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BOUNDARY CONTROL OF FLEXIBLE MECHANICAL SYSTEMS

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Contents

Summary

Flexible systems have many application areas ranging from ocean engineering to aerospace. Driven by theoretical challenges as well as practical demands, the control problem of flexible mechanical systems has received increasing attention in recent decades. The main objective of this thesis is to explore the advanced methodologies for the vibration control of flexible structures with guaranteed stability and alleviate some of the challenges.

In the first part of this thesis, adaptive boundary control is developed for a nonuniform string system under unknown spatiotemporally varying distributed disturbance and time-varying boundary disturbance. The vibrating string is nonuniform since the time-varying tension and mass per unit length are considered in the system. The vibration suppression is first achieved for the flexible nonuniform string by using the model-based boundary control. Adaptive boundary control is then developed to deal with the system parameter uncertainties. The bounded stability of the closed loop system is proved by using the Lyapunov's direct method.

In the second part, the control problem of a coupled nonlinear string system is presented, i.e., not only the transverse displacement of the string system is regarded, but also the axial deformation is under consideration, which leads to a more precise model for the string system. Coupling between longitudinal and transverse dynamic is due to the consideration of the effect of axial elongation. The vibration of the nonlinear string is suppressed and the system parameter uncertainty is handled by the proposed two control laws. The control laws have the simple structure and are easy to implement in practice.

In the third part, the vibration suppression of an Euler-Bernoulli beam system is addressed by using the boundary control technique. By using Lyapunov synthesis, boundary control is first proposed to suppress the vibration and attenuate the effect of the external disturbances. To compensate for the system parametric uncertainties, adaptive boundary control is developed. Furthermore, a novel Integral-Barrier Lyapunov Function is designed for the control of flexible systems with output constraint problems. The employed Integral-Barrier Lyapunov Function guarantees that the boundary output constraint is not violated.

In the last part, modeling and control problem for a Timoshenko beam is discussed. Compared with the Euler-Bernoulli beam, the control design is more difficult for the Timoshenko beam due to its higher order model. Boundary control is proposed to stabilize the system, and the boundary disturbance observers are designed to estimate the time-vary boundary disturbances. The control design is based on the original system model governed by partial differential equations (PDEs), hereby avoiding the spillover instability. By properly selecting the design parameters, the control performance of the closed loop system is ensured.

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List of Symbols

Throughout this thesis, the following notations and conventions have been adopted:

L	length of flexible structures
t	temporal variable
x	spatial variable
$M_s,\ M$	mass of payload
$T,\ T(x,t)$	tension of the flexible structures
$T_0(x)$	initial tension of the flexible structures
$\rho,\ \rho(x)$	uniform and nonuniform mass per unit length of the flexible structures
I_ρ	uniform mass moment of inertia of the cross-section
J	inertia of the payload
E	Young's modulus
k_0	a positive constant that depends on the shape of the cross-section
A	cross section area
G	modulus of elasticity in shear
EI	bending stiffness of the flexible structures
EA	axial stiffness of the flexible structures
$w(x,t)$	transverse displacement of the flexible structures
$v(x,t)$	longitudinal displacement of the flexible structures
$\varphi(x,t)$	cross-section rotation of the flexible structures

$w(L,t)$	end point position of the flexible structures
$\dot{w}(L,t)$	velocity of the tip payload
$w'(L,t)$	slope of the tip payload
$\dot{w}'(L,t)$	slope rate of the tip payload
$f(x,t)$	spatiotemporally varying distributed disturbance
$\bar{f}$	upper bound of the spatiotemporally varying distributed disturbance
$d(t),\ \theta(t)$	time-varying boundary disturbances
$\bar{d},\ \bar{\theta}$	upper bound for the time-varying boundary disturbances
$u(t),\ v(t),\ \tau(t)$	control inputs
E_k	kinetic energy
E_p	potential energy
W_f	work done by external disturbances
W_m	work done by control input
W	total work done on the flexible systems
δ	variation operator
$\lambda_{\min}(A)$	minimum eigenvalue of the matrix A where all eigenvalues are real
$\lambda_{\max}(A)$	maximum eigenvalue of the matrix A where all eigenvalues are real
$\lambda,\ \lambda_a,\ \mu,\ \lambda_1 - \lambda_4,\ \lambda_{a1} - \lambda_{a3},\ \mu_1 - \mu_3, \xi$	positive constants
$\delta_1 - \delta_7$	small positive constants
$\alpha,\ \beta,\ a,\ b$	positive weighting constants

$\Gamma,\ \gamma,\ \gamma_m,\ \gamma_t,\ \gamma_d$	adaptation gains
$r,\ \sigma,\ \varsigma_m,\ \varsigma_t,\ \varsigma_d$	positive constants
$\mathbb{R}$	set of all real numbers
A^T	transpose of vector or matrix A
$\|\|A\|\|$	Euclidean norm of vector A or the induced norm of matrix A
l_0	boundary output constraint of the flexible structure
$\phi(x,t)$	a function defined on $x \in [0,L]$ and $t \in [0,\infty)$
Ω	compact set
$(\hat{*})$	estimate of $(*)$
$(\tilde{*})$	estimate error of $(*)$
$k,\ k_1,\ k_2$	positive control gains
$l_1,\ l_2$	positive estimate gains
$(\dot{*})\ (\ddot{*})$	first and second partial time derivative of $(*)$
$(*)'\ (*)''$	first and second partial space derivative of $(*)$
$(*)''',\ (*)''''$	third and fourth partial space derivatives of $(*)$

Chapter 1

Introduction

1.1 Motivation and Background

Flexible structures can be used to model a large number of mechanical systems in different engineering fields, such as telephone wires, crane cables [1], helicopter blades, robotic arms [2], mooring lines [3], marine risers [4], human DNA and so on. Recently, the vibration control problem of flexible mechanical systems has received great attention due to the large applications in industry [5,6]. Examples of practical applications where flexible structures are exposed to the external disturbance include the flexible manipulator for grasping, industry chains for transmission, crane cables for positioning of the payload, marine risers for gas and oil transportation, etc.. The excessive vibration due to the external disturbances and the flexible property reduces the system quality, leads to limited productivity, results in premature fatigue failure and limits the utility of the flexible mechanical systems. Therefore, vibration suppression is well motivated to improve the performance of the system. In addition, compared with the rigid systems, the advantages of flexible systems such as lightweight, better mobility and lower cost also greatly motivate the applications of flexible mechanical systems in industrial engineering.

1.1.1 Distributed parameter system

From a mathematical point, a system with vibration is often considered as an original distributed parameter system (DPS). Different from a lumped parameter system, a DPS has an infinite-dimensional state space. DPSs cannot be modeled by ordinary differential equations (ODEs) since the motion of such systems is described by variables depending on both time and space [7]. Due to the time and spatial variables, the dynamics of DPSs can be modeled as a coupled PDE-ODE system, and a large number of control methods for the conventional rigid systems cannot be directly used. A common modeling method for DPSs is based on discretization of the PDE into a finite number of ODEs [8–21]. However, the finite dimensional discrete models are approximated by neglecting high order modes, which would result in spillover instability [22, 23], and the requirements of high control performance may not be satisfied. Therefore, researchers have developed several control techniques which the control design were based on the original distributed parameter systems, such as boundary control [24–29], sliding model control [30], energy-based robust control [31, 32], model-free control [33], variable structure control [34], methods derived through the use of bifurcation theory and the application of Poincaré maps [35], and the averaging method [36–40].

1.1.2 Boundary control

Distributed control [41–45] is difficult to implement since it needs more actuators and sensors. Boundary control which is an economical and effective method to control DPSs, has the following merits: (i) providing a more practical alternative since fewer actuators and sensors are needed at the boundary of the system, (ii) boundary control can be derived from a Lyapunov function which is relevant to the mechanical energy based on the dynamics of the system, and (iii) the spillover problem can be removed since boundary control is proposed on the base of the original distributed parameter systems. Therefore, boundary control has received great attention in many research fields such as chemical process control, vibration suppression of flexible mechanical systems, etc.. Recent progress in the boundary control is summarized in [46]. An overview on the boundary control for DPSs is introduced in [47]. In [48,49], boundary control based on Lyapunov techniques is developed to stabilize the vibration. Semi-group theory of the boundary control techniques is introduced in [50]. By integrating the backstepping method, boundary controller and observer are studied in [51–62].

1.1.3 Lyapunov's direct method

Lyapunov theory, the most successfully and widely used tool, provides a means of determining stability without explicit knowledge of system solutions [63]. Besides the stability analysis of the system, Lyapunov theory can also be used to design the control laws of the systems. In addition, compared with the functional analysis based methods, the Lyapunov's direct method requires little background beyond calculus for users to understand the control design and the stability analysis. The Lyapunov's direct method also offers an advantageous technique for PDEs by using well-understood mathematical tools such as integration by parts and integral inequalities. Due to its advantages, the Lyapunov's direct method is widely applied in research. Many remarkable results [1, 64–80] have been presented for the boundary control of flexible mechanical systems based on the Lyapunov's direct method.

Barrier Lyapunov function is a novel concept that can be employed to deal with the control problems with output constraints [81–84]. In [81], a barrier Lyapunov function is employed for the control of SISO nonlinear systems with an output constraint. A novel asymmetric time-varying barrier Lyapunov function is used in [83] to ensure the time-varying output constraint satisfaction for strict feedback nonlinear systems. In the neurocontrol field, two challenging and open problems are addressed in [82] by using a barrier Lyapunov function in the presence of unknown functions. However, in all the papers mentioned above, the barrier Lyapunov functions are designed for linear or nonlinear ODE systems. There is little information about how to handle the constraints for PDEs and there is a need to explore an effective method for the control of flexible systems with constraint problems.

1.2 Previous Works

The applications for boundary control strategies in flexible mechanical systems include second order structures (string) and fourth order structures (beam) [85]. In recent years, boundary control design for string-based structures [86–91] has received much attention among control researchers due to the large applications. The vibration of a moving string with a varying tension is regulated in [92] by developing a robust adaptive boundary control. By using state feedback, the control problem of a moving string is addressed in [93], where the asymptotic and exponential stability is achieved.

Boundary control is designed in [94] for a cable with a gantry crane modeled by a string structure, and the experiment is implemented to verify the control performance. For a nonlinear moving string in [95], exponentially stability is well achieved with a velocity feedback boundary control. The authors proposed a boundary control law for a class of non-linear string-based actuator system [96]. The vibration of a non-linear string system is stabilized by using the boundary control with the negative feedback of the boundary velocity of the string in [97]. In [98], the flexible systems including the string and beam model are stabilized by using the backstepping method with a properly kernel function.

The control problem of beam-based structures [99–101] is also an interesting research topic since it constitutes an important application topic in its own right, such as moving strips [102], marine risers [101], flexible link robots [103]. Exponentially stability is proved in [104] for a beam system with the proposed control. In [105], with ACLD treatments, a boundary control law is constructed to damp the beam's vibration. For a beam in vibration, exponentially stability is achieved with the boundary control strategy in [106], axial tension is also considered. Boundary cooperative control on two flexible beam-like robot arms is employed to realize a grasping task [107]. Exponentially stable controller and observer are designed in [108] for a class of second order DPSs without considering of the distributed damping via Semigroup Theory. For the marine application, the transverse and the angle vibration of the marine riser modeled by a beam system with a distributed load are suppressed by designing a boundary torque in [109]. In [110], backstepping methodology combining with Lyapunov theory is used for proving the uniqueness solutions of the closed loop system of the marine riser. The authors in [101] propose two boundary control laws to regulate both the transverse and the longitudinal vibrations for a marine riser modeled by a coupled non-linear system.

When the beam's length is large in comparison to its cross-sectional dimensions, i.e., the model neglects the rotary inertia of beam, the Euler-Bernoulli beam [111–113] is the most used model since it provides a good description of the beam's dynamic behavior [48]. However, in the beginning of the 20th century, an improvement of the Euler-Bernoulli beam has been proposed by Stephen Timoshenko. Boundary control design for the Timoshenko beam system [114–116] has also been the subject of many investigations. The authors in [117] propose a dynamic boundary control applied at the free end of a clamped-free Timoshenko beam to stabilize vibrations of the system. In [118], the boundary feedback controls for a class of nonself-adjoint operators which the dynamics generators for the systems are governed by the Timoshenko beam model is considered. The Keldysh Theorem applied in [118] is used to prove the completeness for the root subspaces of the beam-like systems with boundary feedbacks in [119]. Backstepping method is also applied to the Slender Timoshenko beam in [120] and [121], where boundary controllers and observers are designed.

Although the extensive research on the flexible systems has been investigated, the external spatiotemporally varying disturbances are neglected in some works. After the consideration of the unknown disturbances, the control problem becomes more difficult. Therefore, the control technique for vibration suppression is desirable for stopping the damage and improving lifespan of flexible structures.

1.3 Thesis Objectives

This thesis is well motivated by the observation of the vibrations in many industrial applications. The general objective of this thesis is to develop constructive methods of designing boundary control for flexible mechanical systems with guaranteed stability and alleviate some of the challenges. More specifically, the objectives of this study are to:

(i) Derive the hybrid PDE-ODE model of flexible systems under unknown disturbances based on Hamilton's principle.

(ii) Propose the constructive boundary control method for suppressing the vibration of the systems and eliminating the effects of the disturbances.

(iii) Investigate the stability of the flexible systems with the proposed boundary control by using Lyapunov's method.

The results of this study may have a significant impact on providing a systematic method for flexible mechanical systems so as to:

(i) Establish a framework of the boundary control method for flexible mechanical systems by the use of the Lyapunov's method.

(ii) In particular, for parametric uncertainties of model, design an adaptive control law to track the system performance in the presence of the parametric uncertainties.

(iii) Design the disturbance observer to reduce the effects of the unknown disturbances.

(iv) Propose a novel Integral-Barrier Lyapunov Function for the control of flexible structures with boundary output constraint.

It is understood that the work presented in this thesis is problem oriented and dedicated to the fundamental academic exploration of boundary control of flexible systems. Thus, the focus is given to the development of the control method. In addition, our studies are focused on the distributed parameter systems, which cover large classes of flexible string and beam systems in mechanical engineering. It would be a future research topic to extend our control design methods to distributed parameter systems in other forms.

1.4 Thesis Organization

The remainder of the thesis is organized as follows.

In Chapter 2, some mathematical preliminaries are introduced. Hamilton's principle are used to derive the dynamic model of the flexible structures, and some inequalities will be applied to analyze the stability of the systems throughout this thesis.

In Chapter 3, we start with the study of modeling and control of a nonuniform string system which is described by a nonlinear nonhomogeneous PDE and two ODEs. The varying tension and mass per unit length is under consideration. Both the model-based boundary control and adaptive boundary control constructed at the right boundary of the nonuniform string can suppress the system's vibration and reduce the effects of the external disturbances. The bounded stability of the nonuniform string system is proved.

In Chapter 4, the boundary control problem of a coupled nonlinear string system under system uncertainties is addressed. The vibrating string is nonlinear due to the coupling between transverse and longitudinal displacements, which provides a more accurate description of the system dynamic model. To reduce the vibrations, boundary control is designed and implemented by two actuators in both longitudinal and transverse directions, respectively. The vibration regulation is well achieved with the proposed control.

In Chapter 5, boundary control is proposed for an Euler-Bernoulli beam under spatiotemporally varying disturbance. By using the Hamilton's principle, the model of the Euler-Bernoulli beam is presented by one PDE and four ODEs. The exact knowledge of the external disturbances including the distributed disturbance and the boundary disturbance is not required in the control design. A novel Integral-Barrier Lyapunov Function is designed for the Euler-Bernoulli beam system with constraint problem.

In Chapter 6, we further investigate the boundary output-feedback problem of a Timoshenko beam by using disturbance observer. Compared with the Euler-Bernoulli beam, the Timoshenko beam model considers shear deformation and rotational inertia effects as it vibrates. Boundary control combined with the disturbance observer is developed to reduce the vibration and deal with the unknown disturbances. The proposed control is implementable with actual instrumentations.

In Chapter 7, conclusions of this thesis and the future research works are presented.

Chapter 2

Preliminaries

In this chapter, for the convenience of stability analysis, we introduce the following mathematical preliminaries, useful technical lemmas and properties which will be extensively used throughout this thesis.

Remark 2.1. *For clarity, notions $(\cdot)' = \partial(\cdot)/\partial x$ and $\dot{(\cdot)} = \partial(\cdot)/\partial t$ are used throughout this thesis.*

Hamilton's principle [122] can be used to derive the model of the flexible systems and represented by

$$\int_{t_1}^{t_2} \delta[E_k(t) - E_p(t) + W(t)]dt \quad = \quad 0, \tag{2.1}$$

where $E_k(t)$ is the kinetic energy of the system, $E_p(t)$ is the potential energy of the system, and $W(t)$ is the total virtual work done on the body. δ is the variational operator, t_1 is the initial time, t_2 is the final time, and $t_1 < t < t_2$. Hamilton's principle provides a methodical manner to derive the system dynamics, with a definite integral involving the kinetic energy $E_k(t)$, the potential energy $E_p(t)$ and the virtual work $W(t)$ of the system.

Lemma 2.1. *[85] Let $\phi_1(x,t)$, $\phi_2(x,t) \in \mathbb{R}$ with $x \in [0,L]$ and $t \in [0,\infty)$, the following inequality holds:*

$$\phi_1\phi_2 \leq |\phi_1\phi_2| \leq \phi_1^2 + \phi_2^2, \quad \forall \phi_1, \phi_2 \in \mathbb{R}. \tag{2.2}$$

Lemma 2.2. *[85] Let $\phi_1(x,t)$, $\phi_2(x,t) \in \mathbb{R}$ with $x \in [0,L]$ and $t \in [0,\infty)$, the following inequality holds:*

$$|\phi_1\phi_2| \quad = \quad \left| \left(\frac{1}{\sqrt{\delta}}\phi_1 \right) (\sqrt{\delta}\phi_2) \right| \leq \frac{1}{\delta}\phi_1^2 + \delta\phi_2^2, \quad \forall \phi_1, \phi_2 \in \mathbb{R} \quad \textit{and} \quad \delta > 0. \tag{2.3}$$

Lemma 2.3. *[82] For all $|\xi| < 1$, and any positive integer p, the following inequality holds.*

$$\ln \frac{1}{1-\xi^{2p}} \leq \frac{\xi^{2p}}{1-\xi^{2p}} \tag{2.4}$$

Lemma 2.4. *[123] Let $\phi(x,t) \in \mathbb{R}$ be a function defined on $x \in [0,L]$ and $t \in [0,\infty)$ that satisfies the boundary condition*

$$\phi(0,t) = 0, \quad \forall t \in [0,\infty), \tag{2.5}$$

then the following inequalities hold:

$$\int_0^L \phi^2 dx \leq L^2 \int_0^L [\phi']^2 dx, \quad \forall x \in [0,L]. \tag{2.6}$$

$$\phi^2 \leq L \int_0^L [\phi']^2 dx, \quad \forall x \in [0,L]. \tag{2.7}$$

If in addition to Eq. (2.5), the function $\phi(x,t)$ satisfies the boundary condition

$$\phi'(0,t) = 0, \quad \forall t \in [0,\infty), \tag{2.8}$$

then the following inequalities also hold:

$$\int_0^L [\phi']^2 dx \leq L^2 \int_0^L [\phi'']^2 dx, \quad \forall x \in [0,L], \tag{2.9}$$

$$\phi^2 \leq L^3 \int_0^L [\phi'']^2 dx, \quad \forall x \in [0,L], \tag{2.10}$$

$$[\phi']^2 \leq L \int_0^L [\phi'']^2 dx, \quad \forall x \in [0,L]. \tag{2.11}$$

Lemma 2.5. *[124] Rayleigh-Ritz theorem: Let $A \in \mathbb{R}^{n\times n}$ be a real, symmetric, positive-definite matrix; therefore, all the eigenvalues of A are real and positive. Let $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalues of A, respectively; then for $\forall x \in \mathbb{R}^n$, we have*

$$\lambda_{\min}||x||^2 \leq x^T A x \leq \lambda_{\max}||x||^2, \tag{2.12}$$

where $||\cdot||$ denotes the standard Euclidean norm.

Chapter 3

Modeling and Control of a Nonuniform Vibrating String under Spatiotemporally Varying Tension and Disturbance

In modern mechanical engineering, a large number of flexible systems [125, 126], such as cables and chains, telephone lines, and human DNA can be modeled as string-based structures. String models and their boundary controls have been studied for decades. Although most of these results are based on linear models, nonlinear string systems are considered in recent results [127–129].

However, in most of these works, the control problems have been addressed by neglecting the unknown spatiotemporally varying distributed disturbance which is the function of both time and space due to the environmental effect. The consideration of the distributed disturbance would lead to a nonhomogeneous PDE string model, and the control design is more difficult than the previous work. Additionally, a constant axial tension and mass per unit length are assumed in most of papers mentioned above. From a practical point of view, many string systems do not have to be uniform and it could have a varying tension and a varying mass per unit length. Thus, another novelty for this work is the consideration of the varying tension and mass per unit length in the boundary control for the nonuniform string system.

In this chapter, a general modeling and control problem for the nonuniform string systems is addressed. Lyapunov's direct method is used to analyze the stability of the closed-loop system. Compared to the existing work, the main contributions of the chapter include:

(i) A coupled PDE-ODE model of the nonuniform string system under unknown disturbances for vibration regulation is derived based on Hamilton's principle. The governing equation of the system is described as a nonlinear nonhomogeneous PDE in which the tension may be an uncertain nonlinear function of both its transverse gradient and the position along its equilibrium. The varying mass per unit length is also considered.

(ii) To eliminate spillover problem, boundary control based on the original infinite dimensional model (PDE) is developed. First, model-based boundary control is proposed for the nonuniform string system when the system parameters are known. Then, adaptive boundary control is developed to deal with system parameter uncertainties.

(iii) A new theorem is presented to illustrate that the Lyapunov-type stability of the closed-loop nonuniform string system is well achieved with the proposed control law and adaption laws.

The structure of this chapter is arranged as follows. In Section 3.1, Hamilton's principle is used to drive the equations of motion for a nonuniform string system under the unknown spatiotemporally varying distributed disturbance and the unknown time-varying boundary disturbance. Then, a boundary control problem is stated. In Section 3.2, model-based boundary control law is formulated with the known system parameters, and the adaptive laws are then developed for the parameter uncertainties case. Stability analysis is on the basis of the Lyapunov's direct method, and all of the internal states of the nonuniform string system are proved to be bounded by using the proposed control. In Section 3.3, numerical simulation results demonstrate the effectiveness of the proposed boundary controller. Conclusions of this chapter are given in Section 3.4.

3.1 Problem Formulation

Fig. 3.1 shows a string-based structure extracted from a class of flexible systems for the control design purpose. $w(x,t)$ is the transverse displacement of the nonuniform string, $w(L,t)$, $\dot{w}(L,t)$ and $\ddot{w}(L,t)$ are the displacement, velocity and acceleration of the tip payload respectively. $w'(L,t)$ and $\dot{w}'(L,t)$ are the slope and the slope rate of the tip payload. The left boundary of the string is fixed at origin, which means $w(0,t)=0$.

The kinetic energy of the nonuniform string system $E_k(t)$ is given as

$$E_k(t) = \frac{1}{2}M_s\left[\dot{w}(L,t)\right]^2 + \frac{1}{2}\int_0^L \rho(x)\left[\dot{w}(x,t)\right]^2 dx, \tag{3.1}$$

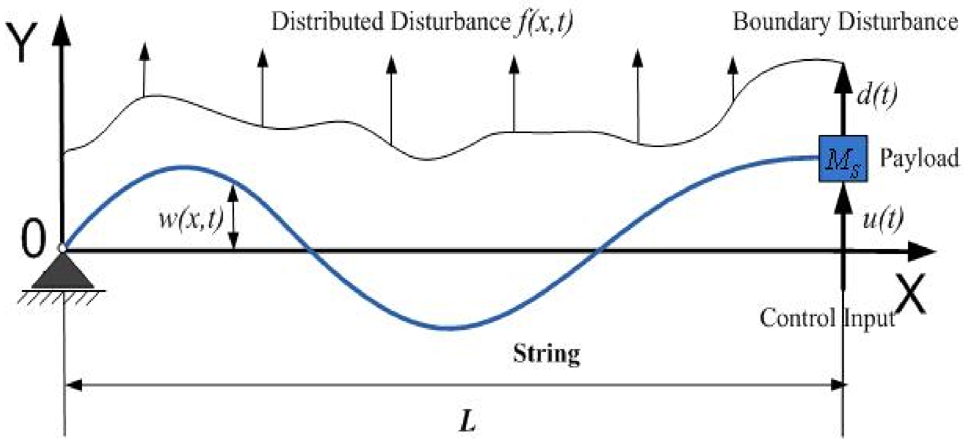

Fig. 3.1: A typical string system.

where L is the length of the string, M_s is the mass of the payload, $\rho(x)$ is the nonuniform mass per unit length of the string, t and x represent the time and spatial variables, respectively.

The potential energy $E_p(t)$ due to a spatiotemporally varying tension $T(x,t)$ can be obtained from

$$E_p(t) \quad = \quad \frac{1}{2}\int_0^L T(x,t)\,[w'(x,t)]^2\,dx, \tag{3.2}$$

where the tension $T(x,t)$ of the string can be expressed as

$$T(x,t) = T_0(x) + \lambda(x)[w'(x,t)]^2, \tag{3.3}$$

where $T_0(x) > 0$ is the initial tension, and $\lambda(x) \geq 0$ is the nonlinear elastic modulus [130].

The virtual work $\delta W(t)$ done on the system can be expressed as

$$\delta W(t) \quad = \quad \delta W_f(t) + \delta W_m(t), \tag{3.4}$$

where the virtual work $\delta W_f(t)$ is done by the unknown distributed disturbance $f(x,t)$ and boundary disturbance $d(t)$, given by

$$\delta W_f(t) \;=\; \int_0^L f(x,t)\delta w(x,t)dx + d(t)\delta w(L,t), \tag{3.5}$$

and the virtual work $\delta W_m(t)$ is done by the control force $u(t)$, which is to be designed to suppress the system vibration of the string, expressed as

$$\delta W_m(t) \;=\; u(t)\delta w(L,t). \tag{3.6}$$

Using the Hamilton's principle Eq. (2.1), the governing equation of the nonuniform string system is obtained as

$$\begin{aligned}&\rho(x)\ddot{w}(x,t) - \left\{T(x,t) + 3\lambda(x)[w'(x,t)]^2\right\} w''(x,t) - T'(x,t)w'(x,t)\\&= f(x,t) + \lambda'(x)[w'(x,t)]^3,\end{aligned} \tag{3.7}$$

$\forall(x,t) \in (0,L)\times[0,\infty)$, and the boundary conditions of the nonuniform string system are given as

$$w(0,t) \;=\; 0, \tag{3.8}$$

$$M_s\ddot{w}(L,t) + T(L,t)w'(L,t) + \lambda(L)[w'(L,t)]^3 \;=\; u(t) + d(t), \tag{3.9}$$

$\forall t \in [0,\infty)$.

Remark 3.1. *With consideration of time-varying tension $T(x,t)$ and mass per unit length $\rho(x)$, the string system Eq. (3.7) is nonuniform, and the control methods for the uniform PDE system can not be used.*

Remark 3.2. *Due to the consideration of the unknown spatiotemporally varying distributed disturbance $f(x,t)$, a nonhomogeneous PDE* (3.7) *is used to describe the governing equation of the nonuniform string system. The nonuniform nonhomogeneous model is different from the string system governed by a homogeneous PDE in [26, 93, 94, 98].*

Property 3.1. *[131]: If the kinetic energy of the system* (3.7) - (3.9)*, given by Eq.* (3.1) *is bounded $\forall t \in [0,\infty)$, then $\dot{w}(x,t)$, $\dot{w}'(x,t)$ and $\dot{w}''(x,t)$ are bounded $\forall (x,t) \in [0,L] \times [0,\infty)$.*

Property 3.2. *[131]: If the potential energy of the system* (3.7) - (3.9)*, given by Eq.* (3.2) *is bounded $\forall t \in [0,\infty)$, then $w'(x,t)$ and $w''(x,t)$ are bounded $\forall (x,t) \in [0,L] \times [0,\infty)$.*

Assumption 3.1. *Assuming that the unknown spatiotemporally varying distributed disturbance $f(x,t)$ and unknown time-varying boundary disturbance $d(t)$ are uniformly bounded, i.e., $|f(x,t)| \leq \bar{f}$, $\forall (x,t) \in [0,L] \times [0,\infty)$ and $|d(t)| \leq \bar{d}$, $\forall t \in [0,\infty)$, where $\bar{f}$ and $\bar{d}$ are two positive constants. The exact values of $f(x,t)$, $d(t)$ and $\bar{f}$ are not required.*

Assumption 3.2. *We assume that $\rho(x)$, $T_0(x)$ and $\lambda(x)$ are bounded by known, constant lower and upper bounds as follows:*

$$\underline{\rho} \leq \rho(x) \leq \overline{\rho}, \tag{3.10}$$

$$\underline{T_0} \leq T_0(x) \leq \overline{T}_0, \tag{3.11}$$

$$\underline{\lambda} \leq \lambda(x) \leq \overline{\lambda}. \tag{3.12}$$

3.2 Control Design

In this section, boundary control combining with the adaption laws are derived to regulate the vibrations of the nonuniform string system as well as to attenuate the effects of the unknown disturbances by use of the Lyapunov's method. Due to the consideration of the spatiotemporally varying distributed disturbance and tension, the control design for the string system governed by a nonlinear nonhomogeneous PDE (3.7) becomes rather difficult. In the following parts, two cases are investigated for the nonuniform string system: (i) model-based boundary control with the known system parameters; and (ii) adaptive boundary control with the unknown system parameters.

3.2.1 Model-based boundary control

For the model-based situation, i.e., with the essential knowledge of system parameters M_s and $T_0(L)$, under the unknown disturbances, boundary control is proposed for the nonuniform string system given by (3.7) - (3.9) as

$$\begin{aligned} u(t) &= -k[w'(L,t)+\dot{w}(L,t)] - M_s\dot{w}'(L,t) + T_0(L)w'(L,t) \\ &\quad -\mathrm{sgn}\left[w'(L,t)+\dot{w}(L,t)\right]\bar{d}, \end{aligned} \tag{3.13}$$

where $k > 0$ is the control gain, $\bar{d}$ is the upper bound of the disturbance $d(t)$, and $\mathrm{sgn}(\cdot)$ denotes the signum function.

The following positive Lyapunov function candidate is considered for the nonuniform string system (3.7) - (3.9) as

$$V(t) = V_1(t) + V_2(t) + \eta(t), \tag{3.14}$$

where the energy term $V_1(t)$, the auxiliary term $V_2(t)$ and the small crossing term $\eta(t)$ are given as

$$\begin{aligned} V_1(t) &= \frac{\beta}{2}\int_0^L \rho(x)[\dot{w}(x,t)]^2 dx + \frac{\beta}{2}\int_0^L T(x,t)[w'(x,t)]^2 dx \\ &= \frac{\beta}{2}\int_0^L \rho(x)[\dot{w}(x,t)]^2 dx + \frac{\beta}{2}\int_0^L T_0(x)[w'(x,t)]^2 dx + \frac{\beta}{2}\int_0^L \lambda(x)[w'(x,t)]^4 dx, \end{aligned} \tag{3.15}$$

$$V_2(t) = \frac{\beta}{2} M_s [w'(L,t) + \dot{w}(L,t)]^2, \tag{3.16}$$

$$\eta(t) = \alpha \int_0^L \rho(x)\varphi(x) x \dot{w}(x,t) w'(x,t) dx, \tag{3.17}$$

α and β are two positive constants, $\varphi(x)$ is a positive scalar function bounded by a known constant, i.e., $\varphi(x) \le \bar{\varphi}$.

Lemma 3.1. *The Lyapunov function equation* (3.14) *is bounded, and given by*

$$0 \le \lambda_1(V_1(t) + V_2(t)) \le V(t) \le \lambda_2(V_1(t) + V_2(t)), \tag{3.18}$$

where two positive constants λ_1 and λ_2 are defined as

$$\lambda_1 = 1 - \frac{2\alpha L \bar{\rho}\bar{\varphi}}{\min(\beta \underline{\rho}, \beta \underline{T_0})} > 0, \tag{3.19}$$

$$\lambda_2 = 1 + \frac{2\alpha L \bar{\rho}\bar{\varphi}}{\min(\beta \underline{\rho}, \beta \underline{T_0})} > 0, \tag{3.20}$$

given that

$$0 < \alpha < \frac{\min(\beta\underline{\rho}, \beta\underline{T_0})}{2L\bar{\rho}\bar{\varphi}}. \tag{3.21}$$

Proof: From Lemma 2.1, we have

$$\begin{aligned} |\eta(t)| &\leq \alpha L\bar{\rho}\bar{\varphi}\int_0^L ([\dot{w}(x,t)]^2 + [w'(x,t)]^2)dx \\ &\leq \alpha_1 V_1(t), \end{aligned} \tag{3.22}$$

where

$$\alpha_1 = \frac{2\alpha L\bar{\rho}\bar{\varphi}}{\min(\beta\underline{\rho}, \beta\underline{T_0})}. \tag{3.23}$$

From Ineq. (3.22), we have

$$-\alpha_1 V_1(t) \leq \eta(t) \leq \alpha_1 V_1(t). \tag{3.24}$$

By considering α satisfying

$$0 < \alpha < \frac{\min(\beta\underline{\rho}, \beta\underline{T_0})}{2L\bar{\rho}\bar{\varphi}}, \tag{3.25}$$

we obtain two positive constants α_2 and α_3 as

$$\alpha_2 = 1 - \alpha_1 = 1 - \frac{2\alpha L\bar{\rho}\bar{\varphi}}{\min(\beta\underline{\rho}, \beta\underline{T_0})} > 0, \tag{3.26}$$

$$\alpha_3 = 1 + \alpha_1 = 1 + \frac{2\alpha L\bar{\rho}\bar{\varphi}}{\min(\beta\underline{\rho}, \beta\underline{T_0})} > 1. \tag{3.27}$$

Furthermore, the following are derived

$$0 \leq \alpha_2 V_1(t) \leq V_1(t) + \eta(t) \leq \alpha_3 V_1(t). \tag{3.28}$$

From the above analysis, the Lyapunov function equation Eq. (3.14) is upper and lower bounded as

$$0 \leq \lambda_1(V_1(t) + V_2(t)) \leq V_1(t) + V_2(t) + \eta(t) \leq \lambda_2(V_1(t) + V_2(t)), \tag{3.29}$$

where two positive constants λ_1 and λ_2 are given as

$$\lambda_1 = \min(\alpha_2, 1) = \alpha_2 \tag{3.30}$$

$$\lambda_2 = \max(\alpha_3, 1) = \alpha_3 \tag{3.31}$$

■

Lemma 3.2. *The time derivation of the Lyapunov function equation* (3.14) *is upper bounded, and given by*

$$\dot{V}(t) \leq -\lambda V(t) + \varepsilon, \tag{3.32}$$

where $\lambda > 0$ and $\varepsilon > 0$.

Proof: Time derivations of the Lyapunov function candidate Eq. (3.14) result in

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{\eta}(t). \tag{3.33}$$

The first term of Eq. (3.33) is rewritten as

$$\dot{V}_1(t) = A_1(t) + A_2(t) + A_3(t), \tag{3.34}$$

where

$$A_1(t) = \beta \int_0^L \rho(x)\dot{w}(x,t)\ddot{w}(x,t)dx, \tag{3.35}$$

$$A_2(t) = \frac{\beta}{2} \int_0^L \dot{T}(x,t)[w'(x,t)]^2 dx, \tag{3.36}$$

$$A_3(t) = \beta \int_0^L T(x,t)w'(x,t)\dot{w}'(x,t)dx. \tag{3.37}$$

Substituting the governing equation (3.7) into $A_1(t)$, we have

$$\begin{aligned} A_1(t) &= \beta T(L,t)\dot{w}(L,t)w'(L,t) - \beta \int_0^L T(x,t)w'(x,t)\dot{w}'(x,t)dx \\ &\quad + \beta\lambda(L)\dot{w}(L,t)[w'(L,t)]^3 - \beta \int_0^L \lambda(x)[w'(x,t)]^3\dot{w}'(x,t)dx \\ &\quad + \beta \int_0^L \dot{w}(x,t)f(x,t)dx. \end{aligned} \tag{3.38}$$

The time derivative of the tension $T(x,t)$ leads to

$$\dot{T}(x,t) = 2\lambda(x)w'(x,t)\dot{w}'(x,t), \tag{3.39}$$

Substituting the above equation into $A_2(t)$, we have

$$A_2(t) = \beta \int_0^L \lambda(x)[w'(x,t)]^3\dot{w}'(x,t)dx. \tag{3.40}$$

Substitution of Eqs. (3.38), (3.40) and (3.37) into Eq. (3.33) yields

$$\begin{aligned}\dot{V}_1(t) &= \beta T(L,t)\dot{w}(L,t)w'(L,t) + \beta\lambda(L)\dot{w}(L,t)[w'(L,t)]^3 \\ &\quad +\beta\int_0^L \dot{w}(x,t)f(x,t)dx. \end{aligned} \tag{3.41}$$

Applying the tension expression Eq. (3.3) and using Ineq. (2.3), we obtain

$$\begin{aligned}\dot{V}_1(t) &\leq \frac{\beta T_0(L)}{2}[\dot{w}(L,t)+w'(L,t)]^2 - \frac{\beta T_0(L)}{2}[\dot{w}(L,t)]^2 - \frac{\beta T_0(L)}{2}[w'(L,t)]^2 \\ &\quad +2\beta\lambda(L)\dot{w}(L,t)[w'(L,t)]^3 + \beta\delta_1\int_0^L[\dot{w}(x,t)]^2dx \\ &\quad +\frac{\beta}{\delta_1}\int_0^L f^2(x,t)dx, \end{aligned} \tag{3.42}$$

where δ_1 is a positive constant.

Substituting boundary condition and the control law into $\dot{V}_2(t)$, we have

$$\begin{aligned}\dot{V}_2(t) &= \beta M_s[w'(L,t)+\dot{w}(L,t)][\dot{w}'(L,t)+\ddot{w}(L,t)] \\ &\leq -k\beta[w'(L,t)+\dot{w}(L,t)]^2 - 2\beta\lambda(L)\dot{w}(L,t)[w'(L,t)]^3 \\ &\quad -2\beta\lambda(L)[w'(L,t)]^4. \end{aligned} \tag{3.43}$$

Similarly, substituting Eq. (3.7) to the third term of Eq. (3.33), we obatin

$$\begin{aligned}
\dot{\eta}(t) &= \alpha \int_0^L \rho(x)\varphi(x)x[\ddot{w}(x,t)w'(x,t) + \dot{w}(x,t)\dot{w}'(x,t)]dx \\
&= \alpha \int_0^L x\varphi(x)\left\{\frac{1}{2}\frac{\partial T(x,t)[w'(x,t)]^2}{\partial x} + \frac{1}{2}T'(x,t)[w'(x,t)]^2\right\}dx \\
&\quad +\alpha \int_0^L x\varphi(x)\left\{\frac{3}{4}\frac{\partial \lambda(x)[w'(x,t)]^4}{\partial x} + \frac{1}{4}\lambda'(x)[w'(x,t)]^4\right\}dx \\
&\quad +\alpha \int_0^L x\varphi(x)w'(x,t)f(x,t)dx \\
&\quad +\frac{\alpha}{2}\int_0^L \rho(x)\varphi(x)x\frac{\partial[\dot{w}(x,t)]^2}{\partial x}dx.
\end{aligned} \tag{3.44}$$

Using integration by parts, we have

$$\begin{aligned}
\dot{\eta}(t) &\leq \frac{\alpha}{2}L\varphi(L)T_0(L)[w'(L,t)]^2 + \frac{3\alpha}{2}L\varphi(L)\lambda(L)[w'(L,t)]^4 \\
&\quad +\frac{\alpha}{2}L\varphi(L)\rho(L)[\dot{w}(L,t)]^2 \\
&\quad -\frac{\alpha}{2}\int_0^L \left[\frac{\partial\varphi(x)x}{\partial x}T_0(x) - x\varphi(x)T_0'(x)\right][w'(x,t)]^2dx \\
&\quad -\frac{\alpha}{2}\int_0^L \frac{\partial\rho(x)\varphi(x)x}{\partial x}[\dot{w}(x,t)]^2dx \\
&\quad -\frac{\alpha}{4}\int_0^L \left[5\frac{\partial\varphi(x)x}{\partial x}\lambda(x) + \frac{\partial\varphi(x)\lambda(x)x}{\partial x} - 3x\varphi(x)\lambda'(x)\right][w'(x,t)]^4dx \\
&\quad +\frac{\alpha L}{\delta_2}\int_0^L f^2(x,t)dx + \alpha\delta_2 L\int_0^L \varphi^2(x)[w'(x,t)]^2dx.
\end{aligned} \tag{3.45}$$

Substituting the results of $\dot{V}_1(t)$, $\dot{V}_2(t)$, and $\dot{\eta}(t)$ into Eq. (3.33), we obtain

$$\begin{aligned}
\dot{V}(t) \leq\; & -\beta\left(k-\frac{T_0}{2}\right)[w'(L,t)+\dot{w}(L,t)]^2 \\
& -\int_0^L\left(\frac{\alpha}{2}\frac{\partial\rho(x)\varphi(x)x}{\partial x}-\beta\delta_1\right)[\dot{w}(x,t)]^2dx \\
& -\frac{\alpha}{4}\int_0^L\left[5\frac{\partial\varphi(x)x}{\partial x}\lambda(x)+\frac{\partial\varphi(x)\lambda(x)x}{\partial x}-3x\varphi(x)\lambda'(x)\right][w'(x,t)]^4dx \\
& -\frac{\alpha}{2}\int_0^L\left[\frac{\partial\varphi(x)x}{\partial x}T_0(x)-x\varphi(x)T_0'(x)-2\delta_2L\varphi^2(x)\right][w'(x,t)]^2dx \\
& -\left[\frac{\beta}{2}T_0(L)-\frac{\alpha}{2}L\varphi(L)T_0(L)\right][w'(L,t)]^2 \\
& -\left[2\beta\lambda(L)-\frac{3\alpha}{2}L\varphi(L)\lambda(L)\right][w'(L,t)]^4 \\
& -\left[\frac{\beta}{2}T_0(L)-\frac{\alpha}{2}L\varphi(L)\rho(L)\right][\dot{w}(L,t)]^2 \\
& +\left(\frac{\beta}{\delta_1}+\frac{\alpha L}{\delta_2}\right)\int_0^L f^2(x,t)dx \\
\leq\; & -\lambda_3\left[V_1(t)+V_2(t)\right]+\varepsilon, \qquad (3.46)
\end{aligned}$$

where the parameters α, β, k, $\varphi(x)$, δ_1 and δ_2 are selected to meet the following requirements:

$$0 < \alpha < \frac{\min(\beta\underline{\rho}, \beta\underline{T_0})}{2L\bar{\rho}\bar{\varphi}}, \tag{3.47}$$

$$\frac{\beta}{2}T_0(L) - \frac{\alpha}{2}L\varphi(L)T_0(L) \geq 0, \tag{3.48}$$

$$2\beta\lambda(L) - \frac{3\alpha}{2}L\varphi(L)\lambda(L) \geq 0, \tag{3.49}$$

$$\frac{\beta}{2}T_0(L) - \frac{\alpha}{2}L\varphi(L)\rho(L) \geq 0, \tag{3.50}$$

$$\sigma_1 = \min\left(\frac{\alpha}{2}\frac{\partial}{\partial x}\left(x\varphi(x)\rho(x)\right) - \beta\delta_1\right) > 0, \tag{3.51}$$

$$\sigma_2 = \min\frac{\alpha}{4}\left[5\frac{\partial}{\partial x}\left(x\varphi(x)\right)\lambda(x) + \frac{\partial}{\partial x}\left(x\varphi(x)\lambda(x)\right) - 3x\varphi(x)\lambda'(x)\right] > 0, \tag{3.52}$$

$$\sigma_3 = \min\frac{\alpha}{2}\left[\frac{\partial}{\partial x}\left(x\varphi(x)\right)T_0(x) - x\varphi(x)T_0'(x) - 2\delta_2 L\varphi^2(x)\right] > 0, \tag{3.53}$$

$$\sigma_4 = \beta\left(k - \frac{T_0(L)}{2}\right) > 0, \tag{3.54}$$

$$\lambda_3 = \min\left(\frac{2\sigma_1}{\beta\underline{\rho}}, \frac{2\sigma_2}{\beta\underline{\lambda}}, \frac{2\sigma_3}{\beta\underline{T_0}}, \frac{2\sigma_4}{\beta M_s}\right) > 0, \tag{3.55}$$

$$\varepsilon = \left(\frac{\beta}{\delta_1} + \frac{\alpha L}{\delta_2}\right)\int_0^L \bar{f}^2 dx \in \mathcal{L}_\infty. \tag{3.56}$$

Combining Ineqs. (3.29) and (3.46), we have

$$\dot{V}(t) \leq -\lambda V(t) + \varepsilon, \tag{3.57}$$

where $\lambda = \lambda_3/\lambda_2 > 0$ and $\varepsilon > 0$. ■

Remark 3.3. *A set of values for constants k, α, β, δ_1 and δ_2 can be found to satisfy the Ineqs.* (3.47) - (3.56).

Theorem 3.1. *Consider the closed-loop nonuniform string system consisting of the* *system dynamics given by (3.7) - (3.9) and boundary control Eq. (3.13), under the Assumptions 3.1 and 3.2, with the bounded initial conditions, then,*

(i) the state $w(x,t)$ of the closed-loop nonuniform string system will stay in Ω_1 given by

$$\Omega_1 := \left\{ w(x,t) \in R \middle| \quad |w(x,t)| \leq D_1, \ \forall (x,t) \in [0,L] \times [0,\infty) \right\}, \tag{3.58}$$

where constant $D_1 = \sqrt{\frac{2L}{\underline{T_0}\lambda_1}\left(V(0) + \frac{\varepsilon}{\lambda}\right)}$.

(ii) the state $w(x,t)$ of the closed-loop nonuniform string system will eventually converge to Ω_2 given by

$$\Omega_2 := \left\{ w(x,t) \in R \middle| \quad \lim_{t\to\infty} |w(x,t)| \leq D_2, \ \forall x \in [0,L) \right\}, \tag{3.59}$$

where constant $D_2 = \sqrt{\frac{2L\varepsilon}{\underline{T_0}\lambda_1\lambda}}$.

Proof: Multiplying Eq. (3.32) by $e^{\lambda t}$ leads to

$$\frac{\partial}{\partial t}(V(t)e^{\lambda t}) \ \leq \ \varepsilon e^{\lambda t}. \tag{3.60}$$

Integration of Ineq. (3.60) yields

$$V(t) \leq \left(V(0) - \frac{\varepsilon}{\lambda}\right) e^{-\lambda t} + \frac{\varepsilon}{\lambda} \leq V(0)e^{-\lambda t} + \frac{\varepsilon}{\lambda} \in \mathcal{L}_\infty. \tag{3.61}$$

Combining Ineq. (2.7) and Eq. (3.15), we obtain

$$\frac{\beta}{2L}\underline{T_0} w^2(x,t) \leq \frac{\beta}{2}\int_0^L T_0(x)[w'(x,t)]^2 dx \leq V_1(t) \leq V_1(t) + V_2(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty. \tag{3.62}$$

From the above inequality, the state $w(x,t)$ can be obtained to be bounded as

$$|w(x,t)| \leq \sqrt{\frac{2L}{\underline{T_0}\lambda_1}\Big(V(0)e^{-\lambda t} + \frac{\varepsilon}{\lambda}\Big)} \leq \sqrt{\frac{2L}{\underline{T_0}\lambda_1}\Big(V(0) + \frac{\varepsilon}{\lambda}\Big)}, \tag{3.63}$$

$\forall (x,t) \in [0,L] \times [0,\infty)$. Furthermore, we have

$$\lim_{t\to\infty} |w(x,t)| \leq \sqrt{\frac{2L\varepsilon}{\underline{T_0}\lambda_1\lambda}}, \ \forall x \in [0,L]. \tag{3.64}$$

■

Remark 3.4. *From Eqs.* (3.54) *and* (3.55)*, it can be seen that increase in the control gain k will lead to a larger λ, which will decrease the values of D_1 and D_2. Therefore, $w(x,t)$ could be set in an small boundedness region by properly choosing the control parameter k and a better vibration control performance can be achieved. However, in practice, the control gains need be chosen properly since increasing k will result in a high gain control scheme.*

Remark 3.5. *From Ineq.* (3.62)*, it can be seen that $V_1(t)$ and $V_2(t)$ are bounded $\forall t \in [0,\infty)$, and then $w'(x,t)$ and $\dot{w}(x,t)$ are bounded $\forall (x,t) \in [0,L] \times [0,\infty)$ and $[\dot{w}(L,t) + w'(L,t)]$ is bounded $\forall t \in [0,\infty)$. Then, we can state that the kinetic energy Eq.* (3.1) *and the potential energy Eq.* (3.2) *are also bounded. From the Properties 3.1, 3.2, we can obtain $w''(x,t)$ and $\dot{w}'(x,t)$ are bounded $\forall (x,t) \in [0,L] \times [0,\infty)$. By using Assumption 3.1 and Eq.* (3.7)*, it is can be concluded that $\ddot{w}(x,t)$ is also bounded $\forall (x,t) \in [0,L] \times [0,\infty)$. Therefore, the boundary control law Eq.* (3.13) *is bounded $\forall t \in [0,\infty)$ due to the boundness of $\dot{w}(x,t)$, $w'(x,t)$, $\dot{w}'(x,t)$, and it guarantees that all the internal system states including $w(x,t)$, $\dot{w}(x,t)$, $\ddot{w}(x,t)$, $\dot{w}'(x,t)$ and $w'(x,t)$ are uniformly bounded.*

3.2.2 Adaptive boundary control

When the system parameters M_s and $T_0(L)$ cannot be obtained directly, adaptive laws are designed to estimate the unknown parameters and update to boundary control law. In this section, the adaptive boundary control is designed using the estimated parameters $\hat{M}_s(t)$ and $\hat{T}_0(L,t)$ as

$$\begin{aligned} u(t) &= -k[w'(L,t)+\dot{w}(L,t)] - \hat{M}_s(t)\dot{w}'(L,t) + \hat{T}_0(L,t)w'(L,t) \\ &\quad -\mathrm{sgn}\left[w'(L,t)+\dot{w}(L,t)\right]\bar{d}, \end{aligned} \tag{3.65}$$

where $\hat{M}_s(t)$, $\hat{T}_0(L,t)$ are the estimate of M_s and $T_0(L)$ respectively, $k>0$ is the control gain. Parameter estimate errors $\tilde{M}_s(t)$ and $\tilde{T}_0(L,t)$ are defined as

$$\tilde{M}_s(t) = M_s - \hat{M}_s(t), \tag{3.66}$$

$$\tilde{T}_0(L,t) = T_0(L) - \hat{T}_0(L,t). \tag{3.67}$$

The adaptation laws are designed as

$$\dot{\hat{M}}_s(t) = \beta\gamma_m\dot{w}'(L,t)[w'(L,t)+\dot{w}(L,t)] - \zeta_m\gamma_m\hat{M}_s(t), \tag{3.68}$$

$$\dot{\hat{T}}_0(L,t) = -\beta\gamma_t w'(L,t)[w'(L,t)+\dot{w}(L,t)] - \zeta_t\gamma_t\hat{T}_0(L,t), \tag{3.69}$$

where γ_m, ζ_m, γ_t and ζ_t are positive constants. Since M_s and $T_0(L)$ are positive constants, from Eqs. (3.66) and (3.67), we have

$$\dot{\tilde{M}}_s(t) = -\beta\gamma_m\dot{w}'(L,t)[w'(L,t)+\dot{w}(L,t)] + \zeta_m\gamma_m\hat{M}_s(t), \tag{3.70}$$

$$\dot{\tilde{T}}_0(L,t) = \beta\gamma_t w'(L,t)[w'(L,t)+\dot{w}(L,t)] + \zeta_t\gamma_t\hat{T}_0(L,t). \tag{3.71}$$

A new Lyapunov function candidate is considered for the nonuniform string system (3.7) - (3.9) as

$$V_a(t) \;=\; V(t) + \frac{1}{2}\gamma_m^{-1}\tilde{M}_s^2(t) + \frac{1}{2}\gamma_t^{-1}\tilde{T}_0^2(L,t), \tag{3.72}$$

where $V(t)$ is defined as Eq. (3.14).

Combining Ineqs. (3.18) and (3.72), we have

$$\begin{aligned} 0 \;&\le\; \lambda_{1a}(V_1(t) + V_2(t) + \tilde{M}_s^2(t) + \tilde{T}_0^2(L,t)) \le V_a(t) \\ &\le\; \lambda_{2a}(V_1(t) + V_2(t) + \tilde{M}_s^2(t) + \tilde{T}_0^2(L,t)), \end{aligned} \tag{3.73}$$

where two positive constants λ_{1a} and λ_{2a} are defined as

$$\lambda_{1a} \;=\; \min(1 - \frac{2\alpha L\bar{\rho}\bar{\varphi}}{\min(\beta\underline{\rho}, \beta\underline{T_0})}, \frac{1}{2\gamma_m}, \frac{1}{2\gamma_t}), \tag{3.74}$$

$$\lambda_{2a} \;=\; \max(1 + \frac{2\alpha L\bar{\rho}\bar{\varphi}}{\min(\beta\underline{\rho}, \beta\underline{T_0})}, \frac{1}{2\gamma_m}, \frac{1}{2\gamma_t}), \tag{3.75}$$

given that

$$0 < \alpha < \frac{\min(\beta\underline{\rho}, \beta\underline{T_0})}{2L\bar{\rho}\bar{\varphi}}. \tag{3.76}$$

Lemma 3.3. *The time derivation of the Lyapunov equation* (3.72) *is upper bounded, and given by*

$$\dot{V}_a(t) \;\le\; -\lambda_a V_a(t) + \varepsilon_a, \tag{3.77}$$

where λ_a and ε_a are two positive constants.

Proof: The differentiation of Eq. (3.72) yileds

$$\dot{V}_a(t) \;=\; \dot{V}(t) + \gamma_m^{-1}\tilde{M}_s(t)\dot{\tilde{M}}_s(t) + \gamma_t^{-1}\tilde{T}_0(L,t)\dot{\tilde{T}}_0(L,t). \tag{3.78}$$

Substituting Eqs. (3.65) and (3.3) into $\dot{V}_2(t)$, we obtain

$$\begin{aligned}\dot{V}_2(t) &= \beta M_s[w'(L,t) + \dot{w}(L,t)][\dot{w}'(L,t) + \ddot{w}(L,t)] \\ &\leq -k\beta[w'(L,t) + \dot{w}(L,t)]^2 - 2\beta\lambda(L)\dot{w}(L,t)[w'(L,t)]^3 \\ &\quad -2\beta\lambda(L)[w'(L,t)]^4 + \beta\tilde{M}_s(t)\dot{w}'(L,t)[w'(L,t) + \dot{w}(L,t)] \\ &\quad -\beta\tilde{T}_0(L,t)w'(L,t)[w'(L,t) + \dot{w}(L,t)].\end{aligned} \tag{3.79}$$

Applying the results of Lemma 3.2 and substituting Eqs. (3.42), (3.79) and (3.45) into Eq. (3.33), we obtain

$$\begin{aligned}\dot{V}(t) &\leq -\lambda_3(V_1(t) + V_2(t)) + \beta\tilde{M}_s(t)\dot{w}'(L,t)[w'(L,t) + \dot{w}(L,t)] \\ &\quad -\beta\tilde{T}_0(L,t)w'(L,t)[w'(L,t) + \dot{w}(L,t)] + \varepsilon,\end{aligned} \tag{3.80}$$

where λ_3 and ε are defined as in Eqs. (3.55), (3.56). Substitution of Ineq. (3.80) into Eq. (3.78) yields

$$\begin{aligned}\dot{V}_a(t) &\leq -\lambda_3(V_1(t) + V_2(t)) + \beta\tilde{M}_s(t)\dot{w}'(L,t)[w'(L,t) + \dot{w}(L,t)] + \gamma_m^{-1}\tilde{M}_s(t)\dot{\tilde{M}}_s(t) \\ &\quad -\beta\tilde{T}_0(L,t)w'(L,t)[w'(L,t) + \dot{w}(L,t)] + \gamma_t^{-1}\tilde{T}_0(L,t)\dot{\tilde{T}}_0(L,t) + \varepsilon.\end{aligned} \tag{3.81}$$

Substituting Eqs. (3.70) and (3.71) into Eq. (3.81), we have

$$\begin{aligned}\dot{V}_a(t) &\leq -\lambda_3(V_1(t)+V_2(t)) + \zeta_m\tilde{M}_s(t)\hat{M}_s(t) + \zeta_t\tilde{T}_0(L,t)\hat{T}_0(L,t) + \varepsilon \\ &\leq -\lambda_{3a}(V_1(t)+V_2(t)+\tilde{M}_s^2(t)+\tilde{T}_0^2(L,t)) + \frac{\zeta_m}{2}{M_s}^2 + \frac{\zeta_t}{2}{T_0}^2(L) + \varepsilon, \end{aligned} \tag{3.82}$$

where a positive constant $\lambda_{3a} = \min(\lambda_3, \frac{\zeta_m}{2}, \frac{\zeta_t}{2})$. From Ineqs. (3.73) and (3.82), we have

$$\dot{V}_a(t) \leq -\lambda_a V_a(t) + \varepsilon_a, \tag{3.83}$$

where $\lambda_a = \lambda_{3a}/\lambda_{2a}$ and $\varepsilon_a = \frac{\zeta_m}{2}{M_s}^2 + \frac{\zeta_t}{2}{T_0}^2(L) + \varepsilon > 0$.

Theorem 3.2. *Consider the closed-loop nonuniform sting system consisting of the system dynamics given by* (3.7) - (3.9)*, boundary control Eq.* (3.13)*, and the adaption laws* (3.68) - (3.69)*, under Assumptions 3.1 and 3.2, with the bounded initial conditions, then,*

(i) the state $w(x,t)$ of the closed-loop nonuniform sting system will stay in Ω_3 given by

$$\Omega_3 := \left\{ w(x,t) \in R \middle| \quad |w(x,t)| \leq D_3,\ \forall (x,t) \in [0,L]\times[0,\infty) \right\}, \tag{3.84}$$

where constant $D_3 = \sqrt{\frac{2L}{\underline{T_0}\lambda_{1a}}\left(V_a(0) + \frac{\varepsilon_a}{\lambda_a}\right)}$.

(ii) the state $w(x,t)$ of the closed-loop nonuniform sting system will eventually converge to Ω_4 given by

$$\Omega_4 := \left\{ w(x,t) \in R \middle| \quad \lim_{t\to\infty} |w(x,t)| \leq D_4,\ \forall x \in [0,L) \right\}, \tag{3.85}$$

where constant $D_4 = \sqrt{\frac{2L\varepsilon_a}{\underline{T_0}\lambda_{1a}\lambda_a}}$.

Proof: Multiplying Eq. (3.77) by $e^{\lambda_a t}$ leads to

$$\frac{\partial}{\partial t}(V_a(t)e^{\lambda_a t}) \quad \leq \quad \varepsilon_a e^{\lambda_a t}. \tag{3.86}$$

Integration of Ineq. (3.86) yields

$$V_a(t) \leq \left(V_a(0) - \frac{\varepsilon_a}{\lambda_a}\right) e^{-\lambda_a t} + \frac{\varepsilon_a}{\lambda_a} \leq V_a(0)e^{-\lambda_a t} + \frac{\varepsilon_a}{\lambda_a} \in \mathcal{L}_\infty. \tag{3.87}$$

Combining Ineq. (2.7) and Eq. (3.15), we obtain

$$\begin{aligned} \frac{\beta}{2L}\underline{T_0}w^2(x,t) \quad &\leq \quad \frac{\beta}{2}\int_0^L T_0(x)[w'(x,t)]^2 dx \leq V_1(t) \\ &\leq \quad V_1(t) + V_2(t) \leq \frac{1}{\lambda_{1a}}V_a(t) \in \mathcal{L}_\infty. \end{aligned} \tag{3.88}$$

From the above inequality, the state $w(x,t)$ is obtained to be bounded as

$$|w(x,t)| \leq \sqrt{\frac{2L}{\underline{T_0}\lambda_{1a}}\left(V_a(0)e^{-\lambda_a t} + \frac{\varepsilon_a}{\lambda_a}\right)} \leq \sqrt{\frac{2L}{\underline{T_0}\lambda_{1a}}\left(V_a(0) + \frac{\varepsilon_a}{\lambda_a}\right)}, \tag{3.89}$$

$\forall (x,t) \in [0,L] \times [0,\infty)$. Furthermore, we have

$$\lim_{t\to\infty} |w(x,t)| \leq \sqrt{\frac{2L\varepsilon_a}{\underline{T_0}\lambda_{1a}\lambda_a}}, \ \forall x \in [0,L]. \tag{3.90}$$

■

Remark 3.6. *Similar to Remark 6.4, it can be concluded that* $w(x,t)$ *can be made in an arbitrarily small boundedness region by properly selecting the design parameters of the proposed adaptive boundary control* (3.65)*.*

Remark 3.7. *In implementing the proposed boundary control* (3.13) *and* (3.65), *and the adaption laws* (3.68) *and* (3.69), *measurement of the velocity* $\dot{w}(L,t)$, *slope* $w'(L,t)$, *slope rate* $\dot{w}'(L,t)$ *of the tip payload are needed. By using a laser displacement sensor and an inclinometer located at the tip payload,* $w(L,t)$ *and* $w'(L,t)$ *can be measured. The backward difference algorithm provides the velocity* $\dot{w}(L,t)$ *and the slope rate* $\dot{w}'(L,t)$, *respectively.*

Remark 3.8. *From Eqs.* (3.73) *and* (3.87), *it can be seen that the parameters estimate errors* $\tilde{M}_s(t)$ *and* $\tilde{T}_0(L,t)$ *are bounded* $\forall t \in [0,\infty)$. *Then it can be concluded that* $\hat{M}_s(t)$, $\hat{T}_0(L,t)$ *are also bounded* $\forall t \in [0,\infty)$. *Similar to Remark 3.5, we can state the proposed control Eq.* (3.65) *is bounded* $\forall t \in [0,\infty)$, *and it guarantees that all the internal system states including* $w(x,t)$, $\dot{w}(x,t)$, $\ddot{w}(x,t)$, $\dot{w}'(x,t)$ *and* $w'(x,t)$ *are uniformly bounded.*

3.3 Numerical Simulations

In this section, the effectiveness of the proposed boundary control is illustrated by numerical simulations using the finite difference method. The system parameters and the external disturbances used in the simulation are referred to [132]. We consider a nonuniform string excited by distributed disturbance $f(x,t)$ and boundary disturbance $d(t)$. The boundary disturbance $d(t)$ on the tip payload is described as

$$d(t) = 1 + 0.2\sin(0.2t) + 0.3\sin(0.3t) + 0.5\sin(0.5t). \tag{3.91}$$

The distributed disturbance $f(x,t)$ on the string is given as

$$f(x,t) = [3 + \sin(\pi x t) + \sin(2\pi x t) + \sin(3\pi x t)]x. \tag{3.92}$$

The initial conditions are

$$w(x,0) = x, \tag{3.93}$$

$$\dot{w}(x,0) = 0. \tag{3.94}$$

Parameters of the nonuniform string system are listed as below table:

Table 1: Parameters of the nonuniform string

Parameter	Description	Value
L	Length of string	$1m$
$\rho(x)$	Mass per unit length	$0.1(x+1)kg/m$
M_s	Mass of the tip payload	$1kg$
$T_0(x)$	Initial tension	$10(x+1)N$
$\lambda(x)$	Elastic modulus	$0.1x$

Fig. 3.2 depicts displacement of the nonuniform string under the external disturbances without control input, i.e. the control input is equal to zero. Displacement of the nonuniform string with the proposed model-based boundary control (3.13), by choosing $k = 5$, is shown in Fig. 3.3. When there are system uncertainties, displacement of the nonuniform string with the proposed adaptive boundary control (3.65) is shown in Fig. 3.4. The adaptive control parameters are chosen as $k = 50$, $\beta = 1$, $\zeta_m = \zeta_t = \zeta_d = 1$ and $\gamma_m = \gamma_t = \gamma_d = 1$. The model-based boundary control input (3.13) and the adaptive boundary control input (3.65) are displayed in Fig. 3.5. As shown in Figs. 3.3 and 3.4, it can be seen that the nonuniform string system can be stabilized with the proposed model-based boundary control (3.13) and adaptive boundary control (3.65).

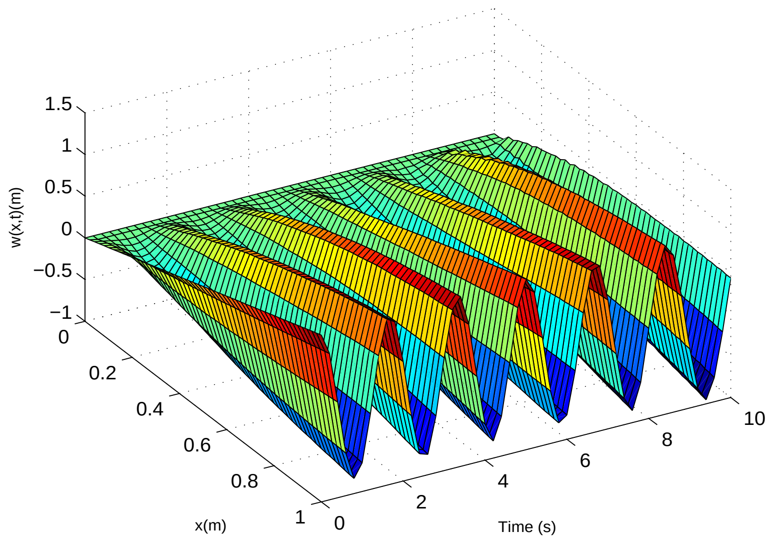

Fig. 3.2: Displacement of the nonuniform string without control.

3.4 Conclusion

In this chapter, the modeling and control problem of a nonuniform string system under unknown spatiotemporally varying tension and disturbance has been investigated. In order to suppress the vibration and avoid spillover problem, model-based boundary control and adaptive boundary control have been discussed. The effectiveness of the proposed control has been verified by numerical simulations.

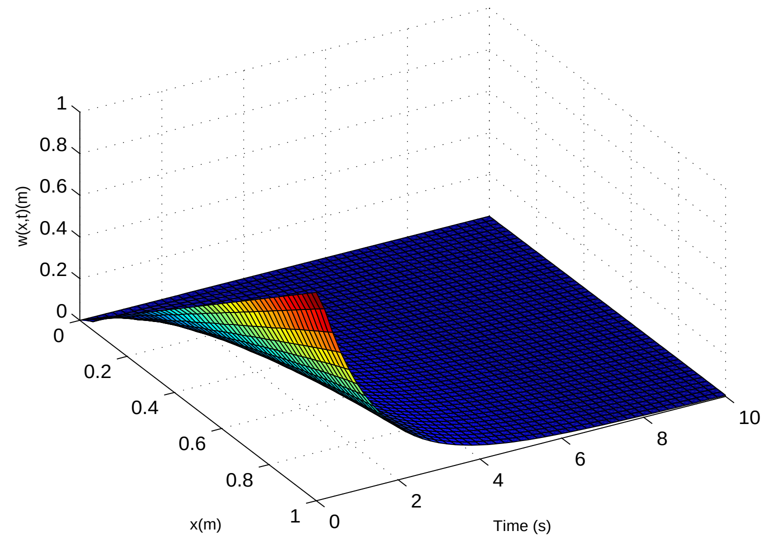

Fig. 3.3: Displacement of the nonuniform string with model-based boundary control.

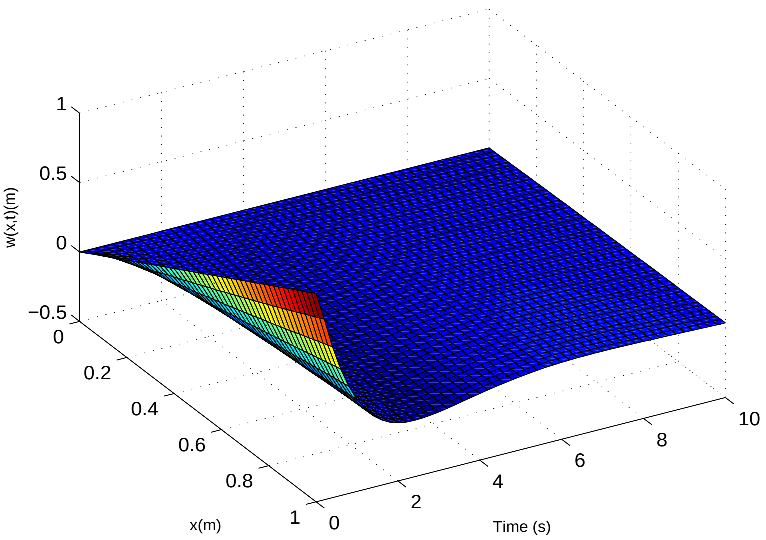

Fig. 3.4: Displacement of the nonuniform string with adaptive boundary control.

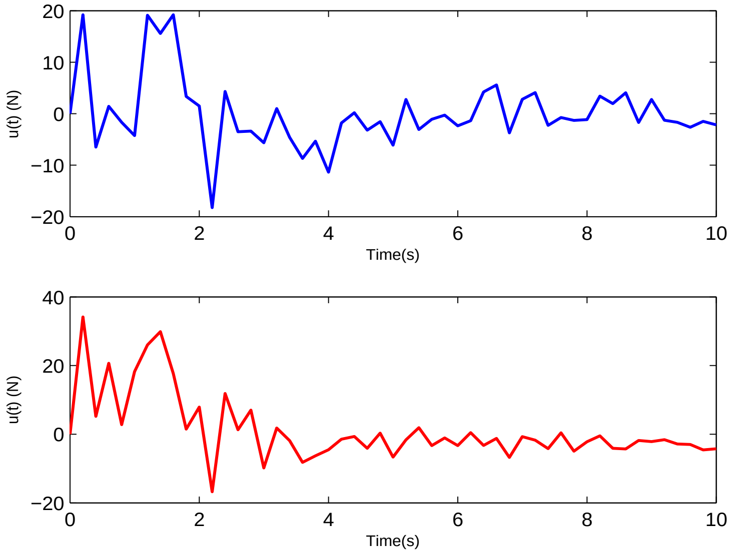

Fig. 3.5: Model-based boundary control input and adaptive boundary control input.

Chapter 4

Vibration Control of a Coupled Nonlinear String System in Transverse and Longitudinal Directions

In some of engineering fields, the string is not only a prototype problem but also constitutes an important application topic. In this chapter, the control problem of a coupled nonlinear string system is considered, i.e., not only the transverse displacement of the string system is regarded, the axial deformation is also under consideration, which leads to a more precise model for the string system. Due to the coupling between longitudinal and transverse dynamics, the control design for the linear model of the string system [48, 80, 132] cannot be straightforwardly used. To the best of our knowledge, the result is the first complete solution of adaptive boundary control to a nonlinear flexible string system for transverse and longitudinal vibrations reduction.

In addition, the control schemes have been designed with considering the unknown spatiotemporally varying distributed disturbance which is the function of both time and space. Compared to the existing work, the main contributions of this chapter include:

(i) The coupled nonlinear dynamic and the uncertainties are admitted in the model. A hybrid PDE-ODE model of the nonlinear string system with consideration of the coupling between transverse and longitudinal displacements is derived based on the Hamilton's principle.

(ii) Based on the original infinite dimensional model, two boundary control laws combining with an adaptation law are designed to regulate the vibration of the coupled nonlinear string and handle the system uncertainties.

(iii) Utilizing the Lyapunov's direct method, uniform ultimate boundedness of the system is proved with the proposed boundary control.

This chapter is structured as follows. In Section 4.1, we derive the dynamic model of the coupled nonlinear string system by using the Hamilton principle. In Section 4.2, boundary control combined with adaption law is designed to stabilize the string system and compensate for the parametric uncertainty. Extensive simulations are provided to illustrate the performance of the control system in Section 4.3. Section 4.4 concludes this chapter.

4.1 Dynamics of the Coupled Nonlinear String System

Fig. 4.1 shows a string-based structure under the distributed disturbance $f(x,t)$ and the boundary disturbance $d(t)$. $w(x,t)$ and $v(x,t)$ are the displacements of the string in transverse direction and the longitudinal direction respectively at the position x for time t. The left boundary of the string is fixed at origin.

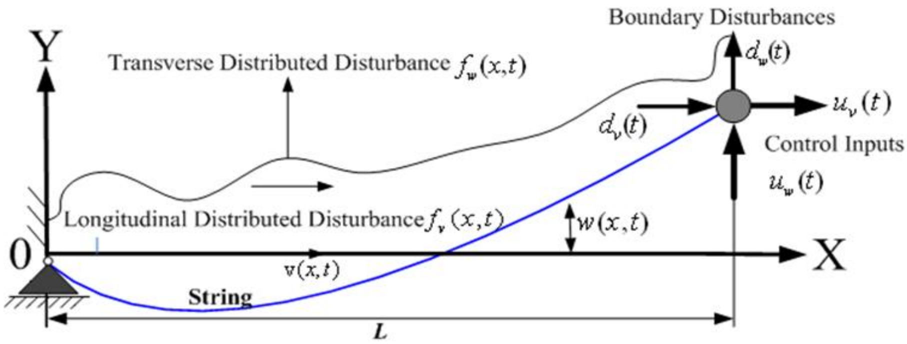

Fig. 4.1: A typical nonlinear string system.

The kinetic energy of the string system $E_k(t)$ is given as

$$E_k(t) = \frac{1}{2}M\left\{[\dot{w}(L,t)]^2 + [\dot{v}(L,t)]^2\right\} + \frac{1}{2}\rho\int_0^L \left\{[\dot{w}(x,t)]^2 + [\dot{v}(x,t)]^2\right\} dx, \quad (4.1)$$

where L is the length of the string, M is the mass of the payload, ρ is the uniform mass per unit length of the string, t and x represent the time and spatial variables, respectively.

The potential energy $E_p(t)$ due to the tension T and the axial stiffness EA can be obtained from

$$E_p(t) = \frac{1}{2}T\int_0^L [w'(x,t)]^2\,dx + \frac{1}{2}EA\int_0^L \left\{v'(x,t) + \frac{1}{2}[w'(x,t)]^2\right\}^2 dx. \quad (4.2)$$

The system suffers to the unknown spatiotemporally varying distributed disturbance $f(x,t) = [f_w(x,t),\ f_v(x,t)]$ on the string and time-varying boundary disturbances $d(t) = [d_w(t),\ d_v(t)]$ on the tip payload. The virtual work done by the external disturbances is given by

$$\delta W_f(t) = \int_0^L [f_w(x,t)\delta w(x,t) + f_v(x,t)\delta v(x,t)]\,dx + d_w(t)\delta w(L,t) + d_v(t)\delta v(L,t). \quad (4.3)$$

The virtual work done by the boundary control force $u(t) = [u_w(t),\ u_v(t)]$, which is introduced to suppress the system vibration of the string, is expressed as

$$\delta W_m(t) = u_w(t)\delta w(L,t) + u_v(t)\delta v(L,t). \quad (4.4)$$

Thus, the total virtual work done to the string system is given by

$$\begin{aligned} \delta W(t) &= \delta W_f(t) + \delta W_m(t) \\ &= \int_0^L [f_w(x,t)\delta w(x,t) + f_v(x,t)\delta v(x,t)]\,dx + [u_w(t) + d_w(t)]\,\delta w(L,t) \\ &\quad + [u_v(t) + d_v(t)]\,\delta v(L,t). \end{aligned} \quad (4.5)$$

Using the Hamilton's principle Eq. (2.1), we obtain the governing equations of the system as

$$
\begin{aligned}
&\rho\ddot{w}(x,t) - Tw''(x,t) - EAw''(x,t)v'(x,t) - EAv''(x,t)w'(x,t) \\
&= f_w(x,t) + \frac{3EA}{2}[w'(x,t)]^2 w''(x,t), \qquad (4.6)\\
&\rho\ddot{v}(x,t) - EAv''(x,t) - EAw'(x,t)w''(x,t) = f_v(x,t), \qquad (4.7)
\end{aligned}
$$

$\forall(x,t) \in (0,L) \times [0,\infty)$, and the boundary conditions of the system as

$$
\begin{aligned}
w(0,t) = v(0,t) &= 0, \qquad (4.8)\\
M\ddot{w}(L,t) + Tw'(L,t) + EAw'(x,t)v'(L,t) + \frac{EA}{2}[w'(L,t)]^3 &= u_w(t) + d_w(t), \qquad (4.9)\\
M\ddot{v}(L,t) + EAv'(L,t) + \frac{EA}{2}[w'(L,t)]^2 &= u_v(t) + d_v(t), \qquad (4.10)
\end{aligned}
$$

$\forall t \in [0,\infty)$.

Remark 4.1. *With consideration of the displacement in both transverse and longitudinal directions, the string equations* (4.6) *and* (4.7) *are coupled and nonlinear. Many the control methods for the linear PDE system can not be used.*

Remark 4.2. *Due to the consideration of unknown distributed disturbances $f_w(x,t)$, $f_v(x,t)$, two nonlinear nonhomogeneous PDEs* (4.6) *and* (4.7) *are used to describe the governing equations of the string system. The nonlinear nonhomogeneous model is different from the string system governed by the homogeneous PDEs or the linear PDEs in [26, 49, 56, 80, 93–95, 97, 98].*

Property 4.1. *[131]: If the kinetic energy of the system* (4.6) - (4.10)*, given by Eq.* (4.1) *is bounded $\forall t \in [0,\infty)$, then $\dot{w}(x,t)$, $\dot{w}'(x,t)$, $\dot{w}''(x,t)$, $\dot{v}(x,t)$, $\dot{v}'(x,t)$ and*

$\dot{v}''(x,t)$ *are bounded* $\forall(x,t) \in [0,L] \times [0,\infty)$.

Property 4.2. *[131]: If the potential energy of the system* (4.6) - (4.10)*, given by Eq.* (4.2) *is bounded* $\forall t \in [0,\infty)$*, then* $w'(x,t)$*,* $w''(x,t)$*,* $v'(x,t)$ *and* $v''(x,t)$ *are bounded* $\forall(x,t) \in [0,L] \times [0,\infty)$.

Assumption 4.1. *Assuming that the unknown spatiotemporally varying distributed disturbances* $f_w(x,t)$*,* $f_v(x,t)$ *and unknown time-varying boundary disturbances* $d_w(t)$*,* $d_v(t)$ *are uniformly bounded, i.e., there exists constants* $\bar{f}_w, \bar{f}_v, \bar{d}_w, \bar{d}_v \in \mathbb{R}^+$*, such that* $|f_w(x,t)| \leq \bar{f}_w$*,* $|f_v(x,t)| \leq \bar{f}_v$*,* $\forall(x,t) \in [0,L] \times [0,\infty)$ *and* $|d_w(t)| \leq \bar{d}_w$*,* $|d_v(t)| \leq \bar{d}_v$*,* $\forall t \in [0,\infty)$.

4.2 Adaptive Boundary Control Design

In this section, boundary control $u_w(t)$ and $u_v(t)$ are designed at the right boundary of the string to stabilize the coupled nonlinear string system in the presence of the unknown spatiotemporally varying disturbances. The Lyapunov's direct method is used to analyze the stability of the closed-loop string system. Since the system parameters are unknown, adaptive boundary control is developed to cope with the system uncertainties as follows

$$u_w(t) = -\hat{M}(t)\dot{w}'(L,t) - 2k_1\dot{w}(L,t) - \text{sgn}[w'(L,t) + \dot{w}(L,t)]\bar{d}_w, \qquad (4.11)$$

$$u_v(t) = -\hat{M}(t)\dot{v}'(L,t) - 2k_2\dot{v}(L,t) - \text{sgn}[w'(L,t) + \dot{w}(L,t)]\bar{d}_v, \qquad (4.12)$$

where k_1 and k_2 are the control gains, $\hat{M}(t)$ is the estimate of M, $\mathrm{sgn}(\cdot)$ denotes the signum function. We define

$$\tilde{M}(t) = M - \hat{M}(t), \tag{4.13}$$

where $\tilde{M}(t)$ is the estimate error. The adaptive law is designed as

$$\dot{\hat{M}}(t) = \alpha\gamma[w'(L,t) + \dot{w}(L,t)] + \alpha\gamma[v'(L,t) + \dot{v}(L,t)] - \sigma\gamma\hat{M}(t), \tag{4.14}$$

where γ and σ are two small positive constants.

Remark 4.3. *The σ modification term is introduced to improve the robustness of the closed-loop system [133]. Without such a modification term, the estimate $\hat{M}(t)$ might drift to very large values, which will result in a variation of a high-gain control scheme [20, 134].*

The following Lyapunov function candidate is considered for the coupled nonlinear string system (4.6) - (4.10) as

$$V(t) \quad = \quad V_1(t) + V_2(t) + V_3(t) + \frac{1}{2}\gamma^{-1}\tilde{M}^2(t), \tag{4.15}$$

where the first term $V_1(t)$ and the second term $V_2(t)$ and the third term $V_3(t)$ are given as

$$
\begin{aligned}
V_1(t) &= \frac{\alpha}{2}\rho\int_0^L \left\{[\dot{w}(x,t)]^2+[\dot{v}(x,t)]^2\right\}dx+\frac{\alpha}{2}T\int_0^L [w'(x,t)]^2\,dx \\
&\quad +\frac{\alpha}{2}EA\int_0^L \left\{v'(x,t)+\frac{1}{2}[w'(x,t)]^2\right\}^2 dx, && (4.16)\\
V_2(t) &= \frac{\alpha}{2}M[w'(L,t)+\dot{w}(L,t)]^2+\frac{\alpha}{2}M[v'(L,t)+\dot{v}(L,t)]^2, && (4.17)\\
V_3(t) &= \beta\rho\int_0^L x\left[\dot{w}(x,t)w'(x,t)+\dot{v}(x,t)v'(x,t)\right]dx, && (4.18)
\end{aligned}
$$

where α and β are two positive constants.

Lemma 4.1. *The Lyapunov function equation Eq.* (4.15) *is bounded, given by*

$$0\le\lambda_1(V_1(t)+V_2(t)+\tilde{M}^2(t))\le V(t)\le\lambda_2(V_1(t)+V_2(t)+\tilde{M}^2(t)), \tag{4.19}$$

where λ_1 and λ_2 are two positive constants.

Proof: From Lemma 2.1, we have

$$|V_3(t)| \le \beta_1 V_1(t), \tag{4.20}$$

where

$$\beta_1=\frac{2\beta\rho L}{\alpha\min(\rho,T,EA)}. \tag{4.21}$$

From Ineq. (4.20), we have

$$-\beta_1V_1(t)\le V_3(t)\le\beta_1V_1(t). \tag{4.22}$$

By considering β satisfying $0 < \beta < \frac{\alpha \min(\rho,T,EA)}{2\rho L}$, we obtain $0 < \beta_1 < 1$, and

$$\beta_2 = 1 - \beta_1 = 1 - \frac{2\beta\rho L}{\alpha \min(\rho, T, EA)} > 0, \tag{4.23}$$

$$\beta_3 = 1 + \beta_1 = 1 + \frac{2\beta\rho L}{\alpha \min(\rho, T, EA)} > 1. \tag{4.24}$$

Then, the following are derived

$$0 \leq \beta_2 V_1(t) \leq V_1(t) + V_3(t) \leq \beta_3 V_1(t). \tag{4.25}$$

From the above analysis, the Lyapunov function candidate Eq. (4.15) is upper and lower bounded as

$$\begin{aligned} 0 &\leq \lambda_1(V_1(t) + V_2(t) + \tilde{M}^2(t)) \leq V_1(t) + V_2(t) + V_3(t) + \frac{1}{2\gamma}\tilde{M}^2(t) \\ &\leq \lambda_2(V_1(t) + V_2(t) + \tilde{M}^2(t)), \end{aligned} \tag{4.26}$$

where two positive constants λ_1 and λ_2 are given as

$$\lambda_1 = \min(\beta_2, 1, \frac{1}{2\gamma}) = \min(\beta_2, \frac{1}{2\gamma}), \tag{4.27}$$

$$\lambda_2 = \max(\beta_3, 1, \frac{1}{2\gamma}) = \max(\beta_3, \frac{1}{2\gamma}). \tag{4.28}$$

■

Lemma 4.2. *The time derivation of the Lyapunov function equation* (4.15) *is upper bounded, give by*

$$\dot{V}(t) \leq -\lambda V(t) + \varepsilon, \tag{4.29}$$

where $\lambda > 0$ *and* $\varepsilon > 0$.

Proof: From the definition of $V_1(t)$, we have

$$\begin{aligned}\dot{V}_1(t) &= \alpha\rho\int_0^L [\dot{w}(x,t)\ddot{w}(x,t)+\dot{v}(x,t)\ddot{v}(x,t)]\,dx+\alpha T\int_0^L w'(x,t)\dot{w}'(x,t)dx\\ &\quad+\alpha EA\int_0^L\left\{v'(x,t)+\frac{1}{2}[w'(x,t)]^2\right\}[\dot{v}'(x,t)+w'(x,t)\dot{w}'(x,t)]\,dx\\ &= A_1(t)+A_2(t)+A_3(t). \end{aligned} \tag{4.30}$$

Substitution of the governing equations (4.6) and (4.7) into $A_1(t)$ yields

$$\begin{aligned}A_1(t) &= \alpha\int_0^L [\rho\dot{w}(x,t)\ddot{w}(x,t)+\rho\dot{v}(x,t)\ddot{v}(x,t)]\,dx\\ &= \alpha\int_0^L \{T\dot{w}(x,t)w''(x,t)+EA\dot{w}(x,t)w''(x,t)v'(x,t)\\ &\quad+EAw'(x,t)\dot{w}(x,t)v''(x,t)+\frac{3}{2}EA[w'(x,t)]^2w''(x,t)\dot{w}(x,t)\\ &\quad+EAv''(x,t)\dot{v}(x,t)+EAw'(x,t)w''(x,t)\dot{v}(x,t)\\ &\quad+f_w(x,t)\dot{w}(x,t)+f_v(x,t)\dot{v}(x,t)\}\,dx. \end{aligned} \tag{4.31}$$

Using integration by parts to $A_2(t)$, and substituting the boundary condition Eq. (4.8), we obtain

$$A_2(t) = \alpha Tw'(L,t)\dot{w}(L,t)-\alpha T\int_0^L \dot{w}(x,t)w''(x,t)dx. \tag{4.32}$$

With the similar process, we obtain the expression of $A_3(t)$ as

$$\begin{aligned} A_3(t) &= \alpha EAv'(L,t)\dot{v}(L,t) - \alpha EA\int_0^L \dot{v}(x,t)v''(x,t)dx \\ &\quad +\alpha EA\dot{w}(L,t)w'(L,t)v'(L,t) + \frac{\alpha EA}{2}[w'(L,t)]^2\dot{v}(L,t) \\ &\quad -\alpha EA\int_0^L \dot{w}(x,t)\left[v''(x,t)w'(x,t) + w''(x,t)v'(x,t)\right]dx \\ &\quad -\alpha EA\int_0^L w'(x,t)w''(x,t)\dot{v}(x,t)dx + \frac{\alpha EA}{2}[w'(L,t)]^3\dot{w}(L,t) \\ &\quad -\frac{3\alpha EA}{2}\int_0^L \dot{w}(x,t)[w'(x,t)]^2w''(x,t)dx. \end{aligned} \tag{4.33}$$

Substituting Eqs. (4.31), (4.32), and (4.33) into Eq. (4.30), and using Ineq. (2.3) in Lemma 2.2, we have

$$\begin{aligned} \dot{V}_1(t) &= \alpha\left\{Tw'(L,t) + EAw'(L,t)v'(L,t) + \frac{EA}{2}[w'(L,t)]^3\right\}\dot{w}(L,t) \\ &\quad +\alpha\delta_1\int_0^L [\dot{w}(x,t)]^2dx + \frac{\alpha}{\delta_1}\int_0^L f_w^2(x,t)dx \\ &\quad +\alpha\left\{EAv'(L,t) + \frac{EA}{2}[w'(L,t)]^2\right\}\dot{v}(L,t) \\ &\quad +\alpha\delta_2\int_0^L [\dot{v}(x,t)]^2dx + \frac{\alpha}{\delta_2}\int_0^L f_v^2(x,t)dx, \end{aligned} \tag{4.34}$$

where δ_1 and δ_2 are two positive constants.

Differentiating $V_2(t)$ and substituting the boundary conditions Eqs. (4.9) and

(4.10), we obtain

$$\begin{aligned}
\dot{V}_2(t) \leq & -\alpha k_1[w'(L,t)+\dot{w}(L,t)]^2+\alpha k_1[w'(L,t)]^2-\alpha k_1[\dot{w}(L,t)]^2 \\
& -\alpha T[w'(L,t)]^2-\frac{\alpha EA}{2}[w'(L,t)]^4 \\
& -\alpha\left\{Tw'(L,t)+EAw'(L,t)v'(L,t)+\frac{EA}{2}[w'(L,t)]^3\right\}\dot{w}(L,t) \\
& -\alpha k_2[v'(L,t)+\dot{v}(L,t)]^2+\alpha k_2[v'(L,t)]^2-\alpha k_2[\dot{v}(L,t)]^2-\alpha EA[v'(L,t)]^2 \\
& -\alpha\left\{EAv'(L,t)+\frac{EA}{2}[w'(L,t)]^2\right\}\dot{v}(L,t)-\frac{3\alpha EA}{2}[w'(L,t)]^2v'(L,t) \\
& -\alpha\tilde{M}(t)\dot{w}'(L,t)[w'(L,t)+\dot{w}(L,t)] \\
& -\alpha\tilde{M}(t)\dot{v}'(L,t)[v'(L,t)+\dot{v}(L,t)].
\end{aligned} \tag{4.35}$$

Differentiating $V_3(t)$ in Eq. (4.18) and substituting the governing equations Eqs. (4.6) and (4.7) yield

$$\dot{V}_3(t) = B_1(t)+B_2(t)+B_3(t)+B_4(t)+B_5(t)+B_6(t), \tag{4.36}$$

where

$$B_1(t) = \beta T \int_0^L xw'(x,t)w''(x,t)dx + \beta EA \int_0^L xv'(x,t)v''(x,t)dx, \quad (4.37)$$

$$B_2(t) = \beta EA \int_0^L xw'(x,t)\left[w''(x,t)v'(x,t) + v''(x,t)w'(x,t)\right]dx, \quad (4.38)$$

$$B_3(t) = \frac{3\beta}{2} EA \int_0^L x[w'(x,t)]^3 w''(x,t)dx, \quad (4.39)$$

$$B_4(t) = \beta EA \int_0^L xv'(x,t)w'(x,t)w''(x,t)dx, \quad (4.40)$$

$$B_5(t) = \beta\rho \int_0^L x\left[\dot{w}(x,t)\dot{w}'(x,t) + \dot{v}(x,t)\dot{v}'(x,t)\right]dx, \quad (4.41)$$

$$B_6(t) = \beta \int_0^L xw'(x,t)f_w(x,t)dx + \beta \int_0^L xv'(x,t)f_v(x,t)dx. \quad (4.42)$$

Using integration by parts to $B_1(t)$ term and the boundary condition, we have

$$\begin{aligned} B_1(t) &= \frac{\beta}{2}T\int_0^L xd[w'(x,t)]^2 + \frac{\beta}{2}EA\int_0^L xd[v'(x,t)]^2, \\ &= \frac{\beta L}{2}T[w'(L,t)]^2 - \frac{\beta}{2}T\int_0^L [w'(x,t)]^2dx + \frac{\beta L}{2}EA[v'(L,t)]^2 \\ &\quad -\frac{\beta}{2}EA\int_0^L [v'(x,t)]^2dx. \end{aligned} \quad (4.43)$$

Following the similar process, we obtain

$$\begin{aligned}
B_2(t) &= \beta LEA[w'(L,t)]^2 v'(L,t) - \beta EA\int_0^L [w'(x,t)]^2 v'(x,t)dx - B_4(t) \\
&\leq \beta LEA[w'(L,t)]^2 v'(L,t) + \beta EA\int_0^L \left(\delta_3[w'(x,t)]^4 + \frac{1}{\delta_3}[v'(x,t))]^2\right) dx \\
&\quad -B_4(t), \qquad (4.44) \\
B_3(t) &= \frac{3\beta L}{8}EA[w'(L,t)]^4 - \frac{3\beta L}{8}EA\int_0^L [w'(x,t)]^4 dx, \qquad (4.45) \\
B_5(t) &= \frac{\beta L}{2}\rho[\dot{w}(L,t)]^2 - \frac{\beta}{2}\rho\int_0^L [\dot{w}(x,t)]^2 dx + \frac{\beta L}{2}\rho[\dot{v}(L,t)]^2 \\
&\quad -\frac{\beta}{2}\rho\int_0^L [\dot{v}(x,t)]^2 dx, \qquad (4.46)
\end{aligned}$$

where δ_3 is a positive constant. Applying Ineq. (2.3) to $B_6(t)$ term, we have

$$\begin{aligned}
B_6(t) &\leq \beta L\delta_4\int_0^L [w'(x,t)]^2 dx + \frac{\beta L}{\delta_4}\int_0^L f_w^2(x,t)dx + \beta L\delta_5\int_0^L [v'(x,t)]^2 dx \\
&\quad +\frac{\beta L}{\delta_5}\int_0^L f_v^2(x,t)dx, \qquad (4.47)
\end{aligned}$$

where δ_4 and δ_5 are two positive constants.

Substituting Eqs. (4.43) - (4.47) to Eq. (4.36), we have

$$\begin{aligned}
\dot{V}_3(t) \leq\ & \frac{\beta L}{2}T[w'(L,t)]^2 + \frac{\beta L}{2}EA[v'(L,t)]^2 + \beta LEA[w'(L,t)]^2 v'(L,t) \\
& + \frac{3\beta L}{8}EA[w'(L,t)]^4 + \frac{\beta L}{2}\rho[\dot{w}(L,t)]^2 + \frac{\beta L}{2}\rho[\dot{v}(L,t)]^2 \\
& - \frac{\beta}{2}T\int_0^L [w'(x,t)]^2 dx - \frac{\beta}{2}EA\int_0^L [v'(x,t)]^2 dx \\
& + \beta EA\int_0^L \left(\delta_3[w'(x,t)]^4 + \frac{1}{\delta_3}[v'(x,t))]^2\right) - \frac{3\beta L}{8}EA\int_0^L [w'(x,t)]^4 dx \\
& - \frac{\beta}{2}\rho\int_0^L [\dot{w}(x,t)]^2 dx - \frac{\beta}{2}\rho\int_0^L [\dot{v}(x,t)]^2 dx + \beta L\delta_4\int_0^L [w'(x,t)]^2 dx \\
& + \frac{\beta L}{\delta_4}\int_0^L f_w^2(x,t)dx + \beta L\delta_5\int_0^L [v'(x,t)]^2 dx + \frac{\beta L}{\delta_5}\int_0^L f_v^2(x,t)dx, \quad (4.48)
\end{aligned}$$

Substituting Eqs. (4.34), (4.35), and (4.48) into Eq. (4.15), and substituting the adaptive law Eq. (4.14), we have

$$
\begin{aligned}
\dot{V}(t) \leq & -\left(\frac{\beta\rho}{2}-\alpha\delta_1\right)\int_0^L [\dot{w}(x,t)]^2 dx - \left(\frac{\beta\rho}{2}-\alpha\delta_2\right)\int_0^L [\dot{v}(x,t)]^2 dx \\
& -\left(\frac{\beta T}{2}-\beta L\delta_4\right)\int_0^L [w'(x,t)]^2 dx \\
& -\left(\frac{\beta EA}{2}-\beta L\delta_5-\beta EA\frac{1}{\delta_3}\right)\int_0^L [v'(x,t)]^2 dx \\
& -\left(\frac{3\beta L}{8}EA-\beta EA\delta_3\right)\int_0^L [w'(x,t)]^4 dx \\
& -\alpha k_1[w'(L,t)+\dot{w}(L,t)]^2-\alpha k_2[v'(L,t)+\dot{v}(L,t)]^2 \\
& +\left(\frac{\alpha}{\delta_1}+\frac{\beta L}{\delta_4}\right)\int_0^L f_w^2(x,t)dx+\left(\frac{\alpha}{\delta_2}+\frac{\beta L}{\delta_5}\right)\int_0^L f_v^2(x,t)dx \\
& -\left(\alpha k_1-\frac{\beta L\rho}{2}\right)[\dot{w}(L,t)]^2-\left(\alpha T-\alpha k_1-\frac{\beta LT}{2}\right)[w'(L,t)]^2 \\
& -\left(\frac{\alpha EA}{2}-\frac{3\beta L}{8}EA-\delta_6\left|\frac{3\alpha}{2}-\beta L\right|EA\right)[w'(L,t)]^4 \\
& -\left(\alpha EA-\alpha k_2-\frac{\beta LEA}{2}-\frac{\left|\frac{3\alpha}{2}-\beta L\right|EA}{\delta_6}\right)[v'(L,t)]^2 \\
& -\left(\alpha k_2-\frac{\beta L\rho}{2}\right)[\dot{v}(L,t)]^2+\frac{\sigma}{2}\tilde{M}(t)\hat{M}(t) \\
\leq & -\lambda_3(V_1(t)+V_2(t))+\varepsilon+\frac{\sigma}{2}\tilde{M}(t)\hat{M}(t) \\
\leq & -\lambda_3(V_1(t)+V_2(t))-\frac{\sigma}{2}\tilde{M}^2(t)+\frac{\sigma}{2}M^2+\varepsilon \\
\leq & -\lambda_4(V_1(t)+V_2(t)+\tilde{M}^2(t))+\frac{\sigma}{2}M^2+\varepsilon, \qquad (4.49)
\end{aligned}
$$

where δ_6 is a positive constant. Other constants k_1, k_2, α, β, δ_1 - δ_6 are selected to satisfy the following conditions:

$$\alpha < \frac{\min(\beta\rho, \beta T)}{2\rho L}, \tag{4.50}$$

$$\alpha k_1 - \frac{\beta L\rho}{2} \geq 0, \tag{4.51}$$

$$\alpha T - \alpha k_1 - \frac{\beta LT}{2} \geq 0, \tag{4.52}$$

$$\frac{\alpha EA}{2} - \frac{3\beta L}{8}EA - \delta_6|\frac{3\alpha}{2} - \beta L|EA \geq 0, \tag{4.53}$$

$$\alpha k_2 - \frac{\beta L\rho}{2} \geq 0, \tag{4.54}$$

$$\alpha EA - \alpha k_2 - \frac{\beta LEA}{2} - \frac{|\frac{3\alpha}{2} - \beta L|EA}{\delta_6} \geq 0, \tag{4.55}$$

$$\sigma_1 = \frac{\beta\rho}{2} - \alpha\delta_1 > 0, \tag{4.56}$$

$$\sigma_2 = \frac{\beta\rho}{2} - \alpha\delta_2 > 0, \tag{4.57}$$

$$\sigma_3 = \frac{\beta T}{2} - \beta L\delta_4 > 0, \tag{4.58}$$

$$\sigma_4 = \frac{\beta EA}{2} - \beta L\delta_5 - \beta EA\frac{1}{\delta_3} > 0, \tag{4.59}$$

$$\sigma_5 = \frac{3\beta L}{8}EA - \beta EA\delta_3 > 0, \tag{4.60}$$

$$\sigma_6 = \alpha k_1 > 0, \tag{4.61}$$

$$\sigma_7 = \alpha k_2 > 0, \tag{4.62}$$

$$\lambda_3 = \min\left(\frac{2\sigma_1}{\alpha\rho}, \frac{2\sigma_2}{\alpha\rho}, \frac{2\sigma_3}{\alpha T}, \frac{\sigma_4}{\alpha EA}, \frac{4\sigma_5}{\alpha EA}, \frac{2\sigma_6}{\alpha M}, \frac{2\sigma_7}{\alpha M}\right) > 0, \tag{4.63}$$

$$\lambda_4 = \min(\lambda_3, \frac{\sigma}{2}), \tag{4.64}$$

$$\begin{aligned}\varepsilon &= \left(\frac{\alpha}{\delta_1} + \frac{\beta L}{\delta_4}\right)\int_0^L f_w^2(x,t)dx + \left(\frac{\alpha}{\delta_2} + \frac{\beta L}{\delta_5}\right)\int_0^L f_v^2(x,t)dx + \frac{\sigma}{2}M^2 \\ &\leq \left(\frac{\alpha L}{\delta_1} + \frac{\beta L^2}{\delta_4}\right)\bar{f}_w^2 + \left(\frac{\alpha L}{\delta_2} + \frac{\beta L^2}{\delta_5}\right)\bar{f}_v^2 + \frac{\sigma}{2}M^2. \end{aligned} \tag{4.65}$$

Combining Ineqs. (4.19) and (4.49), we have

$$\dot{V}(t) \leq -\lambda V(t) + \varepsilon, \tag{4.66}$$

where $\lambda = \lambda_4/\lambda_2 > 0$. ■

Remark 4.4. *A set of values for constants k_1, k_2, α, β, δ_1 - δ_6 can be found to satisfy the Ineqs.* (4.50) - (4.65).

Theorem 4.1. *Consider the closed-loop coupled nonlinear string system consisting of the system dynamics given by* (4.6) - (4.10)*, boundary control Eq.* (4.11) *and* (4.12)*, and the adaption law* (4.14)*, under the Assumption 4.1, with the bounded initial conditions, then,*

(i) the states of the closed-loop coupled nonlinear string system $w(x,t)$ and $v(x,t)$ will stay in the compact set Ω_{w1} and Ω_{v1} defined by

$$\Omega_{w1} := \{w(x,t) \in R| \ \mid w(x,t) \mid \leq D_{w1}\}, \tag{4.67}$$

$$\Omega_{v1} := \{v(x,t) \in R| \ \mid v(x,t) \mid \leq D_{v1}\}, \tag{4.68}$$

$\forall (x,t) \in [0,L] \times [0,\infty)$*, where constants*

$$D_{w1} = \sqrt{\frac{2L}{\alpha T \lambda_1}(V(0) + \frac{\varepsilon}{\lambda})}, \tag{4.69}$$

$$D_{v1} = \sqrt{\frac{2L}{\alpha EA \lambda_1}(V(0) + \frac{\varepsilon}{\lambda})}, \tag{4.70}$$

(ii) the system states $w(x,t)$ and $v(x,t)$ will eventually converge to the compact Ω_{w2} and Ω_{v2} defined by

$$\Omega_{w2} := \{w(x,t) \in R| \ \lim_{t\to\infty} \mid w(x,t) \mid \leq D_{w2}, \}, \tag{4.71}$$

$$\Omega_{v2} := \{v(x,t) \in R| \ \lim_{t\to\infty} \mid v(x,t) \mid \leq D_{v2}, \}, \tag{4.72}$$

$\forall x \in [0,L]$*, where constant* $D_{w2} = \sqrt{\frac{2L\varepsilon}{\alpha T \lambda_1 \lambda}}$ *and* $D_{w2} = \sqrt{\frac{2L\varepsilon}{\alpha EA \lambda_1 \lambda}}$.

Proof: Multiplying Eq. (4.29) by $e^{\lambda t}$ leads to

$$\frac{\partial}{\partial t}(V(t)e^{\lambda t}) \quad \leq \quad \varepsilon e^{\lambda t}. \tag{4.73}$$

Integration of Ineq. (4.73) yields

$$V(t) \leq \left(V(0) - \frac{\varepsilon}{\lambda}\right) e^{-\lambda t} + \frac{\varepsilon}{\lambda} \leq V(0)e^{-\lambda t} + \frac{\varepsilon}{\lambda} \in \mathcal{L}_\infty. \tag{4.74}$$

Applying Ineq. (2.7), we obtain

$$\frac{\alpha}{2L}Tw^2(x,t) \leq \frac{\alpha}{2}T\int_0^L [w'(x,t)]^2 dx \leq V_1(t) \leq V_1(t) + V_2(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty, \tag{4.75}$$

$$\frac{\alpha}{2L}EAv^2(x,t) \leq \frac{\alpha}{2}EA\int_0^L [v'(x,t)]^2 dx \leq V_1(t) \leq V_1(t) + V_2(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty. \tag{4.76}$$

From the above inequalities, the states $w(x,t)$ and $v(x,t)$ can be obtained to be bounded as

$$|w(x,t)| \leq \sqrt{\frac{2L}{\alpha T\lambda_1}\left(V(0)e^{-\lambda t} + \frac{\varepsilon}{\lambda}\right)} \leq \sqrt{\frac{2L}{\alpha T\lambda_1}\left(V(0) + \frac{\varepsilon}{\lambda}\right)}, \tag{4.77}$$

$$|v(x,t)| \leq \sqrt{\frac{2L}{\alpha EA\lambda_1}\left(V(0)e^{-\lambda t} + \frac{\varepsilon}{\lambda}\right)} \leq \sqrt{\frac{2L}{\alpha EA\lambda_1}\left(V(0) + \frac{\varepsilon}{\lambda}\right)}, \tag{4.78}$$

$\forall (x,t) \in [0,L] \times [0,\infty)$. Furthermore, we have

$$\lim_{t\to\infty} |w(x,t)| \leq \sqrt{\frac{2L\varepsilon}{\alpha T\lambda_1\lambda}}, \tag{4.79}$$

$$\lim_{t\to\infty} |v(x,t)| \leq \sqrt{\frac{2L\varepsilon}{\alpha EA\lambda_1\lambda}}. \tag{4.80}$$

$\forall(x,t) \in [0,L]$. ■

Remark 4.5. *From Eqs.* (4.61) - (4.64)*, it can be seen that increase in the control gains k_1, k_2 will lead to a larger λ, which will decrease the values of D_{w2} and D_{v2}. Therefore, $w(x,t)$ and $v(x,t)$ could be set in an small boundedness region by properly selecting the design parameters and a better vibration control performance can be achieved. However, in practice, the control gains should be chosen properly since a larger k will result in a high gain control scheme.*

Remark 4.6. *From Ineqs.* (4.75) *and* (4.76)*, it can be seen that $V_1(t)$ and $V_2(t)$ are bounded $\forall t \in [0,\infty)$ and then $\dot{w}(x,t), w'(x,t), \dot{v}(x,t)$ and $v'(x,t)$ are bounded $\forall(x,t) \in [0,L] \times [0,\infty)$. Then, we can state that the kinetic energy Eq.* (4.1) *and the potential energy Eq.* (4.2) *are also bounded. From the Properties 4.1, 4.2, we can obtain $w''(x,t), v''(x,t), \dot{w}'(x,t)$ and $\dot{v}'(x,t)$ are bounded $\forall(x,t) \in [0,L] \times [0,\infty)$. By using Assumption 4.1 and Eqs.* (4.6) *and* (4.7)*, it can be concluded that $\ddot{w}(x,t)$ and $\ddot{v}(x,t)$ are also bounded $\forall(x,t) \in [0,L] \times [0,\infty)$. Therefore, boundary control Eqs.* (4.11) *and* (4.12) *are bounded $\forall t \in [0,\infty)$, and they guarantee that all the internal system states including $w(x,t)$, $w'(x,t)$, $\dot{w}(x,t)$, $\dot{w}'(x,t)$, $\ddot{w}(x,t)$, $v(x,t)$, $v'(x,t)$, $\dot{v}(x,t)$, $\dot{v}'(x,t)$ and $\ddot{v}(x,t)$ are uniformly bounded.*

4.3 Simulations

Consider a nonlinear string excited by the distributed disturbances $f_w(x,t)$, $f_v(x,t)$ and boundary disturbances $d_w(t)$, $d_v(t)$ described as

$$
\begin{aligned}
f_w(x,t) &= [3+\sin(\pi xt)+\sin(2\pi xt)+\sin(3\pi xt)]x,\\
f_v(x,t) &= [1+\sin(\pi xt)+\sin(2\pi xt)+\sin(3\pi xt)]x,\\
d_w(t) &= 1+0.1\sin(0.1t)+0.3\sin(0.3t)+0.5\sin(0.5t),\\
d_v(t) &= 0.5+0.1\sin(0.1t)+0.3\sin(0.3t)+0.5\sin(0.5t).
\end{aligned}
$$

The initial conditions are given as

$$
\begin{aligned}
w(x,0) &= v(x,0) = x,\\
\dot{w}(x,0) &= \dot{v}(x,0) = 0.
\end{aligned}
$$

Parameters of the string system are referred to [85], and listed in the following table.

Table 1: Parameters of the string

Parameter	Description	Value
L	Length of string	$1m$
M	Mass of the tip payload	$1kg$
ρ	Mass per unit length	$0.1kg/m$
T	Tension	$10N$
EA	Axial stiffness	$2N$

Figs. 4.2 and 4.3 depict the transverse displacement and the longitudinal displacement of the nonlinear string under the external disturbances $f(x,t)$ and $d(t)$ without

control input, i.e. $u(t) = 0$. It is shown that there are large vibrations in both the transverse and longitudinal directions due to the external disturbances. When the system parameter is not available, by choosing $k_1 = 5, k_2 = 2$, $\alpha = 0.1$, $\gamma = 0.1$, $\sigma = 0.001$, the transverse displacement and the longitudinal displacement of the nonlinear string with the proposed boundary control and the designed adaption law are shown in Figs. 4.4 and 4.5 respectively. Figs. 4.4 and 4.5 illustrate that the proposed control is able to regulate the vibration of the coupled nonlinear string with a good performance. The control inputs are shown in Fig. 4.6.

4.4 Conclusion

In this chapter, adaptive boundary control has been proposed to stabilize both the transverse and longitudinal vibration for a coupled nonlinear string system subjected to the distributed disturbances and boundary disturbances. The control design was more difficult due to the coupling between transverse displacement and longitudinal displacement. The spillover problem has been removed since the control design was on the basis of the original distributed parameter systems.

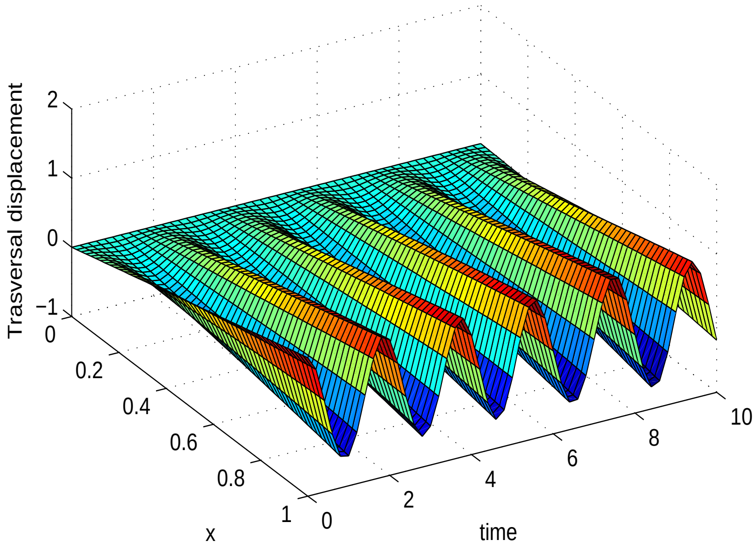

Fig. 4.2: Transverse displacement of the nonlinear string without control.

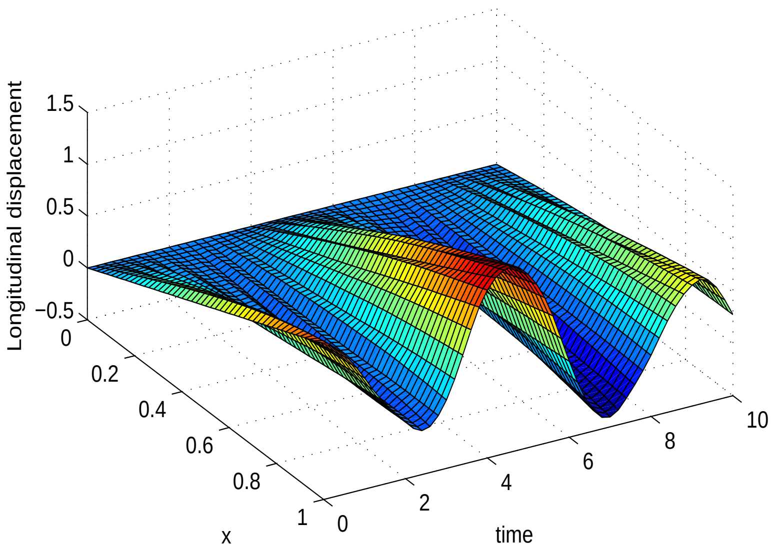

Fig. 4.3: Longitudinal displacement of the nonlinear string without control.

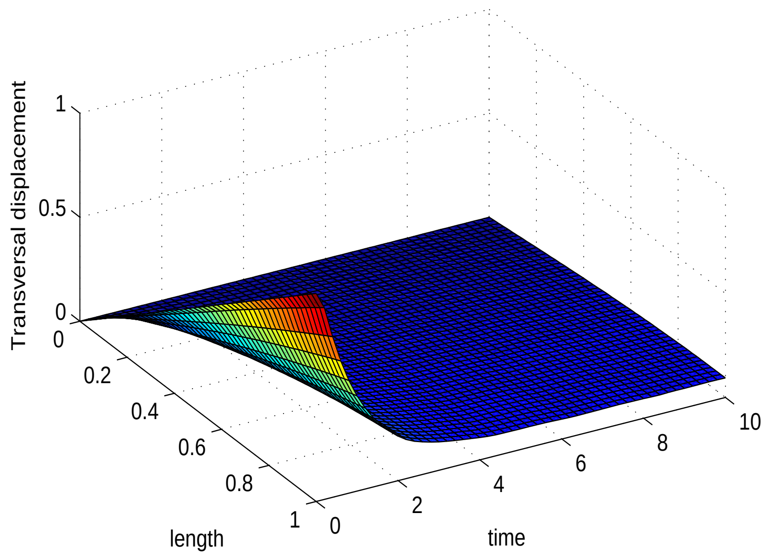

Fig. 4.4: Transverse displacement of the nonlinear string with the proposed boundary control.

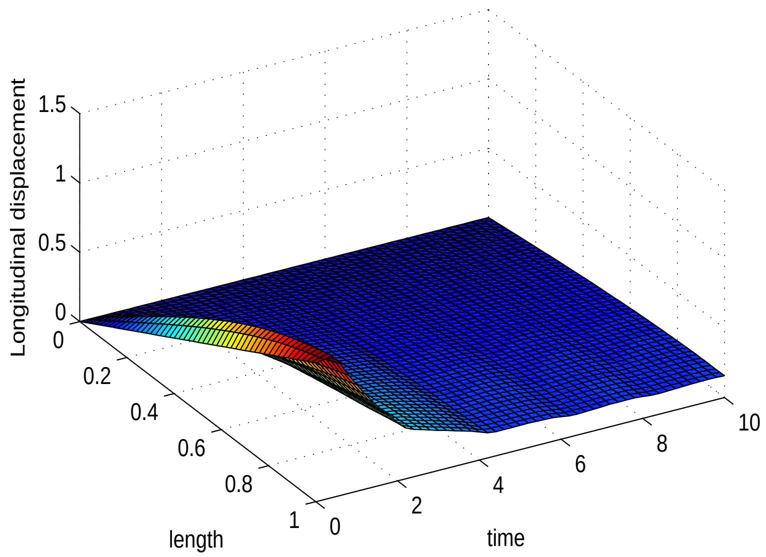

Fig. 4.5: Longitudinal displacement of the nonlinear string with the proposed boundary control.

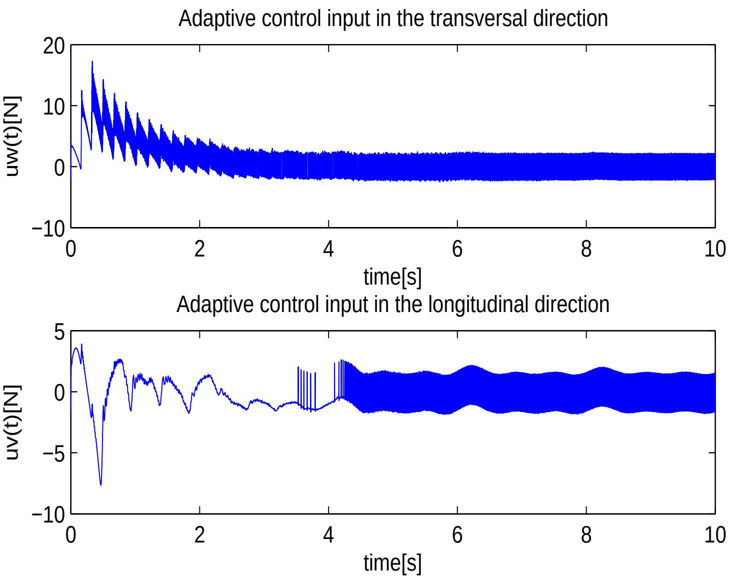

Fig. 4.6: Boundary control inputs $u_w(t)$ and $u_v(t)$.

Chapter 5

Boundary Control of an Euler-Bernoulli Beam under Unknown Spatiotemporally Varying Disturbance

Flexible beams constitute an important benchmark problem in many application areas ranging from aerospace to civil structures [102, 135]. In this chapter, adaptive boundary control is proposed for an Euler-Bernoulli beam in vibration with system parametric uncertainties and external disturbances. The control problem of an Euler-Bernoulli beam with boundary output constraint is also addressed. A novel form of Lyapunov Function that combining both the Integral Lyapunov Function and the Barrier Lyapunov Function is employed for the control design and stability analysis of the system. To the best of our knowledge, this is the first application of Integral-Barrier Lyapunov Function to flexible structure for vibration suppression. The main contributions of this chapter include:

(i) An everywhere-stabilizing boundary control is designed for the beam system when the system parameters are known.

(ii) Adaptive boundary control is developed to compensate for the system uncertainties and suppress the vibration of the system.

(iii) A novel concept Integral-Barrier Lyapunov Function is proposed for the control of flexible systems with output constraint problems. The employed Integral-Barrier Lyapunov Function candidate guarantee that the boundary output constraint is not violated.

The structure of this chapter is organized as follows. In Section 5.1, the Euler-Bernoulli beam equations of motion and the boundary conditions are introduced, and then the problems are formulated. Control strategies including the robust boundary control, adaptive boundary control, and Integral-Barrier Lyapunov function based control are discussed in Section 5.2. The performance of the proposed control is illustrated by the simulations in Section 5.3. The conclusion of this chapter is given in Section 5.4.

5.1 Problem Formulation

Fig. 5.1 shows an Euler-Bernoulli beam model extracted from a class of flexible systems under unknown distributed spatiotemporally varying disturbance $f(x,t)$ and unknown time-varying boundary disturbance $d(t)$. The left boundary of the beam is fixed at the origin. $w(x,t)$ is the displacement of the beam.

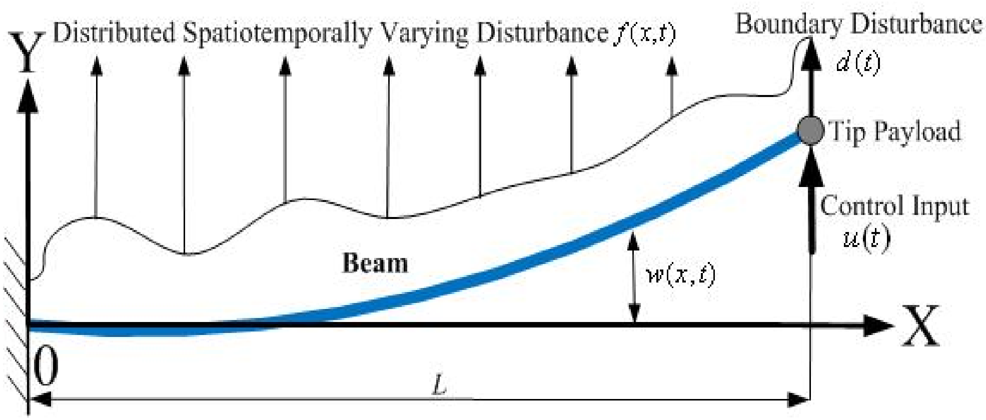

Fig. 5.1: A typical Euler-Bernoulli beam system.

The kinetic energy of the beam $E_k(t)$ is given as

$$E_k(t) = \frac{1}{2}M\left[\dot{w}(L,t)\right]^2 + \frac{1}{2}\rho\int_0^L \left[\dot{w}(x,t)\right]^2 dx, \tag{5.1}$$

where L is the length of the beam, M is the mass of the payload, ρ is the uniform mass per unit length of the beam, t and x represent the time and spatial variables, respectively.

The potential energy $E_p(t)$ due to the bending EI and the tension T can be obtained from

$$E_p(t) = \frac{1}{2}EI\int_0^L \left[w''(x,t)\right]^2 dx + \frac{1}{2}T\int_0^L \left[w'(x,t)\right]^2 dx. \tag{5.2}$$

The virtual work done by distributed disturbance $f(x,t)$ and boundary disturbance $d(t)$ is given by

$$\delta W_d(t) = \int_0^L f(x,t)\delta w(x,t)dx + d(t)\delta w(L,t). \tag{5.3}$$

The virtual work done by the control input $u(t)$ is expressed as

$$\delta W_f(t) = u(t)\delta w(L,t). \tag{5.4}$$

Then, we have the total virtual work done to the system as

$$\begin{aligned}\delta W(t) &= \delta W_d(t) + \delta W_f(t)\\ &= \int_0^L f(x,t)\delta w(x,t)dx + [u(t)+d(t)]\,\delta w(L,t).\end{aligned} \tag{5.5}$$

Applying the Hamilton's principle Eq. (2.1), the governing equation of the Euler-Bernoulli beam system is derived as

$$\rho\ddot{w}(x,t) + EIw''''(x,t) - Tw''(x,t) = f(x,t), \tag{5.6}$$

$\forall(x,t) \in (0,L)\times[0,\infty)$, and the boundary conditions of the system can be obtained as

$$w''(L,t) = 0, \tag{5.7}$$

$$w(0,t) = 0, \tag{5.8}$$

$$w'(0,t) = 0, \tag{5.9}$$

$$-EIw'''(L,t) + Tw'(L,t) = u(t) + d(t) - M\ddot{w}(L,t), \tag{5.10}$$

$\forall t \in [0,\infty)$.

Property 5.1. *[101, 131]: If the kinetic energy of the system* (5.6) - (5.10)*, given by Eq.* (5.1) *is bounded* $\forall t \in [0,\infty)$*, then* $\dot{w}(x,t)$*,* $\dot{w}'(x,t)$*,* $\dot{w}''(x,t)$ *and* $\dot{w}'''(x,t)$ *are bounded* $\forall(x,t) \in [0,L]\times[0,\infty)$*.*

Property 5.2. *[101, 131]: If the potential energy of the system* (5.6) - (5.10)*, given by Eq.* (5.2) *is bounded* $\forall t \in [0, \infty)$*, then* $w''(x,t)$*,* $w'''(x,t)$ *and* $w''''(x,t)$ *are bounded* $\forall (x,t) \in [0, L] \times [0, \infty)$*.*

Remark 5.1. *Due to the consideration of the unknown spatiotemporally varying distributed disturbance* $f(x,t)$*, a nonhomogeneous PDE* (5.6) *is used to describe the governing equation of the Euler-Bernoulli beam system. The nonhomogeneous model is different from the beam system governed by a homogeneous PDE in [26], where the backstepping methods are used.*

Assumption 5.1. *Assuming that the unknown spatiotemporally varying distributed disturbance* $f(x,t)$ *and unknown time-varying boundary disturbance* $d(t)$ *are uniformly bounded, i.e.,* $|f(x,t)| \leq \bar{f}$*,* $\forall (x,t) \in [0, L] \times [0, \infty)$ *and* $|d(t)| \leq \bar{d}$*,* $\forall t \in [0, \infty)$*, where* $\bar{f}$ *and* $\bar{d}$ *are two positive constants. The exact values of* $f(x,t)$*,* $d(t)$ *are not required.*

5.2 Control Design

5.2.1 Robust boundary control with disturbance uncertainties

Given the exact knowledge of the system parameters EI, T, M, we now design a boundary control law for the beam system given by (5.6) - (5.10). The control force is proposed as

$$\begin{aligned} u(t) &= -EIw'''(L,t) + Tw'(L,t) - M[\dot{w}'(L,t) - \dot{w}'''(L,t)] - ku_a(t) \\ &\quad -\mathrm{sgn}\left[\dot{w}(L,t) - w'''(L,t)\right]\bar{d}, \end{aligned} \tag{5.11}$$

where k is the control gain and the auxiliary signal $u_a(t)$ is defined as

$$u_a(t) \quad = \quad \dot{w}(L,t) - w'''(L,t) + w'(L,t). \tag{5.12}$$

We define a vector $P(t)$ and the system parameter vector Φ as

$$\begin{aligned} P(t) &= [w'''(L,t) \qquad -w'(L,t) \qquad \dot{w}'(L,t) - \dot{w}'''(L,t)], & (5.13) \\ \Phi &= [EI \qquad T \qquad M]^T. & (5.14) \end{aligned}$$

Then boundary control (5.11) can be rewritten in the following form

$$u(t) \quad = \quad -P(t)\Phi - ku_a(t) - \mathrm{sgn}\left[\dot{w}(L,t) - w'''(L,t)\right]\bar{d}. \tag{5.15}$$

Remark 5.2. *The proposed control* (5.15) *include a* sgn *term and an auxiliary signal term $u_a(t)$ to deal with the effect of unknown disturbances. The values for distributed disturbance $f(x,t)$ and boundary disturbance $d(t)$ are not required.*

The following positive Lyapunov function candidate is considered for the beam system (5.6) - (5.10) as

$$V_0(t) \quad = \quad V_1(t) + V_2(t) + \Delta(t), \tag{5.16}$$

where the first term $V_1(t)$ and the second term $V_2(t)$ and the third term $\Delta(t)$ are given as

$$V_1(t) = \frac{\beta}{2}\rho\int_0^L [\dot{w}(x,t)]^2 dx + \frac{\beta}{2}EI\int_0^L [w''(x,t)]^2 dx + \frac{\beta}{2}T\int_0^L [w'(x,t)]^2 dx, \quad (5.17)$$

$$V_2(t) = \frac{1}{2}Mu_a^2(t), \quad (5.18)$$

$$\Delta(t) = \alpha\rho\int_0^L x\dot{w}(x,t)w'(x,t)dx, \quad (5.19)$$

where α and β are two small positive constants.

Lemma 5.1. *The Lyapunov function equation* (5.16) *is bounded, given by*

$$0 \leq \lambda_1(V_1(t) + V_2(t)) \leq V_0(t) \leq \lambda_2(V_1(t) + V_2(t)), \quad (5.20)$$

where λ_1 and λ_2 are two positive constants.

Proof: From Lemma 2.1, we have

$$\begin{aligned} |\Delta(t)| &\leq \alpha\rho L\int_0^L ([\dot{w}(x,t)]^2 + [w'(x,t)]^2)dx \\ &\leq \alpha_1 V_1(t), \end{aligned} \quad (5.21)$$

where

$$\alpha_1 = \frac{2\alpha\rho L}{\min(\beta\rho, \beta T)}. \quad (5.22)$$

From Ineq. (5.21), we have

$$-\alpha_1 V_1(t) \leq \Delta(t) \leq \alpha_1 V_1(t). \quad (5.23)$$

By considering α satisfying $0 < \alpha < \frac{\min(\beta\rho,\beta T)}{2\rho L}$, we obtain two positive constants α_2 and α_3 as

$$\alpha_2 = 1 - \alpha_1 = 1 - \frac{2\alpha\rho L}{\min(\beta\rho, \beta T)} > 0, \tag{5.24}$$

$$\alpha_3 = 1 + \alpha_1 = 1 + \frac{2\alpha\rho L}{\min(\beta\rho, \beta T)} > 1. \tag{5.25}$$

Further, the following are derived

$$0 \leq \alpha_2 V_1(t) \leq V_1(t) + \Delta(t) \leq \alpha_3 V_1(t). \tag{5.26}$$

From the above analysis, the Lyapunov function candidate Eq. (5.16) is upper and lower bounded as

$$0 \leq \lambda_1(V_1(t) + V_2(t)) \leq V_0(t) \leq \lambda_2(V_1(t) + V_2(t)), \tag{5.27}$$

where two positive constants λ_1 and λ_2 are given as

$$\lambda_1 = \min(\alpha_2, 1) = \alpha_2, \tag{5.28}$$

$$\lambda_2 = \max(\alpha_3, 1) = \alpha_3. \tag{5.29}$$

Lemma 5.2. *The time derivation of the Lyapunov function equation* (5.16) *is upper bounded, given by*

$$\dot{V}_0(t) \leq -\lambda V_0(t) + \varepsilon_0, \tag{5.30}$$

where $\lambda > 0$ and $\varepsilon_0 > 0$.

Proof: Time derivations of the Lyapunov function candidate (5.16) result in

$$\dot{V}_0(t) \quad = \quad \dot{V}_1(t) + \dot{V}_2(t) + \dot{\Delta}(t). \tag{5.31}$$

$\dot{V}_1(t)$ can be written as

$$\dot{V}_1(t) \quad = \quad A_1(t) + A_2(t) + A_3(t), \tag{5.32}$$

where

$$A_1(t) \quad = \quad \beta\rho \int_0^L \dot{w}(x,t)\ddot{w}(x,t)dx, \tag{5.33}$$

$$A_2(t) \quad = \quad \beta EI \int_0^L w''(x,t)\dot{w}''(x,t)dx, \tag{5.34}$$

$$A_3(t) \quad = \quad \beta T \int_0^L w'(x,t)\dot{w}'(x,t)dx. \tag{5.35}$$

Substituting the governing equation (5.6) into $A_1(t)$, we obtain

$$A_1(t) \quad = \quad \beta \int_0^L \dot{w}(x,t) \left[-EIw''''(x,t) + Tw''(x,t) + f(x,t) \right] dx. \tag{5.36}$$

The integration by parts and the substitution of the boundary condition yields

$$A_2(t) \quad = \quad -\beta EIw'''(L,t)\dot{w}(L,t) + \beta EI \int_0^L \dot{w}(x,t)w''''(x,t)dx. \tag{5.37}$$

Following the similar process, we obtain

$$A_3(t) \quad = \quad \beta Tw'(L,t)\dot{w}(L,t) - \beta T \int_0^L \dot{w}(x,t)w''(x,t)dx. \tag{5.38}$$

Substituting the results of $A_1(t)$, $A_2(t)$ and $A_3(t)$ into Eq. (5.32), we have

$$\dot{V}_1(t) = \beta\left[-EIw'''(L,t) + Tw'(L,t)\right]\dot{w}(L,t) + \beta\int_0^L f(x,t)\dot{w}(x,t)dx. \quad (5.39)$$

Using Eq. (5.12) and Ineq. (2.3), we obtain

$$\begin{aligned}\dot{V}_1(t) \leq\ & \frac{\beta EI}{2}u_a^2(t) - \frac{\beta EI}{2}\{[\dot{w}(L,t)]^2 + [w'''(L,t)]^2 + [w'(L,t)]^2\} \\ & + \frac{\beta}{\delta_1}|T - EI|[\dot{w}(L,t)]^2 + \beta\delta_1|T - EI|[w'(L,t)]^2 + \beta EIw'(L,t)w'''(L,t) \\ & + \beta\delta_2\int_0^L [\dot{w}(x,t)]^2dx + \frac{\beta}{\delta_2}\int_0^L f^2(x,t)dx, \end{aligned} \quad (5.40)$$

where δ_1 and δ_2 are two positive constants.

Differentiating Eq. (5.12) and substituting Eq. (5.10), we obtain

$$\begin{aligned} M\dot{u}_a(t) &= EIw'''(L,t) - Tw'(L,t) + d(t) + M[\dot{w}'(L,t) - \dot{w}'''(L,t)] + u(t) \\ &= P(t)\Phi + d(t) + u(t). \end{aligned} \quad (5.41)$$

Substituting Eq. (5.15) into Eq. (5.41), we obtain the second term of the Eq. (5.31) as

$$\dot{V}_2(t) \leq -ku_a^2(t). \quad (5.42)$$

The third term of the Eq. (5.31) is rewritten as

$$\begin{aligned} \dot{\Delta}(t) &= \alpha\rho\int_0^L (x\ddot{w}(x,t)w'(x,t) + x\dot{w}(x,t)\dot{w}'(x,t))dx \\ &= B_1(t) + B_2(t) + B_3(t) + B_4(t), \end{aligned} \quad (5.43)$$

where

$$B_1(t) = -\alpha \int_0^L EIxw'(x,t)w''''(x,t)dx, \quad (5.44)$$

$$B_2(t) = \alpha \int_0^L Txw'(x,t)w''(x,t)dx, \quad (5.45)$$

$$B_3(t) = \alpha \int_0^L f(x,t)xw'(x,t)dx, \quad (5.46)$$

$$B_4(t) = \alpha\rho \int_0^L x\dot{w}(x,t)\dot{w}'(x,t)dx. \quad (5.47)$$

Integrations of Eq. (5.44) by parts yields

$$B_1(t) = -\alpha EILw'(L,t)w'''(L,t) - \frac{3\alpha EI}{2}\int_0^L [w''(x,t)]^2 dx. \quad (5.48)$$

In a similar manner, we obtain

$$B_2(t) = \frac{\alpha TL}{2}[w'(L,t)]^2 - \frac{\alpha T}{2}\int_0^L [w'(x,t)]^2 dx. \quad (5.49)$$

Applying Ineq. (2.11), we have

$$B_2(t) \le \frac{\alpha TL^2}{2}\int_0^L [w''(x,t)]^2 dx - \frac{\alpha T}{2}\int_0^L [w'(x,t)]^2 dx. \quad (5.50)$$

Similarly, we obtain

$$B_3(t) \le \frac{\alpha L}{\delta_3}\int_0^L f^2(x,t)dx + \alpha L\delta_3 \int_0^L [w'(x,t)]^2 dx, \quad (5.51)$$

where δ_3 is a positive constant. Integrating Eq. (5.47) by parts, we obtain

$$B_4(t) = \frac{\alpha\rho L}{2}[\dot{w}(L,t)]^2 - \frac{\alpha\rho}{2}\int_0^L [\dot{w}(x,t)]^2 dx. \quad (5.52)$$

Substituting Eqs. (5.48), (5.50), (5.51) and (5.52) into Eq. (5.43) and applying the boundary conditions, we obtain

$$\begin{aligned}\dot{\Delta}(t) \leq\ & -\alpha EILw'(L,t)w'''(L,t) - \frac{3\alpha EI}{2}\int_0^L [w''(x,t)]^2 dx + \frac{\alpha TL^2}{2}\int_0^L [w''(x,t)]^2 dx \\ & -\frac{\alpha T}{2}\int_0^L [w'(x,t)]^2 dx + \frac{\alpha L}{\delta_3}\int_0^L f^2(x,t)dx + \alpha L\delta_3\int_0^L [w'(x,t)]^2 dx \\ & +\frac{\alpha\rho L}{2}[\dot{w}(L,t)]^2 - \frac{\alpha\rho}{2}\int_0^L [\dot{w}(x,t)]^2 dx. \qquad (5.53)\end{aligned}$$

Substituting Eqs. (5.40), (5.42) and (5.53) into Eq. (5.31), we obtain

$$\begin{aligned}\dot{V}_0(t) \leq\ & -\left(\frac{3\alpha EI}{2} - \frac{\alpha TL^2}{2}\right)\int_0^L [w''(x,t)]^2 dx - \left(\frac{\alpha T}{2} - \alpha L\delta_3\right)\int_0^L [w'(x,t)]^2 dx \\ & -\left(\frac{\alpha\rho}{2} - \beta\delta_2\right)\int_0^L [\dot{w}(x,t)]^2 dx - \left(k - \frac{\beta EI}{2}\right)u_a^2(t) \\ & +(\beta - \alpha L)EIw'(L,t)w'''(L,t) + \frac{\alpha\rho L}{2}[\dot{w}(L,t)]^2 \\ & -\frac{\beta EI}{2}\{[\dot{w}(L,t)]^2 + [w'''(L,t)]^2 + [w'(L,t)]^2\} \\ & +\frac{\beta}{\delta_1}|T - EI|[\dot{w}(L,t)]^2 + \beta\delta_1|T - EI|[w'(L,t)]^2 \\ & +\left(\frac{\beta}{\delta_2} + \frac{\alpha L}{\delta_3}\right)\int_0^L f^2(x,t)dx. \qquad (5.54)\end{aligned}$$

Applying Ineq. (2.2) and Ineq. (2.11), we have

$$\begin{aligned}
\dot{V}_0(t) \;\leq\; & -\Bigg(\frac{\beta LEI}{2} + \frac{3\alpha EI}{2} - \frac{\alpha TL^2}{2} - \delta_4 EI|\beta - \alpha L|L \\
& -\beta\delta_1 L|T - EI|\Bigg) \int_0^L [w''(x,t)]^2 dx - \left(\frac{\alpha T}{2} - \alpha L\delta_3\right) \int_0^L [w'(x,t)]^2 dx \\
& - \left(\frac{\alpha\rho}{2} - \beta\delta_2\right) \int_0^L [\dot{w}(x,t)]^2 dx - \left(k - \frac{\beta EI}{2}\right) u_a^2(t) \\
& - \left(\frac{\beta EI}{2} - \frac{\beta}{\delta_1}|T - EI| - \frac{\alpha\rho L}{2}\right) [\dot{w}(L,t)]^2 \\
& - \left(\frac{\beta EI}{2} - \frac{EI}{\delta_4}|\beta - \alpha L|\right) [w'''(L,t)]^2 + \left(\frac{\beta}{\delta_2} + \frac{\alpha L}{\delta_3}\right) \int_0^L f^2(x,t) dx \\
\leq\; & -\lambda_3(V_1(t) + V_2(t)) + \varepsilon_0. \qquad (5.55)
\end{aligned}$$

where

$$\varepsilon_0 \;=\; \left(\frac{\beta}{\delta_2} + \frac{\alpha L}{\delta_3}\right) \int_0^L f^2 dx \leq \left(\frac{\beta}{\delta_2} + \frac{\alpha L}{\delta_3}\right) \int_0^L \bar{f}^2 dx \in \mathcal{L}_\infty, \qquad (5.56)$$

and other constants k, α, β, δ_1, δ_2 and δ_3 are selected to meet the following conditions

$$\begin{aligned}
\alpha &< \frac{\min(\beta\rho, \beta T)}{2\rho L}, && (5.57)\\
\frac{\beta EI}{2} &- \frac{\beta}{\delta_1}|T - EI| - \frac{\alpha\rho L}{2} \geq 0, && (5.58)\\
\frac{\beta EI}{2} &- \frac{EI}{\delta_4}|\beta - \alpha L| \geq 0, && (5.59)\\
\sigma_1 &= \frac{\beta LEI}{2} + \frac{3\alpha EI}{2} - \frac{\alpha TL^2}{2} - \delta_4 EI|\beta - \alpha L|L - \beta\delta_1 L|T - EI| > 0, && (5.60)\\
\sigma_2 &= \frac{\alpha T}{2} - \alpha L\delta_3 > 0, && (5.61)\\
\sigma_3 &= \frac{\alpha\rho}{2} - \beta\delta_2 > 0, && (5.62)\\
\sigma_4 &= k - \frac{\beta EI}{2} > 0, && (5.63)\\
\lambda_3 &= \min\left(\frac{2\sigma_1}{\beta EI}, \frac{2\sigma_2}{\beta T}, \frac{2\sigma_3}{\beta\rho}, \frac{2\sigma_4}{M}\right) > 0, && (5.64)\\
\lambda_4 &= \min\left(\lambda_3, \frac{\zeta_d}{2}\right). && (5.65)
\end{aligned}$$

Combining Ineqs. (5.27) and (5.55), we have

$$\dot{V}_0(t) \leq -\lambda V_0(t) + \varepsilon_0, \tag{5.66}$$

where $\lambda = \lambda_4/\lambda_2 > 0$. ■

Remark 5.3. *A set of values for constants k, α, β, δ_1 - δ_3 can be found to satisfy the Ineqs.* (5.56) - (5.65).

Theorem 5.1. *Consider the closed-loop Euler-Bernoulli beam system consisting of the dynamics* (5.6) - (5.10) *and boundary control* (5.15)*, under the Assumption 5.1, with the bounded initial conditions, then,* *(i) the state $w(x,t)$ of the closed-loop beam system will stay in Ω_1 given by*

$$\Omega_1 := \{w(x,t) \in R| \ \mid w(x,t) \mid \leq D_1, \ \forall (x,t) \in [0,L] \times [0,\infty)\}, \tag{5.67}$$

where the constant $D_1 = \sqrt{\frac{2L}{\beta T \lambda_1}(V_0(0) + \frac{\varepsilon_0}{\lambda})}$,
(ii) the state $w(x,t)$ of the system will eventually converge to Ω_2 given by

$$\Omega_2 := \{w(x,t) \in R| \ \lim_{t\to\infty} \mid w(x,t) \mid \leq D_2, \ \forall x \in [0,L]\}, \tag{5.68}$$

where the constant $D_2 = \sqrt{\frac{2L\varepsilon_0}{\beta T \lambda_1 \lambda}}$.

Proof: Multiplying Eq. (5.30) by $e^{\lambda t}$ leads to

$$\frac{\partial}{\partial t}(V_0(t)e^{\lambda t}) \ \leq \ \varepsilon_0 e^{\lambda t}. \tag{5.69}$$

Integration of Ineq. (5.69) yields

$$V_0(t) \leq \left(V_0(0) - \frac{\varepsilon_0}{\lambda}\right) e^{-\lambda t} + \frac{\varepsilon_0}{\lambda} \leq V_0(0)e^{-\lambda t} + \frac{\varepsilon_0}{\lambda} \in \mathcal{L}_\infty, \tag{5.70}$$

Applying Ineq. (2.7), we obtain

$$\frac{\beta}{2L}Tw^2(x,t) \leq \frac{\beta}{2}T\int_0^L [w'(x,t)]^2 dx \leq V_1(t) \leq V_1(t) + V_2(t) \leq \frac{1}{\lambda_1}V_0(t) \in \mathcal{L}_\infty. \tag{5.71}$$

From the above inequality, $w(x,t)$ can be obtained to be bounded as

$$|w(x,t)| \leq \sqrt{\frac{2L}{\beta T \lambda_1}\left(V_0(0)e^{-\lambda t} + \frac{\varepsilon_0}{\lambda}\right)} \leq \sqrt{\frac{2L}{\beta T \lambda_1}\left(V_0(0) + \frac{\varepsilon_0}{\lambda}\right)}, \tag{5.72}$$

$\forall(x,t) \in [0,L] \times [0,\infty)$. Furthermore, from (5.72), we have

$$\lim_{t\to\infty} |w(x,t)| \leq \sqrt{\frac{2L\varepsilon_0}{\beta T \lambda_1 \lambda}},\ \forall(x,t) \in [0,L]. \tag{5.73}$$

■

Remark 5.4. *From Eqs.* (5.63) - (5.65)*, it can be seen that increase in the control gain k will lead to a larger λ, which will decrease the values of D_1 and D_2. Therefore, $w(x,t)$ could be set in an arbitrarily small boundedness region by properly choosing the design parameters and a better vibration control performance can be achieved. However, in practice, the control gains need be chosen properly since increasing k will result in a high gain control scheme.*

5.2.2 Adaptive boundary control with the system parametric uncertainties

An adaption boundary control law is now considered for the case of unknown system parameters EI, T and M. In this section, an adaptive control law is needed to cope with the system uncertainties and update the boundary control law. The adaptive boundary control is proposed as

$$u(t) = -P(t)\hat{\Phi}(t) - ku_a(t) - \mathrm{sgn}\left[\dot{w}(L,t) - w'''(L,t)\right]\bar{d}, \tag{5.74}$$

where $\hat{\Phi}(t) = [\widehat{EI}(t) \quad \widehat{T}(t) \quad \widehat{M}(t)]^T$ is the parameter estimate vector. k is the control gain and the auxiliary signal $u_a(t)$ is defined as Eq. (5.12). The parameter estimate error vector $\tilde{\Phi}(t)$ is defined as

$$\begin{aligned}\tilde{\Phi}(t) &= \Phi - \hat{\Phi}(t) \\ &= [EI - \widehat{EI}(t) \quad T - \widehat{T}(t) \quad M - \widehat{M}(t)]^T \\ &= [\widetilde{EI}(t) \quad \widetilde{T}(t) \quad \widetilde{M}(t)]^T. \end{aligned} \tag{5.75}$$

We design the adaptation law as

$$\dot{\hat{\Phi}}(t) = \Gamma P^T(t)u_a(t) - \zeta_\Phi \Gamma \hat{\Phi}(t), \tag{5.76}$$

where ζ_Φ is a positive constant, and $\Gamma \in \mathbb{R}^{3\times 3}$ is a diagonal positive-definite matrix. Since $\Phi = [EI \quad T \quad M]^T$ is a constant parameter vector, then from Eq. (5.75), we have

$$\dot{\tilde{\Phi}}(t) = -\Gamma P^T(t)u_a(t) + \zeta_\Phi \Gamma \hat{\Phi}(t). \tag{5.77}$$

Remark 5.5. *In implementing the proposed boundary control* (5.15) *and* (5.74)*, and the adaption law* (5.76)*, measurement of the signals $\dot{w}(L,t)$, $w'(L,t)$, $\dot{w}'(L,t)$ $w'''(L,t)$ an $\dot{w}'''(L,t)$ of the tip payload are required. By using a laser displacement sensor and an inclinometer located at the tip payload, $w(L,t)$ and $w'(L,t)$ can be measured. A shear force sensor can be used to measure $w'''(L,t)$. The backward difference algorithm provides $\dot{w}(L,t)$, $\dot{w}'(L,t)$, and $\dot{w}'''(L,t)$ respectively.*

A new Lyapunov function candidate is considered for beam system under system parametric uncertainties as

$$V(t) = V_0(t) + \frac{1}{2}\tilde{\Phi}^T(t)\Gamma^{-1}\tilde{\Phi}(t), \tag{5.78}$$

where $V_0(t)$ is defined in Eq. (5.16). From Lemma 2.5, we have

$$\frac{1}{2\lambda_{max}}||\tilde{\Phi}(t)||^2 \leq \frac{1}{2}\tilde{\Phi}^T(t)\Gamma^{-1}\tilde{\Phi}(t) \leq \frac{1}{2\lambda_{min}}||\tilde{\Phi}(t)||^2, \tag{5.79}$$

where λ_{min} and λ_{max} are the minimum eigenvalue and the maximum eigenvalue of matrix Γ.

Combining Ineqs. (5.27), (5.79) and Eq. (5.78), we have

$$\begin{aligned} 0 &\leq \lambda_{1a}(V_1(t) + V_2(t) + ||\tilde{\Phi}(t)||^2) \leq V(t) \\ &\leq \lambda_{2a}(V_1(t) + V_2(t) + ||\tilde{\Phi}(t)||^2), \end{aligned} \tag{5.80}$$

where two positive constants $\lambda_{1a} = \min(\lambda_1,\ \frac{1}{2\lambda_{max}})$ and $\lambda_{2a} = \max(\lambda_2,\ \frac{1}{2\lambda_{min}})$, given that

$$0 < \alpha < \frac{\min(\beta\rho, \beta T)}{2\rho L}. \tag{5.81}$$

Lemma 5.3. *The time derivation of the Lyapunov equation* (5.78) *is upper bounded, given by*

$$\dot{V}(t) \quad \leq \quad -\lambda_a V(t) + \varepsilon, \tag{5.82}$$

where $\lambda_a > 0$ and $\varepsilon > 0$.

Proof: The differentiation of (5.78) yields

$$\dot{V}(t) = \dot{V}_0(t) + \tilde{\Phi}^T(t)\Gamma^{-1}\dot{\tilde{\Phi}}(t). \tag{5.83}$$

Substituting Eq. (5.74) into Eq. (5.41) and then substituting the results into $\dot{V}_2(t)$, we have

$$\dot{V}_2(t) \leq -ku_a^2(t) + P(t)\tilde{\Phi}(t)u_a(t). \tag{5.84}$$

Substituting Eqs. (5.40), (5.84) and (5.53) into Eq. (5.83), we obtain

$$\begin{aligned}\dot{V}(t) \leq\ & -\left(\frac{\beta LEI}{2} + \frac{3\alpha EI}{2} - \frac{\alpha TL^2}{2} - \delta_4 EI|\beta - \alpha L|L - \beta\delta_1 L|T - EI|\right) \\ & \times \int_0^L [w''(x,t)]^2 dx - \left(\frac{\alpha T}{2} - \alpha L\delta_3\right)\int_0^L [w'(x,t)]^2 dx \\ & - \left(\frac{\alpha\rho}{2} - \beta\delta_2\right)\int_0^L [\dot{w}(x,t)]^2 dx - \left(k - \frac{\beta EI}{2}\right)u_a^2(t) \\ & - \left(\frac{\beta EI}{2} - \frac{\beta}{\delta_1}|T - EI| - \frac{\alpha\rho L}{2}\right)[\dot{w}(L,t)]^2 \\ & - \left(\frac{\beta EI}{2} - \frac{EI}{\delta_4}|\beta - \alpha L|\right)[w'''(L,t)]^2 + \left(\frac{\beta}{\delta_2} + \frac{\alpha L}{\delta_3}\right)\int_0^L f^2(x,t)dx \\ & + \tilde{\Phi}^T(t)\Gamma^{-1}\dot{\tilde{\Phi}}(t) + P\tilde{\Phi}(t)u_a(t).\end{aligned} \tag{5.85}$$

Substituting the adaptive law Eq. (5.77) into Ineq. (5.85), we have

$$\begin{aligned}\dot{V}(t) &\leq -\lambda_4\left[V_1(t) + V_2(t)\right] + \zeta_\Phi\tilde{\Phi}^T(t)\hat{\Phi}(t) + \varepsilon_0 \\ &\leq -\lambda_4\left[V_1(t) + V_2(t)\right] - \frac{\zeta_\Phi}{2}||\tilde{\Phi}(t)||^2 + \frac{\zeta_\Phi}{2}||\Phi||^2 + \varepsilon_0 \\ &\leq -\lambda_{4a}\left[V_1(t) + V_2(t) + ||\tilde{\Phi}(t)||^2\right] + \frac{\zeta_\Phi}{2}||\Phi||^2 + \varepsilon_0,\end{aligned} \tag{5.86}$$

where positive constant $\lambda_{4a} = \min(\lambda_4,\ \frac{\zeta_\Phi}{2})$. Combining Ineqs. (5.80) and (5.86), we obtain

$$\dot{V}(t) \leq -\lambda_a V(t) + \varepsilon, \tag{5.87}$$

where $\lambda_a = \lambda_{4a}/\lambda_{2a} > 0$ and $\varepsilon = \varepsilon_0 + \frac{\zeta_\Phi}{2}||\Phi||^2 > 0$. ■

Theorem 5.2. *Consider the closed-loop Euler-Bernoulli beam system consisting of the system dynamics* (5.6) - (5.10)*, boundary control* (5.15)*, and the adaption law* (5.76)*, under the Assumption 5.1, with the bounded initial conditions, then,*
(i) the state $w(x,t)$ of the closed-loop Euler-Bernoulli beam system will stay in Ω_3 given by

$$\Omega_3 := \{w(x,t) \in R| \ \mid w(x,t) \mid \leq D_3, \ \forall (x,t) \in [0,L] \times [0,\infty)\}, \tag{5.88}$$

where the constant $D_3 = \sqrt{\frac{2L}{\beta T \lambda_{1a}}(V(0) + \frac{\varepsilon}{\lambda_a})}$,
(ii) the state $w(x,t)$ of the beam system will eventually converge to Ω_4 given by

$$\Omega_4 := \{w(x,t) \in R| \lim_{t\to\infty} \mid w(x,t) \mid \leq D_4, \ \forall x \in [0,L]\}, \tag{5.89}$$

where the constant $D_4 = \sqrt{\frac{2L\varepsilon}{\beta T \lambda_{1a}\lambda_a}}$.

Proof: Multiplying Eq. (5.82) by $e^{\lambda_a t}$ leads to

$$\frac{\partial}{\partial t}(V(t)e^{\lambda_a t}) \ \ \leq \ \ \varepsilon e^{\lambda_a t}. \tag{5.90}$$

Integration of above inequality yields

$$V(t) \leq \left(V(0) - \frac{\varepsilon}{\lambda_a}\right) e^{-\lambda_a t} + \frac{\varepsilon}{\lambda_a} \leq V(0)e^{-\lambda_a t} + \frac{\varepsilon}{\lambda_a} \in \mathcal{L}_\infty, \tag{5.91}$$

Applying Ineq. (2.7), we obtain

$$\frac{\beta}{2L}Tw^2(x,t) \leq \frac{\beta}{2}T\int_0^L [w'(x,t)]^2dx \leq V_1(t) \leq V_1(t)+V_2(t) \leq \frac{1}{\lambda_{1a}}V(t) \in \mathcal{L}_\infty. \quad (5.92)$$

From the above inequalities, $w(x,t)$ can be obtained to be bounded as

$$|w(x,t)| \leq \sqrt{\frac{2L}{\beta T\lambda_{1a}}\left(V(0)e^{-\lambda_a t}+\frac{\varepsilon}{\lambda_a}\right)} \leq \sqrt{\frac{2L}{\beta T\lambda_{1a}}\left(V(0)+\frac{\varepsilon}{\lambda_a}\right)}, \quad (5.93)$$

$\forall(x,t) \in [0,L]\times[0,\infty)$. Furthermore, from (5.93), we have

$$\lim_{t\to\infty}|w(x,t)| \quad \leq \quad \sqrt{\frac{2L\varepsilon}{\beta T\lambda_{1a}\lambda_a}},\ \forall(x,t) \in [0,L]. \quad (5.94)$$

■

5.2.3 Integral-Barrier Lyapunov Function based control with boundary output constraint

In this section, the boundary output constraint problem for an Euler-Bernoulli beam is addressed. The novel Integral-Barrier Lyapunov Function is proposed and guarantees that the boundary output constraint is not violated. Fig. 5.2 shows a typical beam-based structure with the boundary output constraint, i.e., $| w(L,t) |< l_0$. The left boundary of the beam is fixed at origin.

Model-based control with boundary output constraint

In order to suppress the vibration of the Euler-Bernoulli beam system governed by Eqs. (5.6) - (5.10) and handle the boundary output constraint $| w(L,t) |< l_0$, the

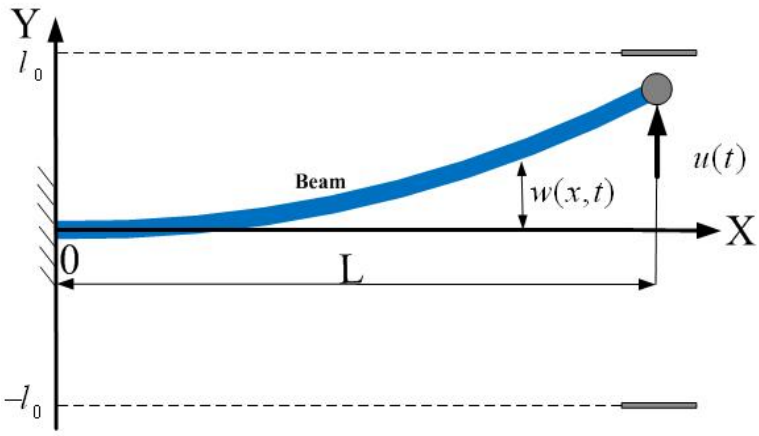

Fig. 5.2: A typical Euler-Bernoulli beam system with boundary output constraint.

Integral-Barrier Lyapunov Function is used to construct the control input $u(t)$ at the right boundary of the flexible beam and analyze the closed-loop stability of the system. The model-based barrier control is proposed as

$$\begin{aligned} u(t) &= \kappa_1\dot{w}(L,t) + \kappa_2 w'''(L,t) - \frac{w(L,t)}{l_0^2 - [w(L,t)]^2} - EIw'''(L,t) + M\dot{w}'''(L,t) \\ &\quad -\mathrm{sgn}\left[\dot{w}(L,t) - w'''(L,t)\right]\bar{d}. \end{aligned} \tag{5.95}$$

Let $k_1 = \frac{\kappa_2-\kappa_1}{2}, k_2 = \frac{\kappa_1+\kappa_2}{2}$, Eq. (5.95) can be written as

$$\begin{aligned} u(t) &= -k_1[\dot{w}(L,t) - w'''(L,t)] + k_2[\dot{w}(L,t) + w'''(L,t)] - \frac{w(L,t)}{l_0^2 - [w(L,t)]^2} \\ &\quad -EIw'''(L,t) + M\dot{w}'''(L,t) - \mathrm{sgn}\left[\dot{w}(L,t) - w'''(L,t)\right]\bar{d}. \end{aligned} \tag{5.96}$$

where k_1 and k_2 are the control gains. We consider the following Lyapunov functional candidate

$$V(t) = V_1(t) + V_2(t) + V_3(t). \tag{5.97}$$

where the energy term $V_1(t)$, the barrier term $V_2(t)$ and the crossing term $V_3(t)$ are defined as

$$V_1(t) = \frac{a}{2}\rho\int_0^L [\dot{w}(x,t)]^2dx + \frac{a}{2}EI\int_0^L [w''(x,t)]^2dx + \frac{a}{2}T\int_0^L [w'(x,t)]^2dx, \tag{5.98}$$

$$V_2(t) = \frac{a}{2}\ln\frac{l_0^2}{l_0^2 - w^2(L,t)} + \frac{a}{2}M[\dot{w}(L,t) - w'''(L,t)]^2, \tag{5.99}$$

$$V_3(t) = b\rho\int_0^L x\dot{w}(x,t)w'(x,t)dx, \tag{5.100}$$

a and b are two positive constants.

Remark 5.6. *The barrier term in Eq.* (5.99) *indicates* $[w(L,t)]^2 < l_0^2$*, and there exists a small positive constant* ϵ *such that* $l_0^2 - [w(L,t)]^2 \geq \epsilon^2$*.*

Lemma 5.4. *The Lyapunov function equation Eq.* (5.97) *is bounded, given by*

$$0 \leq \lambda_1\left(V_1(t) + [\dot{w}(L,t) - w'''(L,t)]^2\right) \leq V(t) \leq \lambda_2\left(V_1(t) + [\dot{w}(L,t) - w'''(L,t)]^2\right), \tag{5.101}$$

where λ_1 *and* λ_2 *are two positive constants.*

Proof: Applying Ineqs. (2.4) and (2.7) to $V_2(t)$, we have

$$\begin{aligned} V_2(t) &\leq \frac{a[w(L,t)]^2}{2\left(l_0^2 - [w(L,t)]^2\right)} + \frac{a}{2}M[\dot{w}(L,t) - w'''(L,t)]^2 \\ &\leq \frac{a[w(L,t)]^2}{2\epsilon^2} + \frac{a}{2}M[\dot{w}(L,t) - w'''(L,t)]^2 \\ &\leq \frac{aL}{2\epsilon^2}\int_0^L [w'(x,t)]^2\,dx + \frac{a}{2}M[\dot{w}(L,t) - w'''(L,t)]^2 \\ &\leq \frac{L}{\epsilon^2 T}V_1(t) + \frac{a}{2}M[\dot{w}(L,t) - w'''(L,t)]^2. \end{aligned} \tag{5.102}$$

Since $\ln \frac{l_0^2}{l_0^2 - w^2(L,t)} \geq 0$, we obtain

$$\frac{a}{2}M[\dot{w}(L,t) - w'''(L,t)]^2 \leq V_2(t) \leq \frac{L}{\epsilon^2 T}V_1(t) + \frac{a}{2}M[\dot{w}(L,t) - w'''(L,t)]^2. \quad (5.103)$$

Utilizing Ineq. (2.2) to Eq. (5.100), we have

$$\begin{aligned} |V_3(t)| &\leq b\rho L \int_0^L ([\dot{w}(x,t)]^2 + [w'(x,t)]^2)dx \\ &\leq \beta_1 V_1(t), \end{aligned} \quad (5.104)$$

where

$$\beta_1 = \frac{2b\rho L}{a\min(\rho, T)}. \quad (5.105)$$

Then, we obtain

$$-\beta_1 V_1(t) \leq V_3(t) \leq \beta_1 V_1(t). \quad (5.106)$$

Considering b is a small positive weighting constant satisfying $0 < b < \frac{a\min(\rho,T)}{2\rho L}$, we obtain $0 < \beta_1 < 1$, and

$$\beta_2 = 1 - \beta_1 = 1 - \frac{2b\rho L}{a\min(\rho, T)} > 0, \quad (5.107)$$

$$\beta_3 = 1 + \beta_1 = 1 + \frac{2b\rho L}{a\min(\rho, T)} > 1. \quad (5.108)$$

Then, we further have

$$0 \leq \beta_2 V_1(t) \leq V_1(t) + V_3(t) \leq \beta_3 V_1(t). \quad (5.109)$$

Given the Lyapunov functional candidate Eq. (5.97), we obtain

$$\begin{aligned} 0 &\leq \lambda_1\left(V_1(t)+[\dot{w}(L,t)-w'''(L,t)]^2\right) \leq V_1(t)+V_2(t)+V_3(t) \\ &\leq \lambda_2\left(V_1(t)+[\dot{w}(L,t)-w'''(L,t)]^2\right), \end{aligned} \tag{5.110}$$

where two positive constants λ_1 and λ_2 are given as

$$\lambda_1 = \min(\beta_2, \frac{aM}{2}), \tag{5.111}$$

$$\lambda_2 = \max(\beta_3 + \frac{L}{\epsilon^2 T}, \frac{aM}{2}). \tag{5.112}$$

Lemma 5.5. *The time derivation of the Lyapunov function equation Eq.* (5.97) *is upper bounded, give by*

$$\dot{V}(t) \leq -\lambda V(t) + \varepsilon_0, \tag{5.113}$$

where $\lambda > 0$*.*

Proof: Differentiating Eq. (5.97) with respect to time leads to

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t). \tag{5.114}$$

The first term of the Eq. (5.114) is written as

$$\dot{V}_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t), \tag{5.115}$$

where

$$V_{11}(t) = a\rho \int_0^L \dot{w}(x,t)\ddot{w}(x,t)dx, \tag{5.116}$$

$$V_{12}(t) = aEI \int_0^L w''(x,t)\dot{w}''(x,t)dx, \tag{5.117}$$

$$V_{13}(t) = aT \int_0^L w'(x,t)\dot{w}'(x,t)dx. \tag{5.118}$$

Substituting the governing equation (5.6) into $A_1(t)$, we obtain

$$V_{11}(t) = a \int_0^L \dot{w}(x,t) \left[-EIw''''(x,t) + Tw''(x,t) + f(x,t)\right] dx. \tag{5.119}$$

Using the boundary conditions and integrating Eq. (5.117) by parts, we obtain

$$V_{12}(t) = -aEIw'''(L,t)\dot{w}(L,t) + aEI \int_0^L \dot{w}(x,t)w''''(x,t)dx. \tag{5.120}$$

Similarly, we obtain

$$V_{13}(t) = aTw'(L,t)\dot{w}(L,t) - aT \int_0^L \dot{w}(x,t)w''(x,t)dx. \tag{5.121}$$

Substituting Eqs. (5.119), (5.120), and (5.121) into Eq. (5.115), we have

$$\begin{aligned} \dot{V}_1(t) &= a\left[-EIw'''(L,t) + Tw'(L,t)\right]\dot{w}(L,t) + a\int_0^L \dot{w}(x,t)f(x,t)dx \\ &= \frac{aEI}{2}[\dot{w}(L,t) - w'''(L,t)]^2 - \frac{aEI}{2}[\dot{w}(L,t)]^2 - \frac{aEI}{2}[w'''(L,t)]^2 \\ &\quad + aTw'(L,t)\dot{w}(L,t) + a\int_0^L \dot{w}(x,t)f(x,t)dx. \end{aligned} \tag{5.122}$$

Using Ineq. (2.3), we obtain

$$\begin{aligned}\dot{V}_1(t) &\leq \frac{aEI}{2}[\dot{w}(L,t)-w'''(L,t)]^2-\frac{aEI}{2}[\dot{w}(L,t)]^2-\frac{aEI}{2}[w'''(L,t)]^2\\ &\quad +aTw'(L,t)\dot{w}(L,t)+a\delta_1\int_0^L[\dot{w}(x,t)]^2dx+\frac{a}{\delta_1}\int_0^L f^2(x,t)dx,\end{aligned}\tag{5.123}$$

where δ_1 is a positive constant.

Differentiating $V_2(t)$ yields

$$\dot{V}_2(t) = \frac{aw(L,t)\dot{w}(L,t)}{l_0^2-[w(L,t)]^2}+aM[\dot{w}(L,t)-w'''(L,t)][\ddot{w}(L,t)-\dot{w}'''(L,t)]\tag{5.124}$$

Substituting the boundary condition (5.9), we have

$$\begin{aligned}\dot{V}_2(t) &= a[\dot{w}(L,t)-w'''(L,t)][u(t)+d(t)-Tw'(L,t)+EIw'''(L,t)-M\dot{w}'''(L,t)]\\ &\quad +\frac{aw(L,t)\dot{w}(L,t)}{l_0^2-[w(L,t)]^2}\end{aligned}\tag{5.125}$$

Using the proposed control law Eq. (5.96), we obtain

$$\begin{aligned}\dot{V}_2(t) &\leq -ak_1[\dot{w}(L,t)-w'''(L,t)]^2+ak_2[\dot{w}(L,t)]^2-ak_2[w'''(L,t)]^2\\ &\quad +a\delta_2T[w'(L,t)]^2+\frac{aT}{\delta_2}[w'''(L,t)]^2+\frac{a[w(L,t)]^2+a[w'''(L,t)]^2}{2\epsilon^2}\\ &\quad -aTw'(L,t)\dot{w}(L,t)\end{aligned}\tag{5.126}$$

where δ_2 is a positive constant.

Utilizing Ineqs. (2.7) and (2.11), we obtain

$$\begin{aligned}\dot{V}_2(t) \leq & -ak_1[\dot{w}(L,t) - w'''(L,t)]^2 + ak_2[\dot{w}(L,t)]^2 - \left(ak_2 - \frac{aT}{\delta_2} - \frac{a}{2\epsilon^2}\right)[w'''(L,t)]^2 \\ & + a\delta_2 TL\int_0^L [w''(x,t)]^2 dx + \frac{aL}{2\epsilon^2}\int_0^L [w'(x,t)]^2 dx - aTw'(L,t)\dot{w}(L,t) \quad (5.127)\end{aligned}$$

The third term of the Eq. (5.114) is written as

$$\begin{aligned}\dot{V}_3(t) &= b\rho\int_0^L [x\ddot{w}(x,t)w'(x,t) + x\dot{w}(x,t)\dot{w}'(x,t)]\, dx \\ &= V_{31}(t) + V_{32}(t) + V_{33}(t) + V_{34}(t), \quad (5.128)\end{aligned}$$

where

$$V_{31}(t) = -bEI\int_0^L xw'(x,t)w''''(x,t)dx, \quad (5.129)$$

$$V_{32}(t) = bT\int_0^L xw'(x,t)w''(x,t)dx, \quad (5.130)$$

$$V_{33}(t) = b\rho\int_0^L x\dot{w}(x,t)\dot{w}'(x,t)dx, \quad (5.131)$$

$$V_{34}(t) = b\int_0^L xw'(x,t)f(x,t)dx, \quad (5.132)$$

After integrating Eq. (5.129) by parts and using the boundary conditions, we obtain

$$\begin{aligned}V_{31}(t) = & -bEILw'(L,t)w'''(L,t) + bEI\int_0^L w'(x,t)w'''(x,t)dx \\ & + bEI\int_0^L xw''(x,t)w'''(x,t)dx. \quad (5.133)\end{aligned}$$

Integrating Eq. (5.133) by parts and using the boundary conditions, we have

$$\begin{aligned} V_{31}(t) &= -\frac{3b}{2}EI\int_0^L [w''(x,t)]^2 dx - bEILw'(L,t)w'''(L,t) \\ &\leq -\frac{3b}{2}EI\int_0^L [w''(x,t)]^2 dx + \frac{bEIL}{\delta_3}[w'''(L,t)]^2 + \delta_3 bEIL[w'(L,t)]^2 \\ &\leq -\frac{3b}{2}EI\int_0^L [w''(x,t)]^2 dx + \frac{bEIL}{\delta_3}[w'''(L,t)]^2 \\ &\quad +\delta_3 bEIL^2\int_0^L [w''(x,t)]^2 dx \end{aligned} \tag{5.134}$$

where δ_3 is a positive constant.

After integrating Eq. (5.130) by parts and using the boundary conditions, we obtain

$$\begin{aligned} V_{32}(t) &= bT\int_0^L xd\left(\frac{[w'(x,t)]^2}{2}\right) = \frac{bTL}{2}[w'(L,t)]^2 - \frac{bT}{2}\int_0^L [w'(x,t)]^2 dx \\ &\leq \frac{bTL^2}{2}\int_0^L [w''(x,t)]^2 dx - \frac{bT}{2}\int_0^L [w'(x,t)]^2 dx. \end{aligned} \tag{5.135}$$

In a similar manner, we obtain

$$V_{33}(t) = \frac{b\rho L}{2}[\dot{w}(L,t)]^2 - \frac{b\rho}{2}\int_0^L [\dot{w}(x,t)]^2 dx. \tag{5.136}$$

Applying Ineq. (2.3), we have

$$V_{34}(t) \leq bL\delta_4\int_0^L [w'(x,t)]^2 dx + \frac{bL}{\delta_4}\int_0^L f^2(x,t)dx. \tag{5.137}$$

where δ_4 is a positive constant.

Substituting Eqs. (5.134), (5.135) and (5.136) into Eq. (5.128), we obtain

$$\begin{aligned}
\dot{V}_3(t) \quad \leq \quad & -\left(\frac{3b}{2}EI - \delta_3 bEIL^2 - \frac{bTL^2}{2}\right)\int_0^L [w''(x,t)]^2 dx \\
& -\left(\frac{bT}{2} - bL\delta_4\right)\int_0^L [w'(x,t)]^2 dx - \frac{b\rho}{2}\int_0^L [\dot{w}(x,t)]^2 dx \\
& +\frac{b\rho L}{2}[\dot{w}(L,t)]^2 + \frac{bEIL}{\delta_3}[w'''(L,t)]^2 + \frac{bL}{\delta_4}\int_0^L f^2(x,t)dx. \qquad (5.138)
\end{aligned}$$

Substituting Eqs. (5.123), (5.127) and (5.138) into Eq. (5.114), we have

$$\begin{aligned}
\dot{V}(t) \quad \leq \quad & -a\left(k_1 - \frac{EI}{2}\right)[\dot{w}(L,t) - w'''(L,t)]^2 \\
& -\left(\frac{3b}{2}EI - \delta_3 bEIL^2 - \frac{bTL^2}{2} - a\delta_2 TL\right)\int_0^L [w''(x,t)]^2 dx \\
& -\left(\frac{bT}{2} - bL\delta_4 - \frac{aL}{2\epsilon^2}\right)\int_0^L [w'(x,t)]^2 dx - \left(\frac{b\rho}{2} - a\delta_1\right)\int_0^L [\dot{w}(x,t)]^2 dx \\
& -\left(ak_2 + \frac{aEI}{2} - \frac{aT}{\delta_2} - \frac{a}{2\epsilon^2} - \frac{bEIL}{\delta_3}\right)[w'''(L,t)]^2 \\
& -\left(\frac{aEI}{2} - \frac{b\rho L}{2} - ak_2\right)[\dot{w}(L,t)]^2 \\
& +\left(\frac{a}{\delta_1} + \frac{bL}{\delta_4}\right)\int_0^L f^2(x,t)dx \\
\leq \quad & -\lambda_3\left(V_1(t) + [\dot{w}(L,t) - w'''(L,t)]^2\right) + \varepsilon_0, \qquad (5.139)
\end{aligned}$$

where $k_1, k_2, \delta_1 - \delta_4$ are chosen to satisfy following conditions:

$$0 < b < \frac{a\min(\rho, T)}{2\rho L}, \tag{5.140}$$

$$ak_2 + \frac{aEI}{2} - \frac{aT}{\delta_2} - \frac{a}{2\epsilon^2} - \frac{bEIL}{\delta_3} \geq 0, \tag{5.141}$$

$$\frac{aEI}{2} - \frac{b\rho L}{2} - ak_2 \geq 0, \tag{5.142}$$

$$\sigma_1 = a\left(k_1 - \frac{EI}{2}\right) > 0, \tag{5.143}$$

$$\sigma_2 = \frac{3b}{2}EI - \delta_3 bEIL^2 - \frac{bTL^2}{2} - a\delta_2 TL > 0, \tag{5.144}$$

$$\sigma_3 = \frac{bT}{2} - bL\delta_4 - \frac{aL}{2\epsilon^2} > 0, \tag{5.145}$$

$$\sigma_4 = \frac{b\rho}{2} - a\delta_1 > 0, \tag{5.146}$$

$$\lambda_3 = \min\left(\frac{2\sigma_1}{aM}, \frac{2\sigma_2}{aEI}, \frac{2\sigma_3}{aT}, \frac{2\sigma_4}{a\rho}\right) > 0, \tag{5.147}$$

$$\varepsilon_0 = \left(\frac{a}{\delta_1} + \frac{bL}{\delta_4}\right)\int_0^L f^2(x,t)dx \leq \left(\frac{aL}{\delta_1} + \frac{bL^2}{\delta_4}\right)\bar{f}^2 \in \mathcal{L}_\infty. \tag{5.148}$$

Combining Ineqs. (5.110) and (5.139), we have

$$\dot{V}(t) \quad \leq \quad -\lambda V(t) + \varepsilon_0, \tag{5.149}$$

where $\lambda = \lambda_3/\lambda_2 > 0$. ■

Theorem 5.3. *Consider the closed-loop system consisting of the system dynamics* (5.6) - (5.10)*, boundary control* (5.96)*, under the Assumption 5.1, with the bounded initial conditions, then we conclude that the closed loop Euler-Bernoulli beam system is uniform ultimate bounded.*

Proof: Multiplying Eq. (5.168) by $e^{\lambda t}$ and integrating of the result, we obtain

Multiplying Eq. (5.168) by $e^{\lambda t}$ yields

$$\frac{\partial}{\partial t}(V(t)e^{\lambda t}) \;\; \leq \;\; \varepsilon_0 e^{\lambda t}. \tag{5.150}$$

Integrating of the above inequality, we obtain

$$V(t) \leq \left(V(0) - \frac{\varepsilon_0}{\lambda}\right) e^{-\lambda t} + \frac{\varepsilon_0}{\lambda} \leq V(0)e^{-\lambda t} + \frac{\varepsilon_0}{\lambda} \in \mathcal{L}_\infty, \tag{5.151}$$

which implies $V(t)$ is bounded. Applying Ineq. (2.7), we have

$$\frac{a}{2L}Tw^2(x,t) \leq \frac{a}{2}T\int_0^L [w'(x,t)]^2 dx \leq V_1(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty. \tag{5.152}$$

Appropriately rearranging the terms of the above inequality, we obtain $w(x,t)$ and $v(x,t)$ are uniformly bounded as follows

$$|w(x,t)| \leq \sqrt{\frac{2L}{aT\lambda_1}\left(V(0)e^{-\lambda t} + \frac{\varepsilon_0}{\lambda}\right)} \leq \sqrt{\frac{2L}{aT\lambda_1}\left(V(0) + \frac{\varepsilon_0}{\lambda}\right)}, \tag{5.153}$$

$\forall (x,t) \in [0,L] \times [0,\infty)$. Furthermore, we have

$$\lim_{t\to\infty} |w(x,t)| \leq \sqrt{\frac{2L\varepsilon_0}{aT\lambda_1\lambda}}, \tag{5.154}$$

$\forall (x,t) \in [0,L]$. ■

Remark 5.7. *In the above analysis, it is clear that the steady system state $w(x,t)$ can be made arbitrarily small provided that the design control parameters are appropriately selected. It is easily seen that the increase in the control gain k_1 will result in a larger λ_3. Then the value of λ will increase, which will produce a better vibration suppression performance. However, increasing k_1 will bring a high gain control* *scheme. Therefore, in practical applications, the design parameters should be adjusted carefully for achieving suitable transient performance and control action.*

Remark 5.8. *For the system dynamics described by Eq.* (5.6) *and boundary conditions* (5.9) *to* (5.10)*, using the proposed control Eq.* (5.96)*, then the exponential stability under the condition $f(x,t)=0$ can be achieved as follows:*

$$|w(x,t)| \leq \sqrt{\frac{2L}{aT\lambda_1}V(0)e^{-\lambda t}}, \tag{5.155}$$

$\forall(x,t) \in [0,L]\times[0,\infty)$*. Furthermore, we have*

$$\lim_{t\to\infty}|w(x,t)| = 0. \tag{5.156}$$

For the case that $f(x,t)=0$, the displacement $w(x,t)$ exponentially converges to zero at the rate of convergence λ as $t\to\infty$.

Adaptive boundary control with boundary output constraint

This section presents the Integral-Barrier Lyapunov Function based adaptive control that ensures the vibration is reduced and the constraint is not violated. When the system parameters T and EI are not available, adaptive barrier control is designed to compensate for the system parameter uncertainties as

$$\begin{aligned} u(t) &= -k_1[\dot{w}(L,t)-w'''(L,t)]+k_2[\dot{w}(L,t)+w'''(L,t)]-\frac{w(L,t)}{l_0^2-[w(L,t)]^2} \\ &\quad -\widehat{EI}(t)w'''(L,t)+\widehat{M}(t)\dot{w}'''(L,t)-\operatorname{sgn}[\dot{w}(L,t)-w'''(L,t)]\,\bar{d}. \end{aligned} \tag{5.157}$$

where $\widehat{EI}(t)$ and $\widehat{M}(t)$ are the estimate of the parameters EI and M. The estimate errors are defined as $\widetilde{EI}(t) = EI - \widehat{EI}(t)$ and $\widetilde{M}(t) = M - \widehat{M}(t)$. In order to compensate for the system parameter uncertainties, the adaptive laws are designed as

$$\dot{\widehat{EI}}(t) = a\eta^{-1}w'''(L,t)[\dot{w}(L,t) - w'''(L,t)] - \eta^{-1}\sigma\widehat{EI}(t), \tag{5.158}$$

$$\dot{\widehat{M}}(t) = -a\zeta^{-1}\dot{w}'''(L,t)[\dot{w}(L,t) - w'''(L,t)] - \zeta^{-1}\gamma\widehat{M}(t), \tag{5.159}$$

where γ and σ are two positive constant. Since the parameter T and EI are constants, we have

$$\dot{\widetilde{EI}}(t) = -a\eta^{-1}w'''(L,t)[\dot{w}(L,t) - w'''(L,t)] + \eta^{-1}\sigma\widehat{EI}(t), \tag{5.160}$$

$$\dot{\widetilde{M}}(t) = a\zeta^{-1}\dot{w}'''(L,t)[\dot{w}(L,t) - w'''(L,t)] + \zeta^{-1}\gamma\widehat{M}(t). \tag{5.161}$$

Consider a new Lyapunov functional candidate

$$V_a(t) = V(t) + \frac{\eta}{2}\widetilde{EI}^2(t) + \frac{\zeta}{2}\widetilde{M}^2(t). \tag{5.162}$$

where $V(t)$ is defined in Eq. (5.97). Combining the Eqs. (5.101) and (5.162), we obtain the following lemmas as

Lemma 5.6. *The Lyapunov function equation Eq.* (5.162) *is bounded, given by*

$$\begin{aligned} 0 \le\ & \mu_1\left(V_1(t) + [\dot{w}(L,t) - w'''(L,t)]^2 + \widetilde{T}^2(t) + \widetilde{EI}^2(t)\right) \le V_a(t) \\ \le\ & \mu_2\left(V_1(t) + [\dot{w}(L,t) - w'''(L,t)]^2 + \widetilde{T}^2(t) + \widetilde{EI}^2(t)\right), \end{aligned} \tag{5.163}$$

where $\mu_1 = \min(\lambda_1, \frac{\eta}{2}, \frac{\zeta}{2})$ *and* $\mu_2 = \max(\lambda_1, \frac{\eta}{2}, \frac{\zeta}{2}))$ *are two positive constants.*

Lemma 5.7. *The time derivation of the Lyapunov function equation Eq.* (5.162) *is* *upper bounded, give by*

$$\dot{V}_a(t) \leq -\mu V_a(t) + \varepsilon, \tag{5.164}$$

where $\lambda > 0$ *and* $\varepsilon > 0$.

Proof: Since the control law is only involved in $V_2(t)$, substituting the adaptive barrier control Eq. (5.157) in to Eq. (5.125) yields

$$\begin{aligned}
\dot{V}_2(t) \leq\ & -ak_1[\dot{w}(L,t) - w'''(L,t)]^2 + ak_2[\dot{w}(L,t)]^2 - \left(ak_2 - \frac{aT}{\delta_2} - \frac{a}{2\epsilon^2}\right)[w'''(L,t)]^2 \\
& + a\delta_2 TL \int_0^L [w''(x,t)]^2 dx + \frac{aL}{2\epsilon^2}\int_0^L [w'(x,t)]^2 dx - aTw'(L,t)\dot{w}(L,t) \\
& + a\widetilde{EI}(t)w'''(L,t)[\dot{w}(L,t) - w'''(L,t)] \\
& - a\widetilde{M}(t)\dot{w}'''(L,t)[\dot{w}(L,t) - w'''(L,t)].
\end{aligned} \tag{5.165}$$

Substituting Eqs. (5.123), (5.165) and (5.138) into Eq. (5.162) yields

$$\begin{aligned}
\dot{V}(t) \leq\ & -\lambda_3\left\{V_1(t) + [\dot{w}(L,t) - w'''(L,t)]^2\right\} + a\widetilde{EI}(t)w'''(L,t)[\dot{w}(L,t) - w'''(L,t)] \\
& - a\widetilde{M}(t)\dot{w}'''(L,t)[\dot{w}(L,t) - w'''(L,t)] + \varepsilon_0,
\end{aligned} \tag{5.166}$$

Substituting the above result into the derivative of Eq. (5.162) and using the adaptive laws (5.160) and (5.161), we have

$$\begin{aligned}
\dot{V}_a(t) &= \dot{V}(t) + \eta\widetilde{EI}(t)\dot{\widetilde{EI}}(t) + \zeta\widetilde{M}(t)\dot{\widetilde{M}}(t) \\
&= -\lambda_3\left\{V_1(t) + [\dot{w}(L,t) - w'''(L,t)]^2\right\} + a\widetilde{EI}(t)w'''(L,t)[\dot{w}(L,t) - w'''(L,t)] \\
&\quad -a\widetilde{M}(t)\dot{w}'''(L,t)[\dot{w}(L,t) - w'''(L,t)] + \eta\widetilde{EI}(t)\dot{\widetilde{EI}}(t) + \zeta\widetilde{M}(t)\dot{\widetilde{M}}(t) + \varepsilon_0 \\
&\le -\lambda_3\left(V_1(t) + [\dot{w}(L,t) - w'''(L,t)]^2\right) - \frac{\sigma}{2}\widetilde{EI}^2(t) + \frac{\sigma}{2}EI^2 - \frac{\gamma}{2}\widetilde{M}^2(t) \\
&\quad +\frac{\gamma}{2}M^2 + \varepsilon_0 \\
&\le -\mu_3\Big(V_1(t) + [\dot{w}(L,t) - w'''(L,t)]^2 + \widetilde{EI}^2(t) + \widetilde{M}^2(t)\Big) + \frac{\gamma}{2}M^2 \\
&\quad +\frac{\sigma}{2}EI^2 + \varepsilon_0,
\end{aligned} \tag{5.167}$$

where $\mu_3 = \min(\lambda_3, \frac{\gamma}{2}, \frac{\sigma}{2})$. Combining Ineqs. (5.163) and (5.167), we have

$$\dot{V}_a(t) \le -\mu V_a(t) + \varepsilon, \tag{5.168}$$

where $\mu = \mu_3/\mu_2 > 0$, and $\varepsilon = \frac{\gamma}{2}M^2 + \frac{\sigma}{2}EI^2 + \varepsilon_0$. ■

Theorem 5.4. *Consider the closed-loop system consisting of the system dynamics* (5.6) - (5.10)*, adaptive boundary control* (5.157) *and adaption laws* (5.158), (5.159)*, under the Assumption 5.1, with the bounded initial conditions, then we conclude that the closed loop Euler-Bernoulli beam system is:*

(i) uniformly bounded: the state of the closed system $w(x,t)$ will remain in the compact set Ω_1 defined by

$$\Omega_1 := \{w(x,t) \in R| \ | w(x,t) | \le D_1\}, \tag{5.169}$$

$\forall(x,t) \in [0,L] \times [0,\infty)$*, where the constant* $D_1 = \sqrt{\frac{2L}{aT\mu_1}(V_a(0) + \frac{\varepsilon}{\mu})}$*,*

(ii) uniformly ultimate bounded: the system state $w(x,t)$ will eventually converge to

the compact Ω_2 *defined by*

$$\Omega_2 := \{w(x,t) \in R| \lim_{t\to\infty} \mid w(x,t) \mid \le D_2, \}, \tag{5.170}$$

$\forall x \in [0, L]$, *where the constant* $D_2 = \sqrt{\frac{2L\varepsilon}{aT\mu_1\mu}}$.

Proof: Multiplying Eq. (5.168) by $e^{\mu t}$ yields

$$\frac{\partial}{\partial t}(V_a(t)e^{\mu t}) \le \varepsilon e^{\mu t}. \tag{5.171}$$

Integrating of the above inequality, we obtain

$$V_a(t) \le \left(V_a(0) - \frac{\varepsilon}{\mu}\right) e^{-\mu t} + \frac{\varepsilon}{\mu} \le V_a(0)e^{-\mu t} + \frac{\varepsilon}{\mu} \in \mathcal{L}_\infty, \tag{5.172}$$

which implies $V_a(t)$ is bounded. Utilizing Ineq. (2.7), we have

$$\frac{a}{2L}Tw^2(x,t) \le \frac{a}{2}T\int_0^L [w'(x,t)]^2 dx \le V_1(t) \le \frac{1}{\mu_1}V_a(t) \in \mathcal{L}_\infty. \tag{5.173}$$

Appropriately rearranging the terms of the above inequality, we obtain $w(x,t)$ and $v(x,t)$ are uniformly bounded as follows

$$|w(x,t)| \le \sqrt{\frac{2L}{aT\mu_1}\left(V_a(0)e^{-\mu t} + \frac{\varepsilon}{\mu}\right)} \le \sqrt{\frac{2L}{aT\mu_1}\left(V_a(0) + \frac{\varepsilon}{\mu}\right)}, \tag{5.174}$$

$\forall (x,t) \in [0, L] \times [0, \infty)$. Furthermore, we have

$$\lim_{t\to\infty} |w(x,t)| \le \sqrt{\frac{2L\varepsilon}{aT\mu_1\mu}}, \tag{5.175}$$

$\forall (x,t) \in [0, L]$. ■

5.3 Numerical Simulations

Consider a beam excited by the distributed disturbance $f(x,t)$ and boundary disturbance $d(t)$. The disturbance $d(t)$ on the tip payload is described as

$$d(t) = 1 + 0.1\sin(0.1t) + 0.3\sin(0.3t) + 0.5\sin(0.5t). \tag{5.176}$$

The distributed disturbance $f(x,t)$ along the beam is described as

$$f(x,t) = [3 + \sin(\pi xt) + \sin(2\pi xt) + \sin(3\pi xt)]x. \tag{5.177}$$

The initial conditions are given as

$$\begin{aligned} w(x,0) &= x, \\ \dot{w}(x,0) &= 0. \end{aligned}$$

Detailed parameters of the Euler-Bernoulli beam are referred to [34], and listed in the following table.

Table 1: Parameters of the beam

Parameter	Description	Value
L	Length of beam	$1m$
EI	Bending stiffness of the beam	$2Nm^2$
T	Tension	$10N$
ρ	Mass per unit length of the beam	$0.1kg/m$
M	Mass of the tip payload	$0.5kg$

Fig. 5.3 depicts displacement of the beam under the disturbances $f(x,t)$ and $d(t)$ without control input, i.e. $u(t) = 0$, where it is shown that there is large vibration due to the external disturbances. By using the proposed control Eq. (5.15), displacement of the beam with the proposed robust boundary control is shown in Fig. 5.4. The design parameters are chosen as $k = 15$, $\zeta_d = 0.01$ and $\gamma = 10$. Fig. 5.5 shows displacement of the beam with the proposed adaptive boundary control (5.74) when there are system parametric uncertainties and disturbances uncertainties, and the design parameters are selected as $k = 50$, $\zeta_\Phi = \zeta_d = 0.01$, and $\gamma = 2$, $\Gamma = \mathrm{diag}\{2, 1, 1\}$. The robust boundary control input (5.15) and the adaptive boundary control input (5.74) are displayed in Fig. 5.6.

Figs. 5.7 display the displacement of the beam with the proposed model-based barrier control (5.96). With the proposed control (5.96), the vibration of the beam can be suppressed greatly within 10 secs, by selecting $a = 0.79, k_1 = 15, k_2 = 10$. For comparison, displacements of the beam with the following boundary control

$$\begin{aligned} u^*(t) &= -k_1[\dot{w}(L,t) - w'''(L,t)] + k_2[\dot{w}(L,t) + w'''(L,t)] - k_p w(L,t) \\ &\quad -EIw'''(L,t) + M\dot{w}'''(L,t) - \mathrm{sgn}\,[\dot{w}(L,t) - w'''(L,t)]\,\bar{d}. \end{aligned} \tag{5.178}$$

is shown in Fig. 5.8. The design parameters are chosen as $a = 0.79, k_1 = 15, k_2 = 10, k_p = 5$. Compared with the proposed model-based barrier control (5.96), the barrier term $\frac{1}{l_0^2 - w^2(L,t)}$ is removed in (5.178). Figs. 5.7 and 5.8 illustrate that both the model-based barrier barrier control (5.96) and the boundary control (5.178) are able to stabilize the beam at the small neighborhood of its equilibrium position. However, as shown in Fig. 5.9, the proposed model-based barrier barrier control (5.96) ensures that the beam's end point position $|\ w(L,t)\ | \leq 0.05$, while boundary control (5.178) cannot guarantee $|\ w(L,t)\ | \leq l_0$. The inputs of the model-based barrier control (5.96) and the boundary control (5.178) are shown in Fig. 5.10.

Displacement of the beam with the proposed adaptive barrier control (5.157) is demonstrated in Fig. 5.11, by choosing $k_1 = 15, k_2 = 10, a = 0.99$. End point position of the beam $w(L,t)$ is draw in Fig. 5.12. It can be seen that the vibration suppression is well achieved without the violation of the boundary constraint. The adaptive barrier control input is shown in Fig. 5.13.

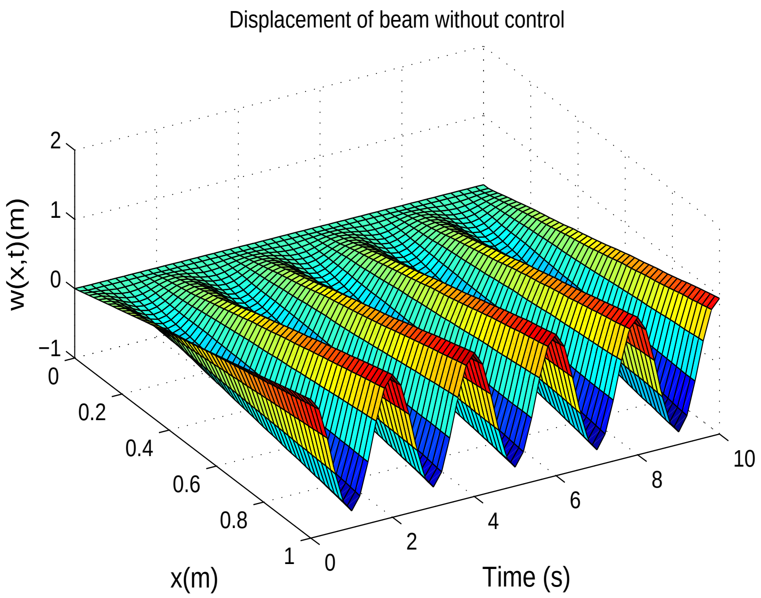

Fig. 5.3: Displacement of the Euler-Bernoulli beam without control.

5.4 Conclusion

In this chapter, three cases for the vibrating beam system under unknown spatiotemporally varying distributed disturbance $f(x,t)$ and unknown time-varying boundary disturbance $d(t)$ have been investigated: (i) robust boundary control for disturbance

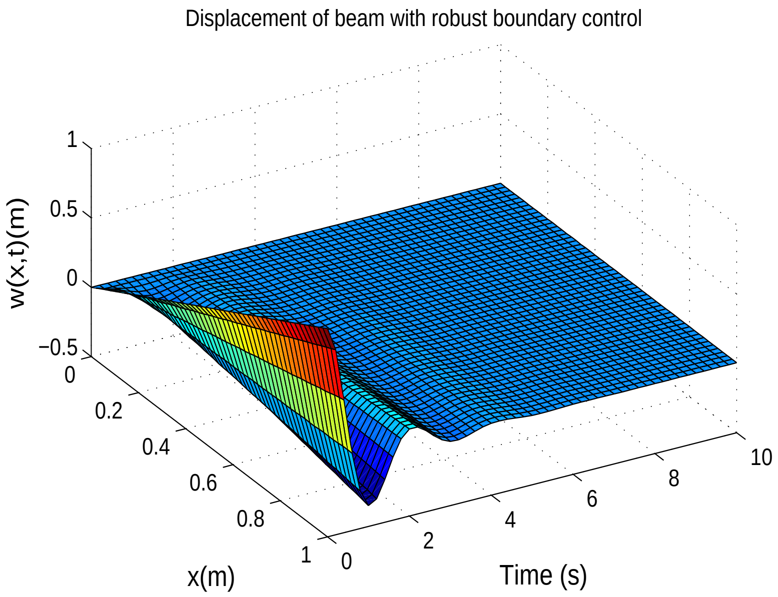

Fig. 5.4: Displacement of the Euler-Bernoulli beam with robust boundary control (5.15).

uncertainties, and (ii) adaptive boundary control for both the system parametric uncertainties and disturbance uncertainties. and (iii) Integral-Barrier Lyapunov Function based control for boundary output constraint.

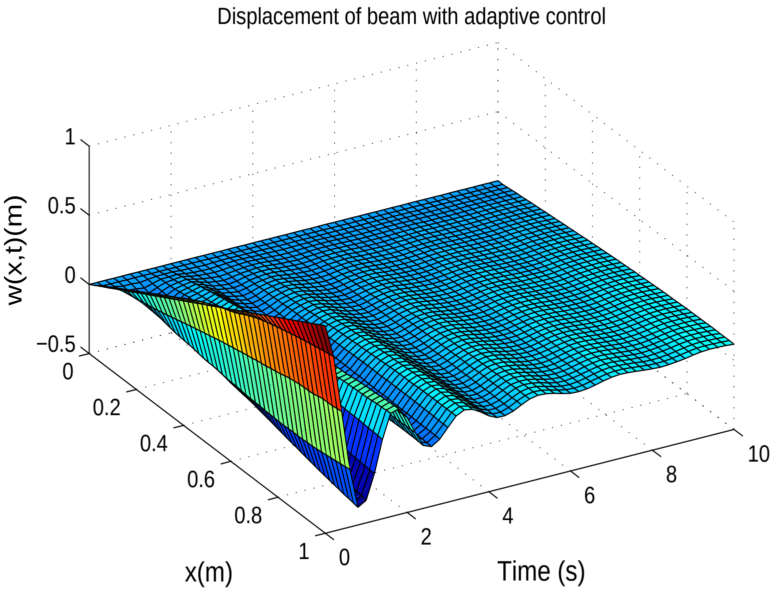

Fig. 5.5: Displacement of the Euler-Bernoulli beam with adaptive boundary control (5.74).

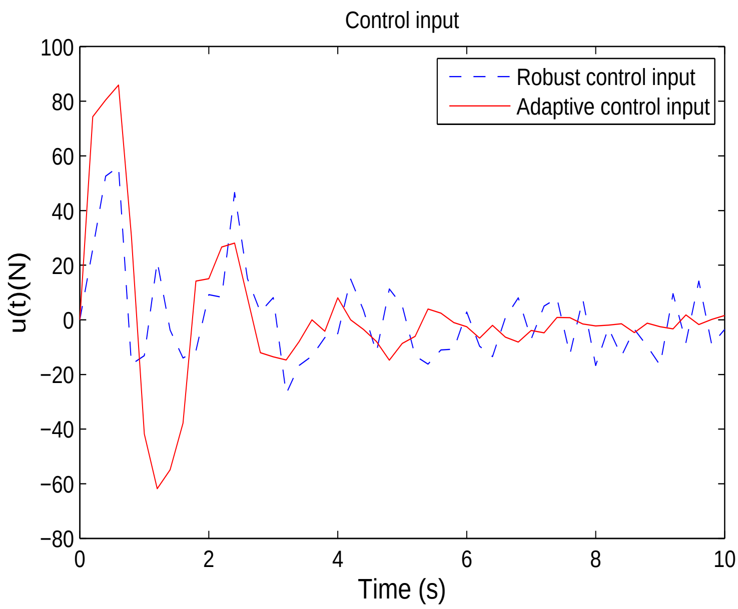

Fig. 5.6: Control inputs (5.15) and (5.74).

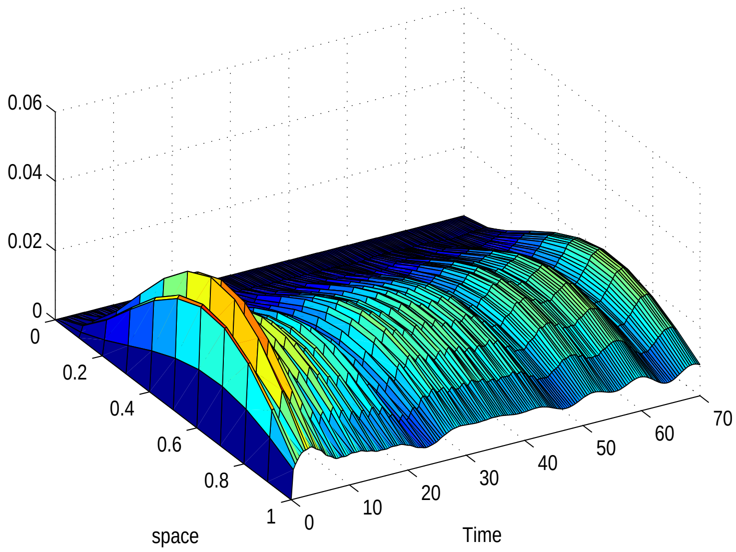

Fig. 5.7: Displacement of the Euler-Bernoulli beam with model-based barrier control (5.96).

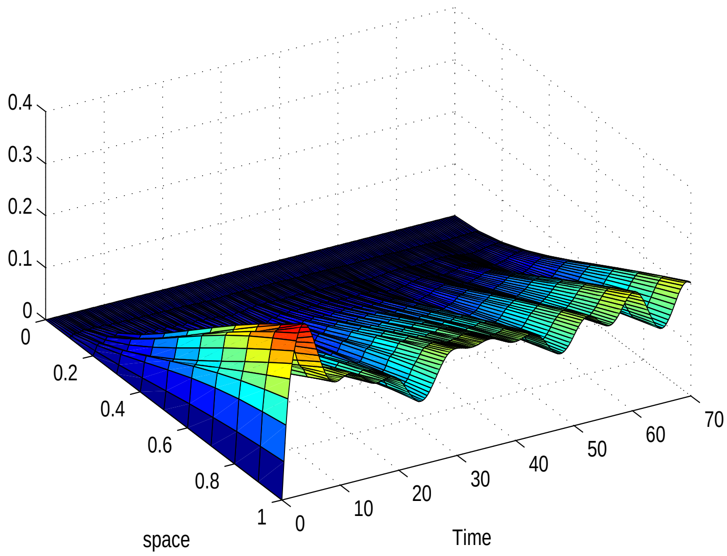

Fig. 5.8: Displacement of the Euler-Bernoulli beam with boundary control (5.178).

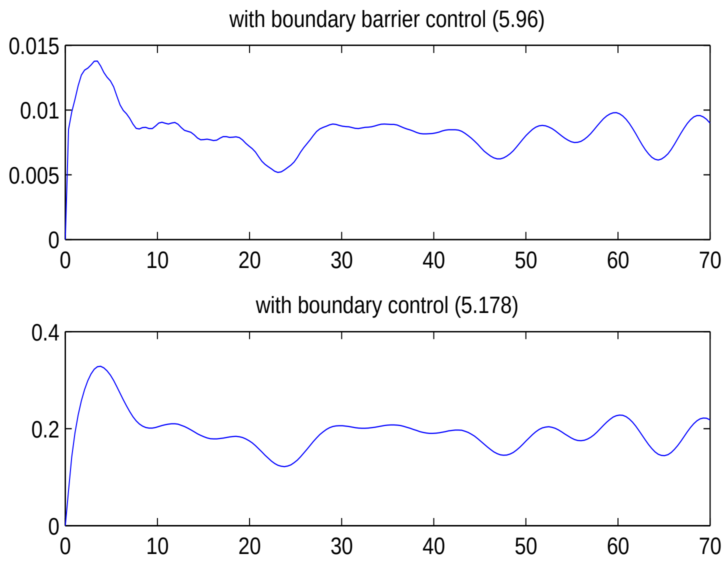

Fig. 5.9: End point position of the Euler-Bernoulli beam with model-based barrier control (5.96) and boundary control (5.178).

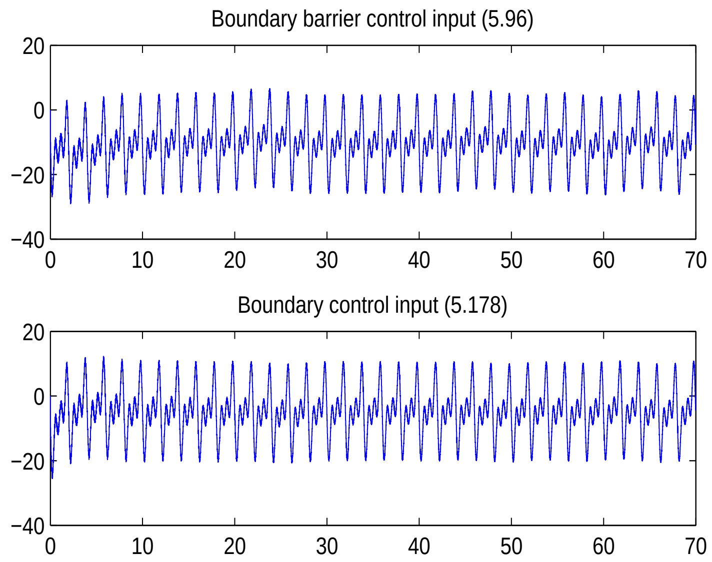

Fig. 5.10: Control inputs (5.96) and (5.178).

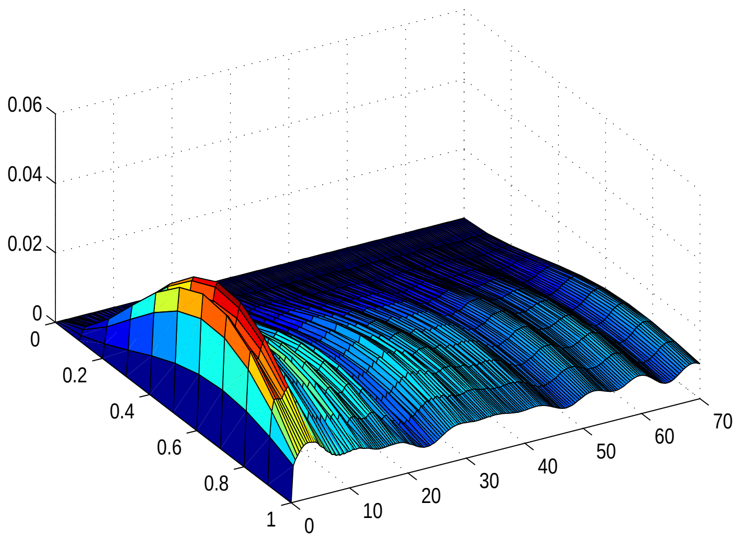

Fig. 5.11: Displacement of the Euler-Bernoulli beam with adaptive barrier control (5.157).

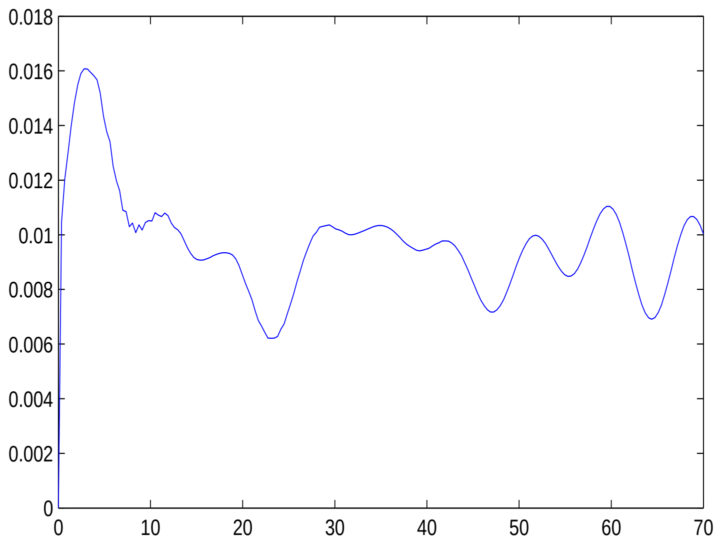

Fig. 5.12: End point position of the Euler-Bernoulli beam with adaptive barrier control (5.157).

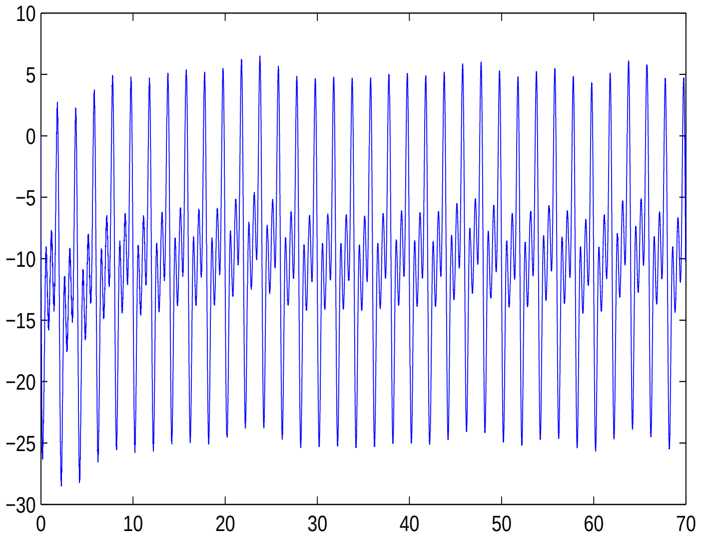

Fig. 5.13: Adaptive barrier control input (5.157).

Chapter 6

Boundary Output-Feedback Stabilization of a Timoshenko Beam Using Disturbance Observer

Timoshenko beam which is an improvement of the Euler-Bernoulli beam system was proposed by Stephen Timoshenko in the beginning of the 20th century. The shearing and rotational inertia of cross-sections effects are included in the Timoshenko beam model. Therefore, the dynamic model of the Timoshenko beam is appropriate for characterizing the behavior of relative short beams or sandwich composite beams. While the Timoshenko beam may be superior to the Euler-Bernoulli beam in predicting the beam response [136], the Timoshenko beam is more difficult to utilize for control design due to its higher order [48]. Thus, the vibration control problem of the Timoshenko beam is important and challenging.

In the literatures of boundary control for the distributed parameter systems, disturbance observers [137] are usually used to handle the unknown boundary disturbances. A disturbance observer is designed in [92] for the an axially moving string with the unknown boundary disturbance. Lyapunov method and Semigroup theory are utilized to prove the stability of the closed-loop system. In [138], the magnitude of unknown boundary disturbance of the axially translating beam is estimated via the disturbance observer. In the above two papers, the bounds of the boundary disturbances are assumed to be uniformly bounded and the observers are used to estimate the values of the bounds. In this research, the conditions for bounds of the disturbance are not required for the proposed disturbance observer.

In this chapter, we study the boundary control problem for the Timoshenko beam system with the unknown boundary disturbances and unknown spatiotemporally varying disturbance. Compared to the existing work, the main contributions of this chapter include:

(i) A Timoshenko beam model under both boundary disturbance and spatiotemporally varying disturbance for vibration suppression is derived based on the Hamilton's principle. The governing equations of the system which can be used for the dynamic analysis of the Timoshenko beam-like structures are described as nonhomogeneous PDEs with the unknown disturbance terms.

(ii) Boundary control is designed for the Timoshenko beam system subjected to the external disturbances. Disturbance observer are designed to estimate the unknown boundary disturbances.

(iii) A new theorem is proposed to illustrate that the Timoshenko beam system is proved to be uniform ultimate bounded with the proposed boundary control.

The rest of this chapter is organized as follows. In Section 6.1, the governing equations and boundary conditions of a Timoshenko beam system are derived by the use of Hamilton's principle. Boundary disturbance observers combined with boundary control are designed in Section 6.2, and the uniform ultimate boundedness of the closed-loop system with the proposed control is also shown. Simulation results are displayed in Section 6.3 to verify the performance of the proposed boundary feedback control. In Section 6.4, conclusions are presented.

6.1 Problem Formulation

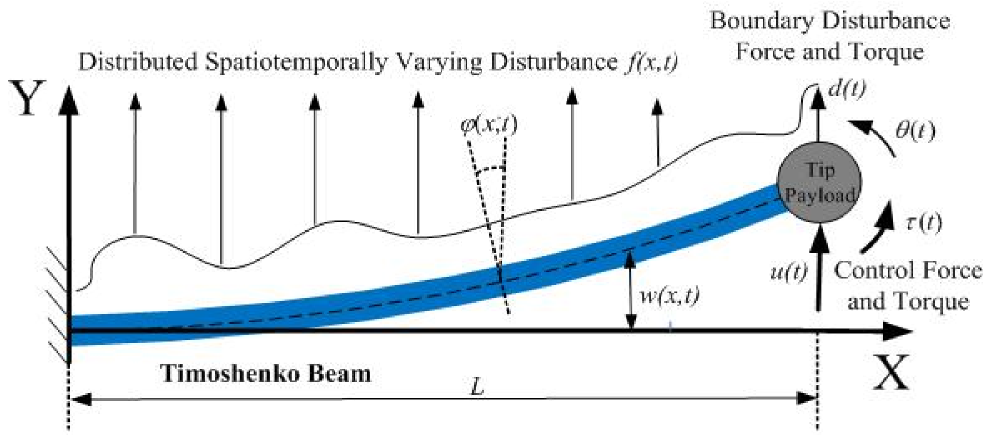

Fig. 6.1: A typical Timoshenko beam system with tip payload.

Fig. 6.1 shows a Timoshenko beam model. The left boundary of the Timoshenko beam is fixed at origin. $w(x,t)$ is the displacement of the Timoshenko beam at the position x for time t, $\varphi(x,t)$ is the rotation of the Timoshenko beam's cross-section owing to bending at the position x for time t.

The kinetic energy of the Timoshenko beam $E_k(t)$ can be represented as

$$E_k(t) = \frac{1}{2}M\dot{w}^2(L,t) + \frac{1}{2}\int_0^L \left[\rho\dot{w}^2(x,t) + I_\rho\dot{\varphi}^2(x,t)\right]dx + \frac{1}{2}J\dot{\varphi}^2(L,t), \quad (6.1)$$

where L is the length of the Timoshenko beam, M is the mass of the payload, ρ is the uniform mass per unit length of the Timoshenko beam, I_ρ is the uniform mass moment of inertia of the Timoshenko beam's cross-section, and J denotes inertia of the payload.

The potential energy $E_p(t)$ due to the bending can be obtained from

$$E_p(t) = \frac{1}{2}EI\int_0^L \left[\varphi'(x,t)\right]^2 dx + \frac{1}{2}K\int_0^L \left[\varphi(x,t) - w'(x,t)\right]^2 dx, \quad (6.2)$$

where EI is the bending stiffness of the Timoshenko beam, $K = k_0GA$ with k_0 is a positive constant that depends on the shape of the Timoshenko beam's cross-section, G is the modulus of elasticity in shear, and A is the cross-sectional area of the Timoshenko beam.

The virtual work done by disturbances including the unknown distributed disturbance $f(x,t)$ along the Timoshenko beam and the boundary disturbances $d(t)$, $\theta(t)$ on the tip payload is described by

$$\delta W_d(t) = \int_0^L f(x,t)\delta w(x,t)dx + d(t)\delta w(L,t) + \theta(t)\delta\varphi(L,t). \quad (6.3)$$

The boundary control force $u(t)$ and boundary control input torque $\tau(t)$ at the boundary of the Timoshenko beam produce the transverse force and torque for vibration suppression. The virtual work done by $u(t)$ and $\tau(t)$ can be written as

$$\delta W_f(t) \quad = \quad u(t)\delta w(L,t) + \tau(t)\delta\varphi(L,t).s \tag{6.4}$$

Then, the total virtual work done to the system can be obtained as

$$\delta W(t) \quad = \quad \delta W_d(t) + \delta W_f(t). \tag{6.5}$$

Using the Hamilton's principle Eq. (2.1), the governing equations of the system are derived as

$$\rho\ddot{w}(x,t) + K[\varphi'(x,t) - w''(x,t)] = f(x,t), \tag{6.6}$$

$$I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) + K[\varphi(x,t) - w'(x,t)] = 0, \tag{6.7}$$

$\forall(x,t) \in (0,L) \times [0,\infty)$, and the boundary conditions of the system are given as

$$w(0,t) = \varphi(0,t) \quad = \quad 0, \tag{6.8}$$

$$M\ddot{w}(L,t) - K[\varphi(L,t) - w'(L,t)] \quad = \quad u(t) + d(t), \tag{6.9}$$

$$J\ddot{\varphi}(L,t) - EI\varphi'(L,t) \quad = \quad \tau(t) + \theta(t), \tag{6.10}$$

$\forall t \in [0,\infty)$.

Remark 6.1. *With consideration of unknown distributed disturbance $f(x,t)$, the governing equations of the Timoshenko beam system are described as a combination of a nonhomogeneous PDE* (6.6) *and a homogeneous PDE* (6.7). *Since the existence of the nonhomogeneous PDE, the model in our paper differs from the Timoshenko beam system governed by a homogeneous PDE in [48, 116]. As a consequence, the control* *schemes in these papers are not suitable for our system. In this paper, we design the boundary control based on the original PDEs of the Timoshenko beam system.*

Assumption 6.1. *We assume that the distributed disturbances $f(x,t)$ is uniformly bounded, i.e., there exists constant $\bar{f} \in \mathbb{R}^+$, such that $|f(x,t)| \leq \bar{f}$, $\forall(x,t) \in [0,L] \times [0,\infty)$.*

Assumption 6.2. *For the boundary disturbances $d(t)$, $\theta(t)$, we assume that their derivatives $\dot{d}(t)$, $\dot{\theta}(t)$ are uniformly bounded, i.e., there exists constants $D \in \mathbb{R}^+$ and $\Theta \in \mathbb{R}^+$, such that $|\dot{d}(t)| \leq D$, $|\dot{\theta}(t)| \leq \Theta$, $\forall(t) \in [0,\infty)$.*

Remark 6.2. *In this chapter, the knowledge of the exact values for $f(x,t)$, $d(t)$, $\theta(t)$ is not required in the control design, which possesses stability to variations of the unknown disturbances.*

Property 6.1. *[48]:If the kinetic energy of the system* (6.6) - (6.10)*, given by Eq.* (6.1) *is bounded* $\forall(x,t) \in [0,L] \times [0,\infty)$*, then $\dot{w}(x,t)$, $\dot{w}'(x,t)$, $\dot{\varphi}(x,t)$ and $\dot{\varphi}'(x,t)$ are bounded* $\forall(x,t) \in [0,L] \times [0,\infty)$.

Property 6.2. *[48]: If the potential energy of the system* (6.6) - (6.10)*, given by Eq.* (6.2) *is bounded* $\forall(x,t) \in [0,L] \times [0,\infty)$*, then $w'(x,t)$, $w''(x,t)$, $\varphi'(x,t)$ and $\varphi''(x,t)$ are bounded* $\forall(x,t) \in [0,L] \times [0,\infty)$.

6.2 Control Design

The control objective is to suppress the vibration of the Timoshenko beam governed by a combination of a nonhomogeneous PDE (6.6) and a homogeneous PDE (6.7) under the unknown disturbances $f(x,t)$, $d(t)$ and $\theta(t)$. Boundary control $u(t)$, $\tau(t)$

are proposed at the right boundary of the flexible Timoshenko beam as

$$
\begin{aligned}
u(t) &= -M\dot{w}'(L,t) + M\dot{\varphi}(L,t) - K\varphi(L,t) + Kw'(L,t) - k_1\eta_1(t) \\
&\quad -\hat{d}(t), && (6.11)\\
\tau(t) &= -J\dot{\varphi}'(L,t) + EI\varphi'(L,t) - k_2\eta_2(t) - \hat{\theta}(t), && (6.12)
\end{aligned}
$$

where k_1 and k_2 are two positive control gains and the auxiliary signals $\eta_1(t)$ and $\eta_2(t)$ are defined as

$$
\begin{aligned}
\eta_1(t) &= \dot{w}(L,t) + w'(L,t) - \varphi(L,t), && (6.13)\\
\eta_2(t) &= \dot{\varphi}(L,t) - \varphi'(L,t). && (6.14)
\end{aligned}
$$

The boundary disturbance observers $\hat{d}(t)$ and $\hat{\theta}(t)$ that estimate of $d(t)$ and $\theta(t)$ respectively, are designed as

$$
\begin{aligned}
\hat{d}(t) &= \xi_d(t) + \xi_1 M\dot{w}(L,t), && (6.15)\\
\hat{\theta}(t) &= \xi_\theta(t) + \xi_2 J\dot{\varphi}(L,t), && (6.16)
\end{aligned}
$$

where ξ_1 and ξ_2 are two positive estimate gains, and $\xi_d(t)$, $\xi_\theta(t)$ are defined as

$$
\begin{aligned}
\dot{\xi}_d(t) &= \eta_1(t) - \xi_1\left\{\xi_d(t) + K[\varphi(L,t) - w'(L,t)] + u(t) + \xi_1 M\dot{w}(L,t)\right\}, && (6.17)\\
\dot{\xi}_\theta(t) &= \eta_2(t) - \xi_2\left\{\xi_\theta(t) + EI\varphi'(L,t) + \tau(t) + \xi_2 J\dot{\varphi}(L,t)\right\}. && (6.18)
\end{aligned}
$$

Differentiating $\hat{d}(t)$ and $\hat{\theta}(t)$ in Eqs. (6.15) and (6.16) yields

$$\begin{aligned}\dot{\hat{d}}(t) &= \eta_1(t) - \xi_1\left\{\xi_d(t) + K[\varphi(L,t) - w'(L,t)] + u(t) + \xi_1 M\dot{w}(L,t)\right\} \\ &\quad + \xi_1 M\ddot{w}(L,t), \qquad (6.19)\\ \dot{\hat{\theta}}(t) &= \eta_2(t) - \xi_2\left\{\xi_\theta(t) + EI\varphi'(L,t) + \tau(t) + \xi_2 J\dot{\varphi}(L,t)\right\} \\ &\quad + \xi_2 J\ddot{\varphi}(L,t). \qquad (6.20)\end{aligned}$$

Substituting the boundary conditions Eqs. (6.9) and (6.10) into Eqs. (6.19) and (6.20) respectively, we obtain

$$\dot{\hat{d}}(t) = \eta_1(t) + \xi_1\tilde{d}(t), \qquad (6.21)$$

$$\dot{\hat{\theta}}(t) = \eta_2(t) + \xi_2\tilde{\theta}(t), \qquad (6.22)$$

where the boundary disturbance estimate errors are defined as $\tilde{d}(t) = d(t) - \hat{d}(t)$, and $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$. Differentiate $\tilde{d}(t)$ and $\tilde{\theta}(t)$ respectively and using Eqs. (6.21) and (6.22) yields

$$\dot{\tilde{d}}(t) = \dot{d}(t) - \xi_1\tilde{d}(t) - \eta_1(t), \qquad (6.23)$$

$$\dot{\tilde{\theta}}_2(t) = \dot{\theta}(t) - \xi_2\tilde{\theta}(t) - \eta_2(t). \qquad (6.24)$$

6.3 Stability Analysis

In this section, Lyapunov's direct method is used to analyze the closed-loop stability of the system. Consider the Lyapunov functional candidate

$$V(t) = V_1(t) + V_2(t) + \Delta(t) + \frac{1}{2}\tilde{d}^2(t) + \frac{1}{2}\tilde{\theta}^2(t), \tag{6.25}$$

where the energy term $V_1(t)$ and an auxiliary term $V_2(t)$ and a small crossing term $\Delta(t)$ are defined as

$$\begin{aligned} V_1(t) &= \frac{1}{2}\rho\int_0^L [\dot{w}(x,t)]^2 dx + \frac{1}{2}I_\rho\int_0^L [\dot{\varphi}(x,t)]^2 dx + \frac{1}{2}EI\int_0^L [\varphi'(x,t)]^2 dx \\ &\quad + \frac{1}{2}K\int_0^L [\varphi(x,t) - w'(x,t)]^2 dx, \end{aligned} \tag{6.26}$$

$$V_2(t) = \frac{1}{2}M\eta_1^2(t) + \frac{1}{2}J\eta_2^2(t), \tag{6.27}$$

$$\Delta(t) = \Delta_1(t) + \Delta_2(t) + \Delta_3(t), \tag{6.28}$$

where

$$\Delta_1(t) = 2\alpha\rho\int_0^L x\dot{w}(x,t)w'(x,t)dx, \tag{6.29}$$

$$\Delta_2(t) = 2\alpha I_\rho\int_0^L x\dot{\varphi}(x,t)\varphi'(x,t)dx, \tag{6.30}$$

$$\Delta_3(t) = \mu\alpha I_\rho\int_0^L \dot{\varphi}(x,t)\varphi(x,t)dx. \tag{6.31}$$

α and μ are positive weighting constants.

Lemma 6.1. *The Lyapunov function equation Eq.* (6.25) *is bounded, given by*

$$0 \leq \lambda_1[V_0(t) + V_2(t) + \tilde{d}^2(t) + \tilde{\theta}^2(t)] \leq V(t)$$
$$\leq \lambda_2[V_0(t) + V_2(t) + \tilde{d}^2(t) + \tilde{\theta}^2(t)], \tag{6.32}$$

where λ_1 *and* λ_2 *are two positive constants and*

$$V_0(t) = \int_0^L ([\dot{w}(x,t)]^2 + [\dot{\varphi}(x,t)]^2 + [\varphi'(x,t)]^2 + [\varphi(x,t)]^2 + [w'(x,t)]^2)dx. \tag{6.33}$$

Proof: From Eq. (6.26), we obtain the upper bound of $V_1(t)$ as

$$V_1(t) \leq \frac{1}{2}\int_0^L (\rho[\dot{w}(x,t)]^2 + I_\rho[\dot{\varphi}(x,t)]^2 + EI[\varphi'(x,t)]^2 + 2K[\varphi(x,t)]^2 + 2K[w'(x,t)]^2)dx$$
$$\leq \frac{1}{2}\max\{\rho, I_\rho, EI, 2K\}V_0(t). \tag{6.34}$$

Applying Ineq. (2.6), we have the lower bound of $V_1(t)$ as follows

$$V_1(t) \geq \frac{1}{2}\int_0^L (\rho[\dot{w}(x,t)]^2 + I_\rho[\dot{\varphi}(x,t)]^2 + \frac{EI}{2}[\varphi'(x,t)]^2 + [\varphi(x,t)\ w'(x,t)]\,A\,[\varphi(x,t)\ w'(x,t)]^T)dx, \tag{6.35}$$

where A is a positive-definite matrix defined as

$$A = \begin{bmatrix} K + \frac{EI}{2L^2} & -K \\ -K & K \end{bmatrix}.$$

From Ineqs. (6.34) and (6.35), we have

$$\frac{1}{2}\min\{\rho, I_\rho, \frac{EI}{2}, \lambda_{\min}(A)\}V_0(t) \le V_1(t) \le \frac{1}{2}\max\{\rho, I_\rho, EI, 2K\}V_0(t). \quad (6.36)$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a matrix. We further have

$$\begin{aligned} |\Delta_1(t)| &\le 2\alpha\rho L\int_0^L([\dot{w}(x,t)]^2 + [w'(x,t)]^2)dx \\ &\le 2\alpha\rho L V_0(t). \end{aligned} \quad (6.37)$$

Then, we obtain

$$-2\alpha\rho L V_0(t) \le \Delta_1(t) \le 2\alpha\rho L V_0(t). \quad (6.38)$$

With the similar process, we have

$$-2\alpha I_\rho L V_0(t) \le \Delta_2(t) \le 2\alpha I_\rho L V_0(t), \quad (6.39)$$

$$-\mu\alpha\rho V_0(t) \le \Delta_3(t) \le \mu\alpha\rho V_0(t). \quad (6.40)$$

Substituting Ineqs.(6.38) - (6.40) into Eq. (6.28), we have

$$-[2\alpha L(\rho + I_\rho) + \mu\alpha\rho]V_0(t) \le \Delta(t) \le [2\alpha L(\rho + I_\rho) + \mu\alpha\rho]V_0(t). \quad (6.41)$$

If α is selected as

$$\alpha < \frac{\frac{1}{2}\min\{\rho, I_\rho, \frac{EI}{2}, \lambda_{\min}(A)\}}{2L(\rho + I_\rho) + \mu\rho}, \quad (6.42)$$

and then substituting Ineqs. (6.36), (6.27) and (6.41) into Eq. (6.25), we obtain

$$0 \leq \min(\gamma_1, 1)(V_0(t) + V_2(t)) \leq V_1(t) + V_2(t) + \Delta(t)$$
$$\leq \max(\gamma_2, 1)(V_0(t) + V_2(t)), \tag{6.43}$$

where

$$\gamma_1 = \frac{1}{2}\min\{\rho, I_\rho, \frac{EI}{2}, \lambda_{\min}\{A\}\} - (2\alpha L(\rho + I_\rho) + \mu\alpha\rho), \tag{6.44}$$
$$\gamma_2 = \frac{1}{2}\max\{\rho, I_\rho, EI, 2K\} + (2\alpha L(\rho + I_\rho) + \mu\alpha\rho). \tag{6.45}$$

Then we have

$$0 \leq \lambda_1[V_0(t) + V_2(t) + \tilde{d}^2(t) + \tilde{\theta}^2(t)] \leq V(t)$$
$$\leq \lambda_2[V_0(t) + V_2(t) + \tilde{d}^2(t) + \tilde{\theta}^2(t)], \tag{6.46}$$

where $\lambda_1 = \min(\gamma_1, \frac{1}{2})$ and $\lambda_2 = \max(\gamma_2, 1)$.

Lemma 6.2. *The time derivation of the Lyapunov function equation Eq.* (6.25) *Eq.* (6.25) *is upper bounded, give by*

$$\dot{V}(t) \leq -\lambda V(t) + \varepsilon, \tag{6.47}$$

where $\lambda > 0$ *and* $\varepsilon > 0$*.*

Proof: Differentiating Eq. (6.25) with respect to time leads to

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{\Delta}(t) + \tilde{d}(t)\dot{\tilde{d}}(t) + \tilde{\theta}(t)\dot{\tilde{\theta}}(t). \tag{6.48}$$

The first term of the Eq. (6.48) is rewritten as

$$\begin{aligned}\dot{V}_1(t) &= \rho\int_0^L \dot{w}(x,t)\ddot{w}(x,t)dx + I_\rho\int_0^L \dot{\varphi}(x,t)\ddot{\varphi}(x,t)dx \\ &+K\int_0^L \left[\varphi(x,t)-w'(x,t)\right]\left[\dot{\varphi}(x,t)-\dot{w}'(x,t)\right]dx \\ &+EI\int_0^L \dot{\varphi}(x,t)\dot{\varphi}'(x,t)dx. \end{aligned} \tag{6.49}$$

Substituting the governing equations of the system Eqs. (6.6) and (6.7) into Eq. (6.49), we obtain

$$\dot{V}_1(t) = A_1(t) + A_2(t) + A_3(t), \tag{6.50}$$

where

$$A_1(t) = \int_0^L f(x,t)\dot{w}(x,t)dx, \tag{6.51}$$

$$A_2(t) = EI\int_0^L \left[\dot{\varphi}(x,t)\varphi''(x,t) + \dot{\varphi}'(x,t)\varphi'(x,t)\right]dx, \tag{6.52}$$

$$\begin{aligned}A_3(t) &= K\int_0^L \left[\dot{w}(x,t)w''(x,t) - \dot{w}(x,t)\varphi'(x,t) - \varphi(x,t)\dot{w}'(x,t)\right. \\ &\left.+w'(x,t)\dot{w}'(x,t)\right]dx. \end{aligned} \tag{6.53}$$

Using the boundary conditions and integrating Eq. (6.52) by parts, we obtain

$$\begin{aligned}A_2(t) &= EI\int_0^L \dot{\varphi}(x,t)\varphi''(x,t)dx + EI\int_0^L \varphi'(x,t)d\dot{\varphi}(x,t) \\ &= EI\varphi'(L,t)\dot{\varphi}(L,t). \end{aligned} \tag{6.54}$$

Similarly, we have

$$A_3(t) = Kw'(L,t)\dot{w}(L,t) - K\dot{w}(L,t)\varphi(L,t). \tag{6.55}$$

Substituting Eqs. (6.54) and (6.55) into Eq. (6.50), we have

$$\begin{aligned}\dot{V}_1(t) &= EI\varphi'(L,t)\dot{\varphi}(L,t) + K\dot{w}(L,t)\left[w'(L,t) - \varphi(L,t)\right] \\ &\quad + \int_0^L f(x,t)\dot{w}(x,t)dx. \end{aligned}\tag{6.56}$$

Substituting Eqs. (6.13) and (6.14) into Eq. (6.56), and using Ineq. (14) in [28] to the $\int_0^L f(x,t)\dot{w}(x,t)dx$ term, we obtain

$$\begin{aligned}\dot{V}_1(t) &\leq -\frac{K}{2}[\dot{w}(L,t)]^2 - \frac{K}{2}\left[w'(L,t) - \varphi(L,t)\right]^2 + \frac{K}{2}\eta_1^2(t) - \frac{EI}{2}[\dot{\varphi}(L,t)]^2 \\ &\quad -\frac{EI}{2}[\varphi'(L,t)]^2 + \frac{EI}{2}\eta_2^2(t) + \delta_1\int_0^L [\dot{w}(x,t)]^2dx + \frac{1}{\delta_1}\int_0^L f^2(x,t)dx \end{aligned}\tag{6.57}$$

where δ_1 is a positive constant.

The derivative of the Eq. (6.27) is given as

$$\dot{V}_2(t) = M\eta_1(t)\dot{\eta}_1(t) + J\eta_2(t)\dot{\eta}_2(t). \tag{6.58}$$

After differentiating the auxiliary signal Eqs. (6.13) and (6.14), multiplying the resulting equation by M and J respectively, and substituting Eqs. (6.9) and (6.10), we obtain

$$M\dot{\eta}_1(t) = M\dot{w}'(L,t) - M\dot{\varphi}(L,t) + K[\varphi(L,t) - w'(L,t)] + d(t) + u(t), \tag{6.59}$$

$$J\dot{\eta}_2(t) = J\dot{\varphi}'(L,t) - EI\varphi'(L,t) + \tau(t) + \theta(t). \tag{6.60}$$

Substituting the propose control Eq. (6.11) and Eq. (6.12) into Eq. (6.59) and Eq. (6.60) respectively, we have

$$M\dot{\eta}_1(t) = -k_1\eta_1 + \tilde{d}(t), \tag{6.61}$$

$$J\dot{\eta}_2(t) = -k_2\eta_2 + \tilde{\theta}(t). \tag{6.62}$$

Substituting the Eqs. (6.61) and (6.62) into Eq. (6.58), we have

$$\dot{V}_2(t) = -k_1\eta_1^2(t) + \tilde{d}(t)\eta_1(t) - k_2\eta_2^2(t) + \tilde{\theta}(t)\eta_2(t). \tag{6.63}$$

The third term of the Eq. (6.48) is given as

$$\dot{\Delta}(t) = \dot{\Delta}_1(t) + \dot{\Delta}_2(t) + \dot{\Delta}_3(t). \tag{6.64}$$

After integrating by parts of the first term of Eq. (6.64) and substituting Eq. (6.6), we have

$$\dot{\Delta}_1(t) = B_1(t) + B_2(t) + B_3(t) + B_4(t), \tag{6.65}$$

where

$$B_1(t) = 2\alpha \int_0^L xw'(x,t)f(x,t)dx, \tag{6.66}$$

$$B_2(t) = 2\alpha K \int_0^L xw'(x,t)w''(x,t)dx, \tag{6.67}$$

$$B_3(t) = -2\alpha K \int_0^L xw'(x,t)\varphi'(x,t)dx, \tag{6.68}$$

$$B_4(t) = 2\alpha\rho \int_0^L x\dot{w}(x,t)\dot{w}'(x,t)dx. \tag{6.69}$$

Using Ineq. (14) in [28] to $B_1(t)$ term, we have

$$B_1(t) \leq 2L \int_0^L \left(\delta_2 [w'(x,t)]^2 + \frac{1}{\delta_2} f^2(x,t) \right) dx, \tag{6.70}$$

where δ_2 is a positive constant.

Integrating $B_2(t)$ by parts and using the boundary conditions, we obtain

$$\begin{aligned} B_2(t) &= \alpha K \int_0^L xd\left([w'(x,t)]^2\right) \\ &= \alpha K L [w'(L,t)]^2 - \alpha K \int_0^L [w'(x,t)]^2 dx. \end{aligned} \tag{6.71}$$

With the similar process

$$\begin{aligned} B_4(t) &= \alpha\rho \int_0^L xd\left([\dot{w}(x,t)]^2\right) \\ &= \alpha\rho L [\dot{w}(L,t)]^2 - \alpha\rho \int_0^L [\dot{w}(x,t)]^2 dx. \end{aligned} \tag{6.72}$$

Substituting Eqs. (6.70), (6.71) and (6.72) into Eq. (6.73), we obtain

$$\begin{aligned} \dot{\Delta}_1(t) \leq{}& \alpha K \left(L[w'(L,t)]^2 - \int_0^L [w'(x,t)]^2 dx \right) + \alpha\rho \left(L[\dot{w}(L,t)]^2 - \int_0^L [\dot{w}(x,t)]^2 dx \right) \\ & -2\alpha K \int_0^L x w'(x,t)\varphi'(x,t)dx + 2L \int_0^L \left(\delta_2 [w'(x,t)]^2 + \frac{1}{\delta_2} f^2(x,t) \right) dx. \end{aligned} \tag{6.73}$$

Integrating by parts of $\Delta_2(t)$ and substituting Eq. (6.7), we have

$$\dot{\Delta}_2(t) = C_1(t) + C_2(t) + C_3(t) + C_4(t), \tag{6.74}$$

where

$$C_1(t) = 2\alpha EI \int_0^L x\varphi'(x,t)\varphi''(x,t)dx$$
$$= \alpha EI \left(L[\varphi'(L,t)]^2 - \int_0^L [\varphi'(x,t)]^2 dx \right), \tag{6.75}$$
$$C_2(t) = 2\alpha K \int_0^L xw'(x,t)\varphi'(x,t)dx, \tag{6.76}$$
$$C_3(t) = -2\alpha K \int_0^L x\varphi'(x,t)\varphi(x,t)dx$$
$$= -\alpha K \left(L[\varphi(L,t)]^2 - \int_0^L [\varphi(x,t)]^2 dx \right), \tag{6.77}$$
$$C_4(t = 2\alpha I_\rho \int_0^L x\dot{\varphi}(x,t)\dot{\varphi}'(x,t)dx$$
$$= \alpha I_\rho \left(L[\dot{\varphi}(L,t)]^2 - \int_0^L [\dot{\varphi}(x,t)]^2 dx \right). \tag{6.78}$$

Substituting Eqs. (6.75) - (6.78) into Eq. (6.74), we obtain

$$\dot{\Delta}_2(t) = \alpha EI \left(L[\varphi'(L,t)]^2 - \int_0^L [\varphi'(x,t)]^2 dx \right) - \alpha K \left(L[\varphi(L,t)]^2 - \int_0^L [\varphi(x,t)]^2 dx \right)$$
$$+\alpha I_\rho \left(L[\dot{\varphi}(L,t)]^2 - \int_0^L [\dot{\varphi}(x,t)]^2 dx \right) + 2\alpha K \int_0^L xw'(x,t)\varphi'(x,t)dx. \tag{6.79}$$

Using Integration by parts, the time derivatives of $\Delta_3(t)$ can be determined to be

$$\dot{\Delta}_3(t) = \mu\alpha EI \int_0^L [\dot{\varphi}(x,t)]^2 dx - \mu\alpha K \int_0^L [\varphi(x,t)]^2 dx + \mu\alpha EI \int_0^L \varphi(x,t)\varphi''(x,t)dx$$
$$+\mu\alpha K \int_0^L \varphi(x,t)w'(x,t)dx$$
$$\leq \mu\alpha EI \left[\delta_3[\varphi(L,t)]^2 + \frac{1}{\delta_3}[\varphi'(L,t)]^2 \right] - \mu\alpha EI \int_0^L [\varphi'(x,t)]^2 dx$$
$$-\mu\alpha K \int_0^L \varphi^2(x,t)dx + \mu\alpha K \int_0^L \left[\delta_4[\varphi(x,t)]^2 + \frac{1}{\delta_4}[w'(x,t)]^2 \right] dx$$
$$+\mu\alpha I_\rho \int_0^L [\dot{\varphi}(x,t)]^2 dx. \tag{6.80}$$

where δ_3 and δ_4 are two positive constants.

Combining Eqs. (6.73), (6.79) and (6.80), we have

$$\begin{aligned}\dot{\Delta}(t) \leq\ & \alpha KL[w'(L,t)]^2 + \alpha\rho L[\dot{w}(L,t)]^2 + \left(\alpha EIL + \frac{\mu\alpha EI}{\delta_3}\right)[\varphi'(L,t)]^2 \\ & - (\alpha KL - \mu\alpha EI\delta_3)[\varphi(L,t)]^2 + \alpha I_\rho L[\dot{\varphi}(L,t)]^2 \\ & - \left(\alpha K - 2L\delta_2 - \frac{\mu\alpha K}{\delta_4}\right)\int_0^L [w'(x,t)]^2 dx - \alpha\rho\int_0^L [\dot{w}(x,t)]^2 dx \\ & -\alpha EI(\mu+1)\int_0^L [\varphi'(x,t)]^2 dx - \alpha K(\mu - 1 - \mu\delta_4)\int_0^L [\varphi(x,t)]^2 dx \\ & -\alpha I_\rho(1-\mu)\int_0^L [\dot{\varphi}(x,t)]^2 dx + \frac{2L}{\delta_2}\int_0^L [f(x,t)]^2 dx. \end{aligned} \tag{6.81}$$

Substituting Ineqs. (6.57), (6.63) and (6.81) into Eq. (6.48), we have

$$\begin{aligned}\dot{V}(t) \leq\ & -(\alpha\rho - \delta_1)\int_0^L [\dot{w}(x,t)]^2 dx - \left(\alpha K - 2L\delta_2 - \frac{\mu\alpha K}{\delta_4}\right)\int_0^L [w'(x,t)]^2 dx \\ & - \left[\alpha EI(\mu+1) - \alpha KL^2\right]\int_0^L [\varphi'(x,t)]^2 dx - \alpha K\mu(1-\delta_4)\int_0^L [\varphi(x,t)]^2 dx \\ & -\alpha I_\rho(1-\mu)\int_0^L [\dot{\varphi}(x,t)]^2 dx - \left(k_1 - \frac{K}{2}\right)\eta_1^2(t) - \left(k_2 - \frac{EI}{2}\right)\eta_2^2(t) \\ & - \left(\frac{EI}{2} - \alpha EIL - \frac{\mu\alpha EI}{\delta_3}\right)[\varphi'(L,t)]^2 - \left(\frac{EI}{2} - \alpha I_\rho L\right)[\dot{\varphi}(L,t)]^2 \\ & - (\alpha KL - \mu\alpha EI\delta_3)[\varphi(L,t)]^2 - \left(\frac{K}{2} - \alpha\rho L\right)[\dot{w}(L,t)]^2 - \frac{K}{2}[w'(L,t)]^2 \\ & + Kw'(L,t)\varphi(L,t) - \frac{K}{2}[\varphi(L,t)]^2 + \alpha KL[w'(L,t)]^2 \\ & + \left(\frac{1}{\delta_1} + \frac{2L}{\delta_2}\right)\int_0^L f^2(x,t)dx + \tilde{d}(t)\dot{\tilde{d}}(t) + \tilde{\theta}(t)\dot{\theta}(t) \\ & + \tilde{d}(t)\eta_1(t) + \tilde{\theta}(t)\eta_2(t). \end{aligned} \tag{6.82}$$

Let

$$\varepsilon_0 = \left(\frac{1}{\delta_1} + \frac{2L}{\delta_2}\right) \int_0^L f^2(x,t)dx \leq \left(\frac{L}{\delta_1} + \frac{2L^2}{\delta_2}\right) \bar{f}^2 \in \mathcal{L}_\infty. \tag{6.83}$$

Then substituting Eqs. (6.23) and (6.24) into Eq. (6.82), we have

$$\begin{aligned}
\dot{V(t)} \leq & -\delta_0 \int_0^L \left([\dot{w}(x,t)]^2 + [w'(x,t)]^2 + [\varphi(x,t)]^2 + [\varphi'(x,t)]^2 + [\dot{\varphi}(x,t)]^2\right) dx \\
& - \left(k_1 - \frac{K}{2}\right) \eta_1^2(t) - \left(k_2 - \frac{EI}{2}\right) \eta_2^2(t) - \left(\frac{EI}{2} - \alpha EIL - \frac{\mu\alpha EI}{\delta_3}\right) [\varphi'(L,t)]^2 \\
& - \left(\frac{EI}{2} - \alpha I_\rho L\right) [\dot{\varphi}(L,t)]^2 - \left(\alpha KL - \mu\alpha EI\delta_3 + \frac{K}{2}\right) [\varphi(L,t)]^2 \\
& - \left(\frac{K}{2} - \alpha\rho L\right) [\dot{w}(L,t)]^2 - \left(\frac{K}{2} - \delta_5 K - \alpha KL\right) [w'(L,t)]^2 \\
& + \left(\frac{1}{\delta_1} + \frac{2L}{\delta_2}\right) \int_0^L f^2(x,t)dx - (\xi_1 - \frac{1}{\delta_6})\tilde{d}^2(t) + \delta_6 \dot{d}^2(t) \\
& -(\xi_2 - \frac{1}{\delta_7})\tilde{\theta}^2(t) + \delta_7 \dot{\theta}^2(t)) \\
\leq & -\lambda_3 \left[V_0(t) + V_2(t) + \tilde{d}^2(t) + \tilde{\theta}^2(t)\right] + \varepsilon,
\end{aligned} \tag{6.84}$$

where

$$\begin{aligned}
\delta_0 =& \min\left\{\alpha\rho - \delta_1, \alpha K - 2L\delta_2 - \frac{\mu\alpha K}{\delta_4}, \alpha K\mu\left(1 - \delta_4\right),\right. \\
& \left.\alpha EI\left(\mu + 1\right) - \alpha KL^2 - \frac{KL}{\delta_5}, \alpha I_\rho\left(1 - \mu\right)\right\} > 0,
\end{aligned} \tag{6.85}$$

$$\varepsilon = \varepsilon_0 + \delta_6 D^2 + \delta_7 \Theta^2, \tag{6.86}$$

other constants $k_1, k_2, \delta_1 - \delta_7$ are chosen to satisfy the following conditions

$$\sigma_1 = k_1 - \frac{K}{2} > 0, \tag{6.87}$$

$$\sigma_2 = k_2 - \frac{EI}{2} > 0, \tag{6.88}$$

$$\lambda_3 = \min\left(\delta_0, \frac{2\sigma_1}{M}, \frac{2\sigma_2}{J}, \xi_1 - \frac{1}{\delta_6}, \xi_2 - \frac{1}{\delta_7}\right) > 0, \tag{6.89}$$

$$\frac{EI}{2} - \alpha EIL - \frac{\mu\alpha EI}{\delta_3} \geq 0, \tag{6.90}$$

$$\frac{EI}{2} - \alpha I_\rho L \geq 0, \tag{6.91}$$

$$\alpha KL - \mu\alpha EI\delta_3 - \frac{K}{\delta_5} + \frac{K}{2} \geq 0, \tag{6.92}$$

$$\frac{K}{2} - \alpha\rho L \geq 0, \tag{6.93}$$

$$\frac{K}{2} - \delta_5 K - \alpha KL \geq 0. \tag{6.94}$$

Combining Ineqs. (6.25) and (6.82), we have

$$\dot{V}(t) \leq -\lambda V(t) + \varepsilon, \tag{6.95}$$

where $\lambda = \lambda_3/\lambda_2 > 0$ and $\varepsilon > 0$. ■

With the above lemmas, we are ready to present the following stability theorem of the closed-loop Timoshenko beam system.

Theorem 6.1. *Consider the closed-loop Timoshenko beam system consisting of the system dynamics* (6.6) - (6.10)*, boundary control* (6.11)*,* (6.12)*, and disturbance observers* (6.15)*,* (6.16)*, under Assumptions 6.1 and 6.2, with the bounded initial conditions, then,*

(i) the state of the closed system $w(x,t)$ and $\varphi(x,t)$ will remain in the compact set

Ω_w *and* Ω_φ *defined by*

$$\Omega_w := \{w(x,t) \in R| \ \mid w(x,t) \mid \leq D_1\}, \tag{6.96}$$

$$\Omega_\varphi := \{w(x,t) \in R| \ \mid \varphi(x,t) \mid \leq D_1\}, \tag{6.97}$$

$\forall (x,t) \in [0,L] \times [0,\infty)$, *and the constant* $D_1 = \sqrt{\frac{L}{\lambda_1}(V(0) + \frac{\varepsilon}{\lambda})}$.

(ii) the system state $w(x,t)$ *and* $\varphi(x,t)$ *will eventually converge to the compact* Ω_{ws} *and* $\Omega_{\varphi s}$ *defined by*

$$\Omega_{ws} := \{w(x,t) \in R| \ \lim_{t\to\infty} \mid w(x,t) \mid \leq D_2\}, \tag{6.98}$$

$$\Omega_{\varphi s} := \{w(x,t) \in R| \ \lim_{t\to\infty} \mid \varphi(x,t) \mid \leq D_2\}, \tag{6.99}$$

$\forall x \in [0,L]$, *and the constant* $D_2 = \sqrt{\frac{L\varepsilon}{\lambda_1 \lambda}}$.

Proof: Multiplying Eq. (6.47) by $e^{\lambda t}$ yields

$$\frac{\partial}{\partial t}(V(t)e^{\lambda t}) \leq \varepsilon e^{\lambda t}. \tag{6.100}$$

Integrating of the above inequality, we obtain

$$V(t) \leq \left(V(0) - \frac{\varepsilon}{\lambda}\right) e^{-\lambda t} + \frac{\varepsilon}{\lambda} \leq V(0)e^{-\lambda t} + \frac{\varepsilon}{\lambda} \in \mathcal{L}_\infty, \tag{6.101}$$

Applying Ineq. (2.7) and Eq. (6.26), we have

$$\frac{1}{L}w^2(x,t) \leq \int_0^L [w'(x,t)]^2 dx \leq V_0(t) \leq V_0(t) + V_2(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty, \tag{6.102}$$

$$\frac{1}{L}\varphi^2(x,t) \leq \int_0^L [\varphi'(x,t)]^2 dx \leq V_0(t) \leq V_0(t) + V_2(t) \leq \frac{1}{\lambda_1}V(t) \in \mathcal{L}_\infty. \tag{6.103}$$

Appropriately rearranging the terms of the above inequality, we obtain $w(x,t)$ is uniformly bounded as follows

$$|w(x,t)| \leq \sqrt{\frac{L}{\lambda_1}\left(V(0)e^{-\lambda t} + \frac{\varepsilon}{\lambda}\right)} \leq \sqrt{\frac{L}{\lambda_1}\left(V(0) + \frac{\varepsilon}{\lambda}\right)}, \tag{6.104}$$

$$|\varphi(x,t)| \leq \sqrt{\frac{L}{\lambda_1}\left(V(0)e^{-\lambda t} + \frac{\varepsilon}{\lambda}\right)} \leq \sqrt{\frac{L}{\lambda_1}\left(V(0) + \frac{\varepsilon}{\lambda}\right)}, \tag{6.105}$$

$\forall(x,t) \in [0,L] \times [0,\infty)$. Furthermore, we have

$$\lim_{t\to\infty} |w(x,t)| \leq \sqrt{\frac{L\varepsilon}{\lambda_1\lambda}}, \tag{6.106}$$

$$\lim_{t\to\infty} |\varphi(x,t)| \leq \sqrt{\frac{L\varepsilon}{\lambda_1\lambda}}, \tag{6.107}$$

$\forall x \in [0,L]$. ■

Remark 6.3. *All the signals in the boundary control* (6.11) *and* (6.12) *can be measured by sensors or obtained by a backward difference algorithm.* $w(L,t)$ *and* $\varphi(L,t)$ *can be sensed by a laser displacement or rotation sensor at the boundary of the beam and* $w'(L,t)$, $\varphi'(L,t)$ *can be measured by an inclinometer. In our proposed control* (6.11) *and* (6.12), $\dot{w}(L,t)$, $\dot{w}'(L,t)$, $\dot{\varphi}(L,t)$, *and* $\dot{\varphi}'(L,t)$*with only one time differentiating with respect to time can be calculated with a backward difference algorithm.*

Remark 6.4. *It can be seen that the increase in the control gains* k_1, k_2 *and the estimated gains* ξ_1, ξ_2 *will result in a larger* λ_3. *Then the value of* λ *will increase, which will reduce the size of* Ω_w *and* Ω_φ *and produce a better vibration suppression performance. In the above analysis, it is clear that the steady system states* $w(x,t)$ *and* $\varphi(x,t)$ *can be made arbitrarily small provided that the design control parameters are appropriately selected. However, increasing* k_1 *and* k_2 *will bring a high gain control scheme. Therefore, in practical applications, the design parameters should be adjusted* *carefully for achieving suitable transient performance and control action.*

Remark 6.5. *From Eq.* (6.102)*, we can state that* $V_0(t)$ *and* $V_2(t)$ *are bounded* $\forall t \in [0, \infty)$*. Since* $V_0(t)$ *and* $V_2(t)$ *are bounded,* $\dot{w}(x,t)$*,* $\dot{\varphi}(x,t)$*,* $\varphi'(x,t)$*,* $\varphi(x,t)$ *and* $w'(x,t)$ *are bounded* $\forall (x,t) \in [0, L] \times [0, \infty)$*, and* $\eta_1(t)$*,* $\eta_2(t)$ *are bounded* $\forall t \in [0, \infty)$*. Then, we can obtain the potential energy Eq.* (6.2) *of the system is bounded. Using Property 6.2, we obtain that* $w''(x,t)$ *and* $\varphi''(x,t)$ *are bounded. Combining Assumption 6.2 and Eqs.* (6.6)*,* (6.7)*, we can state that* $\ddot{w}(x,t)$ *and* $\ddot{\varphi}(x,t)$ *are also bounded* $\forall (x,t) \in [0, L] \times [0, \infty)$*. From the above information, it is shown that the proposed control Eqs.* (6.11) *and* (6.12) *ensure all internal system signals including* $w(x,t)$*,* $\varphi(x,t)$*,* $w'(x,t)$*,* $\varphi'(x,t)$*,* $w''(x,t)$*,* $\varphi''(x,t)$*,* $\ddot{w}(x,t)$ *and* $\ddot{\varphi}(x,t)$ *are uniformly bounded, and we can conclude the boundary control Eqs.* (6.11) *and* (6.12) *are also bounded* $\forall t \in [0, \infty)$*.*

6.4 Numerical Simulations

In this section, the finite difference method is used for numerical simulations. Consider a Timoshenko beam excited by the disturbances $f(x,t)$, $d(t)$ and $\theta(t)$, the initial conditions are

$$
\begin{aligned}
w(x,0) &= x, \\
\dot{w}(x,0) &= 0, \\
\varphi(x,0) &= \arctan 0.5, \\
\dot{\varphi}(x,0) &= 0
\end{aligned}
$$

Parameters of the Timoshenko beam are referred to [48], and listed below:

Table 1: Parameters of the Timoshenko beam

Parameter	Description	Value
L	Length of beam	$1.0m$
EI	Bending stiffness	$7.5Nm^2$
K	kAG	$1.5N$
ρ	Mass per unit length	$1.0kg/m$
I_ρ	Mass moment of inertia	$2kgm$
M	Mass of the tip payload	$0.1kg$
J	Inertia of the payload	$0.1kgm^2$

The boundary disturbances $d(t)$ and $\theta(t)$ on the tip payload and the distributed disturbance $f(x,t)$ on the Timoshenko beam are described as

$$
\begin{aligned}
d(t) &= t + \sin(\pi t), \\
\theta(t) &= 0.1t + 0.1\sin(2\pi t), \\
f(x,t) &= [3 + \sin(\pi x t) + \sin(2\pi x t) + \sin(3\pi x t)]x.
\end{aligned}
$$

Figs. 6.2 and 6.3 show the displacement and the rotation of the Timoshenko beam under disturbances without control input. As shown in Figs. 6.2 and 6.3, there are large vibrations when the control input $u(t) = \tau(t) = 0$. Displacement and rotation of the Timoshenko beam with the proposed boundary control (6.11), (6.12) are demonstrated in Figs. 6.4 and 6.5 respectively. It can be seen that vibrations of the Timoshenko beam can be suppressed greatly within 10 secs, by choosing $k_1 = 10$, $k_2 = 4$, $\xi_1 = 100$, $\xi_2 = 0.01$, which illustrate that the proposed boundary control are able to stabilize the Timoshenko beam at the small neighborhood of its equilibrium position. The corresponding boundary control inputs $u(t)$ and $\tau(t)$ are shown in Fig. 6.6. Boundary disturbance estimate errors are displayed in Fig. 6.7. Although the disturbance estimate errors $\tilde{d}(t)$, $\tilde{\theta}(t)$ can not converge exactly to 0, the proposed boundary control are still able to stabilize the Timoshenko beam system.

6.5 Conclusion

In this chapter, boundary control has been developed for a Timoshenko beam systems under unknown disturbances. The control design and closed-loop stability analysis have been presented in the context of Lyapunov's stability theory and its associated techniques. The boundary disturbance observers have also been designed to deal with the external boundary disturbances.

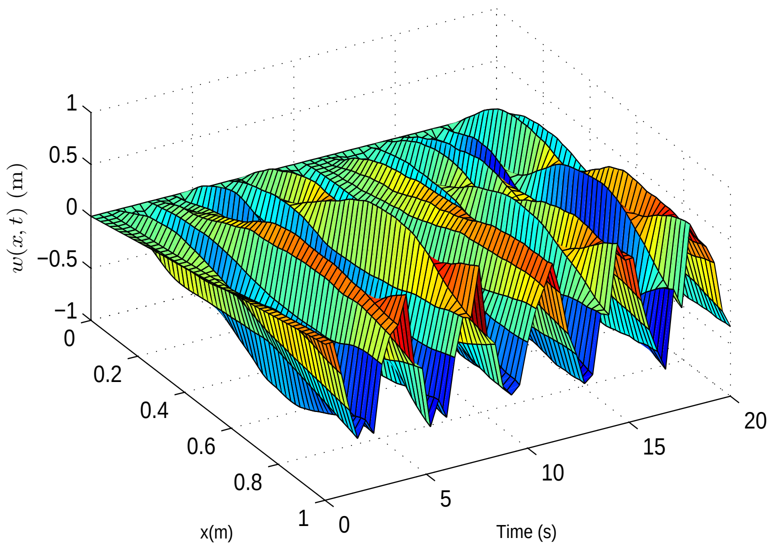

Fig. 6.2: Displacement of the Timoshenko beam without control.

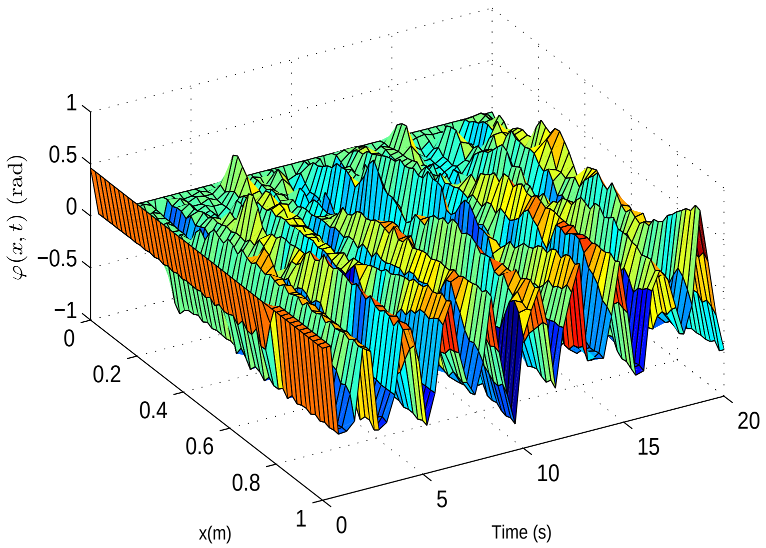

Fig. 6.3: Rotation of the Timoshenko beam without control.

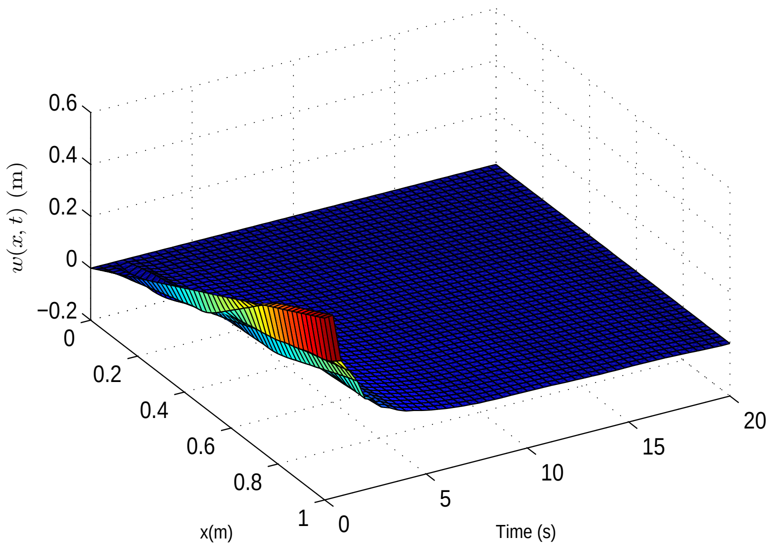

Fig. 6.4: Displacement of the Timoshenko beam with boundary control.

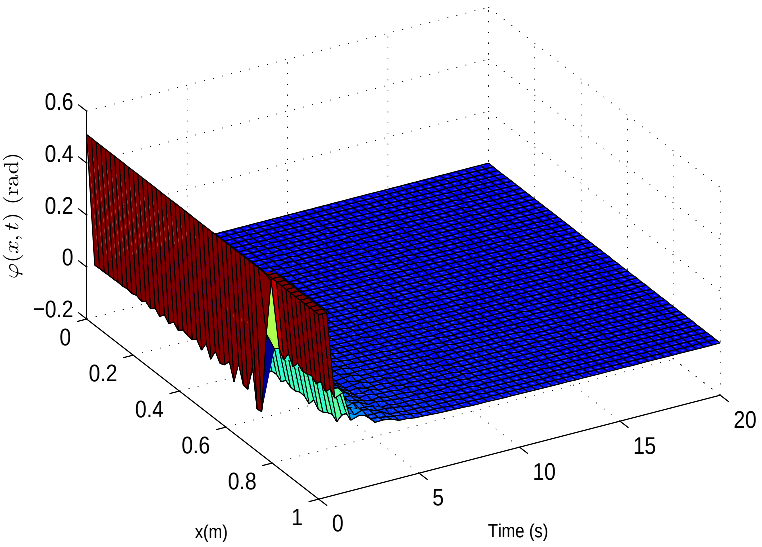

Fig. 6.5: Rotation of the Timoshenko beam with boundary control.

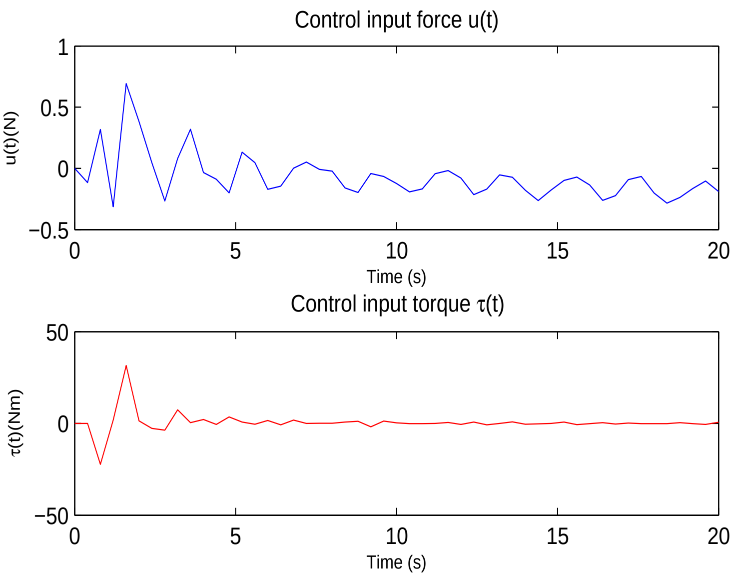

Fig. 6.6: Boundary control inputs $u(t)$ and $\tau(t)$.

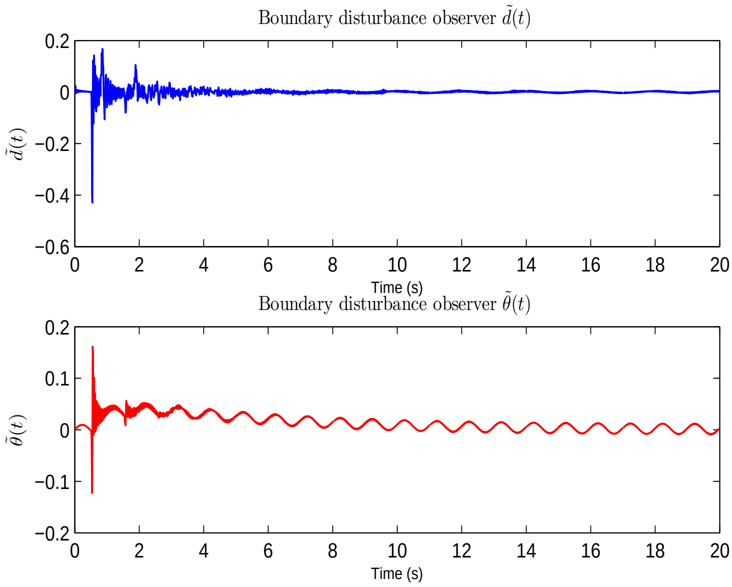

Fig. 6.7: Boundary disturbance estimate error $\tilde{d}(t)$ and $\tilde{\theta}(t)$.

Chapter 7

Conclusions

7.1 Conclusions

The thesis has been dedicated to the boundary control of flexible mechanical systems subjected to unknown disturbances including the distributed disturbance and the boundary disturbance. In order to avoid the spillover instability and improve the accuracy, the study has been based on the original models. This also represents an important step in extending the boundary control theory to distributed parameter systems.

- **Nonuniform string system**
  A nonuniform string model with a varying tension and a varying mass per unit length has been considered in the presence of uncertain dynamics and under unknown spatiotemporally varying distributed disturbance and time-varying boundary disturbance. It has shown that, the proposed boundary control can be designed to ensure the uniformly bounded stability of the closed-loop system

for both model-based control case and adaptive control case.

- **Coupled nonlinear string system**

  A nonlinear string model with coupling between the longitudinal and transversal dynamics has been studied. Using Lyapunov direct method, two adaptive boundary control laws have been derived for this system. In the presence of external disturbances, it has been concluded that the states of the system can stay at a small region of zero by properly selecting the design parameters. Simplicity of the designed control laws is an attractive factor from an implementation point of view.

- **Euler-Bernoulli beam system**

  Boundary control and adaption laws have been proposed for an Euler-Bernoulli beam model to suppress the beam's vibration. It has been proven that, stabilized closed-loop system has a bounded state with the proposed control. The control problem with boundary output constraint has been also discussed. By employing a novel Integral-Barrier Lyapunov Function, the vibration of the beam system has been reduced without violation of the constraint.

- **Timoshenko beam system**

  The control problem of a Timoshenko beam model, which is an improvement of the Euler-Bernoulli beam, has also been addressed. Output-feedback control laws have been designed to attenuate the vibration amplitude. The boundary disturbance observers have also been designed to estimate the boundary disturbances and reduce the effects of the external disturbances. The knowledge of the disturbances is not required for the proposed control.

7.2 Recommendations for Future Research

- **Integral-Barrier Lyapunov Function**

  Barrier Lyapunov Function is a novel concept that can be employed to deal with the control problems with output constraints for ODEs. However, there is little information about how to handle the constraints for PDEs. In this thesis, Integral-Barrier Lyapunov Function is employed to the Euler-Bernoulli beam only, and there is a need to explore an effective method for the control of other flexible structures with constraint problems.

- **Tension control**

  Fatigue problem which is a result of oscillating stress will be a major problem caused by vibrations in flexible systems. Stress variation and large oscillations may lead to cracks to propagate from initial defects in the material. Fatigue problem also occurs when the tension of the system exceeds a given range. Thus, in order to avoid the fatigue problem, not only the displacement is supposed to be controlled, but also tension control should be considered.

- **Experimental results**

  In this thesis, the control performance is verified by using numerical simulations which are made as realistic as the real world. The simulation results in our research support the theoretical analysis effectively. The numerical simulation which imitates the operation of a real-world system to see how the system works, is an effective and extensive used tool in the literature. Numerical simulations have been widely used in a number of research works on the flexible mechanical systems [34, 59, 102, 107, 138–140]. However, the data from the practical word would be a better way to verify the control performance of the proposed scheme.

Due to the limitations in existing facilities and lack of resources, we were not able to conduct the practical validation with good scaling for the proposed controls. As such, we will carry out the experiments in the future when facilities become available.

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Author's Publications

The contents of this thesis are based on the following papers that have been published, accepted, or submitted to the peer-reviewed journals and conferences.

Journal papers:

1. S. S. Ge, S. Zhang, and W. He, "Vibration Control of an Euler-Bernoulli Beam under Unknown Spatiotemporally Varying Disturbance," *International Journal of Control*, vol. 84, no. 5, pp. 947-960, 2011.

2. W. He, S. S. Ge, and S. Zhang, "Adaptive Boundary Control of a Flexible Marine Installation System," *Automatica*, vol. 47, no. 12, pp. 2728-2734, 2011.

3. S. Zhang, W. He, and S. S. Ge, "Modeling and Control of a Nonuniform Vibrating String under Spatiotemporally Varying Tension and Disturbance," *IEEE/ASME Transactions on Mechatronics*, in press, 2011.

4. S. Zhang, S. S. Ge, and W. He, "Boundary Output-Feedback Stabilization of a Timoshenko Beam Using Disturbance Observer," *IEEE Transactions on Industrial Electronics*, under review, 2011.

5. S. S. Ge, S. Zhang, and W. He, "Vibration Control of a Coupled Nonlinear String System in Transverse and Longitudinal Directions," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, under review, 2011.

Conference papers:

1. S. S. Ge, S. Zhang, and W. He, "Modeling and Control of a Vibrating Beam under Unknown Spatiotemporally Varying Disturbance", *the 2011 American Control Conference (ACC 2011)*, San Francisco, CA, USA, June 29 - July 01, 2011.

2. S. S. Ge, S. Zhang, and W. He, "Vibration Control of a Flexible Timoshenko Beam under Unknown External Disturbances", *the 30th Chinese Control Conference (CCC 2011)*, Yantai, China, July 22- 24, 2011.

3. S. Zhang, S. S. Ge, W. He, and K.-S Hong "Modeling and Control of a Nonuniform Vibrating String under Spatiotemporally Varying Tension and Disturbance", *the 2011 IFAC World Congress (IFAC 2011)*, Milano, Italy, August 28- September 02, 2011.

4. S. S. Ge, S. Zhang, and W. He, "Vibration Control of a Coupled Nonlinear String System in Transverse and Longitudinal Directions", *the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC 2011)*, Orlando, Florida, USA, December 12-15, 2011.

5. S. S. Ge, S. Zhang, and W. He, "Modeling and Control of a Flexible Riser with Application to Marine Installation", *the 2012 American Control Conference (ACC 2012)*, Montreal, Canada, June 27-29, 2012.