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Contents

Summary

In this thesis, the distributive proper forcing axiom ($DPFA$) is studied. On one hand, $DPFA$ implies $\mathfrak{c} = \aleph_2$. On the other hand, assume the consistency of the existence of a supercompact cardinal, $DPFA$ is consistently true with many consequences of CH, such as cardinal invariants of the continuum being $\aleph_1$ and the existence of non isomorphic $\aleph_1$- dense subsets of the real line. An application of these results is that $DPFA$ can be separated from other fragments of PFA.

Conventions and Notations

Notations used in this thesis are standard.

$Ord, Card, Lim$ denotes the class of ordinals, cardinals and limit ordinals respectively.

$\mathcal{P}(A)$ denotes the power set of A.

For a cardinal κ, $\mathcal{P}_\kappa(A) = [A]^{<\kappa} = \{X \subseteq A : |X| < \kappa\}$.

$[A]^\kappa = \{X \subseteq A : |X| = \kappa\}$, $[A]^{\leq\kappa} = \{X \subseteq A : |X| \leq \kappa\}$.

If p, q are sequences, then $p \supseteq_e q$ stands for "p end extends q". If p, q are sets of ordinals, then $p \supseteq_e q$ stands for $p \supseteq q$ and $\forall\alpha \in q\ \forall\beta \in p \setminus q(\alpha < \beta)$.

FIN is the Fréchet ideal, which consists of finite subsets of the underlying set.

$A \subseteq^* B$ is short for the statement: $A \setminus B$ is finite.

$\forall^\infty n$ abbreviates for $\exists\, m \in \omega\ \forall\, n \geq m$; dually, $\exists^\infty n$ is an abbreviation for $\forall\, m \in \omega\ \exists\, n \geq m$.

$Fn(I, J, \kappa) = \{f : I \to J \mid |f| < \kappa\}$, $Fn(I, J) = Fn(I, J, \omega)$.

$\mathcal{L}$ denotes the collection of null sets, $\mathcal{B}$ denotes the collection of meager sets.

For a linear order $(A, <)$, (a, b) denotes the interval $\{x \in A \mid a < x < b\}$.

Chapter 1

Introduction

Forcing is powerful in consistency proofs of mathematical statements. Since its birth in the proof of non-provability of continuum hypothesis in the usual formal axiom system of mathematics in 1963 [14] [15], forcing has been studied deeply and various applications have been found in mathematical logic as well as in analysis, algebra and topology. Forcing axioms were sometimes regarded as "black boxes" for people from different branches of mathematics to gain some power of forcing without getting into technical details of consistency proofs. Classical examples of forcing axioms are Martin's Axiom and Proper Forcing Axiom. We will be of special interest in a fragment of PFA which focuses on those partial orders that does not introduce new countable sequence of ordinals. In this chapter, we briefly review forcing, forcing axioms and other related background materials.

1.1 Continuum Hypothesis

The Zermelo-Fraenkel axiom system (ZF for short) is commonly accepted as the foundation of mathematics. An additional axiom, the axiom of choice, is usually required in dealing with infinite objects. It was the analysis of such objects, especially the real line, that led to the research work in set theory. Cantor proved in the 19th century that the size of the real line is uncountable and conjectured that it is the first uncountable cardinal.

This conjecture is known as continuum hypothesis (CH for short). CH simplifies the structure of the real line and hence has wide applications (see [48] for examples). At this point, it is interesting to know whether CH itself is provable or disprovable in ZFC. This problem became the first one in Hilbert's famous list of 23 problems.

For centuries, people used to label mathematical statements with "True" or "False". The situation was changed completely by Gödel's Incompleteness Theorems [20], which implies that if ZFC is consistent, then there are statements neither provable nor disprovable by the ZFC axiom system. Combine this with Gödel's Completeness Theorem [19], it is equivalent to say that such statement survives in one model, but dies in another.

Later, Gödel built a constructible universe L [21] and showed that CH holds in it. L is a class model of set theory. In fact it is the minimal one among all class models. As a corollary, CH is not refutable by ZFC. However, this universe seems to be too idealized. Gödel pointed out several phenomena that indicated the possibility of failure of CH.

This possibility was confirmed in 1960s when Cohen discovered the method of forcing. Generally speaking, forcing is a way to extend models of set theory or its fragments, to add designated "generic objects" to the "ground model" while preserving the class of ordinals. Therefore, CH could be violated by adding enough reals to the ground model (additional arguments are needed, such as the preservation of cardinals). Thus CH is not provable from ZFC. Combining these two together, it can be concluded that it is independent.

1.2 Forcing

In this section, we review some basic facts about forcing. The reader is referred to the standard textbooks [32] and [34] for a complete introduction to forcing. [16] is a wonderful survey on forcing and especially on iterated forcing.

1.2.1 Forcing

A notion of forcing is a pre-ordering. One may also use a partial order or a complete Boolean algebra instead. Any of the three provides essentially equivalent approach to the theory of forcing. Suppose φ is a statement which is true if some witness is found. We wish to build some model with such a witness. Assume we have a "ground model" M, namely a model of ZFC that we start with. Without loss of generality, assume $M \models \neg\varphi$. Find in M a notion of forcing $\mathbb{P}$ which contains information to approximate the witness we need. A filter on $\mathbb{P}$ consists of non-conflicting elements of $\mathbb{P}$. If the filter is "big enough", then the approximation can be achieved. Formally speaking, if a filter G is "generic", then $M[G]$, a larger model of ZFC, can be constructed. Each element of $M[G]$ has a "name" in M. $M[G]$ is the collection of all interpretations of $(M, \mathbb{P})$-names with respect to the generic filter G. $M[G]$ contains the same ordinals as M and additionally the required witness.

However, this does not mean that $M[G] \models \varphi$, since φ may have new explanation in $M[G]$. Either we are lucky enough to obtain the absoluteness for some objects involved in φ and succeed to get the consistency of φ, or we need further forcing to deal with the new situation in $M[G]$. In the former case, we need to argue the preservation of certain properties such as cardinality, cofinality or stationarity. In the latter case, iterated forcing would be needed. There is a third case, namely no matter how hard we work, φ can never be forced to be true.

For example, let $\mathbb{P}$ be the partial order consists of finite partial functions from ω

into 2. The order on $\mathbb{P}$ is reverse extension. Then a filter on $\mathbb{P}$ gives a partial function from ω into 2 and a generic filter "codes" a function from ω into 2. In other words, a generic filter corresponds to a real number. By genericity, this real has to be different from the ones in the ground model. We see that the generic real is approximated by its finite parts, which are elements of $\mathbb{P}$.

We defer the discussion of the second case. As an example for the third case, let φ be the statement $\mathfrak{c} = \aleph_0$. Cantor showed $ZF \vdash \neg\varphi$ and so we can not expect to force φ. If we naively take $\mathbb{P}$ to be the partial order which consists of finite partial functions from ω into the "reals" and is ordered by reverse extension, then we find in the generic extension a surjection from ω into the "reals". However, new reals would appear and the surjection is fake in the generic extension.

We adopt the notation of [34]. For two elements p and q from a notion of forcing $\mathbb{P}$, $p \leq q$ means p contains more information than q. In many cases, we use $\mathbb{V}$ to denote the ground model. We may use $\mathbb{V}^{\mathbb{P}}$, other than $\mathbb{V}[G]$, to denote the generic extension if it is unnecessary to specify the generic filter.

The following facts are frequently quoted.

Fact 1.1. *1. If $p \Vdash \varphi$, then for any generic filter $G \ni p$, $\mathbb{V}[G] \vDash \varphi$.*

2. If $\mathbb{V}[G] \vDash \varphi$, then there is some $p \in G$, $p \Vdash \varphi$.

3. If $p \Vdash \exists x\, \varphi(x)$, then there is a $\mathbb{P}$-name τ in $\mathbb{V}$, $p \Vdash \varphi(\tau)$.

1.2.2 Chain Condition, Distributivity and Closure Property

A notion of forcing has countable chain condition (c.c.c.) if every antichain is countable. A typical example is the collection of nonempty open subsets of the real line ordered by inclusion. This partial order actually satisfies a stronger property: separability. Separable partial orders play the same role as a countable partial order; namely, they add the same generic objects.

c.c.c. partial orders are very useful since they preserve cardinals and cofinalities.

Chain conditions can be generalized to any cardinal κ: a notion of forcing has κ chain condition ($\kappa - c.c.$) if every antichain has cardinality less than κ. Hence, $\aleph_1 - c.c.$ is same as $c.c.c.$

If $\mathbb{P}$ satisfies $\kappa - c.c.$, then every cardinal $\geq \kappa$ is preserved by $\mathbb{P}$.

In order to preserve cardinals, closure or distributivity might also be required.

A partial order is κ-closed, where $\kappa \geq \aleph_1$ is a cardinal, if for every $\lambda < \kappa$, every decreasing sequence $\langle p_\alpha : \alpha < \lambda \rangle \subseteq \mathbb{P}$, there exists a $p \in \mathbb{P}$ such that $\forall \alpha < \lambda$, $p \leq p_\alpha$. We also abuse the notion of ω - closed or countably closed to mean $\aleph_1$-closed.

A partial order is κ-distributive, where $\kappa \geq \aleph_1$ is a cardinal, if it does not introduce new sequence of ordinals with length $< \kappa$. It is distributive iff it is $\aleph_1$- distributive.

If $\mathbb{P}$ is κ-closed, then it is κ-distributive. If $\mathbb{P}$ is κ-distributive, then every cardinal $\leq \kappa$ remains a cardinal in $\mathbb{V}^{\mathbb{P}}$.

1.2.3 Properness

On the other hand, properness concerns about the preservation of stationary sets.

Jech [27] generalized the notion of club subsets and stationary subsets of a regular uncountable cardinal to spaces of the form $\mathcal{P}_\kappa(X)$, where X is an uncountable set and $\kappa \leq |X|$ is a regular cardinal:

Definition 1.2. 1. *$C \subseteq \mathcal{P}_\kappa(X)$ is unbounded, if $\forall x \in \mathcal{P}_\kappa(X)$, $\exists y \in C$ such that $x \subseteq y$.*

2. *$C \subseteq \mathcal{P}_\kappa(X)$ is closed, if for any $\subseteq$-increasing sequence $\langle x_\alpha : \alpha < \mu \rangle \subseteq C$, where $\mu < \kappa$, $\bigcup_{\alpha<\mu} x_\alpha \in C$.*

3. *$C \subseteq \mathcal{P}_\kappa(X)$ is a club, if it is both closed and unbounded.*

4. *$S \subseteq \mathcal{P}_\kappa(X)$ is stationary, if for any club $C \subseteq \mathcal{P}_\kappa(X)$, $S \cap C \neq \emptyset$.*

Shelah [42] introduced properness: A notion of forcing $\mathbb{P}$ is proper, if for any uncountable cardinal λ, for any stationary subset $S \subseteq \mathcal{P}_{\aleph_1}(\lambda)$, S remains stationary in $\mathbb{V}^{\mathbb{P}}$.

There are several equivalent formulations of properness, and the most popular one is:

Definition 1.3. *$\mathbb{P}$ is proper, if for any regular cardinal $\lambda > (2^{|\mathbb{P}|})^+$, there is a club $C \subseteq \mathcal{P}_{\aleph_1}(H_\lambda)$ s.t. for any $M \in C$, M is a countable elementary submodel of $\langle H_\lambda, \in, \mathbb{P}\rangle$ and $\forall p \in M \cap \mathbb{P}, \exists q \leq p$ s.t. $q \Vdash \dot{G} \cap M$ is $(M, \mathbb{P} \cap M)$- generic.*

Remark 1.4. *1. We will use the term "λ is large enough" to mean $\lambda > (2^{|\mathbb{P}|})^+$.*

2. Let "q is $(M, \mathbb{P})$-generic" be short for "$q \Vdash \dot{G} \cap M$ is $(M, \mathbb{P} \cap M)$- generic".

Proper forcing is a wide class, since

Lemma 1.5. *(1) Every c.c.c. forcing is proper.*
(2) Every $\aleph_1$ - closed forcing is proper.

However, distributivity does not imply properness. The forcing "club-shooting" in [9] is distributive but destroys a stationary subset of ω_1, and hence is not proper.

Proper forcing preserves $\aleph_1$, since

Lemma 1.6. *If $\mathbb{P}$ is proper, $A \in \mathbb{V}^{\mathbb{P}}$ is a countable set of ordinals, then there is a $B \in \mathbb{V}$, a countable set of ordinals s.t. $A \subseteq B$.*

1.2.4 Iterated Forcing

Now we discuss the second case in the section 1.2.1. A generic extension $\mathbb{V}^{\mathbb{P}}$ is itself a model of ZFC and thus can also be regarded as a "ground model"; namely, we can pick up a partial order $\mathbb{Q} \in \mathbb{V}^{\mathbb{P}}$ and form the two-step iteration $\mathbb{P} * \mathbb{Q}$ in $\mathbb{V}$ and the corresponding generic extension is $\mathbb{V}^{\mathbb{P}*\mathbb{Q}} = (\mathbb{V}^{\mathbb{P}})^{\mathbb{Q}}$. We can generalise this process to any finite steps without any problem. The main difficulty is how to pass over the limit steps.

Let $\langle(\mathbb{P}_n, \mathbb{Q}_n) : n < \omega\rangle$ be a sequence of partial orders in the above pattern: $\mathbb{Q}_n$ is a notion of forcing in $\mathbb{V}^{\mathbb{P}_n}$ and $\mathbb{V}^{\mathbb{P}_{n+1}} = (\mathbb{V}^{\mathbb{P}_n})^{\mathbb{Q}_n}$.

$\mathbb{P}_n$ consists of sequences with length n. $\mathbb{P}_n$ is an n-step iteration and each Q_k, where $k < n$, is called an iterand of $\mathbb{P}_n$. Notice that $\mathbb{P}_{n+1} \upharpoonright n = \mathbb{P}_n$.

We wish to define $\mathbb{P}_\omega$. One possible approach is that $\mathbb{P}_\omega$ consists of all sequences $\langle q_n : n \in \omega\rangle$ s.t. $\forall m\ \langle q_n : n \in m\rangle \in \mathbb{P}_m$. This collection is the maximal one we can form.

Another approach to define $\mathbb{P}_\omega$ involves the consideration of "compactness". For this purpose, assume each partial order Q_n has a largest element $\mathbb{1}_{Q_n}$. Define for $p = \langle q_n : n \in \omega\rangle$, the support of p, $spt(p) = \{n \in \omega : q_n \neq \mathbb{1}_{Q_n}\}$. Let $\mathbb{P}_\omega$ be the collection of all sequences $p = \langle q_n : n \in \omega\rangle$ s.t. $\forall m\ p \upharpoonright m \in \mathbb{P}_m$ and $spt(p)$ is bounded in ω.

The first approach is called "inverse limit" at stage ω, and the second is called "direct limit" at stage ω. One can easily generalize it to any limit stage. If direct limit is taken at any limit stage, then the support of any condition is finite. Such an iteration is called "finite support iteration".

Solovay and Tennenbaum proved in [50] that it is consistent that there is no Souslin tree. Starting with a model with a Souslin tree T, one can "kill" T by adding an uncountable branch to it. We can also list all Souslin trees in the ground model and eliminate them one by one. However, new Souslin trees may occur while old ones are killed. Fortunately, Solovay and Tennenbaum proved that a finite support iteration of *c.c.c.* forcing is *c.c.c.* (or the property of being *c.c.c.* is preserved under finite support iteration). By using a ω_2-step finite support iteration, with iterands suitably arranged, one can build the model $\mathbb{P}_{\omega_2}$. Each "potential Souslin tree" in $\mathbb{P}_{\omega_2}$ comes from an intermediate step and is therefore killed in latter steps. So there will be no Souslin tree in the last model.

For proper forcing, Shelah [42] proved that properness is preserved under countable support iteration, in which inverse limits are taken at stages with cofinality ω and direct limits are taken elsewhere. This fact is fundamental in many proofs today.

It is worth noting that distributivity is not preserved in many cases. For instance, a finite support iteration of nontrivial iterands will always add Cohen reals. Similarly, a countable support iteration of nontrivial $\aleph_2$ - distributive forcing notions is not $\aleph_2$ - distributive.

1.3 Forcing Axioms

A forcing axiom describes that for any forcing in a certain class, for any collection of dense subsets satisfying a prescribed property, there is a filter meet every dense subset in the collection.

1.3.1 Martin's Axiom

Martin's Axiom was isolated by Martin and Solovay [37].

Definition 1.7. *For a cardinal κ, $MA(\kappa)$ is the following statement: If a nontrivial forcing $\mathbb{P}$ is c.c.c., $\mathfrak{D}$ is a collection of dense subsets of $\mathbb{P}$, $|\mathfrak{D}| \leq \kappa$, then there is a filter $G \subseteq \mathbb{P}$ s.t. $\forall D \in \mathfrak{D}, G \cap D \neq \emptyset$.*

Remark 1.8.

1. *Notice that $MA(\omega)$ is always true and $MA(2^\omega)$ is always false, $MA(\kappa)$ makes sense only when $\omega \leq \kappa < 2^\omega$. MA is the statement: $\forall \kappa < 2^\omega, MA(\kappa)$.*

2. *$MA(\kappa)$ is equivalent to the form in which partial orders are "small": If a nontrivial forcing $\mathbb{P}$ is c.c.c. and have size $\leq \kappa$, $\mathfrak{D}$ is a collection of dense subsets of $\mathbb{P}$, $|\mathfrak{D}| \leq \kappa$, then there is a filter $G \subseteq \mathbb{P}$ s.t. $\forall D \in \mathfrak{D}, G \cap D \neq \emptyset$.*

The class of *c.c.c.* forcing is the most commonly known family of partial orders that arise in the study of analysis and topology; so *MA* is formed naturally. Historically, its formulation was motivated by [50].

Clearly *MA* follows from the continuum hypothesis. It is proved in [37] that *MA* is also consistent with the failure of continuum hypothesis. Many consequences have been found in [37] under the assumption of *MA* and larger continuum. For instance, Souslin's hypothesis holds and Borel's conjecture fails.

On the other hand, *MA* alone is sufficient to extend the properties of countable subsets of the real line to those with cardinality less than the continuum. For instance, the study of cardinal invariants of the reals is trivialized if *MA* is assumed. Martin and Solovay proved that if *MA* holds, then $add(\mathcal{L}) = \mathfrak{c}$ (See Chapter 3 for definition). As a corollary, *MA* implies that $\mathfrak{c}$ is a regular cardinal. Since almost all other cardinal invariants are not less than $add(\mathcal{L})$, they equal to $\mathfrak{c}$ in a model of *MA*.

Martin's Axiom succeeded in opening a new area for set theorists and set-theoretic topologists.

However, weaknesses had been found for the practice of *MA*. For example, Baumgartner [5] showed that it is consistent to have $2^\omega = \omega_2$ with all $\aleph_1$ - dense subsets of the real line are order isomorphic (See Chapter 2 for definition). Avaraham and Shelah [2] showed that $MA + \neg CH$ is not enough to guarantee the isomorphism. We will return to this topic in Chapter 2. It can be concluded that *MA* lacks the power in dealing with more complicated structures.

1.3.2 Proper Forcing Axiom

PFA is the statement: If $\mathbb{P}$ is proper, $\mathfrak{D}$ is a collection of dense subsets of $\mathbb{P}$, $|\mathfrak{D}| = \aleph_1$, then there is a filter $G \subseteq \mathbb{P}$ s.t. $\forall D \in \mathfrak{D}, G \cap D \neq \emptyset$.

Remark 1.9. *1. Since every c.c.c. forcing is proper, PFA is a generalization of* $MA(\omega_1)$.

2. *$PFA(\omega_2)$, if being defined analogically as $MA(\omega_2)$, is false. Since $Fn(\omega_1, \omega_2, \omega_1)$ is ω-closed, it is proper.*

Baumgartner [8] made the first step in studying the effect of proper forcing axiom in various contexts. It was argued that many problems which were proved unsolvable or left unsolved by MA could have a definite answer in the presence of PFA. One significant example found by Baumgartner is that PFA implies that there is only one equivalence class of $\aleph_1$-dense sets of reals under order isomorphism. The result was further developed by Moore [39] to obtain five uncountable linear orders which form a basis for all uncountable linear orders. Moore also introduced the Mapping Reflection Principle (*MRP*) [38], which becomes recently a popular candidate between PFA and its applications.

Besides the structure of the real line, Baumgartner [8] showed that PFA is able to deduce club-isomorphism between any two Aronszajn trees, chain-antichain property of $\mathcal{P}(\omega)$ and to trivialize every automorphism of $\mathcal{P}(\omega)/FIN$. Many other interesting properties were studied in this fruitful work.

PFA also has impact on cardinal arithmetic. Todorčević [11] and Veličković [56] showed that $PFA \vdash c = \aleph_2$. Thus PFA is quite different from its *c.c.c.* fragment, MA. Several principles such as Open Coloring Axiom (*OCA*) [54] and P-Ideal Dichotomy (*PID*) [55] were studied along the way. These principles follow from PFA and capture many consequences of PFA. Later, Viale [58] proved that the Singular Cardinal Hypothesis (SCH) follows from PFA. In fact, Viale showed in his PhD thesis [57] that SCH holds in a model satisfying either *PID* or *MRP*, while the latter two statements are mutually independent. In other words, two provably different approaches were provided in this thesis to show the implication of SCH from PFA.

As pointed out by Baumgartner [8], the power of PFA might be explained by its large cardinal strength. Starting from a model with a supercompact cardinal, a model satisfying PFA could be constructed with the assistance of a Laver function.

For the lower bound, Todorčević [53] proved that the principle of squares fails in the presence of PFA, and hence an inner model with Woodin cardinals would exist.

1.3.3 Fragments of PFA

We will mainly be interested in the following fragments of PFA: $\omega - PFA$, $< \omega_1 - PFA$, $BPFA$ and $DPFA$. Each of these fragment captures many consequences of PFA. We wish to know the relationship between them.

In some applications of PFA, each relevant dense subset has cardinality $\leq \aleph_1$.

Definition 1.10. *[23] The Bounded Proper Forcing Axiom (BPFA) is the statement: If $\mathbb{P}$ is proper, $\mathfrak{D}$ is a collection of dense subsets of $\mathbb{P}$, $|\mathfrak{D}| = \aleph_1$, and each $D \in \mathfrak{D}$, $|D| \leq \aleph_1$, then there is a filter $G \subseteq \mathbb{P}$ s.t. $\forall D \in \mathfrak{D}, G \cap D \neq \emptyset$.*

Remark 1.11. *As the name indicates, BPFA deals with dense subsets whose cardinality have a bound $\aleph_1$. By Remark 1.8, BPFA implies $MA(\omega_1)$.*

We refer to section 1.2.2 for the notion of distributivity.

Lemma 1.12. *Suppose $\mathbb{P}$ is proper, then $\mathcal{P}$ is distributive if and only if it adds no new real.*

Proof: The "only if" part is trivial since each real is a countable sequence of ordinals. For the "if" part, suppose $X \in \mathbb{V}^{\mathbb{P}} \setminus \mathbb{V}$, and $\mathbb{V}^{\mathbb{P}} \models X$ is a countable set of ordinals. Then by Lemma 1.6, there is some $Y \in \mathbb{V}$, a countable set of ordinals such that $X \subseteq Y$. Let $\pi \in \mathbb{V}$ be a bijection between ω and Y, then $\pi^{-1}(X)$ is a real. Since $\pi^{-1} \in \mathbb{V}$, this real must be a new real. □

Definition 1.13. *The Distributive Proper Forcing Axiom (DPFA) is the statement: If $\mathbb{P}$ is distributive and proper, $\mathfrak{D}$ is a collection of dense subsets of $\mathbb{P}$, $|\mathfrak{D}| = \aleph_1$, then there is a filter $G \subseteq \mathbb{P}$ s.t. $\forall D \in \mathfrak{D}, G \cap D \neq \emptyset$.*

Remark 1.14.
- *In [18], a property called "totally proper" was defined and proved to be equivalent to "distributive proper" we use here.*
- *This forcing axiom is originally studied by Magidor and is denoted by "PDFA" in an unpublished note.*

Definition 1.15. *[46] For a partial order $\mathbb{P}$, a regular cardinal λ that is large enough, and an ordinal $0 < \alpha < \omega_1$, say $\langle M_\beta : \beta < \alpha\rangle$ is an α-tower for $\mathbb{P}$ on H_λ, if*

- *Each M_β is a countable elementary model of H_λ and $\mathbb{P} \in M_\beta$,*
- *If $\beta < \gamma < \alpha$, then $\langle M_\delta : \delta \leq \beta\rangle \in M_\gamma$,*
- *If $\gamma < \alpha$ is a limit ordinal, then $M_\gamma = \bigcup_{\beta\in\gamma} M_\beta$.*

Definition 1.16. *$\mathbb{P}$ is α-proper, where $0 < \alpha < \omega_1$, if for any α-tower $\langle M_\beta : \beta < \alpha\rangle$ for $\mathbb{P}$, for any $p \in M_0 \cap \mathbb{P}$, there is a $q \in \mathbb{P}$, $q \leq p$ such that $\forall\beta < \alpha$, q is $(M_\beta, \mathbb{P})-$generic.*

Definition 1.17. *$\mathbb{P}$ is $< \omega_1$-proper, if $\forall\alpha < \omega_1$, $\mathbb{P}$ is α- proper.*

Lemma 1.18.
1. *Properness is 1-properness,*
2. *If $\mathbb{P}$ is α- proper and $0 < \alpha' < \alpha$, then it is α'- proper,*
3. *If $\mathbb{P}$ is α- proper, then it is $\alpha + 1$ -proper,*
4. *If $\alpha = \alpha_1 + \alpha_2$, $\mathbb{P}$ is both α_1- proper and α_2- proper, then it is also α- proper.*

Theorem 1.19. *[46] α-properness is preserved under countable support iteration.*

Definition 1.20. *[7] A notion of forcing $\mathbb{P} = \langle P, \leq\rangle$ satisfies Axiom A, if there is $\langle \leq_n : n \in \omega\rangle$ such that:*

- *Each $\leq_n$ is a partial order on $\mathbb{P}$,*
- $\leq \supseteq \leq_0 \supseteq \leq_1 \supseteq \leq_2 \ \ldots,$

- *If $\langle p_n : n \in \omega \rangle$ is a sequence such that $\forall n\ p_n \geq_n p_{n+1}$, then*
  $\exists\, q \in \mathbb{P} \forall n \in \omega (q \leq_n p_n)$

- *For each $p \in \mathbb{P}$ and $n \in \omega$, if $\Vdash \dot{\alpha}$ is an ordinal, then there is $q \leq_n p$ and a countable set of ordinals X such that $q \Vdash \dot{\alpha} \in X$.*

Lemma 1.21. *The following are true:*

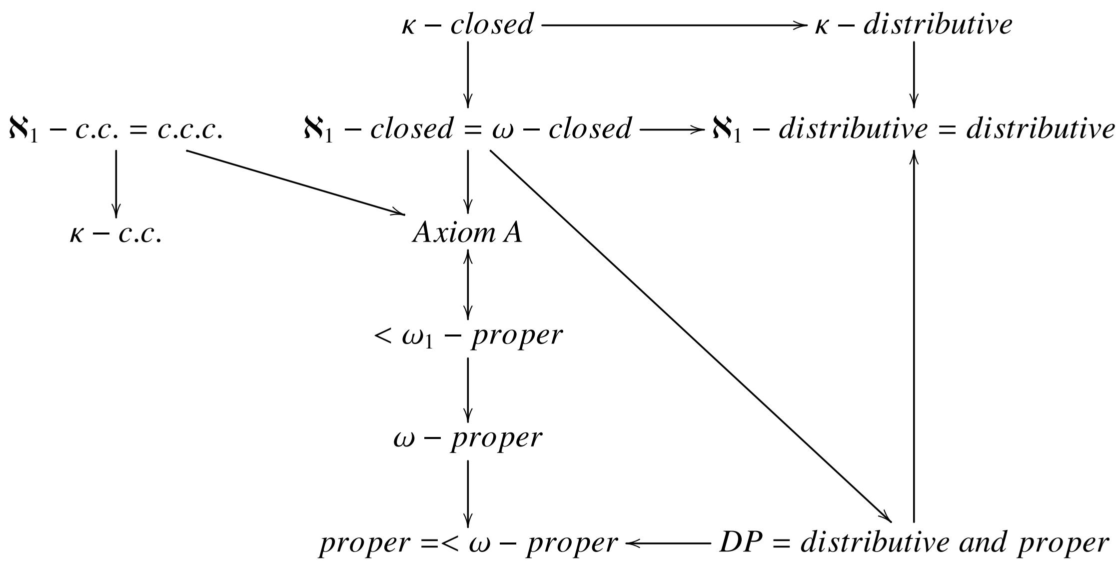
1. *If $\mathbb{P}$ has c.c.c., or is countably closed, then it satisfies Axiom A.*

2. *If $\mathbb{P}$ satisfies Axiom A, then it is proper. In fact, it is $< \omega_1$-proper.*

Ishiu [26] showed that the converse is true:

Theorem 1.22. *If $\mathbb{P}$ is $< \omega_1$-proper, then there is some $\mathbb{Q}$ satisfying Axiom A and $\pi : \mathbb{Q} \to \mathbb{P}$ such that the range of π is dense in $\mathbb{P}$.*

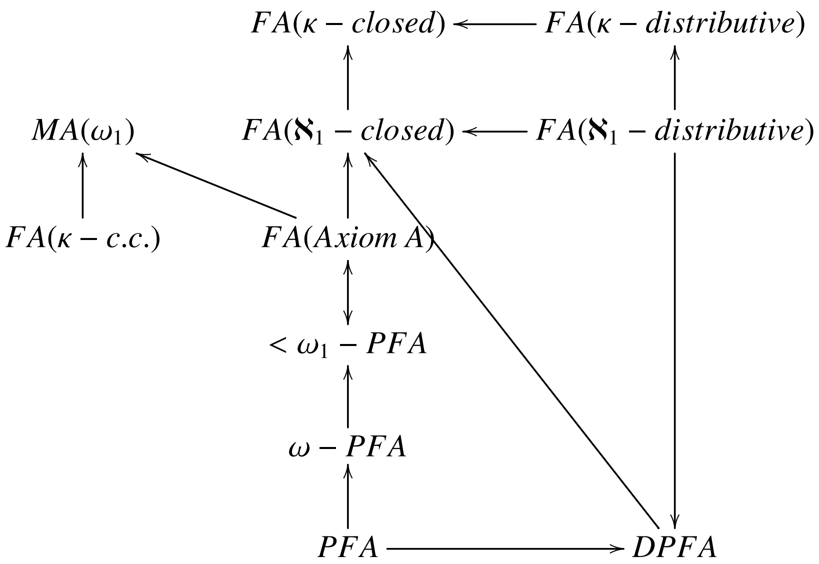
Hence, we obtain the following diagram (where "→" stands for "implication in the sense of forcing" and $\kappa \geq \aleph_1$):



Let C be a class of forcing notions. Define $FA(C)$ to be the statement: If $\mathbb{P} \in C$, $\mathfrak{D}$ is a collection of dense subsets of $\mathbb{P}$, $|\mathfrak{D}| = \aleph_1$, then there is a filter $G \subseteq \mathbb{P}$ s.t. $\forall D \in \mathfrak{D}, G \cap D \neq \emptyset$.

Hence, $FA(c.c.c.)$ is $MA(\omega_1)$, $FA(proper)$ is PFA and $FA(DP)$ is $DPFA$, as defined above. Similarly, let $\omega - PFA$ be $FA(\omega - proper)$ and $< \omega_1 - PFA$ be $FA(< \omega_1 - proper)$.

The forcing axioms of these classes of forcings admit the same diagram, with all arrows inverted:



Remark 1.23. *$FA(\kappa - c.c.)$ makes little sense. If $\mathbb{P} = Fn(\omega, \omega_1)$, then $\mathbb{P}$ has size $\aleph_1$ and hence has $\aleph_2$-c.c. But $FA(\mathbb{P})$ is false!*

Lemma 1.24. *$FA(Axiom\ A)$ holds iff $< \omega_1 - PFA$ holds.*

Proof. The "if" part follows from Lemma 1.21. For the "only if" part, let $\mathbb{P}$ be $< \omega_1$- proper. Let $\langle D_\alpha : \alpha < \omega_1 \rangle$ be a collection of dense subsets of $\mathbb{P}$. By Theorem 1.22, let $\mathbb{Q}$ satisfy Axiom A and $\pi : \mathbb{Q} \to \mathbb{P}$ be a dense embedding. Let $E_\alpha = \{q \in \mathbb{Q} : \exists p \in D_\alpha\ \pi(q) \leq p\}$ for each $\alpha < \omega_1$, then each E_α is dense in $\mathbb{Q}$. Apply $FA(Axiom\ A)$, let $H \subseteq \mathbb{Q}$ be a filter which meets each E_α, say $q_\alpha \in H \cap E_\alpha$. Let $G = \{p \in \mathbb{P} : \exists q \in H(p \geq \pi(q))\}$, then $G \subseteq \mathbb{P}$ is a filter. By definition of E_α, let $p_\alpha \in D_\alpha, \pi(q_\alpha) \leq p_\alpha$. But $q_\alpha \in H$, so $p_\alpha \in G$. Hence $G \cap D_\alpha \neq \emptyset$. □

The more interesting question would be that whether the implications in the diagram can be reversed. We will continue with this topic in Chapter 4.

Before we proceed to the result chapters, a few words about the motivation will be necessary. In a unpublished note, Menachem Magidor proved that it is consistent relative to a supercompact cardinal that $DPFA$ holds and for each infinite cardinal κ, $\square_{\kappa,\aleph_1}$ holds. This provides evidence that though $DPFA$ implies $\neg CH$, there is a chance that $DPFA$ is consistent with some consequences of CH.

Chapter 2

DPFA and Order Type of the Real Line

In this chapter, we consider the effect of forcing axioms on the ordered structure of the real line.

Definition 2.1. *Let A, B be linear orders.*

1. *A and B are isomorphic, denoted by $A \simeq B$, if $\exists f : A \to B$, a bijection which preserves the order.*

2. *A is embedable into B, denoted by $A \trianglelefteq B$, if $\exists f : A \to B$ which is strictly order preserving. In other words, A is isomorphic to a subset of B.*

Definition 2.2. *Let $\mathcal{C}$ be a collection of linear orders, a subset $\mathcal{B} \subseteq \mathcal{C}$ is a basis if $\forall A \in \mathcal{C}\ \exists B \in \mathcal{B}\ (B \trianglelefteq A)$.*

Notation 2.3. *If A is a linear order, let A^* denote the linear order obtained by reverse the order on A.*

For example, let $\mathcal{C}$ be the collection of all infinite linear orders. Let $\mathcal{B} = \{\omega, \omega^*\}$. Then both $\mathcal{B}$ and $\mathcal{C}$ are basis for $\mathcal{C}$.

2.1 Well Known Results

This section records the results on the ordered structure of the real line in ZFC. The real line, denoted by $\mathbb{R}$, is the collection of real numbers with the usual linear order. We mainly follow Baumgartner's survey [6].

Definition 2.4. *$A \subseteq \mathbb{R}$ is $\aleph_1$-dense, if*

- *A has no endpoints and*
- *for any two elements $a < b$ from A, $|\{x \in A \mid a < x < b\}| = \aleph_1$.*

Since the linear order on $\mathbb{R}$ is seperable, we can show:

Fact 2.5. *If A is $\aleph_1$-dense, then $|A| = \aleph_1$. Conversely, if $A \subseteq \mathbb{R}$, $|A| = \aleph_1$, then there is a countable subset of $C \subseteq \mathbb{R}$, $A \setminus C$ is $\aleph_1$-dense.*

To prove this, we list all "rational" intervals in some ω-sequence. Delete all those "rational" intervals that have countable intersection with A. Now, the following is an immediate consequence.

Corollary 2.6. *The collection of $\aleph_1$-dense subsets of the reals forms a basis for the collection of uncountable subsets of the reals.*

Fact 2.7. *We can focus on $\aleph_1$-dense subsets of the interval $(0, 1)$. Also, if we correspond a member of $(0, 1)$ with its binary representation whose string does not end with infinite string of 1's, then the natural order on $(0, 1)$ corresponds to the lexicographic order on Cantor space.*

Definition 2.8. *For $\aleph_1$-dense sets A and B, they are comparable if either $B \trianglelefteq A$ or $A \trianglelefteq B$.*

Theorem 2.9. *[47] Assume CH, then there are incomparable $\aleph_1$-dense sets.*

Shelah generalized this theorem greatly:

Theorem 2.10. *[43] Assume that* $2^{\aleph_0} < 2^{\aleph_1}$*, then*

1. $\forall\, \aleph_1$*-dense A,* $\exists\, \aleph_1$*-dense B such that* $\neg(A \trianglelefteq B)$*.*

2. *There is a family* $\mathcal{F}$ *of mutually incomparable* $\aleph_1$*-dense sets,* $|\mathcal{F}| = 2^{\aleph_1}$*.*

Definition 2.11. *For* $\aleph_1$*-dense* A, B*, they are compatible if there is some* $\aleph_1$*-dense C,* $C \trianglelefteq A \wedge C \trianglelefteq B$*.*

It is directly observed that if A and B are comparable, then they are compatible. However, the converse fails badly.

Theorem 2.12. *[43] It is consistent with* $2^{\aleph_0} < 2^{\aleph_1}$ *that any two* $\aleph_1$*-dense sets are compatible.*

On the other hand, as Baumgartner pointed out:

Fact 2.13. *[6] It is consistent with* $2^{\aleph_0} < 2^{\aleph_1}$ *that there is a family* $\mathcal{F}$ *of mutually incompatible* $\aleph_1$*-dense sets,* $|\mathcal{F}| = 2^{\aleph_1}$*.*

He also argued that with slight modification of the argument above, one can get the consistency of $2^{\aleph_0} = 2^{\aleph_1}$ with such an $\mathcal{F}$.

Now let us turn to the isomorphism type of $\aleph_1$-dense sets of reals.

Theorem 2.14. *[8] Assume CH.* A, B *are* $\aleph_1$*-dense. Then there is a c.c.c. partial order which forces "A and B are isomorphic".*

Iterating this process in ω_2 steps by dealing with all potential pairs of $\aleph_1$-dense sets and interleaving the partial order to force Martin's Axiom, Baumgartner proved:

Theorem 2.15. *[5] It is consistent that* $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ *together with MA and "any two* $\aleph_1$*-dense sets are isomorphic".*

Baumgartner asked if $MA + \neg CH$ guarentees the isomorphism between any two $\aleph_1$-dense sets. This was disproved by Avaraham and Shelah:

Theorem 2.16. *[2] It is consistent with $MA+\neg CH$ that there are two nonisomorphic $\aleph_1$-dense sets.*

However, Baumgartner's proof directly give the following:

Theorem 2.17. *[8] Assume PFA. If A and B are $\aleph_1$-dense, then they are order isomorphic.*

2.2 $DPFA$ and $\aleph_1$-Dense Sets

Theorem 2.18. *If it is consistent to have a supercompact cardinal, then it is also consistent to have $DPFA$ and the existence of two nonisomorphic $\aleph_1$-dense sets.*

Proof: Let $\mathbb{V}$ be a model of ZFC with a supercompact cardinal κ. Let $\mathbb{C} = \mathbb{C}_{\omega_1} \in \mathbb{V}$ be the partial order to add $\aleph_1$ many Cohen reals. Namely,

$$\mathbb{C} = Fn(\omega_1 \times \omega, 2) = \{f \mid f : \omega_1 \times \omega \to 2 \wedge |f| < \omega\}.$$

In $\mathbb{V}^{\mathbb{C}}$, let $C = \{c_\alpha | \alpha < \omega_1\}$ be the set of Cohen reals. We will use the fact that these Cohen reals are mutually generic.

Fact 2.19. $\mathbb{V}^{\mathbb{C}} \models$ *C is $\aleph_1$-dense.*

Proof of Fact: Suppose otherwise, then $\exists a < b$ from C, $(a, b) \cap C$ is countable. Let $\theta < \omega_1$ be such that $(a, b) \cap C \subseteq \{c_\alpha : \alpha < \theta\}$. By density argument, one can force some c'_θ, $\theta' \geq \theta$ to be in (a, b), a contradiction. □

Recall from Notation 2.3, for $X \subseteq \mathbb{R}$, X^* denotes the linear order obtained from X by reversing the order.

Fact 2.20. *In any generic extension of $\mathbb{V}^{\mathbb{C}}$ that preserves $\aleph_1$, both C and C^* are $\aleph_1$-dense.*

Let $\mathbb{C}_2 = Fn(\omega \times 2, 2)$ be the partial order to add two Cohen reals. Notice that $\mathbb{C}_2$ is equivalent to the product forcing $\omega^{<\omega} \times \omega^{<\omega}$. So we will abuse the notation of $\mathbb{C}_2$ to refer to this forcing. In $\mathbb{V}^{\mathbb{C}}$, for $\eta, \xi < \omega_1$, let

$$Z_{\eta,\xi} = \{(s,t) \in \mathbb{C}_2 : s \subseteq c_\eta \wedge t \subseteq c_\xi\}$$

be the filter to add (c_η, c_ξ).

Definition 2.21. *In $\mathbb{V}^{\mathbb{C}}$ or its generic extension, a notion of forcing $\mathbb{Q}$ is "nice" if*

- *$\forall \lambda$ large regular cardinal,*
- *$\forall N \prec H_\lambda$ countable elementary model such that $\mathbb{Q} \in N$,*
- *$\forall \eta \in \omega_1 \setminus N$ such that $\forall \xi \in \omega_1 \setminus N$, $Z_{\eta,\xi}$ is $(N, \mathbb{C}_2)$-generic,*
- *$\forall p \in N \cap \mathbb{Q}$,*

$\exists q \in \mathbb{Q}$ such that:

- *$q \leq p$,*
- *q is $(N, \mathbb{Q})$-generic,*
- *$q \Vdash \forall \xi \in \omega_1 \setminus N,\ Z_{\eta,\xi}$ is $(N[\dot{G} \cap N], \mathbb{C}_2)$-generic.*

Remark 2.22. *1. If q is $(N, \mathbb{Q})$-generic, then $q \Vdash \omega_1 \cap N = \omega_1 \cap N[\dot{G} \cap N]$.*

2. The last requirement on q is the same as
$\forall \xi \in \omega_1 \setminus N$, $q \Vdash Z_{\eta,\xi}$ is $(N[\dot{G} \cap N], \mathbb{C}_2)$-generic.

Lemma 2.23. *In $\mathbb{V}^{\mathbb{C}}$, fix a large regular cardinal λ. Let W be the collection of all countable $N \prec H_\lambda$ such that $\forall \eta, \xi \in \omega_1 \setminus N$, $Z_{\eta,\xi}$ is $(N, \mathbb{C}_2)$-generic. Then W is a club in $[H_\lambda]^\omega$.*

Proof: (i) W is closed. Given any increasing sequence $N_0 \subseteq N_1 \subseteq \cdots$ from W, let $N = \bigcup_{n<\omega} N_n$. We show that $N \in W$. By Tarski's criteria, $N \prec H_\lambda$. Let $\eta, \xi \in \omega_1 \setminus N$, $D \in N$ be any dense subset of $\mathbb{C}_2$. So $\exists n \in \omega$, $D \in N_n$. Since $\eta, \xi \in \omega_1 \setminus N_n$ and $N_n \in W$, $Z_{\eta,\xi} \cap D \cap N_n \neq \emptyset$. So $Z_{\eta,\xi} \cap D \cap N \neq \emptyset$. Hence, $N \in W$.

(ii) W is unbounded. We claim that for any $N \in [H_\lambda]^\omega$, there is $N' \in [H_\lambda]^\omega$ satisfying:$N' \supseteq N$ and $\forall \eta, \xi \in \omega_1 \setminus N'$,$\forall D \in N$ dense open subset of $\mathbb{C}_2$,
$Z_{\eta,\xi} \cap D \cap N' \neq \emptyset$.

Proof of Claim: Let $\{D_k : k \in \omega\}$ list all dense open subsets of $\mathbb{C}_2$ that lie in N. Since $\mathbb{C}_2 \in \mathbb{V}$ and $\mathbb{C}_2$ is countable, each D_k appears in $\mathbb{V}^{\mathbb{C}\restriction\theta_k}$ for some $\theta_k \in \omega_1$. Let $\theta = sup\{\theta_k : k < \omega\}$. Then $\theta < \omega_1$. Let $N' \in [H_\lambda]^\omega$ be such that $\theta \cup N \cup \bigcup_{k\in\omega} D_k \subseteq N'$. We check that N' works. For any $\forall D \in N$, a dense open subset of $\mathbb{C}_2$, we have $D \in \mathbb{V}^{\mathbb{C}\restriction\theta}$ and $D \subseteq N$. For any $\eta, \xi \in \omega_1 \setminus N'$, we have $\eta, \xi \geq \theta$. So $Z_{\eta,\xi}$ is $\mathbb{C}_2$-generic over $\mathbb{V}^{\mathbb{C}\restriction\theta}$, $Z_{\eta,\xi} \cap D \neq \emptyset$. But $D \subseteq N'$, $Z_{\eta,\xi} \cap D \cap N' \neq \emptyset$.

Now we prove the unboundness of W. Given any $N \in [H_\lambda]^\omega$, we construct $N = N_0 \subseteq N_1 \subseteq \cdots$ inductively by the claim. We can require that $N_{n+1} \supseteq N'_n$ and $N_{n+1} \prec H_\lambda$. Let $N^+ = \bigcup_{n\in\omega} N_n$, then $N \subseteq N^+ \prec H_\lambda$. Now it is easy to check that $N^+ \in W$. □

Lemma 2.24. *If* $\mathbb{V}^{\mathbb{C}} \models \mathbb{Q}$ *is nice, then* $\mathbb{V}^{\mathbb{C}*\mathbb{Q}} \models C$ *and* C^* *are not order isomorphic.*

Proof: Suppose otherwise, $\mathbb{V}^{\mathbb{C}*\mathbb{Q}} \models \exists \pi : C \to C$, a reverse order isomorphism. In $\mathbb{V}^{\mathbb{C}}$, let $p \in \mathbb{Q}$ be such that $p \Vdash \dot{\pi} : \check{C} \to \check{C}$ is a reverse order isomorphism. Let $N \in W$ be such that $\{\mathbb{C}, C, \mathbb{Q}, p, \dot{\pi}\} \subseteq N$. Let $\eta = \omega_1 \cap N$. By elementarity, $p \Vdash \dot{\pi} : C \cap N \to C \cap N$ is a bijection. So $p \Vdash \dot{\pi}(c_\eta) \notin N$.

Since $\mathbb{Q}$ is nice, let $q' \leq p$ be such that q' is $(N, \mathbb{Q})$-generic and $\forall \xi \in \omega_1 \setminus N$, $q' \Vdash Z_{\eta,\xi}$ is $(N[\dot{G} \cap N], \mathbb{C}_2)$-generic.

Let $q \leq q'$ decide $\dot{\pi}(c_\eta)$. Namely, $q \Vdash \dot{\pi}(c_\eta) = c_\xi$ for some $\xi \in \omega_1$. So $\xi \in \omega_1 \setminus N$. Moreover, q is $(N, \mathbb{Q})$-generic and $q \Vdash Z_{\eta,\xi}$ is $(N[\dot{G} \cap N], \mathbb{C}_2)$-generic.

Let $G \ni q$ be $(\mathbb{V}^{\mathbb{C}}, \mathbb{Q})$-generic. Let π be the interpretation of $\dot{\pi}$ in $\mathbb{V}^{\mathbb{C}}[G]$. Define in $(H_\lambda)^{\mathbb{V}^{\mathbb{C}}}[G] = (H_\lambda)^{\mathbb{V}^{\mathbb{C}}[G]}$,

$$D = \{(s,t) \in \mathbb{C}_2 : \exists x \in C((x >_{lex} s \wedge \pi(x) >_{lex} t) \vee (x <_{lex} s \wedge \pi(x) <_{lex} t))\}.$$

Claim: D is dense. Proof of claim: Fix any $(s,t) \in \mathbb{C}_2$. We will find some $(s',t') \in D$, such that $s' \supseteq_e s$ and $t' \supseteq_e t$. By density arguments applied to $\mathbb{C}$ in $\mathbb{V}$, we observe that each Cohen real c_α is irrational. In other words,each c_α is neither eventually 0 nor eventually 1. We can also assume that for some $\gamma < \aleph_1$, $c_\gamma \supseteq_e s$.

case1 $\pi(c_\gamma) \supseteq_e t$. There are u, v finite strings of 1's such that $c_\gamma \supseteq_e s^\frown u^\frown 0$ and $\pi(c_\gamma) \supseteq_e t^\frown v^\frown 0$. Let $s' = s^\frown u^\frown 1$ and $t' = t^\frown v^\frown 1$, then $c_\gamma <_{lex} s'$ and $\pi(c_\gamma) <_{lex} t'$.

case2 $\pi(c_\gamma) >_{lex} t$. There is a finite string v such that $\pi(c_\gamma) \supseteq_e v^\frown 1$ and $t \supseteq_e v^\frown 0$. Let u be the finite string of 0's, $c_\gamma \supseteq_e s^\frown u^\frown 1$. Let $s' = s^\frown u^\frown 0$ and $t' = t$.

case3 $\pi(c_\gamma) <_{lex} t$. Similar to case2.

Hence $D \in (H_\lambda)^{\mathbb{V}^{\mathbb{C}}[G]}$ is dense.

Since q is $(N, \mathbb{Q})$-generic and $q \in G$, $N[G \cap N] \prec (H_\lambda)^{\mathbb{V}^{\mathbb{C}}[G]}$. Since $\dot{\pi} \in N$, $\pi \in N[G \cap N]$. Hence $D \in N[G \cap N]$. So $Z_{\eta,\xi} \cap D \cap N[G \cap N] \neq \emptyset$. Let $(s,t) \in D$, $s \subseteq_e c_\eta$ and $t \subseteq_e c_\xi$. Let $x \in C$ witness $(s,t) \in D$. Then either $x >_{lex} c_\eta$ and $\pi(x) >_{lex} c_\xi$, or $x <_{lex} c_\eta$ and $\pi(x) <_{lex} c_\xi$. But $\pi(c_\eta) = c_\xi$, which contradicts that π is reverse order isomorphism. □

Fact 2.25. *If $\mathbb{Q}$ is distributive and proper, then $\mathbb{Q}$ is nice.*

Proof. The reason is that each $D \subseteq \mathbb{C}_2$ is essentially a real. Hence if $\mathbb{Q}$ is distributive, then there will be no new dense subsets of $\mathbb{C}_2$. □

The key is the following preservation theorem:

Theorem 2.26. *In $\mathbb{V}^{\mathbb{C}}$, let $\langle(\mathbb{P}_\alpha, \dot{Q}_\alpha)|\alpha \in \epsilon\rangle$ be a countable support iteration. If $\forall\alpha < \epsilon$, $\Vdash_{\mathbb{P}_\alpha} \dot{Q}_\alpha$ is nice, then $\mathbb{P}_\epsilon$ is nice.*

Lemma 2.27. *[22] Let $\langle(\mathbb{P}_\alpha, \dot{Q}_\alpha)|\alpha \in \epsilon\rangle$ be a countable support iteration. Assume $\alpha_1 < \alpha_2 \le \beta \le \epsilon$, $D \subseteq \mathbb{P}_\beta$ is dense open, and $\Vdash_{\alpha_1} \dot{p}_1 \in \mathbb{P}_\beta$. Then*

$$\Vdash_{\alpha_2} \exists p_2\Big((p_2 \in \mathbb{P}_\beta) \wedge (p_2 \le \dot{p}_1) \wedge (p_2 \in D) \wedge (\dot{p}_1 \restriction_{\alpha_2} \in \dot{G}_{\alpha_2} \to p_2 \restriction_{\alpha_2} \in \dot{G}_{\alpha_2})\Big).$$

Theorem 2.26 is a special case of the following lemma by taking $\alpha = 0, \beta = \epsilon$.

Lemma 2.28. *In $\mathbb{V}^{\mathbb{C}}$, let $\langle(\mathbb{P}_\alpha, \dot{Q}_\alpha)|\alpha \in \epsilon\rangle$ be a countable support iteration of nice forcing notions. Assume that*

- *λ is a large regular cardinal,*
- *$N \prec H_\lambda$ is countable, $\mathbb{P}_\epsilon \in N$,*
- *$\eta \in \omega_1 \setminus N$,*
- *$\alpha < \beta \le \epsilon$ are both in N,*
- *$\dot{p} \in N$ such that $\Vdash_\alpha \dot{p} \in \mathbb{P}_\beta$,*
- *$q \in \mathbb{P}_\alpha$ satisfying*

  $\circledast_1^\alpha$ *q is $(N, \mathbb{P}_\alpha)$-generic,*

  $\circledast_2^\alpha$ *$q \Vdash \dot{p} \restriction_\alpha \in \dot{G}_\alpha \cap N$,*

  $\circledast_3^\alpha$ *$q \Vdash \forall\xi \in \omega_1 \setminus N$, $Z_{\eta,\xi}$ is $(N[\dot{G}_\alpha \cap N], \mathbb{C}_2)$-generic.*

Then there is $q^+ \in \mathbb{P}_\beta$, $q^+ \restriction_\alpha = q$ such that

$\circledast_1^\beta$ *q^+ is $(N, \mathbb{P}_\beta)$-generic,*

$\circledast_2^\beta$ *$q^+ \Vdash \dot{p} \in \dot{G}_\beta \cap N$,*

$\circledast_3^\beta$ *$q^+ \Vdash \forall\xi \in \omega_1 \setminus N,\ Z_{\eta,\xi}$ is $(N[\dot{G}_\beta \cap N], \mathbb{C}_2)$-generic.*

Proof of Lemma 2.28: We prove by induction on the pair (α,β):

case1 $\beta = \alpha + 1$. Since q is $(N, \mathbb{P}_\alpha)$-generic, we have $q \Vdash N[\dot{G}_\alpha \cap N] \prec H_\lambda[\dot{G}_\alpha]$ is a countable elementary submodel and $q \Vdash \eta \in \omega_1 \setminus N[\dot{G}_\alpha \cap N]$.

Also by hypothesis, $q \Vdash \forall\xi \in \omega_1 \setminus N,\ Z_{\eta,\xi}$ is $(N[\dot{G}_\alpha \cap N], \mathbb{C}_2)$-generic. Since $\alpha \in N$, $\dot{Q}_\alpha \in N \wedge \dot{p}_\alpha \in N$. Therefore,

$$q \Vdash \dot{Q}_\alpha \in N[\dot{G}_\alpha \cap N] \wedge \dot{p}_\alpha \in \dot{Q}_\alpha \cap N[\dot{G}_\alpha \cap N]$$

By assumption, $\Vdash \dot{Q}_\alpha$ is nice. So q forces : $\exists x$

1. $x \in \dot{Q}_\alpha \wedge x \le \dot{p}_\alpha$ and
2. x is $(N[\dot{G}_\alpha \cap N], \dot{Q}_\alpha)$-generic, and
3. $x \Vdash \forall\xi \in \omega_1 \setminus N,\ Z_{\eta,\xi}$ is $\Big((N[\dot{G}_\alpha \cap N])[\dot{G}(\alpha) \cap N[\dot{G}_\alpha \cap N]], \mathbb{C}_2\Big)$-generic.

By existence completeness theorem, there is a $\mathbb{P}_\alpha$ name x such that q force x to satisfy the properties above. By computing names, we see that

$$(N[\dot{G}_\alpha \cap N])[\dot{G}(\alpha) \cap N[\dot{G}_\alpha \cap N]] = N[\dot{G}_{\alpha+1} \cap N]$$

$(q, x) \in \mathbb{P}_{\alpha+1}$, $(q, x) \Vdash (\dot{p} \restriction_\alpha, \dot{p}(\alpha)) = \dot{p} \restriction_{\alpha+1} \in \dot{G}_{\alpha+1}$ and (q, x) is $(N, \mathbb{P}_{\alpha+1})$-generic. Thus,

$$(q, x) \Vdash \mathbb{P}_{\alpha+1} \cap N[\dot{G}_{\alpha+1} \cap N] \subseteq N$$

But $\dot{p} \in N$, so $(q, x) \Vdash \dot{p} \in N$.

By the third requirement on x, we see that
$(q, x) \Vdash \forall\xi \in \omega_1 \setminus N,\ Z_{\eta,\xi}$ is $(N[\dot{G}_{\alpha+1} \cap N], \mathbb{C}_2)$-generic.

So $q^+ = (q, x)$ is as required.

case2 $\beta = \gamma + 1$,where $\gamma > \alpha$. By induction hypothesis on the pair (α, γ), we can first extend q to some $q^* \in \mathbb{P}_\gamma$, then use case1 to extend q^* to $q^+ \in \mathbb{P}_\beta$.

case3 β is a limit ordinal. Let $\delta = sup(\beta \cap N)$. Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence of ordinals from $\beta \cap N$, with the supremum δ, and $\alpha_0 = \alpha$.

Let $\{D_n : n \in \omega\}$ list all dense open sets of $\mathbb{P}_\beta$ that lie in N. Let $\{\tau_n : n \in \omega\}$ list all $\mathbb{P}_\beta$ names for dense open sets of $\mathbb{C}_2$ that lie in N.

We build in N inductively $\mathbb{P}_{\alpha_n}$-names $\dot{p}_n$ for conditions in $\mathbb{P}_\beta$. Let $\dot{p}_0 = \dot{p}$. Suppose we have $\Vdash_{\alpha_n} \dot{p}_n \in \mathbb{P}_\beta$. Apply Lemma 2.27 inside N, we obtain $\dot{p}_{n+1} \in N$ such that

$$\Vdash_{\alpha_{n+1}} \dot{p}_{n+1} \in \mathbb{P}_\beta \wedge \dot{p}_{n+1} \leq \dot{p}_n \wedge \dot{p}_{n+1} \in D_n \wedge (\dot{p}_n \restriction_{\alpha_{n+1}} \in \dot{G}_{\alpha_{n+1}} \rightarrow \dot{p}_{n+1} \restriction_{\alpha_{n+1}} \in \dot{G}_{\alpha_{n+1}})$$

By assumption, $\circledast_1^{\alpha_0}, \circledast_2^{\alpha_0}$ and $\circledast_3^{\alpha_0}$ holds for $q_0 = q$ and $\alpha_0 = \alpha$.

By induction hypothesis on the pair (α_0, α_1), we can find $q_1 \in \mathbb{P}_{\alpha_1}, q_1 \restriction_{\alpha_0} = q_0$ satisfying

- q_1 is $(N, \mathbb{P}_{\alpha_1})$-generic,
- $q_1 \Vdash \dot{p}_0 \restriction_{\alpha_1} \in \dot{G}_{\alpha_1} \cap N$,
- $q_1 \Vdash \forall \xi \in \omega_1 \setminus N,\ Z_{\eta,\xi}$ is $(N[\dot{G}_{\alpha_1} \cap N], \mathbb{C}_2)$-generic.

Hence $q_1 \Vdash \dot{p}_1 \restriction_{\alpha_1} \in \dot{G}_{\alpha_1} \cap N$.

Inductively, we get $q_n \in \mathbb{P}_{\alpha_n}$ such that $q_n \restriction_{\alpha_{n-1}} = q_{n-1}$ and

- q_n is $(N, \mathbb{P}_{\alpha_n})$-generic,
- $q_n \Vdash \dot{p}_n \restriction_{\alpha_n} \in \dot{G}_{\alpha_n} \cap N$,
- $q_n \Vdash \forall \xi \in \omega_1 \setminus N,\ Z_{\eta,\xi}$ is $(N[\dot{G}_{\alpha_n} \cap N], \mathbb{C}_2)$-generic.

Notice that $\tau_n \in N$ and $\Vdash_\beta \tau_n \subseteq \mathbb{C}_2$ is dense open. We have

$q_n \Vdash_{\alpha_n} (\tau_n \in N[\dot{G}_{\alpha_n} \cap N]) \wedge (\Vdash_{\alpha_n,\beta} \tau_n \subseteq \mathbb{C}_2$ is dense open).

Let $\dot{T}_n$ be a $\mathbb{P}_{\alpha_n}$-name such that $\Vdash_{\alpha_n} \dot{T}_n = \{u \in \mathbb{C}_2 :\Vdash_{\alpha_n,\beta} u \in \tau_n\}$.

We can find such $\dot{T}_n$ in N. Then

$q_n \Vdash \dot{T}_n \in N[\dot{G}_{\alpha_n} \cap N] \wedge \dot{T}_n$ is dense open in $\mathbb{C}_2$. Thus, $\forall \xi \in \omega_1 \setminus N$, $q_n \Vdash \dot{T}_n \cap Z_{\eta,\xi} \neq \emptyset$. So $(q_n, \mathbb{1}) \Vdash \tau_n \cap Z_{\eta,\xi} \neq \emptyset$.

Let $q^+ \in \mathbb{P}_\beta$ be the limit of $\langle q_n : n \in \omega\rangle$. Namely, and $\forall \theta \in spt(q^+)$, there is some $n \in \omega$, $\theta \in spt(q_n)$, and for any such n, $q^+(\theta) = q_n(\theta)$.

We need to verify that q^+ works. Let $G_\beta \ni q^+$ be any $(\mathbb{V}^{\mathbb{C}}, \mathbb{P}_\beta)$-generic filter and p_n be the interpretation of $\dot{p}_n$ by G_β. We show the following:

(i) $p_n \in N \cap G_\beta$,

(ii) $N \cap G_\beta$ is $(N, \mathbb{P}_\beta)$-generic,

(iii) $\forall \xi \in \omega_1 \setminus N$, $Z_{\eta,\xi}$ is $(N[G_\beta \cap N], \mathbb{C}_2)$-generic.

Let $G_{\alpha_n} = \{r \restriction_{\alpha_n} : r \in G_\beta\}$. It is a $(\mathbb{V}^{\mathbb{C}}, \mathbb{P}_{\alpha_n})$-generic filter.

So $q_n \in G_{\alpha_n}$ and $p_n \restriction_{\alpha_n} \in G_{\alpha_n}$. Since $p_{n+1} \leq p_n$,we have $p_n \restriction_{\alpha_{n+1}} \in G_{\alpha_{n+1}}$. Therefore, $\forall n, m \in \omega$, $p_n \restriction_{\alpha_m} \in G_{\alpha_m}$. By general theory of forcing, we see that $p_n \restriction_\delta \in G_\delta$.

Notice that $p_n \in \mathbb{P}_\beta \cap N[G_{\alpha_n} \cap N] \subseteq N$, $spt(p_n) \in N$. Since N is a countable elementary model, $spt(p_n) \subseteq N \cap \beta \subseteq \delta$. So $p_n \in G_\beta$. This complete the proof of (i).

For (ii), let $D \in N$ be a dense open subset of $\mathbb{P}_\beta$. Say $D = D_n$. By the construction, $p_{n+1} \in D_n$. So $p_{n+1} \in G_\beta \cap N \cap D$. Therefore, $N \cap G_\beta$ is $(N, \mathbb{P}_\beta)$-generic.

For (iii), let $T \in N[G_\beta \cap N]$ be a dense open subset of $\mathbb{C}_2$. Then T has a $\mathbb{P}_\beta$-name in N, say τ_n. Since $(q_n, \mathbb{1}) \Vdash \tau_n \cap Z_{\eta,\xi} \neq \emptyset$ and $(q_n, \mathbb{1}) \in G_\beta$, $T \cap Z_{\eta,\xi} \neq \emptyset$.

Say $u \in T \cap Z_{\eta,\xi}$. Also, by elementarity, $\mathbb{C}_2 \in N \subseteq N[G_\beta \cap N]$, hence $u \in \mathbb{C}_2 \subseteq N[G_\beta \cap N]$. So $T \cap N[G_\beta \cap N] \cap Z_{\eta,\xi} \neq \emptyset$.

This completes the proof of Lemma 2.28. □

A useful fact in forcing with large cardinals is that mild extension does not change large cardinal properties. For instance, the following holds:

Lemma 2.29. *[32, Theorem 21.2] If κ is a supercompact cardinal, $\mathbb{P}$ is a partial order with $|\mathbb{P}| < \kappa$, then $\mathbb{V}^{\mathbb{P}} \models \kappa$ is supercompact.*

Lemma 2.30. $\mathbb{V}^{\mathbb{C}} \models \kappa$ *is supercompact.*

Proof. $|\mathbb{C}| = \aleph_1 < \kappa$ in $\mathbb{V}$. By Lemma 2.29, it will not affect the large cardinal property of κ. In other words, $\mathbb{V}^{\mathbb{C}}$ is a mild extension, in which κ is still supercompact. □

Let us now continue to prove Theorem 2.18.

Work in $\mathbb{V}^{\mathbb{C}}$, let $\mathbb{D}$ be the notion of forcing to force $DPFA$. In other words, we revise the standard consistency proof of PFA from a supercompact cardinal. The reader is referred to Theorem 31.21 of [32] for this proof. Let f be a Laver function [35] witnessing that κ is supercompact. $\mathbb{D} = \langle(\mathbb{P}_\alpha, \dot{Q}_\alpha) : \alpha < \kappa\rangle$ is a κ - step countable support iteration whose iterands are assigned by f. If at step α, $f(\alpha)$ is a "good pair" $(\dot{P}, \dot{\mathfrak{D}})$ in $\mathbb{V}^{\mathbb{P}_\alpha}$, where $\dot{P}$ is distributive and proper, then we adopt it to be $\dot{Q}_\alpha$; otherwise, we use a trivial forcing notion. The rest of the proof goes the same way as that of PFA. So $\mathbb{V}^{\mathbb{C}*\mathbb{D}} \models DPFA$.

Notice that each iterand of $\mathbb{D}$ is distributive and proper. By Lemma 2.25, each iterand is nice. So by Theorem 2.26, $\mathbb{D}$ is nice. By Lemma 2.24, $\mathbb{V}^{\mathbb{C}*\mathbb{D}} \models C$ and C^* are nonisomorphic $\aleph_1$- dense subsets of the reals. This completes the proof. □

Chapter 3

$DPFA$ and Cardinal Invariants

3.1 Review on Cardinal Invariants of the Reals

The material in this section can all be found in Blass's Handbook chapter [12], where "cardinal characteristic" was used instead of "cardinal invariant". As the name suggests, a cardinal invariant characterizes the cardinality of sets satisfying certain given property.

We need the fact that among all cardinal invariants mentioned in [12], the maximal ones are $cof(\mathcal{L})$, $\mathfrak{u}$, $\mathfrak{a}$ and $\mathfrak{i}$.

Notation 3.1. *For $f, g \in \omega^\omega$, denote $f \leq^* g$ if $\forall^\infty n \in \omega$, $f(n) \leq g(n)$.*

Notations such as $\geq^*$, $<^*$, $>^*$ and $=^*$ can be similarly interpreted.

A natural question is: can we have a small family of functions which grow fast enough? The invariants $\mathfrak{d}$ and $\mathfrak{b}$ characterize the least size of a "$\leq^*$-dominating" family and a "$\leq^*$-unbounded" family, respectively. More explicitly,

$$\mathfrak{d} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega : \forall f \in \omega^\omega \exists g \in \mathcal{F}(f \leq^* g)\}$$

$$\mathfrak{b} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega : \forall f \in \omega^\omega \exists g \in \mathcal{F}(g \nleq^* f)\}$$

It can be shown that $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

Definition 3.2. *Let I be an ideal on a set S. Define*

The additivity of I: $add(I) = min\{|\mathcal{F}| : \mathcal{F} \subseteq I, \bigcup \mathcal{F} \notin I\}$;

the covering number of I: $cov(I) = min\{|\mathcal{F}| : \mathcal{F} \subseteq I, \bigcup \mathcal{F} = S\}$;

the uniformity of I: $non(I) = min\{|A| : A \subseteq S \wedge A \notin I\}$;

the cofinality of I: $cof(I) = min\{|\mathcal{F}| : \mathcal{F} \subseteq I, \forall A \subseteq S\, \exists B \in \mathcal{F}(A \subseteq B)\}$, namely, the least cardinality of a basis for I.

We usually assume that

- $S = \mathbb{R}$;
- I contains all the singletons of reals;
- $I \neq \mathcal{P}(\mathbb{R})$;
- I is ω-complete; namely, I is closed under increasing sequence of countable length;
- I has a basis consists of Borel sets.

Under these assumptions, we can show that

$$\aleph_1 \leq add(I) \leq min(cov(I), non(I)) \leq max(cov(I), non(I)) \leq cof(I) \leq \mathfrak{c}$$

Typical examples of such ideals are the collection of null sets and the collection of meager sets, which are denoted by $\mathcal{L}$ and $\mathcal{B}$ respectively, in honor of "Lebesgue" and "Baire". By Baire Category Theorem, $\mathcal{B}$ is ω-complete. Notice that every null set is contained in a G_δ null set and every meager set is a subset of a F_σ meager set.

The invariants $\mathfrak{b}$ and $\mathfrak{d}$ can also be described via ideals. Let $\mathcal{K}_\sigma$ be the collection of all subsets of reals X which can be covered by the union of countably many compact subsets of ω^ω. Then $\mathcal{K}_\sigma$ is an ideal and

$$add(\mathcal{K}_\sigma) = non(\mathcal{K}_\sigma) = \mathfrak{b}, cov(\mathcal{K}_\sigma) = cof(\mathcal{K}_\sigma) = \mathfrak{d}.$$

The relation between these invariants can be shown in the following Cichoń's Diagram: where "→" stands for "≥".

$$\begin{array}{ccccccc}
cov(\mathcal{L}) & \longleftarrow & non(\mathcal{B}) & \longleftarrow & cof(\mathcal{B}) & \longleftarrow & cof(\mathcal{L}) \\
\big\downarrow & & \downarrow & & \downarrow & & \big\downarrow \\
 & & \mathfrak{b} & \longleftarrow & \mathfrak{d} & & \\
\big\downarrow & & \downarrow & & \downarrow & & \big\downarrow \\
add(\mathcal{L}) & \longleftarrow & add(\mathcal{B}) & \longleftarrow & cov(\mathcal{B}) & \longleftarrow & non(\mathcal{L})
\end{array}$$

We will define cardinal invariants $\mathfrak{u}$, $\mathfrak{a}$ and $\mathfrak{i}$ in following sections.

Many cardinal invariants can be defined as "norms" [12]. If A and B are two sets and $R \subseteq A \times B$ is a relation such that $\forall a \in A \exists b \in B\ (a, b) \in R$, then the norm of R (or of the triple (A, B, R)) is:

$$\|(A, B, R)\| = \min\{|C| \mid C \subset B, \forall a \in A \exists b \in C (a, b) \in R\}.$$

3.2 *DPFA* and Cardinal Invariants in Cichoń's Diagram

Theorem 3.3. *It is consistent to have DPFA and* $cof(\mathcal{L}) = \aleph_1$ *relative to the consistency of existence of a supercompact cardinal.*

Definition 3.4. *A function* $s : \omega \to [\omega]^{<\omega}$ *is called a* slalom*, if* $\forall n \in \omega$, $|s(n)| \leq 2^n$.

Notation 3.5. *$\mathcal{S}$ denotes collection of slaloms, i.e.* $\mathcal{S} = \prod_{n\in\omega}[\omega]^{\leq 2^n}$.

Definition 3.6. *Let $s \in \mathcal{S}$, $f : \omega \to \omega$. Define f goes through s, if $\forall^\infty n, f(n) \in s(n)$. Denote this by $f \in^* s$.*

The following characterization is very useful.

Lemma 3.7. *[3] $cof(\mathcal{L}) = \|\omega^\omega, \mathcal{S}, \in^*\|$.*

Definition 3.8. *A notion of forcing $\mathbb{P}$ has Sacks property, if $\forall f \in \omega^\omega \cap \mathbb{V}^{\mathbb{P}}$, $\exists s \in \mathcal{S} \cap \mathbb{V}$, such that $\forall n \in \omega$, $f(n) \in s(n)$.*

The following facts are immediate:

Fact 3.9. *$\mathbb{P}$ has Sacks property, if $\forall f \in \omega^\omega \cap \mathbb{V}^{\mathbb{P}}$, $\exists s \in \mathcal{S} \cap \mathbb{V}$, such that $f \in^* s$.*

Fact 3.10. *If forcing with $\mathbb{P}$ does not add reals, then $\mathbb{P}$ has Sacks property.*

The key is the following preservation theorem:

Lemma 3.11. *[22] Sacks property is preserved by countable support proper iteration.*

Proof of Theorem 3.3: We start with a model $\mathbb{V} \models \exists\kappa(\kappa \text{ is } supercompact)$. First we force CH. The forcing consists of countable partial functions from ω_1 into the ground model reals and is ordered by reverse inclusion. This notion of forcing is small, hence it does not affect the large cardinal property of κ. Without loss of generality, we may assume that $\mathbb{V} \models CH + \exists\kappa(\kappa \text{ is } supercompact)$. Let $\mathbb{D} \in \mathbb{V}$ be the p.o. to force $DPFA$. Each iterand of $\mathbb{D}$ does not add reals, hence has Sacks property. By Lemma 3.11, $\mathbb{D}$ has Sacks property. Let $\mathcal{S}^* = \mathcal{S} \cap \mathbb{V}$, i.e. $\mathcal{S}^*$ is the collection of slaloms in $\mathbb{V}$. For each $f \in \omega^\omega \cap \mathbb{V}^{\mathbb{D}}$, $\exists s \in \mathcal{S}^*$, such that $f \in^* s$. Therefore, by Lemma 3.7,

$$\mathbb{V}^{\mathbb{D}} \models cof(\mathcal{L}) = \|\omega^\omega, \mathcal{S}, \in^*\| \leq |\mathcal{S}^*|$$

Since $\mathbb{V} \models CH$, $\mathbb{V} \models |\mathcal{S}^*| = \aleph_1$. Since $\mathbb{D}$ is proper, $\aleph_1$ is preserved. Thus, $\mathbb{V}^{\mathbb{D}} \models |\mathcal{S}^*| = \aleph_1$. It is concluded that $\mathbb{V}^{\mathbb{D}}$ is a model satisfying DPFA and $cof(\mathcal{L}) = \aleph_1$. □

Remark 3.12. *Since $cof(\mathcal{L})$ is the largest one in Cichoń's diagram, we see that in the model $\mathbb{V}^{\mathbb{D}}$, all cardinal invariants in Cichoń's diagram, including $\mathfrak{d}$, equal to $\aleph_1$.*

3.3 *DPFA* and Ultrafilter Number

Theorem 3.13. *It is consistent to have DPFA and $\mathfrak{u} = \aleph_1$ relative to the consistency of existence of a supercompact cardinal.*

Definition 3.14. *Suppose $B \subseteq \mathcal{P}(\omega)$, we say B is an ultrafilter base, if B generates an ultrafilter, namely $\{x \subseteq \omega : \exists y \in B(y \subseteq x)\}$ is an ultrafilter.*

For example, any principal ultrafilter has a base consists of a singleton.

Definition 3.15. *$\mathfrak{u}$ is the smallest cardinality of any nonprincipal ultrafilter base.*

$\mathfrak{u}$ is a "big" cardinal invariant, $\mathfrak{r} \leq \mathfrak{u}$, for instance. The reader is referred to [12] for more details.

p-points and selective ultrafilters are those ultrafilters with certain completeness properties, more precisely:

Definition 3.16. *A nonprincipal ultrafilter U on ω is a p-point, if $\forall \langle X_n : n \in \omega \rangle \subset U$, $\exists X \in U$ such that $\forall n \in \omega$, $X \subseteq^* X_n$.*

Definition 3.17. *A nonprincipal ultrafilter U on ω is selective (or Ramsey), if $\forall \langle X_n : n \in \omega \rangle \subset U$, $\exists X \in U$, such that $\forall n \in \omega$, $|X \setminus X_n| = 1$.*

It is direct that every selective ultrafilter is a p-point.

Lemma 3.18. *[40] Assume CH, then selective ultrafilter exists.*

Definition 3.19. *A notion of forcing $\mathbb{P}$ preserves p-points (resp. selective ultrafilter), if for any p-point (resp. selective ultrafilter) U, $\mathbb{V}^{\mathbb{P}} \models U$ generates an ultrafilter.*

Use Lemma 1.6, one can prove:

Lemma 3.20. *[42] If $\mathbb{P}$ is proper and preserves p-points, then for any p-points U, $\mathbb{V}^{\mathbb{P}} \models U$ generates a p-points.*

Fact 3.21. *If $\mathbb{P}$ does not add reals, then $\mathbb{P}$ preserves p-points.*

Again, the key is the preservation theorem:

Theorem 3.22. *[13] The property of preserving p-points is preserved under countable support proper iteration.*

Proof of Theorem 3.13: Start with a model $\mathbb{V} \models CH + \exists$ supercompact cardinal. Since $\mathbb{V} \models CH$, there is a p-point $U \in \mathbb{V}$ and moreover $\mathbb{V} \models |U| = \aleph_1$. Let $\mathbb{D} \in \mathbb{V}$ be the notion of forcing to force $DPFA$. Each iterand of $\mathbb{D}$ is proper and adds no new reals, hence preserves p-points. By Theorem 3.22, $\mathbb{D}$ preserves p-points. Thus by Lemma 3.20, $\mathbb{V}^{\mathbb{D}} \models \mathfrak{u} \leq |U|$. Since $\mathbb{D}$ is proper, $\aleph_1$ is preserved. Therefore, $\mathbb{V}^{\mathbb{D}} \models \mathfrak{u} = |U| = \aleph_1 \wedge DPFA$. □

The preservation of selective ultrafilters is more subtle.

Definition 3.23. *A notion of forcing $\mathbb{P}$ is ω^ω-bounding, if for each $f \in \omega^\omega \cap \mathbb{V}^{\mathbb{P}}$, there is $g \in \omega^\omega \cap \mathbb{V}$ such that $\forall n \in \omega\ f(n) \leq g(n)$.*

A typical example for ω^ω-bounding forcing is Solovay's random forcing. If a forcing is ω^ω-bounding, then it will not enlarge the dominating number $\mathfrak{d}$.

Theorem 3.24. *[22] The property of being ω^ω-bounding and proper is preserved under countable support iteration.*

Lemma 3.25. *[42] If $\mathbb{P}$ is proper, ω^ω-bounding and preserves selective ultrafilters, then for any selective ultrafilter U, $\mathbb{V}^{\mathbb{P}} \models U$ generates a selective ultrafilter.*

Fact 3.26. *If $\mathbb{P}$ does not add reals, then $\mathbb{P}$ is ω^ω-bounding and preserves selective ultrafilters.*

Theorem 3.27. *[42] The property of being ω^ω-bounding and preserving selective ultrafilters is preserved under countable support proper iteration.*

Use this preservation theorem, the same proof as Theorem 3.13 enables us to do slightly better:

Corollary 3.28. *It is consistent to have DPFA and $\mathfrak{u} = \mathfrak{d} = \aleph_1$ relative to the consistency of existence of a supercompact cardinal.*

Actually, by combining the preservation theorems in this and the previous sections, we can have $CON(DPFA + cof(\mathcal{L}) = \mathfrak{u} = \aleph_1)$. We will deal with more cardinal invariants in one model in Section 3.6.

The preservation theorems can be generalized to filters. Though we do not need them in this section, they will be useful in the next section.

Definition 3.29. *A nonprincipal filter $\mathcal{F}$ on ω is a p-filter, if for any partition $\{W_n : n \in \omega\}$ of ω, such that for each n, $\omega \setminus W_n \in \mathcal{F}$, then there is some $A \in \mathcal{F}$ such that $|A \cap W_n| < \omega$.*

Definition 3.30. *A non principal filter $\mathcal{F}$ on ω is a q-filter, if for any partition $\{W_n : n \in \omega\}$ of ω, such that for each n, $|W_n| < \omega$, then there is some $A \in \mathcal{F}$ such that $|A \cap W_n| \leq 1$.*

Definition 3.31. *A non principal filter $\mathcal{F}$ on ω is a Ramsey (or selective) filter, if for any partition $\{W_n : n \in \omega\}$ of ω, such that for each n, $\omega \setminus W_n \in \mathcal{F}$, then there is some $A \in \mathcal{F}$ such that $|A \cap W_n| \leq 1$.*

Fact 3.32.

1. *A filter is selective iff it is both a p-filter and a q-filter.*
2. *A p-point, q point or selective ultrafilter is an ultrafilter that is a p-filter, q-filter or selective filter, respectively.*

We have the following preservation theorems which correspond to Theorem 3.22 and Theorem 3.27, respectively.

Theorem 3.33. *Let $\langle(\mathbb{P}_\alpha, Q_\alpha) \mid \alpha < \epsilon\rangle$ be a countable support iteration of proper forcing notions. Assume $\mathcal{F}$ is a filter on ω.*

1. *If $\mathcal{F}$ is a p-filter and each Q_α preserves p-filter, then the limit $\mathbb{V}^{\mathbb{P}_\epsilon} \models \mathcal{F}$ is a p-filter.*

2. *If $\mathcal{F}$ is a selective-filter, each Q_α is ω^ω-bounding and preserves selective filter, then the limit $\mathbb{V}^{\mathbb{P}_\epsilon} \models \mathcal{F}$ is a selective filter.*

The proof is similar to those for Theorem 3.22 and Theorem 3.27. See [42] for example.

3.4 *DPFA* and Independence Number

Theorem 3.34. *It is consistent to have DPFA and $\mathfrak{i} = \aleph_1$ relative to the consistency of existence of a supercompact cardinal.*

This theorem follows from Shelah's work [44]. A maximal independent family was constructed in a ground model satisfying CH such that the maximality is not vialated by any forcing with certain property. The construction will help us to prove Theorem 3.34.

First, some definitions and notations are needed.

Definition 3.35.

1. *For $A \subseteq \omega$, denote $A^1 = A$ and $A^{-1} = \omega \setminus A$.*

2. *For $\mathcal{B} \subseteq \mathcal{P}(\omega)$, $FF(\mathcal{B})$ denotes the collection of finite partial functions from $\mathcal{B}$ into $\{1, -1\}$.*

3. *For $\mathcal{B} \subseteq \mathcal{P}(\omega)$, $h \in FF(\mathcal{B})$, denote $\mathcal{B}^h = \bigcap\{B^{h(B)} : B \in dom(h)\}$. If $h = \emptyset$, then $\mathcal{B}^h = \omega$.*

4. *$\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an independent family, if $\forall h \in FF(\mathcal{A})$, $\mathcal{A}^h$ is infinite.*

5. *$\mathfrak{i}$ is the least cardinality of an independent family which is maximal under inclusion.*

6. *$AP = \{(\mathcal{A}, A) : \mathcal{A}$ is a countable independent family and $A \in [\omega]^\omega$ such that $\forall h \in FF(\mathcal{A}), A \cap \mathcal{A}^h$ is infinite $\}$.*

7. *Assign a partial order on AP: $(\mathcal{A}_1, A_1) \geq (\mathcal{A}_2, A_2)$ iff $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $A_1 \supseteq^* A_2$.*

8. *For any infinite independent family $\mathcal{A}$, let*
   $id_{\mathcal{A}} = \{A \subseteq \omega : \forall g \in FF(\mathcal{A}) \exists\, h \in FF(\mathcal{A})$ s.t. $g \subseteq h$ and $A \cap \mathcal{A}^h$ is finite $\}$.

9. *For any infinite independent family $\mathcal{A}$, let $fil_{\mathcal{A}} = \{A \subseteq \omega : \omega \setminus A \in id_{\mathcal{A}}\}$.*

Fact 3.36. 1. *For an infinite $\mathcal{A} \subseteq \mathcal{P}(\omega)$, $\mathcal{A}$ is independent iff $\forall h \in FF(\mathcal{A})$, $\mathcal{A}^h \neq \emptyset$. Similarly in definition 6 , we can require only that $A \cap \mathcal{A}^h \neq \emptyset$; in definition 8, we can replace "finite" by "empty".*

2. *$id_{\mathcal{A}}$ is an ideal on ω containing all finite subsets; thus $fil_{\mathcal{A}}$ is the dual filter.*

3. *If $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $id_{\mathcal{A}_1} \subseteq id_{\mathcal{A}_2}$ and $fil_{\mathcal{A}_1} \subseteq fil_{\mathcal{A}_2}$.*

4. *If $A \in fil_{\mathcal{A}}$, then $(\mathcal{A}, A) \in AP$.*

5. *$\mathfrak{i}$ is big, since $\mathfrak{r} \leq \mathfrak{i}$ and $\mathfrak{d} \leq \mathfrak{i}$. The reader is referred to [12] for proof.*

To carry out the construction, the following lemma is needed:

Lemma 3.37. *(1) For a decreasing sequence $\langle(\mathcal{A}_n, A_n) : n \in \omega\rangle$ from AP, there is some $A \subseteq \omega$, such that $(\bigcup_n \mathcal{A}_n, A) \in AP$ and is a lower bound for this sequence.*

(2) For any $(\mathcal{A}, A) \in AP$, there is some $B \subseteq A$, $B \notin \mathcal{A}$ such that

$$(\mathcal{A} \cup \{B\}, A) \in AP \ \text{ and } \ (\mathcal{A} \cup \{B\}, A) \leq (\mathcal{A}, A).$$

(3) For any $(\mathcal{A}, A) \in AP$ and finite-to-one $f : \omega \to \omega$, there is some $B \subset A$ such that $(\mathcal{A}, A) \geq (\mathcal{A}, B) \in AP$ and $f \restriction_B$ is one-to-one.

(4) For any $(\mathcal{A}, A) \in AP$, $e : \omega \to \omega$ and $g \in FF(\mathcal{A})$, there is some $B \subset A$, $h \in FF(\mathcal{A})$ such that $(\mathcal{A}, A) \geq (\mathcal{A}, B) \in AP$, $g \subseteq h$ and $e \restriction_{\mathcal{A}^h \cap B}$ is either one-to-one or constant.

The proof is given in detail in [44]. As a corollary, we have:

Corollary 3.38. *If $(\mathcal{A}, A) \in AP$, then there is $\mathcal{B} \supseteq \mathcal{A}$ such that $(\mathcal{B}, A) \in AP$ and $A \in fil_{\mathcal{B}}$.*

Proof: By (2) of Lemma 3.37, we can find $\{B_n : n \in \omega\} \subset \mathcal{P}(A)$ inductively such that $\forall m \in \omega, B_m \notin \mathcal{A} \cup \{B_n : n \in m\}$ *and* $(\mathcal{A} \cup \{B_n : n \in m\}, A) \in AP$. Let $\mathcal{B} = \mathcal{A} \cup \{B_n : n \in \omega\}$. Then $(\mathcal{B}, A) \in AP$. Moreover, if $g \in FF(\mathcal{B})$, then there is some n such that $B_n \notin dom(g)$. Let $h = g \cup (B_n, 1)$, then $\mathcal{B}^h \subseteq B_n \subseteq A$. Thus, $A \in fil_{\mathcal{B}}$. □

If CH holds, then $\mathfrak{i} = \aleph_1$. We will assume CH and construct a "stable" maximal independent family with size $\aleph_1$.

We build inductively $(\mathcal{A}_i, A_i) \in AP$ for each $i \in \omega_1$ and $\mathcal{A}_* = \bigcup_{i \in \omega_1} \mathcal{A}_i$ will be the desired family. We impose the following requirements on the construction:

(i) $\forall i < j < \omega_1, (\mathcal{A}_i, A_i) \geq (\mathcal{A}_j, A_j)$.

(ii) $\forall i \in \omega_1, \exists A \in \mathcal{A}_{i+1} \setminus \mathcal{A}_i, A \subseteq A_i$.

(iii) $\forall e : \omega \to \omega$, $p \in FF(\mathcal{A}_*)$, $\exists q \in FF(\mathcal{A}_*)$ such that $q \supseteq p$ and $e \restriction_{\mathcal{A}_*^q}$ is one-to-one or constant.

(iv) $\forall f : \omega \to \omega$ finite to one, $\exists i$ such that $f \restriction_{A_i}$ is one to one.

(v) $fil_{\mathcal{A}_*}$ is the filter generated by $\{A_i : i < \omega_1\}$; moreover, for each $X \in fil_{\mathcal{A}_*}$, there are unbounded many $i < \omega_1$ such that $X \supseteq A_i$.

(vi) If $n \neq m$ are natural numbers, then there are unbounded many $i < \omega_1$ such that $n \in A_i$ and $m \notin A_i$.

From (iii), $\mathcal{A}_*$ is maximal: Assume $E \notin \mathcal{A}_*$, let $e : \omega \to 2$ be the characteristic function for E, $p = \emptyset$. Let q be given by (iii), then $e \restriction_{\mathcal{A}_*^q}$ is constant function. The constant values 0 or 1 corresponds to the cases $E \cap \mathcal{A}_*^q = \emptyset$ or $\mathcal{A}_*^q \subseteq E$ respectively. Therefore, $\mathcal{A}_* \cup \{E\}$ is not independent.

Though the construction of $\mathcal{A}_*$ is straight forward, we keep record of it as a complementarity of [44].

List $[\omega]^\omega$ in $\langle X_i : i \in \omega_1 \rangle$ such that each infinite subset of ω appears unbounded many times in the list. Let $\langle (e_i, p_i) : i \in \omega_1 \rangle$ be a list of all pairs (e, p) where $e : \omega \to \omega$ is a total function and $p : [\omega]^\omega \to \{-1, 1\}$ is a finite partial function. Let $\langle f_i : i \in \omega_1 \rangle$ list all finite to one total functions from ω to ω.

- Step $i = 0$: take arbitrary $(\mathcal{A}_0, A_0) \in AP$.
- Step $i + 1$: we have $(\mathcal{A}_i, A_i) \in AP$.
  - substep1: By Corollary 3.38, we can find $\mathcal{B}_{i+1} \supseteq \mathcal{A}_i$ such that $(\mathcal{B}_{i+1}, A_i) \in AP$ and $A_i \in fil_{\mathcal{B}_{i+1}}$.
  - substep2: If $X_i \in fil_{\mathcal{B}_{i+1}}$, then let $B_{i+1} = A_i \cap X_i$; otherwise, let $B_{i+1} = A_i$. Therefore, $B_{i+1} \in fil_{\mathcal{B}_{i+1}}$. By Fact 3.36, we have $B_{i+1} \subseteq A_i$ and $(\mathcal{B}_{i+1}, B_{i+1}) \in AP$.
  - substep3: Check if $(\mathcal{B}_{i+1} \cup dom(p_i), B_{i+1}) \in AP$.

    If yes, let $\mathcal{D}_{i+1} = \mathcal{B}_{i+1} \cup dom(p_i)$. So $(\mathcal{D}_{i+1}, B_{i+1}) \in AP$ and $p_i \in FF(\mathcal{D}_{i+1})$. By (4) of Lemma 3.37, there is $q_i \in FF(\mathcal{D}_{i+1})$ and $D_{i+1} \subseteq B_{i+1}$ such that $q_i \supseteq p_i$, $(\mathcal{D}_{i+1}, D_{i+1}) \in AP$ and $e_i \restriction_{\mathcal{D}_{i+1}^{q_i} \cap D_{i+1}}$ is one-to-one or constant.

    If no, let $\mathcal{D}_{i+1} = \mathcal{B}_{i+1}$ and $D_{i+1} = B_{i+1}$.

So we obtained $\mathcal{D}_{i+1} \supseteq \mathcal{B}_{i+1}$ and $D_{i+1} \subseteq B_{i+1}$ such that $(\mathcal{D}_{i+1}, D_{i+1}) \in AP$.

– substep4: By (2) of Lemma 3.37, there is some $Z_{i+1} \subseteq D_{i+1}$ such that $Z_{i+1} \notin \mathcal{D}_{i+1}$ and $(\mathcal{D}_{i+1} \cup \{Z_{i+1}\}, D_{i+1}) \in AP$. Let $\mathcal{A}_{i+1} = \mathcal{D}_{i+1} \cup \{Z_{i+1}\}$.

– substep5: By (3) of Lemma 3.37, there is some $A_{i+1} \subseteq D_{i+1}$ such that $(\mathcal{A}_{i+1}, A_{i+1}) \in AP$ and $f_i \upharpoonright A_{i+1}$ is one to one.

- Step $lim(i)$: By (1) of Lemma 3.37, let $\mathcal{A}_i = \bigcup_{j<i} \mathcal{A}_j$. Let A_i be such that $(\mathcal{A}_i, A_i) \in AP$ and is a lower bound for $\{(\mathcal{A}_j, A_j) : j < i\}$; moreover, if the Cantor Normal Form of i ends with $\omega \cdot 2^n 3^m$, then we require $n \in A_i$ and $m \notin A_i$.

This completes the construction. Let us now verify properties (i)-(vi).

It is easy to check that our construction satisfies (i), (ii), (iv)and(vi).

For (iii), suppose we are given $e : \omega \to \omega$ and $p \in FF(\mathcal{A}_*)$. Say $(e, p) = (e_i, p_i)$. Claim that at substep3 of step $i + 1$, $(\mathcal{B}_{i+1} \cup dom(p_i), B_{i+1}) \in AP$. Let $j > i + 1$ be such that $p_i \in FF(\mathcal{A}_j)$, then $\mathcal{B}_{i+1} \cup dom(p_i) \subseteq \mathcal{A}_j$ and $B_{i+1} \supseteq^* A_j$. So the claim is proved as $(\mathcal{A}_j, A_j) \in AP$. Hence in this substep we get $\mathcal{D}_{i+1}, D_{i+1}$ and q_i such that $e_i \upharpoonright_{\mathcal{D}^{q_i}_{i+1} \cap D_{i+1}}$ is one-to-one or constant . At substep4, we obtain Z_{i+1} such that $Z_{i+1} \notin \mathcal{D}_{i+1}$, hence $Z_{i+1} \notin dom(q_i)$. Let $q = q_i \cup (Z_{i+1}, 1)$, then $q \in FF(\mathcal{A}_*)$ and

$$\mathcal{A}^q_* \subseteq \mathcal{A}^{q_i}_* \cap Z_{i+1} \subseteq \mathcal{D}^{q_i}_{i+1} \cap D_{i+1}$$

Therefore $e_i \upharpoonright \mathcal{A}^q_*$ is one-to-one or constant.

For (v), first notice that for each $i \in \omega_1$, $A_i \in fil_{\mathcal{B}_{i+1}} \subseteq fil_{\mathcal{A}_{i+1}} \subseteq fil_{\mathcal{A}_*}$.

On the other hand, if $X \in fil_{\mathcal{A}_*}$, we claim there is some $j \in \omega_1$, $X \in fil_{\mathcal{A}_j}$. By definition, $\forall h \in FF(\mathcal{A}_*), \exists h \subseteq g \in FF(\mathcal{A}_*), \mathcal{A}^g_* \subseteq X$. List $\mathcal{A}_* = \{W_\alpha : \alpha < \omega_1\}$, we can define a function $\beta \mapsto \theta_\beta$ satisfying $\forall h \in FF(\{W_\alpha : \alpha < \beta\})$, $\exists h \subseteq g \in FF(\{W_\alpha : \alpha < \theta_\beta\}), \mathcal{A}^g_* \subseteq X$. Let $C_0 = \{j \in \omega_1 : \forall \beta < j, \theta_\beta < j\}$ and

$C_1 = \{j \in \omega_1 : \mathcal{A}_j = \{W_\alpha : \alpha < j\}\}$. Both C_0 and C_1 are clubs and any $j \in C_0 \cap C_1$ suffices.

Fix any such j, there are unbounded many $i \in \omega_1$ such that $j \leq i$ and $X = X_i$. For any such i, $X_i \in fil_{\mathcal{A}_j} \subseteq fil_{\mathcal{A}_i}$. Hence at substep2 of step $i+1$, $B_{i+1} \subseteq X_i = X$. But $A_{i+1} \subseteq D_{i+1} \subseteq B_{i+1} \subseteq X$. This completes the verification. □

The key is the following preservation theorem.

Theorem 3.39. *[44] Assume $\langle(\mathbb{P}_\alpha, Q_\alpha) : \alpha < \delta\rangle$ is a countable support iteration. If for each $\alpha < \delta$, the iterand Q_α satisfies the following properties:*

- $\otimes^0_\alpha$ *Q_α is proper in $\mathbb{V}^{\mathbb{P}_\alpha}$.*
- $\otimes^1_\alpha$ *Q_α has Sacks property in $\mathbb{V}^{\mathbb{P}_\alpha}$.*
- $\otimes^2_\alpha$ *Q_α has "dense open set property" in $\mathbb{V}^{\mathbb{P}_\alpha}$. Namely, for each $D \in \mathbb{V}^{\mathbb{P}_\alpha * Q_\alpha}$, $D \subseteq \omega^{<\omega}$ is dense open, there is $E \in \mathbb{V}^{\mathbb{P}_\alpha}$, $E \subseteq \omega^{<\omega}$ is dense open and $E \subseteq D$.*
- $\otimes^3_\alpha$ *If $X \in (fil_{\mathcal{A}_*})^{\mathbb{V}^{\mathbb{P}_\alpha * Q_\alpha}}$, then there is $Y \in (fil_{\mathcal{A}_*})^{\mathbb{V}^{\mathbb{P}_\alpha}}$ with $X \supseteq Y$.*
- $\otimes^4_\alpha$ *In $\mathbb{V}^{\mathbb{P}_\alpha * Q_\alpha}$, for each $g \in FF(\mathcal{A}_*)$ and $A \subseteq \mathcal{A}^g_*$, then either $\exists X \in (fil_{\mathcal{A}_*})^{\mathbb{V}^{\mathbb{P}_\alpha}}$ such that $A \supseteq X \cap \mathcal{A}^g_*$, or $\exists h \in FF(\mathcal{A}_*)$ such that $g \subseteq h$ and $A \cap \mathcal{A}^h_* = \emptyset$.*
- $\otimes^5_\alpha$ *In $\mathbb{V}^{\mathbb{P}_\alpha * Q_\alpha}$, for each $g \in FF(\mathcal{A}_*)$ and $A \subseteq \mathcal{A}^g_*$, then $\exists h \in FF(\mathcal{A}_*)$, $g \subseteq h$ such that either $A \cap \mathcal{A}^h_* = \emptyset$ or $\mathcal{A}^h_* \subseteq A$.*
- $\otimes^6_\alpha$ *Q_α is ω^ω-bounding in $\mathbb{V}^{\mathbb{P}_\alpha}$.*

Then $\mathbb{P}_\delta$ also satisfies the corresponding properties $\otimes^0_\delta - \otimes^6_\delta$ (in $\mathbb{V}$).

The complete proof can be found in [44], we sketch some key points.

Remark 3.40.

1. *It is argued in [44] that $\otimes^2_\alpha \to \otimes^3_\alpha$ and $\otimes^3_\alpha \to (\otimes^4_\alpha \leftrightarrow \otimes^5_\alpha)$.*
2. *If $\mathbb{P}$ does not add reals, then obviously $\mathbb{P}$ satisfies corresponding properties $\otimes^1$-$\otimes^6$.*

3. *The construction of $\mathcal{A}_*$ guarantees that $fil_{\mathcal{A}_*}$ is a selective filter, see requirements (iv) and (v).*

4. *$\otimes^4$ guarantees that $\mathcal{A}_*$ is maximal: let $g = \emptyset$, $A \subseteq \mathcal{A}_*^g = \omega$, then either there is some $X \in (fil_{\mathcal{A}_*})^{\mathbb{V}^{\mathbb{A}}}$ such that $A \supseteq X$ and hence some h such that $A \supseteq X \supseteq \mathcal{A}_*^h$, or there is some h' such that $A \cap \mathcal{A}_*^{h'} = \emptyset$. So $\mathfrak{i} \leq \aleph_1$.*

5. *The preservation of $\otimes^1, \otimes^2$ and $\otimes^6$ under countable support iteration of proper forcings are well known.*

6. *By Theorem 3.33, $\otimes^6$ guarantees that $fil_{\mathcal{A}_*}$ generates a selective filter in $\mathbb{V}^{\mathbb{P}_\alpha}$ for each $\alpha \leq \delta$.*

Sketch of the proof of $\otimes^5_\delta$: Let $T = \{\mathcal{A}_*^h \mid h \in FF(\mathcal{A}_*)\}$ and $J = fil_{\mathcal{A}_*}$ (in the ground model). We show that whenever $p \Vdash_\delta \dot{X} \subset \omega$, then there is some $q \leq p$ such that either

(1)some $A \in T$ satisfies $q \Vdash \dot{X} \cap A = \emptyset$ or

(2)some $A \in J$ satisfies $q \Vdash A \subseteq \dot{X}$.

We may assume w.l.o.g. that $\delta = \omega$. Let $[J]$ denote the filter J generates. Let N be a countable elementary submodel of some H_λ, with $\mathbb{P}_\omega \in N$ etc.

If there is some $p' \leq p$ and $q \in \mathbb{P}_n$ which is $(N, \mathbb{P}_n)$-generic, $q \leq p' \restriction n$ such that

$$q \Vdash_n \{n : p' \nVdash n \notin \dot{X}\} \notin [J]^{\mathbb{V}[\dot{G}_n]}$$

Then by induction hypothesis, we meet the requirement (1).

Otherwise, we define $q_n, p_n, \langle p^n_l; l \in \omega\rangle$ inductively :

When $n = 0$, it is trivial to define q_0 and p_0. However, we define $\langle p^0_l; l \in \omega\rangle \in N$ and $A \in J$ as follows: we inductively find p^0_l such that $p^0_l \leq p^0_{l-1}$ and p^0_l decides $\dot{X} \cap l$ and some $B^0_l \in J \cap N$ such that $p^0_l \Vdash B^0_l \cap l \subseteq \dot{X} \cap l$. This is possible by the requirement (vi) on $J = fil_{\mathcal{A}_*}$. Now since J is selective, let $B^0 \in N \cap J$ selects at most one element from each B^0_l. W.l.o.g, assume that exactly one element $k_l \in B^0_l$ is selected for each $l \in \omega$. Let $A \in J$ "covers" N, namely $\forall Z \in N \cap J, A \subseteq^* Z$, and $A \subseteq B^0$.

When $n > 0$, we require that

1. q_n is $(N, \mathbb{P}_n)$-generic,
2. $p_n \in N$ is a $\mathbb{P}_n$-name for a condition in $\mathbb{P}_\omega$, same for each p^n_l,
3. $B^n \in J \cap N$ such that $A \subseteq B^n$,
4. $q_n \restriction n - 1 = q_{n-1}$,
5. $q_n \Vdash p_n \leq p_{n-1} \wedge p_n \restriction n \in G_n$,
6. $\Vdash p_n \geq p^n_l \geq p^n_{l+1}$
7. $q_n \Vdash \{p^n_l; l \in \omega\} \subseteq \mathbb{P}_\omega/G_n, p^n_l \Vdash_{\mathbb{P}_\omega/G_n} B^n \cap l \subseteq \dot{X} \cap l.$

Briefly, we use the fact that $[J]$ is selective to find B_n. Notice that if we let $q \in \mathbb{P}_\omega$, then $q \vDash A \subseteq \dot{X}$. This is because for each $n \in A$, $q_{n+1} \Vdash p^{n+1}_{n+1} \Vdash n \in \dot{X}$. □

Proof of Theorem 3.34: We start with a model $\mathbb{V} \vDash CH + \exists$ supercompact cardinal. Since $\mathbb{V} \vDash CH$, we can find in $\mathbb{V}$ the maximal independent family $\mathcal{A}_*$ with cardinality $\aleph_1$. Let $\mathbb{D} \in \mathbb{V}$ be the notion of forcing to force $DPFA$. Each iterand of $\mathbb{D}$ is proper and adds no new reals, hence satisfies properties $\otimes^0$-$\otimes^6$. By the preservation theorem 3.39, $\mathbb{D}$ also satisfies these properties. Hence $\mathbb{V}^{\mathbb{D}} \vDash \mathcal{A}_*$ is maximal. So $\mathbb{V}^{\mathbb{D}} \vDash \mathfrak{i} \leq |\mathcal{A}_*|$. Since $\mathbb{D}$ is proper, $\aleph_1$ is preserved. $\mathbb{V}^{\mathbb{D}} \vDash \mathfrak{i} = |\mathcal{A}_*| = \aleph_1$ and $DPFA$. □

3.5 *DPFA* and Almost Disjointness Number

Theorem 3.41. *It is consistent to have DPFA and $\mathfrak{a} = \aleph_1$ relative to the consistency of existence of a supercompact cardinal.*

Definition 3.42. *Let $A, B \in [\omega]^\omega$, they are almost disjoint, if $A \cap B$ is finite.*

Definition 3.43. *$\mathcal{F} \subseteq [\omega]^\omega$ is an almost disjoint family, or AD family for short, if $\mathcal{F}$ is infinite and any two distinct members of $\mathcal{F}$ are almost disjoint.*

Definition 3.44. *$\mathcal{F} \subseteq [\omega]^\omega$ is a maximal almost disjoint family, or MAD family for short, if $\mathcal{F}$ is AD family and it is maximal under inclusion among all AD families.*

Definition 3.45. $\mathfrak{a}$ *is the least cardinality of a MAD family.*

Fact 3.46. $\mathfrak{a}$ *is big, since $\mathfrak{b} \leq \mathfrak{a}$. See [12] for proof.*

Fact 3.47. *If $\mathcal{F}$ is an infinite AD family, then the union of any finite many elements of $\mathcal{F}$ is coinfinite.*

Proof of Theorem 3.41: Let $\mathbb{V}$ be a model of ZFC with a supercompact cardinal κ. First we force a MAD family $\{a_\alpha : \alpha \in \omega_1\}$. We define inductively the forcing $\langle\langle \mathbb{A}_\alpha, B_\alpha\rangle | \alpha \in \omega_1\rangle$. At each successor step $\alpha + 1$, we add a_α as a generic real. To simplify notation, we may assume without loss of generality that $\mathbb{A}_\omega$ is trivial (namely $\mathbb{V}^{\mathbb{A}_\omega} = \mathbb{V}$) and $\mathbb{V}^{\mathbb{A}_\omega} \models \{a_\alpha : \alpha \in \omega\}$ forms a partition of ω.

For successor stage $\beta + 1$, with $\omega \leq \beta < \omega_1$. Assume we have constructed $\mathbb{V}^{\mathbb{A}_\beta}$, and $\mathbb{V}^{\mathbb{A}_\beta} \models \{a_\alpha : \alpha \in \beta\}$ is AD family.

Consider in $\mathbb{V}^{\mathbb{A}_\beta}$ the partial order:

$$B_\beta = \{p = \langle s_p, t_p\rangle | s_p \in [\omega]^{<\omega}, t_p \in [\beta]^{<\omega}\}$$

The partial order on B_β is defined by letting $p \leq q$ iff:

$$s_p \supseteq_e s_q, t_p \supseteq t_q, (s_p \setminus s_q) \cap \bigcup_{\alpha \in t_q} a_\alpha = \emptyset$$

Remark 3.48. *B_β is countable, hence it is essentially Cohen forcing.*

If $G \subseteq B_\beta$ is a $\mathbb{V}^{\mathbb{A}_\beta}$-generic filter, then define $a_\beta = \bigcup\{s_p : p \in G\}$. Conversely, G can be recovered from a_β. $G = \{p : s_p \subseteq_e a_\beta\}$. We will denote this filter by a^*_β.

Lemma 3.49. *If $a \in \mathbb{V}^{\mathbb{A}_\beta} \cap [\omega]^\omega$ satisfies that $\forall t \in [\beta]^{<\omega}$, $a \setminus \bigcup_{\alpha \in t} a_\alpha$ is infinite, then $\mathbb{V}^{\mathbb{A}_{\beta+1}} \vDash |a \cap a_\beta| = \omega$.*

Proof of the lemma : Argue in $\mathbb{V}^{\mathbb{A}_\beta}$. For any given $k \in \omega$, let $D_k = \{p \in B_\beta : |s_p \cap a| > k\}$. We show that D_k is a dense subset of B_β. Fix any $q \in B_\beta$, we have that $a \setminus \bigcup_{\alpha \in t_q} a_\alpha$ is infinite. Let $n_0 < n_1 < \cdots < n_k$ be natural numbers from $a \setminus \bigcup_{\alpha \in t_q} a_\alpha$ such that $n_0 > max(s_q)$. Define $p = \langle s_p, t_p \rangle$ by letting $s_p = s_q \cup \{n_0, \ldots, n_k\}$ and $t_p = t_q$. Then we have $p \le q$ and $p \in D_k$, so D_k is dense. Let $p_k \in D_k \cap a^*_\beta$, then $p_k \Vdash |\dot{a}_\beta \cap a| > k$. Therefore $a \cap a_\beta$ is infinite in $\mathbb{V}^{\mathbb{A}_{\beta+1}}$. □

Corollary 3.50. 1. *ω satisfies the assumption for a in the lemma above.*

So $a_\beta = \omega \cap a_\beta$ is infinite.

2. *If $a \in \mathbb{V}^{\mathbb{A}_\beta} \cap [\omega]^\omega$ satisfying $\forall \alpha < \beta, |a \cap a_\alpha| < \omega$, then $\mathbb{V}^{\mathbb{A}_{\beta+1}} \vDash |a \cap a_\beta| = \omega$.*

Lemma 3.51. $\mathbb{V}^{\mathbb{A}_{\beta+1}} \vDash \{a_\alpha : \alpha \in \beta + 1\}$ *is an AD family.*

Proof of the lemma: Given any $\gamma < \beta$, we show that $|a_\gamma \cap a_\beta| < \omega$.

In $\mathbb{V}^{\mathbb{A}_\beta}$, let $D = \{p \in B_\beta : \gamma \in t_p\}$. Fix $q \in B_\beta$. Define $p = \langle s_p, t_p \rangle$ by letting $s_p = s_q$ and $t_p = t_q \cup \{\gamma\}$. Then we have $p \le q$ and $p \in D$, so D is dense.

Let $p \in D \cap a^*_\beta$, so $\gamma \in t_p$. If $n \in a_\beta \setminus max(s_p)$, then $\exists q \in a^*_\beta$ with $n \in s_q$. We may assume that $q \le p$, then $n \in s_q \setminus s_p \subseteq \omega \setminus \bigcup_{\alpha \in t_p} a_\alpha \subseteq \omega \setminus a_\gamma$. So $n \in a_\beta \setminus a_\gamma$. Thus $a_\beta \setminus max(s_p) \subseteq a_\beta \setminus a_\gamma$ and $a_\beta \cap a_\gamma \subseteq a_\beta \cap max(s_p)$. Since s_p is finite, we see that $a_\beta \cap a_\gamma$ is finite. This completes the proof of lemma. □

At limit stage, we take finite support.

Let $\mathbb{A} = \mathbb{A}_{\omega_1}$ denote the finite support limit of $\langle \langle \mathbb{A}_\alpha, B_\alpha \rangle | \alpha \in \omega_1 \rangle$.

Lemma 3.52. *If X is a real in $\mathbb{V}^{\mathbb{A}}$, then there is some $\alpha < \omega_1$, such that $X \in \mathbb{V}^{\mathbb{A}_\alpha}$. Therefore, CH holds in $\mathbb{V}^{\mathbb{A}}$.*

Proof: Each iterand B_α is countable, therefore has c.c.c. Since we use finite support for the limit, $\mathbb{A}$ has c.c.c. If $\dot{X}$ is a $\mathbb{A}$-name for a real, let for each $n \in \omega$,

$Y_n \subseteq \{p : p \Vdash n \in \dot{X}\}$ be a maximal incompatible set. Then each Y_n is countable. There is some $\alpha < \omega_1$ such that $\forall n \in \omega, \forall p \in Y_n, spt(p) \subseteq \alpha$. So $\dot{X}$ is decided by $\mathbb{A}_\alpha$. So we have

$$\mathcal{P}(\omega) \cap \mathbb{V}^{\mathbb{A}} = \bigcup_{\alpha<\omega_1} \mathcal{P}(\omega) \cap \mathbb{V}^{\mathbb{A}_\alpha}$$

By computing nice names for subsets of ω, we can prove by induction that for each $\alpha < \omega_1$, $\mathbb{V}^{\mathbb{A}_\alpha} \models CH$. Therefore, $\mathbb{V}^{\mathbb{A}} \models CH$. □

Lemma 3.53. $\mathbb{V}^{\mathbb{A}} \models \{a_\alpha : \alpha < \omega_1\}$ *is a MAD family.*

Proof of the lemma : (1) For any $\alpha < \beta < \omega_1$, $\mathbb{V}^{\mathbb{A}_{\beta+1}} \models |a_\alpha \cap a_\beta| < \omega$. This is also true in $\mathbb{V}^{\mathbb{A}}$. Thus, $\{a_\alpha : \alpha < \omega_1\}$ is a AD family.

(2) Maximality: Given $a \subseteq \omega$, there is some $\beta < \omega_1$ by Lemma 3.52 such that $a \in \mathbb{V}^{\mathbb{A}_\beta}$. If there is some $\alpha < \beta$ with $|a_\alpha \cap a| = \omega$, then we are done; otherwise, by Corollary 3.50, $\mathbb{V}^{\mathbb{A}_{\beta+1}} \models |a \cap a_\beta| = \omega$. Thus $\mathbb{V}^{\mathbb{A}} \models |a \cap a_\beta| = \omega$. Therefore, maximality is proved. □

Let's continue the proof of Theorem 3.41. Since in the ground model $\mathbb{V}$, $\mathbb{A}$ is small relative to the supercompact cardinal κ, κ remains supercompact in $\mathbb{V}^{\mathbb{A}}$. Let $\mathbb{D} \in \mathbb{V}^{\mathbb{A}}$ be the partial order to force $DPFA$, all we need is that $\{a_\alpha : \alpha < \omega_1\}$ remains maximal in $\mathbb{V}^{\mathbb{A}*\mathbb{D}}$.

Definition 3.54. *In* $\mathbb{V}^{\mathbb{A}}$ *or its generic extension, define a partial order* $\mathbb{Q}$ *to be "nice" iff* $\forall\lambda$ *large regular cardinal,* $\forall N \prec H_\lambda$ *countable elementary model such that* $\mathbb{Q} \in N$, $\forall \xi \in \omega_1 \setminus N$ *such that* a_ξ *is* (N, B_ξ)*-generic,* $\forall p \in N \cap \mathbb{Q}$, $\exists q \in \mathbb{Q}$ *such that:*

- $q \leq p$,
- q *is* $(N, \mathbb{Q})$*-generic,*
- $q \Vdash a_\xi$ *is* $(N[\dot{G} \cap N], B_\xi)$*-generic.*

Lemma 3.55. *In $\mathbb{V}^{\mathbb{A}}$, let*

$$W = \{N \in [H_\lambda]^\omega : N \prec H_\lambda \wedge \forall \xi \in \omega_1 \setminus N\ (a_\xi \text{ is } B_\xi - \text{generic over } N)\}.$$

Then W contains a club in $[H_\lambda]^\omega$.

Proof: Similar to the one we used in Lemma 2.23. For instance, we sketch the idea in showing the unboundness of W. We claim that for any $N \in [H_\lambda]^\omega$, there is $N' \in [H_\lambda]^\omega$ satisfying: $N' \supseteq N$ and $\forall \xi \in \omega_1 \setminus N'$, $\forall D \in N$ dense open subset of $\mathbb{C}_2$, $a_\xi \cap D \cap N' \neq \emptyset$.

The key is that for each $\xi \in \omega_1$, we can view $\mathbb{V}^{\mathbb{A}}$ as an ω_1-step iteration with ground model $\mathbb{V}^{\mathbb{A}_{\xi+1}}$. If $E \in \mathbb{V}^{\mathbb{A}}$ is a subset of B_ξ, then E is countable. Since B_ξ is in this "ground model", by the chain condition, there is some $\theta < \omega_1$, such that $E \in \mathbb{V}^{\mathbb{A}_\theta}$. The rest of the proof will be the same as in Lemma 2.23. □

Lemma 3.56. *If $\mathbb{V}^{\mathbb{A}} \models \mathbb{Q}$ is nice, then $\mathbb{V}^{\mathbb{A}*\mathbb{Q}} \models \{a_\alpha : \alpha < \omega_1\}$ is MAD.*

Proof: Otherwise, in $\mathbb{V}^{\mathbb{A}}$, there will be some $p \in \mathbb{Q}$, and some $\mathbb{Q}$-name τ such that $p \Vdash \tau$ witnesses that $\{a_\alpha : \alpha < \omega_1\}$ is not maximal. By the previous lemma, let $N \in W$ be such that $\{\tau, p, Q\} \subset N$. Let $\xi \in \omega_1 \setminus N$, then a_ξ is (N, B_ξ)-generic. By the definition of niceness, $\exists q \leq p$, q is $(N, \mathbb{Q})$-generic and $q \Vdash a_\xi$ is $(N[\dot{G} \cap N], B_\xi)$-generic. By a similar argument as in Lemma 3.49, we see that $q \Vdash \exists \alpha \leq \xi\ \ |\tau \cap a_\alpha| = \omega$, a contradiction. □

Fact 3.57. *If $\mathbb{Q}$ is distributive and proper, then $\mathbb{Q}$ is nice.*

The most important part is the preservation theorem below:

Theorem 3.58. *In $\mathbb{V}^{\mathbb{A}}$. Let $\langle(\mathbb{P}_\alpha, \dot{Q}_\alpha) | \alpha \in \epsilon\rangle$ be a countable support iteration. If $\forall \alpha < \epsilon$, $\Vdash_{\mathbb{P}_\alpha} \dot{Q}_\alpha$ is nice, then $\mathbb{P}_\epsilon$ is nice.*

We need Lemma 2.27 and the following inductive lemma:

Lemma 3.59. *In $\mathbb{V}^{\mathbb{A}}$. Let $\langle(\mathbb{P}_\alpha, \dot{Q}_\alpha)|\alpha \in \epsilon\rangle$ be a countable support iteration of nice forcing notions. Assume that λ is a large regular cardinal. $N \prec H_\lambda$ is countable, $\mathbb{P}_\epsilon \in N$. $\xi \in \omega_1 \setminus N$. $\alpha < \beta$ are both in N. $\dot{p} \in N$ such that $\Vdash_\alpha \dot{p} \in \mathbb{P}_\beta$, $q \in \mathbb{P}_\alpha$ satisfying*

$\circledast_1^\alpha$ *q is $(N, \mathbb{P}_\alpha)$-generic,*

$\circledast_2^\alpha$ *$q \Vdash \dot{p} \restriction_\alpha \in \dot{G}_\alpha \cap N$,*

$\circledast_3^\alpha$ *$q \Vdash a_\xi$ is $(N[\dot{G}_\alpha \cap N], B_\xi)$-generic.*

Then there is $q^+ \in \mathbb{P}_\beta$, $q^+ \restriction_\alpha = q$ such that

$\circledast_1^\beta$ *q^+ is $(N, \mathbb{P}_\beta)$-generic,*

$\circledast_2^\beta$ *$q^+ \Vdash \dot{p} \in \dot{G}_\beta \cap N$,*

$\circledast_3^\beta$ *$q^+ \Vdash a_\xi$ is $(N[\dot{G}_\beta \cap N], B_\xi)$-generic.*

Proof: We prove by induction on the pair (α,β):

case1 $\beta = \alpha + 1$.

Since q is $(N, \mathbb{P}_\alpha)$-generic, we have
$q \Vdash N[\dot{G}_\alpha \cap N]$ is a countable elementary submodel of $H_\lambda[\dot{G}_\alpha]$ and
$q \Vdash \xi \in \omega_1 \setminus N[\dot{G}_\alpha \cap N]$.

Also by induction hypothesis, $q \Vdash a_\xi$ is $(N[\dot{G}_\alpha \cap N], B_\xi)$-generic.
Since $\alpha \in N$, $\dot{Q}_\alpha \in N \wedge \dot{p}(\alpha) \in N$. Therefore,

$$q \Vdash \dot{Q}_\alpha \in N[\dot{G}_\alpha \cap N] \wedge \dot{p}(\alpha) \in \dot{Q}_\alpha \cap N[\dot{G}_\alpha \cap N]$$

By assumption, $\Vdash_\alpha \dot{Q}_\alpha$ is nice. Therefore, q forces : $\exists x$

1. $x \in \dot{Q}_\alpha \wedge x \leq \dot{p}(\alpha)$ and

2. x is $(N[\dot{G}_\alpha \cap N], \dot{Q}_\alpha)$-generic, and

3. $x \Vdash a_\xi$ is $\Big((N[\dot{G}_\alpha \cap N])[\dot{G}(\alpha) \cap N[\dot{G}_\alpha \cap N]], B_\xi\Big)$-generic.

By existential completeness theorem, there is a $\mathbb{P}_\alpha$ name x such that q force x to satisfy the properties above. By computing names, we see that

$$(N[\dot{G}_\alpha \cap N])[\dot{G}(\alpha) \cap N[\dot{G}_\alpha \cap N]] = N[\dot{G}_{\alpha+1} \cap N].$$

Therefore, $(q, x) \in \mathbb{P}_{\alpha+1}$, $(q, x) \Vdash (\dot{p} \restriction_\alpha, \dot{p}(\alpha)) = \dot{p} \restriction_{\alpha+1} \in \dot{G}_{\alpha+1}$ and (q, x) is $(N, \mathbb{P}_{\alpha+1})$-generic. Thus, $(q, x) \Vdash \mathbb{P}_{\alpha+1} \cap N[\dot{G}_{\alpha+1} \cap N] \subseteq N$. But $\dot{p} \in N$, so $(q, x) \Vdash \dot{p} \in N$.

By the third requirement on x, we see that
$(q, x) \Vdash a_\xi$ is $(N[\dot{G}_{\alpha+1} \cap N], B_\xi)$-generic.

So $q^+ = (q, x)$ is as required.

case2 $\beta = \gamma + 1$,where $\gamma > \alpha$.

By induction hypothesis on the pair (α, γ), we can first extend q to some $q^* \in \mathbb{P}_\gamma$ satisfying $\circledast_1^\gamma - \circledast_3^\gamma$, then use case1 to extend q^* to $q^+ \in \mathbb{P}_\beta$.

case3 β is a limit ordinal. Let $\delta = sup(\beta \cap N)$. Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence of ordinals from $\beta \cap N$, with the supreme δ, and $\alpha_0 = \alpha$.

Let $\{D_n : n \in \omega\}$ list all dense open sets of $\mathbb{P}_\beta$ that lie in N. Let $\{\tau_n : n \in \omega\}$ list all $\mathbb{P}_\beta$ names for dense open sets of B_ξ that lie in N.

step1 We build in N inductively $\mathbb{P}_{\alpha_n}$-names $\dot{p}_n$ for conditions in $\mathbb{P}_\beta$.

Let $\dot{p}_0 = \dot{p}$. Suppose we have $\Vdash_{\alpha_n} \dot{p}_n \in \mathbb{P}_\beta$. Apply Lemma 2.27 inside N, we obtain $\dot{p}_{n+1} \in N$ such that the following is forced by $\mathbb{1}_{\alpha_{n+1}}$:

$$(\dot{p}_{n+1} \leq \dot{p}_n) \wedge (\dot{p}_{n+1} \in D_n) \wedge (\dot{p}_n \restriction_{\alpha_{n+1}} \in \dot{G}_{\alpha_{n+1}} \rightarrow \dot{p}_{n+1} \restriction_{\alpha_{n+1}} \in \dot{G}_{\alpha_{n+1}})$$

step2 Notice that $\tau_0 \in N$ and $\Vdash_\beta \tau_0 \subseteq B_\xi$ is dense open. Let $\dot{Y}_0$ be a $\mathbb{P}_{\alpha_0}$-name such that $\Vdash_{\alpha_0} \dot{Y}_0 = \{u \in B_\xi : \exists y\ y \Vdash_{\alpha_0,\beta} u \in \tau_0\}$.

Then $\dot{Y}_0$ can be found in N and $\Vdash_{\alpha_0} \dot{Y}_0$ is dense open in B_ξ.

By assumption, $\circledast_1^{\alpha_0}, \circledast_2^{\alpha_0}$ and $\circledast_3^{\alpha_0}$ holds for $q_0 = q$ and $\alpha_0 = \alpha$.

So $q_0 \Vdash \dot{Y}_0 \in N[\dot{G}_{\alpha_0} \cap N] \wedge \dot{Y}_0 \cap a_\xi \neq \emptyset$.

Thus, $q_0 \Vdash \exists x \in a_\xi \exists y\ y \Vdash_{\alpha_0,\beta} x \in \tau_0$.

So $q_0 \Vdash \exists y\ y \Vdash_{\alpha_0,\beta} \exists x \in a_\xi\ x \in \tau_0$. That is, $q_0 \Vdash \exists y\ y \Vdash_{\alpha_0,\beta} a_\xi \cap \tau_0 \neq \emptyset$.

By existential completeness, there is a $\mathbb{P}_{\alpha_0}$ name r_0 for a condition in $\mathbb{P}_{\alpha_0,\beta}$ such that (q_0, r_0) is a condition in $\mathbb{P}_\beta$ and $(q_0, r_0) \Vdash a_\xi \cap \tau_0 \neq \emptyset$.

By induction hypothesis on the pair (α_0, α_1), we can find

$q_1 \in \mathbb{P}_{\alpha_1}, q_1 \restriction_{\alpha_0} = q_0$ satisfying

- q_1 is $(N, \mathbb{P}_{\alpha_1})$-generic,
- $q_1 \Vdash \dot{p}_0 \restriction_{\alpha_1} \in \dot{G}_{\alpha_1} \cap N$,
- $q_1 \Vdash a_\xi$ is $(N[\dot{G}_{\alpha_1} \cap N], B_\xi)$-generic.

Hence $q_1 \Vdash \dot{p}_1 \restriction_{\alpha_1} \in \dot{G}_{\alpha_1} \cap N$. Without loss of generality, it can assumed that $q_1 \leq (q_0, r_0) \restriction_{\alpha_1}$.

By a similar argument as above, we can find a $\mathbb{P}_{\alpha_1}$ name r_1 for a condition in $\mathbb{P}_{\alpha_1,\beta}$ such that (q_1, r_1) is a condition in $\mathbb{P}_\beta$ and $(q_1, r_1) \Vdash a_\xi \cap \tau_1 \neq \emptyset$. We can also assume that $(q_1, r_1) \leq (q_0, r_0)$.

Inductively, we get $q_n \in \mathbb{P}_{\alpha_n}$ such that $q_n \restriction_{\alpha_{n-1}} = q_{n-1}$, $(q_n, r_n) \leq (q_{n-1}, r_{n-1})$ and

- q_n is $(N, \mathbb{P}_{\alpha_n})$-generic,
- $q_n \Vdash \dot{p}_n \restriction_{\alpha_n} \in \dot{G}_{\alpha_n} \cap N$,
- $q_n \Vdash a_\xi$ is $(N[\dot{G}_{\alpha_n} \cap N], B_\xi)$-generic.
- $(q_n, r_n) \Vdash a_\xi \cap \tau_n \neq \emptyset$.

step3 Let $q^+ \in \mathbb{P}_\beta$ be the limit of $\langle q_n : n \in \omega\rangle$. Namely, and $\forall\theta \in dom(q^+)$, there is some $n \in \omega$, $\theta \in dom(q_n)$, and for any such n, $q^+(\theta) = q_n(\theta)$. Then for each $n \in \omega$, $q^+ \leq (q_n, r_n)$.

We need to verify that q^+ works. Let $G_\beta \ni q^+$ be any $(\mathbb{V}^{\mathbb{A}}, \mathbb{P}_\beta)$-generic filter and p_n be the interpretation of $\dot{p}_n$ by G_β. We show the following:

(i) $p_n \in N \cap G_\beta$,

(ii) $N \cap G_\beta$ is $(N, \mathbb{P}_\beta)$-generic,

(iii) a_ξ is $(N[G_\beta \cap N], B_\xi)$-generic.

Let $G_{\alpha_n} = \{r \restriction_{\alpha_n} : r \in G_\beta\}$. It is a $(\mathbb{V}^{\mathbb{A}}, \mathbb{P}_{\alpha_n})$-generic filter.

So $q_n \in G_{\alpha_n}$ and $p_n \restriction_{\alpha_n} \in G_{\alpha_n}$. Since $p_{n+1} \leq p_n$,we have $p_n \restriction_{\alpha_{n+1}} \in G_{\alpha_{n+1}}$. Therefore, $\forall n, m \in \omega$, $p_n \restriction_{\alpha_m} \in G_{\alpha_m}$. Therefore, $p_n \restriction_\delta \in G_\delta$.

Notice that $p_n \in \mathbb{P}_\beta \cap N[G_{\alpha_n} \cap N] \subseteq N$, $dom(p_n) \in N$. Since $dom(p_n)$ is countable and N is a countable elementary model, $dom(p_n) \subseteq N \cap \beta \subseteq \delta$. So $p_n \in G_\beta$. This complete the proof of (i).

For (ii), let $D \in N$ be a dense open subset of $\mathbb{P}_\beta$. Say $D = D_n$. By the construction, $p_{n+1} \in D_n$. So $p_{n+1} \in G_\beta \cap N \cap D_n$. Therefore, $N \cap G_\beta$ is $(N, \mathbb{P}_\beta)$-generic.

For (iii), let $T \in N[G_\beta \cap N]$ be a dense open subset of B_ξ. Then T has a $\mathbb{P}_\beta$-name in N, say τ_n. Since $(q_n, r_n) \Vdash_{\alpha_n,\beta} a_\xi \cap \tau_n \neq \emptyset$ $(q_n, r_n) \in G_\beta$, $a_\xi \cap T \neq \emptyset$. By elementarity, $B_\xi \in N$. But B_ξ is countable. So $T \subseteq B_\xi \subseteq N \subseteq N[G_\beta \cap N]$. So $a_\xi \cap N[G_\beta \cap N] \cap T \neq \emptyset$.

So q^+ satisfies $\circledast_1^\beta - \circledast_3^\beta$.

□

Take $\alpha = 0, \beta = \epsilon$ in the lemma, we prove the Theorem 3.58.

3.6 Conclusion

In this section, we combine the results obtained in the previous sections.

Theorem 3.60. *If it is consistent that a supercompact cardinal exists, then it is consistent that DPFA holds and all the following cardinal invariants are equal to* $\aleph_1$*:*

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{m}, \mathfrak{p}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u},$$

$$add(\mathcal{B}), cov(\mathcal{B}), non(\mathcal{B}), cof(\mathcal{B}), add(\mathcal{L}), cov(\mathcal{L}), non(\mathcal{L}), cof(\mathcal{L}).$$

Remark 3.61. *The reader is referred to [12] for the cardinal invariants not defined in this thesis. The fact that we need is:* $cof(\mathcal{L}), \mathfrak{u}, \mathfrak{i}, \mathfrak{a}$ *are the maximal ones among these cardinal invariants.*

Proof of the Theorem: As before, we start with a model $\mathbb{V}$ with a supercompact cardinal. We may also assume that $\mathbb{V} \models CH$. Force with $\mathbb{A}$, the partial order defined in Section 3.5, to get the MAD family $\{a_\alpha : \alpha < \omega_1\}$. By Lemma 3.52, $\mathbb{V}^{\mathbb{A}} \models CH$. In $\mathbb{V}^{\mathbb{A}}$, there are $\aleph_1$ many Slaloms and moreover we can find a p-point U as in Section 3.3 and construct the independent family $\mathcal{A}_*$ as in Section 3.4. Let $\mathbb{D} \in \mathbb{V}^{\mathbb{A}}$ be the poset to force $DPFA$, then each iterand of $\mathbb{D}$ preserves p-points and satisfies properties $\otimes^0$-$\otimes^6$ which are defined in Section 3.4. By the preservation theorems, $\mathbb{D}$ preserves p-points and satisfies properties $\otimes^0$-$\otimes^6$. In $\mathbb{V}^{\mathbb{A}*\mathbb{D}}$, $DPFA$ holds, each $f : \omega \to \omega$ goes through a Slalom in $\mathbb{V}^{\mathbb{A}}$, the ultrafilter generated by U is a p-point, $\mathcal{A}_*$ is maximal independent and $\{a_\alpha : \alpha < \omega_1\}$ is MAD. □

Chapter 4

Fragments of PFA

In this chapter, we will apply the results obtained in the previous chapters to distinguish fragments of Proper Forcing Axiom.

4.1 Weakness of $DPFA$

Theorem 4.1. *[8] Either $< \omega_1 - PFA$ or $BPFA$ implies that any two $\aleph_1$-dense subsets of the reals are order isomorphic.*

Proof. The partial order Baumgartner used is of the form ω-closed $*c.c.c.$. Namely the partial order is a two-step iteration, the first step being an ω-closed forcing and the second step being a $c.c.c.$ forcing. Hence it satisfies Axiom A, and is $< \omega_1$-proper.

Also, all the dense subsets to meet have size $\leq \aleph_1$, so $BPFA$ suffices.

Combining with Theorem 2.18, we get

Corollary 4.2. 1. *$DPFA$ does not imply $< \omega_1 - PFA$, hence $DPFA$ does not implies $\omega - PFA$,*

2. *$DPFA$ does not imply $BPFA$.*

4.2 Weakness of $BPFA$

Theorem 4.3. *[23] The consistency strength for BPFA is that of a Σ_1-reflecting cardinal.*

Corollary 4.4. *1. BPFA does not imply $< \omega_1 - PFA$,*

2. BPFA does not imply DPFA.

Proof. $< \omega_1 - PFA \vdash PID$, $DPFA \vdash PID$ [55]. PID implies for every cardinal κ with cofinality $> \aleph_1$, $\neg \Box (\kappa)$ [55]. By inner model theory, the failure of squares implies the consistency of many Woodin cardinals [51]. Therefore, The Corollary holds. □

4.3 Alternative Approaches

The Open Coloring Axiom studied in [54] is a nice "Ramsey type" statement.

Theorem 4.5. *Todorčević [54] proved:*

1. *$< \omega_1 - PFA$ implies OCA,*

2. *OCA implies $\mathfrak{b} = \aleph_2$,*

3. *OCA implies that there is no (ω_2, ω_2^*)- gap.*

Therefore, by Theorem 3.3, $DPFA$ does not imply OCA. Besides, it was proved in [1] that $BPFA$ is consistent with the existence of (ω_2, ω_2^*)- gap. Hence, $BPFA$ does not imply OCA. These provides alternative approaches in showing that neither $DPFA$ nor $BPFA$ implies $< \omega_1 - PFA$.

Theorem 4.6. *BPFA implies $\mathfrak{u} = \aleph_2$.*

Proof. Suppose $\{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$. Let $\mathbb{P} = [\omega]^{<\omega}$. $\mathbb{P}$ is ordered by letting $p \leq q$ iff $p \supseteq_e q$. $\mathbb{P}$ is actually Cohen forcing. Let for each $\alpha < \omega_1$,

$$D_\alpha = \{p : \exists k \in A_\alpha \setminus p \ \ (k < max(p))\}$$

D_α is clearly a dense subset of $\mathbb{P}$ as A_α is infinite. Since $\mathbb{P}$ is countable, $BPFA$ guarantees that there is some $G \subseteq \mathbb{P}$, a filter that meets each D_α. Let $Z = \bigcup G$, $p_\alpha \in G \cap D_\alpha$. Then p_α will "force" that $A_\alpha \nsubseteq Z$. Hence $\{A_\alpha : \alpha < \omega_1\}$ is not an ultrafilter base. □

Combine with 3.13, this provides an alternative way to prove that $DPFA$ does not imply $BPFA$.

One can actually do much better. Consider the "localization forcing" [4], $BPFA$ implies $\mathfrak{p} = \aleph_2$ [12]. Since $\mathfrak{p}$ is a relatively small cardinal invariants, many others must equal to $\aleph_2$ under $BPFA$.

In fact, either $BPFA$ or $< \omega_1 - PFA$ proves that $MA + \mathfrak{c} = \aleph_2$, while $DPFA$ does not imply MA_{ω_1}.

4.4 Weakness of $< \omega_1 - PFA$

Definition 4.7. *[45] $\langle c_\alpha : \alpha \in Lim \cap \omega_1 \rangle$ is a club guessing sequence, if*

1. *Each c_α is an unbounded subset of α,*

2. *For each club $C \subseteq \omega_1$, $\{\alpha \in Lim \cap \omega_1 : c_\alpha \subseteq C\}$ is stationary in ω_1.*

Remark 4.8. *The second requirement above is equivalent to the following: For each club $C \subseteq \omega_1$, $\{\alpha \in Lim \cap \omega_1 : c_\alpha \subseteq C\} \neq \emptyset$.*

Let "CG" denote the statement: there is a club guessing sequence.

Theorem 4.9. *If $\mathbb{P}$ is the forcing to add an ω_1- Cohen real, then $\mathbb{V}^{\mathbb{P}} \vDash CG$.*

Proof: The argument is similar to the one showing $\mathbb{V}^{\mathbb{P}} \models \diamond$. We use a substitute of ω_1- Cohen forcing, namely $\mathbb{P} = Fn([\omega_1]^2, 2, \omega_1)$. If $G \subseteq \mathbb{P}$ is generic, for each limit $\alpha < \omega_1$, let $c_\alpha : \alpha \to 2$ be defined such that: $c_\alpha(\eta) = \bigcup G(\{\alpha, \eta\})$. For any $\theta \leq \omega_1$, we identify a function $f : \theta \to 2$ with the set $\{\tau \in \theta : f(\tau) = 1\}$ for simplicity of notation. $\langle c_\alpha : \alpha \in \omega_1 \cap Lim\rangle$ is a $\diamond$ sequence, but not a club guessing sequence. Nevertheless, we can define $d_\alpha = c_\alpha$ whenever c_α is cofinal in α and $d_\alpha = \alpha$ elsewhere.

Claim: $\mathbb{V}^{\mathbb{P}} \models \forall$ club $C \subseteq \omega_1, \{\alpha : c_\alpha$ is cofinal in $\alpha \wedge c_\alpha \subseteq C\}$ is stationary. Suppose otherwise. Then $\mathbb{V}^{\mathbb{P}} \models C$ and D are clubs of ω_1, $\forall \alpha \in D$ (either c_α is bounded in α or $c_\alpha \not\subseteq C$).

Let $p \in \mathbb{P}$ such that $p \Vdash \dot{C}$ and $\dot{D}$ are clubs of ω_1, $\forall \alpha \in \dot{D}$ (either $\dot{c_\alpha}$ is bounded in α or $\dot{c_\alpha} \not\subseteq \dot{C}$). For each $r \in \mathbb{P}$, define $f(r) = min\{\gamma : dom(r) \subseteq [\gamma]^2\}$. We are going to construct $p_n, \beta_n, q_n, \delta_n, b_n$ inductively such that:

1. $p = p_0 \geq q_0 \geq p_1 \geq q_1 \geq \cdots,$
2. $\beta_n = f(p_n),$
3. $\beta_0 \leq \delta_0 \leq \beta_1 \leq \delta_1 \leq \cdots,$
4. $q_n \Vdash \delta_n \in \dot{C} \cap \dot{D},$
5. $b_n \subseteq \delta_n,$
6. $p_{n+1} \Vdash \dot{C} \cap \delta_n = b_n.$

The construction is straight forward. Given p_n, β_n, since $p_n \Vdash \dot{C} \cap \dot{D}$ is a club, there is some $q_n \leq p_n$ and $\delta_n > \beta_n$ such that $q_n \Vdash \delta_n \in \dot{C} \cap \dot{D}$; then there is some $p_{n+1} \leq q_n$ and $b_n \subseteq \delta_n$ such that $p_{n+1} \Vdash \dot{C} \cap \delta_n = b_n$.

Let $p_\omega = \bigcup_{n\in\omega} p_n, b = \bigcup_{n\in\omega} b_n, \delta = \bigcup_{n\in\omega} \delta_n$. Then $\delta = \bigcup_{n\in\omega} \beta_n = f(p_\omega)$, and $p_\omega \Vdash \dot{C} \cap \delta = b$. Therefore, $\forall n \in \omega, \delta_n \in b$. Thus b is unbounded in δ. Let $q = p_\omega \cup \{(\{\delta, \eta\}, 1) : \eta \in b\}$, then $q \Vdash \dot{c_\delta} = b$. Hence, $q \Vdash \delta \in \dot{D} \wedge \dot{c_\delta} \subseteq \dot{C} \wedge \dot{c_\delta}$

is unbounded in δ. Contradicts to the assumption on p. This completes the proof of claim.

For any α, if c_α is unbounded in α and $c_\alpha \subseteq C$, then $d_\alpha = c_\alpha$ and $d_\alpha \subseteq C$. By the lemma, we have: $\mathbb{V}^{\mathbb{P}} \models \forall$ club $C \subseteq \omega_1, \{\alpha \in Lim \cap \omega_1 : d_\alpha \subseteq C\}$ is stationary. So $\langle d_\alpha : \alpha \in Lim \cap \omega_1 \rangle$ is a club guessing sequence in $\mathbb{V}^{\mathbb{P}}$. □

Theorem 4.10. *[25] If $\mathbb{Q}$ is $< \omega_1$-proper, $\langle c_\alpha : \alpha \in Lim \cap \omega_1 \rangle$ is a club guessing sequence, then $\mathbb{V}^{\mathbb{Q}} \Vdash \langle c_\alpha : \alpha \in Lim \cap \omega_1 \rangle$ is a club guessing sequence.*

Proof. Suppose $p \Vdash \dot{D} \subseteq \omega_1$ is a club. Fix a large regular cardinal λ, build an ω_1 tower $\langle M_i : i \in Lim \cap \omega_1 \rangle$ in H_λ such that $\{\mathbb{P}, p, \dot{D}, \omega_1\} \subseteq M_0$. Let $\delta_i = M_i \cap \omega_1$. Let $E = \{\delta_i : i \in Lim \cap \omega_1\}$. Then E is a club, hence $\exists \alpha \in \omega_1, c_\alpha \subset E$. So $\alpha \in E$. Let i be such that $\alpha = \delta_i$. Since $\mathbb{P}$ is i-proper, let $q \leq p$ satisfying $\forall j < i$, q is $(M_j, \mathbb{P})$-generic. Therefore, $q \Vdash \dot{D} \cap M_j$ is unbounded in $\omega_1 \cap M_j = \delta_j$. So $q \Vdash \delta_j \in \dot{D}$. Hence, $q \Vdash E \cap \alpha \subset \dot{D}$. So $q \Vdash c_\alpha \subseteq \dot{D}$. □

Corollary 4.11. *The consistency of "$\exists$ a supercompact cardinal" implies the consistency of $< \omega_1 - PFA + CG$.*

Proof. We start with $\mathbb{V} \models \exists$ a supercompact cardinal, force with $\mathbb{P} = Fn([\omega_1]^2, 2, \omega_1)$. Then $\mathbb{V}^{\mathbb{P}} \models CG + \exists$ a supercompact cardinal. In $\mathbb{V}^{\mathbb{P}}$, let $\mathbb{Q}$ be the forcing to get $< \omega_1 - PFA$. Again, the Laver function will help us. Each iterand of $\mathbb{Q}$ is $< \omega_1$-proper. Now by Theorem 1.19, $\mathbb{Q}$ is $< \omega_1$-proper. So by Theorem 4.10, $\mathbb{V}^{\mathbb{P}*\mathbb{Q}} \models CG+ < \omega_1 - PFA$. □

Definition 4.12. *[24] Let $\omega - CG$ be the statement: there is a sequence $\langle c_\alpha : \alpha \in Lim \cap \omega_1 \rangle$ such that:*

1. *Each c_α is cofinal subset of α with order type ω,*
2. *For each club $D \subseteq \omega_1$, there is $\alpha \in Lim \cap \omega_1$ such that $c_\alpha \subseteq D$.*

Definition 4.13. *[24] Let $\omega - wCG$ be the statement: there is a sequence $\langle c_\alpha : \alpha \in Lim \cap \omega_1 \rangle$ such that:*

1. *Each c_α is cofinal subset of α with order type ω,*

2. *For each club $D \subseteq \omega_1$, there is $\alpha \in Lim \cap \omega_1$ such that $D \cap c_\alpha$ is infinite.*

Remark 4.14. *It is immediate that CG is equivalent to $\omega - CG$ and $\omega - CG$ implies $\omega - wCG$.*

Theorem 4.15. *[38] MRP implies $\neg(\omega - wCG)$.*

Combining these together, we get

$$DPFA \to MPR \to \neg(\omega - wCG) \to \neg(\omega - CG) \leftrightarrow \neg CG.$$

Definition 4.16. *Let "pseudo club guessing (PCG)" be the statement: there is a sequence $\langle c_\alpha : \alpha \in \omega_1 \rangle$ such that:*

1. *Each c_α is an infinite subset of ω_1,*

2. *For each club $D \subseteq \omega_1$, there is $\alpha \in \omega_1$ such that $c_\alpha \subseteq D$.*

Remark 4.17. *It is immediate that CG implies PCG.*

Baumgartner use the forcing "adding a club with finite conditions" to show:

Theorem 4.18. *[8] PFA implies $\neg PCG$.*

Since this forcing has size $\aleph_1$, we have: $BPFA$ implies $\neg PCG$, hence $\neg CG$.

Hence, we have:

Theorem 4.19.
1. *$< \omega_1 - PFA$ does not imply $DPFA$.*
2. *$< \omega_1 - PFA$ does not imply $BPFA$.*

We conclude that $< \omega_1 - PFA$, $BPFA$ and $DPFA$ are mutually independent.

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