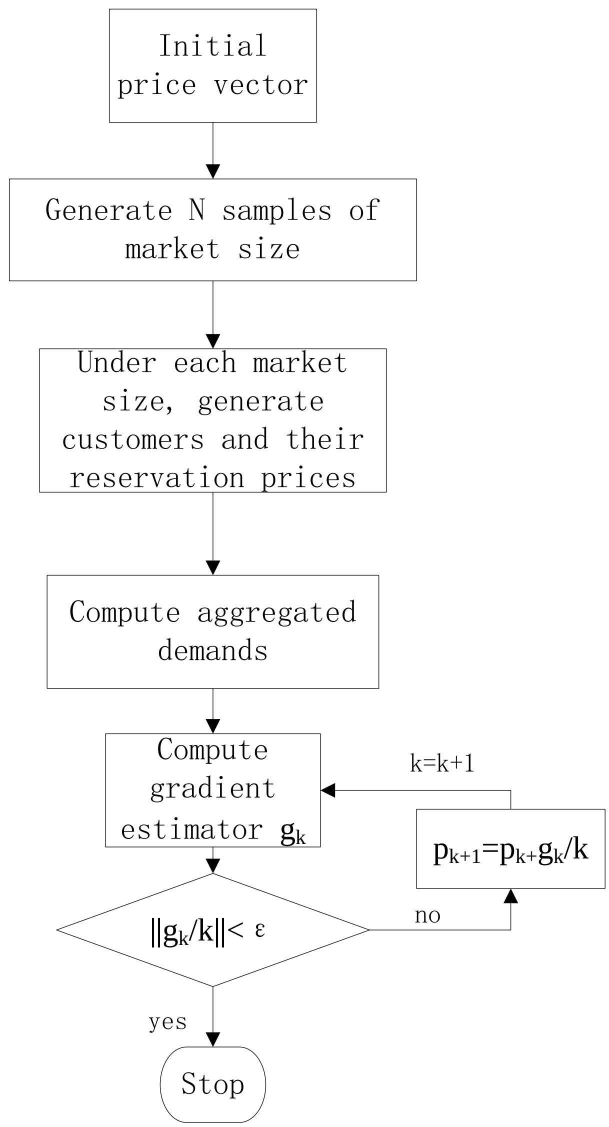


Figure 4-1: Algorithm flowchart

4.5 Numerical experiment and managerial insights

In this study, we propose a simulation method to solve the joint pricing and ordering decisions for mixed bundling problem so as to consider more factors such as correlation coefficient between the reservation prices and degree of contingency which were not covered in our previous work. Therefore, we will conduct extensive numerical study mainly on these two factors in this section to show their impact on the optimal decisions. First of all, we define a base case with parameters as: market size distribution $M \sim N(1000, 200^2)$, joint reservation price distribution $\overline{R} \sim N(\overline{\mu}, \Sigma)$ where $\overline{\mu} = (0.6, 0.6)$ and $\Sigma = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.09 \end{pmatrix}$, degree of contingency $\kappa = 1$ and $c_1 = c_2 = 0.2$. In this base case, the correlation coefficient ρ between the two individual product reservation prices is zero and the reservation price for the bundle is exact the sum of the reservation prices for the two individual products.

4.5.1 Optimal results for base case

In the base case, the reservation prices for the two individual products are independent, i.e., the correlation coefficient ρ is zero. The reservation price for the bundle is exactly the sum of the two individual reservation prices, that is, degree of contingency $\kappa = 1$ for each customer. Coefficient variation of the market size distribution is chosen as 0.2 to represent the market uncertainty. Other parameters settings like reservation price distribution and product unit cost are as stated in the preceding paragraph. Under no bundling strategy, customer behavior is still described by the Multinomial Logit Model for the fair comparison to mixed bundling strategy.

Table 4-1: Comparison between mixed bundling and no bundling strategies

Strategies	(p_1^*, p_2^*, p_b^*)	*Dis*	Q_1^*	Q_2^*	*profit*	*ordering cost*
No bundling	(0.69, 0.69, –)	–	427	427	417.2	170.8

Mixed bundling	(0.88, 0.88,1.24)	29.8%	530	531	422.7	212.2

The simulated optimal results of the base case for both no bundling and mixed bundling strategies are reported in Table 4-1. Compared to the previous study, this model is under more general market assumptions and more realistic customer behavior. The results confirm that mixed bundling outperforms no bundling due to increased aggregated demands for the products, which enables the retailer to tune the pricing and ordering variables for more profit.

4.5.2 Comparison of no bundling and mixed bundling strategies

While mixed bundling performs better than no bundling for the base case, it is not pervasive for all the parameter settings. From various simulations under different parameter settings, we find that correlation coefficient ρ and degree of contingency κ between the two individual products are the two factors that determine the comparison result between no bundling and mixed bundling strategies. Figure 4-1 depicts the performance comparison, which is measured as the relative profit gap with profit under no bundling as the benchmark.

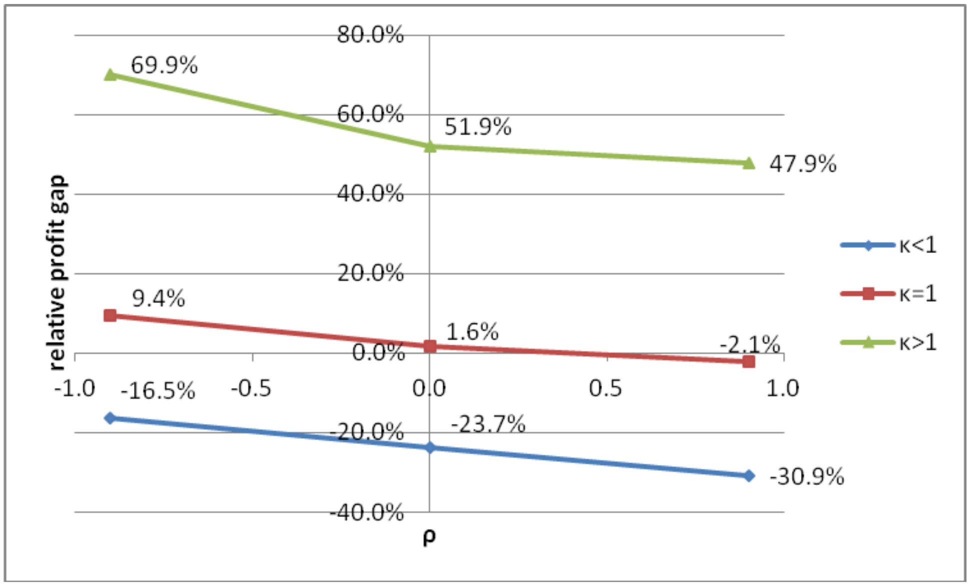


Figure 4-2: Impact of correlation coefficient ρ and degree of contingency κ

As we can see from Figure 4-1, performance of our mixed bundling strategy negatively correlates with correlation coefficient ρ but positively correlates with degree of contingency κ. This finding is consistent with result for conventional mixed bundling strategy without inventory consideration (Schmalansee, 1984). Individual products which are supplementary are more suitable for mixed bundling practice, and this will be further enhanced if customers' reservation prices for them have lower correlation. In the following subsections, we will explain how the factors influence the mixed bundling performance in more details.

4.5.3 Effects of correlation coefficient between the two individual products' reservation prices

Based on the base case, we investigate the factor of correlation coefficient between the individual reservation prices by specifying the value of ρ (-0.9, 0, 0.9) while keeping other parameters same. Table 4-2 shows the simulated results. As this factor is expected to significantly affect the customer choice between individual products and the bundle, respective revenues for the two individual products and the bundle are explicitly recorded in the columns of $rev1$, $rev2$ and $revb$ of Table 4-2.

Table 4-2: Results under various correlation coefficient ρ

ρ	(p_1^*, p_2^*, p_b^*)	*Dis*	Q_1^*	Q_2^*	*profit*	$rev1$	$rev2$	$revb$
-0.9	(0.92, 0.92, 1.14)	38.1%	608	607	455.1	119.9	119.1	459.0
0	(0.88, 0.88, 1.24)	29.8%	530	531	422.7	101.3	101.7	432.0
0.9	(0.87, 0.87, 1.30)	25.5%	501	501	407.2	74.6	75.0	458.0

4.5.4 Effects of degree of contingency

Degree of contingency κ is another significant factor for mixed bundling performance. We generally consider three ranges of κ values to assess their impact, which represent three categories of individual products: substitutable products, independent products and supplementary products. κ values in the three categories are randomly sampled from $U(0.4, 0.8)$, fixed as 1, and randomly sampled from $U(1.2, 1.6)$ respectively. Considering randomness of degree of contingency is more realistic as different customers may have different perceptions for bundle reservation price from same individual products. Table 4-3 demonstrates the simulated optimal results of mixed bundling strategy for the mentioned three categories of individual products with other parameters same as the base case.

Table 4-3: Results under various degree of contingency κ

Degree of contingency	(p_1^*, p_2^*, p_b^*)	Dis	Q_1^*	Q_2^*	$profit$	$rev1$	$rev2$	$revb$
$\kappa < 1$	(0.81, 0.80, 1.04)	35.4%	360	367	317.3	177.8	180.4	104.5
$\kappa = 1$	(0.88, 0.88, 1.24)	29.8%	530	531	422.7	101.3	101.7	432.0
$\kappa > 1$	(0.98, 0.98, 1.55)	21.0%	651	652	631.6	52.0	52.3	788.1

The results imply that the more supplementary the individual products are, the higher total expected profit mixed bundling strategy will result in. Looking into the sources of revenue, we can see that bundle sales rise significantly in the category of supplementary products compared to the other two categories while individual product sales decline at a comparatively lower pace. Higher reservation price for the bundle leads to larger demand, which gives the retailer space to price the bundle higher to capture more consumer surplus. At the same time, demands for the individual products are reduced. This reduction also enables the retailer to set higher prices for the individual products for more consumer surplus. Total inventories needed are increased so as to support the expected selling. We can conclude that supplementary products are more suitable for mixed bundling strategy.

4.5.5 Effects of coefficient variation of the market size distribution

Coefficient variation of the market size distribution is the factor that measures the relative magnitude of market uncertainty. To focus on the effect of coefficient variation, we vary it from 0.1 to 0.4 based on the base case, where $\rho = 0$ and $\kappa = 1$. Table 4-4 summarizes the outputs for various coefficient variations. The results are similar for other values of parameters ρ and κ.

Table 4-4: Results under various coefficient of variance cv

cv	(p_1^*, p_2^*, p_b^*)	Q_1^*	Q_2^*	*profit*
0.1	(0.88,0.88,1.21)	514	514	439.1
0.2	(0.88,0.88,1.24)	530	531	422.7
0.3	(0.89,0.89,1.25)	549	549	395.5
0.4	(0.89,0.89,1.26)	542	543	372.1

It is obvious that the total expected profit decreases when the market becomes more uncertain. This observation is consistent with traditional joint pricing and inventory problems.

4.6 Chapter conclusion

In this chapter, we extend our previous much restricted model for joint pricing and inventory decisions under stochastic market in several directions. First of all, we change the deterministic reservation price model to the probabilistic Multinomial Logit Model to describe the customer choice behavior. This model more realistically represents the actual situation. Secondly, instead of simply assuming independently uniform distribution for reservation prices, we adopt the more general joint reservation price distribution, which allows us to study the effect of correlation coefficient between reservation prices, an important factor widely investigated in the context of bundling. Thirdly, we consider three categories of products (substitutable, independent, and supplementary products) by using the parameter degree of contingency. More specifically, we allow different degree of contingency for different customers if the products not independent. Due to these extensions, direct searching for solutions becomes impossible. Therefore, we utilize the Sample Average Approximation together with IPA (Infinitesimal Perturbation Analysis) gradient estimator for numerical solutions. At last, considerable numerical experiments are conducted to assess the performance of our model and impart of some important factors such as correlation coefficient, degree of contingency and coefficient variation of the market size distribution.

Chapter 5 A Study on Dynamic Pricing for Bundling

5.1 Research motivation

In this chapter, we intend to study the dynamic pricing for bundling where the components are priced separately. In the case of two products, one product is served as "advertising component" and its price directly affects the demand intensity. In the market, we often see the selling advertisements like "Enjoy the buffet with shark fin only $0.99", "book the travel package and enjoy the 5-star hotel at low price of $99", etc. These products or services which are charged at extremely low prices attract customers' attention having the effect of advertisement. This strategy can be viewed as a pure bundling strategy because the components can only be bought together, though they are separately priced. We want to study how to dynamically price the components over a finite horizon to maximize the total revenue given a fixed account of inventories at the beginning over a finite horizon.

5.2 The mathematical model

Consider one retailer who adopts bundling strategy to sell two individual products, notably product *a* and product *b*. That is, product *a* and product *b* are only sold in pairs. Assume there are a fixed amount of inventories at the beginning of the selling season. Demand is stochastic and customer arrival follows a Poisson distribution. Product *a* is served as the advertising component that has influence on the arrival intensity (λ). When the price of product *a* is low (p_a), it will have positive effect on the customer arrival as more customers are expected to attracted by the low price of the advertising component and pay a visit to the retailer's store; vice verse. The advertising effect of product a is assumed to have diminishing return, which is common in the literature. We choose the widely used power model which has constant elasticity for the intensity function. Thus the function for demand intensity is:

$$\lambda = \lambda_0 {p_a}^{-b} \qquad 5\text{-}1$$

where $\lambda_0 > 0$ and $0 < b < 1$.

After the customers are attracted to the shop, we assume that customers will make purchasing decision based on their reservation price for the main component, for which the probability density function and cumulative density function are $f(r)$ and $F(r)$ respectively. For a certain main component price p_b , a customer has probability of $1 - F(p_b)$ to buy the bundle and probability of $F(p_b)$ to leave without purchasing. The retailer dynamically decides the advertising component price p_a and the main component price p_b to control the demand intensity and customer purchasing probability so as to achieve maximal revenue for the limited amount of inventories over the selling period.

5.2.1 Notations

Below is a summary of notations for this study.

Parameters:

T : Selling period

t: Time variable, $0 \le t \le T$

R_b: Reservation price for the main component;

$f(r)$: Probability density function of the customer reservation price distribution for the main component;

$F(r)$: Cumulative density function of the customer reservation price distribution for the main component;

λ: Customer arrival rate, which is a function of the advertising component price p_a and cost c_a, assumed as $\lambda = \lambda_0 p_a{}^{-b}$

N : Available number of bundle units at beginning

Decision variables:

p_a: Price of the advertising component

p_b: Price of the main component

5.2.2 The model

Assume each customer arriving only demands one bundle. The time horizon T is divided into many sufficient small time intervals Δt. Within each interval, at most one customer can arrive with probability $\lambda\Delta t$, while one customer arrival probability is $1-\lambda\Delta t$ and the event of two or more than two customer arrivals has neglectable probability $o(\Delta t)$.

$J_n(t)$ is assumed as the expected revenue of selling bundles with n stock at time t. The retailer needs to set the price of the advertising component p_a and the main component p_b based on the available inventory and remaining time $T-t$ to maximize the total expected revenue.

$$J_n(t) = \max_{p_a, p_b}[(1-\lambda\Delta t)J_n(t+\Delta t) + \lambda\Delta t F(p_x)J_n(t+\Delta t) + \lambda\Delta t(1-F(p_x))(p_a + p_b + J_{n-1}(t+\Delta t))] \quad 5\text{-}2$$

The expected revenue with n stock at time t is generated from three possible events: no customer arrival, one customer arrives but does not purchase, and one customer arrives and purchases one bundle. $J_n(t)$ is weighted sum of resulted outcomes from the three events, where the weight is the according event probability. For the third event, the instant revenue from selling one unit of the bundle would be $p_a + p_b$.

As assumed the customer arrival rate is determined by a power model of the advertising component price in equation 5-1, we substitute it into equation 5-2 and rearrange it by dividing Δt at both sides:

$$-J_n'(t) = \max_{p_a, p_b}[\lambda_0 p_a^{-b}(1-F(p_b))(p_a + p_b - (J_n(t) - J_{n-1}(t)))] \quad 5\text{-}3$$

Under first-order conditions:

$$-\frac{\partial J_n'(t)}{\partial p_a} = -\lambda_0 b p_a^{-b-1}(1-F(p_b))(p_a + p_b - (J_n(t) - J_{n-1}(t))) + \lambda_0 p_a^{-b}(1-F(p_b)) = 0 \quad 5\text{-}4$$

$$-\frac{\partial J_n'(t)}{\partial p_b} = \lambda_0 p_a^{-b}(-f(p_b))(p_a + p_b - (J_n(t) - J_{n-1}(t))) + \lambda_0 p_a^{-b}(1-F(p_b)) = 0 \quad 5\text{-}5$$

We can obtain the optimal solutions for the two price variables:

$$p_a^* = b\frac{1-F(p_b)}{f(p_b)} \quad 5\text{-}6$$

$$p_b^* = (1-b)\frac{1-F(p_b)}{f(p_b)} + (J_n(t) - J_{n-1}(t)) \quad \text{5-7}$$

5.2.3 A specific function for reservation price distribution of the main component

Specifically, we assume an exponential function for the reservation price distribution for the main component as $1-F(p) = e^{-\alpha p}$. After substituting it into equation 5-3, we have:

$$\begin{aligned} -J_n'(t) &= \max_{p_a, p_b}[\lambda_0 p_a^{-b} e^{-\alpha p_b}(p_a + p_b - (J_n(t) - J_{n-1}(t)))] \\ &= \lambda_0 (\frac{b}{\alpha})^{-b} \alpha^{-1} e^{b-1-\alpha(J_n(t)-J_{n-1}(t))} \end{aligned} \quad \text{5-8}$$

Let $\gamma = \lambda_0 (\frac{b}{\alpha})^{-b} \alpha^{-1} e^{b-1}$, then $-J_n'(t) = \gamma e^{-\alpha(J_n(t)-J_{n-1}(t))}$. To solve this differential equation, we need to first identify the boundary conditions, which are $J_n(T) = J_0(t) = 0$. Then we start from $n=1$ to derive solutions for each $J_n(t)$ at any time t as following.

$$J_1(t) = \frac{1}{\alpha}\ln(1+\alpha\gamma(T-t))$$

$$J_2(t) = \frac{1}{\alpha}\ln(1+\alpha\gamma(T-t) + \frac{\alpha^2\gamma^2}{2}(T-t)^2)$$

…

$$J_n(t) = \frac{1}{\alpha}\ln(\sum_{i=0}^{n}\frac{[\alpha\gamma(T-t)]^i}{i!}) \quad \text{5-9}$$

Substituting equation 5-9 into equation 5-7, we can find optimal prices p_b^*. And optimal price p_a^* in equation 5-6 is simplified based on the specific exponential function.

$$p_a^*(t) = \frac{b}{\alpha} \quad \text{5-10}$$

$$p_b^*(t) = \frac{1-b}{\alpha} + \frac{1}{\alpha} \ln \frac{\sum_{i=0}^{n} \frac{[\alpha\gamma(T-t)]^i}{i!}}{\sum_{i=0}^{n-1} \frac{[\alpha\gamma(T-t)]^i}{i!}} \qquad \text{5-11}$$

Interestingly, under the dynamic pricing scheme the optimal price for the advertising component is deterministic with the specific function for main component price distribution. The optimal price for the main component is dynamically changing over the selling horizon, depending on the inventory available. Figure 5-1 is the illustration for the two component prices and inventory level under one sample path (the parameters are set as $\alpha = 0.2, b = 0.2, T = 100, N = 20, \lambda_0 = 8$).

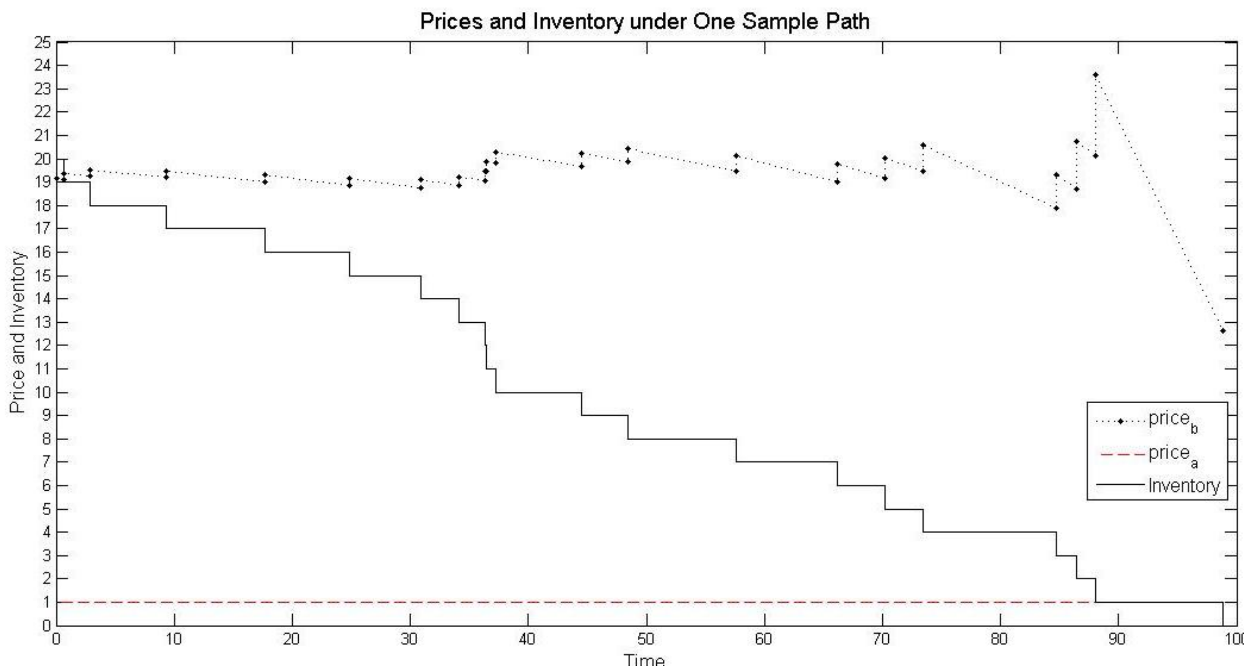


Figure 5-1: One sample path

5.3 The fixed price scheme

Dynamically changing prices brings difficulty in implementation. Therefore, we propose an alternative where prices are fixed as $p_{a,f}$ and $p_{b,f}$. Then we have the below constrained non-linear optimization model to compute the expected revenue at the beginning of the selling period with initial inventory N:

$$\begin{aligned} &\max: \ \lambda_0(p_{a,f})^{-b} e^{-\alpha p_{b,f}} (p_{a,f} + p_{b,f})T \\ &s.t. \quad \lambda_0(p_{a,f})^{-b} e^{-\alpha p_{b,f}} T \le N \end{aligned} \quad \text{5-12}$$

K.K.T. method is adopted to solve above constrained optimization problem.

$K.K.T\ \ conditions:$

$$\begin{aligned} &\lambda_0(p_{a,f})^{-b}(-bp_{a,f}{}^{-1})e^{-\alpha p_{b,f}}(p_{a,f} + p_{b,f})T + \lambda_0(p_{a,f})^{-b} e^{-\alpha p_{b,dc}} T \\ &-\beta\lambda_0(p_{a,f})^{-b}(-bp_{a,f}{}^{-1})e^{-\alpha p_{b,f}} T = 0 \end{aligned} \quad \text{5-13}$$

$$\begin{aligned} &\lambda_0(p_{a,f})^{-b} e^{-\alpha p_{b,f}}(-\alpha)(p_{a,f} + p_{b,f})T + \lambda_0(p_{a,f})^{-b} e^{-\alpha p_{b,f}} T \\ &-\beta\lambda_0(p_{a,f})^{-b} e^{-\alpha p_{b,f}}(-\alpha)T = 0 \end{aligned} \quad \text{5-14}$$

$$-\beta\lambda_0(p_{a,f})^{-b} e^{-\alpha p_{b,f}}(-\alpha)T = 0 \quad \text{5-15}$$

$$\beta(N - \lambda_0(p_{a,f})^{-b} e^{-\alpha p_{b,f}} T) = 0 \quad \text{5-16}$$

$$\lambda_0(p_{a,f})^{-b} e^{-\alpha p_{b,f}} T \le N \quad \text{5-17}$$

$$\beta \ge 0 \quad \text{5-18}$$

The solution is as below:

$$p_{a,f}^{*} = \frac{b}{\alpha} \quad 5\text{-}19$$

$$p_{b,f}^{*} = -\frac{1}{\alpha}\ln\frac{N\alpha^{b}}{\lambda_0 T b^{b}} \quad 5\text{-}20$$

$$\beta = -\frac{1}{\alpha}\ln\frac{N\alpha^{b}}{\lambda_0 T b^{b}} - \frac{1-b}{\alpha} \quad 5\text{-}21$$

The expected revenue within time T is:

$$J_{N,f}(0) = N(\frac{\alpha}{b})^{2b}(\frac{b}{\alpha} - \frac{1}{\alpha}\ln\frac{N\alpha^{b}}{\lambda_0 T b^{b}}) \quad 5\text{-}22$$

In addition, we can also derive the expected revenue at any time t from the following formula as in equation 5-3:

$$-J_{n,f}'(t) = \lambda_0 p_{a,f}^{-b}(1-F(p_{b,f}))(p_{a,f} + p_{b,f} - (J_n(t) - J_{n-1}(t))) \quad 5\text{-}23$$

With the identified the boundary conditions $J_{n,f}(T) = J_{0,f}(t) = 0$, the expression for $J_{n,f}(t)$ is:

$$\begin{aligned} J_{n,f}(t) &= (p_{a,f} + p_{b,f})[n - e^{-D_f(T-t)}(n + (n-1)D_f(T-t))] \\ &= (p_{a,f} + p_{b,f})[n + \frac{D_f(T-t)\Gamma(n, D_f(T-t)) - \Gamma(n+1, D_f(T-t))}{\Gamma(n)}] \end{aligned} \quad 5\text{-}24$$

Where $D_f = \lambda_0 p_{a,f}^{-b}(1-F(p_{b,f}))$. Specifically, $D_f = \lambda_0 p_{a,f}^{-b} e^{-\alpha p_b}$ for exponential function of main component reservation price distribution. Equation 5-24 is equivalent to equation 5-22 when the fixed prices $p_{a,f}$ and $p_{b,f}$ are set as the optimal results in equation 5-19 and equation 5-20 at the beginning of the selling horizon.

The derivation process for equation 5-24 can be found in Appendix B.

5.4 Numerical analysis

In this section, we intend to compare the performance of dynamic pricing and fixed price schemes under different initial inventory levels. The parameters are set same as that in the sample illustration in section 5.2.3, i.e. $\alpha = 0.2, b = 0.2, T = 100, N = 20, \lambda_0 = 8$.

Table 5-1: Comparison between dynamic pricing and fixed pricing

Initial Inventory	Dynamic pricing	Fixed price	Gap
5	131.8793	123.2477	7.0%
10	229.1013	218.8492	4.7%
15	313.2421	302.062	3.7%
20	388.8879	377.0685	3.1%
25	458.217	445.9178	2.8%

The analytical analysis in previous sections shows that the prices for the advertising component are same and deterministic in both dynamic pricing and fixed price schemes. It is the main component price in the two different schemes that causes their performance difference. In the dynamic pricing scheme, the main component price changes according to the inventory level and time lapsed. If no inventory is sold during a certain time period, the main component price continues to drop. It rises after one unit of inventory is purchased by the arrived customer. This dynamic response makes the dynamic pricing scheme perform better than fixed price algorithm at whatever initial inventory levels, as confirmed in Table 5-1. Besides, the table also shows that the performance gap shrinks when the initial inventory increases. That is, dynamic pricing has more value when the initial inventory is low at the beginning of the selling period.

5.5 Two other demand functions

We also try to investigate two other common demand functions, e.g. linear demand function and sigmoid demand function. Under these two functions, the results are similar as that in the power model in terms of solution structure as studied above. Therefore, we just list down the optimal dynamic prices for both components. Other results such as fixed price algorithms should also be similar, which are ignored here.

5.5.1 Linear demand function

Let $\lambda = \lambda_0 (a - b p_a)$

The optimal results are:

$$p_a^* = \frac{a}{b} - \frac{1-F(p_b)}{f(p_b)}$$
$$p_b^* = \frac{2[1-F(p_b)]}{f(p_b)} - \frac{a}{b} + (J_n(t) - J_{n-1}(t))$$

Specifically, when $1-F(p) = e^{-\alpha p}$

$$p_a^* = \frac{a}{b} - \frac{1}{\alpha}$$
$$p_b^* = \frac{2}{\alpha} - \frac{a}{b} + \frac{1}{\alpha}\ln\frac{\sum_{i=0}^{n}\frac{[\alpha\gamma(T-t)]^i}{i!}}{\sum_{i=0}^{n-1}\frac{[\alpha\gamma(T-t)]^i}{i!}}$$

5.5.2 Sigmoid demand model

$$\lambda = \lambda_0(1 - \frac{1}{1+e^{-b(p_a-p_0)}}) = \frac{\lambda_0}{e^{b(p_a-p_0)}+1}$$

$$p_a^* = p_0 - \frac{1}{b}\ln(\frac{b(1-F(p_b))}{f(p_b)} - 1)$$
$$p_b^* = \frac{1-F(p_b)}{f(p_b)} + \frac{1}{b}\ln(\frac{b(1-F(p_b))}{f(p_b)} - 1) - p_0 + (J_n(t) - J_{n-1}(t))$$

Specifically, when $1-F(p) = e^{-\alpha p}$

$$p_a^*(t) = p_0 - \frac{1}{b}\ln(\frac{b}{\alpha} - 1)$$
$$p_b^*(t) = \frac{1}{\alpha} + \frac{1}{b}\ln(\frac{b}{\alpha} - 1) - p_0 + \frac{1}{\alpha}\ln\frac{\sum_{i=0}^{n}\frac{[\alpha\gamma(T-t)]^i}{i!}}{\sum_{i=0}^{n-1}\frac{[\alpha\gamma(T-t)]^i}{i!}}$$

5.6 Chapter conclusion

In this chapter, we studied the dynamic pricing for the bundling problem, where the bundle is defined as consisting of one advertising component and main component. We proposed two different schemes for such situation, namely dynamic pricing and fixed pricing. We derived closed form results for two prices and the total expected revenue under both schemes. Specifically, we compare the results of these two schemes for main component reservation price that follows exponential distribution. The impact of initial inventory on the comparison between dynamic pricing and fixed pricing for bundling here is similar to that under single problem context, that is low initial inventory is a better condition for dynamic pricing scheme.

Chapter 6 Conclusions and Future Research

6.1 Summary and limitations

The first work in this research examined the problem of joint pricing and inventory decisions in mixed bundling under stochastic market. The reservation price model was used to model the relation between prices and demands. Based on these demands, a two-stage stochastic programming model was developed for inventory decisions, including ordering decision at the first stage before the selling season starts and inventory allocation decision when demands are realized. By solving this model, detailed tractable results were derived, thanks to the specific structure of mixed unbundling. A searching algorithm was proposed to find the optimal solution for this multi-variable problem, with Downhill Simplex Method employed as the searching technique. The results of simulation suggest that compared to unbundling, mixed bundling tends to increase prices for the individual products and set a proper price for the bundle. This is consistent with the observations with respect to prices obtained under the stable market where inventory is not under consideration. It indicates that regardless of market conditions, bundling strategy works as a powerful pricing tool. This could be attributed to the rationale of bundling that it can direct customers of various preferences to different alternatives in the way that gives the retailer the best outcomes. Stochastic market makes inventory an important issue which needs to be closely investigated. At the optimal solution, mixed bundling always stocks more inventories for both individual products, resulting in a larger ordering cost, to hedge against market uncertainty. This explanation is supported by the increased aggregated demands through demand reshaping by mixed bundling strategy. Previously, there were results with regard to inventory only under unbundling and pure bundling (McCardle et al., 2007). This study provided a more comprehensive analysis for inventory issue from perspective of mixed bundling, which combines unbundling and pure bundling. Considerable numerical analysis was carried out to examine the effects of relevant factors, such as cost structure and demand variation, on the performance of mixed bundling. These results can serve as guidelines for practitioners who face particular market conditions.

The model was extended to consider the issue of substitution when customers are allowed to switch to their secondary choice if their more preferred one is not available. The results indicated that more inventories should be stocked for each product compared to the case without substitution. This is due to the fact that substitution raises the demand for the product which can be used to satisfy demand originally for other products. In mixed bundling, both individual products can serve as substitutes for bundle demand if this demand is not fully met.

This work is the first attempt to jointly tackle pricing and inventory decisions under mixed bundling, extending previous work from single product to multi-product problem. One key contribution is that tractable results are derived with respect to inventory while such results in other multi-product problems in similar settings are usually unachievable. The results of this thesis have demonstrated the rationale of mixed bundling under stochastic market and provided optimal solutions for the model. The thesis gives managerial implications for practitioners who may adopt bundling strategy in uncertain environments. In addition, this study provides a framework for bundling analysis under stochastic environments and could be a milestone work for any further research.

One major limitation of this work is that the results obtained from the proposed algorithm may not be the global optimum because no similar concavity property with regard to prices has been derived as the one for inventory. This is due to the reservation price model chosen to model product demands, where resulted demands are in quadratic form of prices. Although enumeration in all possible price vectors could ensure global optimal results, it is extremely time-consuming and theoretically unappealing. This problem can be conquered by choosing a proper customer choice model that may lead to attractive properties in prices. Another limitation is that the reservation prices for the two individual products were assumed independently distributed, and correlation in reservation prices is usually an important factor that should be examined in the bundling literature.

The second limitation was tackled in Chapter 4, where we also used a more realistic consumer choice model, i.e., Multinomial Logit Model and considered different degree of contingency between the individual products for each customer. Due to the high complexity, we turned to the simulation optimization method for optimal results, namely, Sample Average Approximation with IPA gradient estimation. The results provided more managerial insights regarding several important factors like correlation coefficient between the reservation prices for the individual products and degree of contingency.

At last, we proposed a study on dynamic pricing for bundling. Closed form results were derived under several common demand functions. Its performance was analyzed by comparison with fixed price algorithms. However, the bundling was different from the conventional concept in bundling literature as we defined the bundling consisting of an advertising component and a main component in terms of pricing effect.

6.2 Future directions

As mentioned, this study can serve as a basic work for any further extensions within the area of mixed bundling under stochastic environments. The following are some possible future research directions from this thesis:

- Extension from two-product mixed bundling to general N products mixed bundling. This is important as most retailers in real market usually handle dozens of products. It is of great interest to build a general model for N-product joint pricing and inventory mixed bundling problem.
- It could be challenging to apply game theory of multi players to the work presented in this thesis. The phenomenon of duopoly and oligopoly and even more fierce competition is common in real business. It may be possible to consider this issue where equilibrium would be reached when multiple retailers make decisions about pricing and inventory under the strategy of mixed bundling.

References

Adams, W.J. and J.L. Yellen, Commodity bundling and the burden monopoly. Quarterly Journal of Economics, 1976. 90(3): p. 475-498.

Anderson, S.P.a.L.L., Why firms may prefer not to price discrimination via mixed bundling. International Journal of Industrial Organization, 1993. 11(1): p. 49-61.

Ansari, A., S. Siddarth, and C.B. Weinberg, Pricing a Bundle of Products or Services: The Case of Nonprofits. Journal of Marketing Research, 1996. 33(1): p. 86-93.

Amiya, B. and V. Padmal, Research Note---Impact of Customer Knowledge Heterogeneity on Bundling Strategy. 2009, INFORMS. p. 792-801.

Arthur, L., Bundling of substitutes or complements. International Journal of Industrial Organization, 1985. 3(1): p. 101-107.

Baker, K.R., M.J. Magazine, and H.L.W. Nuttle, The Effect of Commonality on Safety Stock in a Simple Inventory Model. 1986, INFORMS. p. 982-988.

Bakos, Y. and E. Brynjolfsson, Bundling Information Goods: Pricing, Profits, and Efficiency. Management Science, 1999. 45(12): p. 1613.

Bakos, Y. and E. Brynjolfsson, Bundling and Competition on the Internet. Marketing Science, 2000. 19(1): p. 63-82.

Bitran, G., R. Caldentey. 2003. An overview of pricing models for revenue management. Manufacturing Service Oper. Management 5(3): p.203 – 229.

Bollapragada, r., R. Akella, and R. Srinivasan, Centralized ordering and allocation policies in a two-echelon system with non-identical warehouses. European Journal of Operational Research, 1998. 106(1): p. 74-81.

Bulut, Z., U. Gurler, and A. Sen, Bundle pricing of inventories with stochastic demand. European Journal of Operational Research, 2009. 197(3): p. 897-911.

Carbajo, J., D. de Meza, and D.J. Seidmann, A Strategic Motivation for Commodity Bundling. The Journal of Industrial Economics, 1990. 38(3): p. 283-298.

Chatwin, R.E., 1996. Optimal control of continuous-time terminal-value birth-and-death processes and airline overbooking. Naval Research Logistics 43, 159±168.

Chatwin, R.E., 1998. Multi-period airline overbooking with a single fare class. Operations Research 46, 805±819.

Chatwin, R.E., 1999. Continuous-time airline overbooking with time-dependent fares and refunds. Transportation Science 33, 182±191.

Cheung, K.L., The effects of component commonality in an infinite horizon inventory model. Production Planning & Control, 2002. 13(3): p. 326-333.

Chew, E.K.P., L. Loo Hay, and L. Yew Loon, Component commonality in assembled-to-stock systems. IIE Transactions, 2006. 38(3): p. 239-251.

Chiang, W., Chen, J., Xu, X., 2007. An overview of research on revenue management:Current issues and future research. International Journal of Revenue Management 1 (1), 97 – 128.

Choi, J.P., Bundling new products with old to signal quality, with application to the sequencing of new products. International Journal of Industrial Organization, 2003. 21(8): p. 1179-1200.

Crampes, C. and A. Hollander, Product specification, multi-product screening and bundling: the case of pay TV. Information Economics & Policy, 2005. 17(1): p. 35-59.

Eckalbar, J.C., Exploring Bundling Theory with Geometry. Journal of Economic Education, 2006. 37(1): p. 68-82.

Economides, N.a.S., S., Competition and Integration Among Complements, and Network Market Structure. Journal of Industrial Economics, 1992. 40: p. 105-123.

Elmaghraby, W., P. Keskinocak. 2003. Dynamic pricing: Research overview, current practices and future directions. Management Sci. 49(10) 1287 – 1389.

Eppen, G.a.S., L. , Centralized ordering policies in a multi-warehouse system with lead times and random demand, in Multi-Level Production/Inventory Control Systems: Theory and Practice. Schwarz, L.B. (ed.), North-Holland, Amsterdam, The Netherlands, 1981: p. 51-67.

Gary D. Eppen, Ward A. Hansom, and R. Kipp Martin, Bundling: new products, new markets, low risk. Sloan Management Review, 1991: p. 7-14.

E.P. Chew, L.H.L.a.Y.L.L., Component Allocation and Ordering Policy in a Multi-Component, Multi-product Assembled to Stock System. OR Spectrum, 2010. 32(2): p. 293-317.

Ernst, R. and P. Kouvelis, The Effects of Selling Packaged Goods on Inventory Decisions. Management Science, 1999. 45(8): p. 1142-1155.

Ernst, R.L., A linear inventory model of a monopolistic firm. Ph.D. Dissertation, Department of Economics, University of California, Berkeley, CA., 1970.

Eynan, A. and M.J. Rosenblatt, Component commonality effects on inventory costs. Iie Transactions, 1996. 28(2): p. 93-104.

Feng, Y., and G. Gallego. 1995. Optimal Stopping Times for Promotional Fares and Optimal Starting Times for End- of-Season Sales. Mgmt. Sci. 41, 1371-1391.

Feng, Y. and Gallego, G., 1996. Perishable asset revenue management with Markovian time dependent demand intensities. Department of Industrial Engineering and Operational Research, Columbia University, New York.

Fishburn, P.C., A. Pekec and J.A. Reeds, Subset comparisons for additive linear orders. Mathematics of Operations Research, 2002. 27: p. 227-243.

Gallego, G., G. van Ryzin. 1994. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. Management Sci. 40(8): p.999 – 1020.

Gallego, G. and van Ryzin, G. (1997) A multiproduct dynamic pricing problem and its applications to network yield management. Operations Research 45(1): 24–41.

Gal-Or, E., Evaluating the Profitability of Product Bundling in the Context of Negotiations. Journal of Business, 2004. 77(4): p. 639-373.

Gans, J. and S.P. King, Playing for loyalty: product bundling in oligopoly. Journal of Industrial Economics, 2006. 54(1): p. 43-62.

Gerchak, Y. and M. Henig, An inventory model with component commonality. Operations Research Letters, 1986. 5(3): p. 157-160.

Gerchak, Y., M.J. Magazine, and A.B. Gamble, COMPONENT COMMONALITY WITH SERVICE LEVEL REQUIREMENTS. Management Science, 1988. 34(6): p. 753-760.

Ghosh, B. and S. Balachander, Competitive Bundling and Counterbundling with Generalist and Specialist Firms. Management Science, 2007. 53(1): p. 159-168.

Ginsburgh, V.A. and I. Zang, Bundling by Competitors and the Sharing of Profits. Economics Bulletin, 2007. 12(16).

Grotzinger, S.J., et al., Component Procurement and Allocation for Products Assembled to Forecast - Risk-Pooling Effects. Ibm Journal of Research and Development, 1993. 37(4): p. 523-536.

Gunnar, T.T., A dynamic, nonstationary inventory problem for a price/quantity setting firm. 1975. p. 461-476.

Hanson, W. and R.K. Martin, Optimal Bundle Pricing. Management Science, 1990. 36(2): p. 155-174.

Heidelberger, P., Limitations of infinitesimal perturbation analysis. IBM Resarch Report RC 11891, 1986. No. 53575.

Hillier, M.S., Component commonality in a multiple-period inventory model with service level constraints. International Journal of Production Research, 1999. 37(12): p. 2665.

Hitt, L.M. and C. Pei-yu, Bundling with Customer Self-Selection: A Simple Approach to Bundling Low-Marginal-Cost Goods. Management Science, 2005. 51(10): p. 1481-1493.

Hui, W., et al., Sell by Bundle or Unit?: Pure Bundling versus Mixed Bundling of Information Goods. Decision Support Systems, 2012(In Press).

Jedidi, K., S. Jagpal, and P. Manchanda, Measuring Heterogeneous Reservation Prices for Product Bundles. Marketing Science, 2003. 22(1): p. 107-130.

Karlin, S., C. R. Carr. , Prices and optimal inventory policy. K.J. Arrow, S. Karlin, and H. Scarf (Eds.). Studies in Applied Probability and management Science. Stanford University Press, Stanford, CA, 1962: p. 159-172.

Kim, S., Gradient-based simulation optimization. Proceedings of the 2006 Winter Simulation Conference, 2006.

Kopalle, P.K., A. Krishna, and J.L. Assuncao, The Role of Market Expansion on Equilibrium Bundling Strategies. Managerial and Decision Economics, 1999. 20(7): p. 365-377.

Kyle Y. Lin(2006) Dynamic pricing with real-time demand learning European Journal of Operational Research 174 522 – 538.

L'Ecuyer, P., On the interchange of derivative and expectation for likelihood ration derivative estimators. Management Science, 1995(41): p. 738-748.

Lewbel, A., Bundling of substitutes or complements. International Journal of Industrial Organization, 1985. 3(1): p. 101-107.

MacDonald L. and Rasmussen H. (2010) Revenue management with dynamic pricing and advertising Journal of Revenue and Pricing Management Vol. 9, 1/2, 126–136.

Maglaras, C., Meissner, J., 2006. Dynamic pricing strategies for multi-product revenue management problems. Manufacturing & Service Operations Management 8 (2): p.136 – 148.

Martin, S., Strategic and welfare implications of bundling. Economics Letters, 1999. 62(3): p. 371-376.

Matutes, C. and P. Regibeau, Compatibility and Bundling of Complementary Goods in a Duopoly. The Journal of Industrial Economics, 1992. 40(1): p. 37-54.

McAfee, R.P., J. McMillan, and M.D. Whinston, Multiproduct Monopoly, Commodity Bundling, and Correlation of Values. The Quarterly Journal of Economics, 1989. 104(2): p. 371-383.

McCardle, K.F., K. Rajaram, and C.S. Tang, Bundling retail products: Models and analysis. European Journal of Operational Research, 2007. 177(2): p. 1197-1217.

McFadden, D., Conditional logit analysis of quanlitative choice analysis. P. Zarembka (Ed.). Fronteirs in Econometrics, Academic Press, New York, 1974.

McGill, J., G. van Ryzin. 1999. Revenue management: Research overview and prospects. Transportation Sci. 33(2) 233–256.

M.D., J., H. A., and B. H.H., The effects of price bundling on consumer evaluations of product offerings. International Journal of Research in Marketing, 1999. 16(2): p. 129-142.

Michael D. Johnson, Andreas Herrmann, and H.H. Bauer, The effects of price bundling on consumer evaluations of product offerings. Intern. J. of Research in Marketing, 1999(16): p. 129-142.

Mills, E.S., UNCERTAINTY AND PRICE THEORY. Quarterly Journal of Economics, 1959. 73(1): p. 116-130.

Nelder, J. A., R. Mead. 1965. Downhill Simplex Method in Multidimentions. Comput. J. 7 308-312.

Nalebuff, B., Bundling, tying, and portfolio effects. working paper, 2003.

Olderog, T. and B. Skiera, THE BENEFITS OF BUNDLING STRATEGIES. Schmalenbach Business Review (SBR), 2000. 52(2): p. 137-159.

Peitz, M., Bundling may blockade entry. International Journal of Industrial Organization, 2008. 26(1): p. 41-58.

Petruzzi, N.C. and M. Dada, PRICING AND THE NEWSVENDOR PROBLEM: A REVIEW WITH EXTENSIONS. Operations Research, 1999. 47(2): p. 183-194.

Poter, M., Competitive Advantage: Creating and Sustaining Superior Performance. Free Press, New York, 1985.

R. Suri, M.Z., Perturbation analysis gives strongly consistent estimates for the M/G/1 queue. Management Science, 1988(34): p. 39-64.

Rudi, N., Optimal inventory level in systems with common components. Working paper. The Simon School, University of Rochester, Rochester, NY, 14527, USA, 2000.

Salinger, M.A., A Graphical Analysis of Bundling. Journal of Business, 1995. 68(1): p. 85-98.

Schamalensee, R., Gaussian Demand and Commodity Bundling. Journal of Business, 1984. 57(1): p. 211-230.

Scott, R. and J. Highfill, Mixed Bundling with Profit and Sales Objectives. International Advances in Economic Research, 2001. 7(2): p. 243.

Seidmann, D.J., Bundling as a Facilitating Device: A Reinterpretation of Leverage Theory. Economica, 1991. 58(232): p. 491-499.

Shapiro, A., Monte Carlo sampling methods in Stochastic Programming. Handbooks in Operations Research and Management Scienece, 2003.

Simonin, B.L. and J.A. Ruth, Bundling as a strategy for new product introduction: Effects on consumers' reservation prices for the bundle, the new product, and its tie-in. Journal of Business Research, 1995. 33(3): p. 219-230.

Spector, D., Bundling, tying, and collusion. International Journal of Industrial Organization, 2007. 25(3): p. 575-581.

Stigler, G.J., United States v. Loew's Ink: A note on block booking. Supreme Court Review, 1963. 152.

Stremersch, S. and G.J. Tellis, Strategic Bundling of Products and Prices: A New Synthesis for Marketing. Journal of Marketing, 2002. 66(1): p. 55-72.

Talluri, K. and van Ryzin, G., 1998. An analysis of bid price controls for network revenue management. Management Science 44: p.1577-1593.

Venkatesh, R. and V. Mahajan, A Probabilistic Approach to Pricing a Bundle of Products or Services. Journal of Marketing Research (JMR), 1993. 30(4): p. 494-508.

Venkatesh, R. and W. Kamakura, Optimal Bundling and Pricing under a Monopoly: Contrasting Complements and Substitutes from Independently Valued Products. Journal of Business, 2003. 76(2): p. 211-231.

W. Gong, Y.C.H., Smoothed (conditional) perturbation analysis of discrete event dynamic systems. IEEE Transactions on Automatic Control, 1987. AC-32: p. 858-866.

Xianjun, G., M.B. Stinchcombe, and A.B. Whinston, Bundling Information Goods of Decreasing Value. Management Science, 2005. 51(4): p. 662-667.

Yan, R. and S. Bandyopadhyay, The profit benefits of bundle pricing of complementary products. Journal of Retailing and Consumer Services, 2011. 18(4): p. 355-361.

Y.C. Ho, M.A.E.a.T.T.C., A gradient technique for general storage design in a production line. Int. J. Prod. Res, 1979(17): p. 557-580.

Y.C. Ho, S.L., Extensions of infinitesimal perturbation analysis. IEEE Transactions on Automatic Control, 1987. AC-33: p. 724-438.

Y.C. Ho, X.C.a.C.C., Infinitesimal and finite perburbation analysis for queuing networks. Automatica, 1983(19): p. 439-445.

Young, L., Price, inventory and the structure of uncertain demand. New Zealand Oper. Res. , 1978. 6: p. 157-177.

Zabel, E., Monopoly and Uncertainty. 1970, The Review of Economic Studies Ltd. p. 205-219.

Zhao, W. and Zheng, Y.-S. (2000) Optimal dynamic pricing for perishable assets with nonhomogeneous demand. Management Science 46(3):p. 375–388.

Appendix

Appendix A.

Given $\overline{D}$, which means for a realization of market size M, the possible profits after inventory allocation for a vector (Q_1, Q_2) are as below:

$E_1 = p_1 D_1 + p_2 D_2 + p_b D_b$, when $Q_1 \geq D_1 + D_b, Q_2 \geq D_2 + D_b$

$$\begin{aligned} E_2 &= p_1 D_1 + p_2 (Q_2 - D_b) + p_b D_b \\ &= p_2 Q_2 + p_1 D_1 + (p_b - p_2) D_b \end{aligned}$$, when $Q_1 \geq D_1 + D_b, D_b \leq Q_2 \leq D_2 + D_b$

$E_3 = p_b Q_2 + p_1 D_1$, when $\begin{aligned} &Q_1 \geq D_1 + D_b, Q_2 \leq D_b \\ &Or, Q_1 \leq D_1 + D_b, Q_2 \leq D_2 + D_b, Q_1 - Q_2 \geq D_1 \end{aligned}$

$$\begin{aligned} E_4 &= p_1 (Q_1 - D_b) + p_2 D_2 + p_b D_b \\ &= p_1 Q_1 + p_2 D_2 + (p_b - p_1) D_b \end{aligned}$$, when $D_b \leq Q_1 \leq D_1 + D_b, Q_2 \geq D_2 + D_b$

$E_5 = p_b Q_1 + p_2 D_2$, when $\begin{aligned} &Q_1 \leq D_b, Q_2 \geq D_2 + D_b \\ &Or, Q_1 \leq D_1 + D_b, Q_2 \leq D_2 + D_b, Q_2 - Q_1 \geq D_2 \end{aligned}$

$$\begin{aligned} E_6 &= p_1 D_1 + p_2 (Q_2 - Q_1 + D_1) + p_b (Q_1 - D_1) \\ &= (p_b - p_2) Q_1 + p_2 Q_2 + (p_1 + p_2 - p_b) D_1 \end{aligned}$$, when

$D_1 \leq Q_1 \leq D_1 + D_b, Q_2 \leq D_2 + D_b, D_1 - D_2 \leq Q_1 - Q_2 \leq D_1$

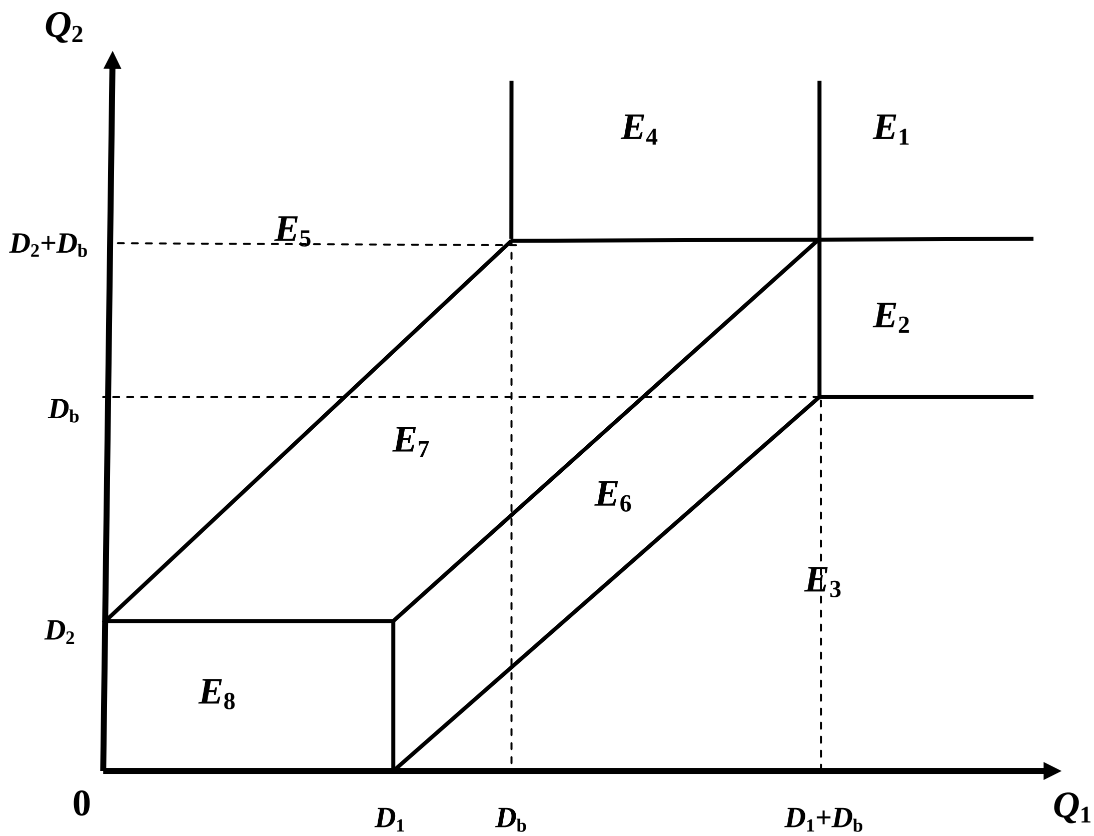
$$\begin{aligned} E_7 &= p_1 (Q_1 - Q_2 + D_2) + p_2 D_2 + p_b (Q_2 - D_2) \\ &= p_1 Q_1 + (p_b - p_1) Q_2 + (p_1 + p_2 - p_b) D_2 \end{aligned}$$, when

$Q_1 \leq D_1 + D_b, D_2 \leq Q_2 \leq D_2 + D_b, D_2 - D_1 \leq Q_2 - Q_1 \leq D_2$

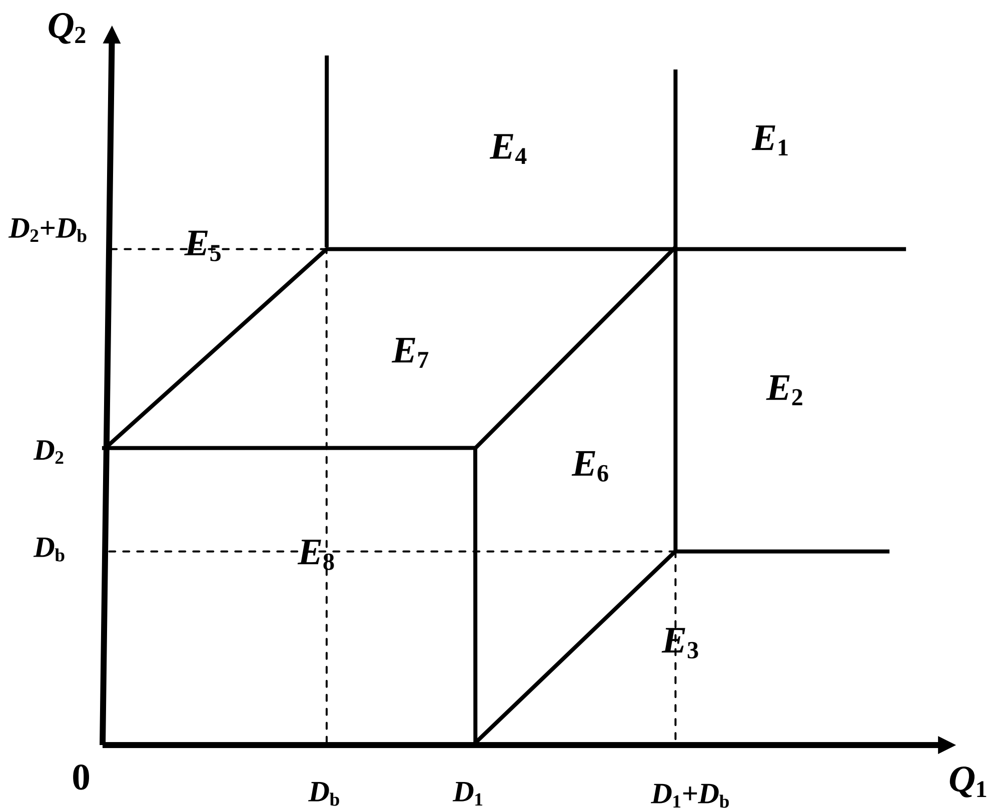
$E_8 = p_1 Q_1 + p_2 Q_2$, when $Q_1 \leq D_1, Q_2 \leq D_2$

4 possible scenarios:

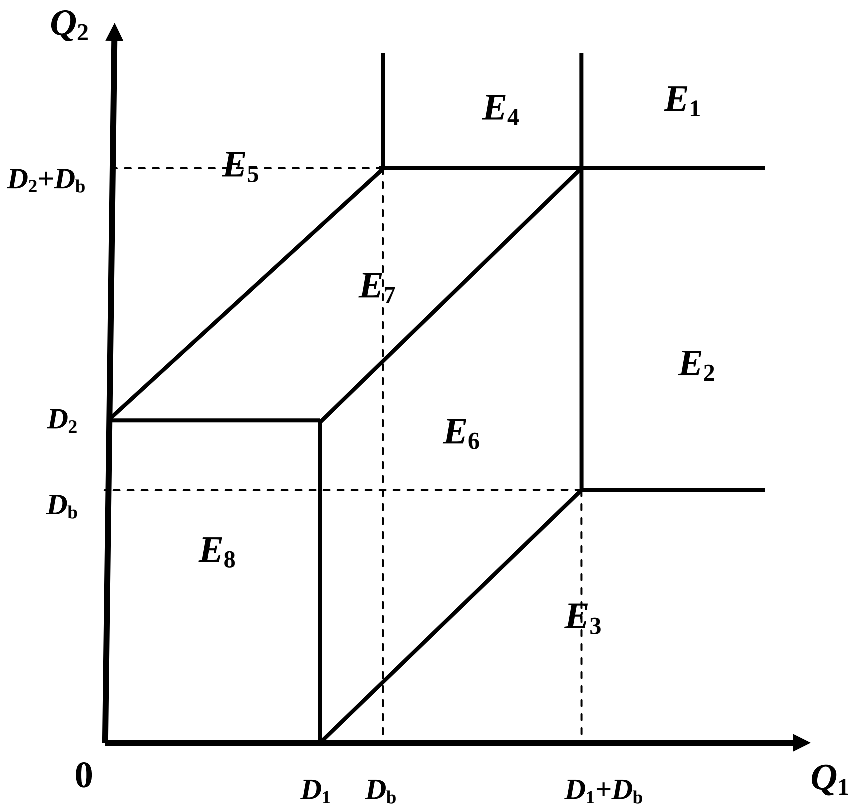
Scenario 1: $D_1 \leq D_b, D_2 \leq D_b$



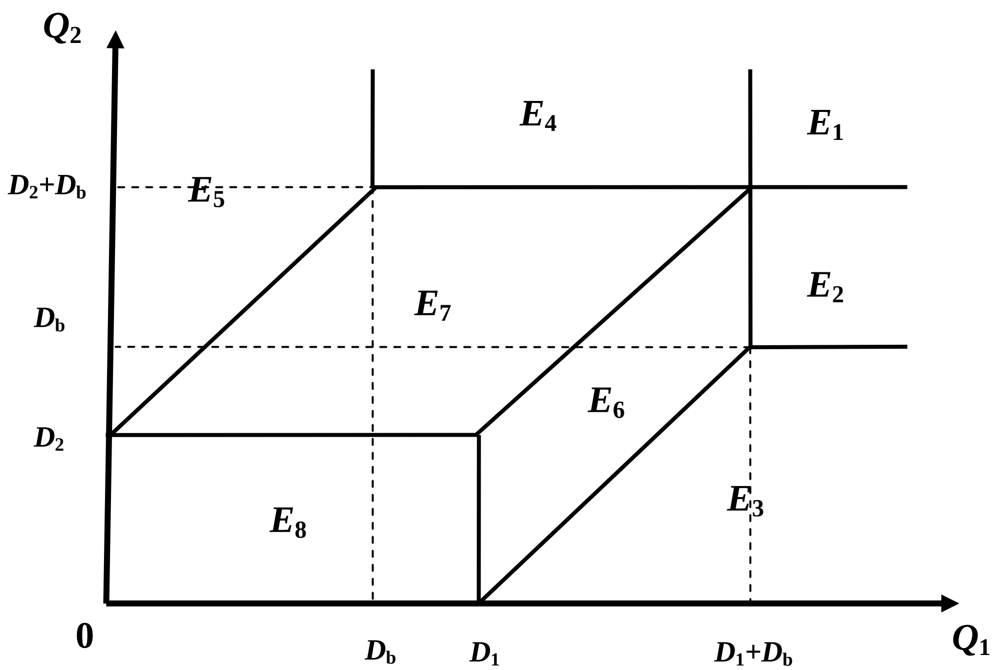
Scenario 2: $D_1 \geq D_b, D_2 \geq D_b$



Scenario 3: $D_1 \leq D_b, D_2 \geq D_b$



Scenario 4: $D_1 \geq D_b, D_2 \leq D_b$



The profit function $E(Q_1,Q_2)|_{\overline{D}} = E_i(Q_1,Q_2)$ if (Q_1,Q_2) satisfies the conditions of area E_i .

We take two steps to prove the concavity of this two-variable piecewise linear function.

Step 1: for any point (Q_1',Q_2') in area E_i , $E_i(Q_1',Q_2') \leq E_j(Q_1',Q_2'), j \neq i, i = \{1,2,...,8\}$

There are total 28 comparisons:

$E_1 - E_2 = -p_2 Q_2' + p_2(D_2 + D_b)$,

$E_1 - E_3 = -p_b Q_2' + p_2 D_2 + p_b D_b$, $E_1 - E_4 = -p_1 Q_1' + p_1(D_1 + D_b)$,

$E_1 - E_5 = -p_b Q_1' + p_1 D_1 + p_b D_b$, $E_1 - E_6 = (p_2 - p_b)Q_1' - p_2 Q_2' + p_2(D_2 - D_1) + p_b(D_1 + D_b)$,

$$E_1 - E_7 = (p_1 - p_b)Q_2' - p_1Q_1' + p_1(D_1 - D_2) + p_b(D_2 + D_b),$$

$$E_1 - E_8 = -p_1Q_1' - p_2Q_2' + p_1D_1 + p_2D_2 + p_bD_b;$$

$$E_2 - E_3 = (p_2 - p_b)Q_2' - (p_2 - p_b)D_b, \; E_2 - E_4 = p_1(D_1 + D_b - Q_1') - p_2(D_2 + D_b - Q_2'),$$

$$E_2 - E_5 = -p_bQ_1' + p_2Q_2' + p_1D_1 - p_2(D_2 + D_b) + p_bD_b,$$

$$E_2 - E_6 = (p_2 - p_b)Q_1' - (p_2 - p_b)(D_1 + D_b),$$

$$E_2 - E_7 = -p_1Q_1' + (p_1 + p_2 - p_b)Q_2' + p_1D_1 + p_bD_b - (p_1 + p_2 - p_b)D_2,$$

$$E_2 - E_8 = -p_1Q_1' + p_1D_1 + (p_b - p_2)D_b;$$

$$E_3 - E_4 = p_bQ_2' - p_1Q_1' + p_1D_1 - p_2D_2 - (p_b - p_1)D_b, \; E_3 - E_5 = p_b(Q_2' - Q_1') + p_1D_1 - p_2D_2,$$

$$E_3 - E_6 = (p_2 - p_b)Q_1' + (p_b - p_2)Q_2' + (p_b - p_2)D_1,$$

$$E_3 - E_7 = -p_1(Q_2' - Q_1') + p_1D_1 - (p_1 + p_2 - p_b)D_2, \; E_3 - E_8 = -p_1Q_1' + (p_b - p_2)Q_2' + p_1D_1;$$

$$E_4 - E_5 = (p_1 - p_b)(Q_1' - D_b),$$

$$E_4 - E_6 = (p_1 + p_2 - p_b)Q_1' - p_2Q_2' + p_2D_2 + (p_b - p_1)D_b - (p_1 + p_2 - p_b)D_1,$$

$$E_4 - E_7 = (p_1 - p_b)Q_2' + (p_b - p_1)(D_2 + D_b), \; E_4 - E_8 = -p_2Q_2' + p_2D_2 + (p_b - p_1)D_b;$$

$$E_5 - E_6 = p_2(Q_1' - Q_2') + p_2D_2 - (p_1 + p_2 - p_b)D_1, \; E_5 - E_7 = (p_b - p_1)(Q_1' - Q_2' + D_2),$$

$$E_5 - E_8 = (p_b - p_1)Q_1' - p_2Q_2' + p_2D_2;$$

$$E_6 - E_7 = (p_b - p_1 - p_2)(Q_1' - Q_2' + D_2 - D_1), \; E_6 - E_8 = (p_b - p_1 - p_2)(Q_1' - D_1);$$

$$E_7 - E_8 = (p_b - p_1 - p_2)(Q_2' - D_2).$$

From the comparisons above, we can conclude that the statement in step 1 holds.

Step 2: choose any two points (Q_1^1, Q_2^1) and (Q_1^2, Q_2^2) in the feasible region, their linear combination is the point $(\theta Q_1^1 + (1-\theta)Q_1^2, \theta Q_2^1 + (1-\theta)Q_2^2)$, where $0 \le \theta \le 1$. Denote these three points locate in region E_i, E_j and E_k respectively (i, j and k are not necessary different).

We have:

$$\begin{aligned}
&\theta E_i(Q_1^1, Q_2^1) + (1-\theta)E_j(Q_1^2, Q_2^2) \\
&\le \theta E_k(Q_1^1, Q_2^1) + (1-\theta)E_k(Q_1^2, Q_2^2) \\
&= E_k(\theta Q_1^1 + (1-\theta)Q_1^2, \theta Q_2^1 + (1-\theta)Q_2^2)
\end{aligned}$$

Therefore, the profit function $E(Q_1, Q_2)|_{\overline{D}}$ is concave in (Q_1, Q_2), so does the expected total profit function $E(Q_1, Q_2)$.

Appendix B.

Equation 5-12 is as below:

$$-J_{n,f}{}'(t) = \max_{p_{a,f}, p_{b,f}} [\lambda_0 p_{a,f}^{-b} e^{-\alpha p_{b,f}} (p_{a,f} + p_{b,f} - (J_n(t) - J_{n-1}(t)))]$$

Let $D = \lambda_0 p_a^{-b} e^{-\alpha p_b}$, then above equation becomes

$$-J_{n,f}{}'(t) = D(p_{a,f} + p_{b,f} - (J_{n,f}(t) - J_{n-1,f}(t)))$$

When n=1:

$$
\begin{aligned}
&-J_{1,f}'(t) = D(p_{a,f} + p_{b,f} - J_{1,f}(t)) \\
&-\frac{\partial J_{1,f}(t)}{p_{a,f} + p_{b,f} - J_{1,f}(t)} = D\partial t \\
&\ln(p_{a,f} + p_{b,f} - J_{1,f}(t)) = Dt + C_1 \\
&C_1 = \ln(p_{a,f} + p_{b,f}) - DT \\
&p_{a,f} + p_{b,f} - J_{1,f}(t) = (p_{a,f} + p_{b,f})e^{-D(T-t)} \\
&J_{1,f}(t) = (p_{a,f} + p_{b,f})(1 - e^{-D(T-t)})
\end{aligned}
$$

When n=2:

$$
\begin{aligned}
-J_{2,f}'(t) &= D(p_{a,f} + p_{b,f} - J_{2,f}(t) + J_{1,f}(t)) \\
&= D(p_{a,f} + p_{b,f} - J_{2,f}(t) + (p_{a,f} + p_{b,f})(1 - e^{-D(T-t)})) \\
&= -DJ_{2,f}(t) + D(p_{a,f} + p_{b,f})(2 - e^{-D(T-t)})
\end{aligned}
$$

Which is $J_{2,f}'(t) - DJ_{2,f}(t) = -D(p_{a,f} + p_{b,f})(2 - e^{-D(T-t)})$

For the first-order linear equations like:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

The general solution is

$$
\begin{aligned}
&y = Ce^{-\int P(x)dx}, && \text{when } Q(x) = 0 \\
&y = \left(\int Q(x)e^{\int P(x)dx}dx + C\right)e^{-\int P(x)dx}, && \text{when } Q(x) \neq 0
\end{aligned}
$$

So we have:

$$
\begin{aligned}
J_{2,f}(t) &= e^{Dt}(-D)(p_{a,f}+p_{b,f})\left(\int (2-e^{-D(T-t)})e^{-Dt}dt + C_2\right) \\
&= e^{Dt}(-D)(p_{a,f}+p_{b,f})\left(\int (2e^{-Dt}-e^{-DT})dt + C_2\right) \\
&= e^{Dt}(-D)(p_{a,f}+p_{b,f})\left(-\frac{2e^{-Dt}}{D} - e^{-DT}t + C_2\right)
\end{aligned}
$$

When t=T, $J_{2,f}(T)=0$. So

$$
-\frac{2e^{-DT}}{D} - e^{-DT}T + C_2 = 0
$$

$$
C_2 = \frac{2e^{-DT}}{D} + e^{-DT}T
$$

Then

$$
\begin{aligned}
J_{2,f}(t) &= e^{Dt}(-D)(p_{a,f}+p_{b,f})\left(-\frac{2e^{-Dt}}{D} - e^{-DT}t + \frac{2e^{-DT}}{D} + e^{-DT}T\right) \\
&= (p_{a,f}+p_{b,f})(2 + Dte^{-D(T-t)} - 2e^{-D(T-t)} - DTe^{-D(T-t)}) \\
&= (p_{a,f}+p_{b,f})(2 - e^{-D(T-t)}(2 + D(T-t))) \\
&= (p_{a,f}+p_{b,f})[2 + e^{-D(T-t)}(D(T-t)(1+D(T-t)) - 2(1 + D(T-t) + \frac{[D(T-t)]^2}{2})] \\
&= (p_{a,f}+p_{b,f})[2 + \frac{D(T-t)\Gamma(2, D(T-t)) - \Gamma(2+1, D(T-t))}{\Gamma(2)}]
\end{aligned}
$$

Then follow the same logic, we can derive the revenue function at the inventory level of n and time t:

$$
\begin{aligned}
J_{n,f}(t) &= (p_{a,f}+p_{b,f})(n - e^{-D(T-t)}(n + (n-1)D(T-t))) \\
&= (p_{a,f}+p_{b,f})[n + \frac{D(T-t)\Gamma(n, D(T-t)) - \Gamma(n+1, D(T-t))}{\Gamma(n)}]
\end{aligned}
$$