# THE EINSTEIN-SCALAR FIELD LICHNEROWICZ EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS

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NATIONAL UNIVERSITY OF SINGAPORE

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# THE EINSTEIN-SCALAR FIELD LICHNEROWICZ EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS

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# Declaration

I hereby declare that this thesis was composed in its entirety by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Besides, I understand that I have duly acknowledged all the sources of information which have been used in the thesis.

Finally, this thesis has also not been submitted for any degree in any university previously.

Ngo Quoc Anh

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Singapore *Ngo Quoc Anh*

# Dedication

*For my ♡ parents...*

# Contents

# Summary

We establish some new existence and multiplicity results for positive solutions of the following Einstein-scalar field Lichnerowicz equations on compact manifolds $(M, g)$ without the boundary of dimension $n \geqslant 3$,

$$-\Delta_g u + hu = f u^{\frac{n+2}{n-2}} + a u^{-\frac{3n-2}{n-2}},$$

with either a negative, a zero, or a positive Yamabe-scalar field conformal invariant $h$. These equations arise from the Hamiltonian constraint equation for the Einstein-scalar field system in general relativity.

The variational method can be naturally adopted to the analysis of the Hamiltonian constraint equations. However, it arises analytical difficulty, especially in the case when the prescribed scalar curvature-scalar field function $f$ may change sign. To our knowledge, such a problem in its most generic case remains open.

Finally, we establish some Liouville type results for a wider class of equations with constant coefficients including the Einstein-scalar field Lichnerowicz equation with constant coefficients.

# List of Notations and Conventions

We list here some notation and conventions which will be used.

| | |
|---|---|
| $\rightharpoonup$, $\to$ | Weak and strong convergences |
| $\Vert\cdot\Vert_X$, $\vert\cdot\vert_g$ | Norm of functions, norm of tensors with respect to the metric $g$ |
| $\Vert u\Vert_{H_h^1}$ | Norm of $u$ given by $\left(\int_M \vert\nabla u\vert^2\,d\mathrm{vol}_g + \int_M hu^2\,d\mathrm{vol}_g\right)^{\frac{1}{2}}$ |
| $f^+, f^-$ | Positive part $\max\{f,0\}$ and negative part $\min\{f,0\}$ of $f$ |
| $2^\star, 2^\flat$ | Critical exponent $\frac{2n}{n-2}$ and the average of 2 and $2^\star$, that is, $\frac{2n-2}{n-2}$ |
| $C_0^\infty(M)$ | Smooth functions with compact support |
| $H_p^k(M)$, $H^k(M)$ | Standard Sobolev spaces with $H^k(M) = H_2^k(M)$ |
| $\langle\cdot,\cdot\rangle$ | Inner product |
| $\nabla_i$, $\partial_i$ | Covariant and partial derivatives |
| $\Gamma_{ij}^k$ | Christoffel symbols |
| $\mathcal{L}_X$ | Lie derivative with respect to the vector field $X$ |
| $\mathbb{L}_g$ | Conformal Killing operator relative to the metric $g$ |
| $L_g$, $L_{g,\psi}$ | Conformal and conformal scalar field Laplacian operators |
| $\Delta_{g,\mathrm{conf}}$ | Second order, self-adjoint, linear, elliptic operator given by $\mathrm{div}_g \circ \mathbb{L}_g$ |
| $d\mathrm{vol}_g$ | Volume element with respect to the metric $g$ |
| $\alpha_{i_1\cdots i_r}{}^{j_1\cdots j_s}$ | A tensor of type $(s,r)$ |
| $\mathrm{Ric}_{ij}, \mathrm{Eins}_{ij}$ | Ricci and Einstein curvature tensors |
| $\mathrm{Scal}_g$ | Scalar curvature with respect to the metric $g$ |

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# 1

# Introduction



The Einstein theory of relativity or general relativity is a geometric theory of gravitation. In this theory, gravity is considered as a geometric property of space and time. Because of the geometric property of space and time, general relativity partially includes both special relativity and the Newton law of universal gravitation as special cases. Theoretically, general relativity describes objects in large scale, like the universe, as Lorentzian manifolds on which gravitation interacts and the universe evolves over time through a system of partial differential equations known as the Einstein equations. Being the central object of the theory, studying the Einstein equations becomes a significant subject in order to understand the whole theory.

In an effort to solve the general Einstein equations, physicists first try to tackle the equations in some simple cases. Fortunately, some remarkably solutions have been found in this direction. Although general relativity nearly coincides with the Newton law of universal gravitation, those known solutions for the Einstein equations in particular cases have led theoretical physicists to predict some new phenomena which deserve investigation carefully. However, much less is known about the solutions of the general Einstein equations. On the other hand, due to the geometric nature of the theory, solving the general Einstein equations turns out to be a wonderful research topic not only for physicists but also for mathematicians, pushing the development of the research rapidly.

However, along with the rapid development of the research, it poses many challenging problems to mathematicians, for example, the initial value problems, the well-posedness problems, the global stability problems, etc. Among these problems, the initial value problems which have their own interest from the mathematical point of view turns out to be the most interesting problem since it involves the theoretical question of the beginning and the end of our universe. When solving the initial value problems, one needs to solve the so-called constraint equations that form an under-determined system of equations which are not easy to solve. For this reason, understanding the constraint equations is a key step to understanding the initial value problems.

This chapter provides a brief description of the constraint equations for the general Einstein equations, while a more detailed discussion of the Einstein equations with field sources is given in Chapter 2. The content of this chapter was basically adapted from [8], see also [30, 7]. Before we describe the constraint equations, we will briefly recall the Einstein equations in general relativity.

## 1.1 The Einstein equations in general relativity

### 1.1.1 The Einstein equations

Let $(V, \mathbf{g})$ be a differentiable manifold of dimension $n+1$ equipped with a non-degenerate, smooth, symmetric metric tensor $\mathbf{g}$ which, unlike a Riemannian metric, needs not be positive-definite, but must be non-degenerate. Such a metric is called a pseudo-Riemannian metric and, clearly, its values can be positive, negative or zero. With a pseudo-Riemannian metric $\mathbf{g}$, the pair $(V, \mathbf{g})$ is called a pseudo-Riemannian manifold, or simply we call $V$ a pseudo-Riemannian manifold if metric $\mathbf{g}$ is already known.

If we further assume that the signature of metric $\mathbf{g}$ is $(1, n)$, that is, $(-, + \cdots +)$, then $\mathbf{g}$ is called a Lorentzian metric and $(V, \mathbf{g})$ becomes a Lorentzian manifold. In general relativity, the main objects of study are spacetimes. A spacetime in general relativity is a Lorentzian manifold. An usual spacetime manifold is of dimension four, but higher dimensions are considered in the aim of unification of gravitation with the other fundamental forces of nature, electromagnetism, weak and strong interactions, and also in super-symmetric theories. There is a priori no restriction on the topology of spacetime.

We assume throughout this chapter that $(V, \mathbf{g})$ is a Lorentzian manifold of dimension $n+1$. Like other field equations in physics, the original Einstein equations can be derived from an action through the principle of least action. More precise, the action

$$\int_V (\mathrm{Scal}_{\mathbf{g}} + L)\, d\mathrm{vol}_{\mathbf{g}} \tag{1.1.1}$$

is required to be stable under compact perturbations of the metric where $L$ is the Lagrangian associated with non-gravitational fields. When $L = 0$, the action (1.1.1) is known as the Einstein-Hilbert action. By computing the variation of (1.1.1) with respect to $\mathbf{g}$, we obtain the field equation

$$\mathrm{Ric}_{\mathbf{g}} - \frac{1}{2}\mathrm{Scal}_{\mathbf{g}}\, \mathbf{g} = T \tag{1.1.2}$$

where the left hand side of (1.1.2) is known as the Einstein tensor which is denoted by $\mathrm{Eins}_{\mathbf{g}}$. If we look at the computation carefully, we then see that the Einstein tensor is a multiple of the variation of $\int_V \mathrm{Scal}_{\mathbf{g}}\, d\mathrm{vol}_{\mathbf{g}}$ with respect to $\mathbf{g}$ while the tensor $T$ is a multiple of the variation of $\int_V L\, d\mathrm{vol}_{\mathbf{g}}$ with respect to $\mathbf{g}$ and is known as the stress-energy tensor. By taking the trace of the Einstein tensor, we arrive at

$$\mathrm{trace}\, \mathrm{Eins}_{\mathbf{g}} = \frac{1-n}{2} \mathrm{Scal}_{\mathbf{g}}\, .$$

A vacuum space time is a Lorentzian manifold $(V, \mathbf{g})$ that satisfies (1.1.2) with $T = 0$. In this case, we can take the trace of (1.1.2) to find $\mathrm{Scal}_{\mathbf{g}} = 0$ and obtain the vacuum Einstein equations $\mathrm{Ric}_{\mathbf{g}} = 0$. The fully Einstein equations (with cosmological constant) may be written in the following form

$$\mathrm{Eins}_{\mathbf{g}} + \Lambda \mathbf{g} = T$$

where $\Lambda$ is constant called the cosmological constant. The cosmological constant term was originally introduced by Einstein to allow for a static universe. It is clear to see that the cosmological term could be absorbed into the stress-energy tensor $T$ as dark energy.

### 1.1.2 Real scalar fields

As can be seen from the previous section, the Einstein equations involve the so called stress-energy tensor $T$ representing the density of all the energies, momentum, and stresses of the sources. On a macroscopic scale, one can couple gravity to either field sources or matter sources such as electromagnetic fields, Yang-Mills sources, and scalar fields. While the latter are more phenomenological, the former, which are deduced directly from special relativity, are one of the simplest non-vacuum models which are the core of studies in recent years. In addition, interest in those models stems partly from recent attempts to use such models to study the observed acceleration of the expansion of the universe. Throughout this thesis, we only focus on the Einstein equations equipped with scalar fields.

In modern cosmology, one can introduce on the spacetime $(V, \mathbf{g})$ a real scalar field $\boldsymbol{\psi}$ with potential $U$ as a smooth function of $\boldsymbol{\psi}$. A particular Einstein field theory is specified by the choice of an action principle with

$$L = -\frac{1}{2}|\nabla \boldsymbol{\psi}|^2_{\mathbf{g}} - U(\boldsymbol{\psi}). \tag{1.1.3}$$

This choice of action principle also includes the well-known massive or massless Klein-Gordon field theory where $U(\boldsymbol{\psi}) = \frac{1}{2} m^2 \boldsymbol{\psi}^2$. In view of the Einstein–Hilbert action and by a fairy standard computation, one can easily deduce that

$$T_{\alpha\beta} = \nabla_\alpha \boldsymbol{\psi} \nabla_\beta \boldsymbol{\psi} - \frac{1}{2} \mathbf{g}_{\alpha\beta} \nabla_\mu \boldsymbol{\psi} \nabla^\mu \boldsymbol{\psi} - \mathbf{g}_{\alpha\beta} U(\boldsymbol{\psi}).$$

It is worth noticing that a cosmological constant $\Lambda$ can be considered as a particular scalar field with potential $\Lambda$. For this reason, we do not consider the cosmological term in our Einstein equation.

In view of the contracted Bianchi identities, the scalar field $\psi$ is supposed to satisfy the following wave equation

$$\nabla^\alpha \nabla_\alpha \psi = U'(\psi).$$

Of course, this equation should be coupled to the field equations (1.1.2).

### 1.1.3 The Cauchy problem for the Einstein equations and the Einstein constraint equations

Since $\mathbf{g}$ is a pseudo-Riemannian metric, that metric naturally splits each tangent space into three regions depending on the sign of $\mathbf{g}$. A vector $X$ is called timelike, spacelike, or null if $\mathbf{g}(X, X)$ is negative, positive, or zero respectively. Therefore, a curve is said to be timelike, spacelike, or null if its tangent vector at every point is timelike, spacelike, or null respectively. Similarly, we say a hypersurface $M$ of $V$ is timelike, spacelike, or null if its tangent space at every point has a spacelike, timelike, or null normal vector respectively.

A spacelike hypersurface $M$ of $V$ is called a Cauchy surface if every inextensible timelike curve in $V$ intersects $M$ once and only once. One can easily observe that not every Lorentzian manifold $V$ admits a Cauchy surface. For those that admit a Cauchy surface are called globally hyperbolic. The globally hyperbolic property of Lorentzian manifold plays an important role since it is well-known that every globally hyperbolic Lorentzian manifold always admits a continuous, globally defined, timelike vector field $F$ so that a time orientation can be defined through $F$. With the choice of time orientation, any timelike vector $X$ is said to be future or past if $\mathbf{g}(F, X)$ is negative or positive respectively.

Since the Einstein equations are geometric equations, one can expect solutions of the Einstein equations verifying the causality principle, that is, the relativistic spacetime future cannot influence the past. Based on two works by Leray [26] and Geroch [17], we know that the globally hyperbolic spacetime $(V, \mathbf{g})$ are topological products $M \times \mathbb{R}$ with each $M \times \{t\}$ intersected once by each inextensible timelike curve. In view of our definition above, such a submanifold $M \times \{t\}$ is a Cauchy surface.

In order to formulate an appropriate Cauchy problem for the Einstein equations, we assume that the globally hyperbolic spacetime $(V, \mathbf{g})$ admits $M$ as a Cauchy surface. We let $\overline{g}$ be the Riemannian metric on $M$ induced by $\mathbf{g}$. Having such a Cauchy surface, we let $n$ be the future pointing timelike unit normal vector to $M$. We also let $\overline{K}$ be the extrinsic curvature of $M$ computed with respect to $n$. We are now able to formulate the Cauchy problem for the Einstein equations intrinsically through the following definition, see [8, Chapter VI].

**Definition 1.1.**

1. *An initial data set is a triplet $(M, \overline{g}, \overline{K})$ where $M$ is an $n$-dimensional smooth manifold, $\overline{g}$ is a properly Riemannian metric on $M$ and $\overline{K}$ a symmetric 2-tensor.*
2. *A Cauchy development of an initial data set is a spacetime $(V, \mathbf{g})$ such that there exists an embedding $\iota : M \to V$ enjoying the following properties*

*(a) The metric $\overline{g}$ is the pullback of $\mathbf{g}$ by $\iota$, that is, $\overline{g} = \iota^*\mathbf{g}$. In other words, if $M$ is identified with its image $\iota(M) = M_0$ in $V$, then $\overline{g}$ is the metric induced by $\mathbf{g}$ on $M_0$.*

*(b) $\iota(\overline{K})$ is the second fundamental form of $\iota(M)$ as submanifold of $(V, g)$.*

*3. A Cauchy development $(V, \mathbf{g})$ of $(M, \overline{g}, \overline{K})$ is called a Einsteinian development if the metric $\mathbf{g}$ satisfies the Einstein equations on $V$.*

We now suppose that $V$ is a Cauchy development of $(M, \overline{g}, \overline{K})$. If it is also true that every Cauchy development of $(M, \overline{g}, \overline{K})$ can be isometrically embedded in $V$, we say $V$ is called the maximal development of $(M, \overline{g}, \overline{K})$. It is easy to see that a maximal development is unique up to isomorphism.

We have seen that in order to generate a Cauchy development the initial data $(M, \overline{g}, \overline{K})$ for the Einstein equations cannot be arbitrary, they must satisfy some conditions. In view of the Gauss-Codazzi equations, those conditions can be rewritten in a form consisting two equations known as the Hamiltonian and momentum constraints as shown below

- The Hamiltonian constraint

$$\mathcal{H}(\overline{g}, \overline{K}) = \mathrm{Scal}_{\overline{g}} - |\overline{K}|^2_{\overline{g}} + (\mathrm{trace}_{\overline{g}}\, \overline{K})^2 - 2T(\mathbf{n}, \mathbf{n}) = 0; \tag{1.1.4}$$

- The momentum constraint

$$\mathcal{M}(\overline{g}, \overline{K}) = \nabla_{\overline{g}} \cdot \overline{K} - \nabla_{\overline{g}}\, \mathrm{trace}_{\overline{g}}\, \overline{K} - T(\cdot, \mathbf{n}) = 0; \tag{1.1.5}$$

where $\mathbf{n}$ is an unit vector normal to the slice $M$.

The quantity $T(\mathbf{n}, \mathbf{n})$, denoted by $\overline{\rho}$, is often called the matter energy density (or simply energy density) while the quantity $T(\cdot, \mathbf{n})$, denoted by $\overline{J}$, is often called the matter momentum density (or simply momentum density) which is a vector on $M$ determined by the projection on $M$ and the normal to $M$, when embedded in a spacetime $(V, \mathbf{g})$, of the tensor $T$ of the sources.

Conversely, we wish that for given initial data verifying those constraint equations above, one can generate a corresponding Cauchy development the for initial data. The following theorem due to Y. Choquet-Bruhat and R. Geroch shows that these constraint equations also give us a sufficient condition.

**Theorem 1.1 (See [9]).** *Given smooth initial data $(M, \overline{g}, \overline{K})$ satisfying the constraint equations, there exists a smooth, maximal, globally hyperbolic Cauchy development of the initial data.*

Consequently, there is a strong connection between the globally hyperbolic Cauchy development of the initial data and the Einstein constraint equations. However, there is no one-to-one correspondence between solutions of the Einstein constraint equations and their globally hyperbolic maximal Cauchy developments in the sense that two distinct solutions of the Einstein constraint equations may generate isometric globally hyperbolic maximal Cauchy developments. Nevertheless, as a first step, it is important to understand solutions of the constraint equations.

## 1.2 The conformal method

As can be seen, the couple of constraint equations consist of $n+1$ equations (a scalar equation and an $n$-vector equation) with more than $n+1$ unknowns (for example, the metric $\overline{g}$ consists of $\frac{n(n+1)}{2}$ components). Since the constraint equations form an underdetermined system, they are in general hard to solve. Fortunately, we have a technique known as the conformal method that can be used in most of cases, see [14].

### 1.2.1 Background of the conformal method

The basic idea of the conformal method is to equalize the number of equations and the number of unknowns in such a way that the resulting system is determined. More specific, the idea of the conformal method is to split the set of initial data $(M, \overline{g}, \overline{K})$ into the following two catalogue

- Conformal data: Degrees of freedom that can be freely chosen; and
- Determined data: Degrees of freedom that are to be found by solving a determined system of partial differential equations

We now discuss the method more precise. The conformal data basically consists of $(g, \sigma, \tau)$ where $g$ is a metric, $\sigma$ is a symmetric 2-tensor, and $\tau$ is a number. In order to reduce the number of unknowns, we select $g$ as a metric of the conformal class $[\overline{g}]$ represented by the metric $\overline{g}$. Concerning to the 2-tensor $\sigma$ and the number $\tau$, they come from the second fundamental form $\overline{K}$. To be exact, they are the weighted traceless part and the trace of $\overline{K}$ with respect to $\overline{g}$. The determined data now consists of a function $u$, the conformal factor, and a vector field $W$. Mathematically, these quantities fulfill the following

$$\begin{aligned} \overline{g} &= u^{\frac{4}{n-2}} g, \\ \overline{K} &= u^{-2}(\sigma + \mathbb{L}_g W) + \frac{\tau}{n} u^{\frac{4}{n-2}} g, \end{aligned} \tag{1.2.1}$$

where $\mathbb{L}$ is the conformal Killing operator relative to $g$.

In the case when gravity is coupled with real scalar fields $\boldsymbol{\psi}$, the conformal data now includes more information, that is $(g, \sigma, \tau, \psi, \pi)$, where $\psi$ is the scalar field restricted to $M$ and $\pi$ is the normalized time derivative of $\boldsymbol{\psi}$ restricted to $M$.

Using this relationship and the change of the scalar curvature under conformal transformations, one can recast the Einstein constraint equations (1.1.4)–(1.1.5) into the following

$$\frac{4(n-1)}{n-2} \Delta_g u - \mathrm{Scal}_g u + |\sigma + \mathbb{L}_g W|_g^2 u^{\frac{-3n+2}{n-2}} - \left( \frac{n-1}{n} \tau^2 - 2\rho \right) u^{\frac{n+2}{n-2}} = 0 \tag{1.2.2}$$

and

$$\mathrm{div}_g(\mathbb{L}_g W) = \frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla_g \tau + u^{\frac{2(n+2)}{n-2}} J. \tag{1.2.3}$$

As can be seen, the question of solving the Einstein constraint equations now becomes solving the recast constraint equations (1.2.2)–(1.2.3) for $u$ and $W$. Unfortunately, without making additional assumptions, solving these recast constraint equations still remains open, even for the vacuum case. In order to seek for additional assumptions, let us consider these recast constraint equations in the vacuum case, that is

$$\frac{4(n-1)}{n-2}\Delta_g u - \mathrm{Scal}_g u + |\sigma + \mathbb{L}_g W|_g^2 u^{\frac{-3n+2}{n-2}} - \frac{n-1}{n}\tau^2 u^{\frac{n+2}{n-2}} = 0 \tag{1.2.4}$$

and

$$\mathrm{div}_g(\mathbb{L}_g W) = \frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla_g \tau. \tag{1.2.5}$$

Since (1.2.4)–(1.2.5) form a coupled system of partial differential equations, the most simplest way to deal with (1.2.4)–(1.2.5) is to decouple (1.2.4)–(1.2.5). Apparently, this can be achieved if one assumes that $\tau$ is constant.

Since $\mathrm{trace}_{\overline{g}}\overline{K} = \tau$, the resulting solutions of the constraint equations are known as constant mean curvature solutions, or CMC solutions. It turns out that this technical assumption on the mean curvature plays a crucial role in the conformal method. To see this more precise, it was remarked in [22] that the conformal method can be effectively applied in most cases. For example, in the vacuum Einstein case, a complete picture of the solvability of the constraint equations on compact manifolds was obtained under the CMC assumption in a remarkable work of Isenberg [22]. Subsequently, several similar results were obtained for asymptotically hyperbolic and asymptotically Euclidean manifolds, see [23, 12]. Equally importantly, CMC solutions of the constraint equations for several field sources such as the Einstein–Maxwell and the Einstein–Yang–Mills have already been achieved.

### 1.2.2 The Einstein-scalar field constraint equations

Unlike those cases mentioned above where we do know exactly which sets of conformal data lead to solutions and which do not, the solvability of the constraint equations with field sources is less understood both in the CMC and the non-CMC cases. This is because when the conformal method is applied in this setting, the constraint equations include terms of types which are not seen in these other cases.

In order to see this, let us mention the decomposition of scalar field initial data $(\overline{\psi}, \overline{\pi})$. Since the decomposition of $(\overline{g}, \overline{K})$ has locally introduced $n+1$ unknowns function ($u$ and the components of $W$), and since there are $n+1$ constraint equations to solve, in the scalar field decomposition we must not introduce any new unknowns. One readily finds that the only decomposition that satisfies these objectives is given as follows

$$\begin{aligned} \overline{\psi} &= \psi, \\ \overline{\pi} &= u^{-\frac{2n}{n-2}}\pi. \end{aligned} \tag{1.2.6}$$

Combining these decompositions, we can write out the constraint equations as the following

$$\begin{aligned} -\frac{4(n-1)}{n-2}\Delta_g u &+ (\mathrm{Scal}_g - |\nabla\psi|_g^2) u \\ &= -\left(\frac{n-1}{n}\tau^2 - 2U(\psi)\right) u^{\frac{n+2}{n-2}} + (|\sigma + \mathbb{L}_g W|_g^2 + \pi^2) u^{\frac{-3n+2}{n-2}} \end{aligned} \tag{1.2.7}$$

and

$$\mathrm{div}_g(\mathbb{L}_g W) = \frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla_g \tau - \pi \nabla_g \psi. \tag{1.2.8}$$

By denoting

$$\begin{aligned}
\mathcal{R}_{g,\psi} &= \frac{n-2}{4(n-1)}(\mathrm{Scal}_g - |\nabla \psi|_g^2),\\
\mathcal{B}_{\tau,\psi} &= \frac{n-2}{4(n-1)}\left(\frac{n-1}{n}\tau^2 - 2U(\psi)\right),\\
\mathcal{A}_{g,W,\pi} &= \frac{n-2}{4(n-1)}(|\sigma + \mathbb{L}_g W|_g^2 + \pi^2),
\end{aligned} \tag{1.2.9}$$

one can see that (1.2.7) simply becomes

$$-\Delta_g u + \mathcal{R}_{g,\psi} u = -\mathcal{B}_{\tau,\psi} u^{\frac{n+2}{n-2}} + \mathcal{A}_{g,W,\pi} u^{-\frac{3n-2}{n-2}}. \tag{1.2.10}$$

Unlike in the vacuum case, one can easily see that, in (1.2.10), the sign of the coefficient $\mathcal{B}_{\tau,\psi}$ does not maintain a constant sign on $M$ since it involves the potential $U$. In addition, the sign of the coefficient $\mathcal{R}_{g,\psi}$ does depend not only on the metric $g$, but also on the scalar field $\psi$. These major differences bring a lot of difficulties for the analysis of solvability.

Recently, in the elegant paper of Choquet-Bruhat et al. [11], the authors discovered the Yamabe-scalar field conformal invariant $\mathcal{Y}_\psi([g])$ which plays a similar role as the Yamabe conformal invariant $\mathcal{Y}([g])$ in the study of the Yamabe problem, see [3]. Having this new conformal invariant, the authors successfully proved in [11] that every Riemannian metric $\overline{g}$ can be conformally transformed to a new metric $g$ in such a way that $\mathcal{R}_{g,\psi}$ is either everywhere positive, everywhere negative, or every zero. Using this important result and the sign of the Yamabe-scalar field conformal invariant $\mathcal{Y}_\psi([g])$, the authors were able to split the set of pairs $([g], \psi)$ into classes corresponding to the positive, zero, and negative value of $\mathcal{Y}_\psi([g])$. Such a division of conformal data for the constraint equations with scalar fields allows us to study these constraint equations easier. In fact, it was showed in [11] that, among other things, for most sets of conformal data, it is known whether the constraint equations can be solved in the compact case. By following this method, similar results have been obtained in the asymptotically Euclidean and asymptotically hyperbolic cases; we refer the reader to [41, 10] for details.

## 1.3 Objective, method, and significance of the study

While, as we have noted, the conformal method can be effectively applied for solving the constraint equations with scalar fields in most cases, it should be pointed out that there are several cases for which either partial result or no result was achieved, see [11] for details. In order to talk about the objectives of the current study, let us go back to [11]. A typical result contained in [11] is a fairly complete picture of sets of CMC conformal data which lead to solutions of the constraint equations with scalar fields and which do not. This picture can be summarized in the following two tables where

- 'Y' indicates that (1.2.10) can be solved for that class of conformal data;

- 'N' indicates that (1.2.10) has no positive solution;
- 'PR' indicates that we have partial results; and
- 'NR' indicates that for this class of initial data we have no results indicating existence or non-existence.

| $\mathcal{B}_{\tau,\psi}$ | other | $\mathcal{B} < 0$ | $\mathcal{B} \leqslant 0$ | $\mathcal{B} \equiv 0$ | $\mathcal{B} \geqslant 0$ | $\mathcal{B} > 0$ |
|---|---|---|---|---|---|---|
| $\mathcal{Y}_\psi < 0$ | NR | N | N | N | PR | Y |
| $\mathcal{Y}_\psi = 0$ | NR | N | N | Y | N | N |
| $\mathcal{Y}_\psi > 0$ | PR | PR | PR | N | N | N |

**Table 1.1.** *Results for the case* $\mathcal{A}_{\gamma,W,\pi} \equiv 0$.

| $\mathcal{B}_{\tau,\psi}$ | other | $\mathcal{B} < 0$ | $\mathcal{B} \leqslant 0$ | $\mathcal{B} \equiv 0$ | $\mathcal{B} \geqslant 0$ | $\mathcal{B} > 0$ |
|---|---|---|---|---|---|---|
| $\mathcal{Y}_\psi < 0$ | NR | N | N | N | PR | Y |
| $\mathcal{Y}_\psi = 0$ | NR | N | N | N | Y | Y |
| $\mathcal{Y}_\psi > 0$ | PR | PR | PR | Y | Y | Y |

**Table 1.2.** *Results for the case* $\mathcal{A}_{\gamma,W,\pi} \not\equiv 0$.

Unlike the Einstein vacuum case so that a complete picture is obtained, one can see from tables 1.1 and 1.2 that there are some sets of conformal data for which it is not known whether (1.2.10) can be solved. These sets of conformal data basically force (1.2.10) to have a sign-changing term.

As a step toward achieving the full answer, the main purpose of this study was to search for some sufficient conditions for the solvability of (1.2.10) in those cases left in [11]. In order to make the study more general, we consider a more general form of (1.2.10) which can be written as the following

$$-\Delta_g u + hu = fu^{\frac{n+2}{n-2}} + au^{-\frac{3n-2}{n-2}}, \tag{1.3.1}$$

where $h$, $f$, and $a$ are smooth functions. Following to what we have already discussed, we assume throughout our study that $a \geqslant 0$. In addition, the function $h$ is assumed to be either everywhere positive, everywhere negative, or everywhere zero.

As was remarked in [11], second order elliptic equations of the type (1.3.1) with the presence of sign-changing $f$ turn out to be difficult to solve. In order to overcome that difficulty, in our study, a careful and deep analysis of (1.3.1) was developed to suit the analysis. Along the line in this study, another purpose of the study was to prove some Liouville type results for positive smooth solution of (1.3.1). In addition, we also derive the Einstein-scalar field constraint equations in the dimension $n = 2$ by adopting those arguments used in [11] and by making use of the conformal method.

In the light of the presented analysis in this study, a class of equations arising from several applied problems could be solved in the same way (see [25, 43]). Hence, the results of the present study may be useful and could provide a new way to tackle related problems in the literature. In addition, it is worth noticing that the basic idea underlying the presented analysis has found a fruitful application in other problems; for interested readers, we refer to [38] and the references therein.

Although the Einstein-scalar field constraint equations were successfully derived by using the conformal method, those constraint equations form an under-determined system of coupled equations, which is hard to solve in its general form. For this reason, this study was limited to the constant mean curvature setting so that those constraint equations are decoupled. Besides, it should also be mentioned that in our study, those constraint equations were considered only on compact manifolds. Hence, the Einstein-scalar field constraint equations for asymptotically hyperbolic and asymptotically Euclidean initial data are out of the scope of this study.

## 1.4 Structure of the present work

Having the discussion of the Einstein constraint equations with field sources in the previous section, a more detailed discussion of the Einstein constraint equations with scalar fields will be presented in Chapter 2. In addition, results in the work of Choquet-Bruhat et al. [11] will also be discussed in details in this chapter.

In Chapter 3, we prove some basic properties of solutions of (1.3.1). Besides, we also present some necessary conditions for the solvability of (1.3.1). In the last part of the chapter, we further study a minimizing problem that was first studied in [38]. This minimizing problem plays an important role in our study.

In Chapter 4, we mainly study (1.3.1) in the negative Yamabe-scalar field conformal invariant, that is $h < 0$.

In Chapter 5, we turn to study (1.3.1) in the null Yamabe-scalar field conformal invariant, namely $h = 0$.

In Chapter 6, we study (1.3.1) in the positive Yamabe-scalar field conformal invariant, that is $h > 0$.

Lastly, we establish in Chapter 7 some Liouville type results for a wider class of equations of the following form

$$-\Delta_g u + hu = \lambda u^q + u^{-q-2} \tag{1.4.1}$$

where $h$ and $\lambda$ are constants and $q > 0$. We show that if the Ricci curvature is bounded from below by some constant to be determined in the sense of quadratic form, then every smooth positive solution of (1.4.1) is constant.

# 2

# Background of the Einstein-scalar field constraint equations in the dimension $n \geqslant 2$



As mentioned earlier, in this chapter, we discuss the derivation of the Einstein-scalar field constraint equations in the dimension $n \geqslant 2$ by using the conformal method. The chapter basically consists of three parts as the following.

In the first part of the chapter, we show how to use the conformal method to derive the recast constraint equations in the dimension $n \geqslant 3$. Since, except the part of scaling the scalar fields that needs more attention, the rest of the procedure is well-known, and therefore, we shall not discuss this procedure in details. The content of this part was basically borrowed from [8], see also [10].

Following this strategy, in the second part of the chapter, we adopt this approach to derive the recast Einstein-scalar field constraint equations in the dimension $n = 2$. To our best of knowledge, the Einstein constraint equations in the dimension $n = 2$ was first computed in the vacuum case in [31]. However, due to the presence of the scalar field, the structure of the Einstein-scalar field constraint equations in the dimension $n = 2$ seems to be richer than that of the vacuum case. We shall address this issue later.

Finally, in the last part of the chapter, we focus on the division of conformal data for the Einstein-scalar field constraint equations in the dimension $n \geqslant 3$ recently found in [11]. Some notations and conventions are also mentioned in this part.

## 2.1 The Einstein-scalar field constraint equations in the dimension $n \geqslant 3$

We are now in a position to apply the conformal method to the Einstein constraint equations with the presence of a real scalar field $\psi$ with potential $U$. Notice that, we always assume throughout this section $n \geqslant 3$, the case $n = 2$ will be treated in the last section of this chapter.

Given a spacetime $(V, \mathbf{g})$, we start by choosing an $(n+1)$-foliation of the spacetime manifold $F_t : M \to V$, $t \in \mathbb{R}$, for which each of the leaves $F_t(M)$ of the foliation, as a level set of the global time function $t$, is presumed spacelike. The future-directed normal unit vector $\mathbf{n}$ can be defined by using the gradient 1-form $\nabla t$ which is nothing but

$$\mathbf{n} = -N\nabla t,$$

where $N$ is the positive definite lapse function.

It is also convenient to choose a threading of the spacetime $V$. The choice of a foliation and a threading together with a choice of coordinates for $M$ automatically induce local coordinates $(x^0 = t, x^1, ..., x^n)$ on $V$. For a natural frame on $M$, we choose

$$\partial_i = \frac{\partial}{\partial x_i} \quad \forall i = \overline{1, n}.$$

The dual coframe $(\theta^i)_{i=0}^n$ is found to be such that

$$\theta^i = dx^i + \beta^i dt \quad \forall i = \overline{1, n},$$

while the 1-form $\theta^0$ is nothing but $dt$. Here $\beta^i$ are the components of the spacelike shift vector $\beta$ which is the difference between $N\mathbf{n}$ and $\partial_t$. To be exact, the shift vector is chosen as follows

$$\beta = \partial_t - N\mathbf{n}.$$

The last thing we need to find is the vector $\partial_0$. To make it suitable, we choose

$$\partial_0 = \partial_t - \beta^j \partial_j.$$

It is now easy to check that $\partial_0 = N\mathbf{n}$ in our case.

By using the coframe $(\theta^i)_{i=0}^n$, we may locally rewrite the metric in the form

$$\mathbf{g} = -N^2\theta^0 \otimes \theta^0 + \overline{g}_{ij}\theta^i \otimes \theta^j,$$

where $\overline{g}_{ij}$ are the components of the spatial metric tensor. We note that for each $t$, $\overline{g}_{ij}(t)dx^i \otimes dx^j$ is the induced Riemannian metric on the leaf $F_t(M)$. Besides, on each leaf $F_t(M)$, we have a tangent vector basis given by $(\partial_0, \partial_1, ..., \partial_n)$. We also use $\overline{K}$ to denote the second fundamental form defined by the foliation. Let us also denote by $\tau$ the mean curvature, that is, $\tau = \text{trace}_{\overline{g}}\overline{K}$.

In this section, we derive constraint equations from (1.1.4) and (1.1.5).

### 2.1.1 The Hamiltonian constraint equation

In order to transform the Hamiltonian constraint (1.1.4) to an elliptic equations, one considers the metric $\overline{g}$ sitting in the conformal class of a given metric $g$; that is,

$$\overline{g} = e^{2\varphi} g$$

with $\varphi$ a function to be determined. It is well-known that the scalar curvatures of the conformal metrics $\overline{g}$ and $g$ are linked by the following rule

$$\mathrm{Scal}_{\overline{g}} = e^{-2\varphi}\big(\mathrm{Scal}_g - 2(n-1)\Delta_g \varphi - (n-1)(n-2) g^{ij}\partial_i\varphi\partial_j\varphi\big). \tag{2.1.1}$$

When $n \geqslant 3$, we set $e^{2\varphi} = u^{2p}$ for some $u > 0$ and we choose $p$ in such a way that the operator on $u$ appearing within the brackets is somewhat linear in $u$; this goal is attained by choosing $p = \frac{2}{n-2}$, that is, $\overline{g} = u^{\frac{4}{n-2}} g$ which we suppose from now on. To see this, one immediately has $u = e^{\frac{\varphi}{p}}$ which implies after a direct computation that

$$\Delta_g u = \frac{1}{p^2} g^{ij} u \partial_i\varphi\partial_j\varphi + \frac{1}{p} u \Delta_g \varphi.$$

Thus,

$$g^{ij}\partial_i\varphi\partial_j\varphi = u^{-1}p^2\Delta_g u - p\Delta_g\varphi.$$

Hence,

$$\begin{aligned} -2(n-1)\Delta_g\varphi - (n-1)(n-2)g^{ij}\partial_i\varphi\partial_j\varphi \\ &= -(n-1)(2-p(n-2))\Delta_g\varphi - (n-1)(n-2)u^{-1}p^2\Delta_g u \\ &= -\frac{4(n-1)}{n-2} u^{-1}\Delta_g u \end{aligned}$$

provided $p = \frac{2}{n-2}$. Thus, equation (2.1.1) transforms to

$$\mathrm{Scal}_{\overline{g}} = u^{-\frac{n+2}{n-2}}\left(u\mathrm{Scal}_g - \frac{4(n-1)}{n-2}\Delta_g u\right).$$

The Hamiltonian constraint (1.1.4) now becomes a semilinear elliptic equation for $u$ given below

$$\frac{4(n-1)}{n-2}\Delta_g u - \mathrm{Scal}_g u + (|\overline{K}|^2_{\overline{g}} - \tau^2 + 2\overline{\rho})u^{\frac{n+2}{n-2}} = 0. \tag{2.1.2}$$

### 2.1.2 The momentum constraint equation

We still use the fact that $\overline{g} = u^{\frac{4}{n-2}} g$. If we denote by $\mathrm{div}_{\overline{g}}$ and $\mathrm{div}_g$ the divergences in metrics $\overline{g}$ and $g$ respectively, we then have the following rule

$$\mathrm{div}_{\overline{g}} P^{ij} = u^{-\frac{2(n+2)}{n-2}} \mathrm{div}_g(u^{\frac{2(n+2)}{n-2}} P^{ij}) - \frac{2}{n-2} u^{-1} g^{ij}\partial_i u \mathrm{trace}_g P \tag{2.1.3}$$

where $P$ is a contravariant $(2,0)$-tensor, see [8, Lemma 3.1]. In view of (2.1.3), it is convenient to split the unknown $\overline{K}$ into a weighted traceless part and its trace, namely, we write

$$\overline{K}^{ij} = u^{-\frac{2(n+2)}{n-2}} \widetilde{K}^{ij} + \frac{\tau}{n}\overline{g}^{ij}.$$

By lowering the indices in $\overline{K}$ and $\widetilde{K}$, one gets

$$\overline{K}_{ij} = u^{-2}\widetilde{K}_{ij} + \frac{\tau}{n}\overline{g}_{ij}.$$

It is clear to see that the tensor $\widetilde{K}$ is symmetric and traceless, that is

$$\text{trace}_g \widetilde{K} = g^{ij}\widetilde{K}_{ij} = u^{-\frac{4}{n-2}}\overline{g}^{ij}\widetilde{K}_{ij} = u^{\frac{2n}{n-2}}\overline{g}^{ij}\left(K_{ij} - \frac{\tau}{n}\overline{g}_{ij}\right) = 0.$$

In view of the momentum constraint (1.1.5) and with the fact that

$$u^{-\frac{4}{n-2}}\tau = \text{trace}_g \overline{K}$$

we have

$$\begin{aligned}
\overline{J}^j + \overline{g}^{ij}\partial_i\tau &= \overline{\nabla}_i\overline{K}^{ij} \\
&= u^{-\frac{2(n+2)}{n-2}}\nabla_i(u^{\frac{2(n+2)}{n-2}}\overline{K}^{ij}) - \frac{2}{n-2}u^{-1}g^{ij}\partial_i u\,\text{trace}_g \overline{K} \\
&= u^{-\frac{2(n+2)}{n-2}}\nabla_i\big(\widetilde{K}^{ij} + \frac{1}{n}u^{\frac{2(n+2)}{n-2}}\overline{g}^{ij}\tau\big) - \frac{2}{n-2}u^{-1}g^{ij}\partial_i u\,\text{trace}_g \overline{K} \\
&= u^{-\frac{2(n+2)}{n-2}}\nabla_i\widetilde{K}^{ij} + \frac{1}{n}u^{-\frac{2(n+2)}{n-2}}\nabla_i\big(u^{\frac{2n}{n-2}}g^{ij}\tau\big) - \frac{2}{n-2}u^{-1}g^{ij}\partial_i u\,\text{trace}_g \overline{K} \\
&= u^{-\frac{2(n+2)}{n-2}}\nabla_i\widetilde{K}^{ij} + \frac{1}{n}g^{ij}u^{-\frac{2(n+2)}{n-2}}\nabla_i\big(u^{\frac{2n}{n-2}}\tau\big) - \frac{2}{n-2}u^{-1}g^{ij}\partial_i u\,\text{trace}_g \overline{K} \\
&= u^{-\frac{2(n+2)}{n-2}}\nabla_i\widetilde{K}^{ij} + \frac{1}{n}\overline{g}^{ij}\partial_i(\text{trace}_g \overline{K}).
\end{aligned}$$

Thus,

$$\nabla_i\widetilde{K}^{ij} = u^{\frac{2(n+2)}{n-2}}\overline{J}^j + \frac{n-1}{n}u^{\frac{2(n+2)}{n-2}}\overline{g}^{ij}\partial_i\tau.$$

Equivalently,

$$\nabla_i\widetilde{K}^{ij} = \frac{n-1}{n}u^{\frac{2n}{n-2}}g^{ij}\partial_i\tau + u^{\frac{2(n+2)}{n-2}}\overline{J}^j.$$

In other words, the momentum constraint (1.1.5) now becomes

$$\text{div}_g \widetilde{K} = \frac{n-1}{n}u^{\frac{2n}{n-2}}\nabla_g\tau + u^{\frac{2(n+2)}{n-2}}\overline{J}. \tag{2.1.4}$$

In particular, if $\overline{J}$ is zero and $\tau$ is constant, the symmetric $(2,0)$-tensor $\widetilde{K}$ is also divergence free. Symmetric $(2,0)$-tensors which are divergence free and trace free are called TT-tensors (tranverse, traceless). Let us go back to the decomposition of $\overline{K}$. Clearly,

$$\begin{aligned}
|\overline{K}|^2_{\overline{g}} &= \overline{g}_{ih}\overline{g}_{jk}\overline{K}^{ij}\overline{K}^{hk} \\
&= u^{-\frac{4n}{n-2}}g_{ih}g_{jk}\widetilde{K}^{ij}\widetilde{K}^{hk} + \frac{\tau^2}{n} \\
&= u^{-\frac{4n}{n-2}}|\widetilde{K}|^2_g + \frac{\tau^2}{n}.
\end{aligned}$$

Therefore, (2.1.2) now reads as follows

$$\frac{4(n-1)}{n-2}\Delta_g u - \text{Scal}_g u + |\widetilde{K}|^2_g u^{\frac{-3n+2}{n-2}} - \left(\frac{n-1}{n}\tau^2 - 2\overline{\rho}\right)u^{\frac{n+2}{n-2}} = 0. \tag{2.1.5}$$

This is a semilinear elliptic equation for $u$.

### 2.1.3 Scaling of the scalar fields

Recall that $M$ is an $n$-dimensional manifold with the spatial metric $g$. We denote by an overbar the values induced on $M$ by the spacetime quantities.

For the scalar field $\boldsymbol{\psi}$, there is no need to do any time and space decomposition. However, the wave equation that $\boldsymbol{\psi}$ fulfills, that is

$$\nabla^\alpha \nabla_\alpha \boldsymbol{\psi} = U'(\boldsymbol{\psi}),$$

suggests that the initial data for the scalar field $\boldsymbol{\psi}$ should be the induced function and normalized time derivative of the function on $M$. Based on this reason and for the sake of simplicity, we use $\overline{\pi}$ to denote the normalized time derivative of $\boldsymbol{\psi}$ restricted to $M$, that is,

$$\overline{\pi} = N^{-1} \partial_0 \psi.$$

where $\partial_0 \psi$ is the value of $\partial_0 \boldsymbol{\psi}$ on $M$. In addition, we also denote by $\psi$ and $\partial_i \psi$ the values of $\boldsymbol{\psi}$ and $\partial_i \boldsymbol{\psi}$ on $M$ respectively.

Due to the fact that the background metric $g$ is unphysical, we associate it to an unphysical lapse $\widetilde{N}$ so that $N$ and $\widetilde{N}$ have the same associated densitized lapse, that is

$$N(\det \overline{g})^{-\frac{1}{2}} = \widetilde{N}(\det g)^{-\frac{1}{2}}. \tag{2.1.6}$$

For a background of the choice of this densitized lapse, we refer the reader to a work by Choquet-Bruhat and Ruggeri [13]. Thanks to $\overline{g} = u^{\frac{4}{n-2}} g$, this condition is equivalent to

$$N = u^{\frac{2n}{n-2}} \widetilde{N}.$$

In this setting, we denote $\pi = \widetilde{N}^{-1} \partial_0 \psi$ for the initial data $\pi$. Therefore, we have the following scaling

$$\overline{\pi} = N^{-1} \partial_0 \psi = u^{-\frac{2n}{n-2}} \pi.$$

In order to get the constraint equations for the initial data, let us first calculate $T(\mathbf{n}, \mathbf{n})$ and $T(\partial_i, \mathbf{n})$.

For the energy density on $M$ of a scalar field $\boldsymbol{\psi}$ with potential $U(\cdot)$, one need to calculate $T(\mathbf{n}, \mathbf{n})$. First, an easy calculation shows that

$$T_{00} = \partial_0 \boldsymbol{\psi} \partial_0 \boldsymbol{\psi} - \frac{1}{2} \mathbf{g}_{00} \sum_{i,j=0}^{n} \mathbf{g}^{ij} \nabla_i \boldsymbol{\psi} \nabla_j \boldsymbol{\psi} - \mathbf{g}_{00} U(\boldsymbol{\psi})$$

and

$$T_{0i} = \partial_0 \boldsymbol{\psi} \partial_i \boldsymbol{\psi} \quad \forall i = \overline{1, n}.$$

Under the choice of our adapted frame, there hold

$$\mathbf{g}_{\alpha\beta} = \begin{pmatrix} \mathbf{g}_{00} & \mathbf{g}_{0j} \\ \mathbf{g}_{i0} & \mathbf{g}_{ij} \end{pmatrix} = \begin{pmatrix} -N^2 & 0 \\ 0 & \overline{g}_{ij} \end{pmatrix}$$

and

$$\mathbf{g}^{\alpha\beta} = \begin{pmatrix} \mathbf{g}^{00} & \mathbf{g}^{0j} \\ \mathbf{g}^{i0} & \mathbf{g}^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{N^2} & 0 \\ 0 & \overline{g}^{ij} \end{pmatrix}.$$

Therefore, we can rewrite

$$\begin{aligned} T_{00} &= N^2\Big(N^{-2}\partial_0\boldsymbol{\psi}\partial_0\boldsymbol{\psi} + \frac{1}{2}\sum_{i,j=0}^{n}\mathbf{g}^{ij}\nabla_i\boldsymbol{\psi}\nabla_j\boldsymbol{\psi} + U(\boldsymbol{\psi})\Big) \\ &= N^2\Big(\frac{1}{2}N^{-2}\partial_0\boldsymbol{\psi}\partial_0\boldsymbol{\psi} + \frac{1}{2}\overline{g}^{ij}\partial_i\boldsymbol{\psi}\partial_j\boldsymbol{\psi} + U(\boldsymbol{\psi})\Big). \end{aligned}$$

Since in our local frame, the unit normal vector $\mathbf{n}$ is nothing but

$$\mathbf{n} = (-\frac{1}{N}, \underbrace{0, ..., 0}_{n \text{ times}}),$$

one can verify that

$$\begin{aligned} T(\mathbf{n}, \mathbf{n}) &= N^{-2}T_{00} \\ &= \frac{1}{2}N^{-2}\partial_0\boldsymbol{\psi}\partial_0\boldsymbol{\psi} + \frac{1}{2}\overline{g}^{ij}\partial_i\boldsymbol{\psi}\partial_j\boldsymbol{\psi} + U(\boldsymbol{\psi}). \end{aligned}$$

Thus, by considering $\boldsymbol{\psi}$ only on $M$, we have just showed that the energy density is nothing but

$$\overline{\rho} = \frac{1}{2}\Big(N^{-2}\partial_0\psi\partial_0\psi + \overline{g}^{ij}\partial_i\psi\partial_j\psi\Big) + U(\psi).$$

In terms of the initial data set, $\overline{\rho}$ becomes

$$\overline{\rho} = \frac{1}{2}\Big(u^{-\frac{4n}{n-2}}\pi^2 + u^{-\frac{4}{n-2}}|\nabla\psi|_g^2\Big) + U(\psi).$$

Hence, we can regroup (2.1.5) as

$$\begin{aligned} &\frac{4(n-1)}{n-2}\Delta_g u - (\mathrm{Scal}_g - |\nabla\psi|_g^2)u \\ &\qquad + (|\widetilde{K}|_g^2 + \pi^2)u^{-\frac{3n-2}{n-2}} - \Big(\frac{n-1}{n}\tau^2 - 2U(\psi)\Big)u^{\frac{n+2}{n-2}} = 0. \end{aligned} \tag{2.1.7}$$

Now for the momentum density, one can see that

$$T(\partial_i, \mathbf{n}) = -\frac{1}{N}T_{0i} = -\frac{1}{N}\partial_0\boldsymbol{\psi}\partial_i\boldsymbol{\psi} \quad \forall i = \overline{1, n}.$$

Therefore, by raising the index $i$ and considering only on $M$, we obtain

$$\overline{J}^i = -N^{-1}\overline{g}^{ij}\partial_j\psi\,\partial_0\psi = -u^{-\frac{2(n+2)}{n-2}}\pi g^{ij}\partial_j\psi.$$

That is

$$\overline{J} = -u^{-\frac{2(n+2)}{n-2}}\pi\nabla_g\psi.$$

Using the formula for $\overline{J}$, (2.1.4) becomes

$$\mathrm{div}_g\widetilde{K} = \frac{n-1}{n}u^{\frac{2n}{n-2}}\nabla_g\tau - \pi\nabla_g\psi. \tag{2.1.8}$$

### 2.1.4 The transverse-traceless decomposition

We consider in this subsection the solvability of (2.1.8) using the transverse-traceless decomposition. Roughly speaking, we search for $\widetilde{K}$ of the form

$$\widetilde{K} = \widetilde{K}_{TT} + \mathbb{L}_g W \tag{2.1.9}$$

where $\widetilde{K}_{TT}$ is a TT-tensor, say the TT-part of $\widetilde{K}$, $W$ is an unknown vector field to be determined, and $\mathbb{L}$ is the conformal Killing operator relative to $g$ defined by

$$(\mathbb{L}_g W)^{ij} = \nabla^i W^j + \nabla^j W^i - \frac{2}{n} g^{ij} \nabla_k W^k. \tag{2.1.10}$$

If the right hand side of (2.1.10) vanishes, the vector field $W$ is called a conformal Killing vector. By definition, any tensor of the form $\mathbb{L}_g Y$ for some vector field $Y$ has trace free. The procedure of solving (2.1.8) is to find $W$ and TT-part of $\widetilde{K}$. This so-called TT-part is not unique in general and we have many ways to extract such a piece of information from $\widetilde{K}$.

We first deal with $W$. In accordance with (2.1.9), we first have

$$(\widetilde{K}_{TT})^{ij} = \widetilde{K}^{ij} - (\mathbb{L}_g W)^{ij}. \tag{2.1.11}$$

The choice of the conformal Killing operator and the fact that $\widetilde{K}$ is tracefree make the right hand side of (2.1.11) trace free. Besides, the transversality requirement $\nabla_i (\widetilde{K}_{TT})^{ij} = 0$ and (2.1.8) lead to covariant equations for $W$ given by

$$\nabla_i (\mathbb{L}_g W)^{ij} = \frac{n-1}{n} u^{\frac{2n}{n-2}} g^{ij} \partial_i \tau - \pi g^{ij} \partial_i \psi.$$

If we formally denote $\Delta_{g,\text{conf}} = \operatorname{div}_g \circ \mathbb{L}_g$, then we can rewrite the equation for $W$ in the vector form as the following

$$\Delta_{g,\text{conf}} W = \frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla_g \tau - \pi \nabla_g \psi. \tag{2.1.12}$$

It is well-known that the operator $\Delta_{g,\text{conf}}$ which is similar to the vector Laplacian is a second order, self-adjoint, linear, elliptic operator whose kernel consists of the space of conformal Killing vector fields, see [8, Appendix II]. Thus under some mild conditions, we can solve (2.1.12) for $W$ up to conformal Killing vector fields. Notice that, any conformal Killing vector field does not constitute any extra information to $\widetilde{K}_{TT}$ in (2.1.11).

We now consider the TT-tensor $\widetilde{K}_{TT}$. The search of such a tensor is somewhat freely. Its procedure can be formulated as follows. We start with a freely chosen traceless 2-tensor $Z$, then we solve for $Y$ from the following equation

$$\Delta_{g,\text{conf}} Y = -\operatorname{div}_g Z. \tag{2.1.13}$$

The existence of some $Y$ from (2.1.13) comes from the fact that $\operatorname{div}_g Z$ is orthogonal to the space of conformal Killing vector fields whose proof is just a simple application of integration by parts as follows

$$\int_M (\nabla_i Z^{ij}) H_j \, \mathrm{dvol}_g = -\int_M Z^{ij} (\mathbb{L}_g H)_{ij} \, \mathrm{dvol}_g = 0$$

where $H$ is a conformal Killing vector field. Since

$$\nabla_j (\mathbb{L}_g Y + Z)^{ij} = (\Delta_{g,\mathrm{conf}} Y)^i + \nabla_j Z^{ij} = 0$$

we know that the traceless tensor $\mathbb{L}_g Y + Z$ is also transverse, thus, a TT-tensor. Let us denote $\mathbb{L}_g Y + Z$ by $\sigma$. In conclusion, we first begin with a freely chosen $Z$ and solve (2.1.13) for $Y$. We then solve (2.1.12) to find $W$. Finally, we find $\widetilde{K} = \sigma + \mathbb{L}_g W$ by means of the decomposition (2.1.9). Such a $\widetilde{K}$ will satisfy (2.1.8).

In summary, let us recall some information that we have already seen from previous sections. By using the conformal method we are able to transform the Einstein-scalar field constraint equations into a couple of two equations given below

$$\begin{aligned} -\frac{4(n-1)}{n-2} \Delta_g u &+ (\mathrm{Scal}_g - |\nabla \psi|_g^2) u \\ &= -\left( \frac{n-1}{n} \tau^2 - 2U(\psi) \right) u^{\frac{n+2}{n-2}} + (|\sigma + \mathbb{L}_g W|_g^2 + \pi^2) u^{-\frac{3n-2}{n-2}} \end{aligned} \tag{2.1.14}$$

and

$$\mathrm{div}_g (\mathbb{L}_g W) = \frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla_g \tau - \pi \nabla_g \psi \tag{2.1.15}$$

where $\tau = \mathrm{trace}_{\overline{g}} \overline{K}$. The main objective of our study here is to determine which choices of the conformal data $(g, \sigma, \tau, \psi, \pi)$ permit one to solve (2.1.14)-(2.1.15) for the determined data $(u, W)$ and which do not. Equation (2.1.14) is a semilinear elliptic equation, called the Einstein-scalar field Lichnerowicz equation, for $u$ provided $\sigma$ and $W$ are already known.

## 2.2 The Einstein-scalar field constraint equations in the dimension $n = 2$

In the previous section, we have shown how the conformal method may be used to derive the constraint equations with scalar fields when the dimension $n \geqslant 3$. In this section, we continue to use the conformal method to construct the constraint equations with scalar fields in the dimension $n = 2$. To the best of our knowledge, the first paper dealing with the Einstein equations in the two dimensional cases is [31]. Subsequently, some generalization of the corresponding situation for the Einstein–Maxwell and Einstein–Maxwell–Higgs equations were also obtained by the author in [32]. Again, no result is known for the case of scalar fields.

In order to fix notations, we keep using $\overline{g} = e^{2\varphi} g$. We first prove the following simple result which was motivated by (2.1.3).

**Lemma 2.1.** *On any 2-dimensional manifold, if $\overline{g} = e^{2\varphi} g$, the covariant derivatives in $\overline{g}$ and $g$ being respectively denoted $\overline{\nabla}$ and $\nabla$, the divergences in the metrics $\overline{g}$ and $g$ of an arbitrary contravariant $(2,0)$-tensor $P^{ij}$ verify the following identity*

$$\mathrm{div}_{\overline{g}} P^{ij} = e^{-4\varphi} \mathrm{div}_g (e^{4\varphi} P^{ij}) - g^{ij} \partial_i \varphi \, \mathrm{trace}_g P. \tag{2.2.1}$$

*Proof.* For the purpose of clarity we may denote the tensor $P$ by

$$P = P^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

Using the Leibniz rule, one easily gets

$$\begin{aligned}(\overline{\nabla} P)\left(\cdot, \cdot, \frac{\partial}{\partial x^k}\right) &= \overline{\nabla}_{\frac{\partial}{\partial x^k}} \left( P^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \right) \\ &= \frac{\partial P^{ij}}{\partial x^k} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} + P^{ij} \left( \overline{\nabla}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} \right) \otimes \frac{\partial}{\partial x^j} + P^{ij} \frac{\partial}{\partial x^i} \otimes \left( \overline{\nabla}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right) \\ &= \left( \frac{\partial P^{ij}}{\partial x^k} + P^{lj} \overline{\Gamma}^i_{lk} + P^{il} \overline{\Gamma}^j_{lk} \right) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.\end{aligned}$$

Therefore, $(2,1)$-tensor $\overline{\nabla} P$, which is of the form

$$\overline{\nabla} P = \overline{\nabla}_k P^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes dx^k = {P^{ij}}_{,k} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes dx^k,$$

verifies

$$\overline{\nabla}_k P^{ij} = {P^{ij}}_{,k} = \partial_k P^{ij} + P^{lj} \overline{\Gamma}^i_{lk} + P^{il} \overline{\Gamma}^j_{lk}.$$

We take the divergence, that is, to use

$$\operatorname{div} P = \delta^k_i {P^{ij}}_{,k} \frac{\partial}{\partial x^j} = \delta^k_i \overline{\nabla}_k P^{ij} \frac{\partial}{\partial x^j} = \overline{\nabla}_i P^{ij} \frac{\partial}{\partial x^j}$$

to arrive at

$$\overline{\nabla}_i P^{ij} = \partial_i P^{ij} + P^{lj} \overline{\Gamma}^i_{li} + P^{il} \overline{\Gamma}^j_{li}.$$

Notice that under the conformal change $\overline{g} = e^{2\varphi} g$, the Christoffel symbols computed with respect to $\overline{g}$ and $g$ verify the following identity

$$\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + (\delta^k_i \partial_j \varphi + \delta^k_j \partial_i \varphi - g_{ij} g^{kl} \partial_l \varphi).$$

Therefore,

$$\begin{aligned}\overline{\nabla}_i P^{ij} =& \partial_i P^{ij} + P^{lj} \Gamma^i_{li} + P^{il} \Gamma^j_{li} \\ & + P^{lj} (\delta^i_l \partial_i \varphi + \delta^i_i \partial_l \varphi - g_{li} g^{im} \partial_m \varphi) \\ & + P^{il} (\delta^j_l \partial_i \varphi + \delta^j_i \partial_l \varphi - g_{li} g^{jm} \partial_m \varphi) \\ =& \nabla_i P^{ij} + (P^{ij} \partial_i \varphi + 2 P^{lj} \partial_l \varphi - P^{mj} \partial_m \varphi) \\ & + (P^{ij} \partial_i \varphi + P^{jl} \partial_l \varphi - P^{il} g_{li} g^{jm} \partial_m \varphi) \\ =& \nabla_i P^{ij} + 4 P^{ij} \partial_i \varphi - g^{jm} \partial_m \varphi \operatorname{trace}_g P \\ =& e^{-4\varphi} \nabla_i (e^{4\varphi} P^{ij}) - g^{jm} \partial_m \varphi \operatorname{trace}_g P.\end{aligned}$$

The proof immediately follows. ∎

### 2.2.1 The momentum constraint equation

Let us consider the momentum constraint. Using Lemma 2.2.1, for a $(2,0)$-tensor $\overline{K}$, we decompose it as follows

$$\overline{K}^{ij} = e^{-4\varphi}\widetilde{K}^{ij} + \frac{\tau}{2}\overline{g}^{ij}.$$

This decomposition obeys the same properties as $n \geqslant 3$, that is, $\widetilde{K}$ is symmetric and traceless. Therefore, one has the following

$$\begin{aligned}
\overline{J}^j + \overline{g}^{ij}\partial_i\tau &= \overline{\nabla}_i\overline{K}^{ij} \\
&= e^{-4\varphi}\nabla_i(e^{4\varphi}\overline{K}^{ij}) - g^{ij}\partial_i\varphi\,\mathrm{trace}_g\,\overline{K} \\
&= e^{-4\varphi}\nabla_i\big(\widetilde{K}^{ij} + \frac{1}{2}e^{4\varphi}\overline{g}^{ij}\tau\big) - g^{ij}\partial_i\varphi\,\mathrm{trace}_g\,\overline{K} \\
&= e^{-4\varphi}\nabla_i\big(\widetilde{K}^{ij} + \frac{1}{2}e^{2\varphi}g^{ij}\tau\big) - g^{ij}\partial_i\varphi\,\mathrm{trace}_g\,\overline{K} \\
&= e^{-4\varphi}\nabla_i\widetilde{K}^{ij} + \frac{1}{2}e^{-4\varphi}g^{ij}\nabla_i(e^{2\varphi}\tau) - g^{ij}\partial_i\varphi\,\mathrm{trace}_g\,\overline{K} \\
&= e^{-4\varphi}\nabla_i\widetilde{K}^{ij} + \frac{1}{2}e^{-2\varphi}g^{ij}\nabla_i\tau.
\end{aligned}$$

In other words,

$$\nabla_i\widetilde{K}^{ij} = e^{4\varphi}\overline{J}^j + \frac{1}{2}e^{4\varphi}\overline{g}^{ij}\partial_i\tau.$$

Thus, the momentum constraint equation that we need is

$$\mathrm{div}_g\,\widetilde{K} = \frac{1}{2}e^{2\varphi}\nabla_g\tau + e^{4\varphi}\overline{J}. \tag{2.2.2}$$

### 2.2.2 The Hamiltonian constraint equation

Let us now consider the Hamiltonian constraint. One first obtains by (2.1.1)

$$\mathrm{Scal}_{\overline{g}} = e^{-2\varphi}(\mathrm{Scal}_g - 2\varDelta_g\varphi).$$

Consequently, the Hamiltonian constraint temporarily reads as the following

$$-2\varDelta_g\varphi + \mathrm{Scal}_g - (|\overline{K}|^2_{\overline{g}} - \tau^2 + 2\overline{\rho})e^{2\varphi} = 0.$$

Obviously,

$$|\overline{K}|^2_{\overline{g}} = \overline{g}_{ih}\overline{g}_{jk}\overline{K}^{ij}\overline{K}^{hk} = e^{-4\varphi}|\widetilde{K}|^2_g + \frac{1}{2}\tau^2$$

which implies that

$$-2\varDelta_g\varphi + \mathrm{Scal}_g - \Big(e^{-4\varphi}|\widetilde{K}|^2_g - \frac{1}{2}\tau^2 + 2\overline{\rho}\Big)e^{2\varphi} = 0. \tag{2.2.3}$$

### 2.2.3 Scaling of the scalar fields

Using (2.1.6) one easily gets $N = e^{2\varphi}\widetilde{N}$ which immediately implies

$$\overline{\pi} = N^{-1}\partial_0\psi = e^{-2\varphi}\pi.$$

Therefore, we can decompose the energy density on $M$ as follows

$$\overline{\rho} = \frac{1}{2}\Big(e^{-4\varphi}\pi^2 + e^{-2\varphi}|\nabla\psi|^2_g\Big) + U(\psi).$$

Now for the momentum density, one can see that

$$\overline{J}^i = -N^{-1}\overline{g}^{ij}\partial_j\psi\,\partial_0\psi = -e^{-4\varphi}\pi g^{ij}\partial_j\psi.$$

That is

$$\overline{J} = -e^{-4\varphi}\pi\nabla_g\psi.$$

Using the formula for $\overline{J}$, we can rewrite (2.2.2) as

$$\operatorname{div}_g \widetilde{K} = \frac{1}{2}e^{2\varphi}\nabla_g\tau - \pi\nabla_g\psi. \tag{2.2.4}$$

We now regroup (2.2.3) as

$$-2\Delta_g\varphi + (\mathrm{Scal}_g - |\nabla\psi|_g^2) = -\Big(\frac{1}{2}\tau^2 - 2U(\psi)\Big)e^{2\varphi} + (\pi^2 + |\widetilde{K}|_g^2)e^{-2\varphi}. \tag{2.2.5}$$

At this stage, the Einstein constraint equations in 2 dimensions become a system of partial differential equations (2.2.5)–(2.2.4). Notice that the coefficient of $e^{-2\varphi}$ is always non-negative in this setting. In the vacuum case, that is $U(\psi) \equiv 0$, the coefficient of $e^{2\varphi}$ is always negative and one can use the method of sub- and super-solutions to prove the existence result, see [31] for details.

## 2.3 Sobolev spaces and related inequalities

Given a smooth compact Riemannian manifold $(M, g)$ of dimension $n$, one easily defines the Sobolev spaces $H_p^k(M)$ for any positive integers $k$ and $p$. To be precise, we define $H_p^k(M)$ as the completion of $C^\infty(M)$ with respect to the following norm

$$\|u\|_{H_p^k} = \sum_{j=1}^k \big\|\nabla^j u\big\|_{L^p}.$$

When $p = 2$, we simply write $H_2^k(M)$ as $H^k(M)$.

By $\mathcal{K}_1$ and $\mathcal{A}_1$ we mean the positive constants such that the Sobolev inequality holds, that is, for all $u \in H^1(M)$,

$$\mathcal{K}_1 \int_M |\nabla u|^2 \,\mathrm{dvol}_g + \mathcal{A}_1 \int_M u^2 \,\mathrm{dvol}_g \geqslant \Big(\int_M |u|^{2^\star} \,\mathrm{dvol}_g\Big)^{\frac{2}{2^\star}}. \tag{2.3.1}$$

We notice that those constants $\mathcal{K}_1$ and $\mathcal{A}_1$ are independent of $u$.

As one may observe from (1.3.1) that the operator appearing in the left hand side of (1.3.1), that is $-\Delta + h$, admits some interesting features when we impose some conditions on the potential $h$. A typical assumption that people usually make is to assume that $-\Delta + h$ is coercive. Roughly speaking, this is equivalent to saying that

$$\inf_{u\in H^1(M)} \frac{\int_M |\nabla u|^2 \,\mathrm{dvol}_g + \int_M hu^2 \,\mathrm{dvol}_g}{\int_M u^2 \,\mathrm{dvol}_g} > 0.$$

In particular, one may see that

$$\|u\|_{H^1_h} = \left(\int_M |\nabla u|^2 \, d\mathrm{vol}_g + \int_M hu^2 \, d\mathrm{vol}_g\right)^{\frac{1}{2}}$$

is an equivalent norm in $H^1(M)$. It is standard to check that if $h > 0$ everywhere then $-\Delta + h$ is coercive.

Another useful inequality appearing in this setting is the following. For all $u \in H^1(M)$, there holds

$$\int_M |\nabla u|^2 \, d\mathrm{vol}_g + \int_M hu^2 \, d\mathrm{vol}_g \geqslant \mathcal{S}_h \left(\int_M |u|^{2^\star} \, d\mathrm{vol}_g\right)^{\frac{2}{2^\star}}, \tag{2.3.2}$$

where the constant $\mathcal{S}_h$ is called the Sobolev constant and is independent of $u$.

## 2.4 The Yamabe-scalar field conformal invariant

Let us assume that $(M, g)$ is a compact Riemannian manifold without boundary. We recall from the study of the Yamabe problem [3, 42, 45] on $(M, g)$ the conformal Laplacian operator $L_g$ acting on a smooth function $u$ is defined by

$$L_g u = \Delta_g u - \frac{n-2}{4(n-1)} \mathrm{Scal}_g \, u. \tag{2.4.1}$$

Operator $L_g$ has the conformal covariance property

$$L_{\widetilde{g}} u = \theta^{-\frac{n+2}{n-2}} L_g(\theta u) \tag{2.4.2}$$

for any $\widetilde{g} = \theta^{\frac{4}{n-2}} g$ for some $\theta > 0$ being a smooth function. Inspired by (2.4.1), the authors in [11] introduced the so-called conformal scalar field Laplacian operator $L_{g,\psi}$ given by

$$L_{g,\psi} u = \Delta_g u - \frac{n-2}{4(n-1)} (\mathrm{Scal}_g - |\nabla \psi|_g^2) u. \tag{2.4.3}$$

It follows from (2.4.1) and (2.4.3) that

$$L_{g,\psi} u = L_g u + \frac{n-2}{4(n-1)} |\nabla \psi|_g^2 u.$$

We wish that $L_{g,\psi}$ also has the conformal covariance property. For this reason, we first have

$$|\nabla \psi|_{\widetilde{g}}^2 = \widetilde{g}^{ij} \partial_i \psi \partial_j \psi = \theta^{-\frac{4}{n-2}} g^{ij} \partial_i \psi \partial_j \psi = \theta^{-\frac{4}{n-2}} |\nabla \psi|_g^2.$$

This and (2.4.2) immediately give

$$\begin{aligned} L_{\widetilde{g},\psi} u &= L_{\widetilde{g}} u + \frac{n-2}{4(n-1)} |\nabla \psi|_{\widetilde{g}}^2 u \\ &= \theta^{-\frac{n+2}{n-2}} L_g(\theta u) + \frac{n-2}{4(n-1)} \theta^{-\frac{4}{n-2}} |\nabla \psi|_g^2 u \\ &= \theta^{-\frac{n+2}{n-2}} L_g(\theta u) + \frac{n-2}{4(n-1)} \theta^{-\frac{n+2}{n-2}} |\nabla \psi|_g^2 (\theta u). \end{aligned}$$

In other words, the operator $L_{g,\psi}$ also verifies the same conformal covariance property in the sense that

$$L_{\widetilde{g},\psi}u = \theta^{-\frac{n+2}{n-2}} L_{g,\psi}(\theta u). \tag{2.4.4}$$

We now define the so-called conformal-scalar field Dirichlet energy of $u$ by

$$\begin{aligned} E_{g,\psi}(u) &= -\frac{4(n-1)}{n-2}\int_M uL_{g,\psi}u \, d\mathrm{vol}_g \\ &= \frac{4(n-1)}{n-2}\int_M \left(|\nabla u|_g^2 + \frac{n-2}{4(n-1)}(\mathrm{Scal}_g - |\nabla\psi|_g^2)u^2\right) d\mathrm{vol}_g \end{aligned}$$

and the conformal-scalar field Sobolev quotient by

$$Q_{g,\psi}(u) = \frac{E_{g,\psi}(u)}{\|u\|^2_{L^{\frac{2n}{n-2}}}}.$$

Using (2.4.4) one has

$$Q_{\widetilde{g},\psi}(u) = Q_{g,\psi}(\theta u) \tag{2.4.5}$$

where $\widetilde{g} = \theta^{\frac{4}{n-2}}g$. We denote by $[g]$ the conformal class of the metric $g$ given by

$$[g] = \{\widetilde{g} = \theta^{\frac{4}{n-2}}g, \theta \in C^\infty(M), \theta > 0\}.$$

Then we define the so-called Yamabe-scalar field conformal invariant by

$$\mathcal{Y}_\psi([g]) = \inf_{u\in H^1(M)} Q_{g,\psi}(u). \tag{2.4.6}$$

By (2.4.5), it is obvious that $\mathcal{Y}_\psi([g])$ is independent of the choice of background metric $g$ in the conformal class used to define it, and is therefore an invariant of the conformal class $[g]$.

We observer from the Hölder inequality and the compactness of $M$ that

$$\left|\int_M \left((\mathrm{Scal}_g - |\nabla\psi|_g^2)u^2\right) d\mathrm{vol}_g\right| \leqslant C\,\|u\|^2_{L^{\frac{2n}{n-2}}}$$

for some positive constant $C$ independent of $u$. Thus, $\mathcal{Y}_\psi([g])$ is finite. Consequently, by using the sign of $\mathcal{Y}_\psi([g])$, we may partition the set of pairs $([g],\psi)$ into three classes which we label $\mathcal{Y}^-$, $\mathcal{Y}^0$, $\mathcal{Y}^+$, and refer to as the negative, zero, and positive Yamabe-scalar field conformal invariants on $M$.

The following important result was proved in [11].

**Proposition 2.1 (See [11]).** *The following conditions are equivalent*

(i) $\mathcal{Y}_\psi([g]) > 0$ *(respectively* $= 0$, $< 0$*);*

(ii) *There exists a metric* $\widetilde{g} \in [g]$ *which satisfies*

$$\mathrm{Scal}_{\widetilde{g}} - |\nabla\psi|^2_{\widetilde{g}} > 0$$

*everywhere on* $M$ *(respectively* $= 0$, $< 0$*);*

(iii) *For any metric* $\widetilde{g} \in [g]$*, the first eigenvalue* $\lambda_1$ *of the self-adjoint elliptic operator* $-L_{\widetilde{g},\psi}$ *is positive (respectively zero, negative).*

For any constant $c > 0$ and any metric $\widetilde{g} \in [g]$, let us consider the following metric $\widehat{g} = c^{\frac{4}{n-2}}\widetilde{g}$. In terms of the metric $g$, one may write $\widehat{g} = (cu)^{\frac{4}{n-2}} g$ provided $\widetilde{g} = u^{\frac{4}{n-2}} g$. Then a direct computation shows that

$$\begin{aligned}
\operatorname{Scal}_{\widehat{g}} - |\nabla\psi|^2_{\widehat{g}} &= \operatorname{Scal}_{\widehat{g}} - c^{-\frac{4}{n-2}} |\nabla\psi|^2_{\widetilde{g}} \\
&= c^{-\frac{4}{n-2}} \operatorname{Scal}_{\widetilde{g}} - \frac{4(n-1)}{n-2} c^{-\frac{n+2}{n-2}} \Delta_{\widetilde{g}}(c) - c^{-\frac{4}{n-2}} |\nabla\psi|^2_{\widetilde{g}} \\
&= c^{-\frac{4}{n-2}} \left( \operatorname{Scal}_{\widetilde{g}} - |\nabla\psi|^2_{\widetilde{g}} \right).
\end{aligned}$$

Therefore, we can extend the preceding proposition as follows.

**Proposition 2.2.** *The following conditions are equivalent*

(i) $\mathcal{Y}_\psi([g]) > 0$ *(respectively* $= 0$, $< 0$*);*

(ii') *There exists a metric* $\widetilde{g} \in [g]$ *which satisfies*

$$\operatorname{Scal}_{c\widetilde{g}} - |\nabla\psi|^2_{c\widetilde{g}} > 0$$

*everywhere on* $M$ *(respectively* $= 0$, $< 0$*) and for any constant* $c > 0$*;*

(iii) *For any metric* $\widetilde{g} \in [g]$*, the first eigenvalue* $\lambda_1$ *of the self-adjoint elliptic operator* $-L_{\widetilde{g},\psi}$ *is positive (respectively zero, negative).*

The advantage of Proposition 2.2 is that it allows us to assume that the manifold $M$ has unit volume by choosing a suitable constant $c > 0$.

One of the most important properties of the Einstein-scalar field Lichnerowicz equations is that they are conformally covariant in the following sense.

**Proposition 2.3 (See [11]).** *Let* $\mathcal{D} = (g, \sigma, \tau, \psi, \pi)$ *be a conformal initial data set for the Einstein-scalar field constraint equations on* $M$*. If* $\widetilde{g} = \theta^{\frac{4}{n-2}} g$ *for a smooth positive function* $\theta$*, then we define the corresponding conformally transformed initial data set by*

$$\widetilde{\mathcal{D}} = (\widetilde{g}, \widetilde{\sigma}, \widetilde{\tau}, \widetilde{\psi}, \widetilde{\pi}) = (\theta^{\frac{4}{n-2}} g, \theta^{-2}\sigma, \tau, \theta^{-\frac{2n}{n-2}}\psi, \pi).$$

*Let* $W$ *be the solution to the conformal form of the momentum constrain equation with respect to the conformal initial data set* $\mathcal{D}$ *(for which we assume that a solution exists), and let* $\widetilde{W}$ *be the solution of the momentum constrain equation with respect to the conformally transformed initial data set* $\widetilde{\mathcal{D}}$ *(which will exist if* $W$ *does). Then* $u$ *is a solution to the Einstein-scalar field Lichnerowicz equation for the conformal data* $\mathcal{D}$ *with* $W$ *if and only if* $\theta^{-1}u$ *is a solution to the Einstein-scalar field Lichnerowicz equation for the transformed conformal data* $\widetilde{\mathcal{D}}$ *with* $\widetilde{W}$*.*

Using Proposition 2.3 above, it turns out that the sign of $\mathcal{Y}_\psi([g])$ plays an important role in the study because we can first perform a conformal transformation on the conformal initial data from $(g, \sigma, \tau, \psi, \pi)$ to $(\theta^{\frac{4}{n-2}} g, \theta^{-2}\sigma, \tau, \theta^{-\frac{2n}{n-2}}\psi, \pi)$ in such a way that $\mathcal{R}_{\widetilde{g},\widetilde{\psi}}$ has a fixed sign by means of Proposition 2.1. Therefore, it suffices to study the solvability of the Einstein-scalar field Lichnerowicz equation for the transformed data $(\theta^{\frac{4}{n-2}} g, \theta^{-2}\sigma, \tau, \theta^{-\frac{2n}{n-2}}\psi, \pi)$ rather than to study to solvability the Einstein-scalar field Lichnerowicz equation for the original data $(g, \sigma, \tau, \psi, \pi)$.

## 2.5 A classification of Choquet-Bruhat–Isenberg–Pollack

### 2.5.1 Solving the momentum constraints

As we have already known that the operator $\varDelta_{g,\mathrm{conf}}$ is a second order, self-adjoint, linear, elliptic operator whose kernel consists of the space of conformal Killing vector fields, see [8, Appendix II]. It follows from the Fredholm alternative that for a given set of functions $(u, \tau, \psi, \pi)$ we may solve the momentum constraint

$$\varDelta_{g,\mathrm{conf}} W = \frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla_g \tau - \pi \nabla_g \psi$$

if either

- $(M, g)$ admits no conformal Killing vector fields, and thus, $W$ is unique;

or

- $\frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla_g \tau - \pi \nabla_g \psi$ is orthogonal in the $L^2$ sense to the space of conformal Killing vector fields.

In the case of constant mean curvature and $(M, g)$ admits conformal Killing vector fields, it suffices to require that $\pi \nabla_g \psi$ is orthogonal to the space of conformal Killing vector fields. Notice that, under the constant mean curvature assumption, the momentum constraint equations consist of only scalar field $\psi$. Therefore, we can solely solve it to obtain $W$. Having the existence of $W$ and the fact that our system of constraint equations are decoupled, we may solely solve the conformal form of the Hamiltonian constraint for $u$.

### 2.5.2 Solving the Hamiltonian constraints

Unlike the Einstein equations in the vacuum case where we know exactly which sets of vacuum constant mean curvature conformal data permit the corresponding Lichnerowicz equation to be solved and which do not, the analysis for the Einstein-scalar field Lichnerowicz equation is more complicated, primarily because there are more relevant possibilities for the signs of the coefficients in (2.1.14). Let us recall from the previous chapter that the corresponding Lichnerowicz equation is simply given by

$$-\varDelta_g u + \mathcal{R}_{g,\psi} u = -\mathcal{B}_{\tau,\psi} u^{\frac{n+2}{n-2}} + \mathcal{A}_{g,W,\pi} u^{-\frac{3n-2}{n-2}}$$

where coefficients $\mathcal{R}_{g,\psi}$, $\mathcal{B}_{\tau,\psi}$, and $\mathcal{A}_{g,W,\pi}$ are given in (1.2.9).

In [11], their classification only depends on the sign of $\mathcal{R}_{g,\psi}$ and $\mathcal{B}_{\tau,\psi}$ since $\mathcal{A}_{g,W,\pi} \geqslant 0$. As for $\mathcal{B}_{\tau,\psi}$, there are six different possibilities, namely, this coefficient can be strictly positive, greater than or equal to zero, identically zero, less than or equal to zero, strictly negative, or of changing sign. For $\mathcal{R}_{g,\psi}$, in view of Proposition 2.1, under a suitable conformal change, we can fix its sign, thus, there are three possibilities, namely, this could be negative, identically zero, or positive. These classification of sign combined with the two options $\mathcal{A}_{g,W,\pi} \equiv 0$ and $\mathcal{A}_{g,W,\pi} \not\equiv 0$ gives us a total of 36 classes of data, see Tables 1.1 and 1.2. Based on this division, the authors in [11] proved for almost all cases, we do know which sets of data permit the Einstein-scalar field Lichnerowicz equation to be solved and which do not. For a detailed statement of this result, we prefer to [11, Theorems 1 and 2].

## 2.6 The Lichnerowicz equations with $\mathcal{R}_{g,\psi}$ being constant

By Proposition 2.1, we know that, after a suitable conformal transformation, the function $h$ is a smooth function having a fixed sign on $M$. Again, by Proposition 2.1, the function $h$ vanishes if we are in the null Yamabe-scalar field conformal invariant, that is equivalent to saying that $h$ is constant (which is equal to zero) in this case. In this section, we show that in fact if we are in the negative Yamabe-scalar field conformal invariant, namely $\mathcal{Y}_\psi([g]) < 0$, we still can perform another conformal transformation on the conformal data from $(g, \sigma, \tau, \psi, \pi)$ to $(u^{\frac{4}{n-2}} g, u^{-2}\sigma, \tau, u^{-\frac{2n}{n-2}}\psi, \pi)$ in such a way that $h$ is a negative constant. Thanks to Proposition 2.3, we may freely assume that $h$ is a negative constant on $M$.

**Proposition 2.4.** *There exists a smooth function $u > 0$ such that under the transformed data $(u^{\frac{4}{n-2}} g, u^{-2}\sigma, \tau, u^{-\frac{2n}{n-2}}\psi, \pi)$ obtained from data $(g, \sigma, \tau, \psi, \pi)$ through the conformal change $\widetilde{g} = u^{\frac{4}{n-2}} g$, coefficient $\mathcal{R}_{\widetilde{g},\widetilde{\psi}}$ is a negative constant.*

*Proof.* First, in view of Proposition 2.1 we may assume that function $\mathcal{R}_{g,\psi} < 0$ in the original data, that is,

$$\mathrm{Scal}_g - |\nabla \psi|_g^2 < 0.$$

Notice that if $\widetilde{g} = u^{\frac{4}{n-2}} g$, coefficient $\mathcal{R}_{\widetilde{g},\widetilde{\psi}}$ verifies the following rule

$$\begin{aligned} \mathcal{R}_{\widetilde{g},\widetilde{\psi}} &= \frac{n-2}{4(n-1)}(\mathrm{Scal}_{\widetilde{g}} - u^{-\frac{4}{n-2}}|\nabla\psi|_g^2) \\ &= \frac{n-2}{4(n-1)}\left(u^{-\frac{4}{n-2}}\,\mathrm{Scal}_g - \frac{4(n-1)}{n-2}u^{-\frac{n+2}{n-2}}\Delta_g u - u^{-\frac{4}{n-2}}|\nabla\psi|_g^2\right) \\ &= u^{-\frac{n+2}{n-2}}(-\Delta_g u + \mathcal{R}_{g,\psi} u), \end{aligned} \tag{2.6.1}$$

which yields

$$-\Delta_g \varphi + \mathcal{R}_{g,\psi}\varphi = \mathcal{R}_{\widetilde{g},\widetilde{\psi}}\varphi^{\frac{n+2}{n-2}}.$$

Therefore, in terms of our notation, it suffices to prove that the following equation with $h$ and $\widetilde{h}$ being negative and $\widetilde{h}$ is constant

$$-\Delta_g u + hu = \widetilde{h} u^{\frac{n+2}{n-2}} \tag{2.6.2}$$

always admits a smooth positive solution $u$. We use the method of sub- and super-solutions to seal this issue.

*Existence of a sub-solution.* This is obvious since a sufficiently small, positive constant $\underline{u}$ will serve. To see this, one can choose any $\underline{u}$ satisfying the following

$$\underline{u} \leqslant \left(\frac{\sup_M h}{\widetilde{h}}\right)^{\frac{n-2}{4}}.$$

With this, we immediately have $-\Delta_g \underline{u} + h\underline{u} \leqslant \widetilde{h}\underline{u}^{\frac{n+2}{n-2}}$. By definition, $\underline{u}$ is a sub-solution of (2.6.2).

*Existence of a super-solution.* We also show that a sufficiently large positive constant $\overline{u}$ will serve. Indeed, similarly to the argument above, one can show that any positive constant $\overline{u}$ satisfying

$$\overline{u} \geqslant \left(\frac{\inf_M h}{\widetilde{h}}\right)^{\frac{n-2}{4}}$$

is a super-solution to (2.6.2) in the sense that $-\Delta_g \overline{u} + h\overline{u} \geqslant \widetilde{h}\overline{u}^{\frac{n+2}{n-2}}$.

Finally, it is an easy task to select those $\underline{u}$ and $\overline{u}$ so that $\underline{u} < \overline{u}$. The method of sub- and super-solutions now guarantees that (2.6.2) admits a positive solution which turns out to be smooth by a simple regularity argument as in the study of the Yamabe problem. The proof is complete. ■

# 3

# Basic properties of solutions of the Lichnerowicz equations



Let us recall from the previous chapter that the general form of the Einstein-scalar field Lichnerowicz equations can be written as follows

$$-\Delta_g u + hu = f u^{\frac{n+2}{n-2}} + a u^{-\frac{3n-2}{n-2}}, \quad u > 0, \tag{3.0.1}$$

where coefficients $h$, $f$, are $a$ are smooth functions. Due to the presence of a term with a critical exponent and a term with a negative exponent, at the beginning, we try to solve the following subcritical equation

$$-\Delta_g u + hu = f|u|^{q-2}u + \frac{au}{(u^2+\varepsilon)^{\frac{q}{2}+1}} \tag{3.0.2}$$

where $q \in (2, 2^\star)$ and $\varepsilon > 0$ are fixed. Obviously, (3.0.2) goes back to (3.0.1) if $q = 2^\star$ and $\varepsilon = 0$ and $u > 0$.

In order to achieve that goal, in this chapter, we first study some basic properties of solutions of (3.0.2) such as non-existence results, regularity, and uniqueness properties. Since the sign of $h$ does affect the analysis of (3.0.2), we basically split our analysis into two cases, one is of the case that $h \leqslant 0$ is constant and the other is of the case $h > 0$.

In the next part of the chapter, we study an auxiliary minimizing problem related to the case that the function $f$ might change sign. It should be mentioned that the basic idea underlying this part was borrowed from Rauzy [38]. In the last part of the chapter, we derive some necessary condition for $f$ and $h$ in order for (3.0.1) to have positive smooth solutions.

Finally, all results represented within this chapter was mainly adapted from [34, 35, 36].

## 3.1 Basic properties of positive solutions of (3.0.2)

We start with a result saying that positive $C^2$ solutions of (3.0.2) actually have a strictly positive lower bound.

### 3.1.1 A lower bound for positive solutions

Our purpose here was to derive a lower bound for a positive $C^2$ solution $u$ of (3.0.2). In the first part of this subsection, we prove that if $h < 0$ is constant, then any $C^2$ solution of (3.0.2) always admits a positive lower bound independent of the function $a$. This is the content of the following.

**Lemma 3.1.** *Let $u$ be a positive $C^2$ solution of (3.0.2) with $h$ a negative constant. Then, there holds*

$$\min_M u \geqslant \min\left\{\left(\frac{h}{\inf_M f}\right)^{\frac{1}{2^\flat-2}}, 1\right\} > 0 \tag{3.1.1}$$

*for any $q \in (2^\flat, 2^\star)$ and any $\varepsilon > 0$.*

*Proof.* Let us assume that $u$ achieves its minimum value at $x_0$. For the sake of simplicity, we denote $u(x_0)$, $f(x_0)$, and $a(x_0)$ by $u_0$, $f_0$, and $a_0$ respectively. Notice that $u_0 > 0$ since $u$ is a positive solution. We then have $\Delta_g u|_{x_0} \geqslant 0$; in particular,

$$hu_0 \geqslant f_0(u_0)^{q-1} + \frac{a_0 u_0}{((u_0)^2+\varepsilon)^{\frac{q}{2}+1}} \geqslant f_0(u_0)^{q-1}.$$

Consequently, we get $f_0 < 0$ and thus $0 < \frac{h}{f_0} \leqslant (u_0)^{q-2}$ which immediately implies

$$\min_M u \geqslant \left(\frac{h}{\inf_M f}\right)^{\frac{1}{q-2}} \geqslant \min\left\{\left(\frac{h}{\inf_M f}\right)^{\frac{1}{2^\flat-2}}, 1\right\}$$

for any $q \in (2^\flat, 2^\star)$ and any $\varepsilon > 0$. This proves our lemma. ■

Next we consider (3.0.2) with $h \geqslant 0$. Unlike the case $h < 0$ where we assume no condition on $a$, for the case $h \geqslant 0$, we do require $\inf_M a > 0$ since the lower bound for any positive $C^2$ solution $u$ depends on $\inf_M a > 0$. We first consider the case $h \equiv 0$. We prove the following lemma.

**Lemma 3.2.** *Let $u$ be a positive $C^2$ solution of (3.0.2) with $h \equiv 0$. Then, there holds*

$$\min_M u \geqslant \frac{1}{2}\min\left\{\left(\frac{\inf_M a}{-\inf_M f}\right)^{\frac{1}{22^\flat}}, 1\right\} \tag{3.1.2}$$

*for any* $q \in (2^\flat, 2^\star)$ *and any*

$$\varepsilon < \frac{1}{2} \min\left\{ \left( \frac{\inf_M a}{-\inf_M f} \right)^{\frac{1}{2^\flat}}, 1 \right\}. \tag{3.1.3}$$

*Proof.* Let us assume that $u$ achieves its minimum value at $x_0$. For simplicity, let us denote $u(x_0)$, $f(x_0)$, and $a(x_0)$ by $u_0$, $f_0$, and $a_0$ respectively. Notice that $u_0 > 0$ since $u$ is a positive solution. We then have $\Delta_g u|_{x_0} \geqslant 0$; in particular,

$$f_0 (u_0)^{q-1} + \frac{a_0 u_0}{((u_0)^2 + \varepsilon)^{\frac{q}{2}+1}} \leqslant 0. \tag{3.1.4}$$

Consequently, we get that $f_0 < 0$. Using (3.1.4) we can see that

$$a_0 \leqslant -f_0 (u_0)^{q-2} ((u_0)^2 + \varepsilon)^{\frac{q}{2}+1} \leqslant -f_0 ((u_0)^2 + \varepsilon)^q$$

which implies that

$$(u_0)^2 + \varepsilon \geqslant \left( \frac{a_0}{-f_0} \right)^{\frac{1}{q}} \geqslant \left( \frac{\inf_M a}{-\inf_M f} \right)^{\frac{1}{q}}.$$

Thus, one can conclude that $u_0$ satisfies (3.1.2) for any $q \in [2^\flat, 2^\star)$ and any $\varepsilon$ verifying the condition (3.1.3). The proof is complete. ■

In the last step, we mainly consider the case $h > 0$. Since such a result was proved in [21], here we just calculate a precise lower bound for positive $C^2$ solutions of (3.0.2).

**Lemma 3.3.** *Let $u$ be a positive $C^2$ solution of (3.0.2) with $h > 0$. Then there holds*

$$\min_M u \geqslant \min\left\{ \left( \frac{1}{2^{2^\flat}} \frac{\inf_M a}{\sup_M h + \sup_M |f|} \right)^{\frac{1}{2^\flat+2}}, 1 \right\} \tag{3.1.5}$$

*for any* $q \in (2^\flat, 2^\star)$ *and any*

$$\varepsilon < \min\left\{ \left( \frac{1}{2^{2^\flat}} \frac{\inf_M a}{\sup_M h + \sup_M |f|} \right)^{\frac{2}{2^\flat+2}}, 1 \right\}. \tag{3.1.6}$$

*Proof.* Following [21], we let $\delta > 0$ be the unique positive solution of the following algebraic equation

$$\delta^{q+2} \left( \sup_M h + (\sup_M |f|) \delta^{q-2} \right) = \frac{1}{2^{2^\flat}} \inf_M a. \tag{3.1.7}$$

Since $\delta$ depends on $q$, we shall prove that for $q \in (2^\flat, 2^\star)$, $\delta$ has a strictly positive lower bound. We have the following two cases.

**Case 1**. Suppose

$$\sup_M h + \sup_M |f| \geqslant \frac{1}{2^{2^\flat}} \inf_M a.$$

In this case, there holds $\delta \leqslant 1$. Consequently, we can estimate

$$\frac{1}{2^{2^\flat}} \inf_M a \leqslant \delta^{q+2} \left( \sup_M h + \sup_M |f| \right)$$

which immediately gives us

$$\delta \geqslant \left(\frac{1}{2^{2^\flat}}\frac{\inf_M a}{\sup_M h + \sup_M |f|}\right)^{\frac{1}{q+2}} \geqslant \left(\frac{1}{2^{2^\flat}}\frac{\inf_M a}{\sup_M h + \sup_M |f|}\right)^{\frac{1}{2^\flat+2}}.$$

**Case 2.** Suppose

$$\sup_M h + \sup_M |f| < \frac{1}{2^{2^\flat}} \inf_M a.$$

In this case, there holds $\delta \geqslant 1$ which immediately gives us a lower bound for $\delta$.

Combining two cases above, we conclude that

$$\delta \geqslant \min\left\{\left(\frac{1}{2^{2^\flat}}\frac{\inf_M a}{\sup_M h + \sup_M |f|}\right)^{\frac{1}{2^\flat+2}}, 1\right\}.$$

Suppose that $u$ is a positive $C^2$ solution of (3.0.2) with $\varepsilon > 0$ satisfying the condition (3.1.6) above, that is,

$$\frac{\Delta_g u}{u} + h = f u^{q-2} + \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}+1}}.$$

Let us assume that $u$ achieves its minimum value at $x_0$, then we have

$$h(x_0) + (-f(x_0))u(x_0)^{q-2} \geqslant \frac{a(x_0)}{(u(x_0)^2+\varepsilon)^{\frac{q}{2}+1}}. \tag{3.1.8}$$

We assume $u(x_0) < \delta$. From the choice of $\delta$, one can verify that

$$\sup_M |h| + (\sup_M |f|)\delta^{q-2} \geqslant h(x_0) + (-f(x_0))u(x_0)^{q-2}. \tag{3.1.9}$$

Since $\varepsilon < \delta^2$ and $u(x_0) < \delta$, it is easy to see that

$$\frac{a(x_0)}{(u(x_0)^2+\varepsilon)^{\frac{q}{2}+1}} \geqslant \frac{\inf_M a}{(\sqrt{2}\delta)^{q+2}} > \frac{1}{2^{2^\flat}}\frac{\inf_M a}{\delta^{q+2}}. \tag{3.1.10}$$

Using (3.1.8), (3.1.9), and (3.1.10), we easily get a contradiction, thus proving that $u(x_0) \geqslant \delta$. In particular, there holds $u \geqslant \delta$ in $M$. This proves our lemma. ■

### 3.1.2 Regularity for non-negative weak solutions of (3.0.2)

This subsection is devoted to the regularity of weak solutions of (3.0.2). Despite the fact that $h$ can be chosen as a constant in the non-positive Yamabe-scalar field conformal invariant, in this subsection, we allow $h$ to be non-constant. As such, we assume that $h$, $f$ and $a \geqslant 0$ are smooth functions and that the function $h$ has a fixed sign on $M$.

**Lemma 3.4.** *Assume that $u \in H^1(M)$ is an almost everywhere non-negative weak solution of Equation (3.0.2). We assume further that $\inf_M a > 0$ in the case when $h \geqslant 0$. Then*

*(a) If $\varepsilon > 0$, then $u \in C^\infty(M)$. In particular, $u \geqslant 0$ in $M$.*

*(b) If $\varepsilon = 0$ and $u^{-1} \in L^p(M)$ for all $p \geqslant 1$, then $u \in C^\infty(M)$.*

*Proof.* We first rewrite (3.0.2) as

$$-\Delta_g u + b(x)(1+u) = 0$$

with

$$b(x) = \frac{u(x)}{1+u(x)} \left( h(x) - \frac{a(x)}{(u(x)^2+\varepsilon)^{\frac{q}{2}+1}} - f(x)|u(x)|^{q-2} \right).$$

By the Sobolev embedding, we know that $u \in L^q(M)$ for any $q \in (2^\flat, 2^\star]$. This and the conditions in both cases (a) and (b) imply

$$h(x) - \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}+1}} - f|u|^{q-2} \in L^{\frac{q}{q-2}}(M).$$

Notice that from $q \leqslant 2^\star$ there holds $\frac{q}{q-2} \geqslant \frac{n}{2}$. We now use the Brezis–Kato estimate [40, Lemma B.3] to conclude that $u \in L^s(M)$ for any $s > 0$. Thus the Caldéron–Zygmund inequality implies that $u \in H^p(M)$ for any $p > 1$. The Sobolev embedding again implies that $u$ is in $C^{0,\alpha}(M)$ for some $\alpha \in (0,1)$. Thus, we know from the Schauder theory that $u \in C^{2,\alpha}(M)$ for some $\alpha \in (0,1)$. In particular, $u$ has a strictly positive lower bound by means of Lemmas 3.1, 3.2, and 3.3. Since $u$ stays away from zero, we can iterate this process to conclude $u \in C^\infty(M)$. ∎

### 3.1.3 Non-existence results for smooth positive solutions of (3.0.1) of finite $H^1$-norm in the case $h \leqslant 0$

Let $u$ be a smooth positive solution of (3.0.1). The main result of this subsection was to derive a necessary condition for $a$ such that $\|u\|_{H^1}$ is bounded by a given constant. More precise, assume that $\alpha$ and $\beta$ are two positive constant, we wish to estimate $\int_M a^\alpha |f^-|^\beta \, d\mathrm{vol}_g$ from above in terms of $\|u\|_{H^1}$. Such a result is basically due to Hebey–Pacard–Pollack [21]. We note that we are in the case $h \leqslant 0$.

In the first part of this subsection, we consider a simple case, that is, $\alpha = \frac{2n}{5n-2}$ and $\beta = 0$. By integrating (3.0.1) over $M$ we get that

$$\int_M hu \, d\mathrm{vol}_g = \int_M f u^{2^\star - 1} \, d\mathrm{vol}_g + \int_M \frac{a}{u^{2^\star+1}} \, d\mathrm{vol}_g. \tag{3.1.11}$$

Let $\beta = \frac{2^\star}{22^\star+1}$. First, the Hölder inequality implies

$$\begin{aligned} \int_M a^\beta \, d\mathrm{vol}_g &= \int_M \left(\frac{a}{u^{2^\star+1}}\right)^\beta (u^{2^\star+1})^\beta \, d\mathrm{vol}_g \\ &\leqslant \left(\int_M \frac{a}{u^{2^\star+1}} \, d\mathrm{vol}_g\right)^\beta \left(\int_M (u^{2^\star+1})^{\frac{\beta}{1-\beta}} \, d\mathrm{vol}_g\right)^{1-\beta} \\ &= \left(\int_M \frac{a}{u^{2^\star+1}} \, d\mathrm{vol}_g\right)^\beta \left(\int_M u^{2^\star} \, d\mathrm{vol}_g\right)^{1-\beta}. \end{aligned} \tag{3.1.12}$$

Thanks to $h \leqslant 0$, one can easily see that the second term in the right of (3.1.11) can be bounded as

$$\begin{aligned}\int_M \frac{a}{u^{2^\star+1}}\,\mathrm{dvol}_g &= \int_M hu\,\mathrm{dvol}_g - \int_M fu^{2^\star-1}\,\mathrm{dvol}_g\\ &\leqslant -\int_M fu^{2^\star-1}\,\mathrm{dvol}_g\\ &\leqslant \int_M |f|u^{2^\star-1}\,\mathrm{dvol}_g,\end{aligned} \tag{3.1.13}$$

while the first term can be controlled as

$$\begin{aligned}\int_M |f|u^{2^\star-1}\,\mathrm{dvol}_g &\leqslant \Big(\int_M |f|^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{1}{2^\star}}\Big(\int_M (u^{2^\star-1})^{\frac{2^\star}{2^\star-1}}\,\mathrm{dvol}_g\Big)^{\frac{2^\star-1}{2^\star}}\\ &= \Big(\int_M |f|^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{1}{2^\star}}\Big(\int_M u^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{2^\star-1}{2^\star}}.\end{aligned} \tag{3.1.14}$$

Combine (3.1.11)–(3.1.14) to get

$$\begin{aligned}\int_M a^\beta\,\mathrm{dvol}_g &\leqslant \Big(\int_M |f|^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{\beta}{2^\star}}\Big(\int_M u^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{2^\star-1}{2^\star}\beta+1-\beta}\\ &= \Big(\int_M |f|^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{\beta}{2^\star}}\Big(\int_M u^{2^\star}\,\mathrm{dvol}_g\Big)^{1-\frac{\beta}{2^\star}}.\end{aligned} \tag{3.1.15}$$

We now consider the general case. In order to state the result, let us assume that $u$ is a smooth positive solution of (3.0.1) and, as always, $\alpha\in(0,1)$, $\beta>0$ are constant. The exact condition for $\alpha$ is given as follows

$$0<\alpha<\frac{2^\star}{22^\star+1}.$$

By the Hölder inequality, we get that

$$\int_M a^\alpha|f^-|^\beta\,\mathrm{dvol}_g \leqslant \Big(\int_M \frac{a}{u^{2^\star+1}}\,\mathrm{dvol}_g\Big)^\alpha\Big(\int_M |f^-|^{\frac{\beta}{1-\alpha}}u^{\frac{(2^\star+1)\alpha}{1-\alpha}}\,\mathrm{dvol}_g\Big)^{1-\alpha}.$$

By integrating both sides of (3.0.1), we get

$$\int_M \frac{a}{u^{2^\star+1}}\,\mathrm{dvol}_g = -\int_M fu^{2^\star-1}\,\mathrm{dvol}_g \leqslant \int_M |f^-|u^{2^\star-1}\,\mathrm{dvol}_g. \tag{3.1.16}$$

Again, by the Hölder inequality, we can estimate

$$\int_M |f^-|u^{2^\star-1}\,\mathrm{dvol}_g \leqslant \Big(\int_M |f^-|^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{1}{2^\star}}\Big(\int_M u^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{2^\star-1}{2^\star}}.$$

Therefore, we have

$$\begin{aligned}\int_M a^\alpha|f^-|^\beta\,\mathrm{dvol}_g \leqslant &\Big(\int_M |f^-|^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{\alpha}{2^\star}}\Big(\int_M u^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{(2^\star-1)\alpha}{2^\star}}\\ &\Big(\int_M |f^-|^{\frac{\beta}{1-\alpha}}u^{\frac{(2^\star+1)\alpha}{1-\alpha}}\,\mathrm{dvol}_g\Big)^{1-\alpha}.\end{aligned}$$

From that choice of $\alpha$, we immediately see that $\frac{(2^\star+1)\alpha}{1-\alpha}<2^\star$. Therefore, by the Hölder inequality, we further have

$$\int_M |f^-|^{\frac{\beta}{1-\alpha}} u^{\frac{(2^\star+1)\alpha}{1-\alpha}} \,\mathrm{dvol}_g \leqslant \left(\int_M |f^-|^{\frac{2^\star\beta}{2^\star-(22^\star+1)\alpha}} \,\mathrm{dvol}_g\right)^{\frac{2^\star-(22^\star+1)\alpha}{2^\star(1-\alpha)}} \left(\int_M u^{2^\star} \,\mathrm{dvol}_g\right)^{\frac{(2^\star+1)\alpha}{2^\star(1-\alpha)}}$$

which helps us to conclude that

$$\begin{aligned}\int_M a^\alpha |f^-|^\beta \,\mathrm{dvol}_g \leqslant & \left(\int_M |f^-|^{2^\star} \,\mathrm{dvol}_g\right)^{\frac{\alpha}{2^\star}} \left(\int_M u^{2^\star} \,\mathrm{dvol}_g\right)^{\frac{(2^\star-1)\alpha}{2^\star}} \\ & \left(\int_M |f^-|^{\frac{2^\star\beta}{2^\star-(22^\star+1)\alpha}} \,\mathrm{dvol}_g\right)^{\frac{2^\star-(22^\star+1)\alpha}{2^\star}} \left(\int_M u^{2^\star} \,\mathrm{dvol}_g\right)^{\frac{(2^\star+1)\alpha}{2^\star}}.\end{aligned}$$

Thus, we have proved that

$$\begin{aligned}\int_M a^\alpha |f^-|^\beta \,\mathrm{dvol}_g \leqslant & \left(\int_M |f^-|^{2^\star} \,\mathrm{dvol}_g\right)^{\frac{\alpha}{2^\star}} \\ & \left(\int_M |f^-|^{\frac{2^\star\beta}{2^\star-(22^\star+1)\alpha}} \,\mathrm{dvol}_g\right)^{\frac{2^\star-(22^\star+1)\alpha}{2^\star}} \left(\int_M u^{2^\star} \,\mathrm{dvol}_g\right)^{2\alpha}.\end{aligned} \tag{3.1.17}$$

Next, let us consider the case when $\frac{(2^\star+1)\alpha}{1-\alpha} = 2^\star$. In this context, we immediately have $\alpha = \frac{2^\star}{22^\star+1}$. Making use of (3.1.15), we get that

$$\begin{aligned}\int_M a^{\frac{2n}{5n-2}} |f^-|^\beta \,\mathrm{dvol}_g \leqslant & (\max_M |f^-|^\beta) \\ & \left(\int_M |f^-|^{2^\star} \,\mathrm{dvol}_g\right)^{\frac{1}{22^\star+1}} \left(\int_M u^{2^\star} \,\mathrm{dvol}_g\right)^{\frac{22^\star}{22^\star+1}}.\end{aligned} \tag{3.1.18}$$

If we now adopt the convention that, in the case $\alpha = \frac{2^\star}{22^\star+1}$,

$$\left(\int_M |f^-|^{\frac{2^\star\beta}{2^\star-(22^\star+1)\alpha}} \,\mathrm{dvol}_g\right)^{\frac{2^\star-(22^\star+1)\alpha}{2^\star}} = \left\| |f^-|^\beta \right\|_{L^\infty},$$

then by collecting (3.1.17) and (3.1.18), we have the following result.

**Proposition 3.1.** *Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $n \geqslant 3$. Let also $a, f$ be smooth functions on $M$ with $a \geqslant 0$ in $M$. If*

$$\begin{aligned}\int_M a^\alpha |f^-|^\beta \,\mathrm{dvol}_g > & \left(\int_M |f^-|^{2^\star} \,\mathrm{dvol}_g\right)^{\frac{\alpha}{2^\star}} \\ & \left(\int_M |f^-|^{\frac{2^\star\beta}{2^\star-(22^\star+1)\alpha}} \,\mathrm{dvol}_g\right)^{\frac{2^\star-(22^\star+1)\alpha}{2^\star}} (\mathcal{K}_1 + \mathcal{A}_1)^{2^\star\alpha} \Lambda^{22^\star\alpha}\end{aligned}$$

*for some $\Lambda > 0$, some $0 < \alpha \leqslant \frac{2n}{5n-2}$, and some $\beta \geqslant 0$, then the Einstein-scalar field Lichnerowicz equation (3.0.1) does not possess smooth positive solutions with $\|u\|_{H^1} \leqslant \Lambda$.*

*Proof.* Let $u$ be a smooth positive solution of (3.0.1) such that $\|u\|_{H^1} \leqslant \Lambda$. By the Sobolev inequality, there holds

$$\left(\int_M u^{2^\star} \,\mathrm{dvol}_g\right)^{2\alpha} \leqslant (\mathcal{K}_1 + \mathcal{A}_1)^{2^\star\alpha} \|u\|_{H^1}^{22^\star\alpha}.$$

This and our calculation above show that

$$\begin{aligned}\int_M a^\alpha |f^-|^\beta \, \mathrm{dvol}_g \leqslant & \Big(\int_M |f^-|^{2^\star} \, \mathrm{dvol}_g\Big)^{\frac{\alpha}{2^\star}} \\ & \Big(\int_M |f^-|^{\frac{2^\star \beta}{2^\star - (22^\star+1)\alpha}} \, \mathrm{dvol}_g\Big)^{\frac{2^\star - (22^\star+1)\alpha}{2^\star}} (\mathcal{K}_1 + \mathcal{A}_1)^{2^\star \alpha} \Lambda^{22^\star \alpha},\end{aligned}$$

which immediately concludes the result. ■

It is worth noting that a similar result was obtained in [21, Theorem 2.1] for the non-negative case $h \geqslant 0$. We shall use this result somewhere in the case $h > 0$.

### 3.1.4 The uniqueness of positive smooth solutions for (3.0.1)

We now turn to the question of uniqueness of positive smooth solutions of (3.0.1). We prove in this section that if $f \leqslant 0$, then (3.0.1) admits the uniqueness property of positive smooth solutions. To this purpose, we prove by way of contradiction and our approach is based on the method of sub- and super-solutions. Before doing so, we need the following result.

**Lemma 3.5.** *For any smooth functions $h$, $f$, and $a \geqslant 0$ with $\int_M a \, \mathrm{dvol}_g > 0$ and $h$ has a fixed sign in $M$. We further assume that there exists a continuous super-solution $\overline{u} > 0$ to (3.0.1). Then there always exists a sub-solution $\underline{u} > 0$ to (3.0.1) such that $\underline{u} < \overline{u}$.*

*Proof.* In order to prove the existence of a sub-solution, we introduce the following trick. Depending on the sign of $h$, we naturally have two cases.

**Case 1**. Suppose $h \leqslant 0$. In this case, the function $f$ always admits negative values in $M$. We first observe that

$$\int_M \Big(a + \frac{\int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^-\Big) \, \mathrm{dvol}_g = 0.$$

Consequently, there exists a function $u_0 \in H^1(M)$ solving

$$-\Delta_g u_0 = a + \frac{\int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^- \tag{3.1.19}$$

Since the right hand side of (3.1.19) is of class $L^p(M)$ for any $p < +\infty$, the Caldéron–Zygmund inequality tells us that the solution $u_0$ is of class $W^{2,p}(M)$ for any $p < +\infty$. Thanks to the Sobolev Embedding theorem [3, Theorem 2.10], we can conclude that $u_0 \in C^{0,\alpha}(M)$ for some $\alpha \in (0,1)$. In particular, the solution $u_0$ is continuous. Therefore, by adding a sufficiently large constant $C$ to the function $u_0$ if necessary, we can always assume that $\min_M u_0 > 1$.

We now find the sub-solution $\underline{u}$ of the form $\varepsilon u_0$ for small $\varepsilon > 0$ to be determined. To this purpose, we first observe that

$$-\Delta_g \underline{u} = \varepsilon a + \frac{\varepsilon \int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^-. \tag{3.1.20}$$

Since $\max_M u_0 < +\infty$, it is easy to see that, for any $0 < \varepsilon \leqslant (\max_M u_0)^{-\frac{2^\star+1}{2^\star+2}}$, we immediately have

$$\varepsilon a \leqslant \frac{a}{\varepsilon^{2^\star+1} u_0^{2^\star+1}}. \tag{3.1.21}$$

Besides, since $f^- \leqslant 0$ and $2^\star > 2$, it is not difficult to see that the following inequality

$$\frac{\varepsilon \int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f^-$$

holds provided

$$\varepsilon \leqslant \left( \frac{\int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} \right)^{\frac{1}{2^\star-2}} (\max_M u_0)^{-\frac{2^\star-1}{2^\star-2}} \tag{3.1.22}$$

In particular, the following

$$\frac{\varepsilon \int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f \tag{3.1.23}$$

holds provided (3.1.22) holds. Combining (3.1.20), (3.1.21), and (3.1.23), we conclude that for small $\varepsilon$

$$-\Delta_g \underline{u} \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f + \frac{a}{\varepsilon^{2^\star+1} u_0^{2^\star+1}}.$$

In other words, and thanks to $h \leqslant 0$, we have showed that

$$-\Delta_g \underline{u} + h\underline{u} \leqslant f \underline{u}^{2^\star-1} + \frac{a}{\underline{u}^{2^\star+1}}.$$

Finally, since $\overline{u}$ has a strictly positive lower bound, we can choose $\varepsilon > 0$ sufficiently small such that $\underline{u} \leqslant \overline{u}$.

**Case 2**. Suppose $h > 0$. In this context, our approach is basically the same as that used in Case 1. We consider the following equation

$$-\Delta_g u + (h - f^-) u = a. \tag{3.1.24}$$

Since $h - f^- > 0$, $a \geqslant 0$, $a \not\equiv 0$, and the manifold $M$ is compact without the boundary, the standard argument shows that (3.1.24) always admits a weak solution, say $u_0$. By a standard regularity result, one can easily deduce that $u_0$ is at least continuous. Thus, by the Maximum Principle, we conclude $u_0 > 0$.

As before, we now find the sub-solution $\underline{u}$ of the form $\varepsilon u_0$ for small $\varepsilon > 0$ to be determined. To this purpose, we first write

$$-\Delta_g \underline{u} + h\underline{u} = \varepsilon a + f^- \underline{u}. \tag{3.1.25}$$

Notice that the term involving $\varepsilon a$ can be controlled using (3.1.21). Besides, since $f^- \leqslant 0$ and $2^\star > 2$, it is not difficult to see that the following inequality

$$\varepsilon u_0 f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f^-$$

holds provided $\varepsilon \leqslant \frac{1}{\max u_0}$. In particular, the following

$$\varepsilon u_0 f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f \tag{3.1.26}$$

holds provided $\varepsilon \leqslant \frac{1}{\max u_0}$. Combining all estimates above, we conclude that for small $\varepsilon$

$$-\Delta_g \underline{u} + h\underline{u} \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f + \frac{a}{\varepsilon^{2^\star+1} u_0^{2^\star+1}}.$$

In other words, we have showed that $\underline{u}$ is a sub-solution of (3.0.1). Finally, since $\overline{u}$ has a strictly positive lower bound, we can choose $\varepsilon > 0$ sufficiently small such that $\underline{u} \leqslant \overline{u}$. We conclude the proof. ∎

We now prove a uniqueness property of positive smooth solutions of (3.0.1).

**Lemma 3.6.** *For any smooth functions $h$, $f \leqslant 0$, and $a \geqslant 0$ with $\int_M a \, d\mathrm{vol}_g > 0$ and $h$ has a fixed sign in $M$. We further assume that there exists a continuous super-solution $\overline{u} > 0$ to (3.0.1). Then (3.0.1) admits at most one positive smooth solution.*

*Proof.* Suppose that there exists two positive smooth solution $u_1$ and $u_2$ of (3.0.1) with $u_1 \neq u_2$. We may assume that $u_1 \geqslant u_2$ in $M$. Indeed, if $u_1 \ngeqslant u_2$ and $u_2 \ngeqslant u_1$ in $M$, we then set

$$\overline{u}(x) = \min\{u_1(x), u_2(x)\}, \quad x \in M.$$

It is easy to see that $\overline{u} > 0$ in $M$ and from [37, Proposition 1] we know that $\overline{u}$ is a super-solution of (3.0.1), that is

$$-\Delta_g \overline{u} + h\overline{u} \geqslant f\overline{u}^{2^\star-1} + \frac{a}{\overline{u}^{2^\star+1}}.$$

In addition, since $u_1$ and $u_2$ are smooth, it is clear that $\overline{u}$ is at least continuous, thus showing that $\inf_M \overline{u} > 0$. By Lemma 3.5, there exists a continuous, positive sub-solution $\underline{u}$ of (3.0.1) with $\underline{u} < \overline{u}$. By the sub- and super-solutions method, there is a solution $v$ of (3.0.1) satisfying

$$\underline{u} \leqslant v \leqslant \overline{u} \quad \text{on } M.$$

So we may choose $v$ to replace $u_2$. In other words, we can assume that $u_1 \geqslant u_2$ on $M$. We now show that in fact $u_1 > u_2$ on $M$. Indeed, we first recall that the following

$$\begin{aligned} -\Delta_g u_1 + hu_1 &= f u_1^{2^\star-2} + a u_1^{-2^\star-1}, \\ -\Delta_g u_2 + hu_2 &= f u_2^{2^\star-2} + a u_2^{-2^\star-1}, \end{aligned} \tag{3.1.27}$$

hold. We now suppose that there exists a point $x_0 \in M$ such that $u_1(x_0) = u_2(x_0)$, by setting $w(x) = u_1(x) - u_2(x)$ with $x \in M$, we arrive at

$$\begin{cases} -\Delta_g w + hw = \left( f\dfrac{u_1^{2^\star-1} - u_2^{2^\star-1}}{u_1 - u_2} + a\dfrac{u_1^{-2^\star-1} - u_2^{-2^\star-1}}{u_1 - u_2} \right) w, & \text{on } M, \\ w \geqslant 0, & \text{on } M, \\ w = 0, & \text{at } x_0. \end{cases}$$

By using the Strong Maximum Principle, there holds $w > 0$ in $M$ which contradicts to the fact that $w(x_0) = 0$. Thus, we have proved that $u_1 > u_2$ in $M$. Since $u_1$ and $u_2$ are solutions of (3.0.1), from (3.1.27), we easily obtain

$$\int_M \nabla u_1 \nabla u_2 \, d\mathrm{vol}_g + \int_M h u_1 u_2 \, d\mathrm{vol}_g = \int_M f u_1^{2^\star-1} u_2 \, d\mathrm{vol}_g + \int_M \frac{a u_2}{u_1^{2^\star+1}} \, d\mathrm{vol}_g,$$
$$\int_M \nabla u_2 \nabla u_1 \, d\mathrm{vol}_g + \int_M h u_2 u_1 \, d\mathrm{vol}_g = \int_M f u_2^{2^\star-1} u_1 \, d\mathrm{vol}_g + \int_M \frac{a u_1}{u_2^{2^\star+1}} \, d\mathrm{vol}_g .$$

By subtracting, we get that

$$\int_M f u_1 u_2 (u_1^{2^\star-2} - u_2^{2^\star-2}) \, d\mathrm{vol}_g + \int_M a u_1 u_2 \left( \frac{1}{u_1^{2^\star+2}} - \frac{1}{u_2^{2^\star+2}} \right) d\mathrm{vol}_g = 0,$$

which is a contradiction since $u_1 > u_2 > 0$, $f \leqslant 0$, and $a \geqslant 0$ with $a \not\equiv 0$. Such a contradiction implies that $u_1 \equiv u_2$, thus proving the uniqueness of positive smooth solution of (3.0.1). ■

## 3.2 An auxiliary minimizing problem

The following next two subsections are basically due to Rauzy [38]. Here we just relax some conditions in the Rauzy arguments for future benefit. Besides, since our functional energy is different from the one in [38], it is worth to reproduce several parts in order to make the thesis to be self-contained.

### 3.2.1 The number $\lambda_f$

Following [38], we define the following number

$$\lambda_f = \begin{cases} \inf\limits_{u \in \mathcal{A}} \dfrac{\int_M |\nabla u|^2 \, d\mathrm{vol}_g}{\int_M |u|^2 \, d\mathrm{vol}_g}, & \text{if } \mathcal{A} \neq \emptyset, \\ +\infty, & \text{if } \mathcal{A} = \emptyset, \end{cases} \tag{3.2.1}$$

where

$$\mathcal{A} = \left\{ u \in H^1(M) : u \geqslant 0, \ u \not\equiv 0, \ \int_M |f^-| u \, d\mathrm{vol}_g = 0 \right\}. \tag{3.2.2}$$

Functions in $\mathcal{A}$ are to be thought of as functions that vanish on the support of $f^-$. It is clear that $\lambda_f < +\infty$ if and only if the set $\{f \geqslant 0\}$ has positive measure, that is, either $\sup_M f > 0$ or $\sup_M f = 0$ and $\int_{\{f=0\}} 1 \, d\mathrm{vol}_g > 0$.

We now show that in any case $\lambda_f$ is actually strictly positive. To this end, it suffices to consider the case $\lambda_f < \infty$, that is equivalent to saying that the set $M_1 = \{x \in M : f(x) \geqslant 0\}$ has positive measure. Let us recall from [38, Lemme 1] that it was proved that $\lambda_f$ coincides with the first eigenvalue $\lambda_1$ of the associated Dirichlet problem over $M_1$, that is,

$$\lambda_1 = \inf_{u \in \mathcal{A}'} \frac{\int_M |\nabla u|^2 \, d\mathrm{vol}_g}{\int_M |u|^2 \, d\mathrm{vol}_g} \tag{3.2.3}$$

where

$$\mathcal{A}' = \{ u \in C^\infty(M) : u > 0 \text{ in } M_1, \ u = 0 \text{ on } \partial M_1 \}. \tag{3.2.4}$$

Obviously, $\lambda_1 > 0$, thus proving $\lambda_f > 0$.

### 3.2.2 The number $\lambda_{f,\eta,q}$

At the beginning of the subsection we temporarily leave our equation to study another minimizing problem. The proof of our main result depends on $\lambda_{f,\eta,q}$ which will be defined below. This quantity was first introduced by Rauzy [38]. To be precise, we introduce $\mathcal{A}(\eta, q)$, another subspace of $H^1(M)$, which is defined as the following

$$\mathcal{A}(\eta, q) = \left\{ u \in H^1(M) : \|u\|_{L^q} = 1, \int_M |f^-||u|^q \, d\mathrm{vol}_g = \eta \int_M |f^-| \, d\mathrm{vol}_g \right\}. \quad (3.2.5)$$

We assume for a moment that $\mathcal{A}(\eta, q)$ is not empty which will be mentioned later after proving Lemma 3.7 below. We define the number

$$\lambda_{f,\eta,q} = \inf_{u \in \mathcal{A}(\eta,q)} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}. \quad (3.2.6)$$

Clearly, $\lambda_{f,\eta,q} \geqslant 0$. We are going to prove the following result.

**Lemma 3.7.** *Starting from some small $\eta > 0$ and as a function of $\eta$, $\lambda_{f,\eta,q}$ is monotone decreasing.*

In the present case, it is hard to consider the equality sign, nevertheless we study the following problem first

$$\lambda'_{f,\eta,q} = \inf_{u \in \mathcal{A}'(\eta,q)} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}$$

where

$$\mathcal{A}'(\eta, q) = \left\{ u \in H^1(M) : \|u\|_{L^q} = 1, \int_M |f^-||u|^q \, d\mathrm{vol}_g \leqslant \eta \int_M \left|f^-\right| d\mathrm{vol}_g \right\}.$$

With $q$ and $\eta$ being fixed, the set $\mathcal{A}'(\eta, q)$ is not empty since it includes the set of functions $u \in H^1(M)$ such that $\|u\|_{L^q} = 1$ and with supports in the set

$$\{x \in M : f(x) > 0\} \subset \left\{ x \in M, |f^-|(x) < \eta \int_M |f^-| \, d\mathrm{vol}_g \right\}.$$

As can be seen form the definition of $\mathcal{A}'$ that if $\eta_1 \leqslant \eta_2$ then $\mathcal{A}'(\eta_1, q) \subset \mathcal{A}'(\eta_2, q)$; thus proving $\lambda'_{f,\eta_2,q} \leqslant \lambda'_{f,\eta_1,q}$. This amounts to saying that $\lambda'_{f,\eta,q}$ is monotone decreasing.

We are going to show that $\lambda'_{f,\eta,q} = \lambda_{f,\eta,q}$. For that reason, it suffices to show that $\lambda'_{f,\eta,q} \geqslant \lambda_{f,\eta,q}$ since the reverse is trivial. The fact $\mathcal{A}'(\eta, q)$ is not empty implies that $\lambda'_{f,\eta,q}$ is finite. We are now in a position to prove Lemma 3.7.

*Proof of Lemma 3.7.* Before proving the lemma, let us first assume that $\lambda'_{f,\eta,q} > 0$ for some small $\eta$. Otherwise, this is trivial since we always have $\lambda'_{f,\eta,q} \geqslant 0$. We first prove that $\lambda'_{f,\eta,q}$ is achieved.

Let $\{v_j\}_j \subset \mathcal{A}'(\eta, q)$ be a minimizing sequence for $\lambda'_{f,\eta,q}$. Obviously the sequence $\{|v_j|\}_j$ is also a minimizing sequence in $\mathcal{A}'(\eta, q)$ and therefore we can assume from the beginning that $v_j \geqslant 0$ in $M$. By the Hölder inequality, one has $\|v_j\|_{L^2} \leqslant 1$. Then for $j$ sufficiently large, we obtain $\|\nabla v_j\|_{L^2}^2 \leqslant \lambda'_{f,\eta,q} + 1$. Thus $\{v_j\}_j$ is bounded in $H^1(M)$. Being bounded, there exists $v \in H^1(M)$ such that, up to subsequences,

- $v_j \rightharpoonup v$ weakly in $H^1(M)$;
- $\nabla v_j \rightharpoonup \nabla v$ weakly in $L^2(M)$;
- $v_j \to v$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;
- $v_j \to v$ almost everywhere in $M$.

Consequently, $v \geqslant 0$ almost everywhere in $M$ and $\|v\|_{L^q} = 1$. Using the Lebesgue Dominated Convergence Theorem, we can pass to the limit to obtain

$$\int_M |f^-||v|^q \, d\mathrm{vol}_g \leqslant \eta \int_M |f^-| \, d\mathrm{vol}_g.$$

In other words, $v \in \mathcal{A}'(\eta, q)$. We notice that

$$\|\nabla v\|_{L^2}^2 \leqslant \lim_{j\to\infty} \|\nabla v_j\|_{L^2}^2 , \quad \|v\|_{L^2}^2 = \lim_{j\to\infty} \|v_j\|_{L^2}^2 .$$

Therefore, $\|\nabla v\|_{L^2}^2 \|v\|_{L^2}^{-2} \leqslant \lambda'_{f,\eta,q}$. Thus $\lambda'_{f,\eta,q}$ is achieved by $v$. Notice that $\|\nabla v\|_{L^2} > 0$ since $\lambda'_{f,\eta,q} > 0$. Since $|\nabla |v|| = |\nabla v|$ we may assume that $v \geqslant 0$. We are going to prove that $v \in \mathcal{A}(\eta, q)$. Indeed, we assume by contradiction that $v \notin \mathcal{A}(\eta, q)$, that is,

$$\int_M |f^-| v^q \, d\mathrm{vol}_g < \eta \int_M |f^-| \, d\mathrm{vol}_g.$$

Then there exists a suitable constant $\kappa > 0$ such that

$$\int_M |f^-|(v + \kappa)^q \, d\mathrm{vol}_g = \eta \int_M |f^-| \, d\mathrm{vol}_g.$$

It follows from $\|v + \kappa\|_{L^2} > \|v\|_{L^2}$ and $\nabla(v + \kappa) = \nabla v$ that

$$\frac{\|\nabla(v+\kappa)\|_{L^2}^2}{\|v+\kappa\|_{L^2}^2} < \frac{\|\nabla v\|_{L^2}^2}{\|v\|_{L^2}^2}.$$

Observe that

$$\frac{v+\kappa}{\|v+\kappa\|_{L^q}} \in \mathcal{A}'(\eta, q) .$$

Since $\|v + \kappa\|_{L^q} > \|v\|_{L^q} = 1$ and $\|\nabla v\|_{L^2} > 0$, we get that

$$\left\| \nabla \left( \frac{v+\kappa}{\|v+\kappa\|_{L^q}} \right) \right\|_{L^2}^2 \left\| \frac{v+\kappa}{\|v+\kappa\|_{L^q}} \right\|_{L^2}^{-2} = \frac{\|\nabla(v+\kappa)\|_{L^2}^2}{\|v+\kappa\|_{L^2}^2} < \frac{\|\nabla v\|_{L^2}^2}{\|v\|_{L^2}^2},$$

which gives us a contradiction, thus proving that $v \in \mathcal{A}(\eta, q)$. In particular, one easily gets that $\lambda'_{f,\eta,q} = \lambda_{f,\eta,q}$. Consequently, $\lambda_{f,\eta,q}$ is decreasing as a function of $\eta$. ■

*Remark 3.1.* The fact that $\mathcal{A}(\eta, q)$ is not empty is a direct consequence of the proof of Lemma 3.7.

Next we prove that $\lambda_{f,\eta,q} > 0$ if $\eta \neq 1$. Although we shall not use this result, we provide a proof here for completeness.

**Lemma 3.8.** *For each $q \in (2, 2^\star)$ and $0 < \eta \neq 1$ fixed, it holds $\lambda_{f,\eta,q} > 0$.*

*Proof.* We suppose the contrary, that is, $\lambda_{f,\eta,q} = 0$. By the previous lemma, there exists some $v \in \mathcal{A}(\eta, q)$ such that $\|\nabla v\|_{L^2}^2 \|v\|_{L^2}^{-2} = 0$. Since $\|v\|_{L^q} = 1$, there holds $\|\nabla v\|_{L^2} = 0$. Thus $v$ is constant. Making use of the constraint for $\mathcal{A}$ we know that $v = 1$ and thus proving $\eta = 1$, a contradiction. The proof is complete. ∎

We are now in a position to derive a strictly positive lower bound for $\lambda'_{f,\eta,q}$. We prove the following result.

**Lemma 3.9.** *For any $q \in [2^\flat, 2^\star)$, there holds*

$$\lambda'_{f,\eta,q} \geqslant \min\left\{\lambda'_{f,\eta^{2^\flat/2^\star},2^\flat}, \lambda'_{f,\eta,2^\flat}\right\}. \tag{3.2.7}$$

*Proof.* We first pick $u \in \mathcal{A}'(\eta, q)$ arbitrarily. We have two cases depending on whether $\eta \geqslant 1$ or not.

**Case 1.** We assume $\eta \geqslant 1$. Using the Hölder inequality, one gets

$$\begin{aligned}
\int_M |f^-||u|^{2^\flat} \, d\mathrm{vol}_g &\leqslant \left(\int_M |f^-||u|^q \, d\mathrm{vol}_g\right)^{\frac{2^\flat}{q}} \left(\int_M |f^-| \, d\mathrm{vol}_g\right)^{1-\frac{2^\flat}{q}}\\
&\leqslant \left(\eta \int_M |f^-| \, d\mathrm{vol}_g\right)^{\frac{2^\flat}{q}} \left(\int_M |f^-| \, d\mathrm{vol}_g\right)^{1-\frac{2^\flat}{q}}\\
&= \eta^{\frac{2^\flat}{q}} \int_M |f^-| \, d\mathrm{vol}_g\\
&\leqslant \eta \int_M |f^-| \, d\mathrm{vol}_g.
\end{aligned}$$

Hence, we have proved that $\mathcal{A}'(\eta, q) \subset \mathcal{A}'(\eta, 2^\flat)$. By taking the infimum, we arrive at $\lambda'_{f,\eta,2^\flat} \leqslant \lambda'_{f,\eta,q}$.

**Case 2.** We assume $\eta < 1$. Using the Hölder inequality, one still gets

$$\begin{aligned}
\int_M |f^-||u|^{2^\flat} \, d\mathrm{vol}_g &\leqslant \left(\int_M |f^-||u|^q \, d\mathrm{vol}_g\right)^{\frac{2^\flat}{q}} \left(\int_M |f^-| \, d\mathrm{vol}_g\right)^{1-\frac{2^\flat}{q}}\\
&\leqslant \left(\eta \int_M |f^-| \, d\mathrm{vol}_g\right)^{\frac{2^\flat}{q}} \left(\int_M |f^-| \, d\mathrm{vol}_g\right)^{1-\frac{2^\flat}{q}}\\
&= \left(\eta^{\frac{2^\flat}{2^\star}}\right)^{\frac{2^\star}{q}} \int_M |f^-| \, d\mathrm{vol}_g\\
&\leqslant \eta^{\frac{2^\flat}{2^\star}} \int_M |f^-| \, d\mathrm{vol}_g.
\end{aligned}$$

Hence, we have proved that $\mathcal{A}'(\eta, q) \subset \mathcal{A}'(\eta^{2^\flat/2^\star}, 2^\flat)$. By taking the infimum, we arrive at $\lambda'_{f,\eta^{2^\flat/2^\star},2^\flat} \leqslant \lambda'_{f,\eta,q}$.

Combining two cases above, one can conclude the lemma. ∎

Our next lemma describes a comparison between $\lambda_{f,\eta,q}$ and $\lambda_f$. Intuitively, $\mathcal{A}$ is smaller than $\mathcal{A}'(\eta, q)$, thus making $\lambda_{f,\eta,q} \leqslant \lambda_f$. We now prove this affirmatively.

**Lemma 3.10.** *For each $q \in (2, 2^\star)$ and $\eta > 0$ fixed, if $\sup_M f > 0$, then $\lambda_{f,\eta,q} \leqslant \lambda_f$.*

*Proof.* We pick $u \in \mathcal{A}$ arbitrarily. Then there holds $\int_M u \,\mathrm{dvol}_g > 0$, otherwise, $u \equiv 0$. By the definition of $\mathcal{A}$, we know that $\int_M |f^-| u \,\mathrm{dvol}_g = 0$ which also implies that $\int_M |f^-| u^q \,\mathrm{dvol}_g = 0$. Again, from the definition of $\mathcal{A}$ and the fact that $\sup_M f > 0$ we must have $\int_M u^q \,\mathrm{dvol}_g > 0$. We now choose $\varepsilon > 0$ such that

$$\int_M (\varepsilon u)^q \,\mathrm{dvol}_g = 1.$$

This amounts to saying that $\varepsilon u \in \mathcal{A}'(\eta, q)$ which helps us to write

$$\lambda'_{f,\eta,q} \leqslant \|\nabla(\varepsilon u)\|_{L^2}^2 \|\varepsilon u\|_{L^2}^{-2} = \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^{-2}.$$

Since the preceding inequality holds for any $u \in \mathcal{A}$, we may take the infimum on both sides with respect to $u$ to arrive at $\lambda'_{f,\eta,q} \leqslant \lambda_f$. The proof follows easily since we have seen that $\lambda'_{f,\eta,q} = \lambda_{f,\eta,q}$. ■

The next lemma concerns that for each $q \in (2, 2^\star)$ fixed, there holds $\lambda_{f,\eta,q} \to \lambda_f$ as $\eta \to 0$ provided $\sup_M f > 0$.

**Lemma 3.11.** *For each $q \in (2, 2^\star)$ fixed, if $\sup_M f > 0$, then $\lambda_{f,\eta,q} \to \lambda_f$ as $\eta \to 0$.*

*Proof.* It suffices to show that for any sequence $\eta_j \to 0$ as $j \to \infty$, there is a subsequence, still denoted by $\eta_j$ such that $\{\lambda_{f,\eta_j,q}\}$ converges to $\lambda_f$ as $j \to \infty$. In the following, for the simplicity, we just simply omit the sub-index $j$. First, by the proof of Lemma 3.7, we can assume that $\lambda_{f,\eta,q}$ is achieved by some function $v_{\eta,q} \in \mathcal{A}(\eta, q)$. By the Hölder inequality, we easily get $\|v_{\eta,q}\|_{L^2}^2 \leqslant 1$. Also by Lemma 3.10, we obtain $\|\nabla v_{\eta,q}\|_{L^2}^2 \leqslant \lambda_f < +\infty$. Thus, for $\eta$ small, $v_{\eta,q}$ is uniformly bounded in $H^1(M)$. Therefore, there exists $v_q \in H^1(M)$ such that, up to subsequences,

- $v_{\eta,q} \rightharpoonup v_q$ weakly in $H^1(M)$;
- $\nabla v_{\eta,q} \rightharpoonup \nabla v_q$ weakly in $L^2(M)$;
- $v_{\eta,q} \to v_q$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;

as $\eta \to 0$. Consequently, we can pass to the limit to see that $v_q \in \mathcal{A}$. This together with

$$\begin{aligned} \|\nabla v_q\|_{L^2}^2 \leqslant \liminf_{\eta \to 0} \|\nabla v_{\eta,q}\|_{L^2}^2 &\leqslant \limsup_{\eta \to 0} \left(\lambda_{f,\eta,q} \|v_{\eta,q}\|_{L^2}^2\right) \\ &\leqslant \lambda_f \lim_{\eta \to 0} \|v_{\eta,q}\|_{L^2}^2 = \lambda_f \|v_q\|_{L^2}^2 \end{aligned}$$

yield $\|\nabla v_q\|_{L^2}^2 \|v_q\|_{L^2}^{-2} = \lambda_f$. Hence,

$$\liminf_{\eta \to 0} \lambda_{f,\eta,q} = \liminf_{\eta \to 0} \left(\|\nabla v_{\eta,q}\|_{L^2}^2 \|v_{\eta,q}\|_{L^2}^{-2}\right) \geqslant \|\nabla v_q\|_{L^2}^2 \|v_q\|_{L^2}^{-2} = \lambda_f$$

where we have used the fact that $\|\nabla v_q\|_{L^2}^2 \leqslant \liminf_{\eta \to 0} \|\nabla v_{\eta,q}\|_{L^2}^2$. This immediately gives us the desired result. The proof is complete. ■

We now prove an analogous version of Lemma 3.11 for the case when $\lambda_f = +\infty$. Keep in mind that this is the case when either $\sup_M f < 0$ or $\sup_M f = 0$ and $\int_{\{f=0\}} 1 \,\mathrm{dvol}_g = 0$. We now focus on the latter case.

**Lemma 3.12.** *For each $q \in (2, 2^\star)$ fixed, if $\sup_M f = 0$ and $\int_{\{f=0\}} 1 \,\mathrm{dvol}_g = 0$, then $\lambda_{f,\eta,q} \to +\infty$ as $\eta \to 0$.*

*Proof.* We assume by contradiction that $\lambda_{f,\eta,q} \not\to +\infty$, then there exist a positive constant $\mathcal{C}$ and a sequence $\{\eta_j\}_j$ such that $\eta_j \to 0$ and $\lambda_{f,\eta_j,q} < \mathcal{C}$. We denote by $v_{\eta_j,q}$ the minimizer for $\lambda_{f,\eta_j,q}$. Keep in mind that we can always assume that $v_{\eta_j,q} \geqslant 0$ almost everywhere. A similar argument shows that the sequence $\{v_{\eta_j,q}\}_j$ is bounded in $H^1(M)$. Consequently, up to subsequences, $v_{\eta_j,q} \rightharpoonup v_q$ weakly in $H^1(M)$, strongly in $L^q(M)$ for some $v_q \in H^1(M)$. By passing to the limit, one gets

$$\int_M (v_q)^q \, d\mathrm{vol}_g = 1, \qquad \int_M |f^-|(v_q)^q \, d\mathrm{vol}_g = 0.$$

The last equality implies that $v_q = 0$ almost everywhere which contradicts to the fact that $\int_M (v_q)^q \, d\mathrm{vol}_g = 1$. The proof is complete. ■

In the next two lemmas, we point out that $\lambda_{f,\eta,q}$ can be arbitrary close to $\lambda_f$ in any case. First, we consider the case $\lambda_f < +\infty$, we have the following lemma.

**Lemma 3.13.** *For each $\delta > 0$ fixed, there exists $\eta_0 > 0$ such that for all $\eta < \eta_0$, there exists $q_\eta \in (2^\flat, 2^\star)$ so that $\lambda_{f,\eta,q} \geqslant \lambda_f - \delta$ for every $q \in (q_\eta, 2^\star)$.*

*Proof.* We assume by contradiction that there is some $\delta_0 > 0$ such that for every $\eta_0 > 0$, there exist $\eta < \eta_0$ and a monotone sequence $\{q_j\}_j$ converging to $2^\star$ so that $\lambda_{f,\eta,q_j} < \lambda_f - \delta_0$ for every $j$. We can furthermore assume that $\lambda_{f,\eta,q_j}$ is achieved by some $v_{\eta,q_j} \in \mathcal{A}(\eta, q_j)$. We then immediately have

$$\left\|\nabla v_{\eta,q_j}\right\|_{L^2}^2 \left\|v_{\eta,q_j}\right\|_{L^2}^{-2} \leqslant \lambda_f - \delta_0$$

for any $j$. Due to the finite of $\lambda_f$, we can prove the boundedness of $v_{\eta,q_j}$ in $H^1(M)$ which helps us to select a subsequence of $v_{\eta,q_j}$ so that

- $v_{\eta,q_j} \rightharpoonup v_{\eta,2^\star}$ weakly in $H^1(M)$;
- $\nabla v_{\eta,q_j} \rightharpoonup \nabla v_{\eta,2^\star}$ weakly in $L^2(M)$;
- $v_{\eta,q_j} \to v_{\eta,2^\star}$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;

for some $v_{\eta,2^\star} \in H^1(M)$ as $j \to \infty$. Therefore, by sending to the limit we can guarantee the following estimate

$$\|\nabla v_{\eta,2^\star}\|_{L^2}^2 \|v_{\eta,2^\star}\|_{L^2}^{-2} \leqslant \lambda_f - \delta_0.$$

Besides, the Hölder inequality implies $1 \leqslant \|v_{\eta,q_j}\|_{L^{2^\star}}$ for each $j$. Using this and the Sobolev inequality applied to $v_{\eta,q_j}$, we get

$$1 \leqslant \left(\mathcal{K}_1 \frac{\left\|\nabla v_{\eta,q_j}\right\|^2}{\left\|v_{\eta,q_j}\right\|^2} + \mathcal{A}_1\right) \left\|v_{\eta,q_j}\right\|^2 \leqslant \left(\mathcal{K}_1(\lambda_f - \delta_0) + \mathcal{A}_1\right)\left\|v_{\eta,q_j}\right\|^2$$

which yields $(\mathcal{K}_1\lambda_f + \mathcal{A}_1)^{-1} \leqslant \|v_{\eta,q_j}\|_{L^2}^2$. Again after passing to the limit as $j \to \infty$, one obtains

$$\frac{1}{\mathcal{K}_1\lambda_f + \mathcal{A}_1} \leqslant \int_M |v_{\eta,2^\star}|^2 \, d\mathrm{vol}_g.$$

For every $q_j \geqslant 2^\flat$, by the Hölder inequality and the fact that $v_{\eta,q_j} \in \mathcal{A}(\eta, q_j)$ one has

$$\int_M |v_{\eta,q_j}|^{2^\flat} \, d\mathrm{vol}_g \leqslant 1$$

and

$$\begin{aligned}\int_M |f^-||v_{\eta,q_j}|^{2^\flat}\, d\mathrm{vol}_g &\leqslant \left(\int_M |f^-||v_{\eta,q_j}|^{q_{\eta_j}}\, d\mathrm{vol}_g\right)^{\frac{2^\flat}{q_j}} \left(\int_M |f^-|\, d\mathrm{vol}_g\right)^{1-\frac{2^\flat}{q_j}} \\ &= \left(\eta \int_M |f^-|\, d\mathrm{vol}_g\right)^{\frac{2^\flat}{q_j}} \left(\int_M |f^-|\, d\mathrm{vol}_g\right)^{1-\frac{2^\flat}{q_j}} \\ &= \eta^{\frac{2^\flat}{q_j}} \int_M |f^-|\, d\mathrm{vol}_g.\end{aligned}$$

Letting $j \to \infty$, by the Fatou lemma, we deduce that

$$\int_M |v_{\eta,2^\star}|^{2^\flat}\, d\mathrm{vol}_g \leqslant 1$$

and

$$\int_M |f^-||v_{\eta,2^\star}|^{2^\flat}\, d\mathrm{vol}_g \leqslant \eta^{\frac{2^\flat}{2^\star}} \int_M |f^-|\, d\mathrm{vol}_g.$$

Now we let $\eta_0 \to 0$, then clearly $\eta \to 0$. The boundedness of $v_{\eta,2^\star}$ in $H^1(M)$ follows from the fact that $v_{\eta,q_j} \rightharpoonup v_{\eta.2^\star}$ and $\lambda_f$ is finite. Therefore, there exists $v \in H^1(M)$ such that, up to subsequence,

- $v_{\eta,2^\star} \rightharpoonup v$ weakly in $H^1(M)$;
- $v_{\eta,2^\star} \rightharpoonup v$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;
- $v_{\eta,2^\star} \rightharpoonup v$ almost everywhere in $M$.

Before giving out contradiction, we notice that

$$\|\nabla v\|^2_{L^2} \leqslant (\lambda_f - \delta_0)\, \|v\|^2_{L^2}\,. \tag{3.2.8}$$

Then it is enough to see

$$\begin{aligned}0 \leqslant \int_M |f^-||v|^{2^\flat}\, d\mathrm{vol}_g &\leqslant \lim_{\eta\to 0}\left(\int_M |f^-||v_{\eta,2^\star}|^{2^\flat}\, d\mathrm{vol}_g\right) \\ &\leqslant \lim_{\eta\to 0}\left(\eta^{\frac{2^\flat}{2^\star}} \int_M |f^-|\, d\mathrm{vol}_g\right) = 0.\end{aligned}$$

In other words, we would have $\int_M |f^-||v|^{2^\flat}\, d\mathrm{vol}_g = 0$. In particular, $\int_M |f^-||v|\, d\mathrm{vol}_g = 0$. The strong convergence $v_{\eta,2^\star} \to v$ in $L^2(M)$ also implies that $(\mathcal{K}_1\lambda_f + \mathcal{A}_1)^{-1} \leqslant \|v\|^2_{L^2}$. Therefore, $v \not\equiv 0$, and thus $|v| \in \mathcal{A}$. By the definition of $\lambda_f$, we know that

$$\lambda_f\, \|v\|^2_{L^2} \leqslant \|\nabla |v|\|^2_{L^2} = \|\nabla v\|^2_{L^2} \tag{3.2.9}$$

The inequalities (3.2.8) and (3.2.9) obviously provide us a desired contradiction. This proves the lemma. ■

Then, we consider the case $\lambda_f = +\infty$, we have

**Lemma 3.14.** *There exists $\eta_0$ such that for all $\eta < \eta_0$, there exists $q_\eta < 2^\star$ so that $\lambda_{f,\eta,q} > |h|$ for every $q \in (q_\eta, 2^\star)$.*

*Proof.* We assume by contradiction that for every $\eta_0$, there exist $\eta < \eta_0$ and a monotone sequence $\{q_j\}_j$ converging to $2^\star$ so that $\lambda_{f,\eta,q_j} \leqslant |h|$ for all $j$. We assume furthermore that $\lambda_{f,\eta,q_j}$ is achieved by some $v_{\eta,q_j} \in \mathcal{A}(\eta, q_j)$. Then the following estimate holds

$$\left\|\nabla v_{\eta,q_j}\right\|_{L^2}^2 \left\|v_{\eta,q_j}\right\|_{L^2}^{-2} = \lambda_{f,\eta,q_j} \leqslant |h|,$$

for all $j$. As in the proof of Lemma 3.13, there is some $v_{\eta,2^\star} \in H^1(M)$ such that, up to subsequence,

- $\nabla v_{\eta,q_j} \rightharpoonup \nabla v_{\eta,2^\star}$ weakly in $L^2(M)$ and
- $v_{\eta,q_j} \to v_{\eta,2^\star}$ strongly in $L^2(M)$

as $j \to \infty$. This and the fact that

$$\|\nabla v_{\eta,2^\star}\|_{L^2}^2 \leqslant \liminf_{j\to\infty}(\|\nabla v_{\eta,q_j}\|_{L^2}^2)$$

give, after sending $j \to \infty$, the following estimate

$$\|\nabla v_{\eta,2^\star}\|_{L^2}^2 \|v_{\eta,2^\star}\|_{L^2}^{-2} \leqslant |h|.$$

Besides, the Hölder inequality implies that $1 \leqslant \|v_{\eta,q_j}\|_{L^{2^\star}}$. Using this and the Sobolev inequality we get

$$\frac{1}{\mathcal{K}_1|h| + \mathcal{A}_1} \leqslant \int_M |v_{\eta,q_j}|^2 \, d\mathrm{vol}_g \, .$$

By strong convergence in $L^2(M)$, one obtains after passing to the limit

$$\frac{1}{\mathcal{K}_1|h| + \mathcal{A}_1} \leqslant \int_M |v_{\eta,2^\star}|^2 \, d\mathrm{vol}_g.$$

For every $q_j \geqslant 2^\flat$, by the Hölder inequality and the fact that $v_{\eta,q_j} \in \mathcal{A}(\eta, q_j)$ one has

$$\int_M |v_{\eta,q_j}|^{2^\flat} \, d\mathrm{vol}_g \leqslant 1$$

and

$$\int_M |f^-||v_{\eta,q_j}|^{2^\flat} \, d\mathrm{vol}_g \leqslant \eta^{\frac{2^\flat}{q_j}} \int_M |f^-| \, d\mathrm{vol}_g.$$

Followed by the proof of Lemma 3.13, we obtain

$$\int_M |v_{\eta,2^\star}|^{2^\flat} \, d\mathrm{vol}_g \leqslant 1$$

and

$$\int_M |f^-||v_{\eta,2^\star}|^{2^\flat} \, d\mathrm{vol}_g \leqslant \eta^{\frac{2^\flat}{2^\star}} \int_M |f^-| \, d\mathrm{vol}_g \, .$$

Now let $\eta \to 0$, the boundedness of $v_{\eta,2^\star}$ in $H^1(M)$ follows from the fact that $v_{\eta,q_j} \rightharpoonup v_{\eta.2^\star}$ and that $|h|$ is finite. Being bounded, there is some $v \in H^1(M)$ such that, up to subsequence,

- $v_{\eta,2^\star} \rightharpoonup v$ weakly in $H^1(M)$ and
- $v_{\eta,2^\star} \to v$ strongly in $L^2(M)$.

Clearly, $\|\nabla v\|_{L^2}^2 \leqslant |h| \, \|v\|_{L^2}^2$. However,

$$0 \leqslant \int_M |f^-||v|^{2^\flat} \, d\mathrm{vol}_g \leqslant \lim_{\eta \to 0} \left( \int_M |f^-||v_{\eta,2^\star}|^{2^\star} \, d\mathrm{vol}_g \right)$$
$$\leqslant \lim_{\eta \to 0} \left( \eta^{\frac{2^\flat}{2^\star}} \int_M |f^-| \, d\mathrm{vol}_g \right) = 0.$$

Hence, $\int_M |f^-||v|^{2^\flat} \, d\mathrm{vol}_g = 0$ which yields $v = 0$ almost everywhere. So, we cannot use the rest of argument as in the proof of Lemma 3.13. The strong convergence $v_{\eta,2^\star} \to v$ in $L^2(M)$ also implies that $\lim_{\eta \to 2^\star} \|v_{\eta,2^\star}\|_{L^2} = 0$ which provides us a desired contradiction since $\|v_{\eta,2^\star}\|_{L^2}$ has a strictly positive lower bound $\frac{1}{\mathcal{K}_1|h| + \mathcal{A}_1}$. The proof follows. ■

*Remark 3.2.* As can be seen from the preceding proof, a stronger form of Lemma 3.14 can be obtained where $|h|$ is replaced by any given positive constant. However, we don't need that strong one.

## 3.3 Necessary conditions for $f$ and $h$

### 3.3.1 A necessary condition for $\int_M f \, d\mathrm{vol}_g$

The purpose of this subsection was to derive a condition for $\int_M f \, d\mathrm{vol}_g$ so that (3.0.2) admits positive smooth solutions. Our argument was motivated from a same result for the well-known prescribing scalar curvature problem.

**Proposition 3.2.** *Assume that $h \leqslant 0$ is constant. Then then necessary condition for $f$ so that Equation (3.0.2) admits positive smooth solution is $\int_M f \, d\mathrm{vol}_g < 0$. In particular, the necessary condition for (3.0.1) to have positive smooth solution is $\int_M f \, d\mathrm{vol}_g < 0$.*

*Proof.* We assume that $u > 0$ is a smooth solution of (3.0.2). By multiplying both sides of (3.0.2) by $u^{1-q}$, one gets

$$(-\Delta_g u) u^{1-q} + h u^{3-q} = f + \frac{a u^{2-q}}{(u^2 + \varepsilon)^{\frac{q}{2}+1}}.$$

Integrating over $M$ and noticing that $h \leqslant 0$ give

$$\int_M (-\Delta_g u) u^{1-q} \, d\mathrm{vol}_g > \int_M f \, d\mathrm{vol}_g + \int_M \frac{a u^{2-q}}{(u^2 + \varepsilon)^{\frac{q}{2}+1}} \, d\mathrm{vol}_g.$$

By the divergence theorem, one obtains

$$\int_M (-\Delta_g u) u^{1-q} \, d\mathrm{vol}_g = \int_M \nabla u \cdot \nabla (u^{1-q}) \, d\mathrm{vol}_g$$
$$= (1-q) \int_M u^{-q} |\nabla u|^2 \, d\mathrm{vol}_g.$$

This and the fact that $q > 2$ deduce that

$$\int_M f \, d\mathrm{vol}_g + \int_M \frac{a u^{2-q}}{(u^2 + \varepsilon)^{\frac{q}{2}+1}} \, d\mathrm{vol}_g < 0.$$

Obviously, there holds $\int_M f \, d\mathrm{vol}_g < 0$ as claimed. ■

It is important to note that in the case $h > 0$, there is no such a condition on $\int_M f \, d\mathrm{vol}_g$.

### 3.3.2 A necessary condition for $h$

In this subsection, we show that the condition $|h| < \lambda_f$ is necessary if $\lambda_f < +\infty$ in order for (3.0.1) to have a positive smooth solution. In the light of the condition $a \geqslant 0$, one may go through [38, Section III.3] to conclude this necessary condition. However, here we provide a different proof which is shorter than the proof in [38, Section III.3]. Our argument depends on a Picone type identity for integrals [1] whose proof makes use of the density. We believe that such an identity has its own interest.

**Lemma 3.15.** *Assume that $v \in H^1(M)$ with $v \geqslant 0$ and $v \not\equiv 0$. Suppose that $u > 0$ is a smooth function. Then we have*

$$\int_M |\nabla v|^2 \,\mathrm{dvol}_g = -\int_M \frac{\Delta u}{u} v^2 \,\mathrm{dvol}_g + \int_M u^2 \Big|\nabla \left(\frac{v}{u}\right)\Big|^2 \mathrm{dvol}_g.$$

*Proof.* By density, there exist a family of regular functions $\{v_j\}_j$ such that

$$v_j \to v \text{ strongly in } H^1(M), \quad v_j \to v \text{ a.e. in } M, \quad v_j \in C^1(M).$$

The standard Picone identity tells us that

$$\int_M |\nabla v_j|^2 \,\mathrm{dvol}_g = -\int_M \frac{\Delta u}{u} v_j^2 \,\mathrm{dvol}_g + \int_M u^2 \Big|\nabla \left(\frac{v_j}{u}\right)\Big|^2 \mathrm{dvol}_g. \tag{3.3.1}$$

Since $u > 0$ is smooth and $v_j \to v$ strongly in $H^1(M)$ we immediately have

$$\int_M |\nabla v_j|^2 \,\mathrm{dvol}_g \to \int_M |\nabla v|^2 \,\mathrm{dvol}_g$$

and

$$\int_M \frac{\Delta u}{u} v_j^2 \,\mathrm{dvol}_g \to \int_M \frac{\Delta u}{u} v^2 \,\mathrm{dvol}_g$$

as $j \to +\infty$. Again using the smoothness of $u$, we can check that $\frac{v_j}{u} \to \frac{v}{u}$ strongly in $H^1(M)$. Therefore,

$$\int_M u^2 \Big|\nabla \left(\frac{v_j}{u}\right)\Big|^2 \mathrm{dvol}_g \to \int_M u^2 \Big|\nabla \left(\frac{v}{u}\right)\Big|^2 \mathrm{dvol}_g$$

as $j \to +\infty$. The proof now follows by taking the limit in (3.3.1) as $j \to +\infty$. ■

We now provide a different proof for necessary condition $|h| < \lambda_f$.

**Proposition 3.3.** *If Equation (3.0.1) has a positive smooth solution, it is necessary to have $|h| < \lambda_f$.*

*Proof.* We only need to consider the case $\lambda_f < +\infty$ since otherwise it is trivial. We let $v \in \mathcal{A}$ arbitrary and assume that $u$ is a positive smooth solution to (3.0.1). Using Lemma 3.15 and (3.0.1), we find that

$$\begin{aligned}
\int_M |\nabla v|^2 \,\mathrm{dvol}_g =& -\int_M \frac{\Delta u}{u} v^2 \,\mathrm{dvol}_g + \int_M u^2 \Big|\nabla \left(\frac{v}{u}\right)\Big|^2 \mathrm{dvol}_g \\
=& |h| \int_M v^2 \,\mathrm{dvol}_g + \int_M f u^{2^\star - 2} v^2 \,\mathrm{dvol}_g \\
&+ \int_M \frac{a v^2}{u^{2^\star + 2}} \,\mathrm{dvol}_g + \int_M u^2 \Big|\nabla \left(\frac{v}{u}\right)\Big|^2 \mathrm{dvol}_g \\
\geqslant& |h| \int_M v^2 \,\mathrm{dvol}_g + \int_M u^2 \Big|\nabla \left(\frac{v}{u}\right)\Big|^2 \mathrm{dvol}_g.
\end{aligned}$$

In other words, there holds

$$\frac{\int_M |\nabla v|^2 \, d\mathrm{vol}_g}{\int_M v^2 \, d\mathrm{vol}_g} \geqslant |h| + \frac{\int_M u^2 \left|\nabla \left(\frac{v}{u}\right)\right|^2 d\mathrm{vol}_g}{\int_M v^2 \, d\mathrm{vol}_g}. \tag{3.3.2}$$

In particular, $\lambda_f \geqslant |h| > 0$ by taking the infimum with respect to $v$. Notice that

$$\begin{aligned} \frac{\int_M u^2 \left|\nabla \left(\frac{v}{u}\right)\right|^2 d\mathrm{vol}_g}{\int_M v^2 \, d\mathrm{vol}_g} &= \frac{\int_M u^2 \left|\nabla \left(\frac{v}{u}\right)\right|^2 d\mathrm{vol}_g}{\int_M u^2 \left(\frac{v}{u}\right)^2 d\mathrm{vol}_g} \\ &\geqslant \left(\frac{\inf_M u}{\sup_M u}\right)^2 \frac{\int_M \left|\nabla \left(\frac{v}{u}\right)\right|^2 d\mathrm{vol}_g}{\int_M \left(\frac{v}{u}\right)^2 d\mathrm{vol}_g} \geqslant \lambda_f \left(\frac{\inf_M u}{\sup_M u}\right)^2 \end{aligned}$$

since $\frac{v}{u} \in \mathcal{A}$. Having this, we can check from (3.3.2) that

$$\frac{\int_M |\nabla v|^2 \, d\mathrm{vol}_g}{\int_M v^2 \, d\mathrm{vol}_g} \geqslant |h| + \lambda_f \left(\frac{\inf_M u}{\sup_M u}\right)^2.$$

By taking the infimum with respect to $v$, we obtain

$$\lambda_f \geqslant |h| + \lambda_f \left(\frac{\inf_M u}{\sup_M u}\right)^2.$$

This and the fact that $\lambda > 0$ give us the desired result. ■

From the above proof, one can observe that the function $a$ plays no role but $a \geqslant 0$. In fact, the proof is still valid if $a \equiv 0$, thus providing a different proof for the Rauzy result [38, III.3. Condition nécessaire].

# 4

# The Lichnerowicz equations in the negative Yamabe-scalar field conformal invariant



In this chapter, we are interested in the existence (if possible, the multiplicity and the uniqueness) of positive solutions to the Einstein-scalar field Lichnerowicz equations (3.0.1) in the negative Yamabe-scalar field conformal invariant, that is the case $h < 0$.

As we have noted before, in order to study (3.0.1), we follow the subcritical approach. Since the subcritical equation (3.0.2) is variational, we study (3.0.2) by using the variational method [2, 40]. Our main procedure is to show that solutions of (3.0.1) exist as first $\varepsilon \to 0$ and then $q \to 2^\star$ under various assumptions.

The content of this chapter consists of two main parts depending of the sign of $f$. Thanks to the necessary condition $\int_M f \,\mathrm{dvol}_g < 0$ that we have already proved before, in the first part of the chapter, we mainly consider the case when the function $f$ takes both positive and negative values. Our main theorem for this part can be stated as follows.

**Theorem 4.1.** *Let $(M, g)$ be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Assume that $f$ and $a \geqslant 0$ are smooth functions on $M$ such that $\int_M f \,\mathrm{dvol}_g < 0$, $\sup_M f > 0$, $\int_M a \,\mathrm{dvol}_g > 0$, and $|h| < \lambda_f$ where $\lambda_f$ is given in (3.2.1) below. Let us also suppose that the integral of $a$ satisfies*

$$\int_M a \,\mathrm{dvol}_g < \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1}\left(\frac{|h|}{\int_M |f^-| \,\mathrm{dvol}_g}\right)^n \int_M |f^-| \,\mathrm{dvol}_g \qquad (4.0.1)$$

*where $f^-$ is the negative part of $f$. Then there exists a number $\mathcal{C} > 0$ to be specified such that if*

$$\frac{\sup_M f}{\int_M |f^-| \, d\mathrm{vol}_g} < \mathcal{C}, \tag{4.0.2}$$

*Equation* (3.0.1) *possesses at least two smooth positive solutions.*

To be precise, the constant $\mathcal{C}$ appearing in (4.0.2) is given in (4.4.25) below, see also Remark 4.5. Roughly speaking, for the existence part, the constant $\mathcal{C}$ depends only on the negative part of $f$. However, for the multiplicity part, $\mathcal{C}$ also depends on the positive part of $f$. The question of whether we can find a explicit formula for $\mathcal{C}$ turns out to be difficult, even for the prescribed scalar curvature equation, for interested readers, we refer to [4].

If we assume that $f$ does not change sign in the sense that $f \leqslant 0$ in $M$, we obtain necessary and sufficient solvability conditions as pointed out by Choquet-Bruhat, Isenberg, and Pollack [11] in the case of (1.2.10). That is the content of the second part of the chapter.

**Theorem 4.2.** *Let* $(M, g)$ *be a smooth compact Riemannian manifold without boundary of dimension* $n \geqslant 3$. *Let* $h < 0$ *be a constant,* $f$ *and* $a$ *be smooth functions on* $M$ *with* $a \geqslant 0$ *in* $M$, $f \leqslant 0$ *but not strictly negative. Then Equation* (3.0.1) *possesses one positive solution if and only if* $|h| < \lambda_f$.

In addition, our approach can be used to handle the case $\sup_M f < 0$ although this has been done in [11] by using the method of sub- and super-solutions. We shall address this issue later. Finally, we should mention that the content of this chapter was adapted from [34].

## 4.1 The analysis of the energy functionals

### 4.1.1 The functional setting

For each $q \in [2^\flat, 2^\star)$ and $k > 0$, we introduce $\mathcal{B}_{k,q}$, a hyper-surface of $H^1(M)$, which is defined by

$$\mathcal{B}_{k,q} = \left\{ u \in H^1(M) : \|u\|_{L^q} = k^{\frac{1}{q}} \right\}. \tag{4.1.1}$$

Notice that for any $k > 0$, our set $\mathcal{B}_{k,q}$ is non-empty since it always contains $k^{\frac{1}{q}}$. Now we construct the energy functional associated to problem (3.0.2). For each $\varepsilon > 0$, we consider the functional $F_q^\varepsilon : H^1(M) \to \mathbb{R}$ defined by

$$\begin{aligned} F_q^\varepsilon(u) = \frac{1}{2} \int_M |\nabla u|^2 \, d\mathrm{vol}_g + \frac{h}{2} \int_M u^2 \, d\mathrm{vol}_g \\ - \frac{1}{q} \int_M f |u|^q \, d\mathrm{vol}_g + \frac{1}{q} \int_M \frac{a}{(u^2 + \varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g. \end{aligned}$$

By a standard argument, $F_q^\varepsilon$ is continuously differentiable on $H^1(M)$, see Lemma 4.1 below, and thus weak solutions of (3.0.2) correspond to critical points of the functional $F_q^\varepsilon$. Now we set

$$\mu_{k,q}^\varepsilon = \inf_{u \in \mathcal{B}_{k,q}} F_q^\varepsilon(u).$$

By the Hölder inequality, it is not hard to see that $F_q^\varepsilon|_{\mathcal{B}_{k,q}}$ is bounded from below by $-k \sup_M f + \frac{h}{2} k^{\frac{2}{q}}$ and thus $\mu_{k,q}^\varepsilon > -\infty$ if $k$ is finite. On the other hand, using the test function $u = k^{\frac{1}{q}}$, we get

$$\mu_{k,q}^\varepsilon \leqslant \frac{h}{2} k^{\frac{2}{q}} - \frac{k}{q} \int_M f \, \mathrm{dvol}_g + \frac{1}{q} \int_M \frac{a}{\left(k^{\frac{2}{q}} + \varepsilon\right)^2} \, \mathrm{dvol}_g \tag{4.1.2}$$

which tells us that $\mu_{k,q}^\varepsilon < +\infty$. Our aim was to find critical points of the functional $F_q^\varepsilon$. In order to support our aim, we prove the following result.

**Lemma 4.1.** *The first variation of $F_q^\varepsilon$ at a point $u$ in a direction $v$ is given by*

$$\begin{aligned} \delta F_q^\varepsilon(u)(v) = & \int_M \nabla u \cdot \nabla v \, \mathrm{dvol}_g + h \int_M uv \, \mathrm{dvol}_g \\ & - \int_M f |u|^{q-2} uv \, \mathrm{dvol}_g - \int_M \frac{au}{(u^2+\varepsilon)^{\frac{q}{2}+1}} v \, \mathrm{dvol}_g \end{aligned}$$

*where $\nabla u \cdot \nabla v$ stands for the pointwise scalar product of $\nabla u$ and $\nabla v$ with respect to the metric $g$.*

*Proof.* The proof is simple, we include it here for completeness. In fact, for any smooth function $v$, there holds

$$\begin{aligned} \delta F_q^\varepsilon(u)(v) = & \frac{d}{dt} F_q^\varepsilon(u+tv) \Big|_{t=0} \\ = & \frac{d}{dt} \left[ \frac{1}{2} \int_M |\nabla(u+tv)|^2 \, \mathrm{dvol}_g + \frac{h}{2} \int_M (u+tv)^2 \, \mathrm{dvol}_g \right] \Bigg|_{t=0} \\ & + \frac{d}{dt} \left[ -\frac{1}{q} \int_M f |u+tv|^q \, \mathrm{dvol}_g + \frac{1}{q} \int_M \frac{a}{\left((u+tv)^2+\varepsilon\right)^{\frac{q}{2}}} \, \mathrm{dvol}_g \right] \Bigg|_{t=0} \\ = & \int_M (-\Delta_g u) v \, \mathrm{dvol}_g + h \int_M uv \, \mathrm{dvol}_g \\ & - \int_M f |u|^{q-2} uv \, \mathrm{dvol}_g - \int_M \frac{au}{(u^2+\varepsilon)^{\frac{q}{2}+1}} v \, \mathrm{dvol}_g, \end{aligned}$$

which provides the desired result. ■

### 4.1.2 $\mu_{k,q}^\varepsilon$ is achieved

The purpose of this subsection was to show that, if $k$, $q$, and $\varepsilon$ are fixed, then $\mu_{k,q}^\varepsilon$ is achieved by a smooth positive function, say $u_\varepsilon$. The proof is standard and is based on the so-called direct methods in the calculus of variations.

**Lemma 4.2.** *For each $k > 0$, $q \in (2, 2^\star)$, and $\varepsilon > 0$ fixed, the number $\mu_{k,q}^\varepsilon$ is achieved by some smooth positive function.*

*Proof.* Let $\{u_j\}_j \subset \mathcal{B}_{k,q}$ be a minimizing sequence for $\mu_{k,q}^\varepsilon$, that is,

$$\int_M |u_j|^q \, \mathrm{dvol}_g = k$$

for any $j$ and

$$\lim_{j\to+\infty} F_q^\varepsilon(u_j) = \mu_{k,q}^\varepsilon.$$

Since $F_q^\varepsilon(u_j) = F_q^\varepsilon(|u_j|)$, we may assume from the beginning that $u_j \geqslant 0$ for all $j$. Observe that $\frac{q-2}{q} + \frac{2}{q} = 1$, so for any $j$, by the Hölder inequality we get

$$\int_M (u_j)^2 \,\mathrm{dvol}_g \leqslant \left(\int_M (u_j)^q \,\mathrm{dvol}_g\right)^{\frac{2}{q}} = k^{\frac{2}{q}}. \tag{4.1.3}$$

Therefore, for $j$ sufficiently large such that $F_q^\varepsilon(u_j) < \mu_{k,q}^\varepsilon + 1$, one has

$$\begin{aligned}
\frac{1}{2}\int_M |\nabla u_j|^2 \,\mathrm{dvol}_g \leqslant & \frac{1}{2}\int_M |\nabla u_j|^2 \,\mathrm{dvol}_g + \frac{h}{2}\left(\int_M (u_j)^2 \,\mathrm{dvol}_g - k^{\frac{2}{q}}\right)\\
& - \frac{1}{q}\left(\int_M f|u_j|^q \,\mathrm{dvol}_g - k|\sup_M f|\right)\\
\leqslant & \frac{1}{2}\int_M |\nabla u_j|^2 \,\mathrm{dvol}_g + \frac{h}{2}\int_M (u_j)^2 \,\mathrm{dvol}_g - \frac{1}{q}\int_M f|u_j|^q \,\mathrm{dvol}_g\\
& + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} \,\mathrm{dvol}_g - \frac{h}{2}k^{\frac{2}{q}} + \frac{k}{q}|\sup_M f|\\
= & F_q^\varepsilon(u_j) - \frac{h}{2}k^{\frac{2}{q}} + \frac{k}{q}|\sup_M f|\\
< & \mu_{k,q}^\varepsilon - \frac{h}{2}k^{\frac{2}{q}} + \frac{k}{q}|\sup_M f| + 1.
\end{aligned}$$

That is

$$\int_M |\nabla u_j|^2 \,\mathrm{dvol}_g < 2\left(\mu_{k,q}^\varepsilon - \frac{h}{2}k^{\frac{2}{q}} + \frac{k}{q}|\sup_M f| + 1\right). \tag{4.1.4}$$

The inequalities (4.1.3) and (4.1.4) imply that $\{u_j\}_j$ is bounded in $H^1(M)$. Being bounded, there exists $u_\varepsilon \in H^1(M)$ such that, up to subsequences,

- $u_j \rightharpoonup u_\varepsilon$ weakly in $H^1(M)$;
- $u_j \to u_\varepsilon$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;
- $u_j \to u_\varepsilon$ almost everywhere in $M$.

Consequently, $u_\varepsilon \geqslant 0$ almost everywhere in $M$ and $\lim_{j\to\infty} \|u_j\|_{L^q} = \|u_\varepsilon\|_{L^q} = k^{\frac{1}{q}}$. In particular, $u_\varepsilon \in \mathcal{B}_{k,q}$. Clearly, it follows from the definition of $\mu_{k,q}^\varepsilon$ that

$$F_q^\varepsilon(u_\varepsilon) \geqslant \mu_{k,q}^\varepsilon. \tag{4.1.5}$$

On the other hand, since $u_j \to u_\varepsilon$ almost everywhere, we also have $((u_j)^2+\varepsilon)^{-\frac{q}{2}} \to ((u_\varepsilon)^2+\varepsilon)^{-\frac{q}{2}}$ almost everywhere in $M$ as $j \to \infty$. Notice that functions $a((u_j)^2+\varepsilon)^{-\frac{q}{2}}$ and $a((u_\varepsilon)^2+\varepsilon)^{-\frac{q}{2}}$ are bounded by the function $a\varepsilon^{-\frac{q}{2}}$ which is of class $L^1(M)$. By the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_M \frac{a}{((u_j)^2+\varepsilon)^{\frac{q}{2}}} \,\mathrm{dvol}_g \to \int_M \frac{a}{((u_\varepsilon)^2+\varepsilon)^{\frac{q}{2}}} \,\mathrm{dvol}_g$$

as $j \to \infty$. We now observe that the part

$$\frac{1}{2}\int_M |\nabla u_j|^2 \,\mathrm{dvol}_g + \frac{h}{2}\int_M u_j^2 \,\mathrm{dvol}_g - \frac{1}{q}\int_M f|u_j|^q \,\mathrm{dvol}_g$$

is weakly lower semi-continuous. We therefore get

$$\begin{aligned}
\mu^{\varepsilon}_{k,q} =& \lim_{j\to+\infty} F^{\varepsilon}_{q}(u_j) \\
\geqslant& \liminf_{j\to+\infty}\left[\frac{1}{2}\int_M |\nabla u_j|^2 \, d\mathrm{vol}_g + \frac{h}{2}\int_M (u_j)^2 \, d\mathrm{vol}_g - \frac{1}{q}\int_M f|u_j|^q \, d\mathrm{vol}_g\right] \\
&\quad + \frac{1}{q}\lim_{j\to+\infty}\int_M \frac{a}{((u_j)^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g \\
\geqslant& \frac{1}{2}\int_M |\nabla u_\varepsilon|^2 \, d\mathrm{vol}_g + \frac{h}{2}\int_M (u_\varepsilon)^2 \, d\mathrm{vol}_g \\
&\quad - \frac{1}{q}\int_M f|u_\varepsilon|^q \, d\mathrm{vol}_g + \frac{1}{q}\int_M \frac{a}{((u_\varepsilon)^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g \\
=& F^{\varepsilon}_{q}(u_\varepsilon).
\end{aligned}$$

In other words, we would have

$$\mu^{\varepsilon}_{k,q} \geqslant F^{\varepsilon}_{q}(u_\varepsilon). \tag{4.1.6}$$

In view of the inequalities (4.1.5) and (4.1.6), there holds $\mu^{\varepsilon}_{k,q} = F^{\varepsilon}_{q}(u_\varepsilon)$. This and the fact that $u_\varepsilon \in \mathcal{B}_{k,q}$ prove that $\mu^{\varepsilon}_{k,q}$ is achieved by $u_\varepsilon$.

It leaves out to prove the smoothness and positivity of $u_\varepsilon$. Keep in mind that the Euler-Lagrange equation for functional $F^{\varepsilon}_{q}$ with the constraint (4.1.1) is

$$-\Delta_g u_\varepsilon + h u_\varepsilon - f|u_\varepsilon|^{q-2}u_\varepsilon - \frac{a u_\varepsilon}{((u_\varepsilon)^2+\varepsilon)^{\frac{q}{2}+1}} - \lambda |u_\varepsilon|^{q-2}u_\varepsilon = 0$$

for some constant $\lambda$. Equivalently, $u_\varepsilon$ solves

$$-\Delta_g u_\varepsilon + h u_\varepsilon = (f+\lambda)|u_\varepsilon|^{q-2}u_\varepsilon + \frac{a u_\varepsilon}{((u_\varepsilon)^2+\varepsilon)^{\frac{q}{2}+1}} \tag{4.1.7}$$

in the weak sense. The regularity result, Lemma 3.4(a), can be applied to (4.1.7). It follows that $u_\varepsilon \in C^\infty(M)$ and $u_\varepsilon \geqslant 0$ in $M$. The Strong Maximum Principle [3, Proposition 3.75] can be applied to conclude that either $u_\varepsilon \equiv 0$ or $u_\varepsilon > 0$ in $M$. Since $\int_M (u_\varepsilon)^q \, d\mathrm{vol}_g = k \neq 0$, we know that $u_\varepsilon \not\equiv 0$. Thus, $u_\varepsilon$ is a smooth positive solution of (4.1.7). Therefore, we have shown that $\mu^{\varepsilon}_{k,q}$ is achieved by smooth positive function $u_\varepsilon$ and the claim follows. ■

### 4.1.3 The continuity of $\mu^{\varepsilon}_{k,q}$ with respect to $k$

The objective of this subsection was to prove the following result.

**Proposition 4.1.** *For $\varepsilon > 0$ fixed, $\mu^{\varepsilon}_{k,q}$ is continuous with respect to $k$.*

*Proof.* Since $\mu^{\varepsilon}_{k,q}$ is well-defined at any point $k$, we have to verify that for each $k$ fixed and for any sequence $k_j \to k$ there holds $\mu^{\varepsilon}_{k_j,q} \to \mu^{\varepsilon}_{k,q}$ as $j \to +\infty$. This is equivalent to showing that there exists a subsequence of $\{k_j\}_j$, still denoted by $k_j$, such that $\mu^{\varepsilon}_{k_j,q} \to \mu^{\varepsilon}_{k,q}$ as $j \to +\infty$.

We suppose that $\mu^{\varepsilon}_{k,q}$ and $\mu^{\varepsilon}_{k_j,q}$ are achieved by $u \in \mathcal{B}_{k,q}$ and $u_j \in \mathcal{B}_{k_j,q}$ respectively. Keep in mind that $u$ and $u_j$ are positive smooth functions on $M$. Our aim was to prove the boundedness of $\{u_j\}_j$ in $H^1(M)$. It then suffices to control $\|\nabla u_j\|_{L^2}$. As in (4.1.4), we have

$$\int_M |\nabla u_j|^2 \, \mathrm{dvol}_g < 2\left(\mu^\varepsilon_{k_j,q} - \frac{h}{2}k_j^{\frac{2}{q}} + \frac{k_j}{q}\sup_M f\right).$$

Thus, we have to control $\mu^\varepsilon_{k_j,q}$. By the homogeneity we can find a sequence of positive numbers $\{t_j\}_j$ such that $t_j u \in \mathcal{B}_{k_j,q}$. Since $k_j \to k$ as $j \to +\infty$ and $k_j^{\frac{2}{q}} = \|t_j u\|_{L^q} = t_j k^{\frac{2}{q}}$, we immediately see that $t_j \to 1$ as $j \to +\infty$. Now we can use $t_j u$ to control $\mu^\varepsilon_{k_j,q}$. Indeed, from the definition of $\mu^\varepsilon_{k_j,q}$ and by using the function $t_j u$, we know that

$$\begin{aligned}
\mu^\varepsilon_{k_j,q} \leqslant & F^\varepsilon_q(t_j u)\\
=& t_j^2\left(\frac{1}{2}\int_M |\nabla u|^2 \, \mathrm{dvol}_g + \frac{h}{2}\int_M u^2 \, \mathrm{dvol}_g\right)\\
& - \frac{1}{q}t_j^q \int_M f u^q \, \mathrm{dvol}_g + \frac{1}{q}\int_M \frac{a}{((t_j u)^2+\varepsilon)^{\frac{q}{2}}} \, \mathrm{dvol}_g\\
\leqslant & t_j^2\left(\frac{1}{2}\int_M |\nabla u|^2 \, \mathrm{dvol}_g + \frac{h}{2}\int_M u^2 \, \mathrm{dvol}_g\right)\\
& - \frac{1}{q}t_j^q \int_M f u^q \, \mathrm{dvol}_g + \frac{1}{q}\varepsilon^{-\frac{q}{2}}\int_M a \, \mathrm{dvol}_g.
\end{aligned}$$

Notice that $u$ is fixed and $t_j$ belongs to a neighborhood of 1 for large $j$. Thus, $\{\mu^\varepsilon_{k_j,q}\}_j$ is bounded which also implies by the preceding estimate that $\{\|\nabla u_j\|_{L^2}\}_j$ is bounded. Hence $\{u_j\}_j$ is bounded in $H^1(M)$. Being bounded, there exists $\overline{u} \in H^1(M)$ such that, up to subsequences,

- $u_j \rightharpoonup \overline{u}$ weakly in $H^1(M)$;
- $u_j \to \overline{u}$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;

Consequently, $\lim_{j\to+\infty}\|u_j\|_{L^q} = \|\overline{u}\|_{L^q} = k^{\frac{2}{q}}$, that is, $\overline{u} \in \mathcal{B}_{k,q}$. In particular, $F^\varepsilon_q(u) \leqslant F^\varepsilon_q(\overline{u})$. We now use weak lower semi-continuity property of $F^\varepsilon_q$ to deduce that

$$F^\varepsilon_q(u) \leqslant F^\varepsilon_q(\overline{u}) \leqslant \liminf_{j\to+\infty} F^\varepsilon_q(u_j).$$

We now use our estimate for $\mu^\varepsilon_{k_j,q}$ above to see that

$$\limsup_{j\to+\infty} \mu^\varepsilon_{k_j,q} \leqslant F^\varepsilon_q(u).$$

This is due to the Lebesgue Dominated Convergence Theorem and the fact that $t_j \to 1$ as $j \to +\infty$. In summary, $\lim_{j\to+\infty}\mu^\varepsilon_{k_j,q} = \mu^\varepsilon_{k,q}$ which proves the continuity of $\mu^\varepsilon_{k,q}$. ■

## 4.2 Asymptotic behavior of $\mu^\varepsilon_{k,q}$ in the case $\sup_M f > 0$

In this section we investigate the behavior of $\mu^\varepsilon_{k,q}$ when both $k$ and $\varepsilon$ vary. Under some suitable conditions, we show among other things that there exist $k_\star < 1$, $k_0$, and $k_{\star\star} > 1$ with $k_0 \in (k_\star, k_{\star\star})$ such that

(a) $\mu^\varepsilon_{k_\star,q} > 0$ for any $\varepsilon \leqslant k_\star$;

(b) $\mu^{\varepsilon}_{k_0,q} < 0$ for any $\varepsilon > 0$;
(c) $\mu^{\varepsilon}_{k,q} < 0$ for any $\varepsilon > 0$ and any $k \geqslant k_{\star\star}$;

It is worth noticing that the assertions (b) and (c) play different role in our argument. In fact, we prove in the next section that $\mu^{\varepsilon}_{k,q} < 0$ for some $k \in (k_0, k_{\star\star})$ and for any $\varepsilon > 0$.

First, we investigate the behavior of $\mu^{\varepsilon}_{k,q}$ when both $k$ and $\varepsilon$ are sufficiently small.

**Lemma 4.3.** *There holds* $\lim_{k\to 0+} \mu^{k^{\frac{2}{q}}}_{k,q} = +\infty$. *In particular, there is some $k_\star$ sufficiently small and independent of both $q$ and $\varepsilon$ such that $\mu^{\varepsilon}_{k_\star,q} > 0$ for any $\varepsilon \leqslant k_\star$.*

*Proof.* The way that $\varepsilon$ comes and plays immediately shows us that $\mu^{\varepsilon}_{k,q}$ is strictly monotone decreasing in $\varepsilon$ for fixed $k$ and $q$. For any $\varepsilon \leqslant k^{\frac{2}{q}}$, any $1 < \frac{q}{2} < \frac{2^\star}{2}$, and any $u \in \mathcal{B}_{k,q}$, we have the following estimate,

$$\begin{aligned}
\int_M \sqrt{a}\, d\mathrm{vol}_g &\leqslant \left(\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\, d\mathrm{vol}_g\right)^{\frac{1}{2}} \left(\int_M (u^2+\varepsilon)^{\frac{q}{2}}\, d\mathrm{vol}_g\right)^{\frac{1}{2}}\\
&\leqslant \left(\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\, d\mathrm{vol}_g\right)^{\frac{1}{2}} \left(\int_M (u^2+k^{\frac{2}{q}})^{\frac{q}{2}}\, d\mathrm{vol}_g\right)^{\frac{1}{2}}\\
&\leqslant \left(\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\, d\mathrm{vol}_g\right)^{\frac{1}{2}} \left(2^{\frac{q}{2}-1}\int_M (|u|^q+k)\, d\mathrm{vol}_g\right)^{\frac{1}{2}}\\
&= 2^{\frac{q}{4}}\sqrt{k}\left(\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\, d\mathrm{vol}_g\right)^{\frac{1}{2}}.
\end{aligned}$$

Squaring both sides, we get

$$\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\, d\mathrm{vol}_g \geqslant \frac{1}{2^{\frac{2^\star}{2}}k}\left(\int_M \sqrt{a}\, d\mathrm{vol}_g\right)^2.$$

This helps us to conclude that

$$\begin{aligned}
F^{\varepsilon}_q(u) = \frac{1}{2}\int_M |\nabla u|^2\, d\mathrm{vol}_g &+ \frac{h}{2}\int_M u^2\, d\mathrm{vol}_g\\
&- \frac{1}{q}\int_M f|u|^q\, d\mathrm{vol}_g + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\, d\mathrm{vol}_g\\
\geqslant k^{\frac{2}{q}}\frac{h}{2} - \frac{k}{q}|\sup_M f| &+ \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\, d\mathrm{vol}_g\\
\geqslant k^{\frac{2}{q}}\frac{h}{2} - \frac{k}{q}|\sup_M f| &+ \frac{1}{2^{\frac{2^\star}{2}}qk}\left(\int_M \sqrt{a}\, d\mathrm{vol}_g\right)^2,
\end{aligned}$$

which proves that $\mu^{k^{\frac{2}{q}}}_{k,q} \to +\infty$ as $k \to 0+$. We wish now to find some small $k_\star < 1$ independent of both $q$ and $\varepsilon$ such that

$$k_\star^{\frac{2}{q}}\frac{h}{2} - \frac{k_\star}{q}|\sup_M f| + \frac{1}{2^{\frac{2^\star}{2}}qk_\star}\left(\int_M \sqrt{a}\, d\mathrm{vol}_g\right)^2 > 0.$$

The preceding inequality is equivalent to

$$\frac{1}{2^{\frac{2^\star}{2}}q}\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2 > k_\star\left(\frac{k_\star}{q}|\sup_M f| + k_\star^{\frac{2}{q}}\frac{|h|}{2}\right).$$

Since $k_\star < 1$, it is clear to see that

$$k_\star\left(\frac{k_\star}{q}|\sup_M f| + k_\star^{\frac{2}{q}}\frac{|h|}{2}\right) < k_\star\left(\frac{1}{q}|\sup_M f| + \frac{|h|}{2}\right).$$

Summing up, it suffices to choose $k_\star < 1$ such that

$$\frac{1}{2^{\frac{2^\star}{2}}q}\frac{\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2}{\left(\frac{1}{q}|\sup_M f| + \frac{|h|}{2}\right)} \geqslant k_\star.$$

Since $2 < q < 2^\star$ we know that

$$\frac{1}{2^{\frac{2^\star}{2}}q}\frac{\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2}{\left(\frac{1}{q}|\sup_M f| + \frac{|h|}{2}\right)} > \frac{1}{2^{\frac{2^\star}{2}}2^\star}\frac{\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2}{\left(\frac{1}{2}|\sup_M f| + \frac{|h|}{2}\right)}.$$

Therefore, we can choose $k_\star$ as

$$k_\star = \min\left\{\frac{\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2}{2^{\frac{2^\star}{2}-1}2^\star\left(|\sup_M f| + |h|\right)}, \left(\frac{|h|}{\int_M |f^-|\,\mathrm{dvol}_g}\right)^{n-1}, 1\right\}. \tag{4.2.1}$$

For such a choice of $k_\star$, we notice that $k_\star \leqslant k_\star^{\frac{2}{q}}$. Clearly, $k_\star$ given by (4.2.1) is independent of both $q$ and $\varepsilon$. ∎

We now investigate the behavior of $\mu_{k,q}^\varepsilon$ as $k \to +\infty$. A direct use of constant functions as in (4.1.2) gives us nothing since $f$ changes its sign. To avoid this difficulty we need to construct a new suitable test function, to this end we have to control $f^-$ by using a suitable cut-off function which is supported in the positive part of $f$.

**Lemma 4.4.** *There holds $\mu_{k,q}^\varepsilon \to -\infty$ as $k \to +\infty$ if $\sup_M f > 0$.*

*Proof.* We first choose a point, say $x_0 \in M$, such that $f(x_0) > 0$. For example, one can choose $x_0$ such that $f(x_0) = \sup_M f$. By the continuity of $f$, there exists some $r_0 > 0$ sufficiently small such that $f(x) > 0$, for any $x \in \overline{B}_{r_0}(x_0)$ and $f(x) \geqslant 0$ for any $x \in \overline{B}_{2r_0}(x_0)$. Let $\varphi : [0, +\infty) \to [0,1]$ be a smooth non-negative function such that

$$\varphi(t) = \begin{cases} 1, & 0 \leqslant t \leqslant r_0^2, \\ 0, & t \geqslant 4r_0^2. \end{cases}$$

For small $r_0$, the function $\varphi$ is clearly smooth. We then define

$$w(x) = \varphi(\mathrm{dist}(x, x_0)^2), \quad x \in M,$$

and set

$$g(t) = \int_M f e^{tw}\,\mathrm{dvol}_g, \quad t \in \mathbb{R}.$$

Obviously, $g$ is continuous and $g(0) < 0$ by assumption $\int_M f \, d\text{vol}_g < 0$. For arbitrary $t$, we have

$$\begin{aligned} g(t) &= \int_M f^+ e^{tw} \, d\text{vol}_g + \int_M f^- e^{tw} \, d\text{vol}_g \\ &\geqslant (\min_{\overline{B}_{r_0}(x_0)} f^+) \int_{B_{r_0}(x_0)} e^{tw} \, d\text{vol}_g + \int_M f^- e^{tw} \, d\text{vol}_g \\ &= (\min_{\overline{B}_{r_0}(x_0)} f^+) \,\text{vol}(B_{r_0}(x_0)) e^t + \int_M f^- e^{tw} \, d\text{vol}_g. \end{aligned}$$

Keep in mind that $\int_M f^- e^{tw} \, d\text{vol}_g$ is bounded since

$$\left| \int_M f^- e^{tw} \, d\text{vol}_g \right| \leqslant \int_M |f^-| e^{tw} \, d\text{vol}_g = \int_{M \setminus B_{2r_0}(x_0)} |f^-| \, d\text{vol}_g < \infty.$$

It follows that

$$g(t) \geqslant (\min_{\overline{B}_{r_0}(x_0)} f^+) \,\text{vol}(B_{r_0}(x_0)) e^t - \int_{M \setminus B_{2r_0}(x_0)} |f^-| \, d\text{vol}_g.$$

Thus, there exists some $t_0$ sufficiently large such that $g(t_0) \geqslant 1$. The monotonicity property of $g$, that can be seen from

$$g'(t) = \int_M f w e^{tw} \, d\text{vol}_g = \int_{B_{2r_0}(x_0)} f^+ w e^{tw} \, d\text{vol}_g > 0,$$

allows us to conclude that $g(t) \geqslant 1$ for any $t \geqslant t_0$. We now take a positive function $v \in C^1$ of the following form

$$v(x) = c e^{t_0 w(x)}, \quad x \in M$$

where $c$ is a positive constant chosen in such a way that $\int_M v^q \, d\text{vol}_g = 1$. By our construction above, the function $e^{t_0 w(x)}$ is independent of both $q$ and $\varepsilon$. Therefore,

$$\int_M f v^q \, d\text{vol}_g = c^q \int_M f e^{q t_0 w} \, d\text{vol}_g = c^q g(q t_0) > c^q g(t_0) > 0. \tag{4.2.2}$$

Since $k^{\frac{1}{q}} v \in \mathcal{B}_{k,q}$, a direct computation leads us to

$$\begin{aligned} F_q^\varepsilon(k^{\frac{1}{q}} v) =& \frac{1}{2} \int_M |\nabla (k^{\frac{1}{q}} v)|^2 \, d\text{vol}_g + \frac{h}{2} \int_M \left(k^{\frac{1}{q}} v\right)^2 d\text{vol}_g \\ & - \frac{1}{q} \int_M f \left(k^{\frac{1}{q}} v\right)^q d\text{vol}_g + \frac{1}{q} \int_M \frac{a}{\left(\left(k^{\frac{1}{q}} v\right)^2 + \varepsilon\right)^{\frac{q}{2}}} \, d\text{vol}_g \\ \leqslant & \frac{1}{2} k^{\frac{2}{q}} \left[ \int_M |\nabla v|^2 \, d\text{vol}_g + h \int_M v^2 \, d\text{vol}_g - \frac{2}{q} k^{1-\frac{2}{q}} \int_M f v^q \, d\text{vol}_g \right] \\ & + \frac{1}{q} \varepsilon^{-\frac{q}{2}} \int_M a \, d\text{vol}_g. \end{aligned}$$

With the help of (4.2.2) we deduce $F_q^\varepsilon(k^{\frac{1}{q}} v) \to -\infty$ by sending $k \to +\infty$ in the preceding inequality, thus proving our claim. ■

We are going to show that there exists $k_0$ such that $\mu_{k_0,q}^\varepsilon < 0$ and $\mu_{k,q}^\varepsilon > 0$ for some $k > k_0$. These results together with Lemmas 4.3 and 4.4 give us a full description of the asymptotic behavior of $\mu_{k,q}^\varepsilon$. First we prove the existence of such a $k_0$.

**Lemma 4.5.** *There exists some $k_0 > 0$ independent of $\varepsilon$ such that $\mu^{\varepsilon}_{k_0,q} \leqslant 0$ for any $\varepsilon \geqslant 0$ provided*

$$\int_M a \,\mathrm{dvol}_g \leqslant \left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-| \,\mathrm{dvol}_g}\right)^{\frac{q+2}{q-2}} \frac{|h|}{4}(q-2). \tag{4.2.3}$$

*In particular, $k_0 > k_\star$.*

*Proof.* Since $k^{\frac{1}{q}} \in \mathcal{B}_{k,q}$, by removing the negative term involving $f^+$, we know from (4.1.2) that, for any $\varepsilon > 0$,

$$\begin{aligned}
F^{\varepsilon}_q(k^{\frac{1}{q}}) &= \frac{hk^{\frac{2}{q}}}{2} - \frac{k}{q}\int_M f \,\mathrm{dvol}_g + \frac{1}{q}\int_M \frac{a}{(k^{\frac{2}{q}}+\varepsilon)^{\frac{q}{2}}} \,\mathrm{dvol}_g \\
&\leqslant \frac{hk^{\frac{2}{q}}}{2} - \frac{k}{q}\int_M f \,\mathrm{dvol}_g + \frac{1}{q}\int_M \frac{a}{k} \,\mathrm{dvol}_g \\
&\leqslant \frac{hk^{\frac{2}{q}}}{2} + \frac{k}{q}\int_M |f^-| \,\mathrm{dvol}_g + \frac{1}{qk}\int_M a \,\mathrm{dvol}_g.
\end{aligned}$$

Clearly, the non-positivity of the right hand side of this inequality is equivalent to

$$\int_M a \,\mathrm{dvol}_g \leqslant \frac{|h|q}{2} k^{\frac{q+2}{q}} - k^2 \int_M |f^-| \,\mathrm{dvol}_g. \tag{4.2.4}$$

By a simple calculation, at

$$k_0 = \left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-| \,\mathrm{dvol}_g}\right)^{\frac{q}{q-2}}$$

the right hand side of (4.2.4) is equal to

$$\left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-| \,\mathrm{dvol}_g}\right)^{\frac{q+2}{q-2}} \frac{|h|}{4}(q-2).$$

Thus, by definition, we claim that $\mu^{\varepsilon}_{k_0,q} \leqslant 0$ provided $\int_M a \,\mathrm{dvol}_g$ satisfies (4.2.3). The fact that $k_0 > k_\star$ can be seen from Lemma 4.3. ■

Now we have the following remark which also plays some role in our argument.

*Remark 4.3.* It follows from $q \in [2^\flat, 2^\star)$ that

$$\min\left\{\left(\frac{|h|}{\int_M |f^-| \,\mathrm{dvol}_g}\right)^{n-1}, 1\right\} \leqslant k_0$$

since $\frac{2+q}{4} > 1$ and the function $\frac{q}{q-2}$ is monotone decreasing. Moreover, if we keep the term involving $f^+$ in the proof of Lemma 4.5, we immediately see that

$$F^{\varepsilon}_q(k_0^{\frac{1}{q}}) \leqslant -\frac{k_0}{q}\int_M f^+ \,\mathrm{dvol}_g.$$

Thus, we can easily control the growth of $\mu^{\varepsilon}_{k_0,q}$ as below

$$\mu^{\varepsilon}_{k_0,q} \leqslant -\frac{1}{2^\star}\min\left\{\left(\frac{|h|}{\int_M |f^-|\,\mathrm{dvol}_g}\right)^{n-1}, 1\right\}\int_M f^+\,\mathrm{dvol}_g \tag{4.2.5}$$

for any $\varepsilon > 0$. Keep in mind that the right hand side of (4.2.5) is strictly negative and is independent of both $q \geqslant q_0$ and $\varepsilon$ provided $\sup_M f > 0$ which is always the case in this section. Furthermore, from the choice of $k_\star$ as in the proof of Lemma 4.3 we have $k_\star < k_0$.

Since the right hand side of (4.2.3) depends on $q$, its behavior for $q$ near $2^\star$ is needed in future argument. In fact under the condition (4.2.6) below we show that it is monotone increasing.

**Lemma 4.6.** *As a function of $q$,*

$$\left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-|\,\mathrm{dvol}_g}\right)^{\frac{q+2}{q-2}}\frac{|h|}{4}(q-2)$$

*is monotone increasing in $(2, 2^\star)$ provided*

$$\frac{2^\star|h|}{2} \leqslant \int_M |f^-|\,\mathrm{dvol}_g. \tag{4.2.6}$$

*Proof.* This is elementary. Let

$$\beta(q) = \frac{q+2}{q-2}\log\left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-|\,\mathrm{dvol}_g}\right) + \log\left(\frac{|h|}{4}(q-2)\right).$$

Our condition (4.2.6) implies that

$$\frac{2+q}{4}\frac{|h|}{\int_M |f^-|\,\mathrm{dvol}_g} \leqslant 1.$$

Hence we have

$$\beta'(q) = -\frac{4}{(q-2)^2}\log\left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-|\,\mathrm{dvol}_g}\right) + \frac{2}{q-2} > 0,$$

if $q > 2$. The conclusion follows. ■

*Remark 4.4.* The preceding proof shows that $\beta'(q)$ is non-negative for any $2 < q < 2^\star$. Also, a simple calculation shows that

$$\lim_{q\to 2^\star} e^{\beta(q)} = \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1}\left(\frac{|h|}{\int_M |f^-|\,\mathrm{dvol}_g}\right)^n \int_M |f^-|\,\mathrm{dvol}_g$$

since $\frac{2^\star+2}{2^\star-2} = n-1$. This suggests that a good condition for $\int_M a\,\mathrm{dvol}_g$ could be (4.0.1).

Notice that, so far our estimate on $\mu^{\varepsilon}_{k,q}$ is still not enough for our purpose. We need finer estimates. We prove that, as a function of $k$ where $k \geqslant k_0$, $\mu^{\varepsilon}_{k,q}$ is bounded from above by a constant independent of $q \in [2^\flat, 2^\star)$ and $\varepsilon > 0$.

**Lemma 4.7.** *Assume that (4.0.1) holds. Then there exists some constant $\mu$ independent of $q$ and $\varepsilon$ such that $\mu^{\varepsilon}_{k,q} \leqslant \mu$ for any $\varepsilon > 0$, $q \in [2^\flat, 2^\star)$ and $k \geqslant k_0$. In other words, $\mu^{\varepsilon}_{k,q}$ has an upper bound when $k$ is large.*

*Proof.* Thanks to the proof of Lemma 4.4, we can conclude our lemma by taking a positive function $v$ of the following form $v(x) = ce^{t_0 w(x)}$, $x \in M$ where $c$ is a positive constant chosen so that $\int_M v^q \, d\mathrm{vol}_g = 1$, namely,

$$c = \left( \int_M e^{qt_0 w} \, d\mathrm{vol}_g \right)^{-\frac{1}{q}}.$$

With this choice of $c$, we immediately have

$$\left( \int_M e^{2^\star t_0 w} \, d\mathrm{vol}_g \right)^{-1} \leqslant c^q \leqslant 1.$$

Therefore,

$$\begin{aligned}
F_q^\varepsilon(k^{\frac{1}{q}} v) =& \frac{1}{2} \int_M |\nabla (k^{\frac{1}{q}} v)|^2 \, d\mathrm{vol}_g + \frac{h}{2} \int_M \left(k^{\frac{1}{q}} v\right)^2 d\mathrm{vol}_g \\
& - \frac{1}{q} \int_M f \left(k^{\frac{1}{q}} v\right)^q d\mathrm{vol}_g + \frac{1}{q} \int_M \frac{a}{\left(\left(k^{\frac{1}{q}} v\right)^2 + \varepsilon\right)^{\frac{q}{2}}} \, d\mathrm{vol}_g \\
\leqslant & \frac{1}{2} k^{\frac{2}{q}} c^2 \left[ \int_M |\nabla (e^{t_0 w})|^2 \, d\mathrm{vol}_g + h \int_M \left(e^{t_0 w}\right)^2 d\mathrm{vol}_g \right] \\
& - \frac{k}{q} \int_M f v^q \, d\mathrm{vol}_g + \frac{1}{qk} \int_M a v^{-q} \, d\mathrm{vol}_g \\
\leqslant & \frac{1}{2} k^{\frac{2}{q}} c^2 \int_M |\nabla (e^{t_0 w})|^2 \, d\mathrm{vol}_g - \frac{k}{q} \int_M f v^q \, d\mathrm{vol}_g + \frac{1}{qk} \int_M a v^{-q} \, d\mathrm{vol}_g.
\end{aligned}$$

We first observe that

$$\int_M f v^q \, d\mathrm{vol}_g = c^q g(qt_0) \geqslant \left( \int_M e^{2^\star t_0 w} \, d\mathrm{vol}_g \right)^{-1}.$$

For the term $\frac{1}{qk} \int_M a v^{-q} \, d\mathrm{vol}_g$, notice that

$$v^q \geqslant c^q \geqslant \left( \int_M e^{2^\star t_0 w} \, d\mathrm{vol}_g \right)^{-1}.$$

Therefore,

$$\frac{1}{qk} \int_M a v^{-q} \, d\mathrm{vol}_g \leqslant \frac{1}{2k} \left( \int_M e^{2^\star t_0 w} \, d\mathrm{vol}_g \right) \left( \int_M a \, d\mathrm{vol}_g \right).$$

We still have to analyze the last integral. Thanks to $c \leqslant 1$, clearly

$$c^2 \int_M |\nabla (e^{t_0 w})|^2 \, d\mathrm{vol}_g \leqslant \int_M |\nabla (e^{t_0 w})|^2 \, d\mathrm{vol}_g.$$

Putting all the estimates together, we deduce that

$$\begin{aligned}
F_q^\varepsilon(k^{\frac{1}{q}} v) \leqslant & \frac{1}{2} (k+1)^{\frac{2}{q}} c^2 \int_M |\nabla (e^{t_0 w})|^2 \, d\mathrm{vol}_g - \frac{k}{q} \int_M f v^q \, d\mathrm{vol}_g + \frac{1}{qk} \int_M a v^{-q} \, d\mathrm{vol}_g \\
\leqslant & \frac{1}{2} (k+1)^{\frac{2}{2^\flat}} \int_M |\nabla (e^{t_0 w})|^2 \, d\mathrm{vol}_g \\
& - \frac{k}{q} \left( \int_M e^{2^\star t_0 w} \, d\mathrm{vol}_g \right)^{-1} + \frac{1}{2k} \left( \int_M e^{2^\star t_0 w} \, d\mathrm{vol}_g \right) \left( \int_M a \, d\mathrm{vol}_g \right).
\end{aligned} \tag{4.2.7}$$

As a function of $k$ and with $k \geqslant k_0$, it is clear that the right hand side of (4.2.7) achieves its maximum, say $\mu$ due to the fact that $2^\flat > 2$. This helps us to complete the proof. ∎

In order to take the limit as $q \to 2^\star$ we need to control $L^q$-norm of the mountain pass solutions. Since our mountain pass solutions have non-negative energy, what we really need is to show that there is an upper bound $k_{\star\star} > \max\{k_0, 1\}$ independent of $\varepsilon$ and $q$ such that $\mu^{\varepsilon}_{k,q} < 0$ for any $k \geqslant k_{\star\star}$. This is done by the following lemma.

**Lemma 4.8.** *There is some $k_{\star\star}$ sufficiently large and independent of both $q$ and $\varepsilon$ such that $\mu^{\varepsilon}_{k,q} < 0$ for any $k \geqslant k_{\star\star}$.*

*Proof.* From the proof of Lemma 4.7, it is easy to see that the right hand side of (4.2.2), being considered as a function of $k$, is continuous and independent of $q$ and $\varepsilon$. Again, thanks to $\frac{2}{2^\flat} < 1$, we know that the function on the right hand side of (4.2.2) goes to $-\infty$ as $k \to +\infty$. Consequently, there is some $k_{\star\star} > \max\{k_0, 1\}$ sufficiently large and independent of both $q$ and $\varepsilon$ such that $\mu^{\varepsilon}_{k,q} < 0$ for any $k \geqslant k_{\star\star}$ and any $\varepsilon > 0$. ∎

Now, we prove that, for any $\varepsilon > 0$ and for some $k > k_0$, $\mu^{\varepsilon}_{k,q} > 0$. A similar result was studied in [38, Proposition 2].

**Proposition 4.2.** *Suppose that $|h| < \lambda_f$ and $\sup_M f > 0$. Then there exists $\eta_0 > 0$ sufficiently small and its corresponding $q_{\eta_0}$ sufficiently close to $2^\star$ such that*

$$\delta = \frac{\lambda_{f,\eta_0,q} + h}{2} > \frac{3}{8}(\lambda_f + h) \tag{4.2.8}$$

*for any $q \in [q_{\eta_0}, 2^\star)$. For such a choice of $\delta$, we denote*

$$C_q = \frac{\eta_0}{4|h|} \underbrace{\min\left\{\frac{\delta}{(\mathscr{A}_1 + 2\mathscr{K}_1(|h| + 2\delta))}, \frac{|h|}{2}\right\}}_{m}. \tag{4.2.9}$$

*If*

$$\frac{\sup_M f}{\int_M |f^-| \, \mathrm{dvol}_g} < C_q, \tag{4.2.10}$$

*then there exists an interval $I_q = [k_{1,q}, k_{2,q}]$ so that for any $k \in I_q$, any $\varepsilon > 0$, and any $u \in \mathcal{B}_{k,q}$, there holds*

$$F^{\varepsilon}_q(u) > \frac{1}{2} m k^{\frac{2}{q}}.$$

*In particular, $\mu^{\varepsilon}_{k,q} > 0$ for any $k \in I_q$ and any $\varepsilon > 0$.*

*Proof.* It follows from Lemma 3.13 that there exist some $0 < \eta_0 < 2$ and its corresponding $q_{\eta_0} \in [2^\flat, 2^\star)$ such that

$$0 \leqslant \lambda_f - \lambda_{f,\eta_0,q} < \frac{1}{4}(\lambda_f - |h|)$$

for any $q \in (q_{\eta_0}, 2^\star)$. This immediately confirms (4.2.8). We now let

$$k_{1,q} = \left(\frac{|h| q}{\eta_0 \int_M |f^-| \, \mathrm{dvol}_g}\right)^{\frac{q}{q-2}}. \tag{4.2.11}$$

From Remark 4.3 and the fact that $\eta_0 < 2$ we deduce that $k_0 < k_{1,q}$. We assume from now on that $k \geqslant k_{1,q}$. We write

$$F_q^\varepsilon(u) = G_q(u) - \frac{1}{q}\int_M f^+|u|^q \,\mathrm{dvol}_g + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\,\mathrm{dvol}_g,$$

where

$$G_q(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{h}{2}\|u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q\,\mathrm{dvol}_g.$$

Then, there are two possible cases.

**Case 1.** Assume that

$$\int_M |f^-||u|^q\,\mathrm{dvol}_g \geqslant \eta_0 k \int_M |f^-|\,\mathrm{dvol}_g.$$

In this case, the term $G_q$ can be estimated from below as follows

$$\begin{aligned} G_q(u) &= \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{h}{2}\|u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q\,\mathrm{dvol}_g \\ &\geqslant \frac{h}{2}\|u\|_{L^2}^2 + \frac{\eta_0 k}{q}\int_M |f^-|\,\mathrm{dvol}_g \\ &\geqslant \frac{|h|}{2}k^{\frac{2}{q}}\left(\underbrace{\frac{2\eta_0\int_M |f^-|\,\mathrm{dvol}_g}{|h|q}k^{1-\frac{2}{q}}}_{\geqslant 2} -1\right) \\ &\geqslant \frac{|h|}{2}k^{\frac{2}{q}}, \end{aligned} \tag{4.2.12}$$

where in the last inequality we have used the fact that $k \geqslant k_{1,q}$ and (4.2.11).

**Case 2.** Assume that

$$\int_M |f^-||u|^q\,\mathrm{dvol}_g < \eta_0 k \int_M |f^-|\,\mathrm{dvol}_g.$$

Under this condition, it is clear that $k^{-\frac{1}{q}}u \in \mathcal{A}'(\eta_0, q)$ which implies $\|\nabla u\|_{L^2}^2\|u\|_{L^2}^{-2} \geqslant \lambda_{f,\eta_0,q}$ by the definition of $\lambda_{f,\eta_0,q}$. Therefore, we can estimate $G_q(u)$ as follows

$$\begin{aligned} G_q(u) &\geqslant \frac{1}{2}(\lambda_{f,\eta_0,q}+h)\|u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q\,\mathrm{dvol}_g \\ &= \delta\|u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q\,\mathrm{dvol}_g. \end{aligned}$$

Clearly,

$$\|u\|_{L^2}^2 = \frac{2}{|h|}\left(\frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q\,\mathrm{dvol}_g - G_q(u)\right).$$

Now if we write $\delta\|u\|_{L^2}^2$ as $\delta\|u\|_{L^2}^2 = (\alpha+\beta)\|u\|_{L^2}^2$ where $\alpha = \frac{\beta\mathcal{A}_1}{2|h|\mathcal{K}_1}$ and $\alpha+\beta=\delta$, we then get

$$G_q(u) \geqslant \alpha\|u\|_{L^2}^2 + \frac{2\beta}{|h|}\left(\frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q\,\mathrm{dvol}_g - G_q(u)\right)$$

$$\begin{aligned}&+\frac{1}{q}\int_M |f^-||u|^q \, d\mathrm{vol}_g\\ \geqslant&\alpha \|u\|_{L^2}^2+\frac{2\beta}{|h|}\left(\frac{1}{2}\|\nabla u\|_{L^2}^2-G_q(u)\right)\end{aligned}$$

which gives

$$\begin{aligned}\left(1+\frac{2\beta}{|h|}\right)G_q(u)&\geqslant \alpha\|u\|_{L^2}^2+\frac{\beta}{|h|}\|\nabla u\|_{L^2}^2\\ &\geqslant \frac{\beta}{|h|}\left(\|\nabla u\|_{L^2}^2+\frac{\alpha|h|}{\beta}\|u\|_{L^2}^2\right).\end{aligned}$$

Using $\mathcal{K}_1\|\nabla u\|_{L^2}^2+\mathcal{A}_1\|u\|_{L^2}^2\geqslant k^{\frac{2}{q}}$ and the fact that $\frac{\alpha|h|}{\beta}=\frac{\mathcal{A}_1}{2\mathcal{K}_1}$, one easily obtains

$$\|\nabla u\|_{L^2}^2+\frac{\alpha|h|}{\beta}\|u\|_{L^2}^2\geqslant\frac{k^{\frac{2}{q}}}{2\mathcal{K}_1}.$$

Since $\beta=\frac{2\mathcal{K}_1|h|\delta}{\mathcal{A}_1+2\mathcal{K}_1|h|}$, we therefore have

$$G_q(u)\geqslant\frac{\beta}{2|h|}\frac{k^{\frac{2}{q}}}{\mathcal{K}_1}\left(1+\frac{2\beta}{|h|}\right)^{-1}=\frac{\delta}{\mathcal{A}_1+2\mathcal{K}_1(|h|+2\delta)}k^{\frac{2}{q}}. \tag{4.2.13}$$

It follows from (4.2.9), (4.2.12), and (4.2.13) that $G_q(u)\geqslant mk^{\frac{2}{q}}$. Thus, we obtain

$$F_q^\varepsilon(u)\geqslant mk^{\frac{2}{q}}-\frac{k}{q}\sup_M f.$$

If we let $k<\left(\frac{mq}{2\sup_M f}\right)^{\frac{q}{q-2}}$ we then get $F_q^\varepsilon(u)>\frac{1}{2}mk^{\frac{2}{q}}>0$. Since

$$\sup_M f\leqslant C_q\int_M|f^-|\,d\mathrm{vol}_g=\frac{m\eta_0}{4|h|}\int_M|f^-|\,d\mathrm{vol}_g,$$

one has, by (4.2.11), the following

$$\left(\frac{mq}{2\sup_M f}\right)^{\frac{q}{q-2}}\geqslant\left(\frac{2q|h|}{\eta_0\int_M|f^-|\,d\mathrm{vol}_g}\right)^{\frac{q}{q-2}}=2^{\frac{q}{q-2}}k_{1,q}.$$

Hence, if we set $k_{2,q}=2^{\frac{n}{2}}k_{1,q}$, then for arbitrary $k\in[k_{1,q},k_{2,q}]$ we always have $F_q^\varepsilon(u)>\frac{1}{2}mk^{\frac{2}{q}}$. In other words, $\mu_{k,q}^\varepsilon>0$ for arbitrary $k\in[k_{1,q},k_{2,q}]$ which completes the present proof. ∎

## 4.3 The Palais-Smale condition

This subsection is devoted to the proof of the Palais–Smale compactness condition. To our knowledge, there is no such a result in the literature since our energy functional contains both critical and negative exponents that cause a lot of difficulty. In addition, the negative constant $h$ also raises several difficulties.

**Proposition 4.3.** *Suppose that the conditions (4.2.8)-(4.2.10) hold. Then for each $\varepsilon > 0$ fixed, the functional $F_q^\varepsilon(\cdot)$ satisfies the Palais–Smale condition.*

*Proof.* Let $\varepsilon > 0$ be fixed. Suppose that $\{v_j\}_j \subset H^1(M)$ is a Palais–Smale sequence, that is, there exists a constant $\mathcal{C}$ such that

$$F_q^\varepsilon(v_j) \to \mathcal{C}, \quad \|\delta F_q^\varepsilon(v_j)\|_{H^{-1}} \to 0$$

as $j \to \infty$, where $H^{-1}(M)$ is the dual space of $H^1(M)$. As the first step, we shall prove that, up to subsequences, $\{v_j\}_j$ is bounded in $H^1(M)$. Without loss of generality, we may assume $\|v_j\|_{H^1} \geqslant 1$ for all $j$. By means of the Palais–Smale sequence, one can derive

$$\begin{aligned}\frac{1}{2}\int_M |\nabla v_j|^2 \, d\mathrm{vol}_g + \frac{h}{2}\int_M |v_j|^2 \, d\mathrm{vol}_g \\ - \frac{1}{q}\int_M f|v_j|^q \, d\mathrm{vol}_g + \frac{1}{q}\int_M \frac{a}{((v_j)^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g = \mathcal{C} + o(1)\end{aligned} \tag{4.3.1}$$

and

$$\begin{aligned}\int_M \nabla v_j \cdot \nabla \xi \, d\mathrm{vol}_g + h\int_M v_j \xi \, d\mathrm{vol}_g - \int_M f|v_j|^{q-2} v_j \xi \, d\mathrm{vol}_g \\ - \int_M \frac{a v_j}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}\xi \, d\mathrm{vol}_g} = o(1)\|\xi\|_{H^1}\end{aligned} \tag{4.3.2}$$

for any $\xi \in H^1(M)$. By letting $\xi = v_j$ in (4.3.2), we obtain

$$\begin{aligned}\int_M |\nabla v_j|^2 \, d\mathrm{vol}_g + h\int_M |v_j|^2 \, d\mathrm{vol}_g - \int_M f|v_j|^q \, d\mathrm{vol}_g \\ - \int_M \frac{a v_j^2}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}} \, d\mathrm{vol}_g = o(1)\|v_j\|_{H^1}.\end{aligned} \tag{4.3.3}$$

Therefore, from (4.3.1) and (4.3.3) we obtain

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{q}\right)\int_M f|v_j|^q \, d\mathrm{vol}_g + \frac{1}{2}\int_M \frac{a v_j^2}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}} \, d\mathrm{vol}_g \\ + \frac{1}{q}\int_M \frac{a}{((v_j)^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g = \mathcal{C} + \frac{1}{2}o(1)\|v_j\|_{H^1} + o(1).\end{aligned} \tag{4.3.4}$$

For the sake of simplicity, let us denote

$$k_j = \int_M |v_j|^q \, d\mathrm{vol}_g.$$

There are two possible cases.

**Case 1.** Assume that there exists a subsequence of $\{v_j\}_j$, still denoted by $\{v_j\}_j$, such that

$$\int_M |f^-||v_j|^q \, d\mathrm{vol}_g \geqslant \eta_0 k_j \int_M |f^-| \, d\mathrm{vol}_g.$$

Using (4.2.9) and (4.2.10), we get that

$$\begin{aligned} F_q^\varepsilon(v_j) =& \frac{1}{2}\|\nabla v_j\|_{L^2}^2 + \frac{h}{2}\|v_j\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||v_j|^q \, d\mathrm{vol}_g \\ & - \frac{1}{q}\int_M f^+|v_j|^q \, d\mathrm{vol}_g + \frac{1}{q}\int_M \frac{a}{((v_j)^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g \\ \geqslant& \frac{h}{2}k_j^{\frac{2}{q}} + \frac{\eta_0 k_j}{q}\int_M |f^-| \, d\mathrm{vol}_g - \frac{1}{q}\int_M f^+|v_j|^q \, d\mathrm{vol}_g \\ \geqslant& \frac{h}{2}k_j^{\frac{2}{q}} + \frac{\eta_0 k_j}{q}\int_M |f^-| \, d\mathrm{vol}_g - \frac{k_j}{q}\sup_M f \\ \geqslant& \frac{h}{2}k_j^{\frac{2}{q}} + \frac{\eta_0 k_j}{q}\int_M |f^-| \, d\mathrm{vol}_g - \frac{k_j}{q}\frac{\eta_0}{8}\int_M |f^-| \, d\mathrm{vol}_g \\ =& \left(\frac{7\eta_0}{8}\int_M |f^-| \, d\mathrm{vol}_g\right)\frac{k_j}{q} - \frac{|h|}{2}k_j^{\frac{2}{q}}. \end{aligned}$$

This and the fact that $F_q^\varepsilon(v_j) \to \mathcal{C}$ imply that $\{k_j\}_j$ is bounded. In other words, $\{v_j\}_j$ is bounded in $L^q(M)$. Hence, the Hölder inequality and (4.3.1) imply that $\{v_j\}_j$ is also bounded in $H^1(M)$.

**Case 2.** In contrast to Case 1, for all $j$ sufficiently large, we assume that

$$\int_M |f^-||v_j|^q \, d\mathrm{vol}_g < \eta_0 k_j \int_M |f^-| \, d\mathrm{vol}_g.$$

Using (4.3.1) and (4.3.3), we obtain

$$\begin{aligned} -\frac{1}{q}\int_M f|v_j|^q \, d\mathrm{vol}_g =& -\frac{2}{q-2}\mathcal{C} + o(1)\|v_j\|_{H^1} + o(1) \\ & + \frac{1}{q-2}\int_M \frac{a v_j^2}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}} \, d\mathrm{vol}_g + \frac{2}{q(q-2)}\int_M \frac{a}{((v_j)^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g. \end{aligned}$$

Therefore, we may rewrite $F_q^\varepsilon$ as follows

$$\begin{aligned} F_q^\varepsilon(v_j) =& \frac{1}{2}\|\nabla v_j\|_{L^2}^2 + \frac{h}{2}\|v_j\|_{L^2}^2 - \frac{1}{q}\int_M f|v_j|^q \, d\mathrm{vol}_g + \frac{1}{q}\int_M \frac{a}{((v_j)^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g \\ \geqslant& \frac{1}{2}\|\nabla v_j\|_{L^2}^2 + \frac{h}{2}\|v_j\|_{L^2}^2 - \frac{2}{q-2}\mathcal{C} + o(1)\|v_j\|_{H^1} + o(1) + A_j \end{aligned} \tag{4.3.5}$$

where

$$A_j = \frac{1}{q-2}\left(\int_M \frac{a(v_j)^2}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}} \, d\mathrm{vol}_g + \int_M \frac{a}{((v_j)^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g\right).$$

Dividing (4.3.5) by $\|v_j\|_{L^2}$ and using the equivalent norm to $\|v_j\|_{H^1} = \|\nabla v_j\|_{L^2} + \|v_j\|_{L^2}$, one obtains

$$\begin{aligned} \frac{F_q^\varepsilon(v_j)}{\|v_j\|_{L^2}} \geqslant& \frac{\|\nabla v_j\|_{L^2}}{\|v_j\|_{L^2}}\left(\frac{1}{2}\|\nabla v_j\|_{L^2} + o(1)\right) + \frac{h}{2}\|v_j\|_{L^2} \\ & - \frac{2}{(q-2)\|v_j\|_{L^2}}\mathcal{C} + o(1) + \frac{o(1)}{\|v_j\|_{L^2}} + \frac{A_j}{\|v_j\|_{L^2}}. \end{aligned} \tag{4.3.6}$$

Observe that, from the definition of $\lambda_{f,\eta_0,q}$, there holds $\|\nabla v_j\|_{L^2}^2 \geqslant \lambda_{f,\eta_0,q}\|v_j\|_{L^2}^2$. Therefore, from (4.3.6) and for $j$ large enough, there holds

$$\begin{aligned}\frac{F_q^\varepsilon(v_j)}{\|v_j\|_{L^2}} \geqslant & \frac{1}{2}(\lambda_{f,\eta_0,q}+h)\|v_j\|_{L^2}+o(1)\sqrt{\lambda_{f,\eta_0,q}}\\ & -\frac{2}{(q-2)\|v_j\|_{L^2}}\mathcal{C}+o(1)+\frac{o(1)}{\|v_j\|_{L^2}}+\frac{A_j}{\|v_j\|_{L^2}}.\end{aligned}$$

If $\|v_j\|_{L^2}\to+\infty$ as $j\to\infty$, then we clearly would reach a contradiction by taking the limit in previous equation as $j\to\infty$ since $\lambda_{f,\eta_0,q}+h>0$ and $A_j>0$ as we notice that $F_q^\varepsilon(v_j)\,\|v_j\|_{L^2}^{-1}\to 0$ as $j\to\infty$. Thus, $\{v_j\}_j$ is bounded in $L^2(M)$. This and (4.3.5) also imply that $\{\nabla v_j\}_j$ is bounded in $L^2(M)$. Consequently, $\{v_j\}_j$ is bounded in $H^1(M)$.

Combining Cases 1 and 2 above, we conclude that there exists a bounded subsequence of $\{v_j\}_j$ in $H^1(M)$, still denoted by $\{v_j\}_j$. This completes the first step.

Being bounded, there exists $v\in H^1(M)$ such that, up to subsequences,

- $v_j\rightharpoonup v$ weakly in $H^1(M)$;
- $\nabla v_j\rightharpoonup\nabla v$ weakly in $L^2(M)$;
- $v_j\to v$ strongly in $L^p(M)$ for any $p\in[1,2^\star)$;
- $v_j\to v$ almost everywhere in $M$.

We now prove that $v_j\to v$ strongly in $H^1(M)$. This can be done once we show that $\nabla v_j\to\nabla v$ strongly in $L^2(M)$. In order to achieve that goal, the following

$$\delta F_q^\varepsilon(v_j)(v_j-v)\to 0,$$

as $j\to\infty$, is crucial in our argument. Indeed, using (4.3.2) with $\xi$ replaced by $v_j-v$, we get

$$\begin{aligned}&\int_M\nabla v_j\cdot\nabla(v_j-v)\,\mathrm{dvol}_g+h\int_M v_j(v_j-v)\,\mathrm{dvol}_g\\ &\quad-\int_M f|v_j|^{q-2}v_j(v_j-v)\,\mathrm{dvol}_g-\int_M\frac{av_j}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}}(v_j-v)\,\mathrm{dvol}_g\to 0\end{aligned}\tag{4.3.7}$$

as $j\to\infty$. Since $v_j\to v$ strongly in $L^2(M)$ and $\{v_j\}_j$ is bounded in $H^1(M)$, the Hölder inequality can be applied to get

$$\int_M v_j(v_j-v)\,\mathrm{dvol}_g\to 0\tag{4.3.8}$$

as $j\to\infty$. For the term involving $f$, we can also use the Hölder inequality as follows

$$\begin{aligned}&\left|\int_M f|v_j|^{q-2}v_j(v_j-v)\,\mathrm{dvol}_g\right|\\ &\quad\leqslant(\sup_M f)\int_M|v_j|^{q-1}|v_j-v|\,\mathrm{dvol}_g\\ &\quad\leqslant(\sup_M f)\Big(\int_M|v_j|^{2^\star}\,\mathrm{dvol}_g\Big)^{\frac{q-1}{2^\star}}\Big(\int_M|v_j-v|^{\frac{2^\star}{2^\star-(q-1)}}\,\mathrm{dvol}_g\Big)^{\frac{2^\star-(q-1)}{2^\star}}.\end{aligned}$$

Notice that $\frac{2^\star}{2^\star-(q-1)}<2^\star$ as long as $q<2^\star$. Hence, thanks to the compact embedding $H^1(M)\hookrightarrow L^{\frac{2^\star}{2^\star-(q-1)}}(M)$, the Sobolev inequality, and the boundedness of $\{v_j\}_j$ in $H^1(M)$, there holds

$$\int_M f|v_j|^{q-2}v_j(v_j - v)\, d\mathrm{vol}_g \to 0 \tag{4.3.9}$$

as $j \to \infty$. For the term involving $a$, we first observe that

$$\left| \int_M \frac{av_j}{((v_j)^2 + \varepsilon)^{\frac{q}{2}+1}}(v_j - v)\, d\mathrm{vol}_g \right| \leqslant \varepsilon^{-\frac{q}{2}-1} \int_M a|v_j(v_j - v)|\, d\mathrm{vol}_g$$

holds true. Using the squeeze theorem, it is immediately to see that

$$\int_M \frac{av_j}{((v_j)^2 + \varepsilon)^{\frac{q}{2}+1}}(v_j - v)\, d\mathrm{vol}_g \to 0 \tag{4.3.10}$$

as $j \to \infty$. Therefore, from (4.3.7), (4.3.8), (4.3.9), and (4.3.10), we obtain

$$\int_M \nabla v_j \cdot \nabla(v_j - v)\, d\mathrm{vol}_g \to 0 \tag{4.3.11}$$

as $j \to \infty$. Making use of the fact that $\nabla v_j \rightharpoonup \nabla v$ weakly in $L^2(M)$ gives

$$\int_M \nabla v \cdot (\nabla v_j - \nabla v)\, d\mathrm{vol}_g \to 0 \tag{4.3.12}$$

as $j \to \infty$. We then obtain immediately from (4.3.11)-(4.3.12) that

$$\int_M |\nabla v_j - \nabla v|^2\, d\mathrm{vol}_g = \int_M \nabla v_j \cdot (\nabla v_j - \nabla v)\, d\mathrm{vol}_g - \int_M \nabla v \cdot (\nabla v_j - \nabla v)\, d\mathrm{vol}_g \to 0$$

as $j \to \infty$. In other words, $\nabla v_j \to \nabla v$ strongly in $L^2(M)$. This completes the proof of the Palais-Smale condition. ■

## 4.4 Proof of Theorem 4.1

We are now in a position to prove Theorem 4.1. This can be done through three steps. First, because of Lemma 4.6, we need to make use of the condition (4.2.6) in order to guarantee the existence of the first solution. This is the content of Proposition 4.4. Next we show that if, in addition, $\sup_M f$ can be controlled by some positive number, then (3.0.1) has at least two positive solutions. In the last step, we remove the condition (4.2.6) by using a scaling argument. It is worth noting that although the condition (4.2.6) is not really important in view of taking the limit as $q \to 2^\star$, we take this chance to perform this trick in order to completely remove (4.2.6).

### 4.4.1 The existence of the first solution

In this section, we obtain the existence of the first solution of (3.0.1). Notice that, we require (4.2.6) to hold. This restriction will be removed by using a scaling argument later. We prove the following result.

**Proposition 4.4.** *Let $(M, g)$ be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Let $h < 0$ be a constant, $f$ and $a \geqslant 0$ be smooth* *functions on $M$ with $\int_M a \,{\rm dvol}_g > 0$, $\int_M f \,{\rm dvol}_g < 0$, $\sup_M f > 0$, and $|h| < \lambda_f$. We further assume (4.2.6) holds and*

$$\int_M a \,{\rm dvol}_g < \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1}\left(\frac{|h|}{\int_M |f^-| \,{\rm dvol}_g}\right)^n \int_M |f^-| \,{\rm dvol}_g .$$

*Then there is a positive number $\mathcal{C}_1$ given by (4.4.1) such that if*

$$\frac{\sup_M f}{\int_M |f^-| \,{\rm dvol}_g} < \mathcal{C}_1,$$

*then (3.0.1) admits at least one smooth positive solution.*

Since our proof is quite long, we divide it into several claims for the sake of clarity.

**Claim 1**. There exists a $q_0 \in [2^\flat, 2^\star)$ such that for all $q \in (q_0, 2^\star)$ and some sufficiently small $\varepsilon$, there will be $k_0$ and $k_\star$ with the following properties: $k_0 < k_\star$ and $\mu^\varepsilon_{k_0,q} \leqslant 0$ while $\mu^\varepsilon_{k_\star,q} > 0$.

*Proof of Claim 1.* We observe that, from Lemma 4.6, the condition (4.2.6), and Remark 4.4, there is some $q_0 \in [2^\flat, 2^\star)$ such that the condition (4.2.3) holds for all $q \in (q_0, 2^\star)$. Hence, by Lemma 4.5, there exists a $k_0 > 0$ small enough such that $\mu^\varepsilon_{k_0,q} \leqslant 0$. Notice that $2^\flat > 2$ for any $n \geqslant 3$. The existence of such a $k_0$ makes it possible for us to select some $k_\star$ such that $k_\star < \min\{k_0, 1\}$ and $\mu^\varepsilon_{k_\star,q} > 0$ for any $\varepsilon \leqslant k_\star$. This settles Claim 1.

**Claim 2**. Equation (3.0.2) with $\varepsilon$ replaced by 0 has two positive solutions, say $u_{1,q}$ and $u_{2,q}$.

*Proof of Claim 2.* By using Proposition 4.2, we have $\eta_0$ and its corresponding $q_{\eta_0} \in [2^\flat, 2^\star)$ such that $\delta = \frac{1}{2}(\lambda_{f,\eta_0,q} + h) > \frac{3}{8}(\lambda_f + h)$ for any $q \in (q_{\eta_0}, 2^\star)$. Thanks to Lemma 3.10, one has $\frac{3}{8}(\lambda_f + h) \leqslant \delta \leqslant \frac{1}{2}(\lambda_f + h)$. Hence

$$\frac{\delta}{\mathcal{A}_1 + 2\mathcal{K}_1(|h| + 2\delta)} \geqslant \frac{\frac{3}{8}(\lambda_f + h)}{\mathcal{A}_1 + 2\mathcal{K}_1(|h| + (\lambda_f + h))} = \frac{3}{8}\frac{\lambda_f + h}{\mathcal{A}_1 + 2\mathcal{K}_1\lambda_f}.$$

Thus, by (4.2.9), we can define

$$\mathcal{C}_1 = \frac{\eta_0}{4|h|}\min\left\{\frac{3}{8}\frac{\lambda_f + h}{\mathcal{A}_1 + 2\mathcal{K}_1\lambda_f}, \frac{|h|}{2}\right\}. \tag{4.4.1}$$

Note that $\mathcal{C}_1$ is independent of $q$ and thus never vanishes for $q \in [q_{\eta_0}, 2^\star)$. Besides, $C_q \geqslant \mathcal{C}_1$ for any $q \in [q_{\eta_0}, 2^\star)$. In addition, one can easily see that

$$\lim_{q\to 2^\star} k_{1,q} = \left(\frac{2^\star|h|}{\eta_0 \int_M |f^-| \,{\rm dvol}_g}\right)^{\frac{n}{2}} = \ell, \qquad \lim_{q\to 2^\star} k_{2,q} = 2^{\frac{n}{2}}\ell.$$

By Proposition 4.2, there exists an interval $I_q = [k_{1,q}, k_{2,q}]$ such that $\mu^\varepsilon_{k,q} > 0$ for any $k \in I_q$. Recall that $k_\star < k_0 < k_{1,q}$, where $k_\star$ is given as in Claim 1.

*The existence of $u^\varepsilon_{1,q}$ with energy $\mu^\varepsilon_{k_1^\varepsilon,q}$.* We first define the number

$$\mu^\varepsilon_{k_1^\varepsilon,q} = \inf_{u\in\mathcal{D}_{k,q}} F^\varepsilon_q(u)$$

where

$$\mathcal{D}_{k,q} = \left\{ u \in H^1(M) : k_\star \leqslant \|u\|_{L^q}^q \leqslant k_{1,q} \right\}.$$

Due to the monotonicity of $k_{1,q}$, we know that $\|u\|_{L^q}^q < \ell$ for any $u \in \mathcal{D}_{k,q}$. It follows from Lemma 4.5 that $\mu^\varepsilon_{k_1^\varepsilon,q}$ is finite and non-positive. Similar arguments to those used before show that $\mu^\varepsilon_{k_1^\varepsilon,q}$ is achieved by some positive smooth function $u^\varepsilon_{1,q}$. In particular, $\mu^\varepsilon_{k_1^\varepsilon,q}$ is the energy of $u^\varepsilon_{1,q}$. Obviously, $u^\varepsilon_{1,q}$ is a solution of (3.0.2). It is not hard to verify that any minimizing sequence for $\mu^\varepsilon_{k_1^\varepsilon,q}$ is bounded in $H^1(M)$. Now the lower semi-continuity of $H^1$-norm implies that $\|u^\varepsilon_{1,q}\|_{H^1}$ is bounded with the bound independent of $q$ and $\varepsilon$. If we denote $\|u^\varepsilon_{1,q}\|^q_{L^q} = k_1^\varepsilon$ we immediately have $k_1^\varepsilon \in (k_\star, k_{\star\star})$.

*The existence of $u_{1,q}$ with strictly negative energy $\mu_{k_1,q}$.* In what follows, we let $\{\varepsilon_j\}_j$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$. For each $j$, let $u^{\varepsilon_j}_{1,q}$ be a smooth positive function in $M$ such that

$$-\Delta_g u^{\varepsilon_j}_{1,q} + h u^{\varepsilon_j}_{1,q} = f(u^{\varepsilon_j}_{1,q})^{q-1} + \frac{a u^{\varepsilon_j}_{1,q}}{((u^{\varepsilon_j}_{1,q})^2 + \varepsilon_j)^{\frac{q}{2}+1}} \tag{4.4.2}$$

in $M$. Being bounded, there exists $u_{1,q} \in H^1(M)$ such that, up to subsequences,

- $u^{\varepsilon_j}_{1,q} \rightharpoonup u_{1,q}$ weakly in $H^1(M)$;
- $u^{\varepsilon_j}_{1,q} \to u_{1,q}$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;
- $u^{\varepsilon_j}_{1,q} \to u_{1,q}$ almost everywhere in $M$.

Using Lemma 3.1, the Lebesgue Dominated Convergence Theorem can be applied to conclude that $\int_M (u_{1,q})^{-p}\, d\mathrm{vol}_g$ is finite for all $p$. Now sending $j \to \infty$ in (4.4.2), we get that $u_{1,q}$ is a weak solution of the following subcritical equation

$$-\Delta_g u_{1,q} + h u_{1,q} = f(u_{1,q})^{q-1} + \frac{a}{(u_{1,q})^{q+1}}. \tag{4.4.3}$$

Thus the regularity result, Lemma 3.4(b), can be applied to (4.4.3). It follows that $u_{1,q} \in C^\infty(M)$. Since $u^{\varepsilon_j}_{1,q} \to u_{1,q}$ strongly in $L^q(M)$ as $j \to \infty$, if we denote $\|u_{1,q}\|^q_{L^q} = k_1$, we still have $k_1 \in (k_\star, k_{\star\star})$. Consequently, there holds $u_{1,q} \not\equiv 0$. With Lemma 3.1 and the Strong Minimum Principle in hand, it is easy to prove that $u_{1,q}$ is strictly positive. From Remark 4.3 and the fact that $u^{\varepsilon_j}_{1,q}$ has strictly negative energy $\mu^{\varepsilon_j}_{k_1^{\varepsilon_j},q}$, by passing to the limit as $j \to \infty$, we know that $u_{1,q}$ also has strictly negative energy $\mu_{k_1,q}$. Thus, we have shown that $u_{1,q}$ is a smooth positive solution of (4.4.3) as claimed. Keep in mind that we still have $\|u_{1,q}\|^q_{L^q} \leqslant k_{\star\star}$ since we have a strong convergence.

*The existence of $u^\varepsilon_{2,q}$ with energy $\mu^\varepsilon_{k_2^\varepsilon,q}$.* Let $k^\star$ be a real number such that

$$\mu^\varepsilon_{k^\star,q} = \max\left\{\mu^\varepsilon_{k,q} : k_{1,q} \leqslant k \leqslant k_{2,q}\right\}.$$

Obviously, $\mu^\varepsilon_{k^\star,q} > 0$. Now we choose $\overline{k}_1 \in (k_0, k_{1,q})$ and $\overline{k}_2 \in (k_{2,q}, k_{\star\star})$ in such a way that $\mu^\varepsilon_{\overline{k}_1,q} = \mu^\varepsilon_{\overline{k}_2,q} = 0$. The existence of $\overline{k}_i$ is guaranteed by Proposition 4.1. Notice that $\mu^\varepsilon_{\overline{k}_1,q}$ and $\mu^\varepsilon_{\overline{k}_2,q}$ have been proved to be achieved, say by $u_{\overline{k}_1,q}$ and $u_{\overline{k}_2,q}$ respectively. We now set

$$\Gamma = \left\{ \gamma \in C([0,1]; H^1(M)) : \gamma(0) = u_{\overline{k}_1,q}, \gamma(1) = u_{\overline{k}_2,q} \right\}.$$

Consider the functional $E(v) = F_q^\varepsilon(u_{\overline{k}_1,q} + v)$ for any non-negative real valued function $v$ with

$$\|v\| = \left(\int_M |u_{\overline{k}_1,q} + v|^q \, d\mathrm{vol}_g\right)^{\frac{1}{q}}.$$

First we have $E(0) = 0$. Let $\rho = (k^\star)^{\frac{1}{q}}$. If $\|v\| = \rho$, then set $u = u_{\overline{k}_1,q} + v$, then $\int_M |u|^q \, d\mathrm{vol}_g = k^\star$. Hence

$$E(v) = F_q^\varepsilon(u) \geqslant \mu_{k^\star,q}^\varepsilon > 0.$$

Next we set $v_1 = u_{\overline{k}_2,q} - u_{\overline{k}_1,q}$, then clearly $E(v_1) = 0$ and $\|v_1\| = (\overline{k}_2)^{\frac{1}{q}} > \rho$. Notice that our functional $E$ satisfies the Palais-Smale condition as we have shown for $F_q^\varepsilon$. Thus, Theorem 6.1 in [40, Chapter II] can be applied to $E$ to conclude that the number

$$\mu_{k_2^\varepsilon,q}^\varepsilon = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} E(\gamma(t) - u_{\overline{k}_1,q})$$

is a critical value of the functional $E$. Clearly, $\mu_{k_2^\varepsilon,q}^\varepsilon > 0$. Thus, there exists a Palais–Smale sequence $\{u_j\}_j \subset H^1(M)$ for the functional $F_q^\varepsilon$ at the level $\mu_{k_2^\varepsilon,q}^\varepsilon$. Since $F_q^\varepsilon(u_j) = F_q^\varepsilon(|u_j|)$ for any $j$, we can assume $u_j \geqslant 0$ for all $j$. Consequently, Proposition 4.3 implies that, up to subsequences, $u_j \to u_{2,q}^\varepsilon$ strongly in $H^1(M)$ for some $u_{2,q}^\varepsilon \in H^1(M)$ as $j \to \infty$. Therefore, the function $u_{2,q}^\varepsilon$ with positive energy $\mu_{k_2^\varepsilon,q}^\varepsilon$ satisfies the following equation

$$-\Delta_g u_{2,q}^\varepsilon + h u_{2,q}^\varepsilon = f(u_{2,q}^\varepsilon)^{q-1} + \frac{a u_{2,q}^\varepsilon}{((u_{2,q}^\varepsilon)^2 + \varepsilon)^{\frac{q}{2}+1}} \tag{4.4.4}$$

in the weak sense where we denote $\|u_{2,q}^\varepsilon\|_{L^q}^q = k_2^\varepsilon$. The non-negativity of $\{u_j\}_j$ implies that $u_{2,q}^\varepsilon \geqslant 0$ almost everywhere, and thus the regularity result, Lemma 3.4(a), can be applied to (4.4.4). It follows that $u_{2,q}^\varepsilon \in C^\infty(M)$ which also implies $u_{2,q}^\varepsilon \geqslant 0$ in $M$. To see $u_{2,q}^\varepsilon$ is not identically zero, thanks to Lemma 4.7 we first know that $\mu_{2,q}^\varepsilon \leqslant \mu < \infty$. Now, if $u_{2,q}^\varepsilon = 0$, then we have $\frac{1}{q}\varepsilon^{-\frac{q}{2}} \int_M a \, d\mathrm{vol}_g = \mu_{2,q}^\varepsilon \leqslant \mu < \infty$ which is impossible if $\varepsilon$ is small enough. Thus, $u_{2,q}^\varepsilon > 0$ on $M$ if $\varepsilon$ is sufficiently small which we will always assume from now on. In view of Lemma 4.8, we know that $k_2^\varepsilon > 0$ is bounded from above by $k_{\star\star}$ independent of both $\varepsilon$ and $q$.

*The existence of $u_{2,q}$ with positive energy $\mu_{k_2,q}$.* We now let $\{\varepsilon_j\}_j$ be a sequence of small positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$. For each $j$, let $u_{2,q}^{\varepsilon_j}$ be a smooth positive function in $M$ such that

$$-\Delta_g u_{2,q}^{\varepsilon_j} + h u_{2,q}^{\varepsilon_j} = f(u_{2,q}^{\varepsilon_j})^{q-1} + \frac{a u_{2,q}^{\varepsilon_j}}{((u_{2,q}^{\varepsilon_j})^2 + \varepsilon_j)^{\frac{q}{2}+1}} \tag{4.4.5}$$

in $M$. The boundedness of $\{k_2^{\varepsilon_j}\}_j$ tells us that sequence $\{u_{2,q}^{\varepsilon_j}\}_j$ is bounded in $H^1(M)$, hence, there exists $u_{2,q} \in H^1(M)$ such that, up to subsequences,

- $u_{2,q}^{\varepsilon_j} \rightharpoonup u_{2,q}$ in $H^1(M)$,
- $u_{2,q}^{\varepsilon_j} \to u_{2,q}$ strongly in $L^2(M)$,
- $u_{2,q}^{\varepsilon_j} \to u_{2,q}$ almost everywhere in $M$.

Consequently, $u_{2,q} \geqslant 0$ almost everywhere in $M$. We now denote $\|u_{2,q}\|_{L^q}^q = k_2$. Since the sequence $\{u_{2,q}^{\varepsilon_j}\}_j$ is bounded from below by means of Lemma 3.1, the Lebesgue

Dominated Convergence theorem can be applied to conclude that $(u_{2,q})^{-1} \in L^p(M)$ for any $p > 0$. By letting $j \to \infty$ in (4.4.5), we get that $u_{2,q}$ is the second weak solution of the following subcritical equation

$$-\Delta_g u_{2,q} + hu_{2,q} = f(u_{2,q})^{q-1} + \frac{a}{(u_{2,q})^{q+1}}. \tag{4.4.6}$$

Now the regularity result, Lemma 3.4(b), can be applied to (4.4.6). It follows that $u_{2,q} \in C^\infty(M)$ and thus $u_{2,q} > 0$ in $M$. Since $u_{2,q}^{\varepsilon_j}$ has positive energy $\mu^{\varepsilon_j}_{k_2^{\varepsilon_j},q}$, by passing to the limit as $j \to \infty$, we know that the energy of $u_{2,q}$ is still non-negative, i.e., $\mu_{k_2,q} > 0$, thus proving $u_{1,q} \not\equiv u_{2,q}$ by means of (4.1.7). Note that $k_2$ is still bounded from above by $k_{\star\star}$ independent of both $\varepsilon$ and $q$. This completes the proof of Claim 2.

**Claim 3**. Equation (3.0.1) has at least one positive solution.

*Proof of Claim 3.* Recall that $\mu_{k_i,q}$ are the energy of $u_{i,q}$ found in Claim 2, i.e.,

$$\begin{aligned}\mu_{k_i,q} = \frac{1}{2}\int_M |\nabla u_{i,q}|^2 \,\mathrm{dvol}_g + \frac{h}{2}\int_M (u_{i,q})^2 \,\mathrm{dvol}_g \\ - \frac{1}{q}\int_M f(u_{i,q})^q \,\mathrm{dvol}_g + \frac{1}{q}\int_M \frac{a}{(u_{i,q})^q} \,\mathrm{dvol}_g.\end{aligned}$$

Keep in mind that by $k_i$ we mean $\|u_{i,q}\|^q_{L^q} = k_i$. We now estimate $\mu_{k_1,q}$ and $\mu_{k_2,q}$. We have noticed that $\mu_{k_1,q} < 0 < \mu_{k_2,q} < \mu$. Since $k_1 \in (k_\star, k_{1,q})$ and $h < 0$, we obtain

$$\begin{aligned}\frac{1}{2}\|\nabla u_{1,q}\|^2_{L^2} &\leqslant \mu_{k_1,q} + \frac{1}{q}\int_M f(u_{1,q})^q \,\mathrm{dvol}_g - \frac{h}{2}k_1^{\frac{2}{q}} \\ &\leqslant \frac{k_1}{q}\sup_M f - \frac{h}{2}k_1^{\frac{2}{q}},\end{aligned}$$

which concludes that the sequence $\{u_{1,q}\}_q$ remains bounded in $H^1(M)$. Similarly, from Lemma 4.7 and the following estimate

$$\begin{aligned}\frac{1}{2}\|\nabla u_{2,q}\|^2_{L^2} &\leqslant \mu_{k_2,q} + \frac{1}{q}\int_M f(u_{2,q})^q \,\mathrm{dvol}_g - \frac{h}{2}k_2^{\frac{2}{q}} \\ &\leqslant \mu + \frac{k_2}{q}\sup_M f - \frac{h}{2}k_2^{\frac{2}{q}},\end{aligned}$$

we know that the sequence $\{u_{2,q}\}_q$ is also bounded in $H^1(M)$. Combining these facts, we get

$$\|u_{i,q}\|^2_{H^1} \leqslant 2\mu + \frac{2k_i}{q}\sup_M f + (1-h)k_i^{\frac{2}{q}}.$$

Thanks to $k_{\star\star} > 1$ and $q > 2^\flat$, if we denote

$$\Lambda = \big(2\mu + (\sup_M f)k_{\star\star} + (1-h)k_{\star\star}^{\frac{2}{2^\flat}}\big)^{\frac{1}{2}}$$

we then see that $\|u_{i,q}\|_{H^1} \leqslant \Lambda$ for $i = 1, 2$. Hence, there exists $u_i \in H^1(M)$ such that, up to subsequences,

- $u_{i,q} \rightharpoonup u_i$ weakly in $H^1(M)$;
- $\nabla u_{i,q} \rightharpoonup \nabla u_i$ weakly in $L^2(M)$;

- $u_{i,q} \to u_i$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;
- $u_{i,q} \to u_i$ almost everywhere in $M$.

Notice that $u_{i,q}$ verifies

$$\begin{aligned}\int_M \nabla u_{i,q} \cdot \nabla v \, \mathrm{dvol}_g + h \int_M u_{i,q} v \, \mathrm{dvol}_g \\ - \int_M f (u_{i,q})^{q-1} v \, \mathrm{dvol}_g - \int_M \frac{a}{(u_{i,q})^{q+1}} v \, \mathrm{dvol}_g = 0\end{aligned} \tag{4.4.7}$$

for any $v \in H^1(M)$. We have already seen in the proof of the Palais-Smale condition that

$$\int_M (\nabla u_{i,q} - \nabla u_i) \cdot \nabla v \, \mathrm{dvol}_g \to 0$$

and

$$\int_M (u_{i,q} - u_i) \, v \, \mathrm{dvol}_g \to 0$$

as $q \to 2^\star$. A strictly positive lower bound for $u_{i,q}$ helps us to conclude that

$$\int_M \frac{a}{(u_{i,q})^{q+1}} v \, \mathrm{dvol}_g \to \int_M \frac{a}{(u_i)^{2^\star+1}} v \, \mathrm{dvol}_g$$

as $q \to 2^\star$. So far, we can pass to the limit every terms on the left hand side of (4.4.8) except the term $\int_M f (u_{i,q})^{q-1} v \, \mathrm{dvol}_g$. Since $u_{i,q} \to u_i$ almost everywhere, $(u_{i,q})^{q-1} \to (u_i)^{2^\star-1}$ almost everywhere. By the Hölder inequality as we have done once, one obtains

$$\begin{aligned}\left\| (u_{i,q})^{q-1} \right\|_{L^{\frac{2^\star}{2^\star-1}}} &= \left( \int_M (u_{i,q})^{\frac{(q-1)2^\star}{2^\star-1}} \, \mathrm{dvol}_g \right)^{\frac{2^\star-1}{2^\star}} \\ &= \left( \left( \int_M (u_{i,q})^{2^\star} \, \mathrm{dvol}_g \right)^{\frac{q-1}{2^\star-1}} \right)^{\frac{2^\star-1}{2^\star}} \\ &= \| u_{i,q} \|_{L^{2^\star}}^{q-1} .\end{aligned} \tag{4.4.8}$$

Making use of the Sobolev inequality, we further obtain

$$\left\| (u_{i,q})^{q-1} \right\|_{L^{\frac{2^\star}{2^\star-1}}} \leqslant (\mathcal{K}_1 + \mathcal{A}_1)^{\frac{q-1}{2}} \| u_{i,q} \|_{H^1}^{q-1} ,$$

which proves the boundedness of $(u_{i,q})^{q-1}$ in $L^{\frac{2^\star}{2^\star-1}}(M)$. According to [3, Theorem 3.45], $(u_{i,q})^{q-1} \rightharpoonup (u_i)^{\frac{2^\star}{2^\star-1}}$ weakly in $L^{\frac{2^\star}{2^\star-1}}(M)$. Thanks to the embedding $H^1(M) \hookrightarrow L^{2^\star}(M)$, we have $v \in L^{2^\star}(M)$ which also implies $fv \in L^{2^\star}(M)$ since $f$ is smooth. Therefore, by definition of weak convergence, there holds

$$\int_M f (u_{i,q})^{q-1} v \, \mathrm{dvol}_g \to \int_M f (u_i)^{2^\star-1} v \, \mathrm{dvol}_g$$

as $q \to 2^\star$. With these information in hand, we are in a position to send $q \to 2^\star$ in (4.4.8) to get the following

$$\begin{aligned}\int_M \nabla u_i \cdot \nabla v \, \mathrm{dvol}_g + h \int_M u_i v \, \mathrm{dvol}_g \\ - \int_M f (u_i)^{2^\star-1} v \, \mathrm{dvol}_g - \int_M \frac{a}{(u_i)^{2^\star+1}} v \, \mathrm{dvol}_g = 0,\end{aligned} \tag{4.4.9}$$

for any $v \in H^1(M)$. In other words, $u_i$ are weak solutions to (3.0.1). It is not hard to see that the regularity result, Lemma 3.4(b), can be applied to (3.0.1). It follows that $u_i \in C^\infty(M)$ and $u_i > 0$ in $M$. In other words, $u_i$ are smooth positive solutions of (3.0.1). The proof of Claim 3 follows.

### 4.4.2 The existence of the second solution

So far we have just shown that $u_i$ are solutions of (3.0.1). However, we have no information enough to guarantee that these solutions are distinct even that

$$\lim_{q \to 2^\star} \left(F_q^0(u_{1,q}) - F_q^0(u_{2,q})\right) \neq 0.$$

Therefore, we have here only the existence part. In the next subsection we show that $u_i$ are in fact different provided $\sup_M f$ is sufficiently small, thus proving Theorem 4.1.

We wish to compare $F_{2^\star}^0(u_1)$ and $F_{2^\star}^0(u_2)$. Recall that

$$\begin{aligned} F_{2^\star}^0(u_i) = \frac{1}{2}\int_M |\nabla u_i|^2 \, \mathrm{dvol}_g + \frac{h}{2}\int_M (u_i)^2 \, \mathrm{dvol}_g \\ - \frac{1}{2^\star}\int_M f(u_i)^{2^\star} \, \mathrm{dvol}_g + \frac{1}{2^\star}\int_M \frac{a}{(u_i)^{2^\star}} \, \mathrm{dvol}_g. \end{aligned}$$

Here we introduce a trick without using any concentration-compactness principle. This can be done once we can show that $\lim_{q\to 2^\star} F_q^0(u_{i,q}) = F_{2^\star}^0(u_i)$ for $i = 1, 2$. If we carefully look at the formula for $F_q^0(u_{i,q})$, the only difficult part is to show that

$$\int_M f(u_{i,q})^q \, \mathrm{dvol}_g \to \int_M f(u_i)^{2^\star} \, \mathrm{dvol}_g \quad \text{as } q \to 2^\star.$$

In contrast to the previous subsection, the bigger exponents generally make us impossible to guarantee such a convergence. To avoid this difficulty, we have to make $\sup_M f$ sufficiently small. Intuitively, such a small $f$ is equivalent to saying, for example, that $f(u_{i,q})^{q-1}$ behaves exactly the same as $f(u_{i,q})^q$. We first prove that following.

**Proposition 4.5.** *Assume that all conditions in Proposition 4.6 below hold true, then the following $\|\nabla u_{i,q}\|_{L^2} \to \|\nabla u_i\|_{L^2}$ holds as $q \to 2^\star$.*

*Proof.* This is elementary. It suffices to prove that $\nabla u_{i,q} \to \nabla u_i$ strongly in $L^2(M)$. Using (4.4.7) with $v$ replaced by $u_{i,q} - u_i$, we arrive at

$$\begin{aligned} \int_M \nabla u_{i,q} \cdot \nabla(u_{i,q} - u_i) \, \mathrm{dvol}_g + h\int_M u_{i,q}(u_{i,q} - u_i) \, \mathrm{dvol}_g \\ - \int_M f(u_{i,q})^{q-1}(u_{i,q} - u_i) \, \mathrm{dvol}_g - \int_M \frac{a}{(u_{i,q})^{q+1}}(u_{i,q} - u_i) \, \mathrm{dvol}_g = 0. \end{aligned} \tag{4.4.10}$$

From (4.4.10), in order to pass to the limit to get

$$\int_M \nabla u_{i,q} \cdot \nabla(u_{i,q} - u_i) \, \mathrm{dvol}_g \to 0,$$

we still need to estimate

$$\int_M f(u_{i,q})^{q-1}(u_{i,q}-u_i)\,\mathrm{dvol}_g.$$

This can be done once we apply Proposition 4.6 which is assumed here to be true at the moment. Indeed, we just write

$$\begin{aligned}\int_M f(u_{i,q})^{q-1}(u_{i,q}-u_i)\,\mathrm{dvol}_g =& \int_M f(u_{i,q})^q\,\mathrm{dvol}_g - \int_M f(u_i)^{2^\star}\,\mathrm{dvol}_g\\ &-\left(\int_M f(u_{i,q})^{q-1}u_i\,\mathrm{dvol}_g - \int_M f(u_i)^{2^\star}\,\mathrm{dvol}_g\right).\end{aligned}$$

Since $u_i$ are fixed, $(u_{i,q})^{q-1} \rightharpoonup (u_i)^{2^\star-1}$ weakly in $L^{\frac{2^\star}{2^\star-1}}(M)$, and $fu_i \in L^{2^\star}$, we get that

$$\int_M f(u_{i,q})^{q-1}u_i\,\mathrm{dvol}_g \to \int_M f(u_i)^{2^\star}\,\mathrm{dvol}_g$$

as $q \to 2^\star$. Using this the fact that $\nabla u_{i,q} \rightharpoonup \nabla u_i$ weakly in $L^2(M)$, we obtain

$$\int_M |\nabla(u_{i,q}-u_i)|^2\,\mathrm{dvol}_g \to 0$$

as $q \to 2^\star$. In other words, $\nabla u_{i,q} \to \nabla u_i$ strongly in $L^2(M)$. ∎

Now we conclude that $\int_M f(u_{i,q})^q\,\mathrm{dvol}_g \to \int_M f(u_i)^{2^\star}\,\mathrm{dvol}_g$ as $q \to 2^\star$. We prove the following proposition.

**Proposition 4.6.** *We assume that all requirements in Proposition 4.4 are fulfilled. We further assume that $f$ verifies*

$$\sup_M f < \mathcal{C}_2,$$

*where the number $\mathcal{C}_2 > 0$ is given in (4.4.16) below. Then*

$$\int_M f(u_{i,q})^q\,\mathrm{dvol}_g \to \int_M f(u_i)^{2^\star}\,\mathrm{dvol}_g$$

*as $q \to 2^\star$.*

*Proof.* In (4.4.7), we choose $v = (u_{i,q})^{1+2\delta}$ for some $\delta > 0$ to be determined later, we arrive at

$$\begin{aligned}\int_M \nabla u_{i,q}\cdot\nabla(u_{i,q}^{1+2\delta})\,\mathrm{dvol}_g + h\int_M (u_{i,q})^{2+2\delta}\,\mathrm{dvol}_g&\\ -\int_M f(u_{i,q})^{q+2\delta}\,\mathrm{dvol}_g - \int_M \frac{a}{(u_{i,q})^{q-2\delta}}\,\mathrm{dvol}_g &= 0.\end{aligned} \tag{4.4.11}$$

Let $w_{i,q} = (u_{i,q})^{1+\delta}$, Equation (4.4.11) can be rewritten as

$$\begin{aligned}\frac{1+2\delta}{(1+\delta)^2}\int_M |\nabla w_{i,q}|^2\,\mathrm{dvol}_g =& |h|\int_M (w_{i,q})^2\,\mathrm{dvol}_g\\ &+\int_M f(w_{i,q})^2(u_{i,q})^{q-2}\,\mathrm{dvol}_g + \int_M \frac{a}{(u_{i,q})^{q-2\delta}}\,\mathrm{dvol}_g.\end{aligned}$$

This and the Sobolev inequality applied to $w_{i,q}$ tell us that

$$\begin{aligned}\|w_{i,q}\|_{L^{2^\star}}^2 \leqslant &\left(\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}|h|+\mathcal{A}_1\right)\|w_{i,q}\|_{L^2}^2\\ &+\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}\left(\int_M f^+(w_{i,q})^2(u_{i,q})^{q-2}\,\mathrm{dvol}_g+\int_M\frac{a}{(u_{i,q})^{q-2\delta}}\,\mathrm{dvol}_g\right).\end{aligned} \tag{4.4.12}$$

We now use the Hölder inequality one more time

$$\int_M (w_{i,q})^2(u_{i,q})^{q-2}\,\mathrm{dvol}_g \leqslant \left(\int_M (w_{i,q})^{2^\star}\,\mathrm{dvol}_g\right)^{\frac{2}{2^\star}}\left(\int_M (u_{i,q})^{\frac{(q-2)2^\star}{2^\star-2}}\,\mathrm{dvol}_g\right)^{1-\frac{2}{2^\star}}.$$

Notice that $\frac{(q-2)2^\star}{2^\star-2}<q$ so long as $q<2^\star$. Again, by the Hölder and Sobolev inequalities, one gets

$$\begin{aligned}\int_M (u_{i,q})^{\frac{(q-2)2^\star}{2^\star-2}}\,\mathrm{dvol}_g &\leqslant \left(\int_M (u_{i,q})^{2^\star}\,\mathrm{dvol}_g\right)^{\frac{q-2}{2^\star-2}}\\ &\leqslant (\mathcal{K}_1+\mathcal{A}_1)^{\frac{2^\star(q-2)}{2(2^\star-2)}}\,\|u_{i,q}\|_{H^1}^{\frac{2^\star(q-2)}{2^\star-2}}.\end{aligned}$$

Therefore,

$$\int_M (w_{i,q})^2(u_{i,q})^{q-2}\,\mathrm{dvol}_g \leqslant \|w_{i,q}\|_{L^{2^\star}}^2(\mathcal{K}_1+\mathcal{A}_1)^{\frac{q-2}{2}}\,\|u_{i,q}\|_{H^1}^{q-2}.$$

Using (4.4.12) and our calculation above, it is obvious that

$$\begin{aligned}\|w_{i,q}\|_{L^{2^\star}}^2 \leqslant &\left(\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}|h|+\mathcal{A}_1\right)\|w_{i,q}\|_{L^2}^2\\ &+\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}(\sup_M f)(\mathcal{K}_1+\mathcal{A}_1)^{\frac{q-2}{2}}\,\|u_{i,q}\|_{H^1}^{q-2}\,\|w_{i,q}\|_{L^{2^\star}}^2\\ &+\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}\int_M\frac{a}{(u_{i,q})^{q-2\delta}}\,\mathrm{dvol}_g.\end{aligned} \tag{4.4.13}$$

We wish to impose condition of $\sup_M f$ so that

$$\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}(\sup_M f)(1+\mathcal{K}_1+\mathcal{A}_1)^{\frac{2^\star-2}{2}}\Lambda^{2^\star-2}<\frac{1}{2} \tag{4.4.14}$$

fulfills. This can be done for a suitable choice of small $\delta>0$ that will be fixed provided $\sup_M f$ verifies

$$\mathcal{K}_1(\sup_M f)(1+\mathcal{K}_1+\mathcal{A}_1)^{\frac{2^\star-2}{2}}\Lambda^{2^\star-2}<\frac{1}{2}. \tag{4.4.15}$$

Notice that $\Lambda$ also contains $\sup_M f$, therefore a straightforward calculation shows us that it is enough for (4.4.15) to assume $\sup_M f<\mathcal{C}_2$ where

$$\mathcal{C}_2=\min\left\{\frac{1}{2\mathcal{K}_1}(1+\mathcal{K}_1+\mathcal{A}_1)^{-\frac{2^\star-2}{2}}\left(2\mu+k_{\star\star}+(1-h)k_{\star\star}^{\frac{2}{2^\flat}}\right)^{-\frac{2^\star-2}{2}},1\right\}. \tag{4.4.16}$$

In view of (4.4.13), we get from (4.4.14) that

$$\|w_{i,q}\|_{L^{2^\star}}^2 \leqslant 2\left(\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}|h|+\mathcal{A}_1\right)\|w_{i,q}\|_{L^2}^2+2\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}\int_M\frac{a}{(u_{i,q})^{q-2\delta}}\,\mathrm{dvol}_g.$$

By the choice of $\delta$ satisfying (4.4.14) and $1+\delta < \frac{2^\star}{2}$, we can verify that

$$\|w_{i,q}\|_{L^2} = \|(u_{i,q})^{1+\delta}\|_{L^2} = \|u_{i,q}\|_{L^{2(1+\delta)}}^{1+\delta} \leqslant \|u_{i,q}\|_{L^{2^\star}}^{1+\delta}.$$

This and the Sobolev inequality imply that $\|w_{i,q}\|_{L^2}$ can be controlled by some constant depending on $\Lambda$. On the other hand, $\int_M a(u_{i,q})^{-(q-2\delta)}\, d\mathrm{vol}_g$ is bounded from above since $q-2\delta>0$ and $u_{i,q}$ has a strictly positive constant lower bound independent of $q$.

All discussion above shows that $\{\|w_{i,q}\|_{L^{2^\star}}\}_q$ is bounded, that is, $\{\|u_{i,q}\|_{L^{2^\star(1+\delta)}}\}_q$ is bounded. We are now in a position to make use [3, Theorem 3.45]. First, by the Hölder inequality as in (4.4.8), one obtains $\|(u_{i,q})^q\|_{L^{1+\delta}} \leqslant \|u_{i,q}\|_{L^{2^\star(1+\delta)}}^q$, that means $(u_{i,q})^q$ is bounded in $L^{1+\delta}(M)$. This and the fact that $(u_{i,q})^q \to (u_i)^{2^\star}$ almost everywhere in $M$ imply $(u_{i,q})^q \rightharpoonup (u_i)^{2^\star}$ weakly in $L^{1+\delta}(M)$. Therefore, by definition of weak convergence and the fact that $L^{1+\frac{1}{\delta}}(M)$ is the dual space of $L^{1+\delta}(M)$, there holds

$$\int_M f(u_{i,q})^q\, d\mathrm{vol}_g \to \int_M f(u_i)^{2^\star}\, d\mathrm{vol}_g$$

as $q \to 2^\star$ since $f \in L^{1+\frac{1}{\delta}}(M)$. ■

We are now in a position to compare the energy of solutions.

**Proposition 4.7.** *We assume that all requirements in Proposition 4.6 are fulfilled. Then Equation (3.0.1) possesses at least two smooth positive solutions, one has strictly negative energy and the other has positive energy.*

*Proof.* It suffices to compare the energies of $u_i$. Using Propositions 4.5 and 4.6, we send $q \to 2^\star$ in the preceding equalities to reach $\lim_{q\to 2^\star} F_q^0(u_{i,q}) = F_{2^\star}^0(u_i)$, $i=1,2$. In view of (4.2.5), there holds $F_{2^\star}^0(u_1) < 0 \leqslant F_{2^\star}^0(u_2)$. Thus, $u_i$ have different energies. This completes the proof. ■

### 4.4.3 A scaling argument

In this part, we use the scaling technique to complete the proof of Theorem 4.1 by removing the condition (4.2.5) mentioned in Proposition 4.4. We first observe that under the variable change $\widetilde{u} = \frac{u}{c}$, where $c$ is a suitable constant to be determined later, Equation (3.0.1) becomes

$$-\Delta_g \widetilde{u} + h\widetilde{u} = c^{2^\star-2} f \widetilde{u}^{2^\star-1} + \frac{1}{c^{2^\star+2}} \frac{a}{\widetilde{u}^{2^\star+1}}. \tag{4.4.17}$$

We wish to find a suitable constant $c>0$ such that our new coefficients $\widetilde{f}$ and $\widetilde{a}$ verify the conditions in Propositions 4.4 and 4.5 where

$$\widetilde{f} = c^{2^\star-2} f, \quad \widetilde{a} = \frac{a}{c^{2^\star+2}}. \tag{4.4.18}$$

Clearly, once $u$ is a solution of Equation (4.4.17), then $cu$ will solve Equation (3.0.1) accordingly. Obviously, the coefficient $h$ remains unchanged after the scaling and we also have $\lambda_f = \lambda_{\widetilde{f}}$ since $c>0$. In addition, thanks to (4.4.18), the following conditions

$$|h| < \lambda_{\widetilde{f}}, \quad \widetilde{a} > 0, \quad \int_M \widetilde{f}\, \mathrm{dvol}_g < 0, \quad \sup_M \widetilde{f}^+ > 0$$

are also fulfilled. Besides, it is obvious to see that

$$\frac{\sup_M \widetilde{f}}{\int_M |\widetilde{f}^-|\, \mathrm{dvol}_g} = \frac{\sup_M f}{\int_M |f^-|\, \mathrm{dvol}_g}.$$

We now wish to remove (4.2.5) but still keep other conditions. In other words, we choose a suitable $c$ so that the following conditions

$$\frac{2^\star |h|}{2} \leqslant \int_M |\widetilde{f}^-|\, \mathrm{dvol}_g, \tag{4.4.19}$$

and

$$\sup_M \widetilde{f} < \mathcal{C}_2, \tag{4.4.20}$$

and

$$\int_M \widetilde{a}\, \mathrm{dvol}_g < \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1} \left(\frac{|h|}{\int_M |\widetilde{f}^-|\, \mathrm{dvol}_g}\right)^n \int_M |\widetilde{f}^-|\, \mathrm{dvol}_g \tag{4.4.21}$$

hold. Indeed, (4.4.19) and (4.4.21) can be rewritten as the following

$$\frac{2^\star |h|}{2} \leqslant c^{2^\star - 2} \int_M |f^-|\, \mathrm{dvol}_g \tag{4.4.22}$$

and

$$\begin{aligned} \frac{1}{c^{2^\star+2}} \int_M a\, \mathrm{dvol}_g <& \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1} \\ & \left(\frac{|h|}{c^{2^\star-2}\int_M |f^-|\, \mathrm{dvol}_g}\right)^n c^{2^\star-2} \int_M |f^-|\, \mathrm{dvol}_g. \end{aligned} \tag{4.4.23}$$

Notice that $(2^\star - 2)n = 22^\star$, this is about to say that again the right hand side of (4.4.23) can be rewritten as

$$\frac{1}{c^{2^\star+2}} \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1} \left(\frac{|h|}{\int_M |f^-|\, \mathrm{dvol}_g}\right)^n \int_M |f^-|\, \mathrm{dvol}_g.$$

By canceling the factor $\frac{1}{c^{2^\star+2}}$, one can easily see that the condition (4.0.1) is invariant under the variable change. In view of (4.4.22), we can choose

$$c = \left(\frac{2^\star |h|}{2\int_M |f^-|\, \mathrm{dvol}_g}\right)^{\frac{1}{2^\star-2}}.$$

It suffices to prove that this particular choice of $c$ and the condition (4.0.2) are enough to guarantee (4.4.20). Notice that

$$\begin{aligned} \sup_M \widetilde{f} &= c^{2^\star-2} \sup_M f \\ &= (\sup_M f)\left(\frac{2^\star |h|}{2\int_M |f^-|\, \mathrm{dvol}_g}\right) \\ &= \frac{2^\star |h|}{2} \frac{\sup_M f}{\int_M |f^-|\, \mathrm{dvol}_g}. \end{aligned} \tag{4.4.24}$$

Therefore, if we assume

$$\frac{\sup_M f}{\int_M |f^-| \, d\mathrm{vol}_g} < \frac{2}{2^\star |h|} \mathcal{C}_2,$$

then the condition (4.4.20) holds. In conclusion, if the constant $\mathcal{C}$ in the statement of Theorem 4.1 is equal to

$$\min \left\{ \mathcal{C}_1, \frac{2}{2^\star |h|} \mathcal{C}_2 \right\}, \tag{4.4.25}$$

we know that Equation (3.0.1) has at least two positive smooth solutions. This finishes the proof of Theorem 4.1.

*Remark 4.5.* Before finishing the proof of Theorem 4.1, it is important to note that the existence of the constant $\mathcal{C}_1$ depends only on the negative part of $f$ and the set $\{x \in M : f(x) \geqslant 0\}$, thus, is independent of $\sup_M f$. To see this, let us notice from the definition of the sets $\mathcal{A}$ and $\mathcal{A}(\eta, q)$ that $\lambda_f$ and $\lambda_{f,\eta,q}$ depend only on $f^-$. This ensures that the existence of $\mathcal{C}_1$ given by (4.4.1) depends only on $f^-$. Now one can observe that the condition

$$(\sup_M f) \left( \int_M |f^-| \, d\mathrm{vol}_g \right)^{-1} < \mathcal{C}_1$$

actually makes sense and therefore we do have the existence part. However, since the constant $\mathcal{C}_2$ depends on $\mu$ and $k_{\star\star}$, it is hard to check whether or not the condition

$$(\sup_M f) \left( \int_M |f^-| \, d\mathrm{vol}_g \right)^{-1} < \frac{2}{2^\star |h|} \mathcal{C}_2$$

actually holds but we believe that an example for this case exists. We hope that we will soon see some responses on this issue.

## 4.5 The asymptotic behavior of $\mu^{\varepsilon}_{k,q}$ in the case $\sup_M f \leqslant 0$

According to [11, Proposition 4], if we restrict ourselves to $f \leqslant 0$ but not strictly negative, the solvability of (3.0.1), where $h$, $f$, and $a$ take the form (1.2.2), is equivalent to solving the so-called prescribing scalar curvature-scalar field problem

$$-\Delta_g u + hu = f u^{2^\star - 1}. \tag{4.5.1}$$

The proof of this fact depends heavily on the conformal covariance property of all these coefficients, that cannot be true for general $h$, $f$, and $a$.

Concerning (4.5.1), Rauzy provided, among other things, necessary and sufficient conditions for the solvability of (4.5.1) in the general form, that is, for any $f \leqslant 0$ and $h < 0$ a constant. Based on this point, in this section, we prove that there is a natural extension of the Rauzy result for the prescribing scalar curvature equation (4.5.1) to (3.0.1) which also provides for necessary and sufficient conditions for the solvability of (3.0.1). Notice that we have already proved necessary conditions.

As always, we first need to consider the asymptotic behavior of $\mu^{\varepsilon}_{k,q}$ for small $k$ and for large $k$.

For small $k$, since the function $f$ plays no role in the argument used in the proof of Lemma 4.3, we can go through Lemma 4.3 without any difficulty, that is, for small $\varepsilon$, $\mu_{k,q}^{\varepsilon} \to +\infty$ as $k \to 0$ in the sense of Lemma 4.3.

Now we want to study the behavior of $\mu_{k,q}^{\varepsilon}$ for large $k$. As can be seen, if $f$ has zero value somewhere in $M$, then in order to control $\mu_{k,q}^{\varepsilon}$ for large $k$, we must study $\lambda_{f,\eta,q}$. Depending on how large the set $\{f = 0\}$ is, there are two possible cases.

**Proposition 4.8.** *Suppose* $\sup_M f = 0$. *If either*

- $\int_{\{f=0\}} 1 \,\mathrm{dvol}_g = 0$ *or*
- $\int_{\{f=0\}} 1 \,\mathrm{dvol}_g > 0$ *and* $\lambda_f > |h|$

*holds, then* $\mu_{k,q}^{\varepsilon} \to +\infty$ *as* $k \to +\infty$ *for any* $\varepsilon \geqslant 0$ *sufficiently small and any* $q$ *sufficiently close to* $2^\star$ *but all are fixed.*

*Proof.* We begin to prove that there is some $\eta_0 > 0$ sufficiently small and its corresponding $q_{\eta_0} \in [2^\flat, 2^\star)$ sufficiently close to $2^\star$ such that $\delta_0 = \frac{1}{2}(\lambda_{f,\eta_0,q} + h) > 0$ for any $q \in [q_{\eta_0}, 2^\star)$. We consider two cases separately.

**Case 1.** Suppose that $\sup_M f = 0$ and $\int_{\{f=0\}} 1 \, d\mathrm{vol}_g = 0$. Under this case, there holds $f < 0$ almost everywhere which implies that the set $\mathcal{A}$ is empty, therefore $\lambda_f = +\infty$.

Since $h$ is fixed, we know from Lemma 3.12 that we can find some $\eta_0$ sufficiently small and its corresponding $q_{\eta_0} \in [2^\flat, 2^\star)$ such that $\lambda_{f,\eta_0,q} + h \gg 0$ for all $q \in [q_{\eta_0}, 2^\star)$, and thus proving the positivity of $\delta_0$.

**Case 2.** Suppose that $\sup_M f = 0$ and $\int_{\{f=0\}} 1 \,\mathrm{dvol}_g > 0$. Under this case, $\lambda_f$ is well-defined and finite. Notice that $\lambda_f + h > 0$.

As in the proof of Proposition 4.2, there exist some $\eta_0 < 2$ and its corresponding $q_{\eta_0} \in [2^\flat, 2^\star)$ such that $0 \leqslant \lambda_f - \lambda_{f,\eta_0,q} < \frac{1}{4}(\lambda_f - |h|)$ for any $q \in (q_{\eta_0}, 2^\star)$. Therefore, $\delta_0 > \frac{3}{8}(\lambda_f + h)$.

Now having the strictly positivity of $\delta_0$ we can easily go through the proof of Proposition 4.2, hence we get $G_q(u) \geqslant mk^{\frac{2}{q}}$ where $m$ is given as in (3.2.5) which implies that $F_q^{\varepsilon}(u) \geqslant mk^{\frac{2}{q}}$ due to $\sup_M f = 0$. Since $\delta_0$ has a strictly positive lower bound, so does $m$. The proof now follows easily. ∎

## 4.6 Proof of Theorem 4.2

From now on, we restrict ourselves to the case $q \in [q_{\eta_0}, 2^\star)$. Let us first do some calculation. Since

$$F_q^{\varepsilon}(k^{\frac{1}{q}}) = \frac{h}{2}k^{\frac{2}{q}} - \frac{k}{q}\int_M f \,\mathrm{dvol}_g + \frac{1}{q}\frac{1}{(k^{\frac{2}{q}} + \varepsilon)^{\frac{q}{2}}}\int_M a \,\mathrm{dvol}_g,$$

we know by solving

$$\frac{h}{2}k^{\frac{2}{q}} - \frac{k}{q}\int_M f \,\mathrm{dvol}_g = 0$$

that
$$\mu^{\varepsilon}_{k_0,q} = F^{\varepsilon}_q(k_0^{\frac{1}{q}}) \leqslant \frac{1}{q}\frac{1}{(k_0^{\frac{2}{q}}+\varepsilon)^{\frac{q}{2}}}\int_M a\ \mathrm{dvol}_g < \frac{1}{2k_0}\int_M a\ \mathrm{dvol}_g$$
where
$$k_0 = \left(\frac{q}{2}\frac{h}{\int_M f\ \mathrm{dvol}_g}\right)^{\frac{q}{q-2}}.$$
Having $k_0$, it is easy to find upper and lower bounds for $k_0$. For example, one can check that with
$$k_1 = \min\left\{\left(\frac{h}{\int_M f\ \mathrm{dvol}_g}\right)^{n-1}, \left(\frac{h}{\int_M f\ \mathrm{dvol}_g}\right)^{\frac{2^\star}{2^\star-2}}\right\}$$
and
$$k_2 = \left(\frac{2^\star}{2}\frac{h}{\int_M f\ \mathrm{dvol}_g}+1\right)^{n-1},$$
we immediately have $k_1 < k_0 < k_2$. Consequently, we can bound $\mu^{\varepsilon}_{k_0,q}$ from above as the following
$$\mu^{\varepsilon}_{k_0,q} < \frac{1}{2k_1}\int_M a\ \mathrm{dvol}_g.$$
It is important to note that in the proof of Theorem 4.1, we bound $\mu^{\varepsilon}_{k_0,q}$ from above by 0, see Lemma 4.5. Since $k_1$ and $k_2$ are independent of both $\varepsilon$ and $q$, this new bound for $\mu^{\varepsilon}_{k_0,q}$ is also independent of both $\varepsilon$ and $q$.

We are now in a position to prove Theorem 4.2 whose proof is similar to the proof of Theorem 4.1, therefore we just sketch it and omit in details.

**Proposition 4.9.** *If $|h| < \lambda_f$ and $\sup_M f = 0$, then Equation (3.0.1) admits a unique positive solution $u$.*

*Sketch of proof.* Since the uniqueness part comes from Lemma 3.6, it suffices to study the existence part. We have to prove the existence of $k_\star$ and $k_{\star\star}$ independent of $\varepsilon$ and $q$ with
$$k_\star < k_1 < k_0 < k_2 < k_{\star\star}$$
such that
$$\frac{1}{2k_1}\int_M a\ \mathrm{dvol}_g < \min\left\{\mu^{\varepsilon}_{k_\star,q}, \mu^{\varepsilon}_{k_{\star\star},q}\right\},$$
for any $\varepsilon \leqslant k_\star$. The existence of such a $k_{\star\star} > k_2$ is clear since $F^{\varepsilon}_q(u) \geqslant \frac{h}{2}k^{\frac{n-2}{n-1}}$ for any $u \in \mathcal{B}_{k,q}$, any $\varepsilon > 0$, and any $k > 1$. For the existence of $k_\star < k_1$, using $h < 0$ and $\frac{2}{q} \in (1, \frac{n-2}{n-1})$, there holds
$$\begin{aligned}F^{\varepsilon}_q(u) &\geqslant \frac{h}{2}k^{\frac{2}{q}} + \frac{k}{q}|\sup_M f| + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\ \mathrm{dvol}_g\\ &\geqslant \frac{hk}{2} + \frac{1}{2^\star}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\ \mathrm{dvol}_g\end{aligned}$$
for any $u \in \mathcal{B}_{k,q}$ and any $0 < k < 1$. Thanks to the proof of Lemma 4.3, for $\varepsilon \leqslant k^{\frac{2}{q}}$ we know that

$$\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\,\mathrm{dvol}_g \geqslant \frac{1}{2^{\frac{2^\star}{2}}k}\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2.$$

Therefore if we choose $k_\star < \min\{1, k_1\}$ in such a way that

$$\frac{1}{2}\frac{1}{2^\star}\frac{1}{2^{\frac{2^\star}{2}}k_\star}\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2 > \max\left\{\frac{1}{2k_1}\int_M a\,\mathrm{dvol}_g, \frac{|h|k_\star}{2}\right\}$$

we then have

$$F_q^\varepsilon(u) \geqslant \frac{1}{2k_1}\int_M a\,\mathrm{dvol}_g$$

for any $u \in \mathcal{B}_{k_\star,q}$. Taking infimum over $B_{k_\star,q}$ we obtain $\mu^\varepsilon_{k_\star,q} > \mu^\varepsilon_{k0,q}$ for any $\varepsilon \leqslant k_\star$ as claimed. Having the existence of $k_\star$ and $k_{\star\star}$ independent of $\varepsilon$ and $q$ we can define

$$\mu^\varepsilon_{k_1,q} = \inf_{u\in\mathcal{D}_{k,q}} F_q^\varepsilon(u)$$

for each $\varepsilon$ and $q$ fixed where

$$\mathcal{D}_{k,q} = \left\{u \in H^1(M) : k_\star \leqslant \|u\|^q_{L^q} \leqslant k_{\star\star}\right\}.$$

It then turns out that $\mu^\varepsilon_{k_1,q}$ is achieved by a smooth positive function $u^\varepsilon_q$ which is exactly the smooth solution to (3.0.2). Since $\|u^\varepsilon_q\|_{L^q}$ is uniformly bounded, by using a sequence $\{\varepsilon_j\}_j$ of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$ we can prove, up to subsequences, that $u_q^{\varepsilon_j} \rightharpoonup u_q$ in $H^1(M)$ as $j \to \infty$. We then show that $u_q$ is smooth positive solution to (3.0.2) with $\varepsilon$ replaced by 0. Finally, we send $q \to 2^\star$ and do the same argument to claim that (3.0.1) admits a smooth positive solution. ■

In order to make the work unique, let us mention here the case $\sup_M f < 0$ although this has been done in [11] by using the method of sub- and super-solutions. Suppose $\sup_M f < 0$. It suffices to study the asymptotic behavior of $\mu^\varepsilon_{k,q}$ for large $k$. Clearly, for any $u \in \mathcal{B}_{k,q}$,

$$F_q^\varepsilon(u) \geqslant \left(\frac{h}{2} + \frac{1}{2^\star}k^{1-\frac{2}{q}}|\sup_M f|\right)k^{\frac{2}{q}}.$$

It is then immediate to deduce that $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to +\infty$ since $1 - \frac{2}{q} > 0$. Hence we can easily prove the existence of at least one positive smooth solution to (3.0.1). More precisely, we prove

**Proposition 4.10.** *If $\sup_M f < 0$, then Equation (3.0.1) admits a unique positive smooth solution $u$.*

*Sketch of proof.* First, the uniqueness part comes from Lemma 3.6. For the existence part, the proof is similar to the proof of Proposition 4.9. The way to find $k_\star$ is exactly the same as in the proof of Proposition 4.9. The existence of $k_{\star\star}$ can be found as in the proof of Proposition 4.9 since there still holds $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to +\infty$. Having the existence of $k_\star$ and $k_{\star\star}$ independent of $\varepsilon$ and $q$ we can go through the proof of Proposition 4.9 to reach the existence of a smooth solution to our equation (3.0.1). ■

# 5

# The Lichnerowicz equations in the null Yamabe-scalar field conformal invariant



In this chapter, we continue to study some quantitative properties of positive, smooth solutions of the following Einstein-scalar field Lichnerowicz equation (3.0.1) in the null Yamabe-scalar field conformal invariant, that is the case $h = 0$.

In the previous chapter, we have already proven that, in the case $h < 0$, a suitable balance between coefficients $h$, $f$, $a$ of the Einstein-scalar field Lichnerowicz equations is enough to guarantee the existence of one positive smooth solution. In addition, it was found that under some further conditions we may or we may not have the uniqueness property of solutions of the Einstein-scalar field Lichnerowicz equations. This chapter is a continuation of the previous chapter where we consider the case when $h = 0$ which was also left as open question in the classification of [11], that is, we are interested in the following simple partial differential equation

$$-\Delta_g u = f u^{2^\star - 1} + \frac{a}{u^{2^\star + 1}}, \quad u > 0. \tag{5.0.1}$$

As always, we assume hereafter that $f$ and $a \geqslant 0$ are smooth functions on $M$ with $\int_M a \, d\mathrm{vol}_g > 0$. In addition, the manifold $M$ has unit volume. Besides, it is worth recalling that in this case, the condition $\int_M f \, d\mathrm{vol}_g < 0$ is necessary as we have proven before.

Following the subcritical approach, as a first step to tackle (5.0.1), we look for positive smooth solutions of the following subcritical problem

$$-\Delta_g u = f|u|^{q-2}u + \frac{au}{(u^2+\varepsilon)^{\frac{q}{2}+1}}. \tag{5.0.2}$$

Again, our main procedure is to show that the limit exists first as $\varepsilon \to 0$ and then as $q \to 2^\star$ under various assumptions.

The content of this chapter consists of three main parts. In the first part of the chapter, we mainly consider the case $\sup_M f > 0$. In this context, we was able to show that if $\sup_M f$ and $\int_M a \, d\mathrm{vol}_g$ are small, then (5.0.1) possesses at least one smooth positive solutions. The first main theorem can be stated as follows.

**Theorem 5.1.** *Let $(M,g)$ be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Assume that $f$ and $a \geqslant 0$ are smooth functions on $M$ such that $\int_M a \, d\mathrm{vol}_g > 0$, $\int_M f \, d\mathrm{vol}_g < 0$, and $\sup_M f > 0$. Then there exist two positive numbers $\eta_0$ and $\lambda$ depending only on the negative part $f^-$ of $f$ such that if*

$$\sup_M f \leqslant \frac{\eta_0}{2}\int_M |f^-| \, d\mathrm{vol}_g \tag{5.0.3}$$

*and*

$$\int_M a \, d\mathrm{vol}_g < \frac{\lambda^n}{4(\eta_0)^{n-2}}\left(\frac{2n}{n-2}\right)^{n-2}\left(\int_M |f^-| \, d\mathrm{vol}_g\right)^{1-n} \tag{5.0.4}$$

*hold, then (5.0.1) possesses at least one smooth positive solution.*

By a simple comparison, one can easily see that except the multiplicity part, we was successful to carry the conclusion of Theorem 4.1 for the case $h < 0$ to the case $h = 0$. However, since the Lichnerowicz equations in the case $h = 0$ take a form simpler than that of the case $h < 0$, we might expect that the above two conditions (5.0.3) and (5.0.4) could be weakened. Surprisingly, we was able to prove that the condition (5.0.3) can be relaxed. Unfortunately, for the price we pay, the estimate for $\int_M a \, d\mathrm{vol}_g$ needs to be replaced by another estimate for $\sup_M a$. This is the content of our second result.

**Theorem 5.2.** *Let $(M,g)$ be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Assume that $f$ and $a \geqslant 0$ are smooth functions on $M$ such that $\int_M a \, d\mathrm{vol}_g > 0$, $\int_M f \, d\mathrm{vol}_g < 0$, and $\sup_M f > 0$. Then if $a$ is small in the sense of (5.4.1) below, then (5.0.1) possesses at least one smooth positive solution.*

In the last part of the chapter, we focus our attention to the case $\sup_M f \leqslant 0$. It should mention that in the statement of Theorem 5.1, $\sup_M f$ is nothing but $\sup_M f^+$ where $f^+$ is the positive part of $f$. Therefore, if we assume $f \leqslant 0$, we then see that the condition (5.0.3) is fulfilled for any small $\eta_0$. However, one can immediately observe that the right hand side of (5.0.4) goes to $+\infty$ as $\eta_0 \to 0$. This suggests that under the case $\sup_M f \leqslant 0$, there is no other condition for $\int_M a \, d\mathrm{vol}_g$ than $\int_M a \, d\mathrm{vol}_g > 0$. That is the content of our next result.

**Theorem 5.3.** *Let $(M,g)$ be a smooth compact Riemannian manifold without boundary of dimension $n \geqslant 3$. Let $f$ and $a$ be smooth functions on $M$ with $a \geqslant 0$ in $M$, $\int_M a \, d\mathrm{vol}_g > 0$, and $f \leqslant 0$. Then Equation (5.0.1) always possesses one positive solution. In addition, this solution is unique.*

Concerning Theorem 5.3, it is worth noticing that it generalizes the same result obtained in [11] when our equation takes the form (1.2.10). Roughly speaking, it was proved in [11] by the method of sub- and super-solutions that (5.0.1) always possesses one positive solution so long as the functions $f$ and $a$ take the form $f = -\mathcal{B}_{\tau,\psi}$ and $a = \mathcal{A}_{g,W,\pi}$ with $f \leqslant 0$ and $a \geqslant 0$. The main ingredient of the proof in [11] is the conformal invariant property of $\mathcal{B}_{\tau,\psi}$ and $\mathcal{A}_{g,W,\pi}$. Apparently, this property is no longer available in our general case.

Before doing so and thanks to $\lambda_f > 0$, we may denote the following

$$\lambda = \begin{cases} \frac{1}{2}\left(\mathscr{K}_1 + \frac{2\mathscr{A}_1}{\lambda_f}\right)^{-1}, & \text{if } \lambda_f < +\infty, \\ \frac{1}{2}\left(\mathscr{K}_1 + \mathscr{A}_1\right)^{-1}, & \text{if } \lambda_f = +\infty. \end{cases}$$

In view of Lemmas 3.13 and 3.14, there exist two numbers $\eta_0 \in (0,1)$ and $q_{\eta_0} \in [2^\flat, 2^\star)$ so that the following estimate

$$\lambda_{f,\eta_0,q} \geqslant \begin{cases} \frac{\lambda_f}{2}, & \text{if } \lambda_f < +\infty, \\ 1, & \text{if } \lambda_f = +\infty, \end{cases} \tag{5.0.5}$$

holds for every $q \in (q_{\eta_0}, 2^\star)$. In addition, in the case $\lambda_f = +\infty$, we may assume that

$$\eta_0 < \frac{2n}{n-2}\left(\frac{\lambda^n}{4}\right)^{\frac{1}{n-2}}\left(\int_M a \, d\mathrm{vol}_g\right)^{\frac{1}{2-n}}\left(\int_M |f^-| \, d\mathrm{vol}_g\right)^{\frac{1-n}{n-2}}$$

since we may take $\eta_0$ as small as we wish. This choice of $\eta_0$ is equivalent to saying that the condition (4.0.1) is fulfilled. It is important to note that, in the case $\sup f > 0$, the number $\eta_0$ depends only on the negative part $f^-$ of $f$. Unless otherwise stated, from now on, we fix such an $\eta_0$ and we only consider $q \in [q_{\eta_0}, 2^\star)$. Notice that, from (5.0.5) and the choice of $\lambda$, we always have

$$\frac{1}{2}\left(\mathscr{K}_1 + \frac{\mathscr{A}_1}{\lambda_{f,\eta_0,q}}\right)^{-1} \geqslant \lambda \tag{5.0.6}$$

for all $q \in (q_{\eta_0}, 2^\star)$. Finally, we let

$$k_{1,q} = \frac{\eta_0}{2q}\left(\frac{\lambda q}{\eta_0 \int_M |f^-| \, d\mathrm{vol}_g}\right)^{\frac{q}{q-2}}, \quad k_{2,q} = \left(\frac{\lambda q}{\eta_0 \int_M |f^-| \, d\mathrm{vol}_g}\right)^{\frac{q}{q-2}}. \tag{5.0.7}$$

From the choice of $\eta_0$, one can see that $k_0 < k_1$ for any $q \in [2^\flat, 2^\star)$. In addition, one can easily bound $k_0$ from below and $k_1$ from above, that is, there exists two positive numbers $\underline{k} < 1$ and $\overline{k} > 1$ independent of $q$ and $\varepsilon$ such that $\underline{k} \leqslant k_0 < k_1 \leqslant \overline{k}$, for example, one can choose

$$\underline{k} = \frac{\eta_0}{2} \min\left\{\left(\frac{\lambda}{\eta_0 \int_M |f^-| \, d\mathrm{vol}_g}\right)^{\frac{2^\flat}{2^\flat - 2}}, 1\right\} \tag{5.0.8}$$

and

$$\overline{k} = \left(\frac{2n}{n-2}\right)^{\frac{2^\flat}{2^\flat - 2}} \max\left\{\left(\frac{\lambda}{\eta_0 \int_M |f^-| \, d\mathrm{vol}_g}\right)^{\frac{2^\flat}{2^\flat - 2}}, 1\right\}. \tag{5.0.9}$$

## 5.1 The analysis of the energy functionals

Our aim was to derive a suitable analysis of the energy functionals associated to our problem. Thanks to what we have already built up in the previous chapter, we shall not mention our argument in details in several parts of the current study.

### 5.1.1 The functional setting

As always, for each $q \in (2, 2^\star)$ and $k > 0$, we still use the hyper-surface $\mathcal{B}_{k,q}$ of $H^1(M)$ which is defined in (4.1.1). Again, the set $\mathcal{B}_{k,q}$ is non-empty. Concerning the energy functional associated to problem (5.0.2), we can continue to use that of the previous chapter, that is,

$$F_q^\varepsilon(u) = \frac{1}{2}\int_M |\nabla u|^2 \, \mathrm{dvol}_g - \frac{1}{q}\int_M f|u|^q \, \mathrm{dvol}_g + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} \, \mathrm{dvol}_g,$$

for $\varepsilon > 0$ small satisfying (3.1.3). There is no doubt to claim that $F_q^\varepsilon$ is differentiable on $H^1(M)$ and $F_q^\varepsilon|_{\mathcal{B}_{k,q}}$ is bounded from below by $-k|\sup f|$. As such, we can define

$$\mu_{k,q}^\varepsilon = \inf_{u \in \mathcal{B}_{k,q}} F_q^\varepsilon(u).$$

Since critical points of $F_q^\varepsilon$ are weak solutions of (5.0.2), we wish to find critical points of the functional $F_q^\varepsilon$.

### 5.1.2 $\mu_{k,q}^\varepsilon$ is achieved

Recall that, it was already proved in the previous chapter that in the case $h < 0$, the corresponding $\mu_{k,q}^\varepsilon$ is achieved by smooth positive function $u_\varepsilon$. In the present case, $h = 0$, the above proof still works since the energy functional is simpler than that of the case $h < 0$. Therefore, we omit any proof here.

### 5.1.3 The continuity of $\mu_{k,q}^\varepsilon$ with respect to $k$

Once again, as we have already shown in the previous chapter, in the case $h < 0$, the corresponding function $\mu_{k,q}^\varepsilon$ is continuous with respect to $k$ for each $\varepsilon$ fixed. Since the energy functional of the case $h = 0$ is simpler than that of the case $h < 0$, we conclude that $\mu_{k,q}^\varepsilon$ is also continuous with respect to $k$ in the case $h = 0$.

## 5.2 Asymptotic behavior of $\mu_{k,q}^\varepsilon$ in the case $\sup_M f > 0$

Following the same procedure as in the previous chapter, in this subsection, we investigate the behavior of $\mu_{k,q}^\varepsilon$ when both $k$ and $\varepsilon$ vary. We first study the behavior of $\mu_{k,q}^\varepsilon$ as $k \to +\infty$. We notice that our argument here is not new since the loss of $h$ does not affect the behavior of $\mu_{k,q}^\varepsilon$ for large $k$. Following the proof of Lemma 4.4, we can easily prove the following result.

**Lemma 5.1.** *There holds* $\mu_{k,q}^\varepsilon \to -\infty$ *as* $k \to +\infty$ *if* $\sup_M f > 0$.

Now we are going to show that $\mu^\varepsilon_{k_{1,q},q} < \mu^\varepsilon_{k_{2,q},q}$ where $k_{1,q}$ and $k_{2,q}$ are given in (5.0.7). To serve the purpose better, we first need a rough estimate for $\mu^\varepsilon_{k_{1,q},q}$.

**Lemma 5.2.** *There holds*

$$\mu^\varepsilon_{k_{1,q},q} \leqslant -\frac{k_{1,q}}{q}\int_M f \,\mathrm{dvol}_g + \frac{1}{qk_{1,q}}\int_M a \,\mathrm{dvol}_g \tag{5.2.1}$$

*where $k_{1,q}$ is given in (5.0.7).*

*Proof.* This is trivial since $\mu^\varepsilon_{k_{1,q},q} \leqslant F^\varepsilon_q(k_{1,q}^{\frac{1}{q}})$. The proof follows. ∎

As can be seen, the right hand side of (5.2.1) is always positive. In order to make $\mu^\varepsilon_{k_{2,q},q} > \mu^\varepsilon_{k_{1,q},q}$ with $k_{2,q} > k_{1,q}$, we need $\sup_M f$ to be small. We now study the asymptotic behavior of $\mu^\varepsilon_{k,q}$ as $k \to 0$. This result together with Lemmas 5.1 and 5.4 give us a full picture of the asymptotic behavior of $\mu^\varepsilon_{k,q}$.

**Lemma 5.3.** *There holds $\lim_{k\to 0+} \mu^{k^{\frac{2}{q}}}_{k,q} = +\infty$. In particular, there is some $k_\star$ sufficiently small and independent of both $q$ and $\varepsilon$ such that*

$$\mu^\varepsilon_{k_\star,q} \geqslant -\overline{k}\int_M f \,\mathrm{dvol}_g + \frac{1}{\underline{k}}\int_M a \,\mathrm{dvol}_g$$

*for any $\varepsilon \leqslant k_\star$. In particular, there holds $\mu^\varepsilon_{k_\star,q} > \mu^\varepsilon_{k_{1,q},q}$.*

*Proof.* The role that $\varepsilon$ plays immediately shows that $\mu^\varepsilon_{k,q}$ is strictly monotone decreasing in $\varepsilon$ for fixed $k$ and $q$. Following the proof of Lemma 4.3, for any $\varepsilon \leqslant k^{\frac{2}{q}}$ and $1 < \frac{q}{2} < \frac{2^\star}{2}$, we can estimate the integral involving $a$. In fact, for any $u \in \mathcal{B}_{k,q}$, we have

$$\int_M \sqrt{a} \,\mathrm{dvol}_g \leqslant 2^{\frac{q}{4}}\sqrt{k}\left(\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} \,\mathrm{dvol}_g\right)^{\frac{1}{2}}. \tag{5.2.2}$$

Squaring (5.2.2), we get

$$\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} \,\mathrm{dvol}_g \geqslant \frac{1}{2^{\frac{2^\star}{2}}k}\left(\int_M \sqrt{a} \,\mathrm{dvol}_g\right)^2.$$

This helps us to conclude

$$F^\varepsilon_q(u) \geqslant -\frac{k}{q}\sup_M f + \frac{1}{2^{\frac{2^\star}{2}}qk}\left(\int_M \sqrt{a} \,\mathrm{dvol}_g\right)^2.$$

Consequently, there holds $\mu^{k^{\frac{2}{q}}}_{k,q} \to +\infty$ as $k \to 0$. It is a simple task to find some small $k_\star < 1$ independent of both $q$ and $\varepsilon$ such that

$$-\frac{k_\star}{q}\sup_M f + \frac{1}{2^{\frac{2^\star}{2}}qk_\star}\left(\int_M \sqrt{a} \,\mathrm{dvol}_g\right)^2 \geqslant -\overline{k}\int_M f \,\mathrm{dvol}_g + \frac{1}{\underline{k}}\int_M a \,\mathrm{dvol}_g. \tag{5.2.3}$$

In order to find such a $k_\star$, we first let $k_\star < 1$. Since $q > 2$, it suffices to select $k_\star$ such that

$$\frac{1}{2^{\frac{2^\star}{2}}2^\star k_\star}\left(\int_M \sqrt{a}\, \mathrm{dvol}_g\right)^2 \geqslant -\overline{k}\int_M f\, \mathrm{dvol}_g + \frac{1}{\underline{k}}\int_M a\, \mathrm{dvol}_g + \frac{1}{2}\sup_M f.$$

Hence, one can choose $k_\star$ as

$$\begin{aligned} k_\star = \min\Bigg\{ &\frac{1}{2^{\frac{2^\star}{2}}2^\star}\left(\int_M \sqrt{a}\, \mathrm{dvol}_g\right)^2 \\ &\left(-\overline{k}\int_M f\, \mathrm{dvol}_g + \frac{1}{\underline{k}}\int_M a\, \mathrm{dvol}_g + \frac{1}{2}\sup_M f\right)^{-1}, \underline{k}, 1\Bigg\}. \end{aligned} \tag{5.2.4}$$

Since $k_\star \leqslant 1$, we always have $k_\star \leqslant k_\star^{\frac{2}{q}}$. Besides, using $\int_M a\, \mathrm{dvol}_g > 0$, $\int_M f\, \mathrm{dvol}_g < 0$, and Lemma 5.1, we can easily check that $\mu^\varepsilon_{k_\star,q} > \mu^\varepsilon_{k_{1,q},q}$, thus concluding the lemma with $\varepsilon \leqslant k_\star$. ■

Notice that, in the proof of Lemma 5.3, we have used $\underline{k}$ in the formula for $k_\star$. The reason is that we wish to ensure that $k_\star < k_{1,q}$ in any case.

In the next result, we conclude that the function $\mu^\varepsilon_{k,q}$ is continuous with respect to $k$ for each $\varepsilon$ fixed. Since a similar result has been proved before, we omit its proof.

**Proposition 5.1.** *For $\varepsilon > 0$ fixed, the function $\mu^\varepsilon_{k,q}$ is continuous with respect to $k$.*

With the information of $\lambda_{f,\eta,q}$ that we have already discussed above, let us go back to our energy functional. In the rest of this section, our aim here was to study $\mu^\varepsilon_{k,q}$ for $k \geqslant k_{1,q}$, in particular, $\mu^\varepsilon_{k_{1,q},q} < \mu^\varepsilon_{k_{2,q},q}$ provided $\sup_M f$ is sufficiently small. To this end, we need to estimate $\mu^\varepsilon_{k,q}$ for $k \geqslant k_{1,q}$.

**Proposition 5.2.** *There exists two numbers $\eta_0 > 0$ sufficiently small and its corresponding $q_{\eta_0}$ sufficiently close to $2^\star$ such that the estimate (5.0.5) holds for every $q \in [q_{\eta_0}, 2^\star)$. Having the existence of both $\eta_0$ and $q_{\eta_0}$, for any $u \in \mathcal{B}_{k,q}$ with $k \geqslant k_{2,q}$, any $q \in [q_{\eta_0}, 2^\star)$, and any $\varepsilon > 0$, there holds*

$$F^\varepsilon_q(u) \geqslant \lambda k^{\frac{2}{q}} - \frac{k}{q}\sup_M f.$$

*In particular, for any $u \in \mathcal{B}_{k_{2,q},q}$, there holds*

$$F^\varepsilon_q(u) \geqslant \frac{\lambda}{2}(k_{2,q})^{\frac{2}{q}}$$

*provided*

$$\frac{\sup_M f}{\int_M |f^-|\, \mathrm{dvol}_g} \leqslant \frac{\eta_0}{2}. \tag{5.2.5}$$

*Proof.* Suppose $u \in \mathcal{B}_{k,q}$ where $k \geqslant k_{2,q}$ is arbitrary. We now estimate $F^\varepsilon_q(u)$. We first write

$$F^\varepsilon_q(u) = G_q(u) - \frac{1}{q}\int_M f^+|u|^q\, \mathrm{dvol}_g + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\, \mathrm{dvol}_g,$$

where

$$G_q(u) = \frac{1}{2}\int_M |\nabla u|^2\, \mathrm{dvol}_g + \frac{1}{q}\int_M |f^-||u|^q\, \mathrm{dvol}_g.$$

We first estimate $G_q$ from below. Then there are two possible cases.

**Case 1.** Assume that

$$\int_M |f^-||u|^q \, d\mathrm{vol}_g \geqslant \eta_0 k \int_M |f^-| \, d\mathrm{vol}_g.$$

In this case, the term $G_q$ can be estimated from below as follows

$$G_q(u) \geqslant \frac{\eta_0 k}{q} \int_M |f^-| \, d\mathrm{vol}_g \geqslant \lambda k^{\frac{2}{q}} \tag{5.2.6}$$

where in the last inequality we have used the fact that $k \geqslant k_{2,q}$ and (5.0.7).

**Case 2.** Assume that

$$\int_M |f^-||u|^q \, d\mathrm{vol}_g < \eta_0 k \int_M |f^-| \, d\mathrm{vol}_g.$$

Under this condition, one can easily check that $k^{-\frac{1}{q}} u \in \mathcal{A}'(\eta_0, q)$ which implies that $\|\nabla u\|^2_{L^2} \|u\|^{-2}_{L^2} \geqslant \lambda_{f,\eta_0,q}$ by the definition of $\lambda_{f,\eta_0,q}$. Using this and the estimate $\mathcal{K}_1 \|\nabla u\|^2_{L^2} + \mathcal{A}_1 \|u\|^2_{L^2} \geqslant k^{\frac{2}{q}}$, we get that

$$\left( \mathcal{K}_1 + \frac{\mathcal{A}_1}{\lambda_{f,\eta_0,q}} \right) \|\nabla u\|^2_{L^2} \geqslant \mathcal{K}_1 \|\nabla u\|^2_{L^2} + \mathcal{A}_1 \|u\|^2_{L^2} \geqslant k^{\frac{2}{q}}.$$

Thus, from the definition of $\lambda$ and thanks to $q \in [q_{\eta_0}, 2^\star)$, we obtain

$$\begin{aligned} G_q(u) &\geqslant \frac{1}{2} \int_M |\nabla u|^2 \, d\mathrm{vol}_g \\ &\geqslant \frac{1}{2} \left( \mathcal{K}_1 + \frac{\mathcal{A}_1}{\lambda_{f,\eta_0,q}} \right)^{-1} k^{\frac{2}{q}} \\ &\geqslant \lambda k^{\frac{2}{q}}. \end{aligned} \tag{5.2.7}$$

It now follows from (5.2.6)-(5.2.7) that $G_q(u) \geqslant \lambda k^{\frac{2}{q}}$. Therefore, we can estimate $F^{\varepsilon}_q(u)$ as follows

$$F^{\varepsilon}_q(u) \geqslant \lambda k^{\frac{2}{q}} - \frac{k}{q} \sup_M f$$

for any $u \in \mathcal{B}_{k,q}$. In particular, for any $u \in \mathcal{B}_{k_{2,q},q}$, there holds

$$F^{\varepsilon}_q(u) \geqslant \lambda (k_{2,q})^{\frac{2}{q}} - \frac{k_{2,q}}{q} \sup_M f.$$

Thus, we obtain $F^{\varepsilon}_q(u) \geqslant \frac{\lambda}{2} (k_{2,q})^{\frac{2}{q}}$ for any $u \in \mathcal{B}_{k_{2,q},q}$ provided $\frac{k_{2,q}}{q} \sup_M f \leqslant \frac{\lambda}{2} (k_{2,q})^{\frac{2}{q}}$ which is equivalent to the requirement that $\sup_M f \leqslant \frac{\eta_0}{2} \int_M |f^-| \, d\mathrm{vol}_g$. The proof is complete. ■

*Remark 5.6.* Unlike the case of the positive Yamabe-scalar field invariant, see [21], the case of the non-positive Yamabe-scalar field invariant requires a control of $\sup_M f$. This is basically due to the fact that we probably loss information corresponding to the $L^2$-norm. Indeed, when $h \leqslant 0$, the following

$$\left( \int_M |\nabla u|^2 \, d\mathrm{vol}_g + \int_M h u^2 \, d\mathrm{vol}_g \right)^{\frac{1}{2}}$$

is not an equivalent norm of $H^1(M)$.

**Lemma 5.4.** *Assume that* (5.2.5) *and that*

$$\int_M a \, \mathrm{dvol}_g < \frac{\lambda \eta_0}{4q} \left( \frac{\lambda q}{\eta_0 \int_M |f^-| \, \mathrm{dvol}_g} \right)^{\frac{q+2}{q-2}}. \tag{5.2.8}$$

*Then, there holds*

$$\mu^\varepsilon_{k_{1,q},q} < \min\{\mu^\varepsilon_{k_\star,q}, \mu^\varepsilon_{k_{2,q},q}\}$$

*for any* $q \in [q_{\eta_0}, 2^\star)$ *and any* $\varepsilon \in (0, k_\star)$.

*Proof.* First, in view of Lemma 5.3, it suffices to prove $\mu^\varepsilon_{k_{1,q},q} < \mu^\varepsilon_{k_{2,q},q}$ for all $q \in [q_{\eta_0}, 2^\star)$ and $\varepsilon \leq k_\star$. First, by Lemma 5.2 and Proposition 5.2, we have the following facts

$$\mu^\varepsilon_{k_{1,q},q} < \frac{k_{1,q}}{2} \int_M |f^-| \, \mathrm{dvol}_g + \frac{1}{2k_{1,q}} \int_M a \, \mathrm{dvol}_g$$

and

$$\frac{\lambda}{2} (k_{2,q})^{\frac{2}{q}} < \mu^\varepsilon_{k_{2,q},q}.$$

Therefore, it suffices to prove for any $q \in [q_{\eta_0}, 2^\star)$ that

$$k_{1,q} \int_M |f^-| \, \mathrm{dvol}_g + \frac{1}{k_{1,q}} \int_M a \, \mathrm{dvol}_g \leqslant \lambda (k_{2,q})^{\frac{2}{q}},$$

or equivalently, for any $q \in [q_{\eta_0}, 2^\star)$, there holds

$$\int_M a \, \mathrm{dvol}_g \leqslant -(k_{1,q})^2 \int_M |f^-| \, \mathrm{dvol}_g + \lambda (k_{2,q})^{\frac{2}{q}} k_{1,q}. \tag{5.2.9}$$

From the choice of $k_{1,q}$ and $k_{2,q}$, it is clear to see that

$$-(k_{1,q})^2 \int_M |f^-| \, \mathrm{dvol}_g + \lambda (k_{2,q})^{\frac{2}{q}} k_{1,q} = \frac{\lambda \eta_0}{4q} \left( \frac{\lambda q}{\eta_0 \int_M |f^-| \, \mathrm{dvol}_g} \right)^{\frac{q+2}{q-2}}. \tag{5.2.10}$$

The proof follows easily by comparing (5.2.8), (5.2.9), and (5.2.10). ∎

## 5.3 Proof of Theorem 5.1

In this section, we prove Theorem 5.1. The proof that we provide here consists of two steps. First, in view of Lemma 3.2, we need to make use of the condition $\inf_M a > 0$ in order to guarantee the existence of one solution. Second, by using a simple sub- and super- solutions argument, we prove that Equation (5.0.1) still admits one positive smooth solution even that $\inf_M a = 0$.

### 5.3.1 The case $\inf_M a > 0$

In this subsection, we obtain the existence of one solution of (5.0.1) under the assumption $\inf_M a > 0$. For the sake of clarity, we divide the proof into several claims.

**Claim 1.** There holds

$$\mu^\varepsilon_{k_{1,q},q} < \min\{\mu^\varepsilon_{k_\star,q}, \mu^\varepsilon_{k_{2,q},q}\}$$

for all $q \in [q_{\eta_0}, 2^\star)$ and for all $\varepsilon \in (0, k_\star)$ satisfying (3.1.3).

*Proof of Claim 1.* This is a consequence of Lemma 5.4. In order to apply Lemma 5.4, we have to derive (5.2.8) for suitable $q$ sufficiently close to $2^\star$. Let us first rewrite the assumption (5.0.4) into the following form

$$\int_M a \,\mathrm{dvol}_g < \frac{(2^\star)^{n-2}\lambda\eta_0}{4}\left(\frac{\lambda}{\eta_0\int_M |f^-|\,\mathrm{dvol}_g}\right)^{n-1}.$$

Observe that

$$\lim_{q\to 2^\star}\frac{q+2}{q-2} = n-1, \quad \lim_{q\to 2^\star} q^{\frac{4}{q-2}} = (2^\star)^{n-2}.$$

Hence, we can choose $q_{\eta_0} \in [2^\flat, 2^\star)$ sufficiently close to $2^\star$ such that (5.0.5) and

$$\int_M a \,\mathrm{dvol}_g < \frac{\lambda\eta_0}{4q}\left(\frac{\lambda q}{\eta_0\int_M |f^-|\,\mathrm{dvol}_g}\right)^{\frac{q+2}{q-2}}, \tag{5.3.1}$$

hold for any $q \in [q_{\eta_0}, 2^\star)$. This settles Claim 1.

It is important to note that $q_{\eta_0}$ is independent of $q$ and $\varepsilon$. Thus, from now on, we only consider $q \in [q_{\eta_0}, 2^\star)$.

**Claim 2**. Equation (5.0.2) with $\varepsilon$ replaced by 0 has a positive solution, say $u_{1,q}$, that is, $u_{1,q}$ solves the following subcritical equation

$$-\varDelta_g u_{1,q} = f(u_{1,q})^{q-1} + \frac{a}{(u_{1,q})^{q+1}}, \tag{5.3.2}$$

where $q \in [q_{\eta_0}, 2^\star)$.

*Proof of Claim 2.* We now define

$$\mu_q^\varepsilon = \inf_{u\in\mathcal{D}_q} F_q^\varepsilon(u)$$

where

$$\mathcal{D}_q = \left\{u \in H^1(M) : k_\star \leqslant \|u\|_{L^q}^q \leqslant k_{2,q}\right\}.$$

It follows from $k_{1,q} \in (k_\star, k_{2,q})$ that

$$\mu_q^\varepsilon \leqslant \mu_{k_{1,q},q}^\varepsilon \leqslant -\frac{\overline{k}}{2}\int_M f\,\mathrm{dvol}_g + \frac{1}{2\underline{k}}\int_M a\,\mathrm{dvol}_g\,.$$

In other words, we have proved that $\mu_q^\varepsilon$ is bounded from above. By using the Ekeland Variational Principle, one can show that there exists a minimizing sequence for $\mu_q^\varepsilon$ in $\mathcal{D}_q$. Standard arguments show that any minimizing sequence for $\mu_q^\varepsilon$ in $\mathcal{D}_q$ is bounded in $H^1(M)$. Therefore, a similar argument to that we have used before shows that $\mu_q^\varepsilon$ is achieved by some positive function $u_{1,q}^\varepsilon \in \mathcal{D}_q$. Notice that one can claim $u_{1,q}^\varepsilon \in \mathcal{D}_q$ since $q < 2^\star$. Obviously, $u_{1,q}^\varepsilon$ is a weak solution of (5.0.2). By applying Lemma 3.4(a) to (5.0.2), we conclude that $u_{1,q}^\varepsilon \in C^\infty(M)$. Since $\|u_{1,q}^\varepsilon\|_{L^q} > (k_\star)^{\frac{1}{2^\flat}}$, it is clear to see $u_{1,q}^\varepsilon \not\equiv 0$. With Lemma 3.2 and the Strong Minimum Principle in hand, it is easy to prove that $u_{1.q}^\varepsilon > 0$.

Next, in order to send $\varepsilon \to 0$, we need a uniformly boundedness of $u_{1,q}^{\varepsilon}$ in $H^1(M)$. Using the Hölder inequality and the fact that $\|u_{1,q}^{\varepsilon}\|_{L^2} \leqslant \|u_{1,q}^{\varepsilon}\|_{L^q}$, it is not hard to prove that $\|u_{1,q}^{\varepsilon}\|_{H^1}$ is bounded from above with the bound independent of $q$ and $\varepsilon$. In what follows, we let $\{\varepsilon_j\}_j$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$. For each $j$, let $u_{1,q}^{\varepsilon_j}$ be a smooth positive function in $M$ such that

$$-\Delta_g u_{1,q}^{\varepsilon_j} = f(u_{1,q}^{\varepsilon_j})^{q-1} + \frac{a u_{1,q}^{\varepsilon_j}}{((u_{1,q}^{\varepsilon_j})^2 + \varepsilon_j)^{\frac{q}{2}+1}} \tag{5.3.3}$$

in $M$. Being bounded, there exists $u_{1,q} \in H^1(M)$ such that, up to subsequences,

- $u_{1,q}^{\varepsilon_j} \rightharpoonup u_{1,q}$ in $H^1(M)$,
- $u_{1,q}^{\varepsilon_j} \to u_{1,q}$ strongly in $L^2(M)$,
- $u_{1,q}^{\varepsilon_j} \to u_{1,q}$ almost everywhere in $M$.

Using Lemma 3.2, the Lebesgue Dominated Convergence Theorem can be applied to conclude that $\int_M (u_{1,q})^{-p}\, d\mathrm{vol}_g$ is finite for all $p$. Now sending $j \to \infty$ in (5.3.3), we get that $u_{1,q}$ is a weak solution of (5.3.2). Again, by applying Lemma 3.4(b) to (5.3.2), we conclude that $u_{1,q} \in C^\infty(M)$. Using the strong convergence in $L^p(M)$ and the fact that $\|u_{1,q}^{\varepsilon_j}\|_{L^q} \geqslant (k_\star)^{\frac{1}{q}}$, one can see that $u_{1,q} \not\equiv 0$. Therefore, $u_{1,q} > 0$ by using Lemma 3.2 and the Strong Minimum Principle. Keep in mind that we still have $\|u_{1,q}\|_{L^q} \leqslant (k_{2,q})^{\frac{1}{q}}$. This settles Claim 2.

**Claim 3**. Equation (5.0.1) has at least one positive solution.

*Proof of Claim 3.* Let us denote by $\mu_{k_{1,q}}$ the energy of $u_{1,q}$ found in Claim 2. We now estimate the $H^1$-norm of the sequence $\{u_{1,q}\}_q$. Since $k_1 \in [k_\star, k_{2,q}]$, we obtain

$$\begin{aligned}\frac{1}{2}\|\nabla u_{1,q}\|_{L^2}^2 &\leqslant \mu_{k_{1,q}} + \frac{1}{q}\int_M f(u_{1,q})^q\, d\mathrm{vol}_g \\ &\leqslant -\frac{\overline{k}}{2}\int_M f\, d\mathrm{vol}_g + \frac{1}{2\underline{k}}\int_M a\, d\mathrm{vol}_g + \frac{\overline{k}}{2}\sup_M f.\end{aligned}$$

This and the fact that

$$\|u_{1,q}\|_{L^2}^2 \leqslant \|u_{1,q}\|_{L^q}^2 = k_1^{\frac{2}{q}} \leqslant (\overline{k})^{\frac{2}{2^\flat}}$$

imply that the sequence $\{u_{1,q}\}_q$ remains bounded in $H^1(M)$. Thus, up to subsequences, there exists $u_1 \in H^1(M)$ such that

- $u_{1,q} \rightharpoonup u_1$ in $H^1(M)$,
- $u_{1,q} \to u_1$ strongly in $L^2(M)$,
- $u_{1,q} \to u_1$ almost everywhere in $M$,

as $q \to 2^\star$. Recall that $u_{1,q}$ solves (5.3.2) in the weak sense, that is, the following

$$\int_M \nabla u_{1,q} \cdot \nabla v\, d\mathrm{vol}_g - \int_M f(u_{1,q})^{q-1} v\, d\mathrm{vol}_g - \int_M \frac{a}{(u_{1,q})^{q+1}} v\, d\mathrm{vol}_g = 0 \tag{5.3.4}$$

holds for any $v \in H^1(M)$. Observe that, as $q \to 2^\star$,

$$\int_M (\nabla u_{1,q} - \nabla u_1) \cdot \nabla v \, \mathrm{dvol}_g \to 0, \qquad \int_M (u_{1,q} - u_1) \, v \, \mathrm{dvol}_g \to 0. \tag{5.3.5}$$

While the latter immediately follows from the strong convergence in $L^2(M)$, the former can be proved easily since $\nabla u_{1,q} \rightharpoonup \nabla u_1$ weakly in $L^2(M)$. In addition, thanks to $\inf_M a > 0$ and Lemma 3.2, a strictly positive lower bound for $u_{1,q}$ helps us to conclude that as $q \to 2^\star$

$$\int_M \frac{a}{(u_{1,q})^{q+1}} v \, \mathrm{dvol}_g \to \int_M \frac{a}{(u_1)^{2^\star+1}} v \, \mathrm{dvol}_g. \tag{5.3.6}$$

So far, we can pass to the limit every terms on the left hand side of (5.3.4) except the term involving $f$. By the Hölder inequality, one obtains

$$\big\| (u_{1,q})^{q-1} \big\|_{L^{\frac{2^\star}{2^\star-1}}} \leqslant \left( \left( \int_M (u_{1,q})^{2^\star} \, \mathrm{dvol}_g \right)^{\frac{q-1}{2^\star-1}} \right)^{\frac{2^\star-1}{2^\star}} = \| u_{1,q} \|_{L^{2^\star}}^{q-1}. \tag{5.3.7}$$

Making use of the Sobolev inequality (2.3.1) and (5.3.7), we can prove the boundedness of $(u_{1,q})^{q-1}$ in $L^{\frac{2^\star}{2^\star-1}}(M)$. In addition, since $u_{1,q} \to u_1$ almost everywhere, $(u_{1,q})^{q-1} \to (u_1)^{2^\star-1}$ almost everywhere. According to [3, Theorem 3.45], we conclude that $(u_{1,q})^{q-1} \rightharpoonup (u_1)^{\frac{2^\star}{2^\star-1}}$ weakly in $L^{\frac{2^\star}{2^\star-1}}(M)$. Therefore, by definition of weak convergence and the smoothness of $f$, one has

$$\int_M f (u_{1,q})^{q-1} v \, \mathrm{dvol}_g \to \int_M f (u_1)^{2^\star-1} v \, \mathrm{dvol}_g \tag{5.3.8}$$

as $q \to 2^\star$. Combining (5.3.5), (5.3.6), and (5.3.8), one can see, by sending $q \to 2^\star$ in (5.3.4), that $u_1$ are weak solutions to (5.0.1). Using Lemma 3.4(b) we conclude that $u_1 \in C^\infty(M)$ and $u_1 > 0$ in $M$.

### 5.3.2 The case $\inf_M a = 0$

Under this context, making use of the method of sub- and super-solutions is the key argument. Thanks to [20], from that we learn this approach. However, it is worth mentioning that our construction of sub-solutions is different from that of [20]. We let $\varepsilon_0 > 0$ sufficiently small and then fix it so that the following inequality

$$\int_M a \, \mathrm{dvol}_g + \varepsilon_0 < \frac{\lambda^n}{4(\eta_0)^{n-2}} \left( \frac{2n}{n-2} \right)^{n-2} \left( \int_M |f^-| \, \mathrm{dvol}_g \right)^{1-n} \tag{5.3.9}$$

still holds. Since the manifold $M$ has unit volume, we can conclude that from (5.3.9), the function $a + \varepsilon_0$ verifies all assumptions in the previous subsection, thus showing that there exists a positive smooth function $\overline{u}$ solving the following equation

$$-\Delta_g \overline{u} = f \overline{u}^{2^\star-1} + \frac{a + \varepsilon_0}{\overline{u}^{2^\star+1}}.$$

Obviously, $\overline{u}$ is a super-solution to (5.0.1), that is

$$-\Delta_g \overline{u} \geqslant f \overline{u}^{2^\star-1} + \frac{a}{\overline{u}^{2^\star+1}}.$$

Our aim is to find a sub-solution to (5.0.1). Indeed, since

$$\int_M \left( a + \frac{\int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^- \right) \mathrm{dvol}_g = 0,$$

there exists a function $u_0 \in H^1(M)$ solving

$$-\Delta_g u_0 = a + \frac{\int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^- \tag{5.3.10}$$

Since the right hand side of (5.3.10) is of class $L^p(M)$ for any $p < +\infty$, the Caldéron-Zygmund inequality tells us that the solution $u_0$ is of class $W^{2,p}(M)$ for any $p < +\infty$. Thanks to the Sobolev Embedding theorem [3, 2.10], we can conclude that $u_0 \in C^{0,\alpha}(M)$ for some $\alpha \in (0,1)$. In particular, the solution $u_0$ is continuous. Therefore, by adding a sufficiently large constant $C$ to the function $u_0$ if necessary, we can always assume that $\min_M u_0 > 1$. We now find the sub-solution $\underline{u}$ of the form $\varepsilon u_0$ for small $\varepsilon > 0$ to be determined. To this end, we first write

$$-\Delta_g \underline{u} = \varepsilon a + \frac{\varepsilon \int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^-. \tag{5.3.11}$$

Since $\max_M u_0 < +\infty$, it is easy to see that, for any $0 < \varepsilon \leqslant (\max_M u_0)^{-\frac{2^\star+1}{2^\star+2}}$, we immediately have

$$\varepsilon a \leqslant \frac{a}{\varepsilon^{2^\star+1} u_0^{2^\star+1}}. \tag{5.3.12}$$

Besides, since $f^- \leqslant 0$ and $2^\star > 2$, it is not difficult to see that the following inequality

$$\frac{\varepsilon \int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f^-$$

holds provided

$$\varepsilon \leqslant \left( \frac{\int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} \right)^{\frac{1}{2^\star-2}} (\max_M u_0)^{-\frac{2^\star-1}{2^\star-2}}. \tag{5.3.13}$$

In particular, the following

$$\frac{\varepsilon \int_M a \, \mathrm{dvol}_g}{\int_M |f^-| \, \mathrm{dvol}_g} f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f \tag{5.3.14}$$

holds provided (5.3.13) holds. Combining (5.3.11), (5.3.12), and (5.3.14), we conclude that for small $\varepsilon$

$$-\Delta_g \underline{u} \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f + \frac{a}{\varepsilon^{2^\star+1} u_0^{2^\star+1}}.$$

In other words, we have showed that

$$-\Delta_g \underline{u} \leqslant f \underline{u}^{2^\star-1} + \frac{a}{\underline{u}^{2^\star+1}}.$$

Finally, since $\overline{u}$ has a strictly positive lower bound, we can choose $\varepsilon > 0$ sufficiently small such that $\underline{u} \leqslant \overline{u}$. Using the sub- and super-solutions method, see [25, Lemma 2.6], we can conclude the existence of a positive solution $u$ to (5.0.1). By a regularity result developed in [25], we know that $u$ is smooth.

## 5.4 Proof of Theorem 5.2

The proof we provide here is based on the method of the sub- and super-solutions, see [25, 24].

We first construct a positive super-solution $\overline{u}$ for (5.0.1). By using the change of variable $u = e^v$, we get that

$$\Delta u + f u^{2^\star - 1} + a u^{-2^\star - 1} = e^v(\Delta v + |\nabla v|^2) + f e^{(2^\star-1)v} + a e^{-(2^\star+1)v}.$$

Hence, it suffices to find $v$ satisfying

$$\Delta v + |\nabla v|^2 + f e^{(2^\star-2)v} + a e^{-(2^\star+2)v} \leqslant 0.$$

In order to do this, thanks to $\int_M f \, d\mathrm{vol}_g < 0$, we can pick $b > 0$ small enough such that

$$|e^{(2^\star-2)b\varphi} - 1| \leqslant -\frac{1}{4 \sup\limits_M f} \int_M f \, d\mathrm{vol}_g$$

and

$$b|\nabla \varphi|^2 < -\frac{1}{4} \int_M f \, d\mathrm{vol}_g,$$

where $\varphi$ is a positive smooth solution of the following equation

$$\Delta \varphi = \int_M f \, d\mathrm{vol}_g - f.$$

We now find the function $v$ of the form

$$v = b\varphi + \frac{\log b}{2^\star - 2}.$$

Indeed, by calculations, we have

$$\begin{aligned}
\Delta v + &|\nabla v|^2 + f e^{(2^\star-2)v} + a e^{-(2^\star+2)v} \\
&= \Delta \left( b\varphi + \frac{\log b}{2^\star - 2} \right) + \left| \nabla \left( b\varphi + \frac{\log b}{2^\star - 2} \right) \right|^2 \\
&\quad + f e^{(2^\star-2)\left(b\varphi + \frac{\log b}{2^\star-2}\right)} + a e^{-(2^\star+2)\left(b\varphi + \frac{\log b}{2^\star-2}\right)} \\
&= b\Delta\varphi + b^2|\nabla\varphi|^2 + b f e^{(2^\star-2)b\varphi} + a e^{-(2^\star+2)b\varphi} b^{-\frac{2^\star+2}{2^\star-2}} \\
&= b \int_M f \, d\mathrm{vol}_g + b^2|\nabla\varphi|^2 + b f (e^{(2^\star-2)b\varphi} - 1) + a e^{-(2^\star+2)b\varphi} b^{-\frac{2^\star+2}{2^\star-2}} \\
&\leqslant b \int_M f \, d\mathrm{vol}_g - \frac{b}{4} \int_M f \, d\mathrm{vol}_g + b(\sup_M f)|e^{(2^\star-2)b\varphi} - 1| + a e^{-(2^\star+2)b\varphi} b^{-\frac{2^\star+2}{2^\star-2}} \\
&\leqslant \frac{b}{2} \int_M f \, d\mathrm{vol}_g + a e^{-(2^\star+2)b\varphi} b^{-\frac{2^\star+2}{2^\star-2}}.
\end{aligned}$$

Therefore, if we assume that the function $a$ verifies the following estimate

$$\sup_M a < -\frac{b^{\frac{22^\star}{2^\star-2}}}{4} e^{(2^\star+2)b\varphi} \int_M f \, d\mathrm{vol}_g, \tag{5.4.1}$$

we then get that

$$\Delta v + |\nabla v|^2 + f e^{(2^\star-2)v} + a e^{-(2^\star+2)v} \leqslant \frac{b}{4}\int_M f \, d\mathrm{vol}_g < 0,$$

which concludes the existence of a super-solution $\overline{u}$. We now turn to the existence of a sub-solution. Before doing so, we can easily check that

$$\overline{u} = e^{b\varphi + \frac{\log b}{2^\star-2}} > e^{\frac{\log b}{2^\star-2}} = b^{\frac{1}{2^\star-2}}.$$

Since $\overline{u}$ has a strictly positive lower bound and thanks to the second stage of the proof of Theorem 5.1, we can easily construct a sup-solution $\underline{u}$ with $\underline{u} < \overline{u}$. It is important to note that the existence of a sub-solution depends heavily on the conditions $a \geqslant 0$ and $a \not\equiv 0$; and here is the only place we make use of that fact in the proof. The proof of the theorem is now complete.

*Remark 5.7.* Having this theorem in hand, one can observe that our problem (5.0.1) possesses the same phenomena of the Brezis-Nirenberg problem [6]. Although we do not know, under the conditions $\sup_M f > 0$ and $\int_M f \, d\mathrm{vol}_g < 0$, whether the prescribing scalar curvature equations in the null case, see [16],

$$-\Delta_g u = f u^{2^\star-1}, \quad u > 0,$$

always admit one positive smooth solution or not, but by adding a term with a negative exponent, that is,

$$-\Delta_g u = f u^{2^\star-1} + \lambda u^{-2^\star-1}, \quad u > 0, \quad \lambda > 0,$$

the perturbed equation always has at least one positive solution provided the constant $\lambda$ is small enough. In addition, although our construction of sub- and super-solutions employed here is simple but it is strong enough to deal with additional nonlinear negative power terms, for example,

$$-\Delta_g u = f u^{2^\star-1} + \sum_j a_j u^{-\alpha_j}, \quad u > 0,$$

where $\alpha_j > 0$.

## 5.5 Asymptotic behavior of $\mu_{k,q}^\varepsilon$ in the case $\sup_M f \leqslant 0$

According to [11, Proposition 3], if we restrict ourselves to $f \leqslant 0$ but not strictly negative, the solvability of (5.0.1), where $h$, $f$, and $a$ take the form (1.2.2), was already proved. The proof of this fact depends heavily on the conformal covariance property of all these coefficients, that cannot be true for general $h$, $f$, and $a$. Based on this point, in this section, we extend the above result for the Lichnerowicz-scalar field equation (5.0.1). To be precise, we prove that (5.0.1) always admits one positive smooth solution provided $\sup_M f \leqslant 0$ and $\int_M f \, d\mathrm{vol}_g < 0$.

As we have already seen that the behavior of $\mu_{k,q}^\varepsilon$ for small $k$ and small $\varepsilon$ depends strongly on the term involving $a$. Despite the fact that we are under the case $\sup_M f \leqslant 0$, we can still go through Lemma 5.3 without any difficulty, that is, for small $\varepsilon$, $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to 0$.

We now study the behavior of $\mu^\varepsilon_{k,q}$ for $k \to +\infty$ when $\sup_M f \leqslant 0$. As can be seen from the proof of Proposition 4.8 that if $f$ has zero value somewhere in $M$, then in order to control $\mu^\varepsilon_{k,q}$ for large $k$, we have to study $\lambda_{f,\eta,q}$. Depending on how large the set $\{f = 0\}$ is, there are two possible cases. We prove the following result.

**Proposition 5.3.** *Suppose* $\sup_M f = 0$, *then* $\mu^\varepsilon_{k,q} \to +\infty$ *as* $k \to +\infty$ *for any* $\varepsilon > 0$ *sufficiently small and any* $q$ *sufficiently close to* $2^\star$ *but all are fixed.*

*Proof.* As we have noticed that there exist two numbers $\eta_0 \in (0,1)$ and $q_{\eta_0} \in [2^\flat, 2^\star)$ so that the estimate (5.0.5) holds true for every $q \in [q_{\eta_0}, 2^\star)$. In particular, (5.0.6) holds for any $q \in [q_{\eta_0}, 2^\star)$. Depending on the size of the set $\{f = 0\}$, we consider two cases separately.

**Case 1.** Suppose $\sup_M f = 0$ and $\int_{\{f\geqslant 0\}} 1 \, d\mathrm{vol}_g = 0$. Under this case, it is obvious to see that $\lambda_f = +\infty$. Following the proof of Proposition 5.2, for any $k \geqslant k_{2,q}$, any $u \in \mathcal{B}_{k,q}$, any $q \in [q_{\eta_0}, 2^\star)$, and any $\varepsilon > 0$, there holds

$$F^\varepsilon_q(u) = G_q(u) + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g \geqslant \lambda k^{\frac{2}{q}}, \tag{5.5.1}$$

thus proving the fact that $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to +\infty$.

**Case 2.** Suppose $\sup_M f = 0$ and $\int_{\{f\geqslant 0\}} 1 \, d\mathrm{vol}_g > 0$. Under this case, $\lambda_f$ is well-defined and finite. Again, from the choice of $\lambda$, we know that (5.0.6) still holds. As in Case 1 above, the estimate (5.5.1) remains true for any $k \geqslant k_{2,q}$, for any $u \in \mathcal{B}_{k,q}$, any $q \in [q_{\eta_0}, 2^\star)$, and any $\varepsilon > 0$. This proves that $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to +\infty$. ∎

Our next lemma gives a full picture for $\mu^\varepsilon_{k,q}$ similarly to that proved in Section 3.

**Lemma 5.5.** *There holds*

$$\mu^\varepsilon_{k_{1,q},q} < \min\{\mu^\varepsilon_{k_\star,q}, \mu^\varepsilon_{k_{2,q},q}\}$$

*for any* $\varepsilon \in (0, k_\star)$ *and any* $q \in (q_{\eta_0}, 2^\star)$.

*Proof.* As in the proof of Lemma 5.4, the proof is similar and straightforward. ∎

## 5.6 Proof of Theorem 5.3

In view of Lemma 3.6, we can conclude the uniqueness part.

**Proposition 5.4.** *If* $|h| < \lambda_f$ *and* $\sup_M f = 0$, *then Equation* (3.0.1) *admits a unique positive solution* $u$.

*Sketch of proof.* The proof of the existence part of Theorem 5.3 consists of two parts.

In the first stage of the proof, we assume that $\inf_M a > 0$ and $\varepsilon \in (0, k_\star)$ satisfying (3.1.3). Following the first stage of the proof of Theorem 5.1, we first define

$$\mu^\varepsilon_{k_{2,q},q} = \inf_{u \in \mathcal{D}_q} F^\varepsilon_q(u)$$

where the set $\mathcal{D}_q$ is nothing but

$$\mathcal{D}_q = \left\{ u \in H^1(M) : k_\star \leqslant \|u\|^q_{L^q} \leqslant k_{2,q} \right\}.$$

With information that we have already proved in Lemma 5.5, we can easily go through Claims 1, 2, and 3 in the first stage of the proof of Theorem 5.1. In other words, we can prove the existence of at least one positive smooth solution to (5.0.1). Since there is no difference in the proofs, we omit the details here.

In the second stage of the proof, we assume $\inf_M a = 0$. Since we have no control on $\int_M a \, d\mathrm{vol}_g$, we can freely add small $\varepsilon_0 > 0$ to $a$ as in the proof of Theorem 5.1. Since the trick that was used in the proof of Theorem 5.1 still works in this new context, a sub- and super-solutions argument as used before concludes that (5.0.1) has at least one positive smooth solution. ■

In order to make the work unique, let us mention here the case $\sup_M f < 0$ although this has been done in [11] by using the method of sub- and super-solution. Suppose $\sup_M f < 0$. It suffices to study the asymptotic behavior of $\mu^\varepsilon_{k,q}$ for large $k$. Clearly, for any $u \in \mathcal{B}_{k,q}$, we can estimate $\mu^\varepsilon_{k,q}$ directly as the following

$$F^\varepsilon_q(u) \geqslant \frac{1}{2^\star} |\sup_M f| k.$$

It is then immediate to deduce that $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to +\infty$. Hence we can easily prove the existence of at least one positive smooth solution to (3.0.1). More precisely, we prove

**Proposition 5.5.** *If $\sup_M f < 0$, then Equation (3.0.1) admits a unique positive smooth solution $u$.*

*Sketch of proof.* First, the uniqueness part comes from Lemma 3.6. For the existence part, the proof is similar to the proof of Proposition 5.4. ■

## 5.7 Some remarks

### 5.7.1 Construction of smooth, sign-changing functions $f$ with small $\sup_M f$

As can be seen from Theorems 4.1 and 5.1, both theorems involve some upper bound for $\sup_M f$. In this subsection, we provide some functions $f$ such that either the condition (4.0.2) or the condition of (5.0.3) is fulfilled. We take a smooth function $f$ with $\sup_M f > 0$ and $\int_M f \, d\mathrm{vol}_g < 0$. The idea is to lower $\sup_M f$ but still keep the negative part $f^-$ of $f$.

For the sake of simplicity, let us only consider the condition (5.0.3). For each number $\eta > 0$, let us denote

$$\Omega_\eta = \{x \in M : f(x) > \eta\}.$$

By the Morse–Sard theorem, there exist two numbers $\xi$ and $\eta$ with $0 < \xi < \eta \leqslant \frac{\eta_0}{4}\int_M |f^-| \, \mathrm{dvol}_g$ such that

$$|\Omega_\xi \backslash \Omega_\eta| > 0, \quad \mathrm{dist}(\partial\Omega_\xi, \partial\Omega_\eta) > 0.$$

We then take $\phi : M \to [0,1]$ to be a (smooth) cut-off function such that

$$\phi(x) = \begin{cases} 0, & \text{if } x \in M\backslash\Omega_\xi, \\ 1, & \text{if } x \in \Omega_\eta. \end{cases}$$

Having such a cut-off function $\phi$, we construct the function $\widetilde{f}(x) = f(x)e^{-t\phi(x)}$ where $t > 0$ is a parameter to be determined later. Obviously, $\widetilde{f}|_{\{M\backslash\Omega_\xi\}} \equiv f|_{\{M\backslash\Omega_\xi\}}$. In particular, there holds $\widetilde{f}|_{\{f\leqslant 0\}} \equiv f|_{\{f\leqslant 0\}}$. From the choice of $\eta$, for any $x \in \Omega_\xi\backslash\Omega_\eta$, there holds

$$\widetilde{f}(x) < f(x) \leqslant \eta \leqslant \frac{\eta_0}{4}\int_M |f^-| \, \mathrm{dvol}_g.$$

For $x \in \Omega_\eta$, since $\widetilde{f}(x) = f(x)e^{-t}$, one can choose $t$ sufficiently large such that $\widetilde{f}(x) \leqslant \frac{\eta_0}{4}\int_M |f^-| \, \mathrm{dvol}_g$. Notice that, this choice of $t$ is independent of $x$, for example,

$$t = \ln\left(1 + \frac{4}{\eta_0}(\max_M f)\left(\int_M |f^-| \, \mathrm{dvol}_g\right)^{-1}\right).$$

It is now clear to see that the function $\widetilde{f}$ satisfies all conditions in Theorem 5.1.

### 5.7.2 A relation between $\sup_M f$ and $\int_M a \, \mathrm{dvol}_g$

Throughout this subsection, we always assume $\sup_M f > 0$. We spend this subsection to point out a connection between $\sup_M f$ and $\int_M a \, \mathrm{dvol}_g$. To be precise, we conclude that if we lower $\sup_M f$ but still keep $f^-$, then we may find a better upper bound for $\int_M a \, \mathrm{dvol}_g$. We note that although in the statement of Theorem 5.1, the right hand side of (5.0.4) only depends on the negative part $f^-$, there is no contradiction to what we are going to discuss here because (5.0.4) is just a sufficient condition for the solvability of (5.0.1). More than that, this connection explains why in the case $\sup_M f \leqslant 0$, we require no condition on $\int_M a \, \mathrm{dvol}_g$ rather than its positivity.

In order to see this, let us first introduce a scaling constant $\tau > 1$. We assume that $\sup_M f$ satisfies the following

$$\sup_M f \leqslant \frac{\eta_0}{2\tau}\int_M |f^-| \, \mathrm{dvol}_g. \tag{5.7.1}$$

Since $\tau > 1$, it is clear that (5.7.1) is stronger than (5.0.3). We also introduce the following

$$\overline{k}_{1,q} = \frac{\eta_0}{2\tau q}\left(\frac{\tau\lambda q}{\eta_0 \int_M |f^-| \, \mathrm{dvol}_g}\right)^{\frac{q}{q-2}}$$

and

$$\overline{k}_{2,q} = \left(\frac{\tau\lambda q}{\eta_0 \int_M |f^-| \, \mathrm{dvol}_g}\right)^{\frac{q}{q-2}}.$$

Clearly, $k_{1,q} < \overline{k}_{1,q} < \overline{k}_{2,q}$ and $\overline{k}_{2,q} > k_{2,q}$. Notice that $\eta_0$ and $\lambda$ remain unchanged since $f^-$ is being kept. Following Proposition 5.2, it follows from (5.7.1) that, for any $u \in \mathcal{B}_{\overline{k}_{2,q},q}$,

$$F_q^\varepsilon(u) \geqslant \frac{\lambda}{2}(\overline{k}_{2,q})^{\frac{2}{q}}.$$

Therefore, in view of Lemmas 5.3 and 5.4, it suffices to compare $\mu^\varepsilon_{\overline{k}_{1,q},q}$ and $\mu^\varepsilon_{\overline{k}_{2,q},q}$. Indeed, a simple calculation shows that $\mu^\varepsilon_{\overline{k}_{1,q},q} < \mu^\varepsilon_{\overline{k}_{2,q},q}$ provided

$$\int_M a \, \mathrm{dvol}_g < \frac{\lambda\eta_0}{4\tau q}\left(\frac{\tau\lambda q}{\eta_0 \int_M |f^-| \, \mathrm{dvol}_g}\right)^{\frac{q+2}{q-2}}.$$

By sending $q$ to $2^\star$, one arrives at

$$\int_M a \, \mathrm{dvol}_g < \frac{\tau^{n-2}\lambda^n}{4(\eta_0)^{n-2}}\left(\frac{2n}{n-2}\right)^{n-2}\left(\int_M |f^-| \, \mathrm{dvol}_g\right)^{1-n}. \tag{5.7.2}$$

Obviously, (5.7.2) is better than (5.0.4) since $n \geqslant 3$ and $\tau > 1$.

# 6

# The Lichnerowicz equations in the positive Yamabe-scalar field conformal invariant



In this chapter, we again continue our study of some quantitative properties of positive smooth solutions to the Einstein-scalar field Lichnerowicz equations (3.0.1) in the positive Yamabe-scalar field conformal invariant, that is when $h > 0$.

As always, we assume hereafter that $f$, $h > 0$, and $a \geqslant 0$ are smooth functions on $M$ with $\int_M a \, d\mathrm{vol}_g > 0$. For the sake of simplicity, it is important to note that we can freely choose a background metric $g$ such that manifold $M$ has unit volume.

As far as we know, Equation (3.0.1) with $h > 0$ was first considered in [21] by using variational methods. In that elegant paper, Hebey–Pacard–Pollack proved, among other things, a fundamental existence result which roughly says that a suitable control of $\int_M a \, d\mathrm{vol}_g$ from above is enough to guarantee the existence of one positive smooth solution. Their result basically makes use of the fact that the operator $-\Delta_g + h$ is coercive. Although the coerciveness property is slightly weaker than the condition $h > 0$, however as one can see from Chapter 2 that this condition is enough to guarantee that $\|\cdot\|_{H_h^1}$ is an equivalent norm on $H^1(M)$. The advantage of this setting is that the first eigenvalue of the operator $-\Delta_g + h$ is strictly positive, and thus, various goods properties of the theory of weighted Sobolev spaces can be applied. Using our notations, their result can be restated as follows: There exists a constant $C = C(n)$, $C > 0$ depending only on $n$, such that if

$$\|\varphi\|_{H_h^1}^{2^\star} \int_M \frac{a}{\varphi^{2^\star}} \, d\mathrm{vol}_g \leqslant \frac{C}{(\mathcal{S}_h \sup\limits_M |f|)^{n-1}} \tag{6.0.1}$$

and

$$\int_M f\varphi^{2^\star}\,\mathrm{dvol}_g > 0 \tag{6.0.2}$$

for some smooth function $\varphi > 0$ in $M$, then the Einstein-scalar field Lichnerowicz equation (3.0.1) possesses a smooth positive solution in the case $a > 0$.

As can be seen from (6.0.2), the condition $\sup_M f > 0$ is crucial. Therefore, it is not clear whether (3.0.1) possesses a smooth positive solution or not in the case $\sup_M f \leqslant 0$. Moreover, it is necessary to have $a > 0$ in $M$ in order to get a positive lower bound for smooth solutions of (3.0.1), see Lemma 3.3. Besides, the condition (6.0.1) involves not only $\sup_M f^+$ but also $\inf_M f^-$. In other words, for given $a$, the negative part $f^-$ of $f$ cannot be too negative. This restriction basically reflects the fact that the energy functional has to verify the mountain pass geometry as their solution was found as a mountain pass point.

The present chapter was also motivated by a recent preprint by Ma–Wei [28]. In their paper, provided $\underline{u}$ is a positive smooth solution, Ma–Wei proved the existence of a mountain pass solution of (3.0.1) of the form $\underline{u} + v$ for some smooth function $v > 0$. More precise, they proved that in the case $3 \leqslant n < 6$ and that the first eigenvalue of the following operator

$$-\Delta + h - \frac{n+2}{n-2} f \underline{u}^{\frac{4}{n-2}} + \frac{3n-2}{n-2} a \underline{u}^{\frac{4n-4}{n-2}} \tag{6.0.3}$$

is positive, (3.0.1) possesses a mountain pass, smooth, positive solution.

It is easy to see that the positivity of the first eigenvalue of the operator given in (6.0.3) immediately implies that the solution $\underline{u}$ is strictly stable. Therefore, it is natural to seek for positive smooth solutions of (3.0.1) as local minimizers.

Another reason that supports this approach is to look at the profile of the functional associated to (3.0.1). Due to the presence of the term $au^{-2^\star-1}$, the energy of $u$ is very large when $\max_M u$ is small. Clearly, in the case $f \leqslant 0$, the energy of $u$ is also large when $\max_M u$ is large. Consequently, a local minimizer of the energy functional should exist which could provide a possible solution. Similarly, if one assumes that $\sup_M f > 0$ and that the energy functional admits some mountain pass geometry, a local minimizer of the energy functional again exists.

While searching for positive smooth solutions of (3.0.1), we found that the method used in the case of non-positive Yamabe-scalar field invariant still works in this context. While the non-positive Yamabe-scalar field invariant $h \leqslant 0$ involves more conditions and our analysis of solvability of the Lichnerowicz-scalar field equations strongly depends on the ratio between $\sup_M f$ and $\int_M |f^-|\,\mathrm{dvol}_g$, the positive Yamabe-scalar field invariant $h > 0$ requires fewer conditions than the non-positive case. In fact, as we shall see later, in the case $\sup_M f > 0$, no condition for $f$ is imposed and we are able to show that if $\int_M a\,\mathrm{dvol}_g$ is small, then (3.0.1) possesses at least one smooth positive solutions since the condition for $\sup_M f$ can be absorbed to the condition for $\int_M a\,\mathrm{dvol}_g$. The first main theorem can be stated as follows.

**Theorem 6.1.** *Let $(M,g)$ be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Assume that $f$, $h > 0$, and $a \geqslant 0$ are smooth functions on $M$ such that $\int_M a \,\mathrm{dvol}_g > 0$ and $\sup_M f > 0$. We assume further that there exists a constant $\tau > \max\{1, (\frac{2}{\mathcal{S}_h}\int_M h \,\mathrm{dvol}_g)^{\frac{2^\star}{2}}\}$ such that*

$$\int_M a \,\mathrm{dvol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1}n^n}\frac{\mathcal{S}_h}{\tau}\left(\frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f \,\mathrm{dvol}_g}\right)^{n-1} \tag{6.0.4}$$

*holds. Then (3.0.1) possesses at least one smooth positive solution.*

Observe from (6.0.4) that $\tau$ plays no role but a scaling factor. Therefore, for given $\int_M a \,\mathrm{dvol}_g$, we could select $\tau$ sufficiently large and $\sup_M f$ sufficiently small in such a way that (6.0.4) is fulfilled. This suggests that under the case when $\sup_M f$ is small, the condition for $\int_M a \,\mathrm{dvol}_g$ appearing in (6.0.4) can be relaxed. In the second part of the present paper, we prove this affirmatively. That is the content of the following.

**Theorem 6.2.** *Let $(M,g)$ be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Let $f$, $h$, and $a$ be smooth functions on $M$ with $h > 0$, $a \geqslant 0$ in $M$, $\int_M a \,\mathrm{dvol}_g > 0$, and $\sup_M f > 0$. Then there exists a positive constant $\mathcal{C}$ to be specified later such that if $\sup_M f < \mathcal{C}$, then Equation (3.0.1) possesses one positive smooth solution.*

Apparently, Theorem 6.2 provides a slightly stronger result than that of Hebey–Pacard–Pollack as the negative part $f^-$ of the function $f$ could be arbitrarily small. Besides, in view of Theorems 5.2 and 6.1, one may expect that, at least, when $\sup_M a$ is very large, Equation (3.0.1) does not possess any positive smooth solution in the case $h \geqslant 0$. Unfortunately, we have no evidence in order to support this point.

In the third part of the present paper, we focus our attention to the case when $\sup_M f \leqslant 0$. In this context, we are able to get a complete characterization of the existence of solutions of (3.0.1) in the case when $f \leqslant 0$. Roughly speaking, it should mention that in the statement of Theorem 6.1, $\sup_M f$ is exactly $\sup_M f^+$ where $f^+$ is the positive part of $f$. Therefore, without any $\sup_M f$, one can immediately observe that the right hand side of (6.0.4) goes to $+\infty$ as $\tau \to +\infty$. This suggests that under the condition $\sup_M f \leqslant 0$, no condition is imposed.

**Theorem 6.3.** *Let $(M,g)$ be a smooth compact Riemannian manifold without boundary of dimension $n \geqslant 3$. Let $f$, $h$, and $a$ be smooth functions on $M$ with $h > 0$, $a \geqslant 0$ in $M$, $\int_M a \,\mathrm{dvol}_g > 0$, and $f \leqslant 0$. Then Equation (3.0.1) always possesses one and only one positive smooth solution.*

Concerning Theorem 6.3, it is worth noticing that it generalizes the same result obtained in [11] when our equation takes the form (1.2.10). Roughly speaking, it was proved in [11] by the method of sub- and super-solutions that (3.0.1) always possesses one positive solution so long as the functions $f$ and $a$ take the form $f = -\mathcal{B}_{\tau,\psi}$ and $a = \mathcal{A}_{g,W,\pi}$ with $f \leqslant 0$ and $a \geqslant 0$. The main ingredient of the proof in [11] is the conformal invariant property of $\mathcal{B}_{\tau,\psi}$ and $\mathcal{A}_{g,W,\pi}$. Apparently, this property is no longer available in our general case.

As a standard routine, in the first step to tackle (3.0.1), we look for positive smooth solutions of the following subcritical problem

$$-\Delta_g u + hu = f|u|^{q-2}u + \frac{au}{(u^2+\varepsilon)^{\frac{q}{2}+1}}. \tag{6.0.5}$$

It is worth noticing that in [21], the authors just considered (6.0.5) with $q$ replaced by $2^\star$. This deference somehow reflects the fact that we need the compact embedding $H^1(M) \hookrightarrow L^q(M)$ while searching for minimum points.

Observe that $\tau > 1$ and therefore $\tau \sup_M f > \int_M f \, d\mathrm{vol}_g$. Having this, we then introduce the following numbers

$$k_{1,q} = \left( \frac{q+2}{4q} \frac{\mathcal{S}_h \tau^{\frac{2}{q}}}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g} \right)^{\frac{q}{q-2}}, \quad k_{2,q} = \tau k_{1,q}. \tag{6.0.6}$$

One can observe that $k_{1,q} < k_{2,q}$. Moreover, one can easily bound $k_{1,q}$ from below and $k_{2,q}$ from above, that is, there exists two positive numbers $\underline{k} < 1$ and $\overline{k} > 1$ independent of $q$ and $\varepsilon$ such that $\underline{k} \leqslant k_{1,q} < k_{2,q} \leqslant \overline{k}$. In order to find such bounds, one first note that

$$\tau k_{1,q} = \left( \frac{q+2}{4q} \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g} \right)^{\frac{q}{q-2}}.$$

Therefore, we can choose

$$\underline{k} = \frac{1}{\tau} \min \left\{ \left( \frac{1}{4} \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g} \right)^{\frac{2^\flat}{2^\flat - 2}}, 1 \right\} \tag{6.0.7}$$

and

$$\overline{k} = \max \left\{ \left( \frac{1}{2} \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g} \right)^{\frac{2^\flat}{2^\flat - 2}}, 1 \right\}. \tag{6.0.8}$$

## 6.1 The analysis of the energy functionals

Our aim was to derive a suitable analysis of the energy functionals associated to our problem. Thanks to what we have already built up in the previous two chapters, we shall not mention our argument in details in several parts of the current study.

### 6.1.1 The functional setting

As always, for each $q \in (2, 2^\star)$ and $k > 0$, we still use the hyper-surface $\mathcal{B}_{k,q}$ of $H^1(M)$ which is defined in (4.1.1). Again, the set $\mathcal{B}_{k,q}$ is non-empty. Concerning the energy functional associated to problem (6.0.5), we can continue to use that of the previous chapter, that is,

$$\begin{aligned} F_q^\varepsilon(u) = \frac{1}{2} \int_M |\nabla u|^2 \, d\mathrm{vol}_g + \frac{1}{2} \int_M h|u|^2 \, d\mathrm{vol}_g \\ - \frac{1}{q} \int_M f|u|^q \, d\mathrm{vol}_g + \frac{1}{q} \int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g, \end{aligned}$$

for $\varepsilon > 0$ small satisfying (3.1.6). There is no doubt to claim that $F^\varepsilon_q$ is differentiable on $H^1(M)$ and $F^\varepsilon_q|_{\mathcal{B}_{k,q}}$ is bounded from below by $-k|\sup_M f|$. As such, we can define

$$\mu^\varepsilon_{k,q} = \inf_{u \in \mathcal{B}_{k,q}} F^\varepsilon_q(u).$$

Since critical points of $F^\varepsilon_q$ are weak solutions of (6.0.5), we wish to find critical points of the functional $F^\varepsilon_q$.

### 6.1.2 $\mu^\varepsilon_{k,q}$ is achieved

Recall that, it was already proved in Chapter 4 that in the case $h < 0$, the corresponding $\mu^\varepsilon_{k,q}$ is achieved by smooth positive function $u_\varepsilon$. In the present case, $h > 0$, the above proof still works since $h$ is smooth. Therefore, we omit its proof here.

### 6.1.3 The continuity of $\mu^\varepsilon_{k,q}$ with respect to $k$

Once again, as we have already shown in Chapter 4, in the case $h < 0$, the corresponding function $\mu^\varepsilon_{k,q}$ is continuous with respect to $k$ for each $\varepsilon$ fixed. Since the order of the term involving $h$ in the energy functional of the case $h > 0$ is lower than $2^\star$, we conclude that $\mu^\varepsilon_{k,q}$ is also continuous with respect to $k$ in the case $h > 0$.

## 6.2 Asymptotic behavior of $\mu^\varepsilon_{k,q}$ in the case $\sup_M f > 0$

Following the same procedure as in the previous two chapters, in this section, we investigate the behavior of $\mu^\varepsilon_{k,q}$ when both $k$ and $\varepsilon$ vary.

We first study the behavior of $\mu^\varepsilon_{k,q}$ as $k \to +\infty$. We notice that our argument here is not new since the order of the term involving $h$ in $F^\varepsilon_q$ is lower than $2^\star$ that does not affect the behavior of $\mu^\varepsilon_{k,q}$ for large $k$. Following the proof of Lemma 4.4, we can easily prove the following lemma.

**Lemma 6.1.** *There holds $\mu^\varepsilon_{k,q} \to -\infty$ as $k \to +\infty$ if $\sup_M f > 0$.*

We are going to show that $\mu^\varepsilon_{k_{1,q},q} < \mu^\varepsilon_{k_{2,q},q}$ where $k_{1,q}$ and $k_{2,q}$ are given in (6.0.6). To this purpose, we first need a rough estimate for $\mu^\varepsilon_{k_{1,q},q}$.

**Lemma 6.2.** *There holds*

$$\mu^\varepsilon_{k_{1,q},q} \leqslant \frac{k_{1,q}^{\frac{2}{q}}}{2} \int_M h \, \mathrm{dvol}_g - \frac{k_{1,q}}{q} \int_M f \, \mathrm{dvol}_g + \frac{1}{q k_{1,q}} \int_M a \, \mathrm{dvol}_g \tag{6.2.1}$$

*where $k_{1,q}$ is given in (6.0.6).*

*Proof.* This is trivial since $\mu^\varepsilon_{k_{1,q},q} \leqslant F^\varepsilon_q(k_{1,q}^{\frac{1}{q}})$. The proof follows. ∎

As a consequence of Lemma 6.2 and thanks to $\underline{k} < 1 \leqslant \overline{k}$, we can bound $\mu^\varepsilon_{k_{1,q},q}$ with the bound independent of $q$ and $\varepsilon$ as follows

$$\mu^{\varepsilon}_{k_{1,q},q} \leqslant \frac{\overline{k}^{\frac{2}{2^\flat}}}{2}\int_M h \,\mathrm{dvol}_g + \frac{\overline{k}}{2}\sup_M |f| + \frac{1}{2\underline{k}}\int_M a \,\mathrm{dvol}_g.$$

We now study the asymptotic behavior of $\mu^{\varepsilon}_{k,q}$ as $k \to 0$. This result together with Lemmas 6.1 and 6.4 give us a full description of the asymptotic behavior of $\mu^{\varepsilon}_{k,q}$.

**Lemma 6.3.** *There holds* $\lim_{k\to 0+}\mu^{k^{\frac{2}{q}}}_{k,q} = +\infty$. *In particular, there is some* $k_\star$ *sufficiently small and independent of both* $q$ *and* $\varepsilon$ *such that*

$$\mu^{\varepsilon}_{k_\star,q} \geqslant -\overline{k}\int_M f \,\mathrm{dvol}_g + \frac{1}{\underline{k}}\int_M a \,\mathrm{dvol}_g$$

*for any* $\varepsilon \leqslant k_\star$. *In particular, there holds* $\mu^{\varepsilon}_{k_\star,q} > \mu^{\varepsilon}_{k_{1,q},q}$.

*Proof.* The way that $\varepsilon$ comes and plays immediately shows us that $\mu^{\varepsilon}_{k,q}$ is strictly monotone decreasing in $\varepsilon$ for fixed $k$ and $q$. Following the proof of Lemma 4.3, for any $\varepsilon \leqslant k^{\frac{2}{q}}$, any $1 < \frac{q}{2} < \frac{2^\star}{2}$, any any $u \in \mathcal{B}_{k,q}$, we have

$$\int_M \sqrt{a}\,\mathrm{dvol}_g \leqslant 2^{\frac{q}{4}}\sqrt{k}\left(\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\,\mathrm{dvol}_g\right)^{\frac{1}{2}} \tag{6.2.2}$$

By squaring (6.2.2), we get that

$$\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\,\mathrm{dvol}_g \geqslant \frac{1}{2^{\frac{2^\star}{2}}k}\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2.$$

This helps us to conclude

$$F^{\varepsilon}_q(u) \geqslant -\frac{k}{q}\sup_M f + \frac{1}{2^{\frac{2^\star}{2}}qk}\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2,$$

which proves that $\mu^{k^{\frac{2}{q}}}_{k,q} \to +\infty$ as $k \to 0$. Since the right hand side of the preceding inequality is independent of $u$, in order to get the desired estimate, it suffices to find some small $k_\star < 1$ independent of both $q$ and $\varepsilon$ such that the following inequality

$$\begin{aligned} -\frac{k_\star}{q}\sup_M f &+ \frac{1}{2^{\frac{2^\star}{2}}qk_\star}\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2 \\ &\geqslant \overline{k}^{\frac{2}{2^\flat}}\int_M h\,\mathrm{dvol}_g + \overline{k}\sup_M |f| + \frac{1}{\underline{k}}\int_M a\,\mathrm{dvol}_g \end{aligned} \tag{6.2.3}$$

holds. In order to find such a $k_\star$, we first let $k_\star < 1$. Since $q > 2$, it suffices to select $k_\star$ such that

$$\frac{1}{2^{\frac{2^\star}{2}}2^\star k_\star}\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2 \geqslant \overline{k}^{\frac{2}{2^\flat}}\int_M h\,\mathrm{dvol}_g + \overline{k}\sup_M |f| + \frac{1}{\underline{k}}\int_M a\,\mathrm{dvol}_g + \frac{1}{2}\sup_M f$$

which is equivalent to

$$\begin{aligned} k_\star \leqslant \frac{1}{2^{\frac{2^\star}{2}}2^\star}&\left(\int_M \sqrt{a}\,\mathrm{dvol}_g\right)^2 \\ &\left(\overline{k}^{\frac{2}{2^\flat}}\int_M h\,\mathrm{dvol}_g + \overline{k}\sup_M |f| + \frac{1}{\underline{k}}\int_M a\,\mathrm{dvol}_g + \frac{1}{2}\sup_M f\right)^{-1}. \end{aligned}$$

Hence, one can choose $k_\star$ as

$$\begin{aligned} k_\star = \min\Bigg\{ & \frac{1}{2^{\frac{2^\star}{2}} 2^\star} \left( \int_M \sqrt{a}\, \mathrm{dvol}_g \right)^2 \\ & \left( \overline{k}^{\frac{2}{2^\flat}} \int_M h\, \mathrm{dvol}_g + (\overline{k} + \frac{1}{2}) \sup_M |f| + \frac{1}{\underline{k}} \int_M a\, \mathrm{dvol}_g \right)^{-1}, \underline{k}, 1 \Bigg\}. \end{aligned} \tag{6.2.4}$$

Since $k_\star \leqslant 1$, we always have $k_\star \leqslant k_\star^{\frac{2}{q}}$. By Lemma 6.1, we can check that $\mu^\varepsilon_{k_\star,q} > \mu^\varepsilon_{k_{1,q},q}$, thus concluding the lemma with $\varepsilon \leqslant k_\star$. ■

Notice that, we have used $\underline{k}$ in the formula for $k_\star$. The reason is that we wish to ensure that $k_\star < k_{1,q}$ in any case.

In the next result, we claim that the function $\mu^\varepsilon_{k,q}$ is continuous with respect to $k$ for each $\varepsilon > 0$ and $q \in (2^\flat, 2^\star)$ fixed. Since a similar result has been proved before, we omit its proof.

**Proposition 6.1.** *For $\varepsilon > 0$ and $q \in [2^\flat, 2^\star)$ fixed, the function $\mu^\varepsilon_{k,q}$ is continuous with respect to $k$.*

In the rest of this section, our aim here was to study $\mu^\varepsilon_{k,q}$ for $k \geqslant k_{1,q}$. It was found that $\mu^\varepsilon_{k_{1,q},q} < \mu^\varepsilon_{k_{2,q},q}$ provided $\int_M a\, \mathrm{dvol}_g$ is sufficiently small. To this end, we need to estimate $\mu^\varepsilon_{k,q}$ for $k \geqslant k_{1,q}$.

**Proposition 6.2.** *For any $u \in \mathcal{B}_{k,q}$ with $k \geqslant k_{1,q}$, any $q \in [2^\flat, 2^\star)$, and any $\varepsilon > 0$, there holds*

$$F^\varepsilon_q(u) \geqslant \frac{1}{2} \mathcal{S}_h k^{\frac{2}{q}} - \frac{k}{q} \sup_M f.$$

*In particular, there holds*

$$\mu^\varepsilon_{k,q} \geqslant \frac{1}{2} \mathcal{S}_h k^{\frac{2}{q}} - \frac{k}{q} \sup_M f$$

*for any $k \geqslant k_{1,q}$.*

*Proof.* Suppose $u \in \mathcal{B}_{k,q}$ where $k \geqslant k_{1,q}$ is arbitrary. We now estimate $F^\varepsilon_q(u)$ from below. To this purpose, from (2.3.2) and the Hölder inequality, we first have

$$\int_M |\nabla u|^2\, \mathrm{dvol}_g + \int_M h u^2\, \mathrm{dvol}_g \geqslant \mathcal{S}_h k^{\frac{2}{q}}$$

It is then easy to estimate $F^\varepsilon_q$ from below as follows

$$F^\varepsilon_q(u) \geqslant \frac{1}{2} \left( \int_M |\nabla u|^2\, \mathrm{dvol}_g + \int_M h u^2\, \mathrm{dvol}_g \right) - \frac{1}{q} \int_M f^+ |u|^q\, \mathrm{dvol}_g.$$

In particular, there holds

$$F^\varepsilon_q(u) \geqslant \frac{1}{2} \mathcal{S}_h k^{\frac{2}{q}} - \frac{k}{q} \sup_M f.$$

Thus, we can conclude the lemma by taking the infimum with respect to $u \in \mathcal{B}_{k,q}$. ■

In order to prove the existence of a local minimum point, the following lemma plays an important role in our analysis.

**Lemma 6.4.** *Assume that*

$$\int_M a \, d\mathrm{vol}_g < \frac{q-2}{4q}\frac{\mathcal{S}_h}{\tau}\left(\frac{q+2}{4q}\frac{\mathcal{S}_h\tau}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g}\right)^{\frac{q+2}{q-2}}. \tag{6.2.5}$$

*Then there holds*

$$\mu^{\varepsilon}_{k_{1,q},q} < \min\{\mu^{\varepsilon}_{k_\star,q}, \mu^{\varepsilon}_{k_{2,q},q}\}$$

*for any* $q \in [2^\flat, 2^\star)$ *and any* $\varepsilon \in (0, k_\star)$.

*Proof.* First, in view of Lemma 6.3, it suffices to prove $\mu^{\varepsilon}_{k_{1,q},q} < \mu^{\varepsilon}_{k_{2,q},q}$ for all $q \in [2^\flat, 2^\star)$. By making use of Lemma 6.2 and Proposition 6.2, we obtain the following facts

$$\mu^{\varepsilon}_{k_{1,q},q} < \frac{k_{1,q}^{\frac{2}{q}}}{2}\int_M h \, d\mathrm{vol}_g - \frac{k_{1,q}}{2}\int_M f \, d\mathrm{vol}_g + \frac{1}{2k_{1,q}}\int_M a \, d\mathrm{vol}_g$$

and

$$\frac{1}{2}\mathcal{S}_h k_{2,q}^{\frac{2}{q}} - \frac{k_{2,q}}{q}\sup_M f \leqslant \mu^{\varepsilon}_{k_{2,q},q}.$$

Therefore, it suffices to prove that

$$k_{1,q}^{\frac{2}{q}}\int_M h \, d\mathrm{vol}_g - k_{1,q}\int_M f \, d\mathrm{vol}_g + \frac{1}{k_{1,q}}\int_M a \, d\mathrm{vol}_g \leqslant \mathcal{S}_h k_{2,q}^{\frac{2}{q}} - k_{2,q}\sup_M f,$$

for any $q \in [2^\flat, 2^\star)$. Notice that, from the choice of $\tau$, that is,

$$\tau > \max\left\{1, \left(\frac{2}{\mathcal{S}_h}\int_M h \, d\mathrm{vol}_g\right)^{\frac{2^\star}{2}}\right\},$$

we can verify that $\mathcal{S}_h \tau^{\frac{2}{q}} \geqslant 2\int_M h \, d\mathrm{vol}_g$. This amounts to saying that

$$k_{1,q}^{\frac{2}{q}}\int_M h \, d\mathrm{vol}_g \leqslant \frac{1}{2}\mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{\frac{2}{q}} = \frac{1}{2}\mathcal{S}_h k_{2,q}^{\frac{2}{q}}.$$

Therefore, it suffices to show that

$$-k_{1,q}\int_M f \, d\mathrm{vol}_g + \frac{1}{k_{1,q}}\int_M a \, d\mathrm{vol}_g \leqslant \frac{1}{2}\mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{\frac{2}{q}} - \tau k_{1,q}\sup_M f$$

or equivalently,

$$\begin{aligned}\int_M a \, d\mathrm{vol}_g &\leqslant \frac{1}{2}\mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{1+\frac{2}{q}} - k_{1,q}^2\left(\tau\sup_M f - \int_M f \, d\mathrm{vol}_g\right)\\ &= k_{1,q}^2\left(\frac{1}{2}\mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{\frac{2-q}{q}} - \left(\tau \sup_M f - \int_M f \, d\mathrm{vol}_g\right)\right),\end{aligned} \tag{6.2.6}$$

for any $q \in [2^\flat, 2^\star)$. Again, from the choice of $k_{1,q}$, it is clear to see that

$$\begin{aligned}\tau^{\frac{2}{q}} k_{1,q}^{\frac{2-q}{q}} &= \tau^{\frac{2}{q}}\left(\frac{q+2}{4q}\frac{\mathcal{S}_h \tau^{\frac{2}{q}}}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g}\right)^{-1}\\ &= \frac{4q}{q+2}\frac{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g}{\mathcal{S}_h}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\mathcal{S}_h\tau^{\frac{2}{q}}k_{1,q}^{\frac{2-q}{q}} - \Big(\tau \sup_M f - \int_M f \,\mathrm{dvol}_g\Big) &= \frac{q-2}{q+2}\Big(\tau \sup_M f - \int_M f \,\mathrm{dvol}_g\Big)\\ &= \frac{q-2}{4q}\mathcal{S}_h\tau^{\frac{2}{q}}\left(\frac{q+2}{4q}\frac{\mathcal{S}_h\tau^{\frac{2}{q}}}{\tau\sup_M f - \int_M f\,\mathrm{dvol}_g}\right)^{-1}.\end{aligned}$$

By using this identity, (6.2.6) is equivalent to

$$\begin{aligned}\int_M a\,\mathrm{dvol}_g &\leqslant \frac{q-2}{4q}\mathcal{S}_h\tau^{\frac{2}{q}}\left(\frac{q+2}{4q}\frac{\mathcal{S}_h\tau^{\frac{2}{q}}}{\tau\sup_M f - \int_M f\,\mathrm{dvol}_g}\right)^{\frac{q+2}{q-2}}\\ &= \frac{q-2}{4q}\frac{\mathcal{S}_h}{\tau}\left(\frac{q+2}{4q}\frac{\mathcal{S}_h\tau}{\tau\sup_M f - \int_M f\,\mathrm{dvol}_g}\right)^{\frac{q+2}{q-2}}.\end{aligned} \tag{6.2.7}$$

The proof follows easily by comparing (6.2.5) and (6.2.7). ■

## 6.3 Proof of Theorem 6.1

In this section, we prove Theorem 6.1. Similar to the proof of Theorem 5.1, the proof that we provide here consists of two steps. First, in view of Lemma 3.3 we need to make use of the condition $\inf_M a > 0$ in order to guarantee the existence of one solution. Second, by using a simple sub- and super- solutions argument, we prove that (3.0.1) still admits one positive smooth solution even that $\inf_M a = 0$. Apparently, the result for the case $\inf_M a = 0$ is new since this case was left open in [21].

### 6.3.1 The case $\inf_M a > 0$

In this subsection, we obtain the existence of one solution of (3.0.1) under the assumption $\inf_M a > 0$. For the sake of clarity, we divide the proof into three claims.

**Claim 1**. There exists a $q_0 \in (2^\flat, 2^\star)$ such that for all $q \in [q_0, 2^\star)$ and for all $\varepsilon \in (0, k_\star)$ satisfying (3.1.6), there will be $k_{1,q}$, $k_{2,q}$, and $k_\star$ with the following properties $k_\star < k_{1,q} < k_{2,q}$ and

$$\mu^\varepsilon_{k_{1,q},q} < \min\{\mu^\varepsilon_{k_\star,q}, \mu^\varepsilon_{k_{2,q},q}\}.$$

*Proof of Claim 1.* This is a consequence of Lemma 6.4. In order to apply Lemma 6.4, we have to derive (6.2.5) for suitable $q$ close enough to $2^\star$. Observe that

$$\lim_{q\to 2^\star}\frac{q+2}{q-2} = n-1, \quad \lim_{q\to 2^\star}\frac{q-2}{4q}\left(\frac{q+2}{4q}\right)^{\frac{q+2}{q-2}} = \frac{(2n-1)^{n-1}}{2^{2n-1}n^n}.$$

Hence, we can choose $q_0 \in [2^\flat, 2^\star)$ sufficiently close to $2^\star$ such that the condition $\mathcal{S}_h\tau^{\frac{2}{q}} \geqslant 2\int_M h\,\mathrm{dvol}_g$ and the following inequality

$$\int_M a \,\mathrm{dvol}_g \leqslant \frac{q-2}{4q}\frac{\mathcal{S}_h}{\tau}\left(\frac{q+2}{4q}\frac{\mathcal{S}_h\tau}{\tau\sup_M f - \int_M f\,\mathrm{dvol}_g}\right)^{\frac{q+2}{q-2}} \tag{6.3.1}$$

holds for any $q \in [q_0, 2^\star)$. Thanks to (6.3.1), we can now use Lemma 6.4 to finish the proof of this claim.

It is important to note that $q_0$ is independent of $q$ and $\varepsilon$. Thus, from now on, we only consider $q \in [q_0, 2^\star)$.

**Claim 2**. Equation (6.0.5) with $\varepsilon$ replaced by 0 has a positive solution, say $u_{1,q}$, that is, $u_{1,q}$ solves the following subcritical equation

$$-\Delta_g u_{1,q} + h u_{1,q} = f(u_{1,q})^{q-1} + \frac{a}{(u_{1,q})^{q+1}}, \tag{6.3.2}$$

where $q \in [q_0, 2^\star)$.

*Proof of Claim 2.* For the sake of clarity, we divide our proof into several steps.
**Step 1**. *The existence of $u_{1,q}^\varepsilon$ with energy $\mu_{k_1^\varepsilon,q}^\varepsilon$.* We now define

$$\mu_{k_1^\varepsilon,q}^\varepsilon = \inf_{u\in\mathcal{D}_q} F_q^\varepsilon(u)$$

where

$$\mathcal{D}_q = \left\{u \in H^1(M) : k_\star \leqslant \|u\|_{L^q}^q \leqslant k_{2,q}\right\}.$$

It follows from $k_{1,q} \in (k_\star, k_{2,q})$ and Lemma 6.2 that

$$\mu_{k_1^\varepsilon,q}^\varepsilon \leqslant \mu_{k_{1,q},q}^\varepsilon \leqslant \overline{k}^{\frac{2}{2^\flat}} \int_M h\,\mathrm{dvol}_g + \overline{k}\sup_M |f| + \frac{1}{\underline{k}}\int_M a\,\mathrm{dvol}_g\,.$$

In other words, we have proved that $\mu_{k_1^\varepsilon,q}^\varepsilon$ is bounded from above.

By a standard argument and the Ekeland Variational Principle, one can show that there exists a $H^1$-bounded minimizing sequence for $\mu_{k_1^\varepsilon,q}^\varepsilon$ in $\mathcal{D}_q$. Therefore, a similar argument to that we have used before shows that $\mu_{k_1^\varepsilon,q}^\varepsilon$ is achieved by some positive function $u_{1,q}^\varepsilon \in \mathcal{D}_q$. Notice that one can claim $u_{1,q}^\varepsilon \in \mathcal{D}_q$ since $q < 2^\star$. Obviously, $u_{1,q}^\varepsilon$ is a weak solution of (6.0.5). Thus, the regularity result, Lemma 3.3(a), developed in Section 2 can be applied to (6.0.5). It follows that $u_{1,q}^\varepsilon \in C^\infty(M)$. If we denote $\|u_{1,q}^\varepsilon\|_{L^q} = k_1^\varepsilon$, we then see that $k_1^\varepsilon > (k_\star)^{\frac{1}{2^\flat}}$. Consequently, there holds $u_{1,q}^\varepsilon \not\equiv 0$. With Lemma 3.3 and the Strong Minimum Principle in hand, it is easy to prove that $u_{1,q}^\varepsilon$ is strictly positive.
**Step 2**. *The existence of $u_{1,q}$ with energy $\mu_{k_1,q}$.* Next, in order to send $\varepsilon \to 0$, we need a uniform bound for $u_{1,q}^\varepsilon$ in $H^1(M)$. Using the Hölder inequality and the fact that $\|u_{1,q}^\varepsilon\|_{L^2} \leqslant \|u_{1,q}^\varepsilon\|_{L^q}$, it is not hard to prove that $\|u_{1,q}^\varepsilon\|_{H^1}$ is bounded from above with the bound independent of $q$ and $\varepsilon$.

In what follows, we let $\{\varepsilon_j\}_j$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$. For each $j$, let $u_{1,q}^{\varepsilon_j}$ be a smooth positive function in $M$ such that

$$-\Delta_g u_{1,q}^{\varepsilon_j} + h u_{1,q}^{\varepsilon_j} = f(u_{1,q}^{\varepsilon_j})^{q-1} + \frac{a u_{1,q}^{\varepsilon_j}}{((u_{1,q}^{\varepsilon_j})^2 + \varepsilon_j)^{\frac{q}{2}+1}} \tag{6.3.3}$$

in $M$. Being bounded in $H^1(M)$, there exists $u_{1,q} \in H^1(M)$ such that, up to subsequences, as $j \to \infty$,

- $u_{1,q}^{\varepsilon_j} \rightharpoonup u_{1,q}$ weakly in $H^1(M)$;
- $u_{1,q}^{\varepsilon_j} \to u_{1,q}$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;
- $u_{1,q}^{\varepsilon_j} \to u_{1,q}$ almost everywhere in $M$.

Using Lemma 3.3, the Lebesgue Dominated Convergence Theorem can be applied to conclude that $\int_M (u_{1,q})^{-p}\, \mathrm{dvol}_g$ is finite for all $p$. Now sending $j \to \infty$ in (6.3.3), we get that $u_{1,q}$ is a weak solution of the subcritical equation (6.3.2). Thus Lemma 3.4(b) can be applied to (6.3.2). It follows that $u_{1,q} \in C^\infty(M)$. For simplicity, let us denote $\|u_{1,q}\|_{L^q}^q = k_1$. Using the strong convergence in $L^p(M)$ and the fact that $k_1^\varepsilon \geqslant k_\star$, one can see that $k_1 \geqslant k_\star$, thus proving $u_{1,q} \not\equiv 0$. With Lemma 3.3 and the Strong Minimum Principle in hand, it is easy to prove that $u_{1,q}$ is strictly positive. Keep in mind that we still have $k_1 \leqslant k_{2,q}$ since we still have a strong convergence. This settles Claim 2.

**Claim 3**. Equation (3.0.1) has at least one positive solution.

*Proof of Claim 3.* Let us denote by $\mu_{k_1,q}$ the energy of $u_{1,q}$ found in Claim 2, i.e.,

$$\begin{aligned}\mu_{k_1,q} = \frac{1}{2}\int_M |\nabla u_{1,q}|^2\, \mathrm{dvol}_g + \frac{1}{2}\int_M h(u_{1,q})^2\, \mathrm{dvol}_g \\ - \frac{1}{q}\int_M f(u_{1,q})^q\, \mathrm{dvol}_g + \frac{1}{q}\int_M \frac{a}{(u_{1,q})^q}\, \mathrm{dvol}_g.\end{aligned}$$

Keep in mind that by $k_1$ we mean $\|u_{1,q}\|_{L^q}^q = k_1$. Since $q < 2^\star$, by strong convergences, we have

$$\begin{aligned}\mu_{k_1,q} &= \limsup_{j\to\infty} \mu^{\varepsilon_j}_{k_1^{\varepsilon_j},q} \\ &\leqslant \limsup_{j\to\infty} \mu^{\varepsilon_j}_{k_{1,q},q} \\ &\leqslant \frac{\overline{k}^{\frac{2}{2^\flat}}}{2}\int_M h\, \mathrm{dvol}_g + \frac{\overline{k}}{2}\sup_M |f| + \frac{1}{2\underline{k}}\int_M a\, \mathrm{dvol}_g.\end{aligned} \tag{6.3.4}$$

We now estimate the $H^1$-norm of the sequence $\{u_{1,q}\}_q$. Clearly, since $h > 0$ and $a \geqslant 0$, we get that

$$\begin{aligned}\frac{1}{2}\int_M |\nabla u_{1,q}|^2\, \mathrm{dvol}_g =& \mu_{k_1,q} - \frac{1}{2}\int_M h(u_{1,q})^2\, \mathrm{dvol}_g \\ &+ \frac{1}{q}\int_M f(u_{1,q})^q\, \mathrm{dvol}_g - \frac{1}{q}\int_M \frac{a}{(u_{1,q})^q}\, \mathrm{dvol}_g \\ \leqslant& \mu_{k_1,q} + \frac{1}{q}\int_M f(u_{1,q})^q\, \mathrm{dvol}_g \\ \leqslant& \mu_{k_1,q} + \frac{k_1}{2}\sup_M |f|.\end{aligned}$$

Since $k_1 \in [k_\star, k_{2,q}]$, we then easily obtain

$$\frac{1}{2}\int_M |\nabla u_{1,q}|^2\, \mathrm{dvol}_g \leqslant \frac{\overline{k}^{\frac{2}{2^\flat}}}{2}\int_M h\, \mathrm{dvol}_g + \overline{k}\sup_M |f| + \frac{1}{2\underline{k}}\int_M a\, \mathrm{dvol}_g.$$

This and the fact that $\|u_{1,q}\|_{L^2}^2 \leqslant (\overline{k})^{\frac{2}{2^\flat}}$ imply that the sequence $\{u_{1,q}\}_q$ remains bounded in $H^1(M)$. Thus, up to subsequences, there exists $u_1 \in H^1(M)$ such that, as $q \to 2^\star$,

- $u_{1,q} \rightharpoonup u_1$ weakly in $H^1(M)$;
- $u_{1,q} \to u_1$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$;
- $u_{1,q} \to u_1$ almost everywhere in $M$.

Notice that $u_{1,q}$ verifies

$$\begin{aligned} \int_M \nabla u_{1,q} \cdot \nabla v \, \mathrm{dvol}_g + \int_M h u_{1,q} v \, \mathrm{dvol}_g \\ - \int_M f (u_{1,q})^{q-1} v \, \mathrm{dvol}_g - \int_M \frac{a}{(u_{1,q})^{q+1}} v \, \mathrm{dvol}_g = 0 \end{aligned} \tag{6.3.5}$$

for any $v \in H^1(M)$. By similar arguments to those used before, one immediately has the following

$$\begin{aligned} \int_M (\nabla u_{1,q} - \nabla u_1) \cdot \nabla v \, \mathrm{dvol}_g \to 0, \\ \int_M (u_{1,q} - u_1) \, v \, \mathrm{dvol}_g \to 0, \\ \int_M \frac{a}{(u_{1,q})^{q+1}} v \, \mathrm{dvol}_g \to \int_M \frac{a}{(u_1)^{2^\star+1}} v \, \mathrm{dvol}_g, \end{aligned} \tag{6.3.6}$$

as $q \to 2^\star$. So far, we can pass to the limit every terms on the left hand side of (6.3.5) except the term involving $f$. By the Hölder inequality, one obtains

$$\left\| (u_{1,q})^{q-1} \right\|_{L^{\frac{2^\star}{2^\star-1}}} \leqslant \left( \left( \int_M (u_{1,q})^{2^\star} \, \mathrm{dvol}_g \right)^{\frac{q-1}{2^\star-1}} \right)^{\frac{2^\star-1}{2^\star}} = \| u_{1,q} \|_{L^{2^\star}}^{q-1}. \tag{6.3.7}$$

Making use of the Sobolev inequality and (6.3.7), we can prove the boundedness of $(u_{1,q})^{q-1}$ in $L^{\frac{2^\star}{2^\star-1}}(M)$. In addition, since $u_{1,q} \to u_1$ almost everywhere, $(u_{1,q})^{q-1} \to (u_1)^{2^\star-1}$ almost everywhere. According to [3, Theorem 3.45], we can conclude that $(u_{1,q})^{q-1} \rightharpoonup (u_1)^{\frac{2^\star}{2^\star-1}}$ weakly in $L^{\frac{2^\star}{2^\star-1}}(M)$. Therefore, by definition of weak convergence and the smoothness of $f$, one has

$$\int_M f (u_{1,q})^{q-1} v \, \mathrm{dvol}_g \to \int_M f (u_1)^{2^\star-1} v \, \mathrm{dvol}_g \tag{6.3.8}$$

as $q \to 2^\star$. Combining (6.3.6) and (6.3.8), one can see, by sending $q \to 2^\star$ in (6.3.5), that $u_1$ are weak solutions to (3.0.1). Using Lemma 3.4(b) we conclude that $u_1 \in C^\infty(M)$ and $u_1 > 0$ in $M$.

### 6.3.2 The case $\inf_M a = 0$

Under this context, making use of the method of sub- and super-solutions is the key argument. Thanks to [20], from that we learn this approach. However, it is worth mentioning that our construction of sub-solutions is different from that of [20]. We let $\varepsilon_0 > 0$ sufficiently small and then fix it so that the following inequality

$$\int_M a \, \mathrm{dvol}_g + \varepsilon_0 < \frac{(2n-1)^{n-1}}{2^{2n-1} n^n} \frac{\mathcal{S}_h}{\tau} \left( \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f \, \mathrm{dvol}_g} \right)^{n-1} \tag{6.3.9}$$

still holds. Since the manifold $M$ has unit volume, we can conclude that from (6.3.9), the function $a + \varepsilon_0$ verifies all assumptions in the previous subsection, thus showing that there exists a positive smooth function $\overline{u}$ solving the following equation

$$-\Delta_g \overline{u} + h\overline{u} = f\overline{u}^{2^\star - 1} + \frac{a+\varepsilon_0}{\overline{u}^{2^\star+1}}.$$

Obviously, $\overline{u}$ is a super-solution to (3.0.1), that is

$$-\Delta_g \overline{u} + h\overline{u} \geqslant f\overline{u}^{2^\star - 1} + \frac{a}{\overline{u}^{2^\star+1}}.$$

Our aim is to find a sub-solution to (3.0.1). In this context, we consider the following equation

$$-\Delta_g u + (h - f^-)u = a. \tag{6.3.10}$$

Since $h - f^- > 0$, $a \geqslant 0$, $a \not\equiv 0$, and the manifold $M$ is compact without boundary, the standard argument shows that (6.3.10) always admits a weak solution, say $u_0$. By a standard regularity result, one can easily deduce that $u_0$ is at least continuous. Thus, by the Maximum Principle, we conclude $u_0 > 0$.

As before, we now find the sub-solution $\underline{u}$ of the form $\varepsilon u_0$ for small $\varepsilon > 0$ to be determined. To this purpose, we first write

$$-\Delta_g \underline{u} + h\underline{u} = \varepsilon a + f^- \underline{u}. \tag{6.3.11}$$

Since $\max_M u_0 < +\infty$, it is easy to see that, for any $0 < \varepsilon \leqslant (\max_M u_0)^{-\frac{2^\star+1}{2^\star+2}}$, we immediately have

$$\varepsilon a \leqslant \frac{a}{\varepsilon^{2^\star+1} u_0^{2^\star+1}}. \tag{6.3.12}$$

Besides, since $f^- \leqslant 0$ and $2^\star > 2$, it is not difficult to see that the following inequality

$$\varepsilon u_0 f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f^-$$

holds provided $\varepsilon \leqslant (\max_M u_0)^{-1}$. In particular, the following

$$\varepsilon u_0 f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f \tag{6.3.13}$$

holds provided $\varepsilon \leqslant (\max_M u_0)^{-1}$. Combining all estimates (6.3.11), (6.3.12), and (6.3.13) above, we conclude that for small $\varepsilon$, there holds

$$-\Delta_g \underline{u} + h\underline{u} \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f + \frac{a}{\varepsilon^{2^\star+1} u_0^{2^\star+1}}.$$

In other words, we have shown that $\underline{u}$ is a sub-solution of (3.0.1). Finally, since $\overline{u}$ has a strictly positive lower bound, we can choose $\varepsilon > 0$ sufficiently small such that $\underline{u} \leqslant \overline{u}$. Using the sub- and super-solutions method, see [25, Lemma 2.6], we can conclude the existence of a positive solution $u$ to (3.0.1). By a regularity result developed in [25], we know that $u$ is smooth.

## 6.4 Proof of Theorem 6.2

In order to prove Theorem 6.3, we need to show that the condition (6.0.4) is fulfilled. Although we have not assumed any upper bound for $\int_M a \, d\text{vol}_g$, we are able to show that we can recover the condition (6.0.4) provided $\sup_M f$ is sufficiently small. As usual, we first assume $\inf_M a > 0$. Depending on the sign of $\int_M f \, d\text{vol}_g$, we have two cases.

**Case 1**. Suppose $\int_M f \, d\text{vol}_g \geqslant 0$. In this context, we can easily verify that

$$\frac{\mathcal{S}_h}{\sup_M f} \leqslant \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f \, d\text{vol}_g}.$$

Therefore, it suffices to show that

$$\int_M a \, d\text{vol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1} n^n} \frac{\mathcal{S}_h}{\tau} \left( \frac{\mathcal{S}_h}{\sup_M f} \right)^{n-1},$$

which is equivalent to

$$\sup_M f < \left( \frac{(2n-1)^{n-1}}{2^{2n-1} n^n} \frac{\mathcal{S}_h^n}{\tau \int_M a \, d\text{vol}_g} \right)^{\frac{1}{n-1}}.$$

**Case 2**. Suppose $\int_M f \, d\text{vol}_g < 0$. In this context, we assume for a moment that $\sup_M f > 0$ is small in such a way that we can select

$$\tau = \frac{1}{\sup_M f} > \max \left\{ 1, \left( \frac{2}{\mathcal{S}_h} \int_M h \, d\text{vol}_g \right)^{\frac{2^\star}{2}} \right\}.$$

Then, thanks to $f = f^+ + f^-$, we have

$$\begin{aligned} \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f \, d\text{vol}_g} &= \frac{1}{\sup_M f} \frac{\mathcal{S}_h}{1 - \int_M f \, d\text{vol}_g} \\ &\geqslant \frac{1}{\sup_M f} \frac{\mathcal{S}_h}{1 + \int_M |f^-| \, d\text{vol}_g}. \end{aligned}$$

Therefore, it suffices to show that

$$\int_M a \, d\text{vol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1} n^n} \mathcal{S}_h \sup_M f \left( \frac{1}{\sup_M f} \frac{\mathcal{S}_h}{1 + \int_M |f^-| \, d\text{vol}_g} \right)^{n-1},$$

which is equivalent to

$$\sup_M f < \left( \frac{(2n-1)^{n-1}}{2^{2n-1} n^n} \frac{\mathcal{S}_h^n}{(1 + \int_M |f^-| \, d\text{vol}_g) \int_M a \, d\text{vol}_g} \right)^{\frac{1}{n-2}}.$$

From our calculation above, we conclude that there exists some positive constant $\mathcal{C} > 0$ depending only on $a$, $h$, and $f^-$ such that if $0 < \sup_M f < \mathcal{C}$, our equation (3.0.1) always admits at least one positive smooth solution.

It remains to consider the case $\inf_M a = 0$. However, since the size of $a$ plays no role in the above calculation, we can freely add a small constant $\varepsilon_0$ to $a$ as in the second stage of the proof of Theorem 6.1. This procedure ensures that we always get a super-solution of (3.0.1) with a strictly positive lower bound and this is enough since a suitable positive sub-solution always exists.

## 6.5 Asymptotic behavior of $\mu^\varepsilon_{k,q}$ in the case $\sup_M f \leqslant 0$

Unlike the case when $\sup_M f > 0$ that forces $\mu^\varepsilon_{k,q} \to -\infty$ as $k \to \infty$, it is found that, in the case $\sup_M f \leqslant 0$, we always have $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to \infty$ and this is enough to guarantee the existence of at least one solution.

Before doing so, one can observe that, thanks to $f^+ \equiv 0$, $k_{1,q}$ and $k_{2,q}$ simply become

$$k_{1,q} = \tau^{\frac{2}{q-2}} \left( \frac{q+2}{4q} \frac{\mathcal{S}_h}{-\int_M f \, d\mathrm{vol}_g} \right)^{\frac{q}{q-2}}, \quad k_{2,q} = \tau k_{1,q}, \tag{6.5.1}$$

where $\tau$ is a suitable scaling constant to be determined later.

As we have already seen that the behavior of $\mu^\varepsilon_{k,q}$ for small $k$ and small $\varepsilon$ depends strongly on the term involving $a$. Despite the fact that we are under the case $\sup_M f \leqslant 0$, we can still go through Lemma 6.3 without any difficulty, that is, for small $\varepsilon$, $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to 0$. We now study the behavior of $\mu^\varepsilon_{k,q}$ for $k \to +\infty$ when $\sup_M f \leqslant 0$.

**Proposition 6.3.** *Suppose $\sup_M f \leqslant 0$, then $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to +\infty$ for any $\varepsilon > 0$ and any $q \in [2^\flat, 2^\star)$ but all are fixed.*

*Proof.* By using (2.3.2) and the Hölder inequality, for any $u \in \mathcal{B}_{k,q}$, any $q \in [2^\flat, 2^\star)$, and any $\varepsilon > 0$, there holds

$$\begin{aligned} F^\varepsilon_q(u) =& \frac{1}{2} \int_M (|\nabla u|^2 + h u^2) \, d\mathrm{vol}_g \\ & - \frac{1}{q} \int_M f |u|^q \, d\mathrm{vol}_g + \frac{1}{q} \int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} \, d\mathrm{vol}_g \\ \geqslant & \frac{1}{2} \mathcal{S}_h k^{\frac{2}{q}}, \end{aligned}$$

which immediately implies that $\mu^\varepsilon_{k,q} \geqslant \frac{1}{2} \mathcal{S}_h k^{\frac{2}{q}}$. Thus, we have shown that $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to +\infty$. ■

Our next lemma gives a full description for $\mu^\varepsilon_{k,q}$ similarly to that proved for the case $\sup_M f > 0$. The only difference is that we do not require $q$ to be close to $2^\star$ since we have no condition on $\int_M a \, d\mathrm{vol}_g$. In order to avoid any condition on $\int_M a \, d\mathrm{vol}_g$, thanks to Proposition 6.3 above, we just have to select a suitable large $\tau$.

**Lemma 6.5.** *There holds*

$$\mu^\varepsilon_{k_{1,q},q} < \min\{\mu^\varepsilon_{k_\star,q}, \mu^\varepsilon_{k_{2,q},q}\}$$

*for any $\varepsilon \in (0, k_\star)$ and any $q \in [2^\flat, 2^\star)$.*

*Proof.* As in the proof of Lemma 6.4, the proof is similar and straightforward. To see this, for new $k_{1,q}$ and $k_{2,q}$, we can also bound by $\underline{k}$ and $\overline{k}$. Therefore, we can define $k_\star$ as in (6.2.4). Having such a $k_\star$, the estimate $\mu^\varepsilon_{k_{1,q},q} \leqslant \mu^\varepsilon_{k_\star,q}$ still holds; therefore, it suffices to prove

$$\mu^\varepsilon_{k_{1,q},q} \leqslant \mu^\varepsilon_{k_{2,q},q}$$

by choosing a suitable $\tau \gg 1$. Equivalently, we need to prove that

$$k_{1,q}^{\frac{2}{q}} \int_M h \,\mathrm{dvol}_g - k_{1,q} \int_M f \,\mathrm{dvol}_g + \frac{1}{k_{1,q}} \int_M a \,\mathrm{dvol}_g \leqslant \mathcal{S}_h k_{2,q}^{\frac{2}{q}},$$

for any $q \in [q_{\eta_0}, 2^\star)$. From the choice of $\tau$, we only need to prove that

$$\int_M a \,\mathrm{dvol}_g \leqslant k_{1,q}^2 \left( \frac{1}{2} \mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{\frac{2-q}{q}} + \int_M f \,\mathrm{dvol}_g \right). \tag{6.5.2}$$

A simple calculation shows that (6.5.2) is equivalent to

$$\int_M a \,\mathrm{dvol}_g \leqslant \tau^{\frac{4}{q-2}} \frac{q-2}{4q} \mathcal{S}_h \left( \frac{q+2}{4q} \frac{\mathcal{S}_h}{-\int_M f \,\mathrm{dvol}_g} \right)^{\frac{q+2}{q-2}}.$$

Hence, by choosing $\tau$ sufficiently large, one easily gets the desired result. ■

## 6.6 Proof of Theorem 6.3

This section is devoted to the proof of Theorem 6.3. In view of Lemma 3.6, we can conclude the uniqueness part.

**Proposition 6.4.** *If* $\sup_M f \leqslant 0$ *and* $h > 0$*, then Equation* (3.0.1) *admits a unique positive solution* $u$*.*

*Sketch of proof.* The proof of the existence part of Theorem 6.3 consists of two parts.

In the first stage of the proof, we assume that $\inf_M a > 0$ and $\varepsilon \in (0, k_\star)$ satisfying (3.1.3). Following the first stage of the proof of Theorem 5.1, we first define

$$\mu^\varepsilon_{k^\varepsilon_1, q} = \inf_{u \in \mathcal{D}_q} F^\varepsilon_q(u)$$

where the set $\mathcal{D}_q$ is nothing but

$$\mathcal{D}_q = \left\{ u \in H^1(M) : k_\star \leqslant \|u\|^q_{L^q} \leqslant k_{2,q} \right\}.$$

With information that we have already proved in Lemma 6.5, we can easily go through Claims 1, 2, and 3 in the first stage of the proof of Theorem 6.1. In other words, we can prove the existence of at least one positive smooth solution to (3.0.1). Since there is no difference in the proofs, we omit the details here.

In the second stage of the proof, we assume $\inf_M a = 0$. Since we have no control on $\int_M a \,\mathrm{dvol}_g$, we can freely add small $\varepsilon_0 > 0$ to $a$ as in the proof of Theorem 6.1. Since the trick that was used in the proof of Theorem 6.1 still works in this new context, a sub- and super-solutions argument as used before concludes that (3.0.1) has at least one positive smooth solution. ■

## 6.7 Some remarks

In view of Theorem 6.2, Equation (3.0.1) always admits at least one positive smooth solution provided $\sup_M f \leqslant 0$. Let us now assume that the smooth function $f$ verifies $\sup_M f \leqslant 0$. For each $\lambda \in \mathbb{R}$, we denote

$$f_\lambda(x) = f(x) + \lambda, \quad x \in M.$$

Clearly, functions in the class $\{f_\lambda\}_\lambda$ are to be thought of as translation of the function $f$ by the value $\lambda$. Let us now consider the following equation

$$-\Delta_g u + hu = f_\lambda u^{2^\star - 1} + \frac{a}{u^{2^\star+1}}, \quad u > 0. \tag{6.7.1$_\lambda$}$$

Obviously, in view of Theorem 6.3 and the condition $\sup_M f \leqslant 0$, (6.7.1)$_\lambda$ always admits one positive smooth solution provided $\lambda \leqslant 0$. We are now interested in the case $\lambda > 0$. It was found that there is a critical number $\lambda^\star > 0$ which affects the number of positive smooth solutions of (6.7.1)$_\lambda$ when $\lambda$ crosses $\lambda^\star$. That is the content of the following theorem.

**Theorem 6.4.** *There exists a constant $\lambda^\star > 0$ such that*

(i) *Problem (6.7.1)$_\lambda$ has no positive smooth solution if $\lambda > \lambda^\star$.*
(ii) *Problem (6.7.1)$_\lambda$ has at least one positive smooth solution if $\lambda < \lambda^\star$.*

We now sketch a proof of this theorem.

*Proof.* In order to prove this theorem, let us observe from Theorem 6.2 that Equation (6.7.1)$_\lambda$ has at least one positive smooth solution for some small $\lambda > 0$ since $f_\lambda$ depends continuously on $\lambda$. In order to see this, our aim was to make use of Theorem 6.1.

Let us first observe that $\sup_M f_\lambda = \lambda$. Since $\int_M f \,\mathrm{dvol}_g < 0$, we can select $\lambda > 0$ small such that $\int_M f_\lambda \,\mathrm{dvol}_g < 0$. As usual, let us first suppose $\inf_M a > 0$. Now we show that there exists some $\tau > \max\{1, (\frac{2}{\mathcal{S}_h}\int_M h \,\mathrm{dvol}_g)^{\frac{2^\star}{2}}\}$ and some $\lambda \in (0,1)$ small enough such that

$$\int_M a \,\mathrm{dvol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1}n^n}\frac{\mathcal{S}_h}{\tau}\left(\frac{\mathcal{S}_h\tau}{\tau\sup f_\lambda - \int_M f_\lambda \,\mathrm{dvol}_g}\right)^{n-1}.$$

Indeed, we can start with small $\lambda$ such that $\frac{1}{\lambda} > \max\{1, (\frac{2}{\mathcal{S}_h}\int_M h \,\mathrm{dvol}_g)^{\frac{2^\star}{2}}\}$ and $\int_M f_\lambda \,\mathrm{dvol}_g < 0$. In particular, we can choose $\tau = \frac{1}{\lambda}$ and observe that

$$0 < 1 - \int_M f_\lambda \,\mathrm{dvol}_g = 1 - \lambda - \int_M f \,\mathrm{dvol}_g < 1 - \int_M f \,\mathrm{dvol}_g.$$

Therefore, a simple calculation shows that it suffices to show that

$$\int_M a \,\mathrm{dvol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1}n^n}\frac{\mathcal{S}_h^n}{\lambda^{n-2}}\left(\frac{1}{1 - \int_M f \,\mathrm{dvol}_g}\right)^{n-1},$$

or equivalently,

$$\lambda < \left(\frac{(2n-1)^{n-1}}{2^{2n-1}n^n}\frac{\mathcal{S}_h^n}{\int_M a \,\mathrm{dvol}_g}\right)^{\frac{1}{n-2}}\left(\frac{1}{1 - \int_M f \,\mathrm{dvol}_g}\right)^{\frac{n-1}{n-2}},$$

which proves the existence of some small $\lambda$ as claimed in the case $\inf_M a > 0$. In the case $\inf_M a = 0$, as in the second stage of the proof of Theorem 6.1, we simply replace $a$ by $a + \varepsilon_0$ for some small $\varepsilon_0 > 0$ and repeat the above procedure to obtain a super-solution. Since a sub-solution always exists, the existence result for small $\lambda$ follows.

Therefore, we can define

$$\lambda^\star = \sup_{\lambda \in \mathbb{R}} \{ \text{ (6.7.1)}_\lambda \text{ has at least one positive smooth solution } \}.$$

We now prove the following comparison: if $0 < \lambda_1 < \lambda_2 < \lambda^\star$ such that Problem $(6.7.1)_{\lambda_2}$ has at least one positive smooth solution, then Problem $(6.7.1)_{\lambda_1}$ also has at least one positive smooth solution. Indeed, suppose that $u_2$ is a positive smooth solution of $(6.7.1)_{\lambda_2}$, we then see that $u_2$ is a super-solution of $(6.7.1)_{\lambda_1}$ since $f_{\lambda_2} > f_{\lambda_1}$ pointwise. Having such an $u_2$, one can easily construct a sub-solution $u_1$ of $(6.7.1)_{\lambda_1}$ with $u_1 < u_2$. By the method of sub- and super-solutions, one can prove the existence of at least one positive smooth solution of $(6.7.1)_{\lambda_1}$. This concludes the comparison result.

In order to see why should we have $\lambda^\star < +\infty$, we make use of [21, Theorem 2.1]. Indeed, for sufficiently large $\lambda$, we obviously have $f_\lambda > 0$. Moreover, the following estimate

$$\left( \frac{n^n}{(n-1)^{n-1}} \right)^{\frac{n+2}{4n}} \int_M a^{\frac{n+2}{4n}} f_\lambda^{\frac{3n-2}{4n}} \,\mathrm{dvol}_g > \int_M h^{\frac{n+2}{4}} f_\lambda^{\frac{2-n}{4}} \,\mathrm{dvol}_g$$

holds provided $\lambda$ is sufficiently large, which immediately proves the finiteness of $\lambda^\star$ since $n \geqslant 3$. ■

Before closing this chapter, we should mention here the role of the size of $h$ in our study. To be exact, for given $a \geqslant 0$ and $f$ with $\int_M a \,\mathrm{dvol}_g > 0$, if $h$ is large enough, Equation (3.0.1) always possesses one positive smooth solution. Indeed, following the proof of Theorem 5.2, one can observe that (3.0.1) admits $\overline{u} = 1$ as a constant super-solution provided

$$-h + f + a \leqslant 0.$$

Therefore, if $h$ verifies

$$h \geqslant \sup_M |f| + \sup_M a \quad \text{ in } M,$$

one concludes that $\overline{u} = 1$ as a constant super-solution for (3.0.1). Since a sub-solution $\underline{u}$ with $\underline{u} \in (0,1)$ always exists, we have shown that (3.0.1) always possesses one positive smooth solution.

# 7

# The Liouville type theorem for positive solutions of the Lichnerowicz equations



The purpose of this chapter was to derive a Liouville type result for positive, smooth solutions of (3.0.1). For simplicity, in this chapter, we only consider the cases when $h$, $f$, and $a$ are constants with $a > 0$. Our aim here was to give some sufficient conditions so that (3.0.1) has only constant solution. By a scaling argument, we may assume that $a = 1$. Therefore, in this chapter, we are interested in the following model equation

$$-\Delta_g u + hu = \lambda u^q + u^{-q-2}, \quad u > 0, \tag{7.0.1}$$

where $h$ and $\lambda$ are constants. It is worth noticing that after using a scaling the sign of $h$ still matches the sign of the Yamabe-scalar field conformal invariant since $a > 0$. We also notice that the exponent $q > 0$ here is arbitrary and will be specified later. It is worth noting that we may consider the case $q < 0$ if physical problems motivate it.

Our result was inspired by a couple of recent papers by Ma et al. [29, 27]. In these papers, the authors considered the following model equation

$$-\Delta_g u = -u^q + u^{-q-2}, \quad u > 0, \tag{7.0.2}$$

in $\mathbb{R}^n$ with the standard metric where $q \in (1, 2^\star)$, that is, $h = 0$ and $\lambda = -1$. First, they proved in [29] that smooth positive solutions of (7.0.2) are uniformly bounded. Then by using the idea from Redheffer [39], Ma [27, Theorem 1] was able to prove that any smooth positive solution of (7.0.2) is constant, hence, is equal to 1. In [5], Brezis used a different approach to establish, among other things, such a Liouville type result. Besides, it was shown in [27, Theorem 2] that the similar Liouville type result is also true for smooth positive solutions for (7.0.2) in a complete non-compact Riemannian manifold with the Ricci curvature bounded from below.

Motivated by all discussion above, we prove in this chapter that any smooth positive solution for (7.0.1) in a complete compact Riemannian manifold with the Ricci curvature bounded from, whose the bound will be determined, is constant. To be precise, we now state our main result.

**Theorem 7.1.** *Let $(M, g)$ be a smooth closed Riemannian manifold of dimension $n \geqslant 3$. Let $h$, $\lambda$, and $q > 0$ be constants. Then there is a constant $K(n, q, h)$ depending only on $n$, $q$ and $h$ so that if $\mathrm{Ric}_g \geqslant K$ in the sense of quadratic forms, then every smooth positive solution of (7.0.1) is constant provided that in the case $h > 0$, $\lambda > 0$, we have to restrict $q \leqslant \frac{n+2}{n-2}$.*

Our first result can be formulated as in the following table.

| $h$ | $\lambda$ | $q$ | $\mathrm{Ric}_g \geqslant K$ |
|---|---|---|---|
| $h < 0$ | $\lambda < 0$ | $q \geqslant 1$ | $\mathrm{Ric}_g \geqslant 0$ |
| | | $q < 1$ | $\mathrm{Ric}_g \geqslant \frac{n-1}{n}(q-1)h$ |
| $h = 0$ | $\lambda < 0$ | $q > 0$ | $\mathrm{Ric}_g \geqslant 0$ |
| $h > 0$ | $\lambda < 0$ | $q > 0$ | $\mathrm{Ric}_g > -\frac{n-1}{n}h$ |
| | $\lambda > 0$ | $q \leqslant \frac{n+2}{n-2}$ | $\mathrm{Ric}_g \geqslant \frac{n-1}{n}(q-1)h$ |

**Table 7.1.** *The Liouville type result in terms of the Ricci curvature.*

Surprisingly enough, as can seen from the above table, the constant $K$ does not depend on $\lambda$. It is worth noticing that by integrating both sides of (7.0.2) over $M$, one easily gets that Equation (7.0.2) has no positive solution if $h \leqslant 0$ and $\lambda \geqslant 0$. The papers by Gidas–Spruck [19] and M. Véron–L. Véron [44] were sources of inspiration.

Finally, as usual, we should mention that the content of this chapter was adapted from [33].

## 7.1 Some basic computations

From now on, we shall prove Theorem 7.1. For simplicity, we shall use $\int_M$ to denote the integral with respect to the measure induced by the metric $g$. First, we need a preparation. Let us denote $u = v^{-\beta}$ for some $\beta \neq 0$ to be determined later. A direct computation shows that

$$\Delta_g v = (\beta+1)\frac{|\nabla v|^2}{v} + \frac{1}{\beta}(-\Delta_g u)v^{\beta+1}.$$

This and the fact that

$$-\Delta_g u = -hv^{-\beta} + \lambda v^{-\beta q} + v^{\beta(q+2)}$$

give

$$\Delta_g v = (\beta+1)\frac{|\nabla v|^2}{v} + \frac{1}{\beta}(-hv + \lambda v^{1-\beta(q-1)} + v^{1+\beta(q+3)}). \tag{7.1.1}$$

Applying the Bochner-Lichnerowicz-Weitzenböck formula [3] to function $v$, one obtains

$$\frac{1}{2}\Delta_g(|\nabla v|^2) = |\mathrm{Hess}(v)|^2 + \langle \nabla \Delta_g v, \nabla v\rangle + \mathrm{Ric}_g(\nabla v, \nabla v). \tag{7.1.2}$$

Multiply both sides of (7.1.2) by $v^\gamma$ where $\gamma \in \mathbb{R}$ will be chosen later and integrate on $M$, to obtain

$$A + B + C + D = 0, \tag{7.1.3}$$

where

$$A = \frac{1}{n}\int_M v^\gamma (\Delta_g v)^2 + \int_M v^\gamma \left(|\mathrm{Hess}(v)|^2 - \frac{1}{n}(\Delta_g v)^2\right), \tag{7.1.4}$$

$$B = \int_M v^\gamma \langle \nabla \Delta_g v, \nabla v\rangle,$$

$$C = -\frac{1}{2}\int_M v^\gamma \Delta_g(|\nabla v|^2),$$

and

$$D = \int_M v^\gamma \mathrm{Ric}_g(\nabla v, \nabla v).$$

We notice that $\gamma$ may not necessarily be nonzero. Besides, there holds

$$J = \int_M v^\gamma \left(|\mathrm{Hess}(v)|^2 - \frac{1}{n}(\Delta_g v)^2\right) \geqslant 0$$

since it is well-known that

$$|\mathrm{Hess}(v)|^2 - \frac{1}{n}(\Delta_g v)^2 \geqslant 0.$$

### 7.1.1 Computations of $A$, $B$, and $C$

We treat the first term of $A$ in the following way. In fact, using (7.1.1), one obtains

$$\begin{aligned}
&\frac{1}{n}\int_M v^\gamma (\Delta_g v)^2 \\
&= \frac{1}{n}\int_M v^\gamma (\Delta_g v)\left((\beta+1)\frac{|\nabla v|^2}{v} + \frac{1}{\beta}(-hv + \lambda v^{1-\beta(q-1)} + v^{1+\beta(q+3)})\right) \\
&= \frac{\beta+1}{n}\int_M v^{\gamma-1}|\nabla v|^2\left((\beta+1)\frac{|\nabla v|^2}{v} + \frac{1}{\beta}(-hv + \lambda v^{1-\beta(q-1)} + v^{1+\beta(q+3)})\right) \\
&\quad + \frac{1}{n\beta}\int_M (-hv^{\gamma+1} + \lambda v^{\gamma+1-\beta(q-1)} + v^{\gamma+1+\beta(q+3)})(\Delta_g v) \\
&= \frac{(\beta+1)^2}{n}\int_M v^{\gamma-2}|\nabla v|^4 \\
&\quad + \frac{\beta+1}{n\beta}\int_M |\nabla v|^2(-hv^\gamma + \lambda v^{\gamma-\beta(q-1)} + v^{\gamma+\beta(q+3)}) \\
&\quad + \frac{1}{n\beta}\int_M (-hv^{\gamma+1} + \lambda v^{\gamma+1-\beta(q-1)} + v^{\gamma+1+\beta(q+3)})(\Delta_g v).
\end{aligned}$$

Therefore, this and (7.1.4) imply that

$$
\begin{aligned}
A =& J + \frac{(\beta+1)^2}{n} \int_M v^{\gamma-2}|\nabla v|^4 \\
&+ \frac{\beta+1}{n\beta} \int_M |\nabla v|^2(-hv^\gamma + \lambda v^{\gamma-\beta(q-1)} + v^{\gamma+\beta(q+3)}) \\
&+ \frac{1}{n\beta} \int_M (-hv^{\gamma+1} + \lambda v^{\gamma+1-\beta(q-1)} + v^{\gamma+1+\beta(q+3)})(\Delta_g v).
\end{aligned} \tag{7.1.5}
$$

By the divergence theorem, it holds

$$
\begin{aligned}
\int_M v^{\gamma+1}\Delta_g v &= -(\gamma+1)\int_M v^\gamma |\nabla v|^2, \\
\int_M v^{\gamma+1-\beta(q-1)}\Delta_g v &= -(\gamma+1-\beta(q-1))\int_M v^{\gamma-\beta(q-1)}|\nabla v|^2, \\
\int_M v^{\gamma+1+\beta(q+3)}\Delta_g v &= -(\gamma+1+\beta(q+3))\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2.
\end{aligned}
$$

Therefore, we can further simplify (7.1.5) as follows

$$
\begin{aligned}
A =& J + \frac{(\beta+1)^2}{n} \int_M v^{\gamma-2}|\nabla v|^4 \\
&+ \frac{\beta+1}{n\beta} \int_M |\nabla v|^2(-hv^\gamma + \lambda v^{\gamma-\beta(q-1)} + v^{\gamma+\beta(q+3)}) \\
&- \frac{\lambda}{n\beta}(\gamma+1-\beta(q-1)) \int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 \\
&+ \frac{1}{n\beta}(\gamma+1)\int_M hv^\gamma|\nabla v|^2 - \frac{1}{n\beta}(\gamma+1+\beta(q+3))\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2.
\end{aligned}
$$

Thus, finally, we have

$$
\begin{aligned}
A =& J + \frac{(\beta+1)^2}{n}\int_M v^{\gamma-2}|\nabla v|^4 + \frac{h(\gamma-\beta)}{n\beta}\int_M v^\gamma|\nabla v|^2 \\
&- \frac{\lambda(\gamma-\beta q)}{n\beta}\int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 - \frac{\gamma+\beta(q+2)}{n\beta}\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2.
\end{aligned} \tag{7.1.6}
$$

For the term $B$, again using (7.1.1), we have

$$
\begin{aligned}
B =& \int_M v^\gamma \left\langle \nabla\left((\beta+1)\frac{|\nabla v|^2}{v} + \frac{1}{\beta}(-hv + \lambda v^{1-\beta(q-1)} + v^{1+\beta(q+3)})\right), \nabla v\right\rangle \\
=& (\beta+1)\int_M \left\langle \nabla\left(\frac{|\nabla v|^2}{v}\right), v^\gamma \nabla v\right\rangle \\
&- \frac{h}{\beta}\int_M v^\gamma|\nabla v|^2 + \lambda\frac{1-\beta(q-1)}{\beta}\int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 \\
&+ \frac{1+\beta(q+3)}{\beta}\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2.
\end{aligned}
$$

Notice that

$$
\int_M \left\langle \nabla\left(\frac{|\nabla v|^2}{v}\right), v^\gamma\nabla v\right\rangle = -\int_M \frac{|\nabla v|^2}{v}\nabla\cdot(v^\gamma\nabla v)
$$

$$= -\gamma \int_M v^{\gamma-2}|\nabla v|^4 - \int_M v^{\gamma-1}|\nabla v|^2 \Delta_g v.$$

Therefore,

$$\begin{aligned} B = & -(\beta+1)\gamma \int_M v^{\gamma-2}|\nabla v|^4 - (\beta+1)\int_M v^{\gamma-1}|\nabla v|^2 \Delta_g v \\ & - \frac{h}{\beta}\int_M v^\gamma |\nabla v|^2 + \lambda \frac{1-\beta(q-1)}{\beta}\int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 \\ & + \frac{1+\beta(q+3)}{\beta}\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2. \end{aligned}$$

Again we use (7.1.1) to reach at

$$\begin{aligned} B = & -(\beta+1)\gamma \int_M v^{\gamma-2}|\nabla v|^4 \\ & -(\beta+1)\int_M v^{\gamma-1}|\nabla v|^2 \left((\beta+1)\frac{|\nabla v|^2}{v} + \frac{1}{\beta}(-hv + \lambda v^{1-\beta(q-1)} + v^{1+\beta(q+3)})\right) \\ & - \frac{h}{\beta}\int_M v^\gamma |\nabla v|^2 + \lambda \frac{1-\beta(q-1)}{\beta}\int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 \\ & + \frac{1+\beta(q+3)}{\beta}\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2. \end{aligned}$$

By simplifying the right hand side of the above identity, one gets

$$\begin{aligned} B = & -(\beta+1)(\gamma+\beta+1)\int_M v^{\gamma-2}|\nabla v|^4 \\ & + h\int_M v^\gamma|\nabla v|^2 - \lambda q \int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 + (q+2)\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2. \end{aligned} \tag{7.1.7}$$

For the term $C$, we first observe that

$$\int_M v^\gamma \Delta_g(|\nabla v|^2) = \int_M \Delta_g(v^\gamma)|\nabla v|^2.$$

Therefore,

$$\begin{aligned} C = & -\frac{1}{2}\int_M |\nabla v|^2 \Delta_g(v^\gamma) \\ = & -\frac{1}{2}\int_M |\nabla v|^2 \left(\gamma v^{\gamma-1}\Delta_g v + \gamma(\gamma-1)v^{\gamma-2}|\nabla v|^2\right) \\ = & -\frac{1}{2}\int_M \gamma v^{\gamma-1}|\nabla v|^2 \left((\beta+1)\frac{|\nabla v|^2}{v} + \frac{1}{\beta}(-hv + \lambda v^{1-\beta(q-1)} + v^{1+\beta(q+3)})\right) \\ & - \frac{\gamma(\gamma-1)}{2}\int_M v^{\gamma-2}|\nabla v|^4. \end{aligned}$$

In other words,

$$\begin{aligned} C = & \frac{\gamma h}{2\beta}\int_M v^\gamma|\nabla v|^2 - \frac{\gamma\lambda}{2\beta}\int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 - \frac{\gamma}{2\beta}\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2 \\ & - \frac{\gamma(\gamma+\beta)}{2}\int_M v^{\gamma-2}|\nabla v|^4. \end{aligned} \tag{7.1.8}$$

We now have enough information to treat (7.1.3).

### 7.1.2 The transformed equation

By using (7.1.6)-(7.1.8), one can see that (7.1.3) reduces to

$$\begin{aligned}
J &+ \left(\frac{(\beta+1)^2}{n} - (\beta+1)(\gamma+\beta+1) - \frac{\gamma(\gamma+\beta)}{2}\right)\int_M v^{\gamma-2}|\nabla v|^4 \\
&+ \left(\frac{h(\gamma-\beta)}{n\beta} + h + \frac{\gamma h}{2\beta}\right)\int_M v^{\gamma}|\nabla v|^2 \\
&+ \left(-\frac{\lambda(\gamma-\beta q)}{n\beta} - \lambda q - \frac{\gamma\lambda}{2\beta}\right)\int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 \\
&+ \left(-\frac{\gamma+\beta(q+2)}{n\beta} + (q+2) - \frac{\gamma}{2\beta}\right)\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2 \\
&+ \int_M v^{\gamma}\mathrm{Ric}_g(\nabla v, \nabla v) = 0.
\end{aligned} \tag{7.1.9}$$

For simplicity, we rewrite Equation (7.1.9) as

$$\begin{aligned}
J + a\int_M v^{\gamma-2}|\nabla v|^4 &+ \int_M v^{\gamma}(c|\nabla v|^2 + \mathrm{Ric}_g(\nabla v, \nabla v)) \\
+ b\int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 &+ d\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2 = 0,
\end{aligned}$$

where

$$\begin{aligned}
a &= -\frac{1}{2}\left(\gamma^2 + (3\beta+2)\gamma + 2\frac{n-1}{n}(\beta+1)^2\right), \\
b &= \frac{\lambda(n+2)}{2n}\left(-\frac{\gamma}{\beta} - 2q\frac{n-1}{n+2}\right), \\
c &= \frac{n+2}{2n}\left(\frac{\gamma}{\beta} + 2\frac{n-1}{n+2}\right)h, \\
d &= \frac{n+2}{2n}\left((q+2)\frac{2(n-1)}{n+2} - \frac{\gamma}{\beta}\right).
\end{aligned}$$

Next we wish to describe the method used in the present paper. Our goal was to find $\beta \neq 0$ and $\gamma \in \mathbb{R}$ such that

$$a \geqslant 0, \quad b \geqslant 0, \quad \mathrm{Ric}_g + cg \geqslant 0, \quad d \geqslant 0. \tag{7.1.10}$$

Having (7.1.10), our result follows easily since $|\nabla v| = 0$ since $a \geqslant 0$ forces $d > 0$. Thus, the key point is $a \geqslant 0$. To better serve this purpose, we set $y = 1 + \frac{1}{\beta}$ and $\delta = -\frac{\gamma}{\beta}$ where $y \neq 1$ and $\delta \in \mathbb{R}$. Thus the set of conditions in (7.1.10) becomes

$$2\frac{n-1}{n}y^2 - 2\delta y + \delta^2 - \delta \leqslant 0, \tag{7.1.11}$$

$$\lambda\left(\delta - 2q\frac{n-1}{n+2}\right) \geqslant 0, \tag{7.1.12}$$

$$\frac{2n}{n+2}\mathrm{Ric}_g \geqslant h\left(\delta - 2\frac{n-1}{n+2}\right)g, \tag{7.1.13}$$

and

$$\delta \geqslant -2(q+2)\frac{n-1}{n+2}. \tag{7.1.14}$$

In view of (7.1.11), it is necessary to have

$$\frac{\delta}{n}(2(n-1)-(n-2)\delta) \geqslant 0$$

which is equivalent to

$$0 \leqslant \delta \leqslant \frac{2(n-1)}{n-2}. \tag{7.1.15}$$

With (7.1.15) in hand, one can see that (7.1.14) is automatically satisfied. Moreover, $d > 0$ provided $\delta \geqslant 0$. Thus, our set of conditions now reduces to (7.1.12), (7.1.13), and (7.1.15). Notice that if inequalities in (7.1.15) are strict, then we can always find some $y \neq 1$ verifying (7.1.11).

## 7.2 Proof of Theorem 7.1

For the sake of clarity, we split our studying into four cases depending on the sign of $h$ and $\lambda$.

### 7.2.1 The case $h < 0$

In this case, it is necessary to have $\lambda < 0$. Then the condition (7.1.12) and the lower bound for $\delta$ in (7.1.15) imply

$$0 \leqslant \delta \leqslant 2q\frac{n-1}{n+2}. \tag{7.2.1}$$

Combining (7.1.15) and (7.2.1) gives

$$0 \leqslant \delta \leqslant \min\left\{2q\frac{n-1}{n+2}, \frac{2(n-1)}{n-2}\right\}. \tag{7.2.2}$$

There are two possible sub-cases.

*Case 1.* Suppose $q \geqslant \frac{n+2}{n-2}$. Then we claim that, with $K = 0$, we can always select $\delta$ such that it satisfies (7.1.12), (7.1.13), and (7.1.15). To this end, we notice that the right hand side of (7.1.13) is always non-negative. In order to see that (7.1.12), (7.1.13), and (7.1.15) hold, we have to select $\delta = \frac{2(n-1)}{n+2}$. Then we have to choose $y$ such that Equation (7.1.11) holds. However we are left without many choices but one, that is, $y = \frac{n}{n+2} \neq 1$. This is enough to serve our purpose since the left hand side of (7.1.11) equals $-\frac{8(n-1)}{(n+2)^2}$ when $y$ is equal to $y = \frac{n}{n+2}$.

*Case 2.* Suppose $q < \frac{n+2}{n-2}$. Then $\delta$ needs to satisfy (7.2.1). With this region for $\delta$, the right hand side of (7.1.13) is not smaller than $2(q-1)\frac{n-1}{n+2}hg$. Thus, we select

$$K = \frac{n-1}{n}(q-1)h.$$

Similarly as above, then we may choose $\delta = \frac{2q(n-1)}{n+2}$ to make sure that (7.1.13), and (7.1.15) hold. Now it is easy to find some $y \neq 1$ satisfying (7.1.11) since the solution of

(7.1.11) is an interval. For example, when $q \ne \frac{n+2}{n}$, we may choose $y = \frac{qn}{n+2}$ since the left hand side of (7.1.11) now equals

$$\frac{2q(n-1)(q(n-2)-(n+2))}{(n+2)^2}.$$

Otherwise, we may choose $y = 1 - \sqrt{\frac{2}{n}}$, in this case, the left hand side of (7.1.11) now vanishes.

### 7.2.2 The case $h = 0$

Under this case, the right hand side of (7.1.13) vanishes, thus it is enough to take $K = 0$. Besides, it is necessary to have $\lambda < 0$. Therefore, $\delta$ must satisfy (7.2.2). It is now a simple task to find some $\delta$ and $y \ne 1$ verifying both conditions (7.1.12) and (7.1.15).

### 7.2.3 The case $h > 0$, $\lambda \leqslant 0$

Again, in this context, $\delta$ has to satisfy (7.2.2). First we show that $K = -\frac{n-1}{n}h$ is enough. Indeed, this condition can be rewritten as

$$\frac{2n}{n+2}\mathrm{Ric}_g \geqslant h\left(-2\frac{n-1}{n+2}\right)g. \tag{7.2.3}$$

Under the condition (7.2.3), we have to select $\delta = 0$. In order to see how could this choice of $\delta$ work, we just go back to (7.1.9) to get

$$\begin{aligned} J &- \frac{n-1}{n}(\beta+1)^2\int_M \frac{|\nabla v|^4}{v^2} \\ &+ \frac{n-1}{n}h\int_M |\nabla v|^2 - \frac{n-1}{n}\lambda q\int_M v^{-\beta(q-1)}|\nabla v|^2 \\ &+ \frac{n-1}{n}(q+2)\int_M v^{\beta(q+3)}|\nabla v|^2 + \int_M \mathrm{Ric}_g(\nabla v, \nabla v) = 0. \end{aligned} \tag{7.2.4}$$

Clearly, we have no choice but $\beta = -1$ or equivalently, $y = 0$. With this choice of $y$, we immediately see that the left hand side of (7.2.4) is non-negative. This forces $\nabla v = 0$ thus giving us the desired result.

### 7.2.4 The case $h > 0$, $\lambda > 0$

Under this case, it follows from (7.1.12) and (7.1.15) that

$$2q\frac{n-1}{n+2} \leqslant \delta \leqslant \frac{2(n-1)}{n-2}.$$

In other words, it is necessary to have $q \leqslant \frac{n+2}{n-2}$. Our choice for $K$ is that

$$K = \frac{n-1}{n}(q-1)h.$$

We will see how this condition is enough for our argument.

*Case 1.* Suppose $q < \frac{n+2}{n-2}$. We rewrite the condition for Ricci curvature in the following way

$$\frac{2n}{n+2}\mathrm{Ric}_g \geqslant h\left(2q\frac{n-1}{n+2} - 2\frac{n-1}{n+2}\right)g.$$

Thus, we may choose $\delta = \frac{2q(n-1)}{n+2}$. Consequently, the conditions (7.1.12) and (7.1.13) clearly hold. Therefore, we may select $y \neq 1$ verifying (7.1.11) since $\delta \in (0, \frac{2(n-1)}{n-2})$ as we have already done in the second case when $h < 0$.

*Case 2.* Suppose $q = \frac{n+2}{n-2}$. Then necessarily $\delta = \frac{2(n-1)}{n-2}$ which verifies (7.1.12). The condition for Ricci curvature can be rewritten as

$$\frac{2n}{n+2}\mathrm{Ric}_g \geqslant h\left(2\frac{n-1}{n-2} - 2\frac{n-1}{n+2}\right)g = \left(\frac{8h(n-1)}{n^2-4}\right)g.$$

Thus, we can pick $K = \frac{4h(n-1)}{n(n-2)}$ and clearly (7.1.13) holds. It suffices to find some $y \neq 1$ verifying (7.1.11). Due to the fact that $q = \frac{n+2}{n-2}$, we only have one choice for $y$, that is, $y = \frac{nq}{n+2}$. Thanks to $q = \frac{n+2}{n-2}$, we immediately see that $y = \frac{n}{n-2} \neq 1$. With this, the left hand side of (7.1.11) vanishes as required.

### 7.2.5 Proof of Theorem 7.1 completed

Finally let us assume that $u$ is a smooth positive solution of Equation (7.0.2). From our discussion above, we know that all inequalities in (7.1.10) are fulfilled. In fact, we have already shown that $d > 0$. Consequently,

$$\int_M v^{\gamma+\beta(q+3)}|\nabla v|^2 = 0$$

which implies that $v$, hence $u$ is a constant.

# 8

# Conclusion

In an effort to understand the Einstein equations with sources, the primary goal of this thesis was to study some quantitative properties of solutions of the following Lichnerowicz-scalar field constraint equations,

$$-\Delta_g u + hu = fu^{\frac{n+2}{n-2}} + au^{-\frac{3n-2}{n-2}}, \quad u > 0,$$

such as the existence, the non-existence, the multiplicity, and the Liouville property of positive solutions in all cases where the corresponding Yamabe-scalar field invariant may take either negative, zero, or positive sign. These Lichnerowicz-scalar field constraint equations naturally arise in the Einstein-scalar field constraint equations for the Cauchy problem for the Einstein field equations.

Our main results, included in Theorems 4.1, 4.2, 5.1, 5.2, 5.3, 6.1, 6.2, 6.3, 7.1, demonstrated that a suitable balance between coefficients $h$, $f$, $a$ of the Lichnerowicz-scalar field equations is enough to guarantee the existence of one positive smooth solution. In addition, it was found that under some further conditions we may or we may not have the uniqueness property of solutions of the Lichernowicz-scalar field equations.

In order to seek for solutions, variational methods were used to examine the Lichnerowicz-scalar field equations. This is because these equations have variational structures. However, since the constraint equations include two terms, the term $au^{-\frac{3n-2}{n-2}}$ with a negative exponent and the term $fu^{\frac{n+2}{n-2}}$ with a sign changing function $f$, it is well-known that standard variational methods do not work in this context. This is because the term $au^{-\frac{3n-2}{n-2}}$ forces the associated energy functional to be unbounded from above while the term $fu^{\frac{n+2}{n-2}}$ forces the functional to be unbounded from below. In addition, when making use of those variational methods, the compactness property of the embedding $H^1(M) \hookrightarrow L^{2^\star}(M)$ is crucial. Such a compactness property is no longer available in our study due to the presence of the term $fu^{\frac{n+2}{n-2}}$ with a critical exponent in the Lichnerowicz-scalar field equations.

To avoid these difficulties, we developed a suitable variational method. Along with the development of this new method, some basic and essential results for standard variational methods were successfully carried into the new method. An important feature of this new approach is that it is possible to deal with equations whose forms have more than three terms. Such an equation, for example, comes from the Einstein-Maxwell-scalar field equations or Einstein-scalar field equations with the cosmological constant.

Having these theorems in hand, one can observe that two aspects of the structure of the Lichernowicz-scalar field equations can be drawn.

First, it was shown that the term $fu^{\frac{n+2}{n-2}}$ with a critical exponent, represented by the dependence of the potential of the scalar field source, plays a central role in the analysis of solvability when $f$ may change sign. This fact partially tells us that any small perturbation with either a negative exponent or a subcritical exponent does not affect the solvability of original equation.

Compared with other problems such as the prescribing scalar curvature equations, which have simpler form than those of the Lichernowicz-scalar field equations, this fact also suggests that the structure of the Lichernowicz-scalar field equations may also depend on the topology of $(M, g)$, and thus could be studied using topology methods. We leave this topic for our study in the future.

Second, it was also shown that there is a certain difference between conformal classes with different Yamabe-scalar field invariants. While the non-positive Yamabe-scalar field invariant $h \leqslant 0$ involves more conditions and our analysis of solvability of the Lichnerowicz-scalar field equations strongly depends on the ratio between $\sup_M f$ and $\int_M |f^-| \, d\mathrm{vol}_g$, the positive Yamabe-scalar field invariant $h > 0$ requires fewer conditions than the non-positive case. To be precise, our main result for the case $h > 0$ showed that the Lichnerowicz-scalar field equations always admit one positive smooth solution so long as $\int_M a \, d\mathrm{vol}_g$ is sufficiently small in the sense that $\int_M a \, d\mathrm{vol}_g$ is bounded from above by some constant depending on $f$. One of the by-products of this result is an interesting result arising from the prescribing scalar curvature equations in the positive case. To be precise, it can be proved that the there always exists a positive, smooth, super-solution to the prescribing scalar curvature equations in the positive case

$$-\Delta_g u + hu = fu^{\frac{n+2}{n-2}}, \quad u > 0.$$

In order to complete the proof, it suffices to find a suitable sub-solution. To the best of our knowledge, such a sub-solution to this equation has long been believed to exist.

Overall, our findings are of crucial importance and intriguing because they not only give us an answer to the question of which sets of conformal data lead to solutions and which do not in the Einstein field equations, but also introduce a new approach to study other similar equations. However, in view of the theory of the Einstein field equations, our findings are still far from the fully understanding of the solutions of the Einstein-scalar field constraint equations since we always assume throughout our study that $(M, g)$ has constant mean curvature. This assumption on the mean curvature basically reflects the complexity of the structure of the Einstein-scalar field constraint equations. Despite several progress that recently have been made [15, 18], in general, we unfortunately do not know what happens if $(M, g)$ does not have constant mean curvature. For this reason, the more we understand the non-constant mean curvature case, the more we understand the whole Einstein-scalar field equations. We also leave this topic for our study in the future.

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