REALIZING AN AD^+ MODEL AS A DERIVED MODEL OF A PREMOUSE

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DECLARATION

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis. This thesis has also not been submitted for any degree in any university previously.

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4 July 2012

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Summary

Assuming $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$, and there is no proper class inner model containing all the reals that satisfies $AD_{\mathbb{R}}$ + "θ is regular", and assuming $\mathrm{cf}(\theta)$ is not singular of uncountable cofinality, we prove that in some forcing extension, either V is a derived model of a premouse or V embeds into a derived model of a premouse.

Chapter 1

Introduction

Inner models are of the form $L[\vec{E}]$, where $\vec{E}$ codes a coherent sequence of extenders. They are supposed to produce detailed information of large cardinals. The study of inner models has entered the region of many Woodin cardinals. Neeman [5] constructed an inner model with a Woodin limit of Woodin cardinals assuming there is a Woodin limit of Woodin cardinals in V. Steel [18] showed that the core model exists assuming there is no inner model with a Woodin cardinal. Computation of the core model and its relatived versions can be used to produce many Woodin cardinals as a consistency lower bound from other axioms such as PFA. In that region, the main obstacle of producing inner models with higher large cardinals is the iterability problem. It is hard to define a canonical iteration strategy when Woodin cardinals are overlapped by extenders. Woodin's derived model theorem plays a important role in analysis of premice with Woodin cardinals. Models of determinacy appears when we reach Woodin cardinals.

Given a set $A \subseteq X^\omega$, the game G_A is played as follows. Two players take turns to play elements of X as in the following diagram. I picks $x(i)$ for even i and II picks $x(i)$ for odd i. Player I wins G_A if the outcome of the play, x, is in C. G_A is determined, or A is determined, if either of the players has a winning strategy.

AD, or the axiom of determinacy, is the statement that for every $A \subseteq \omega^\omega$, G_A is determined. In this thesis, $\mathbb{R}$ refers to the Baire space ω^ω.

$$\begin{array}{c|cccccccc} \mathrm{I} & x(0) & & x(2) & & x(4) & & \cdots & \\ \hline \mathrm{II} & & x(1) & & x(3) & & x(5) & & \cdots \end{array}$$

Woodin defines AD^+, a strengthening of AD. A set of reals A is ∞-Borel if there is a set of ordinals S, an ordinal σ and a formula ϕ such that

$$x \subseteq A \leftrightarrow L_\alpha[S, x] = \phi[S, x].$$

If λ is an ordinal and $A \subseteq \lambda^\omega$, then A is determined if either of the two players has a winning strategy in the game G_A. Ordinal determinacy is the statement that for any $\lambda < \theta$, any continuous function $f : \lambda^\omega \to \omega^\omega$, for any set $A \subseteq \omega^\omega$, the set $\pi^{-1}(A)$ is determined.

Definition 1.1 (Woodin)**.** AD^+ is the following statement.

1. $ZF + AD + DC_{\mathbb{R}}$.

2. Every set of reals is ∞-Borel.

3. Ordinal determinacy.

AD^+ has many nice consequences. A set of reals $A \subseteq \omega^\omega$ is λ-Suslin if there is a tree $T \subseteq \omega^\omega \times \lambda^\omega$ such that $A = p[T] = \{x \in \omega^\omega : \exists y \in \lambda^\omega (x, y) \in T\}$. λ is a Suslin cardinal if there is $A \subseteq \omega^\omega$ such that A is λ Suslin but not γ-Suslin for every $\gamma < \lambda$. $AD_{\mathbb{R}}$ is the statement that for each $A \subseteq \mathbb{R}^\omega$, the game G_A is determined.

Theorem 1.2 (Woodin)**.** *Assume AD^+.*

1. *The set of Suslin cardinals is closed.*

2. $AD_{\mathbb{R}}$ *holds iff there is no largest Suslin ordinal.*

AD contradicts the axiom of choice, but models of AD has fruitful contents, because they are naturally associated to models of large cardinals. The derived model theorem establishes the relationship between AD^+ and large cardinals.

Theorem 1.3 (Derived model theorem I, Woodin, [10, 3])**.** *Let λ be a limit of Woodin cardinals. Let G be V-generic over $Coll(\omega, <\lambda)$.*

$$\mathbb{R}^*_G = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \restriction \alpha],$$

$$\begin{aligned} Hom^*_G = \{A \subset \mathbb{R}^*_G : \exists \alpha < \lambda \exists T, U \in V[G \restriction \alpha](A = p[T] \cap \mathbb{R}^*_G \\ \wedge V[G \restriction \alpha] \models T, U \textit{ are } < \lambda\textit{-complementing trees}), \end{aligned}$$

$$\mathcal{A}^*_G = \{B \subset \mathbb{R}^*_G : B \in V(\mathbb{R}^*_G) \textit{ and } L(B, \mathbb{R}^*_G) \models AD^+\}.$$

Then

1. *For $B, C \in \mathcal{A}^*_G$, either $L(B, \mathbb{R}^*_G) \subset L(C, \mathbb{R}^*_G)$ or $L(C, \mathbb{R}^*_G) \subset L(B, \mathbb{R}^*_G)$.*

2. *$L(\mathcal{A}^*_G, \mathbb{R}^*_G) \models AD^+$.*

3. *For each $B \in \mathcal{P}(\mathbb{R}^*_G) \cap V(\mathbb{R}^*_G)$, the following are equivalent*

   (a) B is Suslin-co-Suslin in $V(\mathbb{R}^*_G)$.

   (b) $B \in \mathcal{A}^*_G$ and B is Suslin-co-Suslin in $L(\mathcal{A}^*_G, \mathbb{R}^*_G)$.

   (c) $B \in Hom^*_G$.

The model $L(\mathcal{A}^*_G, \mathbb{R}^*_G)$ is called the derived model at λ.

Theorem 1.4 (Derived model theorem II, Woodin, [13, 12])**.** *Suppose AD^+. Then in some forcing extension over V, either V is a derived model or V embeds into a derived model.*

Descriptive set theory can be used in analysis of the derived model of a premouse. This leads to a completely new approach of investigating inner model theory. Among those descriptive set theoretic tools, the Solovay sequence often characterizes the complexity of an AD^+ model. We define $\theta = \sup\{\alpha : \text{there is a surjection } f : \mathbb{R} \to \alpha\}$.

Definition 1.5. Assume AD^+. The Solovay sequence is a closed increasing sequence $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined as follows.

1. $\theta_0 = \sup\{\alpha : \text{there is a surjection } f : \mathbb{R} \to \alpha \text{ such that } f \text{ is } OD\}$.
2. if $\theta_\beta < \theta$ then

$$\theta_{\beta+1} = \sup\{\alpha : \text{there is a surjection } f : \mathcal{P}(\theta_\beta) \to \alpha \text{ such that } f \text{ is } OD\}.$$

3. if λ is a limit, then $\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha$.

It follows that $\theta_\Omega = \theta$.

Theorem 1.6 (Woodin,[11, 3])**.** *Assume AD^+.*

1. *If $\theta_\alpha < \theta$, then θ_α is a Suslin cardinal.*
2. *If $\theta_\alpha < \theta$, then $\theta_{\alpha+1}$ is a Woodin cardinal in HOD.*

Hence $AD_{\mathbb{R}}$ holds if and only if the length of the Solovay sequence is a limit ordinal. A hierarchy of determinacy axioms can be obtained by measuring the length of the Solovay sequence. The following are the first few theories of this hierarchy. Here

$T_1 <_{con} T_2$ means $Con(T_2) \vdash Con(T_1)$ but $Con(T_1) \nvdash Con(T_2)$.

$$\begin{aligned}&AD^+ <_{con} AD^+ + \theta_1 = \theta <_{con} AD^+ + \theta_2 = \theta <_{con} AD^+ + \theta_\omega = \theta <_{con} \cdots \\ &<_{con} AD^+ + \theta_{\omega_1} = \theta <_{con} AD^+ + \theta_{\omega_1+1} = \theta <_{con} \cdots \\ &<_{con} AD_{\mathbb{R}} + \text{“}\theta \text{ is regular”} <_{con} \cdots\end{aligned}$$

Earlier results demonstrate a correspondence between some of those determinacy axioms and large cardinal axioms.

Theorem 1.7 (Woodin).

1. *$Con(AD^+) \leftrightarrow Con(ZFC+$"there are infinitely many Woodin cardinals"$)$.*

2. *$Con(AD^+ + \theta_1 = \theta) \leftrightarrow Con(ZFC + \exists\lambda\exists\kappa < \lambda(\lambda$ is a limit of Woodins and κ is $< \lambda$-strong$))$.*

3. *$Con(AD_{\mathbb{R}}) \leftrightarrow Con(ZFC + \exists\lambda(\lambda$ is a limit of Woodins and a limit of $< \lambda$-strongs$))$.*

The HOD computation, among other applications, builds a bridge between premice and AD^+ models. Steel and Woodin [17, 15] showed that $HOD^{L(\mathbb{R})}$ has fine structure, assuming AD holds in $L(\mathbb{R})$. Sargsyan [7] extended their results by carrying out a detailed analysis of HOD of AD^+ models below $AD_{\mathbb{R}}$ + “θ is regular”.

Theorem 1.8 (Sargsyan,[7]). *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and suppose that there is no proper class inner model containing the reals and satisfying $AD_{\mathbb{R}}$ + “θ is regular”. Then V_θ^{HOD} is a hod premouse.*

A hod premouse is a special kind of layered hybrid premouse. The reader might refer to [7] on the definition of hod premouse and related concepts. If $\mathcal{P}$ is a hod premouse, its Woodin cardinals and limits of Woodin cardinals are enumerated as $\langle \delta_\alpha^{\mathcal{P}} : \alpha < \lambda^{\mathcal{P}} \rangle$ in the increasing order. The hierarchy between $\lambda_\alpha^{\mathcal{P}}$ and $\lambda_{\alpha+1}^{\mathcal{P}}$ are of the type $L[\vec{E}, \Sigma_\alpha^{\mathcal{P}}]$, where $\Sigma_\alpha^{\mathcal{P}}$ is the iteration strategy of $\mathcal{P}(\alpha)$. Here $P(\alpha) = \mathcal{P}|\mu_\alpha$, where $\langle \mu_\alpha : \alpha < \lambda^{\mathcal{P}} \rangle$ is a part of the language of $\mathcal{P}$, and has the property that $\mathcal{P} \models \mathcal{P}(\alpha) = \mathrm{Lp}_\omega^{\oplus \Sigma_{<\alpha}}(P|\delta_\alpha)$. All those $\delta_\alpha^{\mathcal{P}}$'s are strong cutpoints, namely no extender E on the $\mathcal{P}$ sequence with $\mathrm{crt}(E) \leq \delta_\alpha^{\mathcal{P}} < lh(E)$. This makes the large cardinal structure in HOD much simpler than the mouse giving arise to the corresponding AD^+ model. Many of the complexities are absorbed into the iteration strategies coded in hod mice. Therefore, hod mice are much easier to analyze than mice. $(\mathcal{P}, \Sigma)$ is a hod pair if Σ is an iteration strategy for $\mathcal{P}$ with hull condensation. We are interested in hod pairs $(\mathcal{P}, \Sigma)$ where Σ is fullness preserving and has branch condensation. Any two such hod pairs can be compared to another such hod pair. The comparison maps commute and form a direct limit. The direct limit is exactly $HOD|\theta_\alpha$ if there is a largest Suslin cardinal, or $HOD|\theta$ if $AD_{\mathbb{R}}$ holds. Those $\delta_\alpha^{\mathcal{P}}$'s map exactly to members the Solovay sequence.

The main idea of one direction of theorem 1.7, from strong AD^+-hypotheses to large cardinals, expressed in terms of hod mice, is to translate the strategies coded in the HOD sequence into extenders that overlap Woodins cardinals of HOD. Because of the success of study of HOD in stronger AD^+ models, it is a natural project to generalize the translation to the region we understand HOD. Intuitively, stronger AD^+ models have more complicated HOD's, and hence their strategies should give more extenders overlapping Woodins. In this paper we prove the fine-structural refinement of theorem 1.4.

Theorem 1.9. *Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and suppose that there is no proper class inner model containing the reals and satisfying $AD_{\mathbb{R}}$ + "θ is regular".*

1. *Suppose there is a largest Suslin cardinal. Then there is a forcing $\mathbb{P}$ such that in the $\mathbb{P}$-generic extension, there is a premouse $\mathcal{N}$ such that letting $\lambda = \omega_1^V$,*

   (a) $\mathcal{N} \models \lambda$ is a limit of Woodin cardinals.

(b) V is a derived model of $\mathcal{N}$ at λ.

2. *Suppose $AD_{\mathbb{R}}$ holds and $\mathrm{cf}(\theta) = \omega \vee$ "θ is regular". Then there is a forcing $\mathbb{P}$ such that in the $\mathbb{P}$-generic extension, there is a premouse $\mathcal{N}$ and a map j such that*

   (a) $N \models \lambda$ is a limit of Woodin cardinals.

   (b) $j : V \to M$ is elementary, where M is a derived model of $\mathcal{N}$ at θ.

Theorem 1.9 mostly answers the fundamental question: what are models of $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ when V is below $AD_{\mathbb{R}}$ + "θ is regular"? In particular, if V is the minimum model of $AD_{\mathbb{R}}$ + "θ is regular", then there is a premouse $\mathcal{N}$ such that V embeds into the derived model of $\mathcal{N}$. Besides, the translation procedure that is used in the proof effectively gets rid of extenders on mice over Woodins and essentially reshape them into strategies, thus reducing the complexity of the iterability problem. The connection that is drawn between mice, which represents large cardinals, and hod mice, which can be easily iterated, contributes to the understanding of inner models with Woodins and HOD of AD^+ models.

We assume familiarity with [7]. The main idea in proving theorem 1.9 is a translation procedure between extenders that overlap certain Woodins and strategies. Sections 2 and 3 handles the case $\theta = \theta_{\alpha+1}$. In Chapter 2, we define the S-operators, which are intended to code fragments of the iteration strategy while at the same time corresponding to extenders that overlap Woodins. We shall work with a fixed hod pair $(\mathcal{P}, \Sigma)$ such that Σ is fullness preserving and has branch condensation, and Σ corresponds to the largest Suslin pointclass. We shall demonstrate how fragments of Σ are computed from those S-operators. In Chapter 3, we define a translation procedure, which turns extenders that overlap a certain Woodin cardinal into an S-operator and vice versa. Section 3.4 concludes the proof of the $\theta = \theta_{\alpha+1}$ case, using the translation procedure and a reflection argument. Chapter 4 handles the

$AD_{\mathbb{R}}$ case. The S-operators defined in Section 2 and the translation defined in Section 3 applies to the $AD_{\mathbb{R}}$ case with slight modification. So we will be sketchy there and hopefully the reader can fill out the details.

The premouse we get from Theorem 1.9 is well below a Woodin limit of Woodins. Starting from a typical strong determinacy hypothesis, such as $AD^+ + \theta = \theta_{\omega_2+1}$, or $AD_{\mathbb{R}}$ + "θ is regular", one could possibly investigate the exact large cardinal strength of the premouse we get from Theorem 1.9, thus obtaining a lower bound of that strong determinacy hypothesis. A more interesting question is to generalize Theorem 1.9 beyond $AD_{\mathbb{R}}$+"θ is regular". Sargsyan in an unpublished work carried out the HOD analysis of AD^+ models below LST (the largest θ is a Suslin cardinal). The translation is likely to generalize as long as HOD of an AD^+ model is well understood. A plausible conjecture is that starting from LST, we may get a mouse with a Woodin limit of Woodins.

Conjecture 1.10. *[6, Open problem 2] Con(LST) $\rightarrow$ Con("there is a Woodin limit of Woodins").*

Chapter 2

The S-operators

In this chapter, we define the S-operators. Suppose for the moment we have a hod pair $(\mathcal{P}, \Sigma)$. An S-operator will code a fragment of Σ. We shall build S-premice, by enhancing premice with an additional predicate S. S-premice are essentially Σ-premice, but strategies are regrouped in a very careful way. We recall that in a Σ-premouse, at each step in the relativized Gödel construction, we throw in the $\Sigma(\vec{\mathcal{T}})$ into the next few steps, where $\vec{\mathcal{T}}$ is the least stack that is not told the strategy. However, an S-operator, in cases of interest, tells a part of Σ that will correspond exactly to an extender. The main job is to cut Σ into pieces in a way that each piece correspond to an extender in the future. There is a difficulty in the case when $\mathcal{P} \models \mathrm{cf}(\lambda^{\mathcal{P}})$ is measurable, since by hitting that measure, we create more Woodins and thus have to take care of those new Woodins. This difficulty is resolved by rearranging stacks, which is done in Section 2.2.

2.1 Preliminaries

Following the notation of [4], an iteration tree is a tuple $\mathcal{T} = \langle T, deg, D, \langle E_\alpha, \mathcal{M}^*_{\alpha+1} : \alpha + 1 < \eta\rangle\rangle$. We let $\mathcal{M}^{\mathcal{T}}_\alpha$ be the αth model of $\mathcal{T}$, $E^{\mathcal{T}}_\alpha$ be the αth extender of $\mathcal{T}$,

$\kappa_\alpha^\mathcal{T} = \mathrm{crt}(E_\alpha^\mathcal{T})$, $\nu_\alpha^\mathcal{T} = lh(E_\alpha^\mathcal{T})$, $i_{\beta\alpha}^\mathcal{T} : \mathcal{M}_\beta^\mathcal{T} \to \mathcal{M}_\alpha^\mathcal{T}$ be the iteration map when $\beta <^\mathcal{T} \alpha$, $(\beta, \alpha] \cap D^\mathcal{T} = \emptyset$.

We fix our terminologies. In this paper, an iteration tree is always a normal tree. By a stack, we mean a stack of iteration trees. Stacks are usually denoted by $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$, etc, with a vector symbol on top.

Let $\vec{\mathcal{T}}$ be a stack on $\mathcal{P}$. Let $\nu < \lambda^\mathcal{P}$. We let $\vec{\mathcal{T}} \restriction \mathcal{P}(\nu)$ be the substack of $\vec{\mathcal{T}}$ by throwing away essential components that are above the image of $\mathcal{P}(\nu)$. We say that $\vec{\mathcal{T}}$ lives below $\mathcal{P}(\nu)$ if $\vec{\mathcal{T}} = \vec{\mathcal{T}} \restriction \mathcal{P}(\nu)$. We say that $\vec{\mathcal{T}}$ lives above $\mathcal{P}(\nu)$ if all extenders of $\vec{\mathcal{T}}$ are above $\mathcal{P}(\nu)$.

If $\mathcal{P}, \mathcal{Q}$ are hod premice, $\mathcal{P} \lhd_{hod} \mathcal{Q}$, $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$, we let $\vec{\mathcal{T}}(\mathcal{Q})$ be the stack on $\mathcal{Q}$ with the same tree structure, extenders, degree sequence as $\vec{\mathcal{T}}$ has, if every model is wellfounded.

If $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are hod premice such that $\mathcal{P} \lhd_{hod} \mathcal{R}$, $\mathcal{P} \lhd_{hod} \mathcal{Q}$, $\vec{\mathcal{T}}$ is a stack on $\vec{\mathcal{R}}$ such that $\vec{\mathcal{T}}$ is based on $\mathcal{P}$, then we let $\vec{\mathcal{T}}(\mathcal{Q}) = (\vec{\mathcal{T}} \restriction \mathcal{P})(\mathcal{Q})$.

The following observation will be useful, whose proof is straightforward.

Lemma 2.1. *Let $\mathcal{P}$ be a hod premouse, $\nu < \lambda^\mathcal{P}$. Let $\vec{\mathcal{T}}$ be a stack on $\mathcal{P}(\nu)$ with last model $\mathcal{Q}$ such that $i^{\vec{\mathcal{T}}}$ exists. Suppose that $\vec{\mathcal{T}}(\mathcal{P})$ is defined. Let $\mathcal{R}$ be the last model of $\vec{\mathcal{T}}(\mathcal{P})$. Then $\mathcal{R}$ is the ultrapower of $\mathcal{P}$ by the long extender derived from $i^{\vec{\mathcal{T}}} \restriction \delta_\nu^\mathcal{P}$. $i^{\vec{\mathcal{T}}(\mathcal{P})} : \mathcal{P} \to \mathcal{R}$ is the ultrapower map.*

Suppose that $j : \mathcal{M} \to \mathcal{N}$ is Σ_1-elementary. Given a stack $\vec{\mathcal{T}}$ on $\mathcal{M}$, we let $k\vec{\mathcal{T}}$ be the copying stack on $\mathcal{N}$. If Σ is an iteration strategy on $\mathcal{N}$, let Σ^j be the pullback strategy on $\mathcal{M}$. If Σ is an iteration strategy on $\mathcal{N}$, $\vec{\mathcal{T}}$ is a stack on $\mathcal{N}$ according to Σ, let $\Sigma_{\vec{\mathcal{T}}}$ be the tail of Σ defined by $\Sigma_{\vec{\mathcal{T}}}(\vec{\mathcal{U}}) = \Sigma(\vec{\mathcal{T}}^\frown\vec{\mathcal{U}})$.

2.2 Rearranging stacks

Lemma 2.2. *Suppose that $\mathcal{P}$ is a hod premouse. Let $\zeta < \lambda^{\mathcal{P}}$ be an ordinal. Suppose that $\mathcal{T}$ is an iteration tree on $\mathcal{P}$ above $\mathcal{P}(\zeta)$ with last model $\mathcal{Q}_1$, $\mathcal{U}$ is an iteration tree on $\mathcal{P}$ with last model $\mathcal{Q}_2$ below $\mathcal{P}(\zeta)$ such that $i^{\mathcal{U}}$ exists. Let $\mathcal{R}_1$ be the last model of the tree $\mathcal{U}(\mathcal{Q}_1)$ on $\mathcal{Q}_1$. Suppose that all models of the copying tree $i^{\mathcal{U}}\mathcal{T}$ on $\mathcal{Q}_2$ are wellfounded. Let $\mathcal{R}_2$ be the last model of $i^{\mathcal{U}}\mathcal{T}$. Let $l : \mathcal{Q}_1 \to \mathcal{R}_2$ be the copying map, $j : \mathcal{Q}_1 \to \mathcal{R}_1$ be the associated tree embedding. Then there is a* $\deg(\mathcal{T})$*-embedding $\pi : \mathcal{R}_1 \to \mathcal{R}_2$ such that $\pi \circ j = l$.*

Proof. We assume $\zeta = 0$ for simplicity. Since $\mathcal{T}$ is above $\mathcal{P}(0)$ and $\mathcal{U}$ is above $\mathcal{P}(0)$, we may apply $\mathcal{U}$ to every model of $\mathcal{T}$. For $\alpha < lh(\mathcal{T})$, let $\langle \mathcal{N}^{\xi}_{\alpha} : \xi < lh(\mathcal{U})\rangle$ be models of $\mathcal{U}(\mathcal{M}^{\mathcal{T}}_{\alpha})$. For $\alpha < lh(\mathcal{T})$ a successor, let $\langle (\mathcal{N}^{\xi}_{\alpha})^* : \xi < lh(\mathcal{U})\rangle$ be models of $\mathcal{U}(\mathcal{M}^{*\mathcal{T}}_{\alpha})$. For $\alpha < lh(\mathcal{T})$, let $j^{\eta\xi}_{\alpha} : \mathcal{M}^{*\mathcal{U}(\mathcal{M}^{\mathcal{T}}_{\alpha})}_{\nu} \to \mathcal{N}^{\xi}_{\alpha}$ be the tree embedding when $(\eta, \xi]_{\mathcal{U}} \cap D^{\mathcal{U}} = \emptyset$. When $\gamma = T - pred(\beta) \leq_T \alpha$ and $(\gamma, \alpha] \cap D^T = \emptyset$, it is easy to see that $\mathcal{U}(\mathcal{M}_{\alpha})$ is the copying tree of $\mathcal{U}(\mathcal{M}^*_{\beta})$ according to $i^{\mathcal{T}}_{\gamma\alpha} : \mathcal{M}^{*\mathcal{T}}_{\beta} \to \mathcal{M}^{\mathcal{T}}_{\alpha}$. Let $\sigma^{\xi}_{\gamma\alpha} : (\mathcal{N}^{\xi}_{\beta})^* \to \mathcal{N}^{\xi}_{\alpha}$ be the copying maps for $\xi < lh(\mathcal{U})$. For $\beta < \alpha < lh(\mathcal{T})$, because $\mathcal{M}^{\mathcal{T}}_{\beta}$ and $\mathcal{M}^{\mathcal{T}}_{\alpha}$ agree up to $\lambda^{\mathcal{T}}_{\beta}$ which is a cardinal in both models, we have

$$j^{\nu\xi}_{\beta} \restriction j^{0\nu}_{\beta}(\nu^{\mathcal{T}}_{\beta}) = j^{\nu}_{\alpha} \restriction j^{0\nu}_{\beta}(\nu^{\mathcal{T}}_{\beta})$$

whenever $\nu \leq_U \xi$ and $[0, \xi]_U$ has no drop. It is easy to see that when $[0, \xi]_U$ has a drop, then $\mathcal{N}^{\xi}_{\alpha} = \mathcal{M}^{\mathcal{U}}_{\xi}$.

For $\xi < lh(\mathcal{U})$, if $[0, \xi]_U$ has no drop, let $\langle \mathcal{K}^{\xi}_{\alpha} : \alpha < lh(\mathcal{T})\rangle$ be the copying tree $i^{\mathcal{U}}_{0\xi}\mathcal{T}$ based on $\mathcal{M}^{\mathcal{U}}_{\xi}$. Let $\langle s^{\xi}_{\gamma\alpha} : \gamma, \alpha \leq lh(\mathcal{T}), \gamma <^T \alpha, (\gamma, \alpha] \cap D^T = \emptyset\rangle$ be tree embeddings of $i^{\mathcal{U}}_{\xi}\mathcal{T}$. Let $k^{0\xi}_{\alpha} : \mathcal{M}^{\mathcal{T}}_{\alpha} \to \mathcal{K}^{\xi}_{\alpha}$ be copying maps. Note that for $\nu <^U \xi$, if $[0, \xi]_U$ has no drop, then $\langle \mathcal{K}^{\xi}_{\alpha} : \alpha < lh(\mathcal{T})\rangle$, $\langle s^{\xi}_{\gamma\alpha} : \gamma, \alpha < lh(\mathcal{T}), \gamma <^T \alpha\rangle$ are also models and embeddings of $i^{\mathcal{U}}_{\nu\xi} i^{\mathcal{U}}_{0\nu}\mathcal{T}$, the copying tree of $i_{0\nu}\mathcal{T}$ according to $i^{\mathcal{U}}_{\nu\xi} : \mathcal{M}^{\mathcal{U}}_{\nu} \to \mathcal{M}^{\mathcal{U}}_{\xi}$. Let $k^{\nu\xi}_{\alpha} : \mathcal{K}^{\nu}_{\alpha} \to \mathcal{K}^{\xi}_{\alpha}$ be the copying maps for $\alpha < lh(\mathcal{T})$. If $[0, \xi]_U$ has a drop, then let $\mathcal{K}^\xi_\alpha = \mathcal{M}^\mathcal{U}_\xi$, $s^\xi_{\gamma\alpha} = id$, $k^\xi_\alpha = i^\mathcal{U}_\xi$, $k^{\nu\xi}_\alpha = i^\mathcal{U}_{\nu\xi}$. It is not hard to see that $\langle k^{\nu\xi}_\alpha : \nu \leq^U \xi, (\nu, \xi]_U \cap D^U = \emptyset \rangle$ form a commuting system.

Claim 2.3. *Let $\alpha < lh(\mathcal{T})$. Let $\xi < lh(\mathcal{U})$ be a limit ordinal. Then $\langle \mathcal{K}^\xi_\alpha, k^{\nu\xi}_\alpha : \nu \leq^U \xi, (\nu, \xi]_U \cap D^U = \emptyset \rangle$ is the direct limit of $\langle \mathcal{K}^\nu_\alpha, k^{\nu\eta}_\alpha : v \leq^U \eta <^U \xi, (\nu, \xi]_U \cap D^U = \emptyset \rangle$.*

Proof. We show by induction on α.

When $[0, \xi]_U$ has a drop, then by definition, $\mathcal{K}^\xi_\alpha = \mathcal{M}^\mathcal{U}_\xi$, $k^{\nu\xi}_\alpha = i^\mathcal{U}_{\nu\xi}$. The claim follows.

Assume from now on that $[0, \xi]_U$ has no drop. When $\alpha = 0$, we also have $\mathcal{K}^\xi_\alpha = \mathcal{M}^\mathcal{U}_\xi$, $k^{\nu\xi}_\alpha = i^\mathcal{U}_{\nu\xi}$, so the claim follows. When $\alpha = \beta + 1$, let $\gamma = T - pred(\alpha)$. We already know that $\langle k^{\nu\xi}_\alpha : \nu <^U \xi \rangle$ form a commuting system. All we need to see is that for all $c \in \mathcal{K}^\xi_\alpha$, there are $\nu <^U \xi$ and $b \in \mathcal{K}^\nu_\alpha$ such that $c = k^{\nu\xi}_\alpha(b)$. We assume for simplicity that $[0, \alpha]_T$ has no drop. The fine ultrapower case is similar.

Fix $c \in \mathcal{K}^\xi_\alpha$. Let $f \in \mathcal{K}^\xi_\gamma$, $a \in [k^\xi_\beta(\nu^\mathcal{T}_\beta)]^{<\omega}$ be such that $c = s_{\gamma\alpha}(f)(a)$. By induction, there is $\nu <^U \xi$ and $g, b \in \mathcal{K}^\nu_\gamma$ such that $f = k^{\nu\xi}_\gamma(g)$, $a = k^{\nu\xi}_\gamma(b)$. Then

$$\begin{aligned} c &= s^\xi_{\gamma\alpha}(k^{\nu\xi}_\gamma(g))(k^{\nu\xi}_\beta(b)) \\ &= k^{\nu\xi}_\alpha(s^\nu_{\gamma\alpha}(g))(k^{\nu\xi}_\beta(b)) \\ &= k^{\nu\xi}_\alpha(s^\nu_{\gamma\alpha}(g))(k^{\nu\xi}_\alpha(b)) \qquad \text{by agreement in copying} \\ &= k^{\nu\xi}_\alpha(s^\nu_{\gamma\alpha}(g)(b)). \end{aligned}$$

Suppose then α is a limit. Fix $c \in \mathcal{K}^\xi_\alpha$. Let $\gamma <^T \alpha$, $b \in \mathcal{K}^\xi_\gamma$ be such that $c = s^\xi_{\gamma\alpha}(a)$. By induction, there is $\nu <^U \xi$ and $b \in \mathcal{K}^\nu_\gamma$ such that $a = k^{\nu\xi}_\gamma(b)$. So

$$\begin{aligned} c &= s^\xi_{\gamma\alpha}(k^{\nu\xi}_\gamma(b)) \\ &= k^{\nu\xi}_\alpha(s^\nu_{\gamma\alpha}(b)). \end{aligned}$$

$\square$

Claim 2.4. *Suppose that* $\alpha < lh(\mathcal{T}), \nu \leq^U \xi, [0,\xi]_U \cap D^U = \emptyset$. *Then*

$$j^{\nu\xi}_\alpha \restriction \mathcal{N}^\nu_\alpha(0) = k^{\nu\xi}_\alpha \restriction \mathcal{K}^\nu_\alpha(0),$$

Proof. We show by induction on α. When $\alpha = 0$, the claim is immediate by definition. When $\alpha = \beta + 1$ is a successor, let $\gamma = T - pred(\alpha)$. Fix some $c \in \mathcal{K}^\nu_\alpha(0)$. Suppose that $c = s_{\gamma\alpha}(f)(a)$, for some $f \in \mathcal{K}^\nu_\gamma(0)$, $a \in j^{0\nu}_\beta(\lambda^{\mathcal{T}}_\beta)$. Then

$$\begin{aligned} k^{\nu\xi}_\alpha(a) &= k^{\nu\xi}_\beta(a) && \text{by agreement in copying} \\ &= j^{\nu\xi}_\beta(a) && \text{by induction} \\ &= j^{\nu\xi}_\alpha(a). \end{aligned}$$

So

$$\begin{aligned} k^{\nu\xi}_\alpha(c) &= k^{\nu\xi}_\alpha(s_{\gamma\alpha}(f)(a)) \\ &= s^\xi_{\gamma\alpha}(k^{\nu\xi}_\gamma(f))(k^{\nu\xi}_\alpha(a)) \\ &= s^\xi_{\gamma\alpha}(j^{\nu\xi}_\gamma(f))(j^{\nu\xi}_\alpha(a)) && \text{by induction hypothesis} \\ &= \sigma^\xi_{\gamma\alpha}(j^{\nu\xi}_\gamma(f))(j^{\nu\xi}_\alpha(a)) && \text{since } \sigma^\xi_{\gamma\alpha}, s^\xi_{\gamma\alpha} \text{ agree below } \mathcal{N}^\xi_\gamma(0) \\ &= j^{\nu\xi}_\alpha(\sigma^\nu_{\gamma\alpha}(f))(j^{\nu\xi}_\alpha(a)) && \text{by copying} \\ &= j^{\nu\xi}_\alpha(\sigma^\nu_{\gamma\alpha}(f)(a)) \\ &= j^{\nu\xi}_\alpha(s^\nu_{\gamma\alpha}(f)(a)) && \text{since } \sigma^\nu_{\gamma\alpha}, s^\nu_{\gamma\alpha} \text{ agree below } \mathcal{N}^\nu_\gamma(0) \\ &= j^{\nu\xi}_\alpha(c). \end{aligned}$$

Suppose now α is a limit. Fix $c \in \mathcal{K}^\nu_\alpha(0)$. Let $\gamma <^{\mathcal{T}} \alpha$ and $b \in \mathcal{K}^\nu_\gamma(0)$ be such that

$c = s^{\nu}_{\gamma\alpha}(b)$. Then

$$\begin{aligned} k^{\nu\xi}_{\alpha}(c) &= k^{\nu\xi}_{\alpha}(s^{\nu}_{\gamma\alpha}(b)) \\ &= s^{\xi}_{\gamma\alpha}(k^{\nu\xi}_{\gamma}(b)) \\ &= s^{\xi}_{\gamma\alpha}(j^{\nu\xi}_{\gamma}(b)) \qquad \text{by induction} \\ &= \sigma^{\xi}_{\gamma\alpha}(k^{\nu\xi}_{\gamma}(b)) \\ &= j^{\nu\xi}_{\alpha}(\sigma^{\nu}_{\gamma\alpha}(b)) \\ &= j^{\nu\xi}_{\alpha}(s^{\nu}_{\gamma\alpha}(b)) \\ &= j^{\nu\xi}_{\alpha}(c). \end{aligned}$$

□

We plan to define $\langle t^{\xi}_{\alpha} : \alpha < lh(\mathcal{T}), \xi < lh(\mathcal{U})\rangle$ with the following properties.

1. $t^{\xi}_{\alpha} : \mathcal{N}^{\xi}_{\alpha} \to \mathcal{K}^{\xi}_{\alpha}$ is an embedding. When $deg^{\mathcal{T}}(\alpha) = \omega$, t^{ξ}_{α} is fully elementary. When $deg^{\mathcal{T}}(\alpha) = n < \omega$, t^{ξ}_{α} is $r\Sigma_{n+1}$-elementary.
2. $t^{0}_{\alpha} = id_{\mathcal{M}^{\mathcal{T}}_{\alpha}}$,
3. $t^{\xi}_{0} = id_{\mathcal{M}^{\mathcal{U}}_{\xi}}$,
4. If $[0,\xi]_U$ has a drop, then $t^{\xi}_{\alpha} = id_{\mathcal{M}^{\mathcal{U}}_{\xi}}$,
5. If $[0,\xi]_U$ has no drop, then $t^{\xi}_{\alpha} \upharpoonright \mathcal{N}^{\xi}_{\alpha}(0) = id_{\mathcal{N}^{\xi}_{\alpha}(0)}$,
6. If $\nu <^U \xi$ and $(\nu,\xi]_U$ has no drop, then $k^{\nu\xi}_{\alpha} \circ t^{\nu}_{\alpha} = t^{\xi}_{\alpha} \circ j^{\nu\xi}_{\alpha}$,
7. If $\gamma <^T \alpha$ and $(\gamma,\alpha]_T$ has no drop, then $s^{\xi}_{\gamma\alpha} \circ t^{\xi}_{\gamma} = t^{\xi}_{\alpha} \circ \sigma^{\xi}_{\gamma\alpha}$,
8. If $\gamma < \alpha$ and $[0,\xi]_U$ has no drop, then $t^{\xi}_{\gamma} \upharpoonright j^{0\xi}_{\gamma}(\nu^{\mathcal{T}}_{\gamma}) = t^{\xi}_{\alpha} \upharpoonright j^{0\xi}_{\gamma}(\nu^{\mathcal{T}}_{\gamma})$,
9. If $[0,\xi]_U$ has no drop, $\alpha = \beta + 1$ is a successor, $\gamma = T - pred(\alpha)$, then for all $c \in \mathcal{N}^{\xi}_{\alpha}$, one of the following holds.

(a) $deg^{\mathcal{T}}(a) = \omega$ and there are $b \in [j^{\xi}_{\beta}(\nu^{\mathcal{T}}_{\beta})]^{<\omega}$, $g \in (\mathcal{N}^{\xi}_{\alpha})^*$ such that $t^{\xi}_{\alpha}(c) = [t^{\xi}_{\beta}(b), t^{\xi}_{\gamma}(g)]_{k^{0\xi}_{\beta}(E^{\mathcal{T}}_{\beta})}$.

(b) $deg^{\mathcal{T}}(a) = n < \omega$ and there are $b \in [j^{\xi}_{\beta}(\nu^{\mathcal{T}}_{\beta})]^{<\omega}$, g a $r\Sigma_{n+1}$-Skolem term in $(\mathcal{N}^{\xi}_{\alpha})^*$ such that $t^{\xi}_{\alpha}(c) = [t^{\xi}_{\beta}(b), t^{\xi}_{\gamma}(g)]_{k^{0\xi}_{\beta}(E^{\mathcal{T}}_{\beta})}$.

Figure 2.1 illustrates the interactions among the maps arising from rearranging a tree. We define t^{ξ}_{α} by induction on lexicographic ordering on (α, ξ).

When $\alpha = 0$, let $t^{\xi}_{\alpha} = id_{\mathcal{M}^{\mathcal{U}}_{\xi}}$ for all $\xi < lh(\mathcal{U})$.

When $\alpha = \beta + 1$ is a successor, let $\gamma = T - pred(\alpha)$. We shall define t^{ξ}_{α} by a subinduction on ξ. When $\xi = 0$, let $t^{\xi}_{\alpha} = id_{\mathcal{M}^{\mathcal{T}}_{\alpha}}$. When $\xi = \eta + 1$, denote $\nu = U - pred(\xi)$. In case $[0, \xi]_U$ has a drop, we let $t^{\xi}_{\alpha} = id_{\mathcal{M}^{\mathcal{U}}_{\xi}}$. Assume now $[0, \xi]_U$ has no drop. We also assume for simplicity that $[0, \alpha]_T$ has no drop. Otherwise we deal with $r\Sigma_{n+1}$-Skolem terms instead. We define $t^{\xi}_{\alpha} : \mathcal{N}^{\xi}_{\alpha} \to \mathcal{K}^{\xi}_{\alpha}$ as follows. Fix $c \in \mathcal{N}^{\xi}_{\alpha}$. There is $a \in [\nu^{\mathcal{U}}_{\eta}]^{<\omega}$ and $f \in \mathcal{N}^{\nu}_{\alpha}$ such that $c = [a, f]^{\mathcal{U}}_{E_{\eta}}$. By property 10 of t^{ν}_{α}, there is $b \in [j^{\nu}_{\beta}(\nu^{\mathcal{T}}_{\beta})]^{<\omega}$ and $g \in (\mathcal{N}^{\nu}_{\alpha})^*$ such that $t^{\nu}_{\alpha}(f) = [t^{\nu}_{\beta}(b), t^{\nu}_{\gamma}(g)]_{k^{0\nu}_{\beta}(E^{\mathcal{T}}_{\beta})}$. Let h be the transpose of g, i.e. $h(x)(y) = g(y)(x)$ for all x, y. We set

$$t^{\xi}_{\alpha}(c) = [t^{\xi}_{\beta}(j^{\nu\xi}_{\beta}(b)), t^{\xi}_{\gamma}([a, h]_{E^{\mathcal{U}}_{\eta}})]_{k^{0\xi}_{\beta}(E^{\mathcal{T}}_{\beta})}$$

We should check that t^{ξ}_{α} is well-defined and elementary. Take a formula $\phi(\cdot)$ with

$$
\begin{tikzcd}
\mathcal{M}_\alpha^{*\mathcal{T}} \arrow[rrr, "i^{\mathcal{T}}_{\gamma\alpha}", "\text{tree embedding of } \mathcal{T}"'] \arrow[dr, dotted, "j^\nu_\gamma"] \arrow[dd, "k^\nu_\gamma"] & & & \mathcal{M}_\alpha^{\mathcal{T}} \arrow[dr, "j^\nu_\gamma"] \arrow[dd, "k^\nu_\alpha"] & \\
& (\mathcal{N}^\nu_\alpha)^* \arrow[rrr, dotted, "\sigma^\nu_{\gamma\alpha}", "\text{copying map}"'] \arrow[dl, dotted, "t^\nu_\gamma"] \arrow[dd, dotted, "j^{\nu\xi}_\gamma"] & & & \mathcal{N}^\nu_\alpha \arrow[dl, "t^\nu_\alpha"] \arrow[dd, "j^{\nu\xi}_\alpha"] \\
(\mathcal{K}^\nu_\alpha)^* \arrow[rrr, "s^\nu_{\gamma\alpha}", "\text{tree embedding of } k^\nu_0\mathcal{T}"'] \arrow[dd, "k^{\nu\xi}_\gamma"] & & & \mathcal{K}^\nu_\alpha \arrow[dd, "k^{\nu\xi}_\alpha"] & \\
& (\mathcal{N}^\xi_\alpha)^* \arrow[rrr, dotted, "\sigma^\xi_{\gamma\alpha}", "\text{copying map}"'] \arrow[dl, dotted, "t^\xi_\gamma"] \arrow[dd, dotted, "\text{tree embeddings of } \mathcal{U}(\mathcal{M}^*_\alpha)^{\mathcal{T}}"] & & & \mathcal{N}^\xi_\alpha \arrow[dl, "t^\xi_\alpha"] \arrow[dd, "\text{tree embeddings of } \mathcal{M}^{\mathcal{T}}_\alpha"] \\
(\mathcal{K}^\xi_\alpha)^* \arrow[rrr, "s^\xi_{\gamma\alpha}", "\text{tree embedding of } k^\xi_0\mathcal{T}"'] \arrow[d, "\text{copying maps}"'] & & & \mathcal{K}^\xi_\alpha \arrow[d, "\text{copying maps}"'] & \\
{} & {} & & {} & {}
\end{tikzcd}
$$

Figure 2.1: Rearranging a stack

one free variable as an example,

$$
\begin{aligned}
&\mathcal{N}_\alpha^\xi \models \phi(c)\\
&\to \exists A \in (E_\eta^{\mathcal{U}})_a \forall x \in A\ \mathcal{N}_\alpha^\nu \models \phi(f(x))\\
&\to \exists A \in (E_\eta^{\mathcal{U}})_a \forall x \in A\ \mathcal{K}_\alpha^\nu \models \phi(t_\alpha^\nu(f)(x)) \qquad \text{since } t_\gamma^\nu \restriction \mathcal{N}_\alpha^\nu(0) = id \text{ by 5}\\
&\to \exists A \in (E_\eta^{\mathcal{U}})_a \forall x \in A\ \{y < \mathrm{crt}(k_\beta^{0\nu}(E_\beta^{\mathcal{T}})) : (\mathcal{K}_\alpha^\nu)^* \models \phi(t_\gamma^\nu(g)(y)(s_{\gamma\alpha}^\nu(x)))\} \in (k_\beta^{0\nu}(E_\beta^{\mathcal{T}}))_{t_\beta^\nu(b)}\\
&\to \exists A \in (E_\eta^{\mathcal{U}})_a \forall x \in A\ \{y < \mathrm{crt}(k_\beta^{0\nu}(E_\beta^{\mathcal{T}})) : (\mathcal{K}_\alpha^\nu)^* \models \phi(t_\gamma^\nu(g)(y)(x))\} \in (k_\beta^{0\nu}(E_\beta^{\mathcal{T}}))_{t_\beta^\nu(b)}\\
&\qquad (\text{since } s_{\gamma\alpha}^\nu \restriction (\mathcal{K}_\alpha^\nu)^*(0) = id \text{ and } \mathrm{crt}(E_\eta^{\mathcal{U}}) < o((\mathcal{K}_\alpha^\nu)^*(0)))\\
&\to \exists A \in (E_\eta^{\mathcal{U}})_a \forall x \in A\ \{y < \mathrm{crt}(j_\beta^{0\nu}(E_\beta^{\mathcal{T}})) : (\mathcal{N}_\alpha^\nu)^* \models \phi(g(y)(x))\} \in (j_\beta^{0\nu}(E_\beta^{\mathcal{T}}))_b\\
&\qquad (\text{since } t_\gamma^\nu \restriction \nu_\gamma^{\mathcal{T}} = t_\gamma^\nu \restriction \nu_\gamma^{\mathcal{T}} \text{ by 8})\\
&\to \{y < \mathrm{crt}(j_\beta^{0\nu}(E_\beta^{\mathcal{T}})) : (\mathcal{N}_\alpha^\xi)^* \models \phi(j_\gamma^{\nu\xi}(g)(j_\gamma^{\nu\xi}(y))(a))\} \in (j_\beta^{0\nu}(E_\beta^{\mathcal{T}}))_b\\
&\to \{y < \mathrm{crt}(j_\beta^{0\xi}(E_\beta^{\mathcal{T}})) : (\mathcal{N}_\alpha^\xi)^* \models \phi(j_\gamma^{\nu\xi}(g)(y)(a))\} \in (j_\beta^{0\xi}(E_\beta^{\mathcal{T}}))_{j_\beta^{\nu\xi}(b)}\\
&\qquad \text{since } j_\beta^{\nu\xi} \restriction j_\gamma^{0\nu}(\nu_\beta^{\mathcal{T}}) = j_\gamma^{\nu\xi} \restriction j_\gamma^{0\nu}(\nu_\beta^{\mathcal{T}})\\
&\to \{y < \mathrm{crt}(k_\beta^{0\xi}(E_\beta^{\mathcal{T}})) : (\mathcal{K}_\alpha^\xi)^* \models \phi(t_\gamma^\xi(j_\gamma^{\nu\xi}(g))(y)(a))\} \in (k_\beta^{0\xi}(E_\beta^{\mathcal{T}}))_{t_\beta^\xi(j_\beta^{\nu\xi}(b))}\\
&\qquad (\text{by elementarity of } t_\gamma^\xi, \text{ and } t_\beta^\xi \restriction j_\gamma^\xi(\nu_\gamma^{\mathcal{T}}) = t_\gamma^\xi \restriction j_\gamma^\xi(\nu_\gamma^{\mathcal{T}}) \text{ by 8})\\
&\to \{y < \mathrm{crt}(k_\beta^{0\xi}(E_\beta^{\mathcal{T}})) : (\mathcal{K}_\alpha^\xi)^* \models \phi(t_\gamma^\xi(j_\gamma^{\nu\xi}(h))(a)(y))\} \in (k_\beta^{0\xi}(E_\beta^{\mathcal{T}}))_{t_\beta^\xi(j_\beta^{\nu\xi}(b))}\\
&\to \{y < \mathrm{crt}(k_\beta^{0\xi}(E_\beta^{\mathcal{T}})) : (\mathcal{K}_\alpha^\xi)^* \models \phi(t_\gamma^\xi(j_\gamma^{\nu\xi}(h)(a))(y))\} \in (k_\beta^{0\xi}(E_\beta^{\mathcal{T}}))_{t_\beta^\xi(j_\beta^{\nu\xi}(b))}\\
&\qquad (\text{by 5 on } t_\gamma^\xi)\\
&\to \mathcal{K}_\alpha^\xi \models \phi([t_\beta^\xi(j_\beta^\nu \xi(b)), t_\gamma^\xi([a,h]_{E_\eta^{\mathcal{U}}})]_{k_\beta^{0\xi}(E_\beta^{\mathcal{T}})}).
\end{aligned}
$$

So t_α^ξ is well-defined and elementary. We need to verify that t_α^ξ has properties 1-9. 9 is clear by definition. The rest are easy except 5,8. For 5, let $c < \mathcal{N}_\alpha^\xi(0)$. We may write $c = [a,f]_{E_\eta^{\mathcal{U}}}$, where $a \in [\nu_\eta^{\mathcal{U}}]^{<\omega}$, $f \in \mathcal{N}_\alpha^\nu$, $f : \kappa_\eta^{\mathcal{U}} \to \mathcal{N}_\alpha^\nu(0)$. By property 9 on t_α^ν, there is $b \in [j_\beta^\nu(\nu_\beta^{\mathcal{T}})]^{<\omega}$ and $g \in (\mathcal{N}_\alpha^\nu)^*$ such that $t_\alpha^\nu(f) = [t_\beta^\nu(b), t_\gamma^\nu(g)]_{k_\beta^{0\xi}(E_\beta^{\mathcal{T}})}$.

It is easy to check by definition that $t^\xi_\alpha \restriction \nu^{\mathcal{U}}_\eta = id$. Therefore

$$\begin{aligned} t^\xi_\alpha(c) &= t^\xi_\alpha(j^{\nu\xi}_\alpha(f)(a)) \\ &= k^{\nu\xi}_\alpha(t^\nu_\alpha(f))(t^\xi_\alpha(a)) \\ &= k^{\nu\xi}_\alpha(f)(t^\xi_\alpha(a)) \qquad \text{by property 5 on } t^\nu_\alpha \\ &= k^{\nu\xi}_\alpha(f)(a) \\ &= j^{\nu\xi}_\alpha(f)(a) \qquad \text{by Claim 2.4} \\ &= c. \end{aligned}$$

To see 8, we need to check that $t^\xi_\beta \restriction j^{0\xi}_\beta(\nu^{\mathcal{T}}_\beta) = t^\xi_\alpha \restriction j^{0\xi}_\beta(\nu^{\mathcal{T}}_\beta)$. Fix $c < j^{0\xi}_\beta(\nu^{\mathcal{T}}_\beta)$. We may write $c = j^{\nu\xi}_\alpha(f)(a)$, where $a \in [\nu^{\mathcal{U}}_\eta]^{<\omega}$, $f \in \mathcal{N}^\nu_\alpha$, $f : \kappa^{\mathcal{U}}_\eta \to j^{0\nu}_\beta(\nu^{\mathcal{T}}_\beta)$. Then

$$\begin{aligned} t^\xi_\alpha(c) &= t^\xi_\alpha(j^{\nu\xi}_\alpha(f)(a)) \\ &= k^{\nu\xi}_\alpha(t^\nu_\alpha(f))(t^\xi_\alpha(a)) \\ &= k^{\nu\xi}_\alpha(t^\nu_\beta(f))(t^\xi_\alpha(a)) \qquad \text{By property 9 on } t^\nu_\alpha \\ &= k^{\nu\xi}_\alpha(t^\nu_\beta(f))(a) \\ &= t^\xi_\beta(j^{\nu\xi}_\beta(f))(a) \\ &= t^\xi_\beta(j^{\nu\xi}_\beta(f)(a)) \\ &= t^\xi_\beta(c). \end{aligned}$$

This finishes definition of t^ξ_α when ξ is a successor. When ξ is a limit ordinal, if $[0,\xi]_U$ has a drop, we again let $t^\xi_\alpha = id_{\mathcal{M}^{\mathcal{T}}_\alpha}$. Assume now $[0,\xi]_U$ has no drop. We know $\mathcal{N}^\xi_\alpha$ is the direct limit of $\mathcal{N}^\nu_\alpha$ for $\nu <^U \xi$ under $i^{\nu\eta}_\alpha$ for $\nu <^U \eta <^U \xi$. By Claim 2.3, $\mathcal{K}^\xi_\alpha$ is the direct limit of $\mathcal{N}^\nu_\alpha$ under $k^{\nu\eta}_\alpha$ for $\nu <^U < \eta <^U \xi$. We then let $t^\xi_\alpha : \mathcal{N}^\xi_\alpha \to \mathcal{K}^\xi_\alpha$ be the natural embedding. Properties 1-9 are immediate so let us check 10. Fix $c \in \mathcal{N}^\xi_\alpha$. Let $\nu <^{\mathcal{U}} \xi$ and $a \in \mathcal{N}^\nu_\alpha$ such that $c = j^{\nu\xi}_\alpha(a)$. By property

10 of t^ν_α, there are b, g such that $t^\nu_\alpha(a) = s^\nu_{\gamma\alpha}(t^\nu_\gamma(g))(t^\nu_\beta(b))$. So

$$\begin{aligned}
t^\xi_\alpha(c) &= k^{\nu\xi}_\alpha(t^\nu_\alpha(a))\\
&= k^{\nu\xi}_\alpha(s^\nu_{\gamma\alpha}(t^\nu_\gamma(g))(t^\nu_\beta(b)))\\
&= s^\xi_{\gamma\alpha}(t^\xi_\gamma(j^{\nu\xi}_\gamma(g)))(k^{\nu\xi}_\beta(t^\nu_\beta(b)))\\
&= s^\xi_{\gamma\alpha}(t^\xi_\gamma(j^{\nu\xi}_\gamma(g)))(t^\xi_\beta(j^{\nu\xi}_\beta(b)))\\
&= [t^\xi_\beta(j^{\nu\xi(b)}_\beta), t^\xi_\gamma(j^{\nu\xi}_\gamma(g))]_{k^{0\xi}_\beta(E^{\mathcal{T}}_\beta)}.
\end{aligned}$$

This finishes the definition of t^ξ_α for all ξ when α is a successor. When α is a limit, we know that for all $\xi < lh(\mathcal{U})$, $\mathcal{K}^\xi_\alpha$ is the direct limit of $\mathcal{K}^\xi_\gamma$ under $s^\xi_{\gamma\beta}$ for $\gamma <^T \beta <^T \alpha$, $(\gamma, \alpha]_T$ does not drop. A similar proof as in Claim 2.3 shows that $\mathcal{N}^\xi_\alpha$ is the direct limit of $\mathcal{N}^\xi_\gamma$ under $\sigma^\xi_{\gamma\beta}$ for $\gamma <^T \alpha$, $(\gamma, \alpha]_T$ does not drop. We then let $t^\xi_\alpha : \mathcal{N}^\xi_\alpha \to \mathcal{K}^\xi_\alpha$ be the natural embedding $t^\xi_\alpha(\sigma^\xi_{\gamma\alpha}(b)) = s^\xi_{\gamma\alpha}(t^\xi_\gamma(b))$. Properties 1-9 are easily verified. Property 10 is vacuous.

Finally note that $\mathcal{K}^{lh(\mathcal{U})-1}_{lh(\mathcal{T})-1} = \mathcal{R}_1$, $\mathcal{N}^{lh(\mathcal{U})-1}_{lh(\mathcal{T})-1} = \mathcal{R}_2$, and $t^{lh(\mathcal{U})-1}_{lh(\mathcal{T})-1}$ is an embedding from $\mathcal{R}_1$ to $\mathcal{R}_2$. $t^{lh(\mathcal{U})-1}_{lh(\mathcal{T})-1}$ is the desired map π as in Lemma 2.2. □

It is not hard to show the following extension of Lemma 2.2.

Lemma 2.5. *Suppose that $\mathcal{P}$ is a hod premouse. Let $\zeta < \lambda^{\mathcal{P}}$ be an ordinal. Suppose that $\vec{\mathcal{T}}$ is an stack on $\mathcal{P}$ above $\mathcal{P}(\zeta)$ with last model $\mathcal{Q}_1$ such that $i^{\vec{\mathcal{T}}}$ exists, $\vec{\mathcal{U}}$ is a stack on $\mathcal{P}$ with last model $\mathcal{Q}_2$ below $\mathcal{P}(\zeta)$ such that $i^{\vec{\mathcal{U}}}$ exists. Let $\mathcal{R}_1$ be the last model of the stack $\vec{\mathcal{U}}(\mathcal{Q}_1)$ on $\mathcal{Q}_1$. Suppose that all models of the copying stack $i^{\vec{\mathcal{U}}}\vec{\mathcal{T}}$ on $\mathcal{Q}_2$ are wellfounded. Let $\mathcal{R}_2$ be the last model of $i^{\vec{\mathcal{U}}}\vec{\mathcal{T}}$. Let $l : \mathcal{Q}_1 \to \mathcal{R}_2$ be the copying map, $j : \mathcal{Q}_1 \to \mathcal{R}_1$ and $k : \mathcal{Q}_2 \to \mathcal{R}_2$ be associated tree embeddings. Then there is an elementary embedding $\pi : \mathcal{R}_1 \to \mathcal{R}_2$ such that $\pi \circ j = l$.*

Definition 2.6 (Simple rearrangement of a stack)**.** Suppose that $\mathcal{P}$ is a hod premouse, $\lambda^{\mathcal{P}}$ has measurable cofinality in $\mathcal{P}$. Let ζ be an least ordinal such that

$\delta^{\mathcal{P}}_\zeta > \mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$. Suppose that $\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}$ is a stack on $\mathcal{P}$ with last model $\mathcal{R}$. Suppose that $\vec{\mathcal{T}}$ is above $\mathcal{P}(\zeta)$, $\vec{\mathcal{U}}$ is below $\mathcal{P}(\zeta)$. Denote $\vec{\mathcal{W}} = \vec{\mathcal{U}}(\mathcal{P})$. Suppose that $i^{\vec{\mathcal{W}}}$ exists and that all models of the copying tree $i^{\vec{\mathcal{W}}}\vec{\mathcal{T}}$ are wellfounded. Let $\mathcal{R}^*$ be the last model of $i^{\vec{\mathcal{W}}}\vec{\mathcal{T}}$. Let $\pi : \mathcal{R} \to \mathcal{R}^*$ be as in Lemma 2.2. We say that $(\vec{\mathcal{W}}, \mathcal{Q}^*, i^{\vec{\mathcal{W}}}\vec{\mathcal{T}}, \mathcal{R}^*, \pi)$ is the simple rearrangement of $\vec{\mathcal{T}}^\frown\vec{\mathcal{U}}$ with respect to ζ.

Definition 2.7 (Rearrangement of a stack)**.** Suppose that $\mathcal{P}$ is a hod premouse, $\lambda^{\mathcal{P}}$ has measurable cofinality in $\mathcal{P}$. Let ζ be an ordinal such that $\delta^{\mathcal{P}}_\zeta > \mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$. Suppose that $\vec{\mathcal{T}}$ is a stack on $\mathcal{P}$ with last model such that $i^{\vec{\mathcal{T}}}(\mathcal{P}(\zeta))$ is defined. Let $\langle \mathcal{M}_\alpha, \mathcal{M}^*_\alpha, \vec{\mathcal{T}}_\alpha, i_{\alpha\beta} : \alpha < \beta \leq \eta\rangle$ be the essential components of $\vec{\mathcal{T}}$. Then the rearrangement of $\vec{\mathcal{T}}$ with respect to ζ is a sequence $\langle \vec{\mathcal{U}}_\alpha, \mathcal{Q}_\alpha, \vec{\mathcal{V}}_\alpha, \mathcal{R}_\alpha, \sigma_{\alpha\beta}, \pi_{\alpha\beta}, \psi_\alpha : \alpha < \beta \leq \eta\rangle$ with the following properties.

1. For each $\alpha < \eta$, $\vec{\mathcal{U}}_\alpha$ is a stack on $\mathcal{P}$ below $\mathcal{P}(\zeta)$ with last model $\mathcal{Q}_\alpha$ such that $i^{\vec{\mathcal{U}}_\alpha}$ exists, $\vec{\mathcal{V}}_\alpha$ is a stack on $\mathcal{Q}_\alpha$ above $\mathcal{P}(\zeta)$ with last model $\mathcal{R}_\alpha$.

2. $\sigma_{\alpha\beta} : \mathcal{Q}_\alpha \to \mathcal{Q}_\beta$, $\pi_{\alpha\beta} : \mathcal{R}_\alpha \to \mathcal{R}_\beta$, $\psi(\alpha) : \mathcal{M}_\alpha \to \mathcal{R}_\alpha$ are sufficiently elementary embeddings.

3. For each $\alpha < \eta$, if $\vec{\mathcal{T}}_\alpha$ is above $i_{0\alpha}(\mathcal{P}(\zeta))$, then $\vec{\mathcal{U}}_{\alpha+1} = \vec{\mathcal{U}}_\alpha$, $\sigma_{\alpha\alpha+1} = id$, $\vec{\mathcal{V}}_{\alpha+1} = \vec{\mathcal{V}}_\alpha{}^\frown\psi_\alpha\vec{\mathcal{T}}_\alpha$, $\pi_{\alpha\alpha+1} = i^{\psi_\alpha\vec{\mathcal{T}}_\alpha}$, $\psi_{\alpha+1} : \mathcal{M}_{\alpha+1} \to \mathcal{R}_{\alpha+1}$ is the copying map.

4. For each $\alpha < \eta$, if $\vec{\mathcal{T}}_\alpha$ is below $i_{0\alpha}(\mathcal{P}(\zeta))$, let $\psi^* : \mathcal{M}_{\alpha+1} \to \mathcal{R}_{\alpha+1}$ be the copying map. let $(\vec{\mathcal{U}}^*, \mathcal{Q}^*, \vec{\mathcal{V}}^*, \mathcal{R}^*, \pi^*)$ be the simple rearrangement of $\vec{\mathcal{V}}_\alpha{}^\frown\psi_\alpha\vec{\mathcal{T}}_\alpha$, then $\vec{\mathcal{U}}_{\alpha+1} = \vec{\mathcal{U}}_\alpha{}^\frown\vec{\mathcal{U}}^*$, $\sigma_{\alpha\alpha+1} = i^{\vec{\mathcal{U}}^*}$, $\vec{\mathcal{V}}_{\alpha+1} = i^{\vec{\mathcal{U}}^*}\vec{\mathcal{V}}_\alpha$, $\pi_{\alpha\alpha+1} = \pi^* \circ i^{\psi_\alpha\vec{\mathcal{T}}_\alpha}$, $\psi_{\alpha+1} = \pi^* \circ \psi^*$.

5. For each $\alpha < \eta$ limit, we have $\vec{\mathcal{U}}_\alpha = \bigcup_{\gamma<\alpha}\vec{\mathcal{U}}_\gamma$, $\sigma_{\gamma\alpha} : \mathcal{Q}_\gamma \to \mathcal{Q}_\alpha$ is the direct limit map. $\vec{\mathcal{V}}_\alpha = \bigcup_{\gamma<\alpha}\sigma_{\gamma\alpha}\vec{\mathcal{V}}_\gamma$, $\pi_{\gamma\alpha} : \mathcal{R}_\gamma \to \mathcal{R}_\alpha$ is the direct limit map. $\psi_\alpha : \mathcal{M}_\alpha \to \mathcal{R}_\alpha$ is the direct limit map.

We say that $(\vec{\mathcal{U}}_\eta, \mathcal{Q}_\eta, \vec{\mathcal{V}}_\eta)$ is the result of the rearrangement of $\vec{\mathcal{T}}$ with respect to ζ.

It may happen that at some point, the copying stack $\pi_\alpha\vec{\mathcal{T}}_\alpha$ or $\sigma_{\gamma\alpha}\vec{\mathcal{U}}_\gamma$ is not wellfounded. If any model of the copying stack is illfounded, we leave the rearrangement of $\vec{\mathcal{T}}$ undefined.

In the next two lemmas we show that given a hod pair $(\mathcal{P},\Sigma)$ such that Σ is fullness preserving and has branch condensation, $\lambda^{\mathcal{P}}$ has measurable cofinality in $\mathcal{P}$, ζ is an ordinal such that $\delta^{\mathcal{P}}_\zeta > \mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$, then Σ can be recovered from

$$\begin{aligned}\{(\mathcal{R},\Sigma_{\mathcal{R}}):\ &\text{There is a stack } \mathcal{T} \text{ on } \mathcal{P}(\nu) \text{ with last model such that } i^{\mathcal{T}} \text{ exists,}\\ &\text{and } \mathcal{R} \text{ is a hod initial segment of the last model of } \mathcal{T}(\mathcal{P})\}.\end{aligned}$$

by rearranging stacks. The technique of rearranging stacks enables us to define the $S^{\mathfrak{P}}$-operator in Sections 2.5 and 2.6 in the measurable cofinality case by induction.

Lemma 2.8. *Suppose that $(\mathcal{P},\Sigma)$ is a hod pair, Σ is fullness preserving and has branch condensation. Let $\vec{\mathcal{T}}$,$\vec{\mathcal{U}}$ be two stacks on $\mathcal{P}$ with last model $\mathcal{Q},\mathcal{R}$. Let $\vec{\mathcal{T}}^0 = \vec{\mathcal{T}}\restriction$ nondrop, $\vec{\mathcal{U}}^0 = \vec{\mathcal{U}}\restriction$ nondrop with last models $\mathcal{Q}^0,\mathcal{R}^0$ respectively. Suppose that there is an elementary embedding $\pi^0:\mathcal{R}^0\to\mathcal{Q}^0$ such that $i^{\vec{\mathcal{T}}^0} = \pi\circ i^{\vec{\mathcal{U}}^0}$. Suppose in addition, if $i^{\vec{\mathcal{T}}}$ does not exist, then there is a* $\deg(\vec{\mathcal{T}})$*-embedding $\pi:\mathcal{R}\to\mathcal{Q}$ such that $\pi\restriction\mathcal{R}^- = \pi_0\restriction\mathcal{R}^-$. Then $\Sigma_{\mathcal{R}} = (\Sigma_{\mathcal{Q}})^\pi$.*

Proof. Note that Σ is positional from [7, Lemma 3.6.1]. So $\Sigma_{\mathcal{R}}$, $\Sigma_{\mathcal{Q}}$ makes sense. Suppose the conclusion is false. Let γ be least such that $\Sigma_{\mathcal{R}_\gamma} \neq (\Sigma_{\mathcal{Q}_{\pi(\gamma)}})^\pi$. We may and shall assume by wellfoundedness that there is no $\vec{\mathcal{T}}_1$, $\vec{\mathcal{U}}_1$, $\mathcal{Q}_1$, $\mathcal{R}_1$, π_1, π^0_1, γ_1 such that $(\vec{\mathcal{T}}^\frown\vec{\mathcal{T}}_1,\vec{\mathcal{U}}^\frown\vec{\mathcal{U}}_1,\pi_1)$ consistute another counter example to the lemma in place of $(\vec{\mathcal{T}},\vec{\mathcal{U}},\pi^0)$, $\Sigma_{\mathcal{R}^1_\gamma} \neq (\Sigma_{\mathcal{Q}^1_\pi(\gamma)})^{\pi_1}$, but $i^{\vec{\mathcal{T}}_1}(\gamma) > \gamma_1$.

Thus there is a stack $\vec{\mathcal{V}}$ on $\vec{\mathcal{R}}(\gamma)$ according to $\Sigma_{\mathcal{R}(\gamma)}$ without last model such that $\pi\vec{\mathcal{V}}$ is according to $\Sigma_{\mathcal{Q}(j(\gamma))}$, but $\Sigma_{\mathcal{Q}(j(\gamma))}(\pi\vec{\mathcal{V}}) = b \neq c = \Sigma_{\mathcal{R}(\gamma)}(\vec{\mathcal{V}})$. Let $j:\mathcal{M}^{\vec{\mathcal{V}}}_b \to \mathcal{M}^{j\vec{\mathcal{V}}}_b$ be the canonical map.

Let $\langle\mathcal{M}_\alpha,\mathcal{M}^*_\alpha,\vec{\mathcal{V}}_\alpha,i_{\alpha\beta}:\alpha<\beta\leq\eta+1\rangle$ be the essential components of $\vec{\mathcal{V}}$.

Case 1. $\vec{\mathcal{T}}^0 = \vec{\mathcal{T}}$.

If $i_b^{\pi\vec{\mathcal{U}}}$ is defined, then because $i_b^{\pi\vec{\mathcal{V}}} \circ i^{\vec{\mathcal{T}}} = i_b^{\pi\vec{\mathcal{V}}} \circ \pi \circ i^{\vec{\mathcal{U}}} = j \circ i^{\vec{\mathcal{V}} \circ i^{\vec{\mathcal{U}}}}$ and both $\vec{\mathcal{T}}^\frown \pi\vec{\mathcal{V}}$ and $\vec{\mathcal{U}}^\frown \vec{\mathcal{V}}$ are according to $\Sigma_{\mathcal{P}}$, we know that $\Sigma(\vec{\mathcal{U}}^\frown \vec{\mathcal{V}}) = b$ by branch condensation. Hence $\Sigma_{\mathcal{R}}(\vec{\mathcal{V}}) = b$. Contradiction.

So it must be that there is a drop along b. By minimality of γ, γ is a successor and $(\vec{\mathcal{V}} \restriction \eta) \restriction \mathcal{R}(\gamma - 1)$ has no drop. Let $\mathcal{N} = (\mathcal{M}_\eta^*)^-$. Then $\mathcal{T}_\eta$ is below $\mathcal{M}_\eta$ and above $\mathcal{N}$. If $\mathcal{M}_b^{\vec{\mathcal{V}}}$ is a $\Sigma_{\mathcal{N}}$-iterable $\Sigma_{\mathcal{N}}$-premouse using the pullback strategy $(\Sigma_{\mathcal{M}_b^{\pi\vec{\mathcal{V}}}})^\pi$, then j ensures that $\mathcal{M}_b^{\vec{\mathcal{U}}}$ is the correct $\mathcal{Q}$-structure. So $\Sigma_{\mathcal{R}}(\vec{\mathcal{V}}) = b$, contradiction. Therefore there must be a stack $\vec{\mathcal{W}}$ on $\mathcal{M}_b^{\vec{\mathcal{V}}}$ above $\mathcal{N}$ according to $(\Sigma_{\mathcal{M}_b^{\pi\vec{\mathcal{V}}}})^\pi$ with last model $\mathcal{N}_1$ such that $\mathcal{N}_1$ is not a $\Sigma_{\mathcal{N}}$-premouse. This means there is $(\vec{\mathcal{S}}, d) \in \mathcal{N}_1$ such that $\vec{\mathcal{S}}$ is a stack on $\mathcal{N}$ according to $\Sigma_{\mathcal{N}}$, $\vec{\mathcal{S}}^\frown d$ is according to $\mathcal{N}_1$'s strategy, but $\Sigma_{\mathcal{N}}(\vec{\mathcal{S}}) \neq d$. Now let $\mathcal{N}_2$ be the last model of $j\vec{\mathcal{W}}$, $k : \mathcal{N}_1 \to \mathcal{N}_2$ be the copying map. Then $k(\vec{\mathcal{S}})^\frown k(d)$ is a stack on $k(\mathcal{N})$ according to $\mathcal{N}_2$'s strategy. Since $\vec{\mathcal{T}}^\frown \pi\vec{\mathcal{V}}^\frown j\vec{\mathcal{W}}$ is a stack according to Σ, $\mathcal{N}_2$ sees $\Sigma_{k(\mathcal{N})}$ correctly. So $\Sigma_{k(\mathcal{N})}(k(\vec{\mathcal{S}})) = k(d)$. As $k\vec{\mathcal{S}}^\frown d$ is a hull of $k(\vec{\mathcal{S}})^\frown k(d)$, $\Sigma_{k(\mathcal{N})}(k\vec{\mathcal{S}}) = d$ by hull condensation of $\Sigma_{k(\mathcal{N})}$. It follows that $\Sigma_{\mathcal{N}} \neq (\Sigma_{k(\mathcal{N})})^k$. Therefore, letting $\mathcal{T}^1 = \pi\vec{\mathcal{V}} \restriction \eta$, $\mathcal{U}^1 = \vec{\mathcal{V}} \restriction \eta$, $\mathcal{Q}^1$ be the last model of $\vec{\mathcal{T}}^\frown \vec{\mathcal{T}}^1$, $\mathcal{R}^1$ be the last model of $\vec{\mathcal{U}}^\frown \vec{\mathcal{U}}^1$, $\pi^1 : \mathcal{R}^1 \to \mathcal{Q}^1$ be the copying map, γ_1 be such that $\mathcal{R}^1(\gamma_1) = \mathcal{N}$, we get a smaller counterexample. Contradiction.

Case 2. $\vec{\mathcal{T}}_0 \neq \vec{\mathcal{T}}$.

By minimality of γ, $(\vec{\mathcal{V}} \restriction \eta) \restriction \mathcal{R}(\gamma - 1)$ has no drop. A similar argument as in Case 1 gives that if we let $\mathcal{T}^1 = \pi\vec{\mathcal{V}} \restriction \eta$, $\mathcal{U}^1 = \vec{\mathcal{V}} \restriction \eta$, $\mathcal{Q}^1$ be the last model of $\vec{\mathcal{T}}^\frown \vec{\mathcal{T}}^1$, $\mathcal{R}^1$ be the last model of $\vec{\mathcal{U}}^\frown \vec{\mathcal{U}}^1$, $\pi^1 : \mathcal{R}^1 \to \mathcal{Q}^1$ be the copying map, γ_1 be such that $\mathcal{R}^1(\gamma_1) = \mathcal{N}$, then we get a smaller counterexample. Contradiction. $\square$

Lemma 2.9. *Suppose that $(\mathcal{P}, \Sigma)$ is a hod pair, Σ is fullness preserving and has branch condensation, $\lambda^{\mathcal{P}}$ has measurable cofinality in $\mathcal{P}$. Let ζ be an ordinal such that $\delta_\zeta^{\mathcal{P}} > \mathrm{cf}^{\mathcal{P}}(\lambda^{\mathcal{P}})$. Let $\vec{\mathcal{T}}$ be a stack on $\mathcal{P}$ according to Σ. Then the rearrangement* *of $\vec{\mathcal{T}}$ with respect to ζ is defined and the result of the rearrangement is also according to Σ.*

Proof. We show by induction on initial segments of essential components of $\vec{\mathcal{T}}$. Suppose we already know that for all $\xi < \eta$, $\vec{\mathcal{T}} \upharpoonright \xi$ has a rearrangement $\langle \vec{\mathcal{U}}^\xi_\alpha, \mathcal{Q}^\xi_\alpha, \vec{\mathcal{V}}^\xi_\alpha, \mathcal{R}^\xi_\alpha, \sigma^\xi_{\alpha\beta}, \pi^\xi_{\alpha\beta} : \alpha < \beta \leq \xi \rangle$ such that $(\vec{\mathcal{U}}^\xi_\xi, \mathcal{Q}^\xi_\xi, \vec{\mathcal{V}}^\xi_\xi, \mathcal{R}^\xi_\xi)$ is according to Σ. It is easy to see by definition that for all $\alpha \leq \beta < \xi < \mu < \eta$, $\vec{\mathcal{U}}^\xi_\alpha = \vec{\mathcal{U}}^\mu_\alpha$, $\mathcal{Q}^\xi_\alpha = \mathcal{Q}^\mu_\alpha$, $\vec{\mathcal{V}}^\xi_\alpha = \vec{\mathcal{V}}^\mu_\alpha$, $\mathcal{R}^\xi_\alpha = \mathcal{R}^\mu_\alpha$, $\sigma^\xi_{\alpha\beta} = \sigma^\mu_{\alpha\beta}$. So we will omit superscripts from now on.

If η is a limit ordinal, then the rearrangement of $\vec{\mathcal{T}}$, if defined, has result $(\bigcup_{\alpha<\eta} \vec{\mathcal{U}}_\alpha, \bigcup_{\alpha<\eta} \sigma_{\alpha\eta} \vec{\mathcal{V}}_\alpha)$. All we need to see is that $\bigcup_{\alpha<\eta} \vec{\mathcal{U}}_\alpha {}^\frown \bigcup_{\alpha<\eta} \sigma_{\alpha\eta} \vec{\mathcal{V}}_\alpha$ is according to Σ. Because each $\vec{\mathcal{U}}_\alpha$ is according to Σ by induction, $\bigcup_{\alpha<\eta} \vec{\mathcal{U}}_\alpha$ is according to Σ. By Lemma 2.8, each $\sigma_{\alpha\eta} \vec{\mathcal{V}}_\alpha$ is according to $\Sigma_{\cup_{\alpha<\eta} \vec{\mathcal{U}}_\alpha}$. So $\bigcup_{\alpha<\eta} \sigma_{\alpha\eta} \vec{\mathcal{V}}_\alpha$ is according to $\Sigma_{\cup_{\alpha<\eta} \vec{\mathcal{U}}_\alpha}$

Assume now $\eta = \xi + 1$ is a successor ordinal. Let $\langle \vec{\mathcal{U}}_\alpha, \mathcal{Q}_\alpha, \vec{\mathcal{V}}_\alpha, \mathcal{R}_\alpha, \sigma_{\alpha\beta}, \pi_{\alpha\beta} : \alpha < \beta \leq \xi \rangle$ be the rearrangement of $\vec{\mathcal{T}}_\xi$. Then $\pi_{0\xi} \vec{\mathcal{T}}_\xi$ is according to $\Sigma_{\mathcal{R}_\xi}$ by Lemma 2.8. If $\vec{\mathcal{T}}_\xi$ is above $i_{0\xi}(\mathcal{P}(\zeta))$, then $\pi_{0\xi} \vec{\mathcal{T}}_\xi$ will be appended to $\vec{\mathcal{V}}_\xi$ to form the result of rearrangement of $\vec{\mathcal{T}}$. So the lemma holds. If $\vec{\mathcal{T}}_\xi$ is below $i_{0\xi}(\mathcal{P}(\zeta))$, then $\pi_{0\xi} \vec{\mathcal{T}}_\xi$ applies to $\mathcal{Q}_\xi$. So $\mathcal{Q}_\xi {}^\frown \pi_{0\xi} \vec{\mathcal{T}}_\xi(\mathcal{Q}_\xi)$ is according to Σ. Let $\mathcal{Q}^*$ be the last model of $\pi_{0\xi} \vec{\mathcal{T}}_\xi$. Let $\sigma^* = i^{\pi_{0\xi} \vec{\mathcal{T}}_\xi(\mathcal{Q}_\xi)}$. Then $\sigma^* \vec{\mathcal{V}}_\xi$ is according to $\Sigma_{\mathcal{Q}^*}$by Lemma 2.8. Therefore the result of the rearrangement of $\vec{\mathcal{T}}$ is according to Σ.

□

2.3 The $S^{*,[0]}$-operator

Throughout the rest of chapter 2 and chapter 3 we assume $\theta_{\alpha+1} = \theta$. We fix a hod pair $(\mathcal{P}, \Sigma)$ such that Σ is super-fullness preserving and has branch condensation, $\delta_\infty(\mathcal{P}, \Sigma) = \theta_\alpha$, and whenever $\mathcal{R} \in pI(\mathcal{P}, \Sigma) \cup pB(\mathcal{P}, \Sigma)$ is such that $\lambda^\mathcal{R}$ is a successor, then there is a sequence $\vec{B} = \langle B_i : i < \omega \rangle \subseteq (\mathbb{B}(\mathcal{R}^-, \Sigma_{\mathcal{R}^-})^{L(\Gamma(\mathcal{R}, \Sigma_\mathcal{R}), \mathbb{R})}$

such that $\vec{B}$ strongly guides Σ.

Given a triple $(\mathcal{Q}, \beta^*, \beta)$ and an ordinal ν such that

1. $\mathcal{Q}$ is an Σ-iterate of $\mathcal{P}$.

2. $\nu < \beta \leq \beta^* \leq \lambda^{\mathcal{Q}}$,

3. $\mathcal{Q} \models \mathrm{cf}(\beta^*)$ is measurable,

we call $(\mu, \tau, \mathcal{R}, \gamma^*, \gamma)$ a one-step blow-up of $(\mathcal{Q}, \beta^*, \beta)$ above $\mathcal{Q}(\nu)$ if

1. μ is least such that
$$\delta_\mu^{\mathcal{Q}} > \max(\delta_\nu^{\mathcal{Q}}, \mathrm{cf}^{\mathcal{Q}}(\beta^*)),$$

2. $\tau : \mathcal{Q} \to \mathcal{R}$ is an iteration map below $Q(\mu)$ that is according to $\Sigma_{\mathcal{Q}}$,

3. $\sup \tau''\beta \leq \gamma \leq \gamma^* < \tau(\beta^*)$,

4. if $\gamma < \gamma^*$, then γ, γ^* are limit ordinals, $\sup \tau''\beta < \gamma$, $\mathcal{R} \models$ "$\mathrm{cf}(\gamma^*)$ is measurable, but $\mathrm{cf}(\gamma)$ is not measurable".

Let $\mathcal{I}$ be the set of $\mathfrak{P} = \langle \mathfrak{P}_i : i \leq n \rangle = \langle (\zeta_i, \pi_i, \mathcal{P}_i, \alpha_i^*, \alpha_i) : i \leq n \rangle$ such that

1. $\zeta_0 = \pi_0 = \emptyset$, $\mathcal{P}_0 = \mathcal{P}$, $\alpha_0^* = \alpha_0 \leq \lambda^{\mathcal{P}}$,

2. for all $0 \leq i < n$, $(\zeta_{i+1}, \pi_{i+1}, \mathcal{P}_{i+1}, \alpha_{i+1}^*, \alpha_{i+1})$is a one-step blow-up of $(\mathcal{P}_i, \alpha_i^*, \alpha_i)$ above $\mathcal{P}_i(\pi_i(\zeta_i))$.

In Sections 2.3 and 2.6, we are going to define the $S^{\mathfrak{P}}$-operators for $\mathfrak{P}$ in the index set $\mathcal{I}$. For such $\mathfrak{P}$, we denote $\mathrm{final}(\mathfrak{P}) = \mathcal{P}_n(\alpha_n^*)$. Essentially, the $S^{\mathfrak{P}}$-operator encodes $\Sigma_{\mathrm{final}(\mathfrak{P})}$ in a very special way. This section is devoted to the special case when $\mathfrak{P} = [0] = \langle (\emptyset, \emptyset, \mathcal{P}, 0, 0) \rangle$.

Given a countable transitive self-wellordered, we let $a_+ = L_\xi[a]$, where ξ is the least such that $L_\xi[a] \models ZFC$. So a_+ is the minimum model of ZFC containing a as an element. We denote $\text{Cone}(a) = \{b : b \text{ is countable transitive swo}, a \in b_+\}$.

Fix a real e_0 which codes $\mathcal{P}$. Fix an enumeration $e : \omega \leftrightarrow |\mathcal{P}|$ such that $e \leq_T e_0$. When $\mathcal{M} = \langle M, \in, etc, \pi\rangle$ is an $\mathcal{L}_m$-structure, we let $\mathcal{M}^-$, or the reduct of $\mathcal{M}$, be the $\mathcal{L}_m \setminus \{\dot{\pi}\}$-structure $\langle M, \in, etc\rangle$.

Definition 2.10. Given a $\mathcal{L}_m$-structure $\mathcal{M} = \langle M, \in, a, etc, \mathcal{R}, \pi\rangle$ such that $\mathcal{M}^- \models ZFC$, $\mathcal{M}^- = Hull^{\mathcal{M}^-}(a \cup \{a\} \cup \pi)$, let $\gamma_n = (\sup Hull^{\mathcal{M}^-}(b \cup \{b\} \cup (\pi \circ e) \restriction n)$, so that $\sup_{n<\omega} \gamma_n = o(\mathcal{M})$.

1. Suppose that $\pi : \mathcal{P} \to \mathcal{R}$ is elementary. The e-amenable code of $\mathcal{M}$ is the structure
$$\langle M, \in, a, etc, \mathcal{R}, \{(\gamma_n, (\pi \circ e) \restriction n : n < \omega\}\rangle,$$
2. Suppose that $\sigma : \mathcal{P} \to \mathcal{Q}$ is an iteration map in $\mathcal{M}$ whose generators are below $\delta^{\mathcal{Q}}_\beta$, $\beta < \lambda^{\mathcal{Q}}$. Suppose $\pi : \mathcal{Q}(\beta+1) \to \mathcal{R}$ is elementary such that $\pi \restriction \mathcal{Q}(\beta) = id$ and π is the ultrapower map of $\text{Ult}(\mathcal{Q}(\beta+1), \pi \restriction \mathcal{Q}|\delta^{\mathcal{Q}}_{\beta+1})$. Then the (e, σ)-amenable code of $\mathcal{M}$ is the structure
$$\langle M, \in, a, etc, \mathcal{R}, \{(\gamma_n, \pi(A_n)) : n < \omega\}\rangle$$
where
$$A_n = \{(i, \eta, \sigma(e(i))(\eta)) : \sigma(e(i)(\eta)) \in \delta^{\mathcal{Q}}_{\beta+1}, i \leq n, \eta < \delta^{\mathcal{Q}}_\beta\}$$

It is easy to see that amenable codes are amenable. The original structure is not amenable, so the purpose here is defining an amenable version of them. Essentially, we cut π into ω many pieces, pick an increasing cofinal sequence in $\mathcal{M}$ of length ω, and glue those pieces to those ordinals. We make sure no information is lost when passing to amenable codes. Obviously, $\mathcal{M}$ is recoverable from the e-amenable code of $\mathcal{M}$. In the second case, $\mathcal{M}$ is also recoverable from the (e,σ)-code of $\mathcal{M}$. Well, from the regularity of $\delta^{\mathcal{Q}}_{\beta+1}$, each A_n as in the definition is in $\mathcal{Q}|\delta^{\mathcal{Q}}_{\beta+1}$, so $\pi(A_n)$ does make sense. Each ordinal α less than $\delta^{\mathcal{Q}}_{\beta+1}$ is of the form $\sigma(e(n))(\eta)$ for some $\eta < \delta^{\mathcal{Q}}_{\beta}$. Hence $\pi(\alpha) = \pi(A_n)(n,\eta)$. After that, we can recover the whole π from $\pi \restriction \delta^{\mathcal{Q}}_{\beta+1}$ since π is the ultrapower map of $\mathrm{Ult}(\mathcal{Q}, \pi \restriction \delta^{\mathcal{Q}}_{\beta+1})$.

The following little lemma will be useful. It confirms that the recovery process passes to Σ_1-elementary substructures.

Lemma 2.11. *Let $\mathcal{M} = \langle M, \in, etc, \mathcal{R}, \pi\rangle$ be as in Definition 2.10.*

1. *Suppose that π is a function on $\mathcal{P}$. Let $\mathcal{M}^e$ be the e-amenable code of $\mathcal{M}$. Suppose that $j : \mathcal{K} \to \mathcal{M}^e$ is Σ_1-elementary, $j(e,\mathcal{P},\bar{\mathcal{R}}) = (e,\mathcal{P},\mathcal{R})$. Then there is $\mathcal{N}$ such that $\mathcal{K}$ is the e-amenable code of $\mathcal{N}$, $\pi^{\mathcal{N}} : \mathcal{P} \to \bar{\mathcal{R}}$ is elementary, $j''\dot{\pi}^{\mathcal{N}} = j \circ \dot{\pi}^{\mathcal{N}} = \dot{\pi}^{\mathcal{M}}$.*

2. *Suppose that $\sigma : \mathcal{P} \to \mathcal{Q}$ is an iteration map in $\mathcal{M}$ whose generators are below $\delta^{\mathcal{Q}}_{\beta}$, $\beta < \lambda^{\mathcal{Q}}$. Suppose $\pi : \mathcal{Q}(\beta+1) \to \mathcal{R}$ is elementary such that $\pi \restriction Q(\beta) = id$ and π is the ultrapower map of $\mathrm{Ult}(\mathcal{Q}(\beta+1, \pi\restriction\delta^{\mathcal{Q}}_{\beta+1}))$. Suppose that $\mathrm{Ult}(\mathcal{Q},\pi)$ is wellfounded. Let $\psi : \mathcal{Q} \to \mathrm{Ult}(\mathcal{Q},\pi)$ be the ultrapower map. Let $\mathcal{M}^{e,\sigma}$ be the (e,σ)-amenable code of $\mathcal{M}$. Suppose that $j : \mathcal{K} \to \mathcal{M}^{e,\sigma}$ is Σ_1-elementary, $j(e,\mathcal{P},\bar{\sigma},\bar{\mathcal{Q}},\bar{\beta},\bar{\mathcal{R}}) = (e,\mathcal{P},\sigma,\mathcal{Q},\beta,\mathcal{R})$. Then there is $\mathcal{N}$ such that $\mathcal{K}$ is the $(e,\bar{\sigma})$-amenable code of $\mathcal{N}$. Moreover, $\mathrm{Ult}(\bar{\mathcal{Q}},\pi^{\mathcal{N}})$ is also wellfounded. Letting $\bar{\psi} : \bar{\mathcal{Q}} \to \mathrm{Ult}(\bar{\mathcal{Q}},\pi^{\mathcal{N}})$ be the ultrapower map, then there is $k : \mathrm{Ult}(\bar{\mathcal{Q}},\pi^{\mathcal{N}}) \to \mathrm{Ult}(\mathcal{Q},\pi)$ such that $j\restriction\bar{\mathcal{R}} \subseteq k$, $k \circ \bar{\psi} \circ \bar{\sigma} = \psi \circ \sigma$.*

Proof. We only show part 2. Existence of $\mathcal{N}$ follows from defining

$$\pi^{\mathcal{N}}(x) = j^{-1}(\pi(j(x)))$$

for $x \in \bar{\mathcal{Q}}(\bar{\beta}+1)$. We show that $\mathcal{K}$ is the amenable of $\mathcal{N}$. Let $\langle \gamma_n, A_n : n < \omega \rangle$ be as in definition of the (e, σ)-amenable code of $\mathcal{M}$. Let $\bar{\gamma}_n = j^{-1}(\gamma_n)$, $\bar{A}_n = j^{-1}(A_n)$. Clearly $\langle \bar{\gamma}_n, \bar{A}_n : n < \omega \rangle$ are the corresponding objects of defining the $(e, \bar{\sigma})$-amenable code of $\mathcal{N}$, and $\pi^{\mathcal{N}}(\bar{A}_n) = j^{-1}(\pi(A_n))$. For the "moreover" part, The map

$$k : \mathrm{Ult}(\bar{\mathcal{Q}}, \pi^{\mathcal{N}}) \to \mathrm{Ult}(\mathcal{Q}, \pi)$$

is defined in the canonical way. For $a \in [\bar{\mathcal{R}}]^{<\omega}$, $f \in \bar{\mathcal{Q}}$, f is a function from γ to $\bar{\mathcal{Q}}$, $\gamma < \delta^{\bar{\mathcal{Q}}}_{\bar{\beta}+1}$ let

$$k([a, f]^{\bar{\mathcal{Q}}}_{\pi^{\mathcal{N}}}) = [j(a), j(f)]^{\mathcal{Q}}_{\pi}.$$

We check that k is well-defined and elementary. Take a formula ϕ with only one free variable as an example.

$$\begin{aligned}
&\mathrm{Ult}(\bar{\mathcal{Q}}, \pi^{\mathcal{N}}) \models \phi([a, f]^{\bar{\mathcal{Q}}}_{\pi^{\mathcal{N}}}) \\
\leftrightarrow &a \in \pi^{\mathcal{N}}(\{u < \gamma : \bar{\mathcal{Q}} \models \phi(f(u))\}) \\
\leftrightarrow &j(a) \in \pi(\{u < j(\gamma) : \mathcal{Q} \models \phi(j(f)(u))\}) \\
\leftrightarrow &\mathrm{Ult}(\mathcal{Q}, \pi) \models \phi(j(f)(j(a))).
\end{aligned}$$

The facts that $k \circ \bar{\psi} \circ \bar{\sigma} = \psi \circ \sigma$ and $j \restriction \bar{\mathcal{R}} \subseteq k$ are easy to verify. $\square$

Definition 2.12. Suppose that a is countable transitive self-wellordered. Let $(\mathcal{Q}, \Lambda)$ be a hod pair such that Λ is fullness preserving and has branch condensation. Suppose that $\mathcal{R}$ is a Λ-premouse over a. We say that $\mathcal{R}$ is $\Sigma^2_1(\Lambda)$-suitable if there is δ such that

1. $\mathcal{R} \models \delta$ is the unique Woodin cardinal,
2. $\mathcal{R} = \mathrm{Lp}^{\Lambda}_{\omega}(\mathcal{R}|\delta)$ [1],

[1]For b countable transitive self-wellordered such that Λ acts on a model in b, $\mathrm{Lp}^{\Lambda}_0(b) = b$,

3. If $\xi < \delta$, then $Lp^\Lambda(\mathcal{R}|\xi) \models \xi$ is not Woodin.

For $\Sigma^2_1(\Lambda)$-suitable $\mathcal{R}$, we let $\delta^{\mathcal{R}}$ be the unique Woodin cardinal of $\mathcal{R}$.

Definition 2.13. Let $\mathcal{R}$ be a $\Sigma^2_1(\Lambda)$-suitable. We say that $\mathcal{R}$ is Λ-good if $\mathcal{R}$ is short-tree iterable and Λ-iterable for maximal trees.

We assume that e_0 has sufficiently high Turing degree so that for all $x \geq_T e_0$, there is a Σ-good $\mathcal{R}$ over x. (cf. [14, 8])

Definition 2.14 (The $S^{*,[0]}$-operator)**.** Suppose that a is countable transitive self-wellordered such that $e \in a_+$. We will define $S^{*,[0]}(a)$ as follows. Let $\mathcal{Q}$ be Σ-good over a. Let $\mathcal{N} = L[\vec{E}][a]^{\mathcal{Q}|\delta^{\mathcal{Q}}}$. By the proof of MSC (see [7]), there is $\mathcal{R} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{R}} \restriction \mathcal{N} \in \mathcal{N}_+$. Let $F_{\mathcal{N}}$ be the direct system

$$\{\mathcal{R}, \pi_{\mathcal{R}\mathcal{R}'} : \mathcal{R}, \mathcal{R}' \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}, \pi_{\mathcal{R}\mathcal{R}'} \text{ is a } \Sigma\text{-iteration map.}\}$$

Let $Q^\infty_{\mathcal{N}}$ be the direct limit of $F_{\mathcal{N}}$ and $\pi^\infty_{\mathcal{N}} : \mathcal{P} \to Q^\infty_N$ be the direct limit map, so that $Q^\infty_N \in \mathcal{N}_+$. Let $\mathcal{M}$ be the transitive collapse of the structure

$$\langle Hull^{\mathcal{N}_+}(a \cup \{a\} \cup \pi^\infty_{\mathcal{N}}), \in, a, \vec{E}^{\mathcal{N}}, \emptyset, Q^\infty_{\mathcal{N}}, \pi^\infty_{\mathcal{N}} \rangle.$$

Then $S^{*,[0]}(a)$ is the e-amenable code of $\mathcal{M}$.

The $S^{*,[0]}$-operator is well defined, because any two Σ-good Σ-mice over a coiterates to a common Σ-good Σ-mouse. We are unable to show that $S^{*,[0]}$ has condensation in general. However, by reducing it to another operator $H^{[0]}$, we can show that $S^{*,[0]}$ does have condensation above a fixed real. The reduction is similar to the interdefinability between $\mathcal{H}^{\vec{\mathcal{A}}}_0$ and $\mathcal{H}^{\vec{\mathcal{A}}}_1$ in [19, Section 12]. In what follows we define the $H^{[0]}$-operator.

$\mathrm{Lp}^\Lambda_{n+1}(b)$ is a Λ-mouse over b which is the stack of sound Λ-mice over b that project to $\mathrm{Lp}^\Lambda_n(b)$. $\mathrm{Lp}^\Lambda_\omega(b) = \cup_{n<\omega} \mathrm{Lp}^\Lambda_n(b)$.

Fix a countable transitive swo. Let $\mathcal{N}$ be $\Sigma_{\mathcal{P}(0)}$-good over a. By the universality proof ([19, Lemma 11.1]), $\mathcal{N}$ is full, in the sense that for any transitive swo $b \in \mathcal{N}$, the maximal $L[\vec{E}][b]^{\mathcal{N}}$ constructions reaches $\mathrm{Lp}(b)$. Therefore $\mathcal{N}$ can define the short-tree iteration strategy of $\mathcal{P}(0)$ by choosing branches whose $\mathcal{Q}$-structure is an initial segment of some model of the maximal $L[\vec{E}]$ construction over common part of the tree. Let $\mathcal{U}$ be the generic genericity iteration tree on $\mathcal{P}(0)$ attempting to make all reals $y \leq_T x_g$ generic over the extender algebra of the final model, whenever x_g codes (a, g), g is a $Coll(\omega, a)$ generic filter over the final model (cf. [8]). We assume that $\mathcal{U}$ has maximal possible length, i.e. either $\mathcal{U}$ has a last model $\mathcal{R}$ such that all $y \leq_T x_g$ are generic over the extender algebra of $\mathcal{R}$, whenever x_g codes (a, g), g is a $Coll(\omega, a)$ generic filter over $\mathcal{R}$, or $\mathcal{U}$ is maximal.

We claim that $\mathcal{U}$ must be maximal, $lh(\mathcal{U}) = (|a|^+)^{\mathcal{N}}$, and $\Sigma_{\mathcal{P}(0)}(\mathcal{U}) \notin \mathcal{N}$. For otherwise let $\mathcal{R}$ be the result of the generic genericity iteration, then $\mathcal{R} \in \mathcal{N}$ and $\delta^{\mathcal{R}}$ is singular in $\mathcal{N}$. Let g be $Coll(\omega, a)$ generic over $\mathcal{N}$. Let x_g be the real that codes (a, g). Then $\mathbb{R} \cap \mathcal{N}[x_g] = \{y \in \mathbb{R} : y \text{ is } OD(x_g)\}$ by fullness of $\mathcal{N}$. But $\mathbb{R} \cap \mathcal{R}[x_g] = \{y \in \mathbb{R} : y \text{ is } OD(x_g)\}$ because Σ is fullness preserving. Since the extender algebra of $\mathcal{R}$ is $\delta^{\mathcal{R}}$-c.c., $\mathcal{R}[x_g] \models \delta^{\mathcal{R}}$ is regular. Contradiction.

So $\mathcal{N}$ is able to define the last model of $\mathcal{U}$ as $\mathcal{R} = \mathrm{Lp}_\omega(\mathcal{M}(\mathcal{U}))$, but $\Sigma_{\mathcal{P}(0)}(\mathcal{U}) \notin \mathcal{N}$. Let's say the formula "$\dot{U}$ is the correct maximal genericity iteration tree on $\dot{P}$ with last model R with respect to $\dot{a}$" expresses the conjunction of the following

1. ZFC+"$\dot{U}$ is a generic genericity iteration tree on $\dot{P}$ attemping to make all reals recursive in a real generically coding $\dot{a}$ according the short-tree strategy as certified by my maximal $L[\vec{E}]$-construction,"

2. $\dot{U}$ is maximal,

3. $\dot{R} = \mathrm{Lp}_\omega(\mathcal{M}(\dot{U}))$ as certified by my maximal $L[\vec{E}]$ construction,

4. $\Vdash^{Coll(\omega,\dot{a})}$ all $y \leq_T x_g$ are generic over the extender algebra of $\dot{R}$ at $\delta^{\dot{R}}$. "

Let $\pi = i^{\mathcal{U}^\frown\Sigma_{\mathcal{P}(0)}(\mathcal{U})}$ be the iteration map. Let $\mathcal{M}$ be the least initial segment of $\mathcal{N}$such that $\mathcal{U}, \mathcal{R} \in \mathcal{M}$ and $\mathcal{M}$ thinks "$\mathcal{U}$ is the correct maximal genericity iteration tree on $P(0)$ with last model $\mathcal{R}$ with respect to a". Then $H^{[0]}(a)$ is the $(e, id_{\mathcal{P}})$-amenable code of the transitive collapse of

$$\langle Hull^{\mathcal{M}}(a \cup \{a\} \cup \pi), \in, a, E^{\mathcal{M}}, \emptyset, \mathcal{U}, \mathcal{R}, \pi\rangle.$$

Note that $\delta(\mathcal{U}) = (|a|^+)^{\mathcal{N}} = (|a|^+)^{\mathcal{M}}$, $o(\mathcal{R}) = (|a|^{+\omega})^{\mathcal{N}} = (|a|^{+\omega})^{\mathcal{M}}$. Let $j_a : H^{[0]}(a) \to \langle \mathcal{M}, \in, \pi, etc\rangle$ be the associated anticollapse map. Let $\langle B_i : i < \omega\rangle \subset \mathbb{B}(\emptyset, \emptyset)$ which strongly guides $\Sigma_{\mathcal{P}(0)}$. Then since $\mathrm{ran}(\pi) \subseteq \mathrm{ran}(j_a)$, by strong branch condensation, $j_a^{-1}(\mathcal{R})$ is full. We claim that this implies $j_a \restriction (|a|^{+\omega})^{\mathcal{M}} = id$. Firstly note that $j_a \restriction (|a|^+)^{\mathcal{M}} = id$. For otherwise, let $\kappa = \mathrm{crt}(j) < (|a|^+)^{\mathcal{M}}$. Then $j_a(\kappa) = (|a|^+)^{\mathcal{M}}$. So $j_a^{-1}(\mathcal{U}) = \mathcal{U} \restriction \kappa$. The fullness of ${j_a}^{-1}(\mathcal{R})$ ensures that $\mathcal{U} \restriction \kappa$ is maximal. Contradiction. So $j_a \restriction (|a|^+)^{\mathcal{M}} = id$. So $j_a \restriction \mathcal{M}(\mathcal{U}) = id$. Since $j_a^{-1}(\mathcal{R})$ is full, $j_a^{-1}(\mathcal{R}) = \mathcal{R}$. So $j_a \restriction \mathcal{R} = id$.

Definition 2.15. Let S be any operator defined on an HC-cone. Let z be a real. We say that S has condensation above z if for all $a \in \mathrm{Cone}(z)$,

$$S(a) \text{ is defined,}$$

and whenever

$$j : \mathcal{N} \to S(a)$$

is Σ_1-elementary, $j(b, z) = (a, z)$, then

$$\mathcal{N} = S(b).$$

Lemma 2.16. *$H^{[0]}$ has condensation above e.*

Proof. Suppose that $j : \bar{\mathcal{H}} \to H^{[0]}(a)$ is Σ_1-elementary. Set $\mathcal{U} = \dot{\mathcal{U}}^{H^{[0]}(a)}$, $\mathcal{R} = \dot{\mathcal{R}}^{H^{[0]}(a)}$, $\pi = \dot{\pi}^{H^{[0]}(a)}$. Suppose that $j(e, \bar{a}, \bar{\mathcal{U}}, \bar{\mathcal{R}}) = (e, a, \mathcal{U}, \mathcal{R})$, $j''\bar{\pi} = \pi$. We want to show that $\bar{\mathcal{H}} = H^{[0]}(\bar{a})$. Since $\mathrm{ran}(\pi) \subseteq \mathrm{ran}(j)$, by strong branch condensation, $\bar{\mathcal{R}}$ is full. Since $\bar{\mathcal{H}}$ is iterable using the strategy induced by $j_a \circ j : \bar{\mathcal{H}} \to \mathcal{M}$, the short-tree strategy of $\mathcal{P}(0)$ as defined in $\bar{\mathcal{H}}$ is the correct short-tree strategy. Since $\bar{\mathcal{H}} \models$ "$\bar{\mathcal{U}}$ is the correct maximal generic iteration tree with last model $\bar{\mathcal{R}}$ with respect to $\bar{a}$", $\bar{\mathcal{U}}$ is indeed maximal in V and $\bar{\mathcal{R}} = \mathrm{Lp}_\omega(\mathcal{M}(\mathcal{U}))$ is the result of the generic genericity iteration. This means that $\bar{\mathcal{U}} = \mathcal{U}^{H^{[0]}(a)}$, $\bar{\mathcal{R}} = \mathcal{R}^{H^{[0]}(a)}$. So $(|\bar{a}|^{+n})^{\bar{\mathcal{H}}} = (|\bar{a}|^{+n})^{H^{[0]}(\bar{a})}$ for all $n < \omega$.

In this paragraph we show that $\bar{\pi} = \pi^{H^{[0]}(\bar{a})}$. This requires a bit care. Denote $\Sigma_{\mathcal{P}(0)}(\mathcal{U}) = b$. From the predicate $\dot{\pi}^{H^{[0]}(a)}$ we can define partial branches $\langle b_i : i < \omega\rangle \subseteq H^{[0]}(a)$ such that $\cup_{i<\omega} b_i = b$. This is because some $\vec{B} \subseteq \mathbb{B}(\emptyset, \emptyset)$ guides $\Sigma_{P(0)}$. By elementarity we get $\bar{b}_i = j^{-1}(b_i)$ in $\bar{\mathcal{H}}$. Denote $\bar{b} = \cup_{i<\omega}\bar{b}_i$. All we need to show is that $\bar{b} = \Sigma_{\mathcal{P}(0)}(\bar{\mathcal{U}})$. By branch condensation, it suffices to show that there is $k : \mathcal{M}^{\bar{\mathcal{U}}}_{\bar{b}} \to \mathcal{M}^{\mathcal{U}}_{b}$ such that $\pi = k \circ i_{\bar{b}}$. But for each i, $j \restriction \mathcal{M}^{\bar{\mathcal{U}}}_{\max(\bar{b}_i)}$ an embedding from $\mathcal{M}^{\bar{\mathcal{U}}}_{\max(\bar{b}_i)}$ into $\mathcal{M}^{\mathcal{U}}_{\max(b_i)}$. Thus we get $k : \mathcal{M}^{\bar{\mathcal{U}}}_{\bar{b}} \to \mathcal{M}^{\mathcal{U}}_{b}$ as the embedding between direct limits.

Let us compare $\bar{\mathcal{H}}$ and $H^{[0]}$ to line up their extender sequence, using strategies induced by $j_a \circ j$ and $j_{\bar{a}}$ respectively. Let $\mathcal{T}_1$ be the tree on the $\bar{\mathcal{H}}$-side, with last model $\mathcal{H}_1$, $\mathcal{T}_2$ be the tree on the $H^{[0]}(\bar{a})$-side, with last model $\mathcal{H}_2$. Suppose for example the $H^{[0]}(\bar{a})$-side does not drop. Then all critical point of $\mathcal{T}_1$ are above $(|\bar{a}|^{+\omega})^{\bar{\mathcal{H}}}$, so $\bar{\mathcal{U}}, \bar{\mathcal{R}}$ are not moved in $\mathcal{T}_1$. We claim that all critical points of $\mathcal{T}_2$ are also above $(|\bar{a}|^{+\omega})^{\bar{\mathcal{H}}}$. For otherwise, let ξ be least such that $\kappa^{\mathcal{T}_2}_\xi < (|\bar{a}|^{+\omega})^{\bar{\mathcal{H}}}$, then $\bar{\mathcal{H}} \models \kappa^{\mathcal{T}_2}_\xi$ is measurable, which is absurd.

The discussion as in the last paragraph shows that $(\bar{\mathcal{U}}, \bar{\mathcal{R}})$ is not moved during the comparison. Since both $H^{[0]}(\bar{a})$ and $\bar{\mathcal{H}}$ thinks "$\dot{S}$ is the least level such that $\mathcal{U}, \mathcal{R} \in \dot{S}$ and $\dot{S} \models \bar{\mathcal{U}}$ is the correct generic genericity iteration tree on $\mathcal{P}(0)$ with last model $\bar{\mathcal{R}}$ with respect to a", neither side has a drop and $\mathcal{H}_1 = \mathcal{H}_2$. As $H^{[0]}(\bar{a})$ and $\bar{\mathcal{H}}$ are both the $\bar{\pi}$-sound, i.e. the Skolem hull of $\bar{a} \cup \{\bar{a}\} \cup \bar{\pi}$ in either model is the model itself, $H^{[0]}(\bar{a})$ and $\bar{\mathcal{H}}$ are both isomorphic to $Hull^{\mathcal{H}_1}(\bar{a} \cup \{\bar{a}\} \cup \bar{\pi})$. So $H^{[0]}(\bar{a}) = \bar{\mathcal{H}}$. □

Definition 2.17. Let S be any operator defined on an HC-cone. Let z be a real. We say that S extends naturally to generic extensions above z if for all a is countable transitive swo such that $z \in a_+$, for all $g \subseteq Coll(\omega, a)$ generic over $S(a)$, let x_g be a real which codes (g, a), then

$$S(x_g) = S(a)[x_g]$$

Here $S(a)[x_g]$ is understood as follows. We define $\mathcal{M}[x_g]$ for $\mathcal{M} = \langle M, \in, a, \vec{E}, S, \mathcal{Q}, \pi, etc\rangle$, g $Coll(\omega, a)$-generic over $\mathcal{M}$, and x_g coding g, by induction on $o(\mathcal{M})$.

$$\mathcal{M}[x_g] = \langle M[x_g], \in, a, \vec{E}[x_g], S[x_g], \mathcal{Q}, \pi, etc\rangle.$$

Here, $\vec{E}[g]$ is the extender sequence obtained by extending each extender on the $\mathcal{E}$-sequence to the small generic extension by adding x_g. $S[x_g]$ is part of the inductive definition.

Lemma 2.18. $H^{[0]}$ *extends naturally to generic extensions above* e_0.

Proof. Let $a \in \text{Cone}(e_0)$. Let $H^{[0]}(a) = \langle H, \in, \pi, a, \mathcal{U}, \mathcal{R}, \vec{E}\rangle$. Suppose that g is $Coll(\omega, a)$-generic over $H^{[0]}(a)$. We want to show that $H^{[0]}(x_g) = \langle H[x_g], \in, \pi, x_g, \mathcal{U}, \mathcal{R}, \vec{E}^*\rangle$, where $\vec{E}^*$ is the canonical extension of $\vec{E}$ to a small generic extension. Let $\mathcal{Q}, \mathcal{N}$ be as in definition of $H^{[0]}(a)$. Then $\mathcal{Q}[x_g]$ is $\Sigma_{\mathcal{P}(0)}$-good over x_g, $\mathcal{N}[x_g]$ is the output of the $L[\vec{E}][x_g]$-construction of $\mathcal{Q}[x_g]|\delta^{\mathcal{Q}[x_g]}$.

Let $\mathcal{V} = \mathcal{U}^{H^{[0]}(x_g)}$. We claim that $\mathcal{U} = \mathcal{V}$. Since $\text{rank}(a_+) = \text{rank}((x_g)_+)$, the linear iteration part on hitting the bottom measure are the same. It is then straightforward to check by induction that for all $\xi \leq lh(\mathcal{U})$, $\mathcal{M}^{\mathcal{U}}_\xi[x_g] = \mathcal{M}^{\mathcal{V}}_\xi$, $(E^{\mathcal{U}}_\xi)^* = E^{\mathcal{V}}_\xi$, where $(E^{\mathcal{U}}_\xi)^*$ is the canonical extension of $E^{\mathcal{U}}_\xi$ on the generic extension by adding x_g.

Once we know $\mathcal{U} = \mathcal{V}$, it is then easy to verify that $H^{[0]}(x_g) = \langle H[x_g], \in, \pi, x_g, \mathcal{U}, \mathcal{R}, \vec{E}^*\rangle$. □

Lemma 2.19. $S^{*,[0]}$ *extends naturally to generic extensions above* e_0*.*

Proof. Let $a \in \text{Cone}(e_0)$. Let $S^{*,[0]}(a) = \langle M, \in, a, \vec{E}, \emptyset, \emptyset, \pi, \mathcal{Q}\rangle$. Suppose that g is $Coll(\omega, a)$-generic over $S^{*,[0]}(a)$. We want to show that $S^{*,[0]}(x_g) = \langle M[x_g], \in, a, \vec{E}^g, \emptyset, \emptyset, \pi, \mathcal{Q}\rangle$. Let $\mathcal{Q}, \mathcal{N}$ be as in definition of $S^{*,[0]}(a)$. Then $\mathcal{Q}[x_g]$ is Σ-good over x_g, $\mathcal{N}[x_g]$ is the output of the $L[\vec{E}][x_g]$-construction of $\mathcal{Q}[x_g]|\delta^{\mathcal{Q}[x_g]}$. It suffices to show that $Q^\infty_{\mathcal{N}} = Q^\infty_{\mathcal{N}[x_g]}$.

Clearly $F_{\mathcal{N}} \subset F_{\mathcal{N}[x_g]}$. Let $j : Q^\infty_{\mathcal{N}} \to Q^\infty_{\mathcal{N}[x_g]}$ be the canonical map induced by the inclusion $F_{\mathcal{N}} \subset F_{\mathcal{N}[x_g]}$. For all $\mathcal{R} \in F_{\mathcal{N}[x_g]}$, by doing the generic comparison as in [7], there is $\mathcal{R}' \in pI(\mathcal{R}, \Sigma) \cap F_{\mathcal{N}}$. So j is onto. So $Q^\infty_{\mathcal{N}} = Q^\infty_{\mathcal{N}[x_g]}$. □

For each $n < \omega$, $x \in \mathbb{R}$ such that $e_0 \in x_+$, let

$$S^{*,[0]}_n(x) = Th^{S^{*,[0]}(x)|\xi_n}(\{x\} \cup (\pi^{S^{*,[0]}(x)} \circ e)\restriction n),$$

$$H^{[0]}_n(x) = Th^{H^{[0]}(x)|\eta_n}(\{x\} \cup (\pi^{H^{[0]}(x)} \circ e)\restriction n).$$

where ξ_n, η_n are ordinals that $\pi^{S^{*,[0]}(x)} \circ e)\restriction n$ and $\pi^{H^{[0]}(x)} \circ e)\restriction n$ glue to. Then $S^{*,[0]}_n$, $H^{[0]}_n$ are uniformly Turing invariant operators. We will show that $\langle S^{*,[0]}_n : n < \omega\rangle$ and $\langle H^{[0]}_n : n < \omega\rangle$ are cofinal in each other.

For each $n < \omega$, $x \in \mathbb{R} \cap \text{Cone}(e)$, we claim that there is $k < \omega$ such that $S^{*,[0]}_n(x) \leq_T H^{[0]}_k(x)$, $H^{[0]}_n(x) \leq_T S^{*,[0]}_k(x)$. Clearly $S^{*,[0]}_n(x), H^{[0]}_n(x) \in \text{Lp}(x)$. We need to show the converse direction, for all $y \in \text{Lp}(x)$, there is k such that

$y \leq_T S^{*,[0]}_k(x)$, $y \leq_T H^{[0]}_k(x)$. By soundness of $S^{*,[0]}(x)$ and $H^{[0]}(x)$, it suffices to show that $\mathrm{Lp}(x) \subseteq S^{*,[0]}(x) \cap H^{[0]}(x)$. Let $\mathcal{Q}, \mathcal{N}$ be as in definition of $S^{*,[0]}(x)$. Let $j : S^{*,[0]}(x) \to \langle \mathcal{N}_+, \in, etc \rangle$ be the uncollapsing map when $S^{*,[0]}(x)$ is defined. By the proof of MSC (see [7]), there is $\mathcal{R} \in pI(\mathcal{P}(0), \Sigma) \cap \mathcal{N} \cap \mathrm{ran}(j)$ such that $\Sigma_{\mathcal{R}} \restriction \mathcal{N} \in \mathrm{ran}(j)$ and x is $Coll(\omega, \delta^{\mathcal{R}})$-generic over $\mathcal{R}$. Since $\mathrm{ran}(i_{\mathcal{PR}}) \subseteq \mathrm{ran}(j)$, $j^{-1}(\mathcal{R})$ is full. So $j^{-1}(\mathcal{R})[x]$ is full. So $\mathrm{Lp}(x) \in S^{*,[0]}(x)$. A similar proof shows that $\mathrm{Lp}(x) \in H^{[0]}(x)$.

According to [16], there is a real z and functions $f_0, f_1, g_0, g_1 : \omega \to \omega$ such that for all $n < \omega$, for all x such that $z \leq_T x$,

$$S^{*,[0]}_n(x) = \{f_1(n)\}^{H^{[0]}_{f_0(n)}(x)},$$

$$H^{[0]}_n(x) = \{g_1(n)\}^{S^{*,[0]}_{g_0(n)}(x)}.$$

Let z_0 be a real which codes $e_0, z_1, f_0, f_1, g_0, g_1$. As in the discussion of [19, Section 12], we can obtain effective maps $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^h \urcorner$ and $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^s \urcorner$ such that for all countable transitive swo a such that $z_0 \in a_+$, for all $c_0, \ldots, c_n \in a$, for all Σ_1 formula $\phi(v_0, \ldots, v_n)$,

$$S^{*,[0]}(a) \models \phi(c_0, \ldots, c_n) \leftrightarrow H^{[0]}(a) \models \phi^h(c_0, \ldots, c_n, z_0),$$

$$H^{[0]}(a) \models \phi(c_0, \ldots, c_n) \leftrightarrow S^{*,[0]}(a) \models \phi^s(c_0, \ldots, c_n, z_0).$$

We sketch the map $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^h \urcorner$ here, in preparation for a more complicated version of these effective maps in Section 2.9. Suppose

$$S^{*,[0]}(a) \models \phi(c_0, \ldots, c_k).$$

Then for all $g \subseteq Coll(\omega, a)$ generic over both $S^{*,[0]}(a)$ and $H^{[0]}(a)$ such that $g(i) = c_i$

for all $i \leq k$,

$$S^{*,[0]}(x_g) \models \text{“}\underbrace{S^{*,[0]}(a) \models \phi(g(0), \ldots, g(k))}_{\text{call this } \phi_1}\text{”}.$$

Hence

$$\exists l \ulcorner \phi_1 \urcorner \in S_l^{[0]}(x_g).$$

Let $\phi_2(z_0)$ be the formula

z_0 codes f_0, f_1, g_0, g_1, and there is $l \in \omega$ such that $\ulcorner \phi_1 \urcorner \in \{f_1(l)\}^{H^{\mathfrak{R}}_{f_0(l)}(x_g)}$.”

Then

$$H^{[0]}(x_g) \models \phi_2(z_0)$$

Let $\phi^h(v_0, \ldots, v_k, z)$ be the formula

“for all $g \subseteq Coll(\omega, a)$ generic over $H^{[0]}(a)$ such that $g(i) = v_i$ for all $i \leq k$, then $V[x_g] \models \phi_2(z)$.”

Then

$$H^{[0]}(a) \models \phi^h(c_0, \ldots, c_k, z).$$

In a similar way we can define the map $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^s \urcorner$.

The upcoming Lemma 2.20 is essentially [19, Theorem 12.8]. Lemma 2.18 and Lemma 2.19 show up in its proof when we look into $S^{*,[0]}$ and $H^{[0]}$ operated on reals coding a generic enumeration of a, where the reduction can be applied.

Lemma 2.20. *$S^{*,[0]}$ has condensation above z_0.*

Proof. Let $j : \mathcal{M} \to S^{*,[0]}(a)$ be Σ_1-elementary, $j(z_0, b) = (z_0, a)$. We want to show that $\mathcal{M} = S^{*,[0]}(b)$. Firstly we show that

$$Hull_1^{H^{[0]}(a)}(j''b) \cap a = j''b.$$

Suppose that $c \in Hull_1^{H^{[0]}(a)}(j''b) \cap a$. Let ϕ be a Σ_1-formula and $b_1, \ldots, b_n \in b$ be such that $v \in c$ iff

$$H^{[0]}(a) \models \phi(v, j(b_1), \ldots, j(b_n)).$$

Then $v \in c$ iff

$$S^{*,[0]}(a) \models \phi^s(v, j(b_1), \ldots, j(b_n), z_0).$$

Let

$$d = \{v \in b : \mathcal{M} \models \phi^s(v, b_1, \ldots, b_n, z_0)\}.$$

Then $j(d) = c$. So $c \in j''b$.

It follows that there is a Σ_1 elementary $k : \mathcal{N} \to H^{[0]}(a)$ such that $k \restriction b \cup \{b\} = j \restriction b \cup \{b\}$. By Lemma 2.16, $\mathcal{N} = H^{[0]}(b)$. Hence for all Σ_1 formula ϕ and $b_1, \ldots, b_n \in b$,

$$\begin{array}{ll} & \mathcal{M} \models \phi(b_1, \ldots, b_n) \\ \leftrightarrow & S^{*,[0]}(a) \models \phi(j(b_1), \ldots, j(b_n)) \\ \leftrightarrow & H^{[0]}(a) \models \phi^h(j(b_1), \ldots, j(b_n), z_0) \\ \leftrightarrow & H^{[0]}(b) \models \phi^h(b_1, \ldots, b_n, z_0) \\ \leftrightarrow & S^{*,[0]}(b) \models \phi(b_1, \ldots, b_n). \end{array}$$

So $\mathcal{M} = S^{*,[0]}(b)$ by soundness of $\mathcal{M}$ and $S^{*,[0]}(b)$. $\square$

2.4 S-premouse

The general $S^{\mathfrak{P}}$-operator inherits a structure called S-premouse.An S-premouse is roughly an ordinary $L[\vec{E}]$-structure with an additional unary predicate S. In this section, we deal with the abstract concept of potential S-premouse. We will define fine structural objects of a potential S-premouse.

Definition 2.21. Let

$$\mathcal{L}_m = \{\in, \dot{a}, \dot{E}, \dot{F}, \dot{S}, \dot{b}, \dot{Q}, \dot{\pi}\}$$

be the language extending the language of set theory where $\dot{a}, \dot{b}, \dot{Q}$ are constant symbols, $\dot{E}, \dot{F}, \dot{\pi}$ are unary predicate symbols, $\dot{S}$ is a unary predicate symbol. Let $a \in \text{Cone}(z)$. A potential S-premouse over a is a structure

$$\mathcal{N} = \langle N, \in, a, \vec{E}, F, S, b, Q, \pi \rangle$$

in the language of $\in, \dot{a}, \dot{E}, \dot{F}, \dot{S}, \dot{b}, \dot{Q}, \dot{\pi}$ with the following properties.

1. $N = J_\xi^{\vec{E},S}[a]$ for some ξ.
2. $\mathcal{N}$ is an acceptable J-structure.
3. $\vec{E}$ is a partial unary function.
4. For all $y \in S$, y is a $\mathcal{L}_m$-structure. For $\eta < \xi$, let $\mathcal{N}|\eta$ be the initial segment of $\mathcal{N}$ given by

$$\mathcal{N}|\eta = \langle J_\eta^{\vec{E},S}[a], \in, a, \vec{E}\restriction\eta, E_\eta, S \cap J_\eta^{\vec{E},S}[a], b^\eta, \mathcal{Q}^\eta, \pi^\eta \rangle$$

   where

$$(b^\eta, \mathcal{Q}^\eta, \pi^\eta) = \begin{cases} (b^y, \dot{\mathcal{Q}}^y, \dot{\pi}^y), & \text{if } y \in S \text{ is unique such that } o(y) = \eta. \\ (\emptyset, \emptyset, \emptyset), & \text{otherwise.} \end{cases}$$

5. For all $y \in S$, $y = \mathcal{N}|o(y)$. (Henceforth, if $y, y' \in S$ and $o(y) = o(y')$, then $y = y'$.)
6. $\vec{E}^\frown F$ is a fine extender sequence in the sense of [4], whose levels are understood as $\mathcal{N}|\eta$.

The unary predicate $\dot{S}$ is intended to be the range of various *successor* S-operators on its sequence. The constant $\dot{b}$ is the preimage of the S-operator of $\mathcal{N}$, i.e. there is an $\mathfrak{P}$ such that $\mathcal{N} = S^{\mathfrak{P}}(b)$. $\mathcal{Q}$ and $\dot{\pi}$ are special objects of the S-operator. They are intended to be the direct limit map $\pi : \mathcal{P} \to \mathcal{Q}$. We devote some effort on fine structure of a potential S-premouse. This much details play a part in the translation procedure of chapter 3, as a potential S-premouse must have, to some extent, equivalent fine structure with some premouse. Several parameters are essential to the study of S-premouse. The reader may recall the definition of $\mu^{\mathcal{M}}, \nu^{\mathcal{M}}, \gamma^{\mathcal{M}}$ for an ordinary Mitchell-Steel premouse $\mathcal{M}$ (cf. [4, 20]). We will define $\mu^{\mathcal{M}}, \nu^{\mathcal{M}}, \gamma^{\mathcal{M}}$ for a potential S-premouse $\mathcal{M}$ in parallel.

Definition 2.22. Let $\mathcal{M} = \langle M, \in, a, \vec{E}, F, S, b, Q, \pi \rangle$ be a potential S-premouse. We say $\mathcal{M}$ is E-active if $F \neq \emptyset$, S-active if $b \neq \emptyset$, and passive otherwise.

1. If $\mathcal{M}$, is E-active then letting $\nu = \nu(F)$ and $\kappa = \text{crt}(F)$, we say $\mathcal{M}$ is of E-type I if $\nu = (\kappa^+)^{\mathcal{M}}$, $\mathcal{M}$ is of E-type II if ν is a successor ordinal, and $\mathcal{M}$ is of E-type III if ν is a limit ordinal $> (\kappa^+)^{\mathcal{M}}$. Set

$$\mu^{\mathcal{M}} = \kappa$$
$$\nu^{\mathcal{M}} = \nu$$

If $\mathcal{M}$ is of E-type II, let G be the longest non-type-Z proper initial segment of F. We let

$$\gamma^{\mathcal{M}} = \text{ the unique } \xi \in \text{dom}(\vec{E}) \text{ such that } G = E_\xi.$$

if there is such a ξ. If there is no such ξ, then setting $\eta = \nu(F)$, we let

$$\gamma^{\mathcal{M}} = (\eta, a, f), \text{ where } F = [a, f]^{\mathcal{M}|\eta}_{E_\eta}$$

and (a,f) is least in the order of construction on $M|\eta$ with this property. If $\mathcal{M}$ is of E-type I or III, then set $\gamma^{\mathcal{M}} = 0$.

2. If $\mathcal{M}$ is S-active, then set

$$\begin{aligned}\mu^{\mathcal{M}} &= 0\\ gen^{\mathcal{M}} &= \{o(a) < \xi < o(b) : \xi \notin Hull^{\mathcal{M}^-}(\xi \cup \pi)\}\\ \nu^{\mathcal{M}} &= \sup\{\xi + 1, o(a) : \xi \in gen^{\mathcal{M}}\}.\end{aligned}$$

We say that $\mathcal{M}$ is of S-type II if $\nu^{\mathcal{M}}$ is a successor ordinal, S-type III if $\nu^{\mathcal{M}}$ is a limit ordinal$>$ $o(a)$, S-type IV if $\nu^{\mathcal{M}} = o(a)$. For $o(a) \leq \xi < \nu^{\mathcal{M}}$, set $X^{\xi} = Hull^{\mathcal{M}^-}(\xi \cup \pi)$, $\mathcal{M}^{\xi}$ be the transitive collapse of $\langle X^{\xi}, \in, etc, \pi\rangle$.

If $\mathcal{M}$ is of S-type II, we say $\mathcal{M}$ is of S-type-Z if $\nu^{\mathcal{M}}$ is a limit ordinal, $\nu^{\mathcal{M}} - 1 = \sup(gen^{\mathcal{M}} \cap \nu^{\mathcal{M}} - 1)$, and $((\nu^{\mathcal{M}})^+)^{\mathcal{M}} = ((\nu^{\mathcal{M}})^+)^{\mathcal{M}^{\nu^{\mathcal{M}}-1}}$.

If $\mathcal{M}$ is of S-type II, let $\mu = \sup\{\xi + 1 : \xi \in gen^{\mathcal{M}} \cap (\nu^{\mathcal{M}} - 1)\}$, If $\mathcal{M}^{\mu}$ is not of S-type-Z, then set

$$\gamma^{\mathcal{M}} = o(\mathcal{M}^{\mu}).$$

If $\mathcal{M}^{\mu}$ is of S-type-Z, then let

$$\gamma^{\mathcal{M}} = o(\mathcal{M}^{\mu-1}).$$

3.

$$l^{\mathcal{M}} = \{\xi \in o(\mathcal{K}) : \text{There is no } y \in S^K \text{ such that } o(b^y) < \xi < o(y)\}.$$

$l^{\mathcal{M}}$ is called the levels of $\mathcal{M}$.

4.

$$I^{\mathcal{M}} = \{\eta < o(\mathcal{M}) : \forall y \in S(o(y) > \eta \to \nu^{\mathcal{M}|o(y)} \geq \eta)\}.$$

When $\mathcal{M}$ comes translating a premouse $\mathcal{K}$, $I^{\mathcal{M}}$ will be steps which get translated back into an initial segment of $\mathcal{K}$. Some levels of $\mathcal{M}$ come from an initial segment of an ultrapower of $\mathcal{K}$. $I^{\mathcal{M}}$ identifies initial segments of $\mathcal{M}$.

In item 2, when $\mathcal{M}$ is S-active, $\pi^{\mathcal{M}}$ is intended to code a Jensen-type extender, namely, the full embedding associated to the extender. Those parameters $\nu^{\mathcal{M}}, \gamma^{\mathcal{M}}$ are essentially Mitchell-Steel parameters of the extender that $\pi^{\mathcal{M}}$ codes. We stick to Mitchell-Steel indexing scheme because the universality proof works only under this indexing. Similar parameters also come up when translating a Jensen-premouse into a Mitchell-Steel premouse (cf. [2]).

In an ordinary premouse over a, every ordinal bigger than the rank of a marks a level. A successor level is the rudimentary closure of the previous one. However, In a potential S-premouse, only ordinals in $l^{\mathcal{K}}$ marks levels. A successor level is usually an S-operator acted on the previous one. (cf. $\oplus\vec{A}$-premouse, [19])

Definition 2.23. Let $\mathcal{M}, \mathcal{N}$ be potential S-premice. We call $\mathcal{M}$ an initial segment of $\mathcal{N}$, or $\mathcal{M} \trianglelefteq \mathcal{N}$, if there is η such that $\mathcal{M} = \mathcal{N}|\eta$. We call $\mathcal{M}$ a proper initial segment of $\mathcal{N}$, or $\mathcal{M} \triangleleft \mathcal{N}$, if $\mathcal{M} \trianglelefteq \mathcal{N}$ but $\mathcal{M} \neq \mathcal{N}$.

Let $\mathcal{L}_c$ be the language $\mathcal{L}_m$ expanded by a unary predicate symbol $\dot{I}$ and additional constant symbols $\dot{\mu}, \dot{\nu}, \dot{\gamma}$. Let

$$M = \langle M, \in, a, \vec{E}, F, S, b, \mathcal{Q}, \pi \rangle$$

be a potential S-premouse over a. Then the Σ_0-code of M, or $\mathfrak{C}_{00}(M)$, is the $\mathcal{L}_c$-structure given by

1. If $\mathcal{M}$ is passive, then

$$\mathfrak{C}_0(\mathcal{M}) = \langle M, \in, a, \vec{E}, \emptyset, S, \emptyset, \emptyset, \emptyset, I^{\mathcal{M}}, \emptyset, \emptyset, \emptyset \rangle$$

2. If $\mathcal{M}$ is E-active of E-type I or II, then

$$\mathfrak{C}_0(\mathcal{M}) = \langle M, \in, a, \vec{E}, F^c, S, \emptyset, \emptyset, \emptyset, I^{\mathcal{M}}, \mu^{\mathcal{M}}, \nu^{\mathcal{M}}, \gamma^{\mathcal{M}}\rangle.$$

   where F^c is the amenable coding of F.

3. If $\mathcal{M}$ is E-active of E-type III, then

$$\mathfrak{C}_0(\mathcal{M}) = \langle |\mathcal{M}|\nu^{\mathcal{M}}|, \in, a, \vec{E}\restriction\nu^{\mathcal{M}}, F, S\cap|\mathcal{M}|\nu^{\mathcal{M}}|, \emptyset, \emptyset, \emptyset, I^{\mathcal{M}}\cap\nu^{\mathcal{M}}, \mu^{\mathcal{M}}, \nu^{\mathcal{M}}, \gamma^{\mathcal{M}}\rangle.$$

4. If $\mathcal{M}$ is S-active of S-type II or IV, then

$$\mathfrak{C}_0(\mathcal{M}) = \langle M, \in, a, \vec{E}, \emptyset, S, b, \mathcal{Q}, \pi, I^{\mathcal{M}}, 0, \nu^{\mathcal{M}}, \gamma^{\mathcal{M}}\rangle.$$

5. If M is S-active of S-type III, then

$$\mathfrak{C}_0(\mathcal{M}) = \langle |\mathcal{M}|\nu^{M}|, \in, a, \vec{E}\restriction\nu^{M}, \emptyset, S\cap|\mathcal{M}|\nu^{M}|, b, \mathcal{Q}, \tilde{\pi}, I^{\mathcal{M}}, 0, 0, 0\rangle.$$

   where $\tilde{\pi} = \{(\xi, \pi^{\xi}) : o(a) \leq \xi < \nu^{\mathcal{M}}\}$, π^{ξ} is the image of π under the transitive collapse $X_{\xi} \to M_{\xi}$.

The Σ_0 code $\mathfrak{C}_0(M)$ is always amenable. Projecta $\rho_n(\mathcal{M})$, standard parameters $p_n(\mathcal{M})$ of a potential S-premouse are defined over $\mathfrak{C}_0(\mathcal{M})$ just as those of a premouse are defined over their Σ_0-codes, cf.[20].

If $\langle \mathcal{N}^{\eta} : \eta < \xi\rangle$, ξ limit, is a sequence of S-structures such that $\eta < \mu < \xi \to \mathcal{N}^{\eta} \lhd \mathcal{N}^{\mu}$, then

$$\bigsqcup_{\eta<\xi} \mathcal{N}^{\eta} = \langle \bigcup_{\eta<\xi} |\mathcal{N}^{\eta}|, \in, a, \bigcup_{\eta<\xi} \dot{E}^{\mathcal{N}^{\eta}}, \emptyset, \bigcup_{\eta<\xi} \dot{S}^{\mathcal{N}^{\eta}}, \emptyset, \emptyset, \emptyset\rangle$$

is the canonical "union" of the $\mathcal{N}^{\eta}$'s

If $\mathcal{N}$ is a premouse over a, $\mathcal{N} = \langle J_{\alpha}^{\vec{E}}[a], \in, a, \vec{E}, F\rangle$, then let $\mathfrak{N}(\mathcal{N}) = \langle J_{\alpha+1}^{\vec{E}^\frown\langle F\rangle}[a], \in$

$, a, \vec{E}^\frown\langle F\rangle, \emptyset\rangle$.

If $\mathcal{N}$ is a potential S-premouse over a, $\mathcal{N} = \langle J_\alpha^{\vec{E},S}[a], \in, a, \vec{E}, F, S, b, \mathcal{Q}, \pi\rangle$, then let $\mathfrak{N}(\mathcal{N}) = \langle J_{\alpha+1}^{\vec{E}^\frown\langle F\rangle,S,\pi}[a], \in, a, \vec{E}^\frown\langle F\rangle, S', \emptyset, \emptyset, \emptyset\rangle$, where

$$S' = \begin{cases} S, & \text{if } b = \emptyset, \\ S \cup \{\mathcal{N}\}, & \text{if } b \neq \emptyset. \end{cases}$$

2.5 The $S^{[0]}$-operator

In this short section, we define the $S^{[0]}$-operator from the $S^{*,[0]}$-operator. In fact, the $S^{[0]}$-operator is not much different from the $S^{*,[0]}$-operator. The only reason why we develop the $S^{[0]}$-operator lies in the fine-structural matter. In the future, we are going to define the $S^{[0]}$-premouse, whose successor level is $S^{[0]}$ of the previous level. The naive way is to think of an $S^{[0]}$-premouse as a stack the $S^{*,[0]}$-operators plus some extenders at certain levels. In order to get a nice iteration theory for the $S^{[0]}$-premice, we want every initial segment of an $S^{[0]}$-premouse be sound, just as an ordinal premouse. However, if we treat $S^{*,[0]}(S^{*,[0]}(a))$ as a potential S-premouse over a, its initial segment of length bigger than $o(S^{*,[0]}(a))$ may project to a, and hence may not be sound. We solve this problem by pausing a while before we apply $S^{*,[0]}$ to $S^{*,[0]}(a)$. The next $S^{*,[0]}$ will be applied to the stack of all S-mice that extends $S^{*,[0]}(a)$, projects across $S^{[0]}(a)$, and is pure $L[\vec{E}]$-mice above $S^{[0]}(a)$.

Definition 2.24. Let a be countable transitive swo. Let $\mathcal{K}$ be a potential S-premouse over a. Then

$$\begin{aligned} SSM(\mathcal{K}) = \bigsqcup\{\mathcal{M} : & \mathcal{M} \text{ is a sound potential } S\text{-premouse extending } \mathcal{K}, \\ & o(\mathcal{K}) \text{ is a strong cutpoint of } \mathcal{M}, \\ & \forall y \in S^{\mathcal{M}}(o(y) \leq o(\mathcal{K})), \\ & \mathcal{M} \text{ is iterable when hitting extenders above } o(\mathcal{K}), \\ & \rho_\omega(\mathcal{M}) \leq o(\mathcal{K}).\} \end{aligned}$$

(SSM stands for "stack of S-mouse").

Definition 2.25 ($S^{[0]}$-operator)**.** Let a be countable transitive. Let $\mathcal{K}$ be a potential S-premouse over a. Suppose $S^{*,[0]}(SSM(\mathcal{K})) = \langle M, \in, SSM(\mathcal{K}), E, S, Q, \pi\rangle$, and suppose that $\langle M, \in, a, E^{SSM(\mathcal{K})} \cup E, S^{SSM(\mathcal{K})} \cup S, \mathcal{K}, Q, \pi\rangle$ is a potential S-premouse over a. Then let $S^{[0]}(\mathcal{K}) = \langle M, \in, a, E^{SSM(\mathcal{K})} \cup E, S^{SSM(\mathcal{K})} \cup S, \mathcal{K}, Q, \pi\rangle$.

$S^{[0]}$ applies only on potential S-premouse. We say an operator S has condensation for potential S-premouse above z if whenever $\mathcal{K}$ is a potential S-premouse above some $a \in \mathrm{Cone}(z)$ such that $S(\mathcal{K})$ exists, and $j : \mathcal{M} \to S(\mathcal{K})$ is Σ_1-elementary, $z \in \mathrm{ran}(j)$, then $\mathcal{M} = S(j^{-1}(\mathcal{K}))$.

Lemma 2.26. *$S^{[0]}$ has condensation for potential S-premouse above z.*

Proof. Let $j : \mathcal{M} \to S^{[0]}(\mathcal{K})$ be Σ_1-elementary. By Lemma 2.20, $\mathcal{M}$, when treated as a potential S-premouse over $j^{-1}(SSM(\mathcal{K}))$, is equal to $S^{*,[0]}(j^{-1}(SSM(\mathcal{K})))$. It suffices to show that $j^{-1}(SSM(\mathcal{K})) = SSM(j^{-1}(\mathcal{K}))$. Clearly, $j^{-1}(SSM(\mathcal{K})) \trianglelefteq SSM(j^{-1}(\mathcal{K}))$. Suppose towards a contradiction that $j^{-1}(SSM(\mathcal{K})) \triangleleft SSM(j^{-1}(\mathcal{K}))$. Pick some $j^{-1}(SSM(\mathcal{K})) \triangleleft \mathcal{N} \triangleleft SSM(j^{-1}(\mathcal{K}))$, such that $\rho_\omega(\mathcal{N}) \leq o(\mathcal{K})$. Then $\mathcal{N}$ is $OD(\mathcal{K})$. By fullness of $S^{*,[0]}(j^{-1}(SSM(\mathcal{K})))$, $\mathcal{N} \in S^{*,[0]}(j^{-1}(SSM(\mathcal{K})))$.

Since $(o(j^{-1}(\mathcal{K}))^+)^{\mathcal{M}} = o(j^{-1}(SSM(\mathcal{K})))$, by acceptability, $\mathcal{N} \in j^{-1}(SSM(\mathcal{K}))$. Contradiction. □

.

2.6 The S-operators and the S^*-operators

We introduce a few notations that will come up in the measurable cofinality case. Suppose that a countable transitive swo, $\mathcal{R} \in a_+$ is a hod premouse, $\mathcal{M}$ is a hybrid premouse over b, $\nu < \lambda^{\mathcal{R}}$, and $\mathcal{R}, \Lambda \in \mathcal{M}$ such that

$$\mathcal{M} \models \Lambda \text{ is an iteration strategy for } \mathcal{R}(\nu) \text{ which is commuting and positional.}$$

Working in $\mathcal{M}$, we let $\mathcal{F}_a^{\mathcal{M},\Lambda}(\mathcal{R})$ be the direct system

$$\{\mathcal{R}_1, \pi_{\mathcal{R}_1\mathcal{R}_2} : \mathcal{R}_1, \mathcal{R}_2 \text{ are iterates of } \mathcal{R} \text{ below } \mathcal{R}(\nu) \text{ of size} \leq |a| \text{ according to } \Lambda,$$
$$\pi_{\mathcal{R}_1\mathcal{R}_2} \text{ is an iteration map}\}.$$

We call

$$\mathrm{dirlim}_a^{\mathcal{M},\Lambda}(\mathcal{R})$$

the direct limit of $\mathcal{F}_a^{\mathcal{M},\Lambda}(\mathcal{R})$,

$$\pi_a^{\mathcal{M},\Lambda}(\mathcal{R}) : \mathcal{R} \to Q_a^{\mathcal{M},\Lambda}(\mathcal{R})$$

the direct limit map.

If $\mathcal{Q}$ is an Σ-iterate of $\mathcal{P}$, $\beta \leq \beta^* \leq \lambda^{\mathcal{Q}}$, $j : \mathcal{Q} \to \mathcal{R}$ is an iteration according to $\Sigma_{\mathcal{Q}}$, then we call

$$(\mathcal{R}, j(\beta^*), \sup j''\beta)$$

the j-promotion of (Q, β^*, β).

Given $\mathfrak{P} = \langle (\zeta_i, \pi_i, \mathcal{P}_i, \alpha_i^*, \alpha_i) : i \leq n\rangle \in \mathcal{I}$. Let a countable transitive swo. We say that $\mathfrak{P}$ is a promotable index for a if $\mathcal{P} \in a_+$ and there are

$$\mathfrak{Q} = \langle (\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i^*, \beta_i) : i \leq n\rangle \in \mathcal{I},$$
$$\vec{j} = \langle j_i : i \leq n\rangle, \vec{\mathcal{N}} = \langle \mathcal{N}_i : i \leq n\rangle$$

such that $\beta_0 = \alpha_0$, $j_0 = id_{\mathcal{P}}$, $\mathcal{N}_0 = a$, and for all $i < n$,

1. $\mathcal{N}_{i+1} = \mathcal{M}_1^{\#, \Sigma_{\mathcal{Q}_i(\nu_{i+1})}}(\mathcal{N}_i)$.
2. $\mathcal{Q}_{i+1} = \mathrm{dirlim}_a^{\mathcal{N}_{i+1}, \Sigma_{\mathcal{Q}_i(\nu_{i+1})}}(\mathcal{Q}_i)$,
3. $\sigma_{i+1} = \pi_a^{\mathcal{N}_{i+1}, \Sigma_{\mathcal{Q}_i(\nu_{i+1})}}(\mathcal{Q}_i)$,
4. $j_{i+1} : \mathcal{P}_{i+1} \to \mathcal{Q}_{i+1}$ is an iteration map below $\mathcal{P}_{i+1}(\pi_{i+1}(\nu_{i+1}))$,
5. $(Q_{i+1}, \beta_{i+1}^*, \beta_{i+1})$ is the j_{i+1}-promotion of $(P_{i+1}, \alpha_{i+1}^*, \alpha_{i+1})$.

For $\mathfrak{P}, \mathfrak{Q}, a, \vec{j}, \vec{\mathcal{N}}$ as above, we say that $\mathfrak{Q}$ is the promotion of $\mathfrak{P}$ for a, $\vec{\mathcal{N}}$ is the $\mathcal{M}_1^{\#}$-sequence of $\mathfrak{P}$ for a, $\vec{j}$ is the lifting map sequence of $\mathfrak{P}$ with respect to a. We denote $\mathfrak{Q} = \mathrm{pro}(\mathfrak{P}, a)$. Clearly if $\mathfrak{P}$ is an index for a, then there is a unique promotion, a unique $\mathcal{M}_1^{\#}$-sequence, and a unique lifting map sequence for a. The following picture shows the process of promoting an index.

$$\begin{tikzcd}
& \mathcal{Q}_1 \arrow[r, "\sigma_2"] & \mathcal{Q}_2 \arrow[r, dashed] & {} \arrow[r, "\sigma_n"] & \mathcal{Q}_n \\
\mathcal{P} = \mathcal{P}_0 = \mathcal{Q}_0 \arrow[ur, "\sigma_1"] \arrow[r, "\pi_1"] & \mathcal{P}_1 \arrow[u, "j_1"] \arrow[r, "\pi_2"] & \mathcal{P}_2 \arrow[u, "j_2"] \arrow[r, dashed] & {} \arrow[r, "\pi_n"] & \mathcal{P}_n \arrow[u, "j_n"]
\end{tikzcd}$$

We say that $\mathfrak{P}$ is a promoted index over a if $\mathfrak{P}$ is a promoted index over a and $\mathfrak{P} = \mathrm{pro}(\mathfrak{P}, a)$. Suppose that $\mathfrak{P} = \langle (\zeta_i, \pi_i, \mathcal{P}_i, \alpha_i^*, \alpha_i) : i \leq n\rangle$ and $\mathfrak{Q} =$

$\langle(\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i^*, \beta_i) : i \leq m\rangle$ are promoted indices over a. Let

$$\mathfrak{P} <_a^{\mathcal{I}} \mathfrak{Q}$$

if

1. There is $i \leq n$ such that either $(i \leq m \wedge \mathfrak{Q}_i \neq \mathfrak{P}_i)$ or $i > m$.
2. Let i be least as in 1. Then for all $i \leq j \leq n$, $\alpha_j^* = \alpha_j$. If in addition $i \leq m$, then $\alpha_i < \beta_i$.

As the reader might expect, definition of $S^{\mathfrak{P}}(a)$ is based on $S^{\mathfrak{Q}}(a)$ for $\mathfrak{Q} <_a^{\mathcal{I}} \mathfrak{P}$. In fact, for any promoted index $\mathfrak{P}$ over a, $\{\mathfrak{Q} : \mathfrak{Q} <_a^{\mathcal{I}} \mathfrak{P}\}$ is well-ordered under $<_a^{\mathcal{I}}$.

Let $\mathfrak{P} = \langle(\zeta_i, \pi_i, \mathcal{P}_i, \alpha_i^*, \alpha_i) : i \leq n\rangle$ be an index. We say $\mathfrak{P}$ is a successor index if α_i is a successor ordinal, $\mathfrak{P}$ is a limit index if α_i is a limit ordinal. If $\mathfrak{P}$ is a limit index, we say $\mathfrak{P}$ is of type A if

$$\mathcal{P}_n \models \text{“}\operatorname{cf}(\alpha_n^*) \text{ is not measurable”} \wedge (n = 0 \vee (n > 0 \wedge \alpha_n > \sup \pi_n''\alpha_{n-1})).$$

$\mathfrak{P}$ is of type B if

$$n > 0 \wedge \alpha_n = \sup \pi_n''\alpha_{n-1}.$$

$\mathfrak{P}$ is of type C if

$$\mathcal{P}_n \models \text{“}\operatorname{cf}(\alpha_n^*) \text{ is measurable”}.$$

We are ready to define the $S^{\mathfrak{P}}$-operator. It is an inductive definition on the hod mouse prewellordering of final($\mathfrak{P}$). To simplify notations, when $\mathfrak{Q} = \langle(\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i^*, \beta_i) : i \leq n\rangle \in \mathcal{I}$, $\epsilon \leq \epsilon^* \leq \beta_n$, we let

$$\mathfrak{Q}[\epsilon] = \mathfrak{Q}\restriction n^\frown\langle\nu_n, \sigma_n, \mathcal{Q}_n, \epsilon, \epsilon\rangle,$$

If in addition β_n^* is a successor, then let

$$\mathfrak{Q} - 1 = \mathfrak{Q}[\beta_n^* - 1].$$

We also let $[\gamma] = \langle(\emptyset, \emptyset, \mathcal{P}, \gamma, \gamma)\rangle$. So $[0] = \langle(\emptyset, \emptyset, \mathcal{P}, 0, 0)\rangle$, which agrees with the notation in Sections 2.3 and 2.5.

The successor $S^{\mathfrak{P}}$-operators are defined to be roughly an $S^{\mathfrak{P}-1}$-mouse with an additional predicate indicating the direct limit map of all iterates of $\mathcal{P}$. The $S^{\mathfrak{P}-1}$-mouse that will be used comes from the $L[\vec{E}, S^{\mathfrak{P}-1}]$-construction. We define the construction in general as follows. We will borrow a formula ϕ_t from Section 2.7. ϕ_t looks for the indices of the S-operators of an S-premouse and there corresponding strategies. For the time being, we just take it to be a first-order formula. We make sure the definitions are not circular.

Definition 2.27 (The $L[\vec{E}, S^{\mathfrak{P}}]$-construction)**.** Let $\mathcal{Q}$ be Σ-good. Let $\mathcal{K} \in \mathcal{Q}|\delta^{\mathcal{Q}}$ be an potential S-premouse over a which is closed under $S^{\mathfrak{P}+1}$. The maximal $L[\vec{E}, S^{\mathfrak{P}}]$-construction over a is a sequence $\langle \mathcal{N}_\xi, \mathcal{M}_\xi : \xi < \mathrm{Ord}\rangle$ with the following properties.

1. $\mathcal{N}_0 = \langle \mathrm{rud}(a), \in, a, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle$.
2. If $\mathcal{M}_\xi = \langle J_\alpha^{\vec{E},S}, \in, a, \vec{E}, \emptyset, S, \emptyset, \emptyset, \emptyset\rangle$, $\alpha < \delta^{\mathcal{Q}}$, and there are an total extender F^* on the $\mathcal{Q}$-sequence, an extender F over $\mathcal{M}_\xi$, an ordinal $\nu < \alpha$ such that

$$V_{\nu+\omega} \subseteq \mathrm{Ult}(V, F^*)$$

and

$$F \restriction \nu = F^* \cap ([\nu]^{<\omega} \times J_\alpha^{\vec{E}})$$

and

$$\mathcal{N}_{\xi+1} = \langle J_\alpha^{\vec{E},S}, \in, a, \vec{E}, F, S, \emptyset, \emptyset, \emptyset\rangle$$

is a potential S-premouse with $\nu = \nu^{\mathcal{N}_{\xi+1}}$, then $\mathcal{N}_{\xi+1}$ is as above,

3. Let $\mathcal{M}_\xi = \langle J_\alpha^{\vec{E},S}, \in, a, \vec{E}, F, S, b, Q, \pi\rangle$. If $\alpha = \delta$, let $\mathbb{T} = (\text{base}, \text{deg}, \text{drop}, \text{lift}, \Lambda)$ be unique such that $(\mathcal{Q}|\delta^{\mathcal{Q}})_+ \models \phi_t(\mathcal{M}_\xi, \mathbb{T})$. Let $d = \max(\text{drop}(\mathcal{M}_\xi))$. If it is *not* the case that $d = o(a) \wedge \text{deg}(\mathcal{M}_\xi | \min(l^{\mathcal{M}_\xi} \setminus o(a)+1)) = \text{pro}(\mathfrak{P}, a)$, let $\mathcal{Q}^\infty_{\mathcal{M}_\xi}$ be the direct limit of all Σ-iterates of $\mathcal{P}$ that are in M_ξ. Let $\pi^\infty_{\mathcal{M}_\xi} : \mathcal{P} \to \mathcal{Q}^\infty_{\mathcal{M}_\xi}$ be the direct limit map. Then $\mathcal{N}_{\xi+1}$ is the e-amenable code of the transitive collapse of

$$\langle Hull^{(\mathcal{M}_\xi)_+}(\mathcal{M}_\xi|d \cup \pi^\infty_{\mathcal{M}_\xi}), \in, a, \vec{E}, \emptyset, S, \mathcal{M}_\xi|d, \mathcal{Q}^\infty_{\mathcal{M}_\xi}, \pi^\infty_{\mathcal{M}_\xi}\rangle$$

4. Let $\mathcal{M}_\xi = \langle J_\alpha^{\vec{E},S}, \in, a, \vec{E}, F, S, b, Q, \pi\rangle$. If neither of the above happens, then

$$\mathcal{N}_{\xi+1} = \mathfrak{N}(M_\xi).$$

5. If $\alpha = \delta$, but $d = o(a) \wedge \text{deg}(\mathcal{M}_\xi | \min(l^{\mathcal{M}_\xi} \setminus o(a) + 1)) = \text{pro}(\mathfrak{P}, a)$, we terminate the definition, and say $\mathcal{N}_\xi$ is the output of the $L[\vec{E}, S^{\mathfrak{P}}][a]$-construction.

6. For all ξ, if $\mathcal{N}_\xi$ is defined, then

$$\mathcal{M}_\xi = \text{core}_\omega(\mathcal{N}_\xi).$$

7. when ξ is a limit ordinal, then

$$\mathcal{N}_\xi = \liminf_{\mu \to \xi} \mathcal{M}_\mu,$$

Definition 2.28 (The $S^{*,\mathfrak{P}}$ operator)**.** Suppose that $\mathfrak{P} = \langle(\zeta_i, \pi_i, \mathcal{P}_i, \alpha_i^*, \alpha_i) : i \leq n\rangle$ is a promotable successor index over a. We define $S^{*,\mathfrak{P}}(a)$ as follows. Let $\mathcal{Q}$ be Σ-good over a. Assume that the $L[\vec{E}, S^{\mathfrak{P}-1}][a]$-construction in $\mathcal{Q}|\delta^{\mathcal{Q}}$ converges to a $S^{\mathfrak{P}-1}$-mouse over a. Let $\mathcal{N}$ be the output. By the proof of MSC (cf. [7]), there is $\mathcal{R} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{R}} \restriction \mathcal{N} \in \mathcal{N}_+$. Let $F_{\mathcal{N}}$ be the direct system

$$\{\mathcal{R}, \sigma_{\mathcal{R}\mathcal{R}'} : \mathcal{R}, \mathcal{R}' \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}, \sigma_{\mathcal{R}\mathcal{R}'} \text{ is a } \Sigma\text{-iteration map.}\}$$

Let $Q^{\infty}_{\mathcal{N}}$ be the direct limit of $F_{\mathcal{N}}$ and

$$\pi^{\infty}_{\mathcal{N}} : \mathcal{P} \to \mathcal{Q}^{\infty}_{\mathcal{N}}$$

be the direct limit map, so that $Q^{\infty}_{\mathcal{N}} \in \mathcal{N}_+$. Let $\mathcal{M}$ be the transitive collapse of the structure

$$\langle Hull^{\mathcal{N}_+}(a \cup \{a\} \cup \pi^{\infty}_{\mathcal{N}}), \in, a, E^{\mathcal{N}}, S^{\mathcal{N}}, Q^{\infty}_{\mathcal{N}}, \pi^{\infty}_{\mathcal{N}}\rangle.$$

Then $S^{*,\mathfrak{P}}(a)$ is the e-amenable code of $\mathcal{M}$.

We only define the $S^{*,\mathfrak{P}}$-operator when $\mathfrak{P}$ is a successor index. In what follows, we plan to define the $S^{\mathfrak{P}}$-operator in general. When $\mathfrak{P}$ is a successor index, the $S^{\mathfrak{P}}$-operator relies on the $S^{*,\mathfrak{P}}$-operator and the concept of "stack of the $S^{\mathfrak{P}-1}$-mouse". When $\mathfrak{P}$ is a limit index, the $S^{\mathfrak{P}}$-operator relies on the $S^{\mathfrak{Q}}$-operators, for $\mathfrak{Q} <^{\mathcal{I}}_a \text{pro}(\mathfrak{P}, a)$. We deal with successor indices first. To settle that, we give the definition of the $S^{\mathfrak{P}}$-premouse, presuming the $S^{\mathfrak{P}}$-operator is defined.

Definition 2.29. Suppose that $\mathfrak{P}$ is a promotable index over a.

1. ($S^{\mathfrak{P}}$-premouse when restricted above γ). Let $\mathcal{K}$ be a potential S-premouse over a. Let $\gamma \in [o(a), o(\mathcal{K}))$. We say $\mathcal{K}$ is an $S^{\mathfrak{P}}$-premouse over a when restricted above γ if
   (a) If $\eta < \mu$ are consecutive elements of $l^{\mathcal{K}} \setminus \gamma$, then $\mathcal{K}|\mu = S^{\mathfrak{P}}(\mathcal{K}|\eta)$,
   (b) If $\max(l^{\mathcal{K}}) = \mu$ exists, then $\mathcal{K} = S^{\mathfrak{P}}(\mathcal{K}|\mu)$.

2. (Stack of $S^{\mathfrak{P}}$-mouse). Let $\mathcal{K}$ be an $S^{\mathfrak{P}}$-premouse over a. Then

$$SSM^{\mathfrak{P}}(\mathcal{K}) = \bigsqcup\{\mathcal{M} : \mathcal{M} \text{ is a sound potential } S^{\mathfrak{P}}\text{-premouse extending } \mathcal{K}. \text{ when restricted above } o(\mathcal{K}), o(\mathcal{K}) \text{ is a strong cutpoint of } \mathcal{M}. \mathcal{M} \text{ is } S^{\mathfrak{P}}\text{-iterable when hitting extenders above } o(\mathcal{K}).\ \rho_\omega(\mathcal{M}) \leq o(\mathcal{K}).\}$$

($SSM^{\mathfrak{P}}$ stands for "stack of $S^{\mathfrak{P}}$-mouse").

Definition 2.30. Suppose that $\mathfrak{P}$ is a promotable index over a. $\mathcal{K}$ is a potential S-premouse over a.

1. If $\mathfrak{P}$ is a successor index, $S^{*,\mathfrak{P}}(SSM^{\mathfrak{P}-1}(\mathcal{K})) = \langle M, \in, SSM^{\mathfrak{P}-1}(\mathcal{K}), E, S, Q, \pi\rangle$ is defined, and $\langle M, \in, a, E^{SSM^{\mathfrak{P}-1}(\mathcal{K})} \cup E, S^{SSM^{\mathfrak{P}-1}(\mathcal{K})} \cup S, Q, \pi\rangle$ is a potential S-premouse, then $S^{\mathfrak{P}}(\mathcal{K}) = \langle M, \in, a, E^{SSM^{\mathfrak{P}-1}(\mathcal{K})} \cup E, \emptyset, S^{SSM^{\mathfrak{P}-1}(\mathcal{K})} \cup S, \mathcal{K}, Q, \pi\rangle$. Otherwise, we leave $S^{\mathfrak{P}}(\mathcal{K})$ undefined.

2. If $\mathfrak{P}$ is a limit index, and for all $\mathfrak{Q} <^{\mathcal{I}}_{\mathcal{K}} \text{pro}(\mathfrak{P}, \mathcal{K})$, $S^{\mathfrak{Q}}(\mathcal{K})$ is defined, and for all $\mathfrak{Q} <^{\mathcal{I}}_{\mathcal{K}} \text{pro}(\mathfrak{P}, \mathcal{K})$, $S^{\mathfrak{Q}}(\mathcal{K}) \lhd S^{\mathfrak{R}}(\mathcal{K})$, then

$$S^{\mathfrak{P}}(\mathcal{K}) = \langle \bigcup_{\mathfrak{Q} <^{\mathcal{I}}_{\mathcal{K}} \text{pro}(\mathfrak{P}, \mathcal{K})} |S^{\mathfrak{Q}}(\mathcal{K})|, \in, a, \bigcup_{\mathfrak{Q} <^{\mathcal{I}}_{\mathcal{K}} \text{pro}(\mathfrak{P}, \mathcal{K})} \dot{E}^{S^{\mathfrak{Q}}(\mathcal{K})}, \emptyset, \bigcup_{\mathfrak{Q} <^{\mathcal{I}}_{\mathcal{K}} \text{pro}(\mathfrak{P}, \mathcal{K})} \dot{S}^{S^{\mathfrak{Q}}(\mathcal{K})}, \mathcal{K}, \emptyset, \emptyset\rangle.$$

Otherwise, we leave $S^{\mathfrak{P}}(\mathcal{K})$ undefined.

Here is a deeper explanation about the limit case of definition 2.30. We can write down $S^{\mathfrak{P}}(\mathcal{K})$ more explicitly, depending on the type of $\mathfrak{P}$. Clearly, $S^{\mathfrak{P}}(\mathcal{K}) = S^{\text{pro}(\mathfrak{P},\mathcal{K})}(\mathcal{K})$. So we need only consider $S^{\mathfrak{P}}(\mathcal{K})$ when $\mathfrak{P}$ is a promoted index over $\mathcal{K}$. Let $\mathfrak{P} = \langle(\zeta_i, \pi_i, \mathcal{P}_i, \alpha_i^*, \alpha_i) : i < n\rangle$ be a limit promoted index over $\mathcal{K}$. Let $\langle \mathcal{N}_i : i \leq n\rangle$ be its $\mathcal{M}_1^{\#}$-sequence.

1. Suppose $\mathfrak{P}$ is of type A. Let

$$C = \begin{cases} [0, \alpha_0), & \text{if } n = 0 \\ [\sup \pi_n'' \alpha_{n-1}, \alpha_n), & \text{if } n > 0 \end{cases}$$

Then $\{\mathfrak{P}[\gamma] : \gamma \in C\}$ is $<^{\mathcal{I}}_{\mathcal{K}}$-cofinal in $\{\mathfrak{Q} : \mathfrak{Q} <^{\mathcal{I}}_{\mathcal{K}} \mathfrak{P}\}$. So

$$S^{\mathfrak{P}}(\mathcal{K}) = \langle \bigcup_{\gamma \in C} |S^{\mathfrak{P}[\gamma]}(\mathcal{K})|, \in, a, \bigcup_{\gamma \in C} \dot{E}^{S^{\mathfrak{P}[\gamma]}(\mathcal{K})}, \emptyset, \bigcup_{\gamma \in C} \dot{S}^{S^{\mathfrak{P}[\gamma]}(\mathcal{K})}, \mathcal{K}, \emptyset, \emptyset \rangle.$$

2. Suppose $\mathfrak{P}$ is of type B. Let

$$C = \begin{cases} [0, \alpha_0), & \text{if } n = 1 \\ [\sup \pi_{n-1}'' \alpha_{n-2}, \alpha_{n-1}), & \text{if } n > 1 \end{cases}$$

Then $\{(\mathfrak{P} \upharpoonright n)[\gamma] : \gamma \in C\}$ is $<^{\mathcal{I}}_{\mathcal{K}}$-cofinal in $\{\mathfrak{Q} : \mathfrak{Q} <^{\mathcal{I}}_{\mathcal{K}} \mathfrak{P}\}$. So

$$S^{\mathfrak{P}}(\mathcal{K}) = \langle \bigcup_{\gamma \in C} |S^{(\mathfrak{P}\upharpoonright n)[\gamma]}(\mathcal{K})|, \in, a, \bigcup_{\gamma \in C} \dot{E}^{S^{(\mathfrak{P}\upharpoonright n)[\gamma]}(\mathcal{K})}, \emptyset, \bigcup_{\gamma \in C} \dot{S}^{S^{(\mathfrak{P}\upharpoonright n)[\gamma]}(\mathcal{K})}, \mathcal{K}, \emptyset, \emptyset \rangle.$$

3. Suppose $\mathfrak{P}$ is of type C. Let ζ^* be least such that

$$\delta^{\mathcal{Q}_n}_{\zeta^*} > \max(\pi_n(\delta^{\mathcal{P}_{n-1}}_{\zeta_n}), \mathrm{cf}^{\mathcal{P}_n}(\alpha^*_n)).$$

Let $\mathcal{N}^* = \mathcal{M}_1^{\#, \pi_{\mathcal{P}_n(\nu^*)}}(\mathcal{N}_n)$, $\mathcal{P}^* = \mathrm{dirlim}_{\mathcal{K}}^{\mathcal{N}^*, \pi_{\mathcal{P}_n(\zeta^*)}}(\mathcal{P}_n)$, $\pi^* = \pi_{\mathcal{K}}^{\mathcal{N}^*, \pi_{\mathcal{P}_n(\zeta^*)}}(\mathcal{P}_n)$, Let

$$C = [\sup \pi^{*\prime\prime} \alpha_n, \pi^*(\alpha^*_n)).$$

(In other words, for any $\gamma \in C$, $(\zeta^*, \pi^*, \mathcal{P}^*, \gamma, \gamma)$ is a one-step blow-up of $\mathfrak{P}_n$ above $\mathcal{P}_n(\pi_n(\zeta_n))$.) Then $\{\mathfrak{P}^\frown \langle \zeta^*, \pi^*, \mathcal{Q}^*, \gamma, \gamma \rangle : \gamma \in C\}$ is $<^{\mathcal{I}}_{\mathcal{K}}$-cofinal in$\{\mathfrak{Q} : \mathfrak{Q} <^{\mathcal{I}}_{\mathcal{K}} \mathfrak{P}\}$. So

$$S^{\mathfrak{P}}(\mathcal{K}) = \langle \bigcup_{\gamma\in C} |S^{\mathfrak{P}^\frown\langle\zeta^*,\pi^*,\mathcal{Q}^*,\gamma,\gamma\rangle}(\mathcal{K})|, \in, a, \bigcup_{\gamma\in C} \dot{E}^{S^{\mathfrak{P}^\frown\langle\zeta^*,\pi^*,\mathcal{Q}^*,\gamma,\gamma\rangle}(\mathcal{K})}, \emptyset,$$
$$\bigcup_{\gamma\in C} \dot{S}^{S^{\mathfrak{P}^\frown\langle\zeta^*,\pi^*,\mathcal{Q}^*,\gamma,\gamma\rangle}(\mathcal{K})}, \mathcal{K}, \emptyset, \emptyset\rangle.$$

Once again, the only reason why we distinguish the $S^{*,\mathfrak{P}}$-operator from the $S^{\mathfrak{P}}$-operator lies in the fine-structural matter. We want all initial segments of an $S^{\mathfrak{P}}$-premouse to be sound. The reader might ignore this subtle difference when reading Section 2.7, where we are going to define the strategy on an S-premouse but no fine structure is involved.

It is easy to check through the definitions that if $S^{\mathfrak{P}}$ is defined above z, then for all $\mathcal{Q}$ good over a, a countable transitive such that $z \in a_+$, we have $S^{\mathfrak{P}} \restriction (\mathcal{Q}|\delta^{\mathcal{Q}}) \in \mathcal{Q}$ and is uniformly definable. Furthermore, we have the following

Lemma 2.31. *$S^{\mathfrak{P}}$ extends to generic extensions above z.*

2.7 Defining strategy over an S-premouse

If $\mathcal{K}$ is a potential S-premouse over a, we set

$$\tilde{S}^{\mathcal{K}} = \{\mathcal{K}|\xi : \text{either } b^{\mathcal{K}|\xi} \neq \emptyset, \text{ or } \exists\eta(\text{ for cofinal-in-}\xi\text{-many } \mu, \mathcal{K}|\mu \in S^{\mathcal{K}} \text{ and } \mathcal{K}|\eta = b^{\mathcal{K}|\mu})\}$$

$\tilde{S}^{\mathcal{K}}$ is the range of all S-operators on the $\mathcal{K}$-sequence, including limit S-operators. It is possible that $\mathcal{K} \in \tilde{S}^{\mathcal{K}}$.

Definition 2.32. An potential S-premouse $\mathcal{K}$ over a is called an S-premouse if there is a map $y \mapsto \deg(y)$ from $\tilde{S}^{\mathcal{K}}$ into $\mathcal{I}$, called the degree map, such that

1. For each $y \in \tilde{S}^{\mathcal{K}}$, letting $x = b^y$, then $\deg(y)$ is a promoted index over x such

that $y = S^{\deg(x,y)}(x)$.

2. For each $y \in \tilde{S}^{\mathcal{K}}$, if $b^y = a$, then $\deg(a, y) \leq^{\mathcal{I}}_a [\lambda^{\mathcal{P}}]$.

3. For each $y \in \tilde{S}^{\mathcal{K}}$, letting $x = b^y$, if $x \neq a$, then for each $\xi \in l^{\mathcal{K}|o(x)}$, a successor element of $l^{\mathcal{K}|o(x)}$, we have $\mathcal{K}|\xi \in \tilde{S}^{\mathcal{K}}$ and $\deg(y) \leq^{\mathcal{I}}_x \mathrm{pro}(\deg(\mathcal{K}|\xi), x)$.

Clause 3 demands that an S-premouse decreases in degree. The degree sequence deg is always unique. We prove uniqueness of $\deg(x)$ for $y \in \tilde{S}^{\mathcal{K}}$ by induction on $o(y)$, as follows.

Let $y \in S^{\mathcal{K}}$, $x = b^y$. If there is no $z \in \tilde{S}^{\mathcal{K}}$ such that $o(x) < o(z) < o(y)$, obviously, we must have $\deg(x, y) = [0]$. If there is $z \in \tilde{S}^{\mathcal{K}}$ such that $o(x) < o(z) < o(y)$, then $\deg(x, y)$ is a successor index. By definition of the successor $S^{\mathfrak{P}}$-operators, there must be a largest z such that $z \in \tilde{S}^{\mathcal{K}} \wedge x = b^z$. We assume as an induction hypothesis that $\deg(z)$ is unique. If $x = a$, then by clause 2, $\deg(z) \leq^{\mathcal{I}}_a [\lambda^{\mathcal{P}}]$. So $\deg(y)$ is the $<^{\mathcal{I}}_a$-successor of $\deg(z)$ in $\{\mathfrak{Q} : \mathfrak{Q} <^{\mathcal{I}}_a [\lambda^{\mathcal{P}}]\}$. If $x \neq a$, then pick any pair μ, a successor element of $l^{\mathcal{K}|o(x)}$. By clause 3, $\deg(y) \leq^{\mathcal{I}}_x \mathrm{pro}(\mathcal{K}|\mu), x)$. So $\deg(y)$ is the $<^{\mathcal{I}}_x$-successor of $\deg(z)$ in $\{\mathfrak{Q} : \mathfrak{Q} \leq^{\mathcal{I}}_x \mathrm{pro}(\deg(\mathcal{K}|\mu), x)\}$.

Let $y \in \tilde{S}^{\mathcal{K}} \setminus S^{\mathcal{K}}$, $x = b^y$, and suppose that uniqueness of $\deg(v)$ for $o(v) < o(y)$ is proved. If $x = a$, then by clause 2, for any $z \in S^{\mathcal{K}}$ such that $a = b^z$ and $z \in y$, $\deg(z) \leq^{\mathcal{I}}_a [\lambda^{\mathcal{P}}]$. Hence the $\deg(z)$'s, for $a = b^z \wedge z \in y$, form an $<^{\mathcal{I}}_a$-increasing sequence in $\{\mathfrak{Q} : \mathfrak{Q} \leq^{\mathcal{I}}_a [\lambda^{\mathcal{P}}]\}$. Hence $\deg(y)$ is the least upper bound of them, which is unique. If $x \neq a$, pick any μ, a successor element of $l^{\mathcal{K}|o(x)}$. By clause 3, for any $z \in S^{\mathcal{K}}$ such that $x = b^z$ and $z \in y$, $\deg(z) \leq^{\mathcal{I}}_x \mathrm{pro}(\deg(\mathcal{K}|\mu), x)$. Hence the $\deg(z)$'s, for $x = b^z \wedge z \in y$, form an $<^{\mathcal{I}}_x$-increasing sequence in $\{\mathfrak{Q} : \mathfrak{Q} \leq^{\mathcal{I}}_x \mathrm{pro}(\deg(\mathcal{K}|\mu), x)\}$. Hence $\deg(x, y)$ is the least upper bound of them, which is unique.

We denote the degree function by $\deg^{\mathcal{K}}$. $\deg^{\mathcal{K}}$ will have this fixed meaning throughout this paper.

If $\xi \in l^{\mathcal{K}}$ and $\xi \neq a$, we say $\mathcal{K}$ drops at ξ if letting $\mu = \min(l^{\mathcal{K}} \setminus (\xi + 1))$, then $\deg(\mathcal{K}|\mu) <^{\mathcal{I}}_{\mathcal{K}|\xi} \text{pro}(\deg(\mathcal{K}|\xi), \mathcal{K}|\xi)$. We let $D^{\mathcal{K}}$ be the set of drops of $\mathcal{K}$ union $\{o(a)\}$. Every drop of $\mathcal{K}$ represents a drop in the hod mouse prewellordering, so $D^{\mathcal{K}}$ must be finite. We let $\text{drop}^{\mathcal{K}} = \{(y, D^y) : y \in \tilde{S}^{\mathcal{K}}\}$.

For $y \in \tilde{S}^{\mathcal{K}}$ such that $x = b^y \neq a$, we let

$$\text{base}^{\mathcal{K}}(y) = \mathcal{K}| \min(l^x \setminus (\max(D^x) + 1))$$

base(y) retreats to the place from which deg(y) could to be defined. We then let

$$\text{lift}^{\mathcal{K}}(y)$$

be the lifting map from final(deg(z)) to final(deg(y)), where $z = y| \min(l^y \setminus (\text{base}(y) + 1))$. For $y \in \tilde{S}^{\mathcal{K}}$ such that $b^y = a$, we let

$$\text{base}^{\mathcal{K}}(y) = a.$$

$$\text{lift}^{\mathcal{K}}(y) = id_{\mathcal{P}}.$$

For each $y \in \tilde{S}^{\mathcal{K}}$, we let

$$\Lambda^{\mathcal{K}}(y) = \Sigma_{\text{final}(\deg(y))}|b^y.$$

One of the main purposes of defining the $S^{\mathfrak{P}}$-operator is that an $\Sigma_{\text{final}(\mathfrak{P})} \cap \mathcal{K}$ can be defined from $S^{\mathfrak{P}}(\mathcal{K})$ in a uniform way. The defining formula is essentially doing an induction on strategies acted on stacks based on hod premice below final($\mathfrak{P}$). More precisely, we need to inductively keep track of $\Sigma_{\text{final}(\deg(y))}$ for all $y \in \tilde{S}^{S^{\mathfrak{P}}(\mathcal{K})}$. The case when deg($y$) is a successor index is monitored by π^y, which is a part of the language of $S^{\mathcal{K}}(\mathcal{K}|\xi)$. The case when deg($y$) is a limit index of type A or B is easy. When deg(y) is a limit index of type C, the strategy is found by rearranging stacks.

Definition 2.33. Let $\mathcal{K}$ be an S-premouse over some countable transitive swo. We let $\mathbb{T}^{\mathcal{K}} = (\mathrm{deg}, \mathrm{base}, \mathrm{lift}, \mathrm{drop}, \Lambda)$ and say that $\mathbb{T}^{\mathcal{K}}$ keeps track of strategies for stacks in $\mathcal{K}$.

Our goal is to define $\Lambda^{\mathcal{K}}$ from $\mathcal{K}$ over any ZFC model. This is an inductive definition on $o(y)$ for the $y \in \tilde{S}^{\mathcal{K}}$. $\mathrm{deg}, \mathrm{base}, \mathrm{lift}, \mathrm{drop}$ are supplementary to defining $\Lambda^{\mathcal{K}}$. The formula ϕ_t will define $\mathbb{T}^{\mathcal{K}}$.

Definition 2.34. $\phi_t(\mathcal{K}, \mathbb{T})$ is the formula expressing conjunction of all of the following.

1. $\mathcal{K}$ is a potential S-premouse over some transitive swo a.
2. $\mathbb{T}$ is a 5-tuple. Write $\mathbb{T} = (\mathrm{base}, \mathrm{deg}, \mathrm{drop}, \mathrm{lift}, \Lambda)$.
3. $\mathrm{base}, \mathrm{deg}, \mathrm{drop}, \mathrm{lift}, \Lambda$ are functions on $\tilde{S}^{\mathcal{K}}$.
4. For each $y \in \tilde{S}^{\mathcal{K}}$, let $x = b^y$. If $x = a$, then $\mathrm{drop}(y) = \{o(a)\}$. If $x \neq a$, then
    (a) $\mathrm{drop}(y)$ is a finite subset of $l^{\mathcal{K}|o(x)}$.
    (b) $\mathrm{drop}(y) \cap o(x) = \mathrm{drop}(x)$.
5. For each $y \in \tilde{S}^{\mathcal{K}}$, let $x = b^y$. If $x = b^y$, then $\mathrm{base}(a, y) = \emptyset$. If $x \neq a$,

$$\mathrm{base}(y) = \mathcal{K}|\xi$$

   where $\xi = \min(l^x \setminus (\max(\mathrm{drop}(x) + 1))$.
6. For each $y \in \tilde{S}^{\mathcal{K}}$, such that $a = b^y$, $\mathrm{lift}(y)$ is identity map on the third coordinate of the last element of $\mathrm{deg}(y)$.

7. (Base case.) Suppose $y \in \tilde{S}^{\mathcal{K}}$, $x = b^y$, and there is no $z \in y$ such that $z \in \tilde{S}^{\mathcal{K}} \wedge x = b^z$. Then

$$\deg(y) = [0],$$
$$\text{lift}(y) = \langle id_{\mathcal{P}} \rangle.$$
$$o(x) \notin \text{drop}(y) \Leftrightarrow x \neq a \wedge \deg(\text{base}(y)) = [0].$$

For each $(\vec{\mathcal{T}}, b)$ such that $\vec{\mathcal{T}} \in x$, we have that

$$\Lambda(y)(\vec{\mathcal{T}}) = b$$

if and only if letting $\mathcal{U}^\frown c$ be the last normal component of $\vec{\mathcal{T}}^\frown b$, then one of the following holds.

- The maximal $L[\vec{E}]$ constructions in $y|\delta^y$ certifies that $\text{Lp}(\mathcal{M}(\mathcal{U})) \models$ "$\delta(\mathcal{U})$ is not Woodin" or $\mathcal{U}$ has a drop in a way that we cannot undo the drop (cf. [7, Fact 3.5.5]), and c is defined using the maximal $L[\vec{E}]$ constructions in $y|\delta^y$.
- The maximal $L[\vec{E}]$ constructions in $y|\delta^y$ certifies that $\text{Lp}(\mathcal{M}(\mathcal{U})) \models$ "$\delta(\mathcal{U})$ is Woodin" and $\mathcal{U}$ does not drop in a way that we cannot undo the drop, and there is $\sigma : \mathcal{M}_c^{\mathcal{U}} \to \mathcal{Q}^y$ such that $\sigma \circ i^{\vec{\mathcal{T}}} = \pi^y$.

8. (Successor case at the bottom of $\mathcal{K}$.) Let $y \in S^{\mathcal{K}}$ be such that $a = b^y$. Suppose z is $\in$-maximal in y such that $z \in \tilde{S}^{\mathcal{K}} \wedge a = b^z$. Denote $\deg(z) = \langle (\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i, \beta_i) : i \leq m \rangle$. Then

$$\deg(y) = \deg(z) + 1,$$

Furthermore, put $\Phi = \bigcup \{\Lambda(v)^{\text{lift}(v)} : \text{base}(v) = z\}$, then for each $(\vec{\mathcal{T}}, b)$ such that $\vec{\mathcal{T}} \in a$, we have that

$$\Lambda(y)(\vec{\mathcal{T}}) = b$$

if and only if letting $\langle \mathcal{M}^0_\alpha, \mathcal{M}^1_\alpha, \vec{\mathcal{T}}^0_\alpha, \vec{\mathcal{T}}^1_\alpha, i_{\alpha\beta} : \alpha < \beta \leq \rho \rangle$ be the essential components of $\vec{\mathcal{T}}^\frown b$, then $\oplus_{\alpha\leq\rho}\vec{\mathcal{T}}^0_\alpha$ is according to Φ, and if $\mathcal{U}^\frown c$ is the last normal component of $\vec{\mathcal{T}}^1_\rho$, then one of the following holds.

- The maximal $L[\vec{E}, (\Phi)_{\oplus_{\alpha\leq\rho}\vec{\mathcal{T}}^0_\alpha}]$ constructions in $y|\delta^y$ certifies that $\mathrm{Lp}^{(\Phi)_{\oplus_{\alpha\leq\rho}\vec{\mathcal{T}}^0_\alpha}}(\mathcal{M}(\mathcal{U})) \models$ "$\delta(\mathcal{U})$ is not Woodin" or $\mathcal{U}$ has a drop in a way that we cannot undo the drop (cf. [7, Fact 3.5.5]), and c is defined using the maximal $L[\vec{E}]$ constructions in $y|\delta^y$.
- The maximal $L[\vec{E}, (\Phi)_{\oplus_{\alpha\leq\rho}\vec{\mathcal{T}}^0_\alpha}]$ constructions in $y|\delta^y$ certifies that $\mathrm{Lp}^{(\Phi)_{\oplus_{\alpha\leq\rho}\vec{\mathcal{T}}^0_\alpha}}(\mathcal{M}(\mathcal{U})) \models$ "$\delta(\mathcal{U})$ is Woodin" and $\mathcal{U}$ does not drop in a way that we cannot undo the drop, and there is $\sigma : \mathcal{M}^{\mathcal{U}}_c \to \mathcal{Q}^y(\gamma + 1)$ such that $\sigma \circ i^{\vec{\mathcal{T}}}_b = \tilde{\pi}$, where $\mathcal{Q}^y(\gamma)$ is the direct limit of all iterates of $\mathcal{Q}_m(\nu_m)$ according to Φ that are in $y|\delta^y$, and $\tilde{\pi} : \mathcal{Q}_m(\nu_m + 1) \to \mathcal{Q}^y(\gamma + 1)$ is an extension this direct limit map given by $\tilde{\pi}(\sigma_m \circ \cdots \circ \sigma_1(f)(\alpha)) = \pi^y(f)(\tilde{\pi}(\alpha))$, for $f \in \mathcal{P}$, $\alpha < \delta^{\mathcal{Q}}_{\nu_m}$.

9. (limit case at the bottom level of $\mathcal{K}$.) Let $y \in \tilde{S}^{\mathcal{K}} \setminus S^{\mathcal{K}}$ be such that $a = b^y$. Suppose for cofinally many $z \in y$ such that $a = b^y$, there are $\mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, \nu_m, \sigma_m, \mathcal{Q}_m, \beta_z, \beta$ such that

   (a) $\deg(z) \restriction m = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1} \rangle$,

   (b) $(\deg(z))_m = (\nu_m, \sigma_m, \mathcal{Q}_m, \beta_z, \beta_z)$,

   (c) β is the least upper bound of such β_z's, but $\beta \neq \beta_z$ for any such z.

   We split into three subcases.

A. Suppose $\mathcal{Q}_m \models$ "cf(β) is not measurable". We then have

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \beta, \beta)\rangle,$$

We also have

$$\Lambda(y) = \oplus\{\Lambda(z) : \deg(z) = \deg(y)[\gamma] \text{ for some } \gamma\}.$$

B. Suppose that $m \geq 1$ and $\beta = \sigma_m(\beta^*_{m-1})$. Then

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}\rangle,$$

Let $\Phi = \Lambda(z)$, where $\deg(z) = \deg(y)[\nu_m]$. For all $(\vec{\mathcal{T}}, b)$ such that $\vec{\mathcal{T}} \in a$, we have

$$\Lambda(y)(\vec{\mathcal{T}}) = b$$

if and only if one of the following holds

i. $i^{(\vec{\mathcal{T}}^\frown b)\restriction \mathcal{Q}_{m-1}(\nu_m)}$ is not defined. $(\vec{\mathcal{T}}^\frown b)\restriction \mathcal{Q}_{m-1}(\nu_m)$ is according to Φ.

ii. $i^{(\vec{\mathcal{T}}^\frown b)\restriction \mathcal{Q}_{m-1}(\nu_m)}$ is defined. Let $(\vec{\mathcal{U}}, \mathcal{Q}, \vec{\mathcal{V}})$ be the last stack of the re-arrangement of $\vec{\mathcal{T}}^\frown b$ with respect to ν_m. Then $\vec{\mathcal{U}}$ is according to Φ. Let $l : \mathcal{Q} \to \mathcal{Q}_m$ be the iteration map according to Φ. Then $\vec{\mathcal{V}}$ is according to

$$(\oplus\{(\Lambda(z)) : \deg(z) = \deg(y)^\frown\langle(\nu_m, \sigma_m, \mathcal{Q}_m, \gamma, \gamma)\rangle \text{ for some } \gamma\})^l.$$

C. Suppose both A. and B. fails. Let

- ν^* be the least ν such that $\nu > \sigma_m(\nu_{m-1})$ and $\delta_\nu^{\mathcal{Q}_m} > \mathrm{cf}^{\mathcal{Q}_m}(\beta)$.

- $z, w \in y$ be such that $a = b^z = b^w$ and

$$\deg(z) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \nu^*, \nu^*) \rangle$$
$$\deg(w) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \nu^* + 1, \nu^* + 1) \rangle$$

- $\Phi = \bigcup\{\Lambda(v)^{\mathrm{lift}(v)} : \mathrm{base}(v) = (z)\}$.
- $\mathcal{N}^* = \mathcal{M}_1^{\#,\Phi}(a)$ as computed inside w.
- $Q^* = \mathrm{dirlim}^{\mathcal{N},\Phi}_{\mathcal{Q}_m,a}$.
- $\sigma^* = \pi^{\mathcal{N},\Phi}_{\mathcal{Q}_m,a}$.

Then

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \beta, \beta), (\nu^*, \sigma^*, \mathcal{Q}^*, \sup \sigma^{*\prime\prime}\beta_m, \sup \sigma^{*\prime\prime}\beta_m) \rangle,$$
$$\mathrm{lift}(y) = id_{\mathcal{Q}^*}.$$

We also have

$$\Lambda(y) = \oplus\{(\Lambda(z))_{\vec{\mathcal{U}}} : \deg(z) = (\deg(y) \restriction m+1)[\gamma] \text{ for some } \gamma\}.$$

where $\vec{\mathcal{U}}$ is any stack on $\mathcal{Q}_m$ leading to $\mathcal{Q}^*$ according to Φ.

10. (Successor case.) Let $(y) \in S^{\mathcal{K}}$, $x = b^y \neq a$. Suppose z is $\in$-maximal in y such that $z \in \tilde{S}^{\mathcal{K}} \wedge x = b^z$. Denote $\deg(z) = \langle (\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i^*, \beta_i) : i \leq m \rangle$. Then $\beta_m^* = \beta_m$,
$$\deg(y) = \deg(z) + 1.$$

Put $\deg(\mathrm{base}(y)) = \langle (\bar{\nu}_i, \bar{\sigma}_i, \bar{\mathcal{Q}}_i, \bar{\beta}_i^*, \bar{\beta}_i) : i \leq k \rangle$, and let $\langle j_i : i \leq l \rangle$ be the sequence as follows:

- $j_0 = id_{\mathcal{P}}$,

- for each $i < m$, if $i < k \wedge \sup j_i''(\bar{\beta}_i) = \beta_i$, letting $w \in y$ be such that $w \in S^{\mathcal{K}} \wedge x = b^w$ and $\deg(w) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_i \rangle [\nu_{i+1}]$, then j_{i+1} is an iteration map from $\bar{\mathcal{Q}}_{i+1}$ to $\mathcal{Q}_{i+1}$ according to Φ, where $\Phi = \bigcup \{(\Lambda(v))^{\mathrm{lift}(v)} : \mathrm{base}(v) = w\}$. Otherwise, we terminate the definition of $\langle j_i : i \leq l \rangle$.

Then
$$o(x) \notin \mathrm{drop}(y) \Leftrightarrow m = k = l \wedge j_m(\bar{\beta}_m) = \beta_m + 1.$$

If $m = k = l \wedge j_m(\bar{\beta}_m) = \beta_m + 1$, then
$$\mathrm{lift}(y) = j_m.$$

Otherwise,
$$\mathrm{lift}(y) = id_{\mathcal{Q}_m}.$$

Furthermore, put $\Phi = \bigcup \{\Lambda(v)^{\mathrm{lift}(v)} : \mathrm{base}(v) = z\}$, then for each $(\vec{\mathcal{T}}, b)$ such that $\vec{\mathcal{T}} \in x$, we have that
$$\Lambda(y)(\vec{\mathcal{T}}) = b$$

if and only if letting $\langle \mathcal{M}^0_\alpha, \mathcal{M}^1_\alpha, \vec{\mathcal{T}}^0_\alpha, \vec{\mathcal{T}}^1_\alpha, i_{\alpha\beta} : \alpha < \beta \leq \rho \rangle$ be the essential components of $\vec{\mathcal{T}}^\frown b$, then $\oplus_{\alpha \leq \rho} \vec{\mathcal{T}}^0_\alpha$ is according to Φ, and if $\mathcal{U}^\frown c$ is the last normal component of $\vec{\mathcal{T}}^1_\rho$, then one of the following holds.

- The maximal $L[\vec{E}, (\Phi)_{\oplus_{\alpha \leq \rho} \vec{\mathcal{T}}^0_\alpha}]$ constructions in $y|\delta^y$ certifies that $\mathrm{Lp}^{(\Phi)_{\oplus_{\alpha \leq \rho} \vec{\mathcal{T}}^0_\alpha}}(\mathcal{M}(\mathcal{U})) \models$ "$\delta(\mathcal{U})$ is not Woodin" or $\mathcal{U}$ has a drop in a way that we cannot undo the drop (cf. [7, Fact 3.5.5]), and c is defined using the maximal $L[\vec{E}]$ constructions in $y|\delta^y$.
- The maximal $L[\vec{E}, (\Phi)_{\oplus_{\alpha \leq \rho} \vec{\mathcal{T}}^0_\alpha}]$ constructions in $y|\delta^y$ certifies that $\mathrm{Lp}^{(\Phi)_{\oplus_{\alpha \leq \rho} \vec{\mathcal{T}}^0_\alpha}}(\mathcal{M}(\mathcal{U})) \models$ "$\delta(\mathcal{U})$ is Woodin" and $\mathcal{U}$ does not drop in a way that we cannot undo the drop, and there is $\sigma : \mathcal{M}^{\mathcal{U}}_c \to \mathcal{Q}^y(\gamma + 1)$ such

that $\sigma \circ i_b^{\vec{\mathcal{T}}} = \tilde{\pi}$, where $\mathcal{Q}^y(\gamma)$ is the direct limit of all iterates of $\mathcal{Q}_m(\nu_m)$ according to Φ that are in $y|\delta^y$, and $\tilde{\pi} : \mathcal{Q}_m(\nu_m + 1) \to \mathcal{Q}^y(\gamma + 1)$ is an extension this direct limit map given by $\tilde{\pi}(\sigma_m \circ \cdots \circ \sigma_1(f)(\alpha)) = \pi^y(f)(\tilde{\pi}(\alpha))$, for $f \in \mathcal{P}$, $\alpha < \delta^{\mathcal{Q}}_{\nu_m}$.

11. (limit case.) Let $y \in \tilde{S}^{\mathcal{K}} \setminus S^{\mathcal{K}}$ and $x = b^y \neq a$. Suppose that for cofinally many $z \in y$, there are $\mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, \nu_m, \sigma_m, \mathcal{Q}_m, \beta_z, \beta$ such that $b^z = x$ and

    (a) $\deg(z) \restriction m = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1} \rangle$,

    (b) $(\deg(z))_m = (\nu_m, \sigma_m, \mathcal{Q}_m, \beta_z, \beta_z)$,

    (c) β is the least upper bound of such β_z's, but $\beta \neq \beta_z$ for any such z.

    Put $\deg(\text{base}(y)) = \langle (\bar{\nu}_i, \bar{\sigma}_i, \bar{\mathcal{Q}}_i, \bar{\beta}^*_i, \bar{\beta}_i) : i \leq k \rangle$, and let $\langle j_i : i \leq l \rangle$ be defined exactly as in item 10. We split into three subcases.

    A. Suppose *neither* of the following holds:

       i. $\mathcal{Q}_m \models$ "$\mathrm{cf}(\beta)$ is measurable".

       ii. j_m is defined, $\beta = \sup j_m'' \bar{\beta}_m$, $\bar{\mathcal{Q}}_m \models$ "$\mathrm{cf}(\bar{\beta}^*_m)$ is measurable".

       Then we have

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \beta, \beta) \rangle,$$

$$o(x) \notin \mathrm{drop}(y) \Leftrightarrow m = l = k \wedge \beta = \sup j_m'' \bar{\beta}_m.$$

       If $m = l = k \wedge \beta = \sup j_m'' \bar{\beta}_m$, then

$$\mathrm{lift}(y) = j_m.$$

       Otherwise,

$$\mathrm{lift}(y) = id_{\mathcal{Q}_m}.$$

We also have

$$\Lambda(y) = \oplus\{\Lambda(z) : \deg(z) = \deg(y)[\gamma] \text{ for some } \gamma\}.$$

B. Suppose that $m \geq 1$ and $\beta = \sigma_m(\beta^*_{m-1})$. Then

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1} \rangle,$$

If $o(y) \in l^{\mathcal{K}}$, then

$$o(x) \notin \mathrm{drop}(y) \text{ if and only if } m-1 = l = k \wedge \beta^*_{m-1} = j_{m-1}(\bar{\beta}_{m-1}).$$

If $m-1 = l = k \wedge \beta^*_{m-1} = j_{m-1}(\bar{\beta}_{m-1})$, then

$$\mathrm{lift}(y) = j_{m-1}.$$

Otherwise,

$$\mathrm{lift}(y) = id_{\mathcal{Q}_{m-1}}.$$

Furthermore, let $\Phi = \Lambda(z)$, where $x = b^z \wedge \deg(z) = \deg(y)[\nu_m]$. Then for all $(\vec{\mathcal{T}}, b)$ such that $\vec{\mathcal{T}} \in x$, we have

$$\Lambda(y)(\vec{\mathcal{T}}) = b$$

if and only if one of the following holds

- $i^{(\vec{\mathcal{T}}^\frown b) \upharpoonright \mathcal{Q}_{m-1}(\nu_m)}$ is not defined. $(\vec{\mathcal{T}}^\frown b) \upharpoonright \mathcal{Q}_{m-1}(\nu_m)$ is according to Φ.
- $i^{(\vec{\mathcal{T}}^\frown b) \upharpoonright \mathcal{Q}_{m-1}(\nu_m)}$ is defined. Let $(\vec{\mathcal{U}}, \mathcal{Q}, \vec{\mathcal{V}})$ be the last stack of the re-arrangement of $\vec{\mathcal{T}}^\frown b$ with respect to ν_m. Then $\vec{\mathcal{U}}$ is according to Φ. Let $l : \mathcal{Q} \to \mathcal{Q}_m$ be the iteration map according to Φ. Then $\vec{\mathcal{V}}$ is

according to

$$(\oplus\{(\Lambda(z)) : x = b^z \wedge \deg(z) = \deg(y)^\frown\langle(\nu_m, \sigma_m, \mathcal{Q}_m, \gamma, \gamma)\rangle \text{ for some } \gamma\})^l.$$

C. Suppose both A. and B. fails. Let $\nu^*, z, w, \Phi, \mathcal{N}^*, \mathcal{Q}^*, \sigma^*$ be defined exactly as in item 9C. Then

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \beta^*, \beta), (\nu^*, \sigma^*, \mathcal{Q}^*, \sup \sigma^{*\prime\prime}\beta, \sup \sigma^{*\prime\prime}\beta)\rangle,$$

where

$$\beta^* = \begin{cases} j_m(\bar{\beta}^*_m), \text{ if } \beta = \sup j''_m \bar{\beta}_m, \\ \beta, \text{ otherwise.} \end{cases}$$

We have

$$o(x) \in \operatorname{drop}(y) \leftrightarrow m + 1 = l = k \wedge \beta = \sup j''_m \bar{\beta}_m \wedge \bar{\beta}_{m+1} = \sup \bar{\sigma}''_{m+1} \bar{\beta}_m.$$

If $m + 1 = l = k \wedge \beta = \sup j''_m \bar{\beta}_m \wedge \bar{\beta}_{m+1} = \sup \bar{\sigma}''_{m+1} \bar{\beta}_m$, then

$$\operatorname{lift}(y) = \text{ the iteration map from } \bar{\mathcal{Q}}_{m+1} \text{ to } \mathcal{Q}_{m+1} \text{ according to } \Phi^{j_m}.$$

Otherwise,

$$\operatorname{lift}(y) = id_{\mathcal{Q}^*}.$$

We also have

$$\Lambda(y) = \oplus\{(\Lambda(z))_{\vec{\mathcal{U}}} : x = b^z \wedge \deg(z) = (\deg(y) \restriction m + 1)[\gamma] \text{ for some } \gamma\}.$$

where $\vec{\mathcal{U}}$ is any stack on $\mathcal{Q}_m$ leading to $\mathcal{Q}^*$ according to Φ.

Theorem 2.35. *Let $\mathcal{K}$ be an S-premouse over a. Let M be any transitive ZFC* *model containing* $\{\mathcal{K}\}$*. Then*

$$\mathbb{T}^{\mathcal{K}} \in M$$

and

$$\mathbb{T}^{\mathcal{K}} \text{ is the unique } v \text{ such that } M \models \phi_t(\mathcal{K}, v).$$

Proof. We may assume by induction that for any $\mathcal{K}'$ that arose in the $L[\vec{E}, S]$-construction of an S-operator in $\mathcal{K}$, the lemma holds for $\mathcal{K}'$. By that we mean, if $y \in S^{\mathcal{K}}$, $x = b^y$, $\mathfrak{P} = \deg^{\mathcal{K}}(y)$, $\mathcal{H} = SSM^{\mathfrak{P}-1}(x)$, and $\mathcal{H}$ is a model that arose in the $L[\vec{E}, S^{\mathfrak{P}-1}]$-construction over a Σ-good $\mathcal{Q}$, then the theorem holds for $\mathcal{K}$.

We carry out a construction $\mathbb{T} = (\text{base}, \deg, \text{drop}, \Lambda, \text{lift})$ in M. Inductively on $o(y)$ for $y \in \tilde{S}^{\mathcal{K}}$, we will decide values of the following:

- $\text{base}(y)$
- $\deg(y)$
- $\text{drop}(y)$
- $\text{lift}(y)$
- $\Lambda(y)$

and prove that for each $\xi \leq o(\mathcal{K})$, $\{(\text{base}(y), \deg(y), \text{drop}(y), \text{lift}(y), \Lambda(y)) : y \in \tilde{S}^{\mathcal{K}|\xi}\}$ $= \mathbb{T}^{\mathcal{K}|\xi}$.

The base case at the bottom level of $\mathcal{K}$*.*

Let $y \in S^{\mathcal{K}}$ be such that $o(y) = \min\{o(z) : z \in \tilde{S}^{\mathcal{K}}\}$. Let $x = b^y$. We must have $x = a$, since otherwise, $l^{\mathcal{K}|o(x)} = o(x) \setminus o(a)$ would have at least two elements, say (ξ, μ) are consecutive ones, then by clause 3 of definition 2.32, $\mathcal{K}|\mu \in \tilde{S}^{\mathcal{K}}$, which means $\mathcal{K}|\mu \in \tilde{S}^{\mathcal{K}}$, contradiction. Let $\text{base}(y), \text{lift}(y), \Lambda(y)$ be defined as in item 7 of definition 2.34. We need to show that they are the correct objects of $\mathbb{T}^{\mathcal{K}}$, i.e.

, $\mathrm{base}(y) = \mathrm{base}^{\mathcal{K}}(y)$, $\mathrm{lift}(y) = \mathrm{lift}^{\mathcal{K}}(y)$, $\Lambda(y) = \Lambda^{\mathcal{K}}(y)$. Clearly, $y = S^{[0]}(a)$. The only nontrivial fact we have to verify is

$$\Lambda(y) = \Sigma_{\mathcal{P}(0)} \restriction a.$$

The verification of this fact is simliar, but simpler to the general successor case. So we devote our effort to the successor case.

The successor case at the bottom level of $\mathcal{K}$.

Now suppose $y \in S^{\mathcal{K}}$, $a = b^y$ and z is $\in$-maximal in y such that $z \in \tilde{S}^{\mathcal{K}} \wedge a = b^z$. Suppose, by induction, for every $v \in \tilde{S}^{\mathcal{K}}$ such that $o(v) < y$, we have defined $\mathrm{base}(v), \mathrm{deg}(v), \mathrm{lift}(v), \Lambda(v)$, and $\mathrm{base}(v) = \mathrm{base}^{\mathcal{K}}(v)$, $\mathrm{deg}(v) = \mathrm{deg}^{\mathcal{K}}(v)$, $\mathrm{lift}(v) = \mathrm{lift}^{\mathcal{K}}(v)$, $\Lambda(v) = \Lambda^{\mathcal{K}}(v)$. Now let $\mathrm{base}(y) = \emptyset$ and let $\mathrm{deg}(y), \mathrm{lift}(y), \Lambda(y)$ be as in item 8 of definition 2.34. We need to sow that they are the correct objects of $\mathbb{T}^{\mathcal{K}}$, i.e.

$$\mathrm{base}(y) = \mathrm{base}^{\mathcal{K}}(y), \mathrm{deg}(y) = \mathrm{deg}^{\mathcal{K}}(y), \mathrm{lift}(y) = \mathrm{lift}^{\mathcal{K}}(y), \Lambda(y) = \Lambda^{\mathcal{K}}(y).$$

By induction, we know that $\mathrm{deg}(z) = \mathrm{deg}^{\mathcal{K}}(z)$ is the unique $\mathfrak{Q}$ such that $z = S^{\mathfrak{Q}}(a)$ and $\mathfrak{Q} \leq^{\mathcal{I}}_a [\lambda^{\mathcal{P}}]$. Therefore, $y = S^{\mathfrak{Q}+1}(a)$. The only nontrivial fact we are left to verify is

$$\Lambda(y) = \Sigma_{\mathcal{Q}_m(\beta_m+1)} \restriction a,$$

where $\mathrm{deg}(z) = \langle (\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i, \beta_i) : i \leq m \rangle$. In other words, we need to show that letting $\Phi = \bigcup\{\Lambda(u,v)^{\mathrm{lift}(u,v)} : \mathrm{base}(v) = z\}$, then for all $(\vec{\mathcal{T}}, b) \in a$,

$$\Sigma_{\mathcal{Q}_m(\beta_m+1)}(\vec{\mathcal{T}}) = b$$

if and only if letting $\langle \mathcal{M}^0_\alpha, \mathcal{M}^1_\alpha, \vec{\mathcal{T}}^0_\alpha, \vec{\mathcal{T}}^1_\alpha, i_{\alpha\beta} : \alpha < \beta \leq \rho \rangle$ be the essential components of $\vec{\mathcal{T}}^\frown b$, then $\oplus_{\alpha\leq\rho}\vec{\mathcal{T}}^0_\alpha$ is according to Φ, and if $\mathcal{U}^\frown c$ is the last normal component of $\vec{\mathcal{T}}^1_\rho$, then one of the following holds.

1. The maximal $L[\vec{E},(\Phi)_{\oplus_{\alpha\le\rho}\vec{\mathcal{T}}^0_\alpha}]$ constructions in $y|\delta^y$ certifies that $\mathrm{Lp}^{(\Phi)_{\oplus_{\alpha\le\rho}\vec{\mathcal{T}}^0_\alpha}}(\mathcal{M}(\mathcal{U}))\models$ "$\delta(\mathcal{U})$ is not Woodin" or $\mathcal{U}$ has a drop in a way that we cannot undo the drop (cf. [7, Fact 3.5.5]), and c is defined using the maximal $L[\vec{E}]$ constructions in $y|\delta^y$.

2. The maximal $L[\vec{E},(\Phi)_{\oplus_{\alpha\le\rho}\vec{\mathcal{T}}^0_\alpha}]$ constructions in $y|\delta^y$ certifies that $\mathrm{Lp}^{(\Phi)_{\oplus_{\alpha\le\rho}\vec{\mathcal{T}}^0_\alpha}}(\mathcal{M}(\mathcal{U}))\models$ "$\delta(\mathcal{U})$ is Woodin" and $\mathcal{U}$ does not drop in a way that we cannot undo the drop, and there is $\sigma:\mathcal{M}^{\mathcal{U}}_c\to\mathcal{Q}^y(\gamma+1)$ such that $\sigma\circ i^{\vec{\mathcal{T}}}_b=\tilde{\pi}$, where $\mathcal{Q}^y(\gamma)$ is the direct limit of all iterates of $\mathcal{Q}_m(\nu_m)$ according to Φ that are in $y|\delta^y$, and $\tilde{\pi}:\mathcal{Q}_m(\nu_m+1)\to\mathcal{Q}^y(\gamma+1)$ is an extension this direct limit map given by $\tilde{\pi}(\sigma_m\circ\cdots\circ\sigma_1(f)(\alpha))=\pi^y(f)(\tilde{\pi}(\alpha))$, for $f\in\mathcal{P}$, $\alpha<\delta^{\mathcal{Q}}_{\nu_m}$.

In order to see the above, letting $\mathcal{Q},\mathcal{N},\mathcal{Q}^\infty_{\mathcal{N}},\pi^\infty_{\mathcal{N}}$ be as in definition of $S^{\deg(y)}(a)$. Recall from definition 2.30 that $y=S^{\deg(x,y)}(a)$ is the amenable code of a Skolem hull of $\mathcal{N}_+$. If $\vec{\mathcal{T}}$ falls under the case 1, since the clause of the case 1 is first order definable inside $\mathcal{N}_+$, by hull condensation, it suffices to show the first case applies inside $\mathcal{N}_+$ with $y|\delta^y$, $\mathcal{Q}^y$, π^y replaced by $\mathcal{N},\mathcal{Q}^\infty_{\mathcal{N}},\pi^\infty_{\mathcal{N}}$. But $\Phi\restriction(y|\delta^y)=\Sigma_{\mathcal{Q}_m(\beta_m)}\restriction(y|\delta^y)$ by induction. Recall that $\mathcal{N}$ comes from the maximal $L[\vec{E},S^{\deg(z)}]$-construction in a suitable Σ-premouse. By the universality proof, a maximal $L[\vec{E},\Sigma_{\mathcal{Q}_m(\beta_m)}]$ -construction done inside a maximal $L[\vec{E},S^{\deg(z)}]$ construction done inside $\mathcal{Q}$ is full. This means any maximal $L[\vec{E},\Sigma_{\mathcal{Q}_m(\beta_m)}]$ construction done inside $(\mathcal{N})_+$ is full. Hence $\mathcal{N}$ sees strategies for $\vec{\mathcal{T}}$ under case 1 correctly. If $\vec{\mathcal{T}}$ falls under case 2, we take γ and $\tilde{\pi}$ be as in the clause of case 2. We set γ' be such that $\mathcal{Q}^\infty_{\mathcal{N}}(\gamma'+1)$ is the direct limit of $I(\mathcal{Q}_{\nu_m},\Sigma_{\mathcal{Q}_{\nu_m}})\cap\mathcal{N}$ and $\tilde{\pi}':\mathcal{Q}_m(\nu_m+1)\to\mathcal{Q}^\infty_{\mathcal{N}}(\gamma'+1)$ be the direct limit map. Then $\tilde{\pi}$ has the property $\tilde{\pi}'(\sigma_m\circ\cdots\circ\sigma_1(f)(\alpha))=\pi^y(f)(\tilde{\pi}'(\alpha))$, for $f\in\mathcal{P}$, $\alpha<\delta^{\mathcal{Q}}_{\nu_m}$. Thus γ' goes to γ under the transitive collapse map of defining

$y = S^{\deg(y)}(x)$, and $\tilde{\pi}'$ goes pointwise to $\tilde{\pi}$. Since $Q^{\infty}_{\mathcal{N}}$ is the direct limit of iterates of $\mathcal{P}$ that are in $\mathcal{N}$, there is in V an iteration $\psi : \mathcal{M}^{\mathcal{U}}_c \to Q^{\infty}_{\mathcal{N}}$ such that $\psi \circ i^{\vec{\mathcal{T}}}_b = \tilde{\pi}'$. ψ is not in $\mathcal{N}$, but since $\Sigma_{\mathcal{Q}_m(\nu_m+1)}$ is strongly guided by an ω-sequence of OD sets, there exist partial branches c^i, partial maps ψ^i, $i < \omega$, such that $c = \cup_{i<\omega} c^i$, $\psi^i = \cap\{i^d : d$ is a branch through $\mathcal{U}$ whose final model is full,$c^i \subseteq d\}$, $\psi = \cup_{i<\omega}\psi^i$, and each $c^i, \psi^i \in \mathcal{N}_+$. The images of these c^i's and ψ^i's under the transitive collapsing map of defining $y = S^{\deg(y)}(x)$ piece together into the branch c and a map $\sigma : \mathcal{M}^{\mathcal{U}}_c \to \tilde{Q}^y(\gamma+1)$. Therefore, c and σ witnesses the clause of case 2. Moreover, any c' which satisfies this clause must be equal to c by branch condensation.

The limit case at the bottom level of $\mathcal{K}$.

Now suppose $y \in \tilde{S}^{\mathcal{K}} \backslash S^{\mathcal{K}}$ and $a = b^y$. Suppose, by induction, for every $v \in \tilde{S}^{\mathcal{K}}$ such that $o(v) < y$, we have defined $\mathrm{base}(v), \deg(v), \mathrm{lift}(v), \Lambda(v)$, and showed $\mathrm{base}(v) = \mathrm{base}^{\mathcal{K}}(v)$, $\deg(v) = \deg^{\mathcal{K}}(v)$, $\mathrm{lift}(v) = \mathrm{lift}^{\mathcal{K}}(v)$, $\Lambda(v) = \Lambda^{\mathcal{K}}(v)$. Now let $\mathrm{base}(y) = \emptyset$ and let $\deg(y), \mathrm{lift}(y), \Lambda(y)$ be as in item 8 of definition 2.34. We need to sow that they are the correct objects of $\mathbb{T}^{\mathcal{K}}$.

By induction, for each $z \in y$ such that $b^z = a$, $\deg(z) = \deg^{\mathcal{K}}(z)$ is the only $\mathfrak{Q}$ such that $z = S^{\mathfrak{Q}}(a)$ and $\mathfrak{Q} \leq^{\mathcal{I}}_a [\lambda^{\mathcal{P}}]$. So there is $\mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, \nu_m, \sigma_m, \mathcal{Q}_m, \beta_z, \beta$ for cofinally-in-y many z as in the first paragraph of definition 2.34 item 9. Define $\deg(y), \mathrm{base}(y), \Lambda(y)$ as in definition 2.34 item 9, splitting into three subcases.

If we are under subcase A, then clearly

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \beta, \beta)\rangle,$$

is the least index $\mathfrak{R}$ such that

$$\mathfrak{R} \leq^{\mathcal{I}}_a [\lambda^{\mathcal{P}}] \text{ and } \deg^{\mathcal{K}}(z) <^{\mathcal{I}}_a \mathfrak{R} \text{ for any } z \in y \text{ such that } a = b^z.$$

Note that $\deg(y)$ is of type A. Hence, $\deg^{\mathcal{K}}(y) = \deg(y)$. Since $\mathcal{Q}_m \models$ "$\mathrm{cf}(\beta)$ is not measurable", $\Sigma_{\mathcal{Q}_m(\beta)} = \oplus_{\gamma<\beta}\Sigma_{\mathcal{Q}_m(\gamma)}$. Hence $\Lambda(y)$, being the join of $\{\Lambda(z) : z \in y \wedge x = b^z\}$, is equal to $\Sigma_{\mathcal{Q}_m(\beta)}$ by induction.

If we are under subcase B, then

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1} \rangle,$$

is the least index $\mathfrak{R}$ such that

$$\mathfrak{R} \leq_a^{\mathcal{I}} [\lambda^{\mathcal{P}}] \text{ and } \deg^{\mathcal{K}}(z) <_a^{\mathcal{I}} \mathfrak{R} \text{ for any } z \in y \text{ such that } a = b^z.$$

The reason is that every extension of $\deg(y)$ is already taken care of, because by definition of $\mathcal{I}$, to make an index, the ordinal on $\mathcal{Q}_m$ must be $< \sigma_m(\beta^*_{m-1}) = \beta$. Hence $\deg(y) = \deg^{\mathcal{K}}(y)$. Note that $\deg(y)$ is of type B. The fact that $\Lambda(y) = \Lambda^{\mathcal{K}}(y)$ follows from induction and Lemma 2.9.

If we are under subcase C, then it is not hard to see that the $\deg(y)$ defined over there is the least $\mathfrak{R}$ such that

$$\mathfrak{R} \leq_a^{\mathcal{I}} [\lambda^{\mathcal{P}}] \text{ and } \deg^{\mathcal{K}}(z) <_a^{\mathcal{I}} \mathfrak{R} \text{ for any } z \in y \text{ such that } a = b^z.$$

The crucial part is that the final ordinal on Q^* is equal to $\sup \sigma^{*\prime\prime}\beta_m$, the least possible one to make an index. Note that $\deg(y)$ is of type C. The fact that $\Lambda(y) = \Lambda^{\mathcal{K}}(y)$ follows from induction.

At a level of $\mathcal{K}$ higher than $o(a)$.

Suppose now $y \in \tilde{S}^{\mathcal{K}}$, $x = b^y \neq a$. Suppose, by induction, for every $v \in \tilde{S}^{\mathcal{K}}$ such that $o(v) < o(y)$, we have defined $\mathrm{base}(v), \deg(v), \mathrm{lift}(v), \Lambda(v)$, and showed $\mathrm{base}(v) = \mathrm{base}^{\mathcal{K}}(v)$, $\deg(v) = \deg^{\mathcal{K}}(v)$, $\mathrm{lift}(v) = \mathrm{lift}^{\mathcal{K}}(v)$, $\Lambda(v) = \Lambda^{\mathcal{K}}(v)$. Now let $\mathrm{base}(y), \deg(y), \mathrm{lift}(y), \Lambda(y), \mathrm{drop}(y)$be as in item 8 of definition 2.34. We have to show that they are the correct objects of $\mathbb{T}^{\mathcal{K}}$, It is important to observe that base(y) depends only on $S^{\mathcal{K}}$ and drop(x). By induction, drop$(x) = \text{drop}^{\mathcal{K}}(x)$, hence

$$\text{base}(y) = \text{base}^{\mathcal{K}}(y).$$

Let deg(base(y)) $= \bar{\mathfrak{Q}} = \langle(\bar{\nu}_i, \bar{\sigma}_i, \bar{\mathcal{Q}}_i, \bar{\beta}_i^*, \bar{\beta}_i) : i \leq k\rangle$. We let $\mathfrak{R} = \langle \mathfrak{R}_0, \ldots, \mathfrak{R}_k\rangle = \text{pro}(\bar{\mathfrak{Q}}, x)$. Let $\langle j_i : i \leq l\rangle$ be constructed as in item 10. The definition there ensures $\langle j_i : i \leq l\rangle$ is a part of the lifting sequence of deg(base(y)), and moreover, l is the least such that $\mathfrak{R}_l \neq \mathfrak{Q}_l \vee l = m$.

Let's skip the base case that there is no $z \in y$ such that $z \in S^{\mathcal{K}}$.

The successor case

Suppose $(y) \in S^{\mathcal{K}}$. Suppose $z \in y$ is $\in$-maximal such that $z \in \tilde{S}^{\mathcal{K}} \wedge a = b^z$. By induction, $\deg(z) = \deg^{\mathcal{K}}(z)$. Hence

$$\deg(y) = \deg(z) + 1 = \deg^{\mathcal{K}}(z) + 1 = \deg^{\mathcal{K}}(y).$$

The third equality above is because $\deg^{\mathcal{K}}(z) + 1$ is the least promoted index over x which is $>^{\mathcal{I}}_x \deg^{\mathcal{K}}(z)$. We show that the truth value of "$o(x) \in \text{drop}(y)$" agrees with the truth value of "$o(x) \in \text{drop}^{\mathcal{K}}(y)$". This is simply a repetition of definition of drop$^{\mathcal{K}}$:

$$o(x) \notin \text{drop}^{\mathcal{K}}(y)$$

if and only if

$$\deg^{\mathcal{K}}(y) = \text{pro}(\deg^{\mathcal{K}}(\text{base}^{\mathcal{K}}(y)), x)$$

if and only if

$$\deg(y) = \text{pro}(\deg(\text{base}(y)), x)$$

if and only if

$$m = k = l \wedge j_m(\bar{\beta}_m) = \beta_m + 1.$$

if and only if

$$o(x) \notin \mathrm{drop}(y).$$

The fact

$$\mathrm{lift}(y) = \mathrm{lift}^{\mathcal{K}}(y)$$

is also clear. The same proof as in the successor case at the bottom level of $\mathcal{K}$ implies $\Lambda(y) = \Lambda^{\mathcal{K}}(y)$.

The limit case.

Now suppose $y \notin S^{\mathcal{K}}$. Let $\mathfrak{Q}_0, \ldots, \mathfrak{Q}_{n-1}, \nu_m, \sigma_m, \mathcal{Q}_m, \beta$ and $\langle(\bar{\nu}_i, \bar{\sigma}_i, \bar{\mathcal{Q}}_i, \bar{\beta}^*, \bar{\beta}_i) : i \leq k\rangle$, $\langle j_i : i \leq l\rangle$be defined as in item 11 of definition 2.34. The definition shows that $\langle j_i : i \leq l\rangle$ is a part of the lifting maps of defining $\mathrm{pro}(\mathrm{deg}(\mathrm{base}(y)), x)$.

Suppose we are under subcase A, we show that

$$\mathrm{deg}(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \beta, \beta)\rangle$$

is the least index which is $\leq^{\mathcal{I}}_x \mathrm{pro}(\mathrm{deg}(\mathrm{base}(y)), x)$ and $>^{\mathcal{I}}_x \mathrm{deg}(z)$ for any $z \in y$ such that $z \in S^{\mathcal{K}} \wedge x = b^z$. Henceforth $\mathrm{deg}(y) = \mathrm{deg}^{\mathcal{K}}(y)$. The case hypothesis $Q_m \models$ "$\mathrm{cf}(\beta)$ is not measurable" shows $\mathrm{deg}(y)$ is an index of type A. If j_m is not defined, then $\mathrm{pro}(\mathrm{deg}(\mathrm{base}(y)), x)$ and $\mathrm{deg}(y)$ differ at a coordinate $\leq m-1$. Hence by definition of $<^{\mathcal{I}}_x$, any index extending $\langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}\rangle$ whose mth and later coordinate agree on ordinals with or without star is $<^{\mathcal{I}}_x \mathrm{pro}(\mathrm{deg}(\mathrm{base}(y)), x)$. Hence $\mathrm{deg}(y)$ is the index as required. If j_m is defined, then $\mathrm{pro}(\mathrm{deg}(\mathrm{base}(y)), x) \restriction m+1 = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m.\sigma_m, \mathcal{Q}_m, j_m(\bar{\beta}^*_m), \sup j''_m \bar{\beta}_m)\rangle$. The case hypothesis either $\beta < \sup j''_m \bar{\beta}_m)$ or $\bar{\mathcal{Q}}_m \models$ "$\mathrm{cf}(\bar{\beta}^*_m)$ is not measurable" implies $\mathrm{deg}(y) \leq^{\mathcal{I}}_x \mathrm{pro}(\mathrm{deg}(\mathrm{base}(y)), x)$: If $\beta < \sup j''_m \bar{\beta}_m$, then m is the least coordinate where $\mathrm{deg}(y)$

and $\text{pro}(\deg(\text{base}(y)), x)$ disagree, hence by definition, $\deg(y) <^{\mathcal{I}}_x \text{pro}(\deg(\text{base}(y)), x)$. If $\bar{\mathcal{Q}}_m \models$ "$\text{cf}(\bar{\beta}^*_m)$ is not measurable", then j_m is continuous at $\bar{\beta}^*_m = \bar{\beta}_m$, hence $\beta \leq \sup j''_m \bar{\beta}_m = j_m(\bar{\beta}^*_m)$. Hence $\deg(y) \leq^{\mathcal{I}}_x \text{pro}(\deg(\text{base}(y)), x)$. (The case hypothesis is necessary, since otherwise $\beta = \sup j''_m \bar{\beta}_m < j_m(\bar{\beta}^*_m)$, thus the $\deg(y)$ defined here is no longer $<^{\mathcal{I}}_x \text{pro}(\deg(\text{base}(y)), x)$!)

The facts that $\text{lift}(y) = \text{lift}^{\mathcal{K}}(y)$, $o(x) \in \text{drop}(y) \leftrightarrow o(x) \in \text{drop}^{\mathcal{K}}(y)$ $\Lambda(y) = \Lambda^{\mathcal{K}}(y)$, are straightforward to verify.

Suppose now we are in subcase B. The case hypothesis $\beta = \sigma_m(\beta^*_{m-1})$ implies that

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1} \rangle$$

is the least that is $<^{\mathcal{I}}_x \text{pro}(\deg(\text{base}(y)), x)$ but $>^{\mathcal{I}}_x$ every $\deg(z)$, for $z \in y \wedge x = b^z$. In other words, we have reached the "ceiling" of the m-th coordinate, and thus we are forced to step back onto the $m-1$-th coordinate. From the viewpoint of strategies captured, we have collected enough strategies on $\mathcal{Q}_m$, a suitable iteration of $\mathcal{Q}_{m-1}$, and thus we already have the strategy of $\mathcal{Q}_{m-1}(\beta^*_{m-1})$. Therefore $y = S^{\deg(y)}(x)$. $\deg(y) = \deg^{\mathcal{K}}(y)$. Note that $\deg(y)$ is an index of type B.

The facts that $\text{lift}(y) = \text{lift}^{\mathcal{K}}(y)$, $o(x) \in \text{drop}(y) \leftrightarrow o(x) \in \text{drop}(y)$ are straightforward to verify. $\Lambda(y) = \Lambda^{\mathcal{K}}(y)$ is due to Lemma 2.9.

Suppose now we are under subcase C. If

$$j_m \text{ is not defined, } \mathcal{Q}_m \models \text{"cf}(\beta) \text{ is measurable" and } \beta < \sigma_m(\beta^*_{m-1}),$$

then

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \beta, \beta), (\nu^*, \sigma^*, \sup \sigma^{*\prime\prime}\beta, \sup \sigma^{*\prime\prime}\beta) \rangle$$

is $\leq^{\mathcal{I}}_x \text{pro}(\deg(\text{base}(y)), x)$ and is the least $>^{\mathcal{I}}_x$ each the previous $\deg(z)$, $z \in y \wedge x =$

b^z. In other words, we have reached a place of measurable cofinality, $\mathcal{Q}_m(\beta)$. We have already captured the strategy of all $\mathcal{Q}_m(\epsilon)$'s for $\epsilon < \beta$. However, they are not enough capture the strategy of $\mathcal{Q}_m(\beta)$. Therefore, we have to pause, do an iteration of $\mathcal{Q}_m(\beta)$ using strategies we already know yet creating new Woodins, and collect strategies for initial segments of that iterate. (The reader may compare this with subcase B. In subcase B, we have collected enough strategy and thus happy to step backwards.) We are currently in the beginning of extending an index. Thus the last ordinal of the index is $\sup \sigma^{*\prime\prime}\beta$. The reader can also see the reason we defined the one-step blow-up in Section 2.3. If

$$j_m \text{ is defined, } \mathcal{Q}_m \models \text{“}\operatorname{cf}(\beta) \text{ is measurable”},$$

then $\beta < \sup j_m''\bar{\beta}_m = \gamma_m$. This is because when $\bar{\beta}_m = \bar{\beta}_m^*$, then j_m hits the order-zero measure on $\operatorname{cf}^{\mathcal{Q}_m^*}(\bar{\beta}_m)$; when $\bar{\beta}_m < \bar{\beta}_m^*$, then $\bar{\mathcal{Q}}_m \models$ “$\operatorname{cf}(\bar{\beta}_m)$ is not measurable. Thus $\mathcal{Q}_m \models$ “$\operatorname{cf}(\gamma_m)$ is not measurable”. So the $\deg(y)$ as above is $\leq_x^{\mathcal{I}} \operatorname{pro}(\deg(\operatorname{base}(y)), x)$. If

$$\begin{aligned} &j_m \text{ is defined, } \mathcal{Q}_m \models \text{“}\operatorname{cf}(\beta) \text{ is not measurable”},\\ &\beta = \sup j_m''\bar{\beta}_m,\ \bar{\mathcal{Q}}_m \models \text{“}\operatorname{cf}(\bar{\beta}_m^*) \text{ is measurable”}\end{aligned}$$

then we still need to extend the index, as $\langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, \beta, \beta)\rangle$ is no longer $<_x^{\mathcal{I}} \mathfrak{R}$, basically because $\beta = \sup j_m''\bar{\beta}_m < j_m(\bar{\beta}_m^*)$. The least index bigger than each previous $\deg(z)$, $z \in y \wedge x = b^z$ is

$$\deg(y) = \langle \mathfrak{Q}_0, \ldots, \mathfrak{Q}_{m-1}, (\nu_m, \sigma_m, \mathcal{Q}_m, j_m(\bar{\beta}_m^*), \beta), (\nu^*, \sigma^*, \sup \sigma^{*\prime\prime}\beta, \sup \sigma^{*\prime\prime}\beta)\rangle.$$

Therefore $\deg(y) = \deg^{\mathcal{K}}(y)$. The facts that $\operatorname{lift}(y) = \operatorname{lift}^{\mathcal{K}}(y)$, $o(x) \in \operatorname{drop}(y) \leftrightarrow o(x) \in \operatorname{drop}^{\mathcal{K}}(y)$ $\Lambda(y) = \Lambda^{\mathcal{K}}(y)$, are straightforward to verify.

This inductive construction of $\mathbb{T} = (\text{base}, \text{deg}, \text{drop}, \text{lift}, \Lambda)$ as above can be carried out in any ZFC model containing $\mathcal{K}$. Therefore $\mathbb{T}^{\mathcal{K}} \in M$, and $\mathbb{T}^{\mathcal{K}}$ is the unique v such that $M \models \phi_t(\mathcal{K}, v)$. This finished the proof of Theorem 2.35. $\square$

Theorem 2.35 indicates that the $S^{\mathfrak{P}}$-operator has approximately equal information as $\Sigma_{final(\mathfrak{P})}$ does. An $S^{\mathfrak{P}}$-premouse $\mathcal{K}$ can define $\Sigma_{final(\mathfrak{P})} \restriction \mathcal{K}$. On the other hand, an Σ-good $\mathcal{Q}$ can define $S^{\mathfrak{P}} \restriction (\mathcal{Q}|\delta^{\mathcal{Q}})$.

2.8 Iteration theory of S-premice

We are going to develop the iteration theory of S-mice, including comparison, solidity and condensation, just like that of ordinary premice. We say Γ is an S-iteration strategy for an S-premouse $\mathcal{M}$ if every Γ-iterate of $\mathcal{M}$ is an S-premouse. We say $\mathcal{M}$ is S-iterable if it has an S-iteration strategy. An S-mouse is an S-iterable S-premouse. Eventually, we want to show that the $L[\vec{E}, S^{[\lambda^{\mathcal{P}}]}]$-construction in any good universe has an S-iteration strategy which is induced by the background strategy. In order to carry out the proof, we introduce the concept of piecewise S-iterability. We say Γ is a semi-S-iteration strategy if letting $d = \max D^{\mathcal{M}}$, then any Γ iterate that is above d is an S-premouse. $\mathcal{M}$ is semi-S-iterable if $\mathcal{M}$ has a semi-S-iteration strategy. A semi-S-mouse is a semi-S-iterable S-premouse.

We start with a general comparison result. In the comparison between two S-mice, we hit the least disagreement on their E-sequence, just like comparing two ordinary mice. The S-predicate will then automatically line up, since both are S-premouse. There is only one exception, that is, two models may agree up to some point, one side has an S-predicate after that, but the other side does not. In that case, we declare that the comparison is done, and the former one is longer than the latter. We spell out the detailed definition of comparison.

Definition 2.36. Let $\mathcal{M}$ and $\mathcal{N}$ be S-mice, and suppose Γ and Φ are their S-iteration strategies. Then the comparison between $\mathcal{M}$ and $\mathcal{N}$ according to (Γ, Φ) is the pair $(\mathcal{T}, \mathcal{U})$ such that $\mathcal{T}$ is a padded normal iteration tree on $\mathcal{M}$ according to Γ, $\mathcal{U}$ is a padded normal iteration tree on $\mathcal{N}$ according to Φ, $lh(\mathcal{T}) = lh(\mathcal{U}) = \theta$, and for all $\alpha < \theta$, letting λ_α be the least γ such that $\mathcal{M}^{\mathcal{T}}_\alpha|\gamma \neq \mathcal{M}^{\mathcal{U}}_\alpha|\gamma$, then either of $E^{M^{\mathcal{T}}_\alpha}_{\lambda_\alpha}$ or $E^{M^{\mathcal{U}}_\alpha}_{\lambda_\alpha}$ is nonempty, and the extenders applied on both sides are the extenders indexed at λ_α, and either $\mathcal{M}^{\mathcal{T}}_\theta$ and $\mathcal{M}^{\mathcal{U}}_\theta$ are lined up, or letting λ_θ be the least γ such that $\mathcal{M}^{\mathcal{T}}_\alpha|\gamma \neq \mathcal{M}^{\mathcal{U}}_\alpha|\gamma$, then $\mathcal{M}^{\mathcal{T}}_\alpha|\lambda^\theta \in S^{\mathcal{M}^{\mathcal{T}}_\alpha} \leftrightarrow \mathcal{M}^{\mathcal{U}}_\alpha|\lambda_\theta \notin S^{\mathcal{M}^{\mathcal{T}}_\alpha}$.

The comparison must succeed, because otherwise, we keep hitting extenders so as to reach a pair of trees $(\mathcal{T}, \mathcal{U})$ of length $\omega_1 + 1$. The usual argument of comparison gives a contradiction. Given two S-mice $\mathcal{M}$ and $\mathcal{N}$, we say $\mathcal{M}$ is a pseudo-initial segment of $\mathcal{N}$ if either $\mathcal{M} \lhd \mathcal{N}$, or letting λ be the least γ such that $\mathcal{M}|\gamma \neq \mathcal{N}|\gamma$, then $\mathcal{M}|\lambda \notin S^{\mathcal{M}}$ but $\mathcal{N}|\lambda \in S^{\mathcal{N}}$. We say $\mathcal{M}$ is a proper pseudo initial segment of $\mathcal{N}$ if $\mathcal{M}$ is a pseudo initial segment of $\mathcal{N}$ but $\mathcal{M} \neq \mathcal{N}$. So the comparison results in two iterates of the original S-mice that are lined up under pseudo-initial segment. We show that, as in comparison of usual mice, at most one side of the main branches has a drop.

Theorem 2.37. *Let $\mathcal{M}$ and $\mathcal{N}$ be S-mice, and suppose Γ and Φ are their iteration strategies. Let $(\mathcal{U}, \mathcal{T})$ be the comparison between $\mathcal{M}$ and $\mathcal{N}$ according to Γ, Φ. Let $lh(\mathcal{T}) = lh(\mathcal{U}) = \theta$. Then either*

- *$[0, \theta]_T$ does not drop in model or degree, and $\mathcal{M}^{\mathcal{T}}_\theta$ is a pseudo initial segment of $\mathcal{M}^{\mathcal{U}}_\theta$, or*
- *$[0, \theta]_U$ does not drop in model or degree, and $\mathcal{M}^{\mathcal{U}}_\theta$ is a pseudo initial segment of $\mathcal{M}^{\mathcal{U}}_\theta$.*

Proof. The usual proof of comparison shows that if $\mathcal{M}^{\mathcal{T}}_\theta = \mathcal{N}^{\mathcal{T}}_\theta$, then at most one side has a drop. So let's suppose, for instance, $\mathcal{M}^{\mathcal{T}}_\theta$ is a proper pseudo initial segment of $\mathcal{M}_\theta^\mathcal{U}$. We claim that $[0,\theta]_T$ has no drop in model or degree. Otherwise, $\mathcal{M}_\theta^\mathcal{T}$ is not sound. Usual arguments show that some extender $E_\beta^\mathcal{T}$ applied on the $\mathcal{T}$-side agrees with the core map of $\mathcal{M}_\theta^\mathcal{T}$ up to its length. But $\rho(\mathcal{M}_\theta^\mathcal{T}) > \lambda_\theta$. So the comparison on the $\mathcal{T}$-side must have stopped before we apply $E_\beta^\mathcal{T}$. Contradiction. □

If $\mathcal{M}$ and $\mathcal{N}$ are semi-S-mice, and $d = \max D^\mathcal{M} = \max D^\mathcal{N}$, $\mathcal{M}|d = \mathcal{N}|d$, then semi-$S$-iteration strategies suffice to compare $\mathcal{M}$ and $\mathcal{N}$. In fact, we only compare the part above d. Moreover, as there is no drop after d, it can't be that the last model of one side is a proper pseudo initial segment of the other. Thus the proofs of solidity and condensation carries verbatim over to the semi-S-mice case (cf. [4, Theorem 8.1 and 8.2]).

Lemma 2.38 (Solidity)**.** *Let $\mathcal{N}$ be a semi-S-mouse over a. Let $d = \max D^\mathcal{N}$, $\mathfrak{P} = \deg(\mathcal{N}|d)$. Assume that if $\mathcal{K}$ is an S-premouse over a, $a \in \mathrm{Cone}(z)$, $j : \bar{\mathcal{S}} \to S^\mathfrak{P}(\mathcal{K})$ is Σ_1-elementary, $j(a, \mathcal{H}, \mathfrak{P}) = (a, \mathcal{K}, \mathfrak{P})$, then $\bar{\mathcal{S}} = S^\mathfrak{P}(\mathcal{H})$. Then standard parameters of $\mathcal{N}$ are solid and universal.*

Proof. Then $\rho_\omega(\mathcal{N}) \geq d$. So the nth core maps are above d. All comparison arguments that arise are above d, so semi-S-iteration strategy suffices for the proof. The additional assumption of the lemma on condensation of $S^\mathfrak{P}$ is used to guarantee that if $\mathcal{M}$ is a $S^\mathfrak{P}$-premouse, $j : \mathcal{K} \to \mathcal{M}$, $j \restriction d = id$, $\mathcal{N}|d \trianglelefteq \mathcal{K}$, then $\mathcal{K}$ is a $S^\mathfrak{P}$-premouse as well. □

Lemma 2.39 (Condensation)**.** *Let $\mathcal{N}$ be an ω-sound semi-S-mice. Suppose $\pi : \mathcal{H} \to \mathcal{N}$ is fully elementary, and $\mathrm{crt}(\pi) = \rho_\omega(\mathcal{H}) = \rho_\omega(\mathcal{N})$. Let $d = \max D^\mathcal{N}$, $\mathfrak{P} = \deg(\mathcal{N}|d)$. Assume that if $\mathcal{K}$ is an S-premouse over a, $a \in \mathrm{Cone}(z)$, $j : \bar{\mathcal{S}} \to S^\mathfrak{P}(\mathcal{K})$ is Σ_1-elementary, $j(a, \mathcal{H}, \mathfrak{P}) = (a, \mathcal{K}, \mathfrak{P})$, then $\bar{\mathcal{S}} = S^\mathfrak{P}(\mathcal{H})$. Then $\mathcal{H}$ is an initial segment of $\mathcal{N}$.*

Lemma 2.40. *Let $\mathcal{Q}$ be Σ-good. Assume that if $\mathcal{K}$ is an S-premouse over a, $a \in \mathrm{Cone}(z)$, $j : \bar{\mathcal{S}} \to S^{\mathfrak{Q}}(\mathcal{K})$ is Σ_1-elementary, $j(a, \mathcal{H}, \mathfrak{Q}) = (a, \mathcal{K}, \mathfrak{Q})$, and $\mathrm{pro}(\mathfrak{Q}, \mathcal{K}) \leq^{\mathcal{I}}_{\mathcal{K}} \mathrm{pro}(\mathfrak{P}, \mathcal{K})$, then $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(\mathcal{H})$. Then every model of the $L[\vec{E}, S^{\mathfrak{P}}][a]$-construction in $\mathcal{Q}$ is a semi-S-mouse. Consequently, the $L[\vec{E}, S^{\mathfrak{P}}][a]$-construction in $\mathcal{Q}$ converges.*

Proof. We show by induction on ξ that every $\mathcal{N}_\xi, \mathcal{M}_\xi$ is a semi-S-mouse. Assume $\mathcal{M}_\xi$ is a sound semi-S-mouse. If $\mathcal{M}_\xi$ falls under case 2 or case 4 of definition 2.27, then $\mathcal{N}_{\xi+1}$ is immediately a S-premouse. If we let $\Lambda_{\mathcal{N}_{\xi+1}}$ be the strategy of $\mathcal{N}_{\xi+1}$ induced from the background universe, and $\mathcal{K}$ is a non-dropping iterate above $\max D^{\mathcal{N}_{\xi+1}}$, then $\mathcal{K}$ is also an S-premouse. This is because we have a lifting map $k : \mathcal{K} \to j(\mathcal{N}_{\xi+1})$, where $j : \mathcal{Q} \to \mathcal{R}$ is an background iteration map. $j(\mathcal{N}_{\xi+1})$ is an S-premouse by elementarity, and the assumption of the lemma tells us $\mathcal{K}$ is an S-premouse as well.

If $\mathcal{M}_\xi$ falls under case 3, let $\mathbb{T}, d$ be in the definition. Theorem 2.35 tells us $\mathbb{T} = \mathbb{T}^{\mathcal{M}_\xi}$, i.e. $\mathbb{T}$ is the correct object that keeps track of the strategy in $\mathcal{M}_\xi$. Put $d^* = (d^+)^{\mathcal{M}_\xi}$. Then d, d^* are strong cutpoints of $\mathcal{M}_\xi$. We claim that $\mathcal{M}_\xi|d^* = SSM^{\mathfrak{Q}}(\mathcal{M}_\xi|d)$, where $\mathfrak{Q} = \deg^{\mathcal{M}_\xi}(\gamma)$, $\gamma = \min(l^{\mathcal{M}_\xi} \setminus (d+1))$. On one hand, since $\mathcal{M}_\xi$ is iterable, $\mathcal{M}_\xi|d^* \trianglelefteq SSM^{\mathfrak{Q}}(\mathcal{M}_\xi|d)$. On the other hand, every level of $SSM^{\mathfrak{Q}}(\mathcal{M}_\xi|d)$ is $OD(S^{\mathfrak{Q}})$, hence is in $\mathcal{M}_\xi$ by fullness, and hence is in $\mathcal{M}_\xi|d^*$ by acceptability. But $Hull^{(\mathcal{M}_\xi)^+}(\mathcal{M}_\xi|d \cup \pi^\infty_{\mathcal{M}_\xi})$ is full by assumption. Hence the hull contains d^*. Since d^* is a cardinal strong cutpoint of $\mathcal{M}_\xi$, we can treat $\mathcal{M}_\xi$ as a $S^{\mathfrak{Q}}$-premouse over $\mathcal{M}_\xi|\delta^*$. But then, $\mathcal{M}_\xi$ is exactly the output of the $L[\vec{E}, S^{\mathfrak{Q}}][\mathcal{M}_\xi|\delta^*]$-construction in $\mathcal{Q}$. That means $Q^\infty_{\mathcal{M}_\xi}, \pi^\infty_{\mathcal{M}_\xi}$ are exactly the objects used in defining $S^{*,\mathfrak{Q}+1}(\mathcal{M}_\xi|d^*)$. It follows then $\mathcal{N}_{\xi+1} = S^{\mathfrak{Q}+1}(M_\xi|d)$. So $\mathcal{N}_{\xi+1}$ is a $S^{\mathfrak{P}}$-premouse. A similar proof as in the last paragraph shows that the induced strategy of $\mathcal{N}_{\xi+1}$ is a semi-S-iteration strategy.

$\mathcal{M}_{\xi+1}$ is a fine-structural core of $\mathcal{N}_{\xi+1}$. Let $d = \max D^{\mathcal{N}_{\xi+1}}$. Then $\rho_\omega(\mathcal{N}_{\xi+1}) \geq d$.

Thus by Lemma 2.39, $\mathcal{M}_{\xi+1}$ is an S-premouse. A similar proof shows that the induced strategy of $\mathcal{M}_{\xi+1}$ is a semi-S-iteration strategy.

If ξ is a limit, then $\mathcal{N}_\xi$ is a $S^{\mathfrak{P}}$-premouse since each $\mathcal{M}_\mu$, $\mu < \xi$ is a $S^{\mathfrak{P}}$-premouse. The induced strategy of $\mathcal{N}_\xi$ is a semi-S-iteration strategy. $\square$

2.9 Condensation of the S-operators

Condensation of the $S^{\mathfrak{P}}$-operator is important, as the next operator $S^{\mathfrak{P}+1}$ relies on an $L[\vec{E}, S^{\mathfrak{P}}]$-construction. At certain steps of the $L[\vec{E}, S^{\mathfrak{P}}]$-construction, we take fine structural cores. Condensation of the $S^{\mathfrak{P}}$-operator guarantees that an $S^{\mathfrak{P}}$-premouse condenses to an $S^{\mathfrak{P}}$-premouse. So far, we don't know if all $S^{\mathfrak{P}}$-operators are not vacuous!

Once again we will show that each $S^{\mathfrak{P}}$ has condensation on a cone by reducing it to the $H^{\mathfrak{P}}$ operator. The $H^{\mathfrak{P}}$-operator has the same Wadge rank as $S^{\mathfrak{P}}$, when coded as a subset of reals, but has condensation outright. The successor step is just like in Section 2.3. At a limit step with non-measurable cofinality we simply take the intersection of countably many cones. The situation at a limit step with measurable cofinality is harder, where the coding lemma is involved to get the base of the cone.

In the following paragraphs we define the $H^{\mathfrak{P}}$ operators for successor $\mathfrak{P}$'s. Fix a countable transitive swo. Suppose that $\mathfrak{P}$ is an index for a, $\mathrm{dom}(\mathfrak{P}) = n+1$, α_n is a successor ordinal. Let $\mathfrak{Q}$ be the promotion of $\mathfrak{P}$ for a. Let $\vec{\mathcal{N}} = \langle \mathcal{N}_i : i \leq n \rangle$ be the $\mathcal{M}_1^{\#}$-sequence of $\mathfrak{Q}$ for a. Let $\mathcal{N}$ be $\Sigma_{\mathcal{Q}_n(\beta_n-1)}$-good over $\mathcal{N}_n$. By the universality proof, $\mathcal{N}$ is full, in the sense that for any transitive swo $b \in \mathcal{N}$, the maximal $L[\vec{E}, \Sigma_{\mathcal{Q}_n(\beta_n-1)}][b]^{\mathcal{N}}$ constructions reaches $\mathrm{Lp}^{\Sigma_{\mathcal{Q}_n(\beta_n-1)}}(b)$. $\mathcal{N}$ can define the short-tree iteration strategy of $\mathcal{Q}_n(\beta_n)$ by choosing branches whose $\mathcal{Q}$-structure is an initial segment of some model of the $L[\vec{E}, \Sigma_{\mathcal{Q}_n(\beta_n-1)}]$ construction over common part of the tree. Let $\mathcal{U}$ be the generic genericity iteration tree on $\mathcal{Q}_n(\beta_n)$ attempting to make all reals $y\leq_T x_g$ generic over the extender algebra of the final model, whenever x_g codes (a, g), g is a $Coll(\omega, a)$ generic filter over the final model (cf. [8]) according to the short-tree strategy. We assume that $\mathcal{U}$ has maximal possible length, i.e. either $\mathcal{U}$ has a last model $\mathcal{R}$ such that all $y\leq_T x_g$ are generic over the extender algebra of $\mathcal{R}$, whenever x_g codes (a, g), g is a $Coll(\omega, a)$ generic filter over $\mathcal{R}$, or $\mathcal{U}$ is maximal.

We again claim that $\mathcal{U}$ must be maximal, $lh(\mathcal{U}) = (|a|^+)^{\mathcal{N}}$, and $\Sigma_{\mathcal{Q}_n(\beta_n)}(\mathcal{U}) \notin \mathcal{N}$. For otherwise let $\mathcal{R}$ be the result of the generic genericity iteration, then $\mathcal{R} \in \mathcal{N}$ and $\delta^{\mathcal{R}}$ is singular in $\mathcal{N}$. Let g be $Coll(\omega, a)$ generic over $\mathcal{N}$. Let x_g be the real that codes (a, g). Then $\mathbb{R}\cap\mathcal{N}[x_g] = \{y \in \mathbb{R} : y \text{ is } OD(x_g, \Sigma_{\mathcal{Q}_n(\beta_{n-1})})\}$ by fullness of $\mathcal{N}$. But $\mathbb{R} \cap \mathcal{R}[x_g] = \{y \in \mathbb{R} : y \text{ is } OD(x_g, \Sigma_{\mathcal{Q}_n(\beta_{n-1})})\}$ because Σ is super-fullness preserving. Since the extender algebra of $\mathcal{R}$ is $\delta^{\mathcal{R}}$-c.c., $\mathcal{R}[x_g] \models \delta^{\mathcal{R}}$ is regular. Contradiction.

So $\mathcal{N}$ is able to define the last model of $\mathcal{U}$ as $\mathcal{R} = \mathrm{Lp}_\omega^{\Sigma_{\mathcal{Q}_n(\beta_{n-1})}}(\mathcal{M}(\mathcal{U}))$, but $\Sigma_{\mathcal{Q}_n(\beta_n)}(\mathcal{U}) \notin \mathcal{N}$. Let $\pi = i^{\mathcal{U}^\frown\Sigma_{\mathcal{Q}_n(\beta_n)}(\mathcal{U})}$ be the iteration map. Let $\mathcal{M}$ be the least initial segment of $\mathcal{N}$ such that $\mathcal{U}, \mathcal{R} \in \mathcal{M}$ and $\mathcal{M} \models$ "$\mathcal{U}$ is the maximal correct genericity iteration tree with last model $\mathcal{R}$ with respect to a". Then $H^{\mathfrak{P}}(a)$ is the $(e, \sigma_n \circ \cdots \circ \sigma_1)$-amenable code of the transitive collapse of

$$\langle Hull^{\mathcal{M}}(a \cup \{a\} \cup \pi), \in, a, E^{\mathcal{M}}, S^{\mathcal{M}},\mathcal{U}, \mathcal{R}, \pi\rangle.$$

Note again that $\delta(\mathcal{U}) = (|a|^+)^{\mathcal{N}} = (|a|^+)^{\mathcal{M}}$, $o(\mathcal{R}) = (|a|^{+\omega})^{\mathcal{N}} = (|a|^{+\omega})^{\mathcal{M}}$. Let $j_a : H^{[0]}(a) \to \langle \mathcal{M}, \in, etc\rangle$ be the associated anticollapse map. For the similar reason as in Section 2.3, $j_a \restriction (|a|^{+\omega})^{\mathcal{M}} = id$.

The proof of Lemma 2.18 and Lemma 2.16 carries over to the general case, Lemma 2.41 and Lemma 2.42. Lemma 2.42 is a stronger form of condensation. It applies to different H-operators: A Σ_1-elementary substructure of $H^{\mathfrak{Q}}(a)$ condenses to $H^{\mathfrak{P}}(b)$ with the possibility that $\mathfrak{P} \neq \mathfrak{Q}$. We even allow the possibility $\mathcal{P}_n(\alpha_n^*) \not\equiv_{DJ} \mathcal{Q}_n(\beta_n^*)$. This stronger form of condensation is necessary in the condensation proof of $S^{\mathfrak{P}}$ in the measurable cofinality case as well as in the translation procedure of sectrion 3. Again, the proof of Lemma 2.42 relies on Lemma 2.11. Lemma 2.42 is the only reason why we make the effort defining the (e, σ)-amenable code and proving Lemma 2.11.

Lemma 2.41. *Let $\mathfrak{P}$ be an index for a. Then for all $g \subseteq Coll(\omega, a)$ generic over $H^{\mathfrak{P}}(a)$,*

$$H^{\mathfrak{P}}(x_g) = H^{\mathfrak{P}}(a)[x_g].$$

Lemma 2.42. *Suppose that $j : \bar{\mathcal{H}} \to H^{\mathfrak{P}}(a)$ is Σ_1 elementary. $\mathfrak{P}$ is a promoted index over a, $\mathfrak{Q}$ is a promoted index over b. Suppose that*

$$j(\mathfrak{Q}, b, z_0) = (\mathfrak{P}, a, z_0)$$

Then

$$\bar{\mathcal{H}} = H^{\mathfrak{Q}}(b).$$

For each $n < \omega$, $x \in \mathbb{R}$ such that $S^{\mathfrak{P}}(x)$ is defined, let

$$S_n^{\mathfrak{P}}(x) = Th^{S^{\mathfrak{P}}(x)|\xi_n}(\{x\} \cup (\pi^{S^{\mathfrak{P}}(x)} \circ e) \restriction n),$$

$$H_n^{\mathfrak{P}}(x) = Th^{H^{\mathfrak{P}}(x)|\eta_n}(\{x\} \cup (\pi^{H^{\mathfrak{P}}(x)} \circ e) \restriction n).$$

where ξ_n, η_n are ordinals that $(\pi^{S^{\mathfrak{P}}(x)} \circ e) \restriction n$ and $(\pi^{H^{\mathfrak{P}}(x)} \circ e) \restriction n$ glue to. We say that a real z codes a reduction between $S^{\mathfrak{P}}$ and $H^{\mathfrak{P}}$ if z codes (f_0, f_1, g_0, g_1) in a fixed coding system such that for all $x \geq_T z$,

$$S_n^{\mathfrak{P}}(x) = \{f_1(n)\}^{H^{\mathfrak{P}}_{f_0(n)}(x)},$$

$$H_n^{\mathfrak{P}}(x) = \{g_1(n)\}^{S^{\mathfrak{P}}_{g_0(n)}(x)}.$$

To show that $S^{\mathfrak{P}}$ has condensation when $\mathfrak{P}$ is of type C, we will need the notion of coding a reduction on further extensions of $S^{\mathfrak{P}}$. As a motivation, let's consider the typical case when $\mathfrak{P} = [\kappa]$, κ is the least measurable of $\mathcal{P}$. As an induction hypothesis, assume that for all $\langle \mathfrak{P}, \mathfrak{P}_1 \rangle \in \mathcal{I}$ such that α_1 successor, there is a real which codes a reduction between $S^{\langle \mathfrak{P}, \mathfrak{P}_1 \rangle}$ and $H^{\langle \mathfrak{P}, \mathfrak{P}_1 \rangle}$. Let A be the set of $(x, z) \in \mathbb{R}^2$ such that

1. x codes $(\vec{\mathcal{T}}_x, \mathfrak{P}_x)$. Denote $\mathfrak{P}_x = (0, \pi_x, \mathcal{P}_x, \alpha_x, \alpha_x)$. Then α_x is a successor.
2. $\vec{\mathcal{T}}_x$ is a stack on $\mathcal{P}$ below $\mathcal{P}(0)$ according to $\Sigma_{\mathcal{P}(0)}$ with last model $\mathcal{P}_x$ such that $i^{\vec{\mathcal{T}}_x} = \pi_x$.
3. $\langle \mathfrak{P}, \mathfrak{P}_x \rangle \in \mathcal{I}$.
4. z codes a reduction between $S^{\langle \mathfrak{P}, \mathfrak{P}_x \rangle}$ and $H^{\langle \mathfrak{P}, \mathfrak{P}_x \rangle}$.

Let $\leq^*$ be the following prewellordering on $\{x : \exists z (x, z) \in A\}$.

$$x \leq^* y \leftrightarrow \mathcal{P}_x(\alpha_x) \leq_{DJ} \mathcal{P}_y(\alpha_y).$$

Note that by Lemma 2.1, the comparison between $\mathcal{P}_x$ and $\mathcal{P}_y$ is below images of $\delta_0^{\mathcal{P}_x}$ and $\delta_0^{\mathcal{P}_y}$. So $\leq^*$ is actually $\Delta^1_1(\Sigma_{\mathcal{P}_n(0)})$. By the coding lemma, there is $B \subseteq A$ such that B is $\underset{\sim}{\Sigma}^1_1(\Sigma_{\mathcal{P}_n(0)})$, and for all $x \in field(\leq^*)$, there is $y =^* x$ and w such that $(y, w) \in B$. Let $U \subseteq \mathbb{R}^3$ be the universal $\Sigma^1_1(\Sigma_{\mathcal{P}(0)})$ set. Let z be a real of sufficiently high Turing degree such that for all $\epsilon < \kappa$, there is a real recursive in w which codes a reduction between $S^{\mathfrak{P}[\epsilon]}$ and $H^{\mathfrak{P}[\epsilon]}$, and that

$$U_z$$

is the set of (x, w) such that

$$\text{there is } y \text{ such that } y =^* x, (y, w) \in B.$$

We sketch a proof that if $\mathcal{K}$ is an S-premouse over a, $a \in \text{Cone}(z)$, $j : \bar{\mathcal{S}} \to S^{\mathfrak{Q}}(\mathcal{K})$ is Σ_1-elementary, $j(\mathcal{H}, \mathfrak{Q}) = (\mathcal{K}, \mathfrak{Q})$, and $\text{pro}(\mathfrak{Q}, \mathcal{K}) \leq^{\mathcal{I}}_{\mathcal{K}} \text{pro}(\mathfrak{P}, \mathcal{K})$, then $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(\mathcal{H})$. Consequently, the $L[\vec{E}, S^{\mathfrak{P}}]$-construction in any good universe converges. The proof is done by induction on final$(\mathfrak{Q})$. If $\mathfrak{Q} = [\epsilon]$ for some $\epsilon < \lambda^{\mathcal{P}}$, then $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(\mathcal{H})$ follows from choice of z. So let's assume $\mathfrak{Q} >^{\mathcal{I}}_{\mathcal{K}} [\epsilon]$ for all $\epsilon < \lambda^{\mathcal{P}}$. This means $\mathfrak{Q} = [\lambda^P]^\frown\langle 0, \pi^*, \mathcal{Q}^*, \epsilon, \epsilon\rangle$ for some $\pi^*, \mathcal{Q}^*, \epsilon$. The main difficulty is when $\mathfrak{Q}$ is a successor index. Let's assume $\mathfrak{Q}$ is a successor index. Let $\mathcal{N} = \mathcal{M}_1^{\#, \Sigma_{\mathcal{P}(0)}}(b)$.
We show that if $b \in \text{Cone}(z)$, g is $Coll(\omega, b)$-generic over $S^{*,\mathfrak{Q}}(b)$, $\mathcal{N} = \mathcal{M}_1^{\#, \Sigma_{\mathcal{P}(0)}}(b)$, x codes $(\vec{\mathcal{T}}_x, \mathfrak{P}_x)$, $\mathfrak{P}_x = (0, \pi_x, \mathcal{P}_x, \alpha_x, \alpha_x)$, α_x is a successor, $\vec{\mathcal{T}}_x$ is a stack on $\mathcal{P}$ below $\mathcal{P}(0)$ according to $\Sigma_{\mathcal{P}(0)}$ with last model $\mathcal{P}_x$ such that $i^{\vec{\mathcal{T}}_x} = \pi_x$, then there is $w \in \mathcal{N}[g]$ such that (z, x, w) such that $(z, x, w) \in U$. Moreover, for all $w \in \mathcal{N}[g]$ such that $(z, x, w) \in U$, w codes a reduction between $S^{*,\mathfrak{Q}}$ and $H^{\mathfrak{Q}}$ above $S^{*,\mathfrak{Q}}(b)$. Let g, $\mathcal{N}$, x be as above. Let

$$\pi_\infty : \mathcal{N} \to \mathcal{M}_\infty$$

be the direct limit embedding of all iterates of $\mathcal{N} = \mathcal{M}_1^{\#, \Sigma_{\mathcal{P}(0)}}(b)$ according to its unique (ω_1, ω_1)-strategy. We know by induction that

$$\text{there are } y, w \text{ such that } y =^* x \text{ and } (y, w) \in B$$

By choice of z,

$$U_{z,x} = \{w : \exists y\ y =^* x \text{ and } (y, w) \in B\}.$$

Here is the crucial step. By generic interpretability, $\mathcal{M}_\infty$ has definable trees T_0, T_1

on $\omega \times \delta^{\mathcal{M}_\infty}$ such that $p[T_0] = \Sigma_{\mathcal{P}(0)} = \mathbb{R} \setminus p[T_1]$. (cf.[7, Theorem 4.2.5]) Hence there is a tree T definable over $\mathcal{M}_\infty$ such that $p[T] = U$. By absoluteness,

$$\mathcal{M}_\infty[g] \models \exists w \ (z, x, w) \in p[\mathcal{T}].$$

Hence

$$\mathcal{N}[g] \models \exists w \ (z, x, w) \in p[{\pi_\infty}^{-1}(T)].$$

Since $p[{\pi_\infty}^{-1}(T)] \subseteq p[T]$, there is $w \in \mathcal{N}_{n+1}[g]$ such that $(z, x, w) \in U$. Moreover, By choice of z, for all $w' \in \mathcal{N}[g]$ such that $(z, x, w') \in U$, there is $y \in \mathcal{N}[g]$ such that $y =^* x$, w' codes a reduction between $S^{\langle \mathfrak{P}, \mathfrak{P}_y \rangle}$ and $H^{\langle \mathfrak{P}, \mathfrak{P}_y \rangle}$. Since $\mathcal{N} \in S^{\mathfrak{P}[1]}(a)$, whenever $b \in \mathrm{Cone}(S^{\mathfrak{R}}(a))$, we have $\mathrm{pro}(\langle \mathfrak{P}_0, \mathfrak{P}_y \rangle, b) = \mathrm{pro}(\langle \mathfrak{P}_0, \mathfrak{P}_x \rangle, b) = \mathrm{pro}(\mathfrak{R}, b)$. Hence w' codes a reduction between $S^{\mathfrak{R}}(b)$ and $H^{\mathfrak{R}}(b)$ above $S^{\mathfrak{R}}(a)$.

We can then obtain effective maps $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^h_{\mathfrak{Q}} \urcorner$ and $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^s_{\mathfrak{Q}} \urcorner$ such that for all countable transitive swo $b \in \mathrm{Cone}(S^{*,\mathfrak{Q}}(a))$, for all $c_0, \ldots, c_k \in b$, for all Σ_1 formula $\phi(v_0, \ldots, v_k)$,

$$S^{*,\mathfrak{Q}}(b) \models \phi(c_0, \ldots, c_k) \leftrightarrow H^{\mathfrak{Q}}(b) \models \phi^h_{\mathfrak{Q}}(c_0, \ldots, c_k, z, \vec{\mathcal{N}}),$$

$$H^{\mathfrak{Q}}(b) \models \phi(c_0, \ldots, c_k) \leftrightarrow S^{*,\mathfrak{Q}}(b) \models \phi^s_{\mathfrak{Q}}(c_0, \ldots, c_k, z, \vec{\mathcal{N}}).$$

Here is a sketch of definition of $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^h_{\mathfrak{Q}} \urcorner$. Assume that

$$S^{*,\mathfrak{Q}}(b) \models \phi(c_0, \ldots, c_k).$$

Then for all $g \subseteq Coll(\omega, b)$ generic over both $S^{*,\mathfrak{Q}}(b)$ and $H^{\mathfrak{Q}}(b)$ such that $g(i) = c_i$ for all $i \leq k$,

$$S^{*,\mathfrak{Q}}(x_g) \models \text{“}\underbrace{S^{*,\mathfrak{Q}}(b) \models \phi(g(0), \ldots, g(k))}_{\text{call this } \phi_1}\text{”}.$$

Hence

$$\exists l\ \ulcorner\phi_1\urcorner \in S_l^{\mathfrak{Q}}(x_g).$$

Let $\phi_2(z, N)$ be the formula

"Let x be such that

1. x codes $(\vec{\mathcal{T}}_x, \mathfrak{P}_x)$. Denote $\mathfrak{P}_x = (\nu_x, \pi_x, \mathcal{P}_x, \alpha_x, \alpha_x)$.

2. $\langle \mathfrak{P}, \mathfrak{P}_x\rangle \in \mathcal{I}$ as witnessed by $\vec{\mathcal{T}}_x$.

3. $\mathcal{P}_x(\alpha_x)$ and $\mathcal{Q}^*(\epsilon)$ coiterate to the same model.

Let w be such that $(z, x, w) \in p[\pi_\infty^{-1}(T)]$. Then w codes f_0, f_1, g_0, g_1, and there is $l \in \omega$ such that $\ulcorner\phi_1\urcorner \in \{f_1(l)\}^{H^{\mathfrak{Q}}_{f_0(l)}(x_g)}$."

Then

$$H^{\mathfrak{Q}}(x_g) \models \phi_2(z, \mathcal{N}).$$

Let $\phi^h_{\mathfrak{Q}}(v_0, \ldots, v_k, z, \mathcal{N})$ be the formula

"for all $g \subseteq Coll(\omega, b)$ generic over $H^{\mathfrak{Q}}(b)$ such that $g(i) = v_i$ for all $i \leq k$, then $V[x_g] \models \phi_2(z, \vec{\mathcal{N}})$."

Then

$$H^{\mathfrak{Q}}(b) \models \phi^h_{\mathfrak{Q}}(c_0, \ldots, c_k, z, \vec{\mathcal{N}}).$$

In a similar way we can define the map $\ulcorner\phi\urcorner \mapsto \ulcorner\phi^s_{\mathfrak{Q}}\urcorner$. The same proof as in Lemma 2.20 gives that $S^{*,\mathfrak{Q}}$ has condensation above $S^{*,\mathfrak{Q}}(b)$. This, combined with a proof like Lemma 2.26, shows $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(\mathcal{H})$.

Let's turn back to the general case. In general, the promotion of an index over a is sensitive to a. So it is necessary to consider the equivalence class of an index instead of a single one. Given $\mathfrak{P} = \langle(\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i^*, \beta_i) : i \leq n\rangle \in \mathcal{I}$ and $\mathfrak{Q} = \langle(\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i^*, \beta_i) : i \leq n\rangle \in \mathcal{I}$, we say that $\mathfrak{P} \sim^{\mathcal{I}} \mathfrak{Q}$ if for all $i \leq n$,

1. $\mathcal{P}_i(\alpha_i^*) \equiv_{DJ} \mathcal{Q}_i(\beta_i^*)$.

2. For all $\epsilon < \alpha_i$, $\mathcal{P}_i(\epsilon) <_{DJ} \mathcal{Q}_i(\beta_i)$.

3. For all $\epsilon < \beta_i$, $\mathcal{Q}_i(\epsilon) <_{DJ} \mathcal{P}_i(\alpha_i)$.

So $\mathfrak{P}$ and its promotion will be $\sim^{\mathcal{I}}$-equivalent. Let $[\mathfrak{P}]_{\mathcal{I}}$ be the set of $\mathfrak{Q} \in \mathcal{I}$ such that $\mathfrak{P} \sim^{\mathcal{I}} \mathfrak{Q}$. Let

$$\Gamma_{[\mathfrak{P}]_{\mathcal{I}}} = \{(x, y) \in \mathbb{R}^2 : x \text{ code } \mathfrak{Q} \in [\mathfrak{P}]_{\mathcal{I}}, y \in Code(\Sigma_{\mathcal{Q}_n(\nu_n)})\}.$$

Let

$$U_{[\mathfrak{P}]_{\mathcal{I}}} \subseteq \mathbb{R}^3$$

be the canonical universal $\Sigma^1_1(\Gamma_{[\mathfrak{P}]_{\mathcal{I}}})$ set.

Lemma 2.43. *Let $\mathfrak{P} = \langle(\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i^*, \beta_i) : i \leq n\rangle \in \mathcal{I}$. Then $\Gamma_{[\mathfrak{P}]_{\mathcal{I}}}$ is $\Delta^1_1(\Sigma_{\mathcal{P}_n(\zeta_n)})$. Consequently, $U_{[\mathfrak{P}]_{\mathcal{I}}}$ is $\Sigma^1_1(\Sigma_{\mathcal{P}_n(\zeta_n)})$.*

Proof. We inductively show that $\Gamma_{[\mathfrak{P}\restriction i+1]_{\mathcal{I}}}$ is $\Delta^1_1(\Sigma_{\mathcal{P}_i(\zeta_i)})$ for all $i \leq n$. The base case $i = 0$ is trivial. The inductive case follows from Lemma 2.1: Suppose that $\mathfrak{Q} \restriction i + 1 \in [\mathfrak{P} \restriction i + 1]_{\mathcal{I}}$. Then the comparison between $\mathcal{Q}_{i+1}$ and $\mathcal{P}_{i+1}$ is below images of $\delta^{\mathcal{Q}_{i+1}}_{\nu_{i+2}}$ and $\delta^{\mathcal{P}_{i+1}}_{\zeta_{i+2}}$. If $\mathcal{Q}_{i+1}(\beta_{i+1})$ and $\mathcal{P}_{i+1}(\alpha_{i+1})$ iterate to $\mathcal{R}_{i+1}$ via $\mathcal{U}$ and $\mathcal{T}$ respectively, then $\Sigma_{\mathcal{Q}_{i+1}(\nu_{i+1})} = ((\Sigma_{\mathcal{P}_{i+1}(\zeta_{i+1})})_{\mathcal{T}})^{i^{\mathcal{U}}}$. □

Given any $(\mathcal{R}, \Lambda) \in pB(\mathcal{P}, \Sigma) \cap a$, let $\pi_\infty : \mathcal{M}_1^{\#,\Lambda}(a) \to \mathcal{M}_\infty$ be the direct limit map of all iterates of $\mathcal{M}_1^{\#,\Lambda}(a)$ according to its unique strategy. Then $\mathcal{M}_\infty$ has definable trees T_0, T_1 on $\omega \times \delta^{\mathcal{M}_\infty}$ such that $p[T_0] = \Lambda = \mathbb{R} \setminus p[T_1]$ (cf. [7, Theorem 4.2.5]). Let T be the tree on $(\omega \times \omega) \times \delta^{\mathcal{M}_\infty}$ canonically definable from T_0 and T_1 such that $p[T] = U_{[\mathfrak{P}]_{\mathcal{I}}}$. We then set $T^{\mathcal{M}_1^{\#,\Lambda}(a)} = {\pi_\infty}^{-1}(T)$.

Let $\mathfrak{P} \in \mathcal{I}$ be a limit index of type C. Let $\mathrm{dom}(\vec{\mathcal{P}}) = n+1$. We say that a real z codes a reduction on further extensions of $[\mathfrak{P}]_{\mathcal{I}}$ if for all $\mathfrak{Q}$, $\vec{\mathcal{N}}$, a, g, $\langle x_i : i \leq m \rangle$ such that

1. $a \in \mathrm{Cone}(z)$,
2. $\mathfrak{Q} = \langle (\nu_i, \sigma_i, \mathcal{Q}_i, \beta_i^*, \beta_i) : i \leq m \rangle \in \mathcal{I}$ is a promotable successor index over a, $\vec{\mathcal{N}}$ is the $\mathcal{M}_1^{\#}$-sequence of $\mathrm{pro}(\mathfrak{Q}, a)$ for a.
3. $\mathrm{dom}(\mathfrak{Q}) = m+1 > n+1$, $\mathfrak{Q} \restriction n+1 \sim^{\mathcal{I}} \mathfrak{P}$,
4. $g \subseteq Coll(\omega, a)$ is generic over $S^{*,\mathfrak{Q}}(a)$.
5. for all $n < i \leq m$, $\beta_i^* = \beta_i$,
6. for all $i \leq m$, $x_i \in \mathcal{N}_i[g]$, x_i codes $(\vec{\mathcal{T}}_{x_i}, \mathfrak{Q}_i)$.
7. $\vec{\mathcal{T}}_{x_0} = \emptyset$. For all $i < m$, $\vec{\mathcal{T}}_{x_{i+1}}$ is a stack on $\mathcal{Q}_i$ below $\mathcal{Q}_i(\nu_i)$ according to $\Sigma_{\mathcal{Q}_i(\nu_i)}$ with last model $\mathcal{Q}_{i+1}$ such that $i^{\vec{\mathcal{T}}_{x_i}}$ is defined.

there is $\langle w_i : n < i \leq m \rangle \in \mathcal{N}_m[g]$ such that

1. $(z, \oplus_{j \leq n+1} x_j, w_{n+1}) \in U_{[\mathfrak{Q} \restriction n+1]_{\mathcal{I}}}$.
2. For all $n < i < m$, $(w_i, \oplus_{j \leq i+1} x_j, w_{i+1}) \in U_{[\mathfrak{Q} \restriction i+1]_{\mathcal{I}}}$.

Moreover, for all such $\langle w_i : n < i \leq m \rangle$, w_m codes a reduction between $S^{*,\mathfrak{Q}}$ and $H^{\mathfrak{Q}}$ above $S^{\mathfrak{Q}}(a)$.

Definition 2.44. Let $\mathfrak{P} \in \mathcal{I}$, $\mathrm{dom}(\vec{\mathcal{P}}) = n+1$. We say that z is a nice real for $\mathfrak{P}$ if all of the following holds.

1. (Reduction) For all $\epsilon \leq \alpha_n$ successor or 0, if $\mathfrak{P}[\epsilon] \in \mathcal{I}$, then there is $y \leq_T z$ such that y codes a reduction between $S^{*,\mathfrak{P}[\epsilon]}$ and $H^{\mathfrak{P}[\epsilon]}$.

2. (Reduction on further extensions) For all $\epsilon \leq \alpha_n$, if $\mathfrak{P}[\epsilon]$ is a limit index of type C, then there is $y \leq_T z$ such that y codes a reduction on further extensions of $[\mathfrak{P}(\epsilon)]_{\mathcal{I}}$.

3. (Condensation) If $\mathcal{K}$ is an S-premouse over a, $a \in \mathrm{Cone}(z)$, $j : \bar{\mathcal{S}} \to S^{\mathfrak{Q}}(\mathcal{K})$ is Σ_1-elementary, $j(\mathcal{H}, \mathfrak{Q}) = (\mathcal{K}, \mathfrak{Q})$, and $\mathrm{pro}(\mathfrak{Q}, \mathcal{K}) \leq^{\mathcal{I}}_{\mathcal{K}} \mathrm{pro}(\mathfrak{P}, \mathcal{K})$, then $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(\mathcal{H})$. Consequently, the $L[\vec{E}, S^{\mathfrak{P}}]$-construction in any good universe converges.

Theorem 2.45. *For all $\mathfrak{P} \in \mathcal{I}$, there is a nice real for $\mathfrak{P}$.*

Proof. We show by induction on hod mouse prewellordering of final($\mathfrak{P}$). Let $\mathfrak{P} = \langle (\zeta_i, \pi_i, \mathcal{P}_i, \alpha_i^*, \alpha_i) : i \leq n \rangle$

Case 1. $\mathfrak{P} = [0]$

Let z_0 be as in Lemma 2.26. Then clearly z satisfies properties 1 and 3 of definition of a nice real for [0]. 2 is vacuous.

Case 2. $\mathfrak{P}$ is a successor index

Let z_0 be a nice real for $\mathfrak{P} - 1$.

We claim that for all $n < \omega$, $x \in \mathbb{R}$ such that $z_0 \leq_T x$, there is $k < \omega$ such that $S^{\mathfrak{P}}_n(x) \leq_T H^{\mathfrak{P}}_k(x)$, $H^{\mathfrak{P}}_n(x) \leq_T S^{\mathfrak{P}}_k(x)$. Since $S^{\mathfrak{P}}_n(x)$ is a real in an $S^{\mathfrak{P}-1}$-mouse over x, $S^{\mathfrak{P}}_n(x) \in \mathrm{Lp}^{S^{\mathfrak{P}-1}}(x)$. So $S^{\mathfrak{P}}_n(x) \in \mathrm{Lp}^{\Sigma_{\mathcal{P}_n(\alpha_n-1)}}(x)$. Since $H^{\mathfrak{P}}_n(x)$ is a real in a $\Sigma_{\mathcal{P}_n(\alpha_n-1)}$-mouse over x, $H^{\mathfrak{P}}_n(x) \in \mathrm{Lp}^{\Sigma_{\mathcal{P}_n(\alpha_n-1)}}(x)$. On the other hand, we show that for all $y \in \mathrm{Lp}^{\Sigma_{\mathcal{P}_n(\alpha_n-1)}}(x)$, there is k such that $y \leq_T S^{\mathfrak{P}}_k(x)$, $y \leq_T H^{\mathfrak{P}}_k(x)$. By soundness of $S^{\mathfrak{P}}(x)$ and $H^{\mathfrak{P}}(x)$, it suffies to show that $\mathrm{Lp}^{\Sigma_{\mathcal{P}_n(\alpha_n-1)}}(x) \subset S^{\mathfrak{P}}(x) \cap H^{\mathfrak{P}}(x)$. Let $\mathcal{Q}, \mathcal{N}$ be as in definition of $S^{\mathfrak{P}}(x)$. Let $j : S^{\mathfrak{P}}(x) \to \langle \mathcal{N}_+, \in, etc \rangle$ be the uncollapsing map. By the proof of MSC, there is $\mathcal{R} \in pI(\mathcal{P}_n(\alpha_n), \Sigma) \cap \mathcal{N} \cap \mathrm{ran}(j)$ such that $\mathcal{R}$ is an iterate of $\mathcal{P}_n(\alpha_n)$ above $\mathcal{P}_n(\alpha_n - 1)$, $\Sigma_{\mathcal{R}} \restriction \mathcal{N} \in \mathcal{N}_+ \cap \mathrm{ran}(j)$, and x is generic over the extender algebra of $\mathcal{R}$ at $\delta^{\mathcal{R}}$. Since $\mathrm{ran}(i_{\mathcal{PR}}) \subseteq \mathrm{ran}(j)$, by strong branch condensation, $j^{-1}(\mathcal{R})$ is full. So $j^{-1}(\mathcal{R})[x]$ is full by super fullness preservation. So $\mathrm{Lp}^{\Sigma_{\mathcal{P}_n(\alpha_n-1)}}(x) \subseteq S^{\mathfrak{P}}(x)$. For the similar reason, $\mathrm{Lp}^{\Sigma_{\mathcal{P}_n(\alpha_n-1)}}(x) \subseteq H^{\mathfrak{P}}(x)$.

Since $S^{\mathfrak{P}}_n, H^{\mathfrak{P}}_n$ are uniformly Turing invariant operators, according to [16], there is a real $z \geq_T z_0$ which codes a reduction between $S^{\mathfrak{P}}$ and $H^{\mathfrak{P}}$. We show that z is a nice real for $\mathfrak{P}$. 1 and 2 follow from induction.

Condensation of $S^{*,\mathfrak{P}}$ above $\mathrm{Cone}(z)$ follows as in Lemma 2.20 and from 2.19. Condensation of $S^{\mathfrak{P}}$ for S-premice above $a \in \mathrm{Cone}(z)$ follow from a proof similar to 2.26. This and the induction hypothesis shows 3.

Case 3. $\mathfrak{P}$ is a limit index of type A.

Let z be such that for all $\sup \pi_n''\alpha_{n-1} \leq \epsilon < \alpha_n$, there is $y \leq_T z$ such that y is a nice real for $S^{\mathfrak{P}[\epsilon]}$. We claim that z is a nice real for α_n.

We prove that $S^{\mathfrak{P}}$ has condensation for S-premice above any $a \in \mathrm{Cone}(z)$. Let $j : \bar{\mathcal{S}} \to S^{\mathfrak{P}}(\mathcal{K})$ be Σ_1-elementary, where $\mathcal{K}$ is an S-premouse over a, $a \in \mathrm{Cone}(z)$. Suppose that $j(\mathcal{H}) = (\mathcal{K})$. We want to show that $\bar{\mathcal{S}} = S^{\mathfrak{P}}(\mathcal{H})$. Let $\mathfrak{Q} = \mathrm{pro}(\mathfrak{P}, \mathcal{K})$, $\vec{\mathcal{M}}$ be the $\mathcal{M}_1^{\#}$-sequence of $\mathfrak{Q}$ for $\mathcal{K}$, $\vec{k}$ be the lifting sequence of $\mathfrak{P}$ for $\mathcal{K}$, $\mathfrak{R} = \mathrm{pro}(\mathfrak{P}, \mathcal{H})$, $\vec{\mathcal{N}}$ be the $\mathcal{M}_1^{\#}$-sequence of $\mathfrak{R}$ for $\mathcal{H}$, $\vec{l}$ be the lifting sequence of $\mathfrak{P}$ for $\mathcal{H}$. We can show that

$$j(\mathfrak{R}, \vec{\mathcal{N}}) = (\mathfrak{Q}, \vec{\mathcal{M}}).$$

This is done by induction. The base case,

$$j(\mathcal{R}_0, \gamma_0, \mathcal{N}_0) = (\mathcal{Q}_0, \beta_0, \mathcal{M}_0)$$

is because $\mathcal{Q}_0 = \mathcal{Q}_0 = \mathcal{P}$, $\gamma_0 = \beta_0 = \alpha_0$, $\vec{\mathcal{N}_0} = b$, $\vec{\mathcal{M}}_0 = a$. Suppose we already know that

$$j(\mathfrak{R}_i, \mathcal{N}_i) = (\mathfrak{Q}_i, \mathcal{M}_i).$$

Then by hull condensation of Σ, $j^{-1}(\mathcal{M}_1^{\#,\Sigma_{\mathcal{Q}_i(\nu_i)}}(\mathcal{M}_i)) = \mathcal{M}_1^{\#,\Sigma_{\mathcal{R}_i(\mu_i)}}(\mathcal{N}_i)$. Hence

$$j(\mathfrak{R}_{i+1}, \mathcal{N}_{i+1}) = (\mathfrak{R}_{i+1}, \mathcal{M}_{i+1}).$$

This finishes showing that $j(\mathfrak{R}, \vec{\mathcal{N}}) = (\mathfrak{Q}, \vec{\mathcal{M}})$. For every ϵ such that $\mathfrak{P}[\epsilon] \in \mathcal{I}$, $\mathfrak{Q}[k_n(\epsilon)] = \mathrm{pro}(\mathfrak{P}[\epsilon], \mathcal{K})$, $\mathfrak{Q}[l_n(\epsilon)] = \mathrm{pro}(\mathfrak{P}[\epsilon], \mathcal{H})$. So

$$\begin{aligned} j^{-1}(S^{\mathfrak{Q}[k_n(\epsilon)]}(\mathcal{K})) &= j^{-1}(S^{\mathfrak{P}[\epsilon]}(\mathcal{K})) \\ &= S^{\mathfrak{P}[\epsilon]}(\mathcal{H}) && \text{by induction} \\ &= S^{\mathfrak{R}[l_n(\epsilon)]}(\mathcal{H}). \end{aligned}$$

Since $k_n''\alpha_n$ is cofinal in β_n, $l_n''\alpha_n$ is cofinal in γ_n,

$$\bar{\mathcal{S}} = S^{\mathfrak{P}}(b)$$

This and induction hypothesis concludes property 3 of a nice real. 1 and 2 follow from induction.

Case 4. $\mathfrak{P}$ is a limit index of type B.:

Similar to Case 3.

Case 5.$\mathfrak{P}$ is a limit index of type C:

Let ζ^* be least such that $\delta_{\zeta^*}^{\mathcal{Q}_n} > \max(\pi_n(\delta_{\zeta_n}^{\mathcal{P}_{n-1}}), \mathrm{cf}^{\mathcal{P}_n}(\alpha_n^*))$. Let A be the set of $(x, z) \in \mathbb{R}^2$ such that

1. $x = \oplus_{i \le n+1} x_i$. For each $i \le n+1$, x_i codes $(\vec{\mathcal{T}}_{x_i}, \mathfrak{P}_{x_i})$. Denote $\mathfrak{P}_{x_i} = (\zeta_{x_i}, \pi_{x_i}, \mathcal{P}_{x_i}, \alpha_{x_i}^*, \alpha_{x_i})$.
2. $\langle \mathfrak{P}_{x_i} : i \le n+1 \rangle \in \mathcal{I}$ as witnessed by $\langle \vec{\mathcal{T}}_{x_i} : i \le n+1 \rangle$.
3. $\langle \mathfrak{P}_{x_i} : i \le n \rangle \sim^{\mathcal{I}} \mathfrak{P}$. $\mathcal{P}_{x_n}(\zeta_{x_{n+1}}) \equiv_{DJ} \mathcal{P}_n(\zeta^*)$.

4. z is a nice real for $\langle \mathfrak{P}_{x_i} : i \leq n+1 \rangle$.

Let $\leq^*$ be the prewellordering on $\{x : \exists z(x,z) \in A\}$ as follows.

$$x \leq^* y \leftrightarrow \mathcal{P}_{x_{n+1}}(\alpha^*_{x_{n+1}}) \leq_{DJ} \mathcal{P}_{y_{n+1}}(\alpha^*_{y_{n+1}}).$$

Note that by Lemma 2.1, the comparison between $\mathcal{P}_{x_{n+1}}$ and $\mathcal{P}_{y_{n+1}}$ is below images of $\pi_{x_{n+1}}(\delta^{\mathcal{P}_{x_{n+1}}}_{\zeta_{x_{n+1}}})$ and $\pi_{y_{n+1}}(\delta^{\mathcal{P}_{y_{n+1}}}_{\zeta_{y_{n+1}}})$. So $\leq^*$ is actually in $\Delta^1_1(\Sigma_{\mathcal{P}_n(\zeta^*)})$. By the coding lemma, there is $B \subseteq A$ such that B is $\underset{\sim}{\Sigma}^1_1(\Sigma_{\mathcal{P}_n(\zeta^*)})$, and for all $x \in field(\leq^*)$, there is $y \equiv^* x$ and w such that $(y,w) \in B$. Let z be a real of sufficiently high Turing degree such that

1. for all $\epsilon < \alpha_n$ such that $\mathfrak{P}[\epsilon] \in \mathcal{I}$, there is a nice real for $\mathfrak{P}[\epsilon]$ which is recursive in z,
2. $(U_{[\mathfrak{P}]_\mathcal{I}})_z = \{(x,w) : \exists y(y \equiv^* x \wedge (y,w) \in B)\}$

We want to show that z is a nice real for $\mathfrak{P}$. Property 1 follows from induction and choice of z. To get property 2, we just need to show that z codes a reduction on further extensions of $[\mathfrak{P}]_\mathcal{I}$. Let $\mathfrak{Q}, \vec{\mathcal{N}}, a, g, \langle x_i : i \leq m \rangle$ be as in definition of coding a reduction on further extensions of $[\mathfrak{P}]_\mathcal{I}$. Let

$$\pi_\infty : \mathcal{M}_1^{\#, \Sigma_{\mathcal{Q}_n(\mu_n)}}(\mathcal{N}_n) \rightarrow \mathcal{M}_\infty$$

be the direct limit embedding of all iterates of $\mathcal{M}_1^{\#, \Sigma_{\mathcal{Q}_n(\mu_n)}}(\mathcal{N}_n)$ according to its unique (ω_1, ω_1)-strategy. (Note that $\mathcal{M}_1^{\#, \Sigma_{\mathcal{Q}_n(\mu_n)}}(\mathcal{N}_n) = \mathcal{N}_{n+1}$.) Note that $U_{[\mathfrak{Q} \restriction n+1]_\mathcal{I}} = U_{[\mathfrak{P}]_\mathcal{I}}$. We know by induction that

$$\text{there are } y, w \text{ such that } y =^* x_n \text{ and } (y,w) \in B.$$

By choice of z,

$$(U_{[\mathfrak{Q}]_{\mathcal{I}}})_{z,x_n} = \{w : \exists y\ y =^* x_n \text{ and } (y,w) \in B\}.$$

Since $p[\pi_\infty(T^{\mathcal{N}_{n+1}})] = U_{[\mathfrak{Q}]_{\mathcal{I}}}$, by absoluteness,

$$\mathcal{M}_\infty[g] \models \exists w\ (z, x_n, w) \in p[\pi_\infty(T^{\mathcal{N}_{n+1}})].$$

Hence

$$\mathcal{N}_{n+1}[g] \models \exists w\ (z, x_n, w) \in p[T^{\mathcal{N}_{n+1}}].$$

Since $p[T^{\mathcal{N}_{n+1}}] \subseteq p[\pi_\infty(T^{\mathcal{N}_{n+1}})]$, there is $w \in \mathcal{N}_{n+1}[g]$ such that $(z, \oplus_{j \le n} x_j, w) \in U_{[\mathfrak{Q}]_{\mathcal{I}}}$. Moreover, By choice of z, for all $w' \in \mathcal{N}_{n+1}[g]$ such that $(z, \oplus_{j \le n} x_j, w') \in U_{[\mathfrak{Q}]_{\mathcal{I}}}$, there is $y \in \mathcal{N}_{n+1}[g]$ such that $y \equiv^* x$, w' is a nice real for $\langle \mathfrak{P}_{y_i} : i \le n+1 \rangle$. If $m = n+1$, then w' codes a reduction between $S^{*,\langle \mathfrak{P}_{y_i} : i \le n+1 \rangle}$ and $H^{\langle \mathfrak{P}_{y_i} : i \le n+1 \rangle}$. Since $\mathcal{N}_{n+1} \in S^{*,\mathfrak{Q}}(a)$, $\text{pro}(\langle \mathfrak{P}_{y_i} : i \le n+1 \rangle, b) = \text{pro}(\langle \mathfrak{P}_{x_i} : i \le n+1 \rangle, b) = \text{pro}(\mathfrak{Q}, b)$ whenever $b \in \text{Cone}(S^{*,\mathfrak{Q}}(a))$. Hence w' codes a reduction between $S^{*,\mathfrak{Q}}(b)$ and $H^{\mathfrak{Q}}(b)$ whenever $b \in S^{*,\mathfrak{Q}}(a)$.

If $m > n+1$, then w' codes a reduction on further extensions of $[\langle \mathfrak{P}_{y_i} : i \le n+1 \rangle]_{\mathcal{I}}$. This means there is $\langle w_i : n+1 < i \le m \rangle \in \mathcal{N}_m[g]$ such that

1. $(w', \oplus_{j \le n+2} x_j, w_{n+2}) \in U_{[\mathfrak{Q} \restriction n+2]_{\mathcal{I}}}$.

2. For all $n+1 < i < m$, $(w_i, \oplus_{j \le i+1} x_j, w_{i+1}) \in U_{[\mathfrak{Q} \restriction i+1]_{\mathcal{I}}}$.

Hence $\langle w \rangle^\frown \langle w_i : n+1 < i \le m \rangle$ verifies requirement 1 and 2 of coding a reduction on further extensions of $[\mathfrak{P}]_{\mathcal{I}}$. Moreover for all $\langle w'_i : n+1 < i \le m \rangle$ such that

1. $(w', \oplus_{j \le n+2} x_j, w'_{n+2}) \in U_{[\mathfrak{Q} \restriction n+2]_{\mathcal{I}}}$.

2. For all $n+1 < i < m$, $(w'_i, \oplus_{j \le i+2} x_j, w'_{i+1}) \in U_{[\mathfrak{Q} \restriction i+1]_{\mathcal{I}}}$.

w'_m codes a reduction between $S^{\langle \mathfrak{P}_{x_i} : i \leq m \rangle}$ and $H^{\langle \mathfrak{P}_{x_i} : i \leq m \rangle}$ above $S^{*,\mathfrak{Q}}(a)$. Since $\mathcal{N}_m \in S^{*,\mathfrak{Q}}(a)$, $\mathrm{pro}(\langle \mathfrak{P}_{x_i} : i \leq m \rangle, b) = \mathrm{pro}(\mathfrak{Q}, b)$ whenever $b \in \mathrm{Cone}(S^{*,\mathfrak{Q}}(a))$. Thus w'_m codes a reduction between $S^{*,\mathfrak{Q}}$ and $H^{\mathfrak{Q}}$ above $\mathrm{Cone}(S^{*,\mathfrak{Q}}(a))$. This finishes verifying property 2 of a nice real.

We now prove property 3. This is by induction on $\mathrm{final}(\mathfrak{Q})$. Let $a \in \mathrm{Cone}(z)$, $\mathfrak{Q} \in \mathcal{I}$. Let $\mathcal{K}$ be an S-premouse over a. Let $\mathfrak{Q} \in \mathcal{I}$ be such that $\mathrm{pro}(\mathfrak{Q}, \mathcal{K}) \leq^{\mathcal{I}}_{\mathcal{K}} \mathrm{pro}(\mathfrak{P}, \mathcal{K})$. Suppose $j : \bar{\mathcal{S}} \to S^{\mathfrak{Q}}(\mathcal{K})$ is Σ_1-elementary, $j(\mathfrak{Q}, \mathcal{H}) = (\mathfrak{Q}, \mathcal{K})$. We want to prove $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(\mathcal{H})$. If $\mathfrak{Q} \leq^{\mathcal{I}}_{\mathcal{K}} \mathrm{pro}(\mathfrak{P}[\epsilon], \mathcal{K})$ for some $\epsilon < \alpha_n$, then $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(\mathcal{H})$ follows from z being a nice real for $\mathfrak{P}[\epsilon]$. So let's assume $\mathfrak{Q} >^{\mathcal{I}}_{\mathcal{K}} \mathrm{pro}(\mathfrak{P}[\epsilon], \mathcal{K})$ for all $\epsilon < \alpha_n$. This means $\mathfrak{Q} \restriction n+1 \sim^{\mathcal{I}} \mathfrak{P}$, which allows us to use the property of coding a reduction on further extensions of $[\mathfrak{P}]_{\mathcal{I}}$. If $\mathfrak{Q}$ is a successor index, then by the property of coding a reduction on further extensions of $[\mathfrak{P}]_{\mathcal{I}}$, we can obtain effective maps $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^h_{\mathfrak{Q}} \urcorner$ and $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^s_{\mathfrak{Q}} \urcorner$ such that for all countable transitive swo $b \in \mathrm{Cone}(S^{*,\mathfrak{Q}}(a))$, for all $c_0, \ldots, c_k \in b$, for all Σ_1 formula $\phi(v_0, \ldots, v_k)$,

$$S^{*,\mathfrak{Q}}(b) \models \phi(c_0, \ldots, c_k) \leftrightarrow H^{\mathfrak{Q}}(b) \models \phi^h_{\mathfrak{Q}}(c_0, \ldots, c_k, z, \vec{\mathcal{N}}),$$

$$H^{\mathfrak{Q}}(b) \models \phi(c_0, \ldots, c_k) \leftrightarrow S^{*,\mathfrak{Q}}(b) \models \phi^s_{\mathfrak{Q}}(c_0, \ldots, c_k, z, \vec{\mathcal{N}}).$$

Here is a sketch of definition of $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^h_{\mathfrak{Q}} \urcorner$. Assume that

$$S^{*,\mathfrak{Q}}(b) \models \phi(c_0, \ldots, c_k).$$

Then for all $g \subseteq Coll(\omega, b)$ generic over both $S^{*,\mathfrak{Q}}(b)$ and $H^{\mathfrak{Q}}(b)$ such that $g(i) = c_i$ for all $i \leq k$,

$$S^{*,\mathfrak{Q}}(x_g) \models \text{“}\underbrace{S^{*,\mathfrak{Q}}(b) \models \phi(g(0), \ldots, g(k))}_{\text{call this } \phi_1}\text{”}.$$

Hence

$$\exists l \ \ulcorner \phi_1 \urcorner \in S^{\mathfrak{Q}}_l(x_g).$$

Let $\phi_2(z, \vec{\mathcal{N}})$ be the formula

"Let $\langle x_i : n < i \leq m \rangle$ be such that

1. For each $i \leq m$, $x_i \in \mathcal{N}_i[g]$. x_i codes $(\vec{\mathcal{T}}_{x_i}, \mathfrak{P}_{x_i})$. Denote $\mathfrak{P}_{x_i} = (\nu_{x_i}, \pi_{x_i}, \mathcal{P}_{x_i}, \alpha^*_{x_i}, \alpha_{x_i})$.
2. $\langle \mathfrak{P}_{x_i} : i \leq m \rangle \in \mathcal{I}$ as witnessed by $\langle \vec{\mathcal{T}}_{x_i} : i \leq m \rangle$.
3. $\langle \mathfrak{P}_{x_i} : i \leq m \rangle \sim^{\mathcal{I}} \mathfrak{Q}$.

Let $\langle w_i : n < i \leq m \rangle \in \mathcal{N}_m[g]$ be such that $(z, \oplus_{0<j\leq n} x_j, w_{n+1}) \in p[T^{\mathcal{N}_{n+1}}]$, and for all $n < i < m$, $(w_i, \oplus_{0<j\leq i+1} x_j, w_{i+1}) \in p[T^{\mathcal{N}_{i+1}}]$. Then w_m codes f_0, f_1, g_0, g_1, and there is $l \in \omega$ such that $\ulcorner \phi_1 \urcorner \in \{f_1(l)\}^{H^{\mathfrak{Q}}_{f_0(l)}(x_g)}$."

Then

$$H^{\mathfrak{Q}}(x_g) \models \phi_2(z, \vec{\mathcal{N}}).$$

Let $\phi^h_{\mathfrak{Q}}(v_0, \ldots, v_k, z, \vec{\mathcal{N}})$ be the formula

"for all $g \subseteq Coll(\omega, b)$ generic over $H^{\mathfrak{Q}}(b)$ such that $g(i) = v_i$ for all $i \leq k$, then $V[x_g] \models \phi_2(z, \vec{\mathcal{N}})$."

Then

$$H^{\mathfrak{Q}}(b) \models \phi^h_{\mathfrak{Q}}(c_0, \ldots, c_k, z, \vec{\mathcal{N}}).$$

In a similar way we can define the map $\ulcorner \phi \urcorner \mapsto \ulcorner \phi^s_{\mathfrak{Q}} \urcorner$. The same proof as in Lemma 2.20 gives that $S^{*,\mathfrak{Q}}$ has condensation above $S^{*,\mathfrak{Q}}(b)$. This, combined with a proof like Lemma 2.26, shows $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(\mathcal{H})$.

If $\mathfrak{Q}$ is a limit index of type A or B, then $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(\mathcal{H})$ by induction. If $\mathfrak{Q}$ is a limit index of type C, we need to show a stronger form of condensation. Let $\mathfrak{R} = \text{pro}(\mathfrak{Q}, \mathcal{K})$, $\vec{\mathcal{M}}$ be the $\mathcal{M}_1^{\#}$-sequence of $\mathfrak{R}$ for $\mathcal{K}$, $\bar{\mathfrak{R}} = \text{pro}(\mathfrak{R}, \mathcal{H})$, $\vec{\mathcal{N}}$ be the $\mathcal{M}_1^{\#}$-sequence of $\bar{\mathfrak{R}}$ for $\mathcal{H}$. Let $\mathcal{M}^* = \mathcal{M}_1^{\#, \Sigma_{\mathcal{R}_n(\mu^*)}}(\mathcal{M}_m)$, $\mathcal{N}^* = \mathcal{M}_1^{\#, \Sigma_{\bar{\mathcal{R}}_n(\mu^*)}}(\mathcal{N}_m)$.

$\mathcal{R}^* = \text{dirlim}_{\mathcal{K}}^{\mathcal{M}^*}(\mathcal{R}_n)$, $\bar{\mathcal{R}}^* = \text{dirlim}_{\mathcal{H}}^{\mathcal{N}^*}(\bar{\mathcal{R}}_n)$, The same proof as in case 3 shows that

$$j(\bar{\mathfrak{R}}, \vec{\mathcal{N}}, \mathcal{N}^*, \bar{\mathcal{R}}^*) = (\mathfrak{R}, \vec{\mathcal{M}}, \mathcal{M}^*, \mathcal{R}^*).$$

For every successor ϵ such that $\bar{\mathfrak{R}}^\frown\langle\bar{\mu}^*, \bar{\tau}^*, \bar{\mathcal{R}}^*, \epsilon, \epsilon\rangle \in \mathcal{I}$, $S^{\mathfrak{R}}(\mathcal{K})^{Coll(\omega,\mathcal{K})}$ satisfies the following:

" Let x be a real coding $\langle x_i : i \leq n+1\rangle$ such that $x_i \in \mathcal{M}_i[g]$ for all $i \leq n$, $x_{n+1} \in \mathcal{M}^*[g]$, $x_0 = \mathfrak{P}_0$, x_i codes $(\vec{\mathcal{T}}_{x_i}, \mathfrak{P}_{x_i})$, $\langle \mathfrak{P}_{x_i} : i \leq m+1\rangle \sim^{\mathcal{I}} \mathfrak{R}^\frown\langle\mu^*, \tau^*, \mathcal{R}^*, j(\epsilon), j(\epsilon)\rangle$. Then $p[(T^{\mathcal{N}^*})_{z,x}] \cap \mathcal{M}^*[g] \neq \emptyset$. For all $w \in p[(T^{\mathcal{M}^*})_{z,x}] \cap \mathcal{M}^*[g]$, w codes a reduction between $S^{\mathfrak{R}^\frown\langle\mu^*,\tau^*,\mathcal{R}^*,j(\epsilon),j(\epsilon)\rangle}(\mathcal{K})$ and $H^{\mathfrak{R}^\frown\langle\mu^*,\tau^*,\mathcal{R}^*,j(\epsilon),j(\epsilon)\rangle}(\mathcal{K})$"

So $\bar{\mathcal{S}}^{Coll(\omega,\mathcal{H})}$ satisfies the following:

" Let x be a real coding $\langle x_i : i \leq n+1\rangle$ such that $x_i \in \mathcal{N}_i[g]$ for all $i \leq n$, $x_{n+1} \in \mathcal{N}^*[g]$, $x_0 = \mathfrak{P}_0$, x_i codes $(\vec{\mathcal{T}}_{x_i}, \mathfrak{P}_{x_i})$, $\langle \mathfrak{P}_{x_i} : i \leq m+1\rangle \sim^{\mathcal{I}} \bar{\mathfrak{R}}^\frown\langle\bar{\mu}^*, \bar{\tau}^*, \bar{\mathcal{R}}^*, \epsilon, \epsilon\rangle$. Then $p[(T^{\mathcal{N}^*})_{z,x}] \cap \mathcal{N}^*[g] \neq \emptyset$. For all $w \in p[(T^{\mathcal{N}^*})_{z,x}] \cap \mathcal{N}^*[g]$, w codes a reduction between $j^{-1}(S^{\bar{\mathfrak{R}}^\frown\langle\bar{\mu}^*,\tau^*,\bar{\mathcal{R}}^*,\epsilon,\epsilon\rangle}(\mathcal{K}))$ and $j^{-1}(H^{\bar{\mathfrak{R}}^\frown\langle\bar{\mu}^*,\bar{\tau}^*,\bar{\mathcal{R}}^*,\epsilon,\epsilon\rangle}(\mathcal{K}))$."

By Lemma 2.42,

$$j^{-1}(H^{\mathfrak{R}^\frown\langle\mu^*,\tau^*,\mathcal{R}^*,\epsilon,\epsilon\rangle}(\mathcal{K})) = H^{\bar{\mathfrak{R}}^\frown\langle\bar{\mu}^*,\bar{\tau}^*,\bar{\mathcal{R}}^*,\epsilon,\epsilon\rangle}(\mathcal{H}).$$

Since for all $w \in p[(T^{\mathcal{N}^*})_{z,x}]$, w codes a reduction between $S^{\bar{\mathfrak{R}}^\frown\langle\bar{\mu}^*,\bar{\tau}^*,\bar{\mathcal{R}}^*,\epsilon,\epsilon\rangle}(\mathcal{H})$ and $H^{\bar{\mathfrak{R}}^\frown\langle\bar{\mu}^*,\bar{\tau}^*,\bar{\mathcal{R}}^*,\epsilon,\epsilon\rangle}(\mathcal{H})$,

$$j^{-1}(S^{\mathfrak{R}^\frown\langle\mu^*,\tau^*,\mathcal{R}^*,j(\epsilon),j(\epsilon)\rangle}(\mathcal{K})) = S^{\bar{\mathfrak{R}}^\frown\langle\bar{\mu}^*,\bar{\tau}^*,\bar{\mathcal{R}}^*,\epsilon,\epsilon\rangle}(\mathcal{H}).$$

Therefore, $\bar{\mathcal{S}} = S^{\mathfrak{P}}(c)$.

This finishes property 3. Hence z is a nice real for $\mathfrak{P}$. $\square$

From now on we fix a nice real z for $[\lambda^{\mathcal{P}}]$. We end this section with a key condensation result which will be crucial in the translation procedure of chapter 3. Its proof is essentially included in the proof of Theorem 2.45.

Theorem 2.46. *Suppose that $\mathfrak{P}$ is a promoted index over $\mathcal{K}$, $\mathfrak{Q}$ is a promoted index for $\mathcal{H}$, $\mathcal{K}$ is an S-premouse over a, $\mathcal{H}$ is an S-premouse over b, $a, b \in \mathrm{Cone}(z)$. Suppose that $j : \bar{\mathcal{S}} \to S^{\mathfrak{P}}(\mathcal{K})$ is Σ_1 elementary, $j(\mathcal{H}, \mathfrak{Q}) = (\mathcal{K}, \mathfrak{P})$. Then $\bar{\mathcal{S}} = S^{\mathfrak{Q}}(b)$.*

Chapter 3

The translation

In this chapter, we define a translation procedure that turns extenders into S-operators.

3.1 Defining the translation

Let $\mathcal{Q}$ be a Σ-good Σ-premouse over a such that $a \in \mathrm{Cone}(z)$. Let η be a cardinal of $\mathcal{Q}$ such that $\mathcal{Q}|\delta \prec \mathcal{Q}$. Let $\mathcal{N}_0 = L[\vec{E}]^{\mathcal{Q}|\eta}$. Suppose that η is Woodin in $(\mathcal{N}_0)_+$, $\mathcal{Q}|\eta$ is generic over $\mathcal{N}_+$ for $\mathbb{Q}_\eta$, the η-generators extender algebra at η. For $\mathcal{N}$ be a (ω_1, ω_1)-iterable premouse extending $(\mathcal{N}_0)_+$ such that η is Woodin in $\mathcal{N}$, let

$$\begin{aligned} U(\mathcal{N}, \eta) = \{\vec{E} = \langle E_i : i \leq n\rangle : & E_0 \text{ is on the } \mathcal{N}\text{-sequence}, \\ & E_{i+1} \text{ is on the } \mathrm{Ult}(\mathcal{N}, E_i)\text{-sequence}, \\ & \text{for all } i \leq n,\ E_i \text{ overlaps } \eta.\} \end{aligned}$$

let

$$P(\mathcal{N},\eta) = \{\mathcal{K} : \text{either } (\mathcal{N}_0)_+ \trianglelefteq \mathcal{K} \trianglelefteq \mathcal{N}, \\ \text{or there is } \vec{E} \in U(\mathcal{N},\eta) \text{ such that } (\mathcal{N}_0)_+ \trianglelefteq \mathcal{K} \trianglelefteq \mathrm{Ult}(\mathcal{N}, E_n)\}$$

Let $<^{P(\mathcal{N},\eta)}$ be the following binary relation on $P(\mathcal{N},\eta)$. For $\mathcal{K}_1, \mathcal{K}_2 \in P(\mathcal{N},\eta)$, $\mathcal{K}_1 <^{P(N,\eta)} \mathcal{K}_2$ just in case either

$$\mathcal{K}_1 \triangleleft \mathcal{K}_2$$

or

$$\text{there is } \vec{E} = \langle E_i : i \leq n\rangle \in U(\mathcal{K}_2,\eta) \text{ such that } \mathcal{K}_1 \trianglelefteq \mathrm{Ult}(\mathcal{N}, E_n).$$

Lemma 3.1. *$<^{P(\mathcal{N},\eta)}$ is a well-order on $P(\mathcal{N},\eta)$.*

Proof. Obviously, $<^{P(\mathcal{N},\eta)}$ is anti-reflexive.

$<^{P(\mathcal{N},\eta)}$ is total. Suppose that $\mathcal{K}, \mathcal{M} \in P(\mathcal{N},\eta)$, $\mathcal{K} \neq \mathcal{M}$. If both $\mathcal{K}$ and $\mathcal{M}$ are initial segments of $\mathcal{N}$, then one must be a proper initial segment of the other, so they are lined up under $<^{P(\mathcal{N},\eta)}$. If $\mathcal{K} \triangleleft \mathcal{N}$ but $\mathcal{M} \not\triangleleft \mathcal{N}$, let $\vec{F} = \langle F_i : i \leq m\rangle \in U(\mathcal{N},\eta)$ be such that $\mathcal{M} \trianglelefteq \mathrm{Ult}(\mathcal{N}, F_m)$ and $\langle lh(F_i) : i \leq m\rangle$ is lexicographically least with this property. Clearly $\langle lh(F_i) : i \leq m\rangle$ is a strict increasing sequence. If $\mathcal{N}||lh(F_0) \trianglelefteq \mathcal{K}$, then $\vec{F} \in U(\mathcal{K},\eta)$ witnesses that $\mathcal{M}<^{P(\mathcal{N},\eta)}\mathcal{K}$. If $\mathcal{K} \triangleleft \mathcal{N}||lh(F_0)$, then $\mathcal{K} \triangleleft \mathcal{M}$ since $\mathcal{M}|lh(F_0) = \mathcal{N}|lh(F_0)$. So $\mathcal{K}<^{P(\mathcal{N},\eta)}\mathcal{M}$. Assume then $\mathcal{K} \not\triangleleft \mathcal{N}$ and $\mathcal{M} \not\triangleleft \mathcal{N}$. Let $\vec{E} = \langle E_i : i \leq n\rangle \in U(\mathcal{N},\eta)$ be such that $\mathcal{K} \trianglelefteq \mathrm{Ult}(\mathcal{N}, E_n)$ and $\langle lh(E_i) : i \leq n\rangle$ is lexicographically least with this property. Let $\vec{F} = \langle F_i : i \leq m\rangle \in U(\mathcal{N},\eta)$ be such that $\mathcal{M} \trianglelefteq \mathrm{Ult}(\mathcal{N}, F_m)$ and $\langle lh(F_i) : i \leq m\rangle$ is lexicographically least with this property. Let k be maximal such that $\vec{E} \upharpoonright k = \vec{F} \upharpoonright k$. If both E_k and F_k are defined, assume wlog that $lh(E_k) < lh(F_k)$. Then $\mathcal{M}|lh(F_k) = \mathrm{Ult}(\mathcal{N}, E_{k-1})|lh(F_k)$. So $\langle E_k, \ldots, E_n\rangle \in$

$U(\mathcal{M},\eta)$, witnessing that $\mathcal{K} <^{P(\mathcal{N},\eta)} \mathcal{M}$. If E_k is not defined, but F_k is defined, again we have $\mathcal{M}|lh(F_k) = \mathrm{Ult}(\mathcal{N}, E_{k-1})|lh(F_k)$. If $\mathrm{Ult}(\mathcal{N}, E_{k-1}) \lhd \mathcal{K}$, then $\langle F_k, \ldots, F_m\rangle \in U(\mathcal{K},\eta)$ witnesses that $M <^{P(\mathcal{N},\eta)} K$. If $K \lhd \mathrm{Ult}(\mathcal{N}, E_{k-1})||lh(F_k)$, then $\mathcal{K} \lhd \mathcal{M}$. So $\mathcal{K} <^{P(\mathcal{N},\eta)} \mathcal{M}$.

$<^{P(\mathcal{N},\eta)}$ is transitive. Assume that $\mathcal{K}_1 <^{P(\mathcal{N},\eta)} \mathcal{K}_2$, $\mathcal{K}_2 <^{P(\mathcal{N},\eta)} \mathcal{K}_3$ as witnessed by $\vec{E} \in U(\mathcal{K}_2,\eta)$, $\vec{F} \in U(\mathcal{K}_3,\eta)$ respectively, then $\vec{F}^\frown\vec{E} \in U(\mathcal{K}_3,\eta)$ witnesses that $\mathcal{K}_1 <^{P(\mathcal{N},\eta)} \mathcal{K}_3$.

$<^{P(\mathcal{N},\eta)}$ is wellfounded because $\mathcal{N}$ is iterable. □

Let $g \subseteq \mathbb{Q}_\eta$ be the natural $\mathcal{N}$-generic filter which codes $\mathcal{Q}|\eta$. Let Q, D be easily definable functions such that $Q(g) = \mathcal{Q}^\infty_{\mathcal{N}_0|\eta}$, $D(g) = \pi^\infty_{\mathcal{N}_0|\eta}$, where $\pi^\infty_{\mathcal{N}_0|\eta} : \mathcal{P} \to \mathcal{Q}^\infty_{\mathcal{N}_0|\eta}$ is the direct limit map of $I(\mathcal{P},\Sigma) \cap \mathcal{N}_0|\eta$.

Definition 3.2. Tr^g is a function on $P(\mathcal{N},\eta)$ defined by induction on $<^{P(\mathcal{N},\eta)}$.

1. If $\mathcal{K} = (\mathcal{N}_0)_+$, then $Tr^g(\mathcal{K}) = \langle |\mathcal{K}|[g], \in, g, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle$.
2. If $\mathcal{K} = \mathfrak{N}(\mathcal{M})$, then $Tr^g(\mathcal{K}) = \mathfrak{N}(Tr^g(\mathcal{M}))$.
3. If $o(\mathcal{K})$ is a limit, $\mathcal{K}$ is passive, then $Tr^g(\mathcal{K}) = \bigsqcup_{\eta < o(\mathcal{K})} Tr^g(\mathcal{K}|\eta)$.
4. If $\mathcal{K}$ is active with top extender E, $\mathrm{crt}(E) > \eta$, let $E[g]$ be the canonical extension of E to the generic. Let $Tr^g(\mathcal{K})$ be $\bigsqcup_{\eta < o(\mathcal{K})} Tr^g(\mathcal{K}|\eta)$ but adding the top extender $E[g]$.
5. If $\mathcal{K}$ is active with top extender E, $\mathrm{crt}(E) < \eta$, let $Tr^g(\mathrm{Ult}((\mathcal{N}_0)_+, E)) = \langle J^{\vec{E},S}_\alpha[g], \in, g, \vec{E}, \emptyset, S, \emptyset, \emptyset, \emptyset\rangle$. Let d be the last drop of $Tr^g(\mathrm{Ult}(\mathcal{N}_0, E))$. Then $Tr^g(\mathcal{K})$ is the e- amenable code of transitive collapse of the hull of $d \cup i_E \circ D(g)$ over

$$\langle J^{\vec{E},S}_\alpha[g], \in, g, \vec{E}, \emptyset, S, Tr^g(\mathrm{Ult}((\mathcal{N}_0)_+, E))|d, i_E(Q(g)), i_E \circ D(g)\rangle.$$

The translation generalizes the one in [19]. Details of a similar translation in a different context is carried out in [1]. The crucial part is in case 5. We translate an extender overlapping η into an S-operator. Although we develop the translation in an abstract manner without relevance to the S-operators, we are only interested in cases when $Tr^g(\mathcal{K})$ is a mixed S-premouse. So suppose $Tr^g(\mathrm{Ult}(\mathcal{N}, E))$ has largest drop d, and suppose $\mathfrak{P}$ is the degree at d. This means we have reached a maximal $L[\vec{E}, S^{\mathfrak{P}}]$ model over $Tr^g(\mathrm{Ult}(\mathcal{N}, E))$, and we aim to define $Tr^g(\mathcal{K}) = S^{\mathfrak{P}+1}(Tr^g(\mathrm{Ult}(\mathcal{N}, E))|d)$, feeding in some new information and thus raising the degree by one bit. Before proceed into the detailed proof of the interdefinability of the translation, let's sketch why E is recoverable from $Tr^g(\mathcal{K})$ in case 5. Suppose we have obtained $\mathcal{H}$ such that $Tr^g(\mathcal{H})$ is equal to $Tr^g(\mathcal{K})$ without top S-predicate. We may recover E by

$$(A, s) \in E$$

if and only if for some n, $A \subseteq [\kappa]^n$, $s \in [o(\mathcal{K})]^n$, and

$$\text{there is } B \in Hull^{\mathcal{H}}(\kappa \cup \mathrm{ran}(\pi^{Tr^g(\mathcal{K})})) \text{ such that } s \in B \text{ and } B \cap [\kappa]^n = A.$$

It relies on the following fact.

Let S_κ be the transitive collapse of $Hull^{(\mathcal{N}_0)+}(\kappa \cup \pi_{\mathcal{N}}^{\infty})$. Then $\mathcal{P}(\kappa)^{\mathcal{N}} \subseteq S_\kappa$.

This is an important fact about $\mathcal{N}$, being the full $L[\vec{E}]$ construction inside a suitable Σ-premouse. We will prove this fact in Section 3.4. For the mean time, let's grant this fact, and develop basic properties of the translation. Given $(A, s) \in E$, $A \subseteq [\kappa]^n$, $s \in [o(\mathcal{K})]^n$, since $\mathcal{P}(\kappa)^{\mathcal{N}} \subseteq S_\kappa$, there is a Skolem term τ, an ordinal c and $a \in \mathrm{ran}\, \pi_{\mathcal{N}}^{\infty} = D(g)$ such that

$$A = \tau^{(\mathcal{N}_0)+}(c, a) \cap [\kappa]^n.$$

Let $k : Tr^g(\mathcal{K}) \to \langle J_\alpha^{\vec{E},S}[g], \in, g, \vec{E}, \emptyset, S, Tr^g(\text{Ult}((\mathcal{N}_0)_+, E))|d, i_E(Q(g)), i_E \circ D(g)\rangle$ be the uncollapse map. Let $\nu = \text{crt}(k)$. Then we can show $k \restriction \mathcal{H} : \mathcal{H} \to \text{Ult}((\mathcal{N}_0)_+, E)$ is the restriction of k. Note $i_E(a) \in i_E \circ D(g) \subseteq \text{ran}\, k$. Let $B = \tau^{\mathcal{H}}(c, k^{-1}(i_E(a)))$, then $B \cap [\nu]^n = k(B) \cap [\nu]^n = \tau^{\text{Ult}((\mathcal{N}_0)_+,E)}(c, i_E(a)) \cap [\nu]^n = i_E(A) \cap [\nu]^n$. Hence $B \cap [\kappa]^n = A$. Moreover, $s \in i_E(A)$. So $s \in B$ once we have $lh(E) \leq \nu$. However, we always have $\nu(E) \leq \nu$. This is proved in three steps. Firstly, if μ is a generator of E but not the largest, then $d \geq \mu$. Otherwise, let ξ index $E \restriction \mu + 1$ on $\mathcal{K}$. Then $\text{Ult}((\mathcal{N}_0)_+, E \restriction \mu + 1)$ embeds into $\text{Ult}((\mathcal{N}_0)_+, E)$ with critical point $> \mu$. The embedding extends to $l : \text{Ult}((\mathcal{N}_0)_+, E \restriction \mu + 1)[g] \to \text{Ult}((\mathcal{N}_0)_+, E)[g]$, or $l : Tr^g(\text{Ult}((\mathcal{N}_0)_+, E \restriction \mu + 1)) \to Tr^g(\text{Ult}((\mathcal{N}_0)_+, E))$. Let $\mathfrak{P} = \deg(Tr^g(\text{Ult}((\mathcal{N}_0)_+, E)|d)$. Then $l^{-1}(\mathfrak{P}, d) = (\mathfrak{P}, d)$. That means, $Tr^g(\text{Ult}((\mathcal{N}_0)_+, E \restriction \mu + 1))$ has largest drop d whose degree is $\mathfrak{P}$. By definition, $Tr^g(K|\xi)$ reaches degree $\mathfrak{P} + 1$ at d. Hence $Tr^g(\text{Ult}((\mathcal{N}_0)_+, E))$ reaches degree $\mathfrak{P} + 1$ at d. Contradiction! Secondly, if $\mu = \nu(E \restriction (\nu(E) - 1))$, then $(\mu^+)^{\text{Ult}((\mathcal{N}_0)_+,E)} > \nu(E)$. This is because $E \restriction \nu(E) - 1 \in \text{Ult}((\mathcal{N}_0)_+, E)$. Hence $\text{Ult}((\mathcal{N}_0)_+, E)$ has a surjection from μ onto $\nu(E) - 1$. It follows then $(d^+)^{Tr^g(\text{Ult}((\mathcal{N}_0)_+,E))} \geq lh(E)$. Finally, we always have $(d^+)^{Tr^g(\text{Ult}((\mathcal{N}_0)_+,E))} \leq \nu$. This is a property about the S-operator, namely, $SSM^{\mathfrak{P}}(Tr^g(\text{Ult}((\mathcal{N}_0)_+, E))|d) \subseteq S^{\mathfrak{P}+1}(Tr^g(\text{Ult}((\mathcal{N}_0)_+, E))|d)$. Again, we grant this fact before proving interdefinability. So far we are done with one direction. For the opposite direction, suppose $(A, s) \in E$, $A \subseteq [\kappa]^n$, $s \in [o(\mathcal{K})]^n$, and $B \in \tau^{\mathcal{H}}(c, a)$ for some $c < \kappa$, $a \in \text{ran}(\pi^{Tr^g(\mathcal{K})})$ such that $s \in B$ and $B \cap [\kappa]^n = A$. Then $k(B) = \tau^{\text{Ult}((\mathcal{N}_0)_+,E)}(c, k(a)) \in \text{ran}(i_E)$. Hence $i_E^{-1}(k(B)) \cap [\kappa]^n = B \cap [\kappa]^n = A$. Hence $s \in B \cap [o(\mathcal{K})]^n = k(B) \cap [o(\mathcal{K})]^n = i_E(A) \cap [o(\mathcal{K})]^n$. Hence $(A, s) \in E$.

Lemma 3.3. *Suppose that $\mathcal{N}$ is an iterable premouse extending $(\mathcal{N}_0)_+$. Assume that for any $\vec{E} = \langle E_i : i \leq n\rangle$, letting $\kappa = \text{crt}(E_n)$, then*

1. *$\mathcal{P}(\kappa) \cap \mathcal{N} \subseteq S_\kappa$.*

2. *If* $Tr^g(\mathrm{Ult}(\mathcal{N}_0, E))$ *is defined, then it has a drop. Let* d *be the largest drop of* $Tr^g(\mathrm{Ult}(\mathcal{N}, E))$*, then* $(d^+)^{Tr^g(\mathrm{Ult}(\mathcal{N},E))} \subseteq Hull^{Tr^g(\mathrm{Ult}(\mathcal{N},E))}(d \cup i''_E D(g))$*.*

Then for all $\mathcal{M} \in P(\mathcal{N}, \eta)$*,*

1. $Tr^g(\mathcal{M})$ *is defined.* $Tr^g(\mathcal{M}) \in M[g]_+$*. If, in addition,* $\mathcal{M} \models ZFC^-$ *is either passive or* $\mathrm{crt}(F^{\mathcal{M}}) > \eta$*, then* $Tr^g(\mathcal{M})$ *and* $\mathcal{M}[g]$ *have the same universe.*

2. $|\mathcal{M}|$*,* $\dot{E}^{\mathcal{M}}$ *and* $\dot{F}^{\mathcal{M}}$ *are uniformly definable classes in* $Tr^g(\mathcal{M})$*. More precisely, there are formulas* $\phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot, \cdot), \phi_4(\cdot)$ *such that*

$$u \in |\mathcal{M}| \leftrightarrow Tr^g(\mathcal{M}) \models \phi_1(u),$$
$$u \in \dot{E}^{\mathcal{M}} \leftrightarrow Tr^g(\mathcal{M}) \models \phi_2(u),$$

if $\mathcal{M}$ *has top extender* F *with* $\mathrm{crt}(F) = \kappa < \eta$*, then*

$$u \in \dot{F}^{\mathcal{M}} \leftrightarrow Tr^g(\mathcal{M}) \models \phi_3(u, \kappa),$$

Otherwise,

$$u \in \dot{F}^{\mathcal{M}} \leftrightarrow Tr^g(\mathcal{M}) \models \phi_4(u),$$

Proof. $\phi_1, \ldots, \phi_4$ defines the backward translation from $Tr^g(\mathcal{M})$ into $\mathcal{M}$. Let $TrInv(V|\gamma) = u$ be the formula

there is a sequence of premice $\langle \mathcal{K}_\alpha : o((\mathcal{N}_0)_+) \leq \alpha \leq \gamma \rangle$ such that

1. $\mathcal{K}_{o((\mathcal{N}_0)_+)} = (\mathcal{N}_0)_+$,

2. $\mathcal{K}_{\alpha+1} = \mathfrak{N}(\mathcal{K}_\alpha)$,

3. If α is a limit ordinal, there is no $(x, y) \in \dot{S}$ such that $o(y) = \alpha$, then $\mathcal{K}_\alpha = \bigsqcup\{\mathcal{K}_\beta : \beta \in I^{V|\alpha}\} \oplus (E_\alpha \restriction \bigsqcup\{\mathcal{K}_\beta : \beta \in I^{V|\alpha}\})$.

4. If there is (x, y) such that $o(y) = \alpha$, then $\mathcal{K}_\alpha = \mathcal{K}_\beta \oplus G$, where $\beta = \sup I^y$, and G is unique such that for some $\mu < o(\mathcal{N})$,

   (a) G is a $(\mu, o(\mathcal{K}_\beta))$-extender over $\mathcal{K}_\beta$ as defined by

   $$(A, s) \in G$$

   if and only if for some n, $A \subseteq [\mu]^n$, $s \in [o(\mathcal{K}_\beta)]^n$, and

   $$\text{there is } B \in Hull^{\mathcal{H}}(\mu \cup \operatorname{ran}(\pi^y)) \text{ such that } s \in B \text{ and } B \cap [\mu]^n = A.$$

   where $\mathcal{H} = \bigsqcup\{\mathcal{K}_\beta : \beta < \alpha\}$ for some sufficiently big $\beta < \alpha$.

   (b) $\mathcal{H}$ embeds into $\operatorname{Ult}(N_1, G)$ with critical point $\geq o(\mathcal{K})$ such that $\operatorname{ran} \pi^y$ is sent pointwise to $i''_G D(g)$.

5. $u = \mathcal{K}_\gamma$.

Clearly, if $\langle \mathcal{K}_\alpha, \mathcal{K}_\alpha : o((\mathcal{N}_0)_+) \leq \alpha \leq \gamma \rangle$ and u as above exist, then they are unique. Hence the definition makes sense. We let $\phi_1(u)$ be the formula

$$\exists \gamma \in I^V \exists n < \omega \; u \in S_n(TrInv(V|\gamma)).$$

$\phi_2(u)$ be the formula

$$\exists \gamma \in I^V \;\; u \in E^{TrInv(V|\gamma)} {}^\frown \langle F^{TrInv(V|\gamma)} \rangle.$$

$\phi_3(u, v)$ be the formula

$$\begin{gathered} \text{Let } \gamma = \sup I^V. \text{ Then for some } n,\ (u)_0 \subseteq [v]^n, (u)_1 \in [\gamma]^n \text{ and} \\ \exists X \in \dot{\pi} \exists B \in Hull^{V^-}(\dot{b} \cup B)((u)_1 \in B \wedge B \cap [v]^n = (u)_0). \end{gathered}$$

(Recall that V^- is the reduct of V removing the predicate $\dot{\pi}$.) ϕ_4 be the formula

$$x \in \dot{F}.$$

We prove the lemma by induction on $<^{P(\mathcal{N},\eta)}$.

Case 1. $\mathcal{M} = (\mathcal{N}_0)_+$.

By definition, $Tr^g(\mathcal{M})$ and $\mathcal{M}[g]$ have the same universe. Trivially, ϕ_1, ϕ_2, ϕ_4 defines $\mathcal{M}$, $E^{\mathcal{M}}$, $F^{\mathcal{M}}$ over $Tr^g(\mathcal{M})$.

Case 2. $\mathcal{M} = \mathfrak{N}(\mathcal{K})$ for some $\mathcal{K}$.

We show that ϕ_1, ϕ_2, ϕ_4 correctly defines $|\mathcal{M}|$, $E^{\mathcal{M}}$, $F^{\mathcal{M}}$ over $Tr^g(\mathcal{M})$. We have, by induction hypothesis, for each $\mathcal{H} <^{P(\mathcal{N},\eta)} \mathcal{K}$,

$$Tr^g(\mathcal{K}) \models TrInv(Tr^g(\mathcal{H})) = \mathcal{H}.$$

Hence

$$Tr^g(\mathcal{M}) \models TrInv(Tr^g(\mathcal{H})) = \mathcal{H}.$$

If $\mathcal{K} = \mathfrak{N}(\mathcal{H})$ for some $\mathcal{H}$, then, $\vec{\mathcal{H}} = \langle TrInv(Tr^g(\mathcal{M})_\alpha) : \xi_0 \leq \alpha \leq o(Tr^g(\mathcal{H}))\rangle$ witnesses $TrInv(Tr^g(\mathcal{H})) = \mathcal{H}$. Hence $\vec{\mathcal{H}}^\frown\langle Tr^g(\mathcal{K})\rangle$ witnesses $TrInv(Tr^g(\mathcal{K})) = \mathcal{K}$ inside $Tr^g(\mathcal{M})$, simply because $Tr^g(\mathcal{K}) = \mathfrak{N}(Tr^g(\mathcal{H}))$.

If $\mathcal{K}$ is of limit level, we show that $I^{Tr^g(\mathcal{K})} = \{o(Tr^g(\mathcal{H})) : \mathcal{H} \trianglelefteq \mathcal{K}\}$. We first observe that for every $(x, y) \in S^{Tr^g(\mathcal{K})}$, suppose (x, y) comes from the extender G, then $gen^{Tr^g(\mathcal{K})|o(y)} = gen^G \setminus \eta$. Hence $\nu^y = \nu(G)$. Now fix an $\mathcal{H} \triangleleft \mathcal{K}$, we show $o(Tr^g(\mathcal{H})) \in I^{Tr^g(\mathcal{K})}$. Suppose toward a contradiction that for some $(x, y) \in S^{Tr^g(\mathcal{K})}$, $(\nu^+)^{Tr^g(\mathcal{K})|o(y)} < o(Tr^g(H)) < o(y)$. Suppose (x, y) comes from the extender E_n with $\langle E_0, \ldots, E_n\rangle \in U(\mathcal{N}, \eta)$, $lh(E_0) < \cdots < lh(E_n)$. Then $lh(E_n) = (\nu^+)^{Tr^g(\mathcal{K})|o(y)}$. But $o(Tr^g(\mathcal{H})) < o(y)$ implies $o(\mathcal{H}) < lh(E_0)$. Contradiction. On the other hand, if $\xi \neq o(Tr^g(\mathcal{H}))$ for any $\mathcal{H} \triangleleft \mathcal{K}$, letting $\mathcal{H}_0$ be the least initial segment of $\mathcal{K}$ such that $\xi < o(Tr^g(\mathcal{H}_0))$, then $\mathcal{H}_0$ has a top extender G such that $\mathrm{crt}(G) < \eta$. Thus for some x, $(x, Tr^g(\mathcal{H}_0)) \in S^{Tr^g(\mathcal{K})}$. Then $lh(\mathcal{H}_0) = (\nu^+)^{Tr^g(\mathcal{H}_0} < o(Tr^g(H)) < o(Tr^g(\mathcal{H})_0)$. This means $\xi \notin I^{Tr^g(\mathcal{K})}$.

The fact $I^{Tr^g(\mathcal{K})} = \{o(Tr^g(\mathcal{H})) : \mathcal{H} \trianglelefteq \mathcal{K}\}$, together with induction hypothesis, implies that

$$Tr^g(\mathcal{M}) \models TrInv(Tr^g(\mathcal{K})) = \mathcal{K}.$$

when $\mathcal{K}$ is either passive or active with $\mathrm{crt}(F^{\mathcal{K}}) > \eta$. One simply traces through clause 3 of definition of $TrInv$.

When $\mathcal{K}$ is active with $\mathrm{crt}(F^{\mathcal{K}}) = \kappa < \eta$, we have $\phi_3(\cdot, \kappa)$ correctly defines $F^{\mathcal{K}}$ over $Tr^g(\mathcal{K})$. The definition of ϕ_3 fits into clause 4 of definition $TrInv(Tr^g(\mathcal{K}))$, except uniqueness of μ. We present the uniqueness proof here. Suppose there happens to be another $G \neq F^{\mathcal{K}}$ such that clause 4(a)(b) defines a $(\mu, o(\mathcal{K}))$-extender G over $\mathcal{K}$, and, replacing the top extender of $\mathcal{K}$ with G, we also get a premouse. Then $\mu \neq \kappa$. Assume wlog $\mu < \kappa$. Then $G \restriction \kappa \in \mathcal{N}$ by initial segment condition. On the other hand, we have an natural embedding $k : \mathrm{Ult}((\mathcal{N}_0)_+, G \restriction \kappa) \to \mathrm{Ult}((\mathcal{N}_0)_+, G)$ and an embedding $l : \mathcal{H} \to \mathrm{Ult}((\mathcal{N}_0)_+, G)$ such that $l \restriction o(\mathcal{K}) = id$, $l''\pi^{Tr^g(\mathcal{K})} = i''_G \operatorname{ran} \pi^\infty_{\mathcal{N}'}$. Hence $k \circ l^{-1} : \mathrm{Ult}((\mathcal{N}_0)_+, G) \to \mathcal{H}$ is identity on κ and sends $i''_{G\restriction\kappa} \operatorname{ran} \pi^\infty_{\mathcal{N}}$ to $\pi^{Tr^g(\mathcal{K})}$ pointwise. Since $\mathcal{P}(\kappa)^{\mathcal{N}} \subseteq S_\kappa$, every subset A of κ can be written as $A = \tau^{(\mathcal{N}_0)_+}(c, a) \cap \kappa$ for some Skolem term τ, ordinal $c < \kappa$ and $a \in \operatorname{ran} \pi^\infty_{\mathcal{N}}$. Let $j : (\mathcal{N}_0)_+ \to \mathcal{H}$ be the embedding coming from taking an Skolem hull of $\mathrm{Ult}((\mathcal{N}_0)_+, F^{\mathcal{K}})$. Then $j(A) = \tau^{\mathcal{H}}(c, j(a))$. But $j(a) \in \pi^{Tr^g(\mathcal{K})} \subseteq \mathrm{ran}(k \circ l^{-1})$. Hence $(k \circ l^{-1} \circ j)(A) \cap \kappa = A \in \mathrm{Ult}(\mathcal{N}, G \restriction \kappa)$. Thus $\mathcal{P}(\kappa)^{\mathcal{N}} \subseteq \mathrm{Ult}(\mathcal{N}, G \restriction \kappa)$. In particular, $G \restriction \kappa \in \mathrm{Ult}(\mathcal{N}, G \restriction \kappa)$. Contradiction.

Case 3. $\mathcal{M}$ is limit level, either $\mathcal{M}$ is passive or $\mathrm{crt}(F^{\mathcal{M}}) > \eta$.

Correctness of $TrInv$ and ϕ_1, ϕ_2, ϕ_4 over $Tr^g(\mathcal{M})$ is essentially shown in Case 2. We show that if $\mathcal{M} \models ZFC^-$, then $Tr^g(\mathcal{M})$ and $\mathcal{M}[g]$ have the same universe. On one hand, for each $\mathcal{K} \lhd \mathcal{M}$, $Tr^g(\mathcal{K}) \in \mathcal{M}[g]$ since the translation can be carried out in $M[g]$. Hence $|Tr^g(M)| \subseteq |\mathcal{M}[g]|$. On the other hand, for each $\mathcal{K} \lhd \mathcal{M}$, $\mathcal{K}$ is definable over $Tr^g(\mathcal{K})$. Hence $\mathcal{K} \in \mathfrak{N}(Tr^g(\mathcal{K})) = Tr^g(\mathfrak{N}(\mathcal{K})) \subseteq Tr^g(\mathcal{M})$. Hence $|\mathcal{M}[g]| \subseteq |Tr^g(\mathcal{M})|$.

Case 4. $\mathrm{crt}(F^{\mathcal{M}}) = \kappa < \eta$.

Correctness of $TrInv$ and ϕ_1, ϕ_2 over $Tr^g(\mathcal{M})$ is essentially shown in Case 2. The fact $\phi_3(\cdot, \kappa)$ defines $F^{\mathcal{M}}$ over $Tr^g(\mathcal{M})$ is follows from the discussion before Lemma 3.3. If one goes through the argument there, the two assumptions of this lemma comes into play. $\square$

3.2 Fine structure of potential S-premouse

According to Lemma 3.3, $\mathcal{K}$ is always a definable class over $Tr^g(\mathcal{K})$. We will show that the projecta and standard parameters of $\mathcal{K}$ and$Tr^g(\mathcal{K})$ are equal modulo g. So we are left to show the other direction, translating Σ_1-facts of $Tr^g(\mathcal{K})$ to that of $\mathcal{K}$. We wish to encode an isomorphic copy of $Tr^g(\mathcal{K})$ inside $\mathcal{K}[g]$. Sections 3.2 and 3.3 is adaption of[1] to the present context.

Definition 3.4. Fix $\mathcal{M} \in P(\mathcal{N}, \eta)$. We define Shoenfield terms A^ξ, $\sim^\xi$, ϵ^ξ, E^ξ, F^ξ, S^ξ, b^ξ, $\mathcal{Q}^\xi$, $\pi^\xi, I^\xi, \mu^\xi, \nu^\xi, \gamma^\xi$, for each $o((\mathcal{N}_0)_+) \leq \xi \leq o(\mathcal{M})$ by induction on ξ. We will ensure that for any $h \subseteq \mathbb{Q}_\eta$ generic over M, $\sim^\xi_h$ is an equivalence relation on A^ξ_h, $\epsilon^\xi_h, E^\xi_h, F^\xi_h, S^\xi_h, b^\xi_h$, $\mathcal{Q}^\xi_h, \pi^\xi_h, I^\xi_h, \mu^\xi_h, \nu^\xi_h, \gamma^\xi_h$ are relations of an appropriate arity on A^ξ_h that are $\sim^\xi_h$-invariant. Let $\mathfrak{A}^\xi_h$ be the transitive collapse of the structure $\langle A^\xi_h/\sim^\xi_h, \epsilon^\xi_h/\sim^\xi_h, E^\xi_h/\sim^\xi_h, F^\xi_h/\sim^\xi_h, S^\xi_h/\sim^\xi_h, b^\xi_h/\sim^\xi_h, \mathcal{Q}^\xi_h/\sim^\xi_h, \pi^\xi_h/\sim^\xi_h, I^\xi_h/\sim^\xi_h, \mu^\xi_h/\sim^\xi_h, \nu^\xi_h/\sim^\xi_h, \gamma^\xi_h/\sim^\xi_h\rangle$ and let $\mathfrak{u}^\xi_h$ be the collapsing map.

1. For $\xi = o((\mathcal{N}_0)_+)$, let $A^\xi = (\mathcal{N}_0)_+^{Coll(\omega,\eta)}$. $\sim^\xi, \epsilon^\xi$ are standard $\mathbb{Q}_\eta$-names such

that for any $h \subseteq \mathbb{Q}_\eta$ generic over $\mathcal{M}$,

$$\begin{aligned} \sim_h^\xi &= \{(x, x) : x \in (\mathcal{N}_0)_+[h]\} \\ \epsilon_h^\xi &= \{(x, y) : x \in y, y \in (\mathcal{N}_0)_+[h]\}. \end{aligned}$$

We define $E^\xi = F^\xi = S^\xi = b^\xi = \mathcal{Q}^\xi = \pi^\xi = I^\xi = \mu^\xi = \nu^\xi = \gamma^\xi = \emptyset$.

2. Suppose $\mathcal{M}|\xi$ is either of successor level, or of limit level but passive or $\mathrm{crt}(E_\xi^{\mathcal{M}}) > \eta$. A^ξ is the standard $\mathbb{Q}_\eta$-name such that for any $h \subseteq \mathbb{Q}_\eta$ generic over $\mathcal{M}$,

$$A_h^\xi = \{(\xi_1, \ulcorner f \urcorner, x) : \xi_1 < \xi, x \in A^\xi, f \text{ codes a binary rudimentary function}\}$$

$\sigma^\xi, \mathfrak{u}^\xi$ are standard $\mathbb{Q}_\eta$-names such that for any $h \subseteq \mathbb{Q}_\eta$ generic over $\mathcal{M}$,

$$\begin{aligned} \sim_h^\xi = \{((\xi_1, \ulcorner f \urcorner, x), (\xi_2, \ulcorner f' \urcorner, x')) : f(\mathfrak{u}_h^{\xi_1}(x), \mathfrak{A}_h^{\xi_1}) = f'(\mathfrak{u}_h^{\xi_2}(x'), \mathfrak{A}_h^{\xi_2})\}, \\ \mathfrak{u}_h^\xi(\xi_1, \ulcorner f \urcorner, x) = f(\mathfrak{u}_h^{\xi_1}(x), \mathfrak{A}_h^{\xi_1}) \end{aligned}$$

We will ensure that $\mathfrak{u}_h^\xi$ is the transitive collapsing map associated to $A_h^\xi / \sim_h^\xi$. $\epsilon_h^\xi, E_h^\xi, F_h^\xi, S_h^\xi, b_h^\xi, \mathfrak{P}_h^\xi, \mathcal{Q}_h^\xi, \pi_h^\xi, H_h^\xi, \mu_h^\xi, \nu_h^\xi, \gamma_h^\xi$ are standard $\mathbb{Q}_\eta$-names such that for any $h \subseteq \mathbb{Q}_\eta$ generic over $\mathcal{M}$,

$$\epsilon_h^\xi = \{(X,Y) : \mathfrak{u}_h^\xi(X) \in \mathfrak{u}_h^\xi(Y)\}$$
$$E_h^\xi = \{(\xi_1, \ulcorner f \urcorner, x) : \mathfrak{u}_h^\xi(\xi_1, \ulcorner f \urcorner, x) \in E_h^{\xi_2} {}^\frown \langle F_h^{\xi_2} \rangle \text{ for some } \xi_2\}$$
$$F_h^\xi = \{X : \mathfrak{u}_h^\xi(X) \in F[h]\}$$
$$S_h^\xi = \{(\xi_1, \ulcorner f \urcorner, x) : \text{ for some } \xi_2, \text{ either } \mathfrak{u}_h^\xi(\xi_1, \ulcorner f \urcorner, x) \in S_h^{\xi_2}, \text{ or } \operatorname{crt}(E_{\xi_2}^{\mathcal{M}}) < \eta, \mathfrak{u}_h^\xi(\xi_1, \ulcorner f \urcorner, x) = \mathfrak{A}_h^{\xi_2}\}$$
$$b_h^\xi = \mathcal{Q}_h^\xi = \pi_h^\xi = \{X : \mathfrak{u}_h^\xi(X) = \emptyset\}$$
$$I_h^\xi = \{(\xi_1, \ulcorner f \urcorner, x) : \mathfrak{u}_h^\xi(\xi_1, \ulcorner f \urcorner, x) = \mathfrak{A}_h^{\xi_2} \text{ for some } \xi_2 \leq \xi_1\}$$
$$\mu_h^\xi = \{X : \mathfrak{u}_h^\xi(X) = \mu^{\mathcal{M}|\xi}\}$$
$$\nu_h^\xi = \{X : \mathfrak{u}_h^\xi(X) = \nu^{\mathcal{M}|\xi}\}$$
$$\gamma_h^\xi = \{X : \mathfrak{u}_h^\xi(X) = \gamma^{\mathcal{M}|\xi}\}$$

3. Suppose $E_\xi^{\mathcal{M}} < \eta$. Set $F = \dot{E}_\xi^{\mathcal{M}}$, $\kappa = \operatorname{crt}(F)$. Let $[r, x]$ represent $\mathbb{Q}_\eta$ in the ultrapower. Then A^ξ is such a $Coll(\omega, \eta)$-name: for any $\mathbb{Q}_\eta$-generic h, A_h^ξ is the set of $(p, (\ulcorner \tau \urcorner, n, a, f)\check{})$ such that
   (a) $p \in \mathbb{Q}_\eta$,
   (b) τ is a Skolem term,
   (c) $n < \omega$,
   (d) $a \in [lh(F)]^{<\omega}$,
   (e) $f \in \mathcal{M}$ is a function from $\kappa^{|a|}$ to $o((\mathcal{N}_0)_+)$,
   (f) Let $[s, y]$ represent p in the ultrapower. Then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{|a\cup r\cup s|}$, $(\mathcal{N}_0)_+$ satisfies the following: $y(u^s_{a\cup r\cup s})$ forces over $x(u^r_{a\cup r\cup s})$ that letting

$\dot{g}$ be the standard $x(u^r_{a\cup r\cup s})$-name for the generic, then

$$Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}f(u^a_{a\cup r\cup s}) < \text{my largest drop”}$$

$\sim^\xi$ is the set of standard names for ordered pairs $((p, (\ulcorner\tau\urcorner, n, a, f)\check{}\,),$ $(p, (\ulcorner\sigma\urcorner, m, b, e)\check{}\,)) \in (A^\xi)^2$ such that letting $[s, y]$ represent p in the ultrapower, then for $F_{a\cup b\cup r\cup s}$-a.e. $u \in \kappa^{a\cup b\cup r\cup s}$,

$$\begin{aligned}(\mathcal{N}_0)_+ \models & y(u^s_{a\cup b\cup r\cup s}) \Vdash^{x(u^r_{a\cup b\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \\ & \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup b\cup r\cup s})) = \sigma(D(\dot{g}\restriction m), g(u^b_{a\cup b\cup r\cup s}))\text{”}.\end{aligned}$$

ϵ^ξ is the set of standard names for ordered pairs $((p, (\ulcorner\tau\urcorner, n, a, f)\check{}\,),$ $(p, (\ulcorner\sigma\urcorner, m, b, e)\check{}\,)) \in (A^\xi)^2$ such that letting $[s, y]$ represent p in the ultrapower, then for $F_{a\cup b\cup r\cup s}$-a.e. $u \in \kappa^{a\cup b\cup r\cup s}$,

$$\begin{aligned}&(\mathcal{N}_0)_+ \models y(u^s_{a\cup b\cup r\cup s}) \Vdash^{x(u^r_{a\cup b\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \\ &\text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup b\cup r\cup s})) \in \sigma(D(\dot{g}\restriction m), g(u^b_{a\cup b\cup r\cup s}))\text{”}.\end{aligned}$$

E^ξ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{}\,)$ such that letting $[s, y]$ represent p in the ultrapower, then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$(\mathcal{N}_0)_+ \models y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s})) \in \dot{E}\text{”}.$$

F^ξ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{}\,)$ such that letting $[s, y]$ represent p in the ultrapower, then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$(\mathcal{N}_0)_+ \models y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s})) = \emptyset\text{”}.$$

S^ξ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{}\,)$ such that letting $[s, y]$ represent p in the the ultrapower, then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$(\mathcal{N}_0)_+ \models y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s})) \in \dot{S}\text{”}.$$

b^ξ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{})$ such that letting $[s, y]$ represent p in the ultrapower, then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$\begin{aligned}(\mathcal{N}_0)_+ \models y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s}))\\ = V|\text{my last drop”}.\end{aligned}$$

$\mathcal{Q}^\xi$ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{})$ such that letting $[s, y]$ represent p in the ultrapower, then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$(\mathcal{N}_0)_+ \models y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s})) = \mathcal{Q}(\dot{g})\text{”}.$$

π^ξ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{})$ such that letting $[s, y]$ represent p in the ultrapower, then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$\begin{aligned}(\mathcal{N}_0)_+ \models & y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s})) = (X, Y),\\ & \text{where for some } m, X = \sup Hull^V(D(\dot{g})\restriction m \cup V|\text{my last drop}),\\ & Y = D(\dot{g})\restriction m.\text{”}\end{aligned}$$

I^ξ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{})$ such that letting $[s, y]$ represent p in the ultrapower, $[s, z]$ represent ξ in the ultrapower, then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$(\mathcal{N}_0)_+ \models y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s})) \in I^{V|z^s_{a\cup r\cup s}}\text{”}.$$

$\mu^\xi = F^\xi$ (also interpreted as the empty set).

ν^ξ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{})$ such that letting $[s, y]$ represent p in the ultrapower, $[s, z]$ represent ν^F in the ultrapower, then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$(\mathcal{N}_0)_+ \models y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s})) = z^s_{a\cup r\cup s}\text{”}.$$

If $\gamma^{\mathcal{M}|\xi}$ is an ordinal, then γ^ξ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{})$ such that letting $[s, y]$ represent p in the ultrapower, $[s, z]$ represent $\gamma^{\mathcal{M}|\xi}$ in the ultrapower, then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$\begin{aligned}(\mathcal{N}_0)_+ \models & y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} \text{ let } X = o(Tr^{\dot{g}}((\mathcal{N}_0)_+ | z^s_{a\cup r\cup s})), \text{ then} \\ & Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s})) = X\text{”}.\end{aligned}$$

If $\gamma^{\mathcal{M}|\xi}$ is a triple (A, B, C), then I^ξ is the set of $(p, (\ulcorner\tau\urcorner, n, a, f)\check{})$ such that letting $[s, y]$ represent p in the ultrapower, $[s, z]$ represent E_A, $[s, w]$ represent (B, C), then for $F_{a\cup r\cup s}$-a.e. $u \in \kappa^{a\cup r\cup s}$,

$$\begin{aligned}(\mathcal{N}_0)_+ \models & y(u^s_{a\cup r\cup s}) \Vdash^{x(u^r_{a\cup r\cup s})} \text{ let } X = o(Tr^{\dot{g}}(\mathrm{Ult}((\mathcal{N}_0)_+, z^s_{a\cup r\cup s}) | [w^s_{a\cup r\cup s}]^{(\mathcal{N}_0)_+}_{z^s_{a\cup r\cup s}})), \\ & \text{then } Tr^{\dot{g}}((\mathcal{N}_0)_+) \models \text{“}\tau(D(\dot{g}\restriction n), f(u^a_{a\cup r\cup s})) = X\text{”}.\end{aligned}$$

We let $A^{\mathcal{M}}, \sim^{\mathcal{M}}, etc$ stand for $A^{o(\mathcal{M})}, \sim^{o(\mathcal{M})}, etc.$

The next lemma says we can encode a copy of $Tr^g(\mathcal{M})$ in $\mathcal{M}$ in an Σ_1-way (actually, we can show in a Δ_1-way). The proof is more or less a tedious repetition of definition 3.4, so we state the lemma without proving it.

Lemma 3.5. *Let $\mathcal{M} \in P(\mathcal{N}, \eta)$. Let A^ξ, etc be defined as in Definition 3.4. Then for any $h \subseteq \mathbb{Q}_\eta$ generic over $\mathcal{M}$, $\sim^\xi_h$ is an equivalence relation on A^ξ_h, $\epsilon^\xi_h, E^\xi_h, F^\xi_h, S^\xi_h, b^\xi_h, \mathcal{Q}^\xi_h, \pi^\xi_h, I^\xi_h, \mu^\xi_h, \nu^\xi_h, \gamma^\xi_h$ are relations of an appropriate arity on A^ξ_h that are $\sim^\xi_h$-invariant. $\langle A^\xi_g/\sim^\xi_g, \epsilon^\xi_g/\sim^\xi_g, E^\xi_g/\sim^\xi_g, F^\xi_g/\sim^\xi_g, S^\xi_g/\sim^\xi_g, b^\xi_g/\sim^\xi_g, \mathcal{Q}^\xi_g/\sim^\xi_g$*

$,\pi^\xi_g/\sim^\xi_g, I^\xi_g/\sim^\xi_g, \mu^\xi_g/\sim^\xi_g, \nu^\xi_g/\sim^\xi_g, \gamma^\xi_g/\sim^\xi_g\rangle$ *is isomorphic to* $\mathfrak{C}_0(Tr^g(\mathcal{M}))$. *More importantly, the maps*

$$\xi \mapsto A^\xi, \sim^\xi, \epsilon^\xi, E^\xi, F^\xi, S^\xi, b^\xi, \mathcal{Q}^\xi, \pi^\xi, I^\xi, \mu^\xi, \nu^\xi, \gamma^\xi$$

are uniformly Σ_1 *over* $\mathfrak{C}_0(\mathcal{M})$. *Moreover each of the sets*

$$A^\mathcal{M}, \sim^\mathcal{M}, etc$$

are Σ_1 *over* $\mathfrak{C}_0(\mathcal{M})$.

We show that fine structure is preserved under the $*$-transform.

Lemma 3.6. *Let* $\mathcal{M} \in P(\mathcal{N}, \eta)$. *Let* $j \geq 1$ *be a natural number.*

1. *There is an effective map* $* : r\Sigma_j \to r\Sigma_j$ *such that for all* $\phi \in r\Sigma_j$ *and all* $b \in \mathcal{M}$,
$$\mathfrak{C}_0(\mathcal{M}) \models \phi(b) \leftrightarrow \mathfrak{C}_0(Tr^g(\mathcal{M})) \models \phi^*(b, g).$$
We also have an effective map $* : Sk_j \to Sk_j$ *such that, for all* $t \in Sk_j$ *and all* $b \in \mathcal{M}$,
$$t^\mathcal{M}(b) = (t^*)^{Tr^g(\mathcal{M})}(b, g).$$

2. *There is an effective map* $\hat{}: r\Sigma_j \to r\Sigma_j$ *such that for all* $\psi \in r\Sigma_j$, $\alpha \in o(\mathcal{M})$, $b \in Tr^g(\mathcal{M})$, *if* $\tau \in \mathcal{M}$ *is such that* $\mathfrak{u}^\mathcal{M}_g(\tau) = b$, *then*
$$\mathfrak{C}_0(Tr^g(\mathcal{M})) \models \psi(\alpha^g, b) \leftrightarrow \exists q \in g\mathfrak{C}_0(\mathcal{M}) \models q \Vdash^{\mathbb{Q}_\eta} \hat{\psi}(\alpha, \tau).$$
where $\alpha^g = o(Tr^g(\mathcal{M}|\alpha))$. *We also have an effective map* $\dot{}Sk_j \to Sk_j$ *such that, for all* $s \in Sk_j$, $\alpha \in o(\mathcal{M})$, $b \in Tr^g(\mathcal{M})$, *if* $\tau \in \mathcal{M}$ *is such that*

$\mathfrak{u}_g^{\mathcal{M}}(\tau) = b$, then there is $q \in g$ such that

$$\mathfrak{u}_g^{\mathcal{M}}(\hat{s}^{\mathcal{M}}(\alpha, \tau, q)) = s^{Tr^g(\mathcal{M})}(\alpha^g, b).$$

3. *for each $\eta < \alpha < o(\mathcal{M})$, α is a cardinal of $\mathcal{M}$ if and only if α is a cardinal of $Tr^g(\mathcal{M})$.*

4. *$\rho_j(\mathcal{M}) = \rho_j(Tr^g(\mathcal{M}))$,*

5. *$p_j(\mathcal{M}) \setminus \eta = p_j(Tr^g(\mathcal{M}))$,*

6. *$\mathcal{M}$ is j-sound $\to$ $Tr^g(\mathcal{M})$ is j-sound.*

Proof. The case $j > 1$ is not much different from the $j = 1$ case. For the sake of briefness, we only prove the $j = 1$ case. We also assume that $\mathcal{M}$ is not of E-type III in the sense of [4], so that no squash is applied when forming $\mathfrak{C}_0(\mathcal{M})$. The reader should have no problem fulfilling the remaining cases.

1. *Case 1.* $\mathcal{M} = (\mathcal{N}_0)_+$.

trivial.

Case 2: $\mathcal{M}$ is of successor level or limit passive level.

Given $\phi \in \Sigma_1$, $\phi^*(v)$ is the formula

"There is $\gamma \in \dot{I}$ such that $TrInv(V|\gamma) = \mathcal{K}$, and for some $n < \omega$, $S_n(\mathcal{K}) \models \phi(v)$."

Given $t \in Sk_1$,

$$t^*(b, g) = t^{S_n(TrInv(V|\gamma))}(b) \text{ for some } \gamma.$$

Case 3: $\mathcal{M}$ is active, $\mathrm{crt}(F^{\mathcal{M}}) > \eta$.

Given $\phi \in r\Sigma_1$, $\phi^*(v)$ is the formula

"There is $\xi \in \dot{I}$ such that $TrInv(V|\xi) = \mathcal{K}$, and $\mathcal{K} \oplus (\dot{F}^c \cap \mathcal{K}) \models \phi(v)$."

Given $t \in Sk_1$,

$$t^*(b,g) = t^{\mathcal{K}\oplus(\mathcal{F}^c\cap\mathcal{K})}(b),\ \text{for some } \mathcal{K} = TrInv(V|\xi), \xi \in \dot{I}.$$

Case 4: $\mathcal{M}$ is active, $\mathrm{crt}(F^{\mathcal{M}}) < \eta$.

Given $\phi \in r\Sigma_1$, $\phi^*(v)$ is the formula

"Let $TrInv(V|\sup \dot{I}) = K$. Then there is $\gamma < o(\mathcal{K})$, $G \subseteq \mathcal{K}|\gamma$, and $(X,Y) \in \dot{\pi}$ such that

(a) G is the set of quadruples (β, ξ, s, A) such that $\beta < (\kappa)^{+\mathcal{N}}$, $\xi < \gamma$, $s \in [\dot{\nu}]^{<\omega}$, $A \subseteq [\kappa]^{<\omega}$, and letting

$$Z = \{(t,B) : \text{for some } n, t \in [\dot{\nu}]^n, B \in [\kappa]^n \cap \mathcal{K}|\beta, \exists C \in Hull^{TrInv(V|X)}$$
$$(\kappa \cup \operatorname{ran} Y)(s \in C \wedge B = C \cap [\kappa]^n)\},$$

then $Z \in \mathcal{K}|\xi$, $(s,A) \in Z$, and moreover, for each $t \in [\nu]^n$ and $B \in \mathcal{N}|\beta \cap [\kappa]^n$,, either $(t,B) \in Z$ or $(t, \kappa \setminus B) \in Z$.

(b) $\mathcal{K}|\gamma \oplus G \models \phi(v)$."

Given $t \in Sk_1$,

$$t^*(b,g) = t^{\mathcal{K}|\gamma\oplus G}(b),\ \text{where } \mathcal{K} = TrInv(V|\sup \dot{I}),\ \text{and some } G \text{ as in (a) above.}$$

2. Comes from Lemma 3.5. Given $\psi \in \Sigma_1$, $\hat{\psi}(\alpha, v)$ is the formula "$\langle A_h^M / \sim_h^M$ $, etc\rangle \models \psi(\alpha', \tau)$." where $\alpha \mapsto \alpha'$ is the canonical map such that $\mathfrak{u}_g^{\mathcal{M}}(\alpha') = \alpha^g$.

Given $s \in Sk_1$,

$$\hat{s}(\alpha, \tau, q) = w \text{ where } q \Vdash w \text{ is the } <_{A_h^{\mathcal{M}}}\text{-least such that } s(\alpha', \tau).$$

3. If $\alpha < o(\mathcal{M})$ is a cardinal of $\mathcal{M}$, then α is a cardinal of each $\mathcal{K}$, $\mathcal{M}|\alpha <^{P(\mathcal{N},\eta)} \mathcal{N} <^{P(\mathcal{N},\eta)} \mathcal{M}$. Hence $Tr^g(\mathcal{M}|\alpha) \trianglelefteq \mathcal{K}$ for each such $\mathcal{K}$. Since $M|\alpha \models ZFC^-$, $o(Tr^g(\mathcal{M})|\alpha) = \alpha$. By induction, for each such $\mathcal{K}$, $\rho_\omega(Tr^g(\mathcal{K})) = \rho_\omega(\mathcal{K}) \geq \alpha$. Hence α is a cardinal of $Tr^g(\mathcal{M})$. Conversely, if α is a cardinal of $Tr^g(\mathcal{M})$, then since $\mathcal{M} \subseteq Tr^g(\mathcal{M})$, α is a cardinal of $\mathcal{M}$.

4. Some arguments of [9] can be used here. We show by contradiction. Suppose $\rho_1(\mathcal{M}) \neq \rho_1(Tr^g(\mathcal{M}))$.

Case 1. $\rho_1(\mathcal{M}) < \rho_1(Tr^g(\mathcal{M}))$.

Subcase 1.1 $\rho_1(\mathcal{M}) < o(\mathcal{M})$.

Let $\pi : \mathcal{M}^* \to \mathfrak{C}_0(\mathcal{M})$ be the σ_1-core map. Let $\bar{p} = \pi^{-1}(p_1(\mathcal{M}))$. There is then a $\Sigma_1^{\mathfrak{C}_0(\mathcal{M}^*)}(\bar{p})$ prewellorder of $\rho_1(\mathcal{M})$ of order type at least $\rho_1(\mathcal{M})^{+\mathcal{M}^*}$. As $p_1(\mathcal{M})$ is 1-universal, $\rho_1(\mathcal{M}^{+\mathcal{M}}) = \rho_1(\mathcal{M})^{+\mathcal{M}^*}$. By 3, $\rho_1(\mathcal{M})^{+\mathcal{M}} = \rho_1(M)^{+Tr^g(\mathcal{M})}$. But by 1, B is $\Sigma_1^{\mathfrak{C}_0(Tr^g(\mathcal{M}))}(p, g)$, hence $B \in \mathfrak{C}_0(Tr^g(\mathcal{M}))$. Contradiction.

Subcase 1.2 $\rho_1(\mathcal{M}) = o(\mathcal{M})$.

We claim that $\mathcal{M} \notin Tr^g(\mathcal{M})$. (Proof: Suppose that $\mathcal{M}$ is of minimal height such that $\mathcal{M} \in Tr^g(\mathcal{M})$. Clearly $\mathcal{M}$ can't be of limit level either passive or $\mathrm{crt}(F^{\mathcal{M}}) > \eta$. M can't be active limit level with $\mathrm{crt}(F^{\mathcal{M}}) < \eta$ because F^M can't be in $\mathrm{Ult}(M, F^{\mathcal{M}})$. M can't be of successor level because $\mathrm{rud}(x) \in \mathrm{rud}(y)$ implies $x \in y$.) Hence there is a $\Sigma_1^{\mathfrak{C}_0(Tr^g(\mathcal{M}))}$ subset of $o(\mathcal{M})$ which is not in $\mathfrak{C}_0(Tr^g(\mathcal{M}))$.

Case 2 $\rho_1(\mathcal{M}) > \rho_1(Tr^g(\mathcal{M}))$.

Let $A = \{\alpha < \rho_1(Tr^g(\mathcal{M})) : \mathfrak{C}_0(Tr^g(\mathcal{M})) \models \psi(\alpha, b)\}$, ψ Σ_1, such that $A \notin Tr^g(\mathcal{M})$. Let $B = \{\alpha^g : \alpha \in A\}$. So $B \notin Tr^g(\mathcal{M})$. Let τ be such that $\mathfrak{u}_h^{\mathcal{M}}(\tau) = b$. Then

$$\begin{aligned} \alpha \in B &\leftrightarrow \mathfrak{C}_0(Tr^g(\mathcal{M})) \models \psi(\alpha^g, b) \\ &\leftrightarrow \exists q \in g \mathfrak{C}_0(\mathcal{M}) \models q \Vdash \hat{\psi}(\alpha, \tau). \end{aligned}$$

Set $C = \{(q, \alpha) : q \in \mathbb{Q}_\eta, \alpha < \rho_1(Tr^g(\mathcal{M}))\mathfrak{C}_0(\mathcal{M}) \models q \Vdash \hat{\psi}(\alpha, \tau)\}$. Then C is coded into a bounded subset of $\rho_1(\mathcal{M})$. Hence $C \in \mathfrak{C}_0(\mathcal{M})$. Hence $B \in \mathfrak{C}_0(\mathcal{M})[g]$. Hence $B \in \mathfrak{C}_0(Tr^g(\mathcal{M}))$. Contradiction.

5. The proof is similar to 4.

Suppose not. $p_1(\mathcal{M}) \setminus \eta \neq p_1(Tr^g(\mathcal{M}))$.

Case 1. $p_1(\mathcal{M}) \setminus \eta <^* p_1(Tr^g(\mathcal{M}))$.

Subcase 1.1 $\rho_1(\mathcal{M}) < o(\mathcal{M})$.

Let $\pi : \mathcal{M}^* \to \mathfrak{C}_0(\mathcal{M})$ be the σ_1-core map. Let $\bar{p} = \pi^{-1}(p_1(\mathcal{M}))$. There is then a $\Sigma_1^{\mathfrak{C}_0(\mathcal{M}^*)}(\bar{p})$ prewellorder of $\rho_1(\mathcal{M})$ of order type at least $\rho_1(\mathcal{M})^{+\mathcal{M}^*}$. As $p_1(\mathcal{M})$ is 1-universal, $\rho_1(\mathcal{M}^{+\mathcal{M}}) = \rho_1(\mathcal{M})^{+\mathcal{M}^*}$. By 3, $\rho_1(\mathcal{M})^{+\mathcal{M}} = \rho_1(M)^{+Tr^g(\mathcal{M})}$. But by 1, B is $\Sigma_1^{\mathfrak{C}_0(Tr^g(\mathcal{M}))}(p \setminus \eta, g)$, hence $B \in \mathfrak{C}_0(Tr^g(\mathcal{M}))$. Contradiction.

Subcase 1.2 $\rho_1(\mathcal{M}) = o(\mathcal{M})$.

We have $\mathcal{M} \notin Tr^g(\mathcal{M})$. Hence there is a $\Sigma_1^{\mathfrak{C}_0(Tr^g(\mathcal{M}))}$ subset of $o(\mathcal{M})$ which is not in $\mathfrak{C}_0(Tr^g(\mathcal{M}))$. Therefore $p_1(Tr^g(\mathcal{M})) = \emptyset$. Contradiction.

Case 2 $p_1(\mathcal{M}) \setminus \eta >^* p_1(Tr^g(\mathcal{M}))$.

Let $A = \{\alpha < \rho_1(Tr^g(\mathcal{M})) : \mathfrak{C}_0(Tr^g(\mathcal{M})) \models \psi(\alpha, p_1(Tr^g(\mathcal{M})))\}$, ψ Σ_1, such that $A \notin Tr^g(\mathcal{M})$. Let $B = \{\alpha^g : \alpha \in A\}$. So $B \notin Tr^g(\mathcal{M})$. Let τ be $\Sigma_1(p_1(Tr^g(\mathcal{M})), g)$ definable over $\mathfrak{C}_0(\mathcal{M})$ such that $\mathfrak{u}_g^{\mathcal{M}}(\tau) = p_1(Tr^g(\mathcal{M}))$. Then

$$\begin{aligned}\alpha \in B &\leftrightarrow \mathfrak{C}_0(Tr^g(\mathcal{M})) \models \psi(\alpha^g, p_1(Tr^g(\mathcal{M})))\\ &\leftrightarrow \exists q \in g \mathfrak{C}_0(\mathcal{M}) \models q \Vdash \hat{\psi}(\alpha, \tau).\end{aligned}$$

Set $C = \{(q, \alpha) : q \in \mathbb{Q}_\eta, \mathfrak{C}_0(\mathcal{M}) \models q \Vdash \hat{\psi}(\alpha, \tau)\}$. Then C is coded into a $\Sigma_1(p_1(Tr^g(\mathcal{M})), \eta)$ subset of $\rho_1(\mathcal{M})$. Our case assumption says that $C \in \mathfrak{C}_0(\mathcal{M})$. Hence $B \in \mathfrak{C}_0(\mathcal{M})[g]$. Hence $B \in \mathfrak{C}_0(Tr^g(\mathcal{M}))$. Contradiction.

6. Assume $\mathcal{M}$ is 1-sound. We first show that $p_1(Tr^g(\mathcal{M}))$ is 1-universal. Fix $A \subseteq \rho_1(Tr^g(\mathcal{M})) = \rho_1(\mathcal{M})$ such that $A \in Tr^g(\mathcal{M})$. We want to show that $\{A\}$

is $\Sigma_1^{Tr^g(\mathcal{M})}(\rho_1(Tr^g(\mathcal{M})) \cup \{p_1(Tr^g(\mathcal{M})), g\})$. Fix $\tau \in \mathcal{M}$ such that $\mathfrak{u}_g(\tau) = A$. We know, since $\mathcal{M}$ is 1-sound, that $\{\tau\}$ is $\Sigma_1^{\mathfrak{C}_0\mathcal{M}}(\rho_1(\mathcal{M}) \cup \{p_1(\mathcal{M})\})$. Let $t \in Sk_1$, $\alpha < \rho_1(\mathcal{M})$ be such that $\tau = t(\alpha, p_1(\mathcal{M}))$. Thus A is the unique $x \in Tr^g(\mathcal{M})$ such that $\exists\gamma \in \dot{I}^{\mathfrak{C}_0(Tr^g(\mathcal{M}))}\exists Y\ (Y = t^{TrInv(V|\gamma)}(\alpha, p_1(\mathcal{M})) \wedge \mathfrak{u}_g^{TrInv(V|\gamma)}(Y) = x)$. Thus $\{A\}$ is $\Sigma_1^{\mathfrak{C}_0(Tr^g(\mathcal{M}))}(\rho_1(Tr^g(\mathcal{M})) \cup \{p_1(Tr^g(\mathcal{M})), g\})$.

Next we show that $Tr^g(\mathcal{M})$ is 1-solid. Say $p_1(Tr^g(\mathcal{M})) = \langle\alpha_0, \ldots, \alpha_k\rangle = p_1(\mathcal{M})\backslash\eta$. Fix $i \leq k$, $\psi \in \Sigma_1$ and $A = \{\alpha < \rho_1(Tr^g(M)) : \mathfrak{C}_0(Tr^g(M)) \models \psi(\alpha, (\alpha_0, \ldots, \alpha_{i-1}))\}$ be a set of ordinals that is $\Sigma_1^{\mathfrak{C}_0(Tr^g(\mathcal{M}))}(\{\alpha_0, \ldots, \alpha_{i_1}, g\})$. We want to show $A \in Tr^g(\mathcal{M})$. Let $B = \{\alpha^g : \alpha \in A\}$. Let τ be $\Sigma_1^{\mathfrak{C}_0(\mathcal{M})}(\alpha_0, \ldots, \alpha_{i-1}, g)$ such that $u_g^{\mathcal{M}}(\tau) = (\alpha_0, \ldots, \alpha_{i-1})$. Then

$$\begin{aligned}\alpha \in B &\leftrightarrow \mathfrak{C}_0(Tr^g(M)) \models \psi(\alpha^g, (\alpha_0, \ldots, \alpha_{i-1}))\\ &\leftrightarrow \exists q \in g\mathfrak{C}_0(M) \models q \Vdash \hat{\psi}(\alpha, \tau).\end{aligned}$$

Set $C = \{(q, \alpha) : q \in \mathbb{Q}_\eta, \mathfrak{C}_0(M) \models q \Vdash \hat{\psi}(\alpha, \tau)\}$. Then C is coded into a $\Sigma_1(\alpha_0, \ldots, \alpha_{i-1}, \eta)$ subset of $\rho_1(M)$. From solidity of $\mathcal{M}$, we know $C \in \mathfrak{C}_0(M)$. Hence $B \in \mathfrak{C}_0(M)[g]$. Hence $B \in Tr^g(M)$. Hence $A \in Tr^g(M)$.

Finally, the proof that $Tr^g(\mathcal{M})$ is 1-sound is just a repetition of the proof that $p_1(Tr^g(\mathcal{M}))$ is 1-universal. This completes the proof of Lemma 3.6 for $j = 1$. $\square$

Lemma 3.7. *$Tr^g(\mathcal{M})$ is a potential S-premouse.*

Proof. Every proper initial segment of $Tr^g(\mathcal{M})$ is sound by 6 of Theorem 3.6. Therefore, $Tr^g(\mathcal{M})$ is acceptable. $\square$

3.3 Iterability

In this section we show that if $\mathcal{M}$ is iterable via a strategy Σ such that every Σ-iterate of $\mathcal{M}$ translates into an S-premouse, then $Tr^g(\mathcal{M})$ is S-iterable. The phrase "translates into an S-premouse" means the S predicate of the result of the translation expresses the correct S-operators as we have defined in Section 2.

Lemma 3.8. *Suppose $Tr^g(\mathcal{M})$ is defined. Let E be an extender over $Tr^g(\mathcal{M})$, $\hat{E} = E \cap \mathcal{M}$ is an extender over $\mathcal{M}$. Assume furthermore that E is close to $Tr^g(\mathcal{M})$, $\hat{E}$ is close to $\mathcal{M}$. $\kappa = \operatorname{crt}(E) = \operatorname{crt}(\hat{E})$. Let k be the largest j such that $\kappa < \rho_j(\mathcal{M})$. Then for all $j \leq k$, if $Tr^g(\operatorname{Ult}_j(\mathcal{M}, \hat{E}))$ is defined, then*

$$Tr^g(\operatorname{Ult}_j(\mathcal{M}, \hat{E})) = \operatorname{Ult}_j(Tr^g(\mathcal{M}), E),$$

$$i_{\hat{E}} = i_E \restriction \mathcal{M}.$$

and if $\tau \in A^{\mathcal{M}}$, then

$$i_E^{Tr^g(\mathcal{M})}(\mathfrak{u}_g^{\mathcal{M}}(\tau)) = \mathfrak{u}_g^{\operatorname{Ult}_j(\mathcal{M},\hat{E})}(i_{\hat{E}}^{\mathcal{M}}(\tau)).$$

Proof. Say $\psi(\cdot,\mu), \psi_E(\cdot,\mu), \psi_F(\cdot,\mu)$ defines $|\mathcal{M}|, E^{\mathcal{M}}, (F^c)^{\mathcal{M}}$ over $\mathfrak{C}_0(Tr^g(\mathcal{M}))$. Let $\mathcal{K}$ be the premouse whose Σ_0-core is defined by $\psi(\cdot,\mu), \psi_E(\cdot,\mu), \psi_F(\cdot,\mu)$ over $\mathfrak{C}_0(\operatorname{Ult}_j(Tr^g(\mathcal{M}), E))$. Then $i_E \restriction \mathcal{M} : \mathcal{M} \to \mathcal{K}$ is Σ_1-elementary. We show that $\mathcal{K} = \operatorname{Ult}_j(\mathcal{M}, \hat{E})$ and $i_E \restriction \mathcal{M} = i_{\hat{E}}$.

Let $\sigma : \operatorname{Ult}(\mathcal{M}, \hat{E}) \to \mathcal{K}$ be the map

$$\sigma([a,f]_{\hat{E}}^{\mathcal{M}}) = [a,f]_E^{Tr^g(\mathcal{M})}$$

for $a \in [lh(E)]^{<\omega}$, $f : \kappa^{|a|} \to \mathcal{M}$, $j = 0 \to f \in \mathcal{M}$, $j \geq 1 \to f \in r\underset{\sim}{\Sigma}_j^{\mathcal{M}}$. Clearly σ is well defined and Σ_1-elementary, $\sigma \circ i_{\hat{E}} = i_E$, $\sigma \restriction lh(E) = id$. It remains to show that σ is onto.

Well, an element of $\mathcal{K}$ is of the form

$$[a,f]_E^{Tr^g(\mathcal{M})}$$

where $a \in [lh(E)]^{<\omega}$, $f : \kappa^{|a|} \to \mathcal{M}$, $j = 0 \to f \in Tr^g(\mathcal{M})$, $j \geq 1 \to f \in r\underset{\sim}{\Sigma}_j(Tr^g(\mathcal{M}))$. We should find $g : \kappa^{|a|} \to \mathcal{M}$, $j = 0 \to g \in \mathcal{M}$, $j \geq 1 \to g \in r\Sigma_j^{\mathcal{M}}$, such that $g(u) = f(u)$ for E_a-a.e. u.

Case 1. $j = 0$.

Subcase 1.1. $\mathcal{M} = \mathfrak{N}(\mathcal{K})$ for some $\mathcal{K}$.

We may assume that $\mathrm{ran}(f) \subseteq S_n(\mathcal{K})$ for a fixed $n < \omega$. Let $\dot{f} \in A^M$ be such that $\mathfrak{u}_g^M(\dot{f}) = f$. We have

$$\bigcup_{q \in g} \{u \in [\kappa]^{|a|} : S_n(\mathcal{K}) \models \exists q \Vdash \mathfrak{u}^{S_n(\mathcal{K})}(\dot{f})(\check{u}) = \check{y}\} = [\kappa]^{|a|} \in \hat{E}_a.$$

But $|g| \leq \eta < \kappa$. So there must be some $q_0 \in g$ such that

$$A_0 = \{u \in [\kappa]^{|a|} : S_n(\mathcal{K}) \models \exists y q_0 \Vdash u^{S_n(\mathcal{K})}(\dot{f})(\check{u}) = \check{y}\} \in \hat{E}_a.$$

So let $g \in M$ be the function on $[\kappa]^{|a|}$ defined by

$$g(u) = \begin{cases} v, \text{ if } S_n(\mathcal{K}) \models q_0 \Vdash \mathfrak{u}^{S_n(\mathcal{K})}(\dot{f})(\check{u}) = \check{v}. \\ 0, \text{ otherwise.} \end{cases}$$

Then $A_0 \in \hat{E}_a \subseteq E_a$ and $A_0 \subseteq \{u : g(u) = f(u)\}$.

Subcase 1.2 $\mathcal{M}$ is of limit level which is either passive or $\mathrm{crt}(F^{\mathcal{M}}) > \eta$.

Say $\mathfrak{u}_g^{\mathcal{M}}(\dot{f}) = f$. Let $\mathcal{K} \lhd \mathcal{M}$ be such that $\dot{f} \in \mathcal{K}$. A similar argument as before gives $g \in \mathfrak{N}(\mathcal{K})$ such that $g(u) = f(u)$ for E_a-a.e. u.

Subcase 1.3. $\mathrm{crt}(F^{\mathcal{M}}) < \eta$.

$o(\mathcal{M})$ must be a cardinal in $Tr^g(\mathcal{M})$. Hence $f \in Tr^g(\mathcal{K})$ for some $\mathcal{K} \lhd \mathcal{M}$. The rest goes as before.

Case 2. $j \geq 1$.

We assume $j = 1$. The case $j > 1$ is not much different. If $\rho_1(\mathcal{M}) = o(\mathcal{M})$, then 0-ultrapowers agree with 1-ultrapowers, so case 1 applies. We assume now $\rho_1(\mathcal{M}) < o(\mathcal{M})$. Denote $\rho = \rho_1(\mathcal{M}) = \rho_1(Tr^g(\mathcal{M}))$, $p = p_1(\mathcal{M}) \setminus \eta = p_1(Tr^g(\mathcal{M}))$. We know $\mathrm{Ult}_1(Tr^g(\mathcal{M}), E)$ is 1-sound and $\rho_1(\mathrm{Ult}_1(Tr^g(\mathcal{M}), E)) \leq lh(E)$, $p_1(\mathrm{Ult}_1(Tr^g(\mathcal{M}), E)) = i_E(p)$. Thus a general element of $\mathcal{K}$ is of the form

$$s^{\mathfrak{C}_0(\mathrm{Ult}_1(Tr^g(\mathcal{M}),E))}(i_E(p), [a, f]_E^{Tr^g(\mathcal{M})})$$

where $s \in Sk_1$, $a \in [lh(E)]^{<\omega}$, $f \in Tr^g(\mathcal{M})$, $f : \kappa \to \rho$. The identical argument as in case 1 gives $g \in \mathcal{M}$ such that $f(u) = g(u)$ for a.e. $u \in [\kappa]^{|a|}$. By 2 of Theorem 3.6, there is $t \in Sk_1$ such that $t^{\mathfrak{C}_0(\mathcal{M})}(p, g(u)) = s^{\mathfrak{C}_0(Tr^g(\mathcal{M}))}(p, g(u))$. Thus, the element $t^{\mathrm{Ult}_1(\mathcal{M},\hat{E})}(i_{\hat{E}}(p), [a, g]_{\hat{E}}^{\mathcal{M}})$ will be mapped to $s^{\mathfrak{C}_0(\mathrm{Ult}_1(Tr^g(\mathcal{M}),E))}(i_E(p), [a, f]_E^{Tr^g(\mathcal{M})})$.

So far we have finished proving $Tr^g(\mathrm{Ult}_j(\mathcal{M}, \hat{E})) = \mathrm{Ult}_j(Tr^g(\mathcal{M}), E)$, $i_{\hat{E}} = i_E \restriction \mathcal{M}$. $i_E^{Tr^g(\mathcal{M})}(\mathfrak{u}_g^{\mathcal{M}}(\tau)) = \mathfrak{u}_g^{\mathrm{Ult}_j(\mathcal{M},\hat{E})}(i_{\hat{E}}^{\mathcal{M}}(\tau))$ then follows from elementarity of i_E. Take $\tau \in A^{\mathcal{M}}$. Let $\mathfrak{u}_g(\tau) = b$. Then $Tr^g(\mathcal{M}) \models \mathfrak{u}_g^{\mathcal{M}}(\tau) = b$. By elementarity, $\mathrm{Ult}_j(Tr^g(\mathcal{M})) \models \mathfrak{u}_g^{\mathrm{Ult}_j(M,\hat{E})}(i_E(\tau)) = i_E(b)$. We just proved $i_{\hat{E}} = i_E \restriction \mathcal{M}$. So $\mathfrak{u}_g^{\mathrm{Ult}_j(M,\hat{E})}(i_{\hat{E}}(\tau)) = i_E(b)$ □

Theorem 3.9. *Suppose $\mathcal{M}$ has an iteration strategy Σ such that every Σ-iterate of $\mathcal{K}$ translates into an S-premouse. Then $Tr^g(M)$ is S-iterable.*

Proof. Fix an iteration strategy $\Sigma_{\mathcal{M}}$ for $\mathcal{M}$. We wish to inductively define an iteration strategy Γ for $Tr^g(\mathcal{M})$. If we assume we have an iteration tree $\mathcal{T}$ of limit length on $Tr^g(\mathcal{M})$ which is by Γ so far, the next step of the induction is to pick a branch through $\mathcal{T}$ to be $\Gamma(\mathcal{T})$. We do this by translating $\mathcal{T}$ to a tree on $\mathcal{M}$, using $\Sigma_{\mathcal{M}}$ to pick a branch there, and then pulling the branch back to $\mathcal{T}$. So the key to the theorem will be define a translation from iteration trees on $Tr^g(\mathcal{M})$ to iteration trees on $\mathcal{M}$.

Fix a normal iteration tree $\mathcal{T}$ on $Tr^g(\mathcal{M})$. Note that we require all extenders used on $\mathcal{T}$ to have critical points above η, as $Tr^g(\mathcal{M})$ to be an mixed S-premouse. We will inductively build an iteration tree $\mathcal{U}$ on $\mathcal{M}$ and maps

$$\tau, \sigma : lh(\mathcal{T}) \to lh(\mathcal{U}).$$

τ picks the model on the $\mathcal{U}$-side which gets translated into the S-premouse on the $\mathcal{T}$-side, namely, $Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\alpha}) = \mathcal{M}^{\mathcal{T}}_\alpha$. σ picks the model on the $\mathcal{U}$-side which has the extender applied on the $\mathcal{T}$-side, namely, $E^{\mathcal{T}}_\alpha = E^{\mathcal{U}}_{\sigma_\alpha}[g]$. We start by setting $\tau_0 = 0$ and $\mathcal{U} \restriction \tau_0 + 1 = \langle \mathcal{M} \rangle$.

Fix $\lambda > 0$ and assume that we have defined increasing sequences of ordinals $\langle \tau_\gamma : \gamma < \lambda \rangle$ and $\langle \sigma_\gamma : \gamma < \alpha \rangle$, where $\alpha = \lambda$ if λ is a limit ordinal and $\alpha + 1 = \lambda$ otherwise. Say, moreover, that $\langle \tau_\gamma : \gamma < \lambda \rangle$ is a continuous sequence. Let $l_\lambda = \sup\{\tau_\gamma + 1 : \gamma < \lambda\}$ and say we have constructed a normal iteration tree $\mathcal{U} \restriction l_\lambda$ on $\mathcal{M}$ such that for all $\gamma < \lambda$ we have

(a) for all $\xi < \lambda$, if $\gamma < \xi$ then $\tau_\gamma \leq \sigma_\gamma < \tau_\xi$, and if $\tau_\gamma <_{U'} \tau_\xi$ then $\gamma <_U \xi$. Moreover, if $\mathcal{U}$ doesn't drop between τ_γ and τ_ξ, then $\mathcal{T}$ doesn't drop between γ and ξ.

(b) $Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}) = \mathcal{M}^{\mathcal{T}}_\gamma$.

(c) If $\lambda = \alpha + 1$ is a successor ordinal, then for all $\gamma < \alpha$,

$$E^{\mathcal{U}}_{\sigma_\gamma} = E^{\mathcal{T}}_\gamma \cap \mathcal{M}^{\mathcal{U}}_{\sigma_\gamma}.$$

If λ is a limit ordinal, then this holds for all $\gamma < \lambda$.

(d) $\deg^{\mathcal{U}}(\tau_\gamma) = \deg^{\mathcal{T}}(\gamma)$.

(e) For every $\xi < \lambda$, if $\tau_\gamma <_{U'} \tau_\xi$ and there is no dropping between τ_γ and τ_ξ on $\mathcal{U}$, so that $i^{\mathcal{U}}_{\tau_\gamma\tau_\xi} : \mathcal{M}^{\mathcal{U}}_{\tau_\gamma} \to \mathcal{M}^{\mathcal{U}}_{\tau_\xi}$ is defined, then for all $\bar{x} \in A_g^{\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}}$,

$$\mathfrak{u}_g^{\mathcal{M}^{\mathcal{U}}_{\tau_\xi}}(i^{\mathcal{U}}_{\tau_\gamma\tau_\xi}(\bar{x})) = i^{\mathcal{U}}_{\gamma\xi}(\mathfrak{u}_g^{\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}}(\bar{x})).$$

(f) τ_γ is a limit ordinal if and only if γ is a limit ordinal, and

$$\Gamma(\mathcal{T}\restriction\gamma) = \{\xi < \gamma : \tau_\xi \in \Sigma(\mathcal{U}\restriction\tau_\gamma)\}$$

is a cofinal, wellfounded branch through $\mathcal{T}\restriction\gamma$.

We wish to extender our construction to $\langle\tau_\gamma : \gamma \leq \lambda\rangle$, $\langle\sigma_\gamma : \gamma < \lambda\rangle$, and $\mathcal{U}\restriction l_{\lambda+1} = \mathcal{U}\restriction\tau_\lambda + 1$ and show that it still satisfies properties (a)-(f).

Let us first consider the situation where λ is a successor ordinal, say $\lambda = \alpha+1$. We start by defining $\sigma_\alpha(\geq \tau_\alpha)$ and the normal extension $\mathcal{U}\restriction\sigma_\alpha + 1$ using the following sublemma.

Sublemma 3.10. *Say $\mathcal{K}$ is iterable and $Tr^g(\mathcal{K})$ is defined. If E is on the $Tr^g(\mathcal{K})$-sequence indexed at λ, then there is $\vec{F} = \langle F_i : i < n\rangle \in U(\mathcal{K}, \eta)$ such that letting $E' = E_\lambda^{\mathrm{Ult}(\mathcal{N}, F_{n-1})}$, then $E'[g] = E$.*

Proof. If $E^{\mathcal{K}}_\lambda[g] = E$ we are done with $F = \emptyset$. Otherwise, let α_0 be least such that $E = E_\lambda^{Tr^g(\mathcal{K}|\alpha_0)}$. Then it must be that $E^{\mathcal{K}}_{\alpha_0} < \eta$. So $E = E_\lambda^{Tr^g(\mathrm{Ult}(\mathcal{N}, E^{\mathcal{K}}_{\alpha_0}))}$. If $E_\lambda^{\mathrm{Ult}(\mathcal{N}, E^{\mathcal{K}}_{\alpha_0})}[g] = E$, we are done with $F = \langle E^{\mathcal{K}}_{\alpha_0}\rangle$. Otherwise, let α_1 be least such that $E = E_\lambda^{Tr^g(\mathcal{K}|\alpha_1)}$. Then it must be that $E^{\mathcal{K}}_{\alpha_1} < \eta$. So $E = E_\lambda^{Tr^g(\mathrm{Ult}(\mathcal{N}, E^{\mathcal{K}}_{\alpha_1}))}$. Continuing this process, we will reach a finite increasing sequence $\langle\alpha_i : i < n\rangle$ such that letting $F_0 = E^{\mathcal{K}}_{\alpha_0}$, $F_i = E_{\alpha_i}^{\mathrm{Ult}(\mathcal{N}, F_{i-1})}$, then $\vec{F} = \langle F_i : i < n\rangle$ satisfies the sublemma. □

We call $\vec{F}$ as constructed in Sublemma 3.10 the recovery sequence of E with respect to $\mathcal{K}$. Now let $\vec{F} = \langle F_i : i < n\rangle$ be the recovery sequence of $E^{\mathcal{T}}_\alpha$. We claim that $lh(F_0) > lh(E^{\mathcal{U}}_\gamma)$ for any $\gamma < \tau_\alpha$, so that $\mathcal{U} \restriction \tau_\alpha + 1$ can be extended to a normal tree $\mathcal{V}$ by adding $\vec{F}$. Well, if not, then since $lh(E^{\mathcal{U}}_\gamma)$ is a cardinal in $M^{\mathcal{U}}_{\tau_\alpha}$, the whole process of constructing $\vec{F}$ can be carried out inside $M_{\tau^{\mathcal{U}}_\alpha}$. Hence $lh(E^{\mathcal{T}}_\alpha) < lh(E^{\mathcal{U}}_{\sigma_\gamma}) = lh(E^{\mathcal{T}}_\gamma)$, contradicting the normality of $\mathcal{T}$. We then let $U' \restriction \sigma_\alpha + 1 = \mathcal{V}$ and $M^{\mathcal{U}}_{\sigma_\alpha} = \mathcal{R}$ for $\mathcal{V}$ and $\mathcal{R}$ given by the above construction. The extender we choose at stage σ_α of $\mathcal{U}$ will be $E^{\mathcal{U}}_{\sigma_\alpha} = E^{\mathcal{T}}_\sigma \cap M^{\mathcal{U}}_{\sigma_\alpha}$, so hypothesis (c) continues to hold.

Now we need to determine the model on the $\mathcal{U}$-side to which we will apply $E^{\mathcal{U}}_{\sigma_\alpha}$. Say β is least such that $\operatorname{crt}(E^{\mathcal{T}}_\alpha) < \nu(E^{\mathcal{T}}_\beta)$. So in $\mathcal{T}$ we are going to have $M^{\mathcal{T}}_{\alpha+1} = \operatorname{Ult}_{k_{\alpha+1}}(\mathcal{Q}_\beta, E^{\mathcal{T}}_\alpha)$ for the longest possible $\mathcal{Q}_\beta \trianglelefteq M^{\mathcal{T}}_\beta$, where $k_{\alpha+1}$ is the largest $k \leq \omega$ such that $\operatorname{crt}(E^{\mathcal{T}}_\alpha) < \rho_k(\mathcal{Q}_\beta)$. Say that on the $\mathcal{U}$-side, we apply $E^{\mathcal{U}}_{\sigma_\alpha}$ to $\mathcal{Q}'_\beta$. We need that Q'_β is the largest initial segment of $M^{\mathcal{U}}_{\tau_\beta}$ over which $E^{\mathcal{T}}_{\sigma_\beta}$ is an extender, so that by Lemmas 3.6 and 3.8,

$$Tr^g(\mathcal{Q}_{\beta'}) = \mathcal{Q}_\beta$$

and

$$k_{\alpha+1} = \text{the largest } k \leq \omega \text{ such that } \operatorname{crt}(E^{\mathcal{U}}_{\sigma_\alpha}) = \operatorname{crt}(E^{\mathcal{T}}_\alpha) < \rho_k(\mathcal{Q}'_\beta)$$

$$Tr^g(\operatorname{Ult}_{k_{\alpha+1}}(\mathcal{Q}'_\beta, E_{\sigma^{\mathcal{U}'}_\alpha})) = \mathcal{M}^{\mathcal{T}}_{\alpha+1}.$$

Say β' such that $\operatorname{crt}(E^{\mathcal{T}}_\alpha) = \operatorname{crt}(E^{\mathcal{U}}_{\sigma_\alpha}) < \nu(E^{\mathcal{U}}_{\beta'})$. If $\tau_\beta = \beta'$ we are done. If not, let $F_0, \ldots, F_n$ be the recovery sequence of $E^{\mathcal{T}}_\beta$ with respect to $\mathcal{M}^{\mathcal{U}}_{\tau_\beta}$. Then $\nu(E^{\mathcal{U}}_{\tau_\beta}) \leq \operatorname{crt}(E^{\mathcal{U}}_{\sigma_\alpha}) < \nu(F_i)$ for some $i \leq n$. But $\nu(F_i) < (\nu(E^{\mathcal{U}}_{\tau_\beta}))^{+\mathcal{M}^{\mathcal{U}}_{\tau_\beta}}$ by the proof of Sublemma 3.10. This implies that $\operatorname{cf}(E^{\mathcal{T}}_\alpha)$ is not a cardinal of $\mathcal{M}^{\mathcal{T}}_\beta$, so $\mathcal{Q}_\beta$ must be a proper initial segment of $\mathcal{M}^{\mathcal{T}}_\beta$. On the other hand, $Tr^g(\mathcal{Q}'_\beta)$ is an initial segment of $\mathcal{M}^{\mathcal{T}}_\beta$, and $Tr^g(\mathcal{Q}'_\beta)$ itself has an initial segment over which $E^{\mathcal{T}}_\alpha$ is an extender. Therefore, $Tr^g(\mathcal{Q}'_\beta) = \mathcal{Q}_\beta$ and we are done.

Finally, we consider how to continue the construction when λ is a limit ordinal. We first need to check that (f) holds at λ. That is, we need to see that

$$\Gamma(\mathcal{T}\restriction\lambda) = \{\xi < \lambda : \tau_\xi \in \Sigma(\mathcal{U}\restriction\tau_\lambda)\}$$

is a cofinal, wellfounded branch through $\mathcal{T}\restriction\lambda$. Note that by our construction, if $\gamma < \tau_\lambda$ is such that $\mathcal{M}^{\mathcal{U}}_\gamma \neq \mathcal{M}^{\mathcal{U}}_{\tau_\xi}$ for all $\xi < \lambda$, it must be because $\mathcal{M}^{\mathcal{U}}_\gamma$ is a $\mathcal{U}'$-immediate successor of 0. So any cofinal branch through $\mathcal{U}\restriction\tau_\lambda$, in particular, $\Sigma(\mathcal{U}'\restriction\tau_\lambda)$, can contain at most one such γ. Thus, for every $\alpha < \lambda$, there is some β such that $\alpha < \beta < \lambda$ and $\tau_\beta \in \Sigma(\mathcal{U}\restriction\tau_\lambda)$, and thus $\beta \in \Gamma(\mathcal{U}\restriction\lambda)$. So $\Gamma(\mathcal{T}\restriction\lambda)$ is a cofinal branch.

Moreover, letting $b = \Gamma(\mathcal{T}\restriction\lambda)$, and $b' = \Sigma(\mathcal{U}\restriction\tau_\lambda)$, we have

$$\operatorname{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{T}}_\gamma = \operatorname{dirlim}_{\gamma\in b} Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}) = Tr^g(\operatorname{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}) = Tr^g(\operatorname{dirlim}_{\xi\in b'}\mathcal{M}^{\mathcal{U}}_\xi).$$

The first equality is by inductive property (b) at ordinals $< \lambda$, the third is because τ maps λ cofinally into τ_λ, and the second is by the final sublemma.

Sublemma 3.11. $\operatorname{dirlim}_{\gamma\in b} Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}) = Tr^g(\operatorname{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\gamma})$

Proof. The uncollapsed version of $\operatorname{dirlim}_{\gamma\in b} Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\gamma})$ is a structure with universe

$$\{(\gamma, x) : \gamma \in b, x \in Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\gamma})\}/\sim$$

where $(\gamma, x) \sim (\xi, y) \leftrightarrow \exists\theta \in b(i^{\mathcal{U}}_{\gamma\theta}(x) = i^{\mathcal{U}}_{\xi\theta}(y))$. The uncollapsed version of $\operatorname{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}$ is similarly a structure with universe

$$\{(\gamma, x) : \gamma \in b, x \in \mathcal{M}^{\mathcal{U}}_{\tau_\gamma}\}/\approx$$

where $(\gamma, x) \approx (\xi, y) \leftrightarrow \exists\theta \in b(i^{\mathcal{U}}_{\tau_\gamma\tau_\theta}(x)) = i^{\mathcal{U}}_{\tau_\xi\tau_\theta}(y)$, but we already know that this branch is wellfounded, so we identify $\mathrm{dirlim}_{\gamma\in b}^{\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}}$ with its transitive collapse. We will provide an isomorphism h between the uncollapsed version of $\mathrm{dirlim}_{\gamma\in b} Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\gamma})$ and $Tr^g(\mathrm{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\gamma})$.

For every pair (γ, x) with $\gamma \in b$ and $x \in Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\gamma})$, where is a term $\bar{x} \in \mathcal{M}^{\mathcal{U}}_{\tau_\gamma}$ such that $x = \mathfrak{u}_g^{\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}}(\bar{x})$. So we define the map h by

$$h([\gamma, x]_\sim) = \mathfrak{u}_g^{\mathrm{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}}([\tau_\gamma, \bar{x}]_\approx).$$

Hypothesis (e) can be used to show h is welldefined and elementary. The argument is standard. Take $\phi(v)$ be a formula with one free variable as an example, take $(\gamma, x)_\sim$ in the direct limit, and let $\bar{x} \in A_g^{\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}}$ such that $\mathfrak{u}_g^{\mathcal{M}^{\mathcal{U}}_{\tau_\beta}}(\bar{x}) = x$,

$$\begin{aligned}
\mathrm{dirlim}_{\gamma\in b} Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}) \models \phi([\gamma, x]_\sim) &\leftrightarrow \exists\theta \in b\ Tr^g(\mathcal{M}^{\mathcal{U}}_{\tau_\theta}) \models \phi(i_{\gamma\theta}(x)) \\
&\leftrightarrow \exists\theta \in b\ \exists q \in g\ \mathcal{M}^{\mathcal{U}}_{\tau_\theta} \models q \Vdash \hat{\phi}(i^{\mathcal{U}}_{\tau_\gamma\tau_\theta}\bar{x}) \\
&\leftrightarrow \exists q \in g\, \mathrm{dirlim}_{\xi\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\xi} \models q \Vdash \hat{\phi}(i_{\tau_\gamma b'}(\bar{x})) \\
&\leftrightarrow Tr^g(\mathrm{dirlim}_{\xi\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\xi}) \models \phi(\mathfrak{u}_g^{\mathrm{dirlim}_{\xi\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\xi}}(\bar{x})) \\
&\leftrightarrow Tr^g(\mathrm{dirlim}_{\xi\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\xi}) \models \phi([\tau_\gamma, \bar{x}]_\approx)
\end{aligned}$$

It remains to show h is onto. For any $y \in Tr^g(\mathrm{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\gamma})$, we can fix $\bar{y} \in \mathrm{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}$ such that $\mathfrak{u}_g^{\mathrm{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}}(\bar{y}) = y$. But then, since τ is cofinal in the branch b', we can fix ξ and some $\bar{x} \in \mathcal{M}^{\mathcal{U}}_{\tau_\xi}$ such that $\bar{y} = [\tau_\xi, \bar{x}]_\approx$. Therefore,

$$h([\xi, x]_\sim) = ([\tau_\xi, \bar{x}]_\approx)_g = \mathfrak{u}_g^{\mathrm{dirlim}_{\gamma\in b}\mathcal{M}^{\mathcal{U}}_{\tau_\gamma}}(\bar{y}) = y.$$

$\square$

Therefore, since b' is wellfounded, b will be as well. Since

$$\mathcal{M}^{\mathcal{T}}_{\lambda} = \text{dirlim}_{\gamma\in b}\, \mathcal{M}^{\mathcal{T}}_{\gamma} = Tr^{g}(\text{dirlim}_{\xi\in b'}\, \mathcal{M}^{\mathcal{U}}_{\xi}) = Tr^{g}(\mathcal{M}^{\mathcal{U}}_{\tau_\lambda}),$$

the sublemma also shows that hypothesis (b) holds at λ. □

3.4 Finishing the largest-Suslin-cardinal case

Recall that we work under the assumption $AD^{+} + V = L(\mathcal{P}(\mathbb{R})) + \theta = \theta_{\alpha+1}$. We fixed a hod pair $(\mathcal{P}, \Sigma)$ such that $\mathcal{M}_\infty(P, \Sigma) = HOD|\theta_\alpha$. We verify that the requirements of Lemma 3.3 are met.

Fix $\mathcal{Q}$, Σ-good. Let $\mathcal{N} = L[\vec{E}]^{\mathcal{Q}}$. For each $\mathcal{R} \in B(\mathcal{P}, \Sigma) \cup \{\mathcal{P}\}$, let $\kappa_{\mathcal{R}}$ be the least κ such that for some $\mathcal{R}' \in (B(\mathcal{P}, \Sigma) \cup \{\mathcal{P}\}) \cap (H_{\kappa^{++}})^{\mathcal{N}}$,

1. $\mathfrak{R} \leq_{DJ} \mathcal{R}'$.

2. $\Sigma_{\mathcal{R}'} \cap \mathcal{N} \in \mathcal{N}_{+}$

So $\mathcal{R} \mapsto \kappa_{\mathcal{R}}$ is an increasing mapping with respect to the hod mouse prewellordering of $\mathcal{R}$. In fact, any strong cardinal of $\mathcal{N}$ below $\kappa_{\mathcal{P}}$ must be some $\kappa_{\mathcal{R}}$, as shown in the following

Lemma 3.12. *Let $\mathcal{R} \in B(\mathcal{P}, \Sigma) \cup \{\mathcal{P}\}$.*

1. *Suppose $\lambda^{\mathcal{R}} = 0$. Then $\kappa_{\mathcal{R}}$ is the least strong of $\mathcal{N}$.*

2. *Suppose $\lambda^{\mathcal{R}}$ is a successor. Let $\mu = \kappa_{\mathcal{R}^{-}}$. Let κ be the least strong of $\mathcal{N}$ which is $> \mu$. Then $\kappa_{\mathcal{R}} \leq \kappa$.*

3. *Suppose $\lambda^{\mathcal{R}}$ is a limit. Let $\mu = \sup\{\kappa_{\mathcal{R}'} : \mathcal{R}' \in B(\mathcal{R}, \Sigma_{\mathcal{R}})\}$. Then $\kappa_{\mathcal{R}} \leq (\mu^{+})^{\mathcal{N}}$.*

Proof. 1. Denote κ_0 = the least strong of $\mathcal{N}$. The proof of MSC shows that $\kappa_{\mathcal{R}} \leq \kappa_0$, as witnessed by $(HOD|\theta)^{D(\mathcal{N},<\kappa_0)}$. It remains to see that there is no $\mathcal{R}' \in I(\mathcal{P}(0), \Sigma_{\mathcal{P}(0)}) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{R}'} \cap \mathcal{N} \in \mathcal{N}_+$. Suppose towards a contradiction there exists such an $\mathcal{R}'$. Since κ_0 is a limit of cutpoint Woodins of $\mathcal{N}$, we can pick $\xi \in (|R'|^{\mathcal{N}}, \kappa_0)$ which is a cutpoint Woodin. In $\mathcal{N}$, we can iterate $\mathcal{R}'$ to $\mathcal{R}''$ making $\mathcal{N}|\xi$ generic. By fullness preservation, $(\xi^+)^{\mathcal{N}} = (\xi^+)^{\mathcal{R}''[N|\xi]}$. However, $(\xi^+)^{\mathcal{R}''}$ is singular in $\mathcal{N}$, and $(\xi^+)^{\mathcal{R}''} = (\xi^+)^{\mathcal{R}''[N|\xi]}$ by the δ-c.c. of the extender algebra. Contradiction.

2. Let $\mathcal{R}_1 \in (H_{\mu^{++}})^{\mathcal{N}}$ be such that $\Sigma_{\mathcal{R}_1} \cap \mathcal{N} \in \mathcal{N}_+$ and $\mathcal{R}_1$ wins the comparison against $\mathcal{R}^-$. Then by the mouse set proof, $\kappa_{\mathcal{R}}$ is not bigger than the least strong of $L[\vec{E}, \Sigma_{\mathcal{R}_1}][\mathcal{R}_1]^{\mathcal{N}}$. So $\kappa_{\mathcal{R}}$ is not bigger than κ.

3. Let $\mathcal{M}_\mu$ be the direct limit of $(\mathcal{R}', \Lambda')$'s such that $\mathcal{R}' \in (H_{\mu^{++}})^{\mathcal{N}}$, Λ' is an $(o(\mathcal{N}), o(\mathcal{N}))$-iteration strategy for $\mathcal{R}'$ which is fullness preserving and has branch condensation. Then $\mathcal{N}$ captures iteration strategies for all proper hod initial segments of $\mathcal{M}_\infty$.

We know by definition of μ that for any $\mathcal{R}' \in B(\mathcal{R}, \Sigma_{\mathcal{R}})$, there is $\alpha < \lambda^{\mathcal{M}_\mu}$ such that $\mathcal{M}_\mu(\alpha)$ wins the comparison against $\mathcal{R}'$.

Let $\Lambda = \oplus_{\alpha<\lambda^{\mathcal{M}_\mu}} \Sigma_\alpha^{\mathcal{M}_\mu}$ and $\mathcal{M}_\mu^+ = \mathrm{Lp}_\omega^\Lambda(\mathcal{M}_\mu)$. Then $\mathcal{M}_\mu^+ \geq_{DJ} \mathcal{R}$. If $\mathcal{M}_\mu^+ \models$ "$\mathrm{cf}(\lambda^{\mathcal{M}_\mu^+})$ is not measurable", then $\Sigma_{\mathcal{M}_\mu^+} \cap \mathcal{N} \in \mathcal{N}_+$. Hence $\kappa_{\mathcal{R}} \leq \mu^+$. If not, let ν be the order 0 measure on $\mathrm{cf}(\lambda^{\mathcal{M}_\mu^+})$. Set $\mathcal{M}'_\mu = (\mathrm{Ult}(\mathcal{M}_\infty, \nu)(i''_\mu \lambda^{\mathcal{M}_\mu}))$. Then $\Sigma_{\mathcal{M}'_\mu} \cap \mathcal{N} \in \mathcal{N}_+$. Hence again $\kappa_{\mathcal{R}} \leq \mu^+$. □

Theorem 3.13. *Let $\mathcal{Q}$ be Σ-good. Let $\mathcal{N} = L[\vec{E}]^{\mathcal{Q}|\delta^{\mathcal{Q}}}$. For each κ, let S_κ be the transitive collapse of $H_\kappa = Hull^{\mathcal{N}}(\kappa \cup \operatorname{ran} \pi_{\mathcal{N}}^\infty)$. Then for each $\kappa \leq \kappa_P$, if κ is a strong cardinal of $\mathcal{N}$, then $\mathcal{P}(\kappa)^N = \mathcal{P}(\kappa)^{S_\kappa}$.*

Proof. Let $\mathcal{R}$ be of least rank in the hod mouse prewellordering such that $\kappa = \kappa_{\mathcal{R}}$.

Case 1. $\lambda^{\mathcal{R}}$ is a successor cardinal.

In this case, $\mu = \kappa_{\mathcal{R}^-} < \kappa$. Lemma 3.12 implies κ must be the least strong cardinal of $\mathcal{N}$ above μ. Let $\mathcal{M}_\mu, \Lambda, \mathcal{M}_\mu^+$ be as in the proof of Lemma 3.12. Since $\kappa_{\mathcal{R}} > \mu$, $\mathcal{M}_\mu^+ =_{DJ} \mathcal{R}^-$.

Let $\mathcal{M}^* = (HOD|\theta)^{D(L[\vec{E},\Lambda][\mathcal{M}_\infty^+]^{\mathcal{N}},<\kappa)}$. Then by the proof of MSC, $\mathcal{M}^* =_{DJ} \mathcal{R}$ and $\Sigma_{\mathcal{M}^*}\restriction\mathcal{N} \in \mathcal{N}_+$. We claim that $(\kappa^+)^{\mathcal{N}} \subseteq \mathcal{M}^*$. For otherwise, $\mathcal{M}^*$ has cardinality κ in $\mathcal{N}$. Let E be an extender on N with critical point κ. Let E^* be the resurrection of E. Then $\mathcal{M}^*, i_E\restriction\mathcal{M}^* \in i_{E^*}(\mathcal{N})$. By elementarity, $\Sigma_{i_{E^*}(\mathcal{M}^*)}\restriction i_{E^*}(\mathcal{N}) \in i_{E^*}(\mathcal{N}_+)$. Therefore, $\Sigma_{\mathcal{M}^*}\restriction\text{Ult}(\mathcal{N}, E) \in \text{Ult}(\mathcal{N}_+, E)$, by pulling back the strategy of $i_{E^*}(\mathcal{M}^*)$. Hence,

$$\begin{aligned} i_{E^*}(\mathcal{N}_+) \models \text{“ I have captured the iteration strategy of some } \mathcal{M} \text{ of size } < i_{E^*}(\kappa) \\ \text{which is fullness preserving and has branch condensation, and} \\ \mathcal{M} \text{ iterates longer than } \mathcal{M}_\mu\text{”} \end{aligned}$$

By elementarity,

$$\begin{aligned} \mathcal{N}_+ \models \text{“ I have captured the iteration strategy of some } \mathcal{M} \text{ of size} < \kappa \\ \text{which is fullness preserving and has branch condensation, and} \\ \mathcal{M} \text{ iterates longer than } \mathcal{M}_\mu\text{”} \end{aligned}$$

This means $\kappa_{\mathcal{R}} < \kappa$. Contradiction.

But we can show $\mathcal{M}^* \subseteq H_\kappa$. Let

$$\varphi : \mathcal{P} \to P_\kappa$$

be the iteration map whose generators are below κ such that $M^* \lhd_{hod} P_\kappa$. Let

$$\psi : P_\kappa \to \mathcal{Q}_{\mathcal{N}}^\infty$$

be the tail of the direct limit map. κ, being the least strong of $\mathcal{N}$ above μ, is in H_κ. Hence $\mathcal{M}^* \in H_\kappa$. It follows from proof of MSC then $\psi \restriction \mathcal{M}^* \in H_\kappa$. Now fix any $\alpha \in \mathcal{M}^*$. We may find $f \in \mathcal{P}$ and $a \in [\kappa]^{<\omega}$ such that

$$\alpha = \varphi(f)(a).$$

Hence

$$\begin{aligned}
\alpha &= \varphi(f)(a) \\
&= \psi^{-1}(\pi_{\mathcal{N}}^{\infty}(f))(a) \\
&= (\psi \restriction \mathcal{M}^*)^{-1}(\pi_{\mathcal{N}}^{\infty}(f))(a) \\
&\in H_\kappa
\end{aligned}$$

It follows that $\kappa^+ \subseteq H_\kappa$. Hence $\mathcal{P}(\kappa)^{\mathcal{N}} = \mathcal{P}(\kappa)^{S_\kappa}$.

Case 2. $\lambda^{\mathcal{R}}$ is a limit ordinal.

Let $\mathcal{M}_\kappa, \Lambda, \mathcal{M}_\kappa^+$ be as in Lemma 3.12 with κ in place of μ. Observe that $\lambda^{\mathcal{M}_\kappa}$ is a limit ordinal, $\mathcal{M}_\kappa^+$ wins the comparison against $\mathcal{R}$, but every hod initial segment of $\mathcal{M}_\kappa^+$ loses the comparison against $\mathcal{R}$. We have $o(M_\kappa) < (\kappa^+)^{\mathcal{N}}$, as the direct limit system has $\mathcal{N}$-size κ. Let $\mathcal{M}^* = \mathrm{Lp}^{\Lambda}(\mathcal{M}_\kappa)$. Essentially a similar argument as in Case 1 gives that $o(\mathcal{M}^*) = (\kappa^+)^{\mathcal{N}}$: otherwise, let E be an extender on $\mathcal{N}$ with critical point κ. Then $\mathcal{M}^*, i_{E^*} \restriction \mathcal{M}^* \in \mathrm{Ult}(\mathcal{N}, E)$. Let E^* be the resurrection of E. It follows then $\Sigma_{\mathcal{M}^*} \restriction i_{E^*}(\mathcal{N}) \in i_{E^*}(\mathcal{N}_+)$. The fact E^* is a background extender on $\mathcal{Q}$ implies that $\mathcal{M}^*$ iterates to $i_{E^*}(\mathcal{M}^*)$. Here is the reason. $\mathcal{M}^*$, being the direct limit, means that there is a stack $\vec{\mathcal{T}}$ on $\mathcal{P}$ with last model $\mathcal{P}_\kappa$ such that $\mathcal{M}^* \lhd \mathcal{P}_\kappa$. So $i_{E^*}(\vec{\mathcal{T}})$, a continuation of $\vec{\mathcal{T}}$, is a stack on $\mathcal{P}$ with last model $i_{E^*}(\mathcal{P}_\kappa)$ such that $i_{E^*}(\mathcal{M}^*) \lhd \mathcal{P}_\kappa$. Moreover, i_{E^*} agrees with the tail of the iteration map along $i_{E^*}(\vec{\mathcal{T}}) \setminus \vec{\mathcal{T}}$: Pick any $y \in \mathcal{P}_\kappa$, there is $\mathcal{K}$, an model of along $\vec{\mathcal{T}}$, and $x \in \mathcal{K}$ such that

$y = i_{\mathcal{K}\mathcal{P}_\kappa}(x)$. Thus $i_{E^*}(y) = i_{E^*}(i_{\mathcal{K}\mathcal{P}_\kappa}(x)) = i_{\mathcal{K},i_{E^*}(\mathcal{P}_\kappa)}(x) = i_{\mathcal{P}_\kappa,i_{E^*}(\mathcal{P}_\kappa)}(i_{\mathcal{K},\mathcal{P}_\kappa}(x)) = i_{\mathcal{P}_\kappa,i_{E^*}(\mathcal{P}_\kappa)}(y)$. Hence,

$i_{E^*}(\mathcal{N}_+) \models$ " I have captured the iteration strategy of some $\mathcal{M}$ of size $< i_{E^*}(\kappa)$
which is fullness preserving and has branch condensation, and
$\mathcal{M}$ iterates to $i_{E^*}(\mathcal{M}^*)$."

By elementarity,

$\mathcal{N}_+ \models$ " I have captured the iteration strategy of some $\mathcal{M}$ of size $< \kappa$
which is fullness preserving and has branch condensation, and
$\mathcal{M}$ iterates to $\mathcal{M}^*$"

This means $\kappa_{\mathcal{R}} < \kappa$. Contradiction. We continue out proof. In contrast to the κ successor strong case, now we don't necessarily have $\kappa \in H_\kappa$. We split into two cases.

Subcase 2.1 $\kappa \in H_\kappa$.

Let $\varphi : \mathcal{P} \to \mathcal{P}_\kappa$ be the iteration map whose generators are below κ such that $\mathcal{M}_\kappa \trianglelefteq_{hod} \mathcal{P}_\kappa$. Let $\psi : \mathcal{P}_\kappa \to \mathcal{Q}^\infty_{\mathcal{N}}$ be the tail of the direct limit map. We have $\psi \restriction \mathcal{M}_\kappa \in H_\kappa$. (We don't necessarily have $\psi \restriction \mathcal{M}^+_\kappa$ in this hull. $\mathcal{N}$ does capture strategies of all proper hod initial segments of $\mathcal{M}^+_\kappa$, but $\mathcal{N}$ does not capture the full strategy of $\mathcal{M}^+_\kappa$) The same argument as in Case 1 shows $o(\mathcal{M}_\kappa) \subseteq H_\kappa$. Now some more argument is needed in order to get $\kappa^+ \subseteq H_\kappa$. Fix an $A \in \mathcal{P}(o^{\mathcal{M}_\kappa}) \cap \mathcal{M}^+_\kappa$.

We may assume $A = \varphi(f)(a)$, $f \in \mathcal{P}$, $a \in [\kappa]^{<\omega}$. Then for any $\beta \in o(\mathcal{M}_\kappa)$,

$$\begin{aligned}\beta \in A &\leftrightarrow \psi(\beta) \in \psi(A)\\ &\leftrightarrow \psi(\beta) \in \psi(\varphi(f)(a))\\ &\leftrightarrow \psi(\beta) \in \pi^\infty_{\mathcal{N}}(f)(\psi(a))\\ &\leftrightarrow \psi(\beta) \in \pi^\infty_{\mathcal{N}}(f)((\psi \restriction \mathcal{M}_\kappa)(a))\end{aligned}$$

Hence $A \in H_\kappa$. Hence $(\kappa^+)^{\mathcal{N}} = o(\mathrm{Lp}^\Lambda(\mathcal{M}_\kappa)) \subseteq H_\kappa$.

Subcase 2.2 $\kappa \notin H_\kappa$.

κ is a strong limit of strongs of $\mathcal{N}$. If we let $\nu = \min(H_\kappa \setminus \kappa)$, then ν is also a strong limit of strongs of $\mathcal{N}$. Otherwise, the largest strong or strong limit of strong of $\mathcal{N}$ below ν, say μ, is definable from ν over $\mathcal{N}$, hence in the hull, but $\kappa \leq \mu < \nu$, contradiction. The discussion before subcase 2.1 shows that $(\kappa^+)^{\mathcal{N}} = (o(\mathcal{M}_\kappa)^+)^{\mathcal{M}^+_\kappa}$, $(\nu^+)^{\mathcal{N}} = (o(\mathcal{M}_\nu)^+)^{\mathcal{M}^+_\nu}$.

Let $\sigma_1 : \mathcal{P} \to \mathcal{P}_\kappa$ be an iteration map whose generators are below κ such that $\mathcal{M}^+_\kappa \trianglelefteq_{hod} \mathcal{P}_\kappa$. Let $\sigma : \mathcal{P}_\kappa \to \mathcal{P}_\nu$ be an iteration map whose generators are below ν such that $\mathcal{M}^+_\nu \trianglelefteq_{hod} \mathcal{P}_\nu$. Let $\sigma_2 = \sigma \circ \sigma_1$. Let $\psi : \mathcal{P}_\nu \to Q^\infty_{\mathcal{N}}$ be the tail of direct limit map. We firstly show that $H_\kappa \cap (\nu^+)^{\mathcal{N}} + 1 \subseteq \operatorname{ran}\sigma$. Fix an $\beta \in H_\kappa$ such that $\beta \leq (\nu^+)^{\mathcal{N}}$. Suppose $\beta = \sigma_2(f)(\alpha)$, $f \in \mathcal{P}$, $\alpha < \nu$. Let

$$\gamma = \min\{\bar{\alpha} < \nu : \sigma_2(f)(\bar{\alpha}) = \beta\}.$$

Since $\sigma_2(f) \restriction \nu = (\psi \restriction \nu)^{-1}(\pi^\infty_{\mathcal{N}}(f) \restriction \nu) \in H_\kappa$, we have $\gamma \in H_\kappa$. But $\gamma < \nu$, hence $\gamma < \kappa$. Hence $\beta = \sigma_2(f)(\gamma) = \sigma(\sigma_1(f)(\gamma)) \in \operatorname{ran}(\sigma)$.

Observe that $\sigma''o(\mathcal{M}_\kappa) \subseteq o(\mathcal{M}_\nu)$. The above paragraph shows that $(\nu^+)^{\mathcal{N}}$ has a preimage under σ. So $\sigma^{-1}((\nu^+)^{\mathcal{N}}) \geq (o(\mathcal{M}_\kappa)^+)^{\mathcal{M}^+_\kappa} = (\kappa^+)^{\mathcal{N}}$. On the other hand, we can show that $\sigma''(\kappa^+)^{\mathcal{N}} \subseteq H_\kappa$. Fix an $\alpha < o(\mathcal{M}_\kappa)$. Say $\alpha = \sigma_1(f)(a)$, $f \in \mathcal{P}$, $\alpha \in [\kappa]^{<\omega}$. Then

$$\begin{aligned}\sigma(a) &= \sigma(\sigma_1(f)(a))\\ &= \sigma_2(f)(\sigma(a))\\ &= \sigma_2(f)(a)\\ &= (\psi \restriction o(\mathcal{M}_\nu))^{-1}(\pi^\infty_{\mathcal{N}}(f))(a)\\ &\in H_\kappa.\end{aligned}$$

We conclude that $\sigma((\kappa^+)^{\mathcal{N}}) \leq (\nu^+)^{\mathcal{N}}$, $\sigma''(\kappa^+)^{\mathcal{N}} \subseteq H_\kappa$. Hence $(\nu^+)^{\mathcal{N}}$ collapses down to an ordinal $\geq (\kappa^+)^{\mathcal{N}}$ when forming the transitive collapse $H_\kappa \to S_\kappa$. Condensation implies that every level of H_κ projecting to ν collapses down to an initial segment of $\mathcal{N}$ projecting to κ. Hence $S_\kappa|(\kappa^+)^{S_\kappa} \supseteq \mathcal{N}|(\kappa^+)^{\mathcal{N}}$. Hence $\mathcal{P}(\kappa)^{S_\kappa} = \mathcal{P}(\kappa)^{\mathcal{N}}$. □

Lemma 3.14. *Suppose that $j : L[\mathcal{Q}|\eta] \to L[\mathcal{Q}]$ is elementary, $j \restriction \eta = id$, $j(\eta) = \delta_0$. Let $\mathcal{N}|\xi$ be an initial segment of $\mathcal{N}$. Suppose η is Woodin in $\mathcal{N}|\xi$, so that $\mathcal{Q}|\eta$ is generic over $\mathcal{N}|\xi$ over the extender algebra. Let $g = j^{-1}(G) \subseteq \mathbb{Q}_\eta$ be the natural $\mathcal{N}|\xi$-generic filter. Let $\langle E^*_0, \ldots, E^*_n\rangle$ be extenders giving rise to a finite iteration tree $\mathcal{T}$ on $\mathcal{Q}$. Assume that each E^*_i overlaps η. Then $Tr^g(i_{E^*_n}(\mathcal{N}|\eta))$ is defined and is an S-premouse.*

Proof. Since each E^*_i overlaps η, 0 is the predecessor of every other node of $\mathcal{T}$. As there is no infinite iteration tree such that 0 is the predecessor of every other node, we may arrange an induction and assume that for all $\mathcal{U}$ extending $\mathcal{T}$ such that every extender applied on $\mathcal{U}$ overlaps η, $Tr^g(i^{\mathcal{U}}(\mathcal{N}|\eta))$ is defined and is an S-premouse.

By Lemma 3.3, all we need to see is that when $\langle E_{n+1}, \ldots, E_m\rangle \in U(i_E(\mathcal{N}|\eta), \eta)$,

$\kappa = \mathrm{crt}(E_m)$, then letting $S_\delta^{\mathcal{N}|\eta}$ be the transitive collapse of $Hull^{(\mathcal{N}|\eta)_+}(\kappa \cup D(h))$, all of the following holds.

1. $\mathcal{P}(\kappa)^{\mathcal{N}|\eta} \subseteq S_\kappa^{\mathcal{N}|\eta}$.

2. $Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))$ has a drop. Let d be the largest drop of $Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))$, then $(d^+)^{Tr^g(\mathrm{Ult}(\mathcal{N}|\eta,E_m))} \subseteq Hull^{Tr^g(\mathrm{Ult}(\mathcal{N}|\eta,E_m))}(d \cup i''_E D(g))$.

3. Let $\mathcal{K} = \mathrm{Ult}(\mathcal{N}, E_{n-1})|lh(E_m)$ if $m > 1$, or $\mathcal{K} = \mathcal{N}|lh(E_0)$ if $m = 0$. Then $Tr^g(\mathcal{K}) = S^{\mathfrak{P}+1}(Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))|d)$ is an S-premouse.

1 and 2 above will tell us $Tr^g(i_{E_n^*})(\mathcal{N}|\eta)$ is defined; 3 will tell us $Tr^g(i_{E_n^*})(\mathcal{N}|\eta)$ is an S-premouse.

1 comes from Theorem 3.13. If $\kappa \leq \kappa_{\mathcal{P}}$, then $\mathcal{P}(\kappa)^{\mathcal{N}} \subseteq S_\kappa^{\mathcal{N}}$, hence by elementarity, $\mathcal{P}(\kappa)^{\mathcal{N}\eta} \subseteq S_\kappa^{\mathcal{N}|\eta}$. But we always have $\kappa \leq \kappa_{\mathcal{P}}$. Otherwise, κ is a limit of Woodins of $\mathcal{N}$. Pick $\gamma \in (\kappa_{\mathcal{P}}, \kappa)$ Woodin of $\mathcal{N}$. Then γ is $\Sigma_1^2(\Sigma)$-Woodin in $\mathcal{N}$. Hence γ is $\Sigma_1^2(\Sigma)$-Woodin in $\mathcal{Q}$. This contradicts suitability of $\mathcal{Q}$.

We show 2. If $Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))$ has no drop, that means $Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))$ is a $S^{[\lambda^{\mathcal{P}}]}$-premouse, i.e. it has reached the largest degree all the way. Theorem 2.35 tells us a basic property of the S-operators that an $S^{[\lambda^{\mathcal{P}}]}$-premouse defines $\Sigma_{\mathcal{P}}$. Hence $\mathrm{Ult}(N|\eta, E_m)[g]$ knows how to iterate $\mathcal{P}$. By generic comparison argument, there is $\mathcal{R} \in I(\mathcal{P}, \Sigma)$ such that $\mathcal{R} \in \mathrm{Ult}(\mathcal{N}|\eta, E_m)|i_{E_m}(\kappa)$ and $\Sigma_{\mathcal{R}} \restriction \mathrm{Ult}(\mathcal{N}|\eta, E_m) \in \mathrm{Ult}(\mathcal{N}|\eta, E_m)_+$. By elementarity, there is $\mathcal{R} \in I(P, \Sigma)$ such that $\mathcal{R} \in \mathcal{N}|\kappa$ and $\Sigma_{\mathcal{R}} \restriction \mathcal{N} \in \mathcal{N}_+$. This means $\kappa_{\mathcal{P}} < \kappa$. Contradiction. Now let d be the largest drop of $Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))$. Let $d^* = (d^+)^{Tr^g(\mathrm{Ult}(\mathcal{N}|\eta,E_m))}$. To show $d^* \subseteq Hull^{Tr^g(\mathrm{Ult}(\mathcal{N}|\eta,E_m))}(d \cup i''_E D(g))$, it suffices to show that $Tr^g(\mathcal{K}) = S^{\mathfrak{P}+1}(Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))|d)$. We prove this next.

Regarding 3, let $E^*_{n+1}, \ldots, E^*_m, k_{n+1}, \ldots, k_m$ be as follows.

$$E^*_{n+1} = \text{the resurrection of } E_{n+1}$$
$$k_{n+1} : \mathrm{Ult}(\mathcal{N}|\eta, E_{n+1}) \to i_{E^*_n}(\mathcal{N}|\eta) \text{ is the lifting map}$$

When $i > n$,

$$E^*_{i+1} = \text{the resurrection of } k_i(E_i),$$
$$s_{i+1} : \mathrm{Ult}(\mathcal{N}|\eta, k_i(E_i)) \to i_{E^*_i}(\mathcal{N}|\eta) \text{ is the lifting map}$$
$$t_{i+1} : \mathrm{Ult}(\mathcal{N}|\eta, E_i) \to \mathrm{Ult}(\mathcal{N}|\eta, k_i(E_i)) \text{ is the canonical map}$$
$$k_{i+1} = s_{i+1} \circ t_{i+1}.$$

Our induction hypothesis says that $Tr^g(i_{E^*_m}(\mathcal{N}|\eta))$ is defined and is an S-premouse. We know k_m extends to a map from $\mathrm{Ult}(\mathcal{N}|\eta, E_m)_+[g] \to (i_{E^*_m}(\mathcal{N}|\eta))_+[g]$. Still call this k_m. By Lemma 3.3, the universe of $Tr^g(i_{E^*_m}(\mathcal{N}|\eta))$ is equal to the universe of $i_{E^*_m}(\mathcal{N}|\eta)[g]$. The map k_m pulls back the property of well-definedness of translation back to $\mathrm{Ult}(\mathcal{N}|\eta, E_m)$. Thus $Tr^g(\mathrm{Ult}(\mathcal{N}, E_m))$ is well-defined and has universe equal to $\mathrm{Ult}(\mathcal{N}|\eta, E_m)[g]$. k_m is actually an elementary map from $Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))$ to $Tr^g(i_{E^*_m}(\mathcal{N}|\eta))$. By Theorem 2.46, $Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))$ is an S-premouse as well. Let d be the largest drop of $Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))$. Let $\mathfrak{P} = \deg(Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))|d)$. Then $k_m(d)$ is the largest drop of $Tr^g(i_{E^*_m}(\mathcal{N}|\eta))$ and $k_m(\mathfrak{P}) = \deg(Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|k(d))$. We just need to prove the following sublemma

Sublemma 3.15. *$S^{k_m(\mathfrak{P})+1}(Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|k_m(d))$ is the amenable code of the* *transitive collapse of*

$$\langle Hull^{(Tr^g(i_{E^*_m}(\mathcal{N}|\eta)))^+}(d^*), \in, g, \dot{E}^{Tr^g(i_{E^*_m}(\mathcal{N}|\eta))}, \emptyset, S^{Tr^g(i_{E^*_m}(\mathcal{N}|\eta))},$$
$$Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|k_m(d), i_{E^*_m}(\mathcal{Q}^\infty_{\mathcal{N}|\eta}), \mathrm{ran}(i_{E^*_m} \circ \pi^\infty_{\mathcal{N}|\eta})\rangle$$

Proof. By fullness of background constructions, $SSM^{k_m(\mathfrak{P})}(Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|k_m(d)) = Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|d^*$, where $d^* = (k_m(d))^{+Tr^g(i_{E^*_m}(\mathcal{N}|\eta))}$. Hence it suffices to show that $S^{*,k_m(\mathfrak{P})+1}(Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|d^*)$ is the amenable code of the transitive collapse of

$$\langle Hull^{(Tr^g(i_{E^*_m}(\mathcal{N}|\eta)))^+}(k_m(d)), \in, Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|d^*, \dot{E}^{Tr^g(i_{E^*_m}(\mathcal{N}|\eta))}, \dot{S}^{Tr^g(i_{E^*_m}(\mathcal{N}|\eta))},$$
$$i_{E^*_m}(\mathcal{Q}^\infty_{\mathcal{N}|\eta}), \mathrm{ran}(i_{E^*_m} \circ \pi^\infty_{\mathcal{N}|\eta})\rangle$$

Let $\mathcal{R} = (L[\vec{E}, \Sigma][Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|d^*])^{\mathrm{Ult}(\mathcal{Q}, E^*_m)|i_{E^*_m}(\delta^{\mathcal{Q}})}$. Then $\mathcal{R}$ is a good universe of defining the S-operators. Let $\mathcal{H} = L[\vec{E}, S^{\mathfrak{P}}][Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|d^*]^{\mathcal{R}}$. Then $S^{*,k_m(\mathfrak{P})+1}(Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|d^*)$ is the amenable code of the transitive collapse of

$$\langle Hull^{\mathcal{H}_+}(d^*), \in, Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|d^*, \dot{E}^{\mathcal{H}}, \emptyset, \mathcal{S}^{\mathcal{H}}, \mathcal{Q}^\infty_{\mathcal{H}}, \pi^\infty_{\mathcal{H}}\rangle$$

where $\pi^\infty_{\mathcal{H}} : \mathcal{P} \to Q^\infty_{\mathcal{H}}$ is the direct limit map of $I(\mathcal{P}, \Sigma) \cap \mathcal{H}$. Since d^* is a strong cutpoint of $Tr^g(i_{E^*_m}(\mathcal{N}|\eta))$, d^* is a strong cutpoint of $Tr^g(i_{E^*_m}(\mathcal{N}))$ as well. We can view $Tr^g(i_{E^*_m}(\mathcal{N}))$ as an $S^{\mathfrak{P}}$-premouse over $Tr^g(i_{E^*_m}(\mathcal{N}))|d^*$. We may compare the constructions of $Tr^g(i_{E^*_m}(\mathcal{N}))$ versus $\mathcal{H}$, by hitting background extenders of disagreements. Universality of maximal background constructions tells us that there are $\mathcal{S}_1$ and $\mathcal{S}_2$, which are iterates of $\mathcal{Q}$, such that $i_{\mathcal{Q}\mathcal{S}_1}(Tr^g(i_{E^*_m}(\mathcal{N}))) = i_{\mathcal{Q}\mathcal{S}_2}(\mathcal{H})$. Note by elementarity of $i_{E^*_m}$ that $i_{E^*_m}(\pi^\infty_{\mathcal{N}}) : \mathcal{P} \to i_{E^*_m}(\mathcal{Q}^\infty_{\mathcal{N}})$ is the direct limit map of $I(P, \Sigma) \cap Tr^g(i_{E^*_m}(\mathcal{N}))$. By elementarity, $i_{\mathcal{Q}\mathcal{S}_1} \circ i_{E^*_m} \circ \pi^\infty_{\mathcal{N}} = i_{\mathcal{Q}\mathcal{S}_2} \circ \pi^\infty_{\mathcal{H}}$. Hence, $S^{*,k_m(\mathfrak{P})+1}(Tr^g(i_{E^*_m}(\mathcal{N}|\eta))|d^*)$ is the amenable code of the transitive collapse of

$$\langle Hull^{Tr^g(i_{E_m^*}(\mathcal{N}_+))}(d^*), \in, Tr^g(i_{E_m^*}(\mathcal{N}|\eta))|d^*, \dot{E}^{Tr^g(i_{E_m^*}(\mathcal{N}))}, \emptyset, \mathcal{S}^{Tr^g(i_{E_m^*}(\mathcal{N}))},$$
$$i_{E_m^*}(\mathcal{Q}^\infty_{\mathcal{N}}), i_{E_m^*} \circ \pi^\infty_{\mathcal{N}}\rangle.$$

But we have the elementary $j : L[\mathcal{Q}|\eta] \to L[\mathcal{Q}]$ with $j \restriction \eta = id$, $j(\eta) = \delta_0$. This implies

$$Th^{\mathcal{N}_+}(\eta \cup \operatorname{ran} \pi^\infty_{\mathcal{N}} \cup \{\mathcal{Q}^\infty_{\mathcal{N}}\}) = Th^{(\mathcal{N}|\eta)_+}(\eta \cup \operatorname{ran} \pi^\infty_{\mathcal{N}|\eta} \cup \{\mathcal{Q}^\infty_{\mathcal{N}|\eta}\})$$

By applying $i_{E_m^*}$ to the above equality, we get

$$Th^{i_{E_m^*}(\mathcal{N}_+)}(i_{E_m^*}(\eta) \cup \operatorname{ran}(i_{E_m^*} \circ \pi^\infty_{\mathcal{N}}) \cup \{i_{E_m^*}(\mathcal{Q}^\infty_{\mathcal{N}})\}) =$$
$$Th^{i_{E_m^*}((\mathcal{N}|\eta)_+)}(i_{E_m^*}(\eta) \cup \operatorname{ran}(i_{E_m^*} \circ \pi^\infty_{\mathcal{N}|\eta}) \cup \{i_{E_m^*}(\mathcal{Q}^\infty_{\mathcal{N}|\eta})\}).$$

Therefore, $S^{*,k_m(\mathfrak{P})+1}(Tr^g(i_{E_m^*}(\mathcal{N}|\eta))|d^*)$ is the amenable code of the transitive collapse of

$$\langle Hull^{(Tr^g(i_{E_m^*}(\mathcal{N}|\eta)))_+}(k_m(d)), \in, Tr^g(i_{E_m^*}(\mathcal{N}|\eta))|d^*, \dot{E}^{Tr^g(i_{E_m^*}(\mathcal{N}|\eta))}, \dot{S}^{Tr^g(i_{E_m^*}(\mathcal{N}|\eta))},$$
$$i_{E_m^*}(\mathcal{Q}^\infty_{\mathcal{N}|\eta}), \operatorname{ran}(i_{E_m^*} \circ \pi^\infty_{\mathcal{N}|\eta})\rangle$$

This proves the sublemma. □

From the sublemma, k_m induces the embedding from $Tr^g(\mathcal{K}) = \langle Hull^{Tr^g(\mathrm{Ult}(\mathcal{N}|\eta,E_m))_+}(d)$, $\in, g, \dot{E}^{Tr^g(\mathrm{Ult}(\mathcal{N}|\eta,E_m))}, \emptyset, \dot{S}^{Tr^g(\mathrm{Ult}(\mathcal{N}|\eta,E_m))}, Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_m))|d, i_{E_m^*}(\mathcal{Q}^\infty_{\mathcal{N}|\eta}), \operatorname{ran}(i_{E_m} \circ \pi^\infty_{\mathcal{N}|\eta})\rangle$ to $S^{k_m(\mathfrak{P})+1}(Tr^g(i_{E_m^*}(\mathcal{N}|\eta))|k_m(d))$. By Theorem 2.46, $Tr^g(\mathcal{K})$ is an S-premouse. □

Lemma 3.16. *Suppose that $j : L[\mathcal{Q}|\eta] \to L[\mathcal{Q}]$ is elementary, $j \restriction \eta = id$, $j(\eta) = \delta_0$. Let $\mathcal{N}|\xi$ be an initial segment of $\mathcal{N}$. Suppose η is Woodin in $\mathcal{N}|\xi$, so that $\mathcal{Q}|\eta$ is* *generic over $\mathcal{N}|\xi$ over the extender algebra. Let $g = j^{-1}(G) \subseteq \mathbb{Q}_\eta$ be the natural $\mathcal{N}|\xi$-generic filter. Then$Tr^g(\mathcal{N}|\xi)$ is an S-premouse, and is S-iterable.*

Proof. Major arguments are already in Lemma 3.14. To see $Tr^g(\mathcal{N}|\xi)$ is an S-premouse, we need to see when $\langle E_0, \ldots, E_n\rangle \in U(\mathcal{N}|\xi, \eta)$, $\kappa = \mathrm{crt}(E)$, then letting $S_\delta^{\mathcal{N}|\eta}$ be the transitive collapse of $Hull^{(\mathcal{N}_\eta)+}(\kappa \cup D(h))$,

1. $\mathcal{P}(\kappa)^{\mathcal{N}} \subseteq S_\kappa^{\mathcal{N}|\eta}$.

2. $Tr^g(\mathrm{Ult}(\mathcal{N}|\eta, E_n))$ has a drop.

3. Let $\mathcal{K} = \mathrm{Ult}(\mathcal{N}, E_{n-1})|lh(E_n)$ if $n > 1$, or $\mathcal{K} = \mathcal{N}|lh(E_0)$ if $n = 0$. Then $Tr^g(\mathcal{K})$ is an S-premouse.

If $n > 0$, this is what we proved in Lemma 3.14. If $n = 0$, the same proof of Lemma 3.16 goes through. To see that $Tr^g(\mathcal{N}|\xi)$ is S-iterable, according to Theorem 3.9, we need to see that letting Λ the induced strategy of $\mathcal{N}|\xi$, then every λ iterate of $\mathcal{N}|\xi$ above η translates into an S-premouse. Let $\mathcal{M}$ be a Λ-iterate of $\mathcal{N}|\xi$. By lifting the tree on $\mathcal{N}|\xi$ onto $\mathcal{Q}$, we get an iterate $\mathcal{R}$ of $\mathcal{Q}$ above η and a lifting map $k : \mathcal{M} \to \mathcal{K}$, where $\mathcal{K}$ is a model of $L[\vec{E}]$-construction of $\mathcal{R}$, $k \restriction \eta = id$. The map $i_{\mathcal{QR}} \circ j : L[\mathcal{R}|\eta] \to L[\mathcal{R}]$ meets the assumptions of this lemma. The result we just proved gives that $Tr^g(\mathcal{K})$ is an S-premouse. By Theorem 2.46, $Tr^g(\mathcal{M})$ is an S-premouse as well. □

We have done preparation work showing S-iterability of translations of background constructions. Let's finally start proving the main theorem. We define a Prikry forcing as in [14]. If a is countable transitive, $x \in \mathbb{R}$, x is coded by a real recursive in x, let

$$\mathcal{F}_a^x = \{\mathcal{Q}_z : z \leq_T x \wedge \mathcal{Q}_z \text{ is } \Sigma\text{-good over } a\}.$$

If T is a tree projecting to the universal $\Sigma_1^2(\Sigma)$-set, we may simultaneously compare all $\mathcal{Q}_z \in \mathcal{F}_a^x$ inside $L[T, x]$, while at the same time making all reals recursive in x

generic for the extender algebra at the image of the Woodin cardinal. $L[T, x]$ can find the correct branch for short trees, because it can figure out the $\mathcal{Q}$-structures: If $\mathcal{M} \in L[T, x]$ is a Σ-mouse over a with a $\mathcal{Q}$-structure $\mathcal{Q}(\mathcal{M})$, then both $\mathcal{Q}(\mathcal{M})$ and the iteration strategy for $\mathcal{Q}(\mathcal{M})$ are $OD(\Sigma, \mathcal{M})$, so $\mathcal{Q}(\mathcal{M}) \in L[T, x]$. Hence the simultaneous comparison is definable in $L[T, x]$ until one of the trees is maximal. But then suitability of the $\mathcal{Q}_z$'s imply that as soon as one of the trees in the comparison is maximal, the others are also maximal. $L[T, x]$ can therefore figure out the last model of the simultaneous comparison, that is $\mathrm{Lp}^{\Sigma}_{\omega}(\mathcal{M}(\mathcal{T}))$ for one of the comparison trees $\mathcal{T}$, without figuring out the last branch. We then let $\mathcal{Q}^x_a$ be the result of the simultaneous comparison. For $d = [x]_T$ a Turing degree, we denote $\mathcal{Q}^d_a = \mathcal{Q}^x_a$ for any $x \in d$. For $\vec{d} = \{d_0 \leq_T \ldots d_n\}$ we let

$$\mathcal{Q}^{\vec{d}}_0 = \mathcal{Q}^{d_0}_a$$
$$\mathcal{Q}^{\vec{d}}_{i+1} = \mathcal{Q}^{d_{i+1}}_{\mathcal{Q}_i}$$

Let $\nu_{\langle \mathcal{Q}_0, \ldots, \mathcal{Q}_n \rangle}$ be the measure on $\{\mathcal{Q}^d_{\mathcal{Q}_n} : d_n \leq_T d\}$ as induced by the Martin measure on the Turing degrees:

$$\nu_n(A) = 1 \leftrightarrow \text{ for a Turing cone of } d, \mathcal{Q}^d_{\mathcal{Q}_n} \in A.$$

$\mathbb{P}_0$ is tree Prikry forcing whose conditions are $(\langle \mathcal{Q}_0, \ldots, \mathcal{Q}_n \rangle, S)$ with the following properties.

1. for some $\vec{d} = \{d_0 \leq_T \ldots \leq_T d_n\}$, $\langle \mathcal{Q}_0, \ldots, \mathcal{Q}_n \rangle = \vec{\mathcal{Q}}^{\vec{d}}$.
2. for each $v \in S$, either v is an initial segment of $\langle \mathcal{Q}_0, \ldots, \mathcal{Q}_n \rangle$ or $\langle \mathcal{Q}_0, \ldots, \mathcal{Q}_n \rangle$ is an initial segment of v.
3. for each $\langle \mathcal{Q}_0, \ldots, \mathcal{Q}_m \rangle \in S$, $\{\mathcal{R} : \langle \mathcal{Q}_0, \ldots, \mathcal{Q}_m, \mathcal{R} \rangle \in S\} \in \nu_{\langle \mathcal{Q}_0, \ldots, \mathcal{Q}_m \rangle}$.

Let $\mathbb{P}_0$ be the forcing defined as above in $L(U, \mathbb{R})$, U is a $\Sigma^2_1(\Sigma)$-complete set, and let $\mathbb{P}$ be such forcing defined in V. Let $\mathcal{Q}_\infty$ be the Prikry generic Σ-premouse for $\mathbb{P}_0$ over $L(U, \mathbb{R})$. The Prikry condition [3, 14] shows that Woodins cardinals of each stem is still Woodin in $L[\mathcal{Q}_\infty]$. Let $\delta_0 < \delta_1 < \cdots$ list the Woodins of $\mathcal{Q}_\infty$. Build

- $\mathcal{R}_1 = (L[\vec{E}, S^{[\lambda^{\mathcal{P}}]}][\mathcal{Q}_0])^{\mathcal{Q}_\infty|\delta_1}$,
- $\mathcal{R}_{i+1} = (L[\vec{E}, S^{[\lambda^{\mathcal{P}}]}][\mathcal{R}_i])^{\mathcal{Q}_\infty|\delta_{i+1}}$.

Let $\mathcal{R}_\infty = \bigcup_{n<\omega} \mathcal{R}_n$. Then $\mathcal{R}_\infty$ can be viewed as an S-premouse over $\mathcal{Q}_0$.

Lemma 3.17. *Let h be generic over $V[\mathcal{Q}_\infty]$ for the poset whose conditions are $\langle h_0, \ldots, h_n \rangle$ such that $h_i : \omega \to \delta_n$ are generic over $L[\mathcal{R}_\infty][\mathcal{Q}_\infty|\delta_0]$ for $Coll(\omega, \delta_n)$. Then*

1. *The universal $\Sigma^2_1(\Sigma)$ set is Suslin in $L[\mathcal{R}_\infty](\mathbb{R}^V)$.*
2. *$L(U, \mathbb{R}) = (L(\mathcal{A}_h, \mathbb{R}^*_h))^{L[\mathcal{R}_\infty][h]} = (L(Hom^*_h, \mathbb{R}^*_h))^{L[\mathcal{R}_\infty][h]}$.*

Proof. 1. Let T_n be the tree in $L[\mathcal{R}_\infty](\mathbb{R}^*_h)$ attempting to build x, y, z, w such that

(a) $x, y \in \mathbb{R}^*_h$,

(b) z codes $\mathcal{M} \prec \mathcal{R}_n$. $z(0)$ codes $z_0 \in \mathcal{M}$, $z(1)$ codes $z_1 \in \mathcal{M}$, $z(2)$ codes a condition $p \in Coll(\omega, \delta_n)$.

(c) w codes a $\mathcal{M}$-generic filter g for $Coll(\omega, \delta_n)$,

(d) $\mathcal{M} \models z_0, z_1$ are $Coll(\omega, \delta_n)$-names for reals.

(e) $(z_0)_g = x$, $(z_1)_g = y$,

(f) $\mathcal{M} \models p \Vdash_{Coll(\omega,\delta_n)} z_1 \in L[\vec{E}, \Sigma][z_0]$.

Let $T = \cup_{n<\omega} T_n$. Since the maximal $L[\vec{E}, \Sigma]$ construction in $\mathcal{R}_n$ is $OD(\Sigma)$-full , $p[T] = \{(x, y) : y \in OD(\Sigma, x)\}$. So $p[T]$ is universal $\Sigma^2_1(\Sigma)$.

2. By 1, every Suslin-co-Suslin set of reals is in $(L(Hom_h^*, \mathbb{R}_h^*))^{L[\mathcal{R}_\infty][h]}$. So $L[\Gamma, \mathbb{R}^V] \subseteq (L(Hom_h^*, \mathbb{R}_h^*))^{L[\mathcal{R}_\infty][h]}$, where Γ is the pointclass of all Suslin-co-Suslin sets of reals. If $U \notin L(\Gamma, \mathbb{R}^V)$, then $L(\Gamma, \mathbb{R}^V)$ is model of $AD_\mathbb{R} + \theta$ is regular, contradicting our minimality hypothesis. Thus, $L(U, \mathbb{R}^V) \subseteq (L(Hom_h^*, \mathbb{R}_h^*))^{L[\mathcal{R}_\infty][h]}$. As we can't force a sharp from a set forcing, we have $L(U, \mathbb{R}) = (L(Hom_h^*, \mathbb{R}_h^*))^{L[\mathcal{R}_\infty][h]} = (L(\mathcal{A}_h, \mathbb{R}_h^*))^{L[\mathcal{R}_\infty][h]}$. $\square$

Therefore, $L[\mathcal{R}_\infty]$ has a derived model $L(U, \mathbb{R})$. Let $\mathcal{N} = L[\vec{E}]^{\mathcal{Q}_\infty|\delta_0}$, G be the $\mathbb{Q}_{\delta_0}$-generic object which codes $\mathcal{Q}_\infty|\delta_0$. We claim that $\mathcal{R}_\infty$ can be translated backwards modulo G into a premouse $\mathcal{N}_\infty \rhd \mathcal{N}$.

This is a reflection argument. By taking a Skolem hull in $L[\mathcal{R}_\infty]$, we get

$$j : L[\mathcal{S}] \to L[\mathcal{R}_\infty].$$

and $\eta, h, \mathcal{M}$ such that $\mathrm{crt}(\pi) = \eta < \delta_0$, $\pi(\eta) = \delta_0$, $\pi(g, \mathcal{M}, \mathcal{S}) = (G, \mathcal{N}, \mathcal{R}_\infty)$.

Let ξ be the least such that there is a definable failure of Woodinness of η over $\mathcal{N}|\xi$. Then $Tr^g(\mathcal{N}|\xi)$ is defined and iterable by Lemma 3.16. But definably over $Tr^g(\mathcal{N}|\xi)$, there is a failure of Woodinness of η. Let's compare $Tr^g(\mathcal{N}|\xi)$ versus $\mathcal{S}$. According to Theorem 2.37, the comparison terminates. Since $\mathcal{S} \models \eta$ is Woodin, the S-side comes out shorter. But $Tr^g(\mathcal{N}|\xi) \models$ "$\forall\xi TrInv(V|\xi)$ is defined", since $Tr^g(\mathcal{N}|\xi)$ comes from the translation. The formula "$\forall\xi TrInv(V|\xi)$ is defined" is expressible in a Π_1-way. Therefore, $S \models$ "$\forall\xi TrInv(V|\xi)$ is defined". Therefore, $\mathcal{R}_\infty \models$ "$\forall\xi TrInv^G(V|\xi)$ is defined". This finishes the claim and thus there is $\mathcal{N}_\infty$ such that $Tr^G(\mathcal{N}_\infty) = \mathcal{R}_\infty$. Hence the premouse $L[\mathcal{N}_\infty]$ has a derived model $L(U, \mathbb{R})$.

If $V = L(U, \mathbb{R})$, we then have finished the successor case. If $V \neq L(U, \mathbb{R})$, we will put more extenders above $\mathcal{N}_\infty$ to get a premouse whose derived model is V. Since the S-operators have the generic interpretation property, we may define

S-operators acted on arbitrary transitive sets containing z_0. Given a countable transitive such that $z_0 \in a$, we let $S^{\mathfrak{P}}(z_0)$ be the unique structure $\mathcal{M}$ such that for each $g \subseteq Coll(\omega, a)$ generic, $\mathcal{M}[g] = S^{\mathfrak{P}}(g)$. We say $\mathcal{M}$ is mixed S-premouse over $\mathbb{R}$ if every countable elementary substructure $\bar{\mathcal{M}}$ of $\mathcal{M}$ with $z_0 \in \bar{\mathcal{M}}$ is a mixed S-premouse.

Lemma 3.18. *Let $A \in \mathcal{P}(\mathbb{R})$. Then A is in an $S^{[\lambda^{\mathcal{P}}]}$-mouse over $\mathbb{R}$.*

The proof is identical to the proof that every subset of the reals is in a Σ-mouse over $\mathbb{R}$ [8].

Every dense set in $\mathbb{P}$ is predense in $\mathbb{P}$, so $\mathcal{Q}_\infty$ is generic over V. Let $\xi_0 > \theta^{L(U,\mathbb{R})}$ be least such that $L_{\xi_0}(U, \mathbb{R}) \models ZF^-$. Let $\mathcal{S}$ be the $S^{[\lambda^{\mathcal{P}}]}$-mouse over $\mathbb{R}$ such that $\mathcal{P}(\mathbb{R})^{\mathcal{S}} = \mathcal{P}(\mathbb{R})^V$. We may rearrange $\mathcal{S}$ into a $S^{[\lambda^{\mathcal{P}}]}$-mouse $\mathcal{S}_0$ over $L_{\xi_0}(U, \mathbb{R})$ such that $\mathcal{P}(\mathbb{R})^{\mathcal{S}_0} = \mathcal{P}(\mathbb{R})^V$. We then level-by-level translate S_0 into a $S^{[\lambda^{\mathcal{P}}]}$-mouse S_1 over $L_{\xi_0}(U, \mathbb{R})[\mathcal{R}_\infty][H]$. Since $\mathcal{R}_\infty[H]$ is able to $L_{\xi_0}(U, \mathbb{R})$, we may translate $\mathcal{S}_1$ into a $S^{[\lambda^{\mathcal{P}}]}$-mouse $\mathcal{S}_2$ over $\mathcal{R}_\infty[H]$. By inverting the generic extension, we get a $S^{[\lambda^{\mathcal{P}}]}$-mouse $\mathcal{S}_3$ over $\mathcal{R}_\infty$ such that $\mathcal{S}_3[H] = \mathcal{S}_2$. Let $\mathcal{S}_4$ be $\mathcal{S}_3$ rearranged as a $S^{[\lambda^{\mathcal{P}}]}$-mouse over $\mathcal{Q}_\infty | \delta_0$. Exactly the same argument shows that there is a class premouse $\mathcal{N}^* \rhd \mathcal{N}_\infty$ such that $Tr^G(\mathcal{N}^*) = S_4$. So

$$(L(\mathcal{A}_h^*, \mathbb{R}_h^*))^{\mathcal{N}^*[h]} = V.$$

This finishes the $\theta = \theta_{\alpha+1}$ case.

Chapter 4

The $AD_{\mathbb{R}} + (\mathrm{cf}(\theta) = \omega \vee$ "θ is regular") case

We show the second half of the main theorem. We assume $AD^+ + \theta = \theta_\alpha$ for some limit ordinal α, and either $\mathrm{cf}(\theta) = \omega$ or θ is regular. Woodin [13] showed that in this case, V is embeddable into a derived model of HOD at θ. In this chapter, we show that we can translate HOD into a premouse $\mathcal{N} \subseteq HOD$, where all Woodin cardinals of HOD remain Woodin in $\mathcal{N}$, without loss of essential information. We will then show HOD and $\mathcal{N}$ have the same derived model, thus finishing the proof of Theorem 1.9.

The translation uses pretty much the similar idea as in the largest Suslin cardinal case. The difference is, in chapter 2 and 3, we had a largest Suslin pointclass, and thus a largest hod pair $(\mathcal{P}, \Sigma)$ such that Σ is fullness-preserving and has branch condensation. All the S-operators were based on this hod pair. The translations were getting rid of overlapping extenders and replacing them by fragments of Σ. In the current case, however, we don't have such a largest hod pair. Therefore, the S-operators will vary, depending on which hod pair we chose in advance. Most of the ideas in this chapter is from chapter 2 and 3. We will be sketchy and only highlight the new idea.

4.1 The S-operators

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ is fullness preserving and has branch condensation. We define the same objects as in chapter 2, but emphasizing their dependence on $(\mathcal{P}, \Sigma)$. $\mathcal{I}_{(\mathcal{P},\Sigma)}$ is the index set $\mathcal{I}$ as defined at the beginning of Section 2.3, but of course based on this particular $(\mathcal{P}, \Sigma)$. We repeat the definition here: $\mathcal{I}_{\mathcal{P},\Sigma}$ is the set of $\mathfrak{P} = \langle \mathfrak{P}_i : i \leq n \rangle = \langle (\zeta_i, \pi_i, \mathcal{P}_i, \alpha_i^*, \alpha_i) : i \leq n \rangle$ such that

1. $\zeta_0 = \pi_0 = \emptyset$, $\mathcal{P}_0 = \mathcal{P}$, $\alpha_0^* = \alpha_0 \leq \lambda^{\mathcal{P}}$,
2. for all $0 \leq i < n$, $(\zeta_{i+1}, \pi_{i+1}, \mathcal{P}_{i+1}, \alpha_{i+1}^*, \alpha_{i+1})$ is a one-step blow-up of $(\mathcal{P}_i, \alpha_i^*, \alpha_i)$ above $\mathcal{P}_i(\pi_i(\zeta_i))$.

The notion of an index being successor, or limit of type A,B,C are exactly the same as before. For notational convenience, we let $[\alpha] = \langle (\emptyset, \emptyset, \mathcal{P}, \alpha, \alpha) \rangle$. This notation of course depends on $\mathcal{P}$, but we often suppress it when the meaning is clear.

The various S-operators and H-operators are defined pretty much the same as in Sections 2.3, 2.5, 2.6, the only difference is we need "finite layers" of operators. We let $\mathcal{J}$ be the set of $((\mathcal{P}, \Sigma), \alpha_0, e_0, \ldots, \alpha_n, e_n, \mathfrak{P})$ such that

1. $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ is fullness preserving and has branch condensation,
2. $\alpha_0 < \ldots < \alpha_n = \lambda^{\mathcal{P}}$,
3. for each $i \leq n$, $e_i : \omega \to |\mathcal{P}(\alpha_i)|$ is a bijection,
4. $\mathfrak{P} \in \mathcal{I}_{(\mathcal{P},\Sigma)}$. If $n > 0$, then $[\alpha_{n-1}] \leq^{\mathcal{I}_{(\mathcal{P},\Sigma)}} \mathfrak{P}$.

We will define the S^v-operator for $v \in \mathcal{J}$. Suppose $((\mathcal{P}, \Sigma), \alpha_0, e_0, \mathfrak{P}) \in \mathcal{J}$. The operators at this level are exactly the same as in Section 2.3. If $\mathfrak{P} = [0]$, a is countable transitive self-wellordered such that $e_0 \in a_+$, we will define $S^{*,[0]}(a)$ as follows. Let $\mathcal{Q}$ be Σ-good over a. Let $\mathcal{N} = L[\vec{E}][a]^{\mathcal{Q}|\delta^{\mathcal{Q}}}$. By the proof of MSC [7], there is $\mathcal{R} \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}$ such that $\Sigma_{\mathcal{R}} \restriction \mathcal{N} \in \mathcal{N}_+$. Let $F_{\mathcal{N}}$ be the direct system

$$\{\mathcal{R}, \pi_{\mathcal{R}\mathcal{R}'} : \mathcal{R}, \mathcal{R}' \in pI(\mathcal{P}, \Sigma) \cap \mathcal{N}, \pi_{\mathcal{R}\mathcal{R}'} \text{ is a } \Sigma\text{-iteration map.}\}$$

Let Q_N^∞ be the direct limit of $F_{\mathcal{N}}$ and $\pi_{\mathcal{N}}^\infty : \mathcal{P} \to Q_N^\infty$ be the direct limit map, so that $Q_N^\infty \in \mathcal{N}_+$. Let $\mathcal{M}$ be the transitive collapse of the structure

$$\langle Hull^{\mathcal{N}_+}(a \cup \{a\} \cup \pi_{\mathcal{N}}^\infty), \in, a, \vec{E}^{\mathcal{N}}, \emptyset, Q_{\mathcal{N}}^\infty, \pi_{\mathcal{N}}^\infty \rangle.$$

Then $S^{*,[0]}(a)$ is the e-amenable code of $\mathcal{M}$.

The general S-operators, inherits a structure called finitely layered S-premouse. Similar to S-premouse as defined in Section 2.4, with the exception that different layes of S-operators are distinguished. We let

$$\mathcal{L}_l = \{\in, \dot{a}, \dot{E}, \dot{F}, \dot{S}_0, \dot{b}_0, \dot{Q}_0, \dot{\pi}_0, \dot{S}_1, \dot{b}_1, \dot{Q}_1, \dot{\pi}_1, \ldots\}$$

be the language extending the language of set theory where $\dot{a}, \dot{b}_0, \dot{b}_1, \ldots, \dot{Q}_0, \dot{Q}_1, \ldots$ are constant symbols, $\dot{E}, \dot{F}, \dot{\pi}_0, \dot{\pi}_1, \ldots$ are unary predicate symbols, $\dot{S}_0, \dot{S}_1, \ldots$ are unary predicate symbols. A potential finitely layered S-premouse over a is a structure

$$\mathcal{N} = \langle N, \in, a, \vec{E}, F, S_0, b_0, Q_0, \pi_0, S_1, b_1, Q_1, \pi_1, \ldots \rangle$$

in the language of $\mathcal{L}_l$ with the following properties.

1. There is some $n < \omega$ such that for all $m > n$, $S_m = b_m = Q_m = \pi_m = \emptyset$.

2. $N = J_\xi^{\vec{E},S_0,S_1,\ldots}[a]$ for some ξ.

3. $\mathcal{N}$ is an acceptable J-structure.

4. $\vec{E}$ is a partial unary function.

5. For all $i < \omega$, for all $y \in S_i$, y is a $\mathcal{L}_l$-structure. For $\eta < \xi$, let $\mathcal{N}|\eta$ be the initial segment of $\mathcal{N}$ given by

$$\mathcal{N}|\eta = \langle J_\eta^{\vec{E},S_0,S_1,\ldots}[a], \in, a, \vec{E}\restriction\eta, E_\eta, S_0 \cap J_\eta^{\vec{E},S_0,S_1,\ldots}[a], b_0^\eta, Q_0^\eta, \pi_0^\eta,$$
$$S_1 \cap J_\eta^{\vec{E},S_0,S_1,\ldots}[a], b_1^\eta, Q_1^\eta, \pi_1^\eta, \ldots\rangle$$

where

$$(b_i^\eta, \mathcal{Q}_i^\eta, \pi_i^\eta) = \begin{cases} (b^y, \dot{\mathcal{Q}}^y, \dot{\pi}^y), & \text{if } y \in S_i \text{ is unique such that } o(y) = \eta. \\ (\emptyset, \emptyset, \emptyset), & \text{otherwise.} \end{cases}$$

6. For all i, for all $y \in S_i$, $y = \mathcal{N}|o(y)$. (Henceforth, if $y, y' \in S_i$ and $o(y) = o(y')$, then $y = y'$.)

7. $\vec{E}^\frown F$ is a fine extender sequence in the sense of [4], whose levels are understood as $\mathcal{N}|\eta$.

Suppose $\mathcal{N}$ is a potential finitely layered S-premouse. We say $\mathcal{N}$ is n-layered if n is least such that for all $m > n$, $S_m = b_m = Q_m = \pi_m = \emptyset$. For convenience, we will suppress those S_m, b_m, Q_m, π_m for $m > n$ and write

$$\mathcal{N} = \langle N, \in, a, \vec{E}, F, S_0, \ldots, S_{n-1}, b_{n-1}, Q_{n-1}, \pi_{n-1}\rangle.$$

We define fine structural relavent objects of finitely layered S-premice similar to Section 2.4. After those preparations, we can start defining the $S^{(\mathcal{P},\Sigma),\alpha_0,e_0,[0]}$-operator, similar to Section 2.5. If a is countable transitive swo, $\mathcal{K}$ is a potential S-premouse over a, then we let

$$\begin{aligned} SSM(\mathcal{K}) = \bigsqcup\{\mathcal{M} :& \mathcal{M} \text{ is a sound potential } S\text{-premouse extending } \mathcal{K}, \\ & o(\mathcal{K}) \text{ is a strong cutpoint of } \mathcal{M}, \\ & \forall i < \omega\ \forall y \in S_i^{\mathcal{M}}(o(y) \leq o(\mathcal{K})), \\ & \mathcal{M} \text{ is iterable when hitting extenders above } o(\mathcal{K}), \\ & \rho_\omega(\mathcal{M}) \leq o(\mathcal{K}).\} \end{aligned}$$

Suppose $S^{*,(\mathcal{P},\Sigma),\alpha_0,e_0,[0]}(SSM(\mathcal{K})) = \langle M, \in, SSM(\mathcal{K}), E, Q_0, \pi_0\rangle$, and suppose that $\langle M, \in, a, E^{SSM(\mathcal{K})} \cup E, S_0^{SSM(\mathcal{K})}, \mathcal{K}, Q_0, \pi_0, S_1^{SSM(\mathcal{K})}, \emptyset, \emptyset, \emptyset, \ldots\rangle$ is a potential S-premouse over a. Then let $S^{[0]}(\mathcal{K}) = \langle M, \in, a, E^{SSM(\mathcal{K})} \cup E, S_0^{SSM(\mathcal{K})}, \mathcal{K}, Q_0, \pi_0, S_1^{SSM(\mathcal{K})}, \emptyset, \emptyset, \emptyset, \ldots\rangle$. We leave it to the reader defining the successor case and the limit case of $S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\mathfrak{P}}$.

Suppose now $((\mathcal{P},\Sigma), \alpha_0, e_0, \ldots, \alpha_n, e_n, \mathfrak{P}) \in \mathcal{J}$, $n > 0$. We again define by induction by hod mouse prewellordering of final($\mathfrak{P}$). The base case is $\mathfrak{P} = [\alpha_{n-1}]$. We let $S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_{n-1}]}(a) = S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_{n-1},e_{n-1},[\alpha_{n-1}]}(a)$. Notice however the $[\alpha_{n-1}]$ has different meanings in the two superscripts of the equation. We sketch how to define $S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_{n-1}]+1}$, and leave the rest as an exercise to the reader.

If $S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_{n-1}]}$-mice has condensation above (a), then $S^{*,(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_{n-1}]}(a)$ is defined as follows. Let $\mathcal{Q}$ be Σ-good over a. Assume that the $L[\vec{E}, S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_{n-1}]}][a]$-construction in $\mathcal{Q}|\delta^{\mathcal{Q}}$ converges to a $S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_{n-1}]}$-mouse over a. Let $\mathcal{N}$ be the output. By the proof of MSC [7], there is $\mathcal{R} \in$

$pI(\mathcal{P},\Sigma)\cap\mathcal{N}$ such that $\Sigma_{\mathcal{R}}\restriction\mathcal{N}\in\mathcal{N}_+$. Let $F_{\mathcal{N}}$ be the direct system

$$\{\mathcal{R},\sigma_{\mathcal{R}\mathcal{R}'}:\mathcal{R},\mathcal{R}'\in pI(\mathcal{P},\Sigma)\cap\mathcal{N},\sigma_{\mathcal{R}\mathcal{R}'}\text{ is a }\Sigma\text{-iteration map.}\}$$

Let $Q^{\infty}_{\mathcal{N}}$ be the direct limit of $F_{\mathcal{N}}$ and

$$\pi^{\infty}_{\mathcal{N}}:\mathcal{P}\to\mathcal{Q}^{\infty}_{\mathcal{N}}$$

be the direct limit map, so that $Q^{\infty}_{\mathcal{N}}\in\mathcal{N}_+$. Let $\mathcal{M}$ be the transitive collapse of the structure

$$\langle Hull^{\mathcal{N}_+}(a\cup\{a\}\cup\pi^{\infty}_{\mathcal{N}}),\in,a,E^{\mathcal{N}},S^{\mathcal{N}}_0,S^{\mathcal{N}}_1,\ldots,S^{\mathcal{N}}_n,Q^{\infty}_{\mathcal{N}},\pi^{\infty}_{\mathcal{N}}\rangle.$$

Then $S^{*,\mathfrak{P}}(a)$ is the e_n-amenable code of $\mathcal{M}$.

Denote $\mathcal{N}=SSM^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_{n-1}]}(\mathcal{K})$. If $S^{*,(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_{n-1}]+1}(\mathcal{N})=\langle M,\in,\mathcal{N},E,S_0,\ldots,S_m,Q,\pi\rangle$ is defined, and $\langle M,\in,a,E^{\mathcal{N}}\cup E,S^{\mathcal{N}}_0\cup S_0,\emptyset,\emptyset,\emptyset,\ldots,S^{\mathcal{N}}_n\cup S_n,\mathcal{K},Q,\pi,\ldots\rangle$ is a finitely layered potential S-premouse, then $S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_{n-1}]}(\mathcal{K})$ be this finitely layered potential S-premouse. Otherwise, we leave $S^{\mathfrak{P}}(\mathcal{K})$ undefined. So the direct limit map is thrown into the n-th layer.

We again leave it to the reader defining the H-operators and other relavent concepts. Once again, we have a nice real for each index.

Definition 4.1. Let $((\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,\mathfrak{P})\in\mathcal{J}$, $\mathrm{dom}(\mathfrak{P})=n+1$. We say that z is a nice real for $((\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,\mathfrak{P})$ if all of the following holds.

1. (Reduction) For all $\epsilon\le\alpha_n$ successor or 0, if $((\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,\mathfrak{P}[\epsilon])\in\mathcal{J}$, then there is $y\le_T z$ such that y codes a reduction between $S^{*,(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,\mathfrak{P}[\epsilon]}$ and $H^{\mathfrak{P}[\epsilon]}$.

2. (Reduction on further extensions) For all $\epsilon\le\alpha_n$, if $\mathfrak{P}[\epsilon]$ is a limit index

of type C, then there is $y \leq_T z$ such that y codes a reduction on further extensions of $((\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\mathfrak{P}(\epsilon)]_{\mathcal{I}(\mathcal{P},\Sigma)})$.

3. (Condensation) If $\mathcal{K}$ is an finitely layered S-premouse over a, $a \in \text{Cone}(z)$, $j : \bar{\mathcal{S}} \to S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,\mathfrak{Q}}(\mathcal{K})$ is Σ_1-elementary, $j(\mathcal{H},\mathfrak{Q}) = (\mathcal{K},\mathfrak{Q})$, and $\text{pro}(\mathfrak{Q},\mathcal{K}) \leq_{\mathcal{K}}^{\mathcal{I}(\mathcal{P},\Sigma)} \text{pro}(\mathfrak{P},\mathcal{K})$, then $\bar{\mathcal{S}} = S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,\mathfrak{Q}}(\mathcal{H})$. Consequently, the $L[\vec{E},S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,\mathfrak{P}}]$-construction in any good universe converges.

Theorem 4.2. *For all $((\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,\mathfrak{P}) \in \mathcal{J}$, there is a nice real for $\mathfrak{P}$.*

So far, we have finished defining the S-operators. We point out that those definitions can be fully worked out in a hod mouse. If $(\mathcal{P},\Sigma)$ is a hod pair such that Σ is fullness preserving and has branch condensation, then for every $\alpha < \lambda^{\mathcal{P}}$, for every $\gamma < \delta^{\mathcal{P}}$, for every g generic over $\mathcal{P}$ for $Coll(\omega,\gamma)$, if $a \in \mathcal{P}|\delta^{\mathcal{P}}[g]$, then a $\Sigma_{\mathcal{P}(\alpha)}$-good $\Sigma_{\mathcal{P}(\alpha)}$-premouse over a is locally constructible inside $\mathcal{P}$. Hence all the $S^{(P(\alpha),\Sigma_{P(\alpha)}),\alpha_0,e_0,\ldots,\alpha_n,e_n,\mathfrak{P}}$-operators are constructible in $\mathcal{P}[g]$, for g $Coll(\omega,\delta^{\mathcal{P}}_{\alpha})$-generic over $\mathcal{P}$, provided they are defined. We emphasize that although the existence of a nice real is not provable in $\mathcal{P}$, the whole construction of the S-operators is definable in $\mathcal{P}[g]$ provided existence of a nice real in $\mathcal{P}[g]$.

However, nice real is not an issue in $\mathcal{P}[g]$, as it always exists in any $Coll(\omega,\delta^{\mathcal{P}}_{\alpha+1})$-generic extension. This is shown by doing genericity iterations. Suppose $\alpha_0 < \alpha_1 < \cdots < \alpha_n = \alpha < \lambda^{\mathcal{P}}$. We argue that in any $Coll(\omega,\delta^{\mathcal{P}}_{\alpha+1})$-generic extension over $\mathcal{P}$, if $e_0,e_1,\ldots,e_n \in \mathcal{P}[g]$ are enumerations of $\mathcal{P}(\alpha_0),\mathcal{P}(\alpha_1),\ldots,\mathcal{P}(\alpha_n)$ from ω respectively, then there must be a nice real for $((\mathcal{P}(\alpha),\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha])$ in $\mathcal{P}(\alpha+1)[g]$. Take a nice real z. We iterate from $\mathcal{P}$ to $\mathcal{R}$ in the window $[\delta^{\mathcal{P}}_{\alpha},\delta^{\mathcal{P}}_{\alpha+1}]$ to make z generic over the extender algebra of $\mathcal{R}$ at the image of $\delta^{\mathcal{P}}_{\alpha+1}$. Then $\mathcal{R}[g][z]$ thinks that $S^{(\mathcal{P}(\alpha),\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha]}(a)$ is defined whenever $a \in \mathcal{R}|\delta^{\mathcal{R}}[g][z]$ is such that $\mathcal{R}|i_{\mathcal{P}\mathcal{R}}(\delta^{\mathcal{P}}_{\alpha+1})[g][z] \in a$. Because the S-operators extend naturally onto generic extensions, $S^{(\mathcal{P}(\alpha),\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha]}(a)$ is defined whenever $a \in \mathcal{R}|\delta^{\mathcal{R}}[g]$ is such that $\mathcal{R}|i_{\mathcal{P}\mathcal{R}}(\delta^{\mathcal{P}}_{\alpha+1})[g] \in a$. Hence by elementarity, $\mathcal{P}[g]$ thinks that $S^{(\mathcal{P}(\alpha),\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha]}(a)$ is defined whenever $a \in \mathcal{P}|\delta^{\mathcal{P}}[g]$ is such that $\mathcal{P}|\delta^{\mathcal{P}}_{\alpha+1}[g] \in a$. The internal constructibility of the S-operators will be useful in the next section.

4.2 The translation

We would like to define a translation procedure as in chapter 3. In the current context, we will do a finite iteration of translation as done in chapter 3. Every single step stands for a correspondence between one particular layer of S-operators and extenders which overlaps the height of the hod mouse representing that layer.

We work with a hod pair $(\mathcal{P},\Sigma)$ such that Σ is fullness preserving and has branch condensation. Suppose $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_n + 1 = \lambda^{\mathcal{P}}$. Let $h_0, \ldots, h_n$ be such that each h_i is a generic filter over $\mathcal{P}$ for $Coll(\omega, \delta^{\mathcal{P}}_{\alpha_i+1})$, and $h_i \in \mathcal{P}[h_{i+1}]$. Suppose $e_0, \ldots, e_n$ are such that each e_i is a bijection from ω to $|\mathcal{P}(\alpha_i)|$ and $e_i \in \mathcal{P}[h_i]$.

Suppose first $n = 0$. Let $\mathcal{N}_0 = L[\vec{E}]^{\mathcal{P}|\delta^{\mathcal{P}}}$. So $\delta^{\mathcal{P}}$ is Woodin in $(\mathcal{N}_0)_+$. Suppose $\mathcal{N}$ is a mouse extending $(\mathcal{N}_0)_+$ such that $\delta^{\mathcal{P}}$ is still Woodin in $\mathcal{N}$. Thus, $\mathcal{P}|\delta^{\mathcal{P}}$ is generic over $\mathcal{N}$ for the $\delta^{\mathcal{P}}$-extender algebra at $\delta^{\mathcal{P}}$. Let g_0 be the generic filter that codes $\mathcal{P}|\delta^{\mathcal{P}}$. The translation $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{N})$ will be a 1-layered S-premouse over h_0. The definition of $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{N})$ is essentially in chapter 3. We briefly restate it in the current context. $U(\mathcal{N},\delta^{\mathcal{P}})$, $P(\mathcal{N},\delta^{\mathcal{P}})$, $<^{P(\mathcal{N},\delta^{\mathcal{P}})}$ is defined as in Section 3.1. Iterability of $\mathcal{N}$ implies that $<^{P(\mathcal{N},\delta^{\mathcal{P}})}$ is a well-order. $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}$ is a function defined on $\mathcal{K} \in P(\mathcal{N},\delta^{\mathcal{P}})$ by induction on $<^{P(\mathcal{N},\delta^{\mathcal{P}})}$. Let Q_0, D_0 be easily definable functions such that $Q(g_0) = Q^{\infty}_{\mathcal{N}_0}$, $D(g_0) = \pi^{\infty}_{\mathcal{N}_0}$, where $\pi^{\infty}_{\mathcal{N}_0} : \mathcal{P}(\alpha_0) \to \mathcal{Q}^{\infty}_{\mathcal{N}_0}$ is the direct limit map of $I(\mathcal{P}(\alpha_0), \Sigma_{\mathcal{P}(\alpha_0)}) \cap \mathcal{N}_0$. Then

1. If $\mathcal{K} = (\mathcal{N}_0)_+$, then $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{K}) = \langle |\mathcal{K}|[h_0], \in, h_0, \emptyset, \emptyset \rangle$.

2. If $\mathcal{K} = \mathfrak{N}(\mathcal{M})$, then $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{K}) = \mathfrak{N}(Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{M}))$.

3. If $o(\mathcal{K})$ is a limit, $\mathcal{K}$ is passive, then $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{K}) = \bigsqcup_{\eta<o(\mathcal{K})} Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{K}|\eta)$.

4. If $\mathcal{K}$ is active with top extender E, $\mathrm{crt}(E) > \delta^{\mathcal{Q}}$, let $E[g_0]$ be the canonical extension of E to the generic. Let $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{K})$ be $\bigsqcup_{\eta<o(\mathcal{K})} Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{K}|\eta)$ but adding the top extender $E[g_0]$.

5. If $\mathcal{K}$ is active with top extender E, $\mathrm{crt}(E) < \delta^{\mathcal{Q}}$, let $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathrm{Ult}((\mathcal{N}_0)_+, E)) = \langle J^{\vec{E},S}_{\alpha}[h_0], \in, h_0, \vec{E}, \emptyset, S, \emptyset, \emptyset, \emptyset\rangle$. Let d be the last drop of $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathrm{Ult}(\mathcal{N}_0, E))$. Then $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{K})$ is the e_0- amenable code of transitive collapse of the hull of $d \cup i_E \circ D(g_0)$ over

$$\langle J^{\vec{E},S}_{\alpha}[h_0], \in, h_0, \vec{E}, \emptyset, S, Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathrm{Ult}(\mathcal{N}_1, E))|d, i_E(Q(g_0)), i_E \circ D(g_0)\rangle.$$

A reflection argument same as in the largest Suslin cardinal case shows that when $(\mathcal{R}, \Phi)$ is a hod pair such that Φ is fullness preserving and has branch condensation, $\mathcal{P} \lhd_{hod} \mathcal{R}$, and $\mathcal{S}$ is a 1-layer S-mouse over $\mathcal{P}[h_0]$ that is definable over $\mathcal{R}$, then we can translate $\mathcal{S}$ backwards into a premouse $\mathcal{M}_0$. In particular, $\mathcal{M}_0$ has the following property.

1. $Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{M}_0) = \mathcal{S}$

2. $\mathcal{M}_0[h_0]$ and $\mathcal{S}$ have the same universe.

Suppose now $n > 0$. Let $\mathcal{N}_n = (L[\vec{E}, S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_{n-1},e_{n-1}}][\mathcal{P}(\alpha_{n-1}+1)[h_{n-1}]])^{\mathcal{P}}$. So $\delta^{\mathcal{P}}$ is Woodin in $(\mathcal{N}_n)_+$. Suppose $\mathcal{N}$ is a $(n-1)$-layered S-premouse extending $(\mathcal{N}_n)_+$ such that $\delta^{\mathcal{P}}$ is still Woodin in $\mathcal{N}$. Thus, $\mathcal{P}|\delta^{\mathcal{P}}$ is generic over $\mathcal{N}$ for the $\delta^{\mathcal{P}}$-extender algebra at $\delta^{\mathcal{P}}$. Let g_n be the generic filter that codes $\mathcal{P}|\delta^{\mathcal{P}}$. The translation $Tr^{g_0,h_0,\ldots,g_n,h_n}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n}(\mathcal{N})$ will be a n-layered S-premouse. The definition of $Tr^{g_0,h_0,\ldots,g_n,h_n}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n}(\mathcal{N})$ is as follows. As before, we define $U(\mathcal{N}, \delta^{\mathcal{P}})$, $P(\mathcal{N}, \delta^{\mathcal{P}})$,

$<^{P(\mathcal{N},\delta^{\mathcal{P}})}$. Iterability of $\mathcal{N}$ implies that $<^{P(\mathcal{N},\delta^{\mathcal{P}})}$ is a well-order. $Tr^{g_0,h_0,\ldots,g_n,h_n}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n}$ is a function defined on $\mathcal{K} \in P(\mathcal{N},\eta)$ by induction on $<^{P(\mathcal{N},\delta^{\mathcal{P}})}$. Let Q_n, D_n be easily definable functions such that $Q(g_n) = Q^{\infty}_{\mathcal{N}_n}$, $D(g_n) = \pi^{\infty}_{\mathcal{N}_n}$, where $\pi^{\infty}_{\mathcal{N}_n} : \mathcal{P}(\alpha_n) \to \mathcal{Q}^{\infty}_{\mathcal{N}_n}$ is the direct limit map of $I(\mathcal{P}(\alpha_n), \Sigma_{\mathcal{P}(\alpha_n)}) \cap \mathcal{N}_n$.. The only difference from the $n = 0$ case is when $\mathcal{K}$ is active with top extender E, $\mathrm{crt}(E) < \delta^{\mathcal{P}}$. Suppose in this case, $Tr^{g_0,h_0,\ldots,g_n,h_n}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n}(\mathrm{Ult}(\mathcal{N}_n, E)) = \langle J^{\vec{E},S_0,\ldots,S_n}_{\alpha}[g], \in, g, \vec{E}, \emptyset, S_0, \emptyset, \emptyset, \emptyset, \ldots, S_n, \emptyset, \emptyset, \emptyset\rangle$. Let d be the last drop of $Tr^g(\mathrm{Ult}(\mathcal{N}_n, E))$. Then $Tr^g(\mathcal{K})$ is the e_n- amenable code of transitive collapse of the hull of $d \cup i_E \circ D(g)$ over

$$\langle J^{\vec{E},S_0,\ldots,S_n}_{\alpha}[g_n], \in, g_n, \vec{E}, \emptyset, S_0, \emptyset, \emptyset, \emptyset, \ldots, S_n, Tr^g(\mathrm{Ult}(\mathcal{N}_1, E))|d, i_E(Q(g)), i_E \circ D(g)\rangle.$$

That means, we put information about the extender into the π_n-predicate.

Again, a reflection argument shows the following. Suppose $(\mathcal{R}, \Phi)$ is a hod pair such that Φ is fullness preserving and has branch condensation, $\mathcal{P} \lhd_{hod} \mathcal{R}$. Suppose S is an n-layer S-premouse over $\mathcal{P}[h]$, then $\mathcal{S}[h_n]$ can be translated back into an $n-1$-layered premouse $\mathcal{M}_{n-1}$ that extends $\mathcal{N}_n$. In particular,

1. $Tr^{g_0,h_0,\ldots,g_n,h_n}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n}(\mathcal{M}_{n-1}) = \mathcal{S}_{h_n}$
2. $\mathcal{M}_{n-1}[h_n]$ and $\mathcal{S}_{h_n}$ have the same universe.

If $\mathcal{M}_{n-1} \in \mathcal{R}[h_{n-1}]$, then by carrying out one more step of the backward translation, we can get a $n-2$-layered premouse $\mathcal{M}_{n-2}$ over $\mathcal{P}(\alpha_{n-2})[h_{n-2}]$ that extends $\mathcal{N}_{n-1} = (L[\vec{E}, S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_{n-2},e_{n-2}}][\mathcal{P}(\alpha_{n-2}+1)[h_{n-2}]])^{\mathcal{P}|\delta^{\mathcal{P}}_{\alpha_{n-1}+1}}$. That means,

1. $Tr^{g_0,\ldots,g_n,(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_{n-1},e_{n-1}}(\mathcal{M}_{n-2}) = \mathcal{M}_{n-1}$
2. $\mathcal{M}_{n-2}[h_{n-1}]$ and $\mathcal{M}_{n-1}$ have the same universe. Hence, $\mathcal{M}_{n-2}[h_n]$ and $\mathcal{S}$ have the same universe.

The second step of the translation turns S-operators at the $n-1$-st layer into extenders overlapping $\delta^{\mathcal{P}}_{\alpha_{n-1}}$. Continuing in this way, we will eventually get rid of all layers of S-operators and reach a premouse $\mathcal{M}_0$ that extends $\mathcal{N}_0 = (L[\vec{E}])^{\mathcal{P}^{\mathcal{P}}_{\delta_{\alpha_0}+1}}$. $\mathcal{M}_0$ has the following property.

1. $Tr^{g_0,h_0,\ldots,g_n,h_n}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n} \circ Tr^{g_0,h_0,\ldots,g_{n-1},h_{n-1}}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_{n-1},e_{n-1}} \cdots \circ Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}(\mathcal{M}_0) = \mathcal{S}$

2. $\mathcal{M}_0[h_n]$ and $\mathcal{S}$ have the same universe.

We write $ITr^{g_0,h_0,\ldots,g_n,h_n}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n} = Tr^{g_0,h_0,\ldots,g_n,h_n}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n} \circ Tr^{g_0,h_0,\ldots,g_{n-1},h_{n-1}}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_{n-1},e_{n-1}} \cdots \circ Tr^{g_0,h_0}_{(\mathcal{P},\Sigma),\alpha_0,e_0}$ for short. In cases of interest, $\mathcal{S}$ is the output of the $L[\vec{E}, S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_n]}]$ construction up to $\delta^{\mathcal{R}}$ over $\mathcal{P}[h_n]$. But then, the result of the reverse translation of $\mathcal{S}$, $\mathcal{M}_0$, is in the ground model $\mathcal{R}$. The reason is, the only place that depends on the generic filter is in the e_i-amenable codes, which comes from the generic enumeration of the respective $\mathcal{P}(\alpha_i)$'s. These amenable codes contain information about the direct limit map from hod mice to certain direct limits. The direct limit maps is the only crucial information coded in the S-operators, and it is important to notice that the maps are in the ground model. In the translation, the generic enumeration is not important at all, because we decode the direct limit maps from the S-operators. A level-by-level induction on the height of $\mathcal{S}$ shows that $\mathcal{M}_0$ is in the ground model. $\mathcal{M}_0$ is the mouse that we extract from the hod mouse, $\mathcal{P}$. Those kinds of mice will merge into HOD under iteration maps of hod mice.

We give a short remark that independence of the generic extension and the internal constructibility of the S-operators is also applicable to the largest Suslin cardinal case. Suppose we have forced $\mathcal{R}$, an ω-suitable Σ-premouse over $\mathcal{P}$, where $(\mathcal{P}, \Sigma)$ is a hod pair giving rise to the largest Suslin pointclass, then we may let $\mathcal{S} = (L[\vec{E}, S^{\mathcal{P}}][\mathcal{Q}(h)])^{\mathcal{R}[h]}$, where $\mathcal{Q}$ is the initial segment of $\mathcal{R}$ that is Σ-suitable over $\mathcal{P}$, h is $Coll(\omega, \delta^{\mathcal{Q}})$-generic over $\mathcal{R}$. The reverse translation gives a premouse $\mathcal{M}$ such that $Tr^g(\mathcal{M}) = \mathcal{S}_0$. $\mathcal{M}$ will then have derived model V.

Returning to the $AD_{\mathbb{R}}$-case, we let $\mathcal{N}^{\mathcal{R};\alpha_0,\ldots,\alpha_n}$ be the mouse we defined there, i.e.

$$ITr^{g_0,h_0,\ldots,g_n,h_n}_{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n}(\mathcal{N}^{\mathcal{R};\alpha_0,\ldots,\alpha_n}) = (L[\vec{E}, S^{(\mathcal{P},\Sigma),\alpha_0,e_0,\ldots,\alpha_n,e_n,[\alpha_n]}][\mathcal{P}[h_n]])^{\mathcal{R}|\delta^{\mathcal{R}}[h_n]}.$$

So $\mathcal{N}^{\mathcal{R};\alpha_0,\ldots,\alpha_n}$ is first-order definable over $\mathcal{R}$ from $\alpha_0,\ldots,\alpha_n$. Let $f:\omega\to\theta$ be a strictly increasing cofinal map, possibly in the generic extension over V collapsing θ to countable. For each hod pair $(\mathcal{P},\Sigma)$ such that Σ is fullness preserving and has branch condensation, for any $\alpha_0<\alpha_1<\cdots<\alpha_n=\lambda^{\mathcal{P}}$ such that $\mathcal{M}_\infty(\mathcal{P}(\alpha_i))|\delta^{\mathcal{M}_\infty(\mathcal{P}(\alpha_i))} = HOD|\theta_{f(i)}$, we let $\mathcal{H}_n = i_{\mathcal{P}\infty}(\mathcal{N}^{\mathcal{P};\alpha_0,\ldots,\alpha_n})$. Because of the local definability of $\mathcal{N}^{\mathcal{R};\alpha_0,\ldots,\alpha_n}$ and commutativity of iteration maps among hod mice, $\mathcal{H}_n$ is independent of the choice of $(\mathcal{P},\Sigma)$ and hence is in HOD. Let $\mathcal{H}=\cup_{n<\omega}\mathcal{H}_n$. Then $L[\mathcal{H}]$ is a premouse whose Woodins sup to θ.

We prove that we can force an elementary embedding from V to a derived model of $L[\mathcal{H}]$ at θ. We will be using Woodin's proof of V embeds into a derived model of HOD. The main result we will use is summarized in the following theorem.

Theorem 4.3 (Woodin, [13])**.** *Suppose $AD_{\mathbb{R}}$ holds and either* $\mathrm{cf}(\theta)=\omega$ *or θ is regular. Let G be $Coll(\omega,<\theta)$-generic over HOD such that $\forall x\in\mathbb{R}^V\exists\lambda<\theta x\in HOD[G\restriction\lambda])$. Let $\mathbb{R}^*_G, Hom^*_G$ be associated objects of the derived model of HOD. Then there is an elementary $j:V\to L(\mathbb{R}^*_G, Hom^*_G)$. Moreover, $j''\theta$ is cofinal in $\theta^{L(\mathbb{R}^*_G,Hom^*_G)}$. For each $A\in V$, let $\gamma<\theta$ and let $\langle T_\lambda, T^*_\lambda : \gamma<\lambda<\theta\rangle\subseteq HOD[G\restriction\gamma]\cap V$ be trees such that for all $\gamma<\lambda<\theta$, $p[T_\lambda]=A$, $p[T^*_\lambda]=\mathbb{R}\setminus A$, $HOD[G\restriction\gamma]\models T_\lambda, T^*_\lambda$ are λ-complementing trees, then $j(A)=\bigcup_{\gamma<\lambda<\theta}(p[T_\lambda])^{HOD(\mathbb{R}^*_G)}$.*

Let G be $Coll(\omega,<\theta)$-generic over both $L[\mathcal{H}]$ and HOD such that $\forall x\in\mathbb{R}^V\exists\lambda<\theta x\in L[\mathcal{H}][G\restriction\lambda]$ and $\forall\gamma<\theta\exists\lambda<\theta V^{HOD}_\gamma\in L[\mathcal{H}][G\restriction\lambda]$. Let $\mathbb{R}^*_G$ be the reals in the symmetric collapse. Let $(Hom^*_G)^{HOD}$ be the power set of reals in the derived model of HOD and $(Hom^*_G)^{L[\mathcal{H}]}$ be the power set of reals in the derived model of $L[\mathcal{H}]$. By Theorem 4.3, there is $j:V\to L(\mathbb{R}^*_G,(Hom^*_G)^{HOD})$. It remains to show that $(Hom^*_G)^{L[\mathcal{H}]} = (Hom^*_G)^{HOD}$. The $\subseteq$-direction is clear. For the $\supseteq$-direction, as $j''\theta$ is cofinal in $\theta^{L(\mathbb{R}^*_G,(Hom^*_G)^{HOD})}$, it is enough to show that for all $A \in \mathcal{P}(\mathbb{R}) \cap V$, $j(A) \in (Hom^*_G)^{L[\mathcal{H}]}$. By Theorem 4.3, it is enough to get $T, T^* \in L[\mathcal{H}][G\restriction\gamma]$, some $\gamma < \theta$ such that $A \subseteq (p[T])^{L[\mathcal{H}](\mathbb{R}^*_G)}, \mathbb{R} \setminus A \subseteq (p[T^*])^{L[\mathcal{H}](\mathbb{R}^*_G)}$, $L[\mathcal{H}][G\restriction\gamma] \models T, T^*$ are θ-absolute complementing trees, and for each $\lambda < \theta$, $T\restriction\lambda, T^*\restriction\lambda \in V$. We may assume $A = Code(\Sigma)$ for some hod pair $(\mathcal{P}, \Sigma)$ such that Σ is fullness preserving and has branch condensation, $\mathcal{M}_\infty(\mathcal{P},\Sigma)|\delta^{\mathcal{M}_\infty(\mathcal{P},\Sigma)} = HOD|\theta_{\alpha_n}$, for some $n < \omega$. Pick $\gamma < \theta$ big enough such that $w(\Sigma) < \gamma$. Let Φ be the strategy of $HOD|\theta_{\alpha_n}$ coded in HOD, when viewing HOD as a hod mouse. Now over $L[\mathcal{H}][G\restriction\gamma]$, we can carry out the translation $ITr^{g_0,h_0,\ldots,g_n,h_n}_{(HOD|\theta_{\alpha_n},\Phi),\alpha_0,e_0,\ldots,\alpha_n,e_n}(\mathcal{H})$, where h_i is $Coll(\omega, \theta_{\alpha_i+1})$-generic over $\mathcal{H}$ such that $HOD|\theta_{\alpha_i}$ is coded in h_i, g_i is the canonical generic filter over the δ_{α_i+1}-extender algebra of $\mathcal{H}$. The result of the translation will be a n-layered S-premouse over h_n, from which we can define the strategy of $HOD|\theta_{\alpha_n}$, using a formula similar to 2.34.

Thus, we have trees $T_0, T_0^* \in L[\mathcal{H}][G\restriction\gamma]$ which projects to $Code(\Phi)$ and $\mathbb{R} \setminus Code(\Phi)$. But in $L[\mathcal{H}][G\restriction\gamma]$, we have the iteration mapping π from $\mathcal{P}$ to $HOD|\theta_{\alpha_n}$. Hence we have $T, T^* \in L[\mathcal{H}][G\restriction\gamma]$ which projects to $Code(\Phi_\pi)$ and $\mathbb{R}\setminus Code(\Phi_\pi)$. As each proper initial segment of $\mathcal{H}$ is in V, for each $\lambda < \theta$, $T\restriction\lambda$ and $T^*\restriction\lambda$ are in V as well. Since $\Sigma \subseteq j(\Sigma)$, $HOD|\theta_{\alpha_n}$ is a $j(\Sigma)$-iterate of $\mathcal{P}$ from the point of view in $L((Hom^*_G)^{HOD[G]}, \mathbb{R}^*_G)$. Hence $(HOD|\theta_\alpha, \Phi)$ is a tail of $(\mathcal{P}, j(\Sigma))$. Hence $\Phi_\pi = j(\Sigma)$. This implies $j(A) \in (Hom^*_G)^{L[\mathcal{H}][G]}$. Thus j embeds V into a derived model of $L[\mathcal{H}]$.

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