FEEDBACK CONTROL OF QUANTUM SYSTEMS AND ENTANGLEMENT

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Declaration

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not submitted for any degree in any university previously.

Vu Thanh Long

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Contents

Summary

This thesis studies the measurement-based feedback control of quantum systems. From the control point of view, a key property of quantum measurement-based feedback control that has to be taken into account carefully is the measurement backaction: the measurement of quantum systems inevitably changes the system state in a probabilistic way. Due to the probabilistic nature of quantum measurement, the stochastic stability theory is instrumental in analyzing the measured and feedback controlled quantum systems. For the first time, we introduce a *non-smooth Lyapunov function-like theory* for generic stochastic nonlinear systems, which includes a continuous Lyapunov-like theorem, a discontinuous Lyapunov-like theorem, and an 1-time switching Lyapunov-like theorem for stability in probability. This theory provides a powerful tool for the stability analysis and feedback control synthesis of quantum systems. Indeed, because of the inherent symmetric topology of filter state space, i.e., the space of conditional state conditioned on the measurement outcomes, smooth controls synthesized via the smooth Lyapunov stochastic stability theory are difficult to obtain the global stabilizability and deterministic control performance for quantum filters. The non-smooth Lyapunov function-like theory is thus important in the synthesis of global stabilizing and deterministic control for quantum systems.

Though quantum system under measurement is intrinsically non-deterministic, by combining measurement with feedback control, we can *deterministically* generate desired quantum state in the means that the desired state is produced almost surely or with probability-1. In this thesis, we will interchangeably utilize the terms "deterministically", "almost surely", and "with probability-1". Applying the continuous Lyapunov-like theorem and the discontinuous Lyapunov-like theorem, switching control and continuous control in saturation form are constructed to almost surely globally stabilize the desired eigenstate of a general class of quantum filters, without knowledge about the initial state.

In the measurement-based feedback control of quantum systems, due to the very fast dynamics of the quantum mechanical systems, the time to compute the conditional state and the control input is not negligible. Owing to this feature, to implement a measurement-based feedback control strategy in real time, we have to take the computation time explicitly into account. To deal with this problem, we investigate the time delay control approach in which the time to compute the filter-based control input is fully compensated for by the delay time in the control input. A new Lyapunov-LaSalle-like theorem for delay-dependent stochastic stability is presented for a class of time delay stochastic nonlinear systems. Non-smooth time delay control is then constructed to compensate for the computation time, that is known but arbitrarily long, while globally stabilizing the quantum filters almost surely.

Entanglement is another key feature that distinguishes quantum systems from classical (non-quantum) ones and attracts much research attention owing to its potential use as a valuable resource for quantum computation and quantum information. However, it is difficult to produce entanglement by single measurement. As such, we introduce the concept of SWM-(*simultaneous-weak-measurement*)-*induced quantum state reduction* for quantum systems which states that under SWMs of commutative observables, the filter state, i.e., the conditional state, almost surely converges to the common set of these observables' eigenspaces. In the applications of this concept, we probabilistically generate the maximally entangled two-qubit Bell states and multipartite entangled states such as the maximally entangled three-qubit $|GHZ\rangle$ (Greenberger-Horne-Zeilinger) state.

Combining the concept of SWM-(*simultaneous-weak-measurement*)*-induced quantum state reduction* with the 1-time switching Lyapunov-like theorem, a continuous control is designed to almost surely generate the maximally entangled two-qubit Bell states from any initial state. This concept is also utilized together with the time delay bang-bang control to almost surely generate the Bell states and the multipartite entangled states such as the $|GHZ\rangle$ state in the real time.

List of Figures

List of Symbols

Throughout this thesis, the following notations and conventions have been adopted:

ρ	the density matrix;
i	$\sqrt{-1}$ (we use the Roman character i to distinguish the imaginary unit from the index i);
$A^\dagger, A^*$	the conjugate transpose and complex conjugate of the matrix A;
$\mathrm{Tr}(\cdot)$	the trace of matrices;
$\lvert \cdot \rvert$	the Euclidean norm of vectors and the Frobenius norm of matrices, i.e., $\lvert A\rvert = \sqrt{\mathrm{Tr}(A^\dagger A)}$;
$[A, B]$	the commutator of the matrices A, B, i.e., $[A, B] = AB - BA$;
$\otimes$	the tensor product of operators or the Kronecker product of matrices;
$\mathcal{K}, \mathcal{K}_\infty$	the classes of comparison functions;
$\mathbb{R}, \mathbb{C}$	the sets of real and complex numbers, respectively;
$\mathcal{L}$	the infinitesimal generator;
I_n	the identity matrix;
$C([-\tau, 0]; \mathbb{R}^n)$:	the family of continuous $\mathbb{R}^n-$valued functions on $[-\tau, 0], \tau > 0$;
$C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$:	the family of $\mathcal{F}_0-$ measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$ valued random variables $\xi = \{\xi_\theta : -\tau \leq \theta \leq 0\}$.

Chapter 1

Introduction

1.1 Introduction

Quantum dynamical systems describe the evolution of physical systems at atomic and molecular scales. Due to the steady growth of capabilities to manipulate matter and light at those scales as well as the fast development of quantum technology, it is possible to transfer the interest from the interpretation of quantum mechanics to the active control of quantum systems. This area of quantum control has attracted an extensive research effort during the last two decades, and its rapid development in the near future is foreseeable.

This thesis studies the measurement-based feedback control of quantum systems. The objective of the thesis is to introduce a framework to drive the inherently probabilistic nature of quantum measurement towards the deterministic control performance by using feedback control (note that in this thesis, we will interchangeably utilize the terms "deterministically", "almost surely", and "with probability-1"). To achieve this objective, the thesis will address several fundamental problems in quantum feedback control: quantum measurement, stability analysis, deterministic quantum feedback control design, real-time implementation of quantum feedback control, and deterministic generation of entangled states. In the following, we give a brief description of quantum control schemes. Several challenges in quantum feedback control are then analyzed, motivating the study in this thesis.

1.2 Quantum Control

Since the time of the first pioneering contributions to the field [1–4], quantum control has gained a stable development. Some very good reviews of this emerging field have been widely appreciated [5,6]. This section hence does not aim at presenting an overview of this field. Instead, we shall provide a brief description of quantum control paradigms. From the control system point of view, the existing quantum control schemes can be classified into three main groups: open-loop control, measurement-based feedback control, and coherent-feedback control.

Open-loop control is the conceptually simplest but very important type of quantum control, in which open-loop controller acts without obtaining knowledge about the underlying state of the system. Instead, the controller may be provided with some information about the system model and system's initial state. In order to achieve the desired evolution of the system, quantum open-loop control involves basically engineering the system Hamiltonian [7] and the interaction between the quantum system of interest and its environment, i.e., the reservoir engineering [8–11]. The remarkable open-loop quantum control techniques include optimal control [4, 7, 12–14], Lyapunov-based design [15–17], dynamical decoupling [18, 19], and learning control [20, 21].

The advantage of quantum open-loop control lies in its simple implementation, while the disadvantages come from the requirement of exact knowledge about system model as well as initial state, which may cause it ineffective in robust control. Thanks to its easy implementation, open-loop quantum engineering plays a central role in many applications

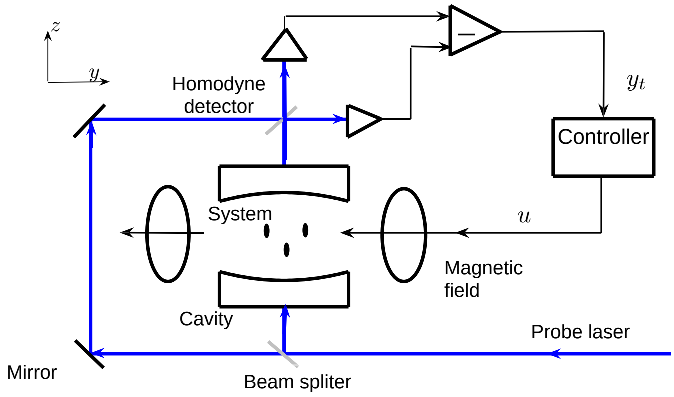


Fig. 1.1: Typical setup for measurement-based quantum feedback control. The quantum system interacts with an optical field produced by a laser. The optical field is detected by homodyne measurement. The measurement outcomes are processed and then fed back via a magnetic field to modify the system Hamiltonian.

including control of electronic [22], rotational and translational degrees of freedom of molecular systems [23], trapped ions, Bose Einstein condensates [24], nuclear and electron spin engineering in nuclear magnetic resonance (NMR) [25] and electron spin resonance (ESR) applications.

The second important type of quantum control is measurement-based feedback control [26, 27]. Typically, this approach also involves Hamiltonian engineering by applying suitable control fields, but in addition, the system is monitored, usually via continuous weak measurements, and information gained from the observation is fed back to the actuators as shown in Fig. 1.1.

Depending on the type of information to be fed back, there are two main techniques in measurement-based quantum feedback control: direct feedback control and estimate feedback control. Direct, or Markovian, feedback control [27, 28], utilizes the physical measurement results to feed back to the system. It has been extensively investigated [29–32] as it promises the real-time control implementation. Estimate, or Bayesian, feedback control [33], is based on feeding back the estimation state conditioned on the measurement outcomes to alter the dynamics of the systems. It is widely appreciated in control community since it provides a greater flexibility in control design than direct feedback control [34, 35].

The advantage of quantum measurement-based feedback control is its flexibility in control design which greatly benefits from a rigorous literature of classical control theory and stochastic stability and synthesis. Its main disadvantages include the difficulty in performing continuous measurement on the system and the real-time implementation, especially with estimate feedback control. In addition, the backaction of quantum measurement is also a challenge for quantum measurement-based feedback control. While open-loop Hamiltonian engineering usually involves control of non-equilibrium dynamics, measurement-based feedback control is very important for control of equilibrium dynamics, including steering the system to a steady state [36] with applications in laser cooling of atomic or molecular motion [37], control of solid-state qubits [38], quantum state reduction [34], and decoherence control [32].

A recently introduced paradigm for quantum control is coherent feedback control [39–45]. Unlike measurement-based feedback control, coherent feedback control does not (at least not directly) involve any classical actuators or measurements. Instead, it relies on indirect control of a target quantum system through its coherent interaction with another quantum system acting as the controller, as in Fig. 1.2.

Quantum coherent feedback control promises a great potential to deal with the real-time control since all the components, i.e., plant and controller, in the control systems are quantum systems with very fast dynamics. However, quantum controllers cannot solve the problem of controlling quantum systems completely as the quantum controller itself needs to be controlled in some form, and this usually requires interaction with a non-quantum system such as classical laboratory equipment at some stage, and thus control strategies such as Hamiltonian engineering or state preparation using measurement-based feedback.

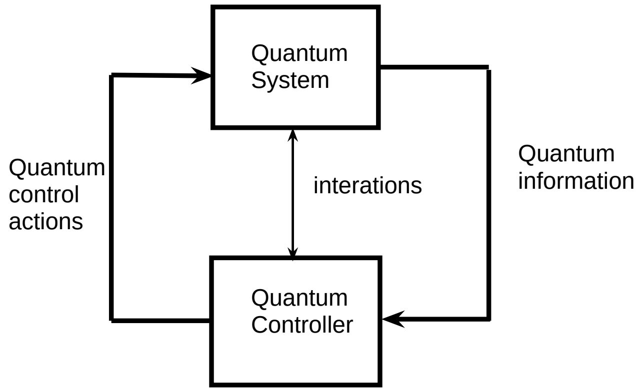


Fig. 1.2: Quantum coherent feedback

1.3 Motivations of the Thesis

This thesis is devoted to the measurement-based feedback control of quantum systems. In the feedback control of quantum systems, the intrinsic feature of measurement backaction, i.e., the stochastic change of system state under measurement, appears and makes the quantum feedback control much challenging. The indeterminism of quantum measurement outcomes has motivated us to the main question of this thesis: is it possible and how to deterministically obtain the control performance for quantum systems by combining measurement and feedback schemes?

To answer this question, we face with several challenges. Due to the probabilistic nature of quantum measurement, the stochastic stability theory is instrumental in analyzing the measured and controlled quantum systems. The first challenge of deterministic quantum feedback control comes from the inherent symmetric topology of filter state space, i.e., the space of system state conditioned on measurement outcomes. Because of this symmetric topology, the smooth controls synthesized via the smooth Lyapunov stochastic stability theory are difficult to obtain the global stabilizability and deterministic control performance for quantum filters. This motivated us to introduce a non-smooth Lyapunov function-like theory for the synthesis of global stabilizing and deterministic control for quantum systems. It will be proved in this thesis that the non-smooth Lyapunov function-like theory provides us with a powerful tool to deterministically control the quantum systems by feedback schemes.

The second challenge of quantum measurement-based feedback control is the real-time implementation. In the measurement-based feedback control of quantum systems, due to the very fast dynamics of the quantum mechanical systems, the time to compute the conditional state and the control input is not negligible. Owing to this feature, to implement a measurement-based feedback control strategy in real time, we have to take the computation time explicitly into account. To deal with this problem, we bring in the method of time delay control. In this method, the delay time is used to compensate for the computation time and thus the real-time implementation of quantum feedback control is guaranteed.

A special and characteristic feature of quantum systems, which attracts a huge research interest in the literature, is entanglement. Recently, the attention on entanglement has been renewed owing to its potential use as a valuable resource for quantum computation and quantum information which outperforms that solely based on classical physics. However, it is difficult to produce entanglement by single measurement. This motivated us to study the effect of simultaneous weak measurements on quantum systems. It turns out that the introduced concept of SWM-(*simultaneous-weak-measurement*)-*induced quantum state reduction* plays an important role in the generation of quantum entangled states.

1.4 Outline of the Thesis

The thesis starts with a brief introduction of quantum control schemes. Several challenging problems in quantum feedback control, which motivated the research of the thesis, are then presented in Chapter 1. For self-containedness, Chapter 2 gives a brief summary of the basic notions in quantum mechanics which are frequently utilized in this thesis.

In Chapter 3, we introduce a *non-smooth Lyapunov function-like theory* for generic stochastic nonlinear systems, which includes a continuous Lyapunov-like theorem, a discontinuous Lyapunov-like theorem, and an 1-time switching Lyapunov-like theorem for stability in probability. This theory provides a necessary and powerful tool for the stability analysis and feedback control synthesis of quantum systems. Indeed, because of the inherent symmetric topology of filter state space, the smooth controls synthesized via the smooth Lyapunov stochastic stability theory are difficult to obtain the global stabilizability and deterministic control performance for quantum filters. The non-smooth Lyapunov function-like theory is thus very important in the the synthesis of global stabilizing and deterministic control for quantum systems.

In Chapter 4, we combine the continuous weak measurement with feedback control to *deterministically* generate the desired quantum state. Applying the continuous Lyapunov-like theorem and the discontinuous Lyapunov-like theorem, switching control and continuous control in saturation form are constructed to deterministically globally stabilize the desired eigenstate of a general class of quantum filters, without knowledge about the initial state.

In Chapter 5, we solve the problem of the real time feedback control of quantum systems by using the time delay control approach in which the time to compute the filter-based control input is fully compensated for by the delay time in the control input. A new Lyapunov-LaSalle-like theorem for delay-dependent stochastic stability is presented for a class of time delay stochastic nonlinear systems. Non-smooth time delay control is then constructed to compensate for the computation time, that is known but arbitrarily long, while globally stabilizing the quantum filters almost surely.

The next two chapters deal with the deterministic generation of entanglement. In Chapter 6, we introduce the concept of SWM-(*simultaneous-weak-measurement*)*-induced quantum state reduction* for quantum systems which states that under SWMs of commutative observables, the filter state, i.e., the estimate state conditioned on the measurement records, almost surely converges to the common set of these observables' eigenspaces. In the applications of this concept, we probabilistically generate the maximally entangled two-qubit Bell states. By combining this concept with the 1-time switching Lyapunov-like theorem, we design the continuous control to deterministically generate the maximally entangled two-qubit Bell states from any initial state.

In Chapter 7, the concept of SWM-(*simultaneous-weak-measurement*)*-induced quantum state reduction* is generalized and utilized to probabilistically prepare the multipartite entangled states such as the maximally entangled three-qubit $|GHZ\rangle$ (Greenberger-Horne-Zeilinger) state. It is also harnessed together with the time delay bang-bang control to deterministically generate the Bell states and multipartite entangled states such as the $|GHZ\rangle$ state in the real time.

1.5 Conclusions

In this chapter, after a brief description of the background of quantum control, several challenging problems in quantum feedback control have been discussed, motivating the research of the thesis and highlighting the importance of this work. Finally, the organization of the thesis is presented with a description of the purposes, contents, and methodologies used in each chapter.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we provides a brief summary of the basic notions in quantum mechanics, which will be frequently utilized in this thesis, including observables, state vectors, density operators, entanglement, quantum system dynamics, and quantum filtering equation.

2.2 Quantum Dynamical Systems

In this section, we present the basic elements of quantum theory for finite-dimensional systems. More details can be seen in the monograph [46] or [7]. Any quantum system $\mathcal{Q}$ is associated to a Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. The dimension of $\mathcal{H}$ depends on the possible outcomes of its variables. In this thesis, we will only consider finite-dimensional quantum systems $\mathcal{H} \simeq \mathbb{C}^N$. A great motivation to study finite-dimensional quantum systems is their importance to the emerging field of quantum information [47]. Let $\mathcal{B}(\mathcal{H})$ denote the set of linear operators on $\mathcal{H}$ and $\mathtt{h}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ denote the real subspace of Hermitian operators.

2.2.1 Observable Quantities and State Vectors

Central to quantum mechanics are the notions of observables, which are mathematical representations of physical quantities that can (in principle) be measured, and state vectors, which summarize the status of physical systems and permit the calculation of expectations of observables.

Any observable is associated to an Hermitian operator $Y \in \mathtt{h}(\mathcal{H})$. When the state of the system is (ideally) known exactly, it can be described by a state vector $|\phi\rangle$ which is a norm-1 vector in the complex N-dimensional Hilbert space $\mathcal{H}$. The state vector $|\phi\rangle$ is living on the unit sphere on the Hilbert space $\mathcal{H} : |\phi\rangle \in \mathbb{S}^{2N-1} \subset \mathcal{H}$.

2.2.2 Density Operators

Density operators are used to describe the state of statistical ensembles, i.e., collections of identical quantum systems, or of a single system in the presence of classical uncertainty. More precisely, assume that $f_j, j = 1, ..., m$, is the fraction of population of some ensemble prepared in the state $|\phi_j\rangle$, with different state vectors not necessarily orthogonal to each other. The associated quantum density operator is defined by

$$\rho = \sum_{j=1}^{m} f_j |\phi_j\rangle\langle\phi_j| \quad s.t. \quad f_j \geq 0, \sum_{j=1}^{m} f_j = 1. \tag{2.1}$$

Clearly, $\rho = \rho^\dagger \geq 0$ and $\mathrm{Tr}(\rho) = 1$. Density operators form a convex set, denoted $\mathcal{S}(\mathcal{H}) \subset \mathtt{h}(\mathcal{H})$. An important subset of density operators is the following: if $f_1 = 1$, the whole ensemble is prepared in the same state $|\phi\rangle$, so that $\rho = |\phi\rangle\langle\phi|$ is a rank-one orthogonal projector. Such a ρ is called a *pure state*. In this case, the description one obtains is completely equivalent to that provided by the state vector $|\phi\rangle$, up to an irrelevant global phase. On the other hand, an ensemble in which at least two of the f_j of Eq. (2.1) are nonzero is called a mixed ensemble, or *mixed state* and does not admit a description in terms of a single state vector.

2.2.3 Closed Quantum Systems

The Schrödinger equation: The basic dynamical postulate of quantum dynamics is that the state vector $|\phi\rangle$ of a closed system obeys the autonomous linear ordinary differential equation called the Schrödinger equation:

$$\begin{aligned} \hbar|\dot{\phi}\rangle &= -\mathrm{i}H_0|\phi\rangle \\ |\phi(0)\rangle &= \phi_0, \end{aligned} \tag{2.2}$$

where H_0 is the Hamiltonian of the system and $\hbar$ is the reduced Planck constant. In this thesis, the units are chosen such that the reduced Planck constant $\hbar = 1$.

The quantum Liouville-von Neumann equation: Given a certain Hamiltonian H_0, the dynamics of ρ is described by the quantum Liouville-von Neumann equation:

$$\begin{aligned} \dot{\rho} &= -\mathrm{i}[H_0, \rho] \\ \rho(0) &= \rho_0. \end{aligned} \tag{2.3}$$

The control of a quantum mechanical system is typically obtained by coupling it with one or more tunable electromagnetic fields. Then, the Hamiltonian H_0 in (2.2) and (2.3) can be replaced by $H_0 + \sum_{i=1}^{k} u_i H_i$ where $H_i = H_i^\dagger$ is the control Hamiltonian and $u_i \in \mathbb{R}$ is the control input. Such controls are usually called *coherent control* as they preserve the unitary evolution of the state vector.

2.3 Entanglement

Consider a bipartite system, i.e., quantum system that is composed of two physically distinct subsystems. It is associated with a Hilbert space $\mathcal{H}$ that is given by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the predefined factor spaces. For pure state, one distinguishes two different kinds of states. A state $|\Psi\rangle$ is called a *product state* or *separable*, if it can be written as a tensor product of subsystem states:

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle, \psi_i \in \mathcal{H}_i \tag{2.4}$$

Such state describes a situation analogous to a classical one as the system state contains exactly the information that is contained in the subsystem states. A state reduction caused by a measurement performed on one subsystem has no influence on the state of the other subsystem. This means that measurement results on the different subsystems are uncorrelated. In contrast to this, they are correlated for *entangled states*, i.e., states that cannot be written as a product of subsystem states as in Eq. (2.4).

For mixed states, the situation is more complicated. A mixed state is entangled if it cannot be represented as a convex sum of separable pure states:

$$\rho \neq \sum_i p_i |\psi_1^i\rangle \otimes |\psi_2^i\rangle, \text{ with } \sum_i p_i = 1 \tag{2.5}$$

Otherwise, it is separable.

Generally, for a n-partite system, a mixed state ρ that cannot be presented in the form

$$\rho = \sum_i p_i |\psi_1^i\rangle \otimes |\psi_2^i\rangle ... \otimes |\psi_n^i\rangle, \text{ with } \sum_i p_i = 1, \tag{2.6}$$

in which $|\psi_j^i\rangle$ is a pure state of $j-th$ subsystem, is an entangled state.

2.4 Quantum Filtering Equation

Consider the quantum systems (2.3), in which H_0 is replaced by $H = H_0 + \sum_{i=1}^{k} u_i H_i$, subject to the continuous weak measurement of the observable L. We have the quantum filtering equation or Stochastic Master Equation (SME) describing the evolution of the conditional state $\rho_t \in \mathcal{S}(\mathcal{H})$ conditioned on the measurement outcomes [34, 48]:

$$d\rho_t = (-\mathrm{i}[H, \rho_t] + \Gamma\mathcal{D}[L]\rho_t)dt + \sqrt{\eta\Gamma}\mathcal{H}[L]\rho_t dw_t, \tag{2.7}$$

$$dy_t = \mathrm{Tr}(L\rho_t)dt + \frac{1}{2\sqrt{\eta\Gamma}}dw_t \tag{2.8}$$

where

- y_t is measurement outcome;
- w_t is the $1-$dimensional Wiener process defined on the classical complete probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, in which Ω is a sample space, $\mathcal{F}$ is a $\sigma-$field, and $\mathbb{P}$ is a probability measure;
- Γ and η are measurement strength and efficiency; and
- $\mathcal{D}[A]\rho := A\rho A^\dagger - \frac{1}{2}(\rho A^\dagger A + A^\dagger A\rho)$ and $\mathcal{H}[A]\rho := A\rho + \rho A^\dagger - [\mathrm{Tr}(A\rho + \rho A^\dagger)]\rho$ are respectively the drift and diffusion parts introduced by the measurement of the field quadrature.

Here $\{\rho_t\}$, the solution of (2.7) given an initial condition ρ_0, exists, is unique, adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the (classical) white noise w_t, and it is $\mathcal{S}(\mathcal{H})$-invariant by construction [34, 49]. We note that the filtering equation (2.7) also holds true when the quantum system is in contact with a Markovian environment and under continuous measurements of multiple observables [48].

2.5 Conclusions

In this chapter, we have presented some basic concepts essential for the thesis, which include observables, state vectors, density operators, entanglement, quantum system dynamics, and quantum filtering equation.

Chapter 3

Nonsmooth Lyapunov Stability

3.1 Introduction

In this thesis, we are interested in the problem of deterministic control for quantum systems by measurement-based feedback control. By the *quantum filtering theory* [2, 49–51], this problem can be seen as a stochastic control problem for quantum filter described by quantum filtering equation [52], which is also called *stochastic master equation* describing the evolution of conditional state conditioned on the measurement outcomes. In particular, we aim at globally stabilizing quantum filters.

One of the main challenges of this problem arises from the symmetric topology of the manifold on which the filter state involves. Generally, the symmetric topology of filter state space makes the smooth controls [34, 35, 53], synthesized via the classical Lyapunov stochastic stability theory [54–56], difficult, if not impossible, to obtain the global stabilizability for a given desired state. The underlying reason is the existence of the so-called *antipodal eigenstates* which, together with the final desired state, are equilibrium points of the closed-loop systems; see [15] for the origin of the name *antipodal eigenstates* from deterministic case. As such, the global stabilization for quantum filters calls for a non-smooth control design method to deal with the symmetric topology of filter state space.

In [34], after the introduction of quantum filters, the global stabilizability was shown to be difficult to obtain even for 2-dimensional quantum filters. Continuous control was then proposed, through computer search, to break the symmetric topology of 2-dimensional filter state space and to globally stabilize the desired state. The design method is, however, computationally involved and the global stability is hard to prove as the design method is not analytical [57]. In [57], control in hysteresis form was introduced to globally drive the expectation of filter state of angular momentum systems to the final desired state, in which the state convergence was proved based on a detailed analysis of the sample paths of filter state.

In this thesis, we present a systematic approach based on Lyapunov stability analysis for the globally stabilizing, non-smooth control synthesis of quantum filters. Since smooth controls are designed via smooth Lyapunov stability theory, the non-smooth control synthesis for quantum filters intuitively calls for a non-smooth Lyapunov stability theory. The need for a non-smooth Lyapunov stability theory also arises from the practical point of view. In practice, many systems in biology, physics, engineering, and information science exhibit stochastic and impulsive dynamical behaviors subjected to stochastic disturbances and/or stochastically abrupt changes at certain instants during the dynamical processes [58–60]. The analysis and control design of such stochastic systems with intrinsically non-smooth energy generally desire a non-smooth Lyapunov stability theory because the system energy is usually a typical candidate for Lyapunov function [61, 62].

Motivated by the above considerations, in this chapter, we introduce a non-smooth Lyapunov-like theory for the stability analysis of generic stochastic nonlinear systems, which will be instrumental in synthesizing non-smooth controls in the following chapters. In particular, in Sections 3.3.1 and 3.3.2, we introduced the continuous Lyapunov-like theorem and the discontinuous Lyapunov-like theorem which will be utilized in the deterministic global stabilization of a class of quantum filters in Chapter 4. The 1-time switching Lyapunov-like theorem for stability in probability is then presented in Section 3.3.3. This theorem is instrumental in controlling of entanglement in Chapter 6.

We note that this non-smooth Lyapunov stability theory for stochastic nonlinear systems whose state evolves on the vector spaces $\mathbb{R}^n, n = 1, 2, ...$, is applicable to quantum filters evolving on the space $\mathcal{S} \subset \mathbb{C}^{n\times n}$ because the density matrix $\rho \in \mathcal{S}$ can be equivalently represented by a vector in the vector space $\mathbb{R}^{n^2-1}$ [34].

3.2 Classical Lyapunov Stochastic Stability

Consider the stochastic nonlinear systems described by

$$dx_t = f(x_t)dt + \sigma(x_t)dw_t, \tag{3.1}$$

where x_t is the state, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n_w}$ are locally bounded, Borel-measurable functions satisfying $f(0) = 0, \sigma(0) = 0$, w_t is an n_w-dimensional standard Wiener process (or Brownian motion) defined on the classical complete probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, in which Ω is a sample space, $\mathcal{F}$ is a $\sigma-$field, and $\mathbb{P}$ is a probability measure.

Definition 3.1. *([56, 63]) The equilibrium $x = 0$ of the system* (3.1) *is*

- *globally stable in probability if $\forall\epsilon > 0$, there exists a $\mathcal{K}-$class function $\gamma(\cdot)$ such that for all $t \geq 0$, for all $x_0 \in \mathbb{R}^n \setminus \{0\}$,*

$$\mathbb{P}\{|x_t| < \gamma(|x_0|)\} \geq 1 - \epsilon. \tag{3.2}$$

- *globally asymptotically stable in probability if it is globally stable in probability and*

$$\mathbb{P}\{\lim_{t\to\infty}|x_t| = 0\} = 1, \quad \forall x_0 \in \mathbb{R}^n. \tag{3.3}$$

We recall the following results.

Lemma 3.1. (Itô's formula [54, 55]) *Let x_t be a stochastic processes defined by*

$$x_t = x_0 + \int_0^t f_s ds + \int_0^t \sigma_s dw_s, \tag{3.4}$$

in which w_t is the standard Weiner process. Suppose that the function $V(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable in $x, \forall x \in \mathbb{R}$, and continuously differentiable in $t, \forall t \in \mathbb{R}$. Consider the process $\{V_t\}_{t\in\mathbb{R}}$ defined by $V_t = V(x_t, t)$.

Then, for any stopping times τ and ρ, we have

$$V_\rho = V_\tau + \int_\tau^\rho \left[\frac{\partial V_s}{\partial t} + \frac{\partial V_s}{\partial x} f_s + \frac{1}{2}\mathrm{Tr}\big(\sigma_s^T \frac{\partial^2 V_s}{\partial x^2}\sigma_s\big)\right] ds + \int_\tau^\rho \frac{\partial V_s}{\partial x}\sigma_s dw_s. \tag{3.5}$$

Theorem 3.1. (Classical Lyapunov stability theorem [56,63]) *For the system* (3.1)*, if there exist a function $V(x) : \mathbb{R}^n \to \mathbb{R}$ positive definite, radially unbounded, and twice continuously differentiable in $x, \forall x \in \mathbb{R}$, and a continuous nonnegative function $W(x) : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\mathcal{L}_{f,\sigma}V(x) \leq -W(x), \forall x \in \mathbb{R}, \tag{3.6}$$

where $\mathcal{L}_{f,\sigma}V$ is the infinitesimal generator of x_t acting on the function V,

$$\mathcal{L}_{f,\sigma}V(x) := \frac{\partial V(x)}{\partial x}f(x) + \frac{1}{2}\mathrm{Tr}\big(\sigma(x)^T \frac{\partial^2 V(x)}{\partial x^2}\sigma(x)\big),$$

then the equilibrium $x = 0$ is globally stable in probability and

$$\mathbb{P}\{\lim_{t\to\infty} W(x_t) = 0\} = 1. \tag{3.7}$$

If, in addition, $W(x)$ is positive definite, then the equilibrium $x = 0$ is globally asymptotically stable in probability.

3.3 Nonsmooth Lyapunov-like Theory

3.3.1 Continuous Lyapunov-like Theorem for Stability in Probability

The continuous Lyapunov-like theorem utilizes the observation that the convergence of stochastic system state is guaranteed, if the Lyapunov function is continuous and decreasing at its smooth segments. Such a Lyapunov function is constructed from multiple smooth functions via a partition of the state space $\mathbb{R}^n$, and is defined as *partition-based Lyapunov function.* It is relevant to note that though being made up of multiple functions, the partition-based Lyapunov function is different from the standard Multiple Lyapunov functions in switched systems [64–66] where each Lyapunov function is constructed for each subsystem on the whole state space.

Definition 3.2. *(Partition-based Lyapunov function)*

The partition-based Lyapunov function is of the form:

$$V(x) = V_i(x), \quad x \in \Phi_i, i \in \{1, 2, ..., r\}, \tag{3.8}$$

where

- $V_i : \mathbb{R}^n \to \mathbb{R}_+$ *is twice continuously differentiable for all* $i \in \{1, 2, ..., r\}$,
- $\{\Phi_i\}_{i=1}^r$ *is a partition of the state space* $\mathbb{R}^n$, *i.e.,* $\bigcup\limits_{i=1}^{r} \Phi_i = \mathbb{R}^n$ *and* $\Phi_i \cap \Phi_j = \emptyset, \forall i \neq j$,

where $\emptyset$ *is the empty set,*

- *the origin is in the interior of* Φ_1, *and*
- $V(x)$ *is continuous, i.e.,* $V_i(x) = V_j(x), \forall x \in \Lambda_{ij}$, *where* $\Lambda_{ij} := \bar{\Phi}_i \cap \bar{\Phi}_j$ *is the boundary between* Φ_i *and* Φ_j, *and* $\bar{\Phi}_i$ *denotes the closure of* $\Phi_i, i \in \{1, 2, ..., r\}$.

To guarantee that the Lyapunov function is decreasing at its smooth segments, we shall apply the Itô's formula for this Lyapunov function in each smooth segment between two consecutive stopping times τ_i and τ_{i+1} at which the system trajectory x_t is on the boundary $\Lambda := \overset{r}{\underset{i \neq j}{\cup}} \Lambda_{ij}$. In order to make $\{\tau_i\}$ be well-defined stopping times, by which Lemma 3.1 applicable, we have the following assumption.

Assumption 3.1. *There is no sliding motion of the system trajectory* x_t *on the boundary* Λ.

By Assumption 3.1, the system trajectory x_t intersects the boundary Λ at separated times $\tau_i, i = 1, 2, ...$, which are all well-defined stopping times (see the proof of Theorem 3.2). In addition, when applied for the control design of quantum filters in Section 4.4.1, Assumption 3.1 helps us to prove the well-posedness of the closed-loop system by joining the solutions in consecutive time intervals (τ_i, τ_{i+1}).

Therefore, in this chapter, the Itô's formula is applied for partially smooth functions, rather than for twice continuously differentiable Lyapunov functions as in classical stability theorems of stochastic systems [54–56, 63].

Theorem 3.2. *Consider the stochastic nonlinear systems described by* (3.1). *Suppose that there exist a Lyapunov function of the form* (3.8) *satisfying Assumption* 3.1, $\mathcal{K}-$*class functions* α_1, α_2, *and a continuous, non-negative function* $W : \mathbb{R}^n \to \mathbb{R}$ *such that*

C.1 $\alpha_1(|x|) \leq V_i(x) \leq \alpha_2(|x|), \forall x \in \mathbb{R}^n, i \in \{1, 2, ..., r\}$,

C.2 $\mathcal{L}_{f,\sigma} V_i(x) \leq -W(x), \forall x \in \Phi_i, i \in \{1, 2, ..., r\}$, *where* $\mathcal{L}_{f,\sigma} V$ *is the infinitesimal* *generator of x_t acting on the function V:*

$$\mathcal{L}_{f,\sigma}V(x) := \frac{\partial V(x)}{\partial x}f(x) + \frac{1}{2}\mathrm{Tr}\big(\sigma(x)^T\frac{\partial^2 V(x)}{\partial x^2}\sigma(x)\big).$$

Then

$$\mathbb{P}\{\lim_{t\to\infty} W(x_t) = 0\} = 1. \tag{3.9}$$

If, in addition, $W(x)$ is positive definite, then from any initial state, the system trajectory x_t converges to the origin almost surely.

Proof: Under Assumption 3.1, the trajectory x_t of system (3.1) intersects the boundary $\Lambda = \overset{r}{\underset{i\neq j}{\cup}} \Lambda_{ij}$ at separated time instants. Let τ_1 be the first time that x_{τ_1} is on the boundary Λ. Let τ_2 be the first time after τ_1 that x_{τ_2} is on Λ. By this procedure, we obtain the sequence of times $\{\tau_i\}_{i=1,2,...}$, at which x_t is on Λ. Under Assumption 3.1, these times are well-defined. Note that these time instants are exit times. Indeed, for example, if x_0 is in Φ_{i_0} for some $i_0 \in \{1, 2, ..., r\}$, then τ_1 is the first time that x_t exits from the open set $\mathring{\Phi}_{i_0}$, where $\mathring{\Phi}_{i_0}$ denotes the interior of Φ_{i_0}. As $\tau_i, i = 1, 2, ...,$ are exit times, they are stopping times [54]. Thus, the following equation holds [55]:

$$\mathbb{E}\left[\int_{\tau_i}^{\tau_j} h(s)dw_s\right] = 0, \tag{3.10}$$

for any process h in $\mathcal{M}^2([a, b]; \mathbb{R})$, the family of process $\{\xi(t)\}_{a\leq t\leq b}$ satisfying that $\mathbb{E}\left[\int_a^b |\xi(t)|^2 dt\right] < \infty$. For any $t > 0$, we denote by $\tau_0 := 0 < \tau_1 < ... < \tau_{n(t)} < t := \tau_{n(t)+1}, n(t) \in [0, \infty]$, the sequence of stopping times between 0 and t, at which x_t is on the boundary Λ. Suppose that in the interval $(\tau_i, \tau_{i+1}), i = 0, ..., n(t)$, the trajectory x_t is in $\Phi_{q_i}, q_i \in \{1, 2, .., r\}$. Hence, the Lyapunov function $V(x_s) = V_{q_i}(x_s)$ for all $s \in (\tau_i, \tau_{i+1})$. As $V_j, j \in \{1, 2, ..., r\}$, are twice continuously differentiable, Lemma 3.1 is applicable. For all $i = 0, ..., n(t)$, we have

$$\begin{aligned} V_{q_i}(x_{\tau_{i+1}}) - V_{q_i}(x_{\tau_i}) = & \int_{\tau_i}^{\tau_{i+1}} \mathcal{L}_{f,\sigma} V_{q_i}(x_s) ds \\ & + \int_{\tau_i}^{\tau_{i+1}} \frac{\partial V_{q_i}}{\partial x_s} \sigma(x_s) dw_s \end{aligned} \tag{3.11}$$

Combining these equations and noting that $V(x)$ is continuous, i.e., $V_{q_i}(x_{\tau_i}) = V_{q_{i-1}}(x_{\tau_i}), \forall i = 1, ..., n(t)$, we obtain

$$\begin{aligned} V(x_t) - V(x_0) &= \sum_{i=0}^{n(t)} \big(V_{q_i}(x_{\tau_{i+1}}) - V_{q_i}(x_{\tau_i}) \big) \\ &= \sum_{i=0}^{n(t)} \int_{\tau_i}^{\tau_{i+1}} \mathcal{L}_{f,\sigma} V_{q_i}(x_s) ds \\ &+ \sum_{i=0}^{n(t)} \int_{\tau_i}^{\tau_{i+1}} \frac{\partial V_{q_i}}{\partial x_s} \sigma(x_s) dw_s \end{aligned} \tag{3.12}$$

From Condition C.2, we have $\mathcal{L}_{f,\sigma} V_{q_i}(x) \leq -W(x), \forall x \in \Phi_{q_i}$. Taking the expectation of both sides of (3.12), it holds that

$$\begin{aligned} \mathbb{E}[V(x_t)] - V_0 \leq & -\mathbb{E}[\sum_{i=0}^{n(t)} \int_{\tau_i}^{\tau_{i+1}} W(x_s) ds] \\ & + \mathbb{E}[\sum_{i=0}^{n(t)} \int_{\tau_i}^{\tau_{i+1}} \frac{\partial V_{q_i}}{\partial x_s} \sigma(x_s) dw_s] \\ = & -\mathbb{E}[\int_0^t W(x_s) ds] \\ & + \mathbb{E}[\sum_{i=0}^{n(t)} \int_{\tau_i}^{\tau_{i+1}} \frac{\partial V_{q_i}}{\partial x_s} \sigma(x_s) dw_s], \end{aligned} \tag{3.13}$$

where $V_0 = V(x_0)$. Letting $t \to \infty$, we obtain

$$\mathbb{E}[\int_0^\infty W(x_s) ds] \leq V_0 + \lim_{t \to \infty} \mathbb{E}[\sum_{i=0}^{n(t)} \int_{\tau_i}^{\tau_{i+1}} \frac{\partial V_{q_i}}{\partial x_s} \sigma(x_s) dw_s] \tag{3.14}$$

We prove that $\lim_{t\to\infty} \mathbb{E}[\sum_{i=0}^{n(t)} \int_{\tau_i}^{\tau_{i+1}} \frac{\partial V_{q_i}}{\partial x_s}\sigma(x_s)dw_s] = 0$. Indeed, if for some sample paths, there are finite number of stoping times τ_i, namely, for example $\tau_0, \tau_1, ..., \tau_k$, then we can add more times as follows: $\tau_i = \tau_k + i, \forall i > k$. As such, we can assume that there are infinite number of stopping times in almost all sample paths. Applying (3.10) yields

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}[\sum_{i=0}^{n(t)} \int_{\tau_i}^{\tau_{i+1}} \frac{\partial V_{q_i}}{\partial x_s}\sigma(x_s)dw_s] \\
&= \mathbb{E}[\sum_{i=0}^{\infty} \int_{\tau_i}^{\tau_{i+1}} \frac{\partial V_{q_i}}{\partial x_s}\sigma(x_s)dw_s] = 0. \qquad (3.15)
\end{aligned}$$

By (3.14) and (3.15), we have

$$\mathbb{E}\left[\int_0^{\infty} W(x_s)ds\right] \leq V_0 < \infty \qquad (3.16)$$

As $W(x)$ is continuous and $f(x), \sigma(x)$ are locally bounded, using the same arguments as in Step 2 of the proof of Theorem 2.1 in [67], from (3.16), we have

$$\mathbb{P}\big\{\lim_{t\to\infty} W(x_t) = 0\big\} = 1. \qquad (3.17)$$

If, in addition, $W(x)$ is positive definite, then (3.17) leads to $\mathbb{P}\big\{\lim_{t\to\infty} |x_t| = 0\big\} = 1$, which means that from any initial state, the system trajectory x_t converges to the origin almost surely. The proof of Theorem 3.2 is completed. $\square$

Remark 3.1. *Though Theorem* 3.2 *is a generation of the standard stochastic stability theorems, see e.g. Theorems* 2.1, 2.2 *in [63], to the case when the Lyapunov function is non-smooth, it serves as an indispensable criterion for the globalizing non-smooth control synthesis of quantum filters in Section* 4.4.1*, Chapter* 4*.*

Remark 3.2. *Although Assumption* 3.1 *is stated on the system trajectory* x_t*, it can be checked* a priori *by the condition that* $\mathcal{L}_{f,\sigma}V_i(x) \neq \mathcal{L}_{f,\sigma}V_j(x), \forall x \in \Lambda_{ij}, \forall i \neq j \in \{1, ..., r\}$*, since the boundary* Λ *satisfies:* $\Lambda = \bigcup_{i\neq j}^{r} \Lambda_{ij} \subset \bigcup_{i\neq j}^{r} \{x \in \mathbb{R}^n : V_i(x) = V_j(x)\}$*. (See Step 1 in*

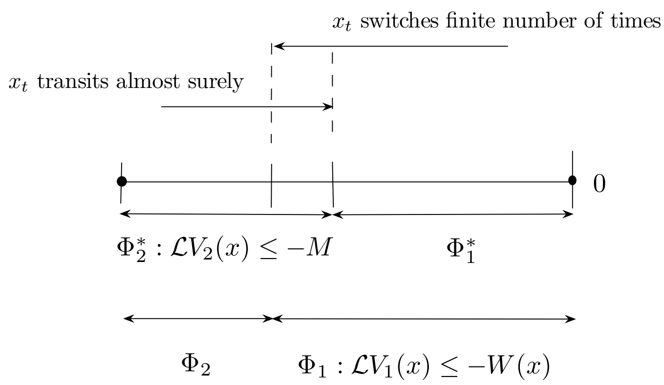


Fig. 3.1: Illustration of Discontinuous Lyapunov-like theorem.

the proof of Theorem 4.1*, Chapter* 4.*)*

3.3.2 Discontinuous Lyapunov-like Theorem for Stability in Probability

This section presents a discontinuous Lyapunov-like theorem for stability in probability of the stochastic nonlinear system (3.1). In this theorem, the Lyapunov function is constructed from a couple of smooth functions via a partition of the state space. Interestingly, these smooth functions are not necessarily equal at the boundary between two regions as in Definition 3.2. The discontinuous Lyapunov-like theorem exploits the observation, showed in Fig. 3.1, that if there exists a partition $\{\Phi_i\}_{i=1,2}$ of the state space such that in the region Φ_2, there exists a Lyapunov function $V_2(x)$ of which the infinitesimal $\mathcal{L}V_2(x)$ is strictly negative, then any trajectory x_t of (3.1) transits to Φ_1 in a finite time almost surely. In Φ_1, if there exists a Lyapunov function $V_1(x)$ such that $\mathcal{L}V_1(x)$ is negative definite, then the trajectory x_t of (3.1) converges to the origin almost surely. Remarkably, as $V_1(x)$ and $V_2(x)$ are not required to be equal at the boundary of Φ_1 and Φ_2, the overall Lyapunov function is not necessarily continuous, making the theorem convenient in applications.

Theorem 3.3. *For the stochastic systems described by* (3.1)*, consider the Lyapunov function*

$$V(x) = \begin{cases} V_1(x), & x \in \Phi_1, \\ V_2(x), & x \in \Phi_2, \end{cases} \tag{3.18}$$

where V_1 and V_2 are twice continuously differentiable functions defined on $\mathbb{R}^n$ and $\{\Phi_i\}_{i=1,2}$ is a partition of the state space $\mathbb{R}^n$ such that the origin is in the interior of Φ_1. If there exist a closed set $\Phi_1^* \subset \Phi_1$ containing the origin, a positive constant M, and a continuous, non-negative function $W(x)$ such that

C.1 $V_i(x) \geq 0, i = 1, 2, \forall x \in \mathbb{R}^n$,

C.2 $\mathcal{L}_{f,\sigma} V_1(x) \leq -W(x), \forall x \in \Phi_1$,

C.3 $\mathcal{L}_{f,\sigma} V_2(x) \leq -M, \forall x \in \mathbb{R}^n \setminus \Phi_1^* := \Phi_2^*$, *and*

C.4 $\sup_{x \in \Phi_1^*} V_1(x) < \inf_{x \in \Lambda} V_1(x)$ *where* $\Lambda := \bar{\Phi}_1 \cap \bar{\Phi}_2$.

Then, the following statements hold:

S.1 From any initial state x_0, after a finite time, the trajectory x_t evolves in Φ_1 permanently.

S.2 The equilibrium $x = 0$ of the system (3.1) *is globally stable in probability and*

$$\mathbb{P}\{\lim_{t \to \infty} W(x_t) = 0\} = 1. \tag{3.19}$$

S.3 If, in addition, $W(x)$ is positive definite in $\bar{\Phi}_1$, then the equilibrium $x = 0$ of the system (3.1) *is globally asymptotically stable in probability.*

Proof: We note that Theorems 2.1 and 2.2 in [63] still hold true when the initial state is a random variable. From these theorems, Statements S.2 and S.3 are corollaries of Statement S.1 and Conditions C.1 and C.2. As such, it is sufficient to prove S.1 only.

Step 1: Firstly, we prove that any trajectory x_t, beginning at x_0, will transit to Φ_1^* in a finite time almost surely. This is obvious if $x_0 \in \Phi_1^*$. Consider the case when $x_0 \in \mathbb{R}^n \setminus \Phi_1^*$ and let $\tau_1(x_0)$ be the first time x_t exits $\mathbb{R}^n \setminus \Phi_1^*$. Applying the Dynkin's formula and combining with Condition C.3, we obtain

$$\begin{aligned}\mathbb{E}[V_2(x_{\tau_1(x_0)})] - V_2(x_0) &= \mathbb{E}[\int_0^{\tau_1(x_0)} \mathcal{L}_{f,\sigma} V_2(x_s) ds] \\ &\leq -M\mathbb{E}[\tau_1(x_0)]. \end{aligned} \tag{3.20}$$

As such, $\mathbb{E}[\tau_1(x_0)] \leq \dfrac{V_2(x_0)}{M} < \infty$, which means that x_t transits to Φ_1^* in a finite time almost surely.

Step 2: Now, we consider the trajectory x_t that begins at $x_0 \in \Phi_1^*$. We will prove that there are almost surely a finite number of switches of the trajectory x_t from Φ_1^* to Φ_2. With x_t beginning at $x_0 \in \Phi_1^*$, there are two probabilities:

$\mathcal{P}_1$: x_t evolves in Φ_1 permanently.

$\mathcal{P}_2$: After a finite time, x_t transits to Φ_2.

Consider $\omega \in \mathcal{P}_2$ and denote by $\tau_2(x_0)$ the first time x_t transits to Φ_2. Note that in Φ_1, we have $\mathcal{L}_{f,\sigma} V_1(x) \leq -W(x) \leq 0$. This, together with Dynkin's formula, leads to

$$\begin{aligned}\mathbb{E}[V_1(x_{\tau_2(x_0)})] &= V_1(x_0) + \int_0^{\tau_2(x_0)} \mathcal{L}_{f,\sigma} V_1(x_s) ds \\ &\leq V_1(x_0). \end{aligned} \tag{3.21}$$

Denote $\alpha = \sup_{x \in \Phi_1^*} V_1(x)$ and $\beta = \inf_{x \in \Lambda} V_1(x)$. Applying Chebyshev's inequality and from (3.21), we have $\mathbb{P}\{\sup_{0 \leq s \leq \tau_2(x_0)} V_1(x_s) \geq \beta\} \leq \dfrac{\mathbb{E}[V_1(x_{\tau_2(x_0)})]}{\beta} \leq \dfrac{V_1(x_0)}{\beta} \leq \dfrac{\alpha}{\beta}$. This implies that

$\mathbb{P}\{\mathcal{P}_2\} \leq \dfrac{\alpha}{\beta}$. Denote the events:

$$\mathcal{A}_n = \big\{x_t \text{ begins at } x_0 \in \Phi_1^* \text{ and transits to } \Phi_2 \ n \text{ times}\big\}$$

with $n = 1, 2, ...$ Then, it holds that $\mathbb{P}\{\mathcal{A}_n\} \leq \left(\dfrac{\alpha}{\beta}\right)^n$. As in Condition C.4, $\alpha < \beta$. Therefore

$$\sum_{n=1}^{\infty} \mathbb{P}\{\mathcal{A}_n\} \leq \sum_{n=1}^{\infty} \left(\frac{\alpha}{\beta}\right)^n < \infty. \tag{3.22}$$

By Borel-Cantelli Lemma, there are almost surely a finite number of switches of the trajectory x_t from Φ_1^* to Φ_2.

Combining two steps, we conclude that any trajectory x_t transits to Φ_1^* in a finite time almost surely and from Φ_1^*, it only switches to Φ_2 almost surely a finite number of times. As such, after a finite time, x_t evolves in Φ_1 permanently, and thus, Statement S.1 is proved. □

3.3.3 $1-$Time Switching Lyapunov-like Theorem for Stability in Probability

To facilitate the design for control of entanglement in Chapter 6, in this section, we provide a non-smooth Lyapunov-like theorem for stability in probability of generic stochastic nonlinear systems. In the non-smooth Lyapunov-like theorems in Sections 3.3.1 and 3.3.2, the switching-number of system state between desired space and undesired space is uncontrollable, resulting in long-time convergence of system state. However, due to the fragileness of entanglement under environmental effect [68], it it desirable to drive the system state to the desired entangled state as fast as possible, before the environment takes dominant effect. Therefore, in the practical control of entanglement, the above feature of switching-number uncontrollability should be removed.

In this section, we introduce an $1-$time switching Lyapunov-like theorem for stability in probability, which assures that the system state switches between desired space and undesired space no more than one time, and thus eliminates the switching-number uncontrollability. Based on this new theorem, the non-smooth control essentially reduces the converge time of system state, while still guaranteeing that the system state deterministically converges to the desired entangled state from any initial state.

Theorem 3.4. *Consider the stochastic systems described by* (3.1) *and the Lyapunov function*

$$V(x) = \begin{cases} V_1(x), & x \in \Phi_1, \\ V_2(x), & x \in \Phi_2, \end{cases} \tag{3.23}$$

where $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}^+$ *are positive, twice continuously differentiable functions and* $\Phi_1 := \{x \in \mathbb{R}^n : V_1(\rho) \leq k\}$ *for some positive real number* k, $\Phi_2 := \mathbb{R}^n \setminus \Phi_1$. *If there exist a positive constant* M, $\mathcal{K}-$*class functions* α_1, α_2, *and a continuous function* $W(x)$ *positive definite on* Φ_1 *such that*

C.1 $\alpha_1(|x|) \leq V_1(x) \leq \alpha_2(|x|), \forall x \in \mathbb{R}^n$,

C.2 $\mathcal{L}_{f,\sigma} V_1(x) \leq -W(x), \forall x \in \Phi_1$,

C.3 $\mathcal{L}_{f,\sigma} V_2(x) \leq -M, \forall x \in \Phi_2$,

where $\mathcal{L}_{f,\sigma} V$ *is the infinitesimal generator associated with* (3.1)*:*

$$\mathcal{L}_{f,\sigma} V(x) := \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \mathrm{Tr}\big(\sigma(x)^T \frac{\partial^2 V(x)}{\partial x^2} \sigma(x)\big).$$

Then, the equilibrium $x = 0$ *of the system* (3.1) *is globally asymptotically stable in probability.*

Proof: Firstly, we prove that there is no sliding motion of the system trajectory on the boundary $\Lambda := \bar{\Phi}_1 \cap \bar{\Phi}_2$ between Φ_1 and Φ_2. Indeed, as Λ is described by

$$\Lambda := \{x \in \mathbb{R}^n : V_1(x) = k\}, \tag{3.24}$$

if there exists sliding motion of x_t on Λ, then, there exist some $x_\tau \in \Lambda$ such that

$$\mathcal{L}_{f,\sigma} V_1(x_\tau) = 0 \tag{3.25}$$

Note that $x_\tau \in \Lambda \subset \Phi_1$. As such, by Condition C.2, we have

$$\mathcal{L}_{f,\sigma} V_1(x_\tau) \leq -W(x_\tau) \tag{3.26}$$

Since the function $W(x)$ is positive definite on Φ_1 and the origin is not on Λ, it holds that $W(x) > 0, \forall x \in \Lambda$. Therefore,

$$\mathcal{L}_{f,\sigma} V_1(x_\tau) \leq -W(x_\tau) < 0 \tag{3.27}$$

which is contradict with (3.25).

Now, we prove that from any initial state x_0, after a finite time, the system trajectory x_t will permanently evolve in Φ_1 almost surely.

Case 1: $x_0 \in \Phi_2$. Let $\tau_1(x_0)$ be the first time x_t at which exits the open set Φ_2. Applying the Dynkin's formula and combining with Condition C.3, we obtain

$$\begin{aligned}\mathbb{E}[V_2(x_{\tau_1(x_0)})] - V_2(x_0) &= \mathbb{E}[\int_0^{\tau_1(x_0)} \mathcal{L}_{f,\sigma} V_2(x_s) ds] \\ &\leq -M\mathbb{E}[\tau_1(x_0)]. \end{aligned} \tag{3.28}$$

As such, $\mathbb{E}[\tau_1(x_0)] \leq \dfrac{V_2(x_0)}{M} < \infty$, which means that x_t exits Φ_2 and transits to Φ_1 in a finite time almost surely.

As there is no sliding motion of x_t on the boundary Λ, there is a time instant $\tau_2(x_0)$ at which x_t meets the boundary Λ of the sets Φ_1 and Φ_2, and after the time instant $\tau_2(x_0)$, the system trajectory x_t will enter the interior of the set Φ_1. Note that there may exist the case that the system trajectory from the set Φ_2 will tend to be tangent to the boundary Λ and then come back the set Φ_2. However, by the analysis similar to (3.28), after a finite time, the system trajectory x_t must enter the interior of Φ_1, from which $\tau_2(x_0)$ is well defined.

We prove that after the time $\tau_2(x_0), x_t$ will evolve in Φ_1 permanently. Define the set

$$\begin{aligned}\mathcal{A} := \{\omega : &\textit{ after the time } \tau_2(x_0), x_t \textit{ enters the interior of} \\ &\Phi_1 \textit{ and then comes back } \Phi_2\}\end{aligned} \tag{3.29}$$

Consider $\omega \in \mathcal{A}$ and let $\tau_3(x_0)$ be the first time after $\tau_2(x_0)$ at which x_t meets the boundary Λ, i.e., $\tau_3(x_0)$ is the first time after $\tau_2(x_0)$ that $x_{\tau_3(x_0)}$ exits the open set $\mathring{\Phi}_1$, which is the interior of Φ_1. For $0 < \mu < k$, we define the compact set

$$\Lambda_\mu := \{x \in \mathbb{R}^n : \mu \leq V_1(x) \leq k\} \tag{3.30}$$

Note that $0 \notin \Lambda_\mu$ and $\Lambda \subset \Lambda_\mu \subset \Phi_1$. As the continuous function $W(x)$ is positive definite on the set Φ_1, there exists

$$W_m := \min_{x \in \Lambda_\mu} W(x) > 0. \tag{3.31}$$

Let $\delta(x_0)$ be the length of duration between $\tau_2(x_0)$ and $\tau_3(x_0)$ in which x_t is in the set Λ_μ. By definition, we have $\delta(x_0) > 0$ for all sample paths that $\omega \in \mathcal{A}$. We define the sets

$$\mathcal{A}_n := \{\omega \in \mathcal{A} : \delta(x_0) \geq \frac{1}{n}\}, n = 1, 2, ... \tag{3.32}$$

Then, for all $\omega \in \mathcal{A}_n$, from (3.31), we have

$$\int_{\tau_2(x_0)}^{\tau_3(x_0)} W(x_s)ds \geq \delta(x_0)W_m \geq \frac{1}{n}W_m > 0. \tag{3.33}$$

On the other hand, applying the Dynkin's formula and combining with Condition C.2, we achieve

$$\begin{aligned}
&\mathbb{E}[I_{\mathcal{A}_n} V_1(x_{\tau_3(x_0)})] - \mathbb{E}[I_{\mathcal{A}_n} V_1(x_{\tau_2(x_0)})] \\
&= \mathbb{E}[I_{\mathcal{A}_n} \int_{\tau_2(x_0)}^{\tau_3(x_0)} \mathcal{L}_{f,\sigma} V_1(x_s)ds] \\
&\leq -\mathbb{E}[I_{\mathcal{A}_n} \int_{\tau_2(x_0)}^{\tau_3(x_0)} W(x_s)ds]
\end{aligned} \tag{3.34}$$

where $I_{\mathcal{A}_n}$ is the indicator function of the set $\mathcal{A}_n$. Note that $V_1(x) = k, \forall x \in \Lambda$. As such $V_1(x_{\tau_2(x_0)}) = V_1(x_{\tau_3(x_0)}) = k$. From (3.33) and (3.34), we have

$$0 \leq -\mathbb{E}[I_{\mathcal{A}_n} \int_{\tau_2(x_0)}^{\tau_3(x_0)} W(x_s)ds] \leq -\mathbb{E}[I_{\mathcal{A}_n}]\frac{1}{n}W_m, \tag{3.35}$$

which implies that $\mathbb{E}[I_{\mathcal{A}_n}] = 0$. Therefore, for any $n = 1, 2, ...$, there is almost surely no sample path in which $\delta(x_0) \geq 1/n$. Letting $n \to \infty$, we conclude that there is almost surely no sample path that enters the interior of the set Φ_1 and then comes back the set Φ_2. As such, after the time $\tau_2(x_0)$, the system trajectory x_t will evolve in Φ_1 permanently.

Case 2: $x_0 \in \Phi_1$. If after a finite time, x_t transits to the set Φ_2 then, by similar analysis as above, we conclude that after a finite time, x_t will evolve in Φ_1 permanently. Otherwise, x_t also evolves in Φ_1 permanently.

Combining two cases, it follows that after a finite time, x_t will permanently evolve in Φ_1 almost surely. From Conditions C.1, C.2, and the positive definiteness of the continuous function $W(x)$ on the set Φ_1, applying Theorem 2.2 in [63], we conclude that the origin is globally asymptotically stable in probability. Theorem 3.4 is proved. □

Remark 3.3. *It can be seen from the proof of Theorem* 3.4 *that, by its conditions, Theorem* 3.4 *guarantees that the system trajectory switches no more than one time between the sets* Φ_1 *and* Φ_2. *This differentiates it from the non-smooth control design in [57, 69, 70] and the non-smooth Lyapunov-like theorems in Sections 3.3.1 and 3.3.2, where the switching-number of system state between the desired space and undesired space is uncontrollable. Therefore, though the proof of Theorem* 3.4 *is inspired by the that of control design in [57, 69, 70] and the proof of Theorem* 3.3*, it enables an important feature of controllable switching-number which will be instrumental in controlling of entanglement in Chapter 6. In comparison to Theorem* 3.3*, we can also observe that Condition* C.4 *of Theorem* 3.3 *is no longer needed in Theorem* 3.4*, and the feature of* $1-$*time switching is guaranteed via exploiting the specific form of the set* Φ_1 *as a level set of the corresponding Lyapunov function* $V_1(x)$.

3.4 Conclusions

The non-smooth Lyapunov-like theory for stability analysis of generic stochastic nonlinear systems has been presented. Exploiting the observation that the stability is achieved, if the Lyapunov function is continuous and decreasing at it smooth segments, a continuous Lyapunov-like theorem for stability in probability has been introduced. A discontinuous Lyapunov-like theorem for stability in probability of generic stochastic nonlinear systems has been also presented. To eliminate the uncontrollable switching numbers in these theorems, we introduced the 1-time switching Lyapunov-like theorem. This theorem enabled the short time convergence of system state, which is essential in the control of entanglement.

Chapter 4

Global Stabilization

4.1 Introduction

In this chapter, we are interested in the problem of global stabilization for quantum filters. The main challenge of this problem arises from the symmetric topology of the manifold on which the filter state involves. Generally, the symmetric topology of filter state space makes the smooth controls [34, 35, 53], synthesized via the classical Lyapunov stochastic stability theory [54–56], difficult, if not impossible, to obtain the global stabilizability for a given desired state. The underlying reason is due to the existence of the so-called *antipodal eigenstates* which, together with the final desired state, are equilibrium points of the closed-loop systems; see [15] for the origin of the term *antipodal eigenstates* from deterministic case. As such, the global stabilization for quantum filters calls for a non-smooth control design method to deal with the symmetric topology of filter state space.

In [34], after the introduction of quantum filters, the global stabilizability was shown to be difficult to obtain even for 2-dimensional quantum filters. Continuous control was then proposed, through computer search, to break the symmetric topology of 2-dimensional filter state space and to globally stabilize the desired state. The design method is, however, computationally involved and the global stability is hard to prove as the design method is not analytical [57]. In [57], control in hysteresis form was introduced to globally drive the expectation of filter state of angular momentum systems to the final desired state, in which the state convergence was proved based on a detailed analysis of the sample paths of filter state.

In this chapter, applying the non-smooth Lyapunov-like theory, non-smooth controls are synthesized to solve the problem of global stabilization for quantum filters. Firstly, based on the continuous Lyapunov-like theorem, switching control is designed to globally asymptotically drive the filter state to the final desired state almost surely. We utilize a continuous Lyapunov function in minimum form to break the symmetric topology of filter state space. The control design is based on a partition of the filter state space with consideration of the sliding motion of filter state on the boundary among the regions. Secondly, based on the discontinuous Lyapunov-like theorem, continuous control in saturation-form is presented. For simplicity, we consider 2-dimensional quantum filters and prove that the proposed control globally asymptotically renders the 2-dimensional filter state to the final desired state almost surely.

In both control designs, the eigenstate-transferring of quantum filters is achieved as a special consequence of global stabilization for quantum filters. This points out the advantage of the non-smooth Lyapunov-based controls over the smooth control approaches for quantum filters [34, 35, 53]. In addition, the fact that control performance is analyzed based on a Lyapunov-like theory distinguishes our approach from the sample path analysis approach in [57]. Moreover, the use of Lyapunov-based theory generally provides a great potential to deal with other control problems such as robust control [71], which is essential when the system dynamics is uncertain. These features together assert the uniqueness and advantages of the non-smooth Lyapunov-based approach for the control of quantum filters.

In Section 4.2, we present the system description and the problem formulation. The necessity of non-smooth Lyapunov theory for global stabilization of quantum filters is analyzed in Section 4.3. The switching control and the saturation-form control designs for quantum filters are then presented in Section 4.4. We show explicitly that these controls globally asymptotically render the filter state to the final desired state almost surely. In Section 4.5, the effectiveness of the proposed controls is demonstrated through the control design for the Spin$-1/2$ systems. Section 4.6 includes concluding remarks.

4.2 Preliminaries

4.2.1 System Description

Consider the finite-dimensional quantum filters whose state is represented by the density matrix ρ evolving on the space

$$\mathcal{S} := \{\rho \in \mathbb{C}^{n\times n} : \rho = \rho^\dagger \geq 0, \mathrm{Tr}(\rho) = 1\}, \tag{4.1}$$

and the time evolution of quantum filter state is described, in units such that $\hbar = 1$, by the following stochastic master equation (SME) [34]

$$\begin{aligned} d\rho_t = &\big(- \mathrm{i}[H_0, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}L^\dagger L\rho_t - \frac{1}{2}\rho_t L^\dagger L\big)dt \\ &- \mathrm{i}[H_1, \rho_t]u_t dt \\ &+ \sqrt{\eta}\big(L\rho_t + \rho_t L^\dagger - \mathrm{Tr}(L\rho_t + \rho_t L^\dagger)\rho_t\big)dw_t \\ &:= f(\rho_t)dt + g(\rho_t)u_t dt + \sigma(\rho_t)dw_t, \end{aligned} \tag{4.2}$$

where

- w_t is the 1$-$dimensional Wiener process defined on the classical complete probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, in which Ω is a sample space, $\mathcal{F}$ is a $\sigma-$field, and $\mathbb{P}$ is a probability measure;

- H_0, H_1 are $n \times n$ Hermitian matrices (or self-adjoint matrices) with entries in $\mathbb{C}$;
- $L \in \mathbb{C}^{n\times n}, \eta \in (0,1]$, and $u_t \in \mathbb{R}$.

The quantum filter (4.2) describes the evolution of conditional state (filter state) of the quantum system with free Hamiltonian H_0, subjected to the continuous measurement of the observable L with the measurement efficiency η, and the coherent control given by the control input u_t and the control Hamiltonian H_1. As $\mathcal{S}$ is an invariant set of (4.2) [34], it is a natural state space of the filter (4.2).

4.2.2 Problem Formulation

From the applications in quantum chemistry and atomic physics, the problem of transferring a quantum system from initial states to desired states is of importance. We have the following control problem:

($\mathcal{P}1$) **Global stabilization by state-feedback:** *design a control law of the form $u_t = u(\rho_t)$ to globally asymptotically render the quantum filter* (4.2) *from any initial state to the final desired state ρ_f almost surely, i.e.,*

$$\mathbb{P}\{\lim_{t\to\infty} \rho_t = \rho_f\} = 1, \forall \rho_0 \in \mathcal{S}. \tag{4.3}$$

Towards a solution to the problem ($\mathcal{P}1$), we consider the following assumptions made on the system (4.2).

Assumption 4.1. *The final desired state ρ_f is an eigenstate of the measurement operator L, i.e., $\rho_f = \psi_f\psi_f^\dagger$ where ψ_f is an eigenvector of L :*

$$L\psi_f = \lambda_f\psi_f. \tag{4.4}$$

Remark 4.1. *It was shown in [34] that under continuous measurement, the filter state* *converges to one of eigenstates of L in an asymptotic manner. This phenomenon was termed* quantum state reduction *due to its compatibility with the usual quantum state reduction postulate [46] in quantum mechanics which states that quantum system state jumps to one of eigenstates of L under projective (discrete time) measurement. From this view, Assumption* 4.1 *is physically plausible.*

Assumption 4.2. *The measurement operator L is self-adjoint, i.e., $L = L^\dagger$, and regular, i.e., the eigenvalues of L are different. The system free Hamiltonian H_0 and L are commutative. As H_0 and L are commutative, we can choose a basis in which H_0 and L are diagonal.*

Remark 4.2. *Assumption* 4.2 *is to guarantee that ρ_f is an equilibrium point of the stochastic master equation* (4.2) *without the control input, i.e., $u = 0$. This assumption is practically reasonable as in many experiment settings, e.g., trapping a cold atomic ensemble in an optical cavity [34, 35, 72], H_0 and L are chosen to be diagonal and L regular. The case when L is not self-adjoint is also of interest; see [73].*

Assumption 4.3. *In the basis that H_0 and L are diagonalized, $H_1 = [h_{kl}]_{n\times n}$ is connected, i.e., $h_{i(i+1)} \neq 0$ for all $i = 1, .., n-1$.*

Lemma 4.1. *Under Assumptions* 4.2 *and* 4.3*, there does not exist $\rho \in \mathcal{S}$ that is an equilibrium of* (4.2) *for all u.*

Proof: Suppose that ρ_e is an equilibrium of (4.2) for all u. Then, $f(\rho_e) = g(\rho_e) = \sigma(\rho_e) = 0$. Multiplying both sides of the equation $f(\rho_e) = 0$ with ρ_e and taking trace of both sides, we obtain $\left|[L, \rho_e]\right|^2 = 0$. It follows that $[L, \rho_e] = 0$. Since L is diagonal and regular, ρ_e must be diagonal. As $g(\rho_e) = 0$ and H_1 is connected, it must hold that $\rho_e = aI_n$ for some constant a. Since $\mathrm{Tr}(\rho_e) = 1$, it follows that $\rho_e = \frac{1}{n} I_n$. Then, $\sigma(\rho_e)$ differs from 0 as L is regular, showing the contradiction. $\square$

Lemma 4.1 guarantees the solvability of the problem of global stabilization for quantum filter (4.2). The class of quantum filters (4.2) with Assumptions 4.1, 4.2, and 4.3

encompasses the angular momentum systems considered in [34, 57, 70].

4.3 Non-smooth Lyapunov Necessity for Global Stabilization of Quantum Filters

In this section, we review the smooth controls synthesized via smooth Lyapunov theory [54–56], which was investigated in [34, 35, 53]. We point out that such smooth controls are difficult to obtain the global stabilizability for quantum filters because of the intrinsic symmetric topology of filter state space. This highlights the importance of non-smooth Lyapunov-like theory for the global stabilization of quantum filters.

Consider the smooth Lyapunov function candidate of the Frobenius norm/variance form [53]:

$$V_S(\rho) = 1 - \mathrm{Tr}(\rho\rho_f) + cU(\rho) \tag{4.5}$$

where $U(\rho) := \mathrm{Tr}(L^2\rho) - \mathrm{Tr}^2(L\rho)$ is the variance of filtering process along the measurement operator L [46] and $c > 0$ is a constant. From Assumption 4.2, a straightforward computation (see Section 1, Appendix A) gives the infinitesimal generator of ρ_t acting on $U(\rho)$ and $V_S(\rho)$:

$$\mathcal{L}_{f+gu,\sigma}U(\rho_t) = g_U(\rho_t)u - 4\eta U(\rho_t)^2, \tag{4.6a}$$

$$\mathcal{L}_{f+gu,\sigma}V_S(\rho_t) = g_S(\rho_t)u - 4c\eta U(\rho_t)^2, \tag{4.6b}$$

where $\mathcal{L}_{f+gu,\sigma}V$ is the infinitesimal generator of function V associated with (4.2):

$$\mathcal{L}_{f+gu,\sigma}V := \frac{\partial V}{\partial \rho}(f(\rho) + g(\rho)u) + \frac{1}{2}\mathrm{Tr}(\sigma^T(\rho)\frac{\partial^2 V}{\partial \rho^2}\sigma(\rho)),$$

and

$$\begin{aligned} g_U(\rho) &= \mathrm{Tr}\big(-\mathrm{i}[H_1,\rho](L^2 - 2L\mathrm{Tr}(L\rho))\big), \\ g_S(\rho) &= \mathrm{Tr}\big(-\mathrm{i}[H_1,\rho](c(L^2 - 2L\mathrm{Tr}(L\rho)) - \rho_f)\big). \end{aligned}$$

With the natural smooth state-feedback control law $u_t = u_S(\rho_t) = -g_S(\rho_t)$ as in [53], the infinitesimal generator of $V_S(\rho)$ along (4.2) becomes

$$\mathcal{L}_{f+gu,\sigma} V_S(\rho_t) = -g_S(\rho_t)^2 - 4c\eta U(\rho_t)^2 \leq 0. \tag{4.7}$$

According to Theorem 2.1 in [63], the filter state almost surely converges to

$$\lim_{t\to\infty} \big(g_S(\rho_t)^2 + 4c\eta U(\rho_t)^2\big) = 0. \tag{4.8}$$

As L is diagonal and regular, $\big(g_S(\rho)^2+4c\eta U(\rho)^2\big) = 0$ iff ρ is an eigenstate of L (see Section 2, Appendix A). As such, (4.8) means that ρ_t almost surely converges to one of the eigenstates of L, which, as L is diagonal and regular, include $\rho_i = \mathrm{diag}\{0, \dots, \underbrace{1}_{i-th}, \dots, 0\},\ i = 1, \dots, n.$ Note that these eigenstates are equilibrium points of the closed-loop systems. Consequently, the smooth control $u_t = u_S(\rho_t)$ fails to render the filter from the *antipodal eigenstates* of ρ_f, i.e., the other eigenstates $\rho_i \neq \rho_f$ of L, to ρ_f. Therefore, the smooth control designed via the smooth Lyapunov stability theory is not sufficient to obtain the global stabilizability for quantum filters. As such, the problem of global stabilization by state-feedback for quantum filters intuitively calls for a non-smooth Lyapunov-based control approach which will be presented in the next section.

4.4 Nonsmooth Lyapunov Function-Based Global Stabilization

4.4.1 Continuous Lyapunov Function-Based Global Stabilization

In this section, we present details of our approach to solution of the problem ($\mathcal{P}1$). Basically, we design the control such that the closed-loop system fulfills conditions of Theorem 3.2 in Chapter 3. The control design is based on a partition of the filter state space into three regions with consideration of the sliding motion of filter state on the boundary among the regions.

1. Control Design

In order to break the symmetric topology of the filter state space, we shall choose the Lyapunov function $V(\rho)$ smooth around the eigenstates of L such that the coefficient of u in $\mathcal{L}_{f+gu}V(\rho)$ is equal to 0 at ρ_f, while being different from 0 at all antipodal eigenstates ρ_i of ρ_f, i.e.,

$$\frac{\partial V(\rho)}{\partial \rho} g(\rho)\Big|_{\rho=\rho_f} = 0, \tag{4.9a}$$

$$\frac{\partial V(\rho)}{\partial \rho} g(\rho)\Big|_{\rho=\rho_i} \neq 0, \forall \rho_i \neq \rho_f. \tag{4.9b}$$

Condition (4.9b) can be fulfilled via the following lemma.

Lemma 4.2. *Under Assumption* 4.3*, there exists* $X = [x_{ij}]_{n\times n} \in \mathbb{C}^{n\times n}$ *self-adjoint and off-diagonal, i.e.,* $x_{ii} = 0, \forall i = 1, .., n$*, such that*

$$-\mathrm{i}[X, H_1] = A = [a_{ij}]_{n\times n}, \tag{4.10}$$

in which $a_{ii} \in \mathbb{R} \setminus \{0\}, \forall i = 1, ..., n$.

Proof: See Section 3, Appendix A. □

A corollary of Lemma 4.2 is that the coefficient of u in $\mathcal{L}_{f+gu}\mathrm{Tr}(X\rho)$ is different from 0 at all antipodal eigenstates ρ_i of ρ_f :

$$\mathrm{Tr}(-\mathrm{i}X[H_1,\rho_i]) = \mathrm{Tr}(A\rho_i) = a_{ii} \neq 0. \tag{4.11}$$

Noting this corollary, we construct a Lyapunov function candidate satisfying (4.9). Let

$$\begin{aligned} &0 < m < \min_{i=1,..,n} a_{ii}^2; 0 < a < \min_{i=1,..,n} a_{ii}^2 - m; \\ &M > \max_{i=1,..,n} a_{ii}^2. \end{aligned} \tag{4.12}$$

Consider the continuous Lyapunov function candidate in minimum form as follows:

$$\begin{aligned} V(\rho) = \min\Big\{&(M-m)(1-\mathrm{Tr}(\rho\rho_f)) + cU(\rho), \\ &M - \mathrm{Tr}(A\rho)^2 + |l\mathrm{Tr}(X\rho) - a| + cU(\rho)\Big\} \end{aligned} \tag{4.13}$$

where the constants $l > 0$ and $c > 0$ are chosen later. The Lyapunov function $V(\rho)$ can be described in the form (3.8):

$$V(\rho) = V_i(\rho), \quad \rho \in \Phi_i, i \in \{1,2,3\}, \tag{4.14}$$

with the smooth functions:

$$V_1(\rho) = (M-m)(1-\mathrm{Tr}(\rho\rho_f)) + cU(\rho), \tag{4.15a}$$

$$V_2(\rho) = M - \mathrm{Tr}(A\rho)^2 + l\mathrm{Tr}(X\rho) - a + cU(\rho), \tag{4.15b}$$

$$V_3(\rho) = M - \mathrm{Tr}(A\rho)^2 - l\mathrm{Tr}(X\rho) + a + cU(\rho), \tag{4.15c}$$

and the partition of the state space $\mathcal{S}$:

$$\begin{aligned}\Phi_1 = \Big\{\rho \in \mathcal{S} :&(M-m)(1-\mathrm{Tr}(\rho\rho_f)) \\ &< M - \mathrm{Tr}(A\rho)^2 + |l\mathrm{Tr}(X\rho) - a|\Big\}, \end{aligned} \tag{4.16a}$$

$$\begin{aligned}\Phi_2 = \Big\{\rho \in \mathcal{S} :&(M-m)(1-\mathrm{Tr}(\rho\rho_f)) \\ &\geq M - \mathrm{Tr}(A\rho)^2 + |l\mathrm{Tr}(X\rho) - a|, \\ l\mathrm{Tr}(X\rho) \geq a\Big\},& \end{aligned} \tag{4.16b}$$

$$\begin{aligned}\Phi_3 = \Big\{\rho \in \mathcal{S} :&(M-m)(1-\mathrm{Tr}(\rho\rho_f)) \\ &\geq M - \mathrm{Tr}(A\rho)^2 + |l\mathrm{Tr}(X\rho) - a|, \\ l\mathrm{Tr}(X\rho) < a\Big\}.& \end{aligned} \tag{4.16c}$$

In the sequel, a function $\phi(\rho)$ is called positive definite with respect to ρ_f on the set Φ containing ρ_f if $\phi(\rho) \geq 0, \forall \rho \in \Phi$, and $\phi(\rho_e) = 0, \rho_e \in \Phi$, iff $\rho_e = \rho_f$. Note that $U(\rho) \geq 0$ for all $\rho \in \mathcal{S}$ (see Section 2, Appendix A) and the function $(1 - \mathrm{Tr}(\rho\rho_f))$ is positive finite with respect to ρ_f on the set $\mathcal{S}$. As $\mathcal{S}$ is compact [34], with M chosen such that $M - \max\limits_{\rho\in\mathcal{S}} \mathrm{Tr}(A\rho)^2 > 0$, then $V(\rho)$ is positive definite with respect to ρ_f on the set $\mathcal{S}$.

As the solution X of (4.10) is off-diagonal, i.e., $x_{ii} = 0, \forall i = 1, .., n$, it follows that $\mathrm{Tr}(X\rho_i) = x_{ii} = 0$. Moreover, $\mathrm{Tr}(\rho_f^2) = 1$ and $\mathrm{Tr}(\rho_i\rho_f) = 0$ for all antipodal eigenstates ρ_i of ρ_f. Therefore, from (4.12) and (4.16), it must hold that $\rho_f \in \Phi_1$ and $\rho_i \in \Phi_3$ for all antipodal eigenstates ρ_i of ρ_f. In addition, there is no eigenstate of L on the boundary $\Lambda := \overset{3}{\underset{i\neq j}{\cup}} \Lambda_{ij}$ where $\Lambda_{ij} = \bar{\Phi}_i \cap \bar{\Phi}_j, i \neq j \in \{1, 2, 3\}$. Hence, ρ_f is in the interior of Φ_1 and all antipodal eigenstates ρ_i of ρ_f are in the interior of Φ_3.

A straightforward computation (see Section 4, Appendix A) gives the infinitesimal generator of ρ_t acting on $V_1(\rho), V_2(\rho)$, and $V_3(\rho)$:

$$\mathcal{L}_{f+gu,\sigma}V_1(\rho) = g_1(\rho)u - 4c\eta U(\rho)^2, \tag{4.17}$$

$$\mathcal{L}_{f+gu,\sigma}V_2(\rho) = g_2(\rho)u + h_2(\rho) - \mathrm{Tr}(A\sigma(\rho))^2 - 4c\eta U(\rho)^2, \tag{4.18}$$

$$\mathcal{L}_{f+gu,\sigma}V_3(\rho) = g_3(\rho)u + h_3(\rho) - \mathrm{Tr}(A\sigma(\rho))^2 - 4c\eta U(\rho)^2, \tag{4.19}$$

where

$$\begin{aligned}
g_1(\rho) &= \mathrm{Tr}(-\mathrm{i}[H_1,\rho](c(L^2 - 2L\mathrm{Tr}(L\rho)) - (M-m)\rho_f)),\\
g_2(\rho) &= l\mathrm{Tr}(A\rho) - 2\mathrm{Tr}(A\rho)\mathrm{Tr}(-\mathrm{i}[H_1,\rho]A)\\
&\quad + c\mathrm{Tr}\big(-\mathrm{i}[H_1,\rho](L^2 - 2L\mathrm{Tr}(L\rho))\big),\\
g_3(\rho) &= -l\mathrm{Tr}(A\rho) - 2\mathrm{Tr}(A\rho)\mathrm{Tr}(-\mathrm{i}[H_1,\rho]A)\\
&\quad + c\mathrm{Tr}\big(-\mathrm{i}[H_1,\rho](L^2 - 2L\mathrm{Tr}(L\rho))\big),\\
h_2(\rho) &= -2\mathrm{Tr}(A\rho)\mathrm{Tr}(Af(\rho)) + l\mathrm{Tr}(Xf(\rho)),\\
h_3(\rho) &= -2\mathrm{Tr}(A\rho)\mathrm{Tr}(Af(\rho)) - l\mathrm{Tr}(Xf(\rho)).
\end{aligned} \tag{4.20}$$

Equation (4.17) suggests us the control law $u = u_1(\rho) = -g_1(\rho), \rho \in \Phi_1$, which yields

$$\mathcal{L}_{f+gu,\sigma}V_1(\rho) = -g_1(\rho)^2 - 4c\eta U(\rho)^2 := -W_1(\rho), \tag{4.21}$$

for all ρ in Φ_1. We now design the control in Φ_2 and Φ_3. From (4.16b) and (4.16c), in Φ_2 and Φ_3, we have

$$M - m \geq (M-m)(1 - \mathrm{Tr}(\rho\rho_f)) \geq M - \mathrm{Tr}(A\rho)^2.$$

Hence, $|\mathrm{Tr}(A\rho)| \geq \sqrt{m}$ for all $\rho \in \Phi_2 \cup \Phi_3$. Let $c > 0$ and $l > 0$ such that

$$\begin{aligned} l\sqrt{m} > \max_{\rho\in\mathcal{S}} \Big| & 2\mathrm{Tr}(A\rho)\mathrm{Tr}(-\mathrm{i}[H_1,\rho]A) \\ & - c\mathrm{Tr}\big(-\mathrm{i}[H_1,\rho](L^2 - 2L\mathrm{Tr}(L\rho))\big)\Big| \\ + \max_{\rho\in\mathcal{S}} & |g_1(\rho)|. \end{aligned} \tag{4.22}$$

Therefore

$$\begin{aligned} l|\mathrm{Tr}(A\rho)| > \max_{\rho\in\mathcal{S}} \Big| & 2\mathrm{Tr}(A\rho)\mathrm{Tr}(-\mathrm{i}[H_1,\rho]A) \\ & - c\mathrm{Tr}\big(-\mathrm{i}[H_1,\rho](L^2 - 2L\mathrm{Tr}(L\rho))\big)\Big| \\ + \max_{\rho\in\mathcal{S}} & |g_1(\rho)|, \forall\rho \in \Phi_2 \cup \Phi_3. \end{aligned}$$

As such, it follows from (4.20) that $|g_2(\rho)| > |g_1(\rho)| \geq 0, \forall\rho \in \Phi_2$, and $|g_3(\rho)| > |g_1(\rho)| \geq 0, \forall\rho \in \Phi_3$. Hence, the coefficient of u in $\mathcal{L}_{f+gu}V(\rho)$ is different from zero for all ρ in $\Phi_2 \cup \Phi_3$, which containing all antipodal eigenstates of ρ_f. Consequently, (4.9b) is satisfied. In addition, the difference between the coefficient $g_1(\rho)$ of u in $\mathcal{L}_{f+gu,\sigma}V_1(\rho)$ and the coefficients $g_2(\rho), g_3(\rho)$ of u in $\mathcal{L}_{f+gu,\sigma}V_2(\rho), \mathcal{L}_{f+gu,\sigma}V_3(\rho)$ will be utilized to avoid the sliding motion of ρ_t on the boundary Λ.

As $g_2(\rho) \neq 0, \forall\rho \in \Phi_2$ and $g_3(\rho) \neq 0, \forall\rho \in \Phi_3$, in Φ_2 and Φ_3, we can choose the control laws:

$$u = u_2(\rho) = \frac{-h_2(\rho) - k}{g_2(\rho)}, k > 0, \forall\rho \in \Phi_2, \tag{4.23a}$$

$$u = u_3(\rho) = \frac{-h_3(\rho) - k}{g_3(\rho)}, k > 0, \forall\rho \in \Phi_3. \tag{4.23b}$$

Substituting (4.23a) into (4.18) and (4.23b) into (4.19), respectively, we obtain

$$\mathcal{L}_{f+gu,\sigma}V_2(\rho) = -k - \mathrm{Tr}(A\sigma(\rho))^2 - 4c\eta U(\rho)^2 < 0,$$
$$\forall \rho \in \Phi_2, \tag{4.24a}$$
$$\mathcal{L}_{f+gu,\sigma}V_3(\rho) = -k - \mathrm{Tr}(A\sigma(\rho))^2 - 4c\eta U(\rho)^2 < 0,$$
$$\forall \rho \in \Phi_3. \tag{4.24b}$$

Combining (4.21) and (4.24), we obtain the infinitesimal generator of ρ_t that acts on $V(\rho)$:

$$\mathcal{L}V(\rho) = \begin{cases} -W_1(\rho), & \rho \in \Phi_1, \\ -k - \mathrm{Tr}(A\sigma(\rho))^2 - 4c\eta U(\rho)^2, & \rho \in \Phi_2, \\ -k - \mathrm{Tr}(A\sigma(\rho))^2 - 4c\eta U(\rho)^2, & \rho \in \Phi_3. \end{cases} \tag{4.25}$$

2. Stability Analysis

In the sequel, we shall show that the control derived above solves the problem ($\mathcal{P}1$).

Theorem 4.1. *Consider the quantum filter* (4.2) *satisfying Assumptions* 4.1*,* 4.2*, and* 4.3*. Then, there exists* $k > 0$ *such that the following switching control renders the filter* (4.2) *from any initial state to the final desired state* ρ_f *almost surely:*

$$u(\rho) = \begin{cases} u_1(\rho) = -g_1(\rho), & \rho \in \Phi_1, \\ u_2(\rho) = (-h_2(\rho) - k)/g_2(\rho), & \rho \in \Phi_2, \\ u_3(\rho) = (-h_3(\rho) - k)/g_3(\rho), & \rho \in \Phi_3. \end{cases} \tag{4.26}$$

Proof: In order to apply Theorem 3.2 in Chapter 3, firstly, we fulfill Assumption 3.1, i.e., to make sure that there is no sliding motion of the system trajectory on the boundary $\Lambda := \bigcup_{i \neq j}^{3} \Lambda_{ij}$, where $\Lambda_{ij} = \bar{\Phi}_i \cap \bar{\Phi}_j, i \neq j \in \{1, 2, 3\}$. Secondly, we show that with each initial state, there exists a unique solution of the closed-loop systems. Lastly, we construct a continuous, non-negative function $W(\rho)$ such that all the conditions of Theorem 3.2 are satisfied.

Step 1: Avoiding the sliding motion of ρ_t on Λ.

The boundary Λ_{ij} between Φ_i and $\Phi_j, i \neq j \in \{1, 2, 3\}$ satisfies

$$\Lambda_{ij} \subset \{\rho \in \mathcal{S} : V_i(\rho) = V_j(\rho)\}. \tag{4.27}$$

Hence, if the system trajectory ρ_t slides on Λ_{ij}, then the infinitesimal generator $\mathcal{L}_{f+gu,\sigma}V_i(\rho_t)$ of V_i and the infinitesimal generator $\mathcal{L}_{f+gu,\sigma}V_j(\rho_t)$ of V_j are equal at some point $\rho_t \in \Lambda_{ij}$. As such, in order to guarantee that there is no sliding motion of ρ_t on Λ, we shall choose $k > 0$ such that for all $i \neq j \in \{1, 2, 3\}$,

$$\mathcal{L}_{f+gu,\sigma}V_i(\rho) \neq \mathcal{L}_{f+gu,\sigma}V_j(\rho), \forall \rho \in \Lambda_{ij}. \tag{4.28}$$

Notice that on the boundary Λ_{1j} between Φ_1 and $\Phi_j, j \in \{2, 3\}$, the control input $u = u_j(\rho)$ is applied because $\Lambda_{1j} \subset \Phi_j$. Hence, on Λ_{1j}, the condition (4.28) is equivalent with $\mathcal{L}_{f+gu_j,\sigma}V_1(\rho) \neq \mathcal{L}_{f+gu_j,\sigma}V_j(\rho), j \in \{2, 3\}$, or from (4.17), (4.18), and (4.19),

$$\begin{aligned} g_1(\rho)\frac{-h_j(\rho) - k}{g_j(\rho)} &\neq -k - \mathrm{Tr}(A\sigma(\rho))^2, \\ &\forall \rho \in \Lambda_{1j}, j \in \{2, 3\}. \end{aligned} \tag{4.29}$$

Indeed, as $|g_j(\rho)| > |g_1(\rho)|, \forall \rho \in \Phi_j, j \in \{2, 3\}$, there exists $k > 0$ such that for all $j \in \{2, 3\}$,

$$k\left(1 - \max_{\rho \in \Phi_j}\left|\frac{g_1(\rho)}{g_j(\rho)}\right|\right) > \max_{\rho \in \Phi_j}\left|\frac{g_1(\rho)h_j(\rho)}{g_j(\rho)} - \mathrm{Tr}(A\sigma(\rho))^2\right|. \tag{4.30}$$

Then, (4.29) holds true. Therefore, there is no sliding motion of ρ_t on Λ_{12} and Λ_{13}.

Similarly, on the boundary $\Lambda_{23} \subset \Phi_2$, as the control input $u = u_2(\rho)$ is applied, (4.28) is equivalent with $\mathcal{L}_{f+gu_2,\sigma}V_2(\rho) \neq \mathcal{L}_{f+gu_2,\sigma}V_3(\rho)$, or from (4.18) and (4.19),

$$g_3(\rho)\frac{-h_2(\rho)-k}{g_2(\rho)} + h_3(\rho) \neq -k, \forall \rho \in \Lambda_{23}. \tag{4.31}$$

Note $|g_2(\rho) - g_3(\rho)| = 2l|\mathrm{Tr}(A\rho)| \leq 2l\sqrt{m}, \forall \rho \in \Phi_2 \cup \Phi_3$. We choose k such that

$$2kl\sqrt{m} > \max_{\rho\in\Lambda_{23}} |g_3(\rho)h_2(\rho) - g_2(\rho)h_3(\rho)|. \tag{4.32}$$

Hence, $k\big|g_2(\rho)-g_3(\rho)\big| > \big|g_3(\rho)h_2(\rho)-g_2(\rho)h_3(\rho)\big|, \forall \rho \in \Lambda_{23}$, and (4.31) follows accordingly. As such, there is no sliding motion of ρ_t on the boundary Λ_{23}. Therefore, with k satisfies (4.30) and (4.32), Assumption 3.1 is fulfilled.

Step 2: Well-posedness.

This subsection proves the well-posedness of the closed-loop system with the proposed control. Since there is no sliding motion of ρ_t on Λ, the system trajectory ρ_t intersects Λ at separated time instants. As in the proof of Theorem 3.2, we denote by $\{\tau_i\}_{i=1,2,...}$ the sequence of stopping times at which ρ_t is on Λ. Let $\tau_0 = 0$. In each period $(\tau_i, \tau_{i+1}), i = 0, 1, ...$, one of the three smooth control laws $u_1(\rho)$, $u_2(\rho)$, and $u_3(\rho)$ is applied; we denote the applied control by $u_{q_i}(\rho), q_i \in \{1, 2, 3\}$. From Proposition 3.5 in [57], in each period $(\tau_i, \tau_{i+1}), i = 0, 1, ...$, with the smooth control $u_{q_i}(\rho)$ and the initial state ρ_{τ_i}, there exists a unique segment $\rho_t(\rho_{\tau_i}, u_{q_i})$ of the system (4.2). Moreover, the evolution of $\rho_t(\rho_{\tau_i}, u_{q_i})$ is on $\mathcal{S}$, which is a compact set. This implies that in each period $(\tau_i, \tau_{i+1}), i = 0, 1, ...$, the segment $\rho_t(\rho_{\tau_i}, u_{q_i})$ is bounded. By joining the segments $\rho_t(\rho_{\tau_i}, u_{q_i})$ in consecutive periods $(\tau_i, \tau_{i+1}), i = 0, 1, ...$, we conclude that with the control law (4.26), for each initial state $\rho_0 \in \mathcal{S}$, there exists a unique solution $\rho_t(\rho_0, u)$ of the system (4.2).

Step 3: Construction of the continuous, positive definite function $W(\rho)$.

In this subsection, we construct a continuous, positive definite function $W(\rho)$ satisfying Condition C.2 of Theorem 3.2, by which the global stability of closed-loop system is guaranteed

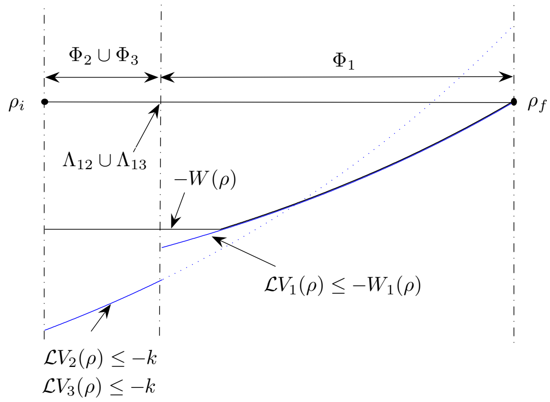


Fig. 4.1: Illustration of the construction of continuous, non-negative function $W(\rho)$.

without the use of some LaSalle's Invariance Principle, which is necessary in the case of semi-positive definite $W(\rho)$. The idea of constructing the continuous, positive definite function $W(\rho)$ is sketched in Fig. 4.1. As there is no eigenstate of L on the boundaries $\Lambda_{12} = \bar{\Phi}_1 \cap \bar{\Phi}_2$ and $\Lambda_{13} = \bar{\Phi}_1 \cap \bar{\Phi}_3$, we have $W_1(\rho) = g_1(\rho)^2 + 4c\eta U(\rho)^2 > 0, \forall \rho \in \Lambda_{12} \cup \Lambda_{13}$. Since $\Lambda_{12} \cup \Lambda_{13}$ is compact, there exists $\min\limits_{\rho \in \Lambda_{12} \cup \Lambda_{13}} W_1(\rho) > 0$. Let $0 < m_w \leq \min\{k, \min\limits_{\rho \in \Lambda_{12} \cup \Lambda_{13}} W_1(\rho)\}$. Then, the non-negative function

$$W(\rho) = \begin{cases} \min\left\{W_1(\rho), m_w\right\}, & \rho \in \Phi_1, \\ m_w, & \rho \in \Phi_2, \\ m_w, & \rho \in \Phi_3, \end{cases} \tag{4.33}$$

is continuous on the whole state space $\mathcal{S}$ and satisfies that

$$W(\rho) \leq \begin{cases} W_1(\rho), & \rho \in \Phi_1, \\ k, & \rho \in \Phi_2, \\ k, & \rho \in \Phi_3. \end{cases} \tag{4.34}$$

This, along with (4.25), leads to

$$\mathcal{L}_{f+gu,\sigma}V(\rho) \leq -W(\rho) \leq 0, \forall \rho \in \mathcal{S}. \tag{4.35}$$

Therefore, Condition C.2 of Theorem 3.2 is satisfied. Note that there is only one eigenstate of L in the set $\bar{\Phi}_1$, that is ρ_f. From Section 2, Appendix A, we conclude that $W_1(\rho_e) = 0$ with $\rho_e \in \bar{\Phi}_1$ only if $\rho_e = \rho_f$. As such, the function $W_1(\rho)$ is positive definite with respect to ρ_f on the set $\bar{\Phi}_1$. Consequently, the function $W(\rho)$ is positive definite with respect to ρ_f on the set $\mathcal{S}$. Applying Theorem 3.2, from (4.35), we conclude that the switching control (4.26) renders the quantum filter (4.2) from any initial state to the final desired state ρ_f almost surely. □

4.4.2 Discontinuous Lyapunov Function-Based Global Stabilization

In this section, we design continuous control in saturation-form, depicted in Fig. 4.2, via the discontinuous Lyapunov-like Theorem 3.3 in Chapter 3 to globally asymptotically render the filter (4.2) from any initial state to the final desired state almost surely. The idea of constructing such a continuous control comes from the observation that we can design switching control such that $\mathcal{L}_{f+gu,\sigma}V(\rho)$ is negative definite in Φ_1 and strictly negative in $\Phi_2 \cup \Phi_3$. Since $\mathcal{L}_{f+gu,\sigma}V(\rho)$ is strictly negative around the boundary Λ, we can transform the switching control in a small neighborhood of Λ to obtain saturation-form control such that $\mathcal{L}_{f+gu,\sigma}V(\rho)$ is still negative definite on the whole state space $\mathcal{S}$. As the control is continuous, it is easier to be implemented in practice. See [74] for another interesting method to transform the discontinuous controls of switched systems to continuous controls.

For the simplicity of presentation and technical proofs, we consider the specific case of 2-dimensional quantum filters. In addition, in Assumption 4.3, we further assume that $h_{11} = h_{22}$. Then, by similar proof with that of Lemma 4.2, there exists $X \in \mathbb{C}^{2\times 2}$ self-adjoint

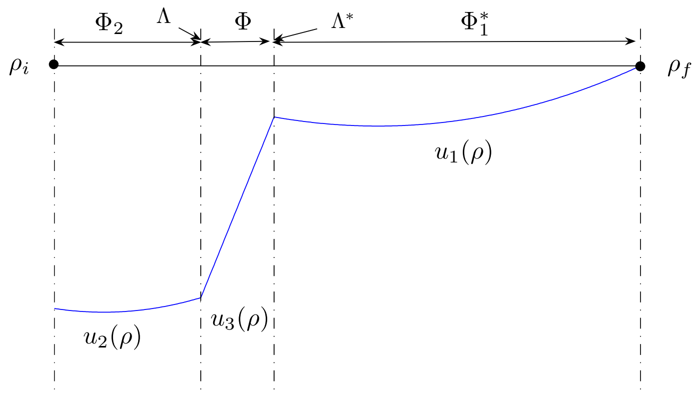


Fig. 4.2: Illustration of the saturation-form control.

and off-diagonal such that

$$-\mathrm{i}[X, H_1] = A = 4\rho_f - 2I_2. \tag{4.36}$$

Consider the Lyapunov function candidates:

$$U_1(\rho) = 1 - \mathrm{Tr}(\rho\rho_f) + cU(\rho), \tag{4.37a}$$

$$U_2(\rho) = h + \mathrm{Tr}(X\rho), \tag{4.37b}$$

where $h > \max_{\rho\in\mathcal{S}} |\mathrm{Tr}(X\rho)|$ and $c > 0$ is chosen such that

$$2cU(\rho) \leq \mathrm{Tr}(\rho\rho_f), \forall\rho \in \mathcal{S}. \tag{4.38}$$

This is possible as there exists $C > 0$ such that $U(\rho) = U(\rho) - U(\rho_a) \leq C(1 - \mathrm{Tr}(\rho\rho_a)) = C\mathrm{Tr}(\rho\rho_f), \forall\rho \in \mathcal{S}$, where ρ_a is the antipodal eigenstate of ρ_f. Hence, we can choose $c = 1/(2C)$. Due to the positivity of $U(\rho)$ and the positive definiteness of the function $(1 - \mathrm{Tr}(\rho\rho_f))$, the positivity of $U_1(\rho)$ is obvious. As $h > \max_{\rho\in\mathcal{S}} |\mathrm{Tr}(X\rho)|$, we have $U_2(\rho) > 0, \forall\rho \in \mathcal{S}$.

For $0 < \gamma < 1/8$, we define the sets:

$$\Phi_1 := \{\rho \in \mathcal{S} : U_1(\rho) < 1 - 1/8 + \gamma\}, \tag{4.39a}$$

$$\Phi_2 := \{\rho \in \mathcal{S} : U_1(\rho) \geq 1 - 1/8 + \gamma\}, \tag{4.39b}$$

$$\Phi_1^* := \{\rho \in \mathcal{S} : U_1(\rho) \leq 1 - 1/8\} \subset \Phi_1, \tag{4.39c}$$

$$\Phi_2^* := \{\rho \in \mathcal{S} : U_1(\rho) > 1 - 1/8\} = \mathcal{S} \setminus \Phi_1^*, \tag{4.39d}$$

$$\Phi := \{\rho \in \mathcal{S} : 1 - 1/8 < U_1(\rho) < 1 - 1/8 + \gamma\}. \tag{4.39e}$$

By (A.5), (A.6), and (A.9) in Section 4, Appendix A, the infinitesimal generator of ρ_t acting on $U_1(\rho)$ and $U_2(\rho)$ gives

$$\begin{aligned}
\mathcal{L}_{f+gu,\sigma} U_1(\rho) &= \mathrm{Tr}(-\mathrm{i}[H_1, \rho](c(L^2 - 2L\mathrm{Tr}(L\rho)) - \rho_f))u \\
&\quad - 4c\eta U(\rho)^2 \\
&:= g_U(\rho)u - 4c\eta U(\rho)^2, &\text{(4.40a)}\\
\mathcal{L}_{f+gu,\sigma} U_2(\rho) &= \mathrm{Tr}(-\mathrm{i}X[H_1, \rho])u + \mathrm{Tr}(Xf(\rho)) \\
&= \mathrm{Tr}(A\rho)u + \mathrm{Tr}(Xf(\rho)). &\text{(4.40b)}
\end{aligned}$$

In Φ_2^*, by (4.38) and (4.39d), we have

$$1 - \frac{1}{2}\mathrm{Tr}(\rho\rho_f) \geq U_1(\rho) \geq 1 - 1/8 \tag{4.41}$$

Hence, $\mathrm{Tr}(\rho\rho_f) \leq 1/4, \forall \rho \in \Phi_2^*$. From (4.36), it follows that

$$\begin{aligned}
\mathrm{Tr}(A\rho) &= \mathrm{Tr}((4\rho_f - 2I_2)\rho) = 4\mathrm{Tr}(\rho\rho_f) - 2 \\
&\leq -1, \forall \rho \in \Phi_2^*. &\text{(4.42)}
\end{aligned}$$

Theorem 4.2. *Consider the 2-dimensional quantum filter* (4.2)*. Suppose that Assumptions* 4.1, 4.2, *and* 4.3 *are satisfied and* H_1 *satisfies that* $h_{11} = h_{22}$*. Then, there exists a sufficiently* *small constant $\gamma > 0$ such that the following saturation-form control globally asymptotically renders the filter* (4.2) *from any initial state to the final desired state ρ_f almost surely:*

1. $u = u_1(\rho) = -g_U(\rho)$, *if* $\rho \in \Phi_1^*$;
2. $u = u_2(\rho) = \dfrac{-\mathrm{Tr}(Xf(\rho)) - k}{\mathrm{Tr}(A\rho)}, k > 0$, *if* $\rho \in \Phi_2$;
3. $u = u_3(\rho) = \dfrac{1 - 1/8 + \gamma - U_1(\rho)}{\gamma} u_1(\rho) + \dfrac{U_1(\rho) - (1 - 1/8)}{\gamma} u_2(\rho)$, *if* $\rho \in \Phi$.

Proof: By (4.42), the above control is well defined. Substituting it into (4.40), we obtain

$$\begin{aligned}\mathcal{L}_{f+gu,\sigma} U_1(\rho) &= -g_U(\rho)^2 - 4c\eta U(\rho)^2 \\ &:= -W_U(\rho) \leq 0, \forall \rho \in \Phi_1^*, \end{aligned} \tag{4.43a}$$

$$\mathcal{L}_{f+gu,\sigma} U_2(\rho) = -k < 0, \forall \rho \in \Phi_2. \tag{4.43b}$$

Notice that $W_U(\rho) = 0$ iff ρ is one of eigenstates of L (see Section 2, Appendix A) and there is no eigenstate of L in the compact set $\Lambda^* := \bar{\Phi}_1^* \cap \bar{\Phi}_2^* = \{\rho \in \mathcal{S} : U_1(\rho) = 1 - 1/8\}$. As such, $W_U(\rho) > 0, \forall \rho \in \Lambda^*$, and consequently, there exists $\min\limits_{\rho \in \Lambda^*} W_U(\rho) > 0$, which is independent with γ. Hence, $\mathcal{L}_{f+gu,\sigma} U_1(\rho)$ is strictly negative on Λ^* and $\mathcal{L}_{f+gu,\sigma} U_2(\rho)$ is strictly negative on $\Lambda := \bar{\Phi}_1 \cap \bar{\Phi}_2$.

On the other hand, as $u(\rho)$ is continuous in ρ on the whole space $\mathcal{S}$, $\mathcal{L}_{f+gu,\sigma} U_1(\rho)$ and $\mathcal{L}_{f+gu,\sigma} U_2(\rho)$ are also continuous in ρ on the whole space $\mathcal{S}$. Therefore, there exists $\gamma > 0$ sufficiently small such that $\mathcal{L}_{f+gu,\sigma} U_1(\rho)$ and $\mathcal{L}_{f+gu,\sigma} U_2(\rho)$ are strictly negative on the small region Φ between Λ^* and Λ in the means that

$$\mathcal{L}_{f+gu,\sigma} U_1(\rho) \leq -\frac{1}{2} \min_{\rho \in \Lambda^*} W_U(\rho), \forall \rho \in \Phi, \tag{4.44a}$$

$$\mathcal{L}_{f+gu,\sigma} U_2(\rho) \leq -\frac{1}{2} k < 0, \forall \rho \in \Phi. \tag{4.44b}$$

Let $m_U = \dfrac{1}{2} \min\limits_{\rho \in \Lambda^*} W_U(\rho)$ and $W(\rho) = \min\{m_U, W_U(\rho)\}$. Then, it follows from (4.43) and

(4.44) that

$$\mathcal{L}_{f+gu,\sigma}U_1(\rho) \leq -W(\rho), \forall \rho \in \Phi_1^* \cup \Phi = \Phi_1, \tag{4.45a}$$

$$\mathcal{L}_{f+gu,\sigma}U_2(\rho) \leq -\frac{1}{2}k, \forall \rho \in \Phi_2 \cup \Phi = \mathcal{S} \setminus \Phi_1^*. \tag{4.45b}$$

Thus, Conditions C.1, C.2, and C.3 of Theorem 3.3 in Chapter 3 are satisfied. In addition, by the definitions of the sets Φ_1, Φ_2, and Φ_1^*, we have

$$\sup_{\rho \in \Phi_1^*} U_1(\rho) \leq 1 - 1/8 < 1 - 1/8 + \gamma = \inf_{\rho \in \Lambda} U_1(\rho). \tag{4.46}$$

As such, Condition C.4 of Theorem 3.3 is satisfied.

On the other hand, from (4.44), we have $\mathcal{L}_{f+gu,\sigma}U_1(\rho) \leq -m_U < 0$ for all $\rho \in \Lambda$ and $\rho \in \Lambda^*$. Hence, there is no sliding motion of ρ_t on Λ and Λ^*. By the same arguments as in Section 4.4.1, we conclude that under the above continuous control, from any initial state, there exists a unique solution of the system (4.2).

Notice that there is no eigenstate of L on the boundary Λ between Φ_1 and Φ_2. As such, $W_U(\rho)$, and then $W(\rho)$, is positive definite with respect to ρ_f on the set $\bar{\Phi}_1$. According to Theorem 3.3, Statement S.3 holds, i.e., the above saturation-form control globally asymptotically renders the filter (4.2) from any initial state to the final desired state ρ_f almost surely. $\square$

Remark 4.3. *In comparison to the continuous Lyapunov-based control design, the discontinuous Lyapunov-based control design is more convenient because we do not need to ensure that $V(\rho)$ is continuous as well as to construct the continuous, non-negative function $W(\rho)$ on the whole state space $\mathcal{S}$.*

4.5 Example: Spin-1/2 Systems

The Spin$-1/2$ systems [75] have been studied by many researchers due to their important role in quantum information processing [47]. The stabilization problem for the Spin$-1/2$ systems was considered in [34]. It is pointed out in [34], that the symmetric topology of the Bloch sphere $\mathbb{S}^2$ makes the smooth controls impossible to obtain the global stabilizability. In this section, we show that the non-smooth Lyapunov-based controls can solve the problem of global stabilization by state-feedback for the Spin$-1/2$ systems. The eigenstate-transferring will be well performed as a special result. With the Spin$-1/2$ systems, the density operator can be represented as follows

$$\rho = \frac{1}{2}(\sigma_0 + x\sigma_x + y\sigma_y + z\sigma_z) \tag{4.47}$$

where $\overrightarrow{s} = (x, y, z) \in \mathbb{R}^3$ is the Bloch vector of ρ, $\sigma_0 = I$, and $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The map $\rho \mapsto \overrightarrow{s}$ described by (4.47) is an isomorphism between the state space $\mathcal{S}$ of ρ and the state space $\mathcal{B}$ of Bloch vectors $\overrightarrow{s}$, in which $\mathcal{B} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$.

In typical experiment settings [34], the Spin$-1/2$ system interacts with a laser field set along the $z-$axis, and with a controlled magnetic field set along the $y-$axis; see Fig. 4.3. Then, the measurement operator is $L = \sigma_z$ and the Hamiltonians are $H_0 = 0, H_1 = \sigma_y$. A straightforward computation gives

$$f(\rho_t) = (-x_t\sigma_x - y_t\sigma_y), \tag{4.48a}$$

$$g(\rho_t) = (z_t\sigma_x - x_t\sigma_z), \tag{4.48b}$$

$$\sigma(\rho_t) = \sqrt{\eta}(-x_t z_t\sigma_x - y_t z_t\sigma_y + (1 - z_t^2)\sigma_z). \tag{4.48c}$$

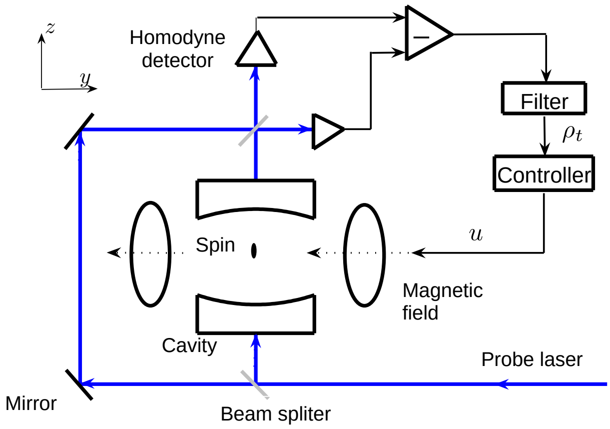


Fig. 4.3: The experiment setup of the continuous quantum measurement and control. The Spin interacts with an optical field produced by a laser. The optical field is detected by homodyne measurement. The measurement outcomes are sent to a filter and the filter state is then fed back via a magnetic field to modify the system Hamiltonian.

Hence, the SME (4.2) becomes

$$\begin{aligned} d\rho_t = &\big((-x_t + uz_t)dt - \sqrt{\eta}x_t z_t dw_t\big)\sigma_x \\ &+ \big(-y_t dt - \sqrt{\eta}y_t z_t dw_t\big)\sigma_y \\ &+ \big(-ux_t dt + \sqrt{\eta}(1 - z_t^2)dw_t\big)\sigma_z. \end{aligned} \tag{4.49}$$

From (4.47) and (4.49), the stochastic differential equation of the Bloch vector is

$$\begin{cases} dx_t &= 2\big((-x_t + uz_t)dt - \sqrt{\eta}x_t z_t dw_t\big) \\ dy_t &= 2\big(-y_t dt - \sqrt{\eta}y_t z_t dw_t\big) \\ dz_t &= 2\big(-ux_t dt + \sqrt{\eta}(1 - z_t^2)dw_t\big). \end{cases} \tag{4.50}$$

When the system is under continuous measurement but without feedback control, i.e., $u = 0$, the *quantum state reduction* assures that the filter state reduces to one of eigenstates of L, which comprise of $\rho_1 = \mathrm{diag}(1, 0)$ and $\rho_2 = \mathrm{diag}(0, 1)$ (See [34]). This means that the

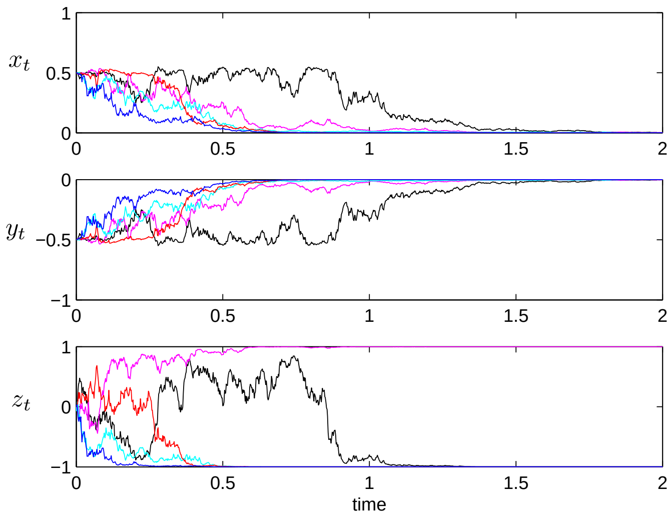


Fig. 4.4: Sample paths of the quantum state reduction in measured system without control

state of (4.50) when $u = 0$ reduces to one of two Bloch vectors $(0, 0, 1)$ and $(0, 0, -1)$.

The simulation data are: $\overrightarrow{s_0} = (0.5, -0.5, 0)$. The measurement efficiency is $\eta = 0.9$. Fig. 4.4 shows 5 arbitrary sample paths of the quantum state reduction. It can be seen from Fig. 4.4 that the quantum filter state reduces stochastically from $\overrightarrow{s_0}$ to one of two states $(0, 0, 1)$ and $(0, 0, -1)$, asserting the *quantum state reduction* phenomenon.

We design non-smooth Lyapunov-based state-feedback controls to render the quantum filter state deterministically to the final desired state $\rho_f = \text{diag}(1, 0)$, which is an eigenstate of the measurement operator L. In this case, the corresponding desired Bloch vector of ρ_f is $\overrightarrow{s_f} = (0, 0, 1)$. As ρ_f is chosen as an eigenvalue of L, Assumption 4.1 is satisfied. Since H_0 and L are diagonal and L is regular, Assumption 4.2 is satisfied. Moreover, $H_1 = \sigma_y$ is connected and has $h_{11} = h_{22} = 0$. Hence, all the conditions of Theorems 4.1 and 4.2 are satisfied.

4.5.1 Continuous Lyapunov Function-Based Control Design

With $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A = 2\text{diag}(1, -1)$ satisfies Lemma 4.2. Choose $M = 6, m = 2$, and $a = 1$. Then, (4.12) holds. As $\text{Tr}(\rho\rho_f) = 0.5(1 + z), \text{Tr}(X\rho) = x, \text{Tr}(A\rho) = 2z$, and $U(\rho) = 1 - z^2$, the partition (4.16a)-(4.16c) of the state space $\mathcal{S}$ corresponds with the following partition of $\mathcal{B}$:

$$\begin{aligned}
\mathcal{B}_1 &= \{(x, y, z) \in \mathcal{B} : 2(1 - z) < 6 - 4z^2 + l|x - \frac{1}{l}|\}, \\
\mathcal{B}_2 &= \big\{(x, y, z) \in \mathcal{B} : 2(1 - z) \geq 6 - 4z^2 + l|x - \frac{1}{l}|, \\
&\qquad lx \geq 1 \big\}, \\
\mathcal{B}_3 &= \big\{(x, y, z) \in \mathcal{B} : 2(1 - z) \geq 6 - 4z^2 + l|x - \frac{1}{l}|, \\
&\qquad lx < 1 \big\},
\end{aligned}$$

and the Lyapunov function (4.13) becomes:

$$V = \begin{cases} 2(1 - z) + c(1 - z^2), & (x, y, z) \in \mathcal{B}_1, \\ 6 - 4z^2 + lx - 1 + c(1 - z^2), & (x, y, z) \in \mathcal{B}_2, \\ 6 - 4z^2 - lx + 1 + c(1 - z^2), & (x, y, z) \in \mathcal{B}_3. \end{cases}$$

Following the control design procedure in Section 4.4.1, we obtain the control law:

$$u = \begin{cases} -(4x + 4cxz), & (x, y, z) \in \mathcal{B}_1, \\ \dfrac{2lx - k}{2lz + (16 + 4c)xz}, & (x, y, z) \in \mathcal{B}_2, \\ \dfrac{-2lx - k}{-2lz + (16 + 4c)xz}, & (x, y, z) \in \mathcal{B}_3. \end{cases} \tag{4.51}$$

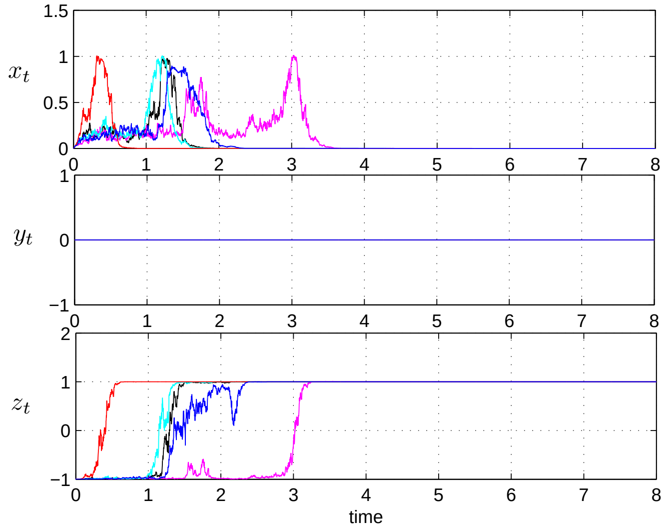
The designed parameters: $c = 0.25, l = 10, k = 10$. Then, (4.22), (4.30), and (4.32) hold. The simulation data: $\overrightarrow{s_0} = (0, 0, -1), \eta = 0.9$. The simulation results with 5 arbitrary sample paths are showed in Fig. 4.5. It can be observed from Fig. 4.5(a) that, in all sample paths, the filter state is driven from the eigenstate $(0, 0, -1)$ to the desired eigenstate $(0, 0, 1)$, i.e., the eigenstate-transferring is well performed. Thus, the eigenstate-transferring is achieved for Spin$-1/2$, distinguishing the proposed control from the classical smooth control methods in [34, 35, 53]. The Lyapunov function $V(\rho_t)$ is shown to be continuous in Fig. 4.5(c). The switchings in the control signal u_t and the infinitesimal $\mathcal{L}V(\rho_t)$ observed in Figs. 4.5(b) and 4.5(c) are due to the evolution of filter state ρ_t through the boundary Λ.

4.5.2 Discontinuous Lyapunov Function-Based Control Design

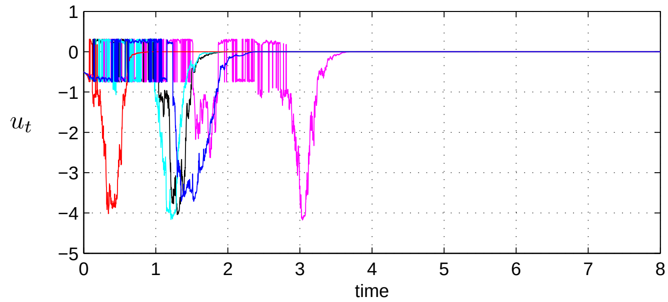
Notice that X and A in Section 4.5.1 also satisfy the Equation (4.36). Following the control design procedure in Section 4.4.2, we obtain the continuous control law in saturation form:

1. $u = u_1 = -(x + 4cxz)$, if $0.5(1-z) + c(1-z^2) \leq 1 - 1/8$;
2. $u = u_2 = \dfrac{2x - k}{2z}$, if $0.5(1-z) + c(1-z^2) \geq 1 - 1/8 + \gamma$;
3. $u = \dfrac{1 - 1/8 + \gamma - (0.5(1-z) + c(1-z^2))}{\gamma} u_1 + \dfrac{0.5(1-z) + c(1-z^2) - (1 - 1/8)}{\gamma} u_2$, if $1 - 1/8 < 0.5(1-z) + c(1-z^2) < 1 - 1/8 + \gamma$.

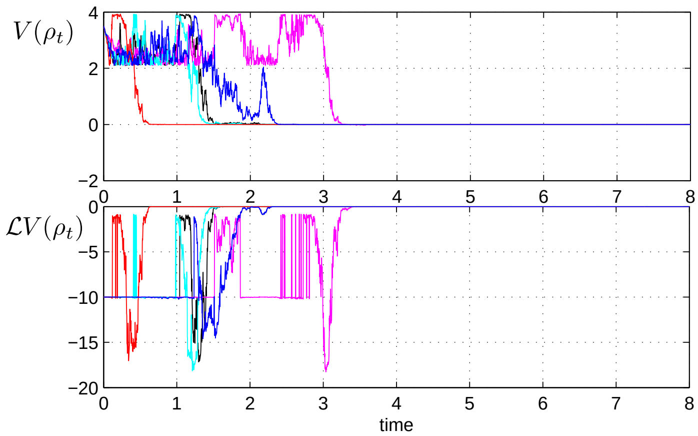
The designed parameters: $c = 0.2, k = 2, \gamma = 0.05$. It can be checked that (4.38) and (4.44) hold. The simulation data: $\overrightarrow{s_0} = (0, 0, -1)$ and $\eta = 0.9$. The simulation results with 5 arbitrary sample paths are shown in Fig. 4.6. It can be seen from Fig. 4.6(a) that under the above control, the filter state is driven from the eigenstate $(0, 0, -1)$ to the desired eigenstate $(0, 0, 1)$ in all sample paths, i.e., the eigenstate-transferring of the Spin-1/2 system is well achieved. The control u_t is shown to be continuous in Fig. 4.6(b), and thus, is easier to be implemented than the switching control in Fig. 4.5(b).



(a) Filter state convergence: from $(0, 0, -1)$ to $(0, 0, 1)$

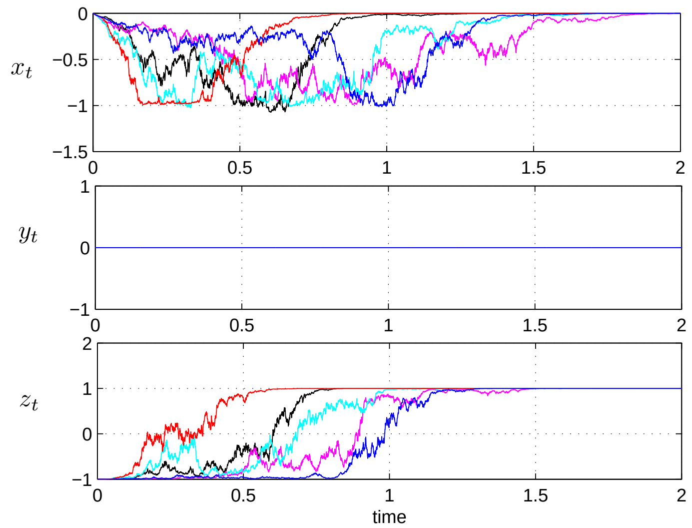


(b) Control input u_t synthesized based on continuous Lyapunov

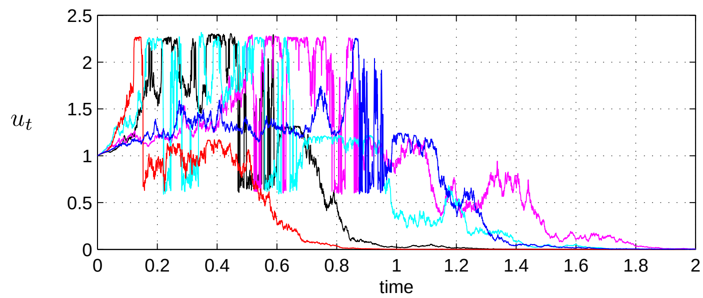


(c) Continuous Lyapunov function $V(\rho_t)$ and negative-definite infinitesimal $\mathcal{L}V(\rho_t)$

Fig. 4.5: Continuous Lyapunov-based stabilization when $c = 0.25, l = 10, k = 10$



(a) Filter state convergence: from $(0, 0, -1)$ to $(0, 0, 1)$



(b) Control input u_t synthesized based on discontinuous Lyapunov

Fig. 4.6: Discontinuous Lyapunov-based stabilization when $c = 0.2, k = 2, \gamma = 0.05$.

4.6 Conclusions

In this chapter, the non-smooth Lyapunov-based control designs for the deterministic generation of the desired eigenstate of a class of quantum filters have been presented. Based on continuous Lyapunov-like theorem, switching control has been constructed to globally stabilize the quantum filter. Applying discontinuous Lyapunov-like theorem, continuous control in saturation form has been designed to successfully deal with the global stabilization problem of 2-dimensional quantum filters. The control design for the Spin$-1/2$ systems has shown that these non-smooth Lyapunov-based controls are effective to cope with the symmetric topology of filter state space and to achieve the global stabilizability for quantum filters.

Chapter 5

Real-time Implementation of Quantum Feedback

5.1 Introduction

In this chapter, we are interested in the problem of real-time implementation of quantum feedback control. In the feedback control of quantum systems, due to the very fast dynamics of the quantum mechanical systems, the time to compute the filter state and the filter-based control input is not negligible. Because of this inherent feature, to implement a filter-based control strategy in real time, we have to take the computation time explicitly into account.

In [37] and [76], a filter-reduction approach was introduced, in which an approximation of the filter state is obtained by using a reduced-dimension filter. This approximation approach reduces the computation time considerably. The limitation of this approach is that the inevitable error between the real filter and the reduced-dimension filter may cause the filter-reduction-based control ineffective. Another approach based on the time delay control was introduced in [77, 78], in which the computation time is compensated for in the control law, i.e., $u_t = u(\rho_{t-\tau})$, where $\tau > 0$ denotes the time spent to compute the filter state. However, the control design in [77, 78] is only applicable for 2−dimensional quantum filters. As the computation time raises a problem when the quantum filter is of high dimension, the real-time quantum measurement-based feedback control calls for a systematic control synthesis for high-dimensional quantum filters.

Motivated by the above considerations, in this chapter, we introduce globally stabilizing, non-smooth time delay control for a class of arbitrary high-dimensional quantum filters to deal with the problem of compensating for the computation time in real-time quantum measurement-based feedback control. Inspired by the control design in [57], the proposed control is of hysteresis form with two modes, of which the constant control almost surely pushes the filter state off all antipodal points of the desired state ρ_f in a finite time, while the nonlinear control drives the filter state to the desired state ρ_f almost surely. The main advantages of our control strategy over the delay-free one in [57] and the time delay ones for 2−dimensional systems in [77, 78] include:

1) The control design is constructive and the obtained control is explicit instead of being guaranteed to exist only as in [57];
2) The known, but arbitrarily long, computation time is compensated for the first time, while the computation time in [78] is required to be small, as can be seen in Section III.B of [78];
3) The proposed control is given in an analytic form and valid for arbitrary high-dimensional quantum filters rather than for 2−dimensional ones as in [77, 78]; and
4) The proposed control encompasses the bang-bang control which can be trivially implemented in practice.

For the stability analysis of the closed-loop stochastic time delay system, we introduce a Lyapunov-LaSalle-like theorem for delay-dependent stability in probability. In time delay systems, due to the inevitable effect of the delay, the increment of the Lyapunov function, which represents system energy, along the system trajectory is unavoidable. Fortunately, when the time delay is constant, the energy caused by the delay can be canceled by the delay-free terms [67, 79]. Exploiting this observation, the stability of the closed-loop system is guaranteed without the use of Lyapunov-Krasovskii-type or Lyapunov-Razumikhin-type conditions as in usual stability theorems for time delay systems [80, 81]. The introduced Lyapunov-LaSalle-like theorem in this chapter is thus much less conservative.

In Section 5.2, we present the system description and the problem formulation. The Lyapunov-LaSalle-like theorem for stability in probability of a class of time delay stochastic nonlinear systems is introduced in Section 5.3. In Section 5.4, we design non-smooth time delay control to globally asymptotically render the quantum filter to the final desired state almost surely. The effectiveness of the proposed control approach is demonstrated through the real-time control design for the Spin$-1/2$ systems in Section 5.5. Section 5.6 includes concluding remarks.

5.2 Problem Formulation

Consider the finite-dimensional quantum filters whose state is represented by the density matrix ρ evolving on the space

$$\mathcal{S} := \{\rho \in \mathbb{C}^{n\times n} : \rho = \rho^\dagger \geq 0, \mathrm{Tr}(\rho) = 1\}, \tag{5.1}$$

and the time evolution of quantum filter state is described, in units such that $\hbar = 1$, by the following stochastic master equation (SME) [34]

$$\begin{aligned} d\rho_t &= f(\rho_t)dt + g(\rho_t)u_t dt + \sigma(\rho_t)dw_t \qquad (5.2)\\ &= \big(- \mathrm{i}[H_0, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}L^\dagger L\rho_t - \frac{1}{2}\rho_t L^\dagger L\big)dt - \mathrm{i}[H_1, \rho_t]u_t dt \\ &+ \sqrt{\eta}\big(L\rho_t + \rho_t L^\dagger - \mathrm{Tr}(L\rho_t + \rho_t L^\dagger)\rho_t\big)dw_t, \end{aligned}$$

where

- w_t is the $1-$dimensional standard Wiener process defined on the classical complete probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, in which Ω is a sample space, $\mathcal{F}$ is a $\sigma-$field, and $\mathbb{P}$ is a probability measure. The Wiener increment $dw_t = dy_t - \sqrt{\eta}\mathrm{Tr}(L\rho_t + \rho_t L^\dagger)dt$, where y_t is the measurement record of the output, appears due to the probabilistic nature of quantum observation;
- $u_t \in \mathbb{R}$ is the control input;
- The free Hamiltonian H_0 and control Hamiltonian H_1 are $n \times n$ Hermitian (or self-adjoint) matrices with entries in $\mathbb{C}$;
- L is the measurement operator (or measured observable), determining how the system interacts with the measurement apparatus; and
- $\eta \in (0, 1]$ is the measurement efficiency.

From the applications in quantum chemistry and atomic physics, the problem of transferring a quantum system from initial states to desired states is of importance. In practice, the time to compute any filter-based control input is not negligible. This is especially essential in quantum systems due to their very fast dynamics. As such, to enable the real-time feedback control, we consider the time delay state-feedback control input $u_t = u(\rho_{t-\tau})$, where $\tau > 0$ denotes the time spent to compute the filter state and control input.

We have the following control problem:

($\mathcal{P}2$) **State-transferring by time delay state-feedback:** *design a time delay control law of the form $u_t = u(\rho_{t-\tau})$, where $\tau > 0$ is known but of arbitrary length, to globally asymptotically render the quantum filter* (5.2) *from any initial data $\{\rho_\theta \in \mathcal{S}, -\tau \leq \theta \leq 0\}$ to the final desired state ρ_f almost surely, i.e.,*

$$\mathbb{P}\{\lim_{t\to\infty} \rho_t = \rho_f\} = 1,$$

for all initial data $\{\rho_\theta \in \mathcal{S}, -\tau \leq \theta \leq 0\}$.

Towards a solution to the problem ($\mathcal{P}2$), we consider the standard assumptions made on the system (5.2) as in Chapter 4.

Assumption 5.1. *The final desired state* ρ_f *is an eigenstate of the measurement operator* L*, i.e.,* $\rho_f = \psi_f \psi_f^\dagger$ *where* ψ_f *is an eigenvector of* L :

$$L\psi_f = \lambda_f \psi_f. \tag{5.3}$$

Assumption 5.2. *The measurement operator* L *is self-adjoint, i.e.,* $L = L^\dagger$*, and regular, i.e., the eigenvalues of* L *are different. The system free Hamiltonian* H_0 *and* L *are commutative. As* H_0 *and* L *are commutative, we can choose a basis in which* H_0 *and* L *are diagonal.*

Assumption 5.3. *In the basis that* H_0 *and* L *are diagonalized,* $H_1 = [h_{kl}]_{n\times n}$ *is connected, i.e.,* $h_{i(i+1)} \neq 0$ *for all* $i = 1, .., n-1$.

With the time delay control $u_t = u(\rho_{t-\tau})$, the closed-loop system (5.2) is a stochastic nonlinear time delay system. To facilitate the control design in Section 5.4, in the next section, we shall present a Lyapunov-LaSalle-like theorem for delay-dependent stochastic stability of a class of stochastic nonlinear time delay systems.

In the typical Lyapunov stability theory [62], to guarantee the system stability (or to drive the system to some desired state), we shall introduce a positive definite function of the system state (or of the distance from system state to the desired state), called Lyapunov function, which is decreasing along the system trajectory. This decrease can be obtained by designing the control such that the derivative of Lyapunov function is negative definite along the system trajectory.

In time delay systems, the derivative of Lyapunov function may contain some positive terms caused by the time delay variables. As such, the stability of time delay systems generally calls for some complex conditions such as Lyapunov-Krasovskii-type or Lyapunov-Razumikhin-type conditions [80, 81]. Fortunately, in our case, since the time delay is constant, the long time effect of positive terms caused by the delay can be canceled by the delay-free terms. Utilizing this observation, the stability of the system is guaranteed without the use of Lyapunov-Krasovskii-type or Lyapunov-Razumikhin-type conditions as in usual stability theorems for time delay systems [80, 81].

5.3 Delay-dependent Stochastic Stability

In this section, we introduce a Lyapunov-LaSalle-like theorem for the stability in probability of a class of stochastic nonlinear systems, which will be instrument to the control design in the next section. This theorem for stochastic nonlinear systems, whose state evolves on the vector spaces $\mathbb{R}^n, n = 1, 2, ...$, can be applied for the quantum filters evolving on the density matrix space $\mathcal{S} \subset \mathbb{C}^{n\times n}$ because the density matrix $\rho \in \mathcal{S}$ can be equivalently represented by a vector in the vector space $\mathbb{R}^{n^2-1}$ [34].

Consider the stochastic nonlinear time delay systems described by

$$dx_t = f(x_t, x_{t-\tau})dt + \sigma(x_t, x_{t-\tau})dw_t, \tag{5.4}$$

where x_t is the state, τ is the constant time delay, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are Borel-measurable, locally bounded, and locally Lipschitz continuous functions, $f(0, 0) = 0, \sigma(0, 0) = 0$, w_t is an $r-$dimensional standard Wiener process (or Brownian motion) defined on the classical complete probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, in which Ω is a sample space, $\mathcal{F}$ is a $\sigma-$field, and $\mathbb{P}$ is a probability measure. The initial data is $\{x_\theta : -\tau \leq \theta \leq 0\} = \xi \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$. From [82], for each initial data $\xi \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, there exists a unique solution of (5.4) denoted as $x(t, \xi) = x_t$.

For the stochastic nonlinear time delay system (5.4), we recall the concept of delay-dependent stability in probability [82].

Definition 5.1. *The equilibrium $x = 0$ of the system* (5.4) *is*

- *(delay-dependent) globally stable in probability if for all $\epsilon > 0$, there exists a $\mathcal{K}-$class function $\gamma(\cdot)$ such that for all $t \geq 0$, for all $\xi \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,*

$$\mathbb{P}\Big\{|x_t| < \gamma\big(\|\xi\|\big)\Big\} \geq 1 - \epsilon$$

where $\|\xi\| = \sup_{\theta \in [-\tau, 0]} |x_\theta|$.

- *(delay-dependent) globally asymptotically stable in probability if it is (delay-dependent) globally stable in probability and*

$$\mathbb{P}\{\lim_{t \to \infty} |x_t| = 0\} = 1, \quad \forall \xi \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n).$$

In the sequel, we refine a Lyapunov-LaSalle-like theorem, proposed in [67], such that it is applicable to the control design of quantum filters in Section 5.4. Noting the compactness property of filter state space, this theorem introduces conditions less conservative than those of Theorem 2.1 in [67]. Moreover, in its proof, we show explicitly the global stability in probability property of the equilibrium $x = 0$, which was omitted in [67].

Theorem 5.1. *Assume that there exist a continuously twice differentiable, non-negative Lyapunov function $V(x)$, $\mathcal{K}_\infty$-class functions $\alpha_1(\cdot), \alpha_2(\cdot)$ satisfying that*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \forall x \in \mathbb{R}^n, \tag{5.5}$$

and continuous, non-negative functions $W_1(x), W_2(x)$ such that along the solution x_t of the system (5.4)*, we have*

$$\mathcal{L}_{f,\sigma} V(x_t, x_{t-\tau}) \leq W_1(x_{t-\tau}) - W_1(x_t) - W_2(x_t), \forall t \geq 0, \tag{5.6}$$

where $\mathcal{L}_{f,\sigma}$ is the infinitesimal operator associated with (5.4)*:*

$$\mathcal{L}_{f,\sigma}V(x_t, x_{t-\tau}) := \frac{\partial V(x_t)}{\partial x_t} f(x_t, x_{t-\tau}) + \frac{1}{2}\mathrm{Tr}\big(\sigma^T(x_t, x_{t-\tau})\frac{\partial^2 V(x_t)}{\partial x_t^2}\sigma(x_t, x_{t-\tau})\big).$$

Then, the equilibrium $x = 0$ of the system (5.4) *is globally stable in probability and*

$$\mathbb{P}\Big\{\lim_{t\to\infty} W_2(x_t) = 0\Big\} = 1, \forall \xi \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n). \tag{5.7}$$

If, in addition, $W_2(x)$ is positive definite, then the equilibrium $x = 0$ of the system (5.4) *is globally asymptotically stable in probability.*

Proof. Applying Dynkin's formula for the continuously twice differentiable function $V(x)$, we have

$$\mathbb{E}[V(x_t)] - V(x_0) = \mathbb{E}\Big[\int_0^t \mathcal{L}_{f,\sigma}V(x_s, x_{s-\tau})ds\Big] \tag{5.8}$$

Combining (5.6) and (5.8), we obtain

$$\begin{aligned}\mathbb{E}[V(x_t)] - V(x_0) &\le \mathbb{E}\Big[\int_0^t \big[-W_2(x_s) - W_1(x_s) + W_1(x_{s-\tau})\big]ds\Big] \\ &= \mathbb{E}\Big[-\int_0^t W_2(x_s)ds - \int_0^t W_1(x_s) + \int_{-\tau}^{t-\tau} W_1(x_s)ds\Big] \\ &\le \mathbb{E}\Big[-\int_0^t W_2(x_s)ds\Big] + \mathbb{E}\Big[\int_{-\tau}^0 W_1(x_s)ds\Big]\end{aligned} \tag{5.9}$$

Since $W_2(x) \ge 0, \forall x \in \mathbb{R}^n$, (5.9) leads to

$$\mathbb{E}[V(x_t)] \le V(x_0) + \tau \sup_{-\tau\le\theta\le 0} W_1(x_\theta) \tag{5.10}$$

Let $a(s) = \sup_{|x|\le s} V(x), b(s) = \sup_{|x|\le s} W_1(x)$, and $c(s) = a(s) + \tau b(s)$. It follows from (5.10) that

$$\mathbb{E}[V(x_t)] \le c(\|\xi\|) \tag{5.11}$$

where $\|\xi\| = \sup_{-\tau\leq\theta\leq 0} |x_\theta|$. Applying Chebyshev's inequality, for any $\mathcal{K}-$class function $\delta(\cdot)$, we have

$$\mathbb{P}\{|V(x_t)| \geq \delta(\|\xi\|)\} \leq \frac{\mathbb{E}[|V(x_t)|]}{\delta(\|\xi\|)} \tag{5.12}$$

It follows from (5.11), (5.12), and the fact that $V(x) \geq 0, \forall x$, that

$$\mathbb{P}\{V(x_t) \geq \delta(\|\xi\|)\} \leq \frac{\mathbb{E}[|V(x_t)|]}{\delta(\|\xi\|)} = \frac{\mathbb{E}[V(x_t)]}{\delta(\|\xi\|)} \leq \frac{c(\|\xi\|)}{\delta(\|\xi\|)}$$

As such

$$\mathbb{P}\{V(x_t) \leq \delta(\|\xi\|)\} \geq 1 - \frac{c(\|\xi\|)}{\delta(\|\xi\|)}. \tag{5.13}$$

For any $\epsilon > 0$, we choose the $\mathcal{K}-$class function $\delta(\cdot)$ such that $\delta(\|\xi\|) \geq \dfrac{c(\|\xi\|)}{\epsilon}$ and let the $\mathcal{K}-$class function $\gamma = \alpha_1^{-1} \circ \delta \circ \alpha_2$. Then, from (5.13), we obtain

$$\mathbb{P}\big\{|x_t| \leq \gamma(\|\xi\|)\big\} \geq \mathbb{P}\big\{V(x_t) \leq \delta(\|\xi\|)\big\} \geq 1 - \frac{c(\|\xi\|)}{\delta(\|\xi\|)} \geq 1 - \epsilon \tag{5.14}$$

Therefore, the equilibrium $x = 0$ is globally stable in probability.

On the other hand, from (5.9), we have

$$\mathbb{E}\Big[\int_0^t W_2(x_s)ds\Big] \leq V(x_0) - \mathbb{E}[V(x_t)] + \mathbb{E}\Big[\int_{-\tau}^0 W_1(x_s)ds\Big] < \infty. \tag{5.15}$$

Letting $t \to \infty$, we obtain that $\mathbb{E}\Big[\int_0^\infty W_2(x_s)ds\Big] < \infty$. From (5.10) and the boundedness of initial state, $\mathbb{E}[V(x_t)]$ is bounded. By (5.15), applying Lemma 3 in [83], we have

$$\mathbb{P}\big\{\lim_{t\to\infty} W_2(x_t) = 0\big\} = 1. \tag{5.16}$$

If, in addition, $W_2(x)$ is positive definite, then, (5.16) leads to

$$\mathbb{P}\big\{\lim_{t\to\infty}|x_t|=0\big\}=1, \tag{5.17}$$

which, together with the global stability in probability, means that the equilibrium $x=0$ is globally asymptotically stable in probability. The proof of Theorem 5.1 is completed. □

Remark 5.1. *The condition* (5.6) *is posed on the solution* x_t *of the system* (5.4)*, instead of on the whole state space as the condition of Theorem* 2.1 *in [67]. Hence, the condition here is less conservative, enabling it applicable to the control design of quantum filters in Section* 5.4 *due to the compactness of filter state space (See Remark* 5.3*).*

Remark 5.2. *Theorem* 5.1 *implies that when conditions* (5.5) *and* (5.6) *are satisfied, then the system trajectory* x_t *converges in probability to the set* $\{x\in\mathbb{R}^n : W_2(x)=0\}$ *regardless of the initial data* $\xi\in\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$.

5.4 Time Delay Control Design

Using results in the previous section, in this section, we solve the problem of global state-transferring by time delay state-feedback control for the quantum filter (5.2). Firstly, we consider time delay smooth controls synthesized based on the smooth Lyapunov theory and show that, similar to the delay-free case [34, 35, 53], such smooth controls are not sufficient to globally render the filter state to the final desired state due to the symmetric topology of the filter state space $\mathcal{S}$. Then, combining this time delay smooth control with a constant control, we obtain time delay control in hysteresis form capable of globally asymptotically rendering the filter state to the desired state almost surely.

5.4.1 Control Design

Consider the Lyapunov function candidate of mixed Frobenius norm/variance form [53]:

$$V(\rho) = 1 - \mathrm{Tr}(\rho\rho_f) + cU(\rho) \tag{5.18}$$

where $U(\rho) = \mathrm{Tr}(L^2\rho) - \mathrm{Tr}^2(L\rho)$ is the variance of filtering process along L and $c > 0$ is a constant to be chosen later. In the Lyapunov function (5.18), the term $1 - \mathrm{Tr}(\rho\rho_f)$ is the distance from the state ρ to the final desired state ρ_f and the term $U(\rho)$ takes into account the probabilistic nature of the system.

Under Assumption 5.2, the stochastic master equation (5.2) becomes

$$\begin{aligned} d\rho_t &= \big(- \mathrm{i}[H_0, \rho_t] + L\rho_t L - \frac{1}{2}L^2\rho_t - \frac{1}{2}\rho_t L^2\big)dt - \mathrm{i}[H_1, \rho_t]u_t dt \\ &+ \sqrt{\eta}\big(L\rho_t + \rho_t L - \mathrm{Tr}(L\rho_t + \rho_t L)\rho_t\big)dw_t \\ &= f(\rho_t)dt + g(\rho_t)u_t dt + \sigma(\rho_t)dw_t. \end{aligned} \tag{5.19}$$

From (5.19) and Assumptions 5.1 and 5.2, a straightforward computation (see Section 1, Appendix B) gives us the infinitesimal operator of ρ_t acting on $V(\rho)$:

$$\mathcal{L}_{f+gu,\sigma}V(\rho_t) = \mathrm{Tr}\left(-\mathrm{i}[H_1, \rho_t]\big(c(L^2 - 2L\mathrm{Tr}(L\rho_t)) - \rho_f\big)\right)u_t - 4c\eta U^2(\rho_t), \tag{5.20}$$

where $\mathcal{L}_{f+gu,\sigma}$ is the infinitesimal operator associated with (5.19):

$$\mathcal{L}_{f+gu,\sigma}V(\rho_t) := \frac{\partial V(\rho_t)}{\partial \rho_t}(f + gu) + \frac{1}{2}\mathrm{Tr}\big(\sigma^T \frac{\partial^2 V(\rho_t)}{\partial \rho_t^2}\sigma\big).$$

Note that the filter state space $\mathcal{S}$ is a compact set since $\mathrm{Tr}(\rho^2) \leq 1$ for all $\rho \in \mathcal{S}$ [34]. As such, there exists

$$\max_{\rho\in\mathcal{S}} \left|\mathrm{Tr}\left(-\mathrm{i}[H_1, \rho_t]\big(c(L^2 - 2L\mathrm{Tr}(L\rho_t)) - \rho_f\big)\right)\right| := M. \tag{5.21}$$

We choose the following smooth time delay control law $u_t = u_S(\rho_{t-\tau})$, where

$$u_S(\rho_{t-\tau}) \leq \frac{U^2(\rho_{t-\tau})}{\bar{M}} = \frac{\Big(\mathrm{Tr}(L^2\rho_{t-\tau}) - \mathrm{Tr}^2(L\rho_{t-\tau})\Big)^2}{\bar{M}}, \bar{M} > 0. \tag{5.22}$$

Then, it follows from (5.20) that

$$\mathcal{L}_{f+gu,\sigma}V(\rho_t, \rho_{t-\tau}) \leq \frac{M}{\bar{M}}U^2(\rho_{t-\tau}) - 4c\eta U^2(\rho_t) \tag{5.23}$$

or equivalently,

$$\mathcal{L}_{f+gu,\sigma}V(\rho_t, \rho_{t-\tau}) \leq \frac{M}{\bar{M}}U^2(\rho_{t-\tau}) - \frac{M}{\bar{M}}U^2(\rho_t) - (4c\eta - \frac{M}{\bar{M}})U^2(\rho_t). \tag{5.24}$$

Therefore, with $0 < \dfrac{M}{\bar{M}} < 4c\eta$, the condition (5.6) of Theorem 5.1 is satisfied with the continuous, non-negative functions $W_1(\rho) = \dfrac{M}{\bar{M}}U^2(\rho)$ and $W_2(\rho) = (4c\eta - \dfrac{M}{\bar{M}})U^2(\rho)$. Note that the condition (5.5), which is used to prove the boundedness of system solution (see the proof of Lemma 3 in [83]), is not necessary in the case of quantum filter as $\mathcal{S}$ is a compact set, implying that the system trajectory is always bounded. Applying Theorem 5.1 (see Remark 5.2), we conclude that when the control input $u_t = u_S(\rho_{t-\tau}) \leq \dfrac{U^2(\rho_{t-\tau})}{\bar{M}}$ is applied, then ρ_t converges in probability to the set

$$\Big\{\rho \in \mathcal{S} : (4c\eta - \frac{M}{\bar{M}})U^2(\rho) = 0\Big\} \tag{5.25}$$

regardless of the initial data.

Remark 5.3. *The density matrix $\rho \in \mathcal{S}$ can be equivalently represented by a vector $\overrightarrow{s}$ in the vector space $\mathbb{R}^{n^2-1}$ [34]. The map $\rho \mapsto \overrightarrow{s}$ is an isomorphism between the set $\mathcal{S}$ and a compact set $\mathcal{B} \subset \mathbb{R}^{n^2-1}$. Notice that the maximum value M in* (5.21) *is only guaranteed to exist in a compact set. Consequently, Inequality* (5.24) *only holds true when $\rho \in \mathcal{S}$, i.e., $\overrightarrow{s} \in \mathcal{B}$. In that view, Theorem* 2.1 *in [67], whose condition is posed on the whole state space $\mathbb{R}^{n^2-1}$, is not applicable. We, however, can apply Theorem* 5.1*, in which the condition* (5.6) *is posed on the system trajectory.*

Note that as L is regular, $(4c\eta - \frac{M}{\bar{M}})U^2(\rho) = 0$ iff ρ is one of eigenstates of L. As such, (5.25) means that ρ_t converges to one of eigenstates of L almost surely. As L is diagonal and regular, its eigenstates are: $\rho_i = \text{diag}\{0, \ldots, \underbrace{1}_{i-th}, \ldots, 0\},\ i = 1, \ldots, n$. These eigenstates are equilibrium points of the closed-loop systems. Consequently, the time delay smooth control $u_t = u_S(\rho_{t-\tau})$ cannot render the filter from the *antipodal eigenstates* of ρ_f, i.e., the eigenstates $\rho_i \neq \rho_f$ of L, to ρ_f. Therefore, similar to the delay-free case [34,35,53], the time delay smooth control is difficult to globally render the filter state to the final desired state due to the symmetric topology of the filter state space $\mathcal{S}$.

In the sequel, inspired by the control synthesis for the delay-free case in [57, 70], we construct time delay control in hysteresis form to render the system trajectory ρ_t from any initial state to the desired ρ_f almost surely. This two-mode control comprises of a constant control, that almost surely pushes the filter state off all antipodal eigenstates of the desired state ρ_f in a finite time, and the time delay control $u_t = u_S(\rho_{t-\tau})$ that drives the filter state to ρ_f almost surely. To present the control, let us denote

$$\begin{aligned}
&M_V := \max_{\rho \in \mathcal{S}} V(\rho), \mathcal{S}_\alpha := \{\rho \in \mathcal{S} : V(\rho) = \alpha\}, \\
&\mathcal{S}_{>\alpha} := \{\rho \in \mathcal{S} : \alpha < V(\rho) \le M_V\}, \mathcal{S}_{\ge\alpha} := \{\rho \in \mathcal{S} : \alpha \le V(\rho) \le M_V\}, \\
&\mathcal{S}_{<\alpha} := \{\rho \in \mathcal{S} : 0 \le V(\rho) < \alpha\}, \mathcal{S}_{\le\alpha} := \{\rho \in \mathcal{S} : 0 \le V(\rho) \le \alpha\}.
\end{aligned}$$

The positive constant c in the Lyapunov function $V(\rho)$ is chosen to be small enough such that the distance $(1 - \text{Tr}(\rho\rho_f))$ from the state ρ to the desired state ρ_f dominates the variance term $cU(\rho)$. Let c satisfy

$$c\big(U(\frac{1}{n}I_n) + \max_{\rho \in \mathcal{S}} U(\rho)\big) < \frac{1}{n}. \tag{5.26}$$

Then, by the definition of M_V, we have $1 \le M_V \le 1 + c \max_{\rho \in \mathcal{S}} U(\rho) < 1 + \frac{1}{n} - cU(\frac{1}{n}I_n)$. As such, there exists $\gamma > 0$ such that

$$2(M_V - 1) < \gamma < M_V - 1 + \frac{1}{n} - cU(\frac{1}{n}I_n). \tag{5.27}$$

Note from (5.23) that the energy increment caused by the time delay is small when gain $\bar{M}$ in the nonlinear control (5.22) is large. Let $\bar{M}$ satisfy

$$0 < \frac{M}{\bar{M}} < 4c\eta, \tag{5.28a}$$

$$\tau\frac{M}{\bar{M}}\max_{\rho\in\mathcal{S}} U^2(\rho) < \frac{\gamma}{2}. \tag{5.28b}$$

Remark 5.4. *The parameters* γ *and* $\bar{M}$ *chosen as in* (5.27) *and* (5.28) *are very important in our control design; see their role in the proofs of Propositions* 5.1 *and* 5.2 *and Theorem* 5.2 *below. The left hand side of* (5.28b) *is the largest energy caused by the initial data (see* (5.9) *and* (B.19) *in Appendix B). As the time delay is constant, the energy caused by the delay in a long time is canceled by the delay-free terms. Inequality* (5.28b) *utilizes the observation that when the largest energy caused by the initial data is sufficiently small, the effect of time delay can be fully eliminated.*

Before presenting the control, let us introduce two technical propositions of which the proofs are found in Appendix B.

Proposition 5.1. *For any initial data with* $\rho_0 \in \mathcal{S}_{>M_V-\gamma}$, *the solution of* (5.2) *with* $u_t = 1$ *exits* $\mathcal{S}_{>M_V-\gamma}$ *in a finite time with probability 1.*

Proof. See Section 2, Appendix B. □

Proposition 5.2. *Let*

$$p := 1 - \frac{M_V - \gamma + \tau\max_{\rho\in\mathcal{S}} W_1(\rho)}{M_V - \gamma/2}.$$

Then, $p > 0$ *and for any initial data with* $\rho_0 \in \mathcal{S}_{\leq M_V-\gamma}$, *the solution* ρ_t *of* (5.2) *with the* *nonlinear control input* $u_t = u_S(\rho_{t-\tau}) \leq U^2(\rho_{t-\tau})/\bar{M}$ *remains in* $\mathcal{S}_{<M_V-\gamma/2}$ *with probability larger or equal to* p.

Proof. See Section 3, Appendix B. □

5.4.2 Convergence Analysis

Theorem 5.2. *Consider the quantum filter* (5.2) *evolving on the state space* $\mathcal{S}$. *Suppose that Assumptions* 5.1, 5.2, *and* 5.3 *are satisfied. Then, the following hysteresis time delay control solves the problem* ($\mathcal{P}$2)*:*

1. $u_t = 1$, *if* $\rho_{t-\tau} \in \mathcal{S}_{\geq M_V-\gamma/2}$;

2. $u_t = u_S(\rho_{t-\tau}) \leq U(\rho_{t-\tau})^2/\bar{M}$, *if* $\rho_{t-\tau} \in \mathcal{S}_{\leq M_V-\gamma}$;

3. *If* $\rho_{t-\tau} \in \Phi := \mathcal{S}_{<M_V-\gamma/2} \cap \mathcal{S}_{>M_V-\gamma}$, *then* $u_t = u_S(\rho_{t-\tau}) \leq U^2(\rho_{t-\tau})/\bar{M}$ *if* $\rho_{t-\tau}$ *last entered* Φ *through the boundary* $\mathcal{S}_{M_V-\gamma}$ *and* $u_t = 1$ *otherwise.*

Proof. We denote by mode $\mathcal{C}$ and mode $\mathcal{N}$ the periods that the constant control input $u_t = 1$ and the nonlinear control input $u_t = u_S(\rho_{t-\tau}) \leq U^2(\rho_{t-\tau})/\bar{M}$ is applied, respectively. For simplicity, the proof of the Theorem 5.2 is divided into three steps with the idea showed in Fig. 5.1:

- Step 1: Showing that a state in mode $\mathcal{C}$ almost surely transits to mode $\mathcal{N}$ in a finite time.
- Step 2: Showing that the system switches between modes $\mathcal{C}$ and $\mathcal{N}$ in a finite number of times and the final mode is $\mathcal{N}$.

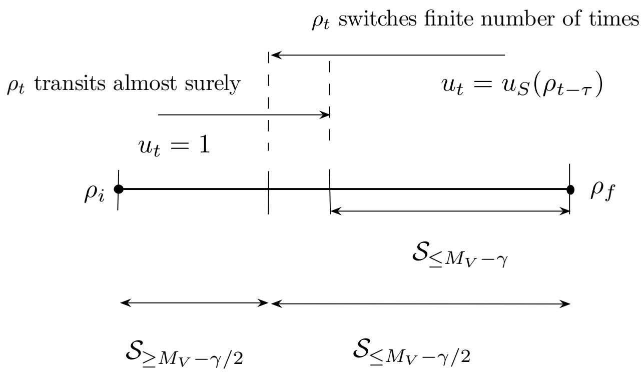


Fig. 5.1: Illustration of the hysteresis control and the proof of Theorem 5.2.

- Step 3: Showing that when the state is in mode $\mathcal{N}$ permanently, it converges to the final desired state ρ_f almost surely.

Proposition 5.1 implies that a state in mode $\mathcal{C}$ almost surely transits to mode $\mathcal{N}$ in a finite time. Step 1 is complete.

Suppose that at a time instant, the mode changes from $\mathcal{C}$ to $\mathcal{N}$. After that time instant, there are two probabilities as follows:

$\mathcal{P}_1$: the state remains in $\mathcal{N}$ permanently.

$\mathcal{P}_2$: the mode changes to $\mathcal{C}$ again.

Proposition 5.2 implies that $\mathcal{P}_2$ occurs with probability smaller than or equal to $(1-p)$. We denote the events

$$\mathcal{B}_n := \{\text{the mode switches from } \mathcal{N} \text{ to } \mathcal{C} \ n \text{ times}\}, n = 1, 2, ...$$

Then, the probability of $\mathcal{B}_n$ satisfies $\mathbb{P}\{\mathcal{B}_n\} \leq (1-p)^n$. Since

$$\sum_{n=1}^{\infty} \mathbb{P}\{\mathcal{B}_n\} \leq \sum_{n=1}^{\infty} (1-p)^n = \frac{1-p}{p} < \infty,$$

Borel-Cantelli's Lemma assures that there exist almost surely finitely many switches from mode $\mathcal{N}$ to mode $\mathcal{C}$. This, together with Step 1, implies that mode $\mathcal{N}$ is the final mode. Step 2 is complete.

As such, after a finite time, only the nonlinear control input $u_t = u_S(\rho_{t-\tau})$ is applied. From (5.24) and Theorem A.1 in [67], with the time delay nonlinear control input $u_t = u_S(\rho_{t-\tau})$, there exists a unique solution of the system (5.19). Employing the similar arguments with those of Lemma 4.10 in [57], we conclude that with the hysteresis control law defined above, for each initial data $\{\rho_\theta : -\tau \leq \theta \leq 0\}$, there exists a unique solution of the system (5.19).

Now, we proceed with Step 3. Note that in mode $\mathcal{N}$, the nonlinear control input $u_t = u_S(\rho_{t-\tau}) \leq U^2(\rho_{t-\tau})/\bar{M}$ is applied. Similar to (5.25), we conclude that the system trajectory ρ_t converges in probability to the set

$$\Big\{\rho \in \mathcal{S} : (4c\eta - \frac{M}{\bar{M}})U^2(\rho) = 0\Big\}, \tag{5.29}$$

which implies that ρ_t almost surely converges to one of eigenstates of L.

On the other hand, the nonlinear control input $u_t = u_S(\rho_{t-\tau})$ is applied only when $\rho_{t-\tau} \in \mathcal{S}_{\leq M_V - \gamma/2}$. As such, when the state is in mode $\mathcal{N}$ permanently, it is in the set $\mathcal{S}_{\leq M_V - \gamma/2}$ permanently. Therefore, ρ_t almost surely converges to one of eigenstates of L in the set $\mathcal{S}_{\leq M_V - \gamma/2}$.

Notice that in the set $\mathcal{S}_{\leq M_V - \gamma/2}$, as $1 - \mathrm{Tr}(\rho\rho_f) + cU(\rho) \leq M_V - \frac{\gamma}{2}$, by (5.27), we have

$$\mathrm{Tr}(\rho\rho_f) \geq 1 + \frac{\gamma}{2} - M_V > 0, \forall \rho \in \mathcal{S}_{\leq M_V - \gamma/2}. \tag{5.30}$$

As L is diagonal and regular, its eigenstates are $\rho_i = \mathrm{diag}\{0, ..., \underbrace{1}_{i-th}, ..., 0\}$, $i = 1, ..., n$. Thus, $\mathrm{Tr}(\rho_i\rho_f) = 0$ for all eigenstates $\rho_i \neq \rho_f$ of L. As such, by (5.30), there is only one eigenstate of L lying in the set $\mathcal{S}_{\leq M_V-\gamma/2}$, that is ρ_f. Therefore,

$$\mathbb{P}\Big\{\lim_{t\to\infty}\rho_t=\rho_f\Big\}=1. \tag{5.31}$$

Combining the three above steps, we conclude that with the time delay control law defined as in Theorem 5.2, there exists a unique solution of (5.19) for each initial data and the desired state ρ_f is globally asymptotically stable in probability. □

Remark 5.5. *As $u_t=0$ is a trivial form of the control $u_t=u_S(\rho_{t-\tau})\leq U^2(\rho_{t-\tau})/\bar{M}$, the proposed control encompasses the bang-bang control, which can be trivially implemented in practice, while still obtaining the same stability for the system. Actually, it can be seen from* (5.23) *that as $\mathcal{L}_{f+gu,\sigma}V(\rho_t,\rho_{t-\tau})$ is non-positive with the control $u_t=0$, the filter state in mode $\mathcal{N}$ converges quicker to the desired state ρ_f. This means that the bang-bang control even achieves the better convergence.*

5.5 Example: Spin-1/2 Systems

The Spin$-1/2$ systems [75] have been studied by many researchers due to their important role in quantum information processing [47]. In this section, we show that the non-smooth time delay control can solve the problem of real-time global state-transferring for the Spin$-1/2$ systems. With the Spin$-1/2$ systems, the density matrix can be represented as follows

$$\rho=\frac{1}{2}(\sigma_0+x\sigma_x+y\sigma_y+z\sigma_z) \tag{5.32}$$

where $\overrightarrow{s}=(x,y,z)\in\mathbb{R}^3$ is the Bloch vector of ρ, $\sigma_0=I_2$, and $\sigma_x,\sigma_y,\sigma_z$ are Pauli matrices

$$\sigma_x=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\sigma_y=\begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix},\sigma_z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

The map $\rho \mapsto \overrightarrow{s}$ described by (5.32) is an isomorphism between the state space $\mathcal{S}$ of ρ and the state space $\mathcal{B}$ of Bloch vectors $\overrightarrow{s}$, in which $\mathcal{B} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$.

In typical experiment settings [34], the Spin$-1/2$ system interacts with a laser field set along the $z-$axis, and with a controlled magnetic field set along the $y-$axis. Then, the measurement operator is $L = \sqrt{\mu}\sigma_z$, where $\mu > 0$ represents the strength of the interaction between the system and the laser field, and the Hamiltonians are $H_0 = 0, H_1 = \sigma_y$. A straightforward computation gives

$$L\rho_t L - \frac{1}{2}L^2\rho_t - \frac{1}{2}\rho_t L^2 = \mu(-x_t\sigma_x - y_t\sigma_y), \tag{5.33a}$$

$$-\mathrm{i}[H_1, \rho_t]u = u(z_t\sigma_x - x_t\sigma_z), \tag{5.33b}$$

$$\sigma(\rho_t) = \sqrt{\eta\mu}(-x_t z_t\sigma_x - y_t z_t\sigma_y + (1 - z_t^2)\sigma_z). \tag{5.33c}$$

Substituting (5.33) into (5.2) and by (5.32), the stochastic differential equation of the Bloch vector is

$$\begin{cases} dx_t &= 2\big((-\mu x_t + u z_t)dt - \sqrt{\eta\mu}x_t z_t dw_t\big) \\ dy_t &= 2\big(- \mu y_t dt - \sqrt{\eta\mu}y_t z_t dw_t\big) \\ dz_t &= 2\big(- u x_t dt + \sqrt{\eta\mu}(1 - z_t^2)dw_t\big). \end{cases} \tag{5.34}$$

When the system is isolated, i.e., $u = 0$, the *quantum state reduction* assures that the filter state reduces to one of eigenstates of L, which comprise of $\rho_1 = \mathrm{diag}(1, 0)$ and $\rho_2 = \mathrm{diag}(0, 1)$. This means that the state of (5.34) when $u = 0$ reduces to one of two Bloch vectors $(0, 0, 1)$ and $(0, 0, -1)$.

The simulation data are: $\overrightarrow{s_0} = (0.5, -0.5, 0), \mu = 1$. The measurement efficiency is $\eta = 0.9$. Fig. 5.2 shows sample paths of the quantum state reduction. It can be seen from Fig. 5.2 that the quantum filter state reduces stochastically to one of two states $(0, 0, 1)$ and $(0, 0, -1)$.

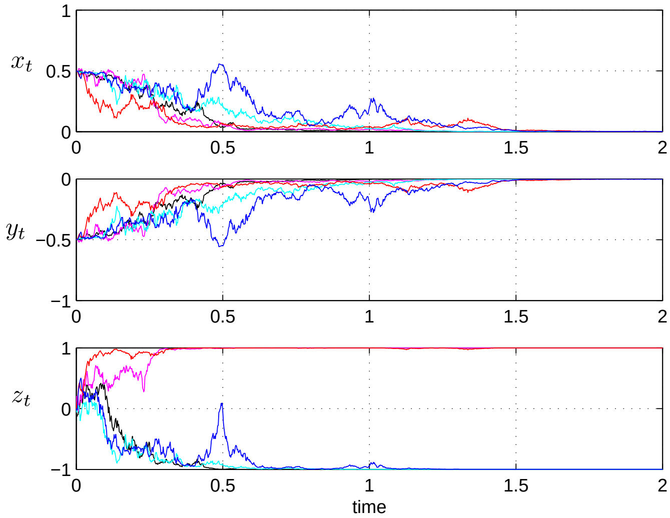


Fig. 5.2: Sample paths of the quantum state reduction in open-loop system.

We design time delay state-feedback control to globally render the quantum filter state almost surely to the final desired state $\rho_f = \text{diag}(1,0)$, i.e., $\overrightarrow{s_f} = (0,0,1)$, which is an eigenstate of the measurement operator L. As ρ_f is chosen as an eigenstate of L, Assumption 5.1 is satisfied. Since H_0, L are diagonal and L is regular, Assumption 5.2 is satisfied. Moreover, as $H_1 = \sigma_y$ is connected, Assumption 5.3 is satisfied. Hence, all the conditions of Theorem 5.2 are satisfied.

Suppose that the computation time is $\tau = 2$. Note that the control law in [78] is not applicable for the computation time $\tau \geq 1$. Following the control design procedure in Section 4, $V(\rho) = \dfrac{1}{2}(1-z) + c(1-z^2)$. The designed parameters: $c = 0.2, \gamma = 0.2, \bar{M} = 25$. Then, $M_V = 1$ and conditions (5.26), (5.27), and (5.28) hold. The proposed time delay control law becomes:

1. $u_t = 1$, if $V(\rho_{t-\tau}) \geq 0.9$;
2. $u_t = u_S(\rho_{t-\tau}) \leq (1 - z_{t-\tau}^2)^2/25$, if $V(\rho_{t-\tau}) \leq 0.8$;
3. If $\rho_{t-\tau} \in \Phi := \{\rho \in \mathcal{S} : 0.8 < V(\rho) < 0.9\}$, then $u_t = u_S(\rho_{t-\tau}) \leq (1 - z_{t-\tau}^2)^2/25$ if $\rho_{t-\tau}$ last entered Φ through the boundary $\mathcal{S}_{0.8}$ and $u_t = 1$ otherwise.

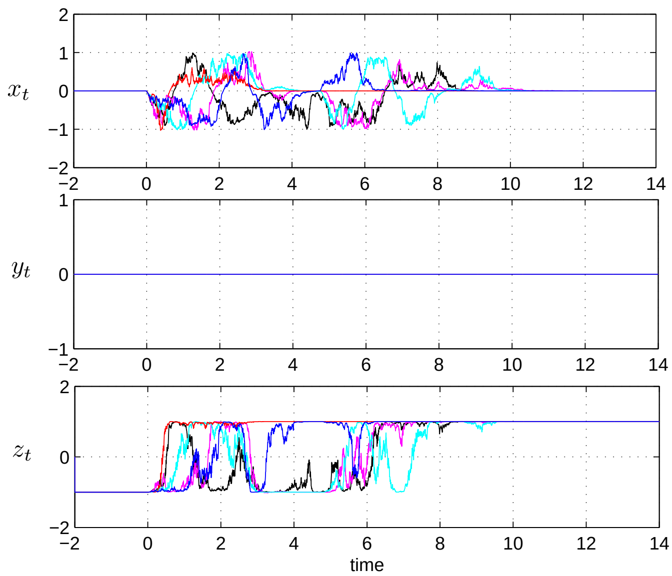


Fig. 5.3: Convergence of filter state: from $(0, 0, -1)$ to $(0, 0, 1)$.

The initial data is: $x_\theta = 0, y_\theta = 0, z_\theta = -1, \forall -\tau \leq \theta \leq 0$. The simulation results, with $u_S(\rho_{t-\tau}) = (1 - z_{t-\tau}^2)^2/25$, are showed in Figs. 5.3 and 5.4. It can be seen from Fig. 5.3 that the convergence of the filter state to the final desired state is well achieved in spite of the effect of time delay τ. The switchings in the control signal observed in Fig. 5.4 are due to the evolution of system state through the boundaries $\mathcal{S}_{0.9}$ and $\mathcal{S}_{0.8}$.

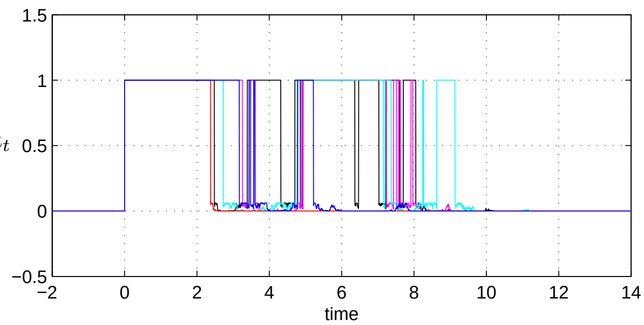


Fig. 5.4: Control input with time delay $\tau = 2$.

5.6 Conclusions

In this chapter, we have solved the problem of real-time measurement-based feedback control for quantum systems by the means of the non-smooth time delay control approach. Exploiting the observation that when the time delay is constant, its effect in a long time can be canceled by the delay-free terms, a Lyapunov-LaSalle-like theorem for delay-dependent stability in probability has been introduced for a class of stochastic time delay nonlinear systems. Based on this theorem and employing the common hysteresis-control design, time delay control in hysteresis form has been constructed to compensate for the known but arbitrarily long computation time, while globally asymptotically rendering the filter state to the final desired state almost surely. The convergence of the filter state of the Spin$-1/2$ systems to the final desired state was well obtained in the presence of long delayed time, showing that the proposed control is effective in dealing with the quantum filters.

Chapter 6

Deterministic Generation of Bell States

6.1 Introduction

6.1.1 Motivations

Entanglement [84] is a key feature that distinguishes quantum systems from classical (non-quantum) ones and has a long standing history initiated by the Einstein-Podolsky-Rosen paradox [85]. Recently, the attention on entanglement has been renewed owing to its potential use as a valuable resource of quantum computation and quantum information which outperforms that solely based on classical physics [47, 86–92]. The Bell states, named after John S. Bell who made the most significant progress towards the resolution of the EPR paradox [93], are two-qubit states possessing maximal entanglement [94], and play an essential role in quantum information science; see [84] for an overview of Bell states as the powerful tool for quantum protocols such as quantum teleportation, quantum cryptography, and quantum dense coding.

Motivated by the above considerations, we naturally raise the question: *is it possible* *to deterministically generate the Bell states by weak measurement and feedback control?* In this chapter, we shall address this question. In particular, we harness simultaneous weak measurements and Lyapunov-based feedback control to deterministically generate any desired Bell state from any initial state, i.e., to globally stabilize the desired Bell state.

6.1.2 Other Works on Feedback Control of Entanglement

The research on entanglement has been very vigorous during the last two decades. The most active area is on the entanglement of two-qubit systems thanks to the existence of the (unique) analytical measure, namely concurrence [94], to quantify two-qubit entanglement. Many works have been done in the direction of generating entanglement for two-qubit systems. Direct, or Markovian, feedback control [27,28], in which the physical measurement results are directly fed back to the system, has been extensively utilized [29–32] as it enables the real-time control implementation. Estimate, or Bayesian, feedback control [33], which is based on feeding back the estimation state conditioned on the measurement results to alter the dynamics of the systems, provides a greater flexibility in control design than direct feedback control [34]. It was used in [35] to almost globally stabilize a special Bell state when the filter state space reduces to the $2-$dimensional space of symmetric, pure states. Estimate feedback control and single measurement of the collective angular momentum operator were also exploited in [57] to generate two Bell states of two-qubit system via a detailed sample path analysis, and in [95] to produce highly entangled Dicke states of an atomic spin ensemble. This chapter will provide a way to deterministically produce maximal entanglement through the global stabilization of any Bell state by utilizing SWMs and estimate feedback control via the Lyapunov-based analysis.

6.1.3 Contributions

A fundamental physical principle states that Local Operations and Classical Communication (LOCC) cannot generate entanglement between initially separable states [96]. As such, to produce entanglement we need some nonlocal effects. For the case of Bell states, we show that if we use single measurement, then it is hard to prepare the Bell states even if the measured observable and the free Hamiltonian are nonlocal. The underlying reason is the degenerateness of the observable usually utilized in the generation of Bell states, the continuous measurement of which renders the filter state to the eigenspace of the measured observable, instead of some eigenstates, and thus the Bell states are hard to obtain.

Motivated by this difficulty, we introduce an interesting property of the quantum systems subjected to simultaneous weak measurements (SWMs). We prove that under SWMs of two commutative observables A_1 and A_2, i.e., $A_1 A_2 = A_2 A_1$, the filter state almost surely converges to the common set of the eigenspaces of A_1 and A_2, which is defined as *SWM-induced space*. We term this property as *SWM-induced quantum state reduction* owing to its consistency with the quantum state reduction postulate in quantum mechanics [46]. The SWM-induced quantum state reduction has great potential in generating quantum states. To produce a desired state, we shall perform the SWMs of two commutative observables A_1 and A_2 such that the desired state is one tangent point between eigenspaces of A_1 and A_2, i.e., the SWM-induced space becomes points including the desired state.

For the Bell states, two possible observables possessing that property are $A_1 = \sigma_1^z \otimes \sigma_2^z$ and $A_2 = \sigma_1^x \otimes \sigma_2^x$, where $\sigma_i^{x,y,z}$ are the Pauli operators and $\otimes$ denotes the tensor product of operators. From the notion of stabilizer code [47], it is known that Bell states are unique common points of eigenspaces of A_1 and A_2 [47]. As such, applying the SWM-induced quantum state reduction, we can utilize the SWMs of observables A_1 and A_2 to generate the Bell states. It is relevant to note the difference between the notions of *SWM-induced quantum state reduction* and *stabilizer code*. The stabilizer code suggests some useful observables when we apply the SWM-induced quantum state reduction to generate some desired states (e.g., Bell states), but it does not imply the SWM-induced quantum state reduction. In addition, the properties and efficiencies of stabilizer code, such as those for quantum error correction, are shown through applying projective (discrete) measurement on quantum systems [47], while the effectiveness of SWM-induced quantum state reduction manifests through the mechanism of continuous weak measurement.

Since measurements can only produce Bell states stochastically, i.e., each Bell state is generated with a positive probability, we move towards with the deterministic generation of any desired Bell state by using both SWM-induced quantum state reduction and estimate feedback control. Due to the symmetric topology of filter state space, there exist points other than the desired Bell state which are also tangent points between eigenspace of A_1 and eigenspace of A_2, and are said to be *antipodal tangent states* of the desired Bell state. Therefore, similar to the case of angular momentum systems [34, 35], the smooth controls as [34, 35], synthesized by classical smooth Lyapunov stability theory, is not sufficient to transfer the filter state from one antipodal tangent state to the desired Bell state because they are all equilibrium points of the closed-loop system. In other words, the smooth controls synthesized by classical smooth Lyapunov stability is difficult to obtain the global stabilizability for the desired Bell state.

From the above analysis, the global stabilization of the desired Bell state intuitively calls for a non-smooth control synthesis. Indeed, the non-smooth controls, based on sample path analysis [57,69,70] and non-smooth Lyapunov analysis as in previous chapters, are standard and efficient for the global stabilization of quantum states. These controls consist of different components in the desired space containing the desired state and the undesired space. A feature of these controls is that the resulting switching-number of system state between desired space and undesired space is uncontrollable (though this number of switchings is finite), which may result in longtime convergence of system state. On the other hand, entanglement is very fragile under the effect of environment. The strange phenomenon of *entanglement sudden death* (ESD) even shows that entanglement may disappear in a finite time due to environmental effect [68]. In the practical control of entanglement, it is thus desirable to drive the system state to the entangled state as fast as possible, because at the longtime instant, the environment may take dominant effect and make the control ineffective. Therefore, to effectively generate the desired Bell state, the feature of switching-number uncontrollability of existing non-smooth controls should be removed.

In this chapter, we continuously utilize the non-smooth Lyapunov-based control technique, but we shall equip it with an important feature relevant to the control of entanglement in practice. In particular, we present a non-smooth control design, based on the 1-time Lyapunov-like theorem for stability in probability, introduced in Chapter 3, which guarantees that the system state deterministically converges to the desired Bell state from any initial state, while essentially reducing the converge time of system state. Unlike the existing non-smooth control designs [57, 69, 70, 97, 98], the 1-time switching Lyapunov-based control in this chapter assures that the system state switches between desired space and undesired space no more than one time, by which the convergence time of system state is reduced considerably. This feature enables the proposed control in this chapter to be suitable with the generation of entanglement in realistic condition. We also note that the non-smooth Lyapunov-based analysis distinguishes our approach from the sample path analysis approach in [57, 69, 70].

In Section 6.2, we present the model utilized in this chapter. Section 6.3 re-investigates the quantum state reduction and points out the difficulty of single weak measurement in the generation of Bell states. The concept of SWM-induced quantum state reduction is introduced in Section 6.4 and then utilized to produce the maximal entanglement via the generation of Bell states in Section 6.5. Section 6.6 combines the SWM-induced quantum state reduction with the non-smooth Lyapunov function-based control design to globally stabilize the desired Bell state. The effectiveness of the proposed schemes is numerically illustrated in Section 6.7. Section 6.8 includes concluding remarks.

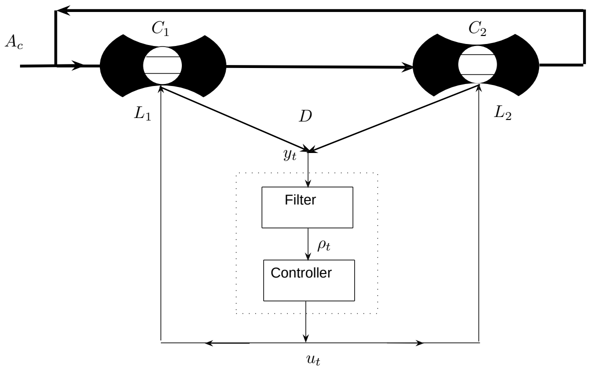


Fig. 6.1: The setup for estimate feedback control of two atoms. Two cavities C_1 and C_2, each of which contains a two-level atom, are connected in a closed loop through optical fibers. The off-resonant driving field A_c generates an effective Hamiltonian H_0. The optical fields are continuously measured by the homodyne detectors $D = \{D_1, ..., D_m\}$. The measurement records $y_t = [y_{1t}, ..., y_{mt}]$ are sent to a filter and the filter state (estimate state) ρ_t is then fed back via the controller u_t and magnetic fields L_1, L_2 to modify the system Hamiltonian.

6.2 The Model

In this chapter, we consider the two-qubit model in [99], which consists of a couple of two-level atoms, 1 and 2. These atoms are placed in two distant cavities and interact through a radiation field in a dispersive way. The two cavities are arranged in a cascade-like configuration such that, given a coherent input field with amplitude A_c in one of them, the output of each cavity enters the other as depicted in Fig. 7.1. After eliminating the radiation fields, the effective interaction Hamiltonian for the internal degrees of the two atoms becomes of Ising type, namely

$$H_0 = 2J\sigma_1^z \otimes \sigma_2^z \tag{6.1}$$

in which $\sigma_i^{x,y,z}$ are the Pauli operators of the qubit $i = 1, 2$, and J is the spin-spin coupling constant dependent on $|A_c|^2$ [99]. We assume that the coupling strength J is fixed.

To produce the Bell states, we utilize the weak measurement and estimate feedback control. A typical experiment set-up for weak measurement and estimate feedback control of two-qubit system is depicted in Fig. 7.1. Weak measurement [48,49,100,101] is modeled by a stochastic master equation (SME) which is obtained by introducing an ancillary system weakly coupled to the system of interest. Then, we make a measurement on the auxiliary system, obtain a stochastic result, and then trace it out. This leaves only the system of interest and a stochastic measurement record y. The filter (or estimate) state ρ, conditioned on the measurement record y, is considered as state of knowledge and will be fed back via estimate feedback controller to modify the dynamics of the system [34].

When considering both estimate feedback control on the system and simultaneous weak measurements over multiple observables $\{A_l\}_{l=1}^{m}$, we have the following SME or filtering equation describing the evolution of the filter state, and the stochastic measurement records [32,48] (in units such that $\hbar = 1$):

$$
\begin{aligned}
d\rho &= -\mathrm{i}[H_0, \rho]dt - \sum_k \mathrm{i}[H_k, \rho]u_k dt \\
&\quad + \sum_l \Gamma_{A_l}\mathcal{D}[A_l]\rho dt + \sum_l \sqrt{\eta_{A_l}\Gamma_{A_l}}\mathcal{H}[A_l]\rho dw_{A_l} \\
dy_l &= \mathrm{Tr}(A_l\rho)dt + \frac{1}{2\sqrt{\eta_{A_l}\Gamma_{A_l}}}dw_{A_l}, l = 1, ..., m
\end{aligned} \tag{6.2}
$$

where

- ρ is the density matrix belonging to the space $\mathcal{S} := \{\rho \in \mathbb{C}^{4\times 4} : \rho = \rho^\dagger \geq 0, \mathrm{Tr}(\rho) = 1\}$;
- H_k is the control Hamiltonian adjusted by the time-dependent control input $u_k \in \mathbb{R}$;
- $\mathcal{D}[A]\rho := A\rho A^\dagger - \frac{1}{2}(\rho A^\dagger A + A^\dagger A\rho)$;
- $\mathcal{H}[A]\rho := A\rho + \rho A^\dagger - [\mathrm{Tr}(A\rho + \rho A^\dagger)]\rho$;
- Γ_{A_l} and η_{A_l} are measurement strengths and efficiencies; and

- $\{dw_{A_l}\}$ are independent Wiener increments, $dw_{A_i} dw_{A_j} = \delta_{ij} dt$, where δ_{ij} is the Kronecker delta.

We would design the weak measurement and feedback control to render the filter state ρ_t from any initial state ρ_0 to a desired Bell state ρ_d, which is one of the Bell states. In the standard basis $\{|0\rangle, |1\rangle\}$, where $|0\rangle$ and $|1\rangle$ are Dirac notations of the two eigenstates of the qubit [75], the Bell states are represented as:

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle); |\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \tag{6.3}$$

or equivalently by density matrices:

$$\phi^{\pm} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{bmatrix}, \psi^{\pm} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.4}$$

6.3 Motivation of Simultaneous Weak Measurements

A fundamental physical principle states that Local Operations and Classical Communication (LOCC) cannot produce entanglement between initially separable states; see [96] for a proof of this principle. As such, to generate entanglement we need some nonlocal effects. In this section, we show that if we use single measurement, then it is hard to prepare the Bell states even if the measured observable and the free Hamiltonian are nonlocal. This difficulty of single measurement will be dealt with by introducing the SWM-induced quantum state reduction in the next section.

Theorem 6.1. *For the quantum system with nonlocal free Hamiltonian $H_0 = 2J\sigma_1^z \otimes \sigma_2^z$, it is impossible to make sure that the filter state almost surely converges to one of the Bell states by the single weak measurement of neither $\sigma_1^z \otimes \sigma_2^z$ nor $\sigma_1^x \otimes \sigma_2^x$.*

Proof: We consider the typical continuous weak measurement of the nonlocal observable $A = \sigma_1^z \otimes \sigma_2^z$. The analysis is similar when we utilize single weak measurement of the nonlocal observable $\sigma_1^x \otimes \sigma_2^x$. We have the following stochastic master equation and measurement record:

$$
\begin{aligned}
d\rho &= -\mathrm{i}[H_0, \rho]dt + \Gamma_A \mathcal{D}[A]\rho dt + \sqrt{\eta_A \Gamma_A}\mathcal{H}[A]\rho dw_t \\
dy &= \mathrm{Tr}(A\rho)dt + \frac{1}{2\sqrt{\eta_A \Gamma_A}} dw_t
\end{aligned}
\tag{6.5}
$$

where Γ_A and η_A are measurement strength and efficiency and dw_t is the Wiener increment. Consider the Lyapunov function candidate $U(\rho) = \mathrm{Tr}(A^2\rho) - \mathrm{Tr}^2(A\rho)$, which is the variance of the filtering process along A. A straightforward computation (see Section 1, Appendix C) gives the infinitesimal generator associated with (6.5) acting on $U(\rho)$:

$$
\mathcal{L}U(\rho_t) = -4\eta_A \Gamma_A U(\rho_t)^2 \leq 0 \tag{6.6}
$$

Applying Theorem 2.1 in Ref. [63], we achieve

$$
\mathbb{P}\{\lim_{t \to \infty} U(\rho_t) = 0\} = 1. \tag{6.7}
$$

As such, the weak measurement of the observable A renders the variance $U(\rho_t)$ of the filtering process along A to 0 almost surely. We prove that this drives the filter state to the eigenspace of A. Indeed, with the density matrix $\rho = [\rho_{ij}]_{4\times 4} \in \mathcal{S}$, then

$$
\begin{aligned}
U(\rho) &= \mathrm{Tr}(A^2\rho) - \mathrm{Tr}^2(A\rho) = \mathrm{Tr}(\rho) - \mathrm{Tr}^2(A\rho) \\
&= 1 - (\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44})^2 \\
&= 4(\rho_{11} + \rho_{44})(\rho_{22} + \rho_{33})
\end{aligned}
\tag{6.8}
$$

Hence, Eq. (6.7) is equivalent to

$$\mathbb{P}\{\lim_{t\to\infty}(\rho_{11}+\rho_{44})(\rho_{22}+\rho_{33})=0\}=1. \tag{6.9}$$

Let $\Phi_A := \{\rho \in \mathcal{S} : U(\rho) = 0\} = \{\rho \in \mathcal{S} : (\rho_{11}+\rho_{44})(\rho_{22}+\rho_{33}) = 0\} = \Phi_A^1 \cup \Phi_A^2$, where $\Phi_A^1 := \{\rho \in \mathcal{S} : \rho_{11}+\rho_{44} = 0\}$ and $\Phi_A^2 := \{\rho \in \mathcal{S} : \rho_{22}+\rho_{33} = 0\}$.

If $\rho \in \Phi_A^1$ then $\rho_{11} = \rho_{44} = 0$ since $\rho_{ii} \geq 0, i = 1, ..., 4$. Due to its positivity, ρ is of the form

$$\rho = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.10}$$

where ρ_{23}^* is the complex conjugate of ρ_{23}. Similarly, if $\rho \in \Phi_A^2$ then $\rho_{22} = \rho_{33} = 0$. From its positivity, ρ is of the form

$$\rho = \begin{bmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{bmatrix}. \tag{6.11}$$

Thus, Φ_A^1 and Φ_A^2 are eigenspaces of A corresponding to eigenvalues -1 and 1. Therefore, under weak measurement of the nonlocal observable A, the filter state converges to the eigenspace $\Phi_A = \Phi_A^1 \cup \Phi_A^2$ of A.

On the other hand, as $H_0 = 2J\sigma_1^z \otimes \sigma_2^z$, H_0 is commutative with all points in the eigenspace Φ_A. As such, any point in the eigenspace Φ_A is an equilibrium of the filter. Therefore, under the single weak measurement of the nonlocal observable A, the filter state can converge to any point in Φ_A, i.e., the limit set is equal to Φ_A. We note that though the eigenspace Φ_A contains all the Bell states, it is impossible to make sure that the filter state converges to one of the Bell states almost surely. Theorem 6.1 is proved. □

From Theorem 6.1, it can be observed that if we use single weak measurement, then it is difficult to generate the Bell states even when the free Hamiltonian and the measured observable are nonlocal. This difficulty motivated us to consider the effect of simultaneous weak measurements of multiple observables in the next section.

6.4 Simultaneous Weak Measurement-Induced Quantum State Reduction

Having pointed out in the previous section the difficulty of single weak measurement in the Bell state generation, in this section, we introduce an interesting property of quantum systems subjected to simultaneous weak measurements, termed as *SWM-induced quantum state reduction*, which will be utilized to generate the Bell states in the next section. We prove that under SWMs of two Hermitian, commutative observables A_1 and A_2, the filter state almost surely converges to the common set of eigenspace of A_1 and eigenspace of A_2.

Consider the quantum system, of arbitrary dimension and with free Hamiltonian H_0, subjected to the simultaneous weak measurements of two self-adjoint observables A_1 and A_2 that are commutative with each other and with H_0. We have the SME and measurement records:

$$\begin{aligned} d\rho &= -\mathrm{i}[H_0,\rho]dt + \Gamma_{A_1}\mathcal{D}[A_1]\rho dt + \sqrt{\eta_{A_1}\Gamma_{A_1}}\mathcal{H}[A_1]\rho dw_{A_1} \\ &\quad + \Gamma_{A_2}\mathcal{D}[A_2]\rho dt + \sqrt{\eta_{A_2}\Gamma_{A_2}}\mathcal{H}[A_2]\rho dw_{A_2} \qquad (6.12) \\ dy_1 &= \mathrm{Tr}(A_1\rho)dt + \frac{1}{2\sqrt{\eta_{A_1}\Gamma_{A_1}}}dw_{A_1} \\ dy_2 &= \mathrm{Tr}(A_2\rho)dt + \frac{1}{2\sqrt{\eta_{A_2}\Gamma_{A_2}}}dw_{A_2} \end{aligned}$$

where dw_{A_1} and dw_{A_2} are independent Wiener increments. Consider the Lyapunov function candidate

$$V(\rho) = U_1(\rho) + U_2(\rho) \tag{6.13}$$

which is a combination of the variances $U_1(\rho)$ and $U_2(\rho)$ of the filtering process along A_1 and A_2 :

$$\begin{aligned} U_1(\rho) &= \mathrm{Tr}(A_1^2\rho) - \mathrm{Tr}^2(A_1\rho), \\ U_2(\rho) &= \mathrm{Tr}(A_2^2\rho) - \mathrm{Tr}^2(A_2\rho). \end{aligned} \tag{6.14}$$

A straightforward computation (see Section 2, Appendix C) gives the infinitesimal generator associated with (6.12) acting on $V(\rho)$:

$$\begin{aligned} \mathcal{L}V(\rho_t) &= -4\eta_{A_1}\Gamma_{A_1}U_1^2(\rho_t) - 4\eta_{A_2}\Gamma_{A_2}U_2^2(\rho_t) \\ &\quad - 4(\eta_{A_1}\Gamma_{A_1} + \eta_{A_2}\Gamma_{A_2})U_{12}^2(\rho_t), \end{aligned} \tag{6.15}$$

where $U_{12}(\rho) := \mathrm{Tr}(A_1A_2\rho) - \mathrm{Tr}(A_1\rho)\mathrm{Tr}(A_2\rho)$. Hence,

$$\begin{aligned} \mathcal{L}V(\rho_t) &\le -4\eta_{A_1}\Gamma_{A_1}U_1^2(\rho_t) - 4\eta_{A_2}\Gamma_{A_2}U_2^2(\rho_t) \\ &\le -\eta_m V(\rho_t)^2 \le 0, \end{aligned} \tag{6.16}$$

where $\eta_m = 2\mathrm{min}\{\eta_{A_1}\Gamma_{A_1}, \eta_{A_2}\Gamma_{A_2}\} > 0$. Similar to Section 6.3, we conclude that

$$\mathbb{P}\lim_{t\to\infty} V(\rho_t) = 0 = 1 \tag{6.17}$$

leading to

$$\mathbb{P}\{\lim_{t\to\infty} U_1(\rho_t) = 0\} = 1, \tag{6.18}$$

and

$$\mathbb{P}\{\lim_{t\to\infty} U_2(\rho_t) = 0\} = 1. \tag{6.19}$$

Therefore, the SWMs of two commutative observables A_1 and A_2 render both variances $U_1(\rho)$ and $U_2(\rho)$ of the filtering process along A_1 and A_2 to 0 almost surely. Let $\Phi_{A_i} = \{\rho \in \mathcal{S} : U_i(\rho) = 0\}, i = 1, 2$, which by the similar analysis as in the previous section, are eigenspaces of A_1 and A_2. Then, Equations (6.18) and (6.19) imply that the filter state ρ_t converges almost surely to the common set $\Phi_{A_1} \cap \Phi_{A_2}$ of eigenspace of A_1 and eigenspace of A_2, which is defined as *simultaneous weak measurement-induced space.* We have the following property of SWMs, termed as *SWM-induced quantum state reduction* because of its consistency with the quantum state reduction postulate in quantum mechanics [46]:

Theorem 6.2. *(SWM-induced quantum state reduction) Consider the system with free Hamiltonian H_0 under simultaneous weak measurements of two Hermitian observables A_1 and A_2 that are commutative with each other and with H_0. Then, the filter state almost surely converges to the SWM-induced space, which is the common set of eigenspace of A_1 and eigenspace of A_2.* □

An illustration of the SWM-induced space is given in Fig. 6.2. It is relevant to note that the SWM-induced quantum state reduction has great potential in generating quantum states. To generate a desired quantum state, we can perform SWMs of two commutative observables such that the desired state is a tangent point of eigenspaces corresponding to these observables, i.e., the SWM-induced space becomes points including the desired state.

6.5 Probabilistic Generation of the Bell States

In Section 6.3, we showed that it is difficult, if not impossible, to generate the Bell states by single weak measurement of either $\sigma_1^z \otimes \sigma_2^z$ or $\sigma_1^x \otimes \sigma_2^x$, where $\sigma_i^{x,y,z}$ are the Pauli operators

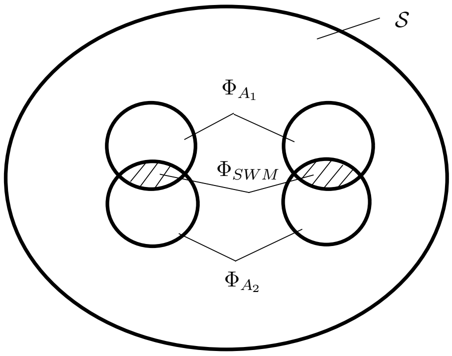


Fig. 6.2: Illustration of the *simultaneous weak measurement-induced space* Φ_{SWM} which is the attractive set of filter state under the SWMs of two commutative observables A_1 and A_2.

of the qubit $i = 1, 2$. In this section, exploiting the SWM-induced quantum state reduction, we show that the Bell states can be produced by the joint measurements of two observables $A_1 = \sigma_1^z \otimes \sigma_2^z$ and $A_2 = \sigma_1^x \otimes \sigma_2^x$. This clearly shows the significance of SWM-induced quantum state reduction.

Theorem 6.3. *Consider the system with free Hamiltonian H_0 as in (6.1), subjected to the SWMs of two commutative observables $A_1 = \sigma_1^z \otimes \sigma_2^z$ and $A_2 = \sigma_1^x \otimes \sigma_2^x$. Then, from any initial state, the filter state ρ_t converges to one of the Bell states almost surely.*

Proof: It can be verified that the observables A_1 and A_2 are Hermitian and A_1, A_2, and $H_0 = 2J\sigma_1^z \otimes \sigma_2^z$ are commutative to each other. As such, all conditions of Property 4.1 are satisfied. From the notion of stabilizer code, we know that Bell states are unique common points of eigenspaces of A_1 and A_2 (page 454 [47]). In other words, the SWM-induced space associated with A_1 and A_2 becomes Bell states. Applying Property 4.1, we conclude that under SWMs of commutative observables $A_1 = \sigma_1^z \otimes \sigma_2^z$ and $A_2 = \sigma_1^x \otimes \sigma_2^x$, the filter state ρ_t converges to one of the Bell states almost surely, i.e., Theorem 6.3 is proved.

For completeness, we provide a detailed proof for the fact that the SWM-induced space Φ_{SWM} associated with A_1 and A_2 reduces to the Bell states. Let the density matrix be $\rho = [\rho_{ij}]_{4\times 4} \in \mathcal{S}$. In the standard basis $\{|0\rangle, |1\rangle\}$, we have

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{6.20}$$

Since $A_1^2 = A_2^2 = I_4$, it holds that

$$U_1(\rho) = \mathrm{Tr}(A_1^2\rho) - \mathrm{Tr}^2(A_1\rho) = 1 - (\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44})^2 \tag{6.21a}$$

$$U_2(\rho) = \mathrm{Tr}(A_2^2\rho) - \mathrm{Tr}^2(A_2\rho) = 1 - (\rho_{14} + \rho_{23} + \rho_{32} + \rho_{41})^2 \tag{6.21b}$$

Noting that $\rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} = \mathrm{Tr}(\rho) = 1$ and $\rho_{ii} \geq 0, \forall i = 1, .., 4$, we conclude that $U_1(\rho) = 0$ iff

$$\rho_{11} = \rho_{44} = 0 \text{ or } \rho_{22} = \rho_{33} = 0. \tag{6.22}$$

Thus, the eigenspace of A_1 is

$$\Phi_{A_1} = \{\rho = [\rho_{ij}]_{4\times 4} \in \mathcal{S} : \rho_{11} = \rho_{44} = 0 \text{ or } \rho_{22} = \rho_{33} = 0\} \tag{6.23}$$

Noticing the positivity and self-adjointness of ρ, we have

$$\begin{aligned}
|\rho_{14} + \rho_{23} + \rho_{32} + \rho_{41}| &\le |\rho_{14} + \rho_{41}| + |\rho_{23} + \rho_{32}| \\
&= 2|\mathrm{Re}(\rho_{14})| + 2|\mathrm{Re}(\rho_{23})| \\
&\le 2|\rho_{14}| + 2|\rho_{23}| \\
&\le 2\sqrt{\rho_{11}\rho_{44}} + 2\sqrt{\rho_{22}\rho_{33}} \\
&\le \rho_{11} + \rho_{44} + \rho_{22} + \rho_{33} = 1
\end{aligned} \tag{6.24}$$

As such, by (6.21b), it holds that $U_2(\rho) = 0$ iff

$$\rho_{14} = \rho_{41} = \pm\rho_{11} = \pm\rho_{44} \text{ and } \rho_{23} = \rho_{32} = \pm\rho_{22} = \pm\rho_{33} \tag{6.25}$$

Hence, the eigenspace of A_2 is

$$\begin{aligned}
\Phi_{A_2} = \{\rho = [\rho_{ij}]_{4\times4} \in \mathcal{S} : &\rho_{14} = \rho_{41} = \pm\rho_{11} = \pm\rho_{44} \text{ and} \\
&\rho_{23} = \rho_{32} = \pm\rho_{22} = \pm\rho_{33}\}
\end{aligned} \tag{6.26}$$

Combining (6.23) with (6.26), we conclude that $\Phi_{SWM} = \Phi_{A_1} \cap \Phi_{A_2} = \{\phi^\pm, \psi^\pm\}$, in which

$$\phi^\pm = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \pm1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm1 & 0 & 0 & 1 \end{bmatrix}, \psi^\pm = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm1 & 0 \\ 0 & \pm1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This mean that the SWM-induced space Φ_{SWM} reduces to the Bell states. $\square$

Remark 6.1. *As all the Bell states possess maximal entanglement, measured by concurrence [94], Theorem 6.3 provides an interesting way based on SWMs to deterministically produce maximal entanglement, which is a valuable resource in both quantum information and quantum computation [47].*

6.6 Non-smooth Lyapunov Function-Based Global Stabilization of the Bell States

As measurements can only stochastically generate Bell states, i.e., each Bell state is generated with a positive probability, in this section, we harness the SWM-induced state reduction and estimate feedback control to deterministically generate any desired Bell state from any initial state, i.e., to globally asymptotically stabilize the desired Bell state. Firstly, we prove that the smooth controls [34, 35], synthesized by smooth Lyapunov theory [54–56], are difficult to obtain the global stabilizability for the desired Bell state because of the existence of the antipodal tangent points of that Bell state. Then, we introduce a discontinuous Lyapunov-like theorem for stability in probability and apply it to design non-smooth controls that globally asymptotically stabilize the desired Bell state.

Consider the SWMs of two commutative observables $A_1 = \sigma_1^z \otimes \sigma_2^z$ and $A_2 = \sigma_1^x \otimes \sigma_2^x$, and the arbitrary feedback control operator, given by the control Hamiltonian H_1, on the system. We have the SME and measurement records:

$$\begin{aligned}
d\rho &= -\mathrm{i}[H_0, \rho]dt - \mathrm{i}[H_1, \rho]udt \\
&\quad + \Gamma_{A_1}\mathcal{D}[A_1]\rho dt + \sqrt{\eta_{A_1}\Gamma_{A_1}}\mathcal{H}[A_1]\rho dw_{A_1} \\
&\quad + \Gamma_{A_2}\mathcal{D}[A_2]\rho dt + \sqrt{\eta_{A_2}\Gamma_{A_2}}\mathcal{H}[A_2]\rho dw_{A_2} \\
dy_1 &= \mathrm{Tr}(A_1\rho)dt + \frac{1}{2\sqrt{\eta_{A_1}\Gamma_{A_1}}}dw_{A_1} \\
dy_2 &= \mathrm{Tr}(A_2\rho)dt + \frac{1}{2\sqrt{\eta_{A_2}\Gamma_{A_2}}}dw_{A_2}
\end{aligned} \tag{6.27}$$

where dw_{A_1} and dw_{A_2} are independent Wiener increments.

6.6.1 Motivation of Non-smooth Lyapunov-Based Control

In Section 6.5, we have presented that SWMs of commutative observables can be used to produce Bell states in the means that under SWMs of two commutative observables, the filter state converges to one of the Bell states almost surely. In this section, however, we show that a combination of SWMs and smooth feedback controls [34, 35], designed via smooth Lyapunov stability theory [54–56], is not sufficient for the global stabilization of the desired Bell state. This intuitively shows the necessity of a non-smooth Lyapunov-like theory to the global stabilization of Bell states.

Consider the smooth Lyapunov function candidate of the distance/variance form [53]

$$V_S(\rho) = 1 - \mathrm{Tr}(\rho\rho_d) + c\big(U_1(\rho) + U_2(\rho)\big) \tag{6.28}$$

Note that the desired Bell state ρ_d is commutative with A_1, A_2, and H_0 because all the Bell states are eigenstates of these operators. As such, $\mathrm{Tr}(\rho_d[H_0, \rho]) = 0, \mathrm{Tr}(\rho_d\mathcal{D}[A_1]\rho) = \mathrm{Tr}(\rho_d\mathcal{D}[A_2]\rho) = 0$. Therefore,

$$\begin{aligned} d\mathrm{Tr}(\rho_t\rho_d) = \mathrm{Tr}(-\mathrm{i}\rho_d[H_1, \rho_t])udt \\ &+ \sqrt{\eta_{A_1}\Gamma_{A_1}}\mathrm{Tr}(\rho_d\mathcal{H}[A_1]\rho_t)dw_{A_1} \\ &+ \sqrt{\eta_{A_2}\Gamma_{A_2}}\mathrm{Tr}(\rho_d\mathcal{H}[A_2]\rho_t)dw_{A_2} \end{aligned} \tag{6.29}$$

As such, the infinitesimal generator associated with (6.27) acting on $\mathrm{Tr}(\rho\rho_d)$ is

$$\mathcal{L}\mathrm{Tr}(\rho_t\rho_d) = \mathrm{Tr}(-\mathrm{i}\rho_d[H_1, \rho_t])u \tag{6.30}$$

From this and similar to (6.15), the infinitesimal generator associated with (6.27) acting on

$V_S(\rho)$ is

$$\begin{aligned}\mathcal{L}V_S(\rho_t) = &-\Big(\mathrm{Tr}(-\mathrm{i}\rho_d[H_1,\rho_t]) + 2c\mathrm{Tr}(A_1\rho_t)\mathrm{Tr}(-\mathrm{i}A_1[H_1,\rho_t])\\ &+ 2c\mathrm{Tr}(A_2\rho_t)\mathrm{Tr}(-\mathrm{i}A_2[H_1,\rho_t])\Big)u\\ &- 4\eta_{A_1}\Gamma_{A_1}U_1^2(\rho_t) - 4\eta_{A_2}\Gamma_{A_2}U_2^2(\rho_t)\\ &- 4(\eta_{A_1}\Gamma_{A_1} + \eta_{A_2}\Gamma_{A_2})U_{12}^2(\rho_t)\\ &:= -g_S(\rho_t)u - 4\eta_{A_1}\Gamma_{A_1}U_1^2(\rho_t) - 4\eta_{A_2}\Gamma_{A_2}U_2^2(\rho_t)\\ &- 4(\eta_{A_1}\Gamma_{A_1} + \eta_{A_2}\Gamma_{A_2})U_{12}^2(\rho_t) \end{aligned} \tag{6.31}$$

where $U_{12}(\rho) := \mathrm{Tr}(A_1A_2\rho) - \mathrm{Tr}(A_1\rho)\mathrm{Tr}(A_2\rho)$. With the natural smooth control $u = lg_S(\rho)$, where $l > 0$, then, $\mathcal{L}V_S(\rho_t)$ becomes

$$\begin{aligned}\mathcal{L}V_S(\rho_t) = &-lg_S^2(\rho_t) - 4\eta_{A_1}\Gamma_{A_1}U_1^2(\rho_t) - 4\eta_{A_2}\Gamma_{A_2}U_2^2(\rho_t)\\ &- 4(\eta_{A_1}\Gamma_{A_1} + \eta_{A_2}\Gamma_{A_2})U_{12}^2(\rho_t) \le 0\end{aligned} \tag{6.32}$$

Similar to Section 6.5, this also implies that the filter state converges to one of the Bell states. However, it can be verified that $[H_0, \rho_B] = \mathcal{D}[A_i]\rho_B = \mathcal{H}[A_i]\rho_B = g_S(\rho_B) = 0$ for any Bell state ρ_B and $i = 1, 2$. Hence, all the Bell states are equilibrium points of the closed-loop system (6.27) composed of the above smooth control. As such, we cannot transfer the system from one antipodal tangent state, i.e., the other Bell state, to the desired Bell state by the above smooth control. Therefore, the smooth controls designed via the classical smooth Lyapunov theory are hard to achieve the global stabilizability for the desired Bell state because of the existence of its antipodal tangent states (the other Bell states). This intuitively calls for non-smooth controls which are synthesized via non-smooth Lyapunov theory.

6.6.2 Non-smooth, Globally Stabilizing Control Design

In this section, we present an application of the SWM-induced quantum state and non-smooth Lyapunov-based control to the global stabilization of the desired Bell state ρ_d. Basically, we aim at making the closed-loop system (6.27) to fulfill conditions of the discontinuous Lyapunov-like Theorem 3.4 in Chapter 3.

1. Control Design

In order to break the symmetric topology of the filter state space, we shall choose the Lyapunov function candidate $V(\rho)$ such that the coefficient of u in $\mathcal{L}V(\rho)$ is equal to 0 at ρ_d, while being different from 0 at all antipodal tangent states ρ_a of ρ_d, i.e.,

$$\frac{\partial V(\rho)}{\partial \rho}[H_1, \rho]\Big|_{\rho=\rho_d} = 0, \tag{6.33a}$$

$$\frac{\partial V(\rho)}{\partial \rho}[H_1, \rho]\Big|_{\rho=\rho_a} \neq 0, \forall \rho_a. \tag{6.33b}$$

Condition (6.33b) can be satisfied via the following lemma.

Lemma 6.1. *If the matrix $H = diag[h_1, ..., h_n] \in \mathbb{C}^{n\times n}$ satisfies that $h_i \neq h_j, \forall i \neq j$, then for any zero-diagonal matrix B, i.e., matrix with all diagonal entries being zero, there exists a matrix $Y \in \mathbb{C}^{n\times n}$ such that*

$$-i[Y, H] = B. \tag{6.34}$$

Proof: Let the matrix $Y = [y_{ij}]_{n\times n}$. As $H = \mathrm{diag}[h_1, ..., h_n]$, we have

$$-\mathrm{i}[Y, H] = -\mathrm{i}[(h_j - h_i)y_{ij}]_{n\times n}. \tag{6.35}$$

Let the zero-diagonal matrix $B = [b_{ij}]_{n\times n}$, in which $b_{11} = ... = b_{nn} = 0$. Then, we can take a solution $Y = [y_{ij}]_{n\times n}$ of Equation (6.34) as follows:

$$y_{ij} = \begin{cases} \mathrm{i}\dfrac{b_{ij}}{h_j - h_i}, & i \neq j \\ \text{arbitrary}, & i = j. \end{cases} \tag{6.36}$$

Lemma 6.1 is proved. $\square$

Let $A = I_4 - 4\rho_d$. Note that $\mathrm{Tr}(A) = 0$. As any square matrix with zero trace is unitarily similar to a square zero-diagonal matrix [102], there exists a unitary matrix P such that the matrix $B = P^{-1}AP$ has all zero diagonal entries. Choose $H = \sigma_1^z \otimes I_2 + 0.5 I_2 \otimes \sigma_2^z = \mathrm{diag}[1.5, 0.5, -0.5, -1.5]$. Applying Lemma 6.1, there exists $Y \in \mathbb{C}^{4\times 4}$ such that

$$-\mathrm{i}[Y, H] = B \tag{6.37}$$

Let the control Hamiltonian be $H_1 = PHP^{-1}$. Let $X = PYP^{-1}$. Then, Equation (6.37) leads to

$$-\mathrm{i}[X, H_1] = PBP^{-1} = A = I_4 - 4\rho_d. \tag{6.38}$$

This equation plays an important role in the design of non-smooth Lyapunov function-based control. A corollary of of Equation (6.38) is that the coefficient of u in $\mathcal{L}\mathrm{Tr}(X\rho)$ is different from 0 at all antipodal tangent states ρ_a of ρ_d :

$$\begin{aligned} \mathrm{Tr}(-\mathrm{i}X[H_1, \rho_a]) &= \mathrm{Tr}(-\mathrm{i}[X, H_1]\rho_a) \\ &= \mathrm{Tr}(\rho_a) - 4\mathrm{Tr}(\rho_d\rho_a) = 1 > 0. \end{aligned} \tag{6.39}$$

Noting this corollary, we construct a Lyapunov function candidate satisfying (6.33).

Consider the discontinuous Lyapunov function candidate

$$V(\rho) = \begin{cases} V_1(\rho), & \rho \in \Phi_1, \\ V_2(\rho), & \rho \in \Phi_2, \end{cases} \tag{6.40}$$

where

$$V_1(\rho) = 1 - \mathrm{Tr}(\rho\rho_d) + c\big(U_1(\rho) + U_2(\rho)\big), \tag{6.41a}$$

$$V_2(\rho) = a + \mathrm{Tr}(X\rho), a > \max_{\rho\in\mathcal{S}} |\mathrm{Tr}(X\rho)|, \tag{6.41b}$$

$$\Phi_1 := \{\rho \in \mathcal{S} : V_1(\rho) \leq k\}, \Phi_2 := \mathcal{S} \setminus \Phi_1, \tag{6.41c}$$

and the positive constants $0 < k < 1$ and c are chosen such that

$$1 - k + c\max_{\rho\in\mathcal{S}}(U_1(\rho) + U_2(\rho)) < 1/4. \tag{6.42}$$

Note that $V_1(\rho_d) = 0$ and $V_1(\rho_a) = 1$ for all antipodal tangent states ρ_a of ρ_d. As such, by (6.41c), we conclude that $\rho_d \in \Phi_1$ and $\rho_a \in \Phi_2$ for all antipodal tangent states ρ_a of ρ_d. In Φ_1, we choose the control law

$$u = u_1(\rho) = lg_S(\rho), l > 0, \rho \in \Phi_1. \tag{6.43}$$

Then, from (6.32), the infinitesimal generator associated with (6.27) acting on $V_1(\rho)$ is

$$\begin{aligned}\mathcal{L}V_1(\rho) &= -lg_S^2(\rho) - 4\eta_{A_1}\Gamma_{A_1}U_1^2(\rho) - 4\eta_{A_2}\Gamma_{A_2}U_2^2(\rho) \\ &\quad - 4(\eta_{A_1}\Gamma_{A_1} + \eta_{A_2}\Gamma_{A_2})U_{12}^2(\rho) \\ &:= -W(\rho) \leq 0, \forall\rho \in \Phi_1. \end{aligned} \tag{6.44}$$

Now, we calculate the infinitesimal generator associated with (6.27) that acts on $V_2(\rho)$.

For convenience, in (6.27), we denote

$$\begin{aligned} f(\rho) &= -\mathrm{i}[H_0, \rho] + \Gamma_{A_1}\mathcal{D}[A_1]\rho + \Gamma_{A_2}\mathcal{D}[A_2]\rho \\ \sigma(\rho) &= \begin{bmatrix} \sqrt{\eta_{A_1}\Gamma_{A_1}}\mathcal{H}[A_1]\rho & \sqrt{\eta_{A_2}\Gamma_{A_2}}\mathcal{H}[A_2]\rho \end{bmatrix} \\ w_t &= \begin{bmatrix} w_{A_1} & w_{A_2} \end{bmatrix}^T \end{aligned}$$

Then, (6.27) is equivalent with

$$d\rho = -\mathrm{i}[H_1, \rho]udt + f(\rho)dt + \sigma(\rho)dw_t \tag{6.45}$$

Note that as dw_{A_1} and dw_{A_2} are independent Wiener increments, dw_t is a standard $2-$dimensional Wiener increment. As such,

$$\begin{aligned} dV_2(\rho) &= d\mathrm{Tr}(X\rho) = \mathrm{Tr}(Xd\rho) \\ &= \mathrm{Tr}\big(X(-\mathrm{i}[H_1, \rho]udt + f(\rho)dt) + \sigma(\rho)dw_t\big) \\ &= \mathrm{Tr}(-\mathrm{i}[X, H_1]\rho)udt + \mathrm{Tr}(Xf(\rho))dt + \mathrm{Tr}(X\sigma(\rho))dw_t \end{aligned}$$

By (6.38), we have

$$\mathrm{Tr}(-\mathrm{i}[X, H_1]\rho) = \mathrm{Tr}((I_4 - 4\rho_d)\rho) = 1 - 4\mathrm{Tr}(\rho\rho_d). \tag{6.46}$$

Therefore, $dV_2(\rho)$ becomes

$$dV_2(\rho) = (1 - 4\mathrm{Tr}(\rho\rho_d))udt + \mathrm{Tr}(Xf(\rho))dt + \mathrm{Tr}(X\sigma(\rho))dw_t$$

As such, the infinitesimal generator associated with (6.27) acting on $V_2(\rho)$ is

$$\mathcal{L}V_2(\rho) = \big(1 - 4\mathrm{Tr}(\rho\rho_d)\big)u + \mathrm{Tr}(Xf(\rho)). \tag{6.47}$$

We prove that the gain $\big(1-4\mathrm{Tr}(\rho\rho_d)\big)$ of u in the infinitesimal generator $\mathcal{L}V_2(\rho)$ is always positive for any density matrix ρ in the set Φ_2. Indeed, by definition, in Φ_2, we have

$$k < V_1(\rho) = 1 - \mathrm{Tr}(\rho\rho_d) + c(U_1(\rho) + U_2(\rho)) \tag{6.48}$$

This, together with (6.42), yields

$$\mathrm{Tr}(\rho\rho_d) < 1 - k + c(U_1(\rho) + U_2(\rho)) < 1/4, \forall \rho \in \Phi_2. \tag{6.49}$$

As such, the control gain $\big(1 - 4\mathrm{Tr}(\rho\rho_d)\big)$ is positive for all ρ in Φ_2. Therefore, in Φ_2, we can choose the control:

$$u = u_2(\rho) = \frac{-M - \mathrm{Tr}(Xf(\rho))}{1 - 4\mathrm{Tr}(\rho\rho_d)}, M > 0, \rho \in \Phi_2, \tag{6.50}$$

by which the infinitesimal generator $\mathcal{L}V_2(\rho)$ in (6.47) becomes

$$\mathcal{L}V_2(\rho) = -M, \forall \rho \in \Phi_2. \tag{6.51}$$

It is relevant to note that the control composed of (6.43) and (6.50) is a switching control. To make the control continuous and easier to be implemented, we define the set

$$\Phi_2^{\epsilon} := \{\rho \in \mathcal{S} : V_1(\rho) \geq k + \epsilon\} \subset \Phi_2, \tag{6.52}$$

with $\epsilon > 0$ sufficiently small. Let $\Phi = \Phi_2 \setminus \Phi_2^{\epsilon} = \{\rho \in \mathcal{S} : k < V_1(\rho) < k + \epsilon\}$. Then, we have the continuous control

$$u(\rho) = \begin{cases} u_1(\rho), & \rho \in \Phi_1, \\ u_2(\rho), & \rho \in \Phi_2^{\epsilon}, \\ u_3(\rho), & \rho \in \Phi, \end{cases} \tag{6.53}$$

where $u_3(\rho) := \frac{1}{\epsilon}\big((k+\epsilon-V_1(\rho))u_1(\rho)+(V_1(\rho)-k)u_2(\rho)\big)$.

2. Stability Analysis

We are ready to prove that a combination of SWM-induced quantum state reduction and the non-smooth Lyapunov-based control is sufficient to deterministically generate the desired Bell state from any initial state.

Theorem 6.4. *Consider the SME* (6.27) *with the control Hamiltonian* H_1 *chosen as in* (6.38) *and the continuous control* (6.53)*. Then, there exists* $\epsilon > 0$ *sufficiently small such that the desired Bell state* ρ_d *of the closed-loop system is globally asymptotically stable in probability.*

Proof: In the sequel, a function $\phi(\rho)$ is called positive definite with respect to ρ_d on a set Ω containing ρ_d if $\phi(\rho) \geq 0, \forall \rho \in \Omega$, and $\phi(\rho_e) = 0, \rho_e \in \Omega$, iff $\rho_e = \rho_d$. It can be checked that $V_1(\rho)$ is positive definite with respect to ρ_d on the set Φ_1. As a is chosen such that $a > \max_{\rho\in\mathcal{S}} |\mathrm{Tr}(X\rho)|$, it holds that $V_2(\rho) = a + \mathrm{Tr}(X\rho) > 0$ for all $\rho \in \Phi_2$. Therefore, $V(\rho)$ is positive definite with respect to ρ_d on the set $\mathcal{S}$. Condition $C.1$ of Theorem 3.4 is satisfied.

By the continuous control (6.53), from (6.44) and (6.51), we have the infinitesimal of $V(\rho)$ along (6.27):

$$\mathcal{L}V(\rho) = \begin{cases} \mathcal{L}V_1(\rho) = -W(\rho) \leq 0, & \rho \in \Phi_1, \\ \mathcal{L}V_2(\rho) = -M < 0, & \rho \in \Phi_2^{\epsilon}. \end{cases} \tag{6.54}$$

As $u(\rho)$ is continuous w.r.t ρ, the infinitesimal generator $\mathcal{L}V_2(\rho)$ is continuous w.r.t ρ also. Therefore, from (6.54), there exists $\epsilon > 0$ sufficiently small such that

$$\mathcal{L}V_2(\rho) < -\frac{M}{2}, \rho \in \Phi \tag{6.55}$$

This, together with (6.54), leads to

$$\mathcal{L}V(\rho) = \begin{cases} \mathcal{L}V_1(\rho) = -W(\rho) \leq 0, & \rho \in \Phi_1, \\ \mathcal{L}V_2(\rho) < -\dfrac{M}{2} < 0, & \rho \in \Phi_2^\epsilon \cup \Phi = \Phi_2. \end{cases} \tag{6.56}$$

Therefore, Conditions C.2 and C.3 of Theorem 3.4 are satisfied.

It can also be verified that the function $W(\rho)$ is positive definite with respect to ρ_d on the set Φ_1 because ρ_d is the unique Bell state in Φ_1. Thus, all the conditions of Theorem 3.4 are satisfied. By the same analysis as in the proof of Theorem 3.4, we conclude that there is no sliding motion of ρ_t on the boundary $\Lambda := \{\rho \in \mathcal{S} : V_1(\rho) = k\}$ between Φ_1 and Φ as well as on the boundary $\Lambda^\epsilon := \{\rho \in \mathcal{S} : V_1(\rho) = k + \epsilon\}$ between Φ_2^ϵ and Φ. As such, the system trajectory ρ_t intersects Λ and Λ^ϵ at separated time instants, which are denoted as $\tau_1, \tau_2, ...$ Similar to Proposition 3.5 in [57], we conclude that in the intervals between these two consecutive time instants, there exists a unique segment ρ_t with the smooth control $u_1(\rho)$ or $u_2(\rho)$ or $u_3(\rho)$. Jointing these consecutive segments, from any initial state ρ_0, we obtain a unique solution ρ_t with the above continuous control (6.53). As such, under the continuous control (6.53), the closed-loop system is well-posed.

As all conditions of Theorem Theorem 3.4 are satisfied, we conclude that the desired Bell state ρ_d of the closed-loop system is globally stable in probability, i.e., the filter state is deterministically driven to the desired Bell state ρ_d from any initial state ρ_0. $\square$

6.7 Numerical Illustration

In this section, we illustrate the effectiveness of the above SWM-induced quantum state reduction and non-smooth Lyapunov function-based control schemes in the generation of the Bell states for two-qubit system. The entanglement of a two-qubit state $\rho \in \mathcal{S}$ is quantified by the concurrence [94] defined as

$$C(\rho) = \max\{\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0\} \tag{6.57}$$

where λ_i are the eigenvalues, in decreasing order, of the matrix $\rho(\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$. ρ^* is the complex conjugate of ρ in the standard basis $\{|0\rangle, |1\rangle\}$ and the Pauli operator

$$\sigma^y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \tag{6.58}$$

in the same basis. Note that $0 \leq C(\rho) \leq 1$ for all $\rho \in \mathcal{S}$ and $C(\rho) = 1$ implies ρ being one of the Bell states.

We quantify the distance between a state ρ and the desired Bell state ρ_d by the function

$$d(\rho) = 1 - \mathrm{Tr}(\rho\rho_d). \tag{6.59}$$

It is obvious that $0 \leq d(\rho) \leq 1, \forall \rho \in \mathcal{S}, d(\rho_d) = 0$, and $d(\rho_a) = 1$ for all antipodal tangent states ρ_a of ρ_d because all the antipodal tangent states ρ_a are Bell states. Therefore, $C(\rho) = 1$ and $d(\rho) = 0$ iff $\rho = \rho_d$, while $C(\rho) = 1$ and $d(\rho) = 1$ iff ρ being one of the antipodal tangent states ρ_a of ρ_d.

6.7.1 SWM-Induced Quantum State Reduction

This section illustrates the effectiveness of the SWM-induced quantum state reduction associated with two commutative observables $A_1 = \sigma_1^z \otimes \sigma_2^z$ and $A_2 = \sigma_1^x \otimes \sigma_2^x$ in the generation of the Bell states. Let the spin-spin coupling constant J be 0.05. Let the measurement strengths $\Gamma_{A_1} = \Gamma_{A_2} = 0.9$ and the measurement efficiencies $\eta_{A_1} = \eta_{A_2} = 0.8$.

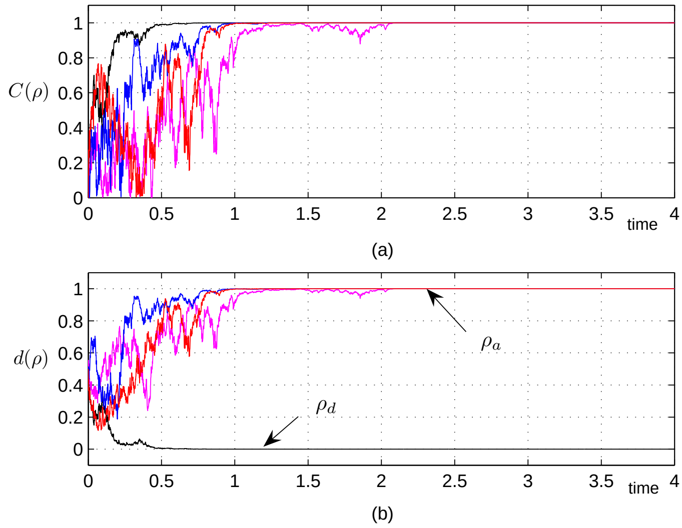


Fig. 6.3: SWM-induced quantum state reduction of four arbitrary sample paths. (a) Concurrence. (b) Distance from ρ_t to ρ_d.

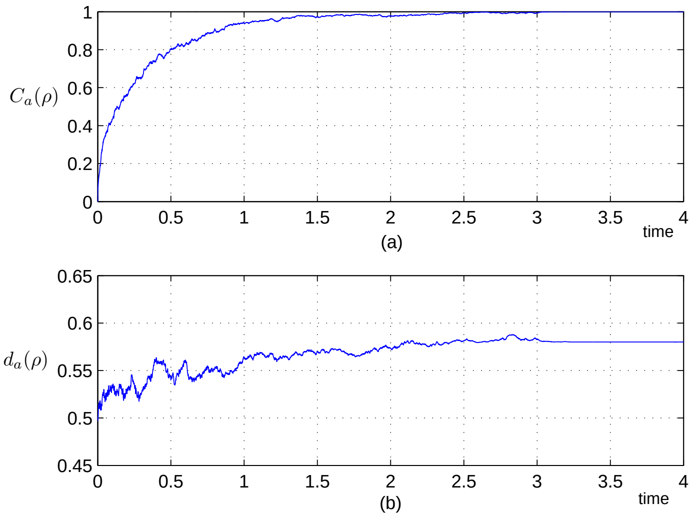


Fig. 6.4: SWM-induced quantum state reduction in average over 100 sample paths. (a) Average concurrence. (b) Average distance.

The numerical illustration is carried out with the initial condition

$$\rho_0 = \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7 \end{bmatrix} \tag{6.60}$$

which is an unentangled state. The simulation results with 4 arbitrary sample paths are showed in Fig. 6.3. It can be observed from Fig. 6.3(a) that, in all sample paths, the filter state is driven from the unentangled initial state ρ_0 to the states with maximal entanglement $C(\rho) = 1$, i.e., to one of the Bell states. As such, the maximal entanglement of two-atom system is well produced as the consequence of the SWM-induced quantum state reduction associated with two commutative observables $A_1 = \sigma_1^z \otimes \sigma_2^z$ and $A_2 = \sigma_1^x \otimes \sigma_2^x$.

We can also observe from Fig. 6.3(b) that the distance $d(\rho)$ is stochastically driven to one of two values 0 and 1. This, together with the fact $C(\rho_\infty) = 1$, shows that the filter state is rendered to one of the Bell states, each Bell state with a positive probability.

Fig. 6.4 shows the averages of concurrence $C(\rho)$ and distance $d(\rho)$ over 100 sample paths. It can be observed from Fig. 6.4 that the entanglement is almost surely enhanced to its maximal value 1, while the distance converges in average to 0.58, meaning that the filter state converges to one of the Bell states, each of which with a positive probability. Therefore, the simulation results in Figs. 6.3 and 6.4 show that the SWMs can be used to deterministically produce the maximal entanglement, though the generation of Bell states is probabilistic.

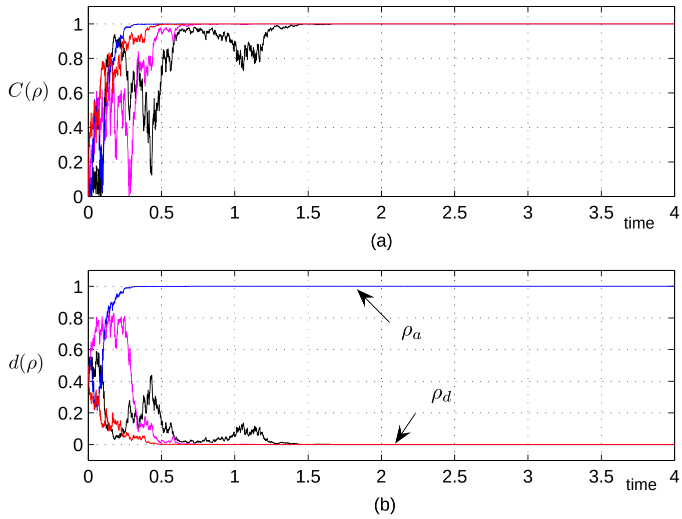


Fig. 6.5: SWM-induced quantum state reduction of four arbitrary sample paths under smooth control. (a) Concurrence. (b) Distance from ρ_t to ρ_d.

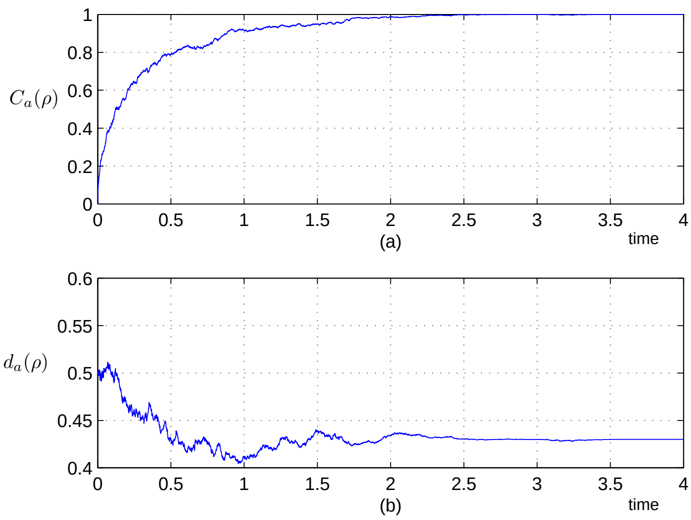


Fig. 6.6: SWM-induced quantum state reduction in average over 100 sample paths under smooth control. (a) Average concurrence. (b) Average distance.

6.7.2 SWM-Induced Quantum State Reduction under Smooth Control

We consider the SWM-induced quantum state reduction under the smooth controls as in Section 6.6.1. The desired Bell state is

$$\rho_d = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \tag{6.61}$$

Choose the standard control Hamiltonian $H_1 = \sigma_1^y \otimes I_2 + I_2 \otimes \sigma_2^y$, which can be physically implemented by applying the magnetic fields to the $y-$axis of both atoms. From Section 6.6.1, we obtain the smooth control:

$$\begin{aligned} u_S = lg_S(\rho) &= l\big(\mathrm{Tr}(-\mathrm{i}\rho_d[H_1,\rho]) + 2c\mathrm{Tr}(A_1\rho)\mathrm{Tr}(-\mathrm{i}A_1[H_1,\rho]) \\ &+ 2c\mathrm{Tr}(A_2\rho)\mathrm{Tr}(-\mathrm{i}A_2[H_1,\rho])\big) \\ &= l\mathrm{Re}(-\rho_{12} - \rho_{13} + \rho_{24} + \rho_{34}). \\ &\big(2 + \frac{2}{5}(\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44}) - \frac{4}{5}\mathrm{Re}(\rho_{14} + \rho_{23})\big), l > 0. \end{aligned} \tag{6.62}$$

The numerical illustration is performed with $l = 4$ and the same simulation data as in Section 6.7.1: $\Gamma_{A_1} = \Gamma_{A_2} = 0.9, \eta_{A_1} = \eta_{A_2} = 0.8, J = 0.05$. Fig. 6.5 shows 4 arbitrary sample paths ρ_t under SWMs and smooth control. It can be seen that though there are more sample paths in which the system trajectory ρ_t converges to the desired state ρ_d, there is still some sample paths in which ρ_d converges to the antipodal tangent states ρ_a. It can be also observed from Fig. 6.6 that almost surely all sample paths tend to the maximal entanglement, and comparison to the case of SWMs, there are more sample paths converge to the desired state ρ_d, showed by the smaller steady state of average distance obtained: $d_a(\rho_\infty) = 0.43 < 0.58$. However, as $0.43 > 0$, there is still some positive probability that ρ_t converges to the antipodal tangent states ρ_a. Therefore, smooth controls increase the

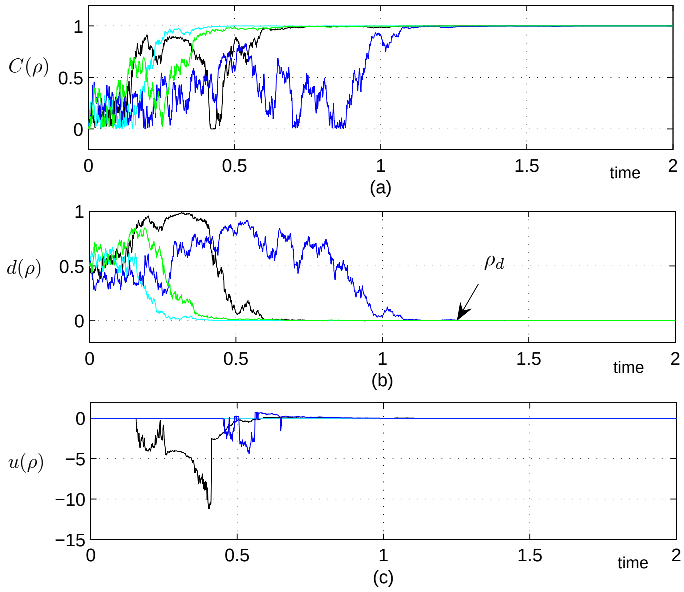


Fig. 6.7: SWM-induced quantum state reduction of four arbitrary sample paths under non-smooth Lyapunov-based control. (a) Concurrence. (b) Distance from ρ_t to ρ_d. (c) Control input $u(\rho_t)$.

probability that ρ_t converges to the desired Bell state ρ_d, but fail to achieve the global stabilization for ρ_d because of the existence of the antipodal tangent states ρ_a.

6.7.3 SWM-Induced Quantum State Reduction and Non-smooth Lyapunov-Based Global Stabilization

This section moves towards with the combination of SWM-induced quantum state reduction and non-smooth Lyapunov-based controls for the global stabilization of the desired Bell state (6.61). We have

$$A = I_4 - 4\rho_d = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix} \tag{6.63}$$

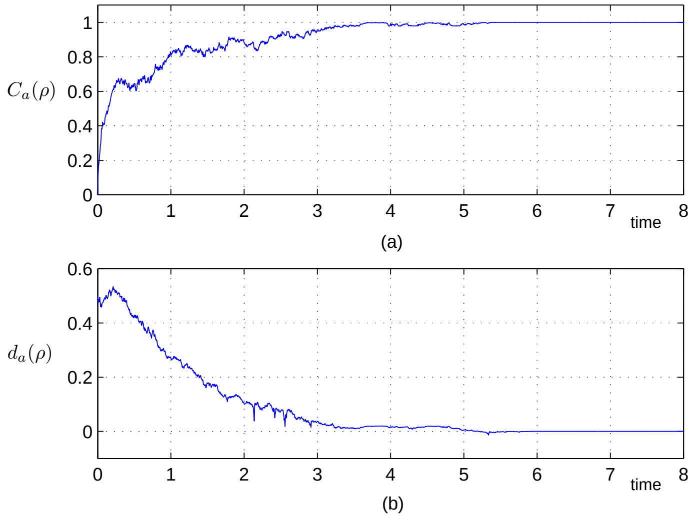


Fig. 6.8: SWM-induced quantum state reduction in average over 100 sample paths under non-smooth Lyapunov-based control. (a) Average concurrence. (b) Average distance.

As such, the unitary matrix P and zero-diagonal matrix B that satisfy $B = P^{-1}AP$ are

$$B = \begin{bmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$P = \frac{1}{5\sqrt{2}} \begin{bmatrix} 0 & 5 & 5 & 0 \\ 3 & -4 & 4 & 3 \\ 4 & 3 & -3 & 4 \\ -5 & 0 & 0 & 5 \end{bmatrix}. \tag{6.64}$$

Thus, the control Hamiltonian H_1 and matrix X that satisfy Equation (6.38) are

$$H_1 = PHP^{-1} = \begin{bmatrix} 0 & -0.4 & 0.3 & 0 \\ -0.4 & 0 & 0 & -0.9 \\ 0.3 & 0 & 0 & -1.2 \\ 0 & -0.9 & -1.2 & 0 \end{bmatrix},$$

$$X = PYP^{-1} = \mathrm{i} \begin{bmatrix} 0 & -0.1 & -1.8 & 0 \\ 0.1 & 0 & 0 & -0.6 \\ 1.8 & 0 & 0 & 1/30 \\ 0 & 0.6 & -1/30 & 0 \end{bmatrix}.$$

Note that $H_1 = -0.8I_2 \otimes \sigma_x + 0.4\sigma_z \otimes \sigma_x - 0.3\sigma_x \otimes I_2 + 0.6\sigma_x \otimes \sigma_z$. As such, in practice, the control Hamiltonian H_1, which is constituted by local and non-local Hamiltonians, can be physically implemented by using appropriate local magnetic fields applied to the $x-$axis of two atoms and nonlocal magnetic fields applied to both x-axis and $z-$axis of two atoms, in which the strengths of these fields are adjusted by the control input $u(\rho_t)$. We note that the local control Hamiltonian were widely utilized in the entangled state generation, e.g., [99, 101, 103]. In addition, the idea of using non-local control Hamiltonian to control the two-qubit entanglement was presented in [32], while the non-local control Hamiltonian was shown to be physically implementable in [104].

The simulation is carried out with the control parameters: $k = 0.85, c = 0.05, M = 3, \epsilon = 0.04, l = 4$. It can be checked that (6.42) and (6.55) hold. The simulation data is as in Section 6.7.1. Fig. 6.7 shows the SWM-induced quantum state reduction of 4 arbitrary sample paths ρ_t under non-smooth Lyapunov function-based control. It can be seen that at some time period, the system trajectory may tend to one of the antipodal tangent states ρ_a, showed by the fact that $C(\rho)$ and $d(\rho)$ tend to 1, but then, the control drives it back to the desired Bell state ρ_d. This makes all the sample paths eventually converge to ρ_d. Fig. 6.8 shows the averages of concurrence $C(\rho)$ and distance $d(\rho)$ over 100 sample paths. It can be observed from Fig. 6.8 that $C_a(\rho_\infty) = 1$ and $d_a(\rho_\infty) = 0$, from which we conclude that the system trajectory ρ_t almost surely asymptotically converges to the desired Bell state ρ_d. This clearly shows the effectiveness of the SWM-induced quantum state reduction and non-smooth Lyapunov-based control in the global stabilization of the desired Bell state as well as the advantage of the non-smooth Lyapunov-based control over the smooth controls.

6.8 Conclusions

In this chapter, we have presented a weak measurement-estimate feedback control scheme to deterministically generate the Bell states of two separated atoms from any initial state. For the first time, the concept of SWM-induced quantum state reduction has been introduced for quantum systems, providing a great potential in generating quantum states. We have harnessed the SWM-induced quantum state reduction to produce maximal entanglement via the stochastic generation of the Bell states. In addition, the SWM-induced quantum state reduction has been utilized together with the non-smooth control, synthesized via the 1-time Lyapunov-like theorem for stability in probability, to deterministically render the filter state from any initial state to the desired Bell state. The 1-time Lyapunov-like theorem enabled the short time convergence of system state, which is essential in the control of entanglement. The numerical illustrations have clearly pointed out the effectiveness of the proposed schemes.

Chapter 7

Real-time Generation of Entangled States

7.1 Introduction

In this chapter, we are interested in the real-time deterministic generation of the Bell states as well as the multipartite entangled states by utilizing feedback control. The main obstacles of realizing quantum estimate feedback control include the measurement back-action, which inevitably disturbs the observed systems, and the real-time implementation, which arises due to the long computation time of filter state and control input in comparison to the very fast dynamics of quantum systems. To deal with the measurement back-action, we have introduced the concept of *simultaneous weak measurements-induced quantum state reduction* in Chapter 6, providing a way for utilizing measurement back-action of two observables to control the observed systems as desired. We also combined it with feedback control to deterministically produce the two-qubit maximally entangled Bell states, without the consideration of real-time control implementation. In this chapter, to enable the generation of multipartite entangled states, we further generalize this concept to the case of multiple observables. In addition, to cope with the real-time implementation challenge, we exploit the time-delay feedback control, in which the computation time of the estimate state and control input is fully compensated for by the (constant) delayed time in the control input [77, 105].

In particular, to deal with the challenge of real-time implementation, we introduce the time-delay bang-bang control, in which the delayed time fully compensates for the computation time, and the bang-bang control structure permits the global stabilization of the desired state. This control structure consists of two modes of which the $1-$mode pushes the system trajectory off all the undesired states in a finite time and then the $0-$mode drives the system trajectory to the desired state almost surely. The simplicity of bang-bang control also enables it to be trivially implemented in practice.

Interestingly, our SWMs and time-delay control schemes can be used as a general mechanism to deterministically produce many multipartite entangled states in real time, such as the stabilizer states [106] and Dicke states [107]. For illustration, we report the use of such mechanism for the generation of the maximally entangled three-qubit $|GHZ\rangle$ state. Firstly, the $|GHZ\rangle$ state is probabilistically produced via the SWMs of three commutative observables $B_1 = \sigma_1^z \otimes \sigma_2^z \otimes I_3, B_2 = \sigma_1^z \otimes I_2 \otimes \sigma_3^z$, and $B_3 = \sigma_1^x \otimes \sigma_2^x \otimes \sigma_3^x$, where $\sigma_i^{x,y,z}$ and I_i are the Pauli operators and identity operator of the qubit i. Then, the SWM-induced quantum state reduction associated with these observables is combined with the time-delay bang-bang control to deterministically generate the $|GHZ\rangle$ state, without knowledge about the initial state.

In Section 7.2, the concept of SWM-induced quantum state reduction is combined with the time delay bang bang control to globally stabilize the desired Bell state. The deterministic generation of the $|GHZ\rangle$ state is presented in Section 7.3. The effectiveness of these measurement and feedback control schemes are illustrated in Section 7.4. Section 7.5 includes concluding remarks.

7.2 Real-time Deterministic Generation of the Desired Bell State

We consider the the two-qubit model in [99], which consists of a couple of two-level atoms, 1 and 2. These atoms are placed in two distant cavities and interact through a radiation field in a dispersive way. The two cavities are arranged in a cascade-like configuration such that, given a coherent input field with amplitude A_c in one of them, the output of each cavity enters the other as depicted in Fig. 7.1. After eliminating the radiation fields, the effective interaction Hamiltonian for the internal degrees of the two atoms becomes of Ising type:

$$H_0 = 2J\sigma_1^z \otimes \sigma_2^z \tag{7.1}$$

in which $\sigma_i^{x,y,z}$ are the Pauli operators of the qubit $i = 1, 2$, and J is the spin-spin coupling strength dependent on $|A_c|^2$ [99]. We assume that the coupling strength J is fixed. To present the idea of the paper, we further assume that we work in the time period in which the spontaneous atomic decay has not happened.

In Chapter 6, we have introduced the concept of *simultaneous weak measurements-induced quantum state reduction* and harnessed it to probabilistically generate the Bell states. As measurements can only stochastically produce Bell states, i.e., each Bell state is generated with a positive probability, we proceed with the combination of SWM-induced quantum state reduction and feedback control to deterministically generate any desired Bell state, without knowledge about the initial state. Unlike Chapter 6, where we have presented a Lyapunov-based feedback control approach for the deterministic generation of Bell states, but without the consideration of real-time control implementation, in this section, we shall utilize the the time-delay bang-bang control to deal with the real-time implementation challenge of quantum feedback control.

Consider the system with free Hamiltonian H_0 as in (7.1), subjected to the SWMs of two

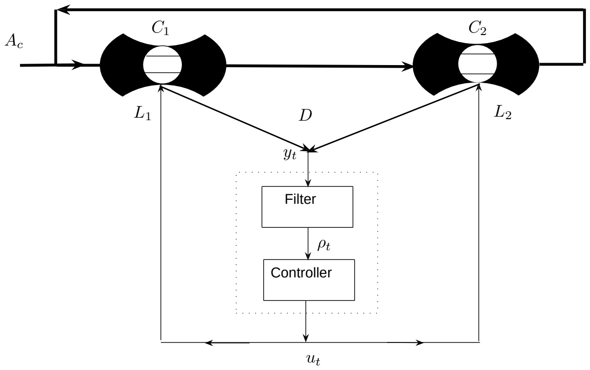


Fig. 7.1: The setup for estimate feedback control of two atoms. Two cavities C_1 and C_2, each of which contains a two-level atom, are connected in a closed loop through optical fibers. The off-resonant driving field A_c generates an effective Hamiltonian H_0. The optical fields are continuously measured by the homodyne detectors $D = \{D_1, ..., D_m\}$. The measurement records $y_t = [y_{1t}, ..., y_{mt}]$ are sent to a filter to extract the information of the system and the filter state (estimate state) ρ_t is then fed back via the controller $u_t = u(\rho_{t-\tau})$ and magnetic fields L_1, L_2 to modify the system Hamiltonian.

commutative observables $A_1 = \sigma_1^z \otimes \sigma_2^z$ and $A_2 = \sigma_1^x \otimes \sigma_2^x$, and the feedback control given by the control Hamiltonian H_1. We have the filtering equation and measurement records:

$$\begin{aligned} d\rho =& - i[H_0, \rho]dt - i[H_1, \rho]udt \\ &+ \Gamma_{A_1}\mathcal{D}[A_1]\rho dt + \sqrt{\eta_{A_1}\Gamma_{A_1}}\mathcal{H}[A_1]\rho dw_{A_1} \\ &+ \Gamma_{A_2}\mathcal{D}[A_2]\rho dt + \sqrt{\eta_{A_2}\Gamma_{A_2}}\mathcal{H}[A_2]\rho dw_{A_2} \\ dy_1 =& \mathrm{Tr}(A_1\rho)dt + \frac{1}{2\sqrt{\eta_{A_1}\Gamma_{A_1}}} dw_{A_1} \\ dy_2 =& \mathrm{Tr}(A_2\rho)dt + \frac{1}{2\sqrt{\eta_{A_2}\Gamma_{A_2}}} dw_{A_2} \end{aligned} \tag{7.2}$$

where dw_{A_1} and dw_{A_2} are independent Wiener increments, and $u \in \mathbb{R}$ is the control input, $\mathbb{R}$ denotes the set of real numbers.

We shall utilize the local control Hamiltonian $H_1 = (\sigma_1^x + \mathrm{i}\sigma_1^y) \otimes I_2/2$. This control Hamiltonian H_1 can be physically implemented by applying the local magnetic fields along the $x-$axis and $y-$axis of the first qubit, in which the strength of these fields is adjusted by the control $u \in \mathbb{R}$ to be designed. The local control Hamiltonians were widely utilized in the entangled state generation, e.g. [99, 101, 103].

Our objective in this section is to design the time-delay feedback control of the form $u_t = u(\rho_{t-\tau})$ to render the filter state ρ_t from any initial state ρ_0 to a desired Bell state ρ_d almost surely, where $\tau > 0$ is known but arbitrarily long. The delayed time τ is used to compensate for the filter state and control computation time, by which the real-time implementation of the proposed control scheme is guaranteed. In the standard basis $\{|0\rangle, |1\rangle\}$, where $|0\rangle$ and $|1\rangle$ are Dirac notations of the two eigenstates of the qubit [75], the Bell states are represented as:

$$\phi^{\pm} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{bmatrix}, \psi^{\pm} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

7.2.1 Control Structure

Notice that all Bell states are mutually orthogonal. As such, all the other Bell states $\rho_a \neq \rho_d$ lie in the set $\{\rho \in \mathcal{S} : D(\rho) = 1\}$, where $D(\rho) = 1 - \mathrm{Tr}(\rho\rho_d)$ is the distance from ρ to ρ_d. Utilizing this observation, we shall design the control that pushes the filter state off the set $\{\rho \in \mathcal{S} : D(\rho) > 1 - \gamma\}$, containing all the other Bell states $\rho_a \neq \rho_d$, in a finite time almost surely, where $\gamma > 0$ sufficiently small. Then, it will push the filter state off all the other Bell states $\rho_a \neq \rho_d$ in a finite time almost surely.

Indeed, inspired by the control design in [57, 105], we shall construct the time delay bang-bang control in hysteresis form with two modes, of which the 1−mode pushes the system trajectory ρ_t off the set $\{\rho \in \mathcal{S} : D(\rho) > 1 - \gamma\}$, and then the 0−mode drives ρ_t to the desired Bell state ρ_d almost surely. To present the control in details, we denote:

$$\begin{aligned} \mathcal{S}_\alpha &:= \{\rho \in \mathcal{S} : D(\rho) = \alpha\}, \\ \mathcal{S}_{>\alpha} &:= \{\rho \in \mathcal{S} : \alpha < D(\rho) \leq 1\}, \\ \mathcal{S}_{\geq\alpha} &:= \{\rho \in \mathcal{S} : \alpha \leq D(\rho) \leq 1\}, \\ \mathcal{S}_{<\alpha} &:= \{\rho \in \mathcal{S} : 0 \leq D(\rho) < \alpha\}, \\ \mathcal{S}_{\leq\alpha} &:= \{\rho \in \mathcal{S} : 0 \leq D(\rho) \leq \alpha\}. \end{aligned} \tag{7.3}$$

We will prove that with $\gamma < 1/4$, the following time-delay bang-bang control renders the filter state to the desired state ρ_d almost surely from any initial state:

1. $u_t = 1$, if $\rho_{t-\tau} \in \mathcal{S}_{\geq 1-\gamma/2}$;

2. $u_t = 0$, if $\rho_{t-\tau} \in \mathcal{S}_{\leq 1-\gamma}$;

3. If $\rho_{t-\tau} \in \Phi := \mathcal{S}_{<1-\gamma/2} \cap \mathcal{S}_{>1-\gamma}$, then $u_t = 0$ if $\rho_{t-\tau}$ last entered Φ through the boundary $\mathcal{S}_{1-\gamma}$ and $u_t = 1$ otherwise.

We can interpret this control as follows. If the system state $\rho_{t-\tau}$ is near one of the undesired Bell states $\rho_a \neq \rho_d$, shown as $\rho_{t-\tau} \in \mathcal{S}_{\geq 1-\gamma/2}$, then we set the control $u_t = 1$ to push the system state off these undesired Bell states. If the system state $\rho_{t-\tau}$ is near the desired Bell state ρ_d, then we switch off the control, and the system state will converge to the desired Bell state as a consequence of Theorem 6.3 in Chapter 6. We note that with the case 3, the above control is of hysteresis form. This form of control prevents the system from the harmful phenomenon of chattering, which arises when the control is of switching form [108].

In order to see clearly the effects of the above control on the system trajectory ρ_t, let us introduce two technical propositions, the proofs of which are found in Sections 1 and 2, Appendix D.

Proposition 7.1. *Let $\gamma < 1/4$. Then, from any initial data with $\rho_0 \in \mathcal{S}_{>1-\gamma}$, the solution of* (7.2) *with $u_t = 1$ exits $\mathcal{S}_{>1-\gamma}$ in a finite time with probability* 1.

Proof. See Section 1, Appendix D. □

Proposition 7.2. *From any initial data with $\rho_0 \in \mathcal{S}_{\leq 1-\gamma}$, the solution ρ_t of* (7.2) *with the control input $u_t = 0$ remains in $\mathcal{S}_{<1-\gamma/2}$ with probability larger or equal to $p := 1 - \dfrac{1-\gamma}{1-\gamma/2}$.*

Proof. See Section 2, Appendix D. □

7.2.2 Convergence Analysis

We are ready to prove the convergence of the system trajectory under the above bang-bang control.

Theorem 7.1. *Consider the quantum filter* (7.2)*. Then, from any initial state, the above time-delay bang-bang control renders the filter state to the desired Bell state* ρ_d *almost surely.*

Proof. The proof of Theorem 7.1 is similar to that of Theorem 5.2 in Chapter 5 and is inspired by that of Theorem 4.2 in [57]. We denote by mode $\mathcal{A}$ and mode $\mathcal{B}$ the periods that the control input $u_t = 1$ and the control input $u_t = 0$ is applied, respectively. For simplicity, the proof of the Theorem 7.1 is divided into three steps:

- Step 1: Showing that a state in mode $\mathcal{A}$ almost surely transits to mode $\mathcal{B}$ in a finite time.
- Step 2: Showing that the system switches between modes $\mathcal{A}$ and $\mathcal{B}$ in a finite number of times and the final mode is $\mathcal{B}$.
- Step 3: Showing that when the state is in mode $\mathcal{B}$ permanently, it converges to the final desired state ρ_d almost surely.

Proposition 7.1 implies that a state in mode $\mathcal{A}$ almost surely transits to mode $\mathcal{B}$ in a finite time. Step 1 is complete.

Suppose that at a time instant, the mode changes from $\mathcal{A}$ to $\mathcal{B}$. After that time instant, there are two probabilities as follows:

$\mathcal{P}_1$: the state remains in $\mathcal{B}$ permanently.

$\mathcal{P}_2$: the mode changes to $\mathcal{A}$ again.

Proposition 7.2 implies that $\mathcal{P}_2$ occurs with probability smaller than or equal to $(1-p)$. We denote the events

$$\mathcal{E}_n := \{\text{the mode switches from } \mathcal{B} \text{ to } \mathcal{A} \text{ in } n \text{ times}\},$$

where $n = 1, 2, ...$ Then, the probability of $\mathcal{E}_n$ satisfies $\mathbb{P}\{\mathcal{E}_n\} \leq (1-p)^n$. Since

$$\sum_{n=1}^{\infty} \mathbb{P}\{\mathcal{E}_n\} \leq \sum_{n=1}^{\infty} (1-p)^n = \frac{1-p}{p} < \infty,$$

Borel-Cantelli's Lemma assures that there exist almost surely a finite number of switches from mode $\mathcal{B}$ to mode $\mathcal{A}$. This, together with Step 1, implies that mode $\mathcal{B}$ is the final mode. Step 2 is complete. Combining Step 1 and Step 2, after a finite time, the system is in the mode $\mathcal{B}$ permanently.

Now, we proceed with Step 3. Note that in mode $\mathcal{B}$, the control input $u_t = 0$ is applied. From Theorem 6.3 in Chapter 6, we conclude that the system trajectory ρ_t converges almost surely to one of Bell states.

On the other hand, the control input $u_t = 0$ is applied only when $\rho_{t-\tau} \in \mathcal{S}_{\leq 1-\gamma/2}$. As such, when the state is in mode $\mathcal{B}$ permanently, it is in the set $\mathcal{S}_{\leq 1-\gamma/2}$ permanently. Therefore, ρ_t almost surely converges to one of Bell states in the set $\mathcal{S}_{\leq 1-\gamma/2}$.

As all Bell states are mutually orthogonal, there is only one Bell state in the set $\mathcal{S}_{\leq 1-\gamma/2}$, that is ρ_d. Therefore,

$$\mathbb{P}\Big\{ \lim_{t\to\infty} \rho_t = \rho_d \Big\} = 1. \tag{7.4}$$

Combining three above steps, we conclude that with the time delay control law defined as in Theorem 7.1, from any initial data, the system trajectory ρ_t converges to the desired Bell state ρ_d almost surely. $\square$

7.3 Real-time Deterministic Generation of Multipartite Entangled States

To facilitate the generation of multipartite entangled states, we first generalize the SWM-induced quantum state reduction, which was stated for the case of two measured observables in Chapter 6, to the case of multiple measured observables.

7.3.1 General SWM-Induced Quantum State Reduction

Theorem 7.2. *Consider the system with free Hamiltonian H_0 under the simultaneous weak measurements of Hermitian observables $A_1, A_2, ...,$ and A_m that are commutative with each other and with H_0. Then, the filter state almost surely converges to the common set of eigenspaces of $A_1, A_2, ...,$ and A_m.*

Proof. Consider the function

$$V(\rho) = U_1(\rho) + U_2(\rho) + ... + U_m(\rho) \tag{7.5}$$

which is a combination of the variances $U_1(\rho), U_2(\rho), ...,$ and $U_m(\rho)$ of the filtering process along $A_1, A_2, ...,$ and A_m, respectively:

$$U_l(\rho) = \mathrm{Tr}(A_l^2 \rho) - \mathrm{Tr}^2(A_l \rho), l = 1, ..., m. \tag{7.6}$$

A straightforward computation (see Section 3, Appendix D) gives the infinitesimal generator of ρ_t acting on $V(\rho)$:

$$\mathcal{L}V(\rho_t) \leq -4 \sum_{l=1}^{m} \eta_{A_1} \Gamma_{A_1} U_1^2(\rho_t) \leq 0, \tag{7.7}$$

where $\mathcal{L}$ denotes the infinitesimal generator of ρ_t. Applying stochastic LaSalle theorem (see e.g. Theorem 2.1 in [63]), we conclude that ρ_t converges with probability 1 to the set in

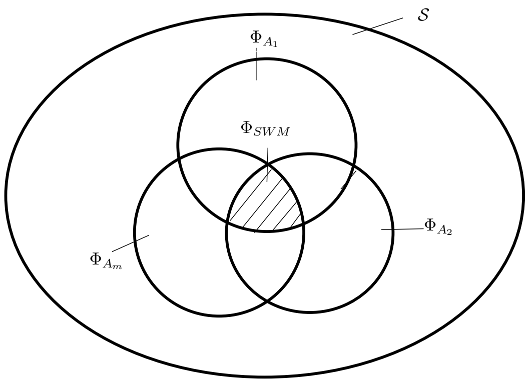


Fig. 7.2: Illustration of the *SWM-induced space* Φ_{SWM} which is the attractive set of filter state under the SWMs of commutative observables $A_1, A_2, ...,$ and A_m.

which $\mathcal{L}V(\rho) = 0$. This, together with Eq. (6.54), implies that

$$\mathbb{P}\{\lim_{t\to\infty} U_l(\rho_t) = 0\} = 1, \forall l = 1, .., m. \tag{7.8}$$

Therefore, the SWMs of commutative observables $A_1, A_2, ...,$ and A_m render the variances $U_1(\rho), U_2, ...,$ and $U_m(\rho)$ of the filtering process along $A_1, A_2, ...,$ and A_m to 0 almost surely. Let $\Phi_{A_l} = \{\rho \in \mathcal{S} : U_l(\rho) = 0\}, l = 1, 2, ..., m$, which are eigenspaces of $A_l, l = 1, 2, ..., m$. Then, Eq. (7.8) states that the filter state ρ_t converges almost surely to the common set $\bigcap_{l=1}^{m} \Phi_{A_l}$ of eigenspaces of $A_1, A_2, ...,$ and A_m. $\square$

We define that common set as the *simultaneous weak measurement-induced space.* An illustration of the SWM-induced space is given in Fig. 7.2.

7.3.2 Deterministic Generation of the $|GHZ\rangle$ State

It is interesting that the above SWMs and time-delay bang-bang control schemes can be used as a general mechanism to deterministically generate a large class of multipartite entangled states. For instance, as the stabilizer states [106] are the eigenstates of commutative observables belonging to the set $\mathfrak{O}$, they can be produced by this mechanism. Indeed, since each stabilizer state lies in the common set of eigenspaces of some commutative observables in $\mathfrak{O}$, we can utilize the simultaneous weak measurements of these observables to probabilistically produce that stabilizer state as a consequence of Theorem 7.2. Then, the deterministic generation of that stabilizer state is obtained by combining the simultaneous weak measurements with feedback control in the same manner as in Section 7.2. Another application of the proposed protocol is the Dicke states that are eigenstates of both commutative observables J_z and $\mathbf{J}^2$ [107], which are combinations of some observables in $\mathfrak{O}$.

In this section, for illustration, we report the deterministic generation of the maximally entangled three-qubit $|GHZ\rangle$ state, i.e., $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Consider the three-qubit systems with free Hamiltonian H_0 that commutes with the observables in the set $\mathfrak{O}$. These systems cover a wide range of systems; for example systems of Ising model and Heisenberg model of which the free Hamiltonian H_0 is a linear combination of some observables in $\mathfrak{O}$. Then, Theorem 7.2 is applicable.

Theorem 7.3. *Consider the three-qubit system with free Hamiltonian H_0 commuting with observables in $\mathfrak{O}$, subjected to the SWMs of three commutative observables $B_1 = \sigma_1^z \otimes \sigma_2^z \otimes I_3, B_2 = \sigma_1^z \otimes I_2 \otimes \sigma_3^z$, and $B_3 = \sigma_1^x \otimes \sigma_2^x \otimes \sigma_3^x$, where $\sigma_i^{x,y,z}$ and I_i are Pauli operators and identity operator of the qubit i. Then, from any initial state, the filter state ρ_t probabilistically converges to the $|GHZ\rangle$ state.*

Proof. Similar to the proof of Theorem 6.3 in Chapter 6, we can prove that the unique common points of $\Phi_{B_i}, i = 1, 2, 3$, are:

$$\begin{aligned}\{&\frac{1}{2}(|000\rangle \pm |111\rangle)(\langle 000| \pm \langle 111|),\\ &\frac{1}{2}(|110\rangle \pm |001\rangle)(\langle 110| \pm \langle 001|),\\ &\frac{1}{2}(|101\rangle \pm |010\rangle)(\langle 101| \pm \langle 010|),\\ &\frac{1}{2}(|011\rangle \pm |100\rangle)(\langle 011| \pm \langle 100|)\}.\end{aligned} \tag{7.9}$$

Therefore, the $|GHZ\rangle\langle GHZ|$ is an isolated point in the SWM-induced space $\Phi_{SWM} =$

$\bigcap_{i=1}^{3} \Phi_{B_i}$. Applying Theorem 7.2, Theorem 7.3 is proved. $\square$

Now, we combine the SWMs of three commutative observables B_1, B_2, and B_3 with the feedback control to deterministically generate the $|GHZ\rangle$ state. We utilize the following local control Hamiltonian

$$H_1 = (\sigma_1^z + \sigma_1^x) \otimes I_2 \otimes I_3 + I_1 \otimes \sigma_2^x \otimes I_3. \tag{7.10}$$

This control Hamiltonian can be physically implemented by applying local magnetic fields along the $z-$axis and $x-$axis of qubits 1 and 2, with the strengths are adjusted by the control input $u \in \mathbb{R}$. Under the SWMs of three commutative observables B_1, B_2, B_3 and the feedback control given by the control Hamiltonian H_1 in (7.10), we have the following filtering equation describing the evolution of filter state, and the stochastic records:

$$\begin{aligned} d\rho =& - i[H_0, \rho]dt - i[H_1, \rho]udt + \sum_{l=1}^{3} \Gamma_{B_l}\mathcal{D}[B_l]\rho dt \\ &+ \sum_{l=1}^{3} \sqrt{\eta_{B_l}\Gamma_{B_l}}\mathcal{H}[B_l]\rho dw_{B_l}, \\ dy_l =& \mathrm{Tr}(B_l\rho)dt + \frac{1}{2\sqrt{\eta_{B_l}\Gamma_{B_l}}}dw_{B_l}, l = 1, 2, 3, \end{aligned} \tag{7.11}$$

where $dw_{B_l}, l = 1, 2, 3$, are independent Wiener increments.

It can be seen that the $|GHZ\rangle\langle GHZ|$ state is orthogonal with all other points in the SWM-induced space (7.9) associated with the SWMs of B_1, B_2, B_3. As such, we can utilize the same control design procedure in Section 7.2 to deterministically produce the $|GHZ\rangle$ state. Let the density matrix $\rho_{GHZ} = |GHZ\rangle\langle GHZ|$, the distance $D_{GHZ}(\rho) =$

$1 - \mathrm{Tr}(\rho\rho_{GHZ})$, and the sets:

$$\mathcal{S}_{\alpha}^{GHZ} := \{\rho \in \mathcal{S} : D_{GHZ}(\rho) = \alpha\},$$
$$\mathcal{S}_{>\alpha}^{GHZ} := \{\rho \in \mathcal{S} : \alpha < D_{GHZ}(\rho) \leq 1\},$$
$$\mathcal{S}_{\geq\alpha}^{GHZ} := \{\rho \in \mathcal{S} : \alpha \leq D_{GHZ}(\rho) \leq 1\},$$
$$\mathcal{S}_{<\alpha}^{GHZ} := \{\rho \in \mathcal{S} : 0 \leq D_{GHZ}(\rho) < \alpha\},$$
$$\mathcal{S}_{\leq\alpha}^{GHZ} := \{\rho \in \mathcal{S} : 0 \leq D_{GHZ}(\rho) \leq \alpha\}.$$

We have the following theorem.

Theorem 7.4. *Consider the quantum filter* (7.11)*. Then, with* $\gamma < 1/8$, *from any initial state, the following time-delay bang-bang control renders the filter state to the* $|GHZ\rangle$ *state almost surely:*

1. $u_t = 1$, if $\rho_{t-\tau} \in \mathcal{S}_{\geq 1-\gamma/2}^{GHZ}$;
2. $u_t = 0$, if $\rho_{t-\tau} \in \mathcal{S}_{\leq 1-\gamma}^{GHZ}$;
3. If $\rho_{t-\tau} \in \Phi^{GHZ} := \mathcal{S}_{<1-\gamma/2}^{GHZ} \cap \mathcal{S}_{>1-\gamma}^{GHZ}$, then $u_t = 0$ if $\rho_{t-\tau}$ last entered Φ^{GHZ} through the boundary $\mathcal{S}_{1-\gamma}^{GHZ}$, and $u_t = 1$ otherwise.

Proof. Similar to the proof of Theorem 7.1 and omitted here. The unique difference is that in comparison to Lemma D.1, now with the control law $u_t = 1$, then the average system trajectory of (7.11) converges to

$$\lim_{t\to\infty} \mathbb{E}[\rho_t] = \frac{1}{8}\mathbb{I}_8,$$

where $\mathbb{I}_8$ is the 8−dimensional identity matrix. As such, in Theorem 7.4, we require that $\gamma < 1/8$, instead of that $\gamma < 1/4$ as in Theorem 7.1. □

7.4 Numerical Illustration

In this section, we numerically demonstrate the efficiency of the SWM-induced quantum state reduction and time delay bang-bang control in the global stabilization of the desired Bell state:

$$\rho_d = |\phi_d\rangle\langle\phi_d| = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \tag{7.12}$$

The simulation data: $\rho_\delta = \rho_0, \forall - \tau \leq \delta \leq 0$. The delayed time: $\tau = 0.25$ and the control parameter: $\gamma = 0.2$. Fig. 7.3 shows the SWM-induced quantum state reduction of 4 arbitrary sample paths ρ_t under the proposed control. It can be seen that at some time period, the system trajectory may tend to one of other Bell states $\rho_a \neq \rho_d$, showed by the fact that $C(\rho)$ and $D(\rho)$ tend to 1, but then, the control drives it back to the desired Bell state ρ_d. This makes all the sample paths eventually converge to ρ_d. Fig. 7.4 shows the averages of concurrence $C(\rho)$ and distance $D(\rho)$ over 100 sample paths. It can be observed from Fig. 7.4 that $C_a(\rho_\infty) = 1$ and $D_a(\rho_\infty) = 0$, from which we conclude that the system trajectory ρ_t almost surely asymptotically converges to the desired Bell state ρ_d. This clearly shows the effectiveness of the SWM-induced quantum state reduction and bang-bang control in the deterministic generation of the desired Bell state.

7.5 Conclusions

We have presented a real-time weak measurement-based feedback control scheme to deterministically generate the Bell states and the maximally entangled three-qubit $|GHZ\rangle$ state. The concept of SWM-induced quantum state reduction has been generalized for quantum systems subjected to SWM of multiple observables. We have harnessed the SWM-induced

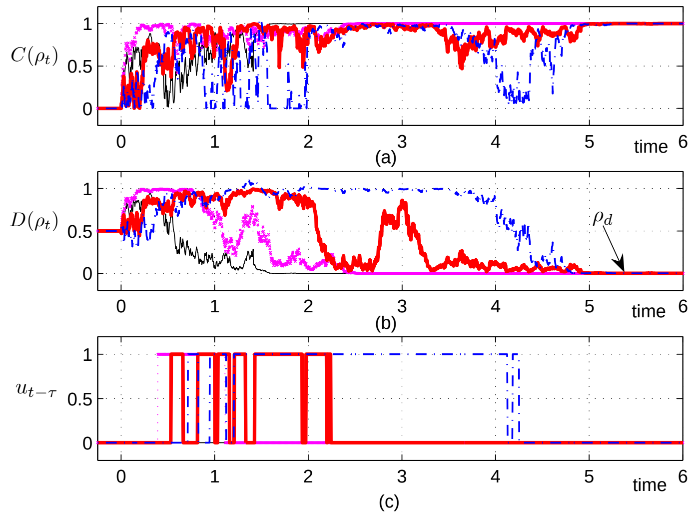


Fig. 7.3: SWM-induced quantum state reduction of four arbitrary sample paths under bang-bang control. (a) Concurrence. (b) Distance from ρ_t to ρ_d. (c) Time delay control input $u(\rho_{t-\tau})$ with $\tau = 0.25, \gamma = 0.2$. The time is in the units such that $\hbar = 1$.

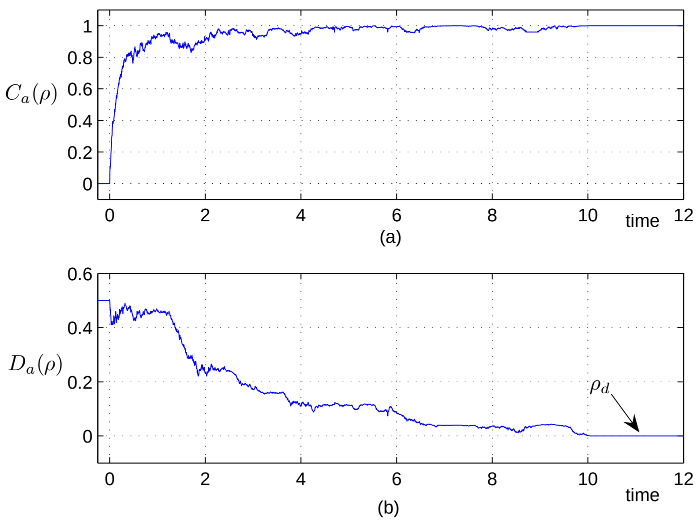


Fig. 7.4: SWM-induced quantum state reduction in average over 100 sample paths under bang-bang control. (a) Average concurrence. (b) Average distance. The time is in the units such that $\hbar = 1$.

quantum state reduction to produce maximal entanglement via the stochastic generation of the Bell states and the $|GHZ\rangle$ state. In addition, the SWM-induced quantum state reduction has been utilized together with the time delay bang-bang control to deterministically render the filter state from any initial state to the desired Bell state and the $|GHZ\rangle$ state. The computation time of filter state and control input was fully compensated for by the time-delay control, enabling the proposed measurement-based feedback control to be implemented in real-time.

Chapter 8

Conclusions

8.1 Conclusions

The measurement-based feedback control of quantum systems was investigated in this thesis. Several challenging problems in quantum feedback control as presented in Chapter 1 were addressed.

The first time introduced *non-smooth Lyapunov function-like theory* for generic stochastic nonlinear systems in Chapter 3 plays a crucial role in the stability analysis and synthesis of globally stabilizing feedback controls for quantum systems. The continuous Lyapunov-like theorem and discontinuous Lyapunov-like theorem for stability in probability were instrumental in designing nonsmooth controls in Chapter 4. The 1-time switching Lyapunov-like theorem for stability in probability provided a way to synthesize nonsmooth control with guaranteed property of short time convergence of the system state. This nice feature was important in the control of entanglement in Chapter 6.

In Chapter 4, we combined the measurement with feedback control to *deterministically* generate the desired quantum states. Applying the continuous Lyapunov-like theorem and the discontinuous Lyapunov-like theorem, switching control and continuous control in saturation form were constructed to deterministically globally stabilize the desired eigenstate of a class of quantum filters, without knowledge about the initial state.

In Chapter 5, we solved the problem of the real time feedback control of quantum systems by using the time delay control approach in which the time to compute the filter state and filter-based control input was fully compensated for by the delay time in the control input. A new Lyapunov-LaSalle-like theorem for delay-dependent stochastic stability was presented for a class of time delay stochastic nonlinear systems. Non-smooth time delay control was then constructed to compensate for the computation time, that is known but arbitrarily long, while globally stabilizing the quantum filters almost surely.

The next two chapters dealt with the deterministic generation of entanglement, for which the introduced concept of SWM-(*simultaneous-weak-measurement*)*-induced quantum state reduction* in Chapter 6 played an important role. Applying the SWM-induced quantum state reduction associated with the commutative observables $\sigma_1^z \otimes \sigma_2^z$ and $\sigma_1^x \otimes \sigma_2^x$, we probabilistically generated the maximally entangled two-qubit Bell states. We then utilized this concept together with the 1-time switching Lyapunov-like theorem to synthesize the continuous control to deterministically generate the maximally entangled two-qubit Bell states from any initial state.

In Chapter 7, the concept of SWM-(*simultaneous-weak-measurement*)*-induced quantum state reduction* was further generalized to enable the generation of multipartite entangled states such as the maximally entangled three-qubit $|GHZ\rangle$ (Greenberger-Horne-Zeilinger) state. It was also harnessed together with the time delay bang-bang control to deterministically generate the Bell states and the multipartite entangled states such as the $|GHZ\rangle$ state in the real time.

8.2 Future Research

Though we have solved several important problems in the feedback control of quantum systems, there are many open questions to be further addressed:

(i) *Control of open quantum systems.*

The feedback control schemes presented in this thesis are devoted to the closed quantum systems, i.e. the systems isolated from environment. Many questions appear when the quantum systems are in contact with the environment: How to model the system-environment interaction? How to deal with the decoherence?

(ii) *Control of quantum non-dynamical semigroup.*

Most existing control schemes for quantum systems deal with the case when the system dynamics can be approximated as Markovian quantum dynamical semigroup. For this type of systems, the well-developed Lyapunov stability theory is an effective tool for the analysis and control synthesis. For the general quantum systems of non-Markovian dynamics, we need another tool for the behavioral analysis and control design.

(iii) *Control of uncertain quantum systems.*

The control schemes in this thesis were presented with the assumption that the system model is known exactly. In practice, there are unavoidable uncertainties in the system model and system parameters. This calls for robust control approaches to deal with the uncertain quantum systems. Some interesting works on this area have appeared; see [109] for a very good introduction of this direction. The extension of our introduced theory and methods in this thesis to the stability analysis and control synthesis of uncertain quantum systems is a promising direction in the future.

(iv) *Control of infinite-dimensional quantum systems*

The area of control for finite-dimensional quantum systems progresses very fast thanks to many tools well-developed in control theory literature. For the infinite-dimensional quantum systems, some results on the controllability have appeared; see [110] for a very good overview of this direction. However, the analysis and control design for infinite-dimensional quantum systems are still much challenging.

Appendix A

Appendices for Chapter 4

1 Proof of Eqs. (4.6a) and (4.6b)

Under Assumption 4.2, the stochastic master equation (4.2) becomes

$$\begin{aligned} d\rho_t = &\big(- \mathrm{i}[H_0, \rho_t] + L\rho_t L - \frac{1}{2}L^2\rho_t - \frac{1}{2}\rho_t L^2\big)dt \\ &- \mathrm{i}[H_1, \rho_t]u dt \\ &+ \sqrt{\eta}\big(L\rho_t + \rho_t L - \mathrm{Tr}(L\rho_t + \rho_t L)\rho_t\big)dw_t \end{aligned} \tag{A.1}$$

Noting the cyclic property of trace: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA), \mathrm{Tr}([A,B]C) = \mathrm{Tr}(A[B,C]$ for all matrices A, B, and C with suitable dimensions. From (A.1) and Assumption 4.2, we have

$$\begin{aligned}
d\mathrm{Tr}(L\rho_t) &= \mathrm{Tr}(Ld\rho_t) \\
&= \mathrm{Tr}(-\mathrm{i}[H_0,\rho_t]L)dt + \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L)udt \\
&+ 2\sqrt{\eta}\mathrm{Tr}(L^2\rho_t - L\mathrm{Tr}(L\rho_t)\rho_t)dw_t \\
&= \mathrm{Tr}(-\mathrm{i}[L,H_0]\rho_t)dt + \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L)udt \\
&+ 2\sqrt{\eta}\big(\mathrm{Tr}(L^2\rho_t) - \mathrm{Tr}^2(L\rho_t)\big)dw_t \\
&= \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L)udt + 2\sqrt{\eta}U(\rho_t)dw_t
\end{aligned} \tag{A.2}$$

This, together with the Itô's product rule, leads to

$$\begin{aligned}
&d\mathrm{Tr}^2(L\rho_t) = 2\mathrm{Tr}(L\rho_t)d\mathrm{Tr}(L\rho_t) + \big(d\mathrm{Tr}(L\rho_t)\big)^2 \\
&= 2\mathrm{Tr}(L\rho_t)\mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L)udt \\
&+ 4\sqrt{\eta}\mathrm{Tr}(L\rho_t)U(\rho_t)dw_t + 4\eta U(\rho_t)^2dt
\end{aligned} \tag{A.3}$$

Similar to (A.2), we have

$$\begin{aligned}
d\mathrm{Tr}(L^2\rho_t) &= \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L^2)udt \\
&\quad + 2\sqrt{\eta}\mathrm{Tr}(L^3\rho_t - L^2\mathrm{Tr}(L\rho_t)\rho_t)dw_t
\end{aligned} \tag{A.4}$$

Combining (A.3) and (A.4), we obtain

$$\begin{aligned}
&dU(\rho_t) = d\mathrm{Tr}(L^2\rho_t) - d\mathrm{Tr}^2(L\rho_t) \\
&= \mathrm{Tr}\left(-\mathrm{i}[H_1,\rho_t]\big(L^2 - 2L\mathrm{Tr}(L\rho_t)\big)\right)udt - 4\eta U(\rho_t)^2dt \\
&+ 2\sqrt{\eta}\big(\mathrm{Tr}(L^3\rho_t - L^2\mathrm{Tr}(L\rho_t)\rho_t) - 2\mathrm{Tr}(L\rho_t)U(\rho_t)\big)dw_t
\end{aligned}$$

Hence, the infinitesimal generator of $U(\rho)$ along (A.1) is

$$\mathcal{L}_{f+gu,\sigma}U(\rho_t) = \mathrm{Tr}\Big(- \mathrm{i}[H_1, \rho_t]\big(L^2 - 2L\mathrm{Tr}(L\rho_t)\big)\Big)u \\ - 4\eta U(\rho_t)^2 \tag{A.5}$$

On the other hand, as H_0 and ρ_f are commutative,

$$\begin{aligned} d\mathrm{Tr}(\rho_t\rho_f) &= \mathrm{Tr}(-\mathrm{i}[H_0, \rho_t]\rho_f)dt + \mathrm{Tr}(-\mathrm{i}[H_1, \rho_t]\rho_f)udt \\ &\quad + 2\sqrt{\eta}\mathrm{Tr}((L - \mathrm{Tr}(L\rho_t))\rho_t\rho_f)dw_t \\ &= \mathrm{Tr}(-\mathrm{i}[H_1, \rho_t]\rho_f)udt \\ &\quad + 2\sqrt{\eta}\mathrm{Tr}((L - \mathrm{Tr}(L\rho_t))\rho_t\rho_f)dw_t \end{aligned}$$

Hence

$$\mathcal{L}_{f+gu,\sigma}\mathrm{Tr}(\rho_t\rho_f) = \mathrm{Tr}(-\mathrm{i}[H_1, \rho_t]\rho_f)u \tag{A.6}$$

From (A.5) and (A.6), then Eqs. (4.6a)-(4.6b) hold true.

2 Prove that $\big(g_S(\rho)^2 + 4c\eta U(\rho)^2\big) = 0$ iff ρ is an eigenstate of L

As L is diagonal and regular, let $L = \mathrm{diag}(l_1, ..., l_n), l_i \neq l_j, \forall i \neq j$. For each $\rho = [\rho_{ij}]_{n\times n} \in \mathcal{S}$, we have

$$U(\rho) = \mathrm{Tr}(L^2\rho) - \mathrm{Tr}(L\rho)^2 = \sum_{i=1}^{n} l_i^2\rho_{ii} - \left(\sum_{i=1}^{n} l_i\rho_{ii}\right)^2$$

Since $\mathrm{Tr}(\rho) = \sum_{i=1}^{n} \rho_{ii} = 1$ and $\rho_{ii} \geq 0, \forall i = 1, .., n$, as $\rho \geq 0$, applying the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{n} l_i^2 \rho_{ii} = \sum_{i=1}^{n} l_i^2 \rho_{ii} \sum_{i=1}^{n} \rho_{ii} \geq \big(\sum_{i=1}^{n} |l_i| \rho_{ii} \big)^2 \geq \big(\sum_{i=1}^{n} l_i \rho_{ii} \big)^2$$

Then, $U(\rho) \geq 0, \forall \rho \in \mathcal{S}$. As $l_i \neq l_j, \forall i \neq j$, the equality happens iff one of ρ_{ii} is equal to 1 and the others are equal to 0, i.e., iff ρ is an eigenstate of L. Therefore, if $\big(g_S(\rho)^2 + 4c\eta U(\rho)^2\big) = 0$, then $U(\rho) = 0$ and thus, ρ is an eigenstate of L. The inverse clause is trivial. $\square$

3 Proof of Lemma 4.2

We choose the solution X self-adjoint and off-diagonal of the form

$$X = \begin{bmatrix} 0 & x_{12} & 0 & \cdots & 0 \\ x_{21} & 0 & x_{23} & \ddots & \vdots \\ 0 & x_{32} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & x_{(n-1)n} \\ 0 & \cdots & 0 & x_{n(n-1)} & 0 \end{bmatrix} \tag{A.7}$$

For the key Equation (4.10), we only concern with the equations associated with $a_{11}, ..., a_{nn}$:

$$\begin{aligned} -\mathrm{i}(x_{12}h_{21} - h_{12}x_{21}) &= a_{11} \neq 0, \\ -\mathrm{i}(x_{21}h_{12} - h_{21}x_{12} + x_{23}h_{32} - h_{23}x_{32}) &= a_{22} \neq 0, \\ &\vdots \\ -\mathrm{i}(x_{n(n-1)}h_{(n-1)n} - h_{n(n-1)}x_{(n-1)n}) &= a_{nn} \neq 0. \end{aligned} \tag{A.8}$$

As $a_{11} + ... + a_{nn} = \mathrm{Tr}(A) = \mathrm{Tr}(-\mathrm{i}[X, H_1]) = 0$, Equations (A.8) are equivalent to $-\mathrm{i}(x_{i(i+1)}h_{(i+1)i} - h_{i(i+1)}x_{(i+1)i}) = a_{11} + ... + a_{ii}$, for all $i = 1, ..., n-1$. Let $h_{i(i+1)} = a_i + \mathrm{i}b_i, x_{i(i+1)} = x_i + \mathrm{i}y_i, \forall i = 1, ..., n-1$. As X and H_1 are self-adjoint, it holds that

$h_{(i+1)i} = a_i - \mathrm{i}b_i$ and $x_{(i+1)i} = x_i - \mathrm{i}y_i, \forall i = 1, ..., n-1$. The above equalities are equivalent to $2(a_i y_i - b_i x_i) = a_{11} + ... + a_{ii}, \forall i = 1, ..., n-1$. As in Assumption 4.3, H_1 is connected, then for all $i = 1, ..., n-1$, or $a_i \neq 0$ or $b_i \neq 0$. If $a_i \neq 0$, we can take $x_i = 0, y_i = \dfrac{a_{11} + ... + a_{ii}}{2a_i}$. If $b_i \neq 0$, we can take $x_i = -\dfrac{a_{11} + ... + a_{ii}}{2b_i}, y_i = 0$. Lemma 4.2 is proved. $\square$

4 Proof of Eqs. (4.17), (4.18), and (4.19)

From (A.5) and (A.6), Eq. (4.17) follows accordingly. On the other hand, it follows from (A.1) and the Itô's product rule that

$$\begin{aligned}
d\mathrm{Tr}(X\rho_t) &= \mathrm{Tr}\big(Xf(\rho_t)\big)dt + \mathrm{Tr}(-\mathrm{i}X[H_1, \rho_t])udt \\
&\quad + \mathrm{Tr}(X\sigma(\rho_t))dw_t, \\
d\mathrm{Tr}(A\rho_t) &= \mathrm{Tr}\big(Af(\rho_t)\big)dt + \mathrm{Tr}(-\mathrm{i}A[H_1, \rho_t])udt \\
&\quad + \mathrm{Tr}(A\sigma(\rho_t))dw_t, \\
d\mathrm{Tr}(A\rho_t)^2 &= 2\mathrm{Tr}(A\rho_t)d\mathrm{Tr}(A\rho_t) + (dTr(A\rho_t))^2 \\
&= 2\mathrm{Tr}(A\rho_t)\mathrm{Tr}(-\mathrm{i}A[H_1, \rho_t])udt \\
&\quad + 2\mathrm{Tr}(A\rho_t)\mathrm{Tr}\big(Af(\rho_t)\big)dt \\
&\quad + \mathrm{Tr}(A\sigma(\rho_t))^2 dt + 2\mathrm{Tr}(A\rho_t)\mathrm{Tr}(A\sigma(\rho_t))dw_t
\end{aligned}$$

Hence, the infinitesimal generator of ρ_t acting on $\mathrm{Tr}(X\rho_t)$ and $\mathrm{Tr}(A\rho_t)^2$ gives

$$\begin{aligned}
\mathcal{L}_{f+gu,\sigma}\mathrm{Tr}(X\rho_t) &= \mathrm{Tr}\big(Xf(\rho_t)\big) + \mathrm{Tr}(-\mathrm{i}X[H_1, \rho_t])u \\
&= \mathrm{Tr}\big(Xf(\rho_t)\big) + \mathrm{Tr}(-\mathrm{i}[X, H_1]\rho_t)u \\
&= \mathrm{Tr}\big(Xf(\rho_t)\big) + \mathrm{Tr}(A\rho_t)u \qquad \text{(A.9)} \\
\mathcal{L}_{f+gu,\sigma}\mathrm{Tr}(A\rho_t)^2 &= 2\mathrm{Tr}(A\rho_t)\mathrm{Tr}(-\mathrm{i}A[H_1, \rho_t])u \\
&\quad + 2\mathrm{Tr}(A\rho_t)\mathrm{Tr}\big(Af(\rho_t)\big) + \mathrm{Tr}(A\sigma(\rho_t))^2 \qquad \text{(A.10)}
\end{aligned}$$

From (A.5), (A.9), and (A.10), Eqs. (4.18) and (4.19) follow accordingly. $\square$

Appendix B

Appendices for Chapter 5

1 Proof of Eq. (5.20)

Noting the cyclic property of trace: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA), \mathrm{Tr}([A,B]C) = \mathrm{Tr}(A[B,C])$ for all matrices A, B, and C with suitable dimensions. From (5.19) and Assumption 5.2, we have

$$
\begin{aligned}
d\mathrm{Tr}(L\rho_t) &= \mathrm{Tr}(Ld\rho_t) \\
&= \mathrm{Tr}(-\mathrm{i}[H_0,\rho_t]L)dt + \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L)u_t dt \\
&+ 2\sqrt{\eta}\mathrm{Tr}(L^2\rho_t - L\mathrm{Tr}(L\rho_t)\rho_t)dw_t \\
&= \mathrm{Tr}(-\mathrm{i}[L,H_0]\rho_t)dt + \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L)u_t dt \\
&+ 2\sqrt{\eta}\big(\mathrm{Tr}(L^2\rho_t) - \mathrm{Tr}^2(L\rho_t)\big)dw_t \\
&= \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L)u_t dt + 2\sqrt{\eta}U(\rho_t)dw_t,
\end{aligned}
\tag{B.1}
$$

This together with the Itô's product rule leads to

$$\begin{aligned} d\mathrm{Tr}^2(L\rho_t) &= 2\mathrm{Tr}(L\rho_t)d\mathrm{Tr}(L\rho_t) + \big(d\mathrm{Tr}(L\rho_t)\big)^2 \\ &= 2\mathrm{Tr}(L\rho_t)\mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L)u_t dt \\ &+ 4\sqrt{\eta}\mathrm{Tr}(L\rho_t)U(\rho_t)dw_t + 4\eta U^2(\rho_t)dt \end{aligned} \tag{B.2}$$

Similar to (B.1), we have

$$d\mathrm{Tr}(L^2\rho_t) = \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]L^2)u_t dt + 2\sqrt{\eta}\mathrm{Tr}(L^3\rho_t - L^2\mathrm{Tr}(L\rho_t)\rho_t)dw_t \tag{B.3}$$

Combining (B.2) and (B.3), we obtain

$$\begin{aligned} dU(\rho_t) &= \mathrm{Tr}\left(-\mathrm{i}[H_1,\rho_t]\big(L^2 - 2L\mathrm{Tr}(L\rho_t)\big)\right)u_t dt - 4\eta U^2(\rho_t)dt \\ &+ 2\sqrt{\eta}\mathrm{Tr}(L^3\rho_t - L^2\mathrm{Tr}(L\rho_t)\rho_t)dw_t - 4\sqrt{\eta}\mathrm{Tr}(L\rho_t)U(\rho_t)dw_t \end{aligned} \tag{B.4}$$

Hence, the infinitesimal operator of ρ_t acting on $U(\rho)$ gives

$$\mathcal{L}_{f+gu,\sigma}U(\rho_t) = \mathrm{Tr}\Big(-\mathrm{i}[H_1,\rho_t]\big(L^2 - 2L\mathrm{Tr}(L\rho_t)\big)\Big)u_t - 4\eta U^2(\rho_t). \tag{B.5}$$

On the other hand, it follows from Assumptions 5.1 and 5.2 and the cyclic property of trace that

$$\begin{aligned} d\mathrm{Tr}(\rho_t\rho_f) &= \mathrm{Tr}(-\mathrm{i}[H_0,\rho_t]\rho_f)dt + \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]\rho_f)u_t dt \\ &+ 2\sqrt{\eta}\mathrm{Tr}((L - \mathrm{Tr}(L\rho_t))\rho_t\rho_f)dw_t \\ &= \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]\rho_f)u_t dt + 2\sqrt{\eta}\mathrm{Tr}((L - \mathrm{Tr}(L\rho_t))\rho_t\rho_f)dw_t \end{aligned} \tag{B.6}$$

Hence, the infinitesimal operator of ρ_t acting on $\mathrm{Tr}(\rho\rho_f)$ gives

$$\mathcal{L}_{f+gu,\sigma}\mathrm{Tr}(\rho_t\rho_f) = \mathrm{Tr}(-\mathrm{i}[H_1,\rho_t]\rho_f)u_t \tag{B.7}$$

From (B.5) and (B.7), the infinitesimal operator of ρ_t acting on $V(\rho)$ gives

$$\mathcal{L}_{f+gu,\sigma}V(\rho_t) = \mathrm{Tr}\left(-\mathrm{i}[H_1,\rho_t]\big(c(L^2 - 2L\mathrm{Tr}(L\rho_t)) - \rho_f\big)\right) u_t - 4c\eta U^2(\rho_t). \tag{B.8}$$

□

2 Proof of Proposition 5.1

We utilize the approach for the proofs of Lemma 4.6 in [57] and Lemma 4.7 in [69].

Lemma B.1. *With the control law $u_t = 1$, we have*

$$\lim_{t\to\infty} \mathbb{E}[\rho_t] = \frac{1}{n}I_n.$$

Proof. Let $\bar{\rho}_t = \mathbb{E}[\rho_t]$. As $u_t = 1$, it follows from (5.19) that the evolution of $\bar{\rho}_t$ is given by

$$\frac{d\bar{\rho}_t}{dt} = -\mathrm{i}[H_0,\bar{\rho}_t] - \frac{1}{2}[L,[L,\bar{\rho}_t]] - \mathrm{i}[H_1,\bar{\rho}_t] \tag{B.9}$$

Consider the Lyapunov function candidate defined on $\mathcal{S}$:

$$W(\rho) = \mathrm{Tr}\big((\rho - \frac{1}{n}I_n)^2\big) = \mathrm{Tr}(\rho^2) - \frac{1}{n} \tag{B.10}$$

By the cyclic property of trace, the derivative of $W(\bar{\rho}_t)$ along the trajectory of (B.9) is

$$\begin{aligned}\frac{dW(\bar{\rho}_t)}{dt} &= -2\mathrm{Tr}(\mathrm{i}[H_0,\bar{\rho}_t]\bar{\rho}_t) - \mathrm{Tr}([L,[L,\bar{\rho}_t]]\bar{\rho}_t) - 2\mathrm{Tr}(\mathrm{i}[H_1,\bar{\rho}_t]\bar{\rho}_t) \qquad \text{(B.11)}\\ &= -2\mathrm{Tr}(\mathrm{i}H_0[\bar{\rho}_t,\bar{\rho}_t]) - \mathrm{Tr}([L,\bar{\rho}_t]^*[L,\bar{\rho}_t]) - 2\mathrm{Tr}(\mathrm{i}H_1[\bar{\rho}_t,\bar{\rho}_t])\\ &= -\big|[L,\bar{\rho}_t]\big|^2 \le 0,\end{aligned}$$

where $|\cdot|$ is the Frobenius norm.

Applying the deterministic LaSalle theorem [111], from (B.11), $\bar{\rho}_t$ converges to the largest invariant set $\mathcal{M}$ contained in the set $\{\rho \in \mathcal{S} : \big|[L,\rho]\big| = 0\}$. As such, for any trajectory $\bar{\rho}_t^{\mathcal{M}}$ of (B.9) in $\mathcal{M}$, we have $\big|[L,\bar{\rho}_t^{\mathcal{M}}]\big| = 0$, and thus, $[L,\bar{\rho}_t^{\mathcal{M}}] = 0$. As in Assumption 5.2, L is diagonal and regular, this leads to $\bar{\rho}_t^{\mathcal{M}}$ is diagonal. Hence, $[H_0,\bar{\rho}_t^{\mathcal{M}}] = 0$ and $d\bar{\rho}_t^{\mathcal{M}}$ is diagonal. By (B.9), it follows that $[H_1,\bar{\rho}_t^{\mathcal{M}}]$ is also diagonal. Since in Assumption 5.3, $H_1 = [h_{ij}]_{n\times n}$ is connected, i.e., $h_{i(i+1)} \neq 0, \forall i = 1,..,n-1$, it must hold that $\bar{\rho}_t^{\mathcal{M}} = aI_n$ for some constant a. As $\mathrm{Tr}(\bar{\rho}_t^{\mathcal{M}}) = 1$, it follows that $\bar{\rho}_t^{\mathcal{M}} = \frac{1}{n}I_n$. Therefore, $\mathcal{M} = \{\frac{1}{n}I_n\}$. Hence

$$\lim_{t\to\infty} \mathbb{E}[\rho_t] = \lim_{t\to\infty} \bar{\rho}_t = \frac{1}{n}I_n. \tag{B.12}$$

□

Lemma B.2. *With the control law $u_t = 1$, there exists $\overline{T} < \infty$ such that*

$$\mathbb{E}[V(\rho_t)] < M_V - \gamma, \forall t \geq \overline{T}, \forall \rho_0 \in \mathcal{S}. \tag{B.13}$$

Proof. Due to the continuity of $V(\rho_t)$ [57] and (B.12), we have

$$\lim_{t\to\infty} \mathbb{E}[V(\rho_t)] = V\big(\lim_{t\to\infty} \mathbb{E}[\rho_t]\big) = V(\frac{1}{n}I_n) = 1 - \frac{1}{n} + cU(\frac{1}{n}I_n). \tag{B.14}$$

As such, with $\epsilon = M_V - \gamma - (1 - \dfrac{1}{n} + cU(\frac{1}{n}I_n)) > 0$, ($\epsilon > 0$ because of (5.27)), there exists $T > 0$ such that $\Big|\mathbb{E}[V(\rho_t)] - \big(1 - \frac{1}{n} + cU(\frac{1}{n}I_n)\big)\Big| < \epsilon, \forall t \geq T$. Consequently, for all $t \geq T$,

$$\mathbb{E}[V(\rho_t)] < \big(1 - \frac{1}{n} + cU(\frac{1}{n}I_n)\big) + \epsilon = M_V - \gamma. \tag{B.15}$$

Define $T(\rho_0) = \inf\{T : \mathbb{E}[V(\rho_t)] < M_V - \gamma, \forall t \geq T\}$ and $\overline{T} = \sup_{\rho_0\in\mathcal{S}} T(\rho_0)$. Due to the continuity of $\mathbb{E}[V(\rho_t)]$ and thus, the continuity of $T(\rho_0)$ with respect to the initial state [57], it must hold that $\overline{T} < \infty$. It follows from the definition of $T(\rho_0)$ that (B.13) holds. The proof of Lemma B.2 is completed. □

Now, we prove the statement of Proposition 5.1. Let $\tau_{\rho_0}(\mathcal{S}_{>M_V-\gamma})$ be the first exit time from the set $\mathcal{S}_{>M_V-\gamma}$ of a solution ρ_t beginning at ρ_0. We need to show that $\tau_{\rho_0}(\mathcal{S}_{>M_V-\gamma}) < \infty$ almost surely. Applying Lemma 4.3, pp.111 in [112], we have

$$\mathbb{E}[\tau_{\rho_0}(\mathcal{S}_{>M_V-\gamma})] \leq \frac{2\overline{T}}{1 - \sup_{\xi\in\mathcal{S}} \mathbb{P}\{\tau_\xi(\mathcal{S}_{>M_V-\gamma}) > 2\overline{T}\}}. \tag{B.16}$$

We will show that $\sup_{\xi\in\mathcal{S}} \mathbb{P}\{\tau_\xi(\mathcal{S}_{>M_V-\gamma}) > 2\overline{T}\} < 1$. Suppose that

$$\sup_{\xi\in\mathcal{S}} \mathbb{P}\{\tau_\xi(\mathcal{S}_{>M_V-\gamma}) > 2\overline{T}\} = 1. \tag{B.17}$$

Obviously, $\mathbb{P}\{\tau_\xi(\mathcal{S}_{>M_V-\gamma}) > 2\overline{T}\} < 1$ for all $\xi \in \mathcal{S}_{\leq M_V-\gamma}$. From (B.17), for each $\epsilon > 0$, there exists $\xi_\epsilon \in \mathcal{S}_{>M_V-\gamma}$ such that $\mathbb{P}\{\tau_{\xi_\epsilon}(\mathcal{S}_{>M_V-\gamma}) > 2\overline{T}\} > 1-\epsilon$. Hence, with $\rho_0 = \xi_\epsilon$, we have

$$\mathbb{E}[V(\rho_t)] > (M_V-\gamma)\mathbb{P}\{\tau_{\xi_\epsilon}(\mathcal{S}_{>M_V-\gamma}) > 2\overline{T}\} > (1-\epsilon)(M_V-\gamma), \forall 0 \leq t \leq 2\overline{T}.$$

Taking $\epsilon \to 0$, there exists $\xi_\infty \in \mathcal{S}_{\geq M_V-\gamma}$ such that with $\rho_0 = \xi_\infty$,

$$\mathbb{E}[V(\rho_t)] \geq M_V - \gamma, \forall 0 \leq t \leq 2\overline{T} \tag{B.18}$$

which is a contradiction with (B.13). Therefore, $\sup_{\xi\in\mathcal{S}} \mathbb{P}\{\tau_\xi(\mathcal{S}_{>M_V-\gamma}) > 2\overline{T}\} < 1$. It follows from (B.16) that $\mathbb{E}[\tau_{\rho_0}(\mathcal{S}_{>M_V-\gamma})] < \infty$ and thus, $\tau_{\rho_0}(\mathcal{S}_{>M_V-\gamma}) < \infty$ almost surely. The proof of Proposition 5.1 is completed. $\square$

3 Proof of Proposition 5.2

From (5.28b), we have

$$\tau \max_{\rho \in \mathcal{S}} W_1(\rho) = \tau \frac{M}{\bar{M}} \max_{\rho \in \mathcal{S}} U^2(\rho) < \gamma/2. \tag{B.19}$$

Hence, $p = 1 - \dfrac{M_V - \gamma + \tau \max_{\rho \in \mathcal{S}} W_1(\rho)}{M_V - \gamma/2} > 0$. As the nonlinear control input $u_t = u_S(\rho_{t-\tau}) \leq U^2(\rho_{t-\tau})/\bar{M}$ is applied, the inequality (5.24) holds. Similar to (5.10), we have

$$\mathbb{E}[V(\rho_t)] \leq V(\rho_0) + \tau \sup_{-\tau \leq \theta \leq 0} W_1(\rho_\theta), \forall t \geq 0 \tag{B.20}$$

Since $V(\rho_0) \leq M_V - \gamma$, it follows that

$$\mathbb{E}[V(\rho_t)] \leq V(\rho_0) + \tau \max_{\rho \in \mathcal{S}} W_1(\rho) \leq M_V - \gamma + \tau \max_{\rho \in \mathcal{S}} W_1(\rho), \forall t \geq 0. \tag{B.21}$$

By this and Chebyshev's inequality, we obtain

$$\mathbb{P}\{\sup_{t \geq 0} V(\rho_t) \geq M_V - \gamma/2\} \leq \frac{M_V - \gamma + \tau \max_{\rho \in \mathcal{S}} W_1(\rho)}{M_V - \gamma/2} = 1 - p. \tag{B.22}$$

Therefore, $\mathbb{P}\{\sup_{t \geq 0} V(\rho_t) < M_V - \gamma/2\} \geq p$, i.e., the system trajectory ρ_t remains in $\mathcal{S}_{<M_V - \gamma/2}$ with probability larger or equal to p. Proposition 5.2 is proved. $\square$

Appendix C

Appendices for Chapter 6

1 Proof of Eq. (6.6)

In the standard basis $\{|0\rangle, |1\rangle\}$, we have

$$\sigma_i^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_i^y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \sigma_i^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$H_0 = 2J \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We note the cyclic property of trace: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA), \mathrm{Tr}([A, B]C) = \mathrm{Tr}(A[B, C])$ for all matrices A, B, and C with suitable dimensions. From (6.5), the self-adjointness of A, and the commutativity of H_0 and A, it holds that

$$\begin{aligned}
d\text{Tr}(A\rho_t) &= \text{Tr}(Ad\rho_t) \\
&= \text{Tr}(-\mathrm{i}A[H_0, \rho_t])dt + \Gamma_A \text{Tr}(A\mathcal{D}[A]\rho_t) \\
&+ \sqrt{\eta_A \Gamma_A} \text{Tr}(A\mathcal{H}[A]\rho_t)dw_t \\
&= +2\sqrt{\eta_A \Gamma_A}\big(\text{Tr}(A^2\rho_t) - \text{Tr}^2(A\rho_t)\big)dw_t \\
&= 2\sqrt{\eta_A \Gamma_A} U(\rho_t)dw_t
\end{aligned} \tag{C.1}$$

This, together with the Itô's product rule, yields

$$\begin{aligned}
d\text{Tr}^2(A\rho_t) &= 2\text{Tr}(A\rho_t)d\text{Tr}(A\rho_t) + \big(d\text{Tr}(A\rho_t)\big)^2 \\
&= 4\sqrt{\eta_A \Gamma_A} \text{Tr}(A\rho_t)U(\rho_t)dw_t + 4\eta_A \Gamma_A U(\rho_t)^2 dt
\end{aligned} \tag{C.2}$$

Similar to (C.1), we have

$$d\text{Tr}(A^2\rho_t) = 2\sqrt{\eta_A \Gamma_A} \text{Tr}(A^3\rho_t - A^2 \text{Tr}(A\rho_t)\rho_t)dw_t \tag{C.3}$$

Combining (C.2) and (C.3), we obtain

$$\begin{aligned}
dU(\rho_t) &= d\text{Tr}(A^2\rho_t) - d\text{Tr}^2(A\rho_t) \\
&= 2\sqrt{\eta_A \Gamma_A} \text{Tr}(A^3\rho_t - A^2 \text{Tr}(A\rho_t)\rho_t)dw_t \\
&- 4\sqrt{\eta_A \Gamma_A} \text{Tr}(A\rho_t)U(\rho_t)dw_t - 4\eta_A \Gamma_A U(\rho_t)^2 dt,
\end{aligned} \tag{C.4}$$

from which (6.6) follows. □

2 Proof of Eq. (6.15)

By the self-adjointness of A_1, A_2 and the commutativity of A_1, A_2, and H_0, it holds that

$$\begin{aligned}
&d\mathrm{Tr}(A_1\rho_t) = \mathrm{Tr}(-\mathrm{i}A_1[H_0,\rho])dt \\
&+ \Gamma_{A_1}\mathrm{Tr}(A_1\mathcal{D}[A_1]\rho)dt + \sqrt{\eta_{A_1}\Gamma_{A_1}}\mathrm{Tr}(A_1\mathcal{H}[A_1]\rho)dw_{A_1} \\
&+ \Gamma_{A_2}\mathrm{Tr}(A_1\mathcal{D}[A_2]\rho)dt + \sqrt{\eta_{A_2}\Gamma_{A_2}}\mathrm{Tr}(A_1\mathcal{H}[A_2]\rho)dw_{A_2} \\
&= 2\sqrt{\eta_{A_1}\Gamma_{A_1}}(\mathrm{Tr}(A_1^2\rho_t) - \mathrm{Tr}^2(A_1\rho_t))dw_{A_1} \\
&+ 2\sqrt{\eta_{A_2}\Gamma_{A_2}}(\mathrm{Tr}(A_1A_2\rho_t) - \mathrm{Tr}(A_1\rho_t)\mathrm{Tr}(A_2\rho_t))dw_{A_2} \\
&= 2\sqrt{\eta_{A_1}\Gamma_{A_1}}U_1(\rho_t)dw_{A_1} + 2\sqrt{\eta_{A_2}\Gamma_{A_2}}U_{12}(\rho_t)dw_{A_2}
\end{aligned} \tag{C.5}$$

This, together with the Itô's product rule, leads to

$$\begin{aligned}
d\mathrm{Tr}^2(A_1\rho_t) &= 2\mathrm{Tr}(A_1\rho_t)d\mathrm{Tr}(A_1\rho_t) + (d\mathrm{Tr}(A_1\rho_t))^2 \\
&= 4\eta_{A_1}\Gamma_{A_1}U_1^2(\rho_t)dt + 4\eta_{A_2}\Gamma_{A_2}U_{12}^2(\rho_t)dt \\
&+ 4\sqrt{\eta_{A_1}\Gamma_{A_1}}\mathrm{Tr}(A_1\rho_t)U_1(\rho_t)dw_{A_1} \\
&+ 4\sqrt{\eta_{A_2}\Gamma_{A_2}}\mathrm{Tr}(A_1\rho_t)U_{12}(\rho_t)dw_{A_2}
\end{aligned} \tag{C.6}$$

Similarly,

$$\begin{aligned}
d\mathrm{Tr}(A_i^2\rho_t) &= \sqrt{\eta_{A_1}\Gamma_{A_1}}\mathrm{Tr}(A_i^2\mathcal{H}[A_1]\rho)dw_{A_1} \\
&+ \sqrt{\eta_{A_2}\Gamma_{A_2}}\mathrm{Tr}(A_i^2\mathcal{H}[A_2]\rho)dw_{A_2}, i = 1, 2
\end{aligned} \tag{C.7a}$$

$$\begin{aligned}
d\mathrm{Tr}^2(A_2\rho_t) &= 4\eta_{A_2}\Gamma_{A_2}U_2^2(\rho_t)dt + 4\eta_{A_1}\Gamma_{A_1}U_{12}^2(\rho_t)dt \\
&+ 4\sqrt{\eta_{A_2}\Gamma_{A_2}}\mathrm{Tr}(A_2\rho_t)U_2(\rho_t)dw_{A_2} \\
&+ 4\sqrt{\eta_{A_1}\Gamma_{A_1}}\mathrm{Tr}(A_2\rho_t)U_{12}(\rho_t)dw_{A_1}
\end{aligned} \tag{C.7b}$$

Therefore, the infinitesimal generator associated with (6.12) acting on $V(\rho)$ is

$$\begin{aligned}\mathcal{L}V(\rho_t) &= \mathcal{L}\mathrm{Tr}(A_1^2\rho_t) - \mathcal{L}\mathrm{Tr}^2(A_1\rho_t) + \mathcal{L}\mathrm{Tr}(A_2^2\rho_t) - \mathcal{L}\mathrm{Tr}^2(A_2\rho_t) \\ &= -4\eta_{A_1}\Gamma_{A_1}U_1^2(\rho_t) - 4\eta_{A_2}\Gamma_{A_2}U_2^2(\rho_t) \\ &\quad - 4(\eta_{A_1}\Gamma_{A_1} + \eta_{A_2}\Gamma_{A_2})U_{12}^2(\rho_t). \qquad \text{(C.8)}\end{aligned}$$

Eq. (6.15) is proved. $\square$

Appendix D

Appendices for Chapter 7

1 Proof of Proposition 7.1

For simplicity, we divide the proof of Proposition 7.1 into the following lemmas to make it easy to follow.

Lemma D.1. *With the control law $u_t = 1$, then the expectation of ρ_t converges to $1/4\mathbb{I}_4$, i.e.,*

$$\lim_{t\to\infty} \mathbb{E}[\rho_t] = \frac{1}{4}\mathbb{I}_4, \tag{D.1}$$

where $\mathbb{I}_4$ is the $4-$dimensional identity matrix.

Before proving this lemma, we recall the well-known LaSalle's theorem (see Theorem 4.4, page 128 [62]). This theorem provides us a powerful tool to prove the asymptotic stability of a system if there exists a positive definite function whose derivative is semi-negative definite.

Theorem D.1. *(LaSalle's theorem) Consider the autonomous system $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Let $\Phi \subset D$ be a* *compact set that is positively invariant with respect to this system. Let $V : D \to \mathbb{R}^+$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Φ. Let E be the set of all points in Φ where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Φ approaches M as $t \to \infty$.*

Proof of Lemma D.1. Let $\bar{\rho}_t = \mathbb{E}[\rho_t]$. As $u_t = 1$, it follows from (7.2) that the evolution of $\bar{\rho}_t$ is given by

$$\dot{\bar{\rho}}_t = -i[H_0 + H_1, \bar{\rho}_t] + \sum_{l=1}^{2} \Gamma_{A_l} \mathcal{D}[A_l]\bar{\rho}_t \tag{D.2}$$

Consider the following function defined on $\mathcal{S}$:

$$W(\rho) = \mathrm{Tr}\big((\rho - \frac{1}{4}\mathbb{I}_4)^2\big) = \mathrm{Tr}(\rho^2) - \frac{1}{4} \tag{D.3}$$

By the cyclic property of trace and the self-adjointness of A_1, A_2, the derivative of $W(\bar{\rho}_t)$ along the solution of (D.2) is

$$\begin{aligned}
\dot{W}(\bar{\rho}_t) &= -2\mathrm{Tr}(i[H_0 + H_1, \bar{\rho}_t]\bar{\rho}_t) - \sum_{l=1}^{2} \mathrm{Tr}([A_l[A_l, \bar{\rho}_t]]\bar{\rho}_t) \\
&= -\sum_{l=1}^{2} \mathrm{Tr}([A_l, \bar{\rho}_t]^\dagger [A_l, \bar{\rho}_t]) \\
&= -\sum_{l=1}^{2} \big|[A_l, \bar{\rho}_t]\big|^2 \leq 0,
\end{aligned} \tag{D.4}$$

where $|\cdot|$ is the Frobenius norm. Hence, $W(\bar{\rho}_t)$ is decreasing along the solution of (D.2). Applying the LaSalle's Theorem D.1, from Eq. (D.4), $\bar{\rho}_t$ converges to the largest invariant set $\mathcal{M}$ contained in the set in which $\dot{W}(\rho) = 0$, i.e., in the set $\big\{\rho \in \mathcal{S} : [A_1, \rho] = [A_2, \rho] = 0\big\}$.

We will prove that $\mathcal{M} = \{1/4\mathbb{I}\}$, from which Lemma D.1 is proved. Indeed, for any trajectory $\bar{\rho}_t^{\mathcal{M}}$ of (D.2) in $\mathcal{M}$, we have $[A_1, \bar{\rho}_t^{\mathcal{M}}] = [A_2, \bar{\rho}_t^{\mathcal{M}}] = 0$. From these equations, $\bar{\rho}_t^{\mathcal{M}}$

must have the form

$$\bar{\rho}_t^{\mathcal{M}} = \begin{bmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{bmatrix}, a, b, c, d \in \mathbb{R}, \tag{D.5}$$

which we call the $X-$form. Then, $\dot{\bar{\rho}}_t^{\mathcal{M}}$ also has the $X-$form, and $[H_0, \bar{\rho}_t^{\mathcal{M}}] = \mathcal{D}[A_1]\bar{\rho}_t^{\mathcal{M}} = \mathcal{D}[A_2]\bar{\rho}_t^{\mathcal{M}} = 0$. These, along with Eq. (D.2), imply that $[H_1, \bar{\rho}_t^{\mathcal{M}}]$ also has the $X-$form. We have

$$H_1 = (\sigma_1^x + \mathrm{i}\sigma_1^y) \otimes I_2/2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$[H_1, \bar{\rho}_t^{\mathcal{M}}] = \begin{bmatrix} 0 & c & b-a & 0 \\ d & 0 & 0 & a-b \\ 0 & 0 & 0 & -c \\ 0 & 0 & -d & 0 \end{bmatrix}$$

As such, if $[H_1, \bar{\rho}_t^{\mathcal{M}}]$ has the $X-$form, it must hold that $a = b = 1/4, c = d = 0$. Therefore, $\bar{\rho}_t^{\mathcal{M}} = 1/4\mathbb{I}_4$ and $\mathcal{M} = \{1/4\mathbb{I}\}$. Lemma D.1 is proved. $\square$

Lemma D.2. *With $\gamma < 1/4$ and the control law $u_t = 1$, there exists $\overline{T} < \infty$ such that*

$$\mathbb{E}[D(\rho_t)] < 1 - \gamma, \forall t \geq \overline{T}, \forall \rho_0 \in \mathcal{S}. \tag{D.6}$$

Proof. Due to the continuity of $D(\rho_t)$ and Lemma D.1, we have

$$\lim_{t\to\infty}\mathbb{E}[D(\rho_t)] = D\big(\lim_{t\to\infty}\mathbb{E}[\rho_t]\big) = D(\frac{1}{4}\mathbb{I}_4) = 1-\frac{1}{4}. \tag{D.7}$$

As such, with $\epsilon = 1-\gamma-(1-1/4) = 1/4-\gamma > 0$, there exists $T > 0$ such that $\big|\mathbb{E}[D(\rho_t)] - (1-1/4)\big| < \epsilon, \forall t \geq T$. Consequently, for all $t \geq T$,

$$\mathbb{E}[D(\rho_t)] < (1-\frac{1}{4}) + \epsilon = 1-\gamma. \tag{D.8}$$

Define $T(\rho_0) = \inf\{T : \mathbb{E}[D(\rho_t)] < 1-\gamma, \forall t \geq T\}$ and $\overline{T} = \sup_{\rho_0\in\mathcal{S}} T(\rho_0)$. Due to the continuity of $\mathbb{E}[D(\rho_t)]$ and thus, the continuity of $T(\rho_0)$ with respect to the initial state [57], it must hold that $\overline{T} < \infty$. It follows from the definition of $T(\rho_0)$ that Eq. (D.6) holds true. The proof of Lemma D.2 is completed. □

Proof of Proposition 7.1. Now, we proceed with the proof of Proposition 7.1. By Lemma D.2, we will prove that from any initial state ρ_0, there is almost surely a finite time such that at that time $D(\rho_t) \leq 1-\gamma$. Let $\tau_{\rho_0}(\mathcal{S}_{>1-\gamma})$ be the first exit time from the set $\mathcal{S}_{>1-\gamma}$ of a solution ρ_t beginning at ρ_0. We need to show that $\tau_{\rho_0}(\mathcal{S}_{>1-\gamma}) < \infty$ almost surely. Notice that the solution of (7.2) is a Markov process. We recall the following lemma from [112] (Lemma 4.3, page 111 [112]):

Lemma D.3. *Let x_t be a Markov process evolving on the space E and let $\tau_{x_0}(S)$ be the first exit time from an open set $S \subset E$. Then, for all $t \geq 0, x_0 \in E$, we have*

$$\mathbb{E}[\tau_{x_0}(S)] \leq \frac{t}{1-\sup_{x_0\in E}\mathbb{P}\{\tau_{x_0}(S) > t\}}. \tag{D.9}$$

Here we stated the lemma in a simpler form than one in [112] to reduce the unnecessary complexity. Lemma D.3 is very effective to prove the finiteness of the exit time. Applying

Lemma D.3, we have

$$\mathbb{E}[\tau_{\rho_0}(\mathcal{S}_{>1-\gamma})] \leq \frac{2\overline{T}}{1 - \sup\limits_{\xi \in \mathcal{S}} \mathbb{P}\{\tau_\xi(\mathcal{S}_{>1-\gamma}) > 2\overline{T}\}}. \tag{D.10}$$

We will show that $\sup\limits_{\xi \in \mathcal{S}} \mathbb{P}\{\tau_\xi(\mathcal{S}_{>1-\gamma}) > 2\overline{T}\} < 1$. Suppose that

$$\sup_{\xi \in \mathcal{S}} \mathbb{P}\{\tau_\xi(\mathcal{S}_{>1-\gamma}) > 2\overline{T}\} = 1. \tag{D.11}$$

Obviously, $\mathbb{P}\{\tau_\xi(\mathcal{S}_{>1-\gamma}) > 2\overline{T}\} < 1$ for all $\xi \in \mathcal{S}_{\leq 1-\gamma}$. From (D.11), for each $\epsilon > 0$, there exists $\xi_\epsilon \in \mathcal{S}_{>1-\gamma}$ such that $\mathbb{P}\{\tau_{\xi_\epsilon}(\mathcal{S}_{>1-\gamma}) > 2\overline{T}\} > 1 - \epsilon$. Hence, with $\rho_0 = \xi_\epsilon$, we have

$$\mathbb{E}[D(\rho_t)] > (1-\gamma)\mathbb{P}\{\tau_{\xi_\epsilon}(\mathcal{S}_{>1-\gamma}) > 2\overline{T}\} > (1-\epsilon)(1-\gamma)$$

for all $0 \leq t \leq 2\overline{T}$. Taking $\epsilon \to 0$, there exists $\xi_\infty \in \mathcal{S}_{\geq 1-\gamma}$ such that with $\rho_0 = \xi_\infty$,

$$\mathbb{E}[D(\rho_t)] \geq 1 - \gamma, \forall 0 \leq t \leq 2\overline{T} \tag{D.12}$$

which is a contradiction with (D.6). Therefore, $\sup\limits_{\xi \in \mathcal{S}} \mathbb{P}\{\tau_\xi(\mathcal{S}_{>1-\gamma}) > 2\overline{T}\} < 1$. It follows from Eq. (D.10) that $\mathbb{E}[\tau_{\rho_0}(\mathcal{S}_{>1-\gamma})] < \infty$ and thus, $\tau_{\rho_0}(\mathcal{S}_{>1-\gamma}) < \infty$ a. s. The proof of Proposition 7.1 is completed. □

2 Proof of Proposition 7.2

Since ρ_d is commutative with H_0, A_1, and A_2, it can be checked that $\mathcal{L}D(\rho_t) = 0$ when $u_t = 0$. This, together with Dynkin's formula, leads to $\mathbb{E}[D(\rho_t)] = D(\rho_0), \forall t \geq 0$. As $\rho_0 \in \mathcal{S}_{\leq 1-\gamma}$, it follows that $\mathbb{E}[D(\rho_t)] = D(\rho_0) \leq 1 - \gamma, \forall t \geq 0$. By this and Chebyshev's inequality, we obtain

$$\mathbb{P}\{\sup_{t\geq 0} D(\rho_t) \geq 1-\gamma/2\} \leq \frac{\mathbb{E}[D(\rho_t)]}{1-\gamma/2} \leq \frac{1-\gamma}{1-\gamma/2} = 1-p.$$

Therefore, $\mathbb{P}\{\sup_{t\geq 0} D(\rho_t) < 1-\gamma/2\} \geq p$, i.e., the system trajectory ρ_t remains in $\mathcal{S}_{<1-\gamma/2}$ with probability larger or equal to p. Proposition 7.2 is proved. $\square$

3 Proof of Eq. (7.7)

By the self-adjointness of $A_1, A_2, ..., A_m$ and the commutativity of $A_1, A_2, ..., A_m$, and H_0, it holds that

$$\begin{aligned}
&d\mathrm{Tr}(A_1\rho_t) = \mathrm{Tr}(-iA_1[H_0,\rho])dt \\
&+ \Gamma_{A_1}\mathrm{Tr}(A_1\mathcal{D}[A_1]\rho)dt + \sqrt{\eta_{A_1}\Gamma_{A_1}}\mathrm{Tr}(A_1\mathcal{H}[A_1]\rho)dw_{A_1} \\
&+ \sum_{l=2}^{m}\big(\Gamma_{A_l}\mathrm{Tr}(A_1\mathcal{D}[A_l]\rho)dt + \sqrt{\eta_{A_l}\Gamma_{A_l}}\mathrm{Tr}(A_1\mathcal{H}[A_l]\rho)dw_{A_l}\big) \\
&= 2\sqrt{\eta_{A_1}\Gamma_{A_1}}(\mathrm{Tr}(A_1^2\rho_t) - \mathrm{Tr}^2(A_1\rho_t))dw_{A_1} \\
&+ 2\sum_{l=2}^{m}\big(\sqrt{\eta_{A_l}\Gamma_{A_l}}(\mathrm{Tr}(A_1A_l\rho_t) - \mathrm{Tr}(A_1\rho_t)\mathrm{Tr}(A_l\rho_t))dw_{A_l}\big) \\
&= 2\sqrt{\eta_{A_1}\Gamma_{A_1}}U_1(\rho_t)dw_{A_1} + 2\sum_{l=2}^{m}\sqrt{\eta_{A_l}\Gamma_{A_l}}U_{1l}(\rho_t)dw_{A_l}
\end{aligned}$$

where $U_{1l}(\rho) := \mathrm{Tr}(A_1 A_l \rho) - \mathrm{Tr}(A_1\rho)\mathrm{Tr}(A_l\rho)$ with $l = 2, ..., m$. This, along with Itô's product rule, leads to

$$\begin{aligned}
d\mathrm{Tr}^2(A_1\rho_t) &= 2\mathrm{Tr}(A_1\rho_t)d\mathrm{Tr}(A_1\rho_t) + (d\mathrm{Tr}(A_1\rho_t))^2 \\
&= 4\eta_{A_1}\Gamma_{A_1}U_1^2(\rho_t)dt + 4\sum_{l=2}^{m}\eta_{A_l}\Gamma_{A_l}U_{1l}^2(\rho_t)dt \\
&+ 4\sqrt{\eta_{A_1}\Gamma_{A_1}}\mathrm{Tr}(A_1\rho_t)U_1(\rho_t)dw_{A_1} \\
&+ 4\sum_{l=2}^{m}\sqrt{\eta_{A_l}\Gamma_{A_l}}\mathrm{Tr}(A_1\rho_t)U_{1l}(\rho_t)dw_{A_l} \qquad \text{(D.13)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
d\mathrm{Tr}(A_k^2\rho_t) &= \sum_{l=1}^{m}\sqrt{\eta_{A_l}\Gamma_{A_l}}\mathrm{Tr}(A_k^2\mathcal{H}[A_l]\rho)dw_{A_l}, k = 1, ..., m \\
d\mathrm{Tr}^2(A_k\rho_t) &= 4\eta_{A_k}\Gamma_{A_k}U_k^2(\rho_t)dt + 4\sum_{l\neq k}^{m}\eta_{A_l}\Gamma_{A_l}U_{kl}^2(\rho_t)dt \\
&+ 4\sqrt{\eta_{A_k}\Gamma_{A_k}}\mathrm{Tr}(A_k\rho_t)U_k(\rho_t)dw_{A_k} \\
&+ 4\sum_{l\neq k}^{m}\sqrt{\eta_{A_l}\Gamma_{A_l}}\mathrm{Tr}(A_k\rho_t)U_{kl}(\rho_t)dw_{A_l}
\end{aligned}$$

Therefore, infinitesimal generator of ρ_t acting on $V(\rho)$ is

$$\begin{aligned}
\mathcal{L}V(\rho_t) &= \sum_{l=1}^{m}\left(\mathcal{L}\mathrm{Tr}(A_l^2\rho_t) - \mathcal{L}\mathrm{Tr}^2(A_l\rho_t)\right) \\
&= -4\sum_{l=1}^{m}\eta_{A_l}\Gamma_{A_l}U_l^2(\rho_t) \\
&- 4\sum_{i<j}^{m}(\eta_{A_i}\Gamma_{A_i} + \eta_{A_j}\Gamma_{A_j})U_{ij}^2(\rho_t), \qquad \text{(D.15)}
\end{aligned}$$

by which Eq. (7.7) holds true.

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Author's Publications

The contents of this thesis are based on the following papers that have been published, accepted, or submitted to the peer-reviewed journals and conferences.

Journal papers:

1. T. L. Vu, S. S. Ge, and C. C. Hang, "Real-Time Deterministic Generation of Maximally Entangled Two-Qubit and Three-Qubit States via Bang-Bang Control," *Physical Review A* 85, 012332 (2012). (*Research Article*)

2. S. S. Ge, T. L. Vu, and C. C. Hang, "Non-smooth Lyapunov Function-based Global Stabilization for Quantum Filters," *Automatica*, Vol. 48, pp. 1031–1044 (2012). (*Regular Paper*)

3. S. S. Ge, T. L. Vu, and T. H. Lee, "Real-Time Quantum Measurement-Based Feedback Control: A Nonsmooth Time Delay Control Approach," *SIAM J. Control and Optimization*, Vol. 50, No. 2, pp. 845–863 (2012). (*Research Article*)

4. T. L. Vu, S. S. Ge, K.-S. Hong, and C. C. Hang, "Deterministic Generation of the Bell States by Local-Nonlocal Measurements and Non-smooth Control," *Automatica*, in revision.

Conference papers:

1. T. L. Vu, S. S. Ge, and C. C. Hang, "Deterministic Generation of The Bell States via Real-Time Quantum Measurement-Based Feedback," Proceedings of the 2012 American Control Conference, June 27-29, 2012, Montréal, Canada.

2. S. S. Ge, T. L. Vu, W. He and C. C. Hang, "Non-smooth Lyapunov Function-Based Global Stabilization for 2-Dimensional Quantum Filters," the 49th IEEE Conference on Decision and Control, pp. 3772-3777, Atlanta, GA, USA, December 15-17, 2010.

3. S. S. Ge, T. L. Vu, and T. H. Lee, "Real-Time Quantum Measurement-Based Feedback Control: A Non-smooth Time Delay Control Approach," Proceedings of IASTED CA, July 15 - 17, 2010, Banff, Alberta, Canada.