DESIGN AND IMPLEMENTATION OF MODEL PREDICTIVE CONTROL APPROACHES

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Contents

Summary

Model Predictive Control (MPC) refers to an ample range of control algorithms which make explicit use of a model of the process for prediction and obtain the control signals by minimizing an objective function. Originated in the late seventies, MPC has been developed considerably and are widely accepted by both the academia and industry due to its generality, optimality, simplicity and the capability to handle constraints. It is a totally open methodology based on certain basic principles which allows for various extension and diverse applications. In this thesis, the MPC principle is applied to three types of specific dynamics and one specific objective function and lastly the implementation issue is discussed.

Firstly, the thesis proposes a simple dead time compensator for linear monovariable systems. A setpoint weighting function, appended with the popular PID feedback loop, is designed to compensate the dead time such that it is removed from the denominator of setpoint response transfer function. The advantage of the proposed approach is that it can improve the control performance while retaining the PID feedback loop. In addition, it is applicable to stable, integrating and unstable systems.

Secondly, a robust minimum time controller for linear periodic systems with external disturbances is proposed. Parallel to the case of LTI systems, the maximal robust periodic positive invariant sets are formulated for linear periodic systems. The state trajectory is designed to evolve from an outer stabilizable set to an inner one step by step, and finally to reach the maximal robust periodic positive invariant sets in spite of external disturbances. The online optimization is efficient since only one step optimization is required. Moreover, the computation can be simplified further if multi-parametric method is adopted for the control law computation.

An output MPC for linear parameter varying systems is proposed next. To handle various uncertainties

in the feedback loop, disturbance invariant tube and quasi-min-max approach are combined: disturbance invariant tube is used to describe the prediction error component caused by external disturbance; parametric model uncertainty is handled by quasi-min-max approach. The resulting optimization is a linear matrix inequality problem and the complexity is similar to MPC for state feedback case.

The fourth contribution of this thesis is two new stabilizing MPC controllers, based on a terminal cost function approach and a stability constraint enforcement approach, for economic optimization. Two methods for the design of terminal cost functions are proposed for the first approach. The second approach enforces closed loop stability by inserting a regulation cost decreasing constraint into the economic optimization problem. The proposed approaches relax the conditions required by previous methods and are applicable to more general dynamic models and economic performance functions.

The last contribution of the thesis is an implementation scheme for robust MPC if the online computation is intolerable. The overall dual-time-scale controller composes of a fast compensator in inner loop and a slow MPC in outer loop. The computation delay is explicitly compensated in the MPC design. Four MPC variants for linear/nonlinear systems in the literature are tailored to fit in the proposed control structure, which provides a practical solution to implement MPC.

The thesis adopts the synthesis MPC approach. Therefore, stability and constraints satisfaction are guaranteed rigorously. Simulation and experimental studies are provided to validate the proposed approaches.

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DOF	Degree of Freedom	MPC	Model Predictive Control
DRTO	Dynamic RTO	MRPIS	Minimal Robust Positively Invariant Set
DTC	Dead Time Compensator	NEC	Neighboring Extremal Control
FCS	Field-bus Control System	NLP	Nonlinear Programming
FOPDT	First Order Plus Dead Time	NMPC	Nonlinear MPC
IPDT	Integrator Plus Dead Time	PLDI	Polytopic Linear Differential Inclusion
ISS	Input-to-state Stable	PMPIS	Periodic Maximal Positive Invariant Sets
IWPp	Invariant With Probability p	PSS	Periodic Stabilizable Sets
LMI	Linear Matrix Inequality	QP	Quadratic Programming
LPV	Linear Parameter Varying	RHC	Receding Horizon Control
LQR	Linear Quadratic Regulator	RTO	Real Time Optimization
LTI	Linear Time Invariant	SOPDT	Second Order Plus Dead Time

Chapter 1

Introduction

This thesis is concerned with the control of systems from predictive control perspective. In particular, Model Predictive Control (MPC) design for specific dynamics–monovariable linear systems with dead time, linear periodic systems and linear parameter varying (LPV) systems, specific performance index–economic optimization, and an implementation scheme of MPC are presented. This chapter provides the background, literature survey, objectives and scope, and the organization of the thesis.

1.1 Background

The analysis and control of physical systems are often centered around mathematical models. State space model has become the most popular framework under which system theory and various control methodologies have been developed since modern control theory flourished in 1960s. However, the complexity of the models and the way how to utilize the models differ in each control methodology. A rather general model in discrete time domain is described by:

$$x(t+1) = f(x(t), u(t)), \tag{1.1}$$

where $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}$ are state variable and control input, respectively.

Most of control methodologies, such as Optimal Control, Robust Control, Nonlinear Control etc, assume certain form of system model for controller design. For example, input-affine structure is normally required for Lyapunov function based synthesis approach; feedback linearization control requires specific differential geometric structure of $f(\cdot,\cdot)$; strict feedback structure of $f(\cdot,\cdot)$ is required for back stepping design. One control methodology that can control a great variety of processes is Model Predictive Control, originated in 1970s and extensively studied since then[2, 4, 6, 9, 12, 13, 15, 42]. It does not designate a specific algorithm but an open methodology, which can be applied to general models with only mild conditions on $f(\cdot,\cdot)$. This advantage is much more significant when the model (1.1) is restricted to satisfy certain constraints on states and control inputs:

$$x \in \mathbb{X} \text{ and } u \in \mathbb{U}, \tag{1.2}$$

where $\mathbb{X} \subset \mathbb{R}^{n_x}, \mathbb{U} \subset \mathbb{R}^{n_u}$ are state and input constraint sets, respectively. These constraints, such as actuator saturation, safety limit, etc, are important model parameters in practice. Simply omitting these in the controller design may lead to constraints violation and hence unpredictable dynamic behaviors or even severe damage to the systems. While most control methodologies are invalid for constrained systems, MPC is perfectly suitable to deal with constraints, which contributes to its popularity in process industry, in which optimal operation near equipment limits is crucial [5].

The capability to address general models and even constrained systems is accredited to MPC's unique design philosophy:

$$\textbf{model} \;\Rightarrow\; \textbf{prediction} \;\Rightarrow\; \textbf{optimization} \;\Rightarrow\; \textbf{control action}$$

As it indicated, prediction and optimization serve as a bridge linking together the model and control action. MPC explicitly makes use of a model to predict the future state trajectory, which is determined by the current state $x(t)$ and the predicted control input $\mathbf{u}(t) = \{u(0|t), u(1|t), \ldots, u(N-1|t)\}$ over a prediction horizon of N steps.

Assumption 1.1. *The state variable $x(t)$ is measurable.*

For any predicted state and input trajectories, a performance function is defined as

$$J(x(t), \mathbf{u}(t)) := \sum_{i=0}^{N-1} l(x(i|t), u(i|t)) + V_f(x(N|t)), \tag{1.3}$$

where $l(\cdot,\cdot)$ and $V_f(\cdot)$ are stage cost function and terminal cost function, respectively. Unlike other controllers directly seeking for an explicit feedback function $\mu(x)$, MPC is less ambitious that at the sampling time t it only provides the value of $\mu(x(t))$, to be determined by a finite horizon optimal control problem, referred to as $\mathscr{P}_N(x(t))$:

$$\arg\min_{\mathbf{u}(t)} \quad J(x(t), \mathbf{u}(t)) \tag{1.4}$$

$$subject\ to \begin{cases} x(0|t) = x(t), \\ x(i+1|t) = f(x(i|t), u(i|t)), \\ x(i|t) \in \mathbb{X}, u(i|t) \in \mathbb{U}, i \leq N-1, \\ x(N|t) \in X_f, \end{cases} \tag{1.5}$$

where $X_f \subset \mathbb{R}^n$ is the terminal constraint set. The optimal control sequence to $\mathscr{P}_N(x(t))$ is denoted as $\mathbf{u}^*(t) = \{u^*(0|t), u^*(1|t), \ldots, u^*(N-1|t)\}$. $u^*(0|t)$, the first element of $\mathbf{u}^*(t)$, is applied to system (1.1) at time t. At the next sampling time $t+1$, the prediction horizon recedes one step. With the new measurement $x(t+1)$, $\mathscr{P}_N(x(t+1))$ is solved and $u^*(0|t+1)$ is applied. That is how feedback is introduced in the control loop. The MPC feedback control law is implicitly constructed as:

$$\mu(x(t)) = u^*(0|t). \tag{1.6}$$

The receding horizon principle earns MPC another name, Receding Horizon Control (RHC). The seminal idea of receding horizon control was expressed in [3] in the context of Optimal Control as a method for obtaining a closed loop controller from open loop trajectory optimization. Most MPC variants in the literature can be cast into the formulation $\mathscr{P}_N(\cdot)$, in which X_f and $V_f(\cdot)$ are important design parameters to endorse the finite horizon formulation closed loop stability. The most popular stabilizing design strategy is summarized by [1] as stability axiom:

1. $X_f \subset \mathbb{X}$, X_f closed, $0 \in X_f$.

2. $\kappa_f(x) \in \mathbb{U}$, $\forall x \in X_f$.

3. $f(x, \kappa_f(x)) \in X_f$, $\forall x \in X_f$.

4. $V_f(f(x, \kappa_f(x))) - V_f(x) + l(x, \kappa_f(x)) \geq 0, \forall x \in X_f$.

If the above assumptions on the design parameters are satisfied, the MPC controller (1.6) asymptotically stabilizes system (1.1).

When the design details is temporarily ignored, three constituting elements[4], model, performance function, and computation and implementation schemes, are identified by inspecting the overall structure of $\mathscr{P}_N(\cdot)$ as shown in Fig.1.1. In fact, the various MPC algorithms basically only differ amongst themselves in the model used to represent the plant, the performance function for optimization and the computation and implementation schemes.

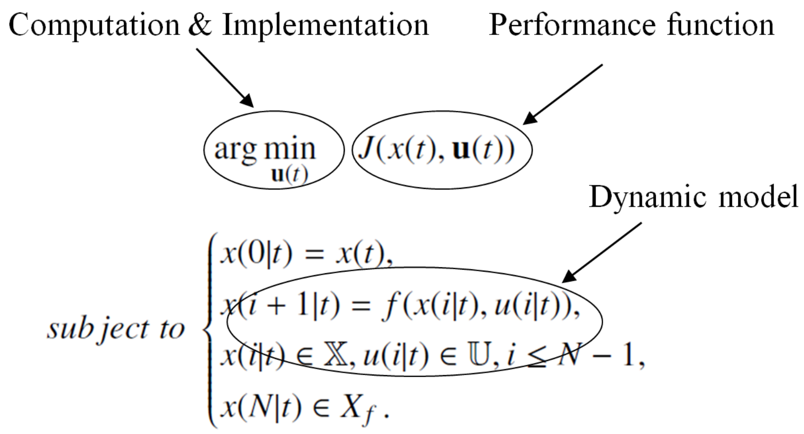

Figure 1.1: Three Constituting Elements in MPC

Model, used to produce prediction, is the cornerstone of MPC. Linear Time Invariant (LTI) model is widely used as a special case of (1.1). Since there always exists discrepancy between a mathematical model, like (1.1), and the real plant, it is desirable to generalize (1.1) by including uncertainty into the model:

$$x(t+1) = f(x(t), u(t), w(t)), \tag{1.7}$$

where $w(t) \in \mathbb{R}^{n_\omega}$ is the uncertain variable, which is not known exactly, but usually is assumed to be bounded by a given set. The nondeterministic model is essential to design MPC with robustness guarantee. Another generalization to (1.1) is to allow the dynamic varying with time:

$$x(t+1) = f(x(t), u(t), w(t), t). \tag{1.8}$$

It is obvious that the choice of model has to capture the dominant dynamic of the real plant and the control performance will be affected by model accuracy.

Performance function is the most important design parameter, deciding the closed loop performance, if the system model is given a priori. To keep the process as close as possible to the reference trajectory, it usually takes the form of a H_2 norm, $1-$norm or $\infty-$ norm functions of the difference between the predicted output and the reference trajectory. Another more intuitive choice is to optimize the dynamic behavior to achieve maximal profit during process operation. MPC, optimizing a economic performance function, is called as Economic MPC.

Computation and implementation decide whether $\mathscr{P}_N(x(t))$ is a practically feasible or only a conceptual solution. The extensive computation to solve the resulting optimization is one of the drawbacks of MPC, often limiting its application only to quite slow processes, like petrochemical and refining industries. Efficient optimization solver and implementation scheme are required to bring MPC into use.

As basic constituting elements for MPC, different options can be chosen for each element giving rise to different algorithms, which deserve individual study to explore the unique characteristics of the diverse dynamic models, performance functions, and computation and implementation schemes. In this thesis, MPC for three dynamic models, monovariable linear time systems with dead time, linear periodic system and linear parameter varying system, economic optimization and an implementation scheme for computation delay compensation are presented. The next section provides a detailed literature survey on each topic.

1.2 Literature Review

1.2.1 Dead Time Compensator

Many processes in process industry, as well as others, exhibit dead times in their dynamic behavior. Dead times are mainly caused by mass, energy or information transportation phenomena. For monovariable systems, the first order plus dead time (FOPDT), integrator plus dead time (IPDT) and second order plus dead time (SOPDT) are the most widely used models to approximate the real plants due to their simplicity and easy availability via simple identification experiments, such as step response and relay test[85].

The presence of dead time in processes greatly complicates the analytical aspects of controller design and imposes fundamental difficulty on achievable control performance[75, 85, 87]. Standard feedback controllers, such as PID controllers, cease to suffice because that the controller tries to correct the error signal of some time before, not the current error. In order to improve the control performance, dead times naturally call for a predictor, that predicts the system output and utilizes it instead of the real outputs for feedback, thus compensating the effect of dead times. The first and the most popular dead time compensator (DTC) is the Smith Predictor proposed in 1950s[77]. It utilizes the dead-time free part to produce the open loop prediction; the closed loop prediction is obtained by correcting the open loop prediction with the difference between the output of the model with dead time and the real output of the plant. The main advantage of Smith Predictor is that dead time is eliminated from the characteristic equation of the closed loop system. Thus the design problem for dead time system can be converted to the one without dead time[85].

One of the largest problems with Smith Predictor is that it always retains the poles of the open loop processes. Thus it is not applicable to unstable and poorly damped systems. It is found that the open loop poles are retained by the predictor structure. Numerous modifications of Smith predictor have been proposed to extend its applications.

Finite Spectrum Assignment approach[84, 83] and process-model approach[82] replace the original dead-time free model by a deliberated modified model for open loop prediction. They can arbitrarily assign the closed loop poles and therefore can be applied to poorly damped and unstable processes. Another approach, filtered Smith Predictor[92], retains the dead-time free model for open loop prediction, but filters the difference between the open loop prediction and the real output of the plant before it is added to form the closed loop prediction. The prediction filter is designed to cancel the unstable and undesirable stable poles. It is noted that the structure of filtered smith predictor was used to improve the robustness of Smith Predictor[88].

All the controllers aforementioned face serious implementation difficulty since a distributed delay element is required. The distributed delay element leads to a closed loop characteristic quasipolynomial of neutral type, which is shown to be the cause of instability. In particular for open loop unstable systems, the controller is internal unstable since unstable pole-zero cancelation, as part of the controller, cannot be eliminated by the use of polynomial division due to the non-rational, quasipolynomial, expression. This issue have been studied in the past decade and various remedies have been proposed[86].

While most DCT are studied in continuous domain, only discrete versions of DTC are used in practice. Moreover, the idea of DTC can be applied to discrete dynamic models other than the traditional continuous processes. A popular discrete DTC derived using a direct synthesis approach is the Dahlin controller[79]. It is noteworthy that the discrete DTCs do not share the implementation difficulty as ones in continuous domain since dead time in discrete domain is represented by a polynomial[11].

Another problem with Smith Predictor is the robustness to model uncertainty. It was pointed out that if the primary controller is not properly tuned, Smith Predictor may become unstable for infinitesimal model error[80]. The robustness of Smith Predictor was thoroughly investigated under the framework of internal model control[81]. The conditions on robust stability and robust performance were derived and used as guideline for controller parameter tuning. The modern solution, such as H_∞ control of dead time systems, is documented in [86] and the references therein.

To avoid the compromise between conflicting objectives, such as setpoint response, load disturbance response and robustness, two degree of freedom (DOF) compensator was proposed for integrating systems[76]. With the additional DOF, the controller decouples the setpoint response from the load response. This simplifies both design and tuning. 2DOF structure was extended to stable and unstable systems subsequently[78].

1.2.2 MPC for Linear Periodic Systems

One extension of LTI model is to allow the system dynamic to vary with time, more specifically to vary periodically. Because of the inherent periodicity in many real and man-made systems, periodic system seems appropriate to model various real world problems[50, 54]. One example is the building climate control problem[45]. The periodicity appears in various parts of the system model with the fluctuation of ambient temperature and the office occupation period as the most prominent features. Except the natural periodicity in time varying systems, another type of periodicity is artificial periodicity, such as asynchronous control, which is intentionally embedded into time invariant systems. It is becoming common that control inputs are updated not simultaneously but according to certain timing sequence due to the communication limitation between controllers and actuators. Asynchronous control may also help alleviate the computation burden, which motivates the study of Multiplexed MPC[52, 46]. Periodic control inputs updating, as a special input parameterization, reduces the number of decision variables and the computational burden. It is hence possible to deploy MPC in some high-bandwidth systems. It is shown that Multiplexed MPC is closely related to distributed MPC[55, 56], thus the advantage of periodicity can be appreciated in a broader viewpoint.

The control of unconstrained linear periodic systems was extensively studied in [54, 50, 53]. In particular, the solution of reverse periodic discrete-time algebraic riccati equations with time dependent dimensions was obtained[53], which is the counterpart of well known LQR controller for LTI systems. The solution is not only elegant in itself, but it also provides a basis for the development of several algorithms for constrained case.

If state or input constraints are considered in linear periodic system models, MPC provides a natural problem formulation. Several approaches based on elliptic terminal sets were proposed in [51]. However, the use of elliptic terminal set and linear matrix inequality(LMI) stability conditions generally results in restrictive control laws and small attraction region. To reduce the conservativeness, maximal periodic controlled positive invariant set was defined and the typical MPC approach for LTI system was generalized to linear periodic systems[44]. Maximal periodic controlled positive invariant set characterizes a necessary and sufficient condition for the existence of an infinite horizon admissible control inputs trajectory, therefore providing a powerful tool for synthesizing MPC. Additionally, the approach in [44] can accommodate time dependent dimensions, which is beyond the scope of previous studies.

Apart from the above approaches for deterministic linear periodic model, MPC with guaranteed robustness, in particular with respect to external disturbances, has been studied[46, 45]. The constraints tightening approach for LTI systems[38] was extended to periodic system in [46]. Its limitation is the possible small feasible region caused by open loop multi-step predictions. A least restrictive MPC was proposed in [45], which characterizes maximal periodic robust controlled invariant sets and uses them in the MPC formulation, resulting in the largest feasible region. However, the closed loop stability with least restrictive MPC is not rigorously proved. Thus new algorithms possessing the properties of both robust stability and sufficiently large feasible region are desirable. Moreover, the computational burden has to be modest in order to be applicable in practice.

1.2.3 MPC for Linear Parameter Varying Systems

Another extension of basic LTI model is the linear parameter varying (LPV) systems. Instead of enforcing the model invariant as for LTI systems or periodic as for linear periodic systems, LPV systems assume that the varying parameters of linear systems are known at the current time, but unknown in the future. Furthermore, the varying parameters, also known as scheduling parameters, are bounded by a given polytope. The term 'linear parameter varying' in control can be traced back in 1950s when gain scheduling approach was proposed for control of nonlinear systems [10]. For gain scheduling controller, the controller is linear parameter varying. Although lots of applications have witnessed its power, even for heuristic designed controllers, it is theoretically hard to analyze the closed loop properties, such as stability. In recognition of the problem of gain scheduling approach, during the last decade, researchers attempted to shift the linear parameter varying from control level to modeling level. With a LPV model, it is possible to systematically design stabilizing controllers for a class of nonlinear systems. The technique that embeds nonlinear systems into LPV models is documented in the work of [63, 64]. Further assumptions, such as bounded variation rate and deterministic input matrix, can be incorporate into the LPV model, in order to narrow down the uncertainty description, hence leading to less conservative control design.

In the early study, control of LPV systems is closely related to robust control of polytopic systems. Quasi-min-max approach was proposed in [24], which optimizes the worst performance of closed loop systems. It makes use of currently known parameters, and treats the future parameters as uncertainty. The first step control input and a linear control for future steps are optimized online with linear matrix inequality (LMI) constraints. It offers significant improvement over robust control methods and was extended in[61] by allowing optimizing the inputs over a control horizon of length N. Bounded parameter variations are exploited in the controller synthesis. Another approach to address bounded parameter variations was proposed in [62], in which a simple estimation method to describe future dynamic uncertainty is used for MPC synthesis.

Due to the polytopic uncertainty description and the feedback policy optimization, the above approaches often associate with heavy online computation burden, which increases significantly with the number of vertices of the polytope and free control inputs. In subsequent studies, several offline MPC approaches were proposed to reduce the online computation burden. Wan & Kothare[28] constructs offline a sequence of nested invariant ellipsoids corresponding to a sequence of linear controllers. At each sampling time, the smallest ellipsoid containing the state is identified and the control law is determined by linear interpolation between linear controllers of two adjacent ellipsoids. The ellipsoid invariant sets are replaced by polytopic sets in [59] resulting in larger attraction region.

Recently, more advanced algorithms[27] have been proposed, which take advantage of the idea that the future scheduling parameter will become known to the controller as time goes. These algorithms are particularly promising since the controllers make fully use of gain scheduling control throughout the prediction horizon, hence providing additional freedoms to design the control inputs and to improve the dynamic response. Moreover, several scenarios, including scheduling parameters of arbitrary switching, bounded rate switching and scheduling parameters irrelevant to input matrix, are studied thoroughly.

Based on the above reviews, it is obvious that there has been great advancements in the MPC design for LPV systems. However, most of the approaches make the assumption that state variables are measurable, while only output variables are measurable in practice. Thus the application of all above state feedback controllers is quite limited and many practical problems remain unsolved. To reduce the discrepancy, MPC based on output variables feedback is desirable. Currently there are only a few results reported in the literature. A control scheme incorporating state observer and quasi-min-max state feedback MPC for output feedback design was proposed in [21]. Although intuitive and simple, the result is questionable as to whether the closed loop stability is assumed in theory by the proposed algorithm. More research therefore is needed to investigate the stability issue and to apply the recently developed state feedback algorithms to output feedback case.

1.2.4 Economic Optimization in MPC

In process control industry, the ultimate objective of process operation is to optimize an economic criteria or profit function of the plant. It is not a new goal for controller design. Actually Optimal Control theory or MPC seems to be a perfect tool to handle the economic optimization problem. Comparing to the conventional optimal control problem for state regulation or tracking, the economic optimization is unique due to two features: infinite horizon and unbounded cost. Infinite horizon is a natural choice for economic optimization since any artificial finite horizon choice is unreasonable. These features pose serious theoretical challenge to the optimal control problem. [7] provides a comprehensive summary of research results on this problem. The most famous result probably is the turnpike theory, which explains asymptotic properties of the optimal trajectory for the economic optimization problem.

The current paradigm of economic optimization in most industrial process control systems is the so called ‘control hierarchy’ that decomposes and distributes the original problem into several layers, including planning, scheduling, real time optimization (RTO) and dynamic control[65]. In particular, RTO is concerned with determining the optimal steady state based on the profit function given by upper layers, nonlinear fundamental model of the plant and estimation of persistent disturbances. Then the optimal steady state is forwarded to dynamic control layer, usually MPC, as setpoints. The MPC regulates the plant dynamic behavior to track the setpoints. The control hierarchy is obviously an indirect solution to the economic problem, and its validity relies on the separated time scale principle. The current statues on its applications is reviewed in [66].

Recently the question on how to use dynamic MPC layer to optimize directly process economics has drawn considerable research attentions[66, 71, 68]. The development is mainly driven by the business competition and enabled by nowadays' computation capability. The proposals are often called Economic MPC[70] or Dynamic RTO (DRTO)[67, 68].

DRTO[67, 68] is to compute a reference trajectory via finite horizon dynamic optimization, instead of computing constant setpoints via a steady state economic optimization. It is noted that the two-layer structure is retained in DRTO in which the regulation layer, usually linear controllers, is used to track the reference trajectory. [69] proposed integrated control algorithms that merges setpoint optimization and dynamic regulation problem into one. The summation of quadratic function of tracking error and the profit function of the steady state setpoint is used as objective function.

While a number of successful applications of Economic MPC were reported, it lacks rigorous stability and optimality study until a number of important papers[70, 71, 72] are published recently. [72, 71] adopt the terminal equality constraint formulation, and determine the control action by optimizing directly the economic performance function. It is shown that the average performance of Economic MPC is no worse than that of the best admissible steady state. The asymptotic stability to the optimal steady state is guaranteed if the system is strictly dissipative. A Lyapunov function is constructed as the optimal value of a rotated cost function, not the original cost function usually used in MPC stability analysis. Economic MPC and its Lyapunov analysis was extended to cyclic systems like simulated moving bed separation and pressure swing absorption, in which non-steady operation is desirable due to the periodic economic performance function or the design of the process[73]. To enlarge the feasible region and ease online optimization, the terminal equality constraint is replaced by a terminal penalty and a terminal set constraint[70]. Terminal penalty function is designed based on whether or not a storage function is known. Both designs take advantage of the linearization around the steady state and the continuity of model and performance function.

In spite of these important progress, Economic MPC is still in its early stage. Current proposals guarantee stability only for dissipative systems. The terminal penalty function design is cumbersome and limited. It is therefore necessary to further investigate stabilizing Economic MPC.

1.2.5 Implementation Issues

As every approach or methodology has its advantages and disadvantages, the online optimization, on one hand, offers MPC the flexibility and optimality, but on other hand, is the main limiting factor of MPC if the real time requirement can not be satisfied due to the extensive computation. The heavy computation burden imposes two challenges on control: 1). the sampling rate is capped to be low enough to permit the optimization complete within one sampling interval; 2). the delay between state information and feedback control input can lead to drastic performance degradation or even to instability of the closed loop system[37]. The former petrochemical application of linear MPC is easy in the sense that the linear MPC results in a quadratic programming (QP) problem, which can be solved efficiently compared to chemical plants' slow dynamic. As MPC is in the progress of penetrating into many new applications, such as vehicle control, paper making and biomedical applications, which are faster processes and usually require Nonlinear MPC (NMPC) due to the strong nonlinearity, large transient movement and tight performance specification, the development of efficient solutions of the non-convex nonlinear programming (NLP) problems is of paramount importance for a wider acceptance of NMPC[13].

Explicit MPC[16] computes the optimal control action offline as an 'explicit', piecewise affine, function of the state via multi-parametric programming technique for QP problem. The control law is saved in a lookup table. The online computation therefore reduces to a simple function evaluation, actually involving partition identification for the current state to determine the active control law entry in the lookup table. Extensive research followed up to improve the online searching efficiency and to reduce the large storage space requirement[48, 43, 40]. It has been reported that Explicit MPC can be applied to small scale electrical systems of bandwidth of $10^3 Hz$.

Some researchers proposed to approximate MPC controller $\mu(x)$ by an easy-to-implement control law $\tilde{\mu}(x)$, which is computed offline. In particular, Artificial Neural Network[4], polynomial function [47], and set membership approach are used to represent the function $\tilde{\mu}(\cdot)$. The approximation error, evaluated at sample points or over the control law function, is minimized. It is shown that $\tilde{\mu}(x)$ can achieve similar control performance to $\mu(x)$ at much less computation cost if the approximation error is small enough.

A common drawback of the offline approaches is the huge offline computation, which may take hours for a modest scale problem, thus they are not as flexible as online optimization approach, especially for the adaptive case. Efficient online optimization strategies and implementation schemes suitable for MPC have been developed.

The complexity of optimization increases with the number of decision variables and constraints. As a way to reduce the number of decision variables, input parameterization has been studied in [39, 42] with the commonly adopted technique, move blocking[41], as a example. For general nonlinear programming problem, the Newton-type algorithm[34] iteratively obtains the solution via linearization and QP; the real time iteration scheme[35] distributes the optimization along the real time axis by allowing only one iteration per sampling time.

In recognition of the computational difficulties, a solution from another angle is to design MPC with the ability to tolerate it. In [31, 32], the necessary computation time is explicitly considered in MPC formulation, and the computation delay is compensated by a predicted optimization problem under the assumption of accurate model. The advantage is that it allows to deploy MPC even in the case that the necessary numerical solution time is significant with nowadays' computation power. Recently, similar idea is adopted by advanced-step MPC[33], which use NLP sensitivity technique to correct the predicted optimal control action, if model uncertainty presents. It is reported that the feedback delay is reduced by two orders of magnitude compared to the online solution of the the full NLP problem. However, the constraints satisfaction is not rigorously guaranteed in advanced-step MPC, therefore the closed loop stability is questionable.

1.3 Objectives and Scope

The diversity of the three constituting elements decides the flexibility of Model Predictive Control. Research efforts have been continuously devoted to apply the basic idea of MPC to various problems, as reviewed in Section 1.2. The progress is greatly affected by the complexity of the choices of the 3 elements. Specific research gaps for the current study of MPC variants reviewed in Section 1.2 are summarized below:

- dead-time compensators are sensitive to modeling error, and its design and implementation for general linear models are difficult.
- current MPC algorithms for linear periodic systems with external disturbances are either too conservative or without stability guarantee;
- the stability of output feedback MPC for LPV systems is not guaranteed;
- the design of terminal constrained Economic MPC is troublesome and under restricted model assumptions;
- there lacks rigorous treatment of the model uncertainty in the MPC strategy with computation delay compensation.

The main aim of this thesis is to propose synthesis design approaches of MPC controllers accommodating the requirements above. The specific objectives of this study are to:

- propose a simple and easy-to-implement dead-time compensator for the purpose of fast set-point tracking;
- propose stabilizing and less conservative MPC controller for linear periodic systems with external disturbances by exploiting the minimal time control approach;
- propose stabilizing output feedback MPC controller for LPV systems by combining Quasi-min-max and Tube MPC approach;
- propose new stabilizing economic MPC by constructing simpler terminal cost function or inserting stability constraints;
- propose robust MPC with computational delay compensation for its real time implementation.

The results of this present study may enable more suitable models, rather than simple LTI model, to be adopted and more challenging task, such as economic optimization, to be considered in the MPC design.

It can be expected that the results may help improve the control performance for certain applications and extend the application domain of MPC. The focus of this thesis is the synthesis approach for MPC design, while the design approach based on posterior stability check is excluded in this thesis because the synthesis approach is more systematic and has gained flavor in academia.

1.4 Organization of Thesis

The thesis is organized as follows.

Chapter 2 presents a simple dead time compensator. The standard PID controller is retained as the feedback controller. The dead time compensation is achieved via setpoint variation. The dead time is eliminated from the denominator of setpoint response transfer function. The motivation to the proposed controller is to keep the simple and user-familiar PID control and to provide an adds-on function block to improve the setpoint response, such that it does not suffer the model sensitivity problem as Smith Predictor. Simulation studies and experimental study on a lab-scale thermal chamber temperature control system are provided for validation.

Chapter 3 presents a robust minimal time controller design for linear periodic systems with external disturbances. The robust maximal periodic positive invariant sets Θ_j for stable linear periodic systems with bounded disturbances is defined and characterized. The s-step stabilizable sets $S_{s,j}$ of all states which can be robustly steered into Θ_j are obtained. The robust minimal time controller identifies the set $S_{m,j}$ in which the current state $x(k)$ is located, and then determines a control input that drive it into $S_{m-1,j}$ robustly. In other word, the controller drives the state toward a neighborhood around the origin step by step as time evolves. The computation is simplified since only one step prediction is used to calculate control action. Furthermore, the control law can be computed offline via Multi-Parametric programming and the online computation is further reduced. The proposed approach also results in a larger feasible region than previous multi-step prediction based MPC approach. Both state regulation and output tracking problems are discussed.

Chapter 4 presents tube based quasi-min-max output feedback MPC for LPV systems. The proposed control scheme incorporates robust observer and robust state feedback control. To handle disturbances and model uncertainty, disturbance invariant tube and quasi-min-max MPC are combined to achieve recursive feasibility and robust stability.

Chapter 5 presents several stabilizing economic MPC algorithms. Two terminal cost function designs are proposed for terminal constrained MPC for strict dissipative systems. For general system model and economic performance functions, a new stabilizing MPC with stability constraints embedded is proposed. Simulation results show that the proposed algorithms can extract more profits from the plants than standard MPC approaches.

Chapter 6 presents an implementation strategy that allows to deploy Robust MPC for even fast processes. A dual time scale control structure is adopted for fast sampling configuration in spite of heavy computation burden. The computation delay is explicitly compensated by MPC algorithms. A prominent advantage of MPC algorithms is that the recursive feasibility and robust stability are rigorously guaranteed.

Chapter 7 summarizes the contributions of this thesis and outlines several possible directions for future research.

Chapter 2

Dead Time Compensation via Setpoint Weighting

Low order plus dead time models, such as FOPDT, IPDT and SOPDT, are often adopted as system models for monovariable plants in process industry[85]. Practical applications demonstrate that they are simple and accurate enough to capture the dominant dynamic of complex industrial plants. PID controllers offer satisfactory performance when the dead time is small, but cease to suffice when the dead time is significant compared to the time constant of dead time free dynamic. In particular, the dead time within the loop requires a much smaller PID controller gain to retain closed loop stability. Normally the response is either extremely sluggish or oscillatory. Predictive Controllers, including DTCs and MPCs, can greatly improve the dynamic performance by removing the dead time from the loop. However, the use of predictive controllers still face resistance towards actual industrial applications and they may not fit within the off-the-shelves industrial controllers. Surveys have shown consistently that PID control is commonly persisted in cases when significant dead time is clearly present[93]. Generations of PID users over the decades have ensured low cost and high operational efficiency, from procurement to operations, maintenance and support. It is therefore highly desirable to yield better performance which is achievable by predictive controllers while retaining the basic PID controller structure. In this chapter the dead time compensation is achieved via weighting the setpoint in a simple manner before it is forwarded to the PID feedback loop. An equivalent PID controller gain higher that what is tolerable within the feedback loop can be achieved by apportioning the excessive gain to the setpoint weight. The proposed approach offers enhanced performance over other reported PID controllers designed for dead time processes[89, 90, 91]. The setpoint weighting function, as a simple add-on to the PID loop, can be accommodated by standard industrial controllers and field-bus control systems.

The rest of the chapter is structured as follows. In section 2.1, the problem considered is defined; the setpoint weighting function is proposed in section 2.2; the simulation and experimental studies are presented in section 2.3 and 2.4, respectively; finally, conclusions are drawn in section 2.5.

2.1 Problem Statement

The process considered is a monovariable LTI system, described by the transfer function

$$y(s) = G(s)u(s) = G_p(s)e^{-sL}u(s), \tag{2.1}$$

where G_p is the dead time free portion of the process, L is the input dead time and y and u are the output and the input of the process respectively. It is assumed that a standard feedback controller $G_c(s)$, such as PID controllers for most applications, is used for controlling the system (2.1). The closed loop transfer function is

$$y(s) = \frac{G_c G_p e^{-sL}}{1 + G_c G_p e^{-sL}} r(s), \tag{2.2}$$

where r is the setpoint variable. It is well known that the dead time term e^{-sL} in the characteristic equation greatly complicates the controller design and reduces the achievable control performance. The objective here is to compensate the dead time by properly exploiting another DOF, which is to weight the setpoint signal $r(t)$ before it is forwarded to (2.2).

2.2 Proposed Scheme for Dead Time Compensation

The proposed scheme is shown in Fig.2.1. The only modification on the standard feedback control structure is the addition of a setpoint weighting function $f_r(s)$ varying the setpoint from r to $\tilde{r}$. Thus, there is no attempt to change the widely accepted and simple control structure which generations of control practitioners will use by default. Setpoint weighting can be accommodated in many industrial controllers. Thus, the proposed scheme is amenable for use with basic and standard control units. We will attempt to increase the margin of achievable performance via the design of f_r only.

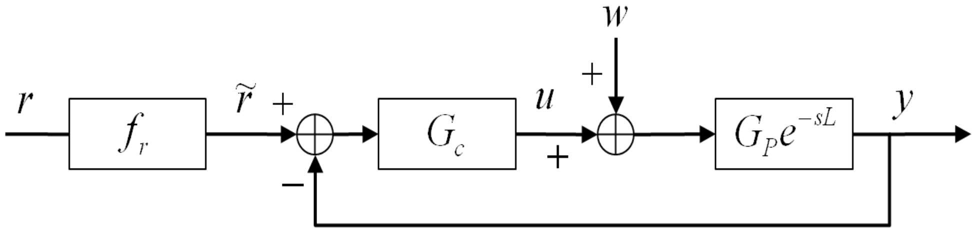

Figure 2.1: Proposed Control Scheme for Dead Time Compensation.

2.2.1 Simple Setpoint Weighting

Let the setpoint weighting function f_r be chosen as

$$f_r = \frac{\tilde{r}(s)}{r(s)} = 1 + \tilde{G}_{ry}(e^{-s\bar{L}} - 1), \tag{2.3}$$

where $\tilde{G}_{ry}$ is the specified desired closed-loop response sans the dead time, and $\bar{L}$ is an estimate of the actual dead time L. Note that the dead time estimation $\bar{L}$ lies outside of the loop, thus posing no stability issue when it is inaccurate. The overall closed-loop transfer function is

$$\frac{y(s)}{r(s)} = \frac{G_c G_p e^{-sL}}{1 + G_c G_p e^{-sL}}(1 + \tilde{G}_{ry}(e^{-s\bar{L}} - 1)). \tag{2.4}$$

Denote $\frac{G_c G_p}{1+G_c G_p}$ as G_{ry}. It can be shown that (2.4) can be written also as

$$\frac{y(s)}{r(s)} = \frac{G_{ry}e^{-sL}(1 + \tilde{G}_{ry}e^{-s\bar{L}} - \tilde{G}_{ry})}{1 + G_{ry}e^{-sL} - G_{ry}}. \tag{2.5}$$

Assume that $\bar{L} = L$ and $G_{ry} = \tilde{G}_{ry}$, then $1 + \tilde{G}_{ry}e^{-s\bar{L}} - \tilde{G}_{ry}$ in the numerator is equal to $1 + G_{ry}e^{-sL} - G_{ry}$ in the denominator. It is noted that $1 + G_{ry}e^{-sL} - G_{ry} = 0$ has the same solutions as $1 + G_c G_p e^{-sL} = 0$ which is the characteristic equation of the feedback loop. If the controller G_c is designed such that the characteristic equation yields solutions with only negative real-parts, the pole-zero cancelation in (2.5) is permissible and the overall closed-loop transfer function between r and y becomes

$$\frac{y(s)}{r(s)} = G_{ry}e^{-sL}. \tag{2.6}$$

Note that (2.6), with the dead time decoupled from a dead time free function, is usually achievable with a DTC.

If the feedback controller G_c is designed for the dead time free part G_p, the setpoint response will be greatly enhanced as it is with DTC. However, the dead time compensation here is only valid for setpoint response since the dead time remains in the feedback loop. The gain of G_c, admissible for $G_p e^{-sL}$, appears to be insufficient for G_p. In other word, although dead time is eliminated from the denominator of the setpoint response transfer function, the response will be too sluggish to be considered as improved compared to the one by using G_c only. This issue is addressed in next section by a general setpoint weighting function.

2.2.2 General Setpoint Weighting

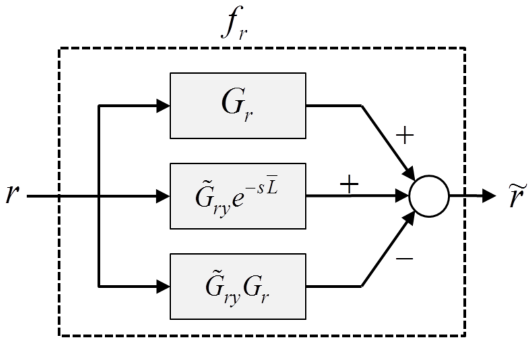

Figure 2.2: General Setpoint Weighting Function f_r.

In order to use a higher equivalent gain more than the maximum permissible within the feedback loop, and to leverage on the control structure to enable a 2DOF design to yield good performance in both setpoint tracking and load disturbance attenuation, a general setpoint weighting function as shown in Fig.2.2 is chosen as

$$f_r = \frac{\tilde{r}(s)}{r(s)} = G_r + \tilde{G}_{ry}(e^{-s\bar{L}} - G_r), \tag{2.7}$$

where G_r can be interpreted as the excessive portion of a controller gain, non-admissible in the feedback loop, apportioned to the setpoint weighting. The overall closed-loop transfer function is

$$\frac{y(s)}{r(s)} = \frac{\hat{G}_{ry}e^{-sL}}{G_r - G_r\hat{G}_{ry} + \hat{G}_{ry}e^{-sL}}(G_r - G_r\tilde{G}_{ry} + \tilde{G}_{ry}e^{-s\bar{L}}), \tag{2.8}$$

where $\hat{G}_{ry} = \frac{G_r G_c G_p}{1+G_r G_c G_p}$. Assume that $\bar{L} = L$ and $\hat{G}_{ry} = \tilde{G}_{ry}$, then the overall closed-loop transfer function between y and r is

$$\frac{y(s)}{r(s)} = \hat{G}_{ry} e^{-sL}. \tag{2.9}$$

It appears that the setpoint response is determined by an equivalent controller $\hat{G}_c = G_r G_c$. If G_r is chosen to be a proportional controller, i.e., $G_r = K$, $\hat{G}_{ry} = \frac{K G_c G_p}{1+K G_c G_p}$. It is clear now that G_r allows the decoupled design for setpoint tracking and load disturbance suppression; the feedback controller G_c is tuned for good disturbance attenuation and yet achieving good setpoint tracking by apportioning the higher gain needed for setpoint tracking to G_r. Thus the potential problem of sluggish response using a simple setpoint weighting function is overcame.

2.2.3 Design Rules

The design of the proposed setpoint weighted control system comprises of two phases. Firstly, the feedback controller G_c is designed for robust regulatory performance against load disturbances. Then, the setpoint weighting function is designed for tracking performance. As adopted in most industrial applications, assume G_c is a PI controller described by

$$G_c(s) = K_c(1 + \frac{1}{T_i s}), \tag{2.10}$$

where K_c and T_i are the proportional gain and integral time, respectively. Various PI control designs can be readily found in the literature. In this paper, a modified design based on the design proposed by Smith and Corripio [89] will be adopted. Empirical study to adapt this design only for regulatory performance results in recommendation to incorporate a 5% to 15% reduction in the proportional gain obtained by Smith and Corripio's approach to yield quick recovery from disturbance upsets with little or no overshoot.

Since smooth output response without overshoot is preferred in certain applications, the setpoint weighting function $f_r(s)$ is designed as

$$G_r = \frac{K {G_p}^{-1}}{K_c + K_c T_i s}, \hat{G}_{ry} = \frac{1}{T_i s/K + 1}, \tag{2.11}$$

to offer first-order type response. It is assumed that G_p does not contain right hand plane zeros. The setpoint response transfer function is

$$\frac{y(s)}{r(s)} = \frac{1}{T_i s/K + 1} e^{-sL}. \tag{2.12}$$

So that K can be simply chosen to achieve a desired closed-loop response signatory of a step response from a first-order transfer function. It may be noted that G_r may not be rational when G_p has a relative order of two or more. In these cases, a sufficient number of fast poles can be appended to f_r to restore rationality.

2.2.4 Robustness Analysis

With the setpoint weighting, dead time compensation is achieved in the nominal case when $\hat{G}_{ry} = \tilde{G}_{ry}$ and $\bar{L} = L$, leading to improved tracking performance. It will be of interest to discuss the situations when the nominal condition does not hold. It is noted that the control structure as presented in Fig. 2.1 is a 2DOF structure. It should be pointed out that f_r is not in the feedback path, so that it does not incrementally affect the closed-loop stability as long as f_r does not contribute any unstable pole. Thus the proposed approach does not share the sensitivity problem with Smith Predictor with respect to stability. With the retained PI feedback loop, personnel's know-how can still be utilized to maintain closed loop stability. However, a deviation from the nominal condition will clearly affect the tracking performance. Following the robustness analysis in [81], a condition to maintain robust tracking performance will be provided. The tracking error $e = r - y$ is given by

$$e = r\left(1 - \frac{G_c G}{1 + G_c G} f_r\right), \tag{2.13}$$

Denote the nominal model as G_n. The real plant model may defer from G_n and it can be represented with a multiplicative uncertainty such that $G = G_n(1 + l_m)$. Robust tracking performance is achieved if

$$\left|\left(1 - \frac{G_c G_n(1 + l_m)}{1 + G_c G_n(1 + l_m)} f_r\right)\omega_r\right| < 1 \quad \forall l_m \in \Lambda_m, \forall \omega, \tag{2.14}$$

where ω_r is the specified robust performance weight and the set $\Lambda_m(i\omega)$ is defined by $\Lambda_m(i\omega) = \{l_m(i\omega) : |l_m(i\omega)| \leq \bar{l}_m(\omega)\}$. With an estimate of the uncertainty bound $\bar{l}_m$, f_r can be designed to satisfy the following condition for robust tracking performance[81]:

$$|f_r| \cdot |G_c G_n(1 + \bar{l}_m)\omega_r| < |1 - |\omega_r|| \cdot |1 + G_c G_n(1 + \bar{l}_m)|. \tag{2.15}$$

2.3 Simulation Studies

Simulation results will be furnished in this section to compare the proposed approach against reported PID algorithms proposed for dead time processes. Four representive models, commonly encountered in process industry, are chosen as test-bed for performance comparison, including a FOPDT model P1, a SOPDT model P2, a high order model P3, and a non-minimum phase model P4. The details about these simulation models are tabulated in Table 2.1. Three well-known PI control designs proposed in Smith and Corripio[89], Hagglund[90], and Hang[91] are adopted to compare the performance with the proposed approach. It is noted that the three approaches[89, 90, 91] are dedicated to processes with dead time. The approach in Smith and Corripio[89] delivers 5% overshoot for dead time processes. Maximum sensitivity is the main design objective considered in Hagglund[90]. A gain margin of 5 and phase margin of 72 degrees are used as the design parameters in Hang[91]. The parameters of PI controllers using different approaches for models P1-P4 are tabulated in Table 2.2. The performance comparison for this study is focused on the dual aspects of the closed-loop response to a setpoint change and a load disturbance change.

Table 2.1: Transfer Function of Simulation Models P1-P4

	P1	P2	P3	P4
$G_{Pi}(s)$	$\frac{1}{s+1}e^{-10s}$	$\frac{1}{s^2+s+1}e^{-5s}$	$\frac{1}{(s+1)^5}$	$\frac{1-s}{(s+1)^3}e^{-5s}$

Table 2.2: PI Controller Parameters Using Different Approaches for Simulation Models

K_c, T_i	$P1$	$P2$	$P3$	$P4$
Proposed approach	0.042, 1	0.12, 1.3	0.45, 2.6	0.15, 2
Smith and Corripio[89]	0.05, 1	0.12, 1.3	0.53, 2.6	0.17, 2
Hagglund[90]	0.04, 0.7	0.096, 0.9	0.42, 1.82	0.14, 1.38
Hang[91]	0.031, 1	0.076, 1.3	0.34, 2.6	0.11, 2

2.3.1 First Order Process with Dead Time

The four approaches are tested with the model P1. For the proposed approach, the setpoint weighting function f_r is designed with $G_r = 23.8$ and $\tilde{G}_{ry} = \frac{1}{s+1}$. Parameters of PI controllers are tabulated in Table 2.2. The setpoint tracking response to an unit change of reference r and load disturbance response to an unit change of disturbance w at the input of the system are shown in Fig.2.3 & 2.4, respectively. It can be observed that the output response using the proposed approach has a fast rise time and no overshoot when the step reference signal is applied to the system, as compared to the other aforementioned approaches. The improvement is due to f_r which changes the setpoint r to the weighted setpoint $\tilde{r}$ as shown in Fig.2.5. For the load disturbance response, the proposed approach offers similar recovery speed as Smith and Corripio[89], but without overshoot. Thus, the proposed setpoint weighting approach performs favorably in the dual aspects of the closed-loop response to a setpoint change and a load disturbance change, as compared to the other approaches.

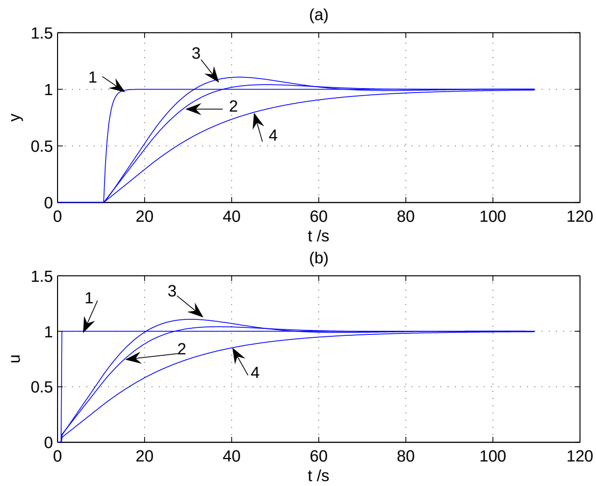

Figure 2.3: Setpoint Response of P1[1]

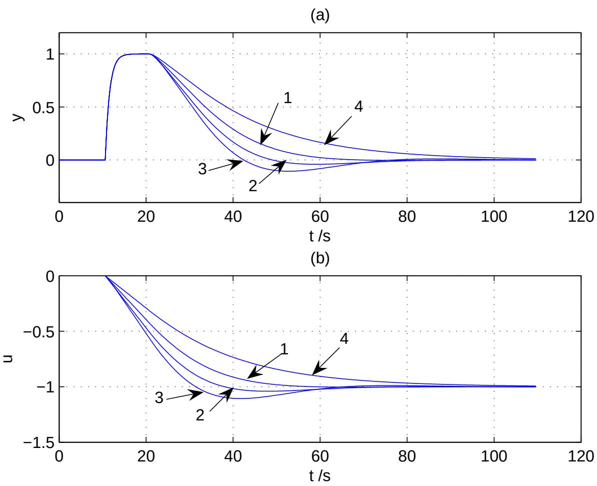

Figure 2.4: Load Response of P1

2.3.2 Second Order Process with Dead Time

For model P2, G_r and $\tilde{G}_{ry}$ are set as $\frac{10(s^2+s+1)}{0.12(1.3s+1)}$ and $\frac{10}{1.3s+10}$. f_r is appended with $\frac{1}{s+1}$ to maintain rationality. The setpoint tracking response and load disturbance response are shown in Fig.2.6 & 2.7, respectively. As

[1](a) output response $y(t)$; (b) control input $u(t)$; line 1:the proposed approach; line 2: Smith and Corripio[89]; line 3: Hagglund[90]; line 4: Hang[91]. The same labels are used in Fig.2.4 & 2.6-2.11.

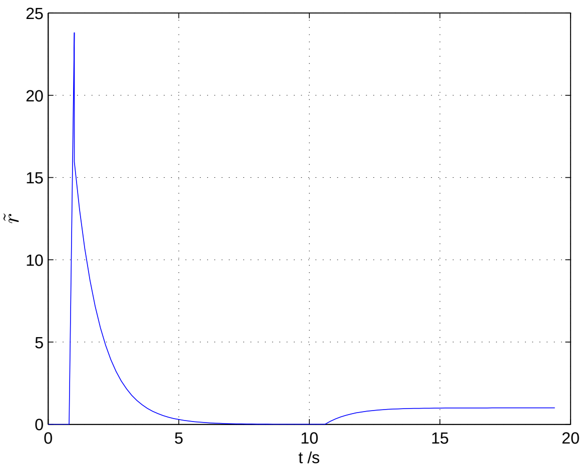

Figure 2.5: Weighted Setpoint $\tilde{r}$.

the previous case with P1, the proposed setpoint weighting approach performs favorably both in setpoint tracking and load regulation.

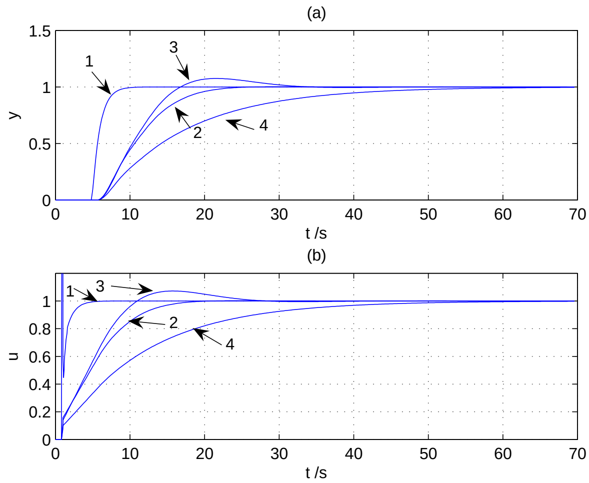

Figure 2.6: Setpoint Response of P2

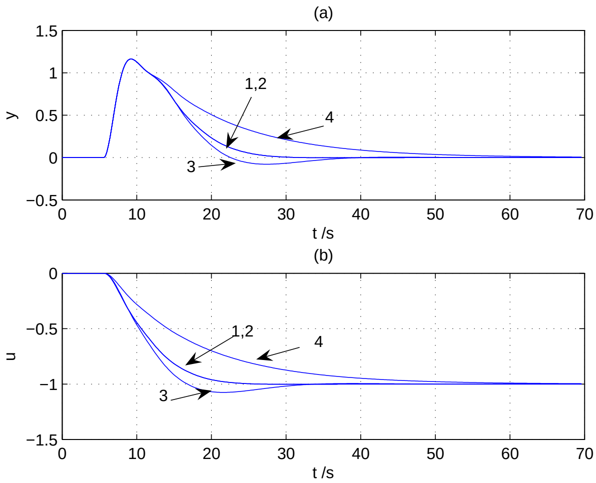

Figure 2.7: Load Response of P2

2.3.3 High Order Process

As no model is perfectly accurate, it is of interest to examine the performance of the proposed approach for some approximated models. It is well known that the dead time is often used to approximate the time lag of a large number of low order system connected in series. Here the 5th order system P3 is modeled as a FOPDT system: $G_{m3} = \frac{1}{2.6s+1}e^{-2.8s}$. G_{m3}, instead of G_{P3}, is used in controllers design. G_r and $\tilde{G}_{ry}$ are set as 4 and $\frac{1}{1.44s+1}e^{-2.8s}$. The setpoint tracking response and load disturbance response are shown in

Fig.2.8 & 2.9, respectively. As can be seen, although the model accuracy is compromised, the proposed setpoint weighting approach still performs favorably both in setpoint tracking and load regulation and offers improved performance over standard PI controllers.

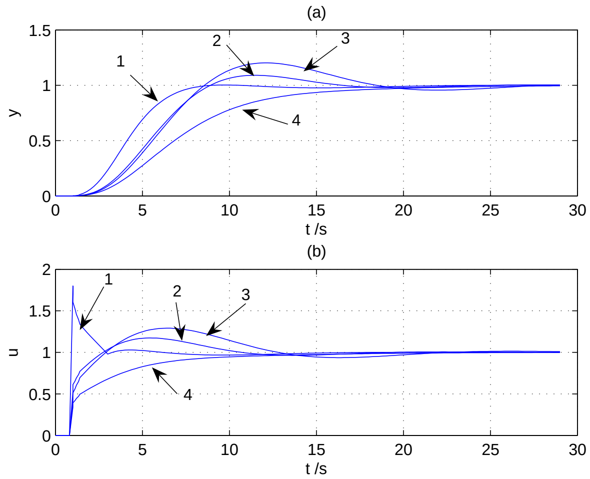

Figure 2.8: Setpoint Response of P3

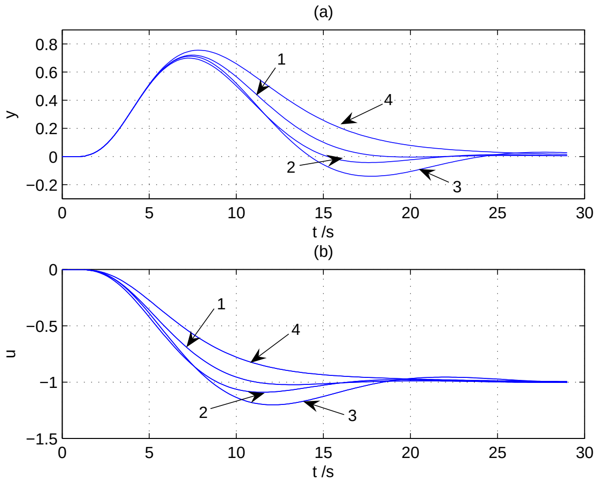

Figure 2.9: Load Response of P3

2.3.4 Non-minimum Phase Process with Dead Time

The non-minimum phase system P4 is again modeled as a FOPDT system: $G_{m4} = \frac{1}{2s+1}e^{-6.8s}$. G_r and $\tilde{G}_{ry}$ are set as 4 and $\frac{0.3}{s+0.3}e^{-6.8s}$. The setpoint tracking response and load disturbance response are shown in Fig.2.10 & 2.11, respectively. The similar improvement is observed. The proposed approach offers enhanced performance using approximated models. Although the simulation results are not extensive and there lacks theoretical performance analysis, the results indeed suggest that it is possible to exploit the unused DOF in the standard PI control structure, setpoint weighting for dead time compensation in this chapter, to improve the control performance even if the model accuracy is not guaranteed.

2.4 Experimental Study

In this section, an experimental study is provided to demonstrate the performance of the proposed approach on actual systems. A thermal chamber platform is used as the test platform for our purpose. In this system, the thermal chamber unit is driven by a National Instruments(NI) control driver platform via a data acquisition PCI card which is slotted in a PC. One NI J-type thermocouple input module is used as the

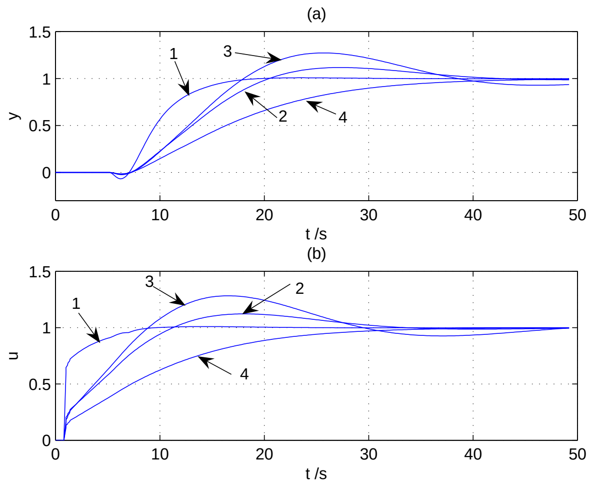

Figure 2.10: Setpoint Response of P4

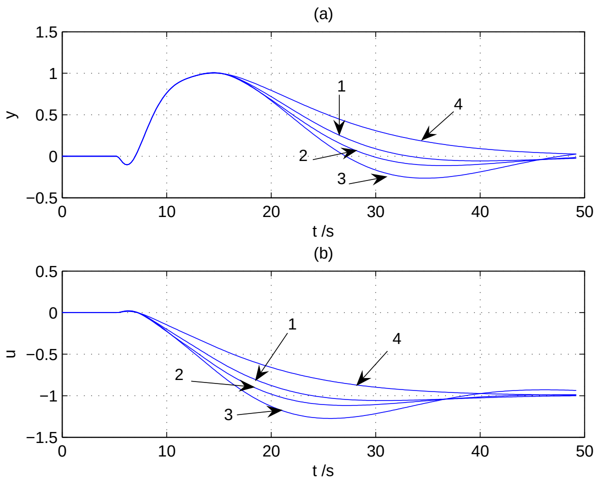

Figure 2.11: Load Response of P4

Table 2.3: PI Controller Parameters Using Different Approaches for Thermal Chamber

	K_c	T_i
Proposed approach	0.048	9.13
Smith and Corripio[89]	0.056	9.13
Hagglund[90]	0.045	6.39
Hang[91]	0.074	9.97

temperature sensor to measure the air temperature in the chamber. Two power amplifier units are used to control the light module and the fan module. The air temperature can be controlled by adjusting the light module and the fan module. In this system, the controlled variable of interest is the temperature inside the enclosed chamber, with the electrical input to the light module acting as the control input. The fan module serves as a disturbance source which is measurable.

The inherent dead time of the thermal chamber is rather small, so an artificial dead time of 10 seconds was inserted into the control loop as input dead time. A FOPTD model was identified as follow:

$$G = \frac{K_p}{T_p s + 1} e^{-sL} = \frac{8}{9.13s + 1} e^{-10.1s}.$$

The parameters of the various PI controllers are tabulated in Table 2.3. Considering the input range is limited within [0,1], setpoint weighting component is designed as $G_r = 6$. The setpoint is shifted from 31° to 29° at $t = 30s$, and a disturbance is added at $t = 210s$. The response of setpoint tracking and disturbance regulation using aforementioned approaches are shown in Fig.2.12 & 2.13, respectively. As can be seen, the temperature of controlled system using the proposed approach decreased fastest, and exhibited about

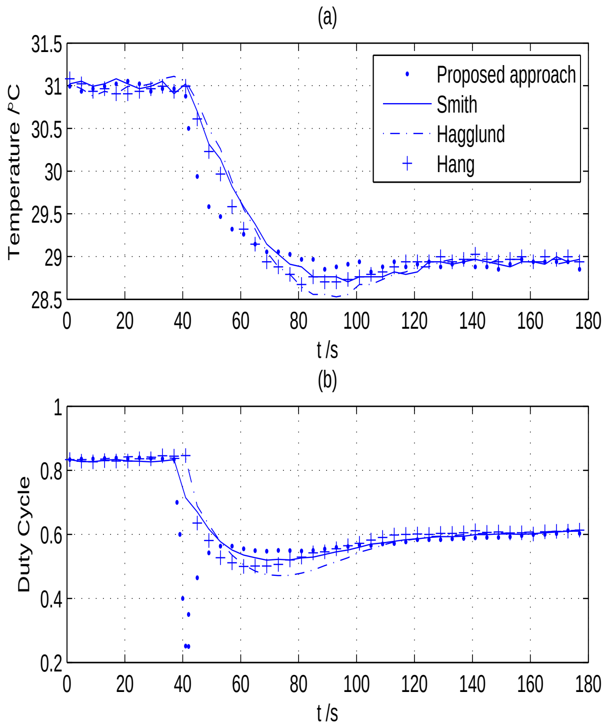

Figure 2.12: Setpoint Response of The Thermal Chamber

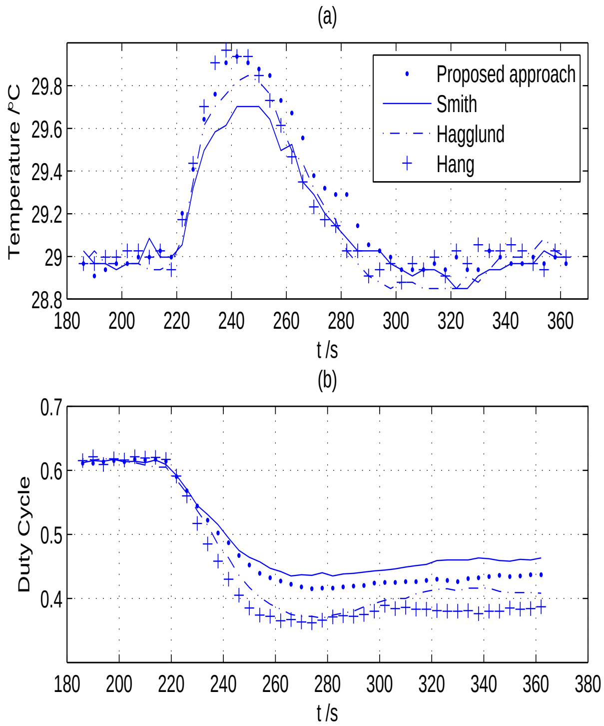

Figure 2.13: Load Response of The Thermal Chamber

10% percent overshoot, smaller than other three output responses. The peaks of output controlled by the proposed approach, Hagglund[90] and Hang[91], caused by disturbance, are almost the same, while PI controller by Smith and Corripio[89] provided the fastest suppression. However, the proposed approach gave the smallest oscillation and settle time. So the proposed approach performs favorably for both setpoint tracking and disturbance regulation. Moreover, if application prefers a quick disturbance suppression, the feedback method could be chosen from other approaches[89] without affecting the setpoint response.

To be applicable to practical problems, the control algorithms have to be implemented in industrial controllers. This is particularly important since most of DTCs proposed in the literature can not be implemented in industrial controllers. For some cases, it is even not possible, especially DTCs for unstable processes. To verity the ease applicability, the proposed control algorithm was implemented on a field-bus control system (FCS). The FCS in Mechatronics & Automation Laboratory, ECE NUS, was purchased from SMAR®. The details about the system architecture are omitted here and only a snap shot of the control algorithm implementation is shown in Fig.2.14. All the function blocks, i.e. DF622-CRTW-1, ∼-PRED-1, ∼-TF-1 and ∼-FMTH-1, used to implement the set-point weighting function, are from the standard function block library provided by Foundation Fieldbus. The graphic programming only involves 'Click', 'Draw', 'Connect' operations and entering in parameter values. Thus it is easy to implement the proposed approach in industrial controllers.

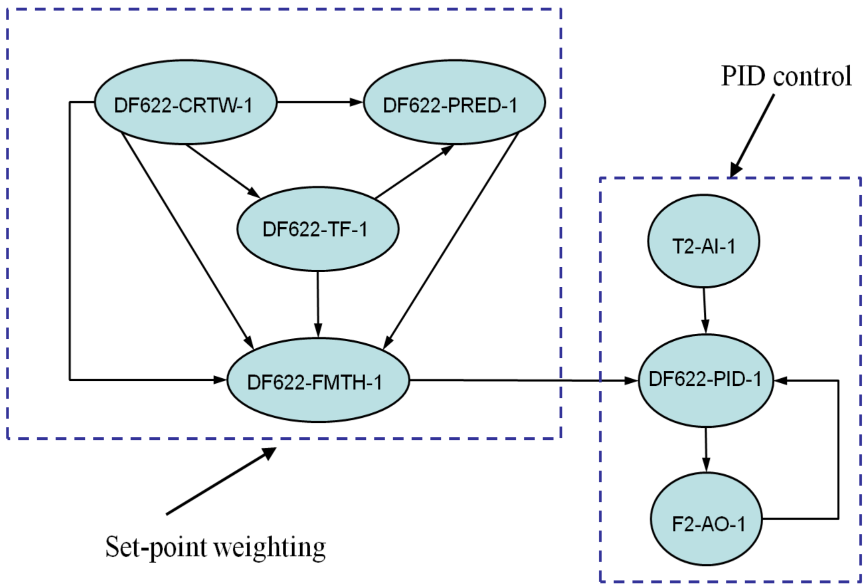

Figure 2.14: Control Algorithm Implementation in FCS

2.5 Conclusion

In this chapter, the use of PI control for dead time processes is revisited. Setpoint weighting functions are introduced which will allow good tracking performance and the PI feedback controller is tuned to give good load regulation. The structure of the proposed control scheme is simple and amenable to practical implementations on industrial controllers since only setpoint variation function is needed. Simulation and experimental results have demonstrated the viability and strengths of the proposed approach.

Chapter 3

Robust Minimal Time Control for Linear Periodic Systems

Minimal Time Control has proven to be a promising alternative to the typical model predictive control for LTI systems in terms of closed loop performance, attraction region and computational efficiency[98, 47]. This chapter presents a robust minimal time controller for linear periodic systems which inherits the advantages of its LTI analogue. The optimization problem can be either solved online or offline via multi-parametric programming.

The rest of the chapter is structured as follows. The problem is defined in 3.1; the definition and determination algorithm for robust periodic maximal positive invariant sets are presented in section 3.2; the controller design for state regulation problem is explained in section 3.3 and followed by an extension to tracking problem in section 3.4; simulation studies are presented in section 3.5; finally, conclusions are drawn in section 3.6.

Notation. A convex set $\Theta \in \mathbb{R}^n$ given by $\Theta := \{z \in \mathbb{R}^n | Az \leq b\}$ is called a *polyhedron* with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Given sets $\Omega \subset \mathbb{R}^n, \Theta \subset \mathbb{R}^n$, the *Pontryagin Difference* between Ω and Θ is defined as $\Omega \ominus \Theta \triangleq \{x \in \mathbb{R}^n | x+y \in \Omega, \forall y \in \Theta\}$; the *Minkowski Sum* of Ω and Θ is defined as $\Omega \oplus \Theta \triangleq \{z \in \mathbb{R}^n | z = x+y, x \in \Omega, y \in \Theta\}$; the *linear mapping* $\Phi \subset \mathbb{R}^m$ of Ω via $M \in \mathbb{R}^{m\times n}$ is defined as $\Phi = \{y \in \mathbb{R}^m | y = Mx, x \in \Omega\}$; the *pre-image* of Ω via the matrix $\Psi \in \mathbb{R}^{n\times n}$ is defined as $Pre(\Psi, \Omega) = \{x \in \mathbb{R}^n | \Psi x \in \Omega\}$. The notation of these set operations are used throughout the thesis.

3.1 Problem Statement

Consider the linear periodic system with bounded external disturbances as

$$x(t+1) = A_j x(t) + B_j u(t) + w(t), \tag{3.1}$$

where step index $t \in \mathbb{N}$, period length $p \in \mathbb{N}_+$, without loss of generality, inter-period step index $j := mod(t, p) \in \mathbb{N}_0^{p-1}$, state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, and disturbance $w \in \mathbb{R}^v$. A_j, B_j are time dependant periodic matrices. The system is subject to periodic constraints on states and input as

$$x(t) \in \mathbb{X}_j, u(t) \in \mathbb{U}_j, \quad j = mod(t, p), \tag{3.2}$$

where $\mathbb{X}_j, \mathbb{U}_j$ are polytopes with non empty interiors. The disturbance is bounded as $w(t) \in \mathbb{W}_j, j = mod(t, p)$, where $\mathbb{W}_j \subset \mathbb{R}^n, j \in \mathbb{N}_0^{p-1}$ are polytopes with the origin as an interior. The control objective is to steer the state to the origin while satisfying the constraints.

Assume a set of p linear controllers are designed for regulation problem as

$$u(t) = K_j x(t), \quad j = mod(t, p). \tag{3.3}$$

The closed loop system is periodic and can be expressed as

$$x(t+1) = \Psi_j x(t) + w(t) = (A_j + B_j K_j)x(t) + w(t), \quad j = mod(t, p). \tag{3.4}$$

Suppose that the set of linear controllers, such as the periodic LQR controller [53], renders a stable closed loop system with satisfactory response. However, the linear control (3.3) is valid only within certain sets around the origin because of the constraints (3.2) imposed on the states and inputs. There are many options to characterize the constraint admissible set, such as ellipsoidal set and polytopic set[58]. The one of the largest volume is the robust periodic maximal positive invariant sets(PMPIS). The counterpart for nominal dynamic without disturbance was formulated in [44], where the algorithm for its determination was presented. In the next section, the set is extended to the robust case, and its determination can be done with the existing algorithm for LTI systems[97]. The robust PMPIS serves as a terminal region for the design of minimal time controller.

3.2 Robust Periodic Maximal Positive Invariant Set

Definition 3.1. *A set Θ_j, $j \in \mathbb{N}_0^{p-1}$, is termed a robust PMPIS for system (3.1) under control (3.3) starting at its jth mode, iff it satisfies the following conditions:*

1. *if the system is at its jth mode and the current state $x(t) \in \Theta_j$, then $x(t+k) \in \mathbb{X}_{mod(j+k,p)}, u(t+k) \in \mathbb{U}_{mod(j+k,p)}, k > 0$, and $x(t+i \times p) \in \Theta_j, \forall i \in \mathbb{N}^+$.*

2. *Θ_j is the largest one of all the sets satisfying condition 1.*

The state and control input evolutions (3.4) and (3.3) within one period of p steps are

$$x(t+k) = \prod_{t_1=1}^{k} \Psi_{mod(j+k-t_1,p)} x(t) + \sum_{t_2=0}^{k-1} \prod_{t_1=k-1}^{t_2+1} \Psi_{mod(j+t_1,p)} w(t+t_2), 1 \le k \le p, \tag{3.5}$$

$$u(t+k) = K_{mod(j+k,p)} x(t+k), 0 \le k \le p-1. \tag{3.6}$$

Based on these prediction equations, the constraints during one period, $x(t+k) \in \mathbb{X}_{mod(j+k,p)}, u(t+k) \in \mathbb{U}_{mod(j+k,p)}, k \in \mathbb{N}_0^{p-1}$, can be entirely projected on the periodic state $x(t)$ with tightened constraints by using polytopic set operations like Pontryagin difference.

$$x(t) \in Pre(\prod_{t_1=1}^{k} \Psi_{mod(j+k-t_1,p)}, \mathbb{X}_{mod(j+k,p)} \ominus (\oplus_{t_2=0}^{k-1} \prod_{t_1=k-1}^{t_2+1} \Psi_{mod(j+t_1,p)} \mathbb{W}_{mod(j+t_2,p)}), 0 \le k \le p-1, \tag{3.7}$$

$$\begin{aligned} x(t) \in & Pre(K_{mod(j+k,p)} \prod_{t_1=1}^{k} \Psi_{mod(j+k-t_1,p)}, \\ & \mathbb{U}_{mod(j+k,p)} \ominus K_{mod(j+k,p)} (\oplus_{t_2=0}^{k-1} \prod_{t_1=k-1}^{t_2+1} \Psi_{mod(j+t_1,p)} \mathbb{W}_{mod(j+t_2,p)})), 0 \le k \le p-1. \end{aligned} \tag{3.8}$$

For simplicity, we represent the above constraints by $x(t) \in \hat{\mathbb{X}}_j$. If $x(t) \in \hat{\mathbb{X}}_j$, the state and input constraints (3.2) for one period, t to $t+p-1$ are satisfied. Next we consider the periodic evolution of $x(t+i \times p)$, whose dynamic is given as

$$\begin{aligned} x(t+(i+1) \times p) &= \hat{\Psi}_j x(t+i \times p) + \hat{w}(t+i \times p) \\ &= \prod_{t_1=1}^{p} \Psi_{mod(j+p-t_1,p)} x(t+i \times p) + \sum_{t_2=0}^{p-1} \prod_{t_1=p-1}^{t_2+1} \Psi_{mod(j+t_1,p)} w(t+i \times p+t_2), i \ge 0. \end{aligned} \tag{3.9}$$

Define $\hat{\mathbb{W}}_j = \oplus_{t_2=0}^{p-1} \prod_{t_1=p-1}^{t_2+1} \Psi_{mod(j+t_1,p)} \mathbb{W}_{mod(j+t_2,p)})$. It is easy to see that Θ_j is robust PMPIS for (3.4) only if it is robust maximal positive invariant set for LTI system:

$$z(i+1) = \hat{\Psi}_j z(i) + \hat{w}(i), z \in \hat{\mathbb{X}}_j, \hat{w} \in \hat{\mathbb{W}}_j. \tag{3.10}$$

The key point of interest is that the problem of characterization of robust PMPIS for linear periodic systems (3.4) is transformed to a standard robust positive invariant set problem for LTI systems (3.10), for which efficient algorithms are readily available[97, 15]. By applying the above operations to each of the p modes, robust PMPISs $\Theta_j, j \in \mathbb{N}_0^{p-1}$ are obtained, which can be used periodically as terminal sets for controller synthesis as shown in the next section.

3.3 Robust Minimal Time Control

3.3.1 Periodic Stabilizable Sets

As per the LTI case, periodic stabilizable sets (PSS) can be calculated based on robust PMPIS Θ_j. The PSS $\mathbb{S}_j^m$ includes all the states which can be robustly steered to $\mathbb{S}_j^{m-1}$ (Θ_j if $m = 1$) via only one step. Mathematically 1-step stabilizable sets $\mathbb{S}_j^1$ is defined as:

$$\mathbb{S}_j^1 = \{x : A_k x + B_k u + w \in \Theta_j, \exists u \in \mathbb{U}_k, \forall w \in \mathbb{W}_k\} = Pre(A_k, \Theta_j \ominus \mathbb{W}_k \oplus (-B_k \mathbb{U}_k)), \tag{3.11}$$

where $k = mod(j-1, p)$. Recursively, the m-step stabilizable sets $\mathbb{S}_j^m$ can be constructed as

$$\mathbb{S}_j^m = \{x : A_k x + B_k u + w \in \mathbb{S}_j^{m-1}, \exists u \in \mathbb{U}_k, \forall w \in \mathbb{W}_k, \} = Pre(A_k, \mathbb{S}_j^{m-1} \ominus \mathbb{W}_k \oplus (-B_k \mathbb{U}_k)) \tag{3.12}$$

where $k = mod(j-m, p)$. Since all the constraints and disturbance set are represented by polytopic sets, the same apply to the stabilizable sets. Thus the PSS is obtainable by existing MATLAB toolbox packages on polytopic set operations[57, 40]. Because no specific feedback policy is assumed in the definition of PSS, it covers all possible states which can be steered to the next stabilizable set with no conservativeness, which implies a larger attraction domain of the proposed algorithm given in subsection 3.3.2.

3.3.2 Controller Design and Implementation

The philosophy behind the control design is to use different controllers for states located in different areas and to steer the state into Robust PMPIS in minimal steps.

Offline: Assume the current state $x(t)$ is in the stabilizable set $\mathbb{S}_j^m$, the objective of the controller is to make the next state fall in $\mathbb{S}_j^{m-1}$, or Θ_j if $m = 1$. This can be done with a one-step-prediction optimization.

$$u^* = \arg\min_u \quad x^T Qx + u^T Ru + (A_k x + B_k u)^T P(A_k x + B_k u) \tag{3.13}$$

$$s.t. \;\; A_k x + B_k u \in \mathbb{S}_j^{m-1} \ominus \mathbb{W}_k, u \in \mathbb{U}_k, k = mod(j - m, p) \tag{3.14}$$

Assume a prediction horizon N is used for the construction of stabilizable sets, the above optimization problem can be performed $N \times p$ times, resulting in controllers $C_j^m, 1 \leq m \leq N, 0 \leq j \leq p-1$. The solution of each optimization problem can be explicitly represented by a PWA function [15], which can be used as a look-up table for the online controller.

Online: An appropriate control law is to be selected from the look-up table for $x(t)$.

- identify the minimal m, such that $x(t) \in \mathbb{S}_j^m, 1 \leq m \leq N, 0 \leq j \leq p - 1, j = mod(t + m, p)$;
- identify the active controller partition r in controller C_j^m;
- apply $u(t) = F_j^m(r)x(t) + G_j^m(r)$.
- if $x(t) \in \Theta_j$, $j = mod(t, p)$, apply $u(t) = K_j x(t)$ directly.

Remark 3.1. *Since the stabilizable set $\mathbb{S}_j^m$ will overlap with each other, the control law should be simplified by discarding the unused partitions. This operation can save the memory and speed up the online search process.*

3.3.3 Stability Analysis

Theorem 3.1. *The online control is recursively feasible if it is feasible at the initial time.*

Proof. If the initial state $x(t) \in \mathbb{S}_j^m$ and the controller C_j^m is used, $x(t + 1)$ will fall in $\mathbb{S}_j^{m-1}$ or Θ_j for any possible disturbance realization $w(t) \in \mathbb{W}_{mod(t,p)}$ since the constraint (3.14) holds. Hence the controller

C_j^{m-1} or K_j will be a feasible candidate controller at the next time instance for $x(t+1)$. If the initial state $x(t) \in \Theta_{mod(t,p)}$, obviously the predefined linear controllers will be used. □

Theorem 3.2. *Under the proposed control, the state will asymptotically converge to periodic minimal robust positively invariant set (MRPIS) Ω_j.*

Proof. It is easy to see that after at most N steps, the controller will switch to predefined linear controllers. After that, as shown in (3.9), the dynamic of state at periodic instances is governed by LTI systems with bounded disturbances. It follows [49] that there exists a MRPIS for LTI system to which the state will asymptotically converge as an attractor. The characterization of MRPIS for LTI systems is given in [49]. The same approach can be applied to (3.9) to obtain periodic MRPIS for mode j. Similarly for all modes, there are p periodic MRPISs $\Omega_j, 0 \le j \le p-1$ which characterize the steady behavior of the closed loop system. □

3.4 Extension to Tracking Problem

For practical problems, output tracking is required instead of state regulation, especially when the setpoint is time varying signal. Since the input number is less than the state number, arbitrary periodic state trajectory tracking may not be achievable. The parametrization of the achievable steady periodic steady state is studied first. The periodic steady state satisfies

$$\bar{x}_0 = \prod_{t_1=1}^{p} A_{p-t_1}\bar{x}_0 + \sum_{t_2=0}^{p-1}\Big(\prod_{t_1=1}^{p-t_2-1} A_{p-t_1}B_{t_2}\bar{u}_{t_2}\Big) \tag{3.15}$$

where the script $\bar{x}_j$ and $\bar{u}_j$ denotes the steady value of x and u at the jth mode. Assume that $I - \prod_{j=0}^{p-1} A_j$ is invertible, $\bar{x}_0 = [\ s_{0,0} \ \ \dots \ \ s_{0,p-1}\ ][\ \bar{u}_0 \ \ \dots \ \ \bar{u}_{p-1}\ ]^T$. Based on the state transition equation $\bar{x}_{j+1} = A_j\bar{x}_j + B_j\bar{u}_j$, the relation between other periodic steady states and input signal can be recursively obtained.

In summary, the periodic steady state is parameterized by periodic steady input as:

$$\begin{bmatrix} \bar{x}_0 \\ \bar{x}_1 \\ \vdots \\ \bar{x}_{p-1} \end{bmatrix} = \begin{bmatrix} s_{0,0} & s_{0,1} & \dots & s_{0,p-1} \\ s_{1,0} & s_{1,1} & \dots & s_{1,p-1} \\ \vdots & \vdots & \vdots & \vdots \\ s_{p-1,0} & s_{p-1,1} & \dots & s_{p-1,p-1} \end{bmatrix} \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \\ \vdots \\ \bar{u}_{p-1} \end{bmatrix}. \tag{3.16}$$

Setpoint tracking can be converted to a state regulation problem if the steady setpoint is defined as the origin. Assume the periodic setpoint falls in the range of periodic input signals, the input is then parameterized as

$$u(t) = K_j(x(t) - \bar{x}_j) + \bar{u}_j, \quad j = mod(t, p). \tag{3.17}$$

The closed loop dynamic is

$$x(t+1) = \Phi_j x(t) + B_j(\bar{u}_j - K_j\bar{x}_j) + w(t), \quad j = mod(t, p), \tag{3.18}$$

where $\Phi_j = A_j + B_jK_j$. Denote $\tilde{x}(t) = [x(t)^T \ \bar{u}^T]^T = [\ x(t)^T \ \ \bar{u}_0 \ \ \dots \ \ \bar{u}_{p-1}\ ]^T$. Then the augmented dynamic is represented by

$$\tilde{x}(t+1) = \tilde{\Phi}_j\tilde{x}(t) + \tilde{B}w(t) = \left(\begin{bmatrix} \Phi_j & -B_jK_js_{j,0} & \dots & -B_jK_js_{j,p-1} \\ 0 & I & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{bmatrix} + P_j \right) \tilde{x}(t) + \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} w(t), \tag{3.19}$$

where

$$P_0 = \begin{bmatrix} 0 & B_0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \dots, P_{p-1} = \begin{bmatrix} 0 & 0 & \dots & B_{p-1} \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, j = mod(t, p). \tag{3.20}$$

The task now is to obtain the robust PMPIS for the augmented dynamic (3.19). Following the procedure in section 3.2, it is easy to convert it to the robust maximum positive invariant set problem for LTI systems. The difference is only the dynamic here is not asymptotical stable but marginal stable, due to the integrator for $\bar{u}_j$. Fortunately, it does not affect the characterization algorithm as analyzed in [97]. The $N \times p$ stabilizable sets can be constructed recursively then.

The decision variable is the perturbation to the linear control law (3.17):

$$u(t) = K_j(x(t) - \bar{x}_j) + \bar{u}_j + v(t), \quad j = mod(t, p). \tag{3.21}$$

The objective function for optimization is modified such that it penalizes the deviation of $x(t), u(t)$ from the reference signal,

$$v^* = \arg\min_{v} \ (u - \bar{u}_k)^T R(u - \bar{u}_k) + (A_k x + B_k u - \bar{x}_{k+1})^T P(A_k x + B_k u - \bar{x}_{k+1}) \tag{3.22}$$

$$s.t. \ \ u = K_k(x - \bar{x}_k) + \bar{u}_k + v, A_k x + B_k u \in \mathbb{S}_j^{m-1} \ominus \mathbb{W}_k, u \in \mathbb{U}_k, k = mod(j - m, p) \tag{3.23}$$

The controllers are saved in a lookup table, and the online implementation is exactly the same as that for regulation problem in section 3.3.2.

Theorem 3.3. *The online control is recursive feasible. Under the proposed control, the state will asymptotically converge to periodic MRPIS Ω_j, thus the tracking error is bounded.*

Proof. The reasoning is similar to Theorem 3.2, thus omitted here. □

Theorem 3.3 requires that the initial state $x(t)$ and setpoint u^{ref} should fall in the feasible region for the controller to work. A potential problem with it is the possible infeasibility if the initial state is far away from the setpoint, which limits its application. To solve the problem, a two-level optimization scheme can be adopted.

Level 1: Solve the steady state optimization problem:

$$\bar{u}^* = \arg\min_{\bar{u}} \ \ |\bar{u} - u^{ref}|^2 \tag{3.24}$$

$$s.t. \ \ \begin{bmatrix} x(t)^T & \bar{u}^T \end{bmatrix}^T \in \mathbb{S}_j^N, j = mod(t + N, p). \tag{3.25}$$

Here $\mathbb{S}_j^N$ denotes the N steps ahead stabilizable set to Θ_j for the augmented state $\tilde{x}(t)$.

Level 2: Take $\bar{u}^*$ as artificial setpoint and calculate the control input for the minimum time control problem above with $\tilde{x}(t) = \begin{bmatrix} x(t)^T & \bar{u}^{*T} \end{bmatrix}^T$

Remark 3.2. *The steady state optimization problem (3.24) is a QP problem, too. It can be solved online or offline via multi-parametric programming. It is noted that its complexity depends on the period p. Thus it does not incur much computation burden if p is small.*

Remark 3.3. *The feasible region is the projection of N step stabilizable set on the x subspace, independent to the desired setpoint, for the two level optimization approach.*

3.5 Simulation Studies

3.5.1 Regulation Problem

Consider a simple periodic system with period length $p = 2$ as

$$x(t+1) = A_j x(t) + B_j u(t) + w(t), \quad j = mod(t, 2), \tag{3.26}$$

where $A_0 = \begin{bmatrix} 1.5 & 0.6 \\ 0 & 1.2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1.0 & 0.6 \\ 0 & 1.2 \end{bmatrix}$, $B_0 = B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Control input u is bounded as $u \in \mathbb{U} = \{u | -1 \le u \le 1\}$, and the disturbance $w(t)$ is bounded by $\mathbb{W}_0 = \mathbb{W}_1 = \{w \| w|_\infty \le 0.1\}$. The linear controllers used are $K_0 = K_1 = [-1.0912 - 0.6113]$ for simplicity. The robust PMPISs were constructed, with the set Θ_2 is shown in Fig 3.1. A prediction horizon $N = 6$ is used in simulation. The stabilizable set associated with Θ_1 is shown in Fig. 3.2. To investigate the advantage of the proposed approach, the feasible regions of it and constraints tightening approach [46] are compared, as shown in Fig. 3.3. It clearly shows that the proposed approach provides a larger attraction domain for the same prediction horizon. As stated in Section 3.3.2, the online controller for every stabilizable set $\mathbb{S}_j^i$ is a PWA function, which needs to be saved and looked up for control action calculation. One instance is shown in Fig. 3.4, which plots the controller $C_1^6(x)$. We use $[-1; -1.63;]$ as the initial state, and the state response and control input trajectory are shown in Fig. 3.5. It is shown that the state is steered to robust PMPIS without violation of constraints on control input.

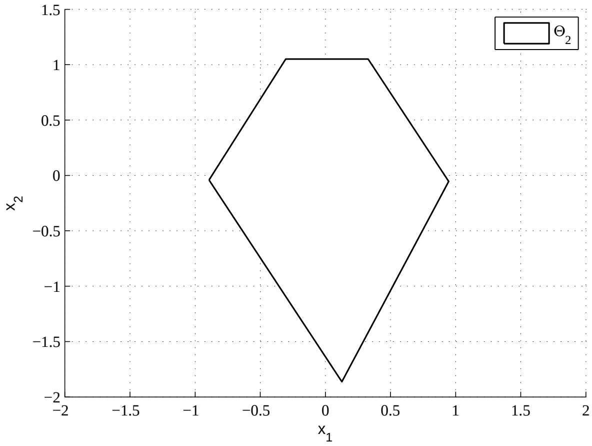

Figure 3.1: Robust PMPIS Θ_2

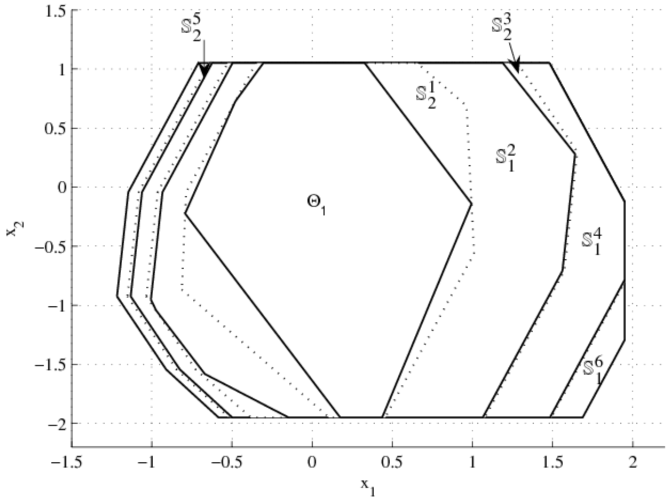

Figure 3.2: Stabilizable Sets Starting from Mode 1

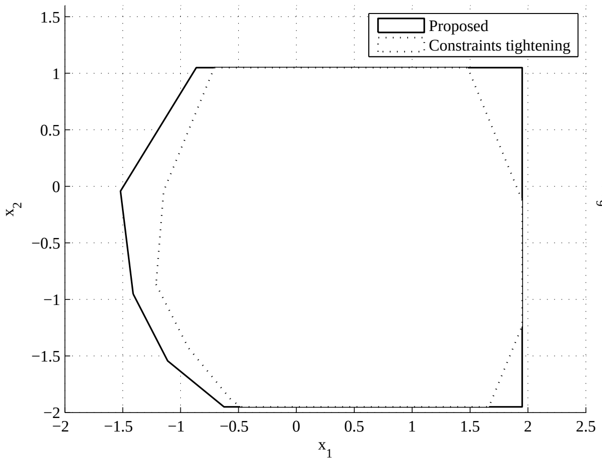

Figure 3.3: Attraction Regions Comparison

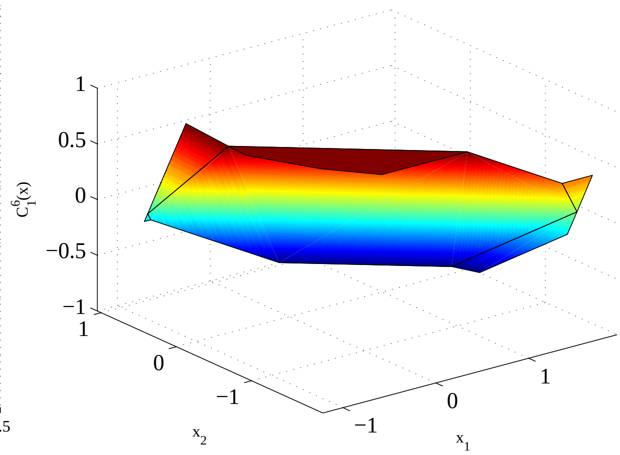

Figure 3.4: Controller $C_1^6(x)$

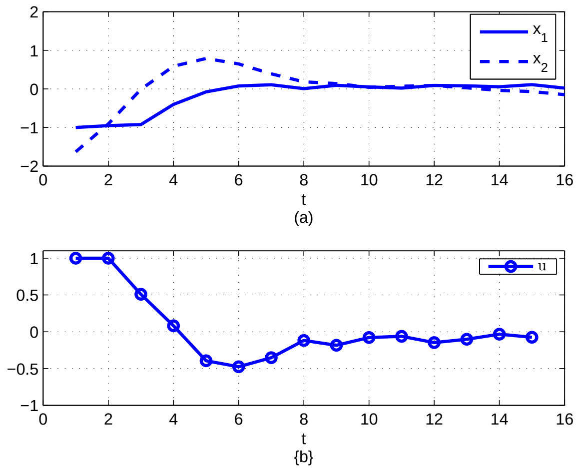

Figure 3.5: Responses with Initial State x=[-1;-1.63;]: (a) State Trajectories; (b) Control Input Trajectory

3.5.2 Tracking Problem

The same model in previous subsection is adopted for tracking problem. The initial state is $x(0) = [1.2\ 2]^T$. During the phase $t = 0$ to $t = 30$ the setpoint switches between $\bar{x}_0 = [1.6\ -1]$ and $\bar{x}_1 = [2\ -1]$ periodically. At $t = 30$, the setpoint changes to $\bar{x}_0 = [2.4\ \ -1.59]$ and $\bar{x}_1 = [3.14\ \ -1.41]$. The state trajectory and control input are depicted in Fig. 3.6 & 3.7, respectively. It is shown that the state converges to the setpoint asymptotically.

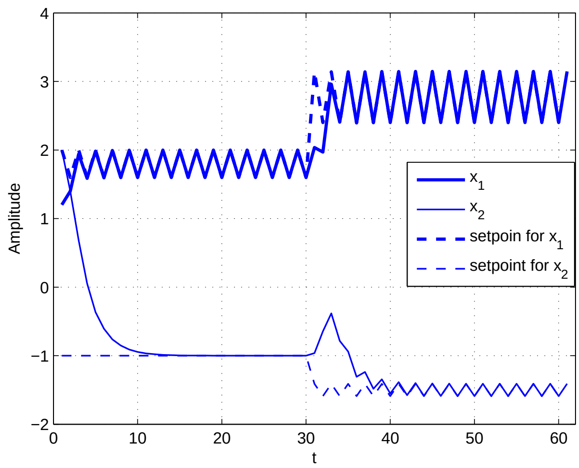

Figure 3.6: State Trajectory for Tracking

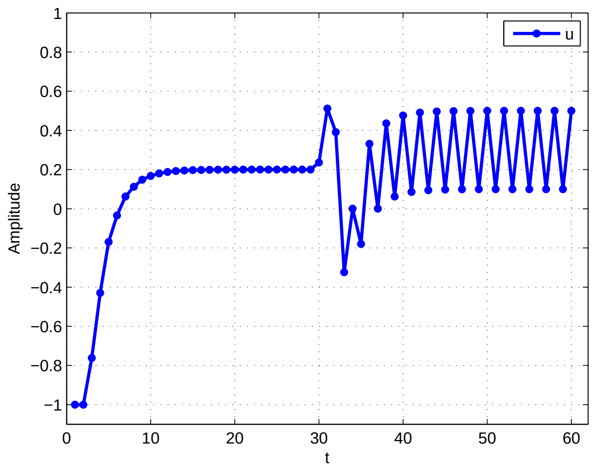

Figure 3.7: Control Input $u(t)$ for Tracking

3.6 Conclusion

In this chapter, a robust minimum time control algorithm is presented for the state regulation and tracking problem of linear periodic systems subject to bounded external disturbance. The approach makes use of stabilizable set and multi-parametric programming technique. The controller can steer the state to the target set in the shortest steps while satisfying state and control inputs constraints. As minimal time approach is adopted here, the resulting control law is simpler than that of MPC algorithm as in LTI case. It is also less conservative in term of feasible region since only one step prediction is required for online control input calculation. The simulation examples provided show the applicability and efficiency of the proposed algorithm.

Chapter 4

Tube-based Quasi-min-max Output Feedback MPC for Linear Parameter Varying Systems

To address model nonlinearity and/or uncertainty, linear parameter varying systems has attracted considerable research attention since 1990s[18, 21, 24, 27, 28]. For LPV systems, the time varying parameters are measured online, but their future evolution is uncertain, but contained in a prescribed bounded set. While great progress has been made in state feedback form, the reality that only output variables are available in practice calls for stabilizing output feedback MPC with reasonable computational complexity. The output feedback introduces additional uncertainty, state estimation error. The interaction between state estimation and control has to be addressed, but is difficult for nonlinear systems. In this chapter, an output feedback MPC for LPV systems combining tube-based robust MPC and quasi-min-max algorithm is proposed. The disturbance invariant tube is used to bound disturbance-like response, while quasi-min-max approach is adopted to handle model parametric uncertainty and to regulate the center trajectory of the tube. The recursive feasibility and robust stability are rigorously guaranteed by design.

The rest of the chapter is structured as follows. In section 4.1, the problem considered is defined; the state observer design is explained in section 4.2; section 4.3 shows the use of tube in controller design;

the online optimization is formulated in section 4.4 and the theoretical properties are analyzed; simulation studies are provided in section 4.5; finally, conclusions are drawn in section 4.6.

Notation. The symbol $*$ is used for convenience to denote

$$\begin{bmatrix} M & * \\ N & H \end{bmatrix} = \begin{bmatrix} M & N^T \\ N & H \end{bmatrix}.$$

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4.1 Problem Statement

The discrete time system to be controlled is described by

$$x(t+1) = A(t)x(t) + B(t)u(t) + w(t), \tag{4.1a}$$

$$y(t) = Cx(t) + v(t), \tag{4.1b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^l$ is the measurement; $w \in \mathbb{R}^n$ and $v \in \mathbb{R}^l$ are persistent disturbances and measurement noises, respectively. w and v are bounded by polytopic sets as $w \in \mathbb{W}$ and $v \in \mathbb{V}$. $A(t)$ and $B(t)$ are time varying matrices with appropriate dimensions, which are assumed to be exactly known at the current moment, and unknown but vary in the convex hull $\Omega = Co\{[A_1|B_1], \ldots, [A_J|B_J]\}$ in the future as

$$[A(t)|B(t)] = \sum_{j=1}^{J} [A_j|B_j]\lambda_j(t), \tag{4.2}$$

where $\sum_{j=1}^{J} \lambda_j(t) = 1$ and $\lambda_j(t) \geq 0$ for $1 \leq j \leq J$. In addition, the input signals $u(t) = \begin{bmatrix} u_1(t) & \ldots & u_m(t) \end{bmatrix}^T$ are component-wisely bounded as

$$|u_j(t)| \leq u_{j,max}, \quad 1 \leq j \leq m, t \geq 0, \tag{4.3}$$

which are decided by the actuator limits. Output constraints may be considered in the same way as input constraints.

Therefore the uncertainty of the system (4.1) includes the unknown time varying system dynamic, external disturbances and sensor noise. Due to various non-vanishing uncertainties, it is difficult to achieve asymptotic stability. The objective of the controller is then to stabilize the system and to steer the state to a neighborhood around the origin, while satisfying the input constraints (4.3). Since only output variables are available, the state observer + state feedback controller scheme, commonly used, is adopted.

4.2 State Observer Design

The following Luenberger state observer is adopted to estimate the state of system:

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B(t)u(t) + L_e(y(t) - C\hat{x}(t)), \tag{4.4}$$

where $\hat{x}(t)$ is the estimation of $x(t)$, and L_e is the observer gain. The estimation error is defined as $e(t) = x(t) - \hat{x}(t)$, and its dynamic is described by

$$e(t+1) = (A(t) + L_e C)e(t) + w(t) + L_e v(t). \tag{4.5}$$

The error dynamic varies within a polytopic set, which is designed to guarantee stability by properly choosing the observer gain L_e (here the stability is defined for the disturbance-free part of (4.5)). Another possibility is the parameter-dependant observer gain as

$$L_e(t) = \sum_{j=1}^{J} L_{e,j}\lambda_j(t). \tag{4.6}$$

The observer gains $L_{e,j}$, $1 \leq j \leq J$ are to satisfy LMI conditions in the following lemma.

Lemma 4.1. *If there exists $P > 0$ and $Y_j = PL_{e,j}$ satisfying the following LMI constraints*

$$\begin{bmatrix} P & * \\ PA_j + Y_j C & P \end{bmatrix} \geq 0, \quad 1 \leq j \leq J, \tag{4.7}$$

the observer dynamic (4.5) with the observer gain as $L_e = \sum_{j=1}^{J} \lambda_j(t) L_{e,j} = \sum_{j=1}^{J} \lambda_j(t) P^{-1} Y_j$ is stable.

Proof. The dynamic of the observer with the parameter-dependant observer gain is

$$e(t+1) = \sum_{j=1}^{J} \lambda_j(t)\Phi_{o,j} e(t) + \tilde{w}(t), \tag{4.8}$$

where $\tilde{w}(t) = w(t) + \sum_{j=1}^{J} L_{e,j}\lambda_j(t)v(t)$ and $\Phi_{o,j} = (A_j + L_{e,j}C)$. Since dynamic (4.8) is linear, the stability is analyzed on the nominal system without disturbances:

$$e(t+1) = \Phi_o e(t) = \sum_{j=1}^{J} \lambda_j(t)\Phi_{o,j}e(t). \tag{4.9}$$

The quadratic stability condition for the error dynamic is equivalent to

$$P - \Phi_o^T P \Phi_o > 0. \tag{4.10}$$

By using the Schur complement, it can be rewritten as

$$\begin{bmatrix} P & * \\ P\Phi_o & P \end{bmatrix} > 0 \tag{4.11}$$

The above LMI condition is satisfied if

$$\begin{bmatrix} P & * \\ P\Phi_{o,j} & P \end{bmatrix} > 0, 1 \leq j \leq J. \tag{4.12}$$

With the introduction of new variables $Y_j = PL_{e,j}$, (4.7) is obtained. The corresponding observer gains are $L_{e,j} = P^{-1}Y_j, 1 \leq j \leq J$. □

For simplicity, the constant gain formulation is adopted hereafter. Since the state observer dynamic is stable and driven by bounded external disturbances, it is possible to characterize the bound of state estimation error as in [19].

Definition 4.1. *[1]: A set $\mathbb{Z}$ is robustly positive invariant for the discrete time system $z^+ = f(z, w)$, where $w \in \mathbb{W}$, if $f(z, w) \in \mathbb{Z}$ for any $z \in \mathbb{Z}$ and $w \in \mathbb{W}$.*

The minimum robustly positive invariant set $\mathbb{E}$ or its outer approximations can be calculated by approaches proposed in [23, 26, 25]. Let us assume that the initial estimation error $e(0) \in \mathbb{E}$, which implies that

$$e(t) \in \mathbb{E}, t \geq 0. \tag{4.13}$$

Thus $\mathbb{E}$ is considered as a bound of state estimation error, which will be used for controller design in following sections.

Remark 4.1. *To calculate $\mathbb{E}$, the LPV system (4.5) is treated as polytopic system, which seems to be the only thing to do, but very conservative. The adoption of parameter-dependant observer gain (4.6) is advantageous as compared to a constant gain L_e. It is helpful to reduce the uncertainty of observer dynamic. In some cases, it is possible to stabilize the observer dynamic, which may be not possible by using a simple constant gain. A less uncertain dynamic is also helpful for the algorithms in [23, 25] to obtain a tighter approximation of $\mathbb{E}$.*

4.3 Disturbance Invariant Tube

Instead of controlling the state directly, output feedback MPC controls the estimated state. The real state will fall in the neighborhood of the estimated state, i.e., $x(t) \in \hat{x}(t) \oplus \mathbb{E}$, as analyzed in section 4.2. The dynamic of the estimated state is

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B(t)u(t) + d(t), \tag{4.14}$$

where $d(t) = L_e Ce(t) + L_e v(t) \in \mathbb{D} = L_e C\mathbb{E} \oplus L_e \mathbb{V}$ is the equivalent disturbance. The presence of d obviously complicates the prediction of the future state estimation. In the following, the disturbance invariant tube technique is exploited to decouple the analysis of nominal prediction and disturbance response. We define the response component contributed by $d(t)$ as $e_{xz}(t) = \hat{x}(t) - z(t)$, where $z(t)$ is the nominal estimated state trajectory without disturbance excitation. Control input is parameterized by

$$u(t) = u_z(t) + u_e(t) = u_z(t) + Ke_{xz}(t) \tag{4.15}$$

It is easy to derive that

$$\hat{x}(t) = z(t) + e_{xz}(t), \tag{4.16}$$

$$z(t+1) = A(t)z(t) + B(t)u_z(t), \tag{4.17}$$

$$e_{xz}(t+1) = \Psi_c(t)e_{xz}(t) + d(t) = (A(t) + B(t)K)e_{xz}(t) + d(t). \tag{4.18}$$

We assume that the controller gain K is properly chosen such that the dynamic of $e_{xz}(t)$ is stable, which can be checked by the following lemma.

Lemma 4.2. *If there exists a matrix $G > 0$ and Y, such that*

$$\begin{bmatrix} G & * \\ A_jG + B_jY & G \end{bmatrix} > 0, 1 \leq j \leq J, \tag{4.19}$$

and controller gain $K = YG^{-1}$, the dynamic (4.18) of e_{xz} is stable.

Proof. The proof is straightforward and similar to it for Lemma 4.1 above, thus omitted here. □

Assumed such a controller gain K exists, then the dynamic of e_{xz} is governed by the polytopic model:

$$e_{xz}(t+1) = \Psi_c(t)e_{xz}(t) + d(t) = \sum_{j=1}^{J} \lambda_j(t)\Psi_{c,j}e_{xz}(t) + d(t), \tag{4.20}$$

where $\Psi_{c,j} = A_j + B_jK$. It is desirable to characterize the bound of e_{xz} excited by the equivalent disturbance $d(t)$. Similar to the previous section, the minimum robustly positive invariant set $\mathbb{E}_{xz}$ of dynamic (4.20) is calculated, such that if $e_{xz}(t) \in \mathbb{E}_{xz}$, then $e_{xz}(t+1) \in \mathbb{E}_{xz}$ for any possible realization of $\Psi_c(t)$ and $d(t) \in \mathbb{D}$. To enforce the input constraints (4.3), $u_z(t)$ has to satisfy a tightened constraints as $u_z(t) \in \mathbb{U}_z = \mathbb{U} \ominus K\mathbb{E}_{xz}$. An inner approximation of $\mathbb{U}_z$ is given as $\tilde{\mathbb{U}}_z$, comprising component-wise constraints as:

$$|u_{z,j}(t)| \leq \tilde{u}_{z,jmax}, 1 \leq j \leq m, t \geq 0, \tag{4.21}$$

such as $\tilde{\mathbb{U}}_z \subseteq \mathbb{U}_z$.

Remark 4.2. *As in section 4.2, it is possible to choose K as parameter dependant controller gain, too. The difference of the resulting dynamic is that the system matrix is not linear, but quadratic function of the time varying variable λ as*

$$e_{xz}(t+1) = \sum_{i=1}^{J} \sum_{j=1}^{J} \lambda_i(t)\lambda_j(t)\Psi_{i,j}e_{xz}(t) + d(t). \tag{4.22}$$

Although it still can be treated as a polytopic system, new techniques of less conservativeness for characterization of the robustly positive invariant set for such systems are desirable.

4.4 Quasi-min-max MPC

In this section, MPC is adopted to regulate the dynamic of the estimated state $\hat{x}$. As in [24] for state feedback case, Quasi-min-max MPC utilizes the infinite horizon objective function

$$J_0^\infty(t) = x^T(0|t)Lx(0|t) + u^T(0|t)Ru(0|t) + \sum_{i=1}^{\infty} x^T(i|t)Lx(i|t) + u^T(i|t)Ru(i|t). \tag{4.23}$$

The online optimization minimizes the upper bound of $J_0^\infty(t)$. For the output feedback case, the approach is not directly applicable. One intuitive solution is to replace state $x(t)$ by the estimated state $\hat{x}(t)$ for the optimization problem. However, since separation principle does not hold for nonlinear systems generally, the simple modification does not provide stability or performance guarantee. The difficulty is introduced by the equivalent disturbance $d(t)$. Thus a remedy to the problem is to deal with the nominal state evolution $z(t)$ and the response caused by non-vanishing disturbances separately. As in section 4.3, the nominal estimated state is given by $z(t)$ as

$$z(t+1) = A(t)z(t) + B(t)u_z(t), \tag{4.24}$$

$u_z(t) \in \mathbb{U}_z$. $z(t)$, the nominal state estimation, is the center trajectory of the state estimation evolution tube $z(t) + e_{xz}(t)$. Instead of considering the disturbance-corrupted dynamic (4.14), Quasi-min-max is designed based on the disturbance-free dynamic (4.24) of the nominal estimated state $z(t)$. The optimization problem is formulated to minimize the upper bound of the nominal cost on $z(t)$ as

$$\min_{z(0|t),u_z(0|t),F(t),P(t)} \max_{A(t+i),B(t+i),i\geq 1} \gamma(t) \tag{4.25}$$

subject to constraints:

$$z(i+1|t) = A(t+i)z(i|t) + B(t+i)u_z(i|t), i \geq 0, \tag{4.26}$$

$$u_z(0|t) \in \mathbb{U}_z, \hat{x}(t) \in z(0|t) \oplus \mathbb{E}_{xz}, \tag{4.27}$$

$$u_z(i|t) = F(t)z(i|t) \in \tilde{\mathbb{U}}_z, i \geq 1, \tag{4.28}$$

$$z^T(0|t)Lz(0|t) + u_z^T(0|t)Ru_z(0|t) + z^T(1|t)P(t)z(1|t) \leq \gamma(t), \tag{4.29}$$

$$z^T(i|t)(P(t) - L - F^T(t)RF(t))z(i|t) \geq z^T(i+1|t)P(t)z(i+1|t),\ i \geq 1. \tag{4.30}$$

Remark 4.3. *In comparison with state feedback Quasi-min-max MPC ([24]), the initial state* $z(0|t)$ *is not a known vector, but a decision variable. It is noted that the nominal trajectory, the center of the tube, is not a single one as in [20] for LTI systems. The center trajectory itself is a tube, comprising of infinite possible trajectories, each corresponding to a particular realization of* $[A(t+i)|B(t+i)], i \geq 1$. *An illustration of the evolution of state estimation is depicted in Fig. 4.1.*

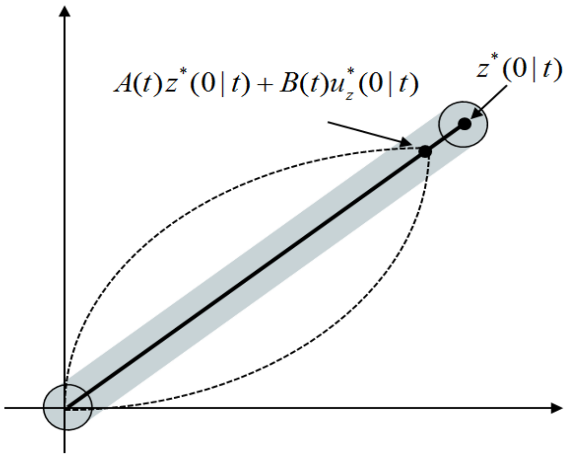

Figure 4.1: Illustration of State Estimation Evolution[1]

The next theorem shows how to convert the optimization problem (4.25-4.30) into a mathematically tractable formulation.

Theorem 4.1. *The mathematical formulation of (4.25-4.30) is*

$$\min_{z(0|t),u_z(0|t),Q(t),Y(t),X(t)} \gamma(t) \tag{4.31}$$

subject to

$$u_z(0|t) \in \mathbb{U}_z,\ \hat{x}(t) \in z(0|t) \oplus \mathbb{E}_{xz}, \tag{4.32}$$

$$\begin{bmatrix} 1 & * & * & * \\ A(t)z(0|t) + B(t)u_z(0|t) & Q(t) & * & * \\ L^{1/2}z(0|t) & 0 & \gamma(t)I & * \\ R^{1/2}u_z(0|t) & 0 & 0 & \gamma(t)I \end{bmatrix} > 0, \tag{4.33}$$

[1]solid line: a trajectory of tube center corresponding to a particular realization $[A(t)|B(t)], t \geq 0$; grey area: the disturbance invariant tube along a particular tube center; area enclosed by dashed line: the bound or tube of the tube center trajectory.

$$\begin{bmatrix} Q(t) & * & * & * \\ A_jQ(t)+B_jY(t) & Q(t) & * & * \\ L^{1/2}Q(t) & 0 & \gamma(t)I & * \\ R^{1/2}Y(t) & 0 & 0 & \gamma(t)I \end{bmatrix} > 0, j \leq J, \tag{4.34}$$

$$\begin{bmatrix} X(t) & * \\ Y(t)^T & Q(t) \end{bmatrix} > 0, \ with\ X_{jj} < \tilde{u}^2_{z,jmax}, j \leq m. \tag{4.35}$$

Proof. It is obvious that (4.32) is identical to (4.27). Then divide (4.29) by $\gamma(t)$ and define $Q^{-1}(t) = \gamma^{-1}(t)P(t)$, it becomes

$$1 - \gamma^{-1}(t)z^T(0|t)Lz(0|t) - \gamma^{-1}(t)u_z^T(0|t)Ru_z^T(0|t) - z^T(t+1)Q^{-1}(t)z(t+1) \geq 0, \tag{4.36}$$

By applying Schur Complements, it is equivalent to

$$\begin{bmatrix} 1 - (z^T(0|t)Lz(0|t) - u_z^T(0|t)Ru_z^T(0|t))/\gamma(t) & * \\ z(1|t) & Q(t) \end{bmatrix} \geq 0. \tag{4.37}$$

This condition can be rewritten as

$$\begin{bmatrix} 1 & * \\ z(i|t) & Q(t) \end{bmatrix} - T^T(t)\gamma^{-1}(t) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} T(t) \geq 0, \tag{4.38}$$

where $T(t) = \begin{bmatrix} L^{1/2}z(0|t) & 0 \\ R^{1/2}u_z(0|t) & 0 \end{bmatrix}$. Repeat to apply the Schur Complement, the above inequality is expressed in LMI form as (4.33). Next we prove that (4.34) implies (4.30). Define

$$M(t) = P(t) - L - F^T(t)RF(t), \tag{4.39}$$

$$N(t+i) = A(t+i) + B(t+i)F(t). \tag{4.40}$$

(4.30) is satisfied if

$$M(t) - N^T(t+i)P(t)N(t+i) \geq 0. \tag{4.41}$$

Multiplying by $\gamma(t)$ and left- and right- multiplying by $P^{-1}(t)$, it becomes

$$H(t) - \gamma(t)P^{-1}(t)N^T(t+i)\gamma^{-1}(t)P(t)N(t+i)(\gamma(t)P^{-1}(t)) \geq 0. \tag{4.42}$$

where $H(t) = \gamma(t)P^{-1}(t)M(t)P^{-1}(t)$. Using Schur Complements, it is converted as

$$\begin{bmatrix} H(t) & * \\ N(t+i)Q(t) & Q(t) \end{bmatrix} \geq 0, \tag{4.43}$$

Define $Y(t) = F(t)Q(t)$, it follows that

$$\begin{bmatrix} Q(t) & * \\ A(t+i)Q(t) + B(t+i)Y(t) & Q(t) \end{bmatrix} - \hat{T}^T(t) \begin{bmatrix} \gamma(t)I & 0 \\ 0 & \gamma(t)I \end{bmatrix}^{-1} \hat{T}(t) \geq 0, \tag{4.44}$$

where $\hat{T}(t) = \begin{bmatrix} L^{1/2}Q(t) & 0 \\ R^{1/2}Y(t) & 0 \end{bmatrix}$. By applying the Schur complement again, it require

$$\begin{bmatrix} Q(t) & * & * & * \\ A(t+i)Q(t) + B(t+i)Y(t) & Q(t) & * & * \\ L^{1/2}Q(t) & 0 & \gamma(t)I & * \\ R^{1/2}Y(t) & 0 & 0 & \gamma(t)I \end{bmatrix} \geq 0. \tag{4.45}$$

This condition is valid for any $[A|B] \in \Omega$, if (4.34) is satisfied. Next the constraints on control input is considered. Since $z^T(i|t)P(t)z(i|t)$ decreases with i and $z^T(1|t)P(t)z(1|t) \leq \gamma(t)$ (due to (4.33,4.34)), $z(i|t)$ will stay in the invariant set $z(i|t)^T Pz(i|t) \leq \gamma(t)$. The peak bound of input is

$$\max_{i\geq 1} |u_{z,j}(i|t)|^2 = \max_{i\geq 1} |(YQ^{-1}z(i|t))_j|^2 \leq \max_{z^T Pz\leq\gamma(t)} |(YQ^{-1}z)_j|^2 \leq |(YQ^{-1/2})_j|^2 = (YQ^{-1}Y^T)_{jj}. \tag{4.46}$$

Then if there exists a symmetrical matrix X such that (4.35) is satisfied, the input constraints satisfaction (4.29) is guaranteed. These complete the proof. □

The online implementation of the proposed controller is as follows:

1. $t = 1$, $\hat{x}(1)$ is given ;
2. solve the optimization problem (4.31-4.35) to obtain the optimal solution $z^*(0|t), u_z^*(0|t)$;
3. apply $u(t) = u_z^*(0|t) + K(\hat{x}(t) - z^*(0|t))$ to the system (4.1);
4. obtain the new estimated state $\hat{x}(t+1)$ via (4.4), $t = t + 1$ and return to step 2.

Remark 4.4. *The differences between the optimization problem (4.31-4.35) and the one for state feedback MPC [24] are the additions of one decision variable $z(t)$ and one linear inequality constraint (4.32). Thus the Quasi-min-max approach is extended from state feedback case to output feedback case with modest increase on computational complexity.*

Remark 4.5. *It is noted that the proposed approach could also be considered as an extension of Tube MPC for LTI system [20] to model uncertain case. It may be less conservative, as compared to the approach in [2], which treats the discrepancy between nominal model prediction and real system response as external disturbances.*

Theorem 4.2. *Assume that the initial state estimation error $e(t) \in \mathbb{E}$. If a feasible solution can be determined at step t, the optimization problem (4.31-4.35) is feasible for all future steps $t + i, i \geq 1$.*

Proof. As in most MPC approaches, it will be shown that the optimal solution $z^*(0|t), u_z^*(0|t), Q^*(t), Y^*(t), X^*(t), \gamma^*(t)$ to (4.31-4.35) at t, is associated with a feasible solution to (4.31-4.35) at $t + 1$. We choose $\tilde{z}(0|t+1) = A(t)z^*(0|t) + B(t)u_z^*(0|t), \tilde{u}_z(0|t+1) = Y^*(t)Q^*(t)^{-1}z(0|t+1), \tilde{Y}(t+1) = Y^*(t), \tilde{Q}(t+1) = Q^*(t), \tilde{X}(t+1) = X^*(t), \tilde{P}(t+1) = P^*(t), \tilde{\gamma}(t+1) = \gamma^*(t), \tilde{F}(t+1) = F^*(t)$ as a candidate for (4.31-4.35) at $t+1$. Consider constraint (4.32) first. Since $e(t) \in \mathbb{E}, \hat{x}(t) \in z^*(0|t) \oplus \mathbb{E}_{xz}$, and $\mathbb{E}$ and $\mathbb{E}_{xz}$ are positive invariant sets, it is inferred that $\hat{x}(t+1) \in \tilde{z}(0|t+1) \oplus \mathbb{E}_{xz}$. Since $\tilde{z}(0|t+1)$ is within the invariant set $z^T(i|t)P^*(t)z(i|t) \leq \gamma^*(t)$ and $F(t)z(i|t) \subset \hat{\mathbb{U}}_z \subset \mathbb{U}_z$ (due to (4.35)), $\tilde{u}_z(0|t+1) \in \mathbb{U}_z$. Thus (4.32) is satisfied at $t + 1$. Secondly, the salification of (4.33,4.34) at $t+1$ requires $\tilde{z}^T(0|t+1)L\tilde{z}(0|t+1) + \tilde{u}_z^T(0|t+1)R\tilde{u}_z^T(0|t+1) + \tilde{z}^T(1|t+1)\tilde{P}(t+1)z(1|t+1) \leq \tilde{\gamma}(t)$ and $\tilde{z}^T(i+1|t+1)\tilde{P}(t+1)\tilde{z}(i+1|t+1) \leq \tilde{z}^T(i|t+1)(\tilde{P}(t+1) - L - \tilde{F}^T(t+1)R\tilde{F}^(t+1))\tilde{z}(i|t+1), i \geq 1$, which are guaranteed to be fulfilled by (4.29, 4.30) at t. Lastly, (4.35) at $t + 1$ is satisfied since it is the same as the one at t. Thus the candidate is a feasible solution to optimization problem (4.31-4.35) at $t + 1$. This completes the proof. □

The system property under the proposed controller (4.15) is explained in the following theorem.

Theorem 4.3. *Assume that the initial state estimation error $e(t) \in \mathbb{E}$. If the optimization problem is feasible initially, the closed system is stable and the state will evolve into a neighborhood of the origin eventually.*

Proof. We first investigate the stability of $z^*(0|t)$, whose dynamic is represented conceptually by

$$z^*(0|t+1) = f(z^*(0|t), \hat{x}(t), d(t)). \tag{4.47}$$

Choose Lyapunov function as $V(z^*(0|t)) = z^{*T}(0|t)Lz^*(0|t) + u_z^{*T}(0|t)Ru_z^*(0|t) + (A(t)z^*(0|t) + B(t)u_z^*(0|t))^T P^*(t)$ $(A(t)z^*(0|t) + B(t)u_z^*(0|t))$. Assume that $L > 0$, it follows that $V(\cdot) \geq 0$. Due to the recursive feasibility property, $V(z^*(0|t+1))$ is upper bounded as:

$$\begin{aligned} V(z^*(0|t+1)) &= z^{*T}Lz^* + u_z^{*T}Ru_z^* + (Az^* + Bu_z^*)^T P^*(t+1)(Az^* + Bu_z^*) \\ &\leq \tilde{z}^T L\tilde{z} + \tilde{u}_z^T R\tilde{u}_z + (A\tilde{z} + B\tilde{u}_z)^T P^*(t)(A\tilde{z} + B\tilde{u}_z). \end{aligned} \tag{4.48}$$

For conciseness, the index $(t+1)$ for A, B and $(0|t+1)$ for $z^*, u_z^*, \tilde{z}$ and $\tilde{u}_z$ are dropped in (4.48). The Lyapunov function decreases monotonically as

$$\Delta V = V(z^*(0|t+1)) - V(z^*(0|t)) \leq -z^{*T}(0|t)Lz^*(0|t) - u_z^{*T}(0|t)Ru_z^*(0|t) < 0, \tag{4.49}$$

due to (4.29, 4.30). Thus $z^*(0|t)$ is asymptotical stable. As shown in section 4.3, $\hat{x}(t), x(t)$ are within a neighborhood of $z^*(0|t)$ as $\hat{x}(t) \in z^*(0|t) \oplus \mathbb{E}_{xz}$ and $x(t) \in \hat{x}(t) \oplus \mathbb{E}$. Thus as $z^*(0|t)$ converges to the origin, $\hat{x}(t)$ and $x(t)$ will asymptotically converge to $\mathbb{E}_{xz}$ and $\mathbb{E}_{xz} \oplus \mathbb{E}$, respectively. □

4.5 Simulation Studies

The following second order system model [21] is used for simulation:

$$\begin{aligned} x(t+1) &= A(\alpha(t))x(t) + B(\beta(t))u(t) + w(t), \\ y(t) &= Cx(t) + v(t), \end{aligned} \tag{4.50}$$

where $A(\alpha(t)) = \begin{bmatrix} 0.872 & -0.0623\alpha(t) \\ 0.0935 & 0.997 \end{bmatrix}$, $B(\beta(t)) = \beta(t)\begin{bmatrix} 0.0935 \\ 0.00478 \end{bmatrix}$, $C = \begin{bmatrix} 0.333 & -1 \end{bmatrix}$. We assume that the scheduling parameters $\alpha(t)$ and $\beta(t)$ belong to the following regions:

$$\alpha(t) \in [1, 5], \quad \beta(t) \in [0.1, 1]. \tag{4.51}$$

The small disturbance and sensor noises are bounded as $w \in \mathbb{W} = \{w| \|w\|_\infty \leq 0.001\}$ and $v \in \mathbb{V} = \{v| |v| \leq 0.001\}$. The input constraint is assumed to be bounded as $u \in \mathbb{U} = \{u| |u| \leq 1\}$. The initial states of the system and the observer are assumed as $x(0) = [-1.5 \quad -0.2]^T$ and $\hat{x}(0) = [-1.3 \quad 0]^T$, respectively. The observer gain $L_e = [-1.2060 \quad -1.7289]^T$ and controller gain $K = [-2.2379 \quad -1.3429]$ are adopted. The system response under the controller (4.15) is shown in Fig. 4.2. The real state $x(t)$ and estimated state $\hat{x}(t)$ are plotted by the blue and red lines, respectively. In simulation, no non-vanishing disturbance was introduced, thus the state converges to the origin asymptotically. The control input at each step is shown in Fig. 4.3. It is clear that the constraints on input signal is enforced by the proposed MPC. Fig. 4.4 shows the worst case upper bound cost $\gamma(t)$. It is obvious that $\gamma(t)$ decreases monotonically. The simple simulation demonstrates that the proposed output MPC stabilizes the LPV systems against various bounded uncertainties.

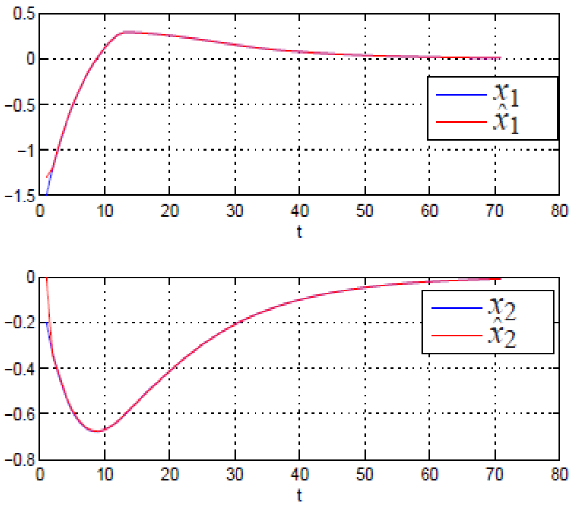

Figure 4.2: State Trajectories

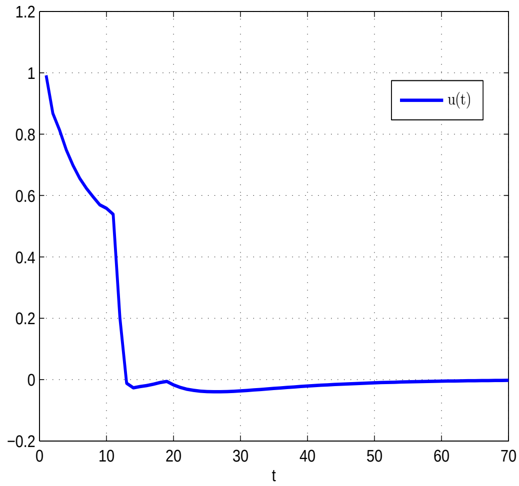

Figure 4.3: Control Input Trajectory

4.6 Conclusion

The tube based Quasi-min-max MPC is presented for control of input constrained LPV systems. The recursive feasibility and robust stability are guaranteed. However, some conservativeness exists. Two directions may be exploited to improve the performance. The first one is to utilize a parameter dependant offline controller K in order to characterize a tighter positive invariant set; the second one is the conservativeness introduced by the ellipsoidal approximation of polyhedral set by the LMI condition. It is desirable to combine more advanced state feedback MPC approaches, such as those proposed in [27] recently, into output

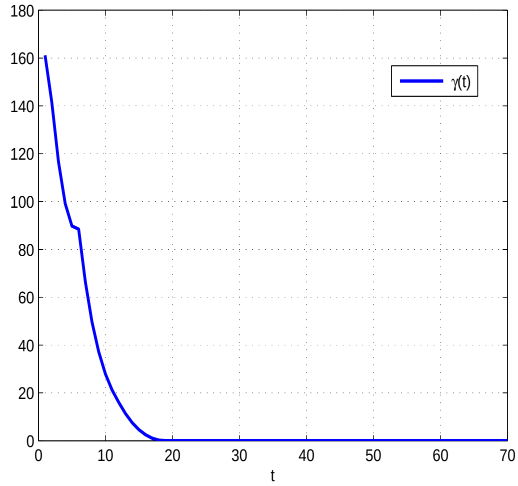

Figure 4.4: Evolution of Upper Bound of Worst Cost $\gamma(t)$

feedback scenario.

Chapter 5

Economic MPC with Stability Assurance

Motivated by optimizing plant economics, Dynamic RTO [68] or Economic MPC[72], which directly optimizes the plant's economic performance in the dynamic regulation problem based on the fundamental dynamic models at high frequency(faster sampling rate compared to RTO), has drawn considerable attention from both the research and industrial community. The main difference between Economic MPC and the standard MPC is the performance function used for online optimization, which gives rise to the stability issue. This chapter presents two stabilizing Economic MPC formulations: terminal cost function approach and stability constraint enforcement approach. As compared to previous approaches, the proposed formulations are applicable to a larger class of system models and performance functions. Enhanced closed loop performance is observed in simulation results.

The rest of this chapter is structured as follows. The problem is defined in section 5.1; a brief introduction to previous results is given in section 5.2; two Economic MPC formulations are presented in section 5.3 and 5.4, respectively; simulation studies are presented in section 5.5; finally, conclusions are drawn in section 5.6.

5.1 Problem Statement

We consider the nonlinear discrete time system

$$x(t+1) = f(x(t), u(t)), \tag{5.1}$$

which is subject to constraints as state $x \in \mathbb{X} \subset \mathbb{R}^n$, control input $u \in \mathbb{U} \subset \mathbb{R}^m$. The objective of MPC controller is to optimize the economic performance, while satisfying state and control input constraints. The economic performance function is given by $l(x, u)$. It is thus profitable to steer the system to the optimal steady state solution (x_s, u_s) associated with $l(x, u)$. The closed loop stability to (x_s, u_s) under the control of Economic MPC is the central theoretical question, which has drawn serious investigation[70, 71, 72] and is also the main design objective of Economic MPC formulations proposed later in this chapter. Since $l(\cdot, \cdot)$ is a general function, a distinctive feature of Economic MPC is that $l(\cdot, \cdot)$ is not positive definite with respect to the optimal steady state (x_s, u_s) as it is assumed in the standard MPC for regulation problem. Thus the analysis tools developed for MPC for regulation problem[1] is not directly applicable to economic MPC. However, since it is relevant to the Economic MPC proposed in section 5.4, a standard stabilizing MPC for regulation problem is presented here first. At every sampling instance t, MPC for regulation problem solves a finite horizon optimal control problem $\mathscr{P}^r(x(t))$:

$$J^{r*}(x(t)) = \min_{\mathbf{u}(t)} J^r(x(t), \mathbf{u}(t)) = \min_{\mathbf{u}(t)} \sum_{i=0}^{N-1} l^r(x(i|t), u(i|t)) + V_f^r(x(N|t)) \tag{5.2}$$

subject to

$$x(0|t) = x(t), \tag{5.3}$$

$$x(i+1|t) = f(x(i|t), u(i|t)), 0 \le i \le N-1, \tag{5.4}$$

$$x(i|t) \in \mathbb{X}, u(i|t) \in \mathbb{U}, 0 \le i \le N-1, \tag{5.5}$$

$$x(N|t) \in X_f^r, \tag{5.6}$$

where $\mathbf{u}(t) = \{u(0|t), u(1|t), \ldots, u(N-1|t)\}$ is the future control input sequence. The stage cost function is normally chosen as $l^r(x, u) = x^T Qx + u^T Ru$ and the terminal set X_f^r and terminal cost function $V_f^r(\cdot)$ satisfy the stability axiom[1]. The first element $u^*(0|t)$ of the optimal solution to $\mathscr{P}^r(x(t))$ is applied to (5.1). The MPC controller is implicitly constructed as $\mu^r(x(t)) = u^*(0|t)$.

5.2 Previous Results

Although the idea to use general economic performance function for online optimization is not new, most proposals are heuristic without rigorous stability analysis. A breakthrough was made by a series of papers[72, 70, 71]. Closed loop stability is obtained for strictly dissipative systems.

Definition 5.1. *(Strictly dissipative system [71]) The system* $x^+ = f(x, u)$ *is strictly dissipative with respect to the supply rate* $s : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ *if there exists a storage function* $\lambda(x) : \mathbb{X} \to \mathbb{R}$ *and a positive definite function* $\rho : \mathbb{X} \to \mathbb{R}$ *such that for all* $x \in \mathbb{X}, u \in \mathbb{U}$

$$\lambda(f(x, u)) - \lambda(x) \leq -\rho(x - x_s) + s(x, u). \tag{5.7}$$

Assumption 5.1. *The system (5.1) is strictly dissipative with respect to the supply rate* $s(x, u) = l(x, u) - l(x_s, u_s)$.

A terminal cost MPC formulation proposed in [70] solves the optimization problem $\mathscr{P}_a^e(x(t))$:

$$J_a^{e*}(x(t)) = \min_{\mathbf{u}(t)} J_a^e(x(t), \mathbf{u}(t)) = \min_{\mathbf{u}(t)} \sum_{i=0}^{N-1} l(x(i|t), u(i|t)) + V_f^a(x(N|t)) \tag{5.8}$$

subject to

$$(5.3) - (5.5) \text{ and } x(N|t) \in X_f^a, \tag{5.9}$$

$\mathscr{P}_a^e(x(t))$ is equivalent to the another optimization problem $\mathscr{P}_b^e(x(t))$:

$$J_b^{e*}(x(t)) = \min_{\mathbf{u}(t)} J_b^e(x(t), \mathbf{u}(t)) = \min_{\mathbf{u}(t)} \sum_{i=0}^{N-1} L(x(i|t), u(i|t)) + V_f^b(x(N|t)), \tag{5.10}$$

subject to (5.9), where $L(x, u) = l(x, u) + \lambda(x) - \lambda(f(x, u)) - l(x_s, u_s)$ and $V_f^b(x) = V_f^a(x) + \lambda(x) - V_f^a(x_s) - \lambda(x_s)$ are rotated stage cost and terminal cost functions. The MPC associated with $\mathscr{P}_a^e(x)$ and $\mathscr{P}_b^e(x)$ are denoted as $\mu_a^e(x) = \mu_b^e(x) = u^*(0|t)$. If there exists a control law $\kappa_f(x)$ such that

$$V_f^a(f(x, \kappa_f(x))) - V_f^a(x) \leq -l(x, \kappa_f(x)) + l(x_s, u_s), \quad \forall x \in X_f^a, \tag{5.11}$$

the rotated terminal cost function satisfies

$$V_f^b(f(x, \kappa_f(x))) - V_f^b(x) \leq -L(x, \kappa_f(x)), \quad \forall x \in X_f^a, \tag{5.12}$$

which is essential to show the monotonically decreasing property of Lyapunov function $J_b^{e*}(x(t))$, therefore the closed loop stability to (x_s, u_s). The remaining problem is how to design the terminal cost function $V_f^a(x)$ satisfying (5.11) or $V_f^b(x)$ satisfying (5.12) and the corresponding terminal set X_f^a. Two approaches were presented in [70] for cases of storage function $\lambda(x)$ unknown and known, respectively.

5.3 Terminal Cost Function Approach

The terminal cost MPC formulation[70] is adopted in this section, and two methods to design the terminal cost function are proposed.

5.3.1 Amrit's Approach [70] for Case of Known $\lambda(x)$

Here we consider the case of known $\lambda(x)$. The function $V_f^b(x)$ satisfying (5.12) is to be constructed. Let us inspect the two inequalities (5.11) and (5.12), it is easy to see that the requirement on $V_f^a(x)$ and $V_f^b(x)$ are exactly the same in structure. Thus the approach for the case of the storage function $\lambda(x)$ unknown is readily applicable to the case of $\lambda(x)$ known without any additional conservatism introduced. The extension of Amrit's approach[70] for $\lambda(x)$ unknown case to the case of $\lambda(x)$ known is summarized below. Interested reader may refer to [70] for details. For simplicity of notation, the origin is shifted to (x_s, u_s).

Assumption 5.2. *The function $f(\cdot,\cdot)$ and $L(\cdot,\cdot)$ are twice continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m$, and the linearized system $x^+ = Ax + Bu$ is stabilizable. The terminal control law $\kappa_f(x) = Kx$ such that $A_K = A + BK$ is stable. The terminal set X_f^a is chosen as an invariant level set of Lyapunov function $V(x) = (1/2)x^T \bar{P} x$.*

Theorem 5.1. *Let Assumption 5.2 hold. Then there exists a matrix Q^* such that*

$$x^T(Q^* - L_{xx}(x))x \geq 0, \quad \forall x \in X_f^a, \tag{5.13}$$

and the function $V_f^b(x) = (1/2)x^T P x + p^T x$, where $P = P^* + \alpha P_1^*$, $A_K^T P^* A_K - P^* = -Q^*$, $A_K^T P_1^* A_K - P_1^* = -I$ and $p^T = L_x^T(0)(I - A_K)^{-1}$, satisfies (5.12) if α is large enough.

Theorem 5.2. *Let Assumptions 5.1 and 5.2 hold, then the MPC controller $\mu_b^e(x)$ associated with $\mathscr{P}_b^e(x)$, in which $V_f^b(x)$ is designed according to Theorem 5.1, stabilizes the system (5.1).*

Since $V_f^b(f(x,\kappa_f(x))) - V_f^b(x) \leq -(l(x,\kappa_f(x)) - l(x_s,u_s) + \lambda(x) - \lambda(f(x,\kappa_f(x))))$, $(V_f^b(f(x,\kappa_f(x))) - \lambda(f(x,\kappa_f(x)))) - (V_f^b(x) - \lambda(x)) \leq -l(x,\kappa_f(x)) + l(x_s,u_s)$. Then $V_f^a(x) = V_f^b(x) - \lambda(x)$ will satisfy (5.11) and be used in $\mathscr{P}_a^e(x)$ for online optimization.

5.3.2 Lyapunov Function Approach

Another way for designing terminal cost functions $V_f^a(x)$ or $V_f^b(x)$ is to exploit the Lyapunov function of the system. This approach is applicable to both cases of $\lambda(x)$ known and unknown. For simplicity, only the formulation for the case of known $\lambda(x)$ is presented. An advantage of the Lyapunov function approach is that Assumption 5.2 does not necessarily hold.

Assume that there exists a terminal control law $\kappa_f(x)$ such that the closed loop system allows a Lyapunov function $V^c(x)$ and the corresponding positive invariant set X_f^c. According the converse Lyapunov Theorem, the following inequality holds:

$$\Delta V^c(x) = V^c(f(x,\kappa_f(x))) - V^c(x) \leq -\kappa_1(|x|), \quad \forall x \in X_f^c, \tag{5.14}$$

where $\kappa_1(\cdot)$ is of class $\mathcal{K}$ function. The candidate terminal cost function is chosen as $V_f^c = \alpha V^c$. we would like V_f^c to satisfy

$$\Delta V_f^c(x) = \alpha \Delta V^c(x) \leq -L(x,u) = -L(x,\kappa_f(x)), \quad \forall x \in X_f^c \tag{5.15}$$

via choosing a sufficient large α. However, this objective may not be satisfied due to the local property of $\Delta V^c(x)$ at $x = 0$. Therefore, a sufficient large α is sought first to enforce (5.15) over X_f^c excluding a neighborhood Ω around the origin. Ω may take the form of a norm function such as $\Omega = \{x | |x| \leq \epsilon\}$. There exists a constant a such that for all $\alpha \geq a$, (5.15) holds for all x satisfying $x \in X_f^c$ and $x \notin \Omega$, since $\Delta V^c(x) < 0$. The admissible α can be checked by the following optimization problem:

$$D = \max_x \quad \alpha \Delta V(x) + L(x,\kappa_f(x)) \tag{5.16}$$

subject to

$$x \in X_f, and\ x \notin \Omega. \tag{5.17}$$

If $D > 0$, try to increase α until $D \leq 0$. The online optimization problem $\mathscr{P}_c^e(x(t))$ is

$$J_c^{e*}(x(t)) = \min_{\mathbf{u}(t)} J_c^e(x(t), \mathbf{u}(t)) = \min_{\mathbf{u}(t)} \sum_{i=0}^{N-1} L(x(i|t), u(i|t)) + V_f^c(x(N|t)), \tag{5.18}$$

subject to

$$(5.3) - (5.5) \textit{ and } x(N|t) \in X_f^c, \tag{5.19}$$

the corresponding MPC controller is $\mu_c^e(x)$. The closed loop property is analyzed via the optimal cost $J_c^{e*}(x(t))$.

$$\begin{aligned} \Delta J_c^{e*}(x(t)) = J_c^{e*}(x(t+1)) - J_c^{e*}(x(t)) = & \sum_{i=0}^{N-1} L(x^*(i|t+1), u^*(i|t+1)) + V_f^c(x^*(N|t+1)) \\ & - \sum_{i=0}^{N-1} L(x^*(i|t), u^*(i|t)) - V_f^c(x^*(N|t)) \\ & \leq \sum_{i=1}^{N-1} L(x^*(i|t), u^*(i|t)) + L(x^*(N|t), \kappa_f(x^*(N|t))) + V_f^c(x^*(N+1|t)) \\ & - \sum_{i=0}^{N-1} L(x^*(i|t), u^*(i|t)) - V_f^c(x^*(N|t)) \\ & \leq -L(x(t), u^*(0|t)) + L(x^*(N|t), \kappa_f(x^*(N|t))) + V_f^c(x^*(N+1|t)) - V_f^c(x^*(N|t)) \end{aligned} \tag{5.20}$$

The first inequality is due to the feasibility of control sequence $\{u^*(1|t), \ldots, u^*(N-1|t), \kappa_f(x^*(N|t))\}$ to the problem $\mathscr{P}_c^e(x(t+1))$. The term $d = L(x^*(N|t), \kappa_f(x^*(N|t))) + V_f^c(x^*(N+1|t)) - V_f^c(x^*(N|t))$ is interpreted as disturbance. It is clear that $d \leq 0$ if $x^*(N|t) \notin \Omega$, and it is bounded as $d \leq \hat{d}$ if $x^*(N|t) \in \Omega$, where

$$\hat{d} = \max_x L(x, \kappa_f(x)) + V_f^c(f(x, \kappa_f(x))) - V_f^c(x), \quad s.t. \;\; x \in \Omega. \tag{5.21}$$

Although the closed loop dynamic is disturbance-free, the virtual disturbance d enables the analysis of the closed loop system under the framework of Input-to-State Stability (ISS)[74].

Definition 5.2. *A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov function in Γ for system $x(t+1) = F(x(t), w(t))$ ($x \in X, w \in W$), if*

1. *Γ is a compact positive invariant set including the origin as an interior point;*

2. *there exist a compact set* $\Xi \in \Gamma$ *(including the origin as an interior point), and a pair of suitable* $\mathcal{K}_\infty$*-functions* α_1, α_2 *such that*

$$V(x) \geq \alpha_1(|x|), \forall x \in \Gamma \tag{5.22}$$

$$V(x) \leq \alpha_2(|x|), \forall x \in \Xi \tag{5.23}$$

3. *there exist a suitable* $\mathcal{K}_\infty$*-function* α_3 *and a* $\mathcal{K}$*-function* σ *such that*

$$\Delta V(x) = V(x(t+1)) - V(x(t)) \leq -\alpha_3(|x|) + \sigma(|w|) \tag{5.24}$$

for all $x \in \Gamma$ *and all* $w \in W$.

Lemma 5.1. *If the system* $x(t+1) = F(x(t), w(t))$ *above allows an ISS-Lyapunov function in* Γ*, the state* $x(t)$ *will converge to a neighborhood* Θ *around the origin.*

It is easy to see that the function $J_c^{e*}(x)$ satisfies the ISS-Lyapunov function assumption, thus the state x will converge to a neighborhood Θ around the origin (optimal steady state). However, the asymptotical stability is not assured by $\mathscr{P}_c^e(x)$. A possible method to enforce asymptotical stability is to switch to a stabilizing controller when the state $x(t)$ is near x_s. To this purpose, another stabilizing MPC controller solves the optimization problem $\mathscr{P}_d^e(x(t))$:

$$J_d^{e*}(x(t)) = \min_{\mathbf{u}(t)} J_d^e(x(t), \mathbf{u}(t)) = \min_{\mathbf{u}(t)} \sum_{i=0}^{N-1} l(x(i|t), u(i|t)) \tag{5.25}$$

subject to

$$V^c(x(1|t)) - V^c(x(t)) \leq \beta V^c(f(x(t), \kappa_f(x(t)))) - V^c(x(t)), \tag{5.26}$$

and (5.9), where $\beta \in (0, 1]$ is a design parameter, which can be tuned to balance between stability and economic performance. The corresponding MPC $\mu_d^e(x)$ is stabilizing controller if $x \in X_f^c$ due to the stabilizing constraint (5.26). A simple logic on online optimization can be implemented to switch between $\mathscr{P}_c^e(x)$ and $\mathscr{P}_d^e(x)$.

Assumption 5.3. *The attraction region* Θ *around* x_s *satisfies* $\Theta \in X_f^c$.

Since $\hat{d} \leq \max_{x\in\Omega} L(x, \kappa_f(x))$, $\hat{d}$ can be arbitrarily small and consequently Assumption 5.3 can always be satisfied by restricting the set $\Omega = \{x||x| \leq \epsilon\}$ small enough. The overall control law $\mu_e^e(x)$ is

$$\mu_e^e(x) = \begin{cases} \mu_c^e(x), \ if x \notin X_f^c \\ \\ \mu_d^e(x), \ if x \in X_f^c \end{cases} \tag{5.27}$$

Theorem 5.3. *Let Assumptions (5.1) and (5.3) hold. The closed loop system under the switched control $\mu_e^e(x)$ is asymptotical stable to (x_s, u_s).*

Proof. If $x(k) \notin X_f^c$, $\mu_c^e(x)$ is on. The state will be steered to X_f^c due to the ISS property and Assumptions 5.1 and 5.3. Then $\mu_d^e(x)$ is on to asymptotical stabilize the system to x_s due to the stability constraint (5.26). □

Remark 5.1. *Comparing to Amrit's approach, the design is simplified, only a single-variate optimization problem needs to be solved.*

Remark 5.2. *The Lyapunov function approach does not rely on Assumption 5.2. The terminal controller can be synthesized by any available approaches. The economic performance function is not necessarily to be twice continuously differentiable. Hence, the economic MPC is applicable to a wider class of problems.*

Remark 5.3. *The terminal region for Lyapunov function approach can be chosen as any control invariant set using the terminal controller. It may be larger than the ellipsoidal set used in Amrit's approach, therefore rendering larger feasible region. One extreme example is for the linear model, the polytopic maximal positive invariant set, definitely larger than ellipsoidal set, can be used as the terminal set, thus resulting in a larger attraction domain for the same prediction horizon length.*

5.4 Stability Constraint Enforcement Approach

In section 5.3, Assumption 5.2 is relaxed in the Lyapunov function approach. Assumption 5.1, that the system is strictly dissipative with respect to $l(x, u) - l(x_s, u_s)$, is still assumed. Otherwise additional regularization terms have to be added to the original $l(\cdot, \cdot)$ so as to make it satisfy (5.7). Therefore the objective function for online optimization will be not the original one, which is necessary but not desirable. It is thus useful to investigate stabilizing Economic MPC formulations for general system models and economic cost functions which are not restricted by Assumption 5.1.

Recall that the MPC formulation for regulation problem $\mathscr{P}^r(x)$ is stabilizing for general model functions, which is assured by the monotonically decreasing property of the optimal regulation cost function $J^{r*}(x(t)) = \min_{\mathbf{u}(t)} J^r(x(t), \mathbf{u}(t))$ as $J^{r*}(x(t+1)) - J^{r*}(x(t)) \leq l^r(x(t), u^*(0|t))$. A hint $\mathscr{P}^r(x)$ offers is that the stability may be assured if the monotonically decreasing property of the regulation cost function is enforced in Economic MPC. In other word, the economic optimization is performed with an additional constraint that forces the regulation cost function, in fact the Lyapunov function, decrease. The online optimization is given by $\mathscr{P}^e_f(x(t))$:

$$J_f^{e*}(x(t)) = \min_{\mathbf{u}(t)} J_f^e(x(t), \mathbf{u}(t)) = \min_{\mathbf{u}(t)} \sum_{i=0}^{N-1} l(x(i|t), u(i|t)) \tag{5.28}$$

subject to

$$\begin{aligned} &(5.3) - (5.6), \\ &\sum_{i=0}^{N-1} l^r(x(i|t), u(i|t)) + V_f^r(x(N|t)) \leq C^*(t), \end{aligned} \tag{5.29}$$

where $C^*(t)$ is the upper bound of regulation cost for $x(t)$ evaluated along the previous predicted state trajectory for $x(t-1)$. The implementation is listed below:

1. Initialize at $t = 0$, $C^*(0) = \infty$;
2. Solve the optimization problem $\mathscr{P}^e_f(x(t))$;
3. Implement the first input $u(t) = \mu^e_f(x(t)) = u^*(0|t)$;
4. $C^*(t+1) = \sum_{i=0}^{N-1} l^r(x^*(i|t), u^*(i|t)) + V^r_f(x^*(N|t)) - \beta l^r(x^*(0|t), u^*(0|t))$;
5. $t = t + 1$, go to step 2.

β, any value within $(0, 1]$, can be used as a tuning parameter to balance the economic optimality and closed loop stability. A smaller β will render a wider margin for $\mathscr{P}^e_f(x)$ to optimize the economic performance function; a larger β results in faster decreasing rate of the regulation cost and better stability. Recursive feasibility and closed loop stability are summarized in the following theorem.

Theorem 5.4. *If $l^r(x,u), V_f^r(x), X_f^r$ are chosen to satisfy the stability axiom[1], the optimization problem (5.28-5.29) is recursive feasible with the N-step stabilizable set to X_f^r as its feasible region. The closed loop system with the control $\mu_f^e(x(t))$ is asymptotically stable. The state x will converge to the optimal steady state (x_s, u_s).*

Proof. The tail of the optimal solution $\mathbf{u}^*(t)$ is chosen as the candidate solution $\bar{\mathbf{u}}(t+1) = \{u^*(1|t), \ldots, u^*(N-1|t), \kappa_f(x^*(N|t))\}$. Since X_f^r is a positive invariant set for the dynamic $x^+ = f(x, \kappa_f(x))$ and $\kappa_f(x) \in \mathbb{U}$ for all $x \in X_f^r$, the constraints (5.3)-(5.6) are fulfilled by $\bar{\mathbf{u}}(t+1)$. $\sum_{i=1}^{N-1} l^r(x^*(i|t), u^*(i|t)) + l^r(x^*(N|t), \kappa_f(x^*(N|t))) + V_f^r(x^*(N+1|t)) \leq \sum_{i=0}^{N-1} l^r(x^*(i|t), u^*(i|t)) + V_f^r(x^*(N|t)) - l^r(x^*(0|t), u^*(0|t)) \leq C^*(t+1)$ implies that (5.29) is satisfied. Hence $\bar{\mathbf{u}}(t+1)$ is a feasible candidate to $\mathscr{P}_f^e(x(t+1))$, which is recursive feasible.

The Lyapunov function is chosen as $J_f^{r*}(x(t)) = J^r(x(t), \mathbf{u}^*(t)) = \sum_{i=0}^{N-1} l^r(x^*(i|t), u^*(i|t)) + V_f^r(x^*(N|t))$. It is easy to see $\Delta J_f^{r*}(t) \leq -\beta l^r(x(t), u^*(0|t)) \leq 0$ due to (5.29). Therefore $\lim_{t\to\infty} l^r(x(t), u^*(0|t)) = 0$ and the state x and control input u will converge to (x_s, u_s). □

Remark 5.4. *The constraint (5.29) in $\mathscr{P}_f^e(x)$ may increase the computational burden significantly. The computation delay compensation strategy in Chapter 6 may be applied to the economic MPC problem if the real time requirement is difficult to be fulfilled.*

5.5 Simulation Studies

In this section, three comparisons are presented to illustrate the advantages of the proposed Economic MPC in terms of applicability, terminal set volume and the economic performance.

5.5.1 Comparison on Applicability

A simple first order linear system is used to show the proposed approach's advantage in applicability.

$$x(k+1) = 0.8x(k) + u(k), \tag{5.30}$$

where $x \in \mathbb{X}$ and $u \in \mathbb{U}$. The economic cost function is given as $l(x,u) = |x| + u$. It is clear that $l(x,u)$ is not twice differentiable due to the term $|x|$, thus Amrit's approach [70] is not applicable to this simple example. It is shown below that it can be still handled by the Lyapunov function approach. The terminal controller is chosen as $\kappa_f(x) = Kx = -0.2x$; the terminal set $X_f^c = \{x|x^2 \le 1\}$. A Lyapunov function $V(x) = 13/8x^2$ satisfies $(A+BK)^T P(A+BK) - P = -Q = -1.04$. The terminal cost function is chosen as $V_f^c(x) = 5V(x)$. The functions of $\Delta V_f^c(x)$ and $-L(x,\kappa_f(x))$ are depicted in Fig. 5.1. It is clear that $-L(x,\kappa_f(x))$ lies below $\Delta V_f^c(x)$ except a neighborhood around the origin. The maximal virtual disturbance $\hat{d}$ is 0.0691. Through the ISS analysis, the state x will converge to the set $\Theta = \{x|13/8x^2 \le 0.18\}$, which is within X_f^c. Thus the state x will evolve to X_f^e and then $\mu_d^e(x)$ is to be switched on to stabilize the state and control input to $(0,0)$.

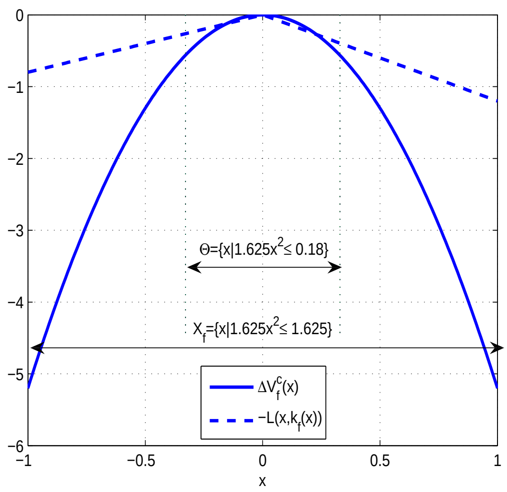

Figure 5.1: Illustration of Functions $\Delta V_f^c(x)$ and $-L(x,\kappa_f(x))$

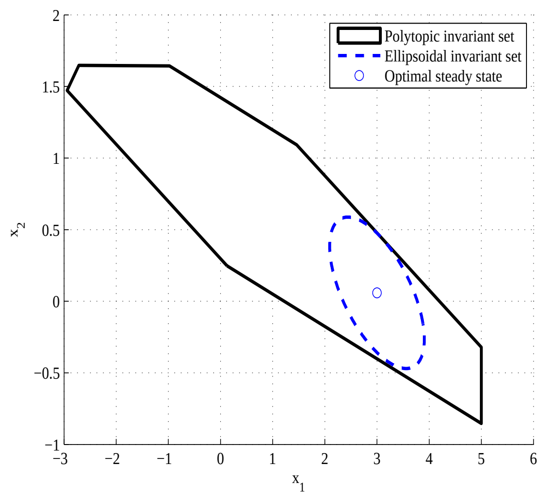

Figure 5.2: Comparison of Terminal Sets

5.5.2 Comparison on Terminal Set

The system considered is the double integrator:

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix} u(k). \tag{5.31}$$

Constraints on state and control input are $x \in \mathbb{X} = \{x||x| \leq 5\}$ and $u \in \mathbb{U} = \{u||u| \leq 0.3\}$. The economic cost function is given by $l(x,u) = (x - \begin{bmatrix} 3 \\ 3 \end{bmatrix})^T(x - \begin{bmatrix} 3 \\ 3 \end{bmatrix}) + 10u^T u$. The optimal steady state is $x_s = [3.0000\ 0.0588]^T$, $u_s = [0.0588\ \ -0.1176]^T$. The double integrator (5.31) is dissipative with respect to $l(x,u) - l(x_s,u_s)$. In this example, only the terminal sets around x_s constructed using different approaches are compared, as shown in Fig. 5.2. Since Amrit's approach[70] restricts X_f is the level set of a quadratic Lyapunov function, X_f of the maximal volume is depicted as an ellipsoidal set. It is well known that for linear systems ellipsoidal invariant set is the inner approximation of the maximal positive invariant set, which is a polytopic set. The proposed approach only requires the terminal set is control invariant, thus the maximal positive invariant set can be adopted for the double integrator model. It is obvious that the proposed approach offers much larger terminal set X_f than Amrit's approach.

5.5.3 Comparison on Economic Performance

Consider the single first order, irreversible chemical reaction in an isothermal continuous stirred tank reactor studied in [71, 70].

$$A \rightarrow B \ \ r = k_r c_A \tag{5.32}$$

in which $k_r = 0.4L/(mol\ min)$ is the rate constant. The material balance model for this process is

$$\begin{cases} \frac{dc_A}{dt} = \frac{u}{V_R}(c_{Af} - c_A) - k_r c_A \\ \frac{dc_B}{dt} = \frac{u}{V_R}(c_{Bf} - c_B) - k_r c_A \end{cases} \tag{5.33}$$

in which c_A and c_B are the molar concentration of A and B respectively, $c_{Af} = 1mol/L$ and $c_{Bf} = 0$ are the feed concentrations of A and B, and the volume of the reactor V_R is $10L$. The manipulated variable is the feed flow rate u, which is limited within $[0,20]L/min$. Economic performance function is $l(c_A, c_B, u) = -(2uc_B - 0.5u)$. The optimal steady state is $c_A = c_B = 0.5mol/L, u = 4L/min$. l is modified as $\bar{l} = -(2uc_B - 0.5u) + 0.1|u-4|^2$ so as to make (5.33) strictly dissipative w.r.t $\bar{l}$. The sampling period is 0.5s.

For MPC design, a terminal LQR controller ($Q = I_2, R = 10$) for the linearized model around the optimal steady state is chosen as $\kappa_f(x) = -Hx$, and the corresponding terminal set is $X_f = \{x|x^T Px =$

$x^T \begin{bmatrix} 3.1261 & 1.8924 \\ 1.8924 & 43.1793 \end{bmatrix} x \leq 1\}$. Four MPC controllers are simulated to test their performance, including standard regulation MPC, Amrit's approach, Lyapunov function approach and the SCE approach. The terminal cost function for Amrit's approach is $V_f^a(x) = 0.5x^T \begin{bmatrix} 0.11 & 0.04 \\ 0.04 & 0.15 \end{bmatrix} x + [-19.82 \; -44]\, x$; the terminal cost function for Lyapunov function approach is chosen as $V_f^c = 10x^T Px$; $\beta = 1$ is adopted in SCE approach. The prediction horizon is set as $N = 8$. The closed loop performance is evaluated via the cumulative economic cost over the first 15 seconds, which are tabulated in Table 5.1. The first 3 approaches produce similar performance, while SCE approach produces the best result. The state trajectories using Amrit's approach and Lyapunov function approach are very similar as depicted in Fig. 5.3 & 5.4. Since both approaches are of the TCF approach, the similarity in terms of closed loop performance suggests Lyapunov function approach is a good alternative to Amrit's approach due to its simplicity and generality.

Next, to evaluate the performance of SCE approach, a group of simulations with different β are done. The design parameters are $Q = I_2, R = 10, N = 8$. The state trajectories are presented in Fig. 5.5.3. The cumulative economic costs are tabulated in Table 5.2. As can be seen, a small β helps extract more profits from the dynamic behavior for a reasonable large prediction horizon; a large β helps the output converge fast to the optimal steady state. Comparing to Amrit's approach and Lyapunov function approach, stability constraint enforcement approach is better in term of the economic function. The state trajectories suggest that periodic operation may provide better economic performance than the optimal steady state operation. This point is better illustrated by another simulation, presented in Fig. 5.6. The design parameters are $Q = 10I_2, R = 1, N = 4, \beta = 0.01$. Although a smaller prediction horizon N is used, it still yields better economic performance, -57.2201, than Amrit's approach or Lyapunov function approach with $N = 8$. The state trajectory is almost periodic with a very weak magnitude decreasing, which is enforced by stability constraint.

Table 5.1: Economic Cost for Different Controllers

	Amrit's approach	Regulation	Lyapunov approach	SCE approach
economic cost	-50.0182	-50.0004	-51.8643	-54.2181

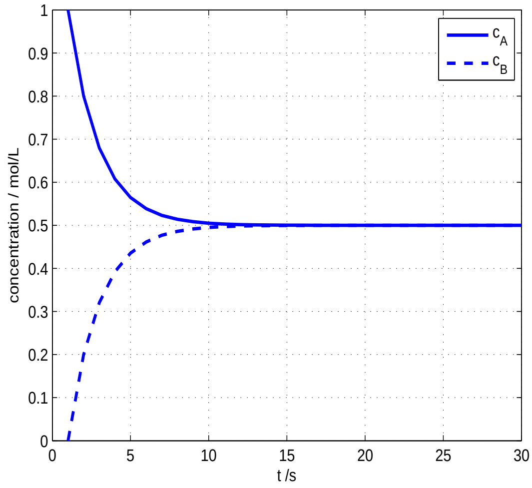

Figure 5.3: State Trajectories using Amrit's Approach

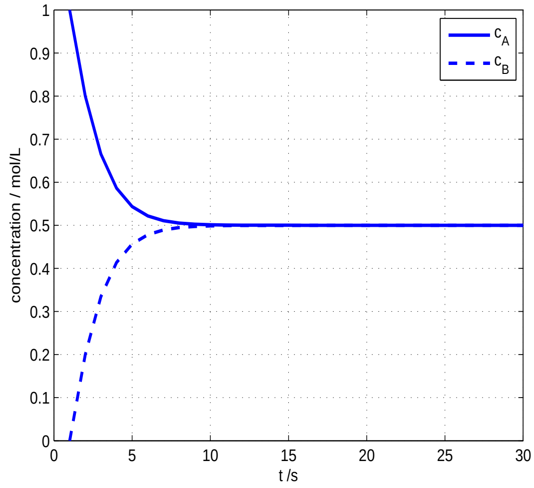

Figure 5.4: State Trajectories using Lyapunov Function Approach

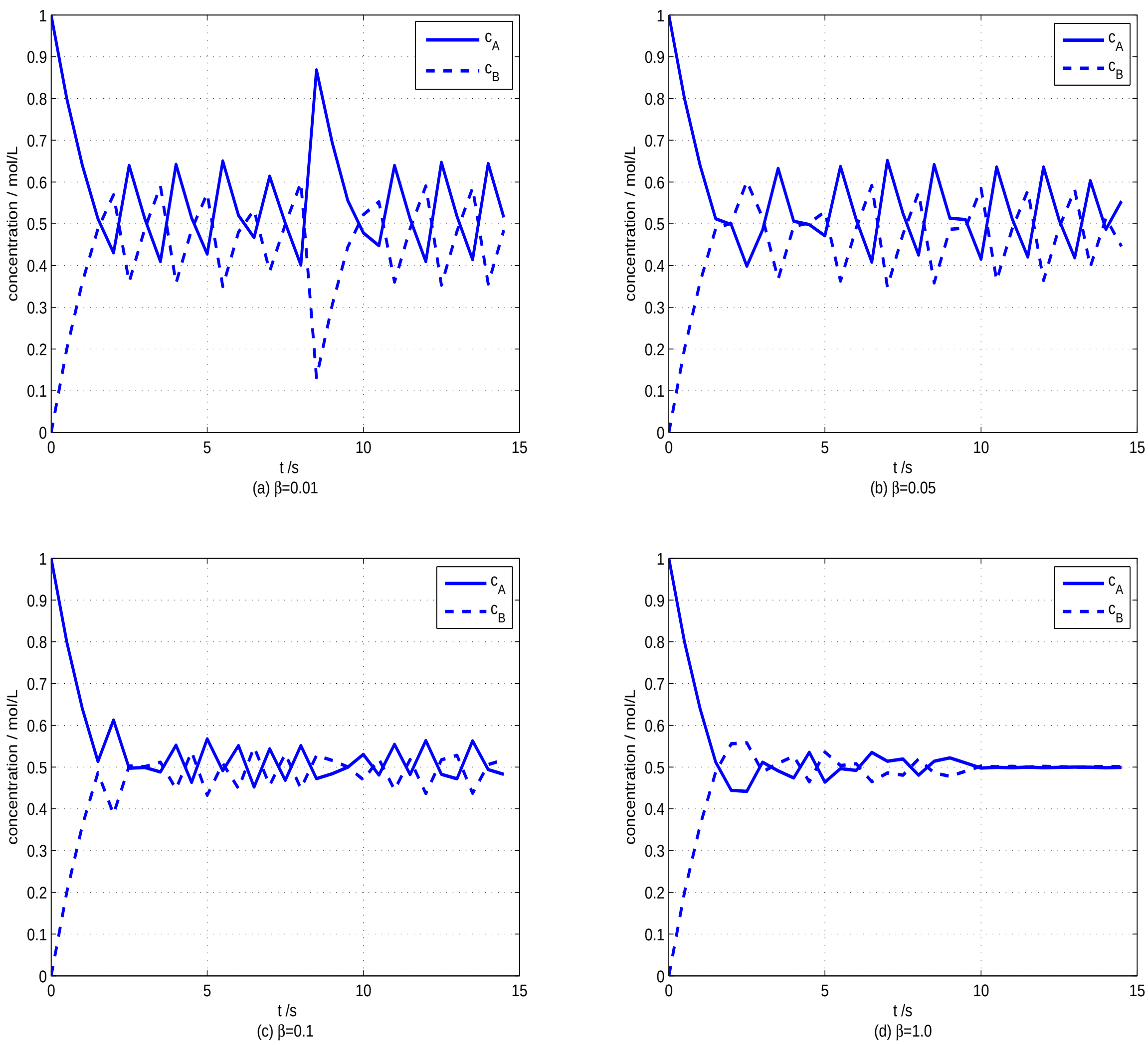

Figure 5.5: State Trajectories using Different β

Table 5.2: Economic Cost for Different β

β	0.01	0.05	0.1	1
economic cost	-64.7016	-60.0640	-55.8428	-54.2181

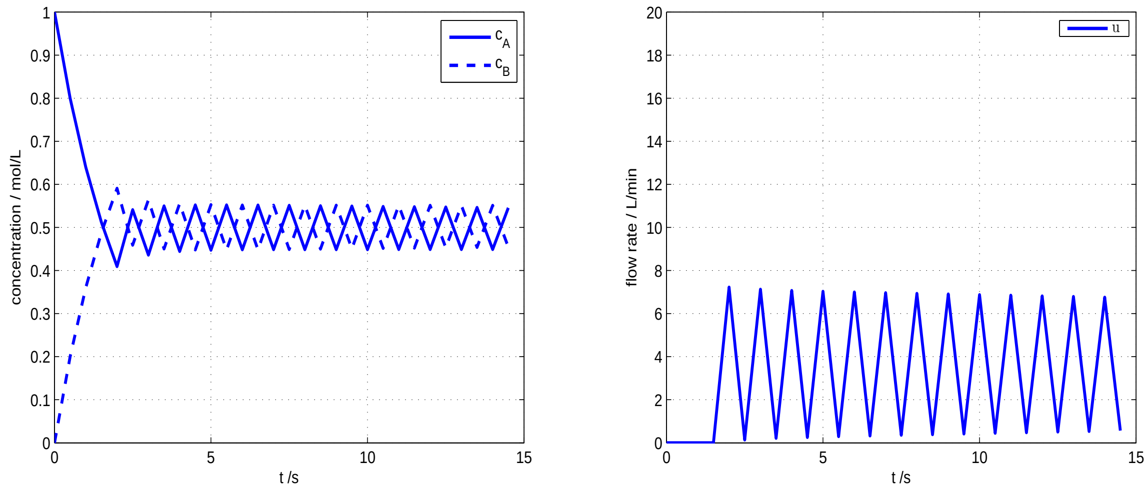

Figure 5.6: State and Input Trajectories ($Q = 10I_2, R = 1, N = 4, \beta = 0.01$)

5.6 Conclusion

Terminal cost function approach and stability constraint enforcement approach are proposed for stabilizing Economic MPC. The design methods for the terminal cost function are simpler and the restriction on the dynamic model, economic performance, terminal controller and terminal set is relaxed. The stability constraint enforcement approach explicitly enforces the closed loop stability by inserting a regulation cost decreasing constraint. It is applicable to general dynamic model and economic performance function. These advantages are validated by simulation results.

Chapter 6

Computation Delay Compensation for Real Time Implementation of Robust Model Predictive Control

Most of MPC variants in the literature[19, 25, 38], hereafter denoted as ideal MPC, implicitly assume that the optimization problem can be ideally solved with negligible computation time. [31, 32, 33] proposed MPC strategies taking into account the computation delay, inevitable in practice, for real time implementation. The uncertainty, inevitable too, is either ignored or implicitly addressed by the inherent robustness of nominal MPC controller. It is recognized that robust MPC that explicitly handles model uncertainty is necessary if a high degree of uncertainty presents. Here the computation delay compensation is extended to robust MPC so as to allow fast sampling rate. The closed loop stability and constraints satisfaction are rigorously guaranteed.

The rest of the chapter is structured as follows. In section 6.1, the problem is defined; the proposed dual time scale control scheme is elaborated in section 6.2; MPC algorithms are presented in section 6.3 and their theoretical properties are analyzed in section 6.4; the possibility to incorporate neighboring extremal control is discussed in section 6.5; simulation studies are provided in section 6.6; finally, conclusions are drawn in section 6.7.

6.1 Problem Statement

The uncertain plant is assumed to be adequately described by the discrete-time system model:

$$x(t+1) = f(x(t), u(t), w(t)), \tag{6.1}$$

where $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}$ and $w \in \mathbb{R}^{n_w}$ are the states, controls and disturbances variables, respectively at time step t. The sampling interval is T seconds. The model function $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}^{n_x}$ satisfies $f(0,0,0) = 0$. The disturbance term w can be used to describe model uncertainties and external disturbances. A simplified version of (6.1) is given by the linear model

$$x(t+1) = Ax(t) + Bu(t) + w(t). \tag{6.2}$$

The system is subject to constraints on the states and on the control inputs as $x \in \mathbb{X}$ and $u \in \mathbb{U}$. $\mathbb{X}$ is a closed set and $\mathbb{U}$ is a compact set, both of which containing the origin as an interior point. The disturbance w is assumed to be bounded as $w \in \mathbb{W}$. The objective is to stabilize the system while satisfying the state and input constraints. Further assumption on $f(\cdot,\cdot,\cdot)$ will be made clear in following sections when the MPC design is presented.

A typical MPC controller is defined as an optimization problem $\mathscr{P}(x)$ (1.4-1.5). The optimization routine solves $\mathscr{P}(x(t))$ at every sampling instant and applies the optimal input to the system. The ideal MPC assumes no delay is incurred by the optimization. For real implementation, the usual approaches to handle the delay are listed as

- T is larger than computation time and simply ignore the computational delay;
- T is larger than computation time and the one step delay is included in the system model.

Both approaches require slow enough sampling rate and introduce one step feedback delay into the closed loop. The performance may be severely degraded by the delay as reported in [37]. The following sections 6.2 and 6.3 present the strategies for fast sampling and delay compensation, respectively.

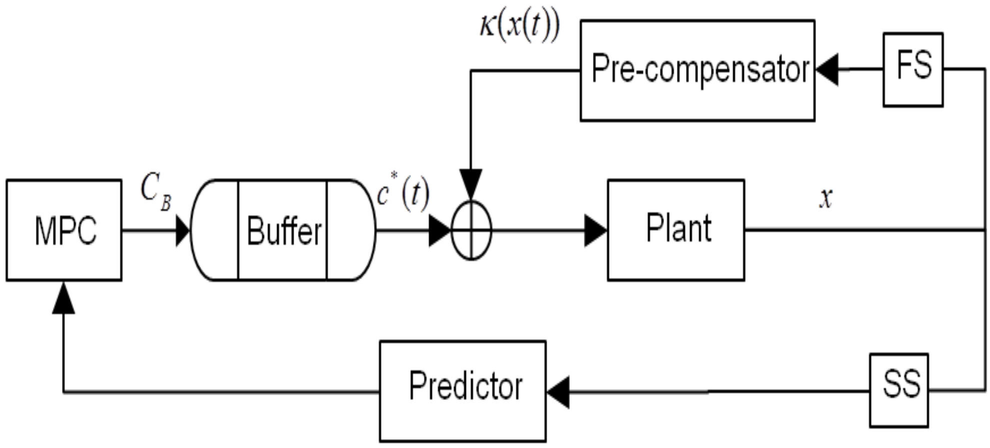

Figure 6.1: Dual Time Scale Control Scheme (FS/SS: fast/slow sampler)

6.2 Dual Time Scale Control for Fast Sampling

As analyzed in section 6.1, the computation burden of standard MPC, if used alone, decides the sampling rate. To allow fast sampling rate and thus sufficient feedback, another feedback signal, provided by a pre-compensator, is used to complement MPC. The proposed dual time scale control scheme is depicted in Fig. 6.1. The control input u is parameterized as

$$u(t) = c(t) + \kappa(x(t)). \tag{6.3}$$

The MPC should be designed for the compensated loop

$$x(t+1) = f(x(t), \kappa(x(t)) + c(t), w(t)) = F(x(t), c(t), w(t)). \tag{6.4}$$

Here we assume that the computation in MPC takes at most s sampling intervals. One useful feature of MPC is that it calculates the input sequences for a future period, instead of the current input signal only. The calculated input sequence C_B is stored in a buffer. During a period of s sampling intervals, the newest inputs $c(t)$ stored in the buffer is applied to the compensated system in order, which is called multi-step implementation of MPC. The input C_B stored in the buffer is updated when the new calculation in MPC is completed. In summary, the combination of multi-step implementation of MPC and the feedback input from the compensator allows fast sampling configuration.

Remark 6.1. *There is no specific requirement on the form of pre-compensator $\kappa(x(t))$. It can be model-*

free PID controller (it is possible to eliminate the tracking error at steady state if the disturbance is time invariant and integrator is included) or model-based linear/nonlinear controllers which are designed for the unconstrained system.

Remark 6.2. *It is noted that the inner compensator does not only provide faster reaction to disturbances, it may also help design less conservative robust MPC for the outer loop if the compensated loop is less uncertain than the open loop system.*

Remark 6.3. *It is worth noting that the dual time scale control scheme resembles the control hierarchy in chemical process in which the control function is decomposed and distributed into several layers, which work on different time scales. For the proposed control scheme, 'dual time scale' refers to how often the feedback is introduced, not to the model used in controller design. Both pre-compensator and MPC are designed based on the same model of fast sampling rate.*

6.3 Robust MPC with Computation Delay Compensation

As for the deterministic systems in [31, 32], the computation delay can be compensated by a predicted optimization, which means the optimization problem for the s steps ahead predicted state. The idea is briefly explained here. The input sequence u for the period of t to $t + s - 1$ is known at initial time t. Thus the predicted state $x(s|t)$ can be obtained via simulation. Instead of calculating $u(t + s)$ with $x(t + s)$ at $t + s$, the optimization problem for $x(s|t)$ starts at t so as to complete before $t + s$ and apply the inputs to the system on time at $t + s$ and onwards. The above procedure repeats every s time steps with the latest state prediction. The same strategy is adopted here for robust design, which is more involved technically. It will be shown in this section that various robust MPC strategies proposed in the literature can be adopted, with necessary modifications, in the dual time scale control scheme. Their theoretical properties will be analyzed in section 6.4.

6.3.1 Constraints Tightening Approach for Linear Systems

In this subsection, MPC is designed for the linear model (6.2). This is due not just to its simplicity, but also due to the actual need for some systems, for which even QP problem is still computation consuming[39]. The constraints tightening approach proposed in [38] enforces the robustness against disturbances by inserting in the predictive controller suitable constraints restrictions. The pre-compensator $\kappa(x)$ is exemplified by linear control as $\kappa(x(t)) = Kx(t)$. The compensated loop is

$$x(t+1) = Ax(t) + B(c(t) + Kx(t)) + w(t) = \Phi x(t) + Bc(t) + w(t), \tag{6.5}$$

where $\Phi = A + BK$ is a stable matrix. The state prediction for the compensated system is

$$x(i|t) = \Phi^i x(t) + \sum_{j=0}^{i-1} \Phi^{i-1-j}(Bc(j|t) + w(j|t)) \tag{6.6}$$

via recursive iteration of (6.5). Denote the nominal prediction and prediction uncertainty as $\bar{x}(i|t) = \Phi^i x(t) + \sum_{j=0}^{i-1} \Phi^{i-1-j} Bc(j|t)$ and $e(i|t) = \sum_{j=0}^{i-1} \Phi^{i-1-j} w(j|t)$. An equivalent form of (6.3,6.5) is given in terms of $\bar{x}$ and e:

$$x(i|t) = \bar{x}(i|t) + e(i|t), \quad u(i|t) = B\bar{x}(i|t) + c(i|t) + Be(i|t). \tag{6.7}$$

It is clear that the prediction uncertainty $e(i|t)$ is bounded as

$$e(i|t) \in \oplus_{j=0}^{i-1} \Phi^j \mathbb{W}. \tag{6.8}$$

Assumption 6.1. *The state $x(t)$ and future input signal $C_B = [c^{*T}(t), \ldots, c^{*T}(t+s-1)]^T$ are available at time t.*

The nominal state trajectory is optimized at t by the optimization problem $\mathscr{P}_1(x(t), C_B)$ over the control sequence $C(t+s) = [c^T(s|t), \ldots, c^T(s+N-1|t)]^T$ as:

$$V_1(x(t), C_B) = \min_{C(t+s)} \sum_{i=0}^{N-1} c^T(s+i|t) \Pi c(s+i|t) \tag{6.9}$$

subject to

$$\bar{x}(s|t) = \Phi^s x(t) + \sum_{j=0}^{s-1} \Phi^{s-1-j} B c^*(t+j), \tag{6.10}$$

$$\bar{x}(s+i+1|t) = \Phi \bar{x}(s+i|t) + Bc(s+i|t), 0 \le i \le N-1, \tag{6.11}$$

$$K\bar{x}(s+i|t) + c(s+i|t) \in \mathbb{U} \ominus K(\oplus_{j=0}^{s+i-1} \Phi^j \mathbb{W}), 0 \le i \le N-1, \tag{6.12}$$

$$\bar{x}(s+N|t) \in \bar{X}_f, \tag{6.13}$$

where $\bar{X}_f = X_f \ominus (\oplus_{j=0}^{s+N-1} \Phi^j \mathbb{W})$ and X_f is the maximal robust positive invariant set for $x^+ = \Phi x + w$. The stage cost and terminal constraint set are chosen to enforce robust constraints satisfaction and closed loop stability. The solution of $\mathscr{P}_1$ is denoted as $C^*(t+s) = [c^{*T}(s|t), \ldots, c^{*T}(s+N-1|t)]^T$. The MPC above is implemented in the dual time scale scheme as:

In background, between time t and $t+s$:

- solve the optimization problem $\mathscr{P}_1(x(t), C_B)$ and update the buffer $C_B = [c^{*T}(t+s), \ldots, c^{*T}(t+2s-1)]^T = [c^{*T}(s|t), \ldots, c^{*T}(2s-1|t)]$.

Online, at sampling instants $t, \ldots, t+s-1$:

- apply $u(t+i) = Kx(t+i) + c^*(t+i), 0 \le i \le s-1$.

6.3.2 Disturbance Invariant Set Approach for Linear Systems

A different robust MPC, tube based MPC [19], was proposed for the linear systems. The optimal control problem utilizes the concept of minimal disturbance invariant set and includes the initial state of the model, whose solution is the center of the tube, as a decision variable. Robust exponential stability of the disturbance invariant set can be achieved. The input is parameterized as

$$u(t) = v(t) + Ke(t) = v(t) + K(x(t) - z(t)) \tag{6.14}$$

where $z(t)$ is the center of the tube, driven by the open loop input $v(t)$ as

$$z(t+1) = Az(t) + Bv(t), \tag{6.15}$$

and $e(t)$ is the difference between the state and the center of the tube. It is easy to see that (6.14) is equivalent to (6.3) if $c(t) = v(t) - Kz(t)$. The design of tube MPC is to construct a tube online with the disturbance invariant set Ω as the intersection. If the initial state is in the tube, then the control law (6.14) will keep its evolution within the tube.

Assumption 6.2. *The state $x(t)$, open loop input sequence $V_B = [v^{*T}(t), \ldots, v^{*T}(t + s - 1)]^T$ and initial value of the tube center $z^*(t)$ are available at time t.*

The center trajectory of the tube is optimized over initial state $z(s|t)$ and the control sequence $V(t+s) = [v(s|t)^T, \ldots, v(s + N - 1|t)^T]^T$ at t by the optimization problem $\mathscr{P}_2(x(t), z^*(t), V_B)$ as:

$$V_2(x(t), z^*(t), V_B) = \min_{z(s|t),V(t+s)} \sum_{i=0}^{N-1} L(z(s + i|t), v(s + i|t)) + V_f(z(s + N|t)) \tag{6.16}$$

subject to

$$\bar{x}(s|t) = A^s z^*(t) + \sum_{j=0}^{s-1} A^{s-1-j} Bv^*(t + j) + \Phi^s e(t), \tag{6.17}$$

$$x(s|t) \in z(s|t) \oplus \Omega, \Leftrightarrow \bar{x}(s|t) \in z(s|t) \oplus \Omega \ominus (\oplus_{j=0}^{s-1} \Phi^j \mathbb{W}), \tag{6.18}$$

$$z(s + i + 1|t) = Az(s + i|t) + Bv(s + i|t),\ 0 \le i \le N - 1, \tag{6.19}$$

$$z(s + i|t) \in \mathbb{X} \ominus \Omega, v(s + i|t) \in \mathbb{U} \ominus K\Omega, 0 \le i \le N - 1, \tag{6.20}$$

$$z(s + N|t) \in Z_f. \tag{6.21}$$

The objective function and the terminal set Z_f are designed as required in [19] to guarantee the closed loop stability. The solution of $\mathscr{P}_2$ is denoted as $V^*(t + s) = [v^{*T}(s|t), \ldots, v^{*T}(s + N - 1|t)]^T$ and $z^*(s|t)$. The MPC above is implemented in the dual time scale scheme as:

In background, between time t and $t + s$:

- solve the optimization problem $\mathscr{P}_2(x(t), z^*(t), V_B)$ and update the buffer $V_B = [v^{*T}(t + s), \ldots, v^{*T}(t + 2s - 1)]^T = [v^{*T}(s|t), \ldots, v^{*T}(2s - 1|t)]^T$ and $z^*(t + s) = z^*(s|t)$.

Online, at sampling instants $t, \ldots, t + s - 1$:

- apply $u(t + i) = v^*(t + i) + K(x(t + i) - z^*(t + i))$ and calculate $z^*(t + i + 1) = Az^*(t + i) + Bv^*(t + i)$, $0 \le i \le s - 1$.

6.3.3 ISS Nominal MPC for Nonlinear Systems

Input-to-state stable (ISS) MPC [36] for nonlinear systems can be interpreted as an analogue of constraint tightening approach for linear systems. The nominal trajectory is optimized with constraints tightened for the purpose of robust constraints satisfaction. The closed loop stability is analyzed in the context of ISS stability. For the nonlinear system (6.1), the control input is parameterized as adds-on to a pre-compensator as

$$u(t) = \kappa(x(t)) + c(t) \;\; or \;\; u(t) = Kx(t) + c(t). \tag{6.22}$$

The compensated system is represented by

$$x(t+1) = f(x(t), u(t)) + w(t) = F(x(t), c(t)) + w(t). \tag{6.23}$$

The pre-compensator is mainly used to regulate the disturbance response and to reduce the Lipschitz constant of the nonlinear system.

Assumption 6.3. *The nominal model $F(x, c)$ is Lipschitz in x uniformly in c as*

$$|F(x_1, c) - F(x_2, c)|_n \leq L_F |x_1 - x_2|_n, \tag{6.24}$$

$\kappa(x)$ is Lipschitz in x as $|\kappa(x_1) - \kappa(x_2)|_n \leq L_\kappa |x_1 - x_2|_n$ and the disturbance w is bounded by $|w|_n \leq r$.

Then the uncertainty of state predictions at i steps ahead will be

$$|e(i|t)|_n = |x(i|t) - \bar{x}(i|t)|_n \leq \sum_{j=0}^{i-1} L_F^j r. \tag{6.25}$$

where $\bar{x}(i|t)$ is the nominal state prediction. Assumption 6.1 applies to the nonlinear case, too. The nominal state trajectory is optimized over the control sequence $C(t+s)$ at t by the optimization problem $\mathscr{P}_3(x(t), C_B)$ as:

$$V_3(x(t), C_B) = \min_{C(t+s)} \sum_{i=0}^{N-1} L(\bar{x}(s+i|t), c(s+i|t)) + V_f(\bar{x}(s+N|t)) \tag{6.26}$$

subject to

$$\bar{x}(0|t) = x(t), \bar{x}(i+1|t) = F(\bar{x}(i|t), c^*(t+i)), 0 \leq i \leq s-1, \tag{6.27}$$

$$\bar{x}(s+i+1|t) = F(\bar{x}(s+i|t), c(s+i|t)), 0 \leq i \leq N-1, \tag{6.28}$$

$$\bar{x}(s+i|t) \in \mathbb{X}_i, \ \ 0 \leq i \leq N-1, \tag{6.29}$$

$$\kappa(\bar{x}(s+i|t)) + c(s+i|t) \in \mathbb{U}_i, \ \ 0 \leq i \leq N-1, \tag{6.30}$$

$$\bar{x}(s+N|t) \in \Omega. \tag{6.31}$$

The constraints above are defined as below:

$$\mathbb{X}_i = \mathbb{X} \ominus B_n(\sum_{j=0}^{s+i-1} L_F^j r), \mathbb{U}_i = \mathbb{U} \ominus B_n(\sum_{j=0}^{s+i-1} L_\kappa L_F^j r), 0 \leq i \leq N-1. \tag{6.32}$$

where $B_n(\cdot)$ is the norm ball and Ω is the terminal set for the nominal prediction. To enforce the recursive feasibility and stability, additional assumptions are made as below.

Assumption 6.4.

I) *There is a Lyapunov function such that* $V_f(F(x(t),0)) - V_f(x(t)) \leq -L(x(t), \kappa(x(t))), \ \forall x(t) \in \Psi$. *And* $V_f(x)$ *is Lipschitz in* Ψ *with a Lipschitz constant* L_v *as* $|V_f(x_1) - V_f(x_2)| \leq L_v|x_1 - x_2|, \forall x_1, x_2 \in \Psi$*;*

II) Ψ *is the positive invariant set given by* $\Psi = \{V_f(x) \leq \alpha\}$*;*

III) $x(t+1) = F(x(t),0)$ *is exponential stable in* Ψ *as* $V_f(x(t+1)) \leq \lambda V_f(x(t)), \lambda < 1$*;*

IV) $u = \kappa(x) \in \mathbb{U}_{N-s+i}, \forall x \in \{V_f(x) \leq \lambda^i \alpha\}, 0 \leq i \leq s-1$*;*

V) *The set* $\Omega = \{V_f(x) \leq \alpha_v\}$ *is such that* $F^s(x,0) \in \Omega, \forall x \in \Psi$*;*

VI) $r \leq \frac{\alpha - \alpha_v}{L_v(L_F^N + \cdots + L_F^{N+s-1})}$.

The MPC above is implemented as same as that in subsection 6.3.1 except for applying $u(t+i) = \kappa(x(t+i)) + c^*(t+i)$.

6.3.4 Disturbance Invariant Set Approach for Nonlinear Systems

The tube MPC was extended to a class of nonlinear systems (6.1) satisfying the following assumption [25].

Assumption 6.5. *Suppose that for all $x \in \mathbb{X}$, $u \in \mathbb{U}$ and $w \in \mathbb{W}$ there exists a matrix $G(x, u, w) \in \Theta$ such that $f(x, u, w) = G(x, u, w)[x\ u\ w]^T$, where Θ is a polytopic linear differential inclusion (PLDI) of the nonlinear system:*

$$\Theta = Co\{[A_1\ B_1\ B_{w1}], \ldots, [A_L\ B_L\ B_{wL}]\}. \tag{6.33}$$

The input is parameterized as

$$u(t) = v(t) + Ke(t) = v(t) + K(x(t) - z(t)) \tag{6.34}$$

where $z(t)$ is the center of the tube, driven by the open loop input $v(t)$ as

$$z(t + 1) = f(z(t), v(t), 0), \tag{6.35}$$

and $e(t)$ is the difference between the state and the center of the tube. Assume there exist a disturbance invariant set Ω such that

$$e(t + 1) \in \Omega, \forall e(t) \in \Omega \text{ and } w(t) \in \mathbb{W}. \tag{6.36}$$

The set Ω can be characterized by the techniques in [25, 23]. We denote $\mathbb{W}_i$ as the bound of the disturbance response at $t+i$ contributed by $w(t)$. Assumption 6.2 applies to the nonlinear case, too. The center trajectory of the state tube is optimized over initial state $z(s|t)$ and the control sequence $V(t+s)$ at t by the optimization problem $\mathscr{P}_4(x(t), z^*(t), V_B)$ as:

$$V_4(x(t), z^*(t), V_B) = \min_{z(s|t), V(t+s)} \sum_{i=0}^{N-1} L(z(s + i|t), v(s + i|t)) + V_f(z(s + N|t)) \tag{6.37}$$

subject to

$$\bar{x}(0|t) = x(t),\ \bar{x}(i + 1|t) = f(\bar{x}(i|t), v^*(t + i) + K(\bar{x}(i|t) - z^*(t + i)), 0), 0 \le i \le s - 1, \tag{6.38}$$

$$z^*(t + i + 1) = f(z^*(t + i), v^*(t + i), 0), 0 \le i \le s - 2, \tag{6.39}$$

$$x(t+s) \in z(s|t) \oplus \Omega, \Leftrightarrow \bar{x}(s|t) \in z(s|t) \oplus \Omega \ominus (\oplus_{j=0}^{s-1} \mathbb{W}_j), \tag{6.40}$$

$$z(s+i+1|t) = f(z(s+i|t), v(s+i|t), 0), 0 \le i \le N-1, \tag{6.41}$$

$$z(s+i|t) \in \mathbb{X} \ominus \Omega, v(s+i|t) \in \mathbb{U} \ominus K\Omega, 0 \le i \le N-1, \tag{6.42}$$

$$z(s+N|t) \in Z_f. \tag{6.43}$$

The objective function and the terminal set Z_f are chosen as required in [25]. The implementation is the same as in subsection 6.3.2.

6.4 Stability Analysis

The algorithms presented in section 6.3 can be categorized as disturbance invariant set approach or constraint tightening approach. For conciseness, two MPC algorithms in subsection 6.3.2 and 6.3.3 are chosen as representees for stability analysis. The other two algorithms can be analyzed in a similar way.

6.4.1 Disturbance Invariant Set Approach

Theorem 6.1. *The optimization problem $\mathscr{P}_2$ is feasible at $t \ge 0$ if it is feasible at $t = 0$ initially.*

Proof. Assume there is an optimal solution $V^*(s) = [v^{*T}(s|0), \ldots, v^{*T}(s+N-1|0)]^T$ and $z^*(s|0)$ of the optimization problem $\mathscr{P}_2(x(0), z^*(0), V_B)$ at $t = 0$. $\tilde{z}(s|s) = A^s z^*(s|0) + \sum_{j=0}^{s-1} A^j B v^*(s+j|0)$ is chosen as a candidate of $z(s|s)$ for $\mathscr{P}_2$ at $t = s$. A candidate for $V(2s)$ is chosen as $\tilde{V}(2s) = [v^{*T}(2s|0), \ldots, v^{*T}(s+N-1|0), \kappa_f^T(z^*(s+N|0)), \ldots, \kappa_f^T(z^*(2s+N-1|0))]^T$, where $\kappa_f(\cdot)$ is the terminal controller used in the MPC design and $z^*(s+N+i+1|0) = Az^*(s+N+i|0) + B\kappa_f(z^*(s+N+i|0))$ for $0 \le i \le s-2$. It is easy to see that the constraints (6.17-6.21) at $t = s$ are satisfied by the candidates. Thus $\mathscr{P}_2$ is feasible at $t = s$, which implies $\mathscr{P}_2$ is recursive feasible. □

Theorem 6.2. *The system controlled by disturbance invariant set MPC approach is robustly stable, and the state will converge to a neighborhood of the origin.*

Proof. Since the state trajectory is around the center of the disturbance invariant tube, it is necessary to analyze the stability of the tube center $z(t)$. The Lyapunov function is, as usual, the objective function of the MPC optimization problem.

$$J(t) = \sum_{i=0}^{N-1}(z^{*T}(s+i|t)Qz^*(s+i|t) + v^{*T}(s+i|t)Rv^*(s+i|t)) + z^{*T}(s+N|t)Pz^*(s+N|t) \tag{6.44}$$

The candidate upper bounds the objective function $J(t+s)$.

$$\Delta J = J(t+s) - J(t) \leq -\sum_{i=0}^{s-1}(z^{*T}(s+i|t)Qz^*(s+i|t) + v^{*T}(s+i|t)Rv^*(s+i|t)) \leq 0. \tag{6.45}$$

Thus $\sum_{i=0}^{s-1} z^{*T}(s+i|t)Qz^*(s+i|t) + v^{*T}(s+i|t)Rv^*(s+i|t)$ will converge to 0 at times $t+s, t+2s, \ldots$, which means that the center of the tube is asymptotically stable. Since $x(t) - z^*(t) \in \Omega$, the real state $x(t)$ will converge to Ω as $t \to \infty$. □

6.4.2 ISS Approach

Theorem 6.3. *The optimization problem $\mathscr{P}_3$ is recursive feasible if $\mathscr{P}_3(x(0), [c^{*T}(0), \ldots, c^{*T}(s-1)]^T)$ is feasible at $t = 0$.*

Proof. Assume that there exists an optimal solution $C^*(s) = [c^{*T}(s|0), \ldots, c^{*T}(s+N-1|0)]^T$ for the optimization problem $\mathscr{P}_3(x(0), C_B)$. Then at $t = s$, the feasibility of $\mathscr{P}_3(x(s), [c^{*T}(s|0), c^{*T}(s+1|0), \ldots, c^{*T}(2s-1|0)]^T)$ is to be examined. One candidate based on the optimal solution of previous optimization problem is $\tilde{C}(2s) = [c^{*T}(2s|0), \ldots, c^{*T}(s+N-1|0), 0^T, \ldots, 0^T]^T$.

1) feasibility of the first portion of $\tilde{C}(2s)$.

Denote $\bar{x}^*(2s+i|0)$ as the nominal prediction using $x(0), [c^{*T}(0), \ldots, c^{*T}(s-1)]^T$ and $C^*(s)$ and $\bar{x}(s+i|s)$ as the nominal prediction using $x(s), [c^{*T}(s|0), \ldots, c^{*T}(2s-1|0)]^T$ and $\tilde{C}(2s)$. The two predictions are related as:

$$\bar{x}(s+i|s) \in \bar{x}^*(2s+i|0) \oplus B_n(\sum_{j=s+i}^{2s+i-1} L_F^j r), \quad 0 \leq i \leq N-s. \tag{6.46}$$

The feasibility of $\mathscr{P}_3(x(0), C_B)$ implies that $\kappa(\bar{x}^*(2s+i|0)) + c^*(2s+i|0) \in \mathbb{U}_{s+i} = \mathbb{U} \ominus B_n(\sum_{j=0}^{2s+i-1} L_\kappa L_F^j r)$. By combining the above two equations, $\kappa(\bar{x}(s+i|s)) + c^*(2s+i|0) \in \kappa(\bar{x}^*(2s+i|0)) \oplus B_n(\sum_{j=s+i}^{2s+i-1} L_\kappa L_F^j r) + c^*(2s+i|0) \in \mathbb{U} \ominus B_n(\sum_{j=0}^{2s+i-1} L_\kappa L_F^j r) \oplus B_n(\sum_{j=s+i}^{2s+i-1} L_\kappa L_F^j r) = \mathbb{U} \ominus B_n(\sum_{j=0}^{s+i-1} L_\kappa L_F^j r) = \mathbb{U}_i$, which means that (6.30) is feasible for $i = 0, \ldots, N-s$.

2) feasibility of the second portion of $\tilde{C}(2s)$.

Due to Assumption 6.4.VI , $\bar{x}(N|s) \in \Psi$. Combining of Assumption 6.4.III and 6.4.IV, the remaining part of $\tilde{C}(2s)$ is feasible. Thus (6.30) is satisfied for $i = 0, \ldots, N-1$.

3) feasibility of the terminal set constraints (6.31).

Due to Assumption 6.4.VI, $\bar{x}(N|s) \in \Psi$. Assumption 6.4.V implies that $\bar{x}(s+N|s) \in \Omega$.

4) feasibility of state constraints (6.29).

It is easy to see the state constraints is satisfied in a similar way.

To summarize, $\mathscr{P}_3(x(s), [c^{*T}(s|0), c^{*T}(s+1|0), \ldots, c^{*T}(2s-1|0)]^T)$ is feasible. Repeating the reasoning forward with time, $\mathscr{P}_3$ is recursive feasible. □

A preliminary study on the stability of system dynamic at the intermittent time instants $t + s \times i, i \geq 0$ is presented as below.

Theorem 6.4. *The state dynamic is ISS stable at the intermittent time instants $t + s \times i, i \geq 0$.*

Proof. The Lyapunov function is chosen as the objective function of problem $\mathscr{P}_3$ as $J(t) = V_3(x(t), C_B)$.

$$\begin{aligned}
\Delta J &= J(t+s) - J(t) = V_3(x(t+s), \cdots) - V_3(x(t), \cdots) \qquad (6.47)\\
&= \sum_{i=0}^{N-1} L(\bar{x}^*(s+i|t+s), c^*(s+i|t+s)) + V_f(\bar{x}^*(s+N|t+s))\\
&- \sum_{i=0}^{N-1} L(\bar{x}^*(s+i|t), c^*(s+i|t)) - V_f(\bar{x}^*(s+N|t))\\
&\leq \sum_{i=0}^{N-s-1} L(\bar{x}(s+i|t+s), c^*(2s+i|t)) + \sum_{i=N-s}^{N-1} L(\bar{x}(s+i|t+s), 0) + V_f(\bar{x}(s+N|t+s))\\
&- \sum_{i=0}^{N-1} L(\bar{x}^*(s+i|t), c^*(s+i|t)) - V_f(\bar{x}^*(s+N|t))\\
&\leq \sum_{i=0}^{N-s-1} L(\bar{x}(s+i|t+s), c^*(2s+i|t)) + V_f(\bar{x}(N|t+s))\\
&- \sum_{i=0}^{N-1} L(\bar{x}^*(s+i|t), c^*(s+i|t)) - V_f(\bar{x}^*(s+N|t))\\
&\leq - \sum_{i=0}^{s-1} L(\bar{x}^*(s+i|t), c^*(s+i|t)) + \sum_{i=0}^{N-s-1} L_L \sum_{j=s+i}^{2s+i-1} L_F^j r + L_v \sum_{j=N}^{s+N-1} L_F^j r.
\end{aligned}$$

The first inequality is due to that $\bar{x}(s+i|t+s)$ is a feasible trajectory; the second inequality is due to Assumption 6.4.I; the last inequality is due to (6.46). Thus $J(t)$ is a ISS Lyapunov function for the predicted state $\bar{x}^*(s|t)$ at $t, t+s, t+2s, \ldots$. Since $x(t+s) \in \bar{x}^*(s|t) \oplus B_n(\sum_{i=0}^{s-1} L_F^i r)$, the real state is bounded. □

6.5 Neighboring Extremal Control

As stated in section 6.2, although there is feedback signal from the pre-compensator at fast sampling rate, from the perspective of outer loop, the compensated loop is controlled by the open loop input sequences from the MPC without any feedback correction. It is noted that the proposed multi-step implementation MPC is a type of intermittent feedback implementation, which sits between standard MPC and optimal control. The standard MPC solves the optimization problem at every sampling time, thus feedback is introduced; optimal control solves the optimization problem and applies the whole open loop input sequence to the plant. Due to the model uncertainty/disturbances, the real state evolution will never be the same as predicted. Thus the optimal input from the proposed MPC or optimal control will not be optimal for the real state. There is a desire that the uncertainty can be handled by a new calculated input. Actually that is the origin of standard MPC, which re-solve the optimization problem with the new state and applies the updated input to the plant. Due to the computational difficulty, standard MPC is discarded here. Since the proposed multi-step implementation MPC and optimal control share the same drawback, it is interesting to check how optimal control handles the uncertainty problem. One possible solution is the so called Neighboring Extremal Control (NEC)[8]. NEC can be considered as an approximate, but computational efficient, approach to resolve the optimization problem with the new state information. Thus it is suitable to incorporate NEC into the dual time scale control scheme. It provides an additional feedback. The overall control structure is depicted in Fig. 6.2. NEC makes use of the linearization and time-varying LQR technique to obtain the approximate solution of optimization problem for state x around the predicted nominal

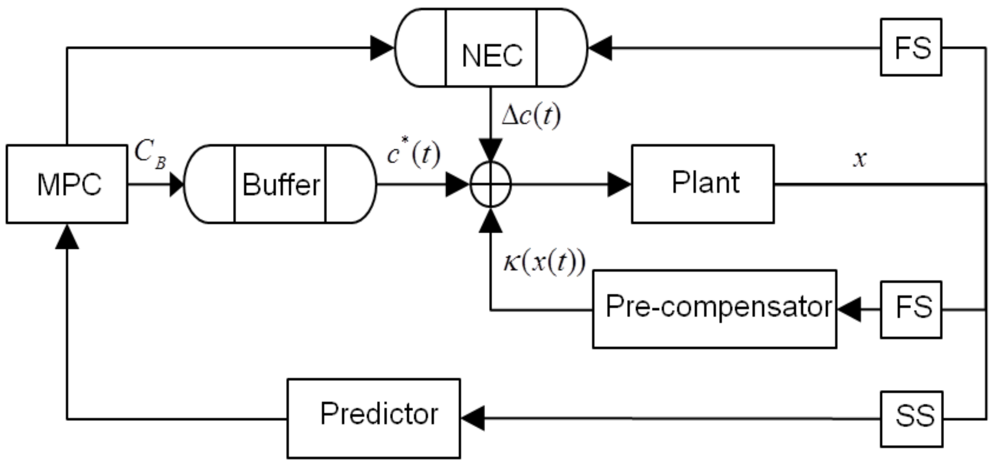

Figure 6.2: Neighboring Extremal Control Structure

trajectory. The controller is given in the form of:

$$\Delta c = -F_k \Delta x, \tag{6.48}$$

$$F_k = (R + F_c^T(k) P_k F_c(k))^{-1} F_c^T(k) P_k F_x(k), \tag{6.49}$$

$$P_{k-1} = Q + F_x^T(k)(P_k - P_k F_c(k)(R + F_c^T(k) P_k F_c(k))^{-1} F_c^T(k) P_k) F_x(k), \tag{6.50}$$

$$P_N = P. \tag{6.51}$$

where $F_x(k) = \frac{\partial F(x,c)}{\partial x}|_{x=x(t+k),c=c(t+k)}$, $F_c(t) = \frac{\partial F(x,c)}{\partial c}|_{x=x(t+k),c=c(t+k)}$ is the linear model around the nominal trajectory. The NEC functions as a regulator that pushes the state towards the predicted nominal trajectory. However, the hard constraints on state and input are not included in the formulation and rigorous stability study is not available.

6.6 Simulation Studies

6.6.1 Double Integrator

The double integrator model in [19] is used for simulation of a linear plant, which is described by

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w. \tag{6.52}$$

The state constraints is $x \in \mathbb{X} = \{x | [0 \;\; 1] x \leq 2\}$; the control constraint is $u \in \mathbb{U} = \{u||u| \leq 1\}$ and the disturbance is bounded as $w \in \mathbb{W} = \{w||w|_\infty \leq 0.1\}$. The cost function is defined with $Q = I, R = 0.01$. The initial state is $x(0) = [-5; \;\; -2]$. It is assumed that the optimization problem can only be done over 2 sampling intervals. Three scenarios are adopted for simulation, including the ideal MPC, standard implementation of RMPC with computation delay of 2 sampling intervals and the proposed approach in subsection 6.3.1. The state and input trajectories are presented in Fig. 6.3 and 6.4, respectively. As can

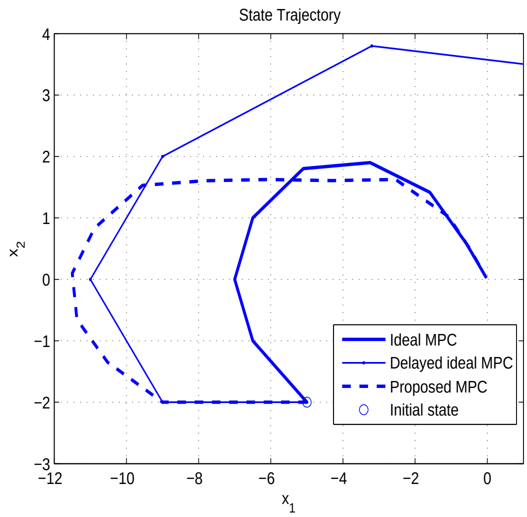

Figure 6.3: State Trajectories of Double Integrator Model

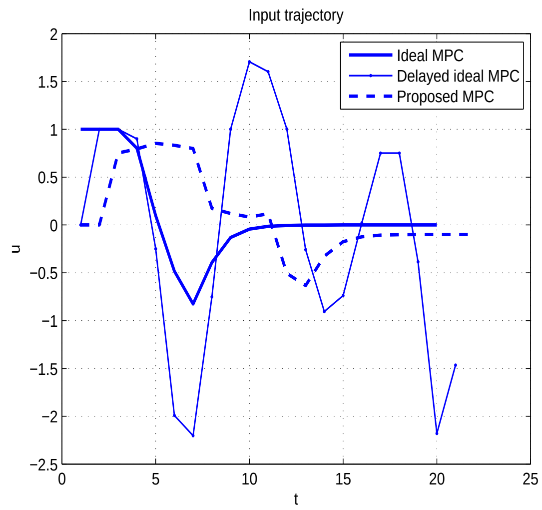

Figure 6.4: Input Trajectories of Double Integrator Model

be seen, the ideal MPC renders the best performance with the fastest convergence rate. The proposed approach can still stabilize the system, although with degraded performance, mainly due to the zero input during the first two steps. It performs significantly better as compared to the second case, which is a realistic implementation of ideal MPC and destabilizes the system.

6.6.2 Nonlinear Model

The same nonlinear model as adopted in [25] is used here. The state space model is

$$\begin{cases} \dot{x}_1 = 0.5x_1 + 0.15x_1^2 + x_2 + 0.6u \\ \dot{x}_2 = x_1 - 0.2x_2^2 + 0.6u + w \end{cases} \tag{6.53}$$

The input is constrained as $u \in \mathbb{U} = \{u| - 2 \leq u \leq 2\}$; disturbance is assumed to be bounded by $w \in \mathbb{W} = \{w||w|_\infty \leq 0.1\}$. $Q = 0.5I, R = 1$ are used as state and input weights. The disturbance invariant set is

$\Omega = \{x | x^T P x < 1\}$ with $P = 10^3 \begin{bmatrix} 2.3907 & 1.7277 \\ 1.7277 & 1.2950 \end{bmatrix}$. The pre-compensator is given by the feedback matrix $K = [-24.9562 \quad -18.7641]$. Prediction horizon is $T_p = 7.5s$ and the sampling rate $\delta = 0.075s$. Initial state is $x(0) = [0.95; -0.95]$. Discrete model for optimization is obtained via Euler approximation. Three scenarios are simulated, including the ideal NMPC, ideal MPC with a computation delay of $0.075 \times 5s$, and the proposed MPC in subsection 6.3.4. The state and input trajectories using these controllers are shown in Fig. 6.5 and 6.6, respectively. For the proposed MPC, it is assumed that the backstage computation takes $0.075 \times 10s$. Similar results to linear case are observed. The performance improvement is obvious.

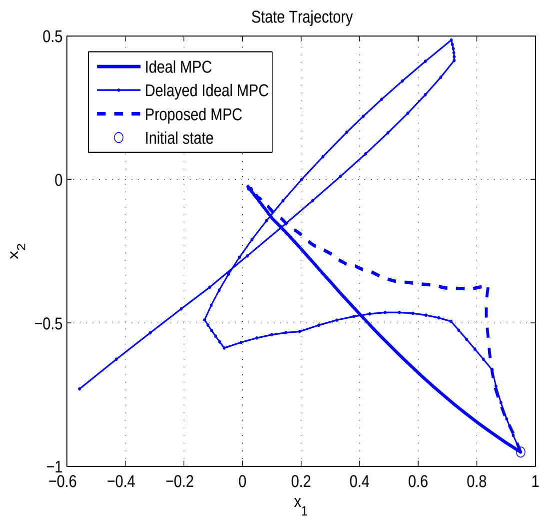

Figure 6.5: State Trajectories of Nonlinear Model

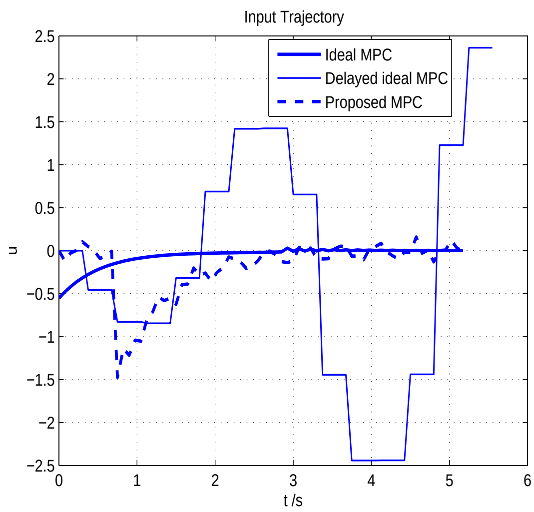

Figure 6.6: Input Trajectories of Nonlinear Model

6.6.3 Nonlinear Model with NEC

Here, the performance of NEC is examined via a simple nonlinear model simulation[36]:

$$\begin{cases} x_1^+ = 0.55x_1 + 0.12x_2 + (0.01 - 0.6x_1 + x_2)u \\ x_2^+ = 0.67x_2 + (0.15 + x_1 - 0.8x_2)u + w \end{cases} \tag{6.54}$$

where w is the disturbance variable and the control action is constrained to $|u| < 0.1$. The cost function parameters are chosen as $Q = 2I, R = 1, P = \begin{bmatrix} 4.01 & 0.39 \\ 0.39 & 4.98 \end{bmatrix}$ and the computation delay equals to 2. The initial state is chosen as $x = [0.5;\ 1.5;]$. The ISS MPC controller and ISS MPC with NEC are examined, and the trajectory of state x_1 is shown in Fig. 6.7. As $t = 30$, a constant disturbance $w = 0.08$ is introduced

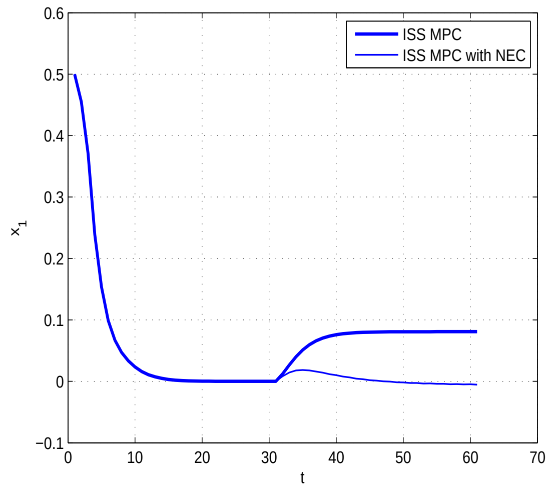

Figure 6.7: Comparison between ISS MPC and ISS MPC with NEC

in the loop. As can be seen, the ISS MPC with NEC can suppress the disturbance better than ISS MPC only.

6.7 Conclusion

A dual time scale control scheme is proposed for uncertain nonlinear systems. The computation delay is compensated in the MPC design, which is tailored to fit in the dual time scale control scheme. The recursive feasibility and stability are guaranteed by design. The neighboring extremal control, developed in the context of optimal control, may be put on the top of the proposed scheme and provide additional degree of freedom to react to uncertainty. The buffer, used in the implementation of MPC, reminds the same mechanism in networked control system. Thus the proposed approach may provide a way to analyze and design robust networked control system.

Chapter 7

Conclusions

7.1 Summary of Contributions

This thesis discussed Model Predictive Control design for three specific dynamic models–monovariable linear systems with dead time, linear periodic systems and linear parameter varying systems, one specific performance index–economic optimization, and one specific implementation scheme–dual time scale control with computation delay compensation. Emphasis is placed on the synthesis approach for MPC design with stability or robustness guarantee. In particular, this thesis contributes towards the five topics stated in section 1.3.

1. **Setpoint Weighting for Dead Time Compensation**

A simple dead time compensation scheme via setpoint weighting for monovariable linear systems with dead time, which are often encountered in process industry, was proposed. The tracking performance is significantly enhanced. Another advantage is that the basic feedback loop, for example PID control, is retained in the proposed control structure. Thus its implementation is rather simple and amenable in industrial controllers.

2. **Robust Minimum Time Control for Linear Periodic Systems**

A robust minimum time control for linear periodic systems with bounded disturbances is proposed in Chapter 3. It reduces the conservativeness of previous approaches. The optimization only involves one step prediction, therefore it is also computational efficient. Moreover, the solution can be obtained via multi-parametric approach which can further reduce the computational burden.

3. **Output MPC for LPV Systems**

The third contribution is a stabilizing output MPC for LPV systems. The model used for design includes disturbances and sensor noise, therefore is more realistic. Stability is guaranteed by exploiting the tube MPC and quasi-min-max approach. The computation burden is similar to state feedback case.

4. **Stabilizing Economic MPC**

The fourth contribution is two stabilizing Economic algorithms, terminal cost function approach with two methods for terminal cost function design and stability constraint enforcement approach. The proposed approaches guarantee the closed loop stability when MPC adopts economic performance function for online optimization. The advantage over previous results is that the restriction on system model and economic performance function is relaxed. Thus they are applicable to a wider class of problems.

5. **Computation delay compensation in Robust MPC**

The last contribution of the thesis is the explicit computation delay compensation in the design of Robust MPC, which allows fast sampling rate setting in spite of significant online computation burden. Four MPC algorithms are proposed for linear/nonlinear systems, leading to closed loop stable systems. The proposed algorithms may enable the real implementation of MPC controller for certain fast processes.

7.2 Suggestions of Future Work

Several directions related to the work presented in this thesis are available for future research and applications.

7.2.1 Enhancement for Output Feedback MPC for LPV Systems

The algorithm proposed in Chapter 4 utilizes the quasi-min-max approach, which considers the future scheduling parameters as bounded uncertain variables. In fact, the characteristic of LPV systems is that the scheduling parameters is unknown in advance but known at due time. Thus the predicted control law can be parameterized as gain scheduling controller, which can greatly reduce the conservativeness and enhance the control performance. This idea is extensively studied in [27] for state feedback case. It is therefore expected that improved control can be achieved via applying the idea in output feedback case.

7.2.2 Comparison on Output Feedback MPC Variants for Offset Free Tracking of Piecewise Constant Setpoint

Piecewise constant setpoint tracking is practically important since it is a basic control requirement for various applications. Commonly this can be achieved with MPC by disturbance modeling, error integration augmentation, delta input formulation, or reference governor approach. It is desirable to compare these approaches and investigate their advantages and shortcomings in terms of generality and computation efficiency. A serious comparison may help to identify the most suitable approach for a particular problem. In addition, the tracking objective can be either inferential variable or output variable tracking or both. While the unbiased output estimation is obtained normally by a Luenberger observer; inferential variable estimation is biased. Therefore it is possibly advantageous to use a high gain robust observer, unknown input observer or sliding model observer for inferential variable tracking. The results for constant setpoint tracking may be extended to other servo dynamics, such as ramp, sinusoid or periodic reference signal. This work is currently underway.

7.2.3 Maximal Positive Invariant Set With Marginal Unstable Dynamic

Maximal positive invariant set is important for stabilizing MPC design as it serves as the terminal constraint set in the optimization formulation. Its characterization for stable dynamic and marginal stable dynamic is thoroughly investigated in [97]. It was shown that the finite determinability is general not guaranteed, except for the case that the marginal stable dynamic is integrator. However, other marginal stable dynamics, such as sinusoidal oscillating and periodic repeat, arise from many practical applications. It is therefore interesting to investigate the finite determinability of maximal positive invariant set for such dynamic.

7.2.4 Robust Economic Optimization in MPC

Economic MPC based on a deterministic model is investigated in Chapter 5. Just like for regulation problem, robustness is important. It is highly desirable to investigate Economic MPC for uncertain systems. On one hand, robust stability is a key requirement no matter what is the specific optimization objective; on another hand, robust economic optimization is useful to ensure the economic optimality. One possible direction to this problem may be the min-max approach for nonlinear systems.

7.2.5 Robust MPC for Networked Control Systems

With the prevalence of communication network technology that allows remote data exchanges, distributed control systems connected through networks are increasing rapidly in various applications and have received extensive research attention[99]. Networked control systems provide many benefits, such as global operation, modularity, wiring savings, and self-configuration. However, the network, inserted in the feedback loop, also introduces serious difficulties on control, including random transmission delay and packet dropout, which may severely degrade the overall control performance if not properly handled. In the literature, buffers are used to handle the above difficulties. The scheme is similar to the computation delay compensation MPC as in Chapter 6. It is therefore interesting to investigate whether the MPC in Chapter 6 can be tailored to fit in networked control systems. In particular, the problem that how to modify the MPC algorithms to accommodate to the random delay, instead of a fixed delay in Chapter 6, and to utilize the freshest feedback signal as much as possible, is worth studying.

Appendix A

Comments on Stochastic MPC

In a recent paper [94], a MPC algorithm for systems with stochastic multiplicative uncertainty was proposed. It offers potentially significant computational advantages and it is less conservative. A modification is presented in this Appendix to strengthen the original result[1]. The notations and the reference labels, excluding the ones in the form of (A.~), follow the convention used in [94], to which the readers may refer.

A.1 Problem Statement

The model considered in [94] is given by

$$x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C x_k \tag{A.1}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$. A_k and B_k are random matrices with distributions defined by

$$A_k = \bar{A} + \sum_{j=1}^{m} \tilde{A}_j q_j(k), \quad B_k = \bar{B} + \sum_{j=1}^{m} \tilde{B}_j q_j(k), \quad [q_1(k) \ \ldots \ q_m(k)]^T \sim \mathcal{N}(0, I), \tag{A.2}$$

where $\mathcal{N}(0, I)$ denotes the normal distribution with zero mean and identity covariance matrix, and where $q_j(k)$ is independent of $q_h(i)$ for all $k \neq i$ and all j, h. It is required that the input remains within the prescribed range with probability no less than p:

$$Pr\{|u_k| \leq \bar{u}\} \geq p. \tag{A.3}$$

[1]Part of the materials in this Appendix was published as Correspondence in *Automatica*. The generous assistance of Dr. Mark Cannon at University of Oxford is sincerely acknowledged.

The predicted inputs are parameterized as

$$u_{k+i|k} = Kx_{k+i|k} + c_i, 0 \le i \le N-1; \;\; u_{k+i|k} = Kx_{k+i|k}, i \ge N. \tag{A.4}$$

The predicted trajectories are generated by the augmented autonomous model

$$z_{k+1} = \Psi_k z_k, \Psi_k = \bar{\Psi} + \sum_{j=1}^{m} \tilde{\Psi}_j q_j(k), \bar{\Psi} = \begin{bmatrix} \bar{\Phi} & \bar{B}\Gamma_u^T \\ 0 & M \end{bmatrix}, \tilde{\Psi}_j = \begin{bmatrix} \tilde{\Phi}_j & \tilde{B}_j\Gamma_u^T \\ 0 & 0 \end{bmatrix}, \tag{A.5}$$

where $z_k = [x_k^T \; f_k^T]^T$ is the augmented state, $f_k = [c_0^T \; c_1^T \; \dots \; c_{N-1}^T]^T$ is the decision variable, $\bar{\Phi} = \bar{A} + \bar{B}K$, $\tilde{\Phi}_j = \tilde{A} + \tilde{B}_j K$, $\Gamma_u^T = [I \; 0]^T$ and M is the time shift matrix. The MPC optimizes the expected performance function $z_k^T \tilde{p} z_k$, whose optimal value is supposed to be the stochastic Lyapunov function of the closed loop system. The input constraints (A.3) is enforced by $z^T \hat{P} z \le 1$, whose projection on x space, $\hat{E}_x = \{x | x^T \hat{P}_x x \le 1\}$, is an ellipsoidal set of Invariant with Probability p (IWPp). The MPC is summarized in Algorithm 4.3 that at $t = k+1$

1. if $x_{k+1} \in \hat{E}_x$, compute

$$f_{k+1}^* = \arg\min_{f_{k+1}} z_{k+1}^T \tilde{P} z_{k+1}, \quad s.t. \quad z_{k+1}^T \hat{P} z_{k+1} \le 1. \tag{27}$$

Then it is stated that 'the objective in (27) . . . ensure that, for any uncertainty realization at time k, the cost at time $k+1$ satisfies

$$J_{k+1}^* \le z_k^{*T} \Psi_k^T \tilde{P} \Psi_k z_k^* \tag{30}$$

'. In fact, if $\Psi_k z_k^*$ is within the set $\hat{E} = \{z : z^T \hat{P} z \le 1\}$, (30) is true at $t = k+1$. However the condition that $x_{k+1} = \Gamma_x^T \Psi_k z_k^*$ is within $\hat{E}_x$ is not sufficient to ensure that $\Psi_k z_k^*$ will be within $\hat{E}$ as well. Intuitively, it seems that tighter constraints, such as the one in (27), can produce f_{k+1}^* such that $J_{k+1}^* > z_k^{*T} \Psi_k^T \tilde{P} \Psi_k z_k^*$. Thus, the IWPp condition imposed on the projection onto x subspace of ellipsoids in the z space does not guarantee that the extension to the next time instant of the currently computed optimal trajectory is feasible and further the supermartingale property of the sequences $\{J_0^*, J_1^*, \dots\}$.

A.2 Modified Algorithm

The main issue lies in that IWPp over x subspace is not sufficient. Thus, hereby the IWPp is extended to z space.

Modified Definition 3.1. *For the augmented autonomous prediction model (A.5), a set $\hat{E}_z$ is said to be IWPp if every state $z \in \hat{E}_z$ is steered(at the next time instant) to a state which remains in the set $\hat{E}_z$ with probability p.*

Then, following similar procedures in [94], for given $r > 0$, let

$$H = r[\tilde{\Psi}_1 z_k \ldots \tilde{\Psi}_m z_k], HH^T = U\Lambda^2 U^T. \tag{A.6}$$

Modified Lemma 3.2. *For given z_k, z_{k+1} evolves with probability p into the confidence ellipsoid:*

$$\Pi_z(z_k) = \{z : (z - \bar{\Psi} z_k)^T U\Lambda^{-2} U^T (z - \bar{\Psi} z_k) \leq 1\} \tag{A.7}$$

where r is the confidence radius for a χ^2 distribution with m degrees of freedom defined as $Pr\{\chi^2(m) \leq r\} = p$.

Modified Theorem 3.3. *$\hat{E}_z$ is IWPp whenever $z_k \in \hat{E}_z$ if there exists scalar $\lambda \in (0, 1)$ satisfying*

$$\begin{bmatrix} \hat{P}^{-1} & \bar{\Psi}\hat{P}^{-1} & [\ \tilde{\Psi}_1 \hat{P}^{-1} \ \ldots \ \tilde{\Psi}_m \hat{P}^{-1}\ ] \\ \hat{P}^{-1}\bar{\Psi}^T & (1-\lambda)\hat{P}^{-1} & 0 \\ \begin{bmatrix} \hat{P}^{-1}\tilde{\Psi}_1^T \\ \vdots \\ \hat{P}^{-1}\tilde{\Psi}_m^T \end{bmatrix} & 0 & \begin{bmatrix} \frac{\lambda}{r^2}\hat{P}^{-1} & & \\ & \ddots & \\ & & \frac{\lambda}{r^2}\hat{P}^{-1} \end{bmatrix} \end{bmatrix} \geq 0. \tag{A.8}$$

The proof follows the same procedures in [94] and it is thus omitted here. Finally, the MPC algorithm is given as follows.

Modified Algorithm 4.3. *Given $x_0 \in \hat{E}_x$, at times $k = 0, 1, \ldots$:*

1. *If $[x_k^T \ (Mf_{k-1}^*)^T]^T \in \hat{E}_z$ or $k = 0$, then compute*

$$f_k^* = \arg\min_{f_k} z_k^T \tilde{P} z_k, \quad s.t. \quad z_k^T \hat{P} z_k \leq 1 \tag{A.9}$$

2. *Otherwise, compute*

$$f_k^* = \arg\min_{f_k} z_k^T \bar{\Psi}^T \Gamma_x \hat{P}_x \Gamma_x^T \bar{\Psi} z_k, \tag{A.10}$$

$$s.t. \quad z_k^T \tilde{P} z_k \leq \begin{bmatrix} x_k \\ Mf_{k-1}^* \end{bmatrix}^T \tilde{P} \begin{bmatrix} x_k \\ Mf_{k-1}^* \end{bmatrix} \tag{A.11}$$

3. Implement the control move $u_k = Kx_k + \Gamma_u^T f_k^*$.

The modified algorithm renders the nonnegative sequence $\{J_0^*, J_1^*, \ldots\}$ a supermartingale, and thus validates Theorem 4.5 that the closed loop system under the modified MPC control law is mean square stable, and $x_k \to 0$ as $k \to \infty$ with probability 1 (w.p.1). Furthermore, if $z_k \in \hat{E}_z$, then $u_k \in [-\bar{u}, \bar{u}]$, and u_{k+1} satisfies the probabilistic constraints.

A.3 Extension for A Larger Feasible Region

The pre-defined linear control gain K is an important design parameter in the stochastic MPC above. The choice of K has to cater for conflicting objectives, such as good dynamic regulation and large feasible region, thus compromise is often encountered. To reduce the difficulty, a more general input parameterization can be adopted as

$$u_{k+i|k} = Kx_{k+i|k} + H\xi_{k+i|k}, \xi_{k+i+1|k} = G\xi_{k+i|k} = (\bar{G} + \sum_{j=1}^{m} \tilde{G}_j q_j(k))\xi_{k+i|k}. \tag{A.12}$$

The closed loop system is

$$z_{k+1} = \begin{bmatrix} x_{k+1} \\ \xi_{k+1} \end{bmatrix} = \Psi_k z_k = (\bar{\Psi} + \sum_{j=1}^{m} \tilde{\Psi}_j q_j(k)) z_k, \tag{A.13}$$

$$\bar{\Psi} = \begin{bmatrix} \bar{\Phi} & \bar{B}H \\ 0 & \bar{G} \end{bmatrix} = \begin{bmatrix} \bar{A} + \bar{B}K & \bar{B}H \\ 0 & \bar{G} \end{bmatrix}, \tilde{\Psi}_j = \begin{bmatrix} \tilde{\Phi}_j & \tilde{B}_j K \\ 0 & \tilde{G}_j \end{bmatrix} = \begin{bmatrix} \tilde{A}_j + \tilde{B}_j K & \tilde{B}_j H \\ 0 & \tilde{G}_j \end{bmatrix}, 1 \leq j \leq m. \tag{A.14}$$

An ellipsoidal set $\hat{E}_z = \{z | z^T \hat{P} z \leq 1\}$ is IWPp if it satisfies the condition in Modified Definition 3.1 with the dynamic model replaced by (A.13-A.14).

Remark A.1. *Modified Theorem 3.3 is still valid for the general input parameterization formulation.*

For a fixed controller gain K, different $H, \bar{G}, \tilde{G}_1, \ldots, \tilde{G}_m$ will result in $\hat{E}_z$ of different volumes. Thus it is desirable to optimize the volume of $\hat{E}_z$ over $H, \bar{G}, \tilde{G}_1, \ldots, \tilde{G}_m$ [95].

Theorem A.1. *Let $n_\xi \geq n_x$, there exist $\lambda, \hat{P}, H, \bar{G}$ and $\tilde{G}_j$ $(j = 1, \ldots, m)$ satisfying LMIs (A.8), if and only if the following matrix in $\lambda, W, X, \mathcal{H}, \bar{\Theta}$ and $\tilde{\Theta}_j$ $(j = 1, \ldots, m)$ is positive definite:*

$$\begin{bmatrix} \begin{bmatrix} W & I \\ I & X \end{bmatrix} & \begin{bmatrix} \bar{\Phi}W + \bar{B}\mathcal{H} & \bar{\Phi} \\ \bar{\Theta} & X\bar{\Phi} \end{bmatrix} & \begin{bmatrix} \tilde{\Phi}_1 W + \tilde{B}_1\mathcal{H} & \tilde{\Phi}_1 \\ \tilde{\Theta}_1 & X\tilde{\Phi}_1 \end{bmatrix} \cdots \begin{bmatrix} \tilde{\Phi}_m W + \tilde{B}_m\mathcal{H} & \tilde{\Phi}_m \\ \tilde{\Theta}_m & X\tilde{\Phi}_m \end{bmatrix} \\ * & (1-\lambda)\begin{bmatrix} W & I \\ I & X \end{bmatrix} & 0 \\ * & * & \begin{bmatrix} \frac{\lambda}{r^2}\begin{bmatrix} W & I \\ I & X \end{bmatrix} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{\lambda}{r^2}\begin{bmatrix} W & I \\ I & X \end{bmatrix} \end{bmatrix} \end{bmatrix} \tag{A.15}$$

Proof. 1). Necessity

Partition $\hat{P}$ and $\hat{P}^{-1}$ into components as $\hat{P} = \begin{bmatrix} X & V \\ V^T & \sim \end{bmatrix}$ and $\hat{P}^{-1} = \begin{bmatrix} W & U \\ U^T & \sim \end{bmatrix}$ with $W, X \in \mathbb{R}^{n_x \times n_x}$ and

$$WX + UV^T = I. \tag{A.16}$$

Define $\Pi_1 = \begin{bmatrix} I & X \\ 0 & V^T \end{bmatrix}, \Pi_2 = \begin{bmatrix} W & I \\ U^T & 0 \end{bmatrix}$, thus $\hat{P}^{-1}\Pi_1 = \Pi_2$. Pre- and post-multiply (A.8) by $diag(\Pi_1^T, \ldots, \Pi_1^T)$ and $diag(\Pi_1, \ldots, \Pi_1)$, we can get the conditions

$$\begin{bmatrix} \Pi_1^T\Pi_2 & \Pi_1^T\bar{\Psi}\Pi_2 & \begin{bmatrix} \Pi_1^T\tilde{\Psi}_1\Pi_2 & \ldots & \Pi_1^T\tilde{\Psi}_m\Pi_2 \end{bmatrix} \\ \Pi_2^T\bar{\Psi}^T\Pi_1 & (1-\lambda)\Pi_1^T\Pi_2 & 0 \\ \begin{bmatrix} \Pi_2^T\tilde{\Psi}_1^T\Pi_1 \\ \vdots \\ \Pi_2^T\tilde{\Psi}_m^T\Pi_1 \end{bmatrix} & 0 & \begin{bmatrix} \lambda/r^2\Pi_1^T\Pi_2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda/r^2\Pi_1^T\Pi_2 \end{bmatrix} \end{bmatrix} \geq 0, \tag{A.17}$$

since

$$\Pi_1^T\Pi_2 = \begin{bmatrix} I & 0 \\ X & V \end{bmatrix}\begin{bmatrix} W & I \\ U^T & 0 \end{bmatrix} = \begin{bmatrix} W & I \\ I & X \end{bmatrix} \tag{A.18}$$

$$\Pi_1^T \bar{\Psi} \Pi_2 = \begin{bmatrix} I & 0 \\ X & V \end{bmatrix} \begin{bmatrix} \bar{\Phi} & \bar{B}H \\ 0 & \bar{G} \end{bmatrix} \begin{bmatrix} W & I \\ U^T & 0 \end{bmatrix} = \begin{bmatrix} \bar{\Phi}W + \bar{B}HU^T & \bar{\Phi} \\ X\bar{\Phi}W + X\bar{B}HU^T + VGU^T & X\bar{\Phi} \end{bmatrix} \tag{A.19}$$

$$\Pi_1^T \tilde{\Psi}_j \Pi_2 = \begin{bmatrix} I & 0 \\ X & V \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_j & \tilde{B}_j H \\ 0 & \tilde{G}_j \end{bmatrix} \begin{bmatrix} W & I \\ U^T & 0 \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}_j W + \tilde{B}_j HU^T & \tilde{\Phi}_j \\ X\tilde{\Phi}_j W + X\tilde{B}_j HU^T + V\tilde{G}_j U^T & X\tilde{\Phi}_j \end{bmatrix}. \tag{A.20}$$

Define $\mathcal{H} = HU^T, \bar{\Theta} = X\bar{\Phi}W + X\bar{B}HU^T + VGU^T, \tilde{\Theta}_j = X\tilde{\Phi}_j W + X\tilde{B}_j HU^T + V\tilde{G}_j U^T \, j = 1, \ldots, m$, (A.8) is converted to (A.15).

2). Sufficiency

For given $W, X, \mathcal{H}, \bar{\Theta}, \tilde{\Theta}_j, 1 \le j \le m$, it is possible to back calculate $\hat{P}, H, \bar{G}, \tilde{G}_j, 1 \le j \le m$. The condition $n_\xi \ge n_x$ guarantees that there exists solutions. □

Input Constraints (A.3) is satisfied if

$$\begin{bmatrix} (e_i^T \bar{u})^2 & e_i^T \hat{K} \hat{P}^{-1} \\ \hat{P}^{-1} \hat{K} e_i & \hat{P}^{-1} \end{bmatrix} \ge 0, i = 1, \ldots, n_u, \hat{K} = [K, H]. \tag{A.21}$$

where e_i^T is the ith column of the identity matrix in $\mathbb{R}^{n_u * n_u}$.

Pre- and post-multiply (A.21) by $diag(I, \Pi_1^T)$ and $diag(I, \Pi_1)$, the constraints satisfaction (A.21) is expressed by

$$\begin{bmatrix} (e_i^T \bar{u})^2 & e_i^T \begin{bmatrix} KW + \mathcal{H} & K \end{bmatrix} \\ * & \begin{matrix} W & I \\ I & X \end{matrix} \end{bmatrix} \ge 0. \tag{A.22}$$

The offline selection for the parameters $H, \bar{G}, \tilde{G}_1, \ldots, \tilde{G}_m$ as

$$\max_{\lambda, W, X, \mathcal{H}, \bar{\Theta}, \tilde{\Theta}_1, \ldots, \tilde{\Theta}_m} \quad logdet(W) \tag{A.23}$$

subject to (A.15) and (A.22) such that the volume of the projection on x space of $\hat{E}_z$ is maximized.

To conclude, by the virtue of the general control input parameterization, K may be chosen for good dynamic regulation, while other parameters, $H, \bar{G}, \tilde{G}_1, \ldots, \tilde{G}_m$ can be used to obtain a IWPp set of maximal volume.

Author's Publications

The author has contributed to the following publications:

[1] Kok Kiong Tan, Kok Zuea Tang, Yang Su, Tong Heng Lee, Chang Chieh Hang, "Deadtime compensation via setpoint variation", *Journal of Process control*, 20(7), 2010, 848-859.

[2] Yang Su, Kok Kiong Tan, Tong Heng Lee, "Comments on "Model predictive control for systems with stochastic multiplicative uncertainty and probabilistic constraints" [Automatica 45 (2009), 167-172]", *Automatica*, 47(2), 2011, 427-428.

[3] Yang Su, Kok Kiong Tan, "Comments on "Output feedback model predictive control for LPV systems based on quasi-min-max algorithm" [Automatica 47 (2011), 2052-2058]", accepted by *Automatica*, 2012.

[4] Yang Su, Kok Kiong Tan, Tong Heng Lee, "Computation delay compensation for real time implementation of robust model predictive control", manuscript submitted to *Journal of Process Control*, 2012.

[5] Yang Su, Kok Kiong Tan, Tong Heng Lee,"Economic MPC with stability guarantee", manuscript submitted to *Journal of Process Control*, 2012.

[6] Yang Su, Kok Kiong Tan and Tong Heng Lee, "Tube based quasi-min-max output feedback MPC for LPV systems", *Proc. IFAC-ADCHEM*, Singapore, 2012, 186–191.

[7] Yang Su, Kok Kiong Tan and Tong Heng Lee, "Computation delay compensation for real time implementation of robust model predictive control", *Proc. IEEE-INDIN*, Beijing, 2012, 242-247.

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