ESSAYS ON CONTESTS DESIGN WITH STOCHASTIC ENTRY AND INFORMATION DISCLOSURE

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Summary

My dissertation contains three essays on optimal contests design with stochastic entry and information disclosure.

The first two chapters study imperfectly discriminatory contests with stochastic entries. As much of the contest literature assumes the number of competing agents is fixed, and this number is known by all participants. While economic activities always involve an uncertain set of participants. Under the assumption that a fixed pool of potential bidders can enter a contest to compete for an indivisible prize, chapter 1 explores how a contest organizer who seeks to maximize participant effort should disclose the information on the actual number of contestants, when each potential contestant has a *fixed* probability of entering the contest. In a setting with risk neutral contestants, the optimal disclosure policy depends crucially on the properties of the characteristic function $H(\cdot)= f(\cdot)/ f'(\cdot)$, where $f(\cdot)$ is the impact function. The contest organizer prefers full disclosure (full concealment) if $H(\cdot)$ is strictly concave (strictly convex). However, the expected equilibrium effort is independent of the prevailing information disclosure policy if a linear $H(\cdot)$ (Tullock Contest) applies.

Chapter 2 differs from chapter 1 in the sense that the probability of entry is endogenous. Each bidder incurs an irreversible fixed cost if he decides to enter. After entering, the bidders then bid for the prize. This setting leads to a two-dimensional discontinuous game (Dasgupta and Maskin, 1986). I establish that a symmetric equilibrium exists in the entry-bidding game, where all potential bidders enter with a

probability. I further identify the conditions for the existence (non-existence) of a symmetric equilibrium with pure-strategy bidding after entry. Based on the equilibrium result, three main issues about optimal contest design are explored: (i) the optimal level of accuracy of the winner selection mechanism (the proper size of r in Tullock contests); (ii) the efficiency implications of shortlisting and exclusion; and (iii) the optimal disclosure policy.

Chapter 3 investigates information disclosure in a perfectly discriminating contest. Early contributions assume that a player's ability, measured by his cost of expending effort, is fixed and common knowledge. While empirically, contestants usually do not know the actual abilities of their rival at the time they make their decision. In chapter 3, I assume the private abilities of the contestants are stochastic and they are observed by the contest organizer who decides whether to disclose this information publicly. The organizer may care about total effort or rent dissipation. I find that concealing the abilities of the contestants elicits higher expected total effort, regardless of the distribution of the abilities. By way of contrast, rent dissipation rate does not depend on the disclosure policy. This finding is robust in a setting with multiple prizes as long as effort cost function is linear. And also robust in generalized settings with endogenous distribution of abilities and endogenous entry of contestants. However, when the cost function is nonlinear, the organizer may prefer disclosure.

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Chapter One
On Disclosure Policy in Contests with Stochastic Entry

1 Introduction

Much of the contest literature makes the assumption that the number of competing agents is fixed, and that this number is known by all participants. Although this paradigm simplifies the analysis significantly, it stands in contrast to numerous contest settings in real-life that involve an uncertain set of participants. For instance, a firm racing to develop an innovation may not know how many other firms are pursuing the same idea. Similarly, a job applicant may be uncertain about the number of competitors for the same post. In a procurement tournament, a seller may not be aware of the number of bidders who are interested in the contract.

In this study, we study contests with a stochastic number of contestants. Our basic setting involves a fixed number of potential contestants, each of whom has a fixed probability of entering the contest. The realized number of participants remains uncertain, but follows a binomial distribution. The participating contestants exert costly and nonrefundable efforts to compete for a single prize. We further assume that their effort accrues to the benefit of the contest organizer. In this scenario, our analysis sets out to address a classical question in the contest literature: *How does the contest organizer choose a disclosure policy that* *maximizes the expected total effort?* That is, *should the contest organizer disclose or conceal the actual number of contestants to participants? Which policy alternative leads to a higher level of expected total effort?*

To address these questions, we consider a three-stage game. In the first stage, the contest organizer chooses her disclosure policy. She either reveals the actual number of contestants, or conceals this information. She announces her policy choice publicly to potential contestants. In the second stage, the actual number of contestants is realized and learnt by the organizer. This information is disclosed to the contestants if the organizer had earlier chosen to do so. In the third stage, contestants submit their effort entries simultaneously in competition for the single prize.

We adopt the well-studied ratio-form contest success function to abstract the underlying stochastic winner selection process.[1] In this setting, a contestant i, who exerts an effort x_i, wins the prize with a probability $p_i(x_i, \mathbf{x}_{-i}) = \frac{f(x_i)}{f(x_i)+\sum_{j\neq i} f(x_j)}$ if there are $N-1$ others who exert effort of $\mathbf{x}_{-i} = (x_1, x_2, ...x_{i-1}, x_{i+1}, ..., x_N)$. The function $f(\cdot)$ has been named the "impact function" by Wärneryd (2001), and it specifies each contestant's production technology in the contest.

The optimal disclosure policy depends crucially on the *characteristic function* of the contest, which is formally defined as $H(x) \equiv \frac{f(x)}{f'(x)}$. The properties of this function determine how each participating contestant responds to various environmental factors in the contest. We show that disclosing the actual number of contestants leads to a higher (lower) level of total effort, relative to concealing the information, if the characteristic function is concave (convex). However, the level of expected total effort is independent of the prevailing disclosure policy, if the characteristic function is linear. We further show that a linear characteristic function is uniquely generated by contests known as Tullock (1980) contests,

[1] The reader is referred to Skaperdas (1996) for the axiomatic foundation of the ratio-form contest success function and Fu and Lu (2008) for the function's micro-foundation that is derived from a noisy-ranking perspective.

which assume $f(x) = x^r$.

Our analysis yields interesting theoretical implications. Despite all contestants being risk-neutral, a strictly concave characteristic function leads contestants to behave as if they were risk-loving when they supply their effort.[2] Conversely, "pseudo" risk-aversion appears when a strictly convex characteristic function applies. With non-Tullock contest technologies, the disclosure policy plays a pivotal role in determining the equilibrium level of effort, because of the "pseudo" risk-loving/averse attitudes that are underpinned by concave/convex characteristic functions.

To check the robustness of our main results and to deepen our analysis, we further generalize our basic setting by allowing the contest organizer to partially disclose the actual number of participants. Under a partial disclosure policy, the organizer does not reveal the exact number of participants, but only the range of this number. *Will the organizer benefit from partial disclosure? How should she structure the optimal partial disclosure policy?* We show that strict concavity (convexity) of the characteristic function must lead to full disclosure (full concealment), and partial disclosure is never optimal. By way of contrast, the disclosure policy does not affect the expected overall effort in a Tullock contest (which has a linear characteristic function), in spite of the numerous possible ways of constructing a partial disclosure policy.

Only a handful of papers have formally investigated contests with stochastic participation. Higgins, Shughart, and Tollison (1985) pioneered this strand of literature by studying a contest in which each rent seeker bears a fixed cost for participation. They established a unique symmetric mixed strategy equilibrium, where each rent seeker randomly enters the contest, and ends up with zero surplus. While Higgins, Shughart and Tollison (1985) investigated endogenous entry strategies, a few other studies have assumed exogenous entry patterns. Myerson and Wärneryd (2006) examined a contest with an infinite number of potential en-

[2] In other words, the individual effort function is convex in terms of the amount of prize.

trants. Both Münster (2006) and Lim and Matros (2009) assumed a finite pool of potential contestants. In their setting, each participating contestant enters the contest with a fixed and independent probability and the number of participating contestants follows a binomial distribution. Münster (2006) focused on the impact of players' risk attitudes on the contestants' incentive to supply effort. In contrast, Lim and Matros (2009) considered a scenario with risk-neutral contestants.

The current study is most closely related to Lim and Matros (2009), who provide a complete account of the bidding equilibrium in a Tullock contest with a stochastic number of contestants. To the best of our knowledge, Lim and Matros (2009) are the first to study optimal disclosure policies in contests. They establish that the disclosure policy (full disclosure or full concealment) does not impact the level of effort. Our analysis allows for more general contest technologies, and we find sharply different results that indicate the "relevance" of disclosure policy when non-Tullock contests are considered. Furthermore, we allow contest organizers to partially disclose information. The "disclosure irrelevance" principle in Tullock contests (with their linear characteristic functions) holds, despite the substantially richer set of candidate disclosure strategies available to organizers. Our study thus complements Lim and Matros (2009) in these regards.

2 Contest with A Stochastic Number of Contestants

Let $M(\geq 2)$ denote the set of risk neutral potential contestants whose probability of participating in the contest is $q \in (0, 1)$. All participating contestants compete for a single prize of value $v > 0$.

Suppose that $N \leq M$ contestants participate and simultaneously commit to their nonnegative rent seeking efforts $x_i, i = 1, 2, ..., N$. The effort is costly and non-refundable, and the contestants incur a unit marginal cost. We assume also that the winner is determined by a ratio-form contest success function. This mechanism has been commonly adopted in the literature, and is axiomatized by Skaperdas (1996). If $N \geq 2$ contestants enter the contest, a participating contestant i wins the prize v with a probability

$$p_i(x_i, \mathbf{x}_{-i}; N) = \frac{f(x_i)}{\sum_{j=1}^{N} f(x_j)}, \tag{1.1}$$

where the function $f(\cdot)$ is strictly increasing, thrice differentiable and weakly log concave, with $f(0) = 0$. The log-concavity, as will be shown, guarantees the uniquness of equilibrium in the contest. Wärneryd (2001) names $f(\cdot)$ the impact function of the contest, which indicates a contestant's production technology. If all contestants make zero effort, we assume that the prize recipient is randomly chosen from the pool. Moreover, we assume that if there is only one participant, then he automatically wins the prize regardless of his effort.

We assume further that the effort exerted by the contestants accrues to the benefit of the contest organizer. The contest organizer is allowed to commit to her disclosure policy - either to disclose the actual number of participants, or to conceal this information - and announces this policy choice publicly. We denote the former policy by D, and the latter by C. Nature then determines N, the actual number of participants. The organizer observes this information, and discloses it if and only if she has committed to a disclosure. The participants then submit their effort entry simultaneously $\mathbf{x} = (x_i)$ to compete for the prize.

2.1 Equilibrium

We now explore the equilibrium of the contest under each policy. We first consider a case where the impact function of $f(\cdot)$ is concave, where a unique equilibrium is readily established. We next study convex impact functions and we show that the contest may still yield a unique symmetric equilibrium.

2.1.1 Concave Impact Functions

Concave impact functions provide a stronger condition than weak log-concavity. It is well known that a concave impact function $f(\cdot)$ is sufficient for the existence and uniqueness of symmetric equilibria in a standard contests. We will show that this condition guarantees the existence and uniqueness of symmetric equilibria in our context regardless of the prevailing disclosure policy.

Contest with Disclosure We first consider the subgame where the contest organizer commits to the policy D. All contestants learn of N before they decide on their effort level. Each contestant i then rationally chooses his effort x_i to maximize the expected payoff

$$\pi_i = p_i(x_i, \mathbf{x}_{-i}; N)v - x_i. \tag{1.2}$$

Consider a subgame where N contestants participate. We now solve for the symmetric equilibrium of the contest. Define $H(x) \equiv \frac{f(x)}{f'(x)}$. As shown below, the equilibrium behavior of each contestant is characterized by the function $H(\cdot)$ and its inverse. It is thus named as the *characteristic function* of the contest for convenience.

The symmetric equilibrium effort x is determined by the first order condition

$$H(x) = \frac{N-1}{N^2}v. \tag{1.3}$$

Because $f(x)$ is concave, we have $H'(x) > 0$. As $H(0) = 0$, there exists a unique $x > 0$ which solves equation (1.3). The solution to (1.3) constitutes a unique symmetric pure-strategy equilibrium, if and only if it globally maximizes a representative contestant i's expected payoff π_i given that all others exert the same effort. We now formally establish the existence and uniqueness of a symmetric pure-strategy equilibrium.

Proposition 1 *Suppose that $N(\geq 2)$ contestants participate in the contest. If they learn the actual number (N) of participants, each contestant in the unique symmetric pure-strategy equilibrium makes an effort*

$$x(N) = H^{-1}(\frac{N-1}{N^2}v) > 0, \tag{1.4}$$

where $H^{-1}(\cdot)$ is the inverse of the characteristic function $H(\cdot)$. The overall effort of the N-person contest is then given by

$$E(N) \equiv Nx(N) = N \cdot H^{-1}(\frac{N-1}{N^2}v). \tag{1.5}$$

Proof. $x(N)$ of (1.4) is derived from the first order condition (1.3). To establish it as a symmetric equilibrium, if suffices to show that a representative contestant i's expected payoff π_i is globally concave in x_i given that all others exert the effort of (1.4). We have $\frac{\partial \pi_i}{\partial x_i} = \frac{(N-1)x(N)f'(x)}{[f(x)+(N-1)x(N)]^2}v - 1$. As $f'(x) \geq 0$ and $f''(x) \leq 0$, $f(x)$ increases and $f'(x)$ decreases with their arguments. Hence, $\frac{\partial \pi_i}{\partial x_i}$ decreases with x_i, i.e. π_i is concave in x_i: π_i increases with x_i when $x_i \leq x(N)$ and π_i decreases with x_i when $x_i \geq x(N)$. A symmetric equilibrium is therefore established where every contestant exerts effort $x(N)$. As (1.3) has a unique solution, the symmetric equilibrium with $x(N)$ is unique. ■

Having obtained the solution to every possible contest with N participants, we are now ready to find the expected total effort of the game when the $D-$policy is adopted. Given the fixed entry probability q, the probability of $N \in \{0, 1, 2, ..., M\}$ contestants showing up is given by $\Pr(N) = C_M^N q^N (1-q)^{M-N}$. Hence, the expected total effort is given by

$$\begin{aligned} TE_D(q) &= \sum_{N=1}^{M} C_M^N q^N (1-q)^{M-N} N x(N) \\ &= \sum_{N=1}^{M} C_M^N q^N (1-q)^{M-N} N H^{-1}[\frac{1}{N}(1-\frac{1}{N})v] \end{aligned}$$

$$= Mq\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} H^{-1}[\frac{1}{N}(1-\frac{1}{N})v]. \quad (1.6)$$

Contest with Concealment We now analyze the subgame in which the actual number of participants is not revealed by the contest organizer. A participant i chooses his effort x_i to maximize the expected payoff

$$\pi_i(x_i, \mathbf{x}_{-i}; q) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} p_i(x_i, \mathbf{x}_{-i}; N)v - x_i.$$

Proposition 2 *If the actual number of participating contestants is not disclosed, each participant exerts an effort*

$$x_C(q) = H^{-1}[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{1}{N}(1-\frac{1}{N})v], \quad (1.7)$$

in the unique symmetric pure-strategy equilibrium, where $H^{-1}(\cdot)$ is the inverse of the characteristic function $H(\cdot)$.

Proof. We first assume that a symmetric equilibrium exists. The first order condition for effort is given by

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{N-1}{N^2}\frac{f'(x)}{f(x)}v - 1 = 0. \quad (1.8)$$

A concave $f(\cdot)$ implies that $\frac{f'(x)}{f(x)}$ must be monotonic. Hence, there exists a unique solution to the function, as given by (1.7). It remains to verify that $x_C(q)$ constitutes an equilibrium. First, note that $p_i(x_i, \mathbf{x}_{-i}; N)$ is concave. $\frac{d^2 p_i(x_i, \mathbf{x}_{-i}; N)}{d^2 x_i} = \frac{f''(x_i)[f(x_i)+\sum_{j\neq i} f(x_j)]-2[f'(x_i)]^2}{[f(x_i)+\sum_{j\neq i} f(x_j)]^3}\sum_{j\neq i} f(x_j)$ is negative because of the concavity. Second, $\pi_i(x_i, \mathbf{x}_{-i}; q)$ is a weighted sum of $p_i(x_i, \mathbf{x}_{-i}; N)$. Hence, $\pi_i(x_i, \mathbf{x}_{-i}; q)$ must be concave in x_i as well. The global concavity ensures that the solution of (1.7) constitutes an equilibrium. ■

Proposition 2 establishes the unique pure-strategy symmetric equilibrium of the contest with concealment. The expected overall effort in the subgame is therefore obtained as

$$
\begin{aligned}
TE_C(q) &= Mqx_C(q) \\
&= MqH^{-1}[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{1}{N}(1-\frac{1}{N})v]. \qquad (1.9)
\end{aligned}
$$

2.1.2 Convex Impact Functions

Symmetric equilibria in a contest do not necessarily require a conave impaction function. A convex impact function would, however, substantially complicate the analysis, because a contestant's payoff function may not be globally concave. In a two-player setting, Baye, Kovenock and de Vries (1994) demonstrates the difficulty in characterizing the equilibria when the impact function gets excessively convex.

The analysis in our context can be further complicated by stochastic entries. We now explore the possible equilibria when convex impact functions are in place. We propose two examples to demonstrate these possibilities. Because of log-concavity of $f(x)$, equations (1.3) and (1.8) would continue to yield unique solution, as given by (1.4) and (1.7), respectively. However, the solutions to first order conditions do not necessarily constitute an equilibrium. In the two examples we discuss below, unique symmetric equilibria do exist and the results established in the previous section (Propositions 1 and 2, (1.6) and (1.9)) continue to apply.

We first consider the popularly studied Tullock contest with impact function $f(x) = x^r$. The following can be obtained.

Claim 1 *When $r \in (1, 1+\frac{1}{M-1}]$, there always exist a unique symmetric pure-strategy equilibrium.*

(a) When N is disclosed, in a N-person contest, each participant exerts an effort $x(N) = \frac{r(N-1)}{N^2}v > 0$.

(b) When N is concealed, each participant exerts an effort

$$x_C(q) = r\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{1}{N}(1-\frac{1}{N})v > 0$$

Proof. Please refer to Appendix A. ∎

It has been well known that when N, the number of participants, is common knowledge, a symmetric equilibrium exists in a Tullock if and only if $r \leq 1+\frac{1}{N-1}$. When r falls below the cutoff $1+\frac{1}{M-1}$, a unique symmetric equilibrium results in a contest with disclosure regardless of the actual number N. We further show that the cutoff also guarantees the existence of a unique symmetric equilibrium in a contest with concealment. The equilibrium effort outlays are adapted from (1.4) and (1.7), respectively. The overall efforts in contests with disclosure and concealment can also be obtained from (1.6) and (1.9), respectively.

Further, we consider another family of convex impact functions that could also yield symmetric equilibrium. Consider the family of impact functions $f(x) = e^{\alpha x} - 1$, with $\alpha \in (0, 1]$. For analytical convenience, we normalize the prize to $v = 1$. We show the following.

Claim 2 *Let $f(x) = e^{\alpha x} - 1$, with $\alpha \in (0, 1]$. When $M \leq 4$, a unique symmetric equilibrium exists in the contest regardless of the prevailing disclosure policy.*

(a) When N is disclosed, in a N-person contest, each participant exerts the equilibrium effort

$$x(N) = -\frac{1}{\alpha}\ln(1-\frac{N-1}{N^2}\alpha). \tag{1.10}$$

(b) When N is concealed, each participant exerts the equilibrium effort

$$x_C(q) = -\frac{1}{\alpha}\ln[1-\alpha\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{N-1}{N^2}]. \tag{1.11}$$

Proof. Please refer to Appendix A. ∎

Remark 2 identifies another possible context where convex impact function render symmetric equilibria. Again, (1.10) and (1.11) are adapted from (1.4) and

(1.7) respectively. The overall efforts in contests with disclosure and concealment can be obtained from (1.6) and (1.9), respectively.

2.2 Optimal Disclosure Policy

We now compare (1.6) with (1.9) to investigate the effort-maximizing disclosure policy. One can conclude by Jensen's Inequality that $TE_D(q) > TE_C(q)$, if and only if $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} H^{-1}[\frac{1}{N}(1-\frac{1}{N})v] > H^{-1}[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{1}{N}(1-\frac{1}{N})v]$, which simply requires $H^{-1}(\cdot)$ to be convex, and therefore the characteristic function $H(\cdot) \equiv \frac{f(\cdot)}{f'(\cdot)}$ to be strictly concave. We summarize our results as follows.

Theorem 1 *Suppose that every contestant independently enters the contest with the same exogenous probability q and symmetric equilibria exist for contests with disclosure and concealment of number of entrants.*

(a) Disclosing the actual number of contestants elicits strictly more (less) effort than concealing the actual number of contestants, if the characteristic function $H(\cdot)$ is strictly concave (convex).

(b) (Disclosure Irrelevance) The resultant expected total effort is independent of the disclosure policy, if the characteristic function $H(\cdot)$ is linear.

We do not lay out a dedicated proof, but briefly interpret the logic that underpins our main result. Note that the function $H^{-1}(\cdot)$ (as well as its inverse $H(\cdot)$) plays a pivotal role in determining the equilibrium effort of each participating contestant. As revealed by (1.3) and (1.4), each contestant's equilibrium effort depends crucially on the properties of the characteristic function (and those of its inverse), which are fundamentally determined by the contest technology $f(\cdot)$. Recall from (1.4) that a contestant exerts an equilibrium effort $x(N) = H^{-1}(\frac{N-1}{N}v)$. The function $H(\cdot)$ thus depicts how contestants respond to the competitive environment of the contest, e.g., how they respond to changes in the number of competitors and/or the value of prize, etc. As illustrated below, a given contest environment would trigger sharply different responses by contestants when the prevailing contest technologies (i.e., the characteristic functions) differ.

When N is to be concealed, each participating contestant exerts a uniform equilibrium effort $x_C(q) = H^{-1}[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{1}{N}(1-\frac{1}{N})v]$ upon entry. By way of contrast, when N is to be disclosed, each participating contestant responds to each realization of N by exerting an effort $x(N) = H^{-1}(\frac{N-1}{N^2}v)$ upon entry. On average, he exerts an effort $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} H^{-1}[\frac{1}{N}(1-\frac{1}{N})v]$.

A larger N implies that a less favorable contest is realized. Hence, when N is disclosed, a contestant exerts more effort when N (≥ 2) is small, while he exerts less effort when N is large.[3] A concave $H(\cdot)$ (i.e., a convex $H^{-1}(\cdot)$) implies that a contestant's equilibrium effort is increasingly elastic with respect to the value of its argument. A contestant tends to respond increasingly sensitively to any given decrease in N (by increasing effort $x(N)$), but less sensitively to any given increase in N. A strictly concave characteristic function leads a contestant to behave as if he were risk-loving when he supplies his effort, in spite of his risk-neutrality: a smaller N (a more favorable contest) incentivizes a contestant more than a larger N (a less favorable contest) disincentivizes him. Consequently, each contestant, on average, exerts more effort when N is disclosed than when it is concealed.

By way of contrast, when $H(\cdot)$ is convex (i.e., $H^{-1}(\cdot)$ is concave) and the realized N is disclosed, a contestant responds more sensitively to an increase in N (by lowering his effort), but less sensitively to a decrease in N. A strictly convex characteristic function leads a contestant to behave as if he were risk-averse: A larger N (a less favorable contest) disincentivizes him more than a smaller N (a more favorable contest) incentivizes him. This leads to the result that his overall expected effort $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} H^{-1}[\frac{1}{N}(1-\frac{1}{N})v]$ falls below $x_C(q)$.

[3]Note that $\frac{1}{N}(1-\frac{1}{N})$ decreases with N.

Theorem 1(b) shows that $TE_D(q) = TE_C(q)$ if $H(x)$ is linear in x. Lim and Matros (2009) establish the "disclosure-irrelevance" principle in a Tullock contest with $f(x) = ax^r$. It can be directly verified that a linear characteristic function results if and only if a Tullock contest prevails. Note $H(0) = 0$ and $H'(\cdot) > 0$. Therefore, we must have $H(x) = tx$ if $H(\cdot)$ is a linear function, with constant $t > 0$. According to the definition of $H(x)$, we have $\frac{f'(x)}{f(x)} = \frac{1}{tx}$. Solving the differential equation yields $\ln f(x) = \ln(x^{\frac{1}{t}}) + b$, where b is a constant. It further leads to $f(x) = e^b x^{\frac{1}{t}}$, which takes the form of a power function. Our result reveals that the "disclosure-irrelevance" principle of Lim and Matros (2009) is essentially underpinned by the linearity of characteristics function $H(\cdot)$ that is associated with a "Tullock" contest.

There are many functional forms of impact function $f(\cdot)$ that guarantee the existence of symmetric equilibria and lead to strictly concave or strictly convex characteristic functions. We present below two examples to illustrate these possibilities.

Example 1 *Consider the family of functions* $f(x) = [\ln(1+x)]^{\alpha}$, *with* $\alpha \in (0,1]$. *Simple calculus verifies* $H(x) = \frac{f(x)}{f'(x)} = \alpha^{-1}(1+x)\ln(1+x)$, *which further leads to* $H'(x) = \alpha^{-1}[1+\ln(1+x)] > 0$ *and* $H''(x) = \frac{1}{\alpha}\frac{1}{1+x} > 0$. *We then conclude that this functional form leads to a convex characteristic function.*

Example 2 *Consider the family of functions in Claim 2 of Section 2.1.2* $f(x) = e^{\alpha x} - 1$, *with* $\alpha \in (0,1]$. *As has been shown there,* $H''(x) = -\alpha e^{-\alpha x} < 0$. *This functional form then yields a concave characteristic function.*

3 Extensions and Discussion

This part of the paper further explores the issue of information disclosure from two additional dimensions. First, an extension that generalizes the disclosure policy in the basic setting by allowing the contest organizer to partially reveal the information on the actual number of participants is considered. Second, the commitment issue of disclosure policy is explored. The impact function take the same forms as in Section 2: Either $f(x)$ is concave, or it belongs to the convex family of Section 2.1.2.

3.1 Imperfect Information Disclosure

We have assumed that the organizer of the contest either fully discloses the number of participating contestants, or completely withholds this information. We now allow the organizer to partially disclose her information.

Let the organizer's information disclosure strategy be depicted by an ordered set $(k_1, k_2, ..., k_I)$, where $k_i \in \{1, \ldots, M\}$ and $1 \leq I \leq M$. We arrange k_is in ascending order and let $k_I = M$. Each $(k_1, k_{2,} ..., k_I)$ thus characterizes a partition of the information space $\{1, 2, \ldots, M\}$. The organizer does not announce the exact realization of N, but discloses that N is in a partition set $\Omega_i = \{k_{i-1} + 1, \ldots, k_i\}$, i.e., $k_{i-1} + 1 \leq N \leq k_i$. For convenience, we assume $k_0 \equiv 0$.

When $I = M$, the finest partition is obtained. The partition strategy converges to a full disclosure strategy and the exact realization of N is revealed. When $I = 1$, the partition strategy is the coarsest, reducing to a concealment policy. The finer the partition, the more information on the actual number of contestants is revealed to contestants. We now investigate the optimal partition strategy of the organizer.

Define $P_i = \sum_{t=k_{i-1}}^{k_i - 1} C_{M-1}^t q^t (1-q)^{(M-1)-t}, i = 1, 2, ..., I$. P_i is the conditional probability that a participant faces a competition where the total number of contestants falls within the range Ω_i.

When a contestant participates in the contest and is informed that $N \in \Omega_i$, he has to form a posterior belief of the number of competitors. He will be competing against t contestants with a probability $\Pr(t|\Omega_i) = \frac{C_{M-1}^t q^t (1-q)^{(M-1)-t}}{P_i}, t = k_{i-1}, ..., k_i - 1$. Similar to (1.8), an entrant would exert an effort $x(\Omega_i) = H^{-1}(\sum\limits_{t=k_{i-1}+1}^{k_i}$

$\Pr(t-1|\Omega_i)\frac{1}{t}(1-\frac{1}{t})v)$. This equilibrium effort is obtained from the first order condition. The second order condition that guarantees that it is a global optimum can be established similarly as in Section 2.1.1 if $f(x)$ is concave; and it can be established similarly as in the analysis for concealment of number of entrants as in Section 2.1.2 if $f(x)$ belongs to the convex families of Section 2.1.2. To save space, we do not repeat these proofs.

We can immediately obtain that each participant on average expends an expected effort

$$Ex = \sum_{i=1}^{I} P_i x(\Omega_i).$$

We then conclude the following.

Theorem 2 *Suppose that every contestant independently enters the contest with the same exogenous probability q, and symmetric equilibria exist for contests with disclosure and concealment of the partition sets.*

(a) If the characteristic function $H(\cdot)$ is strictly concave (strictly convex), the contest organizer fully discloses (fully conceals) the actual number of participating contestants, and partial disclosure is never optimal.

*(b) (**Disclosure Irrelevance**) The resultant expected total effort is independent of the disclosure strategy (i.e., how the partitions are constructed), if the characteristic function $H(\cdot)$ is linear, where a Tullock contest with $f(x) = ax^r$ applies.*

Proof. Let us merge two arbitrary neighbor partition sets Ω_j and Ω_{j+1}. After the merger, we denote $\tilde{\Omega} = \Omega_j \cup \Omega_{j+1}$. Define $\tilde{P} = \sum_{t=k_{j-1}}^{k_{j+1}-1} C_{M-1}^t q^t (1-q)^{(M-1)-t} = P_j + P_{j+1}$. Then $\tilde{P}$ is the conditional probability that a participant would face a competition where the total number of contestants falls in $\tilde{\Omega}$. The expected effort of an entrant is given by

$$\tilde{E}x = \tilde{P}x(\tilde{\Omega}) + \sum_{i\neq j,j+1} P_i x(\Omega_i).$$

To compare Ex and $\tilde{E}x$, we only need to compare $\sum_{i=j}^{j+1} P_i x(\Omega_i)$ with $\tilde{P}x(\tilde{\Omega})$. Note that

$$\begin{aligned}
\tilde{P}x(\tilde{\Omega}) &= (P_j + P_{j+1})H^{-1}(\textstyle\sum_{t=k_{j-1}+1}^{k_{j+1}} \dfrac{C_{M-1}^{t-1} q^{t-1}(1-q)^{M-t}}{P_j + P_{j+1}} \dfrac{1}{t}(1-\dfrac{1}{t})v), \\
\text{and } P_j x(\Omega_j) &= P_j H^{-1}(\textstyle\sum_{t=k_{j-1}+1}^{k_j} \dfrac{C_{M-1}^{t-1} q^{t-1}(1-q)^{M-t}}{P_j} \dfrac{1}{t}(1-\dfrac{1}{t})v).
\end{aligned}$$

If the $H^{-1}(\cdot)$ is strictly concave, i.e., the characteristic function $H(\cdot)$ is strictly convex, then

$$\begin{aligned}
&\textstyle\sum_{i=j}^{j+1} P_i x(\Omega_i) \\
= \;& (P_j + P_{j+1}) \textstyle\sum_{i=j}^{j+1} \dfrac{P_i}{P_j + P_{j+1}} x(\Omega_i) \\
\leq \;& (P_j + P_{j+1})H^{-1}[\textstyle\sum_{i=j}^{j+1}(\dfrac{P_i}{P_j + P_{j+1}} \textstyle\sum_{t=k_{i-1}+1}^{k_i} \dfrac{C_{M-1}^{t-1} q^{t-1}(1-q)^{M-t}}{P_i} \dfrac{1}{t}(1-\dfrac{1}{t})v)] \\
= \;& (P_j + P_{j+1})H^{-1}[\textstyle\sum_{t=k_{j-1}+1}^{k_{j+1}} \dfrac{C_{M-1}^{t-1} q^{t-1}(1-q)^{M-t}}{P_j + P_{j+1}} \dfrac{1}{t}(1-\dfrac{1}{t})v)] \\
= \;& \tilde{P}x(\tilde{\Omega}).
\end{aligned}$$

In this case, a coarser partition strategy always leads to more effort. At the optimum, the organizer creates only one partition set ($I = 1$ and $k_1 = M$), i.e., she discloses no information to participating contestants.

When the characteristic function is strictly concave, the comparison is reversed: the finer the partition strategy, the more effort is expended in the contest. The optimum requires full information disclosure, i.e., $I = M$.

When the characteristic function is linear, where a Tullock contest applies and $f(x)$ takes the form $f(x) = ax^r$, merging the two partitions does not affect equilibrium effort.

We then obtain the results of Theorem 2.

Q.E.D. ■

Theorem 2 strengthens the argument of Theorem 1. The results of Theorem 1 are robust even when a partial disclosure strategy is allowed in the game.

It further verifies that the optimal disclosure policy depends crucially on the concavity of the characteristic function. More importantly, partial disclosure never emerges in the equilibrium if the characteristic function is strictly concave or strictly convex.

We again find that the "disclosure irrelevance" principle applies in the case of linear characteristic functions (i.e., Tullock contests). Theorem 2(b) substantially adds to our knowledge of behavior in this type of contest: the equilibrium level of effort expended in the contest does not depend on whether the contest organizer discloses information and how much information is disclosed, despite there being numerous ways to construct a partition disclosure strategy!

3.2 Commitment of Disclosure Policy

We assume that the contest organizer commits to her disclosure policy prior to the realization of the actual number of contestants. We follow the standard literature on mechanism design, such as Myerson (1981), and assume that the contest organizer has commitment power. Lim and Matros (2009) have also studied a case where the organizer is unable to commit, and can decide whether or not to disclose the actual number of participants after the number has been realized. They showed that the contest organizer would be unable to conceal the information, and she always reveals it in equilibrium. The same result would be obtained in the setting studied in this paper, regardless of the contest technology.[4]

It should be noted that the inability to commit could harm the contest organizer, as it has been shown here that concealing the actual number of contestants can elicit more effort, when the characteristic function $H(\cdot)$ is convex. Hence, it would be theoretically interesting and important to explore the mechanisms that strengthen the commitment power of the contest organizer. A thorough analysis on the commitment issue of disclosure policy is beyond the scope of this study,

[4] A detailed proof is omitted for the sake of brevity, but it is available from the authors upon request.

but will be pursued by the authors in future studies. However, two remarks are in order to address this issue.

First, the contest organizer can seek third parties to maintain the credibility of her disclosure policy. One mechanism for this is to resort to obtaining certification from the relevant authorities, such as notaries, to verify the integrity of the committed contest rules. When the characteristic function is convex, it would be incentive-compatible to exercise such a procedure in order to maintain a concealment policy in the contest, provided it does not entail prohibitive certifying costs. Alternatively, the contest organizer may outsource or delegate the administrative task to independent parties, which carry out the rules of the contest on her behalf.

Second, the contest organizer can carry out a concealment policy more credibly when she sponsors the contest not once but repeatedly over time. Insights can be borrowed from the notion of "reputation equilibria", and the extensive literature on reputation building.[5] Reputation concerns create a trade-off between immediate gains and long-run payoffs, and provide the contest organizer with additional incentives to maintain her concealment policy. Although the contest organizer can be tempted to reveal the actual number of contestants when it turns out to be low (which, if revealed, would incentivize each participant to supply more effort) in a single contest, she may refrain from doing so since it prevents her from establishing her reputation, and the loss can outweigh the temporary advantage. Deviation in one period changes the beliefs of the contestants. By a logic analogous to the full-revelation result in single-period contests (see Lim and Matros, 2009), the organizer may have to reveal the information in all future periods. This necessarily leads to less future effort on average.

[5] Reader is referred to Shapiro (1982 and 1983), Fudenberg and Levine (1989), Fudenberg, Kreps and Maskin (1990), and Kreps (1990).

4 Concluding Remarks

The current study examines the impact of disclosure on expected effort in contests with a stochastic number of contestants. Our analysis provides important insights into the design of a contest with a stochastic number of contestants. We showed that whenever the characteristic function $H(x) = \frac{f(x)}{f'(x)}$ is linear (i.e., Tullock contest technology), the expected total effort in a contest does not depend on how much information on the actual number of contestants is revealed to participants. However, this result does not hold when the characteristic function is nonlinear. The comparison is determined by the concavity of the characteristic function.

Chapter Two
Contests with Endogenous and Stochastic Entry

1 Introduction

Economic agents are often involved in contests. They expend costly effort to compete for a limited number of prizes, while their investments are usually nonrefundable whether they win or lose. A wide variety of economic activities exemplify such competitions. They include rent-seeking, lobbying, political campaigns, R&D races, competitive procurement, college admissions, ascents of organizational hierarchies, and movement in internal labor markets. The vast wealth of literature on contests has delineated economic agents' strategic behaviors in contests from diverse perspectives, and has identified the various institutional elements in contest design that affect bidding incentives.

Most existing studies focus on a setting where a fixed number (n) of bidders participate. These studies, under the "fixed-n paradigm", typically abstract away from the *ex ante* contest participation decisions of bidders and focus on their post-entry activities, assuming that the actual number of active participants is commonly known. In this paper, we complement these studies by explicitly examining a setting where bidders have to make a strategic decision about participating in a contest. They enter contests randomly, so the actual number of participants in a particular contest is uncertain. Participants take into account this uncertainty when placing their bids.[1]

As noted by Konrad (2009), a bidder often bears a nontrivial (fixed) entry cost, which can be explicitly sunk resources or foregone outside opportunities. Incurring the costs allows a bidder to merely participate and is unrelated to their chances of winning.[2] In our setting, a fixed pool of potential bidders decide whether to participate and then sink their bids after entering the contest. Each bidder weighs his expected payoff in future competitions against the entry cost, and participates if and only if the former (at least) offsets the latter. With nontrivial entry costs, we show that a symmetric mixed-strategy equilibrium emerges: each potential bidder enters with the same probability, and adopts the same (possibly mixed) bidding strategy upon entry.

This entry-bidding game complements and enriches the existing literature in several aspects. We elaborate upon its distinct flavors as follows.

First, the strategy of each potential bidder involves two elements in a contest with endogenous entry: (1) whether to enter; and (2) how to bid after entering. This entry-bidding game exemplifies a discontinuous game with two-dimensional actions (Dasgupta and Maskin, 1986). The game distinguishes itself from standard contests that are typically identified as uni-dimensional discontinuous games (Baye, Kovenock and de Vries 1994 and Alcalde and Dahm, 2010), where a player's strategy involves only his bidding action.[3] Due to stochastic entry, the conventional approach to establish equilibrium existence in contests (Baye, Kovenock and de Vries 1994 and Alcalde and Dahm, 2010) does not en-

[1] We show later that the optimal contest in general entails nondisclosure of the actual number of contestants.

[2] An analogy is that while an air ticket enables the American tennis player Venus Williams to arrive at the Australian Open, it does not help her win the championship. Similarly, to participate in a R&D tournament, a research company may need to acquire some necessary laboratory equipment to gather project-specific information, or to turn down other profitable tasks, while its chances of winning depend on its subsequent creative input.

[3] The literature on contests recognizes that (Baye, Kovenock and de Vries 1994, and Alcalde and Dahm, 2010), a well-defined contest success function (e.g., Tullock contest) can be discontinuous at its origin, i.e., when all bidders bid zero.

compass our settings where the number of active players is uncertain.[4] This novel setting entails the application of Dasgupta and Maskin's (1986) general theorem on multi-dimensional discontinuous games, which allows us to establish the existence of a symmetric mixed-strategy equilibrium in the entry-bidding game.[5] To our knowledge, our analysis provides the first application of the existence theorem for multi-dimensional settings in the contest literature.

Second, the bidding behavior in contests with stochastic participation has yet to be explored thoroughly. It is well-known in the literature that a bidder's payoff maximization problem becomes irregular when the contest success function is excessively elastic to effort, e.g. when the discriminatory parameter r in a Tullock contest exceeds certain boundaries. Stochastic entries further complicate the analysis. By taking into account the uncertainty caused by the stochastic entries, a participant chooses his bid to maximize his expected payoff, which amounts to a weighted sum of a series of irregular functions of his bid. Furthermore, the weights of the summation are determined by the endogenously formed entry probabilities. The general property of a bidder's overall expected payoff function cannot be readily discerned. We establish sufficient conditions under which participating bidders do (or do not) randomize their bids upon entry. This result allows us to derive an equilibrium bidding strategy in this game and further analyse the design of contests.

Third, endogenous entry yields rich implications for contest design. We fol-

[4] To solve for the entry-bidding equilibrium, the traditional approach in the auction literature proceeds in two steps. First, the existence of symmetric bidding equilibrium is shown for each given (symmetric) entry probability and solved for the bidders' equilibrium payoffs. Second, a break-even condition characterizes the equilibrium entry. This approach is inappropriate in our setting. The first step (finding the bidding equilibrium when potential bidders enter with fixed probabilities) is solvable in an auction setting, but not when the parameter r for the Tullock contest is big, as in our case. As such, existing results on the existence of equilibria in contests does not apply to contests with random entry and an uncertain number of active players. More detail is provided in Section 3.

[5] One should note that our two-dimensional strategy space of (entry, effort) cannot be reduced to a setting with single dimensional strategy of effort with a positive fixed cost. In our two dimensional setting, if no one enters the contest, no one wins. If everyone enters but exerts zero effort, every one incurs an entry cost and has an equal chance at winning. In the single dimensional setting, if everyone exerts zero effort, no one incurs any costs but has an equal chance at winning.

low the mainstream literature by searching for mechanisms that maximize the expected overall bid in a contest, and examine three issues: (1) whether the contest designer prefers a more precise winner selection mechanism; (2) whether the contest designer should exclude potential bidders, and invite only a subset of them to participate in the competition; and (3) whether the contest designer could improve the contest's design by disclosing the actual number of participating bidders when she can observe it.

- **Precision Could Hurt:** We focus on Tullock contests and regard the discriminatory parameter r as a measure of the level of noise in the winner selection mechanism. A greater r implies that a higher bid can be more effectively translated into a higher likelihood of winning, thereby increasing the marginal return to the bid. Conventional wisdom informs us that a greater r provides higher-powered incentives and intensifies competition. We demonstrate, nevertheless, that in our setting the expected overall bid does not vary monotonically with the size of r. A contest with a smaller r can paradoxically elicit more effort. An immediate trade-off is triggered when r is raised. A more precise contest incentivizes each participant to bid more, while an overheated competition leaves lesser rent for participants, thereby discouraging entries. Moreover, contestants' entry probabilities affect the expected overall bids ambiguously. More active entry expands the contest and tends to amplify the overall supply of bids; while it also leads individual participants to bid more prudently, as they anticipate more potential competitors and a smaller chance of winning. The optimum has to balance out these diverse and possibly conflicting forces.

- **Exclusion Helps:** Based on results on optimal precision,[6] we investigate whether the contest designer is better off when there is a larger pool of potential bidders. Without endogenous entry, the contest literature states

[6] Optimal r is contingent on number of potential contestants.

that the overall bid always increases with the number of bidders. However, our analysis reveals the opposite: contests elicit lesser effort, when a larger pool of potential bidders may enter. Contest designers prefer to limit competition by inviting only a subset of them for participation. The existing studies on shortlisting and exclusion, e.g. those of Baye, Kovenock and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003), usually focus on heterogeneous contestants, and concern themselves with selecting (usually two) players of the "right types." Our result, however, espouses the merit of exclusion in a setting of homogenous players and concerns itself with creating a contest of the "right size."

- **Opaqueness May Pay Off:** We establish that there is no loss of generality when considering the optimal design of contests that do not disclose the actual number of participants. It is in general suboptimal for the contest designer to announce the actual number of participants when she can observe it.

The rest of the paper proceeds as follows. In Section 2, we discuss the relation of our paper to the relevant literature in the rest of this section. In section 3, we set up the model, and establish our main results on equilibrium existence. Optimal contest design is explored in Section 4, and Section 5 concludes the paper.

2 Relation to Literature

Our paper complements the literature on contests and auctions in various aspects.[7] We next discuss the links to these two strands of literature respectively.

[7] Our paper can also be related to the literature on standard oligopolistic competition. Our paper echoes the argument of Dixit and Shapiro (1986) and Shapiro (1989) on firms' behavior in oligopolistic markets. He shows that Bertrand competition, which is fiercer, can be more anti-competitive *ex post* than Cournot competition, which is *ex ante* more subdued, as the latter limits the contestability of the market and discourages entries. We focus on the issue of mechanism design in our particular context. In addition, the level of post-entry competition is

2.1 Contests

Our paper provides a comprehensive and formal account of equilibrium existence in the entry-bidding game. Our paper primarily belongs to the literature on equilibrium existence in contests. Szidarovszky and Okuguchi (1997) establish the existence of pure-strategy equilibria when contestants have concave production functions. The existence and properties of the equilibria remain a nagging problem for contests with less well-behaved technologies. Baye, Kovenock and de Vries (1994) establish the existence of mixed-strategy equilibria in two-player Tullock contests with $r \geq 2$. Alcalde and Dahm (2010) further the literature by showing that all-pay auction equilibria exist under a wide class of contest success functions.[8] Both studies apply the results of Dasgupta and Maskin (1986) on uni-dimensional discontinuous games. Our paper contributes to this literature by introducing bidders' entry decisions while allowing the number of active bidders to be stochastic. These new flavours enrich our analysis by forming a two-dimensional discontinuous game, and provide a novel application of the general result of Dasgupta and Maskin (1986) on multi-dimensional discontinuous games in the contest literature.

The literature on contests with endogenous entry remains scarce. Higgins, Shughart, and Tollison (1985) in their pioneering work study a tournament model in which each rent seeker bears a fixed entry cost, and randomly participates in equilibrium. In an all-pay auction model, Kaplan and Sela (2010) provide a rationale for entry fees in contests. Besides the differing modeling choice and the diverging focus, Kaplan and Sela (2010) differ from the current paper in a few other aspects. First, they allow players to bear privately-known entry costs, while we assume that entry cost is uniform and commonly known. Second, they let participants know who else has entered, while we focus mainly on uninformed participants. However, we also study the ramifications of disclosure policy as an

a continuous variable and is considered as a strategic choice of the contest designer.

[8] Wang (2010) also characterizes the equilibria in two-player asymmetric Tullock contests when r is large.

institutional element of contests.

Two recent experimental studies, Cason, Masters and Sheremeta (2010) and Morgan, Orzen and Sefton (2010), also contribute to this research agenda by studying bidders' entries. Similar to Morgan, Orzen and Sefton's (2010) theoretical model, Fu and Lu (2010) also assume that potential bidders enter sequentially, so neither setting involves stochastic participation.

A handful of papers have examined contests with stochastic participation. The majority of these studies, however, assume exogenous entry patterns. Myerson and Wärneryd (2006) examine a contest with an infinite number of potential entrants, whose entry follows a Poisson process. Münster (2006), Lim and Matros (2009) and Fu, Jiao and Lu (2011) assume a finite pool of potential contestants, with each contestant entering the contest with a fixed and independent probability.

The current study also contributes to the growing literature on contest design by exploring the optimal mechanism in a context with endogenous and stochastic entries.

First, our analysis complements the literature on the proper level of precision in evaluating bidding performance. Conventional wisdom says that a precise contest incentivizes aggressive bidding. A handful of studies, however, espouse low-powered incentives in contests and demonstrate that a less "discriminatory" contest can improve efficiency. One salient example is provided by Lazear (1989), who argues that excessive competition leads to sabotage. A more popular stream in the literature instead stresses the "handicapping" effect of the imprecise performance evaluation mechanism in (two-player) asymmetric contests. When contestants differ in their abilities, a noisier contest balances the playfield. This effect encourages weaker contestants to bid more intensely, and deters the stronger ones from shirking. O'Keeffe, Viscusi and Zeckhauser (1984) are among the first to formalize this logic. This rationale is further elaborated upon by Che and Gale (1997, 2000), Fang (2002), Nti (2004), Amegashie (2009), and Wang (2010). In a recent study, Epstein, Mealem and Nitzan (2011) contend that contest designers still prefer all-pay auctions to Tullock contests if they can strategically discriminate between bidders. In contrast to these studies, our paper adopts a N-player symmetric contest, and stresses the trade-off between *ex post* bidding incentives and *ex ante* entry incentives. Our paper is closely related to Cason, Masters and Sheremeta's (2010) experimental study in this aspect, which compares endogenous entries in all-pay auctions and lottery contests.

Our finding on efficient exclusion echoes a handful of pioneering studies by Baye, Kovenock and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003). These studies typically involve heterogeneous bidders and identify the subset of bidders with the most desirable characteristics. Dasgupta (1990) studies a two-stage procurement tournament. Bidders invest in cost reduction in the first stage, and place their bids in the second. Wider competition may diminish bidders' incentives to engage in R&D. Limiting the number of competing firms may or may not benefit the principal. None of these studies involves entry cost and endogenous entry. In contrast to these studies, an invited (potential) bidder in our setting has to decide whether to enter the subsequent contest, and the entry pattern in the equilibrium remains endogenous and stochastic.

Our study is also related to the literature on efficient disclosure and feedback rules in contests.[9] Lim and Matros (2009) are the first to examine the issue of disclosing the number of contestants, where potential bidders enter with an exogenous probability. They demonstrate the independence of prevailing policy in Tullock contests with $r = 1$ and linear effort costs. Fu, Jiao and Lu (2011) further reveal that the optimal disclosure policy depends on the characteristics of the production functions of contestants. The current paper illustrates the critical role played by the convexity of the bidding cost function and endogeneity of entry.

[9] Aoyagi (2010), Gershkov and Perry (2009), Ederer (2010), Gürtler and Harbring (2010), and Goltsman and Mukherjee (2011) focus on interim performance feedback in dynamic contests. In contrast, our paper looks at interim feedback on entries.

2.2 Auctions with Stochastic Entry

Our paper is also related to the literature on auctions with endogenous entry. Myerson (1981) shows that a second-price or first-price auction with an optimal reserve price is revenue-maximizing when bidders bear zero entry costs. Samuelson (1985), Menezes and Monteiro (2000) and Lu (2009) require that bidders sink entry costs to participate in auctions. Levin and Smith (1994), Shi (2009), Lu (2010) and Moreno and Wooders (2010) allow bidders to make costly investments to learn their valuations of the object for sale. These studies conclude that revenue maximization requires weaker incentives, i.e. lower reservation prices, than that in Myerson (1981), due to the trade-off between the *ex post* incentive to bid and the *ex ante* incentives of entry or information acquisition.

Our study departs subtly from the auction literature in two main aspects. First, the auction design problem addresses an adverse-selection problem: bidders possess private information about their own types and therefore the optimal mechanism screens heterogeneous bidders. Our contest design problem nevertheless concerns itself primarily with a moral hazard problem: the type of player is commonly known, while the optimal mechanism sets out to incentivize effort supply. Second, the auction literature shows that a weaker *ex ante* incentive, i.e. a reserve price lower than Myerson's (1981) zero-entry-cost benchmark, is always necessary whenever entry or information acquisition is costly. By way of contrast, the optimum in our setting could involve either a weaker (i.e. a smaller precision r) or a stronger (i.e. a bigger precision r) *ex ante* incentive than that for the zero-entry-cost benchmark.

Shortlisting and exclusion have long been recognized as an important element in designing auctions with costly entry. Our setting resembles that of Sameulson (1985) and Lu (2009), as both studies assume that bidders bear common entry costs, although the results differ. While Lu (2009) finds that shortlisting is not necessarily optimal, we find that contest designers can always elicit higher overall bids by excluding potential bidders. Levin and Smith (1994) let potential bidders make costly investments to discover their valuations of the object. They establish that the revenue in the optimum decreases with the number of potential bidders to the extent that the information acquisition costs lead to a mixed-strategy entry. Our finding echoes that of Levin and Smith (1994), despite the different settings.

The optimal disclosure policy has also been examined in auctions with a stochastic number of bidders. McAfee and McMillan (1987) and Levin and Ozdenoren (2004) consider exogenous stochastic entry and show that the expected revenue is independent of the disclosure policy when bidders are risk neutral. Our paper allows for endogenous entry and concludes that concealment may elicit strictly higher overall bid under various circumstances.

3 Model and Analysis

In this section, we first set up the model and then conduct the equilibrium analysis.

3.1 Setup

We consider a two-stage game. A fixed pool of $M(\geq 2)$ identical risk-neutral potential bidders demonstrate interest in a contest with a winner's purse $v > 0$. In the first stage, potential bidders simultaneously decide whether or not to participate. In the second stage, all participants simultaneously submit their bids. A winner is selected and awarded the prize.

3.1.1 Winner Selection Mechanism

Suppose that $N \geq 2$ potential bidders enter the contest. They simultaneously submit their bids $x_i, i = 1, 2, ..., N$, to compete for the prize v. The probability of a participating bidder i winning the prize is given by

$$p_N(x_i, \mathbf{x}_{-i}) = \frac{x_i^r}{\sum_{j=1}^N x_j^r}, \text{ if } N \geq 2, \text{ and } \sum_{j=1}^N x_j^r > 0, \tag{2.1}$$

which follows the setup of widely adopted Tullock contest success function. If all participants submit zero bids, the winner is randomly picked from the participants. To the extent that only one bidder enters, he receives the prize v automatically, regardless of his bid. In the event that nobody enters, the designer keeps the prize.

A bid x_i costs a bidder $c(x_i)$, with $c'(\cdot) > 0$ and $c''(\cdot) \geq 0$. For the sake of tractability, we assume that the bidding cost function takes the form $c(x_i) = x_i^\alpha$, with $\alpha \geq 1$.

It should be noted that our main theorem on equilibrium existence in the entry-bidding game applies to contests with more general success functions and cost functions, which will be discussed in more detail later in the paper.

3.1.2 Entry

In the first stage of the game, potential bidders simultaneously decide whether to participate in the contest. Each participant has to sink a fixed cost $\Delta > 0$ if he enters. Entry is irreversible, and the cost Δ cannot be recovered. We impose the following regularity condition on the model.

Assumption 1 $\frac{v}{M} < \Delta < v$.

The assumption requires that the entry cost Δ is nontrivial but not prohibitively high. First, no entry is triggered if it costs more than the winner's purse. Second, the analysis becomes relatively trivial when entry involves little cost, in which case the institutional elements of the contest do not affect bidders' entry incentives significantly. Under Assumption 1, no equilibria exist where all potential bidders participate in the contest with certainty.

In our main analysis, we assume that each participating bidder does not know the actual number N of participants. This setting leads to a two-dimensional discontinuous game and demands a more sophisticated analysis. Two remarks are in order. First, entry often involves hidden actions, which cannot be readily observed or verified by other parties. Second, one may view the public observability of N as an institutional element, which is to be chosen strategically by the contest designer. In Section 4.3, we assume that the contest designer is able to observe N and choose the disclosure policy of the contest. We show that a contest would in general elicit lesser bids when N is to be disclosed.

3.1.3 Some Preliminaries

Before the formal analysis is carried out, we define two cutoff probabilities, which are used repeatedly throughout the analysis.

Definition 1 *Let $\bar{q} \in (0,1)$ be the unique solution to $(1-(1-q)^M)v - Mq\Delta = 0$, and $q_0 \in (0,1)$ be the unique solution to $(1-q)^{M-1}v - \Delta = 0$.*

Comparing the two cutoffs leads to the following.

Lemma 1 *$q_0 < \bar{q}$.*

Proof. See Appendix B. ■

Let us discuss the implications of the two cutoffs briefly, although their implications unfold as the analysis proceeds. The entry-bidding game cannot trigger an equilibrium, where all potential bidders enter with a probability more than $\bar{q}$: they would otherwise end up with negative expected payoff in the game. In contrast, the cutoff q_0 defines a lower bound. If there is an equilibrium where all potential bidders enter with a probability less than q_0, participating bidders must randomize their bids upon entry.

3.2 Existence of Symmetric Equilibrium

A bidder i's behavioral strategy is an ordered pair $(q_i, \mu_i(x_i))$, where q_i is the probability he enters the contest, and x_i is his bid submitted upon entry. We allow him to randomize on his bids. The probability distribution $\mu_i(x_i)$ depicts his behavioral bidding strategy conditional on his entry. It reduces to a singleton when the participant does not randomize his bid.

Assumption 1 implies that potential bidders play a mixed-strategy in the entry stage. Each participant is uncertain about the actual level of competition when placing his bid. He bids based on his rational belief about others' entry patterns. The solution concept of a subgame perfect equilibrium would not apply, because participants possess only imperfect information and no proper subgame exists after the entry stage. We simply use the concept of Nash equilibrium to solve the game. An equilibrium is a strategy combination $\times_{i=1}^{M}(q_i, \mu_i(x_i))$ of all contestants, which requires that the pair strategy $(q_i, \mu_i(x_i))$ of each potential bidder i maximize his expected payoff based on his rational belief and others' strategy profile $\times_{j\neq i}(q_j, \mu_j(x_j)$.

We focus on the symmetric equilibrium of the game where all potential bidders play the same strategy $(q^*, \mu^*(x))$. As aforementioned, a potential bidder's payoff can be discontinuous as the contest success function is discontinuous at origin (see Baye, Kovenock and de Vries, 1994, and Alcalde and Dahm, 2010), i.e. when all participants bid zero. The strategy of each player involves two elements. A conventional approach (in auction literature) to establishing the existence of symmetric equilibria proceeds with two steps, which disentangles the two elements in each player's strategy and simplifies the analysis. In the first step, for each given (symmetric) entry probability, one shows the existence of symmetric bidding equilibrium and solves for the bidders' equilibrium payoffs. In the second step, a break-even condition characterizes the equilibrium entry probability. This "disentangling" approach loses its bite in our setting.

First, Dasgupta and Maskin's (1986) theorem on uni-dimensional games cannot be directly applied to games with an uncertain number of players. The existence of a symmetric bidding equilibrium under a given entry probability q has yet to be established using alternative approaches. Second, similar to contests with deterministic participation, the bidding game may not be directly solvable when the contest success function is excessively elastic, e.g. when the discriminatory parameter r of a Tullock contest is excessively large. As a result, even if an equilibrium exists, it remains difficult to characterize the properties (e.g. continuity and monotonicity) of bidders' expected payoffs.

This game, however, can be viewed as a two-dimensional discontinuous game (Dasgupta and Maskin, 1986). We apply the general result of Dasgupta and Maskin (1986) for a multi-dimensional strategy space to establish the existence of symmetric equilibria.

Theorem 1 *(a) For any $r > 0$, a symmetric equilibrium $(q^*, \mu^*(x))$ exists. In the equilibrium, each potential bidder enters with a probability $q^* \in (0, \bar{q})$ and his bid follows a probability distribution $\mu^*(x)$. (b) Each potential bidder receives an expected payoff of zero in the entry-bidding equilibrium.*

Proof. See Appendix B. ■

To our knowledge, Theorem 1 and its proof provide the first application of Dasgupta and Maskin's (1986) equilibrium existence result on two-dimensional discontinuous games in the literature on contests. A few remarks are in order. First, the equilibrium existence result applies to broader contexts. We explicitly adopt Tullock technologies to economize on our presentation and facilitate subsequent discussion on contest design. However, the proof of the theorem does not rely on the specific properties of Tullock success functions and the particular form of bidding cost functions. The analysis can be readily adapted to contests with more broadly defined success functions, such as those in Alcalde and Dahm (2010), by redefining the discontinuity set slightly. Second, our analysis has yet to provide a more comprehensive account of equilibrium bidding behaviors, which remains one of the central concerns in contest literature. In this entry-bidding game, a participating bidder may randomize on his bid x_i in the equilibrium. We establish the relevant conditions for pure or mixed bidding strategies subsequently.

Before we proceed, it should be noted that multiple entry equilibria exist in the game. With nontrivial entry cost ($\Delta > \frac{v}{M}$ by Assumption 1), there always exist asymmetric entry equilibria, where a subset of potential bidders stay inactive regardless, while the others enter either randomly or deterministically. Throughout this paper, we focus on symmetric entry equilibria for two reasons. First, symmetric equilibria can be arguably viewed as a natural focal point. Second, many asymmetric equilibria that involve only a subset of $M'(< M)$ active players essentially can be analyzed through the symmetric equilibria in a smaller entry-bidding game with a total of $M'(< M)$ potential bidders.

3.3 Existence of Equilibrium with Pure-Strategy Bidding

Suppose that a symmetric equilibrium with pure-strategy bidding exists. Consider an arbitrary potential bidder i who has entered the contest. Suppose that all other potential bidders play a strategy (q, x) with $x > 0$.[10] He chooses his bid x_i to maximize his expected payoff

$$\pi_i(x_i|\, q, x) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}[\frac{x_i^r}{x_i^r + (N-1)x^r} v - x_i^\alpha]. \tag{2.2}$$

Evaluating $\pi_i(x_i|\, q, x)$ with respect to x_i yields

$$\frac{d\pi_i(x_i|\, q, x)}{dx_i} = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{(N-1)r x_i^{r-1} x^r v}{[x_i^r + (N-1)x^r]^2} - \alpha x_i^{\alpha-1}. \tag{2.3}$$

[10] It is impossible to have all participating bidders bid zero deterministically in an equilibrium. When all others bid zero, a participating bidder would prefer to place an infinitely small positive bid, which allows him to win the prize with probability one.

The (pure) bidding strategy in such an equilibrium, if it exists, can be solved for by the first order condition $\frac{d\pi_i(x_i)}{dx_i}|_{x_i=x} = 0$. The following lemma fully characterizes such an equilibrium if it exists.

Lemma 2 *Suppose that a symmetric equilibrium (q^*, x^*) with pure-strategy bidding exists. In such an equilibrium, entry probability q^*must satisfy*

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{v}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}) = \Delta. \tag{2.4}$$

Each participating bidder places a bid

$$x^* = [\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{N-1}{N^2}\frac{rv}{\alpha}]^{\frac{1}{\alpha}}. \tag{2.5}$$

The expected overall bid of the contest obtains as

$$x_T^* = Mq^*x^* = Mq^*[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{N-1}{N^2}\frac{rv}{\alpha}]^{\frac{1}{\alpha}}. \tag{2.6}$$

Proof. See Appendix B. ■

Lemma 2 depicts the main properties of a symmetric equilibrium with pure-strategy bidding, if it exists. We call equation (2.4) the *break-even condition* of the entry-bidding game with pure-strategy bidding. It determines the entry probability q^* in such an equilibrium. The break-even condition leads to the following.

Lemma 3 *(a) For any $r > 0$, there exists a unique $q^* \in (0, \overline{q})$ that satisfies the break-even condition (2.4). Hence, x^* is also uniquely determined for the given r by the break-even condition (2.4).*

(b) The probability q^* strictly decreases with r.

Proof. See Appendix B. ■

Lemma 3 establishes a unique correspondence between r and (q^*, x^*). The symmetric equilibrium with pure bidding strategy must be unique for each given

r, whenever it exists. However, the strategy profile (q^*, x^*) of Lemma 2 may not constitute an equilibrium.

Consider an arbitrary participating bidder's payoff maximization problem. Suppose that all other bidders enter the contest with a probability q and bid x upon entry. The participating bidder i chooses his bid x_i to maximize his expected payoff $\pi_i(x_i|\, q, x)$ in the contest, which is the weighted sum of $\pi_i^N(x_i|\, q, x) = \frac{x_i^r}{x_i^r+(N-1)x^r}v - x_i^\alpha$ over all possible N, i.e. $\pi_i(x_i|\, q, x) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\pi_i^N(x_i|\, q, x)$. Note that each individual component $\pi_i^N(x_i|\, q, x) = \frac{x_i^r v}{x_i^r+(N-1)x^r}$ $-x_i^\alpha$ is simply his expected payoff when he enters a contest in which he competes against $N-1$ other bidders deterministically and each of them bids x. We graphically illustrate the relation between $\pi_i^N(x_i|\, q, x)$ and $\pi_i(x_i|\, q, x)$ in Figure 2.1.

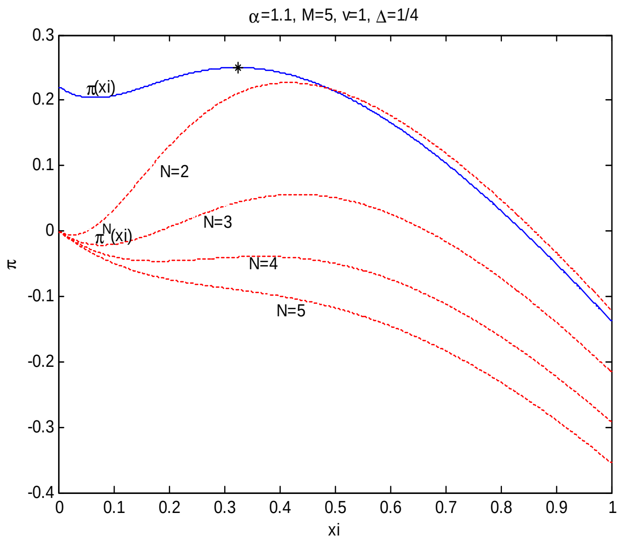

Figure 2.1: The relation between $\pi_i^N(x_i|\, q, x)$ and $\pi_i(x_i|\, q, x)$

The equilibrium analysis is trivial when $r \leq 1$. In that case, each component $\pi_i^N(x_i|\, q, x)$ is concave. Maximizing $\pi_i(x_i|\, q, x)$ is thus a well-behaved concave program. In this case, the hypothetical equilibrium bid x^*, which is determined by the first order condition and the symmetry condition, must maximize $\pi_i(x_i|\, q^*, x^*)$. A strategy profile with all playing (q^*, x^*) must constitute the unique symmetric equilibrium.

When r is large, however, the function $\pi_i^N(x_i|\, q, x)$ is no longer globally concave. It is well-known in the contest literature that maximizing $\pi_i^N(x_i|\, q, x)$ is not a regular program. The irregularity is exacerbated tremendously in our context. First, it is difficult to draw a general conclusion on the properties of the payoff function $\pi_i(x_i|\, q, x)$, which is the weighted sum of a series of not necessarily concave functions. Second, the weights $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}$ literally depend on the entry probability q, which, however, is determined endogenously in the equilibrium. The existing results on equilibrium existence obtained from contests with deterministic participation cannot be carried over.

In subsequent analysis, we derive the upper (lower) bound of r which guarantees the existence (non-existence) of a symmetric equilibrium with pure-strategy bidding. Recall the unique correspondence between r and (q^*, x^*) (Lemma 3). Consider a contest with a given r. Define

$$\tilde{\pi}_i(x_i) = \pi_i(x_i|q^*, x^*) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\left(\frac{x_i^r}{x_i^r + (N-1)x^{*r}} v\right) - x_i^{\alpha}, \tag{2.7}$$

which is a participating bidder i's expected payoff in the contest when all other bidders play the strategy (q^*, x^*), as given by Lemma 2. Clearly, $\tilde{\pi}_i(x_i)$ is continuous on $[0, \infty)$. The strategy profile (q^*, x^*) constitute an equilibrium if and only if x^* is a global maximizer of $\tilde{\pi}_i(x_i)$.

The next result depicts an important property of the payoff function.

Lemma 4 *When $r \in (1, \alpha(1 + \frac{1}{M-2})]$, x^* is the unique inner local maximizer of $\tilde{\pi}_i(x_i)$ over $(0, \infty)$, i.e. $\tilde{\pi}_i(x^*) > \tilde{\pi}_i(x), \forall x \in (0, \infty)$ and $x \neq x^*$. There exists a unique $x_m \in (0, x^*)$ such that $\tilde{\pi}_i(x_i)$ decreases on $[0, x_m]$, increases on $[x_m, x^*]$, and then decreases on $[x^*, \infty)$.*

Proof. See Appendix B. ■

Lemma 4 depicts the property of $\tilde{\pi}_i(x_i)$ when r remains in the range $(1, \alpha(1+\frac{1}{M-2})]$. We define $\alpha(1 + \frac{1}{M-2})$ as $+\infty$ when $M = 2$. Although it is no longer globally concave, the function still demonstrates the regularity as depicted by Lemma 4. For $x \in (0, \infty)$, the function is locally minimized at x_m and then maximized at $x^* \in (x_m, \infty)$. Hence, x^* is its unique maximizer for $x \in (0, \infty)$.

However, x^* has yet to be established as the global maximizer: the equilibrium requires that the boundary condition $\tilde{\pi}_i(x^*) \geq \tilde{\pi}_i(0)$ hold. Recall that x^* is uniquely determined by (2.5) for each given r. A participating bidder's expected payoff in the contest when bidding x^*, i.e. $\tilde{\pi}_i(x^*)$, would amount to exactly Δ. However, the bidder automatically receives a reserve payoff $(1-q^*)^{M-1}v$ from the contest by bidding zero: with a probability $(1-q^*)^{M-1}$, all other potential bidders stay out of the contest, and a rent of $(1-q^*)^{M-1}v$ will accrue to him automatically. Hence, the bidder has an incentive to bid x^* only if $(1-q^*)^{M-1}v \leq \Delta$. The implication of this condition is straightforward: bidding $x^*(>0)$ is rational only if it generates nonnegative additional return (when all others bid x^*) in excess of the reservation payoff from bidding zero. The condition essentially requires that r be bounded from above: the contest must leave sufficient rent to contenders and make sure that each bidder is sufficiently rewarded by bidding x^*.

Recall the cutoff $q_0 \in (0, \overline{q})$ depicted by Definition 1, which uniquely satisfies $(1-q_0)^{M-1}v = \Delta$. The unique correspondence between r and (q^*, x^*), as determined by the break-even condition (2.4), allows us to obtain the following cutoff of r.

Definition 2 *Define $r_0 \in (\alpha(1+\frac{1}{M-1}), 2\alpha]$ to be the unique solution to*

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q_0^{N-1}(1-q_0)^{M-N}\frac{v}{N}(1-\frac{N-1}{N}\frac{r_0}{\alpha}) = \Delta$$

By Lemma 3(b), q^* is inversely related to r. The condition $(1-q^*)^{M-1}v \leq \Delta$ requires $q^* \geq q_0$, which would hold if and only if $r \leq r_0$. We then conclude the following.

Theorem 2 *A symmetric equilibrium with pure-strategy bidding does not exist* *if* $r > r_0$.

Similar to contests with deterministic participation, pure-strategy bidding cannot be sustained when r is excessively large. Theorem 2 provides a sufficient condition under which randomized bidding must occur under endogenous entry. When $r > r_0$, the strategy profile (q^*, x^*) would not constitute an equilibrium. In that case, a bidder, when bidding x^*, receives $\tilde{\pi}_i(x^*) = \Delta$. He would strictly prefer to bid zero, because his expected payoff when bidding zero, $(1 - q^*)^{M-1}v$, must be strictly more than Δ. In other words, x^* is not a part of the best response of player i to (q^*, x^*). The symmetric equilibria must involve randomized bidding.

By Lemma 4 and Definition 2, we define the following cutoff of r.

Definition 3 *Define* $\overline{r} \triangleq \min(r_0, \alpha\frac{M-1}{M-2})$.

The previous analysis leads to the following.

Theorem 3 *For each* $r \in (0, \overline{r}]$, *the strategy profile* (q^*, x^*), *as characterized by Lemma 2, constitutes the unique symmetric equilibrium with pure-strategy bidding during the entry-bidding game.*

When r is bounded from above by both r_0 and $\alpha\frac{M-1}{M-2}$, a unique symmetric equilibrium with pure-strategy bidding emerges. The condition $r \in (0, \overline{r}]$ guarantees: (1) that the payoff function $\tilde{\pi}_i(x_i)$ is well-behaved, in the sense that its curve reaches a unique peak at x^* for $x \in (0, \infty)$; and (2) that the boundary condition $\tilde{\pi}_i(x^*) \geq \tilde{\pi}_i(0)$ is met. We then conclude that x^* is the global maximizer when r is subject to both upper bounds. The strategy profile (q^*, x^*) is established as the unique symmetric equilibrium with pure-strategy bidding accordingly.

Theorem 3 imposes a (conservative) upper limit on r for the existence of such an equilibrium. It should be noted that $r \leq \alpha(1 + \frac{1}{M-2})$ is sufficient but not necessary to establish x^* as the local maximizer of $\tilde{\pi}_i(x_i)$ for $x_i > 0$. Analytical difficulty prevents us from fully characterizing the property of $\tilde{\pi}_i(x_i)$ when r

exceeds $\alpha(1+\frac{1}{M-2})$. It remains less than explicit how the equilibrium would behave if $\alpha(1+\frac{1}{M-2}) < r_0$ and $r \in (\alpha(1+\frac{1}{M-2}), r_0]$. More definitive conclusions can be drawn in more specific contexts with small numbers of potential bidders.

Corollary 1 *When M is small, i.e. $M = 2, 3$, a symmetric equilibrium with pure-strategy bidding exists if and only if $r \leq r_0$.*

In these instances, $(\alpha(1+\frac{1}{M-2}), r_0]$ is empty, because $\alpha(1+\frac{1}{M-2}) > r_0$ regardless of v and Δ. Whenever r falls below r_0, it automatically satisfies the condition $r \leq \alpha(1+\frac{1}{M-2})$, which guarantees that x^* maximizes $\tilde{\pi}_i(x_i)$.

However, technical difficulty prevents us from drawing more general conclusions analytically when M is larger, which may lead r_0 to exceed $\alpha(1+\frac{1}{M-2})$. We resort to a numerical exercise to obtain further insights about the properties of the expected payoff function $\tilde{\pi}_i(x_i)$ when $r \in (\alpha(1+\frac{1}{M-2}), r_0]$. For expositional efficiency, we postpone the presentation and discussion of these observations to Section 4.1.2 as they shed light on the optimal contest design problem explored in that section.

4 Contest Design

The equilibrium behavior of the bidders may depend critically on the institutional elements of the contest. Central to the contest literature is the question of how the contest rules affect equilibrium bidding. As Gradstein and Konrad (1999) argued, "... contest structures result from the careful consideration of a variety of objectives, one of which is to maximize the effort of contenders." Based on the equilibrium analysis, we follow this literature to discuss the optimal design of a contest that maximizes the overall bid. Specifically, we consider three main issues: (1) the optimal level of accuracy of the winner selection mechanism (the proper size of r in Tullock contests); (2) the efficiency implications of shortlisting and exclusion; and (3) the optimal disclosure policy.

4.1 Optimal Accuracy: Choice of r

In a Tullock contest, the parameter r reflects the "discriminatory power" or the level of precision of the winner selection mechanism in the contest. With a higher r, a bidder's win depends more on the level of his bid, rather than other noisy or random factors. The level of precision in a contest is largely subject to the autonomous choice of the contest designer. For instance, the designer can modify the judging criteria of the contest to suit her strategic goals, e.g. adjusting the weights of subjective component in contenders' overall ratings. Alternatively, she can vary the composition of judging committees (experts vs. non-experts).

Following the literature (e.g. Nti, 2004), we let r be chosen strategically by the contest designer. We then consider a three-stage game. The designer chooses r and publicly announces it in the beginning. Next, the entry-bidding game takes place. In the subsequent analysis, we investigate how the size of r affects the equilibrium bids.

Before we proceed, we consider the benchmark case of a contest with a fixed number M of participants. A larger r increases the marginal return of a bid and further incentivizes bidders. It is well known in the contest literature that both individual bids and overall bids strictly increase with r whenever a pure-strategy equilibrium exists, i.e. $r \in [0, \alpha(1 + \frac{1}{M-1})]$. This conventional wisdom, however, loses its bite in our setting with endogenous entry.

4.1.1 Optimum

A contest with endogenous and costly entry involves tremendously more extensive strategic trade-offs. On the one hand, a more discriminatory contest compels participants to bid more, while on the other, the increasing dissipation of rent limits entry. As revealed by Lemma 3, q^* would strictly decrease with r in the symmetric equilibrium with pure-strategy bidding.

This trade-off, however, does not exhaust the intricacy involved in the determination of optimal r. An additional trade-off is triggered at a differing layer. More extensive participation (i.e. a higher q^*) does not necessarily improve the supply of bids in the contest. On one hand, the contest on average engages more bidders, which amplifies the sources of contribution and tends to increase the overall bid. On the other hand, each participant would bid less, as they anticipate more intense competition and therefore expect lesser reward. The overall effect has yet to be explored more formally.

Consider an arbitrary entry-bidding game where potential bidders enter with a probability $q^* \in (0,1)$ in a symmetric equilibrium. The prize v is given away with a probability $1-(1-q^*)^M$. Hence, bidders win an expected overall rent of $[1-(1-q^*)^M]v$; while they on average incur entry cost $Mq^*\Delta$. The following fundamental equality must hold in this symmetric equilibrium:

$$[1-(1-q^*)^M]v \equiv Mq^*(\Delta + E(x^\alpha)). \tag{2.8}$$

The fundamental equality allows us to identify the expected overall bidding cost incurred by the bidders in the equilibrium without explicitly solving for it:

$$Mq^*E(x^\alpha) = [1-(1-q^*)^M]v - Mq^*\Delta. \tag{2.9}$$

The convexity of cost function ($\alpha \geq 1$) further implies that the expected overall bid ($Mq^*E(x)$) must be bounded from above:

$$(Mq^*E(x)) = Mq^*E[(x^\alpha)^{\frac{1}{\alpha}}] \leq Mq^*[E(x^\alpha)]^{\frac{1}{\alpha}}. \tag{2.10}$$

By the fundamental equality (2.8) or (2.9), we further obtain

$$(Mq^*E(x)) \leq [Mq^*]^{\frac{\alpha-1}{\alpha}}\{[1-(1-q^*)^M]v - Mq^*\Delta\}^{\frac{1}{\alpha}}. \tag{2.11}$$

Equation (2.11) yields important implication: Regardless of the equilibrium bidding strategy upon entry, RHS of (2.11) imposes an upper limit on the expected overall bids that an equilibrium with entry probability q^* could elicit. The expected overall bid ($Mq^*E(x)$) reaches the upper limit $[Mq^*]^{\frac{\alpha-1}{\alpha}} \times \{[1-(1-q^*)^M]v - Mq^*\Delta\}^{\frac{1}{\alpha}}$ if and only if: (1) bidders play a pure bidding strategy upon entry; or (2) participants randomize their bids but $\alpha = 1$.

Denote the upper boundary by

$$\overline{x_T}(q) \triangleq (Mq)^{\frac{\alpha-1}{\alpha}} \left\{[1-(1-q)^M]v - Mq\Delta\right\}^{\frac{1}{\alpha}} \tag{2.12}$$

with $q \in (0,1)$. The function $\overline{x_T}(q)$ exhibits the following important properties.

Lemma 5 *(a) There exists a unique $\hat{q} \in (q_0, \bar{q})$, which uniquely maximizes $\overline{x_T}(q)$;*

(b) The function $\overline{x_T}(q)$ strictly increases with q when $q \in (0, \hat{q})$, and strictly decreases when $q \in (\hat{q}, 1)$.

Proof. See Appendix B. ■

As stated by Lemma 5, the function $\overline{x_T}(q)$ varies non-monotonically with q and is uniquely maximized by $\hat{q} \in (q_0, \bar{q})$. This property implies that the overall bid that can be possibly elicited from the contest will never exceed $\overline{x_T}(\hat{q})$, regardless of the prevailing contest rules.

Definition 4 *Define $\overline{x_T}^* \equiv \overline{x_T}(\hat{q})$, which indicates the maximum amount of the overall bid a contest can elicit.*

The key to the design problem unfolds in Lemma 5: a mechanism must be optimal if it achieves the "first best" $\overline{x_T}^*$. We subsequently discuss the possibility of eliciting the "first best" through contest design.

By Lemma 5 and (2.8)-(2.11), the first best $\overline{x_T}^*$ can be achieved in a symmetric equilibrium with an entry probability $\hat{q}$, if there exists a $\hat{r}$ that induces entry probability $\hat{q}$ and (1) participants play a pure bidding strategy upon entry in the equilibrium; or (2) participants randomize their bids but $\alpha = 1$. For any given

r, the exact forms of bidding strategies in equilibria with mixed bidding remain unknown. It is difficult to identify the correspondence between prevailing contest rules and the subsequent equilibrium when it involves randomized bidding. We thus focus on the possibility of inducing the "first best" in equilibria with pure-strategy bidding. Nevertheless, our investigation shows it is rather sufficient to focus on contests that induce pure-strategy bidding.

Recall that Lemma 3 establishes the unique correspondence between r and (q^*, x^*) if a symmetric equilibrium with pure-strategy bidding exists. The equilibrium with entry probability q^* is determined by the break-even condition $v\sum_{N=1}^{M} C_{M-1}^{N-1} q^*(1-q^*)^{M-N}[\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha}] = \Delta$. We highlight the following cutoff.

Definition 5 *Let $r(\hat{q})$ be the unique solution of r to*

$$v\sum_{N=1}^{M} C_{M-1}^{N-1} \hat{q}(1-\hat{q})^{M-N}[\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha}] = \Delta. \tag{2.13}$$

The following result is formally stated.

Theorem 4 *(a) $r(\hat{q}) < r_0$.*

(b) Whenever $r(\hat{q}) \leq \alpha(1 + \frac{1}{M-2})$, the contest designer can elicit the "first best" $\overline{x_T}^*$ by setting $r = r(\hat{q})$. It induces a symmetric equilibrium with pure-strategy bidding. Potential bidders enter the contest with a probability $\hat{q}$ in the symmetric equilibrium.

Proof. See Appendix B. ■

Setting r to $r(\hat{q})$ could allow the contest designer to elicit the "first best" $\overline{x_T}^* \equiv \overline{x_T}(q)$. Because $r(\hat{q}) \in (0, r_0)$, whenever $r(\hat{q})$ falls below $\alpha(1 + \frac{1}{M-2})$, it satisfies the sufficient condition $r \leq \overline{r}$, thereby inducing a symmetric equilibrium with pure-strategy bidding by Theorem 3. In the equilibrium, potential bidders enter the contest with a probability of exactly $\hat{q}$. By Lemma 5, the expected overall bid must strictly decrease when r deviates from $r(\hat{q})$.

Additional discussion is provided as follows to complement our analysis.

4.1.2 Discussion

Two main issues are discussed. First, we compare our results to benchmark cases. Second, we examine the robustness of our result to what extend the condition $r(\hat{q})$ could robustly induce pure-strategy bidding.

Comparison to Benchmark Cases Our results run in sharp contrast to the conventional wisdom in contest literature. In a contest with a fixed number M of participants or free entry, a higher r provides stronger incentives to bidders, and elicits strictly higher bids whenever a pure-strategy equilibrium prevails, i.e. when $r \leq \alpha(1+\frac{1}{M-1})$. The size of r in our setting, however, affects the resultant equilibrium bid non-monotonically.

Despite the various trade-offs between conflicting forces, a softer *ex ante* incentive, i.e. a smaller r, may or may not be optimal. The optimal size of the parameter could either fall below or remain above the benchmark $\alpha(1+\frac{1}{M-1})$. In the left panel of Figure 2.2, the observations demonstrate the incidence of optimal "soft" incentives, with $r(\hat{q}) < \alpha(1+\frac{1}{M-1})$. In the right panel, the results illustrate the possibility of the opposite, with $r(\hat{q}) \in (\alpha(1+\frac{1}{M-1}), \alpha(1+\frac{1}{M-2}))$.

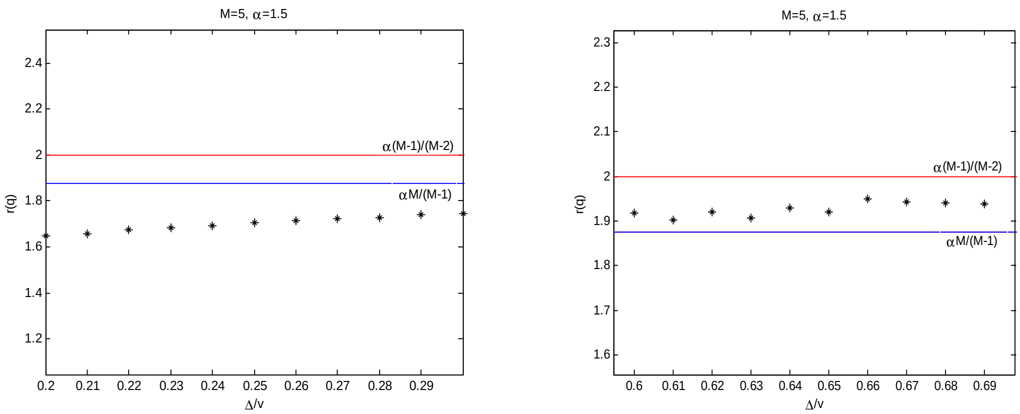

Figure 2.2: The optimal size of $r(\hat{q})$ when $M = 5, \alpha = 1.5$

These observations also contrast the results of related studies in auction literature. A number of studies have been devoted to the optimal design of auctions with costly entry, including Menezes and Monteiro (2000) and Lu (2009), Levin and Smith (1994), Shi (2009), Lu (2010) and Moreno and Wooders (2010). They all espouse the optimality of a "softer" incentive: the optimal reserve price is always lower than in the free-entry benchmark. The insight from auction literature does not extend to our setting. The observations in Figure 2.2 demonstrate that the optimum does not necessarily requires a lower-powered incentive mechanism than the free-entry benchmark level $\alpha(1+\frac{1}{M-1})$.[11]

Robustness and Numerical Exercises Our analysis has been limited so far. The global optimality of $r(\hat{q})$ is conditioned on that it also leads to pure-strategy bidding. It remains to be explored to what extent $r(\hat{q})$ could robustly induce pure-strategy bidding.

Because $r(\hat{q}) < r_0$, pure-strategy bidding can be induced as long as $r(\hat{q})$ falls below $\alpha(1+\frac{1}{M-2})$. A definitive conclusion can be drawn in contests with small pools of potential participants.

Corollary 2 *When the number of contestants contest is small, i.e., $M = 2, 3$, $r(\hat{q}) \leq \alpha(1+\frac{1}{M-2})$ must hold, and a symmetric equilibrium with pure-strategy bidding can always be induced by setting $r = r(\hat{q})$.*

In these cases, $\alpha(1+\frac{1}{M-2}) > r_0$, so the condition $r(\hat{q}) \leq \alpha(1+\frac{1}{M-2})$ is satisfied automatically. Nevertheless, it is less certain when M is large. We further check its robustness through numerical exercises. The condition is found to hold over a large parameter space, and ample incidents $r(\hat{q}) \leq \alpha(1+\frac{1}{M-2})$ can be observed. Figure 2.2, which is provided above, gives a small sample of these observations.

We observe incidents of $r(\hat{q}) > \alpha(1+\frac{1}{M-2})$ as well. However, recall that $r \leq \alpha(1+\frac{1}{M-2})$ is sufficient but not necessary for pure-strategy bidding. It should be noted that pure-strategy bidding can still be induced by $r \in (\alpha(1+\frac{1}{M-2}), r_0]$. As aforementioned, technical difficulty prevents us from drawing definitive conclusion on the property of bidders' expected payoff function $\tilde{\pi}_i(x_i)$ when r exceeds $\alpha(1+\frac{1}{M-2})$. Our numerical exercises, however, yield interesting observations.

[11] In our setting, if entry does not involve fixed entry cost, all the M potential bidders will participate. The conventional wisdom in contest literature would apply, such that $r = \alpha(1+\frac{1}{M-1})$ would emerge as the optimum.

We normalize v to unity. The simulation is run over a large set of the parameters (α, M), which span the entire space of $[1,2] \times \{4, 5, \ldots, 100\}$. For given (α, M), we let r vary over the entire range of $(\alpha(1+\frac{1}{M-2}), r_0]$ if $\alpha(1+\frac{1}{M-2}) < r_0$, and let Δ vary over the interval $(\frac{1}{M}, 1)$ as required by Assumption 1. We observe from our simulation results, with no exception, that all $\tilde{\pi}_i(x_i)$ demonstrates the property depicted by Lemma 4, and is uniquely maximized by x^*, despite that r exceeds $\alpha(1+\frac{1}{M-2})$. In all resultant figures, the curve is regularly shaped as described by Lemma 4. Figure 2.3 provides one example of them. The strategy profile (q^*, x^*) constitutes the unique symmetric equilibrium with pure-strategy bidding in all the simulated settings.

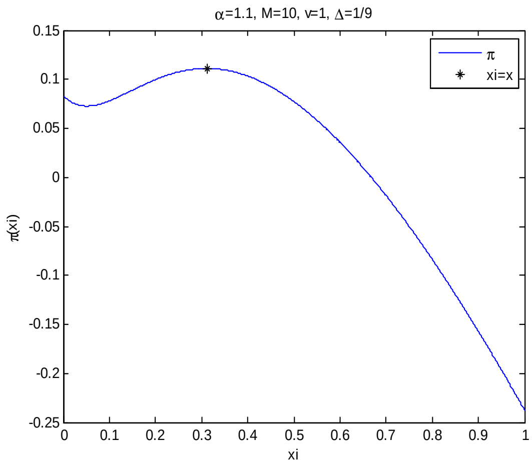

Figure 2.3: The shape of $\pi_i\left(x_i\right)$ when $\alpha = 1.1, M = 10, v = 1, \Delta = 1/9$

Hence, in all the simulated settings, we can elicit the "first best" by setting $r = r(\hat{q})$, regardless of the size of $r(\hat{q})$. Based on these observations, we propose the following conjectures.

Conjecture 1 *(a) A symmetric equilibrium with pure bidding exists if and only if* $r \leq r_0$.

(b) The first best overall bid $\overline{x_T}^*$ *can always be induced in a unique symmetric equilibrium with pure-strategy bidding by setting* r *to* $r(\hat{q})$.

We are unable to prove it analytically. However, all of our numerical exercises lend support to the claim. We leave it to future studies due to its technical difficulty.

4.2 Efficient Exclusion

The equilibrium analysis also allows us to investigate another classical question in the literature on contest design. With a fixed number n of bidders, a Tullock contest elicits an overall bid of $\alpha r \frac{n^2}{n-1}$ whenever a pure-strategy equilibrium exists. The number of overall bids strictly increases with the number of bidders n. A handful of studies, including Baye, Kovenock and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003), demonstrate that a contest designer may benefit from narrowing the slate of potential prize winners and excluding a subset of contestants. This strand of literature conventionally focuses on heterogeneous players and concerns themselves with selecting bidders of proper types. None of these studies involves stochastic and endogenous entry. In what follows, we demonstrate that exclusion can improve the efficiency of the contest in our setting despite the potential bidders being symmetric.

Consider our basic setting where M potential bidders are interested in participating in the competition. We now allow the contest designer to invite only a subset of these bidders for participation. The invited bidders then decide whether to participate in the contest after they observe the rules of the contest, i.e. the size of r set by the contest designer.

Let M' be an arbitrary positive integer. Define $M_0 \triangleq \min(M'| \frac{v}{M'} < \Delta)$ and assume $M_0 < M$. Recall that the amount of overall bid in a given contest is bounded from above by the first best $\overline{x_T}^*$ (see Lemma 5 and Definition 4), which can be achieved when r is set to $r = r(\hat{q})$, and $r(\hat{q})$ leads to pure-strategy bidding. It should be noted that the exact amount of $\overline{x_T}^*$ depends on the number of potential bidders who may enter the contest. Let $\overline{x_T}^*(M')$ be the first best bid for a contest with M' potential bidders. The function $\overline{x_T}^*(M')$ exhibits the following property.

Lemma 6 *$\overline{x_T}^*$ strictly decreases with M' for all $M' \geq M_0$.*

Proof. See Appendix B. ■

Lemma 6 shows that the first best $\overline{x_T}^*$ of a contest strictly declines with M'. The result yields direct implications for the contest design: a contest may have a weaker potential of eliciting bid if it involves a larger pool of potential bidders. We allow the contest designer to set r strategically. Let $r(\hat{q}(M'))$ be the unique solution to (2.13) in a contest with M' potential bidders. We conclude the following.

Theorem 5 *Whenever $r(\hat{q}(M_0)) \leq \alpha(1 + \frac{1}{M_0-2})$, the contest designer will not invite more than M_0 contestants.*

Theorem 5 demonstrates that exclusion improves bidding efficiency. Whenever the condition $r(\hat{q}(M_0)) \leq \alpha(1 + \frac{1}{M_0-2})$ is met, the contest designer will get strictly better off by excluding $M - M_0$ potential bidders from the contest. By inviting M_0 of them, and setting r to $r(\hat{q}(M_0))$, it elicits an overall bid $\overline{x_T}^*(M_0)$, which, by Lemma 6, is unambiguously more than what she can possibly achieve if she engages a greater number of potential bidders. Our result thus provides an alternative rationale for shortlisting and exclusion in a setting with homogeneous bidders but endogenous entry. The logic resembles that on the optimal r. To put it simply, although inviting more bidders may engage more participants to contribute their bid, each of them would enter less often and bid less (if he enters) anticipating a more intense competition and expecting subsequently a smaller share of the rent. Further, more frequent participation may lead to excessive rent dissipation because of the entry costs incurred, which tends to limit bidders' effort supply.

Theorem 5 shows that the optimal number of invited bidders must not exceed M_0. It provides only an upper bound for the possible optimum, and does not pin down exactly how many bidders should be invited in the optimum. When the contest designer invites less than M_0 potential bidders, the overall bid of the contest can elicit would change indefinitely, and the efficiency of the contest may either improve or suffer.[12]

The analysis for a contest with less than M_0 potential bidders is beyond the scope of the current paper, as Assumption 1 no longer holds in that setting. The alternative context in fact renders an even more handy equilibrium analysis. Most of the analysis in the current setting would not lose its bite after slight alteration. However, a general and systematic conclusion on the exact optimum M^* remains difficult. First, the optimization problem requires comparison across integers. Second, bidders behave qualitatively differently across the two contexts, i.e., when the number of potential bidder is above and below M_0. The discontinuity makes the comparison depend sensitively on the specific settings of (v, Δ), which, in general, does not exhibit regularity.

4.3 Disclosure Policy

Our analysis so far assumes that the actual participation level N is unknown to bidders. Firms' actual entry often involves unobservable or unverifiable actions. However, we now consider it as an institutional element. We assume that the actual participation rate is observable to the contest designer, and we explore whether the designer can benefit from disclosing N, i.e. eliciting a higher amount of overall bid.

We let the contest organizer commit to her disclosure policy prior to the entry-bidding game. Upon learning the disclosure policy, bidders enter and bid. Denote by d the policy that commits to announcing the true realization of N to participating bidders and by c the policy that conceals the actual N. Participants learn N before they bid if and only if policy d is chosen. Under the

[12] Examples in specific settings are available from the authors upon request, which demonstrate that the overall bid may either decrease or increase.

former, the actual number of participants N becomes common knowledge upon its realization. Under the latter, our benchmark setting remains.

The same issue has also been investigated in other studies that involve stochastic participation. The question is raised in the auction literature by McAfee and McMillan (1987) and Levin and Ozdenoren (2004). Lim and Matros (2009) and Fu, Jiao and Lu (2010) explore this issue in auctions and contests with exogenous stochastic entries.

4.3.1 Equilibrium When N is Disclosed

Under policy d, the analysis on the entry-bidding game is simplified substantially. Each contest after the entry stage is a proper subgame. With N to be known to participating bidders, each subgame of N-person contest boils down to a uni-dimensional game. The existence theorem of Dasgupta and Maskin (1986) for uni-dimensional discontinuous games allows us to verify the existence of equilibrium in every possible subgame.[13] We then readily establish the existence of symmetric equilibria in the entry-bidding game.

Theorem 6 *For any given $r > 0$, there exist symmetric subgame perfect equilibria $(q_d^*, \{x_N^*, N = 1, 2, ..., M\})$ in the entry-bidding game. All potential bidders enter with a probability $q_d^* \in (0, 1)$, and play a (pure or mixed) bidding strategy x_N^* in each subgame with N entrants. Each potential bidder receives zero expected payoff in the entry-bidding game.*

Proof. See Appendix B. ■

As well known in the contest literature, in a given subgame of an N-person contest, pure-strategy bidding emerges in the equilibrium if and only if $r \leq \alpha(1 + \frac{1}{N-1})$. Denote by $x_T^*(r, t)$ the expected overall bid in a contest with a discriminatory parameter r and under a disclosure policy t. We further obtain the following.

[13] Under policy c, the theorem for uni-dimensional game does not apply as the bidding game involves an uncertain number of bidders.

Lemma 7 *Suppose that the contest designer chooses policy d, and* $r \leq \alpha(1+\frac{1}{M-1})$*. There exists a unique symmetric equilibrium in the entry-bidding game. The equilibrium leads to pure-strategy bidding in all subgames. Each potential bidder enters the contest with a probability* $q_d^* \in (0,1)$*, which uniquely solves*

$$\textstyle\sum_{N=1}^{M} C_{N-1}^{M-1} q_d^{*N-1}(1-q_d^*)^{M-N}\pi_N = \Delta, \tag{2.14}$$

where π_N *is the payoff of an entrant in a subgame with* N *entrants. In the equilibrium, the contest elicits an expected overall bid*

$$x_T^*(r,d) = Mq_d^* \sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{*N-1}(1-q_d^*)^{M-N}\left(\frac{N-1}{N^2}\frac{rv}{\alpha}\right)^{\frac{1}{\alpha}}. \tag{2.15}$$

Proof. See Appendix B. ■

When r exceeds $\alpha(1+\frac{1}{M-1})$, mixed-strategy bidding arises in subgames of large N.

4.3.2 Optimal Disclosure Policy under Pure-Strategy Bidding

The aforementioned existing studies in auction and contest literature are typically based in settings where pure-strategy bidding equilibrium would emerge regardless of the prevailing disclosure policy. To facilitate comparison across the two scenarios, we focus on the setting with $r \leq \alpha(1+\frac{1}{M-1})$. Under this condition, participating bidders would not randomize over their bids regardless of the prevailing disclosure policy.

The expected overall bid under policy c is simply given by (2.6). We compare (2.6) with (2.15), which leads to the following.

Theorem 7 *For given* $r \in (0, \alpha(1+\frac{1}{M-1})]$*, we have* $x_T^*\ (r,c) \geqq x_T^*(r,d)$ *if and only if* $\alpha \geqq 1$*. That is, concealing the actual number of participating bidders allows the contest to elicit a strictly higher amount of expected overall bids if and only if the bidding cost function is strictly convex. In addition, the resultant* *expected overall bid of the contest is independent of the prevailing disclosure policy if and only if the bidding cost function is linear.*

Proof. See Appendix B. ■

A few remarks are in order. First, the same result would continue to hold when the contest designer is allowed to partially disclose the realization of N. That is, she is allowed to disclose the range of N but not its exact realization. We omit it for brevity but the detail is available upon request.

Second, our analysis provides new insights on the well known "disclosure-independence principle" in auction literature (e.g. Levin and Ozdenoren, 2004), and contest literature (Lim and Matros, 2009, and Fu, Jiao and Lu, 2010). As shown in Theorem 7, the resultant expected overall bid is independent of the prevailing disclosure policy if and only if bidding cost is linear, while it depends on the disclosure policy when the bidding cost function is nonlinear. Theorem 7 thus provides another incident of "disclosure-dependence." The logic of this result parallels that of Fu, Jiao and Lu (2010) in explaining why concealment elicits higher overall bid when the characteristics function is strictly concave. Convex bidding cost leads bidding behavior to exhibit "pseudo risk aversion". When N is to be disclosed, bidders "over-react" to "unfavorable contests" (i.e. those with large N) by reducing their bids substantially, but "under-react" to "favorable contests" (i.e. those with small N) by increasing their bids less than proportionally. Concealment alleviates the adverse effect. More detailed discussion on "pseudo risk aversion" can be seen in Fu, Jiao and Lu (2010).

Third, the analysis has focused on the tractable case of $r \leq \alpha(1 + \frac{1}{M-1})$, such that pure-strategy bidding always arises. It remains to be investigated how the prevailing disclosure policy determines the expected overall bid when r exceeds the cutoff and pure-strategy bidding does not necessarily arise in the equilibrium. A direct comparison between the two schemes is limited by existing techniques in delineating symmetric bidding behavior and the resultant rent dissipation when r is large. Baye, Kovenock and de Vries (1994) demonstrate that rent is fully dissipated in two-bidder contests when r exceeds two. Alcalde and Dahm (2010) characterize the asymmetric equilibrium (all-pay auction equilibrium) and resultant equilibrium rent dissipation in n-bidder contest. These findings do not directly shed light on our setting. Furthermore, our analysis is complicated when bidding cost is allowed to be convex. However, the following claim can still be made.

Remark 1 *When $r \in (\alpha(1+\frac{1}{M-1}), \alpha(1+\frac{1}{M-2})]$, the "disclosure-independence principle" does not hold even if bidding cost if linear.*

Proof. See Appendix B. ■

The result imposes a further limit on the scope of the "disclosure-independence principle": $x_T^*(r,c) \neq x_T^*(r,d)$ even if bidding cost is linear when $r \in (\alpha(1+\frac{1}{M-1}), \alpha(1+\frac{1}{M-2})]$. The prevailing disclosure policy does play a role in determining the equilibrium overall bid.[14]

However, when $r > \alpha(1+\frac{1}{M-1})$, mixed-strategy bidding will be definitely involved in the comparison between the two disclosure policies, which makes it technically challenging to determine the optimal disclosure policy. Nevertheless, we next show that from a perspective of contest design, there is no loss of generality to focus on contests with nondisclosure of number of actual contestants for optimal design.

4.3.3 A Broader Perspective: Mechanism Design

Despite the analytical difficulty in comparing $x_T^*(r,c)$ with $x_T^*(r,d)$ directly when r is large, the incompleteness of the direct comparison is of lesser concerns when the issue is examined from the perspective of mechanism design, i.e. when r is allowed to be chosen by the designer.

Theorem 8 *Suppose that $r(\hat{q})$ (as identified in Definition 5) can induce a symmetric equilibrium with pure-strategy bidding under policy c. A contest $(r(\hat{q}), c)$*

[14] Remark 1 is likely to hold for any $r > \alpha(1+\frac{1}{M-1})$.

dominates any contest (r, d) *regardless of* r*, i.e.* $x_T^*(r(\hat{q}), c) \geq x_T^*(r, d), \forall r \in (0, \infty)$.

Proof. See Appendix B. ■

Theorem 8 states that policy d (that discloses the number of participating bidders) would not lead to more efficient bidding when r can be set by the contest designer. The logic underlying Theorem 8, to a large extent, reflects a broad argument from the perspective of mechanism design. It should be noted that the amount of overall bid a contest can possibly elicit can never exceed $\overline{x_T}^*$, regardless of the prevailing disclosure policy. Hence, when a contest $(r(\hat{q}), c)$ can successfully achieve the first best, it must (at least weakly) dominate all other possible mechanisms.

5 Concluding Remarks

In this paper, we provide a thorough account of contests with endogenous and stochastic entries. We show the existence of a symmetric mixed-strategy equilibrium in which potential bidders randomly enter. We also provide a sufficient condition under which participants engage in pure bidding actions. Based on these equilibrium results, we identify relevant institutional elements in contest rules, and we demonstrate that analysis in this setting adds substantially to existing knowledge on optimal contest design.

While our study is one of the first to investigate the subtle and rich strategic interaction that occurs in contests with endogenous entries, our analysis reveals the enormous possibilities for future studies. Due to analytical difficulties, the open conjectures in Section 4 pose a challenge for future research on contests.

Further, our setting (characterized by common entry cost and resultant stochastic entry) is only one way for modeling contests that involve endogenous entry. Other examples include the setting of Kaplan and Sela (2010). They consider all-pay auctions with privately-known entry costs. Another possibility is to allow for non-uniform but commonly-known entry cost. The setting has not been widely studied in contest literature. It would lead to a "stratified" entry pattern, under which a portion of bidders with lower costs participate deterministically while the rest remain inactive. In this case, the cutoff type breaks even while other participants end up with positive rents. The optimal contest rules under this setting deserve more serious exploration, which should also be pursued in the future.

Chapter Three
Disclosure Policy in Contests with Stochastic Abilities

1 Introduction

It has been widely recognized that contestants' incentive to make effort and the resultant rent dissipation crucially depend on the rules of the contest. Most of the received rent-seeking literature deals with how a forward-looking organizer implements an optimal structure to achieve a given objective. Early contributions assume that a player's ability, measured by his or her cost of expending effort, is fixed and common knowledge. However, contestants usually do not know the actual abilities of their rival at the time they make their decision. For example, advertising firms that are vying for a commercial project are not fully aware of each other's advertising ideas and thus unable to fully assess their relative competitiveness. Consider another example whereby a university is actively sourcing for new research professionals to join its teaching faculty. Each prospective candidate is unlikely to be fully aware of other candidates' research background and capabilities, thus seriously challenging the assumptions of common knowledge.

This paper analyzes contests where contestants have private information about their abilities, and are observed by the contest organizer. Following Konrad and Kovenock (2009), a player's ability measured by his or her cost of expending effort is determined as the outcome of a stochastic process. Players with lower cost can be thought of as stronger (more able) players. This assumption is reasonable since in reality, many aspects of a contestant's actual effectiveness or ability have transitory ups and downs, the value of unit cost of effort is not known to the rival contestants, but are easily observed by a organizer. Similar to research tournaments, the organizer can form tentative judgements regarding a firm's ability based on its research proposal. In addition, in the job market, a candidate's curriculum vitae usually pre-signals his ability to the prospective employer. One important consideration when designing a tournament with commitment power concerns the control of information. The organizer should strategically plan whether the information about agents' abilities should be revealed back to them. In other words, we want to compare between two policies: revealing or concealing the abilities of contestants.

On the other hand, most literature on contests design has focused on the case where the contestants compete in order to win a unique prize. More work needs to be done because the more prevalent form of contests in the real world involves multiple rather than single prize. For example, in public recruitment, departments normally offer several identical positions at all ranks to the candidates. We extend the all pay contest models to allow for multiple (homogenous) prizes. Our extension can be used at the theoretical level to examine where established properties of the single prize all pay contests carry over to the more general case.

In the meanwhile, contestants often make costly investment to improve competency prior to the formal competition. For instance, an R&D company may purchase laboratory equipment, which improves the efficiency of subsequent research activities. We endogenize the distribution of abilities by allowing a contestant to reduce his marginal cost by making technological investment prior to the contest. Moreover, taking the time and trouble to enter the contest is a major concern for the contestants. We further assume that potential contestant have to incur a positive entry cost to participate in the contest. Potential bidders simultaneously make symmetric pure-strategy entry decisions so that their expected profits are exactly zero. In this way, we endogenize the entry probabilities. The comparison may differ.

The focus of the paper is to study how disclosure policy would affect the contestants' effort supply and rent dissipation. The private abilities of the contestants are observed by the contest organizer. The organizer may care about total effort or rent dissipation. She decides whether to disclose this information publicly.

Within the auction and contest literature, under disclosure policy, these kinds of all-pay auction with complete information has been carefully studied by Hillman and Riley (1989), Baye, Kovenock and de Vries (1996), Konrad and Kovenock (2009), Clark and Riis (1998) and Siegel (2009). They characterize the unique Nash equilibrium and calculate the expected effort of each agent. Especially, Clark and Riis (1998) extend the complete information single-prize all-pay auction for multiple (homogeneous) prizes. Moreover, Siegel (2009) provide a closed-form formula for players' equilibrium payoffs, and analyze player participation in all-pay contests. Last but not least, Hillman and Riley (1989) provides for the equilibrium under an all-pay auction with incomplete information and concealment policy.

Following the methodology provided by this literature, we find that concealing the abilities of the contestants elicits higher expected total effort, regardless of the distribution of the abilities. For rent dissipation, we find that the rent dissipation rate does not depend on the disclosure policy. To check the robustness of our main results and to deepen our basic analysis, we extend our model to allow any number of prizes. In order to have a less technical exposition, we focus on the organizer's problem in the case where she can award two identical prizes, we find that our findings are robust in this two-prize contest structure. We then study the robustness of our results while allowing endogenous ability distribution and endogenous entry of contestants. We find that our findings remain robust to these generalized settings. Further generalization are taken by exploring nonlinear cost function. We find that our findings are not robust when cost function is strictly concave or convex.

This paper is connected to a few strands of economic literature on contests and tournaments. Firstly, it is inspired by Lim and Matros (2009), Aoyagi (2010), Fu, Jiao and Lu (2011), Kovenock, Morath and Münster (2010) and Denter and Sisak (2011). All of these papers study the information revelation on contest design. Lim and Matros (2009) and Fu, Jiao and Lu (2011) investigate the impact of disclosure policy on expected effort in contests with a stochastic number of contestants. Aoyagi (2010) studies optimal feedback policy about agents' performance in a multi-stage tournament. Kovenock, Morath and Münster (2010) consider the information sharing in a two player all pay contest where firms have independent values and common values of winning the contest. Denter and Sisak (2011) focus on information transmission between lobbying groups and the consequences for disclosure policy in general rent seeking contests. Our analyses consider how disclosure policy would affect the contestants' effort supply and rent dissipation when players' abilities are stochastic.

This paper is also related to the literature on contests with asymmetric information. Hurley and Shogen (1998) explore how one-side asymmetric information over values affects effort levels in a Cournot Nash contest. Wärneryd (2003) studies a model of a common value contest under different assumptions about the information held by the players. In addition, our paper is linked to Fu and Lu (2009, 2010), who study contest with pre-investment and optimal endogenous entry in an imperfectly discriminating contest. Our analysis departs from these papers in that we focus on perfectly discriminating contests (All pay contests).

The paper is organized as follows. Section 2 sets up the general all pay contest model with n contestants, the unique equilibrium is characterized, the total expected effort and rent dissipation rate is calculated under both policies, and the optimal disclosure policy is explored. In section 3, the robustness of optimal disclosure policy is checked in a general all pay contest model with m

prizes. In sections 4 and 5, we check the robustness of two endogenous cases. Section 6 further explores nonlinear cost cases. Some concluding remarks are presented in section 7.

2 A Model with unique prize

We study a contest with n players. A prize normalized to unity is awarded to the winner. The competition for this prize is organized as a perfectly discriminating contest (all-pay auction), in which each player simultaneously expend effort $x_i \geq 0$. The costs of effort are equal to $c_i x_i$. Here, c_i is the marginal effort cost of player i. Assume that unit cost c_i is an independent random variable that is absolutely continuous with finite support $[\underline{c}, \overline{c}]$. The cumulative distribution functions of c_i is $F(c_i)$ with corresponding densities $f(c_i)$. The contest organizer knows all the information about players' cost.

We assume further that the contest organizer is allowed to commit to her disclosure policy - either to disclose the actual ability of participants, or to conceal this information - and announces this policy choice publicly. We denote the former policy by D, and the latter by C. Nature then determines c_i, the actual value of abilities. The organizer observes this information, and discloses it if and only if she has committed to a disclosure. The participants then submit their effort entry simultaneously $\mathbf{x} = (x_i)$ to compete for the prize.

2.1 Disclosure

We assume that when the efforts are chosen, the contest organizer will disclose each player's unit cost to the public; hence, this problem is a perfectly discriminating contest with complete information, the payoff to player i is given by

$$\pi_i(x_1,\ldots,x_n)=\begin{cases}-c_ix_i & \text{if } \exists j \text{ such that } x_j>x_i,\\ \frac{1}{m}-c_ix_i & \text{if } i \text{ ties for the high bid with } m-1 \text{ others},\\ 1-c_ix_i & \text{if } x_i>x_j\ \forall j\neq i.\end{cases}$$

This game has been carefully analyzed by Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1996). They demonstrated that the equilibrium of the perfectly discriminating contest for given values of $c_1, c_2, \ldots c_n$ is unique and this is described as follows.

Proposition 1 *(Baye et al. 1996) The unique equilibrium of n players all pay contest with complete information is a set of mixed strategies. Assume $c_1 = \min\{c_1, c_2, \ldots, c_n\}$, $c_2 = \min\{c_2, c_3, \ldots, c_n\}$. Then the unique equilibrium is for the two players with the lowest cost to compete as if they were the only two players. All other players remain passive. And bids are described by the following cumulative distribution functions:*

$$G_1(x_1)=\begin{cases}c_2x_1 & \textit{for } x_1\in[0,\frac{1}{c_2}),\\ 1 & \textit{for } x_1\geq\frac{1}{c_2};\end{cases}$$

$$G_2(x_2)=\begin{cases}1-\frac{c_1}{c_2}+c_1x_2 & \textit{for } x_2\in[0,\frac{1}{c_2}),\\ 1 & \textit{for } x_2\geq\frac{1}{c_2}.\end{cases}$$

With homogeneous abilities ($c_1 = c_2 = c_3 \ldots = c_n$), there exists a unique symmetric equilibrium and a continuum of asymmetric equilibria. All of the equilibria imply the same expected payoff (zero) for each player, and yield the organizer the same expected revenue.

This result is easy to obtain because in an all-pay contest, one can interpret differences in the c_i's as arising from differences in valuations or differences in abilities of players to convert an entry into a prize. Dividing contestant i's expected payoff by c_i we obtain an affine transformation of the expected payoff given by Hillman and Riley (1989) and Baye et al. (1996). Indeed, in our case, c_i plays the same role as the value v_i in the current literature as the adjusted-value $\frac{1}{c_i}$ is equal to v_i.

Lemma 1 *In an all pay contest with complete information, the total expected effort for contest organizer is*

$$R^D = n(n-1)\int_{\underline{c}}^{\overline{c}}\int_{c_1}^{\overline{c}}\left[\frac{1}{2c_2}+\frac{c_1}{2c_2^2}\right][1-F(c_2)]^{n-2}\,dF(c_2)\,dF(c_1). \tag{3.1}$$

Proof. See Appendix C. ■

We now examine the expected payoff for each player. Take player i with unit cost c_i as a representative contestant and let $\widetilde{c} = \min\{c_{-i}\}$. Then the cumulative distribution function of $\widetilde{c}$ is $1-[1-F(\widetilde{c})]^{n-1}$. Player i can get a positive payoff if and only if $c_i < \widetilde{c}$. Apply the payoff characterization in Siegel (2009), in any equilibrium of a generic contest, only the contestant with minimum cost can get a positive payoff. i.e., $\pi_i(x_i; c_i, c_{-i}) = 1 - \frac{c_i}{\widetilde{c}}$ if and only if $c_i < \widetilde{c}$. [1]

Therefore, the expected payoff of player i is

$$E\pi_i^D = \int_{\underline{c}}^{\overline{c}}\left\{\int_{c_i}^{\overline{c}}\left[1-\frac{c_i}{\widetilde{c}}\right]d\left[1-[1-F(\widetilde{c})]^{n-1}\right]\right\}dF(c_i). \tag{3.2}$$

Assume $\frac{[1-F(\widetilde{c})]^{n-2}}{\widetilde{c}}$ is integrable and define

$$A(\widetilde{c}) = -(n-1)\int\frac{[1-F(\widetilde{c})]^{n-2}}{\widetilde{c}}dF(\widetilde{c}), \tag{*}$$

which will be used repeatedly throughout the analysis.

[1] Following definitions of Siegel (2009), with a unique prize $m=1$, and initial score $a_i = 0$. The payoff is given as $v_i(x_i) = V_i - c_i(x_i) = 1 - c_i x_i$. The contestant with marginal cost $\widetilde{c}$ is the *marginal player*, his *reach* $\widetilde{r} = \max\{x_i | v_i(x_i) = 0\} = \frac{1}{\widetilde{c}}$. And player i's *power* $w_i = v_i(\max\{0,\widetilde{r}\}) = \max\{0, v_i(\widetilde{r})\} = \begin{cases} 1-\frac{c_i}{\widetilde{c}} > 0 & \text{if } c_i < \widetilde{c} \\ 0 & \text{if } c_i \geq \widetilde{c} \end{cases}$. The expected payoff of every player equals the maximum of his power and 0.

2.2 Concealment

In this subsection, we assume when the efforts are chosen, the contest organizer will conceal his information about every player's unit cost, causing each player to know only his own unit cost; hence, at this stage, the problem describes a perfectly discriminating contest with incomplete information.

Take player i with unit cost c_i as a representative contestant, let $\widetilde{c} = \min\{c_{-i}\}$. Then the cumulative distribution functions of $\widetilde{c}$ is $1-[1-F(\widetilde{c})]^{n-1}$. Player i will get the prize if and only if $c_i < \widetilde{c}$.

Assuming all bidder other than i adopt a bidding strategy $x_{-i}(\cdot)$. The payoff of player i given as

$$\begin{aligned}\pi_i(\tilde{x}_i, x_{-i}(\cdot); c_i) &= \Pr(\tilde{x}_i > x_j(c_j), \forall j \neq i)\, 1 - c_i\tilde{x}_i \\ &= \Pr\left(c_j > x_j^{-1}(\tilde{x}_i), \forall j \neq i\right) 1 - c_i\tilde{x}_i \\ &= \left[1 - F\left(x_{-i}^{-1}(\tilde{x}_i)\right)\right]^{n-1} - c_i\tilde{x}_i. \qquad (3.3)\end{aligned}$$

Player i will choose $x_i(c_i)$ to maximize his expected payoff.

Lemma 2 *In an pay contest with incomplete information, the equilibrium bid of each player is*

$$x_i(c_i) = (n-1)\int_{c_i}^{\bar{c}} \frac{[1-F(\widetilde{c})]^{n-2}}{\widetilde{c}} dF(\widetilde{c}),$$

and the total expected effort for contest organizer is

$$R^C = n(n-1)\int_{\underline{c}}^{\bar{c}} \frac{F(c_2)[1-F(c_2)]^{n-2}}{c_2} dF(c_2). \qquad (3.4)$$

Proof. See Appendix C. ■

Given c_i, the expected payoff of each player is given by

$$\begin{aligned}\pi_i^C(x_i, \mathbf{x}_{-i}; c_i) &= \Pr(c_i < \widetilde{c})\, 1 - c_i x_i(c_i) \\ &= [1-F(c_i)]^{n-1} - c_i x_i(c_i).\end{aligned}$$

The expected payoff of each player under concealment policy is given by

$$E\pi_i^C = \int_{\underline{c}}^{\overline{c}} \left\{ [1 - F(c_i)]^{n-1} - c_i x_i(c_i) \right\} dF(c_i). \tag{3.5}$$

2.3 Optimal Disclosure Policy

For a contest organizer who is interested in maximizing the total expected effort, the following equilibrium analysis allows us to investigate the structure of the optimal disclosure policy. Lemma 1 and 2 imply the following.

Theorem 1 *In an N players all pay contests, concealing the abilities of contestants elicits higher expected total revenue to contest organizer.*

Proof. See Appendix C. ■

It follows that expected payoff (3.2) and (3.5) take exactly the same form, the following result can be obtained immediately.

Theorem 2 *Given the number of participant n, both disclosure and concealment policies give each contestant same expected payoff .*

Proof. See Appendix C. ■

Theorems 1 and 2 indicate that contestants' expected payoffs are identical in the two cases. However, the contest organizer would nonetheless like to conceal contestant's ability to induce higher expected effort. Note that the total expected effort and player's expected payoff are both *ex ante*. Before disclosure policy is implemented, each player expects to get a positive payoff if and only if he is the most able player, which produces the same *ex ante* expected payoff, regardless subsequent disclosure policy .

Under concealment policy, each contestant has incomplete knowledge on his competitiveness or lack of comparative advantage over other contestants, thus leading him to have a positive *ex post* expected payoff and do their best in the bidding process. The player with highest ability wins the unique prize with probability one. However, when players' abilities are common knowledge, competition is less fierce as the *ex post* expected payoff of contestants with lower abilities is zero, but they also have positive probability of winning since the equilibrium is in mixed strategies. Intuitively, under disclosure, each player knows his capability and rivals capabilities as well. Everyone will slack off as weaker player will give up if he feels no chance of winning, and stronger player will also slack off (do not work hard) when he see his competitors are weak. Therefore disclosure creates a distort competition. It is never a good idea to make players' competitive strength publicly. This provide an explanation why cover the information of contestants' abilities can elicit higher total expected effort.

We now turn to analyzing the property of contestant's expected payoff, recall

$$E\pi_i(n) = \int_{\underline{c}}^{\overline{c}} \left\{ \int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{\widetilde{c}}\right] d\left[1 - [1 - F(\widetilde{c})]^{n-1}\right] \right\} dF(c_i),$$

where c_i is the unit cost of a representative contestant i when there are $n-1$ other bidders, and $\widetilde{c} = \min\{c_{-i}\}$.

Corollary 1 *$E\pi_i(n) \geq 0$ and monotonically decreases with $n \leq N$.*

Proof. The conditional payoff of a bidder given c_i is

$$\begin{aligned}
& \int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{\widetilde{c}}\right] d\left[1 - [1 - F(\widetilde{c})]^{n-1}\right] \\
= & \left[1 - \frac{c_i}{\widetilde{c}}\right] \left[1 - [1 - F(\widetilde{c})]^{n-1}\right] |_{c_i}^{\overline{c}} - \int_{c_i}^{\overline{c}} \left[1 - [1 - F(\widetilde{c})]^{n-1}\right] d\left[1 - \frac{c_i}{\widetilde{c}}\right] \\
= & (1 - c_i) - 0 - \int_{c_i}^{\overline{c}} \left[1 - [1 - F(\widetilde{c})]^{n-1}\right] \frac{c_i}{\widetilde{c}^2} d\widetilde{c} \text{ since } F(\overline{c}) = 1.
\end{aligned}$$

since $[1 - F(\widetilde{c})]^{n-1}$ decreases with n, then the term $-\int_{c_i}^{\overline{c}} \left[1 - [1 - F(\widetilde{c})]^{n-1}\right] \frac{c_i}{\widetilde{c}^2} d\widetilde{c}$ decreases with n. Therefore $\int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{\widetilde{c}}\right] d\left[1 - [1 - F(\widetilde{c})]^{n-1}\right]$ decreases with n as well.

$E\pi_i(n) \geq 0$ is apparent since both $1-\frac{c_i}{\widetilde{c}}$ and $1-[1-F(\widetilde{c})]^{n-1}$ are nonnegative.

■

This result parallels the finding in a standard all pay contest that the expected payoff decreases in the number of contestants as the competition intensifies.

Following part of the paper further explores the issue of information disclosure from three additional dimensions. First, we generalize the disclosure policy in the basic setting by allowing n participants competing for more than one prizes. Second, we endogenize the distribution of abilities by allowing a contestant to reduce his marginal cost through making technological investment prior to the contest, and endogenize the entry probabilities by assuming that entry incurs a fixed cost to each contestant. Third, an extension that generalizes cost function to nonlinear form is explored.

3 Multi-prize Contests

Consider an all pay auction in which there are m identical prizes to be won. There are n players who are ranked according to their abilities. To simplify the model assume $c_1 < c_2 < \ldots < c_n$ and each prize is normalized to unity $V_1 = V_2 = \ldots V_m = 1$. The players with m highest efforts win these prizes and everyone can only win one prize. Abilities are draw independently of each other from an interval $[\underline{c}, \overline{c}]$ according to the absolutely continuous distribution function $F(c_i)$ which is common knowledge.

3.1 Disclosure

We assume at the point in time when the efforts are chosen, the contest organizer will disclose each player's unit cost to public; In this case, $c_1 < c_2 < \ldots < c_n$ are common knowledge. Hence, this problem is a perfectly discriminating contest with complete information, the payoff to player i is given by

$$P_i(x_1, x_2, \ldots, x_n)\,1 - c_i x_i$$

Let $G_i(x)$ represent the cumulative density function of player i's equilibrium mixed strategy. This game has been carefully analyzed by Clark and Riis (1998). They have shown that the equilibrium of the perfectly discriminating contest for given values of $c_1, c_2, \ldots c_n$ is necessarily in mixed strategies and described as follows.

Proposition 2 *(Clark and Riis 1998) There exists a unique mixed strategy equilibrium of the game in which the* $m+1$ *highest ranked of players bid* $x_i, i = 1, 2, \ldots, m+1$, *from probability distribution functions* $G_i(x)$ *over* $\left[\frac{1}{c_i^l}, \frac{1}{c_{m+1}}\right]$, *with common upper support* $\frac{1}{c_i^u} = \frac{1}{c^u} = \frac{1}{c_{m+1}}$ *and lower supports given by*

$$\frac{1}{c_{m+1}^l} = 0,$$

$$\frac{1}{c_i^l} = \left[1 - \Pi_{j=i}^m \left(\frac{c_i}{c_j}\right)\right] \frac{1}{c_{m+1}} \qquad i = 1, 2, \ldots, m,$$

and where

$$G_i(x) = 1 - \frac{1}{c_i} \Pi_{j=k}^m c_j^{1/(m+1-k)} (1 - c_{m+1} x)^{1/(m+1-k)} \quad i = 1, 2, \ldots, m,$$

where

$$\begin{aligned} k &= 1 \quad \text{if } \frac{1}{c_1^l} \le x \le \frac{1}{c_{m+1}}, \\ k &= s \quad \text{if } \frac{1}{c_s^l} \le x < \frac{1}{c_{s-1}^l}, \\ s &= 2, 3, \ldots, m. \end{aligned}$$

Player $m+1$ *bids* $x_{m+1} > 0$ *with probability* $c_m/c_{m+1}.$ *The conditional distribution* *function of this player is*

$$G_{m+1}\left(x|x>0\right)=1-\frac{1}{c_m}\Pi_{j=k}^{m}c_j^{1/(m+1-k)}\left(1-c_{m+1}x\right)^{1/(m+1-k)}.$$

When there are two prizes $m=2$, the unique equilibrium is for the three players with the lowest cost to compete as if they were the only three players. All other players remain passive. And bids are described by the following cumulative distribution functions:

$$G_1\left(x_1\right)=1-\left(\frac{c_2}{c_1}\right)^{\frac{1}{2}}\left(1-c_3x_1\right)^{\frac{1}{2}} \qquad \text{for } x_1\in[\left(1-\frac{c_1}{c_2}\right)\frac{1}{c_3},\frac{1}{c_3});$$

$$G_2\left(x_2\right)=\begin{cases} c_3x_2 & \text{for } x_2\in[0,\left(1-\frac{c_1}{c_2}\right)\frac{1}{c_3}),\\ 1-\left(\frac{c_1}{c_2}\right)^{\frac{1}{2}}\left(1-c_3x_2\right)^{\frac{1}{2}} & \text{for } x_2\in[\left(1-\frac{c_1}{c_2}\right)\frac{1}{c_3},\frac{1}{c_3});\end{cases}$$

$$G_3\left(x_3\right)=\begin{cases} 1-\frac{c_2}{c_3}+c_2x_3 & \text{for } x_3\in[0,\left(1-\frac{c_1}{c_2}\right)\frac{1}{c_3}),\\ 1-\left(\frac{c_1c_2}{c_3}\right)^{\frac{1}{2}}\left(\frac{1}{c_3}-x_3\right)^{\frac{1}{2}} & \text{for } x_3\in[\left(1-\frac{c_1}{c_2}\right)\frac{1}{c_3},\frac{1}{c_3}).\end{cases}$$

Corollary 2 *The unique equilibrium of n players all pay contest with complete information is in mixed strategies. The unique equilibrium is for the three players with the lowest cost to compete as if they were the only three players. All other players remain passive. The total expected effort for contest organizer is*

$$\begin{aligned} R^D &= n\left(n-1\right)\left(n-2\right) \\ &\quad \int_{\underline{c}}^{\overline{c}}\int_{c_1}^{\overline{c}}\int_{c_2}^{\overline{c}}\left[\left(\frac{3}{2}-\frac{c_1}{3c_2}+\frac{c_1^2}{6c_2^2}\right)\frac{1}{c_3}+\left(c_2+\frac{c_1^2}{3c_2}\right)\frac{1}{2c_3^2}\right] \\ &\quad \left[1-F\left(c_3\right)\right]^{n-3}dF\left(c_3\right)dF\left(c_2\right)dF\left(c_1\right). \end{aligned} \tag{3.6}$$

Proof. See Appendix C. ■

3.2 Concealment

In this case, the ability of contestant i is private information to i. Player i's maximization problem is

$$\max_{x(c)} \sum_{j=0}^{m-1} C_{n-1}^{j} \left[F\left(x^{-1}(c)\right)\right]^{j} \left[1-F\left(x^{-1}(c)\right)\right]^{n-1-j} - c \cdot x(c).$$

Fix agent i, and let $F_s(c)$, $1 \leq s \leq m$, denote the probability that agent i with type $c \in [\underline{c}, \overline{c}]$ meets $n-1$ competitors such that $s-1$ of them have lower types, while $n-s$ have higher types. Hence, F_s is the probability of winning the s'th prize, where $s = 1, 2, \ldots, m$. We now have

$$F_s(c) = \frac{(n-1)!}{(s-1)!\,(n-s)!} \times [1-F(c)]^{n-s} [F(c)]^{s-1}. \tag{3.7}$$

The corresponding derivatives are given by

$$F_1^{'}(c) = -(n-1)(1-F(c))^{n-2} F^{'}(c) \tag{3.8}$$

when $s = 1$,

$$F_s^{'}(c) = \frac{(n-1)!}{(s-1)!\,(n-s)!} \times [1-F(c)]^{n-s-1} [F(c)]^{s-2} F^{'}(c) \times [(1-n)F(c) + (s-1)] \tag{3.9}$$

when $2 \leq s \leq m$.

Moldovanu and Sela (2001) states the symmetric equilibrium as follows.

Proposition 3 *(Moldovanu and Sela 2001) The equilibrium bid under concealment policy for any number of prizes $m \geq 2$ and $n \geq m$ contestants is given by*

$$x_i(c_i) = \sum_{s=1}^{m} \int_{c_i}^{\overline{c}} -\frac{1}{c} F_s^{'}(c)\, dc,$$

where $F_s^{'}(c)$ is given by (3.8) and (3.9).

In the special case where there are two prizes $m = 2$, the contestants with the highest and second-highest effort win the two prizes.

The symmetric equilibrium of the perfectly discriminating contest with incomplete information is described as follows

Corollary 3 *Assume that there are two prizes, $V_1 = V_2 = 1$, and $n \geq 3$ contestants. In a symmetric equilibrium, the bid function of each contestant is given by $x_i(c) = P(c)V_1 + Q(c)V_2 = P(c) + Q(c)$, where*

$$P(c) = (n-1)\int_c^{\overline{c}} \frac{1}{c_i}\left[1 - F(c_i)\right]^{n-2} dF(c_i),$$

$$Q(c) = (n-1)\int_c^{\overline{c}} \frac{1}{c_i}\left[1 - F(c_i)\right]^{n-3}\left[(n-1)F(c_i) - 1\right] dF(c_i),$$

the total expected effort for contest organizer is

$$R^C = n\int_{\underline{c}}^{\overline{c}} x_i(c)\, dF(c). \tag{3.10}$$

3.3 Optimal Disclosure Policy

The goal of the contest designer is to maximize the total expected effort (i.e., the expected sum of the bids) at the contest. In order to keep the analysis as simple and tractable as possible, and in order to compare our results directly, we focus, however, on the total expected effort comparison with *two* identical prizes.[2] Given the characterization of the total expected effort under disclosure and concealment policy, we can now compare (3.6) and (3.10) to address the issue of optimal contest design.

Theorem 3 *In an N players all pay contest with two identical prizes, concealing the abilities of contestants elicits higher total expected revenue to contest organizer.*

[2] It will become clear that none of our qualitative results change if we allow for more than two prizes. We leave the analysis of more general environments to future works.

Proof. See Appendix C. ■

Theorem 3 shows that our findings in Theorem 1 are robust in multi-prizes contests.

3.4 Payoff Equivalent

In this section, we further explore the rent dissipation rate comparison under both policies.

3.4.1 Payoff under Disclosure

We first get the expected payoff of each player under disclosure. Assume there are m identical prizes $V_1 = V_2 = \ldots V_m = 1$. As $c_1 < c_2 < \ldots < c_n$ are common knowledge, the c.d.f of the mth lowest among $n-1$ bidders' costs is

$$H(c) = \sum_{j=m}^{n-1} C_{n-1}^{j} F(c)^{j} [1 - F(c)]^{n-1-j}.$$

Following the result provided by Siegel (2009), only the contestants with marginal cost $c_i < c$ can get a positive payoff.[3]

Given c_i,the conditional expected payoff of each player is given by

$$\pi_i^D(c_i) = \int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{c}\right] dH(c). \tag{3.11}$$

The expected payoff of player i is

$$\begin{aligned} E\pi_i^D &= \int_{\underline{c}}^{\overline{c}} \pi_i^D(c_i)\, dF(c_i) \\ &= \int_{\underline{c}}^{\overline{c}} \left\{ \int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{c}\right] dH(c) \right\} dF(c_i). \end{aligned}$$

note that $F(c = \overline{c}) = 1$ and $H(c = \overline{c}) = 1$ since $c \in [\underline{c}, \overline{c}]$.

[3] Applying the payoff characterization in Siegel (2009), the payoff of player i is given as $v_i(x_i) = 1 - c_i x_i$, and his *reach* is $r_i = \frac{1}{c_i}$, his *power* is $1 - \frac{c_i}{c}$. Since $\frac{1}{c_1} > \frac{1}{c_2} > \ldots > \frac{1}{c_m}$, in this m prizes model, player $m+1$ is the *marginal* player.

3.4.2 Payoff under Concealment

In the following part, we will get the equilibrium bid and expected payoff of each player under concealment. Proposition 3 has stated the equilibrium bid. Since $c_1 < c_2 < \ldots < c_m < c_{m+1} < \ldots < c_n$, player i will get a prize if and only if $c_i < c$. Here c is the mth lowest among $n-1$ bidders' costs, and the c.d.f of c is

$$H(c) = \sum_{j=m}^{n-1} C_{n-1}^{j} F(c)^{j} [1 - F(c)]^{n-1-j}.$$

Given c_i,the conditional expected payoff of each player is given by

$$\pi_i^C(c_i) = \Pr(c_i < c)\, 1 - c_i x_i(c_i)$$

$$= 1 - H(c_i) - c_i x_i(c_i). \tag{3.12}$$

And the expected payoff of player i is

$$\begin{aligned} E\pi_i^C &= \int_0^{\bar{c}} \pi_i^C(c_i)\, dF(c_i) \\ &= \int_{\underline{c}}^{\bar{c}} \{1 - H(c_i) - c_i x_i(c_i)\}\, dF(c_i). \end{aligned}$$

Lemma 3 *Given c_i, both policies give each player same conditional expected payoff.*

Proof. See Appendix C. ■

Since the conditional expected payoffs under both policies are identical, each player will get equal expected payoff finally, i.e., $E\pi_i^D = E\pi_i^C$. The following result can be established immediately.

Theorem 4 *In multi-prize perfectly discriminating contests, each player will get identical expected payoff under both policies, and rent dissipation rates are identical as well.*

Note that rent dissipation rate $= \frac{\text{total effort cost}}{\text{prize value}} = \frac{1-\text{ total expected payoff}}{\text{prize value}}$. Both policies will induce the same rent dissipation rate as long as each contestant gets the same expected payoff. Theorem 4 shows that our findings in Theorem 2 are robust in multi-prizes contests.

Theorem 3 and 4 and their intuition are similar to the corresponding findings in the previous section. In all pay auction with multiple prizes, a policy maker interested in maximizing his total rent-seeking revenues always prefers concealment to disclosure, while disclosure policy does not affect contestant's *ex ante* expected payoff.

The results discussed in the previous sections offer a sharp characterization of the optimal contest design. But to what extent can one generalize our result in a model when the distribution of player's ability and contestant's entry can be endogenized? Also, does the main result continue to hold if a more general class of effort costs becomes feasible? In subsequent analysis, we discuss the robustness of our main results to each of these issues.

4 Endogenous Distribution of Abilities

We have thus far assumed that the cumulative distribution functions of unit effort cost is taken as given. We now consider endogenous distribution of abilities where contestants can independently make investment to improve their distributions of types. Specifically, the distribution of marginal cost c_i of contestant i is determined by her pre-contest investment α_i, with investment cost $I(\alpha_i)$, where $I'(\alpha_i) < 0$ and $I''(\alpha_i) > 0$.

In an N players all pay auction, player i with investment $\alpha_i \in [0,1]$ will have a corresponding cumulative distribution functions $F_i(c_i; \alpha_i)$ on support $[0,1]$. In addition, the investment cost is $I(\alpha_i)$ with $I(1) = 0, I(0) = \infty$. It is clear that contestants' pre-investment affect the distribution of their abilities. In our setting, ability is interpreted as the value of per-unit-of-bid effort cost.

Consider the following game:

Stage 1: The contest organizer commits to and announces publicly her disclosure policy, D or C;

Stage 2: Upon observing the disclosure rule, each player simultaneously engage in technological investment α_i with investment cost $I(\alpha_i)$ in order to lower their marginal costs, so that effort cost with c.d.f $F_i(c_i;\alpha_i)$, and the investments of contestants are private information;

Stage 3: Nature draw abilities c_i, and each player privately learns his own ability, the value of marginal cost is revealed if and only if the organizer committed to policy D;

Stage 4: Players bids x_i simultaneously in the all pay auction.

The subgame perfect equilibrium in stage 4 is the same as in Section 2, except that players have endogenized distribution functions of unit cost.

4.1 Disclosure

Without loss of generality, take player i with unit cost c_i as a representative contestant, assume $\widetilde{c} = \min\{c_{-i}\}$. We concentrate on characterizing a symmetric pure-strategy equilibrium. Assume all bidders other than i adopt an optimal investment level at stage 1, that is, $\alpha_k = \alpha_j = \alpha_D^*$ for any $k \neq j \neq i$. Then the cumulative distribution functions of $\widetilde{c}$ is $1 - [1 - F(\widetilde{c};\alpha_D^*)]^{n-1}$. Player i will get a positive payoff if and only if $c_i < \widetilde{c}$.

We first look for a subgame equilibrium at stage 4. Recall the results in section 2.1, the expected payoff of player i is

$$E\pi_i = \int_{\underline{c}}^{\overline{c}} \left\{ \int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{\widetilde{c}}\right] d\left[1 - [1 - F(\widetilde{c};\alpha_D^*)]^{n-1}\right] \right\} dF_i(c_i;\alpha_i).$$

Then, given c_i, the expected payoff of player i is

$$\begin{aligned}\pi_i^D\left(c_i;\alpha_D^*\right) &= \int_{c_i}^{\overline{c}}\left[1-\frac{c_i}{\widetilde{c}}\right]d\left[1-\left[1-F\left(\widetilde{c};\alpha_D^*\right)\right]^{n-1}\right] \\ &= \left[1-F\left(c_i;\alpha_D^*\right)\right]^{n-1}-c_iA\left(c_i\right), \qquad (3.13)\end{aligned}$$

where $A\left(c_i\right)$ is defined by (*).

Note that for any realization of abilities c_i, $\pi_i^D\left(c_i;\alpha_D^*\right)$ is irrelevant with their pre-contest investment α_i.

The expected payoff of player i at stage 4 is

$$E\pi_i^D\left(c_i;\alpha_D^*,\alpha_i\right)=\int_{\underline{c}}^{\overline{c}}\pi_i^D\left(c_i;\alpha_D^*\right)dF_i\left(c_i;\alpha_i\right).$$

At stage 1, player i will make a pre-contest investment α_i to maximize $\int_{\underline{c}}^{\overline{c}}\pi_i^D\left(c_i;\alpha_D^*\right)dF_i\left(c_i;\alpha_i\right)-I\left(\alpha_i\right)$. Therefore, if α_D^* is a symmetric equilibrium solution, player i's best reply is $\alpha_D^*=\arg\max_{\alpha_i}\left\{\int_{\underline{c}}^{\overline{c}}\pi_i^D\left(c_i;\alpha_D^*\right)dF_i\left(c_i;\alpha_i\right)-I\left(\alpha_i\right)\right\}$.

4.2 Concealment

In this perfectly discriminating contest with incomplete information, take player i with unit cost c_i as a representative contestant, let $\widetilde{c}=\min\left\{c_{-i}\right\}$. Then the cumulative distribution functions of $\widetilde{c}$ is $1-\prod_{j=1,j\neq i}^n\left[1-F_j\left(\widetilde{c}\right)\right]$. Player i will get the prize if and only if $c_i<\widetilde{c}$.

We still concentrate on characterizing a symmetric pure-strategy equilibrium. Assume all bidders adopt an optimal investment level α_C^* at stage 1. Therefore the cumulative distribution functions of $\widetilde{c}$ is $1-\left[1-F\left(\widetilde{c};\alpha_C^*\right)\right]^{n-1}$ since $\alpha_k=\alpha_j=\alpha_C^*$ for any $k\neq j\neq i$.

Then the subgame equilibrium at stage 4 is the same as section 2. Recall the results in section 2.2, a representative contestant's optimal bidding strategy is given as

$$x^*(c_i) = A(c_i) = -(n-1)\int \frac{[1-F(c_i;\alpha_C^*)]^{n-2}}{c_i} dF(c_i;\alpha_C^*).$$

Then, given c_i, the expected payoff of player i at stage 4 is given by

$$\begin{aligned}\pi_i^C(c_i;\alpha_C^*) &= \Pr(c_i < \widetilde{c}) \cdot 1 - c_i x^*(c_i) \\ &= [1-F(c_i;\alpha_C^*)]^{n-1} - c_i x^*(c_i). \qquad (3.14)\end{aligned}$$

Note that for any realization of ability c_i, $x^*(c_i)$ is player i's optimal bidding strategy which maximize his payoff, i.e., $\frac{\partial \pi_i}{\partial \tilde{x}_i}|_{\tilde{x}_i = x^*(c_i)} = 0$. While $x^*(c_i)$ and $\pi_i^C(c_i;\alpha_C^*)$ is irrelevant with his pre-contest investment α_i.

The expected payoff of player i at stage 4 is

$$E\pi_i^C(c_i;\alpha_C^*,\alpha_i) = \int_{\underline{c}}^{\overline{c}} \pi_i^C(c_i;\alpha_C^*)\, dF_i(c_i;\alpha_i).$$

At stage 1, player i will make a pre-contest investment α_i to maximize $\int_{\underline{c}}^{\overline{c}} \pi_i^C(c_i;\alpha_C^*)\, dF_i(c_i;\alpha_i) - I(\alpha_i)$. Therefore, if α_C^* is a symmetric equilibrium solution, player i's best response is $\alpha_C^* = \arg\max_{\alpha_i} \left\{ \int_{\underline{c}}^{\overline{c}} \pi_i^C(c_i;\alpha_C^*)\, dF_i(c_i;\alpha_i) - I(\alpha_i) \right\}$.

4.3 Comparison

At stage 1, a representative contestant i will make investment to maximize his pre-contest expected payoff $\int_{\underline{c}}^{\overline{c}} \pi_i(c_i;\alpha^*)\, dF_i(c_i;\alpha_i) - I(\alpha_i)$. Recall (3.13) and (3.14), for any realization of abilities, his expected payoffs at stage 4 are identical under different policies $\pi_i^D(c_i;\alpha^*) = \pi_i^C(c_i;\alpha^*)$. Therefore, the optimal symmetric investment levels are same under both policies. i.e., $\alpha_D^* = \alpha_C^* = \alpha^*$.

The existence of symmetric equilibrium implies that players will make the same level of investment at stage 1, the endogenized distribution of abilities are parallel under different policies. Therefore both policies give players equivalent pre-contest expected payoff at stage 1. Given the same distribution of abilities, however, hiding the information leads to higher effort according to the benchmark model. Therefore, contest organizer still prefers to conceal the actual value of contestants' abilities to elicit higher total expected effort. The following result can be established immediately.

Theorem 5 *Consider the symmetric equilibria of endogenous distribution of abilities, both policies implement the same level of investment and the same contestant's expected payoff, while concealing the abilities of contestants still elicits higher expected total revenue to contest organizer.*

In addition, one should note that when $\alpha^* = 1$, no one makes pre-contest investment. $F(c_i; \alpha^* = 1) = F(c_i)$, and the distribution of abilities is taken as given, similar to the case in section 2.

4.4 An Example

In a two players all pay auction, player i with investment $\alpha_i \in [0,1]$ will have a corresponding cumulative distribution functions $F_i(c_i; \alpha_i) = c_i^{\alpha_i}$ on support $[0,1]$. And the investment cost is $I(\alpha_i) = \frac{5}{36}\left(\frac{1}{\alpha_i} - 1\right)$ which satisfy $I(1) = 0, I(0) = \infty, I'(\alpha_i) < 0$ and $I''(\alpha_i) > 0$.

Under disclosure policy, apply the results in Konrad and Kovenock (2009), given values $c_1 < c_2$, bidder 1 gets payoff $\pi_1 = 1 - \frac{c_1}{c_2}$ and bidder 2 gets 0; given values $c_2 < c_1$, bidder 2 gets payoff $\pi_2 = 1 - \frac{c_2}{c_1}$ and bidder 1 gets 0.

When $\alpha_2 \in [0,1)$, the expected payoff of player 1 is given by

$$\begin{aligned} E\pi_1^D(\alpha_1, \alpha_2) &= \int_0^1 \left[\int_{c_1}^1 \left(1 - \frac{c_1}{c_2}\right) dF(c_2)\right] dF(c_1) - I(\alpha_1) \\ &= \frac{\alpha_2}{(1+\alpha_1)(\alpha_1+\alpha_2)} - \frac{5}{36}\left(\frac{1}{\alpha_1} - 1\right); \end{aligned}$$

When $\alpha_2 = 1$, the expected payoff of player 1 is given by

$$\begin{aligned}
E\pi_1(\alpha_1,\alpha_2) &= \int_0^1 \left[\int_{c_1}^1 \left(1-\frac{c_1}{c_2}\right) dc_2\right] dF(c_1) - I(\alpha_1) \\
&= 1-\left[1-\frac{1}{\alpha_1+1}\right]^2 - \frac{5}{36}\left(\frac{1}{\alpha_1}-1\right).
\end{aligned}$$

Under concealment policy, with only two players, the payoff of the player 1 given as

$$\begin{aligned}
\pi_1(x_1,x_2;c_1) &= \Pr\left(x_1 > x_2^*(c_2)\right) 1 - c_1 x_1\left(c_1^{'}\right) - I(\alpha_1) \\
&= \Pr\left(c_2^{*} > c_1^{'}\right) 1 - c_1 x_1\left(c_1^{'}\right) - I(\alpha_1) \\
&= \left[1-F_2\left(c_1^{'}\right)\right] - c_1 x_1\left(c_1^{'}\right) - I(\alpha_1).
\end{aligned}$$

Note that $x_i(c_i)$ is decreasing with c_i, the higher cost, the lower effort. Take first order condition with respect to c_1,

$$\frac{\partial \pi_1}{\partial c_1^{'}} = -f_2\left(c_1^{'}\right) - c_1 \frac{dx_1\left(c_1^{'}\right)}{dc_1^{'}}.$$

When $c_1^{'} = c_1^*$, $\frac{\partial \pi_i}{\partial c_i^{'}} = 0$, hence $\frac{dx_1(c_1)}{dc_1} = \frac{-f_2(c_1;\alpha_2)}{c_1}$.

Note that $x_1(c_1 = 1) = 0$, then

$$x_1(c_1^*) = \int_{c_1}^1 \frac{f_2(c_1;\alpha_2)}{c_1} dc_1 = \int_{c_1}^1 \frac{\alpha_2 c_1^{\alpha_2-1}}{c_1} dc_1 = \frac{\alpha_2}{\alpha_2-1}\left(1-c_1^{\alpha_2-1}\right).$$

Then given c_1,the payoff of player 1 is

$$\begin{aligned}
\pi_1(x_1,x_2;c_1) &= \left[1-F_2(c_1)\right] - c_1 x_1(c_1) - I(\alpha_1) \\
&= 1 - c_1^{\alpha_2} - c_1\frac{\alpha_2}{\alpha_2-1}\left(1-c_1^{\alpha_2-1}\right) - \frac{5}{36}\left(\frac{1}{\alpha_1}-1\right) \\
&= 1 - \frac{\alpha_2}{\alpha_2-1}c_1 + \frac{1}{\alpha_2-1}c_1^{\alpha_2} - \frac{5}{36}\left(\frac{1}{\alpha_1}-1\right),
\end{aligned}$$

and the expected payoff of player 1 is given by

$$\begin{aligned}
E\pi_1^C(\alpha_1,\alpha_2) &= \int_0^1 [\pi_1(x_1,x_2;c_1)]\,dF(c_1) - I(\alpha_1) \\
&= \int_0^1 \left[1 - \frac{\alpha_2}{\alpha_2-1}c_1 + \frac{1}{\alpha_2-1}c_1^{\alpha_2}\right] d\alpha_1 c_1^{\alpha_1-1} dc_1 - \frac{5}{36}\left(\frac{1}{\alpha_1}-1\right) \\
&= \frac{\alpha_2}{(1+\alpha_1)(\alpha_1+\alpha_2)} - \frac{5}{36}\left(\frac{1}{\alpha_1}-1\right).
\end{aligned}$$

The expected payoff of each player is the same regardless of the disclosure policy.

And by taking first order condition with respect to α_1, we get

$$\frac{\partial E\pi_1(\alpha_1,\alpha_2)}{\partial \alpha_1} = \frac{\alpha_2}{1-\alpha_2}\left[\frac{1}{(1+\alpha_1)^2} - \frac{1}{(\alpha_2+\alpha_1)^2}\right] + \frac{5}{36\alpha_1^2}.$$

The symmetric equilibrium investment level is $\alpha_1 = \alpha_2 = \frac{1}{2}$ with expected payoff $\frac{7}{36}$.

Figure 3.1 provides the shape of $E\pi_1\left(\alpha_1;\alpha_2=\frac{1}{2}\right) = \frac{1}{(1+\alpha_1)(2\alpha_1+1)} - \frac{5}{36}\left(\frac{1}{\alpha_1}-1\right)$.

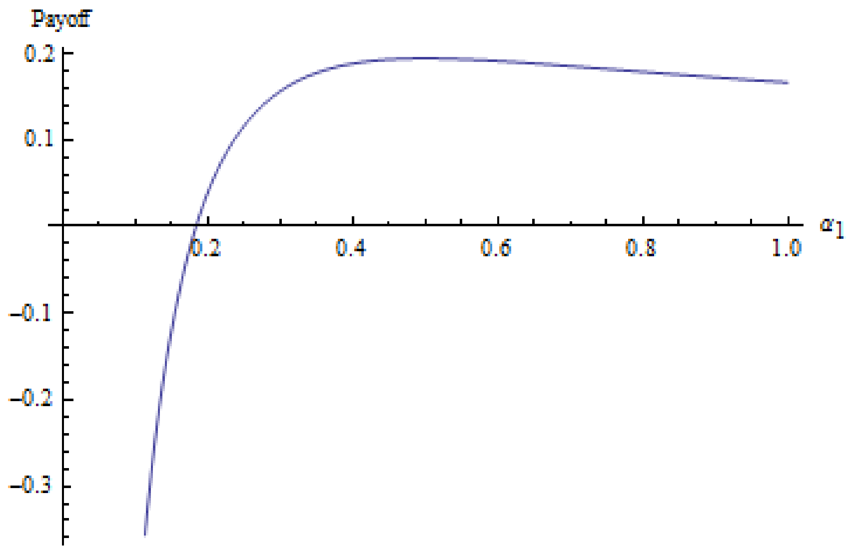

Figure 3.1: Expected payoff $E\pi_1\left(\alpha_1;\alpha_2=\frac{1}{2}\right)$

The symmetric equilibrium pre-contest investment level does exist in this two players all pay contests example.

5 Endogenous Entry

In this subsection, we further explore the role of disclosure policy in rent dissipation. We now consider strategic contestants. Instead of entering the contest with a fixed probability, each contestant makes his entry decision. We assume that entry incurs a fixed positive sunk cost Δ to each contestant, which is irreversible once entry decision has been made. One enters if and only if his expected payoff in the subsequent contest at least offsets the fixed cost.

The game then proceeds as follows. The contest organizer commits to and announces publicly her disclosure policy, D or C. Upon observing the disclosure rule, contestants simultaneously choose their entry strategies, and each participant sinks a fixed cost $\Delta > 0$ upon entry. The value of marginal cost is revealed if and only if the organizer committed to policy D. Participating contestants then simultaneously submit their effort entries.

5.1 Disclosure

Given the number of participants k, the equilibrium expected payoff of each contestant under disclosure policy is solved in section 2.1, where

$$E\pi_i^D(k) = \frac{1}{k} - \int_{\underline{c}}^{\overline{c}} c_i A(c_i, k)\, dF(c_i). \tag{3.15}$$

At the equilibrium entry probability $p \in [0, 1)$, every contestant is indifferent between participation and nonparticipation.

$$\pi(p) = \sum_{k=1}^{n} C_{n-1}^{k-1} p^{k-1} (1-p)^{n-k} E\pi_i^D(k) = \Delta. \tag{3.16}$$

5.2 Concealment

Following the methodology in section 2.2, for each participant i with marginal cost c_i, the probability of k contestants showing up is $C_{n-1}^{k-1}p^{k-1}(1-p)^{n-k}$, the payoff of player i is given as

$$\pi_i(x_i;k,c_i)=\sum_{k=1}^{n}C_{n-1}^{k-1}p^{k-1}(1-p)^{n-k}\left[1-F\left(x_{-i}^{-1}(x_i)\right)\right]^{n-1}-c_ix_i. \tag{3.17}$$

At a symmetric equilibrium, $x_i(\cdot)=x_{-i}(\cdot)=x(\cdot)$. We thus have

$$\frac{dx\left(c_i\right)}{dc_i}=-\sum_{k=1}^{n}(k-1)C_{n-1}^{k-1}p^{k-1}(1-p)^{n-k}f\left(c_i\right)\left[1-F(c_i)\right]^{n-2}/c_i.$$

Apply (*)[4]

$$A\left(c_i;k\right)=-\left(k-1\right)\int\frac{\left[1-F\left(c_i\right)\right]^{k-2}}{c_i}dF\left(c_i\right).$$

Therefore, the equilibrium bid of each player is

$$\begin{aligned}x_i\left(c_i\right)&=\sum_{k=1}^{n}C_{n-1}^{k-1}p^{k-1}(1-p)^{n-k}A\left(c_i\right)\\&=\sum_{k=1}^{n}(k-1)C_{n-1}^{k-1}p^{k-1}(1-p)^{n-k}\int_{c_i}^{\bar{c}}\frac{\left[1-F\left(\widetilde{c}\right)\right]^{n-2}}{\widetilde{c}}dF\left(\widetilde{c}\right).\end{aligned}$$

Given c_i, the expected payoff of each player is given by

$$\begin{aligned}\pi_i^C\left(x_i,\mathbf{x}_{-i};c_i\right)&=\sum_{k=1}^{n}C_{n-1}^{k-1}p^{k-1}(1-p)^{n-k}\Pr\left(c_i<\widetilde{c}\right)1-c_ix_i\left(c_i\right)\\&=\sum_{k=1}^{n}C_{n-1}^{k-1}p^{k-1}(1-p)^{n-k}\left\{\left[1-F\left(c_i\right)\right]^{k-1}-c_iA\left(c_i;k\right)\right\}.\end{aligned}$$

[4]Note that $A\left(\bar{c}\right)=0$ since $F\left(\bar{c}\right)=1$. And $x\left(\bar{c}\right)=0$.

Next, the expected payoff of each player under concealment policy is given by

$$E\pi_i^C = \sum_{k=1}^{n} C_{n-1}^{k-1} p^{k-1} (1-p)^{n-k} \left\{ \frac{1}{k} - \int_{\underline{c}}^{\overline{c}} c_i A\left(c_i; k\right) dF\left(c_i\right) \right\}. \tag{3.18}$$

At the equilibrium entry probability $p \in [0, 1)$, every contestant's expected payoff offsets his entry cost.

$$E\pi_i^C = \Delta. \tag{3.19}$$

Moreover, each potential contestant has 0 expected payoff at the equilibrium[5], and the expected total cost of effort can be written as

$$TEC\left(p\right) = \left[1 - (1-p)^n\right] - np\Delta. \tag{3.20}$$

where $n \geq 2$ is the number of potential contestants.

5.3 Comparison

Note that (3.16) and (3.19) take identical forms, then we get the equivalent equilibrium entry probability and from (3.20) we get the same rent dissipation rate. Given the same entry, however, hiding the information leads to higher effort according to the exogenous entry result. Therefore, contest organizer still prefers to conceal the actual value of contestants' abilities to elicit higher total expected effort. Then we have the following result,

Theorem 6 *When entry is endogenized, both policies induce equal equilibrium entry probability $p(k)$ and the same dissipation rate, while concealing the abilities of contestants still elicits higher expected total revenue to contest organizer.*

Theorem 5 and 6 strengthen the argument of Theorem 1 and 2. The results of Theorem 1 and 2 are robust even when an endogenous ability distribution or

[5] Please refer to Fu and Lu (2010) for detailed interpretation and proof.

endogenous entry is allowed in the game. It further verifies that the expected payoff for each contestant and equilibrium level of total effort cost in the contest do not depend on whether the contest organizer disclose information, while concealing the abilities of the contestants elicits higher expected total effort, despite the endogeneity of ability distribution and entry.

6 Contests with Nonlinear Cost

In this section, we will check whether our result about disclosure policy is robust when cost function is nonlinear. A bid x_i costs a contestant $c(x_i)$, with $c'(\cdot) > 0$ and $c''(\cdot) \geq 0$. For the sake of tractability, we assume that player i's bidding cost function takes the form $c\left(x_i\right) = c_i x_i^{\beta}$. We just focus on the basic model with two contestants and a unique prize.

6.1 Disclosure

We assume at the point in time when the efforts are chosen, the contest organizer will disclose each player's unit cost to public; hence, this problem is a perfectly discriminating contest with complete information, the payoff to player i is given by

$$\pi\left(x_1, x_2; c_1, c_2\right) = P_i\left(x_1, x_2\right) 1 - c_i x_i^{\beta}.$$

Following the method outlined by Hillman and Riley (1999), the equilibrium of the perfectly discriminating contest for given values of c_1 and c_2 is unique and described as follows,

Proposition 4 *The unique equilibrium of the perfectly discriminating contest for given values of c_1 and c_2 $(c_1 < c_2)$ is in mixed strategies and bids are described* *by the following cumulative distribution functions:*

$$G_1(x_1) = \begin{cases} c_2 x_1^{\beta} & \textit{for } x_1 \in [0, \left(\frac{1}{c_2}\right)^{\frac{1}{\beta}}), \\ 1 & \textit{for } x_1 \geq \left(\frac{1}{c_2}\right)^{\frac{1}{\beta}}; \end{cases}$$

$$G_2(x_2) = \begin{cases} 1 - \frac{c_1}{c_2} + c_1 x_2^{\beta} & \textit{for } x_2 \in [0, \left(\frac{1}{c_2}\right)^{\frac{1}{\beta}}), \\ 1 & \textit{for } x_2 \geq \left(\frac{1}{c_2}\right)^{\frac{1}{\beta}}. \end{cases}$$

The expected value of bids for bidder 1 and bidder 2 are

$$Ex_1 = \int_0^{\left(\frac{1}{c_2}\right)^{\frac{1}{\beta}}} \beta c_2 x_1^{\beta} dx_1 = \frac{\beta}{\beta+1}\left(\frac{1}{c_2}\right)^{\frac{1}{\beta}},$$

$$Ex_2 = 0 \times \Pr(x_2 = 0) + \int_{0^+}^{\left(\frac{1}{c_2}\right)^{\frac{1}{\beta}}} \beta c_1 x_2^{\beta-1} x_2 dx_2 = \frac{\beta c_1}{\beta+1}\left(\frac{1}{c_2}\right)^{\frac{\beta+1}{\beta}}.$$

Therefore, with general cumulative distribution function $F(c_i)$ and $c_i \in [\underline{c}, \overline{c}]$, the total expected effort for contest organizer is

$$\begin{aligned} R^D &= 2\int\int_{c_1<c_2} [Ex_1 + Ex_2]\, dF(c_1)\, dF(c_2) \\ &= 2\int_{\underline{c}}^{\overline{c}}\int_{\underline{c}}^{c_2} \left[\frac{\beta}{\beta+1}\left(\frac{1}{c_2}\right)^{\frac{1}{\beta}} + \frac{\beta c_1}{\beta+1}\left(\frac{1}{c_2}\right)^{\frac{\beta+1}{\beta}}\right] dF(c_1)\, dF(c_2) \end{aligned}$$

$$\begin{aligned} &= \frac{4\beta}{\beta+1}\int_{\underline{c}}^{\overline{c}}\left(\frac{1}{c_2}\right)^{\frac{1}{\beta}} F(c_2)\, dF(c_2) \\ &\quad -\frac{2\beta}{\beta+1}\int_{\underline{c}}^{\overline{c}}\left[\left(\frac{1}{c_2}\right)^{\frac{\beta+1}{\beta}}\left(\int_{\underline{c}}^{c_2} F(c_1)\, dc_1\right)\right] dF(c_2). \end{aligned} \tag{3.21}$$

6.2 Concealment

In this subsection, we assume that at the point in time when the efforts are chosen, the contest organizer will conceal information regarding player's unit cost, and each player does not know the rival player's unit cost; hence, at this stage, the problem describes a perfectly discriminating contest with incomplete information.

The payoffs of the player is given as

$$\begin{aligned}\pi_i\left(x_1,x_2,c_1,c_2\right) &= p_i\left(x_i^{'}>x_j\right)1-c_ix_i^{\beta}\left(c_i^{'}\right)\\ &= p_i\left(c_i^{'}<c_j\right)1-c_ix_i^{\beta}\left(c_i^{'}\right)\\ &= \left[1-F\left(c_i^{'}\right)\right]-c_ix_i^{\beta}\left(c_i^{'}\right).\end{aligned} \tag{3.22}$$

Note that $x_i\left(c_i\right)$ is decreasing with c_i,thus the higher the cost, the lower the effort. Take first order condition with respect to c_i we obtain

$$\frac{\partial\pi_i}{\partial c_i^{'}}=-f\left(c_i^{'}\right)-c_i\frac{dx_i^{\beta}\left(c_i^{'}\right)}{dc_i^{'}},$$

when $c_i^{'}=c_i$, $\frac{\partial\pi_i}{\partial c_i^{'}}=0$. Hence

$$\frac{dx_i\left(c_i\right)}{dc_i}=\frac{-f\left(c_i\right)}{\beta c_ix_i^{\beta-1}}.$$

Note that $x_i\left(\overline{c}\right)=0$, therefore, the individual equilibrium effort is given by

$$x_i\left(c_i\right)=\left[\int_{c_i}^{\overline{c}}\frac{f\left(c_i\right)}{c_i}dc_i\right]^{\frac{1}{\beta}}.$$

The total expected effort for contest organizer is

$$\begin{aligned}R^C &= 2\int_{\underline{c}}^{\overline{c}}x_i\left(c_i\right)dF\left(c_i\right)\\ &= 2\int_{\underline{c}}^{\overline{c}}\left(\int_{c_1}^{\overline{c}}\frac{1}{c_2}dF\left(c_2\right)\right)^{\frac{1}{\beta}}dF\left(c_1\right).\end{aligned} \tag{3.23}$$

6.3 Optimal Disclosure Policy

We now compare (3.21) and (3.23) to investigate the effort maximizing disclosure policy.

Compare R^D and R^C,

$$\begin{aligned}
R^D &= \frac{4\beta}{\beta+1}\int_{\underline{c}}^{\overline{c}}\left(\frac{1}{c_2}\right)^{\frac{1}{\beta}}F(c_2)\,dF(c_2) \\
&\quad -\frac{2\beta}{\beta+1}\int_{\underline{c}}^{\overline{c}}\left[\left(\frac{1}{c_2}\right)^{\frac{\beta+1}{\beta}}\left(\int_{\underline{c}}^{c_2}F(c_1)\,dc_1\right)\right]dF(c_2) \\
&= \frac{2\beta}{\beta+1}\int_{\underline{c}}^{\overline{c}}\left\{2\left(\frac{1}{c_2}\right)^{\frac{1}{\beta}}F(c_2)-\left(\frac{1}{c_2}\right)^{\frac{\beta+1}{\beta}}\left(\int_{\underline{c}}^{c_2}F(c_1)\,dc_1\right)\right\}dF(c_2),
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
R^C &= 2\int_{\underline{c}}^{\overline{c}}\left(\int_{c_1}^{\overline{c}}\frac{1}{c_2}dF(c_2)\right)^{\frac{1}{\beta}}dF(c_1) \\
&= 2\int_{\underline{c}}^{\overline{c}}\left(\underset{c_1<c_2}{E}\frac{1}{c_2}\right)^{\frac{1}{\beta}}dF(c_1).
\end{aligned} \tag{3.23}$$

With linear cost $c(x_i)=c_ix_i$, we have shown that $R^C>R^D$ in section 2. However, it is possible that at some level where $\beta\neq 1$, $R^C<R^D$.

Example 1 *In a two player perfectly discriminating contest, when $F(c)=c$, and $c\in[1,2]$, with either convex cost $c(x_i)=c_ix_i^2$ or concave cost $c(x_i)=c_ix_i^{0.5}$, disclosing the abilities of contestants elicits higher total expected revenue to contest organizer.*

Proof. When $F(c)=c, c\in[1,2]$,

(1) With convex cost $\beta=2$,

$$\begin{aligned}
R^D &= \frac{4}{3}\int_1^2\left\{2\left(\frac{1}{c_2}\right)^{\frac{1}{2}}c_2-\left(\frac{1}{c_2}\right)^{\frac{3}{2}}\left(\int_1^{c_2}c_1dc_1\right)\right\}dc_2 \\
&= \frac{4}{3}\left(2^{\frac{3}{2}}-2^{-\frac{1}{2}}\right)=2.8284,
\end{aligned}$$

$$R^C=2\int_1^2\left(\int_{c_1}^2\frac{1}{c_2}dc_2\right)^{\frac{1}{2}}dc_1=1.0325.$$

(2) With concave cost $\beta=\frac{1}{2}$,

$$\begin{aligned}
R^D &= \frac{2}{3}\int_1^2 \left\{ 2\left(\frac{1}{c_2}\right)^2 c_2 - \left(\frac{1}{c_2}\right)^3 \left(\int_1^{c_2} c_1 dc_1\right) \right\} dc_2 \\
&= \ln 2 + \frac{1}{8} = 0.8182,
\end{aligned}$$

$$\begin{aligned}
R^C &= 2\int_1^2 \left(\int_{c_1}^2 \frac{1}{c_2} dc_2\right)^2 dc_1 \\
&= 0.2665.
\end{aligned}$$

Then $R^D > R^C$ in both cases. ■

The following theorem regarding nonlinear cost is therefore obvious,

Theorem 7 *The optimal disclosure policy depends on convexity of cost functions, when cost function is nonlinear, disclosing the abilities of contestants may elicit higher total expected revenue.*

Theorem 7 reveals that the nonlinearity of effort cost function does affect the optimal disclosure policy of contest organizer. In the above analysis, we adopted a power form effort cost function. The analysis shows that the form of the cost function plays a pivotal role in determining the optimal disclosure policy of the contest organizer. With a linear effort cost function, a concealment policy leads to the best outcome in terms of expected aggregate effort, although this need not to be true when effort cost is nonlinear. Hence the effect of optimal disclosure policy is ambiguous and no general results can be obtained to guide contest organizer.

7 Conclusion

This paper investigates the optimal disclosure policy of the contest organizer in a perfectly discriminating contest. The private abilities of the contestants are stochastic and they are observed by the contest organizer who decides whether to disclose this information publicly. The organizer may care about total effort or rent dissipation. We find that in a benchmark model with unique prize and linear effort cost, concealing the abilities of the contestants elicits higher expected total effort, regardless of the distribution of the abilities. For rent dissipation, we find that the rent dissipation rate does not depend on the disclosure policy. We then develop these results in the context of a tractable two-prize model. While we believe that our main insights are robust in settings with multiple prizes as long as effort cost function is linear, we leave the analysis of more generalized environments to future work.

We further study the robustness of our results while allowing endogenous ability distribution and endogenous entry of contestants. We find that our findings are robust to these generalized settings. Another natural extension of our model would be to allow the cost function to be nonlinear. However, the analysis of this extension would depend critically on the cost function form. The finding from the benchmark model is not robust, and the organizer may prefer disclosure instead.

Bibliography

[1] Alcalde, J., and Dahm, M. (2010). Rent seeking and rent dissipation: A neutrality result, *Journal of Public Economics*, 94 (1-2), 1-7.

[2] Amegashie, J.A. (2009). American idol: should it be a singing contest or a popularity contest, *Journal of Cultural Economics*, 33 (4), 265-277.

[3] Aoyagi, M. (2010). Information feedback in a dynamic tournament. *Games and Economic Behavior*, 70 (2), 242-260.

[4] Baik, K.H. (1994). Effort levels in contest with asymmetric players. *Southern Economic Journal,* 61, 367-378.

[5] Baye, M.R., Kovenock, D., and de Vries, C.G. (1993). Rigging the lobbying process: An application of the all-pay auction, *American Economic Review*, 83 (1), 289-294.

[6] Baye, M.R., Kovenock, D., and de Vries, C.G. (1994). The solution to the Tullock rent-seeking game when $R > 2$: mixed strategy equilibria and mean dissipation rates, *Public Choice,* 81, 363–380.

[7] Baye, M.R., Kovenock, D., and de Vries, C.G. (1996). The all-pay auction with complete information. *Economic Theory*, 8, 362-380.

[8] Cason, T.N., Masters, W.A., and Sheremeta, R.M. (2010). Entry into winner-take-all and proportional-prize contests: An experimental study, *Journal of Public Economics*, 94, 604-611.

[9] Che, Y.K. and Gale, I. (1997). Rent dissipation when rent seekers are budget constrained, *Public Choice*, 92, 109-126.

[10] Che, Y.K. and Gale, I. (2000). Difference-form contests and the robustness of all-pay auctions, *Games and Economic Behavior*, 30, 22-43.

[11] Che, Y.K., and Gale, I. (2003). Optimal design of research contests, *American Economics Review*, 93, 646-671.

[12] Clark, D.J., and Riis, C. (1998). Competition over more than one prize, *American Economic Review*, 88, 276-289.

[13] Dasgupta, S. (1990). Competition for procurement contracts and underinvestment, *International Economics Review*, 31(4), 841-865.

[14] Dasgupta, P., and Maskin, E. (1986). The existence of equilibrium in discontinuous economic games, I: Theory, *Review of Economic Studies*, 53 (1), 1–26

[15] Denter, P., and Sisak, D. (2011). "Where ignorance is bliss, 'tis folly to be wise": Transparency in contests, working paper.

[16] Dixit, A.K., and Shapiro, C. (1986). Entry dynamics with mixed strategies, *The Economics of Strategic Planning*, Thomas, L.G. ed., Lexington Press.

[17] Ederer, F. (2010). Feedback and motivation in dynamic tournaments, *Journal of Economics and Management Strategy*, 19, 733-769.

[18] Epstein, G.S., Mealem, Y., and Nitzan, S. (2011). Political culture and discrimination in contests, *Journal of Public Economics*, 95, 88-93.

[19] Fang, H. (2002). Lottery versus all-pay auction models of lobbying, *Public Choice*, 112, 351-371.

[20] Fu, Q., Jiao, Q., and Lu, J. (2011). On disclosure policy in contests with stochastic entry, *Public Choice*, 148 (3), 419-434.

[21] Fu, Q., and Lu, J. (2009). Contests with pre-contest investment, *Economics Letters*, 103, 142-145.

[22] Fu, Q., and Lu, J. (2010). Optimal endogenous entry in tournaments, *Economic Inquiry*, 48, 80-89.

[23] Fu, Q., and Lu J. (2008). A Micro Foundation of Generalized Multi-Prize Contests: A Noisy Ranking Perspective, *Social Choice and Welfare*, forthcoming.

[24] Fudenberg, D., Kreps, D., and Maskin, E. (1990). Repeated Games with Long-run and Short-run Players, *Review of Economic Studies*, 57, 555-573.

[25] Fudenberg, D., and Levin, D. (1989). Reputation and Equilibrium Selection in Games with a Patient Player, *Econometrica*, 57, 759-778

[26] Fullerton, R.L., and McAfee, P.R. (1999). Auctioning entry into tournaments, *Journal of Political Economy*, 107, 573-605.

[27] Gershkov, A. and Perry, M. (2009). Tournament with midterm reviews, *Games and Economic Behavior*, 66 (1), 162-190.

[28] Gilbert, R.J., and Klemperer, P. (2000). An equilibrium theory of rationing, *RAND Journal of Economics*, 31(1), 1-21

[29] Goltsman M., and Mukherjee, A. (2011). Interim performance feedback in multistage tournaments: The optimality of partial disclosure, *Journal of Labor Economics*, 29, 229-265.

[30] Gradstein, M. and Konrad, K.A. (1999). Orchestrating rent seeking contests, *The Economic Journal*, 109, 536–545.

[31] Gürtler, O. and Harbring, C. (2010). Feedback in tournaments under commitment problems: Experimental evidence, *Journal of Economics and Management Strategy*, 19, 771-810.

[32] Higgins, R.S., Shughart W.F.II, and Tollison, R.D., 1985, Free entry and efficient rent seeking, *Public Choice*, 46, 247-258; reprinted in Congleton, R.D., Hillman, A.L., and Konrad, K.A. (eds.), 2008, *40 Years of Research on Rent Seeking 1: Theory of Rent Seeking*, Berlin: Springer, 121-132.

[33] Hillman, A.L, and Riley. J.G. (1989). Politically contestable rents and transfers. *Economics and Politics*, 1 (1), 17-39.

[34] Hurley, T.M., and Shogen, J.F. (1998). Effort levels in a Cournot Nash contest with asymmetric information. *Journal of Public Economics,* 69, 195-210

[35] Kaplan, T.R., and Sela, A. (2010). Effective contests, *Economics Letters*, 106, 38-41.

[36] Konrad, K. (2007). Strategy in contests-an introduction, WZB Discussion Papers.

[37] Konrad, K., (2009). *Strategy and Dynamics in Contests*, Oxford University Press.

[38] Konrad, K., and Kovenock, D. (2008). Contests with stochastic abilities, *Economic Inquiry.* 48 (1), 89-103

[39] Kovenock, D., Morathz, F., and Münster, J. (2010). Information sharing in contests, working paper.

[40] Kreps, D. (1990). *A Course in Microeconomic Theory*, Princeton, Princeton University Press.

[41] Lazear, E.P. (1989). Pay equality and industrial politics, *Journal of Political Economy*, 97 (3), 561-580.

[42] Levin, D., and Ozdenoren, E. (2004). Auctions with uncertain numbers of bidders, *Journal of Economic Theory*, 118 (2), 229-51.

[43] Levin, D., and Smith, J.L. (1994). Equilibrium in auctions with entry, *American Economic Review,* 84, 585–599.

[44] Lim,W., and Matros, A. (2009). Contests with a stochastic number of players, *Games and Economic Behavior*, 67, 584–597.

[45] Lu, J. (2009). Auction design with opportunity cost, *Economic Theory*, 38(1), 73-103.

[46] Lu, J. (2010). Entry coordination and auction design with private costs of information-acquisition, *Economic Inquiry*, 48(2), 274-289.

[47] McAfee, R.P, and McMillan, J. (1987). Auctions and bidding. *Journal of Economic Literature*, 25, 699-738.

[48] McAfee, R.P., McMillan, J. (1987a). Auctions with entry. *Economics Letters* 23, 343–347.

[49] McAfee, R.P., and McMillan, J. (1987b). Auctions with a stochastic number of bidders, *Journal of Economic Theory*, 43, 1–19.

[50] Menezes, F.M., and Monteiro, P.K. (2000). Auction with endogenous participation, *Review of Economic Design*, 5, 71-89.

[51] Moldovanu, B., and Sela, A. (2001). The optimal allocation of prizes in contests. *American Economic Review*. 91 (3), 542-558

[52] Morgan, J., Orzen, H., and Sefton, M. (2010). Endogenous entry in contests, *Economic Theory*, forthcoming.

[53] Moreno, D., and Wooders, J. (2010). Auctions with heterogeneous entry costs, *Rand Journal of Economics*, forthcoming.

[54] Münster, J. (2006). Contests with an unknown number of contestants, *Public Choice*, 129(3-4), 353-368.

[55] Myerson, R.B. (1981). Optimal auction design, *Mathematics of Operations Research*, 6(1), 58-73.

[56] Myerson, R.B. (1998). Population uncertainty and Poisson games, *International Journal of Game Theory,* 27, 375-392.

[57] Myerson, R.B. (1998). Extended Poisson games and the Condorcet jury theorem, *Games and Economic Behavior,* 25, 111-131.

[58] Myerson, R.B. (2000). Large Poisson games, *Journal of Economic Theory,* 94 (1), 7-45.

[59] Myerson, R.B., and Wärneryd, K. (2006). Population uncertainty in contests, *Economic Theory*, 27 (2), 469-474.

[60] Nti, K.O., 2004. Maximum efforts in contests with asymmetric valuations, *European Journal of Political Economy*, 20 (4), 1059-66.

[61] O'Keeffe, M., Viscusi, W.K., and Zeckhauser, R.J. (1984). Economic contests: comparative reward schemes, *Journal of Labor Economics*, 2 (1), 27-56.

[62] Samuelson, W.F. (1985). Competitive bidding with entry costs, *Economics Letters,* 17, 53–57.

[63] Shapiro, C. (1982). Consumer information, product quality, and seller reputation, *Bell Journal of Economics*, 13, 20-35.

[64] Shaprio, C. (1983). Premium for high quality products as returns to reputations, *Quarterly Journal of Economics*, 98, 659-679.

[65] Shapiro, C. (1989). Theories of oligopoly behavior, *Handbook of Industrial Organization*, Schmalensee, R. and Willig, R.D. ed, North-Holland Press, 1989.

[66] Shi, X. (2009). Optimal auctions with information acquisition, Working Paper, University of Toronto.

[67] Siegel, R. (2009). All pay contests, *Econometrica,* 77 (1), 71-92.

[68] Skaperdas, S. (1996). Contest success function, *Economic Theory*, 7 (2), 283-290.

[69] Szidarovszky, F., and Okuguchi, K. (1997). On the existence and uniqueness of pure Nash equilibrium in rent-seeking games, *Games and Economic Behavior*, 18, 135-140.

[70] Taylor, C.R. (1995). Digging for golden carrots: An analysis of research tournaments, *American Economic Review*, 85(4), 872-890.

[71] Tullock, G. (1980). Efficient rent seeking, in: J. Buchanan, R. Tollison and G. Tullock, (eds.), *Towards a Theory of the Rent-Seeking Society*, College Station, Texas A&M University Press, 97-112.

[72] Wang, Z. (2010). The optimal accuracy level in asymmetric contests. *B.E. Journal of Theoretical Economics,* Topics in Theoretical Economics, 10(1).

[73] Wärneryd, K. (2001). Replicating contests, *Economics Letters*, 71 (3), 323-27.

[74] Wärneryd, K. (2003). Information in conflicts, *Journal of Economic Theory.* 110 (1). 121-136.

[75] Wärneryd, K. (2006). Participation in contests with asymmetric information, SSRN Discussion Paper 940169.

Appendix A: Proofs of Chapter One

Proof of Claim 1

Proof. When N is disclosed, it is well known that a unique pure-strategy symmetric equilibrium exists, and the solution is not different from (1.4). The analysis is less explicit in the case where N is concealed. We then examine the payoff function $\pi_i(x_i; q)$. (1.7) still solves equation (1.8), but it has yet to be established as a global maximizer of $\pi_i(x_i; q)$ given that all other participants exert the same effort.

One can verify

$$\begin{aligned}\xi_N(x_i) &= \left.\frac{\partial^2 p_i(x_i, \mathbf{x}_{-i}; N)}{\partial x_i^2}\right|_{x_{-i}=x_C(q)} \\ &= \frac{-(r+1)x_i^r + (r-1)(N-1)(x_C(q))^r}{[x_i^r + (N-1)(x_C(q))^r]^3} r x_i^{r-2}(N-1)(x_C(q))^r.\end{aligned}$$

It implies that $\Phi_N(x_i) = \left.\frac{\partial p_i(x_i, \mathbf{x}_{-i}; N)}{\partial x_i}\right|_{x_{-i}=x_C(q)}$ is not monotonic: It is positive if $x_i^r < \frac{r-1}{r+1}(N-1)(x_C(q))^r$, and negative if $x_i^r > \frac{r-1}{r+1}(N-1)(x_C(q))^r$. Clearly $\frac{r-1}{r+1}(N-1) \le 1$ if and only if $r \le \frac{N}{N-2}$. Because $r \le 1 + \frac{1}{M-1}$, we must have $\frac{r-1}{r+1}(N-1) < 1$ for all $N \le M$.

Let

$$\Phi(x_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{\partial p_i(x_i, \mathbf{x}_{-i}; N)}{\partial x_i}|_{x_{-i}=x_C(q)}$$

and

$$\xi(x_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{\partial^2 p_i(x_i, \mathbf{x}_{-i}; N)}{\partial x_i^2}|_{x_{-i}=x_C(q)}.$$

The above results imply that $x_i^r > \frac{r-1}{r+1}(N-1)(x_C(q))^r$ when $x_i = x_C(q)$ for all

$N \leq M$, which means that $\xi(x_i)|_{x_i = x_C(q)} < 0$. This leads to that

$$\left.\frac{d^2\pi_i(x_i, \mathbf{x}_{-i}; q)}{dx_i^2}\right|_{x_i = \mathbf{x}_{-i} = x_C(q)} = v\, \xi(x_i)|_{x_i = x_C(q)} < 0.$$

Hence, $x_i = x_C(q)$ must be at least a local maximizer of when $x_{-i} = x_C(q)$.

Since when $x_i < [\frac{r-1}{r+1}]^{1/r} x_C(q)$, $\xi_N(x_i) > 0$ for all $N \leq M$, we have $\xi(x_i) > 0$ when $x_i < [\frac{r-1}{r+1}]^{1/r} x_C(q)$, which means that $\Phi(x_i)$ increases when $x_i < [\frac{r-1}{r+1}]^{1/r} x_C(q)$. Similarly, $\xi(x_i) < 0$ when $x_i > [\frac{r-1}{r+1}(M-1)]^{1/r} x_C(q)$, which means that $\Phi(x_i)$ decreases when $x_i > [\frac{r-1}{r+1}(M-1)]^{1/r} x_C(q)$. We next show that there exists a unique $x' \in ([\frac{r-1}{r+1}]^{1/r} x_C(q), [\frac{r-1}{r+1}(M-1)]^{1/r} x_C(q))$ such that $\Phi(x_i)$ increases (decreases) if and only if $x_i < (>)\ x'$. For this purpose, it suffices to show that there exists a unique $x' \in ([\frac{r-1}{r+1}]^{1/r} x_C(q), [\frac{r-1}{r+1}(M-1)]^{1/r} x_C(q))$, such that $\xi(x') = 0$.

First, such x' must exist by continuity of $\xi(x_i)$. As have been revealed, $\xi(x_i) > 0$ when $x_i < [\frac{r-1}{r+1}]^{1/r} x_C(q)$; and $\xi(x_i) < 0$ when $x_i < [\frac{r-1}{r+1}(M-1)]^{1/r} x_C(q)$.

Second, the uniqueness of x' can be verified as below. We have

$$\begin{aligned}
&\left.\frac{\partial^3 p_i(x_i, \mathbf{x}_{-i}; N)}{\partial x_i^3}\right|_{x_{-i} = x_C(q)} \\
= \;& r(N-1)(x_C(q))^r \left\{ \begin{array}{c} (r-2)x_i^{r-3} \frac{-(r+1)x_i^r + (r-1)(N-1)(x_C(q))^r}{[x_i^r + (N-1)(x_C(q))^r]^3} \\ + x_i^{r-2}\left(\frac{-r(r+1)x_i^{r-1}[x_i^r + (N-1)(x_C(q))^r]}{[x_i^r + (N-1)(x_C(q))^r]^4}\right. \\ \left. - \frac{3r x_i^{r-1}[-(r+1)x_i^r + (r-1)(N-1)(x_C(q))^r]}{[x_i^r + (N-1)(x_C(q))^r]^4}\right) \end{array} \right\} \\
= \;& \frac{r(N-1)(x_C(q))^r x_i^{r-3}}{[x_i^r + (N-1)(x_C(q))^r]^3} \left\{ \begin{array}{c} (r-2)[-(r+1)x_i^r + (r-1)(N-1)(x_C(q))^r] \\ + \frac{-r(r+1)x_i^r[x_i^r + (N-1)(x_C(q))^r]}{[x_i^r + (N-1)(x_C(q))^r]} \\ - \frac{3r x_i^r[-(r+1)x_i^r + (r-1)(N-1)(x_C(q))^r]}{[x_i^r + (N-1)(x_C(q))^r]} \end{array} \right\} \\
= \;& \frac{r(N-1)(x_C(q))^r x_i^{r-3}}{[x_i^r + (N-1)(x_C(q))^r]^3} \left\{ \begin{array}{c} (r-2)[-(r+1)x_i^r + (r-1)(N-1)(x_C(q))^r] \\ + \frac{2r x_i^r}{[x_i^r + (N-1)(x_C(q))^r]}[(r+1)x_i^r \\ -(2r-1)(N-1)(x_C(q))^r] \end{array} \right\}.
\end{aligned}$$

Recall $\xi_N(x_i) = \frac{-(r+1)x_i^r+(r-1)(N-1)(x_C(q))^r}{[x_i^r+(N-1)(x_C(q))^r]^3} r x_i^{r-2}(N-1)(x_C(q))^r$. We then have

$$\begin{aligned}
& \left.\frac{\partial^3 p_i(x_i, \mathbf{x}_{-i}; N)}{\partial x_i^3}\right|_{x_{-i}=x_C(q)} \\
= \; & (r-2)x_i^{-1}\xi_N(x_i) \\
& + \frac{2r^2(N-1)(x_C(q))^r x_i^{2r-3}}{[x_i^r + (N-1)(x_C(q))^r]^4}[(r+1)x_i^r - (2r-1)(N-1)(x_C(q))^r].
\end{aligned}$$

We now claim $[(r+1)x_i^r - (2r-1)(N-1)(x_C(q))^r]$ is negative for all $x_i \leq [\frac{r-1}{r+1}(M-1)]^{1/r}x_C(q)$. A detailed proof is as follows. From $x_i \leq [\frac{r-1}{r+1}(M-1)]^{1/r}x_C(q)$, we have $(r+1)x_i^r \leq (r-1)(M-1)(x_C(q))^r$. To show $(r+1)x_i^r - (2r-1)(N-1)(x_C(q))^r < 0$, it suffices to show $(r-1)(M-1) < (2r-1)(N-1)$ when $N = 2$, which requires $r < 1 + \frac{1}{M-3}$. This holds as $r < 1 + \frac{1}{M-1}$.

We thus have at any $x_i \in ([\frac{r-1}{r+1}]^{1/r}x_C(q), [\frac{r-1}{r+1}(M-1)]^{1/r}x_C(q))$ such that $\xi(x_i) = 0$, $\xi(x_i)$ must be locally decreasing, because

$$\begin{aligned}
& \frac{\partial \xi(x_i)}{\partial x_i} \\
= \; & (r-2)x_i^{-1}\sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1-q)^{M-N}\xi_N(x_i) \\
& + \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1-q)^{M-N}A_N(x_i) \\
= \; & (r-2)x_i^{-1}\xi(x_i) + \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1-q)^{M-N}A_N(x_i) \\
= \; & \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1-q)^{M-N}A_N(x_i) < 0
\end{aligned}$$

as

$$A_N(x_i) = \frac{2r^2(N-1)(x_C(q))^r x_i^{2r-3}}{[x_i^r + (N-1)(x_C(q))^r]^4}[(r+1)x_i^r - (2r-1)(N-1)(x_C(q))^r] < 0.$$

We are ready to show the uniqueness of x' by contradiction. Suppose that there exists more than one zero points x' and x'' with $x' \neq x''$ for $\xi(x_i)$. Because $\xi(x_i)$ must be locally decreasing, then there must exist at least another zero point

$x''' \in (x', x'')$ at which $\xi(x_i)$ is locally increasing. Contradiction thus results. Hence, such a zero point x' of $\xi(x_i)$ must be unique.

Recall $\Phi(x_i)$ increases (decreases) if and only if $x_i < (>)\ x'$ and it reaches its maximum at x'. Note $\frac{\partial \pi_i(x_i, \mathbf{x}_{-i}; q)}{\partial x_i}|_{x_{-i}=x_C(q)} = v\Phi(x_i) - 1$ and $\Phi(0) = 0$. $\frac{\partial \pi_i(x_i, \mathbf{x}_{-i}; q)}{\partial x_i}|_{x_{-i}=x_C(q)}$ at most has two zero points. Note $x_i = x_C(q)$ must be a zero point for $\frac{\partial \pi_i(x_i, \mathbf{x}_{-i}; q)}{\partial x_i}|_{x_{-i}=x_C(q)}$ by definition. One can verify that

$$\pi_i(x_C(q), \mathbf{x}_{-i}; q)|_{x_{-i}=x_C(q)} > \pi_i(0, \mathbf{x}_{-i}; q)|_{x_{-i}=x_C(q)} = v(1-q)^{M-1}$$

as follows.We have

$$\begin{aligned}
& \pi_i(x_C(q), \mathbf{x}_{-i}; q)|_{x_{-i}=x_C(q)} \\
= & \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{1}{N} v - x_C(q) \\
= & \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{1}{N} v - r \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{1}{N}(1-\frac{1}{N})v \\
= & v(1-q)^{M-1} + \sum_{N=2}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}[\frac{1}{N} - r\frac{1}{N}(1-\frac{1}{N})]v
\end{aligned}$$

The terms $\frac{1}{N} - r\frac{1}{N}(1-\frac{1}{N})$, $N \geq 2$ are apparently positive because $r(1-\frac{1}{N}) \leq \frac{M}{M-1} \times \frac{N-1}{N} \leqq 1$ if and only if $N \leqq M$.

Since $\pi_i(x_C(q), \mathbf{x}_{-i}; q)|_{x_{-i}=x_C(q)} > \pi_i(0, \mathbf{x}_{-i}; q)|_{x_{-i}=x_C(q)}$, $\frac{\partial \pi_i(x_i, \mathbf{x}_{-i}; q)}{\partial x_i}|_{x_{-i}=x_C(q)}$ must have two zero points, and $x_i = x_C(q)$ is the local maximum point of $\pi_i(x_i, \mathbf{x}_{-i}; q)|_{x_{-i}=x_C(q)}$ and the other is the local minimum point. Hence, $x_i = x_C(q)$ is the global best response. ■

Proof of Claim 2

Proof. We first consider the contest with disclosure. When $N = 1$, the entrant clearly exerts zero effort. When $N \geq 2$, we claim that all entrants exert an equilibrium effort of $x(N) = H^{-1}\left(\frac{N-1}{N^2}\right) = -\frac{1}{\alpha}\ln(1 - \frac{N-1}{N^2}\alpha)$. To prove this claim, we need to show that when $x_j = x(N)$ for $j \neq i$, $x_i = x(N)$ maximizes $\pi_i = p_i(x_i, \mathbf{x}_{-i}; N)v - x_i = \frac{e^{\alpha x_i}-1}{(e^{\alpha x_i}-1)+(N-1)(e^{\alpha x(N)}-1)} - x_i$. $\frac{\partial \pi_i}{\partial x_i} = (N-1)(e^{\alpha x(N)} - 1)\alpha \frac{e^{\alpha x_i}}{[(e^{\alpha x_i}-1)+(N-1)(e^{\alpha x(N)}-1)]^2} - 1 = \frac{\left(\frac{N-1}{N^2}\right)^2 \alpha}{1-\frac{N-1}{N^2}\alpha} \alpha \frac{e^{\alpha x_i}}{[e^{\alpha x_i}-\Phi]^2} - 1$ with $\Phi = 1 - \frac{(\frac{N-1}{N})^2 \alpha}{1-\frac{N-1}{N^2}\alpha} \in (0,1)$. Hence, $\frac{\partial \pi_i}{\partial x_i}|_{x_i=0} = \frac{1-\frac{N-1}{N}\alpha}{(\frac{N-1}{N})^2} > 0$. Let $\Delta(y) = \frac{y}{[y-\Phi]^2}$, where $y \geq 1$. We have $\frac{d\Delta}{dy} = \frac{1}{[y-\Phi]^2} - 2\frac{y}{[y-\Phi]^3} = \frac{-1}{[y-\Phi]^3}[y+\Phi] < 0$, which implies that $\frac{\partial \pi_i}{\partial x_i}$ decreases with x_i. Hence, the solution of $x(N)$ from first order condition (1.3) is the unique global maximizer.

We now consider the contest with concealment. We will show that when $M \leq 4$, all entrants exert an equilibrium effort of $x_C(q)$ of (1.8), i.e. $x_C(q) = -\frac{1}{\alpha}\ln[1 - \alpha \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{N-1}{N^2}]$. Since $\frac{N-1}{N^2} = \frac{1}{N}(1-\frac{1}{N})$ decreases with $N \geq 2$ and $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} = 1$, $x_C(q) \leq x(2) = -\frac{1}{\alpha}\ln[1-\frac{\alpha}{4}q]$. Hence, $f(x_C(q)) = e^{\alpha x_C(q)} - 1 \leq e^{\alpha x(2)} - 1 = \frac{\frac{\alpha}{4}q}{1-\frac{\alpha}{4}q} \leq \frac{1}{3}$, because $\alpha q \in (0,1]$. It further implies that $(N-1)f(x_C(q)) \leq 1$ as long as $N \leq M \leq 4$. We are now ready to show that when $x_j = x_C(q)$ for $j \neq i$, $x_i = x_C(q)$ maximizes $\pi_i(x_i;q) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} p_i(x_i, \mathbf{x}_{-i}; N)v - x_i = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{e^{\alpha x_i}-1}{(e^{\alpha x_i}-1)+(N-1)(e^{\alpha x_C(q)}-1)} - x_i$. It suffices to show that $\pi_i(x_i;q)$ is concave in x_i. $\frac{\partial \pi_i}{\partial x_i} = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{\partial \Psi_N}{\partial x_i}$, where $\Psi_N = \frac{e^{\alpha x_i}-1}{(e^{\alpha x_i}-1)+(N-1)(e^{\alpha x_C(q)}-1)}$. We have $\frac{\partial \Psi_N}{\partial x_i} = (N-1)(e^{\alpha x_C(q)} - 1)\alpha \frac{e^{\alpha x_i}}{[(e^{\alpha x_i}-\Phi_N]^2}$, where $\Phi_N = 1 - (N-1)f(x_C(q)) = 1 - (N-1)(e^{\alpha x_C(q)} - 1) \geq 0$ since $N \leq M \leq 4$. Note that $\frac{e^{\alpha x_i}}{[(e^{\alpha x_i}-\Phi_N]^2}$ decreases with x_i when $\Phi_N \in [0,1]$. The concavity of $\frac{\partial \pi_i}{\partial x_i}$ is thus guaranteed when $x_j = x_C(q)$ for $j \neq i$. Because $x_C(q) > 0$, $\pi_i(x_i, \mathbf{x}_{-i}; q)$ increases with x_i when $x_i \leq x_C(q)$ and $\pi_i(x_i, \mathbf{x}_{-i}; q)$ decreases with x_i when $x_i \geq x_C(q)$, which guarantees that the solution to (1.7) constitutes a symmetric equilibrium. The uniqueness of symmetric equilibrium is implied by the monotonicity of $H(\cdot)$. ■

Appendix B: Proofs of Chapter Two

Proof of Lemma 1

Proof. Let $f_1(q) = [1-(1-q)^M]v - Mq\Delta$, and $f_2(q) = (1-q)^{M-1}v - \Delta$. $\bar{q}$ (> 0) is defined as $f_1(\bar{q}) = 0$. The first order derivative of $f_1(q)$ is $f_1'(q) = Mf_2(q)$, which is a decreasing function of q. $f_1'(q)$ is positive when $q = 0$, and it is negative when $q = 1$.

q_0 is defined as $f_2(q_0) = 0$. Therefore, $f_1(q)$ increases on $[0, q_0]$, and decreases from $[q_0, 1)$. $f_1(q)$ thus has two zero points, i.e. $\{0, \bar{q}\}$, and $q_0 < \bar{q}$. ■

Proof of Theorem 1

Proof. Part (a) Existence of symmetric equilibria: Consider the following extended game. There are M contestants who simultaneously choose their two-dimensional actions, which are denoted by $a_i = (a_{i1}, a_{i2}) = (q_i, x_i) \in A$, $i = 1, 2, ..., M$, where the uniform action space $A = [0,1] \times [0, v^{1/\alpha}]$ is nonempty, convex and compact.

Let $\mathbf{k} = (k_1, k_2, ..., k_i, ..., k_N)$ where k_i is either 0 or 1. Let K to be the set of all possible $\mathbf{k}$. Similarly, we can define $\mathbf{k}_{-i}$ and K_{-i}, $i = 1, 2, ..., M$.

Given action profile $\mathbf{a} = \{a_1, a_2, ..., a_M\}$ of the M players, the payoff of player i is defined as

$$U_i(\mathbf{a}) = q_i\{[\sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} q_j^{k_j}(1-q_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x})]v - x_i^\alpha - \Delta\}, i = 1, 2, ..., M,$$

where $\Pr(i|\mathbf{k}_{-i}, \mathbf{x}) = \frac{x_i^r}{x_i^r + \sum_{j\neq i} k_j x_j^r}$ if $x_i^r + \sum_{j\neq i} k_j x_j^r > 0$, and $\Pr(i|\mathbf{k}_{-i}, \mathbf{x}) = \frac{1}{1+\sum_{j\neq i} k_j}$ if $x_i^r + \sum_{j\neq i} k_j x_j^r = 0$. Note that $\Pr(i|\mathbf{k}_{-i}, \mathbf{x})$ equals to the winning probability of an entrant i when the entry status of others is denoted by $\mathbf{k}_{-i}$ and players' effort is $\mathbf{x}$ if they enter.

Note that this game is a symmetric game as defined by Dasgupta and Maskin (1986) in their Definition 7. We will apply their Theorem 6* in Appendix to establish the existence of symmetric equilibrium in mixed strategy.

In what follows, we show that for each i, the discontinuities of U_i are confined to a subset of a continuous manifold of dimension less than M as required by page 7 of Dasgupta and Maskin (1986). Following the notations on page 22 of Dasguspta and Maskin (1986). Let $Q = \{2\}, D(i) = 1$, and f_{ij}^1 to be an identity function. Following their $(A1)$ of page 22, we define manifold $A^*(i) = \{\mathbf{a} \in A | \exists j \neq i, \exists k \in Q, \exists d, 1 \leq d \leq D(i)$ such that $a_{jk} = f_{ij}^d(a_{ik})\}$. Clearly, $A^*(i)$ is of dimension less than M. The set of discontinuous points for $U_i(\mathbf{a})$ can be written as $A^{**}(i) = \{\mathbf{a} \in A | q_j x_j = 0, \forall j = 1, 2, ..., M; q_i > 0, x_i = 0; \exists j_0 \neq i$, such that $q_{j_0} > 0$ and $x_{j_0} = 0\}$. Clearly, $A^{**}(i) \subset A^*(i)$, since any element in $A^{**}(i)$ must satisfy the following conditions: For $k = 2 \in Q, \exists j_0 \neq i$, such that $x_{j_0} = f_{ij}^1(x_{ik})$, i.e. $a_{j_0 2} = f_{ij}^1(a_{i2})$. According to their Theorem 6*, we need to verify the following conditions hold.

First, as constructed above, $U_i(\mathbf{a})$ is continuous except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by $(A1)$.

Second, clearly, we have $\sum_i U_i(\mathbf{a}) = v[1 - \prod_i (1 - q_i)] - \sum_i q_i(x_i^\alpha + \Delta)$, which is continuous and thus upper semi-continuous.

Third, $U_i(\mathbf{a})$ clearly is bounded on $A = [0, 1] \times [0, v^{1/\alpha}]$.

Fourth, we verify that Property (α^*) of page 24 is satisfied. Define B^2 as the unit circle with the origin as its center, i.e. $B^2 = \{\mathbf{e} = (q, x) \mid q^2 + x^2 = 1\}$. Pick up any continuous density function $v(\cdot)$ on B^2 such that $v(\mathbf{e}) = 0$ iff $e_1 \leq 0$ or $e_2 \leq 0$. Note that $U_i(a_i, \mathbf{a}_{-i})$ is continuous in a_{i1} and lower semi-continuous in a_{i2}. $\forall \mathbf{a} = (\bar{a}_i, \mathbf{a}_{-i}) \in A^{**}(i)$, clearly we have that for any $\mathbf{e}$ such that $v(\mathbf{e}) > 0$ (i.e. $min(e_1, e_2) > 0$), $\liminf_{\theta \to 0^+} U_i(\bar{a}_i + \theta \mathbf{e}, \mathbf{a}_{-i}) > U_i(\bar{a}_i, \mathbf{a}_{-i})$ as $\theta > 0, e_2 > 0$ and $q_i > 0, x_i = 0$ in $\bar{a}_i$. This leads to that $\int_{B^2} [\liminf_{\theta \to 0^+} U_i(\bar{a}_i + \theta \mathbf{e}, \mathbf{a}_{-i}) v(\mathbf{e}) d\mathbf{e}] > U_i(\bar{a}_i, \mathbf{a}_{-i}), \forall \bar{a}_i \in A_i^{**}(i), \mathbf{a}_{-i} \in A_{-i}^{**}(\bar{a}_i)$, where $A_i^{**}(i)$ is the collection of all $\bar{a}_i$ of player i that appear in $A^{**}(i)$, $A_{-i}^{**}(\bar{a}_i)$ is the collection of others' actions $\mathbf{a}_{-i}$ such that $\mathbf{a} = (\bar{a}_i, \mathbf{a}_{-i}) \in A^{**}(i)$. This confirms that Property (α^*) holds for the above game.

Thus according to Theorem 6* of Dasgupta and Maskin (1986), there exists a symmetric mixed strategy equilibrium. Without loss of generality, we use $\mu_1(q)$ to denote the equilibrium probability measure of action q, and use $\mu_2(x)$ to denote the equilibrium probability measure of action x.

Next we show that for any strategy profile of players $\{(\mu_{i1}(q_i), \mu_{i2}(x_i))\}$. The players' payoffs are same from strategy profile of players that is defined as $\{(E_{\mu_{i1}} q_i, \mu_{i2}(x_i))\}$. The expected utility of player i from profile $\{(\mu_{i1}(q_i), \mu_{i2}(x_i))\}$ is

$$
\begin{aligned}
E_{\mathbf{a}} U_i(\mathbf{a}) &= E_{q_i}\{E_{\mathbf{q}_{-i}} E_{\mathbf{x}}[q_i \sum_{\mathbf{k}_{-i} \in K_{-i}} (\prod_{j \neq i} q_j^{k_j} (1-q_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x}))v - x_i^{\alpha} - \Delta]\} \\
&= E_{q_i}\{q_i E_{\mathbf{x}} E_{\mathbf{q}_{-i}}[\sum_{\mathbf{k}_{-i} \in K_{-i}} (\prod_{j \neq i} q_j^{k_j} (1-q_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x}))v - x_i^{\alpha} - \Delta]\} \\
&= E_{q_i}\{q_i E_{\mathbf{x}}[\sum_{\mathbf{k}_{-i} \in K_{-i}} (\prod_{j \neq i} (Eq_j)^{k_j} (1-Eq_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x}))v - x_i^{\alpha} - \Delta]\} \\
&= Eq_i \cdot E_{\mathbf{x}}[\sum_{\mathbf{k}_{-i} \in K_{-i}} (\prod_{j \neq i} (Eq_j)^{k_j} (1-Eq_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x}))v - x_i^{\alpha} - \Delta]\} \\
& \quad \text{for } \forall i \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.1)}
\end{aligned}
$$

The above result means that given others take strategy $(E_{\mu_1} q, \mu_2(x))$, the same strategy is also the best strategy for player i. Otherwise, $(\mu_1(q), \mu_2(x))$ would not be the optimal strategy for player i when others take the same strategy $(\mu_1(q), \mu_2(x))$. Therefore, $(E_{\mu_1} q, \mu_2(x))$ is a *symmetric equilibrium* for the above game.

It is easy to see that $(q^*, \mu^*(x)) = (E_{\mu_1} q, \mu_2(x))$ is a symmetric equilibrium for our original game based on the way the extended game is constructed. $U_i(\mathbf{a})$ equals player i's expected payoffs when he enters with probability q_i and exerts effort x_i when he enters, given that other bidder j enters with probability q_j and exerts effort x_j when he enters. This claim also holds when they adopt any other entry strategies with measure $\{\mu_{i1}(q), i = 1, 2, ..., M\}$ due to (B.1). According to (B.1), only the expected entry probabilities $\{E_{\mu_{i1}} q, i = 1, 2, ..., M\}$ count.

Note we must have $q^* = E_{\mu_1} q \in (0, 1)$. First, $q^* = E_{\mu_1} q = 0$ cannot be an entry equilibrium when $\Delta < v$ (Assumption 1). Second, $q^* = E_{\mu_1} q = 1$ cannot be an entry equilibrium when $\Delta > \frac{v}{M}$ (Assumption 1). The expected equilibrium payoff of players must be nonnegative. Thus we must have $(1-(1-E_{\mu_1}q)^M)v - M(E_{\mu_1}q)[\Delta + E_{\mu_2}x] \geq 0$. This leads to $(1-(1-E_{\mu_1}q)^M)v - M(E_{\mu_1}q)\Delta > 0$. Thus $q^* = E_{\mu_1}q < \bar{q}$ by Definition 1 and proof of Lemma 1.

Part (b): The equilibrium payoff cannot be negative. When $q^* = E_{\mu_1}q \in (0,1)$, we must have the equilibrium payoffs of player to be zero as otherwise it cannot be an equilibrium as the player would enter with probability 1 and earn a positive payoff. ■

Proof of Lemma 2

Proof. If a symmetric equilibrium with pure strategy bidding exists, according to the first order condition $\frac{d\pi_i(x_i)}{dx_i} = 0$ and the symmetry condition $x_i = x$, x^* must solve

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{(N-1)rv}{N^2 x^*} - \alpha x^{*\alpha-1} = 0,$$

which yields

$$x^*(q) = [\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{N-1}{N^2}\frac{rv}{\alpha}]^{\frac{1}{\alpha}}.$$

The equilibrium expected payoff is

$$\begin{aligned}
& \pi^*(x^*(q), q) \\
= & \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{v}{N} - [\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{N-1}{N^2}\frac{rv}{\alpha}] \\
= & \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}).
\end{aligned}$$

By entering the contest and submit the bid $x^*(q)$, every potential contestant

i ends up with an expected payoff

$$\pi^*(x^*(q), q) - \Delta.$$

By Theorem 1 (b), each potential bidder receives a zero expected payoff for the equilibrium entry q^*, i.e. $\pi^*(x^*(q^*), q^*) = \Delta$.

The expected overall effort of the contest (x_T^*) obtains as

$$\begin{aligned} x_T^* &= Mq^* x^*(q^*) \\ &= Mq^*[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{N-1}{N^2}\frac{rv}{\alpha}]^{\frac{1}{\alpha}}. \end{aligned}$$

■

Proof of Lemma 3

Proof. By Lemma 2, q^* satisfies $F(q^*, r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) - \Delta = 0$. Apparently, $F(q^*, r)$ is continuous in and differentiable with both arguments. We first claim that $F(q^*, r)$ strictly decreases with q^*. Define $\pi_N = \frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha})$. Taking its first order derivative yields

$$\begin{aligned} &\frac{F(q^*, r)}{dq^*} \\ =& \sum_{N=1}^{M} C_{M-1}^{N-1}[(N-1)q^{*N-2}(1-q^*)^{M-N} - (M-N)q^{*N-1}(1-q^*)^{M-N-1}]\pi_N \\ =& \sum_{N=1}^{M} C_{M-1}^{N-1}(N-1)q^{*N-2}(1-q^*)^{M-N}\pi_N \\ &- \sum_{N=1}^{M} C_{M-1}^{N-1}(M-N)q^{*N-1}(1-q^*)^{M-N-1}\pi_N \\ =& (M-1)\{\sum_{N=2}^{M} C_{M-2}^{N-2} q^{*N-2}(1-q^*)^{M-N}\pi_N - \sum_{N=1}^{M-1} C_{M-2}^{N-1} q^{*N-1}(1-q^*)^{M-N-1}\pi_N\} \\ =& (M-1)\sum_{N=1}^{M-1} C_{M-2}^{N-1} q^{*N-1}(1-q^*)^{M-N-1}(\pi_{N+1} - \pi_N), \end{aligned}$$

which is obviously negative because $\pi_N = \frac{1}{N}\left[1 - \left(1 - \frac{1}{N}\right)\frac{r}{\alpha}\right] v \geq 0$ and it monotonically decreases with N.

When all other potential contestants play $q = 0$, a potential contestant receives a payoff $v - \Delta > 0$, and he must enter with probability one. When all others play $q = \overline{q}$, a participating contestant receives negative expected payoff if he enters by Definition 1 and Lemma 1 ($(1-\overline{q})^{M-1}v < \Delta$), which cannot constitute an equilibrium either. Hence, a unique $q^* \in (0, \overline{q})$ must exist that solves $\pi^*(x^*, q) = \Delta$. Each potential contestant is indifferent between entering and staying inactive when all others play the strategy. This constitutes an equilibrium.

Moreover, $F(q^*, r)$ strictly decreases with r. Since it also strictly decreases with q^*, the part (b) of the lemma is then verified. ■

Proof of Lemma 4

Proof. Denote $k_i = x_i^{\alpha}$, $k^* = x^{*\alpha}$, $t = \frac{r}{\alpha} \in (0, \frac{M-1}{M-2}]$, then $\tilde{\pi}_i(x_i)$ can be rewritten as

$$\tilde{\pi}_i(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{k_i^t}{k_i^t + (N-1)k^{*t}} v - k_i,$$

Evaluating $\tilde{\pi}_i$ with respect to k_i yields

$$\frac{d\tilde{\pi}_i}{dk_i} = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{(N-1)tk_i^{t-1}k^{*t}v}{[k_i^t + (N-1)k^{*t}]^2} - 1.$$

Note

$$k^* = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{N-1}{N^2} tv.$$

To verify that k^* is the global maximizer of $\tilde{\pi}_i(k_i)$ given that all other participants exert the same effort. Define $p_i(k_i, \mathbf{k}_{-i}; N) = \frac{k_i^t}{k_i^t + (N-1)k^{*t}}$. One can verify $\xi_N(k_i) = \left.\frac{\partial^2 p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i^2}\right|_{k_{-i}=k^*} = \frac{-(t+1)k_i^t + (t-1)(N-1)k^{*t}}{[k_i^t + (N-1)k^{*t}]^3} t k_i^{t-2}(N-1)k^{*t}$. It implies that $\Phi_N(k_i) = \left.\frac{\partial p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i}\right|_{k_{-i}=k^*}$ is not monotonic: It is positive if $k_i^t < \frac{t-1}{t+1}(N-1)k^{*t}$, and negative if $k_i^t > \frac{t-1}{t+1}(N-1)k^{*t}$. Clearly $\frac{t-1}{t+1}(N-1) \leq 1$

if and only if $t \leq \frac{N}{N-2}$. Because $t \leq 1 + \frac{1}{M-2}$, we must have $\frac{t-1}{t+1}(N-1) < 1$ for all $N \leq M$.

Let

$$\Phi(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{\partial p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i}|_{k_{-i}=k^*}$$

and

$$\xi(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{\partial^2 p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i^2}|_{k_{-i}=k^*}$$

The above results imply that $k_i^t > \frac{t-1}{t+1}(N-1)k^{*t}$ when $k_i = k^*$ for all $N \leq M$, which means that $\xi(k_i)|_{k_i=k^*} < 0$. This leads to that $\left.\frac{d^2\tilde{\pi}_i(k_i)}{dk_i^2}\right|_{k_i=\mathbf{k}_{-i}=k^*} = v\,\xi(k_i)|_{k_i=k^*} < 0$. Hence, $k_i = k^*$ must be at least a local maximizer of when $k_{-i} = k^*$.

Since when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$, $\xi_N(k_i) > 0$ for all $N \leq M$, we have $\xi(k_i) > 0$ when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$, which means that $\Phi(k_i)$ increases when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$. Similarly, $\xi(k_i) < 0$ when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$, which means that $\Phi(k_i)$ decreases when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$. We next show that there exists a unique $k' \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$ such that $\Phi(k_i)$ increases (decreases) if and only if $k_i < (>)\ k'$. For this purpose, it suffices to show that there exists a unique $k' \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$, such that $\xi(k') = 0$.

First, such k' must exist by continuity of $\xi(k_i)$. As have been revealed, $\xi(k_i) > 0$ when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$; and $\xi(k_i) < 0$ when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$.

Second, the uniqueness of k' can be verified as below. We have

$$\begin{aligned} & \left.\frac{\partial^3 p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i^3}\right|_{k_{-i}=k^*} \\ = \quad & t(N-1)k^{*t} \left\{ \begin{array}{c} (t-2)k_i^{t-3} \frac{-(t+1)k_i^t + (t-1)(N-1)k^{*t}}{[k_i^t + (N-1)k^{*t}]^3} \\ +k_i^{t-2} \frac{-t(t+1)k_i^{t-1}[k_i^t + (N-1)k^{*t}] - 3tk_i^{t-1}[-(t+1)k_i^t + (t-1)(N-1)k^{*t}]}{[k_i^t + (N-1)k^{*t}]^4} \end{array} \right\} \end{aligned}$$

$$= \frac{t(N-1)k^{*t}k_i^{t-3}}{[k_i^t+(N-1)k^{*t}]^3}\left\{\begin{array}{c}(t-2)[-(t+1)k_i^t+(t-1)(N-1)k^{*t}] \\ +\frac{-t(t+1)k_i^t[k_i^t+(N-1)k^{*t}]-3tk_i^t[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]}{[k_i^t+(N-1)k^{*t}]}\end{array}\right\}$$

$$= \frac{t(N-1)k^{*t}k_i^{t-3}}{[k_i^t+(N-1)k^{*t}]^3}\left\{\begin{array}{c}(t-2)[-(t+1)k_i^t+(t-1)(N-1)k^{*t}] \\ +\frac{2tk_i^t}{[k_i^t+(N-1)k^{*t}]}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}]\end{array}\right\}.$$

Recall $\xi_N(k_i) = \frac{-(t+1)k_i^t+(t-1)(N-1)k^{*t}}{[k_i^t+(N-1)k^{*t}]^3}tk_i^{t-2}(N-1)k^{*t}$. We then have

$$\begin{aligned}&\left.\frac{\partial^3 p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i^3}\right|_{k_{-i}=k^*} \\ = \; &(t-2)k_i^{-1}\xi_N(k_i) \\ &+\frac{2t^2(N-1)k^{*t}k_i^{2t-3}}{[k_i^t+(N-1)k^{*t}]^4}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}].\end{aligned}$$

We now claim $[(t+1)k_i^t-(2t-1)(N-1)k^{*t}]$ is negative for all $k_i \leq [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$. A detailed proof is as follows. From $k_i \leq [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$, we have $(t+1)k_i^t \leq (t-1)(M-1)k^{*t}$. To show $(t+1)k_i^t-(2t-1)(N-1)k^{*t} < 0$, it suffices to show $(t-1)(M-1) < (2t-1)(N-1)$ when $N=2$, which requires $t < 1+\frac{1}{M-3}$. This holds as $t \leq 1+\frac{1}{M-2}$.

We thus have at any $k_i \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$ such that $\xi(k_i)=0$, $\xi(k_i)$ must be locally decreasing, because

$$\begin{aligned}&\frac{\partial \xi(k_i)}{\partial k_i} \\ = \; &(t-2)k_i^{-1}\sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}\xi_N(k_i) \\ &+\sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}A_N(k_i) \\ = \; &(t-2)k_i^{-1}\xi(k_i)+\sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}A_N(k_i)\end{aligned}$$

$$=\sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}A_N(k_i) < 0$$

As $A_N(k_i) = \frac{2t^2(N-1)k^{*t}k_i^{2t-3}}{[k_i^t+(N-1)k^{*t}]^4}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}] < 0$.

We are ready to show the uniqueness of k' by contradiction. Suppose that there exists more than one zero points k' and k'' with $k' \neq k''$ for $\xi(k_i)$. Because $\xi(k_i)$ must be locally decreasing, then there must exist at least another zero point $k''' \in (k', k'')$ at which $\xi(k_i)$ is locally increasing. Contradiction thus results. Hence, such a zero point k' of $\xi(k_i)$ must be unique.

Recall $\Phi(k_i)$ increases (decreases) if and only if $k_i < (>)\ k'$ and it reaches its maximum at k'. Note $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i} = v\Phi(k_i) - 1$ and $\Phi(0) = 0$. Therefore $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}|_{k_i=0}$ < 0. Thus $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ has exactly two zero points with the smaller one (k_s) being the local minimum point of $\tilde{\pi}_i(k_i)$. Note $k_i = k^*$ must be a zero point for $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ by definition. Since $k_i = k^*$ is a local maximum point of $\tilde{\pi}_i(k_i)$, it is higher than other zero point (k_s) of $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ which is a local minimum point of $\tilde{\pi}_i(k_i)$.

Note $x_m = (k_s)^{1/\alpha}$ is the unique local minimum of $\tilde{\pi}_i(x_i)$, and note $x^* = (k^*)^{1/\alpha}$ is the unique inner local maximum of $\tilde{\pi}_i(x_i)$. Note $x_m < x^*$. The results of Lemma 4 are shown. ■

Proof of Lemma 5

Proof. Define an increasing transformation of $\overline{x_T}(q)$:

$$\Psi(q) = [\overline{x_T}(q)]^\alpha = (Mq)^{\alpha-1}\left\{[1-(1-q)^M]v - Mq\Delta\right\}$$

Note that $\Psi(q)|_{q=0} = 0$; and $\Psi(q)|_{q=1} = M^{\alpha-1}(v - M\Delta) < 0$ since $\frac{v}{M} < \Delta$ (Assumption 1). We have

$$\frac{d\Psi(q)}{dq} = f(q)\, q^{\alpha-2} M^{\alpha-1},$$

where

$$f(q) = (\alpha-1)\underbrace{\left\{[1-(1-q)^M]v - Mq\Delta\right\}}_{f_1(q)} + Mq\underbrace{[(1-q)^{M-1}v - \Delta]}_{f_2(q)}.$$

We have

$$f'(q) = Mv(1-q)^{M-2}[\alpha - (M+\alpha-1)q] - \alpha M\Delta.$$

Note that $f'(0) = \alpha Mv - \alpha M\Delta > 0$, $f'(1) = -\alpha M\Delta < 0$ and $f'(q)$ decreases with $q \in (0, \frac{\alpha}{M+\alpha-1}]$. Clearly, $f'(q) < 0$ when $q \in [\frac{\alpha}{M+\alpha-1}, 1]$. Then there exists a unique $q_c \in (0, \frac{\alpha}{M+\alpha-1})$, such that $f'(q_c) = 0$, which means q_c is the maximum point of $f(q)$. Since $f(0) = 0$, $f(q_c) > 0$ and $f(1) = (\alpha-1)v - \alpha M\Delta = \alpha(v - M\Delta) - v < 0$, then there must exist a unique $\hat{q} \in (q_c, 1)$, such that $f(\hat{q}) = 0$. Note that $f'(q) < 0$ on $(q_c, 1)$. Clearly, $f(q) > 0$ when $0 < q < \hat{q}$; and $f(q) < 0$ when $\hat{q} < q < 1$.

Since $\frac{d\Psi(q)}{dq}$ shares the same sign with $f(q)$, we have that $\frac{d\Psi(q)}{dq} > 0$ when $0 < q < \hat{q}$; and $\frac{d\Psi(q)}{dq} < 0$ when $\hat{q} < q < 1$. This implies $\hat{q} = \arg\max_q \Psi(q)$, i.e. $\hat{q} = \arg\max_q \overline{x_T}(q)$.

By the proof of Lemma 1, we know both $f_1(q)$ and $f_2(q)$ are positive when $q \in [0, q_0]$ and both are negative when $q > \bar{q}$. Thus the zero point $(\hat{q})$ of $f(q)$ must fall in $[q_0, \bar{q}]$. ■

Proof of Theorem 4

Proof. Proof of Lemma 3 has shown that $F(q,r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{v}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}) - \Delta$ decreases with both q and r. Thus $F(q,r) = 0$ uniquely defines r as a decreasing function of q. Since $F(q_0, r_0) = 0$ and $\hat{q} > q_0$, we must have $r(\hat{q}) < r_0$. Theorem 3 thus means that contest $r(\hat{q})$ would induce entry equilibrium $\hat{q}$ and pure-strategy bidding whenever $r(\hat{q}) \leq \alpha(1+\frac{1}{M-2})$. Since we have a pure-strategy bidding, an overall effort of $\overline{x_T}(\hat{q})$ clearly is induced at the equilibrium.

Consider any other $r \neq r(\hat{q})$. If r induces equilibrium entry $q(r)$ and pure-strategy bidding, then the total effort induced is $\overline{x_T}(q(r))$. Note that by Lemma 3, equilibrium $q(r)$ decreases with r. Thus $r \neq r(\hat{q})$ means $q(r) \neq \hat{q}$. $\overline{x_T}(q)$ is single peaked at $\hat{q}$ according to Lemma 5. Thus for any $r \neq r(\hat{q})$, we must have

$\overline{x_T}(q(r)) < \overline{x_T}(\hat{q})$. If r induces equilibrium entry $q(r)$ and mixed-strategy bidding, then the total expected effort induced is strictly lower than $\overline{x_T}(q(r))$ when $\alpha > 1$, based on the arguments deriving this boundary in Section 4.1. Therefore the total effort induced must be strictly lower than $\overline{x_T}(\hat{q})$. ■

Proof of Lemma 6

Proof. By definition $\overline{x_T}^*\left(M'\right) = \overline{x_T}(\hat{q}(M'); M')$.

By Envelope Theorem, $\frac{d\overline{x_T}(\hat{q}(M'); M')}{dM'} = \frac{\partial \overline{x_T}(q; M')}{\partial M'}|_{q=\hat{q}(M')}$. We have

$$\begin{aligned}
& \frac{\partial \overline{x_T}(q; M')}{\partial M'}|_{q=\hat{q}(M')}. \\
= \ & \partial\left[(M'\hat{q}(M'))^{\frac{\alpha-1}{\alpha}}\left\{[1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta\right\}^{\frac{1}{\alpha}}\right]/\partial M' \\
= \ & \frac{\alpha-1}{\alpha}M'^{-\frac{1}{\alpha}}\left[\hat{q}(M')\right]^{\frac{\alpha-1}{\alpha}}\left\{[1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta\right\}^{\frac{1}{\alpha}} \\
& +\frac{1}{\alpha}(M'\hat{q}(M'))^{\frac{\alpha-1}{\alpha}}\left\{[1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta\right\}^{\frac{1}{\alpha}-1} \\
& \times[-(1-\hat{q}(M'))^{M'}v\ln(1-\hat{q}(M')) - \hat{q}(M')\Delta],
\end{aligned}$$

which has the same sign as

$$\begin{aligned}
\lambda \ = \ & (\alpha-1)\left\{[1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta\right\} \\
& +M'[-(1-\hat{q}(M'))^{M'}v\ln(1-\hat{q}(M')) - \hat{q}(M')\Delta].
\end{aligned}$$

Because $-\ln(1-\hat{q}(M')) < \frac{\hat{q}(M')}{1-\hat{q}(M')}$, we have $M'[-(1-\hat{q}(M'))^{M'}v\ln(1-\hat{q}(M')) - \hat{q}(M')\Delta] < \hat{q}(M')[M'(1-\hat{q}(M'))^{M'-1}v - M'\Delta]$. Hence, $\lambda < (\alpha-1)\left\{[1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta\right\} + \hat{q}(M')[M'(1-\hat{q}(M'))^{M'-1}v - M'\Delta] = 0$ (by the definition of $\hat{q}(M')$). We then have $\frac{d\overline{x_T}(\hat{q}(M'); M')}{dM'} < 0$. ■

Proof of Theorem 6

Proof. We first show the following claim for a subgame with N players.

Claim: *For $N \leq M$ such that $\frac{N}{N-1} < \frac{r}{\alpha}$, there exists a symmetric mixed strat-*

egy equilibrium for the $N-$player subgame. The equilibrium payoff of a player π_N^d falls in $[0, \frac{v}{N})$.

The proof of this claim replies on Theorem 6 of Dasgupta and Maskin (1986). The application of their Theorem 6 requires four conditions as has been pointed out by Baye et al (1994) who have shown the existence of a symmetric mixed-strategy equilibrium when $N = 2$ and effort costs are linear. However, when effort costs are nonlinear and $N > 2$, the proof is almost identical. Condition (i) requires that the discontinuity set S_i of player i's payoff is confined to a subset of a continuous manifold of dimension less than N. Let this manifold be defined as $A^*(i) = \{\mathbf{x}|x_1 = x_2 = ... = x_N\}$, which has a zero measure. The only discontinuity point of player i's payoff is $(0, 0, ..., 0) \in A^*(i)$. Thus condition (i) holds. Condition (ii) of this theorem requires that the sum of players' payoffs must be upper semi-continuous. From (2.2), we have that this sum is $v - \sum_i x_i^\alpha$, which is continuous and therefore upper semi-continuous. Condition (iii) requires that player i's payoff is bounded. This clearly holds as it falls in $[-v, v]$ when $x_i \in [0, v^{1/\alpha}]$. Note that a player never bids higher than $v^{1/\alpha}$. Condition (iv) requires that player i's payoff must be weakly lower semi-continuous. The only point one needs to check is the discontinuity point $(0, 0, ..., 0)$. At this point, player i's payoff is lower semi-continuous, and thus is weakly lower semi-continuous. Since all four conditions required are satisfied. The existence of a symmetric mixed-strategy equilibrium is guaranteed by Theorem 6 in Dasgupta and Maskin (1986).

In a symmetric equilibrium, every contestant wins the prize v with the same probability, and they incur positive effort costs.[15] Therefore, the equilibrium payoff must be lower than $\frac{v}{N}$.

We now introduce the definition of a symmetric entry equilibrium. Entry

[15] Clearly, exerting a zero effort is not an equilibrium.

probability $q_d^* \in [0,1]$ constitutes a symmetric entry equilibrium if and only if

$$\begin{aligned}\sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{*N-1}(1-q_d^*)^{M-N}\pi_N^d &= \Delta, \text{if } q_d^* \in (0,1),\\ \pi_M^d &\geq \Delta, \text{ if } q_d^* = 1,\\ \pi_1^d &= v < \Delta, \text{ if } q_d^* = 0.\end{aligned}$$

We now are ready to show a symmetric entry equilibrium exists which must fall into $(0,1)$.

Note that with Assumption 1, both $q_d^* = 1$ and $q_d^* = 0$ cannot be an entry equilibrium. The existence of symmetric entry equilibria depends on the existence of the solution of $\sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{*N-1}(1-q_d^*)^{M-N}\pi_N^d = \Delta$. Note the left hand side is continuous in q_d^*. When $q_d^* = 0$, it is lower than the right hand side. When $q_d^* = 1$, it is higher than the right hand side. Therefore, there must exist $q_d^* \in (0,1)$ such that $\sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{*N-1}(1-q_d^*)^{M-N}\pi_N^d = \Delta$. ■

Proof of Lemma 7

Proof. Under policy d, for a given $r \in (0, \alpha(1+\frac{1}{M-1})]$ the subgame boils down to a standard symmetric $N-$ player contest. Whenever $N \geq 2$, each representative participant i chooses his bid x_i to maximize his expected payoff

$$\pi_i = p_N(x_i, \mathbf{x}_{-i})v - x_i^\alpha,$$

where $p_N(x_i, \mathbf{x}_{-i})$ is given by the contest success function (2.1). Standard technique leads to the well known results in contest literature. In the unique symmetric pure-strategy Nash equilibrium, each participant bids

$$x_N = \left(\frac{N-1}{N^2}\frac{rv}{\alpha}\right)^{\frac{1}{\alpha}}$$

Each participating contestant earns an expected payoff

$$\pi_N = \frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}).$$

Note that x_N reduces to zero, and π_N amounts to v if $N = 1$, i.e. nobody else enters the contest. Suppose that all others choose a strategy $q_d \in [0,1]$. A potential contestant i ends up with an expected payoff

$$u_i(q) = \sum_{N=1}^{M} C_{N-1}^{M-1} q_d^{N-1}(1-q_d)^{M-N}\pi_N - \Delta.$$

By proof of Lemma 3, $\pi(q_d)$ strictly decreases with q_d. There must exist a unique $q_d^* \in (0,1)$ that solves $\pi_d^* = \pi_d^*(q_d) = \Delta$. Each potential contestant is indifferent between entering and staying inactive when all others play the strategy. This constitutes an equilibrium.

Since each $N-$ player contest elicits a total bid $N \cdot x_N \equiv N\left(\frac{N-1}{N^2}\frac{rv}{\alpha}\right)^{\frac{1}{\alpha}}$. Hence, expected overall bid is obtained as

$$\begin{aligned} x_T^*(r,d) &= \sum_{N=1}^{M} C_M^N q_d^{*N}(1-q_d^*)^{M-N} N \left(\frac{N-1}{N^2}\frac{rv}{\alpha}\right)^{\frac{1}{\alpha}} \\ &= M q_d^* \sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{*N-1}(1-q_d^*)^{M-N}\left(\frac{N-1}{N^2}\frac{rv}{\alpha}\right)^{\frac{1}{\alpha}}. \end{aligned}$$

■

Proof of Theorem 7

Proof. For a given r, concealment and disclosure yields the same equilibrium entry strategy, i.e., $q_d^* = q^*$. Potential contestants are *ex ante* indifferent between concealment and disclosure. This claim can be directly verified by the proofs of Lemmas 2 and 7. q^* and q_d^* solve the same equations (2.4) and (2.14). By Jensen's inequality, $\frac{1}{\alpha} \leq 1$ implies that $x_T^*\ (r,c) \geqq x_T^{d*}\ (r,d)$ because $[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-$

$q)^{M-N}\frac{N-1}{N^2}\frac{rv}{\alpha}]^{\frac{1}{\alpha}} \geq \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\left(\frac{N-1}{N^2}\frac{rv}{\alpha}\right)^{\frac{1}{\alpha}}$. ■

Proof of Remark 1

Proof. Fix $\alpha = 1$. When $r \in (\alpha(1+\frac{1}{M-1}), \alpha(1+\frac{1}{M-2})]$, the symmetric equilibrium probability $q_c \in (0,1)$ of a contest under policy c is determined by the break-even condition

$$v\sum_{N=1}^{M} C_{M-1}^{N-1} q_c(1-q_c)^{M-N}[\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha}] = \Delta.$$

However, under policy d, the break-even condition that determines equilibrium entry probability is

$$v\sum_{N=1}^{M} C_{M-1}^{N-1} q_d(1-q_d)^{M-N}\pi_N = \Delta,$$

where π_N is the equilibrium payoff of a participating bidder in a subgame with a total of N participants. For small N such that $r \leq \alpha(1+\frac{1}{N-1})$, $\pi_N = \frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha} \geq 0$. However, for sufficiently large N such that $\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha} < 0$, π_N must be strictly greater than $\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha}$, because it must be nonnegative by individual rationality. Hence, we must have

$$v\sum_{N=1}^{M} C_{M-1}^{N-1} q(1-q)^{M-N}\pi_N > v\sum_{N=1}^{M} C_{M-1}^{N-1} q(1-q)^{M-N}[\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha}]$$

for $q \in (0,1)$. This implies generally $q_d \neq q_c$ for the given r, which further means that the total effort induced would generally be different. Note for $\alpha = 1$, the total effort induced is completely determined by the entry probability. ■

Proof of Theorem 8

Proof. First note that at any symmetric equilibrium when the number of bidders is disclosed, every bidder enjoys zero payoff. Therefore, we have $[1-(1-$

$q_d^*)^M]v = Mq_d^*\{\Delta + E_N E[(x_N)^\alpha]\}$, i.e. $E_N E[(x_N)^\alpha] = [Mq_d^*]^{-1}[1-(1-q_d^*)^M]v - \Delta$, where x_N denotes the equilibrium individual effort in a subgame with N contestants. The expected total effort at the equilibrium is $Mq_d^*\ E_N[E(x_N)] = Mq_d^*\ E_N E\{[(x_N)^\alpha]^{1/\alpha}\} \leq Mq_d^*\ E_N\{E[(x_N)^\alpha]\}^{1/\alpha} \leq Mq_d^*\ \{E_N E[(x_N)^\alpha]\}^{1/\alpha} = [Mq_d^*]^{\frac{\alpha-1}{\alpha}} \times \{[1-(1-q_d^*)^M]v - Mq_d^*\Delta\}^{\frac{1}{\alpha}}$ as $\alpha \geq 1$. Note that the last expression is identical to the right hand side of (2.11). When $r(\hat{q})$ induces entry $\hat{q}$ and pure-strategy bidding while the number of bidders is concealed, the maximum of $[Mq_d^*]^{\frac{\alpha-1}{\alpha}} \times \{[1-(1-q_{d)}^*)^M]v - Mq_d^*\Delta\}^{\frac{1}{\alpha}}$ is achieved with concealment policy. Therefore, any contest with number of bidders being disclosed is dominated by a contest $r(\hat{q})$ with the number of bidders being concealed. ■

Appendix C: Proofs of Chapter Three

Proof of Lemma 1

Proof. According to the bidding strategy, the expected value of bids for bidder 1 and bidder 2 are

$$Ex_1 = \int_0^{\frac{1}{c_2}} c_2 x_1 dx_1 = \frac{1}{2c_2},$$

$$Ex_2 = 0 \times \Pr(x_2 = 0) + \int_{0^+}^{\frac{1}{c_2}} c_1 x_2 dx_2 = \frac{c_1}{2c_2^2}.$$

And for all the other remaining $n-2$ players, their marginal costs are above c_2, they will remain passive and exert zero effort.

In a model with n players, there are $n(n-1)$ cases that two of their marginal costs are ranked as the lowest and second lowest. Therefore, with general cumulative distribution function $F(c_i)$ with $c_i \in [\underline{c}, \overline{c}]$, the total expected effort for contest organizer is given as

$$\begin{aligned}
R^D &= n(n-1) \int\int \cdots \int [Ex_1 + Ex_2]\, dF(c_n) \cdots dF(c_3)\, dF(c_2)\, dF(c_1) \\
&= n(n-1) \int_{\underline{c}}^{\overline{c}} \int_{c_1}^{\overline{c}} \left\{ \int_{c_2}^{\overline{c}} \cdots \int_{c_2}^{\overline{c}} \left[\frac{1}{2c_2} + \frac{c_1}{2c_2^2} \right] dF(c_n) \cdots dF(c_3) \right\} \\
&\quad dF(c_2)\, dF(c_1) \\
&= n(n-1) \int_{\underline{c}}^{\overline{c}} \int_{c_1}^{\overline{c}} \left[\frac{1}{2c_2} + \frac{c_1}{2c_2^2} \right] [1 - F(c_2)]^{n-2}\, dF(c_2)\, dF(c_1).
\end{aligned}$$

■

Proof of Lemma 2

Proof. Note that $x_i(c_i)$ is decreasing with c_i, the higher cost, the lower effort.

Take first order condition with respect to c_i,

$$\begin{aligned}\frac{\partial \pi_i}{\partial \tilde{x}_i} &= -(n-1)\left[1-F\left(x_{-i}^{-1}(\tilde{x}_i)\right)\right]^{n-2} f\left(x_{-i}^{-1}(x_i')\right)\frac{dx_{-i}^{-1}(\tilde{x}_i)}{d\tilde{x}_i} - c_i \\ &= -(n-1)\left[1-F\left(x_{-i}^{-1}(\tilde{x}_i)\right)\right]^{n-2} f\left(x_{-i}^{-1}(x_i')\right)[x_{-i}'(x_{-i}^{-1}(\tilde{x}_i))]^{-1} - c_i.\end{aligned}$$

Given $x_{-i}(\cdot)$, $x_i(\cdot)$ is the optimal strategy of i, we must have

$$\begin{aligned}&\frac{\partial \pi_i}{\partial \tilde{x}_i}|_{\tilde{x}_i = x_i(c_i)} \\ =\ & -(n-1)\left[1-F\left(x_{-i}^{-1}(x_i(c_i))\right)\right]^{n-2} f\left(x_{-i}^{-1}(x_i(c_i))\right)[x_{-i}'(x_{-i}^{-1}(x_i(c_i)))]^{-1} - c_i \\ =\ & 0.\end{aligned}$$

At a symmetric equilibrium, $x_i(\cdot) = x_{-i}(\cdot) = x(\cdot)$. We thus have

$$\frac{dx\left(c_i\right)}{dc_i} = \frac{-(n-1)f\left(c_i\right)\left[1-F(c_i)\right]^{n-2}}{c_i}.$$

Therefore, the equilibrium bid of each player is

$$x_i\left(c_i\right) = (n-1)\int_{c_i}^{\bar{c}} \frac{\left[1-F\left(\widetilde{c}\right)\right]^{n-2}}{\widetilde{c}} dF\left(\widetilde{c}\right).$$

Then the total expected effort for contest organizer is

$$\begin{aligned}R^C &= n\int_{\underline{c}}^{\bar{c}} x_i\left(c_i\right) dF\left(c_i\right) \\ &= n\int_{\underline{c}}^{\bar{c}} (n-1)\left\{\int_{c_1}^{\bar{c}} \frac{\left[1-F\left(c_2\right)\right]^{n-2}}{c_2} dF\left(c_2\right)\right\} dF\left(c_1\right) \\ &= n\left(n-1\right)\int_{\underline{c}}^{\bar{c}} \left\{\left[\int_{\underline{c}}^{c_2} dF\left(c_1\right)\right]\frac{\left[1-F\left(c_2\right)\right]^{n-2}}{c_2}\right\} dF\left(c_2\right) \\ &= n\left(n-1\right)\int_{\underline{c}}^{\bar{c}} \frac{F\left(c_2\right)\left[1-F\left(c_2\right)\right]^{n-2}}{c_2} dF\left(c_2\right).\end{aligned}$$

■

Proof of Theorem 1

Proof. Recall (3.1) and (3.4), compare R^D and R^C,

$$
\begin{aligned}
R^D &= n(n-1)\int_{\underline{c}}^{\bar{c}}\int_{c_1}^{\bar{c}}\left[\frac{1}{2c_2}+\frac{c_1}{2c_2^2}\right][1-F(c_2)]^{n-2}\,dF(c_2)\,dF(c_1) \\
&= n(n-1)\int_{\underline{c}}^{\bar{c}}\frac{[1-F(c_2)]^{n-2}}{c_2}F(c_2)\,dF(c_2) \\
&\quad -\frac{n(n-1)}{2}\int_{\underline{c}}^{\bar{c}}\left[\frac{1}{c_2^2}\int_{\underline{c}}^{c_2}F(c_1)\,dc_1\right][1-F(c_2)]^{n-2}\,dF(c_2) \\
&= \frac{n(n-1)}{2}\left\{\begin{array}{c}\int_{\underline{c}}^{\bar{c}}\frac{[1-F(c_2)]^{n-2}}{c_2}F(c_2)\,dF(c_2) \\ +\int_{\underline{c}}^{\bar{c}}\left[\frac{1}{c_2^2}\int_{\underline{c}}^{c_2}c_1 dF(c_1)\right][1-F(c_2)]^{n-2}\,dF(c_2)\end{array}\right\},
\end{aligned}
$$

$$
R^C = n(n-1)\int_{\underline{c}}^{\bar{c}}\frac{F(c_2)[1-F(c_2)]^{n-2}}{c_2}dF(c_2).
$$

Since

$$
\begin{aligned}
&\frac{2}{n(n-1)}\left(R^D-R^C\right) \\
&= \int_{\underline{c}}^{\bar{c}}\left[\frac{1}{c_2^2}\int_0^{c_2}c_1 dF(c_1)\right][1-F(c_2)]^{n-2}\,dF(c_2) \\
&\quad -\int_{\underline{c}}^{\bar{c}}\frac{F(c_2)[1-F(c_2)]^{n-2}}{c_2}dF(c_2) \\
&= \int_{\underline{c}}^{\bar{c}}\frac{1}{c_2}\left[\frac{1}{c_2}\int_0^{c_2}c_1 dF(c_1)-F(c_2)\right][1-F(c_2)]^{n-2}\,dF(c_2) \\
&= \int_{\underline{c}}^{\bar{c}}\frac{1}{c_2}\left[-\frac{1}{c_2}\int_{\underline{c}}^{c_2}F(c_1)\,dc_1\right][1-F(c_2)]^{n-2}\,dF(c_2) \\
&= -\int_{\underline{c}}^{\bar{c}}\frac{1}{c_2^2}\left[\int_0^{c_2}F(c_1)\,dc_1\right][1-F(c_2)]^{n-2}\,dF(c_2) < 0,
\end{aligned}
$$

then $R^D < R^C$. ■

Proof of Theorem 2

Proof. Recall (3.2), the expected payoff of player i under disclosure policy is

$$E\pi_i^D = \int_{\underline{c}}^{\overline{c}} \left\{ \int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{\widetilde{c}}\right] d\left[1 - \left[1 - F\left(\widetilde{c}\right)\right]^{n-1}\right] \right\} dF\left(c_i\right).$$

Then given c_i, the expected payoff of each player is

$$\begin{aligned}
& \pi_i^D\left(x_i; c_i\right) \\
= & \int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{\widetilde{c}}\right] d\left[1 - \left[1 - F\left(\widetilde{c}\right)\right]^{n-1}\right] \\
= & (n-1) \int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{\widetilde{c}}\right] \left[1 - F\left(\widetilde{c}\right)\right]^{n-2} dF\left(\widetilde{c}\right) \\
= & (n-1) \int_{c_i}^{\overline{c}} \left[1 - F\left(\widetilde{c}\right)\right]^{n-2} dF\left(\widetilde{c}\right) - (n-1) c_i \int_{c_i}^{\overline{c}} \frac{\left[1 - F\left(\widetilde{c}\right)\right]^{n-2}}{\widetilde{c}} dF\left(\widetilde{c}\right) \\
= & \left[1 - F\left(c_i\right)\right]^{n-1} + \left[A\left(\overline{c}\right) - A\left(c_i\right)\right] c_i \\
= & \left[1 - F\left(c_i\right)\right]^{n-1} - c_i A\left(c_i\right) \quad \text{since } A\left(\overline{c}\right) = 0, F\left(\overline{c}\right) = 1 \text{ by definition (*).}
\end{aligned}$$

Therefore the equilibrium expected payoff of each player under disclosure policy is given by

$$\begin{aligned}
E\pi_i^D &= \int_{\underline{c}}^{\overline{c}} \left[\left[1 - F\left(c_i\right)\right]^{n-1} - c_i A\left(c_i\right)\right] dF\left(c_i\right) \\
&= \int_{\underline{c}}^{\overline{c}} \left[1 - F\left(c_i\right)\right]^{n-1} dF\left(c_i\right) - \int_{\underline{c}}^{\overline{c}} c_i A\left(c_i\right) dF\left(c_i\right) \\
&= \frac{1}{n} - \int_{\underline{c}}^{\overline{c}} c_i A\left(c_i\right) dF\left(c_i\right).
\end{aligned}$$

Recall (3.5), the expected payoff under concealment policy is given by

$$\begin{aligned}
E\pi_i^C &= \int_{\underline{c}}^{\overline{c}} \left\{\left[1 - F\left(c_i\right)\right]^{n-1} - c_i x_i\left(c_i\right)\right\} dF\left(c_i\right) \\
&= \frac{1}{n} - \int_{\underline{c}}^{\overline{c}} c_i x\left(c_i\right) dF\left(c_i\right).
\end{aligned}$$

Note that

$$\begin{aligned}
x_i\left(c_i\right) &= (n-1) \int_{c_i}^{\overline{c}} \frac{\left[1 - F\left(\widetilde{c}\right)\right]^{n-2}}{\widetilde{c}} dF\left(\widetilde{c}\right) \\
&= -\left[A\left(\overline{c}\right) - A\left(c_i\right)\right] = A\left(c_i\right),
\end{aligned}$$

therefore,

$$E\pi_i^D = E\pi_i^C.$$

■

Proof of Corollary 2

Proof. The expected value of bids for player 1, 2 and 3 are

$$Ex_1 = \frac{1}{c_3}\left(1 - \frac{c_1}{3c_2}\right),$$

$$Ex_2 = \frac{1}{2c_3}\left(1 + \frac{c_1^2}{3c_2^2}\right),$$

$$Ex_3 = \frac{1}{c_3^2}\left(\frac{c_1^2}{6c_2} + \frac{c_2}{2}\right).$$

And for all the other remaining $n-3$ players, their marginal costs are above c_2, they will remain passive and exert zero effort.

In a model with n players, there are $n(n-1)(n-2)$ cases that three of them are the three players with the lowest cost. Therefore, the total expected effort for contest organizer is given as

$$\begin{aligned}
R^D &= n(n-1)(n-2)\int\int\cdots\int\left[Ex_1 + Ex_2 + Ex_3\right] dF(c_n)\cdots dF(c_3) \\
&\quad dF(c_2)\,dF(c_1) \\
&= n(n-1)(n-2)\int_{\underline{c}}^{\bar{c}}\int_{c_1}^{\bar{c}}\int_{c_2}^{\bar{c}} \\
&\quad \{\int_{c_3}^{\bar{c}}\cdots\int_{c_3}^{\bar{c}}\left[\frac{1}{c_3}\left(1-\frac{c_1}{3c_2}\right) + \frac{1}{2c_3}\left(1+\frac{c_1^2}{3c_2^2}\right) + \frac{1}{c_3^2}\left(\frac{c_1^2}{6c_2} + \frac{c_2}{2}\right)\right] \\
&\quad dF(c_n)\cdots dF(c_4)\}dF(c_3)\,dF(c_2)\,dF(c_1) \\
&= n(n-1)(n-2) \\
&\quad \int_{\underline{c}}^{\bar{c}}\int_{c_1}^{\bar{c}}\int_{c_2}^{\bar{c}}\left[\left(\frac{3}{2} - \frac{c_1}{3c_2} + \frac{c_1^2}{6c_2^2}\right)\frac{1}{c_3} + \left(c_2 + \frac{c_1^2}{3c_2}\right)\frac{1}{2c_3^2}\right] \\
&\quad \left[1 - F(c_3)\right]^{n-3} dF(c_3)\,dF(c_2)\,dF(c_1).
\end{aligned}$$

■

Proof of Theorem 3

Proof. Recall (3.6)

$$\begin{aligned} R^D &= n(n-1)(n-2) \\ & \int_{\underline{c}}^{\bar{c}}\int_{c_1}^{\bar{c}}\int_{c_2}^{\bar{c}}\left[\left(\frac{3}{2}-\frac{c_1}{3c_2}+\frac{c_1^2}{6c_2^2}\right)\frac{1}{c_3}+\left(c_2+\frac{c_1^2}{3c_2}\right)\frac{1}{2c_3^2}\right] \\ & [1-F(c_3)]^{n-3}\,dF(c_3)\,dF(c_2)\,dF(c_1). \end{aligned}$$

Under concealment, the equilibrium bid

$$\begin{aligned} x_i(c) &= A(c)+B(c) \\ &= (n-1)\left\{\begin{array}{c} \int_c^{\bar{c}}\frac{1}{c_i}[1-F(c_i)]^{n-2}\,dF(c_i) \\ +\int_c^{\bar{c}}\frac{1}{c_i}[1-F(c_i)]^{n-3}[(n-1)F(c_i)-1]\,dF(c_i) \end{array}\right\} \\ &= (n-1)(n-2)\int_c^{\bar{c}}\frac{F(c_i)}{c_i}[1-F(c_i)]^{n-3}\,dF(c_i). \end{aligned}$$

Then the total expected effort for contest organizer is

$$\begin{aligned} R^C &= n\int_{\underline{c}}^{\bar{c}} x_i(c)\,dF(c) \\ &= n(n-1)(n-2)\int_{\underline{c}}^{\bar{c}}\left\{\int_c^{\bar{c}}\frac{F(c_i)}{c_i}[1-F(c_i)]^{n-3}\,dF(c_i)\right\}dF(c). \end{aligned}$$

It is sufficient to forget the coefficient before integration. By swapping integrations,

$$\begin{aligned} R'^C &= \int_{\underline{c}}^{\bar{c}}\left\{\int_c^{\bar{c}}\frac{F(c_i)}{c_i}[1-F(c_i)]^{n-3}\,dF(c_i)\right\}dF(c) \\ &= \int_{\underline{c}}^{\bar{c}}\left[\int_{\underline{c}}^{c_i}dF(c)\right]\frac{F(c_i)}{c_i}[1-F(c_i)]^{n-3}\,dF(c_i) \\ &= \int_{\underline{c}}^{\bar{c}}\frac{F^2(c_i)}{c_i}[1-F(c_i)]^{n-3}\,dF(c_i). \end{aligned}$$

In addition

$$\begin{aligned}
R'^D &= \int_{\underline{c}}^{\overline{c}}\int_{c_1}^{\overline{c}}\int_{c_2}^{\overline{c}}\left[\left(\frac{3}{2}-\frac{c_1}{3c_2}+\frac{c_1^2}{6c_2^2}\right)\frac{1}{c_3}+\left(c_2+\frac{c_1^2}{3c_2}\right)\frac{1}{2c_3^2}\right] \\
&\quad [1-F(c_3)]^{n-3}\,dF(c_3)\,dF(c_2)\,dF(c_1) \\
&= \int_{\underline{c}}^{\overline{c}}\frac{[1-F(c_3)]^{n-3}}{c_3}dF(c_3)\int_{\underline{c}}^{c_3}\int_{\underline{c}}^{c_2}\left(\frac{3}{2}-\frac{1}{3}\frac{c_1}{c_2}+\frac{1}{6}\left(\frac{c_1}{c_2}\right)^2\right)dF(c_1)\,dF(c_2) \\
&\quad +\int_{\underline{c}}^{\overline{c}}\frac{[1-F(c_3)]^{n-3}}{c_3}dF(c_3)\int_{\underline{c}}^{c_3}\int_{\underline{c}}^{c_2}\left(c_2+\frac{c_1^2}{3c_2}\right)\frac{1}{2c_3}dF(c_1)\,dF(c_2) \\
&< \int_{\underline{c}}^{\overline{c}}\frac{[1-F(c_3)]^{n-3}}{c_3}dF(c_3)\int_{\underline{c}}^{c_3}\int_{\underline{c}}^{c_2}\left(\frac{3}{2}-\frac{1}{3}+\frac{1}{6}\right)dF(c_1)\,dF(c_2) \\
&\quad +\int_{\underline{c}}^{\overline{c}}\frac{[1-F(c_3)]^{n-3}}{c_3}dF(c_3)\int_{\underline{c}}^{c_3}\int_{\underline{c}}^{c_2}\frac{2}{3}dF(c_1)\,dF(c_2) \\
&= \frac{1}{2}\left(\frac{4}{3}+\frac{2}{3}\right)\int_{\underline{c}}^{\overline{c}}\frac{F^2(c_3)}{c_3}[1-F(c_3)]^{n-3}\,dF(c_3)=R'^C,
\end{aligned}$$

where the first inequality is derived as follows: by assumption $\underline{c}\le c_1<c_2<c_3\le \overline{c}$, this easily implies that

$$\frac{3}{2}-\frac{1}{3}\frac{c_1}{c_2}+\frac{1}{6}\left(\frac{c_1}{c_2}\right)^2<\frac{3}{2}-\frac{1}{3}+\frac{1}{6}=\frac{4}{3},$$

$$\left(c_2+\frac{c_1^2}{3c_2}\right)\frac{1}{2c_3}<\frac{c_2}{2c_3}+\frac{c_1^2}{6c_2c_3}<\frac{1}{2}+\frac{1}{6}=\frac{2}{3},$$

and the last equality is derived by

$$\int_{\underline{c}}^{c_3}\int_{\underline{c}}^{c_2}dF(c_1)\,dF(c_2)=\int_{\underline{c}}^{c_3}F(c_2)\,dF(c_2)=\frac{1}{2}F^2(c_3)$$

Therefore $R'^D<R'^C$, and we can conclude $R^D<R^C$. ■

Proof of Lemma 3

Proof. Recall (3.11), given c_i,the conditional expected payoff of each player under disclosure policy is given by

$$
\begin{aligned}
\pi_i^D(c_i) &= \int_{c_i}^{\overline{c}} \left[1 - \frac{c_i}{c}\right] dH(c) \\
&= \int_{c_i}^{\overline{c}} dH(c) - c_i \int_{c_i}^{\overline{c}} \frac{1}{c} dH(c) \\
&= H(\overline{c}) - H(c_i) - c_i \int_{c_i}^{\overline{c}} \frac{1}{c} dH(c) \\
&= 1 - H(c_i) - c_i \int_{c_i}^{\overline{c}} \frac{1}{c} dH(c).
\end{aligned}
$$

Recall (3.12), given c_i,the conditional expected payoff of each player under concealment policy is given by

$$
\begin{aligned}
\pi_i^C(c_i) &= 1 - H(c_i) - c_i x_i(c_i) \\
&= 1 - H(c_i) - c_i \sum_{s=1}^{m} \int_{c_i}^{\overline{c}} -\frac{1}{c} F_s'(c)\, dc \\
&= 1 - H(c_i) - c_i \int_{c_i}^{\overline{c}} \frac{1}{c} d\left[-\sum_{s=1}^{m} F_s(c)\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{s=1}^{m} F_s(c) &= \sum_{s=1}^{m} \frac{(n-1)!}{(s-1)!\,(n-s)!} \times [1-F(c)]^{n-s} [F(c)]^{s-1} \text{ let } s = j+1 \\
&= \sum_{j=0}^{m-1} \frac{(n-1)!}{j!\,(n-1-j)!} F(c)^j [1-F(c)]^{n-1-j} \\
&= \sum_{j=0}^{m-1} C_{n-1}^{j} F(c)^j [1-F(c)]^{n-1-j}.
\end{aligned}
$$

Note that

$$
H(c) = \sum_{j=m}^{n-1} C_{n-1}^{j} F(c)^j [1-F(c)]^{n-1-j} ,
$$

and

$$
\begin{aligned}
& \sum_{j=0}^{n-1} C_{n-1}^{j} F\left(c\right)^{j} \left[1-F\left(c\right)\right]^{n-1-j} \\
= \ & \sum_{j=0}^{m-1} C_{n-1}^{j} F\left(c\right)^{j} \left[1-F\left(c\right)\right]^{n-1-j} + \sum_{j=m}^{n-1} C_{n-1}^{j} F\left(c\right)^{j} \left[1-F\left(c\right)\right]^{n-1-j} \\
= \ & 1,
\end{aligned}
$$

then

$$\sum_{s=1}^{m} F_s\left(c\right) + H\left(c\right) = 1,$$

therefore

$$dH\left(c\right) = d\left[1-\sum_{s=1}^{m} F_s\left(c\right)\right] = d\left[-\sum_{s=1}^{m} F_s\left(c\right)\right].$$

We can conclude $\pi_i^D\left(c_i\right) = \pi_i^C\left(c_i\right)$. ■