Plat closure of braids

by

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Contents

Declaration

I declare that, to the best of my knowledge, the all the material contained in this thesis is original, except where explicitly stated otherwise. I confirm that this thesis has not been submitted for a degree at another university.

Summary

Given a braid $b \in \mathbf{B}_{2n}$ we can produce a link by joining consecutive pairs of strings at the top, forming caps, and at the bottom, forming cups. This link is called the plat closure of b. The set of all braids that fix the caps form a subgroup $\mathbf{H}_{2n}$ and the plat closure of a braid is unchanged after multiplying on the left or on the right by elements of $\mathbf{H}_{2n}$. So plat closure gives a map from the double cosets $\mathbf{H}_{2n} \backslash \mathbf{B}_{2n} / \mathbf{H}_{2n}$ to the set of isotopy classes of non-empty links. As well moving within a double coset there is a stabilisation move which leaves the plat closure unchanged but increases the braid index by two and multiplies on the right by σ_{2n}. Birman [2] has shown that any two braid with isotopic plat closures can be related by a sequence of double coset and stabilisation moves.

In Chapter 1 we show that if we change the way we draw the cups then we can use twisted cabling as the stabilisation move. Moreover, we show that any two braids with equal plat closure can be stabilised until they lie in the same double coset. If we restrict to even braids then we can give the plat closure a well defined orientation. In this case we show that untwisted cabling can be used as the stabilisation move. Assuming an oriented version of Birman's result we construct a groupoid $\mathcal{G}$ and two subgroupoids $\mathcal{H}^+$ and $\mathcal{H}^-$ which satisfy the following. All the even braid groups embed in $\mathcal{G}$. There is a plat closure map on $\mathcal{G}$ which takes the same value on the embedded even braid group. This plat closure is constant on the double cosets $\mathcal{H}^+ \backslash \mathcal{G} / \mathcal{H}^-$ and induces a bijection between double cosets and isotopy classes of non-empty oriented links.

In Chapter 2 we compute a presentation for $\mathbf{H}_{2n}$. To do this we construct a 2-complex $\mathbf{X}_n$ on which $\mathbf{H}_{2n}$ acts. Then we show that this complex is simply connected, the action is transitive on the vertex set and the the number of edge and face orbits is finite. We get generators from each edge orbit and relations from the edge and face orbits. In the final chapter we compute a presentation for the intersection of $\mathbf{H}_{2n}$ and the pure braid group.

Chapter 1

Plat Closure of Braids and the Braid Cabling Groupoid

1.1 Plat closure of braids

Given a braid $b \in \mathbf{B}_{2n}$ on $2n$ strings we define the *plat closure of* b to be the link obtained by joining consecutive pairs of strings at the top, forming caps, and at the bottom, forming cups. We can think of the caps as forming a $(0, 2n)$-tangle a_0^+ and the cups as forming a $(2n, 0)$-tangle a_0^-.

$$a_0^+ = \cap \; \cap \; \cdots \; \cap \qquad\qquad a_0^- = \cup \; \cup \; \cdots \; \cup$$

Using this notation we can write the plat closure in the following way.

$$\mathsf{p}_0(b) = a_0^+ b \, a_0^-$$

As with the closure of braids, we have the following analogue of Alexander's theorem[1].

Proposition 1. *For every link L there exists some braid $b \in \mathbf{B}_{2n}$ such that the plat closure of b is L.* $\square$

Let $\mathbf{H}_{2n}$ be the stabiliser of a_0^+ under the action of $\mathbf{B}_{2n}$ on $(0, 2n)$-tangles, ie

$$\mathbf{H}_{2n} = \left\{ h \in \mathbf{B}_{2n} \,\middle|\, a_0^+ h = a_0^+ \right\}.$$

By reflecting horizontally we see that this is also the stabiliser of a_0^-.

For any braid $b \in \mathbf{B}_{2n}$ we have the following two moves which preserve its plat closure. A double coset move, (A_0), which moves within $\mathbf{H}_{2n} b \mathbf{H}_{2n}$, and a stabilisation move (B_0) which increases the braid index by two.

$$b \to h_1 b\, h_2 \qquad h_1, h_2 \in \mathbf{H}_{2n} \tag{A_0}$$

$$b \to b\sigma_{2n} \in \mathbf{B}_{2n+2} \tag{B_0}$$

In [2] Birman proves the following analogue of Markov's theorem[8].

Theorem 2 (Birman)**.** *Given two braids $b \in \mathbf{B}_{2n}$ and $b' \in \mathbf{B}_{2n'}$ with $\mathsf{p}_0(b) = \mathsf{p}_0(b')$ then there exists a sequence of braids*

$$b = b_0 \to b_1 \to b_2 \to \ldots \to b_N = b'$$

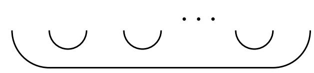
such that the plat closure of each b_i is equal to that of b, $\mathsf{p}_0(b_i) = \mathsf{p}_0(b)$, and such that each move $b_i \to b_{i+1}$ is either a double coset move (A_0), a stabilisation move (B_0) or the inverse of a stabilisation move $(\mathrm{B}_0)^{-1}$. $\square$

In [6] Hilden calculates a set of generators for $\mathbf{H}_{2n}$ and in Chapter 2 we calculate a presentation for $\mathbf{H}_{2n}$.

1.2 Shifted plat closure

If we shift the cups giving a modified form of plat closure, as defined below, then we can use inclusion as the stabilisation move.

Definition 3. Let $a^+ = a_0^+$ and $a^- = \delta a_0^-$ where $\delta = \sigma_1 \sigma_2 \cdots \sigma_{2n-1}$.

$a^+ =$  $a^- =$ 

Define the *shifted plat closure* of a braid $b \in \mathbf{B}_{2n}$ by

$$\mathsf{p}(b) = a^+ b \, a^-.$$

Proposition 4. *Given any link L there exists a braid $b \in \mathbf{B}_{2n}$ such that the shifted plat closure of b is L.*

Proof. This follows from Proposition 1 and the fact that $a^- = \delta a_0^-$. $\square$

The loss of symmetry between the caps and the cups means that we now have two subgroups of $\mathbf{B}_{2n}$, the stabiliser of a^+ and the stabiliser of a^-.

$$\mathbf{H}_{2n}^+ = \left\{ h \in \mathbf{B}_{2n} \,\middle|\, a^+ h = a^+ \right\} \qquad \mathbf{H}_{2n}^- = \left\{ h \in \mathbf{B}_{2n} \,\middle|\, h \, a^- = a^- \right\}$$

As before, we now have two moves which preserve the shifted plat closure. A double coset move (A_1) and a stabilisation move (B_1). For $b \in \mathbf{B}_{2n}$ let $\overline{b}$

denote the inclusion of b in $\mathbf{B}_{2n+2}$, ie $\overline{b}$ is b with two vertical strings added on the right.

$$b \to h^+ b\, h^- \qquad \text{for some } h^+ \in \mathbf{H}^+_{2n},\ h^- \in \mathbf{H}^-_{2n} \tag{A$_1$}$$

$$b \to \overline{b} \tag{B$_1$}$$

Theorem 5. *Given any two braids $b \in \mathbf{B}_{2n}$ and $b' \in \mathbf{B}_{2n'}$ with $\mathsf{p}(b) = \mathsf{p}(b')$ then there exists a sequence of braids*

$$b = b_0 \to b_1 \to b_2 \to \ldots \to b_N = b'$$

such that for each b_i we have that $\mathsf{p}(b_i) = \mathsf{p}(b)$ and that each move $b_i \to b_{i+1}$ is either an (A$_1$), (B$_1$) *or* $(\mathrm{B}_1)^{-1}$ *move.*

Proof. As $a^- = \delta a_0^-$ we have that $\mathsf{p}(b) = \mathsf{p}(b')$ only if $\mathsf{p}_0(b\,\delta) = \mathsf{p}_0(b'\delta)$ so by Theorem 2 there exists a sequence

$$b\,\delta = d_0 \to d_1 \to d_2 \to \ldots \to d_M = b'\delta$$

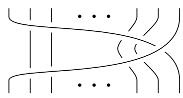
such that for each d_i we have that $\mathsf{p}_0(d_i) = \mathsf{p}(b)$ and that each move $d_i \to d_{i+1}$ is one of (A$_0$), (B$_0$) or $(\mathrm{B}_0)^{-1}$. Hence, if we let $b_i = d_i\,\delta^{-1}$, it suffices to show that for each move $d_i \to d_{i+1}$ there exists a sequence of (A$_1$), (B$_1$) or $(\mathrm{B}_1)^{-1}$ moves from b_i to b_{i+1}.

Suppose that $d_i \to d_{i+1}$ is a move of type (A$_0$). Then $d_{i+1} = h_1 d_i h_2$ for some $h_1, h_2 \in \mathbf{H}^+_{2n}$. Consider the braid $h_1 b_j \delta h_2 \delta^{-1}$, if we multiply by δ on the right we get

$$(h_1 b_i \delta h_2 \delta^{-1})\delta = h_1 b_i \delta h_2 = h_1 d_i h_2 = d_{i+1}.$$

So $b_{i+1} = h_1 b_i \delta h_2 \delta^{-1}$ and, because $h_1 \in \mathbf{H}^+_{2n}$ and $\delta h_2 \delta^{-1} \in \mathbf{H}^-_{2n}$, we have that $b_i \to b_{i+1}$ is a type (A_1) move.

Now suppose that $d_i \to d_{i+1}$ is a move of type (B_1). So we have that $d_{i+1} = \overline{d_i}\,\sigma_{2n}$. Let x be the following element of $\mathbf{B}_{2n+2}$.

$$x = \sigma_1 \sigma_2 \cdots \sigma_{2n} \sigma_{2n+1}^{-1} \sigma_{2n}^{-1} \cdots \sigma_1^{-1} =$$ 

Clearly $x \in \mathbf{H}^-_{2n+2}$ as it is just a half twist of the outer cup. Now consider $\overline{b_i}\,x$, multiplying on the right by δ gives

$$\overline{b_i} x \delta = \overline{b_i} \sigma_1 \sigma_2 \cdots \sigma_{2n} = \overline{(b_i \delta)} \sigma_{2n} = d_i \sigma_{2n}.$$

Hence $b_{i+1} = \overline{b_i}\,x$ and we have the following sequence of moves from b_i to b_{i+1}.

$$b_i \xrightarrow{(\mathrm{B}_1)} \overline{b_i} \xrightarrow{(\mathrm{A}_1)} \overline{b_i}\,x$$

$\square$

1.3 Twisted cabling

Definition 6. For $b \in \mathbf{B}_{2n}$ let $\mathsf{tw}_i(b)$ denote the braid on $2n+2$ strings obtained by twist cabling the ith string of b, that is the braid obtained by replacing the ith string from the top left with three new strings in a neighbourhood of the original such that at each crossing they perform a half twist in the same direction as the crossing. The function tw_i can be defined recursively by $\mathsf{tw}_i(\mathbf{1}) = \mathbf{1}$ and

$$\mathsf{tw}_i\big(\sigma_j^{\epsilon}\, b\big) = \begin{cases} \sigma_j^{\epsilon}\, \mathsf{tw}_i(b) & \text{if } j < i-1 \\ \\ \sigma_{i+2}^{\epsilon}\sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon}\sigma_{i+2}^{\epsilon}\sigma_{i+1}^{\epsilon}\sigma_{i+2}^{\epsilon}\, \mathsf{tw}_{i+1}(b) & \text{if } i = j \\ \\ \sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon}\sigma_{i-1}^{\epsilon}\sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon}\sigma_{i+1}^{\epsilon}\, \mathsf{tw}_{i-1}(b) & \text{if } i = j-1 \\ \\ \sigma_j^{\epsilon}\, \mathsf{tw}_i(b) & \text{if } j > i \end{cases}$$

where $\epsilon = \pm 1$.

Proposition 7. *The shifted plat closure of a braid is preserved by twisted cabling.*

Before we prove this we need the following definitions and lemma.

Let t_i be the $(2n, 2n+2)$–tangle which adds a cap to the right of the ith string if i is even and to the left if i is odd.

$$t_i = \begin{cases} \big|\,\big|\ \cdots\ {\cap}\big|^{i} \cdots\ \big| & \text{if } i \text{ is odd} \\ \big|\,\big|\ \cdots\ \big|^{i}_{\cap}\ \cdots\ \big| & \text{if } i \text{ is even} \end{cases}$$

So $a^+ t_i = a^+$, $t_i\, a^- = a^-$ and, for example, $t_2 = t_3$.

Let s_i be the $(2n+2, 2n)$–tangle which adds a cup to the left of the ith string if i is even and to the right if i is odd.

$$s_i = \begin{cases} \big|\,\big| \;\cdots\; \big|_i^{\cup} \;\cdots\; \big| & \text{if } i \text{ is odd} \\[2em] \big|\,\big| \;\cdots\; {}^{\cup}\big|_i \cdots \; \big| & \text{if } i \text{ is even} \end{cases}$$

So $s_i\, a^- = a^-$, $a^+ s_i = a^+$ and, for example, $s_1 = s_2$.

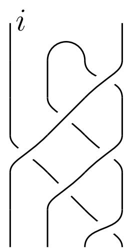
Let π be the natural map from $\mathbf{B}_{2n}$ to the symmetric group $\mathbf{S}_{2n}$, ie π takes σ_i to the transposition $(i\ i{+}1)$. If we write π and the action of the symmetric group on the right then we can define the action of $\mathbf{B}_{2n}$ on the set $\{1, \ldots, 2n\}$ by $i \cdot b = i \cdot (b\pi)$.

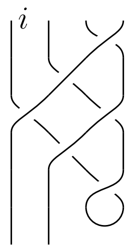
Lemma 8. *For a braid $b \in \mathbf{B}_{2n}$ the following equations hold.*

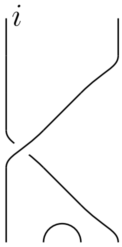
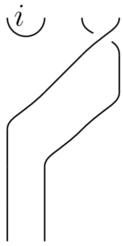
$$t_i\, \mathsf{tw}_i(b) = b\, t_j$$
$$\mathsf{tw}_i(b)\, s_j = s_i\, b$$

where $j = i \cdot b$.

Proof. As tw_i can be defined recursively, it is enough to show this when $b = \sigma_k^{\pm 1}$. We will do the case when $b = \sigma_k$, the case $b = \sigma_k^{-1}$ is similar. If $k < i-1$ or $k > i$ then clearly the equations hold. If $k = i$ and i is even then

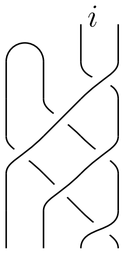
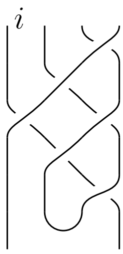
$t_i\, \mathsf{tw}_i(\sigma_i) =$ 

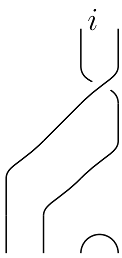
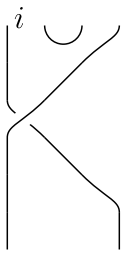
$\mathsf{tw}_i(\sigma_i)\, s_{i+1} =$ 

$=$  $=$ 

$= \sigma_i \, t_{i+1}$ $= s_i \, \sigma_i$

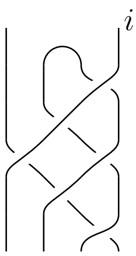
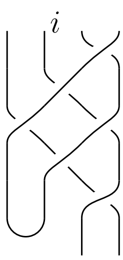
If $k = i$ and i is odd then

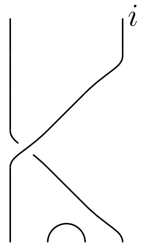
$t_i \, \mathsf{tw}_i(\sigma_i) =$  $\mathsf{tw}_i(\sigma_i) \, s_{i+1} =$ 

$=$  $=$ 

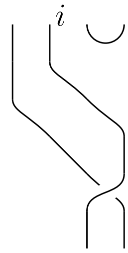
$= \sigma_i \, t_{i+1}$ $= s_i \, \sigma_i$

If $k = i - 1$ and i is odd then

$t_i \, \mathsf{tw}_i(\sigma_{i-1}) =$  $\mathsf{tw}_i(\sigma_{i-1}) \, s_{i-1} =$ 

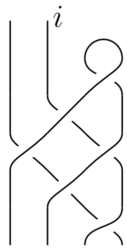
$=$ 

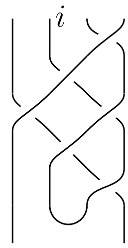
$= \sigma_{i-1}\, t_{i-1}$

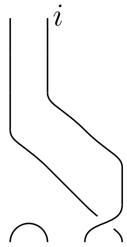
$=$ 

$= s_i\, \sigma_{i-1}$

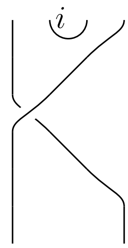
If $k = i - 1$ and i is even then

$t_i\, \mathsf{tw}_i(\sigma_{i-1}) =$ 

$\mathsf{tw}_i(\sigma_{i-1})\, s_{i-1} =$ 

$=$ 

$= \sigma_{i-1}\, t_{i-1}$

$=$ 

$= s_i\, \sigma_{i-1}$

$\square$

Proof of Proposition 7. For $b \in \mathbf{B}_{2n}$ and $j = i \cdot b$ we have the following.

$$\begin{aligned}\mathsf{p}(\mathsf{tw}_i(b)) = a^+ \mathsf{tw}_i(b)\, a^- &= a^+ t_i\, \mathsf{tw}_i(b)\, a^- \\ &= a^+ b\, t_j a^- = a^+ b\, a^- = \mathsf{p}(b)\end{aligned}$$

$\square$

So we have the following two moves which preserve the shifted plat closure of a braid $b \in \mathbf{B}_{2n}$. A double coset move (A_2) and a twisted cabling move (B_2).

$$b \to h^+ b\, h^- \qquad \text{for some } h^+ \in \mathbf{H}^+_{2n}, h^- \in \mathbf{H}^-_{2n} \tag{A_2}$$

$$b \to \mathsf{tw}_i(b) \qquad \text{for some } i \tag{B_2}$$

Theorem 9. *Given two braids $b \in \mathbf{B}_{2n}$ and $b' \in \mathbf{B}_{2n'}$ with $\mathsf{p}(b) = \mathsf{p}(b')$ then there exists a sequence of braids*

$$b = b_0 \to b_1 \to b_2 \to \ldots \to b_N = b'$$

such that the shifted plat closure of each b_i is equal to that of b, $\mathsf{p}(b_i) = \mathsf{p}(b)$, and that each move $b_i \to b_{i+1}$ is either an (A_2), (B_2) or $(\mathrm{B}_2)^{-1}$ move.

Proof. By Theorem 5 there exists a sequence of (A_1), (B_1) and $(\mathrm{B}_1)^{-1}$ moves from b to b'. As $(\mathrm{A}_1) = (\mathrm{A}_2)$ it is enough to show that we can replace each (B_1) move with a sequence of (A_2), (B_2) and $(\mathrm{B}_2)^{-1}$ moves. So suppose that $b \to \overline{b}$ is a (B_1) move and consider $a^+\mathsf{tw}_i(b)$ where $i = 2n \cdot b^{-1}$.

$$a^+\mathsf{tw}_i(b) = a^+ t_i\, \mathsf{tw}_i(b) = a^+ b\, t_{2n} = a^+\overline{b}$$

Therefore $\mathsf{tw}_i(b)$ and $\overline{b}$ lie in the same coset of $\mathbf{H}^+_{2n+2}\backslash\mathbf{B}_{2n+2}$ and hence $\mathsf{tw}_i(b) \to \overline{b}$ is an (A_2) move. So we can replace $b \to \overline{b}$ with the following sequence of moves.

$$b \xrightarrow{(\mathrm{B}_2)} \mathsf{tw}_i(b) \xrightarrow{(\mathrm{A}_2)} \overline{b}$$

$\square$

Proposition 10. *Twisted cabling preserves $\mathbf{H}^+_{2n}$ and $\mathbf{H}^-_{2n}$, ie for all $h^+ \in \mathbf{H}^+_{2n}$ and all $h^- \in \mathbf{H}^-_{2n}$ and for all i we have that $\mathsf{tw}_i(h^+) \in \mathbf{H}^+_{2n+2}$ and $\mathsf{tw}_i(h^-) \in \mathbf{H}^-_{2n+2}$.*

Proof.

$$a^+\mathsf{tw}_i\big(h^+\big) = a^+t_i\,\mathsf{tw}_i\big(h^+\big) = a^+h^+t_j = a^+t_j = a^+$$

where $j = i \cdot h^+$.

$$\mathsf{tw}_i\big(h^-\big)\,a^- = \mathsf{tw}_i\big(h^-\big)\,s_ja^- = s_ih^-a^- = s_i\,a^- = a^-$$

where $j = i \cdot h^-$. $\square$

Lemma 11. *For $i < j$ the braid $\mathsf{tw}_i\mathsf{tw}_j(b)$ is in the same double coset as $\mathsf{tw}_{j+2}\mathsf{tw}_i(b)$.*

Proof. Let $k = i \cdot b$ and $l = j \cdot b$. We will assume that $k < l$, the case when $k > l$ is analogous. It is easy to see that the tangles t_lt_k and t_kt_{l+2} are equal. So we have the following.

$$\begin{aligned} a^+\mathsf{tw}_i\mathsf{tw}_j(b) = a^+t_i\,\mathsf{tw}_i\mathsf{tw}_j(b) &= a^+\mathsf{tw}_j(b)\,t_k \\ &= a^+t_j\,\mathsf{tw}_j(b)\,t_k = a^+b\,t_lt_k \end{aligned}$$

and

$$a^+\mathsf{tw}_{j+2}\mathsf{tw}_i(b) = a^+t_{j+2}\mathsf{tw}_{j+2}\mathsf{tw}_i(b)$$
$$= a^+\mathsf{tw}_i(b)\,t_{l+2} = a^+t_i\,\mathsf{tw}_i(b)\,t_{l+2} = a^+b\,t_kt_{l+2}.$$

Hence $a^+\mathsf{tw}_i\mathsf{tw}_j(b) = a^+\mathsf{tw}_{j+2}\mathsf{tw}_i(b)$ and so $\mathsf{tw}_i\mathsf{tw}_j(b)$ and $\mathsf{tw}_{j+2}\mathsf{tw}_i(b)$ lie in the same coset of $\mathbf{H}^+_{2n+4}\backslash\mathbf{B}_{2n+4}$. ☐

Theorem 12. *The sequence of braids in Theorem 9 can be chosen so that it consists of a sequence of* (B_2) *moves, then an* (A_2) *move, and then a sequence of* $(\mathrm{B}_2)^{-1}$ *moves.*

Proof. First we will show that an (A_2) followed by a (B_2) move can be replaced with a (B_2) followed by an (A_2) move, and hence that a $(\mathrm{B}_2)^{-1}$ followed by an (A_2) can be replaced by an (A_2) followed by a $(\mathrm{B}_2)^{-1}$. Then we will show that a $(\mathrm{B}_2)^{-1}$ followed by a (B_2) can either be eliminated or replaced by a (B_2), an (A_2), and then a $(\mathrm{B}_2)^{-1}$ move. Finally, noting that an (A_2) move followed by another (A_2) move is equivalent to a single (A_2) move, we see that any sequence of moves can be rewritten into one of the required form and hence the theorem holds.

So, suppose that we have an (A_2) move followed by a (B_2) move.

$$b \xrightarrow{(\mathrm{A}_2)} h^+b\,h^- \xrightarrow{(\mathrm{B}_2)} \mathsf{tw}_i\big(h^+b\,h^-\big)$$

Let $j = i\cdot h^+$ and $k = i\cdot(h^+b)$. By Proposition 10 we have that $\mathsf{tw}_i(h^+)\in\mathbf{H}^+_{2n+2}$ and $\mathsf{tw}_k(h^-)\in\mathbf{H}^-_{2n+2}$. So we can replace the sequence of moves with the following.

$$b \xrightarrow{(\mathrm{B}_2)} \mathsf{tw}_j(b) \xrightarrow{(\mathrm{A}_2)} \mathsf{tw}_i\big(h^+\big)\,\mathsf{tw}_j(b)\,\mathsf{tw}_k\big(h^-\big) = \mathsf{tw}_i\big(h^+ b\, h^-\big)$$

Now suppose that we have a $(\mathrm{B}_2)^{-1}$ then a (B_2) move

$$b_1 \xrightarrow{(\mathrm{B}_2)^{-1}} b_2 \xrightarrow{(\mathrm{B}_2)} b_3.$$

So $b_1 = \mathsf{tw}_i(b_2)$ and $b_3 = \mathsf{tw}_j(b_2)$ for some i, j. If $i = j$ then the sequence can be simplified. So without loss of generality we may assume that $i < j$. So, by Lemma 11, we have the following sequence of moves.

$$b_1 = \mathsf{tw}_i(b_2) \xrightarrow{(\mathrm{B}_2)} \mathsf{tw}_{j+2}\mathsf{tw}_i(b_2) \xrightarrow{(\mathrm{A}_2)} \mathsf{tw}_i\mathsf{tw}_j(b_2) \xrightarrow{(\mathrm{B}_2)^{-1}} \mathsf{tw}_j(b_2) = b_3.$$

$\square$

1.4 Cabling

Definition 13. For $b \in \mathbf{B}_{2n}$ let $\mathsf{c}_i(b)$ denote the braid on $2n+2$ strings obtained by replacing the ith string from the top left with three new strings parallel to the original. The function c_i can be defined recursively by $\mathsf{c}_i(\mathbf{1}) = \mathbf{1}$ and

$$
\mathsf{c}_i\left(\sigma_j^{\epsilon} b\right) = \begin{cases} \sigma_j^{\epsilon}\, \mathsf{c}_i(b) & \text{if } j < i-1 \\[2ex] \sigma_{i+2}^{\epsilon}\sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon}\, \mathsf{c}_{i+1}(b) & \text{if } i = j \\[2ex] \sigma_{i-1}^{\epsilon}\sigma_i^{\epsilon}\sigma_{i+1}^{\epsilon}\, \mathsf{c}_{i-1}(b) & \text{if } i = j-1 \\[2ex] \sigma_{j+2}^{\epsilon}\mathsf{c}_i(b) & \text{if } j > i \end{cases}
$$

where $\epsilon = \pm 1$.

The goal is to use cabling as the stabilisation move, however there is an obvious problem. Cabling does not always preserve the plat closure of a braid, for example $\mathsf{p}(\mathsf{c}_1(\sigma_1)) \neq \mathsf{p}(\sigma_1)$.

Proposition 14. *For all $b \in \mathbf{B}_{2n}$, if $i = i \cdot b \mod 2$ then $\mathsf{p}(b) = \mathsf{p}(\mathsf{c}_i(b))$.*

Before we prove this we need the following.

As before, let t_i be the $(2n, 2n+2)$–tangle which adds a cap to the right of the ith string if i is even and to the left if i is odd. Similarly let t_i' be the $(2n, 2n+2)$–tangle which adds a cap to the left of the ith string if i is even and to the right if i is odd.

$$
t_i = \begin{cases} \Big|\Big| \;\cdots\; \cap \Big|^{i} \cdots \;\Big| & \text{if } i \text{ is odd} \\[2ex] \Big|\Big| \;\cdots\; \Big|^{i} \cap \;\cdots\; \Big| & \text{if } i \text{ is even} \end{cases} \qquad t_i' = \begin{cases} \Big|\Big| \;\cdots\; \Big|^{i} \cap \;\cdots\; \Big| & \text{if } i \text{ is odd} \\[2ex] \Big|\Big| \;\cdots\; \cap \Big|^{i} \cdots \;\Big| & \text{if } i \text{ is even} \end{cases}
$$

As before, let s_i be the $(2n+2, 2n)$–tangle which adds a cup to the left of the ith string if i is even and to the right if i is odd. Similarly, let s_i' be the $(2n+2, 2n)$–tangle which adds a cup to the right of the ith string if i is even and to the left if i is odd.

$$s_i = \begin{cases} \Big|\,\Big| \;\cdots\; \Big|_i^{\cup} \;\cdots\; \Big| & \text{if } i \text{ is odd} \\ \\ \Big|\,\Big| \;\cdots\; {}^{\cup}\Big|_i \cdots\; \Big| & \text{if } i \text{ is even} \end{cases} \qquad s'_i = \begin{cases} \Big|\,\Big| \;\cdots\; {}^{\cup}\Big|_i \cdots\; \Big| & \text{if } i \text{ is odd} \\ \\ \Big|\,\Big| \;\cdots\; \Big|_i^{\cup} \;\cdots\; \Big| & \text{if } i \text{ is even} \end{cases}$$

Lemma 15. *For any braid $b \in \mathbf{B}_{2n}$, let $j = i \cdot b$, then we have the following.*

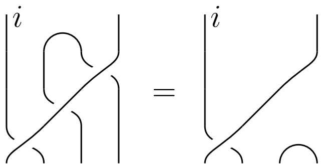
$$t_i\,\mathsf{c}_i(b) = \begin{cases} b\,t_j & \textit{if } i = j \mod 2 \\ \\ b\,t'_j & \textit{if } i \neq j \mod 2 \end{cases} \qquad t'_i\,\mathsf{c}_i(b) = \begin{cases} b\,t'_j & \textit{if } i = j \mod 2 \\ \\ b\,t_j & \textit{if } i \neq j \mod 2 \end{cases}$$

$$\mathsf{c}_i(b)\,s_j = \begin{cases} s_i\,b & \textit{if } i = j \mod 2 \\ \\ s'_i\,b & \textit{if } i \neq j \mod 2 \end{cases} \qquad \mathsf{c}_i(b)\,s'_j = \begin{cases} s'_i\,b & \textit{if } i = j \mod 2 \\ \\ s_i\,b & \textit{if } i \neq j \mod 2 \end{cases}$$

Proof. Because c_i can be defined iteratively it is enough to show this when $b = \sigma^{\pm 1}$, we will only show that $t_i\,\mathsf{c}_i(\sigma_k) = \begin{cases} \sigma_k\,t_j \\ \\ \sigma_k\,t'_j \end{cases}$, the remaining cases are similar.

If $k < i - 1$ or $k > i$ then clearly the equation holds. If $k = i$ and i is even then

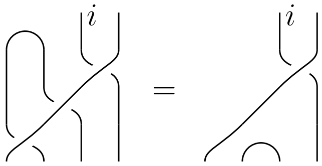
$$t_i\,\mathsf{c}_i(\sigma_i) =$$



$$= \sigma_i\,t'_{i+1}$$

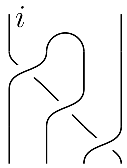
If $k = i$ and i is odd then

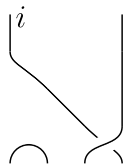
$$t_i\,\mathsf{c}_i(\sigma_i) =$$



$$= \sigma_i\,t'_{i+1}$$

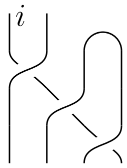
If $k = i-1$ and i is odd then

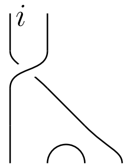
$t_i\,\mathsf{c}_i(\sigma_{i-1}) =$ 

$=$ 

$= \sigma_{i-1}\, t'_{i-1}$

If $k = i-1$ and i is even then

$t_i\,\mathsf{c}_i(\sigma_{i-1}) =$ 

$=$ 

$= \sigma_{i-1}\, t'_{i-1}$

□

Proof of Proposition 14. If $b \in \mathbf{B}_{2n}$ and $i = i\cdot b \mod 2$ then we have

$$\begin{aligned}\mathsf{p}(\mathsf{c}_i(b)) = a^+\mathsf{c}_i(b)\,a^- &= a^+ t_i\,\mathsf{c}_i(b)\,a^- \\ &= a^+ b\, t_{i\cdot b}\, a^- = a^+ b\, a^- = \mathsf{p}(b)\,.\end{aligned}$$

□

So we have the following two types of moves that preserve the shifted plat closure of a braid $b \in \mathbf{B}_{2n}$. A double coset move (A_3) and a cabling move (B_3).

$$b \to h^+ b\, h^- \qquad \text{for some } h^+ \in \mathbf{H}^+_{2n}, h^- \in \mathbf{H}^-_{2n} \tag{A_3}$$

$$b \to \mathsf{c}_i(b) \qquad \text{for } i \text{ such that } i = i\cdot b \mod 2 \tag{B_3}$$

Theorem 16. *Given two braids $b \in \mathbf{B}_{2n}$ and $b' \in \mathbf{B}_{2n'}$ with $\mathsf{p}(b) = \mathsf{p}(b')$ then there exists a sequence of braids*

$$b = b_0 \to b_1 \to b_2 \to \ldots \to b_N = b'$$

such that the shifted plat closure of each b_i is equal to that of b, $\mathsf{p}(b_i) = \mathsf{p}(b)$, and that each move $b_i \to b_{i+1}$ is either an (A_3), (B_3) or $(\mathrm{B}_3)^{-1}$ move.

Proof. By Theorem 5 there exists a sequence of (A_1), (B_1) and $(\mathrm{B}_1)^{-1}$ moves from b to b'. As $(\mathrm{A}_3) = (\mathrm{A}_1)$ it is enough to show that we can replace each (B_1) move with a sequence of (A_1), (B_1) and $(\mathrm{B}_1)^{-1}$ moves.

So suppose that $b \to \overline{b}$ is a (B_1) move. Let $i = 2n \cdot b^{-1}$. If i is even then

$$a^+ \mathsf{c}_i(b) = a^+ t_i\, \mathsf{c}_i(b) = a^+ b\, t_{2n} = a^+ \overline{b}.$$

Therefore $\mathsf{c}_i(b)$ and $\overline{b}$ lie in the same coset of $\mathbf{H}^+_{2n} \backslash \mathbf{B}_{2n}$ and we can replace $b \to \overline{b}$ with

$$b \xrightarrow{(\mathrm{B}_2)} \mathsf{c}_i(b) \xrightarrow{(\mathrm{A}_3)} \overline{b}.$$

If on the other hand i is odd, we have that $\mathsf{c}_{i+1}(\sigma_i b)$ is in the same coset as $\sigma_i \overline{b}$ and we can replace $b \to \overline{b}$ with the following.

$$b \xrightarrow{(\mathrm{A}_3)} \sigma_i b \xrightarrow{(\mathrm{B}_3)} \mathsf{c}_{i+1}(\sigma_i b) \xrightarrow{(\mathrm{A}_3)} \sigma_i \overline{b} \xrightarrow{(\mathrm{A}_3)} \overline{b}$$

□

Proposition 17. *Cabling preserves $\mathbf{H}^+_{2n}$ and $\mathbf{H}^-_{2n}$. In other words, for all $h^+ \in \mathbf{H}^+_{2n}$ and all i such that $i = i \cdot h^+ \mod 2$ we have $\mathsf{c}_i(h^+) \in \mathbf{H}^+_{2n+2}$ and* *for all* $h^- \in \mathbf{H}^-_{2n}$ *and all* i *such that* $i = i \cdot h^- \mod 2$ *we have* $\mathsf{c}_i(h^-) \in \mathbf{H}^-_{2n+2}$.

Proof.

$$a^+ \mathsf{c}_i(h^+) = a^+ t_i \, \mathsf{c}_i(h^+) = a^+ h^+ t_j = a^+ t_j = a^+$$

where $j = i \cdot h^+$.

$$\mathsf{c}_i(h^-) \, a^- = \mathsf{c}_i(h^-) \, s_j \, a^- = s_i \, h^- a^- = s_i \, a^- = a^-$$

where $j = i \cdot h^-$. ☐

Definition 18. Say that a braid $b \in \mathbf{B}_{2n}$ is *even* if for all i we have $i = i \cdot b \mod 2$. The even braids form a subgroup $\mathbf{E}_{2n}$ of $\mathbf{B}_{2n}$.

Proposition 19. *For any* $b \in \mathbf{B}_{2n}$ *there exists* $h^+ \in \mathbf{H}^+_{2n}$ *and* $h^- \in \mathbf{H}^-_{2n}$ *such that* $h^+ b \, h^-$ *is even.*

Proof. Label the end of the strings with the elements of $\{1, 2, \ldots, 2n\} \times \{+, -\}$ so that the top of the strings are labelled $(1, +), (2, +), \ldots, (2n, +)$ and the bottom labelled $(1, -), (2, -), \ldots, (2n, -)$. Each component C of the link $\mathsf{p}(b)$ gives a sequence of labels constructed by starting at a point $(i, +)$ on C then following C down to $(i \cdot b, -)$ and then carrying on along C listing the labels in the order that they occur. We will assume that the starting point $(i, +)$ is chosen so that i is odd. A braid is even if and only if the indices follow the pattern odd, odd, even, even, odd, odd, etc.

Suppose that the component C gives rise to the sequence $(i_k, s_k)_{k=1}^N$. Then $s_k = +$ if $k = 0$ or $1 \mod 4$ and $s_k = -$ if $k = 2$ or $3 \mod 4$. Hence $s_{2k} = s_{2k+1}$. From the construction it follows that the pair i_{2k} and i_{2k+1}

consists of one even number and one odd number. A braid is even if and only if whenever k is odd this pair is of the form odd then even and whenever k is even this pair is even then odd. For each pair $(i_{2k}, s_{2k}), (i_{2k+1}, s_{2k+1})$ that doesn't follow this pattern the half twist of the corresponding cap, if $s_{2k} = +$, or cup, if $s_{2k} = -$, multiplied on the top or the bottom respectively of the braid corrects this pair and moves within the double coset. Applying this to every pair of every component we produce an even braid that lies in the same double coset as the original braid. $\square$

Proposition 20. *If $b \in \mathbf{E}_{2n}$ and $b' \in \mathbf{E}_{2n'}$ with $\mathsf{p}(b) = \mathsf{p}(b')$ then the sequence of braids in Theorem 16 can be chosen so that each b_i is even.*

Proof. We will show that any sequence of length two starting at an even braid can be replaced with a sequence where all but the last braid are even. We can assume that at most one of the moves is an (A_3) move. As (B_3) and $(\mathrm{B}_3)^{-1}$ moves take even braids to even braids we may assume that the first move is an (A_3) move. So we only have two cases, an (A_3) then a (B_3), or an (A_3) then a $(\mathrm{B}_3)^{-1}$.

First suppose that we have an (A_3) move followed by a (B_3) move

$$b \to h^+ b\, h^- \to \mathsf{c}_i(h^+ b\, h^-)$$

where b is even. Let $j = i \cdot h^+$, $k = j \cdot b$ and $l = k \cdot h^-$.

If $i = j \mod 2$ then $k = l \mod 2$ and so $\mathsf{c}_i(h^+) \in \mathbf{H}^+_{2n+2}$ and $\mathsf{c}_k(h^-) \in \mathbf{H}^-_{2n+2}$. Hence we have the following sequence of moves.

$$b \xrightarrow{(\mathrm{B}_3)} \mathsf{c}_j(b) \xrightarrow{(\mathrm{A}_3)} \mathsf{c}_i(h^+)\, \mathsf{c}_j(b)\, \mathsf{c}_k(h^-) = \mathsf{c}_i(h^+ b\, h^-)$$

Suppose that $i \neq j \mod 2$ so we also have that $k \neq l \mod 2$. Let $x = \sigma_j \sigma_{j+1} \sigma_j$ and $y = \sigma_k^{-1} \sigma_{k+1}^{-1} \sigma_k^{-1}$. We have that $\mathsf{c}_j(b) = x\, \mathsf{c}_j(b)\, y$, to see this it is enough consider the cases $b = \sigma_i$ and $b = \sigma_{i-1}$. The following shows that $\mathsf{c}_i(h^+)\, x \in \mathbf{H}_{2n+2}^+$ and $y\, \mathsf{c}_k(h^-) \in \mathbf{H}_{2n+2}^-$.

$$a^+ \mathsf{c}_i(h^+)\, x = a^+ t_i\, \mathsf{c}_i(h^+)\, x = a^+ h^+ t_j'\, x = a^+ h^+ t_j = a^+ t_j = a^+$$

$$y\, \mathsf{c}_k(h^-)\, a^- = y\, \mathsf{c}_k(h^-)\, s_l\, a^- = y\, s_k'\, h^- a^- = s_k\, h^- a^- = s_k\, a^- = a^-$$

So we can replace the original sequence of moves with the following.

$$b \xrightarrow{(\mathrm{B}_3)} \mathsf{c}_j(b) \xrightarrow{(\mathrm{A}_3)} \mathsf{c}_i(h^+)\, x\ \mathsf{c}_j(b)\ y\, \mathsf{c}_k(h^-) = \mathsf{c}_i(h^+ b\, h^-)$$

Now we look at the second case. Suppose that we have an (A_3) and then a $(\mathrm{B}_3)^{-1}$ move

$$b \to h^+ b\, h^- = \mathsf{c}_j(b') \to b'$$

where b is even. By Proposition 19 there exists $x \in \mathbf{H}_{2n-2}^+$ and $y \in \mathbf{H}_{2n-2}^-$ such that $xb'y$ is even. Let $i = j \cdot x^{-1}$, $k = j \cdot b'$ and $l = k \cdot y$.

If $i = j \mod 2$ then we also have that $k = l \mod 2$ and so $\mathsf{c}_i(x) \in \mathbf{H}_{2n}^+$ and $\mathsf{c}_k(y) \in \mathbf{H}_{2n}^-$. Therefore the original sequence can be replaced with

$$b \xrightarrow{(\mathrm{A}_3)} \mathsf{c}_i(x)\, h^+ b\, h^- \mathsf{c}_k(y) = \mathsf{c}_i(xb'y) \xrightarrow{(\mathrm{B}_3)^{-1}} xb'y \xrightarrow{(\mathrm{A}_3)} b'.$$

If $i \neq j \mod 2$ then $k \neq l \mod 2$. So, as in the first case, we have that

$$\mathsf{c}_i(x)\,\sigma_j\sigma_{j+1}\sigma_j \in \mathbf{H}_{2n}^+,$$
$$\sigma_k^{-1}\sigma_{k+1}^{-1}\sigma_k^{-1}\mathsf{c}_k(y) \in \mathbf{H}_{2n}^-,$$

and

$$\sigma_j\sigma_{j+1}\sigma_j\,\mathsf{c}_j(b')\,\sigma_k^{-1}\sigma_{k+1}^{-1}\sigma_k^{-1} = \mathsf{c}_j(b')\,.$$

So the sequence can be replaced with the following.

$$b \xrightarrow{(\mathrm{A}_3)} \mathsf{c}_i(x)\,\sigma_j\sigma_{j+1}\sigma_j\,h^+b\,h^-\sigma_k^{-1}\sigma_{k+1}^{-1}\sigma_k^{-1}\mathsf{c}_k(y) = \mathsf{c}_i(xb'y) \xrightarrow{(\mathrm{B}_3)^{-1}} xb'y \xrightarrow{(\mathrm{A}_3)} b'.$$

$\square$

Theorem 21. *Given two even braids $b \in \mathbf{E}_{2n}$ and $b' \in \mathbf{E}_{2n'}$ with $\mathsf{p}(b) = \mathsf{p}(b')$ then the sequence of braids from b to $'$ can be chosen so that all of the* (B$_3$) *moves come first, we then have an* (A$_3$) *move and then all of the* (B$_3$)$^{-1}$ *moves at the end.*

Note that this can fail if either b or b' is not even. For example, if $b = \sigma_1 \in \mathbf{B}_2$ and $b' = \mathbf{1} \in \mathbf{B}_4$ then $\mathsf{p}(b) = \mathsf{p}(b')$ is the unknot but there are no (B$_3$) moves starting at b.

Proof. By Proposition 20 we may assume that every braid in the sequence is even. So, by the proof of Proposition 20, whenever we have an (A$_3$) move followed by an (B$_3$) move we can replace it with a (B$_3$) followed by an (A$_3$) move.

Suppose that we have a $(\mathrm{B}_3)^{-1}$ then a (B_3),

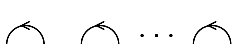
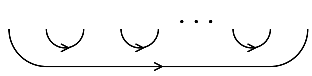
$$\mathsf{c}_i(b) \to b \to \mathsf{c}_j(b)\,.$$

If $i = j$ then clearly this sequence can be removed. Otherwise we may assume that $i < j$ and then this can be replaced by the following sequence.

$$\mathsf{c}_i(b) \xrightarrow{(\mathrm{B}_3)} \mathsf{c}_{j+2}\mathsf{c}_i(b) = \mathsf{c}_i\mathsf{c}_j(b) \xrightarrow{(\mathrm{B}_3)^{-1}} \mathsf{c}_j(b)$$

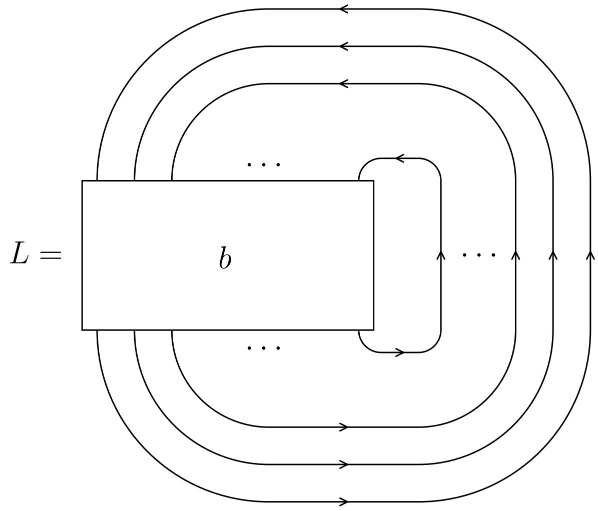
Therefore any sequence can be rewritten to one of the required form. $\square$

For an even braid $b \in \mathbf{E}_{2n}$ the even numbered strings at the top connect to even numbered strings at the bottom and the odd numbered strings at the top connect to odd numbered strings at the bottom. So we have a well defined notion of an even string and an odd string. We can now give the odd strings a downward orientation and the even strings an upward orientation. This orientation is consistent with the shifted plat closure and hence gives a well defined oriented plat closure $\overleftarrow{\mathsf{p}}(b)$. This also gives an orientation on the caps and on the cups and we can write $\overleftarrow{\mathsf{p}}(b) = \overleftarrow{a}^+ b \overleftarrow{a}^-$.

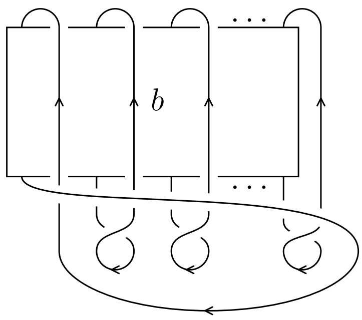
$\overleftarrow{a}^+ =$  $\qquad \overleftarrow{a}^- =$ 

Proposition 22. *Given any oriented link L there exists some braid $b \in \mathbf{B}_{2n}$ such that $L = \overleftarrow{\mathsf{p}}(b)$.*

Proof. By Alexander's theorem[1] L can be expressed as the closure of a braid $b \in \mathbf{B}_n$, ie



This is isotopic to the following, which is clearly the oriented plat closure of some even braid.



$\square$

Proposition 23. *The oriented plat closure of an even braid $b \in \mathbf{E}_{2n}$ is preserved by cabling, ie $\overleftarrow{\mathsf{p}}(b) = \overleftarrow{\mathsf{p}}(\mathsf{c}_i(b))$ for every $1 \leq i \leq 2n$.*

Proof. If we add the appropriate orientation to the t_i and s_i used in the proof of Proposition 14 and Lemma 15 the same arguments hold for the oriented plat closure. $\square$

As with the unoriented case we have two moves which preserve the oriented plat closure of an even braid. A double coset move and a stabilisation move.

$$b \to h^+ b\, h^- \qquad \text{for some } h^+ \in \mathbf{H}^+_{2n} \cap \mathbf{E}_{2n},\ h^- \in \mathbf{H}^-_{2n} \cap \mathbf{E}_{2n} \tag{A$_4$}$$

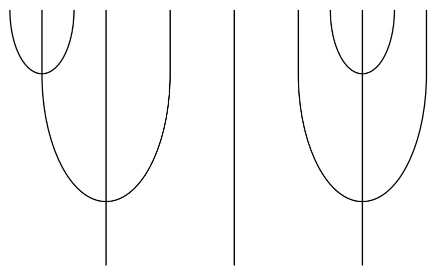
$$b \to \mathsf{c}_i(b) \qquad \text{for some } i \tag{B$_4$}$$

Conjecture 24. *Given two braids* $b \in \mathbf{E}_{2n}$ *and* $b' \in \mathbf{E}_{2n'}$ *with* $\overleftarrow{\mathsf{p}}(b) = \overleftarrow{\mathsf{p}}(b')$ *then there exists a path from* b *to* b' *consisting of* (A_4), (B_4) *and* $(\mathrm{B}_4)^{-1}$ *moves.*

1.5 The Braid Cabling Groupoid

By a *forest* we will mean a sequence of planar rooted ternary trees. For example,

$f =$



Let $F(k,l)$ be the set of all forest with k leaves and l trunks. Our example f lies in $F(11,3)$. We have a map

$$F(k,l) \times F(l,m) \to F(k,m)$$
$$(f,g) \mapsto fg$$

given by gluing the trunks of the first forest onto the leaves of the second. The set $F(k,k)$ contains a single forest which we will call the identity. The forests form a category F with objects $\mathbb{N}$ and morphism $F(k,l)$.

Given a braid $\beta \in \mathbf{B}_n$ and a forest $f \in F(m,n)$ we can glue the trunks of the trees in f to the top of the strings in β. If we then pull all the branching points down through β we get a new braid $\mathsf{c}_f(\beta)$ and a new forest $f \cdot \beta$. The forest $f \cdot \beta$ can also be thought of as the result of permuting the trees of f via the permutation defined by β. If f contains a single branching point on the ith tree then c_f is the same as the cabling map c_i.

Let Γ be the directed graph with vertex set $2\mathbb{N}$ and edges $2n \xrightarrow{\beta} 2n$ for each $\beta \in \mathbf{E}_{2n}$, and edges $2m \xrightarrow{C_f} 2n$ and $2n \xrightarrow{C_f^{-1}} 2m$ for each $f \in F(2m, 2n)$. Let $\mathcal{C}$ be the free category on Γ. In other words, $\mathcal{C}$ is the category with objects $2\mathbb{N}$ and morphisms $\hom(2m, 2n)$ the set of all paths from $2m$ to $2n$ in Γ. Given paths x from $2l$ to $2m$ and y from $2m$ to $2n$ we will write xy for the composite path from $2l$ to $2n$, ie we will write composition of morphisms as a map $\hom(2l, 2m) \times \hom(2m, 2n) \to \hom(2l, 2n)$. We identify the edges given by the identity elements $\mathbf{1} \in \mathbf{E}_{2n}$ and $\mathbf{1} \in F(2n, 2n)$ with the identity morphism $2n \to 2n$. We will write $x \in \Gamma$ if x is an edge of Γ and $x \in \mathcal{C}$ if x is a morphism of $\mathcal{C}$. We will call elements of $\mathcal{C}$ words. Let the braid cabling groupoid $\mathcal{G}$ be the quotient of $\mathcal{C}$ by the following relations.

$$C_f C_f^{-1} = \mathbf{1} = C_f^{-1} C_f \tag{1}$$

$$\alpha\beta = \gamma \qquad \text{for } \alpha, \beta, \gamma \in \mathbf{E}_{2n} \text{ with } \alpha\beta =_{\mathbf{E}_{2n}} \gamma \tag{2}$$

$$C_f C_g = C_{fg} \tag{3}$$

$$C_f \beta = \mathsf{c}_f(\beta)\, C_{f\cdot\beta} \tag{4}$$

The equivalence relation on words generated by (1)–(4) is the same as the reflexive transitive closure of the following system of rewrite rules.

$$C_f \beta \to \mathsf{c}_f(\beta)\, C_{f\cdot\beta} \tag{i}$$

$$\beta\, C_f^{-1} \to C_{f\cdot\beta^{-1}}^{-1} \mathsf{c}_{f\cdot\beta^{-1}}(\beta) \tag{ii}$$

$$C_f C_g \to C_{fg} \tag{iii}$$

$$C_g^{-1} C_f^{-1} \to C_{fg}^{-1} \tag{iv}$$

$$\alpha\beta \to \gamma \qquad \text{where } \alpha\beta =_{\mathbf{E}_{2n}} \gamma \tag{v}$$

$$C_f C_g^{-1} \to C_{g'}^{-1} C_{f'} \qquad \text{where } g'f = f'g \tag{vi}$$

$$C_{fg}^{-1}\, \mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} \to C_g^{-1} \beta C_h \tag{vii}$$

Note that for every $f \in F(2n, 2m)$ and $g \in F(2l, 2m)$ there exists $g' \in F(2k, 2n)$ and $f' \in F(2k, 2l)$ such that $g'f = f'g$.

Proposition 25. *The rules* (i)–(vii) *define a well-founded confluent rewrite system.*

Proof. We need to show that whenever a word can be rewritten in two different ways then the resulting words can be rewritten by a sequence of rewrites to the same word. The following is a list of all words that can be rewritten in two different ways.

$$
\begin{array}{cc}
C_f\beta\, C_g^{-1} & \text{(i), (ii)} \\
C_fC_g\beta & \text{(i), (iii)} \\
C_f\alpha\beta & \text{(i), (v)} \\
C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h}\gamma & \text{(i), (vii)} \\
\beta\, C_g^{-1}C_f^{-1} & \text{(ii), (iv)} \\
\alpha\beta\, C_f^{-1} & \text{(ii), (v)} \\
\alpha\, C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} & \text{(ii), (vii)} \\
C_fC_gC_h & \text{(iii), (iii)} \\
C_fC_gC_h^{-1} & \text{(iii), (vi)} \\
C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h}C_e & \text{(iii), (vii)} \\
C_f^{-1}C_g^{-1}C_h^{-1} & \text{(iv), (iv)} \\
C_fC_g^{-1}C_h^{-1} & \text{(iv), (vi)} \\
C_e^{-1}C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} & \text{(iv), (vii)} \\
\alpha\beta\gamma & \text{(v), (v)} \\
C_eC_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} & \text{(vi), (vii)} \\
C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h}C_e^{-1} & \text{(vi), (vii)} \\
C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} = C_{f'g'}^{-1}\mathsf{c}_{f'}(\beta')\, C_{(f'\cdot\beta')h'} & \text{(vii), (vii)}
\end{array}
$$

Before we continue we will need the following lemma.

Lemma 26. *Cabling and the action of braids on forests satisfies the following equations.*

$$
\begin{aligned}
(fg)\cdot\beta &= \Big(f\cdot \mathsf{c}_g(\beta)\Big)\,(g\cdot\beta)\\
\mathsf{c}_f(\beta)^{-1} &= \mathsf{c}_{f\cdot\beta}(\beta^{-1})\\
\mathsf{c}_f\Big(\mathsf{c}_g(\beta)\Big) &= \mathsf{c}_{fg}(\beta)\\
\mathsf{c}_f(\alpha\beta) &= \mathsf{c}_f(\alpha)\,\mathsf{c}_{f\cdot\alpha}(\beta)
\end{aligned}
$$

□

First we consider the word $C_f\beta\,C_g^{-1}$ which can be rewritten by a (i) rule or a (ii) rule. Suppose that $f'g = g'(f\cdot\beta)$. Applying β^{-1} to both sides of this equation we see that $\Big(f'\cdot\mathsf{c}_g(\beta^{-1})\Big)\,(g\cdot\beta^{-1}) = (g'\cdot\mathsf{c}_{f\cdot\beta}(\beta^{-1}))f$.

$$
\begin{tikzcd}
C_f\beta\,C_g^{-1} \arrow[rr, "\text{(i)}"] \arrow[d, "\text{(ii)}"] & & \mathsf{c}_f(\beta)\,C_{f\cdot\beta}C_g^{-1} \arrow[d, "\text{(iv)}"] \\
C_fC^{-1}_{g\cdot\beta^{-1}}\mathsf{c}_{g\cdot\beta^{-1}}(\beta) \arrow[d, "\text{(vi)}"] & & \mathsf{c}_f(\beta)\,C_{g'}^{-1}C_{f'} \arrow[d, "\text{(ii)}"] \\
C^{-1}_{(g'\cdot\mathsf{c}_{f\cdot\beta}(\beta^{-1}))}C_{f'\cdot\mathsf{c}_g(\beta^{-1})}\mathsf{c}_{g\cdot\beta^{-1}}(\beta) \arrow[rr, "\text{(i)}"] & & C^{-1}_{g'\cdot\mathsf{c}_f(\beta)^{-1}}\mathsf{c}_{(g'\cdot\mathsf{c}_f(\beta^{-1}))f}(\beta)\,C_{f'}
\end{tikzcd}
$$

Now consider $C_f C_g \beta$.

$$\begin{tikzcd}
C_f C_g \beta \arrow[r, "\text{(i)}"] \arrow[d, "\text{(iii)}"] & C_f \mathsf{c}_g(\beta)\, C_{g\cdot\beta} \arrow[d, "\text{(i)}"] \\
C_{fg}\beta \arrow[d, "\text{(i)}"] & \mathsf{c}_{fg}(\beta)\, C_{f\cdot \mathsf{c}_g(\beta)} C_{g\cdot\beta} \arrow[d, "\text{(iii)}"] \\
\mathsf{c}_{fg}(\beta)\, C_{(fg)\cdot\beta} \arrow[r, equal] & \mathsf{c}_{fg}(\beta)\, C_{(f\cdot \mathsf{c}_g(\beta))(g\cdot\beta)}
\end{tikzcd}$$

Now consider $C_f \alpha\beta$, where $\alpha\beta = \gamma$.

$$\begin{tikzcd}
C_f \alpha\beta \arrow[r, "i"] \arrow[d, "\text{(v)}"] & \mathsf{c}_f(\alpha)\, C_{f\cdot\alpha}\beta \arrow[d, "\text{(i)}"] \\
C_f \gamma \arrow[d, "\text{(i)}"] & \mathsf{c}_f(\alpha)\, \mathsf{c}_{f\cdot\alpha}(\beta)\, C_{f\cdot\gamma} \arrow[dl, "\text{(v)}"] \\
\mathsf{c}_f(\gamma)\, C_{f\cdot\gamma} &
\end{tikzcd}$$

Now consider $C_{fg}^{-1} \mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h}\gamma$.

$$\begin{tikzcd}
C_{fg}^{-1} \mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h}\gamma \arrow[r, "\text{(i)}"] \arrow[d, "\text{(vii)}"] & C_{fg}^{-1} \mathsf{c}_f(\beta)\, \mathsf{c}_{(f\cdot\beta)h}(\gamma)\, C_{((f\cdot\beta)h)\cdot\gamma} \arrow[d, "\text{(v)}"] \\
C_g^{-1}\beta\, C_h\gamma \arrow[d, "\text{(i)}"] & C_{fg}^{-1} \mathsf{c}_f(\beta\, \mathsf{c}_h(\gamma))\, C_{((f\cdot\beta)h)\cdot\gamma} \arrow[d, equal] \\
C_g^{-1}\beta\, \mathsf{c}_h(\gamma)\, C_{h\cdot\beta} & C_{fg}^{-1} \mathsf{c}_f(\beta\, \mathsf{c}_h(\gamma))\, C_{(f\cdot\beta\, \mathsf{c}_h(\gamma))(h\cdot\gamma)} \arrow[l, "\text{(vii)}"]
\end{tikzcd}$$

Now consider $\beta\, C_g^{-1} C_f^{-1}$.

$$\begin{tikzcd}
\beta\, C_g^{-1} C_f^{-1} \arrow[rr, "\text{(ii)}"] \arrow[d, "\text{(iv)}"] & & C_{g\cdot\beta^{-1}}^{-1} \mathsf{c}_{g\cdot\beta^{-1}}(\beta)\, C_f^{-1} \arrow[d, "\text{(ii)}"] \\
\beta C_{fg}^{-1} \arrow[d, "\text{(iv)}"] & & C_{g\cdot\beta^{-1}}^{-1} C_{f\cdot \mathsf{c}_g(\beta^{-1})}^{-1} \mathsf{c}_{(f\cdot\mathsf{c}_g(\beta^{-1}))(g\cdot\beta^{-1})}(\beta) \arrow[d, "\text{(iv)}"] \\
C_{(fg)\cdot\beta^{-1}}^{-1} \mathsf{c}_{(fg)\cdot\beta^{-1}}(\beta) \arrow[rr, equal] & & C_{(f\cdot\mathsf{c}_g(\beta^{-1}))(g\cdot\beta^{-1})}^{-1} \mathsf{c}_{(f\cdot\mathsf{c}_g(\beta^{-1}))(g\cdot\beta^{-1})}(\beta)
\end{tikzcd}$$

Now consider $\alpha\beta\, C_f^{-1}$, where $\gamma = \alpha\beta$.

$$\begin{tikzcd}
\alpha\beta\, C_f^{-1} \arrow[r, "\text{(ii)}"] \arrow[d, "\text{(v)}"] & \alpha\, C_{f\cdot\beta^{-1}}^{-1} \mathsf{c}_{f\cdot\beta^{-1}}(\beta) \arrow[d, "\text{(ii)}"] \\
\gamma\, C_f^{-1} \arrow[d, "\text{(ii)}"] & C_{f\cdot\beta^{-1}\alpha^{-1}}^{-1} \mathsf{c}_{f\cdot\beta^{-1}\alpha^{-1}}(\alpha)\, \mathsf{c}_{f\cdot\beta^{-1}}(\beta) \arrow[dl, "\text{(v)}"] \\
C_{f\cdot\gamma^{-1}}^{-1} \mathsf{c}_{f\cdot\gamma^{-1}}(\gamma) &
\end{tikzcd}$$

Now consider $\alpha\, C_{fg}^{-1} \mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h}$. First note that

$$\begin{aligned}
\mathsf{c}_{f\cdot\mathsf{c}_g(\alpha^{-1})}\Big(\mathsf{c}_{g\cdot\alpha^{-1}}(\alpha)\,\beta\Big) &= \mathsf{c}_{f\cdot\mathsf{c}_g(\alpha^{-1})}\Big(\mathsf{c}_{g\cdot\alpha^{-1}}(\alpha)\Big)\, \mathsf{c}_{f\cdot\mathsf{c}_g(\alpha^{-1})\mathsf{c}_{g\cdot\alpha^{-1}}(\alpha)}(\beta) \\
&= \mathsf{c}_{(fg)\cdot\alpha^{-1}}(\alpha)\, \mathsf{c}_f(\beta)
\end{aligned}$$

and

$$(f\cdot\mathsf{c}_g(\alpha^{-1}))\cdot\mathsf{c}_{g\cdot\alpha^{-1}}(\alpha)\,\beta = f\cdot\beta.$$

$$\begin{tikzcd}
\alpha\, C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} \arrow[r, "\text{(vii)}"] \arrow[d, "\text{(ii)}"] & \alpha\, C_g^{-1}\beta\, C_h \arrow[d, "\text{(ii)}"] \\
C^{-1}_{(fg)\cdot\alpha^{-1}}\mathsf{c}_{(fg)\cdot\alpha^{-1}}(\alpha)\,\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} \arrow[d, "\text{(v)}"] & C^{-1}_{g\cdot\alpha^{-1}}\mathsf{c}_{g\cdot\alpha^{-1}}(\alpha)\,\beta\, C_h \arrow[d, "\text{(v)}"] \\
C^{-1}_{(f\cdot\mathsf{c}_g(\alpha^{-1}))(g\cdot\alpha^{-1})}\mathsf{c}_{f\cdot\mathsf{c}_g(\alpha^{-1})}\left(\mathsf{c}_{g\cdot\alpha^{-1}}(\alpha)\,\beta\right) C_{(f\cdot\beta)h} \arrow[r, "\text{(vii)}"] & C^{-1}_{g\cdot\alpha^{-1}}(\mathsf{c}_{g\cdot\alpha^{-1}}(\alpha)\,\beta)\, C_h
\end{tikzcd}$$

Now consider $C_f C_g C_h$.

$$\begin{tikzcd}
C_f C_g C_h \arrow[r, "\text{(iii)}"] \arrow[d, "\text{(iii)}"] & C_{fg} C_h \arrow[d, "\text{(iii)}"] \\
C_f C_{gh} \arrow[r, "\text{(iii)}"] & C_{fgh}
\end{tikzcd}$$

Now consider $C_f C_g C_h^{-1}$. Let f', g', h' and h'' satisfy the following.

$$h'g = g'h$$

$$f'h' = h''f$$

So we have that $h''fg = f'h'g = f'g'h$.

$$\begin{tikzcd}
C_f C_g C_h^{-1} \arrow[r, "\text{(vi)}"] \arrow[d, "\text{(iii)}"] & C_f C_{h'}^{-1} C_{g'} \arrow[d, "\text{(vi)}"] \\
C_{fg} C_h^{-1} \arrow[d, "\text{(vi)}"] & C_{h''}^{-1} C_{f'} C_{g'} \arrow[dl, "\text{(iii)}"] \\
C_{h''}^{-1} C_{f'g'} &
\end{tikzcd}$$

Now consider $C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h}C_e$.

$$\begin{tikzcd}
C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h}C_e \arrow[r, "\text{(vii)}"] \arrow[d, "\text{(iii)}"] & C_g^{-1}\beta\, C_hC_e \arrow[d, "\text{(iii)}"] \\
C_{fg}^{-1}\mathsf{c}_f(\beta)\, C_{(f\cdot\beta)he} \arrow[r, "\text{(vii)}"] & C_g^{-1}\beta\, C_{he}
\end{tikzcd}$$

Now consider $C_f^{-1}C_g^{-1}C_h^{-1}$.

$$\begin{tikzcd}
C_f^{-1}C_g^{-1}C_h^{-1} \arrow[r, "\text{(iv)}"] \arrow[d, "\text{(iv)}"] & C_{gf}^{-1}C_h^{-1} \arrow[d, "\text{(iv)}"] \\
C_f^{-1}C_{hg}^{-1} \arrow[r, "\text{(iv)}"] & C_{hgf}^{-1}
\end{tikzcd}$$

Now consider $C_fC_g^{-1}C_h^{-1}$. Let f', g', h' and f'' satisfy the following

$$f'g = g'f$$

$$h'f' = f''h$$

$$\begin{tikzcd}
C_fC_g^{-1}C_h^{-1} \arrow[r, "\text{(vi)}"] \arrow[d, "\text{(iv)}"] & C_{g'}^{-1}C_{f'}C_h^{-1} \arrow[d, "\text{(vi)}"] \\
C_fC_{hg}^{-1} \arrow[d, "\text{(vi)}"] & C_{g'}^{-1}C_{h'}^{-1}C_{f''} \arrow[dl, "\text{(iv)}"] \\
C_{h'g'}^{-1}C_{f''} &
\end{tikzcd}$$

Now consider $C_e^{-1}C_{fg}^{-1}\mathsf{c}_f(\beta)\,C_{(f\cdot\beta)h}$.

$$\begin{tikzcd}
C_e^{-1}C_{fg}^{-1}\mathsf{c}_f(\beta)\,C_{(f\cdot\beta)h} \arrow[r, "\text{(vii)}"] \arrow[d, "\text{(iv)}"] & C_e^{-1}C_g^{-1}\beta C_h \arrow[d, "\text{(iv)}"] \\
C_{fge}^{-1}\mathsf{c}_f(\beta)\,C_{(f\cdot\beta)h} \arrow[r, "\text{(vii)}"] & C_{ge}^{-1}\beta C_h
\end{tikzcd}$$

For the word $\alpha\beta\gamma$ this follows from the associativity of multiplication in the braid group.

Now consider $C_eC_{fg}^{-1}\mathsf{c}_f(\beta)\,C_{(f\cdot\beta)h}$. Let e', f', g' and e'' satisfy the following.

$$g'e = e'g$$

$$e''f = f'e'$$

$$
\begin{tikzcd}
C_e C_{fg}^{-1} \mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} \arrow[r, "\text{(vii)}"] \arrow[d, "\text{(vi)}"] & C_e C_g^{-1} \beta C_h \arrow[d, "\text{(vi)}"] \\
C_{f'g'}^{-1} C_{e''} \mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} \arrow[d, "\text{(i)}"] & C_{g'}^{-1} C_{e'} \beta C_h \arrow[d, "\text{(i)}"] \\
C_{f'g'}^{-1} \mathsf{c}_{e''f}(\beta)\, C_{e''\cdot \mathsf{c}_f(\beta)} C_{(f\cdot\beta)h} \arrow[d, "\text{(iii)}"] & C_{g'}^{-1} C_{e'} \beta C_h \arrow[d, "\text{(i)}"] \\
C_{f'g'}^{-1} \mathsf{c}_{e''f}(\beta)\, C_{e''\cdot \mathsf{c}_f(\beta)(f\cdot\beta)h} \arrow[d, equal] & C_{g'}^{-1} \mathsf{c}_{e'}(\beta)\, C_{e'\cdot\beta} C_h \arrow[ddd, "\text{(iii)}"] \\
C_{f'g'}^{-1} \mathsf{c}_{f'e'}(\beta)\, C_{((e''f)\cdot\beta)h} \arrow[d, equal] & \\
C_{f'g'}^{-1} \mathsf{c}_{f'e'}(\beta)\, C_{((f'e')\cdot\beta)h} \arrow[d, equal] & \\
C_{f'g'}^{-1} \mathsf{c}_{f'e'}(\beta)\, C_{(f'\cdot \mathsf{c}_{e'}(\beta)(e'\cdot\beta)h} \arrow[r, "\text{(vii)}"] & C_{g'}^{-1} \mathsf{c}_{e'}(\beta)\, C_{(e'\cdot\beta)h}
\end{tikzcd}
$$

Now consider $C_{fg}^{-1} \mathsf{c}_f(\beta)\, C_{(f\cdot\beta)h} C_e^{-1}$. Let e', f', h' and e'' satisfy the following.

$$
\begin{aligned}
e'h &= h'e \\
e''(f\cdot\beta) &= f'e'
\end{aligned}
$$

Note that

$$
\begin{aligned}
((e''\cdot \mathsf{c}_f(\beta)^{-1})f)\cdot\beta &= (e''\cdot \mathsf{c}_f(\beta)^{-1})\cdot \mathsf{c}_f(\beta)\,(f\cdot\beta) \\
&= e''(f\cdot\beta) \\
&= f'e'
\end{aligned}
$$

and so

$$
\begin{aligned}
(e'' \cdot \mathsf{c}_f(\beta)^{-1})f &= (f'e') \cdot \beta^{-1} \\
&= (f' \cdot \mathsf{c}_{e'}(\beta^{-1}))(e' \cdot \beta^{-1}).
\end{aligned}
$$

$$
\begin{tikzcd}
C_{fg}^{-1}\mathsf{c}_f(\beta)\,C_{(f\cdot\beta)h}C_e^{-1} \arrow[r, "\text{(vii)}"] \arrow[d, "\text{(vi)}"] & C_g^{-1}\beta\,C_hC_e^{-1} \arrow[d, "\text{(vi)}"] \\
C_{fg}^{-1}\mathsf{c}_f(\beta)\,C_{e''}^{-1}C_{f'h'} \arrow[d, "\text{(ii)}"] & C_g^{-1}\beta\,C_{e'}^{-1}C_{h'} \arrow[d, "\text{(ii)}"] \\
C_{fg}^{-1}C_{e''\cdot\mathsf{c}_f(\beta)^{-1}}^{-1}\mathsf{c}_{(e''\cdot\mathsf{c}_f(\beta)^{-1})f}(\beta)\,C_{f'h'} \arrow[d, "\text{(iv)}"] & C_g^{-1}C_{e'\cdot\beta^{-1}}^{-1}\mathsf{c}_{e'\cdot\beta^{-1}}(\beta)\,C_{h'} \arrow[dd, "\text{(iv)}"] \\
C_{(e''\cdot\mathsf{c}_f(\beta)^{-1})fg}^{-1}\mathsf{c}_{(e''\cdot\mathsf{c}_f(\beta)^{-1})f}(\beta)\,C_{f'h'} \arrow[d, equal] & \\
C_{(f'\cdot\mathsf{c}_{e'}(\beta^{-1}))(e'\cdot\beta^{-1})g}^{-1}\mathsf{c}_{(f'\cdot\mathsf{c}_{e'}(\beta^{-1}))(e'\cdot\beta^{-1})}(\beta)\,C_{f'h'} \arrow[r, "\text{(vii)}"] & C_{(e'\cdot\beta^{-1})g}^{-1}\mathsf{c}_{e'\cdot\beta^{-1}}(\beta)\,C_{h'}
\end{tikzcd}
$$

Now consider $C_{fg}^{-1}\mathsf{c}_f(\beta)\,C_{(f\cdot\beta)h} = C_{f'g'}^{-1}\mathsf{c}_{f'}(\beta')\,C_{(f'\cdot\beta')h'}$. We have the following.

$$
\begin{tikzcd}
C_{fg}^{-1}\mathsf{c}_f(\beta)\,C_{(f\cdot\beta)h} = C_{f'g'}^{-1}\mathsf{c}_{f'}(\beta')\,C_{(f'\cdot\beta')h'} \arrow[r, "\text{(vii)}"] \arrow[d, "\text{(vii)}"] & C_g^{-1}\beta C_h \\
C_{g'}^{-1}\beta' C_{h'} &
\end{tikzcd}
$$

If we let $F = \mathrm{LCM}(f, f')$ then there exists G such that $FG = fg$ and there exists f_1 and f_1' such that $ff_1 = F = f'f_1'$. So we have that $g = f_1G$ and $g' = f_1'G$. There exists β_0 such that $\mathsf{c}_F(\beta_0) = \mathsf{c}_f(\beta)$. We also have that $\beta = \mathsf{c}_{f_1}(\beta_0)$ and $\beta' = \mathsf{c}_{f_1'}(\beta_0)$. Also $F \cdot \beta = \mathrm{LCM}(f \cdot \beta, f' \cdot \beta')$ hence there exists H such that $(F \cdot \beta_0)H = (f \cdot \beta)h$ and we have that $(f_1 \cdot \beta_0)H = h$ and $(f_1' \cdot \beta_0)H = h'$. So we can complete the above with the following.

$$\begin{tikzcd}
 & C_g^{-1}\beta C_h = C_{f_1 G}\mathsf{c}_{f_1}(\beta_0)\, C^{-1}_{(f_1\cdot\beta_0)H} \arrow[d, "\text{(vii)}"] \\
C_{g'}^{-1}\beta' C_{h'} = C_{f_1' G}\mathsf{c}_{f_1'}(\beta_0)\, C^{-1}_{(f_1'\cdot\beta_0)H} \arrow[r, "\text{(vii)}"] & C_G \beta_0 C_H^{-1}
\end{tikzcd}$$

□

Now that we have a complete rewrite system we have a unique normal form $N(x)$ for any given word x. We have that for each word x there exists f, g and β such that $N(x) = C_f^{-1}\beta C_g$.

Corollary 27. *The maps $\mathbf{E}_{2n} \to \mathcal{G}$ and the functor $F \to \mathcal{G}$ given by $\beta \mapsto \beta$ and $f \mapsto C_f$ respectively are injective.*

For $x \in \mathcal{G}$ we can define its oriented plat closure by $\overleftarrow{\mathsf{p}}(x) = \overleftarrow{\mathsf{p}}(\beta)$ where $N(x) = C_f^{-1}\beta C_g$. Note that once we have written x in the form $C_f^{-1}\beta C_g$ we don't need to do the remaining (vii) rewrite to find $\overleftarrow{\mathsf{p}}(x)$ as the plat closure is invariant under cabling.

Proposition 28. *For every link L there exists some $x \in \mathcal{G}$ such that $\overleftarrow{\mathsf{p}}(x) = L$.*

Proof. If we let i denote the inclusion of $\mathbf{E}_{2n}$ into $\mathrm{Aut}(2n)$ then it is clear that $\overleftarrow{\mathsf{p}} = \overleftarrow{\mathsf{p}} \circ i$. Hence this follows from Proposition 22. □

Let $\mathcal{H}^+$ be the subgroupoid consisting of all morphisms whose normal form is $C_f^{-1}\beta C_g$ for any f and g and any $\beta \in \mathbf{H}_{2n}^+ \cap \mathbf{E}_{2n}$. By Proposition 17 this is closed under multiplication. Similarly let $\mathcal{H}^-$ be the subgroupoid consisting of all morphisms whose normal form is $C_f^{-1}\beta C_g$ where $\beta \in \mathbf{H}_{2n}^- \cap \mathbf{E}_{2n}$.

Proposition 29. *The oriented plat closure map is constant on the double cosets of $\mathcal{H}^+\backslash\mathcal{G}/\mathcal{H}^-$.*

Proof. We will show that multiplying on the left by an element of $\mathcal{H}^+$ preserves the plat closure. Multiplying on the right by elements of $\mathcal{H}^-$ is similar.

Given $x = C_f^{-1}\beta\, C_g \in \mathcal{G}$ and $y = C_p^{-1}\theta\, C_q \in \mathcal{H}^+$, ie $\theta \in \mathbf{H}_{2n}^+$ for some n. We want to show that $\overleftarrow{\mathsf{p}}(x) = \overleftarrow{\mathsf{p}}(yx)$, so we start by calculating the normal form of yx. We have the following where $f'q = q'f$.

$$
\begin{array}{c}
C_p^{-1}\theta\, C_q C_f^{-1}\beta\, C_g \\
\Big\downarrow \text{(vi)} \\
C_p^{-1}\theta\, C_{f'}^{-1} C_{q'}\beta\, C_g \\
\Big\downarrow \text{(ii),(i)} \\
C_p^{-1} C_{f'\cdot\theta^{-1}}^{-1} \mathsf{c}_{f'\cdot\theta^{-1}}(\theta)\, \mathsf{c}_{q'}(\beta)\, C_{q'\cdot\beta} C_g \\
\Big\downarrow \text{(iii),(v),(iv)} \\
C_{(f'\cdot\theta^{-1})p}^{-1}(\mathsf{c}_{f'\cdot\theta^{-1}}(\theta)\, \mathsf{c}_{q'}(\beta))\, C_{(q'\cdot\beta)g}
\end{array}
$$

Then there may be a (vii) move on the end of this sequence. So we have that $\overleftarrow{\mathsf{p}}(yx) = \overleftarrow{\mathsf{p}}\left(\mathsf{c}_{f'\cdot\theta^{-1}}(\theta)\, \mathsf{c}_{q'}(\beta)\right)$. By Proposition 17 $\mathsf{c}_{f'\cdot\theta^{-1}}(\theta)$ is in $\mathbf{H}_{2m}^+$ for some m and by Proposition 14 $\overleftarrow{\mathsf{p}}(\beta) = \overleftarrow{\mathsf{p}}\left(\mathsf{c}_{q'}(\beta)\right)$ therefore $\overleftarrow{\mathsf{p}}(x) = \overleftarrow{\mathsf{p}}(yx)$. ☐

Conjecture 30. *The oriented plat closure map $\overleftarrow{\mathsf{p}}$ from $\mathcal{H}^+\backslash\mathcal{G}/\mathcal{H}^-$ to the set of non-empty oriented links is a bijection.*

Proof given Conjecture 24. Given $x, x' \in \mathcal{G}$ with $\overleftarrow{\mathsf{p}}(x) = \overleftarrow{\mathsf{p}}(x')$. Suppose that $N(x) = C_f^{-1} \beta\, C_g$ and that $N(x') = C_{f'}^{-1} \beta'\, C_{g'}$. So we have that $\overleftarrow{\mathsf{p}}(\beta) = \overleftarrow{\mathsf{p}}(\beta')$. Therefore, by Theorem 21 and Conjecture 24 there exists $i_1, i_2, \dots i_N$, $j_1, j_2, \dots j_M$ and $h^+ \in \mathbf{H}_{2n}^+$ and $h^- \in \mathbf{H}_{2n}^-$ such that

$$\mathsf{c}_{i_1} \mathsf{c}_{i_2} \cdots \mathsf{c}_{i_N}(\beta) = h^+ \mathsf{c}_{j_1} \mathsf{c}_{j_2} \cdots \mathsf{c}_{j_M}(\beta')\, h^-.$$

The sequence of cabling moves can be combined into a single forest cabling move. So for some f_1 and f_2,

$$\mathsf{c}_{f_1}(\beta) = h^+ \mathsf{c}_{f_2}(\beta')\, h^-.$$

We have that

$$\begin{aligned} C_f^{-1} \beta\, C_g &= C_f^{-1} C_{f_1}^{-1} \mathsf{c}_{f_1}(\beta)\, C_{f_1 \cdot \beta} C_g \\ &= C_f^{-1} C_{f_1}^{-1} h^+ \mathsf{c}_{f_2}(\beta')\, h^- C_{f_1 \cdot \beta} C_g \end{aligned}$$

Because $C_{f'}^{-1} C_{f_2}^{-1} (h^+)^{-1} C_{f_1} C_f \in \mathcal{H}^+$ and $C_g^{-1} C_{f_1 \cdot \beta}^{-1} (h^-)^{-1} C_{f_2 \cdot \beta'} C_{g'} \in \mathcal{H}^-$ we have that x is in the same double coset as

$$C_{f'}^{-1} C_{f_2}^{-1} \mathsf{c}_{f_2}(\beta')\, C_{f_2 \cdot \beta'} C_{g'} = C_{f'}^{-1} \beta' C_{g'}.$$

□

Chapter 2

Hilden's group

2.1 Introduction

Let H^3 denote the closed upper half-space of $\mathbb{R}^3$, let $a_1, a_2, \ldots, a_n \subset H^3$ be n pairwise disjoint properly embedded unknotted arcs and let $a_* = a_1 \cup a_2 \cup \cdots \cup a_n$. Viewing the braid group as the mapping class group of the punctured disc, if this disc is included in ∂H^3 with ∂a_* as the punctures, one can define Hilden's group, $\mathbf{H}_{2n}$, to be the subgroup of $\mathbf{B}_{2n}$ consisting of all mapping classes that can be extended to $H^3 \setminus a_*$. Or equivalently, $\mathbf{H}_{2n}$ is the stabiliser of a_* under the action of $\mathbf{B}_{2n}$ on $0, 2n$–tangles.

Hilden[6] found generators for a similar subgroup of the braid group of a sphere. For any given braid b multiplying on either the left or the right by elements of $\mathbf{H}_{2n}$ preserves the plat closure, ie plat closure is constant on each double coset. Birman[2] showed that if two braids have the same plat closure then they can be related by a sequence of these double coset moves and stabilisation moves that changes the braid index by 2.

We calculate a presentation for $\mathbf{H}_{2n}$ using the action of this group on a cellular complex. Hatcher–Thurston[5], Wajnryb[9, 10, 11], Laudenbach[7], etc used the same method to calculate presentations for mapping class groups. We start in Section 2.2 by outlining this method. A similar but more general method is given by Brown [4]. Brendle–Hatcher[3] have calculated a presentation for $\mathbf{H}_{2n}$ using a different method.

In Section 2.3 we define a simply-connected complex $\overline{\mathbf{X}}_n$. In Section 2.4 we remove some of the edges and faces of this complex resulting in a new complex which remains simply-connected but gives a simpler presentation. This presentation is calculated in Section 2.5 and then used to calculate a presentation with generators similar to those found by Hilden.

2.2 The method

In this section we recall the method of Hatcher–Thurston[5]. This section follows §2 “Une Méthode pour présenter G” of Laudenbach[7] and all results in this section are from there.

Suppose that X is a connected simply-connected cellular 2-complex such that each attaching map is injective and that each cell is uniquely determined by its boundary. Suppose that G is a group acting cellularly on the right of X, and that this action is transitive on the vertex set X^0. Pick a vertex $v_0 \in X^0$ as a basepoint and let H denote its stabiliser in G, ie $H = \{g \in G \mid v_0 \cdot g = v_0\}$. Suppose that H has a presentation with generating set S_0 and relations R_0, ie $H = \langle S_0 | R_0 \rangle$.

Given vertices $u, v \in X^0$ such that $\{u, v\}$ is the boundary of an edge of X

we will write (u, v) for this (oriented) edge. Given a sequence $v_1, v_2, \ldots, v_k$ of vertices such that either $v_i = v_{i+1}$ or (v_i, v_{i+1}) forms an edge we will write $(v_1, v_2, \ldots, v_k)$ for the path traversing these edges. Whenever $v_i = v_{i+1}$ we shall say that v_i is a stationary point.

Let E denote the set of all oriented edges starting at v_0, so H acts on E. Suppose that $\{e_\lambda\}_{\lambda\in\Lambda}$ is a set of representatives for the H–orbits of the edges in E, ie $E = \bigcup_{\lambda\in\Lambda} e_\lambda H$ and $e_\lambda H = e_{\lambda'} H$ only if $\lambda = \lambda'$. Since the action of G is transitive on X^0 we can find $r_\lambda \in G$ such that $e_\lambda = (v_0, v_0 \cdot r_\lambda)$. Let $S_1 = \{r_\lambda\}_{\lambda\in\Lambda}$.

The edges $\{e_\lambda\}_{\lambda\in\Lambda}$ also form a set of representatives for the edge orbits of the G–action on X. To see this suppose that two of these edges lie in the same G–orbit, ie $(v_0, v) = (v_0, u) \cdot g$. Then we have that $v_0 = v_0 \cdot g$ therefore $g \in H$.

Suppose that $\{f_\mu\}_{\mu\in M}$ is a set of representatives for the G–orbits of the faces of X. Since the action is transitive on X^0, we may assume that the boundary of each face f_μ contains the vertex v_0.

Definition 1. An *h-product of length k* is a word of the form

$$h_{k+1}\; r_{\lambda_k} h_k\; r_{\lambda_{k-1}} h_{k-1}\; \cdots\; r_{\lambda_1} h_1$$

where each $\lambda_i \in \Lambda$ and each of the h_i are words in H. To each h-product we can associate an edge path $P = (v_0, v_1, \ldots, v_k)$ in X starting at v_0 then visiting the vertices $v_1 = v_0 \cdot r_{\lambda_1} h_1$, $v_2 = v_0 \cdot r_{\lambda_2} h_2\; r_{\lambda_1} h_1$, etc. This means that the edge (v_{i-1}, v_i) is in the orbit of $(v_0, v_0 \cdot r_{\lambda_i})$. Given any edge path starting at v_0 we can choose an h-product to represent it.

We can now choose the following three sets of relations.

R_1: For each edge orbit representative e_λ pick a generating set T for the stabiliser of this edge, ie $\langle T\rangle = \mathrm{Stab}_G(v_0) \cap \mathrm{Stab}_G(v_0 \cdot r_\lambda)$. For each $t \in T$ we have the relation $r_\lambda t r_\lambda^{-1} = h$ for some word $h \in H$.

R_2: For each e_λ we have a relation $r_{\lambda'} h\, r_\lambda = h'$ where the LHS is a choice of h-product for the path $(v_0, v_0 \cdot r_\lambda, v_0)$ and h' is some word in H.

R_3: For each face orbit representative f_μ with boundary $(v_0, v_1, \ldots, v_{k-1}, v_0)$ choose an h-product representing this path and a word $h \in H$ such that $r_{\lambda_k} h_k \cdots r_{\lambda_1} h_1 = h$.

Theorem 2. *The group G has the following presentation.*

$$G = \langle S_0 \cup S_1 | R_0 \cup R_1 \cup R_2 \cup R_3 \rangle$$

Corollary 3. *Suppose that H is finitely presented, that the number of edge and face orbits is finite and that each edge stabiliser is finitely generated. Then G has a finite presentation.*

We prove Theorem 2 in several steps.

Claim 1. *The set $S_0 \cup S_1$ generates G.*

Proof. Given any $g \in G$, let $v = v_0 \cdot g$. Now as X is connected there is an edge path connecting v_0 to v. Choose an h-product $g_1 = h_{k+1}\, r_{\lambda_k} h_k \cdots r_{\lambda_1} h_1$ representing this path. Then $v_0 \cdot g g_1^{-1} = v_0$ so $g = h g_1$ for some $h \in H$. $\square$

Claim 2. *If two h-products, p_1 and p_2, give rise to the same path and are equal in G then they are equivalent modulo $R_0 \cup R_1$.*

Proof. Because p_1 and p_2 represent the same path they must have equal length. Suppose that $p_1 = h_{k+1}\ r_{\lambda_k} h_k \cdots r_{\lambda_1} h_1$ and $p_2 = f_{k+1}\ r_{\lambda'_k} f_k \cdots r_{\lambda'_1} f_1$. Clearly, if the two h-products are of length 0 then they are both words in H and so are equivalent modulo R_0. Now suppose that $k \neq 0$. The fact that p_1 and p_2 represent the same path means that

$$(v_0,\ v_0 \cdot r_{\lambda_1} h_1,\ v_0 \cdot r_{\lambda_2} h_2\ r_{\lambda_1} h_1,\ \ldots) = (v_0,\ v_0 \cdot r_{\lambda'_1} f_1,\ v_0 \cdot r_{\lambda'_2} f_2\ r_{\lambda'_1} f_1,\ \ldots),$$

therefore

$$(v_0, v_0 \cdot r_{\lambda_1}) = (v_0, v_0 \cdot r_{\lambda'_1}) \cdot f_1 h_1^{-1}.$$

So $\lambda_1 = \lambda'_1$ and $f_1 h_1^{-1}$ is in the stabiliser of the edge e_{λ_1}. Hence, for some word f'_2 in H

$$f_{k+1}\ r_{\lambda'_k} f_k \cdots r_{\lambda'_2} f_2\ r_{\lambda'_1} f_1 h_1^{-1} h_1 \quad = \quad f_{k+1}\ r_{\lambda'_k} f_k \cdots r_{\lambda'_2} f'_2\ r_{\lambda_1} h_1$$

modulo R_1. By induction the two shorter h-products $h_{k+1}\ r_{\lambda_k} h_k \cdots r_{\lambda_2} h_2$ and $f_{k+1}\ r_{\lambda'_k} f_k \cdots r_{\lambda'_2} f'_2$ are equivalent modulo $R_0 \cup R_1$, and so $p_1 = p_2$ modulo $R_0 \cup R_1$. $\square$

Claim 3. *Suppose that two h-products represent the same element of G and induce edge paths that are equivalent modulo backtracking. Then they are equivalent modulo $R_0 \cup R_1 \cup R_2$.*

Proof. It is enough to show that any h-product is equivalent to an h-product that represents a path without any backtracking. Furthermore, if we proceed by induction on the length of the h-product, it is enough to show that any h-product whose associated path has backtracking at the end is equivalent to a shorter h-product.

Suppose that $g = h_{k+3}\ r_{\lambda_{k+2}} h_{k+2}\ r_{\lambda_{k+1}} h_{k+1}\ g_k$ is such an h-product, ie

$$\begin{aligned} v_k &= v_0 \cdot g_k \\ v_{k+1} &= v_0 \cdot r_{\lambda_{k+1}} h_{k+1}\ g_k \\ v_{k+2} = v_k &= v_0 \cdot r_{\lambda_{k+2}} h_{k+2}\ r_{\lambda_{k+1}} h_{k+1}\ g_k \end{aligned}$$

and g_k is a shorter h-product. So, multiplying by $g_k^{-1} h_{k+1}^{-1}$, we find that $r_{\lambda_{k+2}} h_{k+2}\ r_{\lambda_{k+1}}$ is an h-product with associated path $(v_0, v_0 \cdot r_{\lambda_{k+1}}, v_0)$. Suppose that $r_{\lambda'} h r_\lambda = h'$ is the R_2 relation corresponding to this path. Then $\lambda = \lambda_{k+1}$ and $v_0 \cdot r_{\lambda'} h = v_0 \cdot r_{\lambda_{k+2}} h_{k+2}$. So $\lambda' = \lambda_{k+2}$ and $h_{k+2} h^{-1}$ is in the stabiliser of the edge $e_{\lambda_{k+1}}$. Therefore there exists a word f in H such that

$$h_{k+3}\ r_{\lambda_{k+2}} h_{k+2}\ r_{\lambda_{k+1}} h_{k+1}\ g_k = h_{k+3} f\ r_{\lambda'} h\ r_\lambda h_{k+1}\ g_k$$

modulo R_1. Hence modulo R_2 this is equal to $h_{k+3} f h' h_{k+1} g_k$, a shorter h-product. ☐

Claim 4. *Any h-product equal to the identity in G is equivalent to the identity modulo $R_0 \cup R_1 \cup R_2 \cup R_3$.*

Proof. Given any h-product g_k equal to the identity in G its associated edge path must be a loop. Since X is simply-connected this loop is the boundary of a union of faces of X. So choose one of these faces f touching the loop at a vertex v then modulo $R_0 \cup R_1 \cup R_2$ we can add backtracking starting at v going around the boundary of f. Modulo R_3 we can remove one pass round ∂f. This leaves a new loop that can be spanned by one less face, which, by induction on the minimum number of faces needed to span a loop, is equivalent to the identity. $\square$

Proof of Theorem 2. Given any word in the generators, $S_0 \cup S_1$, that is equal to the identity in G then modulo R_2 it is equivalent to an h-product and so by Claim 4 is equivalent to the identity modulo $R_0 \cup R_1 \cup R_2 \cup R_3$. $\square$

2.3 The complex $\overline{\mathbf{X}}_n$

An embedded disc $D \subseteq H^3$ is said to *cut out* a_i if the interior of D is disjoint from a_*, the arc a_i is contained in the boundary of D and the boundary of D lies in $a_i \cup \partial H^3$, ie $a_i \subset \partial D$ and $\partial D \subset a_i \cup \partial H^3$. A *cut system for* a_* is the isotopy class of n pairwise disjoint discs $\langle D_1, D_2, \ldots D_n\rangle$ where each D_i cuts out the arc a_i. Say that two cut systems $\langle D_1, D_2, \ldots, D_n\rangle$ and $\langle E_1, E_2, \ldots, E_n\rangle$ differ by a simple i-move if $D_i \cap E_i = a_i$ and $D_j = E_j$ for all $j \neq i$. If this is the case we will suppress the non-changing discs and write $\langle D_i\rangle \to \langle E_i\rangle$.

Definition 4. Define the cut system complex $\overline{\mathbf{X}}_n$ as follows. The set of all cut systems for a_* forms the vertex set $\overline{\mathbf{X}}_n^0$. Two vertices are connected by a single edge iff they differ by a simple move. Finally, glue faces into every loop of the following form, giving triangular and rectangular faces.

$$\begin{array}{ccc} \langle D_i\rangle & \text{———} & \langle D_i'\rangle \\ & \diagdown \quad \diagup & \\ & \langle D_i''\rangle & \end{array} \qquad \begin{array}{ccc} \langle D_i, D_j\rangle & \text{——} & \langle D_i', D_j\rangle \\ \Big| & & \Big| \\ \langle D_i, D_j'\rangle & \text{——} & \langle D_i', D_j'\rangle \end{array}$$

Define the basepoint to be $v_0 = \langle d_1, d_2, \ldots, d_n \rangle$ where the d_i are vertical discs below the a_i, see Figure 2.1. Sometimes it is convenient to think of the

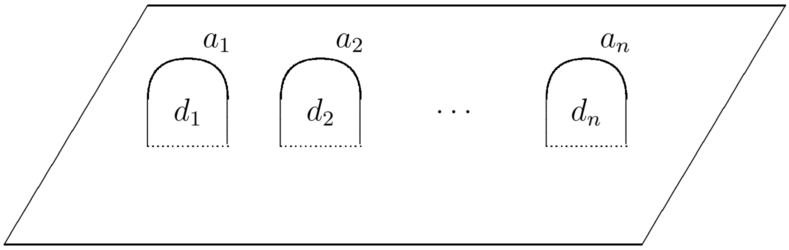


Figure 2.1: The arcs a_i and the discs d_i

a_i and d_i rotated by a quarter turn.

Before we prove that this complex is simply connected we need the following lemma about substituting one disc for another.

Suppose that $v = \langle D_1, D_2, \ldots D_n \rangle$ is a vertex of $\overline{\mathbf{X}}_n$ and that D and D^* are two discs cutting out the arc a_i. We will say that the tuple (v, D, D^*) forms a *valid substitution* if either $D \neq D_i$ for any i, or if there exists some i such that $D = D_i$ and that for all $j \neq i$ we have that $D_j \cap D^* = \emptyset$. In other words if D is in v then (v, D, D^*) forms a valid substitution if there exists an edge $\langle D = D_i \rangle$—$\langle D^* \rangle$. If (v, D, D^*) forms a valid substitution then we can replace D with D^* to get a vertex v^*, ie

$$v^* = \begin{cases} v & \text{if } D_i \neq D, \\ \langle D^* \rangle & \text{if } D_i = D. \end{cases}$$

Similarly, for any edge path P with a choice of discs representing each vertex, we say (P, D, D^*) forms a valid substitution if for each vertex v of P the tuple (v, D, D^*) forms a valid substitution and for each edge (v_i, v_{i+1}) of

P there is an edge (v_i^*, v_{i+1}^*). If (P, D, D^*) forms a valid substitution then we can replace each occurrence of D with D^*, ie replace each vertex v with v^*, giving a new path P^*.

Lemma 5. *If (P, D, D^*) forms a valid substitution, where $P = (v_1, \ldots, v_k)$, then P^* is a path and the loop*

$$\begin{array}{ccc} v_1 & \overset{P}{\text{———}} & v_k \\ | & & | \\ v_1^* & \overset{P^*}{\text{———}} & v_k^* \end{array}$$

is homotopic to a point. Moreover, if P is a loop then so is P^* and they are homotopic as loops.

Proof. Clearly we may assume that D and D^* are not isotopic, otherwise $P = P^*$. Suppose that D and D^* cut out the arc a_i. For each vertex v of P we have that either $v = v^*$ or (v, v^*) is an edge of $\overline{\mathbf{X}}_n$.

For each edge (u, v) in P, where $u = \langle D_j \rangle$ and $v = \langle D_j' \rangle$, we have the following possibilities. If D is not in u nor in v then $(u, v) = (u^*, v^*)$. Otherwise we have two cases depending on whether $i = j$ or not.

If $i = j$ then only one of either u or v contains D. Suppose that $D \in u$, ie $D_j = D$. If $D^* = D_j'$ then $u^* = v^* = v$ and (u, v) is homotopic to (u^*, v^*) in $\overline{\mathbf{X}}_n^1$. Otherwise, if $D^* \neq D_j$, we have the following face of $\overline{\mathbf{X}}_n$.

$$\begin{array}{ccc} u & \overset{P}{\text{———}} & v = v^* \\ | & \diagup_{P^*} & \\ u^* & & \end{array}$$

If $i \neq j$ the we have the following face of $\overline{\mathbf{X}}_n$.

$$\begin{tikzcd}
\langle D, D_j \rangle \arrow[r, dash, "P"] \arrow[d, dash] & \langle D, D'_j \rangle \arrow[d, dash] \\
\langle D^*, D_j \rangle \arrow[r, dash, "P^*"] & \langle D^*, D'_j \rangle
\end{tikzcd}$$

In either case there is a homotopy from (u, v) to (u^*, v^*) that agrees with the homotopies between the vertices of P and P^*. Therefore P is homotopic to P^*. $\square$

Theorem 6. *The complex $\overline{\mathbf{X}}_n$ is connected and simply connected.*

Proof. It suffices to show that any loop is homotopic to the constant loop at v_0. Given a loop in $\overline{\mathbf{X}}_n$ it is homotopic to an edge path P. Now choose discs to represent each vertex of P. We shall write $D \in P$ if D is one of the discs chosen as a representative of some vertex of P.

Claim. *The path P is homotopic to a path whose vertices admit representative discs which intersect the discs $d_1, d_2, \ldots, d_n$ only in the arcs $a_1, a_2, \ldots, a_n$.*

Assuming that the intersection of the discs $D \in P$ with $d_1 \cup d_2 \cup \ldots \cup d_n$ isn't only $a_1, a_2, \ldots, a_n$ we can carry out the following procedure.

For some i the union of the discs in P intersects d_i in a non-empty collection of arcs. Pick an arc α of this intersection that is lowest in the sense that it doesn't separate the entirety of any other arc from $\partial H^3 \cap d_i$. For example, see Figure 2.2 where α and γ are lowest but β is not.

The arc α comes from some $D \in P$. Now cut D along α, discard the section not containing a_i and glue in a disc parallel to d_i. This results in a new disc D^* whose intersection with d_i contains at least one less arc.

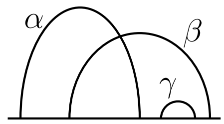


Figure 2.2: Lowest arcs α and γ

Any disc $E \in P$ for which $E \cap D = a_j$ or $\emptyset$ also has $E \cap D^* = a_j$ or $\emptyset$ respectively; if not E must intersect D^* in the section parallel to d_i and this contradicts the condition that α is a lowest arc. Therefore the triple (P, D, D^*) form a valid substitution and, by Lemma 5, we can replace D with D^* to get a new homotopic loop P^*.

We now have a homotopic loop P^* that has fewer intersections with $d_1 \cup d_2 \cup \ldots \cup d_n$. So by induction on the number of intersections we have proved the claim.

So we may assume that the path P meets $d_1, d_2, \ldots, d_n$ only in the arcs $a_1, a_2, \ldots, a_n$. Therefore, for each $D \in P$ cutting out the arc a_i, the triple (P, D, d_i) forms a valid substitution and so by in turn replacing each $D \in P$ with d_i we see that P is homotopic to the constant path v_0. The connectedness of $\overline{\mathbf{X}}_n$ follows by taking P to be a constant loop. $\square$

Up to homotopy the group $\mathbf{H}_{2n}$ acts on (H^3, a_*) by homeomorphisms, therefore it takes cut systems to cut systems. The edges and faces of $\overline{\mathbf{X}}_n$ are determined by the intersections of pairs of discs, hence this action on $\overline{\mathbf{X}}_n^0$ extends to a cellular action on $\overline{\mathbf{X}}_n$.

Theorem 7. *The action of* $\mathbf{H}_{2n}$ *on* $\overline{\mathbf{X}}_n^0$ *is transitive.*

Proof. Given a vertex $\langle D_1, D_2, \ldots, D_n \rangle$ of $\overline{\mathbf{X}}_n$, if we take each i in turn and look at the intersection of D_i with ∂H^3. We see that this defines a path from one end of a_i to the other. If we now move one end around this path until it is close to the other and then move it straight back to its starting point we have an element of $\mathbf{H}_{2n}$ that moves D_i to d_i. Combining all of these we see that $\langle D_1, D_2, \ldots, D_n\rangle$ is in the orbit of v_0, ie the action is transitive on $\overline{\mathbf{X}}_n^0$. $\square$

2.4 The complex $\mathbf{X}_n$

We now construct a subcomplex $\mathbf{X}_n$ of $\overline{\mathbf{X}}_n$ with the same vertex set but with fewer edges and faces.

Given an edge $e = (\langle D\rangle, \langle D'\rangle)$ of $\overline{\mathbf{X}}_n$ define its length, $l(e)$, to be the number of arcs underneath $D \cup D'$. In other words, since $H^3 \setminus D \cup D'$ has two components, one bounded and one unbounded, we can define the length as follows

$$l(e) = \#\{i \mid a_i \text{ is contained in the bounded component of } H^3 \setminus D \cup D'\}.$$

Given two edges e and e' with the same length there exists an element of $\mathbf{H}_{2n}$ taking e to e'.

We will say that a rectangle $(\langle D, E\rangle, \langle D', E\rangle, \langle D', E'\rangle, \langle D, E'\rangle)$ is *nested* if $E \cup E'$ lies in the bounded component of $H^3 \setminus D \cup D'$ or vice versa, ie if one pair of changing discs lies underneath the other.

For $i \leq j$ let $\mathcal{T}_{ij}$ denote the subcomplex consisting of all triangular faces of $\overline{\mathbf{X}}_n$ with shortest two edges of length i and j. Note, this implies that the remaining edge has length $i + j$. Given a rectangular face of $\overline{\mathbf{X}}_n$ we have two cases depending on whether it is nested or not. Let $\mathcal{R}_{ij}$ denote the subcomplex consisting of all rectangular nested faces with inner edge of length i and outer edge of length j. For $i \leq j$ let $\mathcal{S}_{ij}$ denote the subcomplex consisting of all non-nested rectangular faces with edges of length i and j.

Definition 8. Let $\mathbf{X}_n$ be the subcomplex of $\overline{\mathbf{X}}_n$ with the same vertex set, all edges of length 1 and 2 and all faces from $\mathcal{R}_{12}$, $\mathcal{S}_{11}$ and $\mathcal{T}_{11}$, ie $\mathbf{X}_n = \mathcal{R}_{12} \cup \mathcal{S}_{11} \cup \mathcal{T}_{11}$. As the length of an edge is invariant under the action of $\mathbf{H}_{2n}$ on $\overline{\mathbf{X}}_n$ this action preserves each $\mathcal{T}_{ij}$, $\mathcal{R}_{ij}$ and $\mathcal{S}_{ij}$ and so preserves $\mathbf{X}_n$.

A vertex $v = \langle D_1, \ldots, D_n \rangle$ is completely determined by the intersection of the discs D_i with ∂H^3. Using this we can define the vertices x_i for $0 \leq i \leq n-1$, y_{ij} for $0 \leq i \leq n-2$ and $j = 0$ or $i < j \leq n-1$ and z_{ij} for $0 \leq i, j, i+j \leq n-2$ as in Figure 2.3. So we have $l(v_0, x_i) = i$, $l(v_0, y_{0j}) = j$, $l(v_0, y_{i0}) = i$, $l(v_0, z_{i0}) = i$ and $l(v_0, z_{0j}) = j$. Note, there is some redundancy in this notation, ie $x_i = y_{0i}$ and $x_0 = y_{00} = z_{00} = v_0$.

We now define the faces $R_{ij} \in \mathcal{R}_{ij}$, $S_{ij} \in \mathcal{S}_{ij}$, $T_{ij} \in \mathcal{T}_{ij}$ of $\overline{\mathbf{X}}_n$ as follows.

$$R_{ij} = \begin{tikzcd} y_{00} \arrow[r, dash] \arrow[d, dash] & y_{i0} \arrow[d, dash] \\ y_{0j} \arrow[r, dash] & y_{ij} \end{tikzcd} \qquad S_{ij} = \begin{tikzcd} z_{00} \arrow[r, dash] \arrow[d, dash] & z_{i0} \arrow[d, dash] \\ z_{0j} \arrow[r, dash] & z_{ij} \end{tikzcd} \qquad T_{ij} = \begin{tikzcd} x_0 \arrow[rr, dash] \arrow[dr, dash] & & x_i \arrow[dl, dash] \\ & x_{i+j} & \end{tikzcd}$$

For every face in $\overline{\mathbf{X}}_n$ there is an element of $\mathbf{H}_{2n}$ taking it to one of these representatives.

Theorem 9. *The complex $\mathbf{X}_n$ is simply connected.*

Proof. Figure 2.4 shows that the boundary of each of the faces R_{ij} for $1 < i < j$ and S_{ij} for $1 < i, j$ can be expressed as the boundary of a union of

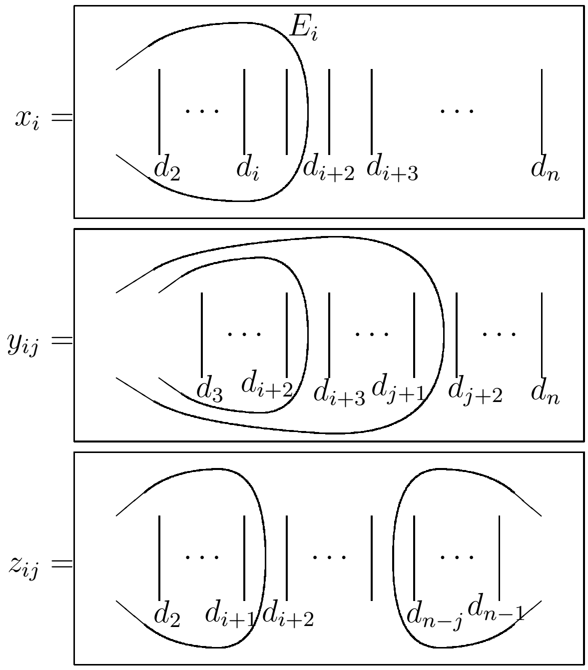


Figure 2.3: The vertices x_i, y_{ij} and z_{ij}

faces with shorter edges. The first column shows how to replace faces where the first index is not 1. Then the second column can be used to reduce the second index to either 2 or 1 respectively.

As each of the rectangular faces can be moved to one of R_{ij} or S_{ij} by some element of $\mathbf{H}_{2n}$ it follows that every loop in $\mathbf{X}_n$ is null-homotopic in

$$\mathcal{R}_{12} \cup \mathcal{S}_{11} \cup \bigcup_{1 \le i \le j \le n} \mathcal{T}_{ij}.$$

Let the E_i be the discs as shown in Figure 2.3, ie $x_i = \langle E_i, d_2, d_3, \ldots, d_n \rangle$. For $j > 2$ let A_j the be full subcomplex of $\overline{\mathbf{X}}_n$ containing all the vertices

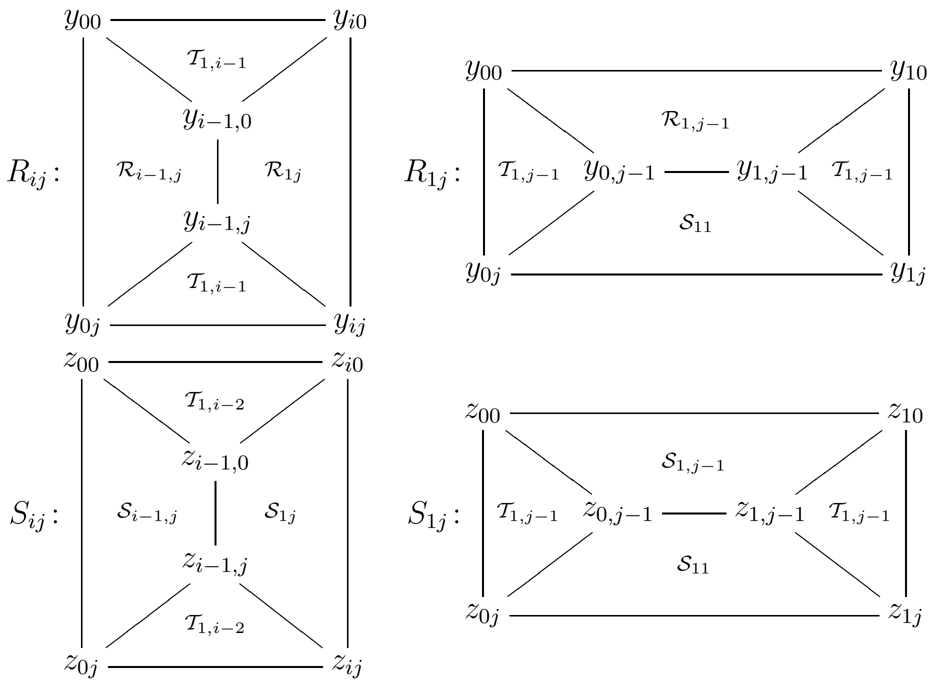


Figure 2.4: Decomposing rectangular faces

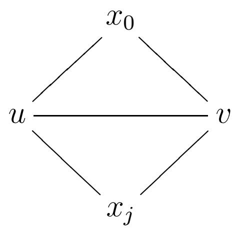
"between" x_0 and x_j, ie

$$A_j^0 = \{\langle D, d_2, d_3, \ldots, d_n\rangle \in \overline{\mathbf{X}}_n^0$$
$$\mid D \neq d_0 \text{ or } E_j, \text{interior of } D \subset \text{bounded component of } H^3 \setminus E_0 \cup E_j\}.$$

Choose x_1 as a base point of A_j.

For every edge (u, v) of A_j we have the following two triangles in $\overline{\mathbf{X}}_n$.

Note that all edges have length less than j.



Lemma 10. *The subcomplex A_j is path connected.*

Proof. Given a vertex $v = \langle D, d_2, \ldots d_n \rangle \in A_j^0$. First suppose that for some $2 < i \leq j$ there exists a path γ on ∂H^3 from d_i to E_j such that γ does not cross E_1, D or d_l for $l \neq i$. Let D' be a disc parallel to E_j except in a neighbourhood of γ where we glue in the boundary of a neighbourhood of $\gamma \cup d_i$. Then there is a path (v, v', x_1) in A_j where $v' = \langle D' \rangle$. See Figure 2.5.

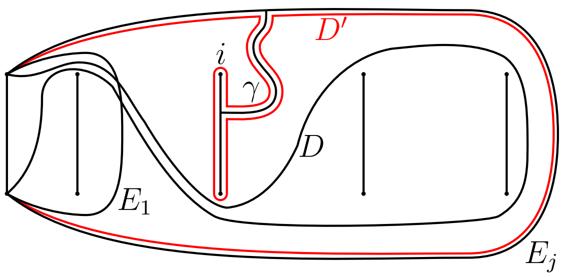
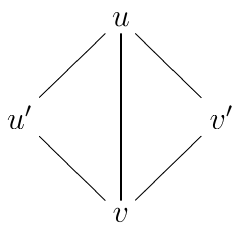


Figure 2.5: Tunnelling along γ

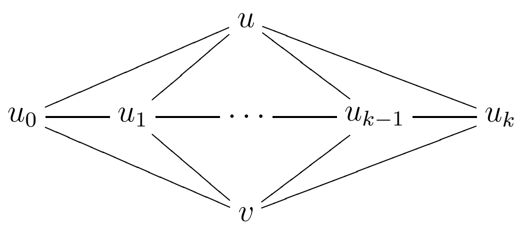
Now suppose that no such path exists on ∂H^3. Each vertex $u = \langle D_u \rangle$ of A_j partitions the set $\{d_2, d_3, \ldots d_{j+1}\}$ into two non-empty subsets. The first containing those discs that are between d_1 and D_u, the second those between D_u and E_j. (If one of these sets were empty then we would have that either $D_u = d_1$ or $D_u = E_j$.) As $j > 2$ at least one of these sets contains more than one disc. Choose an $i \neq 1$ such that d_i is in this set.

Now draw a path γ on ∂H^3 from d_i to E_j that doesn't intersect d_l for $l = 3, \ldots, j$ or E_1 and only intersects D transversely. Starting at d_i move along γ and label the successive points of $\gamma \cap D$ as $p_1, p_2, \ldots, p_k$. Now we can construct a sequence of discs $D = D^0, D^1, \ldots, D^k$ where each D^{l+1} is parallel to D^l except in a neighbourhood of p_{l+1} where we glue in the boundary of a sufficiently small neighbourhood of the disc d_i and the segment of γ up to p_{l+1}. With each successive D^l the disc d_i moves from one side of the partition to the other. At each step neither side of the partition is empty so $\langle D^l \rangle$ is a vertex of A_j. This gives a path $(v = \langle D^0 \rangle, \langle D^1 \rangle, \ldots, \langle D^k \rangle)$ in A_j. Now, $\langle D^k \rangle$ satisfies the hypothesis above, therefore this path can be continued to the base point x_1. $\square$

We can now complete the proof of Theorem 9. So far we have shown that any loop in $\mathbf{X}_n$ is the boundary of a union of faces in $\mathcal{R}_{12} \cup \mathcal{S}_{11} \cup \bigcup_{1 \leq i \leq j \leq n} \mathcal{T}_{ij}$. For a given loop take an edge (u, v) of maximal length j in this union. If $j > 2$ then the faces on either side of (u, v) must be triangular with the remaining edges of length less than j. So we have the following situation for some $u', v' \in \overline{\mathbf{X}}_n^0$.



By Lemma 10 we can replace these two triangles with the following.



Where $u_0 = u'$ and $u_k = v'$. Here each edge has length less than j. Therefore all edges of length greater that 2 can be replaced and so the loop is null-homotopic in $\mathbf{X}_n$. $\square$

2.5 Calculating the presentation

By Section 2.4 we have an $\mathbf{H}_{2n}$–action on a simply connected cellular complex. So we can now follow the method given in Section 2.2.

Using the fact that $\mathbf{H}_{2n}$ is a subgroup of $\mathbf{B}_{2n}$, we can define the following elements of $\mathbf{H}_{2n}$ in terms of $\sigma_1, \ldots, \sigma_{2n-1}$ the generators of $\mathbf{B}_{2n}$.

$$
\begin{aligned}
r_1 &= \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1} \\
r_2 &= \sigma_4\sigma_3\sigma_2\sigma_1\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1} \\
s_i &= \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i} && \text{for } i \in \{1, \ldots, n-1\} \\
t_i &= \sigma_{2i-1} && \text{for } i \in \{1, \ldots, n\}
\end{aligned}
$$

So r_1 is the first arc passing through the second, r_2 is the first two arcs passing through the third, s_i is the ith and $i+1$st arcs crossing and t_i is the ith arc performing a half twist. Subsequently we will prove that these generate $\mathbf{H}_{2n}$.

Proposition 11. *The stabiliser of the vertex v_0 is isomorphic to the framed braid group and hence has a presentation $\langle S_0 \mid R_0 \rangle$ where*

$$\begin{array}{rcl} S_0 & = & \{s_1, s_2, \ldots, s_{n-1},\ t_1, t_2, \ldots, t_n\} \\ R_0 & = & \{ \quad \begin{array}{rcll} s_i s_j &=& s_j s_i & \textit{for } |i-j| > 1, \\ s_i s_j s_i &=& s_j s_i s_j & \textit{for } |i-j| = 1, \\ t_i t_j &=& t_j t_i & \textit{for all } i, j, \\ s_i t_j &=& t_j s_i & \textit{if } j \notin \{i, i+1\}, \\ s_i t_j &=& t_k s_i & \textit{if } \{i, i+1\} = \{j, k\} \quad \} \end{array} \end{array}$$

Proof. If we restrict to ∂H^3, elements of $\mathbf{H}_{2n}$ can be thought of as motions of the end points of the a_i. For elements of the vertex stabiliser this motion moves the $d_i \cap \partial H^3$ among themselves, ie this is the fundamental group of configurations of n line segments in the plain, the framed braid group. $\square$

We have two edge orbits, one consisting of edges of length 1 and the other consisting of edges of length 2. Note that our choice of r_1 and r_2 mean that

$$(v_0, v_0 \cdot r_1) \in l^{-1}(1)$$
$$(v_0, v_0 \cdot r_2) \in l^{-1}(2).$$

For $i = 1,\ 2$, let I_i denote the stabiliser of the edge $(v_0, v_0 \cdot r_i)$, ie the subgroup of all elements that fix both v_0 and $v_0 \cdot r_i$.

Proposition 12. *The subgroups I_1 and I_2 are generated as follows.*

$$I_1 = \langle t_2, t_3, \dots, t_n,\ s_3, s_4, \dots, s_{n-1},\ s_1 s_1 t_1 t_1,\ s_2 s_1 s_1 s_2 \rangle$$

$$I_2 = \langle t_2, t_3, \dots, t_n,\ s_2,\ s_4, s_5, \dots, s_{n-1},\ s_1 s_2 s_2 s_1 t_1 t_1,\ s_3 s_2 s_1 s_1 s_2 s_3 \rangle$$

Proof. For I_1 [I_2] the motion of the d_i outside of $d_1 \cup E_2$ [$d_1 \cup E_3$] is generated by $t_3, t_4, \dots, t_n, s_3, s_4, \dots, s_{n-1}$ and $s_2 s_1 s_1 s_2$ [$t_4, t_5, \dots, t_n$, $s_4, s_5, \dots, s_{n-1}$ and $s_3 s_2 s_1 s_1 s_2 s_3$], the motion of the d_i inside $d_1 \cup E_2$ [$d_1 \cup E_3$] is generated by t_2 [t_2, t_3, s_2] and the motion of $d_1 \cup E_2$ [$d_1 \cup E_3$] is generated by $s_1 s_1 t_1 t_1$ [$s_1 s_2 s_2 s_1 t_1 t_1$]. $\square$

We are now ready to calculate relations for R_1, R_2 and R_3. The following relations are easily verifiable, in fact most of them take place in $\mathbf{B}_8$.

The R_1 relations

To calculate the R_1 relations we have to find, for each edge orbit representative $(v_0, v_0 \cdot r_i)$ and each generator t of I_i, a word h in S_0 such that $r_i t r_i^{-1} = h$.

One possibility is the following.

$$r_1 t_2 r_1^{-1} = t_1 \qquad (R_1 1)$$

$$r_1 t_k r_1^{-1} = t_k \qquad \text{for } k > 2 \qquad (R_1 2)$$

$$r_1 s_k r_1^{-1} = s_k \qquad \text{for } k > 2 \qquad (R_1 3)$$

$$r_1 s_1 s_1 t_1 t_1 r_1^{-1} = s_1 s_1 t_2 t_2 \qquad (R_1 4)$$

$$r_1 s_2 s_1 s_1 s_2 r_1^{-1} = s_2 s_1 s_1 s_2 \qquad (R_1 5)$$

$$r_2 t_2 r_2^{-1} = t_1 \qquad (R_1 6)$$

$$r_2 t_3 r_2^{-1} = t_2 \qquad (R_1 7)$$

$$r_2 t_k r_2^{-1} = t_k \qquad \text{for } k > 3 \qquad (R_1 8)$$

$$r_2 s_2 r_2^{-1} = s_1 \qquad (R_1 9)$$

$$r_2 s_k r_2^{-1} = s_k \qquad \text{for } k > 3 \qquad (R_1 10)$$

$$r_2 s_1 s_2 s_2 s_1 t_1 t_1 r_2^{-1} = s_2 s_1 s_1 s_2 t_3 t_3 \qquad (R_1 11)$$

$$r_2 s_3 s_2 s_1 s_1 s_2 s_3 r_2^{-1} = s_3 s_2 s_1 s_1 s_2 s_3 \qquad (R_1 12)$$

The R_2 relations

To calculate the R_2 relations we need to find, for each edge orbit representative $(v_0, v_0 \cdot r_i)$, an h-product $r_i h\ r_i$ for the path $(v_0, v_0 \cdot r_i, v_0)$ and a word h' in S_0 such that $r_i h\ r_i = h'$.

$$r_1 t_1 s_1\ r_1 = s_1 t_1 \qquad (R_2 1)$$

$$r_2 s_1 t_2 s_2\ r_2 = s_2 s_1 t_1 \qquad (R_2 2)$$

The R_3 relations

To calculate the R_3 relations we need to find, for each edge orbit, an h-product representing the boundary of a face in the orbit and an equivalent word in S_0. The following are such relations for the $\mathcal{S}_{11}$, $\mathcal{R}_{12}$ and $\mathcal{T}_{11}$ orbits respectively.

$$\begin{aligned} &r_1s_1s_2s_3s_1s_2\ r_1s_1s_2s_3s_1s_2t_2t_4\ r_1s_2s_3s_1s_2\ r_1 \\ &\quad = s_1s_2s_3s_1s_2s_1s_2s_1s_3s_2s_2s_3s_1s_2t_1t_3 \end{aligned} \tag{R_31}$$

$$r_1\ r_2s_1s_2s_1t_2t_3\ r_1\ r_2 = s_2s_1s_2t_1t_2 \tag{R_32}$$

$$r_2s_1t_2\ r_1s_2s_1\ r_1 = s_1s_2s_1t_1 \tag{R_33}$$

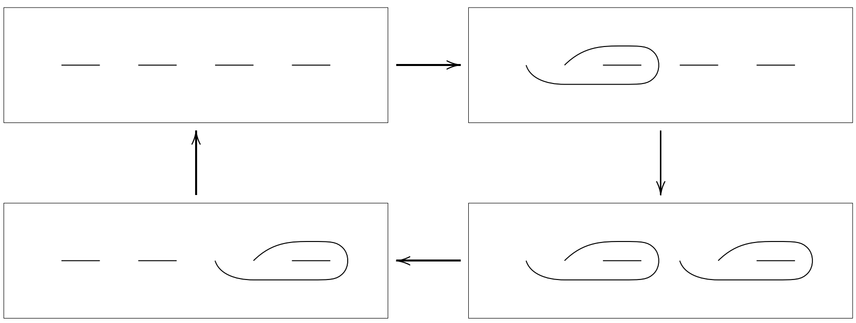


Figure 2.6: The path given by the h-product on the LHS of (R_31)

If we use a different set of generators, similar to those found by Hilden, then we can get a more braid like presentation. Let $p_i = \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}^{-1}\sigma_{2i}^{-1}$ for $1 \le i < n$. So p_i is the ith arc passing under the $i+1$st arc, see Figure 2.7.

Theorem 13. *The group $\mathbf{H}_{2n}$ has a presentation with generators p_i, s_j and*

t_k *for* $1 \leq i, j < n$ *and* $1 \leq k \leq n$ *and the following relations.*

$$p_i p_j = p_j p_i \qquad \textit{for } |i-j| > 1 \tag{P1}$$

$$p_i p_j p_i = p_j p_i p_j \qquad \textit{for } |i-j| = 1 \tag{P2}$$

$$s_i s_j = s_j s_i \qquad \textit{for } |i-j| > 1 \tag{P3}$$

$$s_i s_j s_i = s_j s_i s_j \qquad \textit{for } |i-j| = 1 \tag{P4}$$

$$p_i s_j = s_j p_i \qquad \textit{for } |i-j| > 1 \tag{P5}$$

$$p_i s_{i+1} s_i = s_{i+1} s_i p_{i+1} \tag{P6}$$

$$p_{i+1} p_i s_{i+1} = s_i p_{i+1} p_i \tag{P7}$$

$$p_{i+1} s_i s_{i+1} = s_i s_{i+1} p_i \tag{P8}$$

$$p_i t_i s_i p_i = s_i t_i \tag{P9}$$

$$p_i t_j = t_j p_i \qquad \textit{for } j \neq i, \ \textit{or } i+1 \tag{P10}$$

$$p_i t_{i+1} = t_i p_i \tag{P11}$$

$$s_i t_j = t_j s_i \qquad \textit{if } j \neq i \ \textit{or } i+1 \tag{P12}$$

$$s_i t_j = t_k s_i \qquad \textit{if } \{i, i+1\} = \{j, k\} \tag{P13}$$

$$t_i t_j = t_j t_i \qquad \textit{for } 1 \leq i, j \leq n \tag{P14}$$

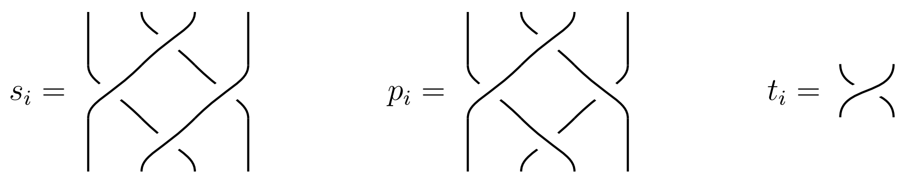


Figure 2.7: Generators of $\mathbf{H}_{2n}$

These generators and relations can be represented pictorially as in Figure 2.8 and Figure 2.9.

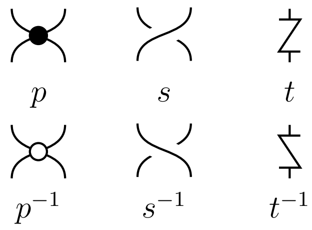


Figure 2.8: Pictorial representation of the p, s, t and their inverses

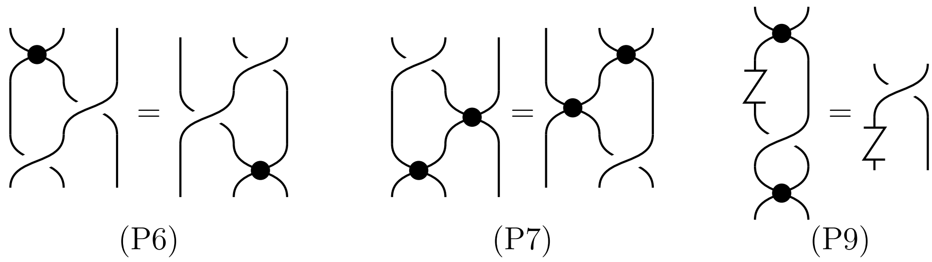


Figure 2.9: Pictorial representation of (P6), (P7) and (P9)

Proof. Since $\mathbf{H}_{2n}$ is a subgroup of the braid group it is easy to check that these relations all hold. So it remains to prove that each of the relations in R_0, R_1, R_2 and R_3 can be deduced from (P1)–(P14) using the fact that $r_1 = p_1$ and $r_2 = p_2 p_1$. First note that R_0 is a subset of these relations. The relations ($R_1$1), ($R_1$2), ($R_1$3) and ($R_2$1) follow directly from (P11), (P10), (P5) and (P9) respectively. The remaining relations can be deduced as follows. Some of these relations are quite long and are perhaps better understood using pictorial representations. For the longest, ($R_3$1), see Figure 2.10 for a pictorial version.

$$
\begin{aligned}
(R_1 4): \qquad r_1 s_1 s_1 t_1 t_1 r_1^{-1} &= p_1 \underline{s_1 s_1 t_1} t_1 p_1^{-1} && (\text{P13})^2 \\
&= \underline{p_1 t_1 s_1} s_1 t_1 p_1^{-1} && (\text{P9}) \\
&= s_1 t_1 \underline{p_1^{-1} s_1 t_1 p_1^{-1}} && (\text{P9}) \\
&= s_1 \underline{t_1 t_1 s_1} && (\text{P13})^2 \\
&= s_1 s_1 t_2 t_2
\end{aligned}
$$

$$(R_15): \qquad r_1s_2s_1s_1s_2r_1^{-1} = \underline{p_1s_2s_1}s_1s_2p_1^{-1} \qquad \text{(P6)}$$
$$= s_2s_1\underline{p_2s_1s_2}p_1^{-1} \qquad \text{(P8)}$$
$$= s_2s_1s_1s_2$$

$$(R_16): \qquad r_2t_2r_2^{-1} = p_2\underline{p_1t_2}p_1^{-1}p_2^{-1} \qquad \text{(P11)}$$
$$= \underline{p_2t_1}p_2^{-1} \qquad \text{(P10)}$$
$$= t_1$$

$$(R_17): \qquad r_2t_3r_2^{-1} = p_2\underline{p_1t_3}p_1^{-1}p_2^{-1} \qquad \text{(P10)}$$
$$= \underline{p_2t_3}p_2^{-1} \qquad \text{(P11)}$$
$$= t_2$$

$$(R_18): \qquad r_2t_kr_2^{-1} = p_2\underline{p_1t_k}p_1^{-1}p_2^{-1} \qquad \text{(P10)}$$
$$= \underline{p_2t_k}p_2^{-1} \qquad \text{(P10)}$$
$$= t_k$$

$$(R_19): \qquad r_2s_2r_2^{-1} = \underline{p_2p_1s_2}p_1^{-1}p_2^{-1} \qquad \text{(P7)}$$
$$= s_1$$

$$(R_110): \qquad r_2s_kr_2^{-1} = \underline{p_2p_1s_k}p_1^{-1}p_2^{-1} \qquad \text{(P5)}^2$$
$$= s_k$$

To deduce $(R_1 11)$ we make use of the following deduction.

$$\begin{aligned}
&\underline{p_2p_1}s_1s_2t_3p_2p_1 && \text{(P7)} \\
&= s_1^{-1}p_2p_1\underline{s_2s_1s_2}t_3p_2p_1 && \text{(P4)} \\
&= s_1^{-1}p_2p_1\underline{s_1s_2s_1t_3}p_2p_1 && \text{(P12)(P13)}^2 \\
&= s_1^{-1}p_2p_1t_1s_1\underline{s_2s_1p_2}p_1 && \text{(P6)} \\
&= s_1^{-1}p_2\underline{p_1t_1s_1p_1}s_2s_1p_1 && \text{(P9)} && (\star) \\
&= \underline{s_1^{-1}p_2}s_1t_1s_2s_1p_1 && \text{(P8)} \\
&= s_2p_1s_2^{-1}\underline{t_1s_2}s_1p_1 && \text{(P12)} \\
&= s_2\underline{p_1t_1s_1p_1} && \text{(P9)} \\
&= s_2s_1t_1
\end{aligned}$$

$$\begin{aligned}
(R_1 11):\quad r_2s_1s_2s_2s_1t_1t_1r_2^{-1} &= p_2p_1s_1s_2\underline{s_2s_1t_1}t_1p_1^{-1}p_2^{-1} && \text{(P13)}^2 \\
&= p_2p_1s_1s_2t_3\underline{s_2s_1t_1p_1^{-1}p_2^{-1}} && (\star) \\
&= \underline{p_2p_1s_1s_2t_3p_2p_1}s_1s_2t_3 && (\star) \\
&= s_2s_1\underline{t_1s_1s_2}t_3 && \text{(P13)}^2 \\
&= s_2s_1s_1s_2t_3t_3
\end{aligned}$$

$$\begin{aligned}
(R_1 12):\quad r_2s_3s_2s_1s_1s_2s_3r_2^{-1} &= p_2\underline{p_1s_3}s_2s_1s_1s_2s_3p_1^{-1}p_2^{-1} && \text{(P5)} \\
&= p_2s_3\underline{p_1s_2s_1}s_1s_2s_3p_1^{-1}p_2^{-1} && \text{(P6)} \\
&= p_2s_3s_2s_1\underline{p_2s_1s_2}s_3p_1^{-1}p_2^{-1} && \text{(P8)} \\
&= p_2s_3s_2s_1s_1s_2\underline{p_1s_3}p_1^{-1}p_2^{-1} && \text{(P5)} \\
&= \underline{p_2s_3s_2}s_1s_1s_2s_3p_2^{-1} && \text{(P6)} \\
&= s_3s_2\underline{p_3s_1s_1}s_2s_3p_2^{-1} && \text{(P5)}^2 \\
&= s_3s_2s_1s_1\underline{p_3s_2s_3}p_2^{-1} && \text{(P8)}
\end{aligned}$$

$$= s_3 s_2 s_1 s_1 s_2 s_3$$

$$(R_2 2): \quad r_2 s_1 t_2 s_2 r_2 = p_2 p_1 s_1 \underline{t_2 s_2 p_2} p_1 \qquad \text{(P9)}$$
$$= p_2 p_1 s_1 p_2^{-1} s_2 t_2 \underline{p_1} \qquad \text{(P9)}$$
$$= p_2 p_1 s_1 p_2^{-1} s_2 \underline{t_2 s_1^{-1}} t_1^{-1} p_1^{-1} s_1 t_1 \qquad \text{(P13)}$$
$$= p_2 p_1 \underline{s_1 p_2^{-1}} s_2 s_1^{-1} p_1^{-1} s_1 t_1 \qquad \text{(P6)}$$
$$= p_2 p_1 s_2^{-1} p_1^{-1} \underline{s_2 s_1 s_2} s_1^{-1} p_1^{-1} s_1 t_1 \qquad \text{(P4)}$$
$$= p_2 p_1 s_2^{-1} p_1^{-1} \underline{s_1 s_2 p_1^{-1}} s_1 t_1 \qquad \text{(P8)}$$
$$= p_2 p_1 \underline{s_2^{-1} p_1^{-1} p_2^{-1}} s_1 s_2 s_1 t_1 \qquad \text{(P7)}$$
$$= s_2 s_1 t_1$$

$$(R_3 1): \quad r_1 s_1 s_2 s_3 s_1 s_2 r_1 s_1 s_2 s_3 s_1 s_2 t_2 t_4 r_1 s_2 s_3 s_1 s_2 r_1$$
$$= p_1 s_1 s_2 s_3 s_1 s_2 p_1 s_1 s_2 s_3 \underline{s_1 s_2 t_2} t_4 p_1 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P13)(P12)}$$
$$= p_1 s_1 s_2 s_3 s_1 s_2 p_1 \underline{s_1 s_2 s_3 t_3} s_1 s_2 t_4 p_1 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P13)(P12)}^2$$
$$= p_1 s_1 s_2 s_3 \underline{s_1 s_2 p_1 t_4} s_1 s_2 s_3 s_1 s_2 t_4 p_1 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P10)(P12)}^2$$
$$= p_1 \underline{s_1 s_2 s_3 t_4} s_1 s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 p_1 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P13)}^3$$
$$= p_1 t_1 s_1 s_2 s_3 s_1 s_2 p_1 s_1 s_2 s_3 s_1 s_2 \underline{t_4 p_1} s_2 s_3 s_1 s_2 p_1 \qquad \text{(P10)}$$
$$= p_1 t_1 s_1 s_2 s_3 s_1 s_2 p_1 s_1 \underline{s_2 s_3 s_1 s_2 p_1} t_4 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P8)}^2$$
$$= p_1 t_1 s_1 s_2 s_3 s_1 s_2 \underline{p_1 s_1 p_3} s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P5)(P1)}$$
$$= p_1 t_1 s_1 s_2 \underline{s_3 s_1} s_2 p_3 p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P3)}$$
$$= p_1 t_1 s_1 s_2 s_1 \underline{s_3 s_2 p_3} p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P6)}$$
$$= p_1 t_1 s_1 \underline{s_2 s_1 p_2} s_3 s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P6)}$$
$$= p_1 t_1 s_1 p_1 s_2 \underline{s_1 s_3} s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P3)}$$
$$= \underline{p_1 t_1 s_1 p_1} s_2 s_3 s_1 s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 s_2 s_3 s_1 s_2 p_1 \qquad \text{(P9)}$$
$$= s_1 t_1 s_2 s_3 s_1 s_2 p_1 s_1 s_2 s_3 s_1 s_2 t_4 \underline{s_2 s_3 s_1 s_2 p_1} \qquad \text{(P8)}^2$$

$$
\begin{aligned}
&= s_1t_1s_2s_3s_1s_2p_1\underline{s_1s_2s_3s_1s_2t_4}p_3s_2s_3s_1s_2 && (\text{P12})^2(\text{P13})^3\\
&= s_1t_1s_2s_3s_1s_2p_1t_1s_1s_2\underline{s_3s_1}s_2p_3s_2s_3s_1s_2 && (\text{P3})\\
&= s_1t_1s_2s_3s_1s_2p_1t_1s_1\underline{s_2s_1s_3s_2p_3}s_2s_3s_1s_2 && (\text{P6})^2\\
&= s_1t_1s_2s_3s_1s_2\underline{p_1t_1s_1p_1}s_2s_1s_3s_2s_2s_3s_1s_2 && (\text{P9})\\
&= s_1t_1s_2s_3s_1s_2s_1\underline{t_1s_2s_1}s_3s_2s_2s_3s_1s_2 && (\text{P12})(\text{P13})\\
&= s_1t_1s_2s_3s_1s_2s_1s_2s_1\underline{t_2s_3s_2s_2}s_3s_1s_2 && (\text{P12})(\text{P13})^2\\
&= s_1t_1s_2s_3s_1s_2s_1s_2s_1s_3s_2s_2\underline{t_2s_3s_1}s_2 && (\text{P12})(\text{P13})\\
&= s_1t_1s_2s_3s_1s_2s_1s_2s_1s_3s_2s_2s_3s_1\underline{t_1s_2} && (\text{P12})\\
&= s_1\underline{t_1s_2s_3s_1s_2}s_1s_2s_1s_3s_2s_2s_3s_1s_2t_1 && (\text{P12})^2(\text{P13})^2\\
&= s_1s_2s_3s_1s_2\underline{t_3s_1s_2s_1}s_3s_2s_2s_3s_1s_2t_1 && (\text{P12})(\text{P13})^2\\
&= s_1s_2s_3s_1s_2s_1s_2s_1\underline{t_1s_3s_2s_2s_3s_1s_2}t_1 && (\text{P12})^4(\text{P13})^2\\
&= s_1s_2s_3s_1s_2s_1s_2s_1s_3s_2s_2s_3s_1s_2\underline{t_3t_1} && (\text{P14})\\
&= s_1s_2s_3s_1s_2s_1s_2s_1s_3s_2s_2s_3s_1s_2t_1t_3
\end{aligned}
$$

$$
\begin{aligned}
(R_32):\quad r_1r_2s_1s_2s_1t_3t_2r_1r_2 &= p_1p_2p_1\underline{s_1s_2s_1t_3}t_2p_1p_2p_1 && (\text{P12})(\text{P13})^2\\
&= p_1p_2\underline{p_1t_1s_1}s_2s_1t_2p_1p_2p_1 && (\text{P9})\\
&= p_1p_2s_1t_1\underline{p_1^{-1}s_2s_1}t_2p_1p_2p_1 && (\text{P6})\\
&= p_1p_2s_1t_1s_2s_1p_2^{-1}t_2\underline{p_1p_2p_1} && (\text{P7})\\
&= p_1p_2s_1t_1s_2s_1p_2^{-1}\underline{t_2p_2}p_1p_2 && (\text{P11})\\
&= p_1p_2s_1\underline{t_1s_2}s_1\underline{t_3p_1}p_2 && (\text{P12})(\text{P10})\\
&= p_1p_2s_1s_2\underline{t_1s_1p_1}t_3p_2 && (\text{P9})\\
&= p_1p_2\underline{s_1s_2p_1^{-1}}s_1t_1t_3p_2 && (\text{P8})\\
&= p_1\underline{s_1s_2s_1}t_1t_3p_2 && (\text{P4})\\
&= \underline{p_1s_2s_1}s_2t_1t_3p_2 && (\text{P6})\\
&= s_2s_1p_2s_2\underline{t_1t_3p_2} && (\text{P14})(\text{P10})
\end{aligned}
$$

$$
\begin{aligned}
&= s_2 s_1 \underline{p_2 s_2 t_3 p_2} t_1 && \text{(P9)}\\
&= s_2 s_1 s_2 t_2 t_1
\end{aligned}
$$

$$
\begin{aligned}
(R_3 3): \qquad r_2 s_1 t_2 r_1 s_2 s_1 r_1 &= p_2 \underline{p_1 s_1 t_2 p_1} s_2 s_1 p_1 && \text{(P9)}\\
&= p_2 s_1 \underline{t_1 s_2} s_1 p_1 && \text{(P10)}\\
&= \underline{p_2 s_1 s_2} t_1 s_1 p_1 && \text{(P8)}\\
&= s_1 s_2 \underline{p_1 t_1 s_1 p_1} && \text{(P9)}\\
&= s_1 s_2 s_1 t_1
\end{aligned}
$$

$\square$

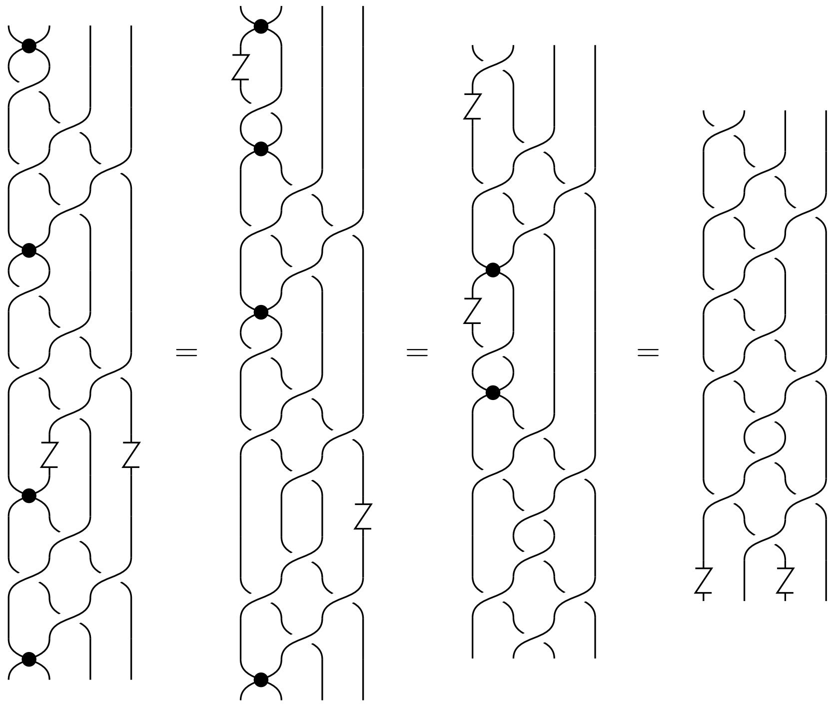


Figure 2.10: Deducing the $(R_3 1)$ relation

Chapter 3

The Pure Hilden group

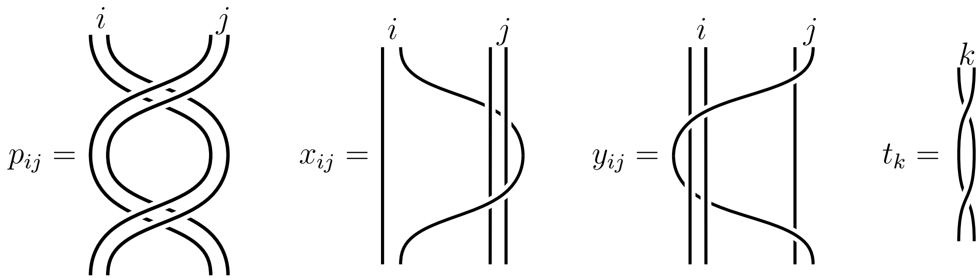
Let $\mathbf{PH}_{2n}$ denote the pure Hilden group on $2n$ strings, ie the intersection of the Hilden group with the pure braid group.

$$\mathbf{PH}_{2n} = \mathbf{H}_{2n} \cap \mathbf{P}_{2n}$$

In this chapter we will compute a presentation for $\mathbf{PH}_{2n}$ using the method and the complex given in the previous chapter.

3.1 The presentation

Let the elements $p_{ij} = p_{ji}$, $x_{ij} = x_{ji}$, $y_{ij} = y_{ji}$ and $t_k \in \mathbf{PH}_{2n}$ for $1 \leq i < j \leq n$ and $1 \leq k \leq n$ be as follows. Here all of the other strings lie behind those shown.



Let S denote the set of all these elements.

$$S = \{p_{ij},\ x_{ij},\ y_{ij},\ t_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}$$

Let R denote the following relations.

$$p_{ij}\, t_k = t_k\, p_{ij} \tag{C-pt}$$

$$t_i\, t_j = t_j\, t_i \tag{C-tt}$$

$$x_{ij}\, t_k = t_k\, x_{ij} \qquad i < j \qquad k \neq i \tag{C-xt}$$

$$y_{ij}\, t_k = t_k\, y_{ij} \qquad i < j \qquad k \neq j \tag{C-yt}$$

$$\alpha_{ij}\, \beta_{kl} = \beta_{kl}\, \alpha_{ij} \qquad \begin{array}{c} \alpha, \beta \in \{p, x, y\}, \\ (i, j, k, l) \text{ cyclically ordered} \end{array} \tag{C1}$$

$$\alpha_{ij}\ \beta_{ik}\, \gamma_{jk} = \beta_{ik}\, \gamma_{jk}\ \alpha_{ij} \qquad \begin{array}{c} (i, j, k) \text{ cyclically ordered,} \\ \alpha, \beta, \gamma \text{ as in Table 3.1} \end{array} \tag{C2}$$

$$\alpha_{ik}\ p_{jk}\, \beta_{jl}\, p_{jk}^{-1} = p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\ \alpha_{ik} \qquad \begin{array}{c} \alpha, \beta \in \{p, x, y\}, \\ (i, j, k, l) \text{ cyclically ordered} \end{array} \tag{C3}$$

$$x_{ij}\, p_{ij}\, t_i = p_{ij}\, t_i\, x_{ij} \qquad i < j \tag{M-x}$$

$$y_{ij}\, p_{ij}\, t_j = p_{ij}\, t_j\, y_{ij} \qquad i < j \tag{M-y}$$

$i<j<k$	(p,p,p)	(p,y,y)	(x,p,p)	(x,x,p)
	(x,y,y)	(y,p,p)	(y,p,x)	(y,y,y)
$j<k<i$	(p,p,p)	(p,x,y)	(x,p,p)	(x,p,x)
	(x,x,y)	(y,p,p)	(y,x,y)	(y,y,p)
$k<i<j$	(p,p,p)	(p,x,x)	(x,p,p)	(x,x,x)
	(x,y,p)	(y,p,p)	(y,p,y)	(y,x,x)

Table 3.1: The values of (α, β, γ) for which (C2) holds

Theorem 1. *The pure Hilden group has a presentation with generating set S and relations R.*

$$\mathbf{PH}_{2n} = \langle S \mid R \rangle$$

3.2 Vertex stabiliser

Recall that the complex $\mathbf{X}_n$ given in the previous chapter comes with an $\mathbf{H}_{2n}$-action on it and that we have a prefered basepoint $v_0 = \langle d_1, d_2, \ldots, d_n \rangle$. As the pure Hilden group is a subgroup of the Hilden group the action of $\mathbf{H}_{2n}$ on $\mathbf{X}_n$ restricts to an action of $\mathbf{PH}_{2n}$ on $\mathbf{X}_n$. The proof of Theorem 7 of Chapter 2 shows that the action of $\mathbf{PH}_{2n}$ on $\mathbf{X}_n^0$ remains transitive.

Proposition 2. *The stabiliser of the vertex v_0 is the framed pure braid group $\mathbf{FP}_n$ and so is isomorphic to $\mathbf{P}_n \times \mathbb{Z}^n$.*

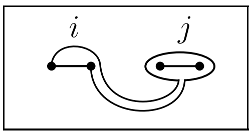
Proof. By Proposition 11 of Chapter 2 the stabiliser of the action of $\mathbf{H}_{2n}$ is the framed braid group on n strings. If we intersect this with the pure braid group on $2n$ strings we get the framed pure braid group on n strings. (Note that for $\mathbf{FP}_n$ the number of twists on each string must be an integer and not a half integer as in the case of $\mathbf{FB}_n$.) $\square$

From this we see that the vertex stabiliser is generated by the p_{ij} and t_k, that all relations between these elements follow from (C-pt), (C-tt), (C1), (C2) and (C3) (with $\alpha = \beta = \gamma = p$), and hence the R_0 relations are included in R.

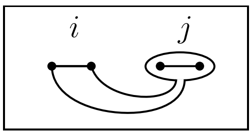
3.3 Edge orbits

Let E denote the set of all oriented edges that start at v_0 the basepoint of $\mathbf{X}_n$. We will now find a representative of each orbit of the $\mathbf{FP}_n$ action on E. Given an edge $(v_0, v) \in E$, because $v = \langle D_1, D_2, \ldots D_n \rangle$ differs from v_0 by a simple move, there exists a unique i such that $D_i \neq d_i$.

If the edge is of length one then there is a unique d_j under $D_i \cup d_i$. All of the remaining discs, d_k for $k \neq i, j$, can be moved by an element of $\mathbf{FP}_n$ away from $D_i \cup d_i$ and then back from behind to their original positions. After applying t_i^p for some p we have one of the following possibilities, each of which lie in a different orbit.

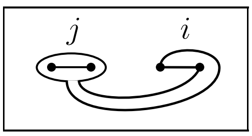


$(v_0, v_0 \cdot x_{ij})$

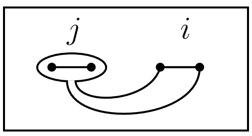


$(v_0, v_0 \cdot x_{ij}^{-1})$

for $i < j$



$(v_0, v_0 \cdot y_{ij})$

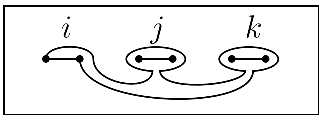


$(v_0, v_0 \cdot y_{ij}^{-1})$

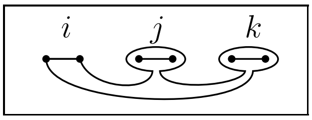
for $j < i$

Similarly, if the edge is of length two then there exists two discs d_j and

d_k, under $d_i \cup D_i$. We may assume that $j < k$. As in the previous case there is an element of $\mathbf{FP}_n$ which takes (v_0, v) to one of the following possibilities, each of which lie in different orbits.

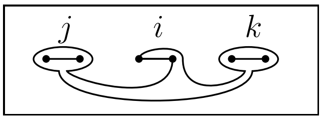


$(v_0, v_0 \cdot x_{ij}\, x_{ik})$

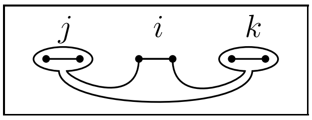


$(v_0, v_0 \cdot x_{ik}^{-1}\, x_{ij}^{-1})$

for $i < j < k$

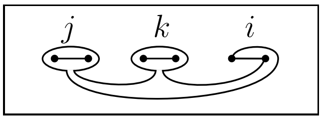


$(v_0, v_0 \cdot x_{ik}\, y_{ij})$

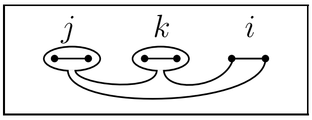


$(v_0, v_0 \cdot y_{ij}^{-1}\, x_{ik}^{-1})$

for $j < i < k$



$(v_0, v_0 \cdot y_{ij}\, y_{ik})$



$(v_0, v_0 \cdot y_{ik}^{-1}\, y_{ij}^{-1})$

for $j < k < i$

Proposition 3. *The pure Hilden group* $\mathbf{PH}_{2n}$ *is generated by* p_{ij}, t_i, x_{ij} *and* y_{ij}.

$$\mathbf{PH}_{2n} = \langle S \rangle$$

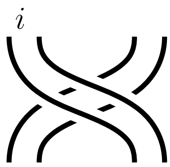
Proof. By the method of Chapter 2 the group $\mathbf{PH}_{2n}$ is generated by the generators of the vertex stabiliser and $\{r_\lambda\}$. We have that

$$\{r_\lambda\} = \left\{ \begin{array}{cc|c} x_{ij}, & x_{ij}^{-1} & \\ & & i < j \\ y_{ij}, & y_{ij}^{-1} & \end{array} \right\} \cup \left\{ \begin{array}{cc|c} x_{ij}\, x_{ik}, & x_{ik}^{-1}\, x_{ij}^{-1} & \\ x_{jk}\, y_{ij}, & y_{ij}^{-1}\, x_{ik}^{-1} & i < j < k \\ y_{ik}\, y_{jk}, & y_{jk}^{-1}\, y_{ik}^{-1} & \end{array} \right\}$$

and so all of these generators either are contained in S or can be written in terms of the elements of S. $\square$

3.4 Action of the framed braid group

We have an embedding of the framed braid group on n strings $\mathbf{FB}_n$ in the braid group on $2n$ strings given as follows.

$\sigma_i =$ 

$\tau_j =$ 

This makes $\mathbf{FB}_n$ a subgroup of $\mathbf{H}_{2n}$. It is clear that conjugation by elements of $\mathbf{FB}_n$ preserves the pure Hilden group and hence we have a left action of $\mathbf{FB}_n$ on $\mathbf{PH}_{2n}$. In fact this action can be defined on the level of reduced words as well. In other words we have an action of $F\langle \sigma_i,\, \tau_j \rangle$, the free group on the letters σ_i and τ_j, on $F\langle p_{ij},\, x_{ij},\, y_{ij},\, t_k \rangle$, the free group on the letters p_{ij}, x_{ij}, y_{ij}, t_k. So we have a homomorphism

$$\begin{aligned} F\langle \sigma_i,\, \tau_j \rangle &\longrightarrow \mathrm{Aut}(F\langle p_{ij},\, x_{ij},\, y_{ij},\, t_k \rangle) \\ g &\longmapsto \Phi_g \end{aligned}$$

In Section 3.8 we will construct Φ and then show that it satisfies the following properties. For any word $g \in F\langle \sigma_i,\, \tau_j \rangle$,

(A) for each x we have that $\Phi_g(x) =_{\mathbf{B}_{2n}} g\, x\, g^{-1}$.

(B) for any word $h \in F\langle p_{ij},\, t_k \rangle$ we have that $\Phi_g(h) \in F\langle p_{ij},\, t_k \rangle$.

(C) for each r_λ we have that $\Phi_g(r_\lambda) =_R h_1 r_{\lambda'} h_2$ for some h_1, h_2 and $r_{\lambda'}$.

(D) for any relation $x =_R y$ we have a relation $\Phi_g(x) =_R \Phi_g(y)$.

3.5 The R_1 relations

The R_1 relations consist of a relation of the form $r_\lambda\, t r_\lambda^{-1} = h$ for each edge orbit representative $(v_0, v_0 \cdot r_\lambda)$, for each t in a generating set of the stabiliser of this edge and for some word h in $\mathbf{FB}_n$.

Proposition 4. *The stabiliser of the edge $(v_0, v_0 \cdot x_{12})$ is generated as follows.*

$$\mathrm{Stab}(v_0, v_0 \cdot x_{12}) = \left\langle \begin{array}{cc} p_{ij} & i,j > 2 \\ t_k & k > 1 \\ p_{12}t_1 & \\ p_{1k}\ p_{2k} & k > 2 \end{array} \right\rangle$$

Proof. As $\mathrm{Stab}(v_0, v_0 \cdot x_{12})$ is a subgroup of $\mathrm{Stab}(v_0) = \mathbf{FP}_n$ we can view the elements of $\mathrm{Stab}(v_0, v_0 \cdot x_{12})$ as motions of line segments. If we draw a line L between the second and third line segments then this motion can be broken into section consisting only of motions of the segments to the right of L, sections consisting only of motions to the left of L and the motion of a single segment across L around both the first and second segment and then back across L. The motions to the right are generated by p_{ij} for $i, j > 2$ and t_k for $k > 2$. The motions to the left are generated by t_2 and $p_{12}\ t_1$. And the motions across L are of the form $p_{1k}\ p_{2k}$ for $k > 2$. $\square$

So the R_1 relations can be chosen as follows.

$$x_{12}\ p_{ij}\ x_{12}^{-1} = p_{ij} \qquad \text{for } i,j > 2 \tag{1}$$

$$x_{12}\ t_k\ x_{12}^{-1} = t_k \qquad \text{for } k > 1 \tag{2}$$

$$x_{12}\ p_{12}\, t_1\ x_{12}^{-1} = p_{12}\, t_1 \tag{3}$$

$$x_{12}\ p_{1k}\, p_{2k}\ x_{12}^{-1} = p_{1k}\, p_{2k} \qquad \text{for } k > 2 \tag{4}$$

Relation (1) follows from (C1), relation (2) follows from (C-xt), relation (3) follows from (M-x) and relation (4) follows from (C2).

For the edge orbit representative $(v_0, v_0 \cdot x_{12}\, x_{13})$ we can draw a line L between the third and fourth line segment. Motion of the segments to the right is generated by p_{ij} for $i,j > 3$ and t_k for $k > 3$. Motion of the segments to the left is generated by $p_{12}\, p_{13}\, t_1$, t_2, t_3 and p_{23}. Finally the elements $p_{1k}\ p_{2k}\ p_{3k}$ give the motion between the two halves. Therefore we have the following.

Proposition 5. *The stabiliser of the edge $(v_0, v_0 \cdot x_{12}\, x_{13})$ is generated as follows.*

$$\mathrm{Stab}(v_0, v_0 \cdot x_{12}\, x_{13}) = \left\langle \begin{array}{cc} p_{23} & \\ p_{ij} & i,j > 3 \\ t_k & k > 1 \\ p_{12}\, p_{13}\, t_1 & \\ p_{1k}\, p_{2k}\, p_{3k} & k > 3 \end{array} \right\rangle$$

□

Hence the R_1 relations can be chosen as follows.

$$x_{12}\, x_{13}\ p_{23}\ (x_{12}\, x_{13})^{-1} = p_{23} \tag{5}$$

$$x_{12}\, x_{13}\ p_{ij}\ (x_{12}\, x_{13})^{-1} = p_{ij} \qquad \text{for } i, j > 3 \tag{6}$$

$$x_{12}\, x_{13}\ t_k\ (x_{12}\, x_{13})^{-1} = t_k \qquad \text{for } k > 1 \tag{7}$$

$$x_{12}\, x_{13}\ p_{12}\, p_{13}\, t_1\ (x_{12}\, x_{13})^{-1} = p_{12}\, p_{13}\, t_1 \tag{8}$$

$$x_{12}\, x_{13}\ p_{1k}\, p_{2k}\, p_{3k}\ (x_{12}\, x_{13})^{-1} = p_{1k}\, p_{2k}\, p_{3k} \qquad \text{for } k > 3 \tag{9}$$

Relation (5) follows from (C2), relation (6) follows from two applications of (C1), relation (7) follows from two applications of (C-xt). Relation (8) follows from the following.

$$\begin{aligned}
& x_{12}\, x_{13}\, \underline{p_{12}\, p_{13}}\, t_1 && \text{(C2)} \\
&\quad = x_{12}\, x_{13}\, p_{13}\, p_{23}\, p_{12}\, p_{23}^{-1}\, \underline{t_1} && \text{(C-}pt)^3 \\
&\quad = x_{12}\, \underline{x_{13}\, p_{13}\, t_1}\, p_{23}\, p_{12}\, p_{23}^{-1} && \text{(M-}x) \\
&\quad = x_{12}\, p_{13}\, t_1\, \underline{x_{13}\, p_{23}\, p_{12}}\, p_{23}^{-1} && \text{(C2)} \\
&\quad = x_{12}\, p_{13}\, \underline{t_1}\, p_{23}\, p_{12}\, x_{13}\, p_{23}^{-1} && \text{(C-}pt)^2 \\
&\quad = \underline{x_{12}\, p_{13}\, p_{23}}\, p_{12}\, t_1\, x_{13}\, p_{23}^{-1} && \text{(C2)} \\
&\quad = p_{13}\, p_{23}\, \underline{x_{12}\, p_{12}\, t_1}\, x_{13}\, p_{23}^{-1} && \text{(M-}x) \\
&\quad = p_{13}\, p_{23}\, p_{12}\, t_1\, \underline{x_{12}\, x_{13}\, p_{23}^{-1}} && \text{(C2)} \\
&\quad = p_{13}\, p_{23}\, p_{12}\, \underline{t_1\, p_{23}^{-1}}\, x_{12}\, x_{13} && \text{(C-}pt) \\
&\quad = \underline{p_{13}\, p_{23}\, p_{12}\, p_{23}^{-1}}\, t_1\, x_{12}\, x_{13} && \text{(C2)} \\
&\quad = p_{12}\, p_{13}\, t_1\, x_{12}\, x_{13}
\end{aligned}$$

Finally (9) follows from the following.

$$
\begin{aligned}
&\underline{x_{13}\,p_{1k}}\,p_{2k}\,p_{3k} && \text{(C2)}\\
&\qquad = p_{1k}\,p_{3k}\,x_{13}\,\underline{p_{3k}^{-1}\,p_{2k}\,p_{3k}} && \text{(C2)}\\
&\qquad = p_{1k}\,p_{3k}\,\underline{x_{13}\,p_{23}\,p_{2k}\,p_{23}^{-1}} && \text{(C3)}\\
&\qquad = p_{1k}\,\underline{p_{3k}\,p_{23}\,p_{2k}\,p_{23}^{-1}}\,x_{13} && \text{(C2)}\\
&\qquad = p_{1k}\,p_{2k}\,p_{3k}\,x_{13}
\end{aligned}
$$

$$
\begin{aligned}
&\underline{x_{12}\,p_{1k}\,p_{2k}}\,p_{3k} && \text{(C2)}\\
&\qquad = p_{1k}\,p_{2k}\,\underline{x_{12}\,p_{3k}} && \text{(C1)}\\
&\qquad = p_{1k}\,p_{2k}\,p_{3k}\,x_{12}
\end{aligned}
$$

Now consider the edge orbit representative $(v_0, v_0 \cdot r_\lambda)$ for $r_\lambda \neq x_{12}$ or $x_{12}\,x_{13}$. There exists some $g \in \mathbf{FB}_n$ such that $(v_0, v_0 \cdot r_1) \cdot g = (v_0, v_0 \cdot r_\lambda)$, where $r_1 = x_{12}$ or $x_{12}\,x_{13}$. By property (A) of Φ

$$\Phi_{g^{-1}}(r_1) =_{\mathbf{B}_{2n}} g^{-1} r_1 g$$

and by property (C) there exists words $h_1, h_2 \in \mathbf{FP}_n$ and some $r_{\lambda'}$ such that

$$\Phi_{g^{-1}}(r_1) =_R h_1\, r_{\lambda'}\, h_2. \tag{3.1}$$

Combining these we see that $v_0 \cdot r_1\, g = v_0 \cdot r_{\lambda'}\, h_2$ and hence that $\lambda = \lambda'$ and $h_2 \in \mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$.

Let T be the choice of generators for $\mathrm{Stab}(v_0, v_0 \cdot r_1)$ chosen above. So for all $t \in T$ there exists $h \in \mathbf{FP}_n$ such that

$$r_1\, t\, r_1^{-1} =_R h.$$

So by property (D) we have

$$\Phi_{g^{-1}}(r_1\, t\, r_1^{-1}) =_R \Phi_{g^{-1}}(h). \tag{3.2}$$

Property (B) implies that $\Phi_{g^{-1}}(t) \in \mathbf{FP}_n$ and $\Phi_{g^{-1}}(h) \in \mathbf{FP}_n$. Combining (3.1) and (3.2) we get

$$h_1\, r_\lambda\, h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1}\, r_\lambda^{-1}\, h_1^{-1} =_R \Phi_{g^{-1}}(h)$$

and so $h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1} \in \mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$.

Claim 5. *The set* $\{h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1} \mid t \in T\}$ *generates* $\mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$.

Proof. As $h_2 \in \mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$ the set $\{h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1} \mid t \in T\}$ generates $\mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$ if and only if the set $\{\Phi_{g^{-1}}(t) \mid t \in T\}$ generates $\mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$. This is equivalent to saying that for any $s \in \mathrm{Stab}(v_0, v_0 \cdot r_\lambda)$ we can find $t_1, \ldots, t_k \in T$ such that $s = \Phi_{g^{-1}}(t_1 \cdots t_k)$, in other words that $\Phi_g(s) \in \mathrm{Stab}(v_0, v_0 \cdot r_1)$. Now

$$\begin{aligned}
(v_0 \cdot r_1) \cdot \Phi_g(s) &= v_0 \cdot r_1\, g\, s\, g^{-1} \\
&= v_0 \cdot r_\lambda\, s\, g^{-1} \\
&= v_0 \cdot r_\lambda\, g^{-1} \\
&= v_0 \cdot r_1
\end{aligned}$$

Therefore the claim holds. $\square$

So for our R_1 relation we can choose the following

$$r_\lambda\, h_2\, \Phi_{g^{-1}}(t)\, h_2^{-1}\, r_\lambda^{-1} = h_1^{-1}\, \Phi_{g^{-1}}(h)\, h_1$$

and hence we can choose our R_1 relations so that they all follow from R.

3.6 The R_2 relations

The R_2 relations consist of a relation of the form $r_{\lambda'}\, h\, r_\lambda = h'$ for each edge orbit representative, where the LHS is an h-product for the path $(v_0, v_0{\cdot}r_\lambda, v_0)$ and $h' \in \mathbf{FB}_n$. For each edge $(v_0, v_0 \cdot r_\lambda)$ the edge $(v_0, v_0 \cdot r_\lambda^{-1})$ is in a different orbit. Our choice of r_λ mean that for all λ there exists λ' such that $r_\lambda^{-1} = r_{\lambda'}$. This means that for all the R_2 relations we can choose $r_\lambda^{-1}\, r_\lambda = 1$, ie they are all trivial.

3.7 The R_3 relations

The R_3 relations consist of a relation of the form $r_{\lambda_k} h_k \cdots r_{\lambda_1} h_1 = h$ for each edge orbit representative, where the LHS is an h-product that represents the boundary of the face and $h \in \mathbf{FP}_n$. As with the R_1 relations, we will calculate the relations for some specific orbits first then use Φ for the general case.

We will start with the triangular face $(v_0, v_0{\cdot}x_{12}\, x_{13}, v_0{\cdot}x_{12})$. An h-product for this path is $x_{13}^{-1}\, x_{12}^{-1}\, (x_{12}\, x_{13})$. So the R_3 relations is

$$x_{13}^{-1}\, x_{12}^{-1}\, (x_{12}\, x_{13}) = 1$$

and so it is trivial.

Next consider the non-nested rectangular face $(v_0, v_0 \cdot x_{12}, v_0 \cdot x_{34}\, x_{12}, v_0 \cdot x_{34})$. An h-product that represents this path is $x_{34}^{-1}\ x_{12}^{-1}\ x_{34}\ x_{12}$. So the R_3 relations is

$$x_{34}^{-1}\ x_{12}^{-1}\ x_{34}\ x_{12} = 1$$

which follows from (C1).

Now consider the nested rectangular face

$$(v_0,\ v_0 \cdot x_{23},\ v_0 \cdot x_{12}\, x_{13}\, x_{23},\ v_0 \cdot x_{12}\, x_{13}).$$

An h-product that represents this path is

$$(x_{12}\, x_{13})^{-1} x_{23}^{-1}\, (x_{12}\, x_{13})\, x_{23}.$$

So the R_3 relations is

$$(x_{12}\, x_{13})^{-1}\ x_{23}^{-1}\, (x_{12}\, x_{13})\, x_{23} = 1$$

which follows from (C2).

Given any other face orbit representative $(v_0 = u_0, u_1, \ldots, u_k = v_0)$ there exists some $g \in \mathbf{FB}_n$ such that

$$(u_0, u_1, \ldots, u_k) = (v_0, v_1, \ldots, v_k) \cdot g$$

where $(v_0, v_1, \ldots, v_k)$ is the boundary of one of the three faces whose R_3 relations we calculated above. Suppose the relation from $(v_0, v_1, \ldots, v_k)$ is the following.

$$r_{\lambda_k} h_k \cdots r_{\lambda_1} h_1 = h$$

By property (C), for each r_{λ_i} there exists $h_{i1}, h_{i2} \in \mathbf{FP}_n$ and $r_{\lambda'_i}$ such that

$$\Phi_{g^{-1}}(r_{\lambda_i}) =_R h_{i1}\, r_{\lambda'_i}\, h_{i2}$$

Claim 6. *The following h-product represents the path* $(u_0, u_1, \ldots, u_k)$.

$$r_{\lambda'_k}\, h_{k2}\, \Phi_{g^{-1}}(h_k)\, h_{(k-1)1}\ \cdots\ r_{\lambda'_1}\, h_{11}\, \Phi_{g^{-1}}(h_1)$$

Proof. The ith vertex of the path associated to the h-product is given as follows.

$$\begin{aligned}
& v_0 \cdot r_{\lambda'_i}\, h_{i2}\, \Phi_{g^{-1}}(h_i)\, h_{(i-1)1}\ \cdots\ r_{\lambda'_1}\, h_{11}\, \Phi_{g^{-1}}(h_1) \\
= \quad & v_0 \cdot \Phi_{g^{-1}}(r_{\lambda_i},\ h_i \cdots r_{\lambda_1}\, h_1) \\
= \quad & v_0 \cdot r_{\lambda_i}\, h_i \cdots r_{\lambda_1}\, h_1\, g \\
= \quad & v_i \cdot g \\
= \quad & u_i
\end{aligned}$$

$\square$

Therefore for our R_3 relation we may choose the following

$$r_{\lambda'_k}\, h_{k2}\, \Phi_{g^{-1}}(h_k)\, h_{(k-1)1}\ \cdots\ r_{\lambda'_1}\, h_{11}\, \Phi_{g^{-1}}(h_1) = h_{k1}^{-1}\, \Phi_{g^{-1}}(h)$$

which follows from R by property (D).

3.8 Definition and properties of Φ

All that remains to prove Theorem 1 is to construct Φ and show that it satisfies properties (A)–(D).

Define Φ, the action of $F\langle \sigma_i,\ \tau_j\rangle$ on $F\langle\, p_{ij},\ x_{ij},\ y_{ij},\ t_k\rangle$, as follows. For $\alpha \in \{p, x, y\}$

$$
\begin{aligned}
\Phi_{\sigma_i}(\alpha_{kl}) &= \alpha_{kl} && \text{for } i \neq k-1, k, l-1, l \\
\Phi_{\sigma_i}(\alpha_{ij}) &= \alpha_{i+1,j} && \text{for } i+1 < j \\
\Phi_{\sigma_i}(\alpha_{i+1,j}) &= p_{i,i+1}\, \alpha_{ij}\, p_{i,i+1}^{-1} && \text{for } i+1 < j \\
\Phi_{\sigma_j}(\alpha_{i,j+1}) &= p_{j,j+1}\, \alpha_{ij}\, p_{j,j+1}^{-1} && \text{for } i+1 < j \\
\Phi_{\sigma_j}(\alpha_{ij}) &= \alpha_{i,j+1} && \text{for } i+1 < j \\
\Phi_{\sigma_i}(p_{i,i+1}) &= p_{i,i+1} \\
\Phi_{\sigma_i}(x_{i,i+1}) &= t_{i+1}^{-1}\, y_{i,i+1}\, t_{i+1} \\
\Phi_{\sigma_i}(y_{i,i+1}) &= x_{i,i+1} \\
\Phi_{\sigma_i}(t_j) &= \begin{cases} t_j & \text{if } j \neq i,\ i+1 \\ t_{j+1} & \text{if } j = i \\ t_i & \text{if } j = i+1 \end{cases} \\
\Phi_{\tau_i}(p_{kl}) &= p_{kl} \\
\Phi_{\tau_i}(x_{kl}) &= \begin{cases} x_{kl} & \text{if } i \neq k \\ x_{kl}^{-1}\, p_{kl} & \text{if } i = k \end{cases} && \text{for } k < l \\
\Phi_{\tau_i}(y_{kl}) &= \begin{cases} y_{kl} & \text{if } i \neq l \\ y_{kl}^{-1}\, p_{kl} & \text{if } i = l \end{cases} && \text{for } k < l \\
\Phi_{\tau_i}(t_j) &= t_j
\end{aligned}
$$

Proposition 6. *The map Φ is a well defined action of $F\langle\, \tau_i,\, \sigma_i \,\rangle$ on $F\langle\, p_{ij},\, t_i,\, x_{ij},\, y_{ij}\rangle$.*

Proof. All that needs to be checked is that Φ_{σ_i} and Φ_{τ_i} are invertible. The inverses are as follows.

$$\begin{array}{lll}
\Phi_{\sigma_i^{-1}}(\alpha_{kl}) & = \alpha_{kl} & \text{for } i \neq k-1, k, l-1, l \\
\Phi_{\sigma_i^{-1}}(\alpha_{ij}) & = p_{i,i+1}^{-1}\, \alpha_{i+1,j}\, p_{i,i+1} & \text{for } i+1 < j \\
\Phi_{\sigma_i^{-1}}(\alpha_{i+1,j}) & = \alpha_{ij} & \text{for } i+1 < j \\
\Phi_{\sigma_j^{-1}}(\alpha_{i,j+1}) & = \alpha_{ij} & \text{for } i+1 < j \\
\Phi_{\sigma_j^{-1}}(\alpha_{ij}) & = p_{j,j+1}^{-1}\, \alpha_{i,j+1}\, p_{j,j+1} & \text{for } i+1 < j \\
\Phi_{\sigma_i^{-1}}(p_{i,i+1}) & = p_{i,i+1} & \\
\Phi_{\sigma_i^{-1}}(x_{i,i+1}) & = y_{i,i+1} & \\
\Phi_{\sigma_i^{-1}}(y_{i,i+1}) & = t_i\, x_{i,i+1}\, t_i^{-1} & \\
\Phi_{\sigma_i^{-1}}(t_j) & = \begin{cases} t_j & \text{if } j \neq i,\ i+1 \\ t_{j+1} & \text{if } j = i \\ t_{j-1} & \text{if } j = i+1 \end{cases} & \\
\Phi_{\tau_i^{-1}}(p_{kl}) & = p_{kl} & \\
\Phi_{\tau_i^{-1}}(x_{kl}) & = \begin{cases} x_{kl} & \text{if } i \neq k \\ p_{kl}\, x_{kl}^{-1} & \text{if } i = k \end{cases} & \text{for } k < l \\
\Phi_{\tau_i^{-1}}(y_{kl}) & = \begin{cases} y_{kl} & \text{if } i \neq l \\ p_{kl}\, y_{kl}^{-1} & \text{if } i = l \end{cases} & \text{for } k < l \\
\Phi_{\tau_i^{-1}}(t_j) & = t_j &
\end{array}$$

□

It is easy to check the Φ satisfies property (A), ie that for every word $g \in F\langle \sigma_i,\, \tau_j \rangle$ and for each $x \in F\langle\, p_{ij},\, t_i,\, x_{ij},\, y_{ij} \rangle$ we have that $\Phi_g(x) = g\, x\, g^{-1}$ as braids. It is also clear that Φ satisfies property (B). That is that for any word $g \in F\langle \sigma_i,\, \tau_j \rangle$ and for any word $h \in F\langle\, p_{ij},\, t_k \rangle$ we have $\Phi_g(h) \in F\langle\, p_{ij},\, t_k \rangle$.

Proposition 7. *The map Φ satisfies property* (C). *In other word for any word h in $F\langle\, p_{ij},\, t_k \rangle$ and any*

$$r_\lambda \in \left\{ \begin{array}{cc|c} x_{ij}, & x_{ij}^{-1} & \\ & & i<j \\ y_{ij}, & y_{ij}^{-1} & \end{array} \right\} \cup \left\{ \begin{array}{cc|c} x_{ij}\, x_{ik}, & x_{ik}^{-1}\, x_{ij}^{-1} & \\ x_{jk}\, y_{ij}, & y_{ij}^{-1}\, x_{ik}^{-1} & i<j<k \\ y_{ik}\, y_{jk}, & y_{jk}^{-1}\, y_{ik}^{-1} & \end{array} \right\}$$

we have a relation $\Phi_g(r_\lambda) = h_1 r_{\lambda'} h_2$ that can be deduced from the relations in R, for some $h_1, h_2 \in F\langle\, p_{ij},\, t_k \rangle$ and some $r_{\lambda'}$.

Proof. First note that for each word h in $F\langle\, p_{ij},\, t_k \rangle$, by property (B), the map Φ_g takes h to another word in $F\langle\, p_{ij},\, t_k \rangle$. Therefore we only need to check Φ_g where $g = \tau^{\pm 1}, \sigma^{\pm 1}$.

For $r_\lambda = x_{ij}, x_{ij}^{-1}, y_{ij}, y_{ij}^{-1}$ this follows immediately from the definition of Φ given above.

Now consider $\Phi_{\sigma_m}(r_\lambda)$ for $r_\lambda = x_{ij}\, x_{ik},\ x_{jk}\, y_{ij}$ or $y_{ik}\, y_{jk}$. The only cases when $\Phi_{\sigma_m}(r_\lambda) \neq r_\lambda$ are $m = i-1$, $m = i$ and $j = i+1$, $m = i$ and $j > i+1$, $m = j-1$ and $i < j-1$, $m = j$ and $k = j+1$, $m = j$ and $k > j+1$, $m = k-1$ and $j < k-1$, and $m = k$. We now show that $\Phi_{\sigma_m}(r_\lambda) =_R h_1\, r_{\lambda'}\, h_2$.

$m = i-1$

$$\Phi_{\sigma_{i-1}}(x_{ij}\, x_{ik}) = p_{i-1,i}\ x_{i-1,j}\, x_{i-1,k}\ p_{i-1,i}^{-1}$$

$$\Phi_{\sigma_{i-1}}(x_{jk}\, y_{ij}) = \underline{x_{jk}\, p_{i-1,i}}\, y_{i-1,j}\, p_{i-1,i}^{-1} \tag{C1}$$

$$
\begin{aligned}
&= p_{i-1,i}\ x_{jk}\,y_{i-1,j}\ p_{i-1,i}^{-1} \\[1ex]
\Phi_{\sigma_{i-1}}(y_{ik}\,y_{jk}) &= p_{i-1,i}\,y_{i-1,k}\,\underline{p_{i-1,i}^{-1}\,y_{jk}} && \text{(C1)} \\
&= p_{i-1,i}\ y_{i-1,k}\,y_{jk}\ p_{i-1,i}^{-1}
\end{aligned}
$$

$m = i$ and $j = i+1$

$$
\begin{aligned}
\Phi_{\sigma_i}(x_{ij}\,x_{ik}) &= t_j^{-1}\,y_{ij}\,\underline{t_j\,x_{jk}} && \text{(M-}x\text{)} \\
&= t_j^{-1}\,\underline{y_{ij}\,p_{jk}^{-1}}\,x_{jk}\,p_{jk}t_j && \text{(C2)} \\
&= t_j^{-1}\,p_{jk}^{-1}\,\underline{p_{ik}^{-1}\,y_{ij}\,p_{ik}\,x_{jk}}\,p_{jk}\,t_j && \text{(C2)} \\
&= t_j^{-1}\,p_{jk}^{-1}\ x_{jk}\,y_{ij}\ p_{jk}\,t_j \\[1ex]
\Phi_{\sigma_i}(x_{jk}\,y_{ij}) &= \underline{p_{ij}\,x_{ik}\,p_{ij}^{-1}}\,x_{ij} && \text{(C2)} \\
&= p_{jk}^{-1}\,\underline{x_{ik}\,p_{jk}\,x_{ij}} && \text{(C2)} \\
&= p_{jk}^{-1}\ x_{ij}\,x_{ik}\ p_{jk} \\[1ex]
\Phi_{\sigma_i}(y_{ik}\,y_{jk}) &= \underline{y_{jk}\,p_{ij}\,y_{ik}\,p_{ij}^{-1}} && \text{(C2)} \\
&= y_{ik}\,y_{jk}
\end{aligned}
$$

$m = i$ and $j > i+1$

$$
\begin{aligned}
\Phi_{\sigma_i}(x_{ij}\,x_{ik}) &= x_{i+1,j}\,x_{i+1,k} \\[1ex]
\Phi_{\sigma_i}(x_{jk}\,y_{ij}) &= x_{jk}\,y_{i+1,j} \\[1ex]
\Phi_{\sigma_i}(y_{ik}\,y_{jk}) &= y_{i+1,k}\,y_{jk}
\end{aligned}
$$

$m = j-1$ and $i < j-1$

$$
\begin{aligned}
\Phi_{\sigma_{j-1}}(x_{ij}\,x_{ik}) &= p_{j-1,j}\,x_{i,j-1}\,\underline{p_{j-1,j}^{-1}\,x_{ik}} && \text{(C1)} \\
&= p_{j-1,j}\ x_{i,j-1}\,x_{ik}\ p_{j-1,j}^{-1} \\[1ex]
\Phi_{\sigma_{j-1}}(x_{jk}\,y_{ij}) &= p_{j-1,j}\ x_{j-1,k}\,y_{i,j-1}\ p_{j-1,j}^{-1} \\[1ex]
\Phi_{\sigma_{j-1}}(y_{ik}\,y_{jk}) &= \underline{y_{ik}\,p_{j-1,j}}\,y_{j-1,k}\,p_{j-1,j}^{-1} && \text{(C1)} \\
&= p_{j-1,j}\ y_{ik}\,y_{j-1,k}\ p_{j-1,j}^{-1}
\end{aligned}
$$

$m = j$ and $k = j+1$

$$\begin{aligned}
\Phi_{\sigma_j}(x_{ij}\, x_{ik}) &= \underline{x_{ik}\, p_{jk}\, x_{ij}\, p_{jk}^{-1}} && \text{(C2)}\\
&= x_{ij}\, x_{ik}
\end{aligned}$$

$$\begin{aligned}
\Phi_{\sigma_j}(x_{jk}\, y_{ij}) &= t_k^{-1}\, \underline{y_{jk}}\, t_k\, y_{ik} && \text{(M-}y\text{)}\\
&= \underline{t_k^{-1}\, p_{jk}\, t_k}\, y_{jk}\, \underline{t_k^{-1}\, p_{jk}^{-1}\, t_k}\, y_{ik} && \text{(C-}pt\text{)}^2\\
&= p_{jk}\, y_{jk}\, \underline{p_{jk}^{-1}\, y_{ik}} && \text{(C2)}\\
&= p_{jk}\, \underline{y_{jk}\, p_{ij}\, y_{ik}\, p_{ij}^{-1}}\, p_{jk}^{-1} && \text{(C2)}\\
&= p_{jk}\ y_{ik}\, y_{jk}\ p_{jk}^{-1}
\end{aligned}$$

$$\begin{aligned}
\Phi_{\sigma_j}(y_{ik}\, y_{jk}) &= \underline{p_{jk}\, y_{ij}\, p_{jk}^{-1}}\, x_{jk} && \text{(C2)}\\
&= \underline{p_{ik}^{-1}\, y_{ij}\, p_{ik}\, x_{jk}} && \text{(C2)}\\
&= x_{jk}\, y_{ij}
\end{aligned}$$

$m = j$ and $k > j+1$

$$\begin{aligned}
\Phi_{\sigma_j}(x_{ij}\, x_{ik}) &= x_{i,j+1}\, x_{ik}\\
\Phi_{\sigma_j}(x_{jk}\, y_{ij}) &= x_{j+1,k}\, y_{i,j+1}\\
\Phi_{\sigma_j}(y_{ik}\, y_{jk}) &= y_{ik}\, y_{j+1,k}
\end{aligned}$$

$m = k-1$ and $j < k-1$

$$\begin{aligned}
\Phi_{\sigma_{k-1}}(x_{ij}\, x_{ik}) &= \underline{x_{ij}\, p_{k-1,k}}\, x_{i,k-1}\, p_{k-1,k}^{-1} && \text{(C1)}\\
&= p_{k-1,k}\ x_{ij}\, x_{i,k-1}\ p_{k-1,k}^{-1}
\end{aligned}$$

$$\begin{aligned}
\Phi_{\sigma_{k-1}}(x_{jk}\, y_{ij}) &= p_{k-1,k}\, x_{j,k-1}\, \underline{p_{k-1,k}^{-1}\, y_{ij}} && \text{(C1)}\\
&= p_{k-1,k}\ x_{j,k-1}\, y_{ij}\ p_{k-1,k}^{-1}
\end{aligned}$$

$$\Phi_{\sigma_{k-1}}(y_{ik}\, y_{jk}) = p_{k-1,k}\ y_{i,k-1}\, y_{j,k-1}\ p_{k-1,k}^{-1}$$

$m = k$

$$\begin{aligned}
\Phi_{\sigma_k}(x_{ij}\, x_{ik}) &= x_{ij}\, x_{i,k+1}\\
\Phi_{\sigma_k}(x_{jk}\, y_{ij}) &= x_{j,k+1}\, y_{ij}
\end{aligned}$$

$$\Phi_{\sigma_k}(y_{ik}\, y_{jk}) = y_{i,k+1}\, y_{j,k+1}$$

For Φ_{τ_m} we only have three cases where $\Phi_{\tau_m}(r_\lambda) \neq r_\lambda$ these are when $m = i$ and $r_\lambda = x_{ij}\, x_{ik}$, $m = j$ and $r_\lambda = x_{jk}\, y_{ij}$, and $m = k$ and $r_\lambda = y_{ik}\, y_{jk}$.

$$\begin{aligned}
\Phi_{\tau_i}(x_{ij}\, x_{ik}) &= x_{ij}^{-1}\, \underline{p_{ij}\, x_{ik}^{-1}}\, p_{ik} && \text{(C2)}\\
&= \underline{x_{ij}^{-1}\, p_{jk}^{-1}\, x_{ik}^{-1}\, p_{jk}}\, p_{ij}\, p_{ik} && \text{(C2)}\\
&= x_{ik}^{-1}\, x_{ij}^{-1}\, p_{ij}\, p_{ik}
\end{aligned}$$

$$\begin{aligned}
\Phi_{\tau_j}(x_{jk}\, y_{ij}) &= x_{jk}^{-1}\, \underline{p_{jk}\, y_{ij}^{-1}}\, p_{ij} && \text{(C2)}\\
&= \underline{x_{jk}^{-1}\, p_{ik}^{-1}\, y_{ij}^{-1}\, p_{ik}}\, p_{jk}\, p_{ij} && \text{(C2)}\\
&= y_{ij}^{-1}\, x_{jk}^{-1}\, p_{jk}\, p_{ij}
\end{aligned}$$

$$\begin{aligned}
\Phi_{\tau_k}(y_{ik}\, y_{jk}) &= y_{ik}^{-1}\, \underline{p_{ik}\, y_{jk}^{-1}}\, p_{jk} && \text{(C2)}\\
&= \underline{y_{ik}^{-1}\, p_{ij}^{-1}\, y_{jk}^{-1}\, p_{ij}}\, p_{ik}\, p_{jk} && \text{(C2)}\\
&= y_{jk}^{-1}\, y_{ik}^{-1}\, p_{ik}\, p_{jk}
\end{aligned}$$

Now consider $\Phi_{\sigma_m^{-1}}(r_\lambda)$ the cases where $\Phi_{\sigma_m^{-1}}(r_\lambda) \neq r_\lambda$ are the same as for $\Phi_{\sigma_m}(r_\lambda)$.

$m = i - 1$

$$\Phi_{\sigma_{i-1}^{-1}}(x_{ij}\, x_{ik}) = x_{i-1,j}\, x_{i-1,k}$$

$$\Phi_{\sigma_{i-1}^{-1}}(x_{jk}\, y_{ij}) = x_{jk}\, y_{i-1,j}$$

$$\Phi_{\sigma_{i-1}^{-1}}(y_{ik}\, y_{jk}) = y_{i-1,k}\, y_{jk}$$

$m = i$ and $j = i + 1$

$$\begin{aligned}
\Phi_{\sigma_i^{-1}}(x_{ij}\, x_{ik}) &= y_{ij}\, \underline{p_{ij}^{-1}\, x_{jk}\, p_{ij}} && \text{(C2)}\\
&= \underline{y_{ij}\, p_{ik}\, x_{jk}\, p_{ik}^{-1}} && \text{(C2)}\\
&= x_{jk}\, y_{ij}
\end{aligned}$$

$$
\begin{aligned}
\Phi_{\sigma_i^{-1}}(x_{jk}\, y_{ij}) &= x_{ik}\, \underline{t_i\, x_{ij}\, t_i^{-1}} && \text{(M-}x\text{)}\\
&= \underline{x_{ik}\, p_{ij}^{-1}}\, x_{ij}\, p_{ij} && \text{(C2)}\\
&= p_{ij}^{-1}\, p_{jk}^{-1}\, \underline{x_{ik}\, p_{jk}\, x_{ij}}\, p_{ij} && \text{(C2)}\\
&= p_{ij}^{-1}\, p_{jk}^{-1}\; x_{ij}\, x_{ik}\; p_{jk}\, p_{ij}
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{\sigma_i^{-1}}(y_{ik}\, y_{jk}) &= \underline{p_{ij}^{-1}\, y_{jk}\, p_{ij}\, y_{ik}} && \text{(C2)}\\
&= y_{ik}\, y_{jk}
\end{aligned}
$$

$m = i$ and $j > i+1$

$$
\Phi_{\sigma_i^{-1}}(x_{ij}\, x_{ik}) = p_{i+1\,i}^{-1}\; x_{i+1,j}\, x_{i+1,k}\; p_{i+1\,i}
$$

$$
\begin{aligned}
\Phi_{\sigma_i^{-1}}(x_{jk}\, y_{ij}) &= \underline{x_{jk}\, p_{i,i+1}^{-1}}\, y_{i+1,j}\, p_{i,i+1} && \text{(C1)}\\
&= p_{i,i+1}^{-1}\; x_{jk}\, y_{i+1,j}\; p_{i,i+1}
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{\sigma_i^{-1}}(y_{ik}\, y_{jk}) &= p_{i,i+1}^{-1}\, y_{i+1,k}\, \underline{p_{i,i+1}\, y_{jk}} && \text{(C1)}\\
&= p_{i,i+1}^{-1}\; y_{i+1,k}\, y_{jk}\; p_{i,i+1}
\end{aligned}
$$

$m = j-1$ and $i < j-1$

$$
\Phi_{\sigma_{j-1}^{-1}}(x_{ij}\, x_{ik}) = x_{i,j-1}\, x_{ik}
$$

$$
\Phi_{\sigma_{j-1}^{-1}}(x_{jk}\, y_{ij}) = x_{j-1,k}\, y_{i,j-1}
$$

$$
\Phi_{\sigma_{j-1}^{-1}}(y_{ik}\, y_{jk}) = y_{ik}\, y_{j-1,k}
$$

$m = j$ and $k = j+1$

$$
\begin{aligned}
\Phi_{\sigma_j^{-1}}(x_{ij}\, x_{ik}) &= p_{jk}^{-1}\, \underline{x_{ik}\, p_{jk}\, x_{ij}} && \text{(C2)}\\
&= p_{jk}^{-1}\; x_{ij}\, x_{ik}\; p_{jk}
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{\sigma_j^{-1}}(x_{jk}\, y_{ij}) &= y_{jk}\, \underline{p_{jk}^{-1}\, y_{ik}\, p_{jk}} && \text{(C2)}\\
&= \underline{y_{jk}\, p_{ij}\, y_{ik}\, p_{ij}^{-1}} && \text{(C2)}\\
&= y_{ik}\, y_{jk}
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{\sigma_j^{-1}}(y_{ik}\, y_{jk}) &= y_{ij}\, \underline{t_j\, x_{jk}\, t_j^{-1}} && \text{(M-}x\text{)}\\
&= \underline{y_{ij}\, p_{jk}^{-1}}\, x_{jk}\, p_{jk} && \text{(C2)}
\end{aligned}
$$

$$
\begin{aligned}
& & &= p_{jk}^{-1}\, \underline{p_{ik}^{-1}\, y_{ij}\, p_{ik}\, x_{jk}}\, p_{jk} && \text{(C2)}\\
& & &= p_{jk}^{-1}\; x_{jk}\, y_{ij}\; p_{jk}
\end{aligned}
$$

$$
\begin{aligned}
&m = j \text{ and } k > j+1 \quad & \Phi_{\sigma_j^{-1}}(x_{ij}\, x_{ik}) &= p_{j,j+1}^{-1}\, x_{i,j+1}\, \underline{p_{j,j+1}\, x_{ik}} && \text{(C1)}\\
& & &= p_{j,j+1}^{-1}\; x_{i,j+1}\, x_{ik}\; p_{j,j+1}\\
& & \Phi_{\sigma_j^{-1}}(x_{jk}\, y_{ij}) &= p_{j,j+1}^{-1}\; x_{j+1,k}\, y_{i,j+1}\; p_{j,j+1}\\
& & \Phi_{\sigma_j^{-1}}(y_{ik}\, y_{jk}) &= \underline{y_{ik}\, p_{j,j+1}^{-1}}\, y_{j+1,k}\, p_{j,j+1} && \text{(C1)}\\
& & &= p_{j,j+1}^{-1}\; y_{ik}\, y_{j+1,k}\; p_{j,j+1}
\end{aligned}
$$

$$
\begin{aligned}
&m = k-1 \text{ and } j < k-1 \quad & \Phi_{\sigma_{k-1}^{-1}}(x_{ij}\, x_{ik}) &= x_{ij}\, x_{i,k-1}\\
& & \Phi_{\sigma_{k-1}^{-1}}(x_{jk}\, y_{ij}) &= x_{j,k-1}\, y_{ij}\\
& & \Phi_{\sigma_{k-1}^{-1}}(y_{ik}\, y_{jk}) &= y_{i,k-1}\, y_{j,k-1}
\end{aligned}
$$

$$
\begin{aligned}
&m = k \quad & \Phi_{\sigma_k^{-1}}(x_{ij}\, x_{ik}) &= \underline{x_{ij}\, p_{k,k+1}^{-1}}\, x_{i,k+1}\, p_{k,k+1} && \text{(C1)}\\
& & &= p_{k,k+1}^{-1}\; x_{ij}\, x_{i,k+1}\; p_{k,k+1}\\
& & \Phi_{\sigma_k^{-1}}(x_{jk}\, y_{ij}) &= p_{k,k+1}^{-1}\, x_{j,k+1}\, \underline{p_{k,k+1}\, y_{ij}} && \text{(C1)}\\
& & &= p_{k,k+1}^{-1}\; x_{j,k+1}\, y_{ij}\; p_{k,k+1}\\
& & \Phi_{\sigma_k^{-1}}(y_{ik}\, y_{jk}) &= p_{k,k+1}^{-1}\; y_{i,k+1}\, y_{j,k+1}\; p_{k,k+1}
\end{aligned}
$$

As with Φ_{τ_m}, for $\Phi_{\tau_m^{-1}}$ we only have three cases where $\Phi_{\tau_m^{-1}}(r_\lambda) \neq r_\lambda$ these are when $m = i$ and $r_\lambda = x_{ij}\, x_{ik}$, $m = j$ and $r_\lambda = x_{jk}\, y_{ij}$, and $m = k$ and $r_\lambda = y_{ik}\, y_{jk}$.

$$
\begin{aligned}
\Phi_{\tau_i^{-1}}(x_{ij}\, x_{ik}) &= p_{ij}\, \underline{x_{ij}^{-1}\, p_{ik}}\, x_{ik}^{-1} && \text{(C2)}\\
&= p_{ij}\, p_{ik}\, p_{jk}\, \underline{x_{ij}^{-1}\, p_{jk}^{-1}\, x_{ik}^{-1}} && \text{(C2)}
\end{aligned}
$$

$$= p_{ij}\, p_{ik}\, p_{jk}\ x_{ik}^{-1}\, x_{ij}^{-1}\ p_{jk}^{-1}$$

$$\begin{aligned}
\Phi_{\tau_j^{-1}}(x_{jk}\, y_{ij}) &= p_{jk}\, \underline{x_{jk}^{-1}\, p_{ij}}\, y_{ij}^{-1} && \text{(C2)}\\
&= p_{jk}\, p_{ij}\, p_{ik}\, \underline{x_{jk}^{-1}\, p_{ik}^{-1}\, y_{ij}^{-1}} && \text{(C2)}\\
&= p_{jk}\, p_{ij}\, p_{ik}\ y_{ij}^{-1}\, x_{jk}^{-1}\ p_{ik}^{-1}
\end{aligned}$$

$$\begin{aligned}
\Phi_{\tau_k^{-1}}(y_{ik}\, y_{jk}) &= p_{ik}\, \underline{y_{ik}^{-1}\, p_{jk}}\, y_{jk}^{-1} && \text{(C2)}\\
&= p_{ik}\, p_{jk}\, p_{ij}\, \underline{y_{ik}^{-1}\, p_{ij}^{-1}\, y_{jk}^{-1}} && \text{(C2)}\\
&= p_{ik}\, p_{jk}\, p_{ij}\ y_{jk}^{-1}\, y_{ik}^{-1}\ p_{ij}^{-1}
\end{aligned}$$

For $r_\lambda = x_{ik}^{-1}\, x_{ij}^{-1}$, $y_{ij}^{-1}\, x_{ik}^{-1}$ and $y_{jk}^{-1}\, y_{ik}^{-1}$ we have shown that for some $h_1, h_2 \in \mathbf{FP}_n$ and some $r_{\lambda'}^{-1}$ we have that $\Phi_g(r_\lambda^{-1}) =_R h_1\ r_{\lambda'}^{-1}\ h_2$. Hence we have $\Phi_g(r_\lambda) =_R h_2^{-1}\ r_{\lambda'}\ h_1^{-1}$. □

Proposition 8. *The map Φ satisfies property* (D). *In other words, for any word $g \in F\langle \sigma_i,\, \tau_j\rangle$ and any relation $x =_R y$ we have that $\Phi_g(x) =_R \Phi_g(y)$.*

Proof. This is equivalent to saying that for each relation $x = y$ in R and each $g \in \{\sigma_i,\, \sigma_i^{-1},\, \tau_i,\, \tau_i^{-1}\}$ the relation $\Phi_g(x) = \Phi_g(y)$ follows from those in R. For any relation only involving p_{ij}'s and t_k's the image under Φ_g will still only involve p_{ij}'s and t_k's and hence, by Proposition 2, the new relation will follow from those in R.

We will now considering action of Φ_{σ_q} and Φ_{τ_q} on each of the relations. For any relation $x =_R y$ we will say that the deduction of $\Phi_g(x) = \Phi_g(y)$ is trivial if $\Phi_g(x) = \Phi_g(y)$ is a relation in R of the same type.

(C-xt) $\qquad x_{ij}\, t_k = t_k\, x_{ij} \qquad k \neq i,\ i < j$

First consider Φ_{σ_q}. If we start with $q = 1$ and increase it the first non-trivial case is when $q = i - 1$. The next case is when $q = i$ and this is only non-trivial if $j = i + 1$. The next case is when $q = j - 1$ and $j \neq i + 1$. The remaining values are all trivial.

When $q = i - 1$ we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq i - 1$.

$$\begin{aligned}
\Phi_{\sigma_q}(x_{ij}\, t_k) &= p_{i-1,i}\, x_{i-1,j}\, \underline{p^{-1}_{i-1,i}\, t_{k'}} && \text{(C-}pt\text{)}\\
&= p_{i-1,i}\, \underline{x_{i-1,j}\, t_{k'}}\, p^{-1}_{i-1,i} && \text{(C-}xt\text{)}\\
&= \underline{p_{i-1,i}\, t_{k'}}\, x_{i-1,j}\, p^{-1}_{i-1,i} && \text{(C-}pt\text{)}\\
&= t_{k'}\, p_{i-1,i}\, x_{i-1,j}\, p^{-1}_{i-1,i}\\
&= \Phi_{\sigma_q}(t_k\, x_{ij})
\end{aligned}$$

When $q = i$ and $j = i + 1$ we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq j$.

$$\begin{aligned}
\Phi_{\sigma_q}(x_{ij}\, t_k) &= t_j^{-1}\, y_{ij}\, \underline{t_j\, t_{k'}} && \text{(C-}tt\text{)}\\
&= t_j^{-1}\, \underline{y_{ij}\, t_{k'}}\, t_j && \text{(C-}yt\text{)}\\
&= \underline{t_j^{-1}\, t_{k'}}\, y_{ij}\, t_j && \text{(C-}tt\text{)}\\
&= t_{k'}\, t_j^{-1}\, y_{ij}\, t_j\\
&= \Phi_{\sigma_q}(t_k\, x_{ij})
\end{aligned}$$

When $q = j - 1$ and $j \neq i + 1$ we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq i$.

$$\begin{aligned}
\Phi_{\sigma_q}(x_{ij}\, t_k) &= p_{j-1,j}\, x_{i,j-1}\, \underline{p^{-1}_{j-1,j}\, t_{k'}} && \text{(C-}pt\text{)}\\
&= p_{j-1,j}\, \underline{x_{i,j-1}\, t_{k'}}\, p^{-1}_{j-1,j} && \text{(C-}xt\text{)}\\
&= \underline{p_{j-1,j}\, t_{k'}}\, x_{i,j-1}\, p^{-1}_{j-1,j} && \text{(C-}pt\text{)}\\
&= t_{k'}\, p_{j-1,j}\, x_{i,j-1}\, p^{-1}_{j-1,j}\\
&= \Phi_{\sigma_q}(t_k\, x_{ij})
\end{aligned}$$

Now consider Φ_{τ_q}, the only non-trivial case is when $q = i$.

$$\begin{aligned}
\Phi_{\tau_q}(x_{ij}\, t_k) &= x_{ij}^{-1}\, \underline{p_{ij}\, t_k} && \text{(C-}pt\text{)}\\
&= \underline{x_{ij}^{-1}\, t_k}\, p_{ij} && \text{(C-}xt\text{)}\\
&= t_k\, x_{ij}^{-1}\, p_{ij}\\
&= \Phi_{\tau_q}(t_k\, x_{ij})
\end{aligned}$$

(C-yt) $$y_{ij}\, t_k = t_k\, y_{ij} \qquad k \neq j,\ i < j$$

First consider Φ_{σ_q}, the non-trivial cases are $q = i-1$, $q = i$ and $j = i+1$, and $q = j-1$ and $j \neq i+1$.

When $q = i-1$ we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq j$.

$$\begin{aligned}
\Phi_{\sigma_q}(y_{ij}\, t_k) &= p_{i-1,i}\, y_{i-1,j}\, \underline{p_{i-1,i}^{-1}\, t_{k'}} && \text{(C-}pt\text{)}\\
&= p_{i-1,i}\, \underline{y_{i-1,j}\, t_{k'}}\, p_{i-1,i}^{-1} && \text{(C-}xt\text{)}\\
&= \underline{p_{i-1,i}\, t_{k'}}\, y_{i-1,j}\, p_{i-1,i}^{-1} && \text{(C-}pt\text{)}\\
&= t_{k'}\, p_{i-1,i}\, y_{i-1,j}\, p_{i-1,i}^{-1}\\
&= \Phi_{\sigma_q}(t_k\, y_{ij})
\end{aligned}$$

When $q = i$ and $j = i+1$ we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq i$.

$$\begin{aligned}
\Phi_{\sigma_q}(y_{ij}\, t_k) &= \underline{x_{ij}\, t_{k'}} && \text{(C-}xt\text{)}\\
&= t_{k'}\, x_{ij}\\
&= \Phi_{\sigma_q}(t_k\, y_{ij})
\end{aligned}$$

When $q = j-1$ and $j \neq i+1$ we have that $\Phi_{\sigma_q}(t_k) = t_{k'}$ where $k' \neq j-1$.

$$\begin{aligned}
\Phi_{\sigma_q}(y_{ij}\, t_k) &= p_{j-1,j}\, y_{i,j-1}\, \underline{p_{j-1,j}^{-1}\, t_{k'}} && \text{(C-}pt\text{)}\\
&= p_{j-1,j}\, \underline{y_{i,j-1}\, t_{k'}}\, p_{j-1,j}^{-1} && \text{(C-}yt\text{)}
\end{aligned}$$

$$
\begin{aligned}
&= \underline{p_{j-1,j}\, t_{k'}}\, y_{i,j-1}\, p_{j-1,j}^{-1} && \text{(C-}pt\text{)}\\
&= t_{k'}\, p_{j-1,j}\, y_{i,j-1}\, p_{j-1,j}^{-1}\\
&= \Phi_{\sigma_q}(t_k\, y_{ij})
\end{aligned}
$$

Now consider Φ_{τ_q}, the only non-trivial case is when $q = j$.

$$
\begin{aligned}
\Phi_{\tau_q}(y_{ij}\, t_k) &= y_{ij}^{-1}\, \underline{p_{ij}\, t_k} && \text{(C-}pt\text{)}\\
&= \underline{y_{ij}^{-1}\, t_k}\, p_{ij} && \text{(C-}yt\text{)}\\
&= t_k\, y_{ij}^{-1}\, p_{ij}\\
&= \Phi_{\tau_q}(t_k\, y_{ij})
\end{aligned}
$$

(C1) $\qquad \alpha_{ij}\, \beta_{kl} = \beta_{kl}\, \alpha_{ij} \qquad (i, j, k, l)$ cyclically ordered

First consider Φ_{σ_q}. The non-trivial cases are $q = i-1$ and $i \neq l+1$, $q = i$ and $j = i+1$, $q = j-1$ and $j \neq i+1$, $q = j$ and $k = j+1$, $q = k-1$ and $j \neq k-1$, $q = k$ and $l = k+1$, $p = l-1$ and $l \neq k+1$, and $p = l$ and $i = l+1$.

When $q = i-1$ and $i \neq l+1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, \beta_{kl}) &= p_{i-1,i}\, \alpha_{i-1,j}\, \underline{p_{i-1,i}^{-1}\, \beta_{kl}} && \text{(C1)}\\
&= p_{i-1,i}\, \underline{\alpha_{i-1,j}\, \beta_{kl}}\, p_{i-1,i}^{-1} && \text{(C1)}\\
&= \underline{p_{i-1,i}\, \beta_{kl}}\, \alpha_{i-1,j}\, p_{i-1,i}^{-1} && \text{(C1)}\\
&= \beta_{kl}\, p_{i-1,i}\, \alpha_{i-1,j}\, p_{i-1,i}^{-1}\\
&= \Phi_{\sigma_q}(\beta_{kl}\, \alpha_{ij})
\end{aligned}
$$

When $q = i$ and $j = i+1$ the only non-trivial case is when $\alpha = x$.

$$
\Phi_{\sigma_q}(x_{ij}\, \beta_{kl}) = t_j^{-1}\, y_{ij}\, \underline{t_j\, \beta_{kl}} \qquad \text{(C-}\beta t\text{)}
$$

$$\begin{aligned}
&= t_j^{-1}\, \underline{y_{ij}\, \beta_{kl}}\, t_j && \text{(C1)} \\
&= \underline{t_j^{-1}\, \beta_{kl}}\, y_{ij}\, t_j && \text{(C-}\beta t\text{)} \\
&= \beta_{kl}\, t_j^{-1}\, y_{ij}\, t_j \\
&= \Phi_{\sigma_q}(\beta_{kl}\, x_{ij})
\end{aligned}$$

When $q = j - 1$ and $j \neq i + 1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, \beta_{kl}) &= p_{j-1,j}\, \alpha_{i,j-1}\, \underline{p_{j-1,j}^{-1}\, \beta_{kl}} && \text{(C1)} \\
&= p_{j-1,j}\, \underline{\alpha_{i,j-1}\, \beta_{kl}}\, p_{j-1,j}^{-1} && \text{(C1)} \\
&= \underline{p_{j-1,j}\, \beta_{kl}}\, \alpha_{i,j-1}\, p_{j-1,j}^{-1} && \text{(C1)} \\
&= \beta_{kl}\, p_{j-1,j}\, \alpha_{i,j-1}\, p_{j-1,j}^{-1} \\
&= \Phi_{\sigma_q}(\beta_{kl}\, \alpha_{ij})
\end{aligned}$$

When $q = j$ and $k = j + 1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, \beta_{kl}) &= \underline{\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}} && \text{(C3)} \\
&= p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik} \\
&= \Phi_{\sigma_q}(\beta_{kl}\, \alpha_{ij})
\end{aligned}$$

When $q = k - 1$ and $j \neq k - 1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, \beta_{kl}) &= \underline{\alpha_{ij}\, p_{k-1,k}}\, \beta_{k-1,l}\, p_{k-1,k}^{-1} && \text{(C1)} \\
&= p_{k-1,k}\, \underline{\alpha_{ij}\, \beta_{k-1,l}}\, p_{k-1,k}^{-1} && \text{(C1)} \\
&= p_{k-1,k}\, \beta_{k-1,l}\, \underline{\alpha_{ij}\, p_{k-1,k}^{-1}} && \text{(C1)} \\
&= p_{k-1,k}\, \beta_{k-1,l}\, p_{k-1,k}^{-1}\, \alpha_{ij} \\
&= \Phi_{\sigma_q}(\beta_{kl}\, \alpha_{ij})
\end{aligned}$$

When $q = k$ and $l = k + 1$ the only non-trivial case is when $\beta = x$.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, x_{kl}) &= \underline{\alpha_{ij}\, t_l^{-1}}\, y_{kl}\, t_l && \text{(C-}\alpha t\text{)}\\
&= t_l^{-1}\, \underline{\alpha_{ij}\, y_{kl}}\, t_l && \text{(C1)}\\
&= t_l^{-1}\, y_{kl}\, \underline{\alpha_{ij}\, t_l} && \text{(C-}\alpha t\text{)}\\
&= t_l^{-1}\, y_{kl}\, t_l\, \alpha_{ij}\\
&= \Phi_{\sigma_q}(x_{kl}\, \alpha_{ij})
\end{aligned}
$$

When $q = l - 1$ and $l \neq k + 1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, \beta_{kl}) &= \underline{\alpha_{ij}\, p_{l-1,l}}\, \beta_{k,l-1}\, p_{l-1,l}^{-1} && \text{(C1)}\\
&= p_{l-1,l}\, \underline{\alpha_{ij}\, \beta_{k,l-1}}\, p_{l-1,l}^{-1} && \text{(C1)}\\
&= p_{l-1,l}\, \beta_{k,l-1}\, \underline{\alpha_{ij}\, p_{l-1,l}^{-1}} && \text{(C1)}\\
&= p_{l-1,l}\, \beta_{k,l-1}\, p_{l-1,l}^{-1}\, \alpha_{ij}\\
&= \Phi_{\sigma_q}(\beta_{kl}\, \alpha_{ij})
\end{aligned}
$$

Finally, when $q = l$ and $i = l + 1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, \beta_{kl}) &= \underline{p_{il}\, \alpha_{jl}\, p_{il}^{-1}\, \beta_{ik}} && \text{(C3)}\\
&= \beta_{ik}\, p_{il}\, \alpha_{jl}\, p_{il}^{-1}\\
&= \Phi_{\sigma_q}(\beta_{kl}\, \alpha_{ij})
\end{aligned}
$$

Now consider Φ_{τ_q}, there are two non-trivial cases. In the first case $\Phi_{\tau_q}(\alpha_{ij}) = \alpha_{ij}^{-1} p_{ij}$ and we have the following.

$$
\begin{aligned}
\Phi_{\tau_q}(\alpha_{ij}\, \beta_{kl}) &= \alpha_{ij}^{-1}\, \underline{p_{ij}\, \beta_{kl}} && \text{(C1)}\\
&= \underline{\alpha_{ij}^{-1}\, \beta_{kl}}\, p_{ij} && \text{(C1)}\\
&= \beta_{kl}\, \alpha_{ij}^{-1}\, p_{ij}\\
&= \Phi_{\tau_q}(\beta_{kl}\, \alpha_{ij})
\end{aligned}
$$

In the second case $\Phi_{\tau_q}(\beta_{kl}) = \beta_{kl}^{-1} p_{kl}$ and we have the following.

$$\begin{aligned}
\Phi_{\tau_q}(\alpha_{ij}\,\beta_{kl}) &= \underline{\alpha_{ij}\,\beta_{kl}^{-1}}\,p_{kl} && \text{(C1)}\\
&= \beta_{kl}^{-1}\,\underline{\alpha_{ij}\,p_{kl}} && \text{(C1)}\\
&= \beta_{kl}^{-1}\,p_{kl}\,\alpha_{ij}\\
&= \Phi_{\tau_q}(\beta_{kl}\,\alpha_{ij})
\end{aligned}$$

(C2) $\qquad \alpha_{ij}\;\beta_{ik}\,\gamma_{jk} = \beta_{ik}\,\gamma_{jk}\;\alpha_{ij} \qquad$ (i, j, k) cyclically ordered, (α, β, γ) as in Table 3.1

First consider Φ_{σ_q}. The only non-trivial cases are when $q = i-1$ and $i \neq k+1$, $q = i$ and $j = i+1$, $q = j-1$ and $j \neq i+1$, $q = j$ and $k = j+1$, $q = k-1$ and $k \neq j+1$, and $q = k$ and $i = k+1$.

When $q = i-1$ and $i \neq k+1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= p_{i-1,i}\,\alpha_{i-1,j}\,\beta_{i-1,k}\,\underline{p_{i-1,i}^{-1}\,\gamma_{jk}} && \text{(C1)}\\
&= p_{i-1,i}\,\underline{\alpha_{i-1,j}\,\beta_{i-1,k}\,\gamma_{jk}}\,p_{i-1,i}^{-1} && \text{(C2)}\\
&= p_{i-1,i}\,\beta_{i-1,k}\,\underline{\gamma_{jk}}\,\alpha_{i-1,j}\,p_{i-1,i}^{-1} && \text{(C1)}\\
&= p_{i-1,i}\,\beta_{i-1,k}\,p_{i-1,i}^{-1}\,\gamma_{jk}\,p_{i-1,i}\,\alpha_{i-1,j}\,p_{i-1,i}^{-1}\\
&= \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}$$

When $q = i$ and $j = i+1$ we have two cases. Except for when $i < j < k$ and $(\alpha, \beta, \gamma) = (x, x, p)$ or $k < i < j$ and $(\alpha, \beta, \gamma) = (x, y, p)$ we have the following deduction. Let $\bar{t}_j$ and $\bar{\alpha}_{ij}$ be defined as follows.

$$\bar{t}_j = \begin{cases} t_j & \text{if } \alpha = x \\ 1 & \text{if } \alpha \neq x \end{cases} \qquad \bar{\alpha}_{ij} = \begin{cases} p_{ij} & \text{if } \alpha = p \\ y_{ij} & \text{if } \alpha = x \\ x_{ij} & \text{if } \alpha = y \end{cases}$$

So we have that $\Phi_{\sigma_q}(\alpha_{ij}) = \bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\bar{t}_j$.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\underline{\bar{t}_j\,\beta_{jk}\,p_{ij}\,\gamma_{ik}\,p_{ij}^{-1}} && \text{(C-}\beta t\text{) (C-}pt\text{) (C-}\gamma t\text{) (C-}pt\text{)}\\
&= \bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\underline{\beta_{jk}\,p_{ij}\,\gamma_{ik}\,p_{ij}^{-1}}\,\bar{t}_j && \text{(C2)}\\
&= \bar{t}_j^{-1}\,\underline{\bar{\alpha}_{ij}\,\gamma_{ik}\,\beta_{jk}}\,\bar{t}_j && \text{(C2)}\\
&= \underline{\bar{t}_j^{-1}\,\gamma_{ik}\,\beta_{jk}}\,\bar{\alpha}_{ij}\,\bar{t}_j && \text{(C-}pt\text{) (C-}pt\text{)}\\
&= \underline{\gamma_{ik}\,\beta_{jk}\,p_{ij}}\,p_{ij}^{-1}\,\bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\bar{t}_j && \text{(C2)}\\
&= \beta_{jk}p_{ij}\,\gamma_{ik}\,p_{ij}^{-1}\,\bar{t}_j^{-1}\,\bar{\alpha}_{ij}\,\bar{t}_j\\
&= \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}$$

When $i < j < k$ and $(\alpha, \beta, \gamma) = (x, x, p)$ or $k < i < j$ and $(\alpha, \beta, \gamma) = (x, y, p)$ we have the following deduction with $\beta = x$ or y respectively.

$$\begin{aligned}
\Phi_{\sigma_q}(x_{ij}\,\beta_{ik}\,p_{jk}) &= t_j^{-1}\,y_{ij}\,t_j\,\underline{\beta_{jk}\,p_{ij}\,p_{ik}}\,p_{ij}^{-1} && \text{(C2)}\\
&= \underline{t_j^{-1}\,y_{ij}\,t_j p_{ij}}\,p_{ik}\,\beta_{jk}\,p_{ij}^{-1} && \text{(M-}y\text{)}\\
&= p_{ij}\,\underline{y_{ij}\,p_{ik}\,\beta_{jk}}\,p_{ij}^{-1} && \text{(C2)}\\
&= \underline{p_{ij}\,p_{ik}\,\beta_{jk}}\,y_{ij}\,p_{ij}^{-1} && \text{(C2)}\\
&= \beta_{jk}\,p_{ij}\,p_{ik}\,y_{ij}\,p_{ij}^{-1} && \text{(C-}pt\text{)}\\
&= \beta_{jk}\,p_{ij}\,p_{ik}\,p_{ij}^{-1}\,t_j^{-1}\,\underline{p_{ij}\,t_j\,y_{ij}}\,p_{ij}^{-1} && \text{(M-}y\text{)}\\
&= \beta_{jk}\,p_{ij}\,p_{ik}\,p_{ij}^{-1}\,t_j^{-1}\,y_{ij}\,\underline{p_{ij}\,t_j}\,p_{ij}^{-1} && \text{(C-}pt\text{)}\\
&= \beta_{jk}\,p_{ij}\,p_{ik}\,p_{ij}^{-1}\,t_j^{-1}\,y_{ij}\,t_j\\
&= \Phi_{\sigma_q}(\beta_{ik}\,p_{jk}\,x_{ij})
\end{aligned}$$

When $q = j - 1$ and $j \neq i + 1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= p_{j-1,j}\,\alpha_{i,j-1}\,\underline{p_{j-1,j}^{-1}\,\beta_{ik}\,p_{j-1,j}}\,\gamma_{j-1,k}\,p_{j-1,j}^{-1} && \text{(C1)}\\
&= p_{j-1,j}\,\underline{\alpha_{i,j-1}\,\beta_{ik}\,\gamma_{j-1,k}}\,p_{j-1,j}^{-1} && \text{(C2)}\\
&= \underline{p_{j-1,j}\,\beta_{ik}}\,\gamma_{j-1,k}\,\alpha_{i,j-1}\,p_{j-1,j}^{-1} && \text{(C1)}
\end{aligned}$$

$$
\begin{aligned}
&= \beta_{ik}\, p_{j-1,j}\, \gamma_{j-1,k}\, \alpha_{i,j-1}\, p_{j-1,j}^{-1} \\
&= \Phi_{\sigma_q}(\beta_{ik}\, \gamma_{jk}\, \alpha_{ij})
\end{aligned}
$$

When $q = j$ and $k = j+1$ we have two cases. Except for when $i < j < k$ and $(\alpha, \beta, \gamma) = (y, p, x)$ or $j < k < i$ and $(\alpha, \beta, \gamma) = (x, p, x)$ we have the following. Here

$$
\bar{\gamma}_{jk} = \begin{cases} p_{jk} & \text{if } \gamma = p \\ y_{jk} & \text{if } \gamma = x \\ x_{jk} & \text{if } \gamma = y \end{cases}
$$

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, \beta_{ik}\, \gamma_{jk}) &= \alpha_{ik}\, \underline{p_{jk}\beta_{ij}\, p_{jk}^{-1}}\, \bar{\gamma}_{jk} && \text{(C2)} \\
&= \alpha_{ik}\, p_{ik}^{-1}\, \underline{\beta_{ij}\, p_{ik}\, \bar{\gamma}_{jk}} && \text{(C2)} \\
&= \underline{\alpha_{ik}\, \bar{\gamma}_{jk}\, \beta_{ij}} && \text{(C2)} \\
&= \underline{\bar{\gamma}_{jk}\, \beta_{ij}}\, \alpha_{ik} && \text{(C2)} \\
&= \underline{p_{ik}^{-1}\, \beta_{ij}\, p_{ik}}\, \bar{\gamma}_{jk}\, \alpha_{ik} && \text{(C2)} \\
&= p_{jk}\, \beta_{ij}\, p_{jk}^{-1}\, \bar{\gamma}_{jk}\, \alpha_{ik} \\
&= \Phi_{\sigma_q}(\beta_{ik}\, \gamma_{jk}\, \alpha_{ij})
\end{aligned}
$$

When $i < j < k$ and $(\alpha, \beta, \gamma) = (y, p, x)$ or when $j < k < i$ and $(\alpha, \beta, \gamma) = (x, p, x)$ we have

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, p_{ik}\, x_{jk}) &= \alpha_{ik}\, p_{jk}\, p_{ij}\, \underline{p_{jk}^{-1}\, t_k^{-1}\, y_{jk}\, t_k} && \text{(M-}y\text{)} \\
&= \underline{\alpha_{ik}\, p_{jk}\, p_{ij}}\, y_{jk}\, p_{jk}^{-1} && \text{(C2)} \\
&= p_{jk}\, p_{ij}\, \underline{\alpha_{ik}\, y_{jk}}\, p_{jk}^{-1} && \text{(C2)} \\
&= p_{jk}\, p_{ij}\, y_{jk}\, \underline{p_{ij}\, \alpha_{ik}\, p_{ij}^{-1}\, p_{jk}^{-1}} && \text{(C2)} \\
&= p_{jk}\, p_{ij}\, \underline{y_{jk}\, p_{jk}^{-1}}\, \alpha_{ik} && \text{(M-}y\text{)}
\end{aligned}
$$

$$
\begin{aligned}
&= p_{jk}\, p_{ij}\, p_{jk}^{-1}\, t_k^{-1}\, y_{jk}\, t_k\, \alpha_{ik} \\
&= \Phi_{\sigma_q}(p_{ik}\, x_{jk}\, \alpha_{ij})
\end{aligned}
$$

When $q = k-1$ and $k \neq j+1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, \beta_{ik}\, \gamma_{jk}) &= \underline{\alpha_{ij}\, p_{k-1,k}}\, \beta_{i,k-1}\, \gamma_{j,k-1}\, p_{k-1,k}^{-1} && \text{(C1)} \\
&= p_{k-1,k}\, \underline{\alpha_{ij}\, \beta_{i,k-1}\, \gamma_{j,k-1}}\, p_{k-1,k}^{-1} && \text{(C2)} \\
&= p_{k-1,k}\, \beta_{i,k-1}\, \gamma_{j,k-1}\, \underline{\alpha_{ij}\, p_{k-1,k}^{-1}} && \text{(C1)} \\
&= p_{k-1,k}\, \beta_{i,k-1}\, \gamma_{j,k-1}\, p_{k-1,k}^{-1}\, \alpha_{ij} \\
&= \Phi_{\sigma_q}(\beta_{ik}\, \gamma_{jk}\, \alpha_{ij})
\end{aligned}
$$

Finally, when $q = k$ and $i = k+1$ we have the following two cases. If $\beta \neq x$ then we have the following. Here

$$
\bar{\beta}_{ik} = \begin{cases} p_{jk} & \text{if } \beta = p \\ y_{jk} & \text{if } \beta = x \end{cases}
$$

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, \beta_{ik}\, \gamma_{jk}) &= \underline{p_{ik}\, \alpha_{jk}\, p_{ik}^{-1}}\, \bar{\beta}_{ik}\, \gamma_{ij} && \text{(C2)} \\
&= \underline{p_{ij}^{-1}\, \alpha_{jk}\, p_{ij}\, \bar{\beta}_{ik}}\, \gamma_{ij} && \text{(C2)} \\
&= \bar{\beta}_{ik}\, \underline{\alpha_{jk}\, \gamma_{ij}} && \text{(C2)} \\
&= \bar{\beta}_{ik}\, \gamma_{ij}\, p_{ik}\, \alpha_{jk}\, p_{ik}^{-1} \\
&= \Phi_{\sigma_q}(\beta_{ik}\, \gamma_{jk}\, \alpha_{ij})
\end{aligned}
$$

And if $\beta = x$ then we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ij}\, x_{ik}\, \gamma_{jk}) &= p_{ik}\, \alpha_{jk}\, \underline{p_{ik}^{-1}\, t_i^{-1}}\, y_{ik}\, t_i\, \gamma_{ij} && \text{(C-}pt\text{)} \\
&= p_{ik}\, \alpha_{jk}\, \underline{t_i^{-1}\, p_{ik}^{-1}\, y_{ik}}\, t_i\, \gamma_{ij} && \text{(M-}y\text{)}
\end{aligned}
$$

$$
\begin{aligned}
&= p_{ik}\,\alpha_{jk}\,y_{ik}\,\underline{t_i^{-1}\,p_{ik}^{-1}\,t_i}\,\gamma_{ij} && \text{(C-}pt\text{)}\\
&= p_{ik}\,\alpha_{jk}\,y_{ik}\,\underline{p_{ik}^{-1}\,\gamma_{ij}} && \text{(C2)}\\
&= p_{ik}\,\alpha_{jk}\,\underline{y_{ik}\,p_{jk}\,\gamma_{ij}\,p_{jk}^{-1}}\,p_{ik}^{-1} && \text{(C2)}\\
&= p_{ik}\,\underline{\alpha_{jk}\,\gamma_{ij}\,y_{ik}}\,p_{ik}^{-1} && \text{(C2)}\\
&= p_{ik}\,\underline{\gamma_{ij}\,y_{ik}}\,\alpha_{jk}\,p_{ik}^{-1} && \text{(C2)}\\
&= p_{ik}\,y_{ik}\,\underline{p_{jk}\,\gamma_{ij}\,p_{jk}^{-1}}\,\alpha_{jk}\,p_{ik}^{-1} && \text{(C2)}\\
&= p_{ik}\,y_{ik}\,\underline{p_{ik}^{-1}}\,\gamma_{ij}\,p_{ik}\,\alpha_{jk}\,p_{ik}^{-1} && \text{(C-}pt\text{)}\\
&= p_{ik}\,\underline{y_{ik}\,t_i^{-1}\,p_{ik}^{-1}}\,t_i\,\gamma_{ij}\,p_{ik}\,\alpha_{jk}\,p_{ik}^{-1} && \text{(M-}y\text{)}\\
&= \underline{p_{ik}\,t_i^{-1}\,p_{ik}^{-1}}\,y_{ik}\,t_i\,\gamma_{ij}\,p_{ik}\,\alpha_{jk}\,p_{ik}^{-1} && \text{(C-}pt\text{)}\\
&= t_i^{-1}\,y_{ik}\,t_i\,\gamma_{ij}\,p_{ik}\,\alpha_{jk}\,p_{ik}^{-1}\\
&= \Phi_{\sigma_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}
$$

Now consider Φ_{τ_q}, the non-trivial cases are as follows.

$q=i$	$i<j<k$	(x,p,p)	(x,y,y)	(x,x,p)
	$j<k<i$	(y,p,p)	(y,x,y)	(y,y,p)
	$k<i<j$	(x,p,p)	(x,x,x)	(x,y,p)
$q=j$	$i<j<k$	(y,p,p)	(y,y,y)	(y,p,x)
	$j<k<i$	(x,p,p)	(x,x,y)	(x,p,x)
	$k<i<j$	(y,p,p)	(y,x,x)	(y,p,y)
$q=k$	$i<j<k$	(p,y,y)	(x,y,y)	(y,y,y)
	$j<k<i$	(p,x,y)	(x,x,y)	(y,x,y)
	$k<i<j$	(p,x,x)	(x,x,x)	(y,x,x)

For the first two columns of the cases $q=i$ and $q=j$ we have the following.

$$
\begin{aligned}
\Phi_{\tau_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \alpha_{ij}^{-1}\,\underline{p_{ij}\,\beta_{ik}\,\gamma_{jk}} && \text{(C2)}\\
&= \underline{\alpha_{ij}^{-1}\,\beta_{ik}\,\gamma_{jk}}\,p_{ij} && \text{(C2)}\\
&= \beta_{ik}\,\gamma_{jk}\,\alpha_{ij}^{-1}\,p_{ij}\\
&= \Phi_{\tau_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}
$$

For the third column in the case $q = i$ we have the following.

$$
\begin{aligned}
\Phi_{\tau_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \alpha_{ij}^{-1}\,p_{ij}\,\beta_{ik}^{-1}\,\underline{p_{ik}\,\gamma_{jk}} && \text{(C2)}\\
&= \alpha_{ij}^{-1}\,\underline{p_{ij}\,\beta_{ik}^{-1}\,p_{ij}^{-1}}\,p_{ik}\,\gamma_{jk}\,p_{ij} && \text{(C2)}\\
&= \underline{\alpha_{ij}^{-1}\,p_{jk}^{-1}\,\beta_{ik}^{-1}\,p_{jk}}\,p_{ik}\,\gamma_{jk}\,p_{ij} && \text{(C2)}\\
&= \beta_{ik}^{-1}\,\underline{\alpha_{ij}^{-1}\,p_{ik}}\,\gamma_{jk}\,p_{ij} && \text{(C2)}\\
&= \beta_{ik}^{-1}\,p_{ik}\,\gamma_{jk}\,\alpha_{ij}^{-1}\,p_{ij}\\
&= \Phi_{\tau_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}
$$

For the third column in the case $q = j$ we have the following.

$$
\begin{aligned}
\Phi_{\tau_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \alpha_{ij}^{-1}\,\underline{p_{ij}\,\beta_{ik}\,\gamma_{jk}^{-1}}\,p_{jk} && \text{(C2)}\\
&= \alpha_{ij}^{-1}\,\gamma_{jk}^{-1}\,\underline{p_{ij}\,\beta_{ik}\,p_{jk}} && \text{(C2)}\\
&= \underline{\alpha_{ij}^{-1}\,\gamma_{jk}^{-1}}\,\beta_{ik}\,p_{jk}\,p_{ij} && \text{(C2)}\\
&= \beta_{ik}\,\gamma_{jk}^{-1}\,\underline{\beta_{ik}^{-1}\,\alpha_{ij}^{-1}\,\beta_{ik}\,p_{jk}}\,p_{ij} && \text{(C2)}\\
&= \beta_{ik}\,\gamma_{jk}^{-1}\,p_{jk}\,\alpha_{ij}^{-1}\,p_{ij}\\
&= \Phi_{\tau_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}
$$

For the case when $q = k$ we have the following.

$$
\begin{aligned}
\Phi_{\tau_q}(\alpha_{ij}\,\beta_{ik}\,\gamma_{jk}) &= \alpha_{ij}\,\beta_{ik}^{-1}\,\underline{p_{ik}\,\gamma_{jk}^{-1}}\,p_{jk} && \text{(C2)}\\
&= \alpha_{ij}\,\underline{\beta_{ik}^{-1}\,p_{ij}^{-1}\,\gamma_{jk}^{-1}\,p_{ij}}\,p_{ik}\,p_{jk} && \text{(C2)}\\
&= \underline{\alpha_{ij}\,\gamma_{jk}^{-1}\,\beta_{ik}^{-1}}\,p_{ik}\,p_{jk} && \text{(C2)}
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma_{jk}^{-1}\,\beta_{ik}^{-1}\,\underline{\alpha_{ij}\,p_{ik}\,p_{jk}} && \text{(C2)}\\
&= \underline{\gamma_{jk}^{-1}\,\beta_{ik}^{-1}}\,p_{ik}\,p_{jk}\,\alpha_{ij} && \text{(C2)}\\
&= \beta_{ik}^{-1}\,\underline{p_{ij}^{-1}\,\gamma_{jk}^{-1}\,p_{ij}\,p_{ik}}\,p_{jk}\,\alpha_{ij} && \text{(C2)}\\
&= \beta_{ik}^{-1}\,p_{ik}\,\gamma_{jk}^{-1}\,p_{jk}\,\alpha_{ij}\\
&= \Phi_{\tau_q}(\beta_{ik}\,\gamma_{jk}\,\alpha_{ij})
\end{aligned}
$$

(C3) $\qquad \alpha_{ik}\;p_{jk}\,\beta_{jl}\,p_{jk}^{-1} = p_{jk}\,\beta_{jl}\,p_{jk}^{-1}\;\alpha_{ik} \qquad (i,j,k,l)$ cyclically ordered

First consider Φ_{σ_q}. As before the only non-trivial cases are when $q = i-1$ and $i \neq l+1$, $q = i$ and $j = i+1$, $q = j-1$ and $j \neq i+1$, $q = j$ and $k = j+1$, $q = k-1$ and $k \neq j+1$, $q = k$ and $l = k+1$, $p = l-1$ and $l \neq k+1$, and $p = l$ and $i = l+1$.

When $q = i-1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}) &= p_{i-1,i}\,\alpha_{i-1,k}\,\underline{p_{i-1,i}^{-1}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}} && \text{(C1)(C1)(C1)}\\
&= p_{i-1,i}\,\underline{\alpha_{i-1,k}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}}\,p_{i-1,i}^{-1} && \text{(C3)}\\
&= \underline{p_{i-1,i}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}}\,\alpha_{i-1,k}\,p_{i-1,i}^{-1} && \text{(C1)(C1)(C1)}\\
&= p_{jk}\,\beta_{jl}\,p_{jk}^{-1}\,p_{i-1,i}\,\alpha_{i-1,k}\,p_{i-1,i}^{-1}\\
&= \Phi_{\sigma_q}(p_{jk}\,\beta_{jl}\,p_{jk}^{-1}\,\alpha_{ik})
\end{aligned}
$$

When $q = i$ and $j = i+1$ we have the following. (Here the (C2)s hold because we are in either of the bottom two rows of Table 3.1, both of which contain (α, p, p) for $\alpha = p$, x, and y.)

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\,p_{jk}\,\beta_{jl}\,p_{jk}^{-1}) &= \underline{\alpha_{jk}\,p_{ij}\,p_{ik}}\,\beta_{il}\,p_{ik}^{-1}\,p_{ij}^{-1} && \text{(C2)}\\
&= p_{ij}\,p_{ik}\,\underline{\alpha_{jk}\,\beta_{il}}\,p_{ik}^{-1}\,p_{ij}^{-1} && \text{(C1)}\\
&= p_{ij}\,p_{ik}\,\beta_{il}\,\underline{\alpha_{jk}\,p_{ik}^{-1}\,p_{ij}^{-1}} && \text{(C2)}
\end{aligned}
$$

$$
\begin{aligned}
&= p_{ij}\, p_{ik}\, \beta_{il}\, p_{ik}^{-1}\, p_{ij}^{-1}\, \alpha_{jk} \\
&= \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}
$$

When $q = j-1$ and $j \neq i+1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \underline{\alpha_{ik}\, p_{j-1,j}}\, p_{j-1,k}\, \beta_{j-1,l}\, p_{j-1,k}^{-1}\, p_{j-1,j}^{-1} && \text{(C1)} \\
&= p_{j-1,j}\, \underline{\alpha_{ik}\, p_{j-1,k}\, \beta_{j-1,l}\, p_{j-1,k}^{-1}}\, p_{j-1,j}^{-1} && \text{(C3)} \\
&= p_{j-1,j}\, p_{j-1,k}\, \beta_{j-1,l}\, p_{j-1,k}^{-1}\, \underline{\alpha_{ik}\, p_{j-1,j}^{-1}} && \text{(C1)} \\
&= p_{j-1,j}\, p_{j-1,k}\, \beta_{j-1,l}\, p_{j-1,k}^{-1}\, p_{j-1,j}^{-1}\, \alpha_{ik} \\
&= \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}
$$

When $q = j$ and $k = j+1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= p_{jk}\, \underline{\alpha_{ij}\, \beta_{kl}}\, p_{jk}^{-1} && \text{(C1)} \\
&= p_{jk}\, \beta_{kl}\, p_{jk}^{-1}\, p_{jk}\, \alpha_{ij}\, p_{jk}^{-1} \\
&= \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}
$$

When $q = k-1$ and $k \neq j+1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= p_{k-1,k}\, \alpha_{i,k-1}\, p_{j,k-1}\, \underline{p_{k-1,k}^{-1}\, \beta_{jl}}\, p_{k-1,k}\, p_{j,k-1}^{-1}\, p_{k-1,k}^{-1} && \text{(C1)} \\
&= p_{k-1,k}\, \underline{\alpha_{i,k-1}\, p_{j,k-1}\, \beta_{jl}\, p_{j,k-1}^{-1}}\, p_{k-1,k}^{-1} && \text{(C3)} \\
&= p_{k-1,k}\, p_{j,k-1}\, \underline{\beta_{jl}}\, p_{j,k-1}^{-1}\, \alpha_{i,k-1}\, p_{k-1,k}^{-1} && \text{(C1)} \\
&= p_{k-1,k}\, p_{j,k-1}\, p_{k-1,k}^{-1}\, \beta_{jl}\, p_{k-1,k}\, p_{j,k-1}^{-1}\, \alpha_{i,k-1}\, p_{k-1,k}^{-1} \\
&= \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik})
\end{aligned}
$$

When $q = k$ and $l = k+1$ we have the following. (Here the (C2)s hold because we are in either of the top two rows of Table 3.1, both of which contain (β, p, p) for $\beta = p$, x, and y.)

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \alpha_{il}\, \underline{p_{jl}\, p_{kl}\, \beta_{jk}\, p_{kl}^{-1}\, p_{jl}^{-1}} && \text{(C2)}\\
&= \underline{\alpha_{il}\, \beta_{jk}} && \text{(C1)}\\
&= \underline{\beta_{jk}}\, \alpha_{il} && \text{(C2)}\\
&= p_{jl}\, p_{kl}\, \beta_{jk}\, p_{kl}^{-1}\, p_{jl}^{-1}\, \alpha_{il} &&\\
&= \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik}) &&
\end{aligned}$$

When $q = l - 1$ and $l \neq k + 1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \underline{\alpha_{ik}\, p_{jk}\, p_{l,l-1}}\, \beta_{j,l-1}\, \underline{p_{l,l-1}^{-1}\, p_{jk}^{-1}} && \text{(C1)(C1)(C1)}\\
&= p_{l,l-1}\, \underline{\alpha_{ik}\, p_{jk}\, \beta_{j,l-1}\, p_{jk}^{-1}}\, p_{l,l-1}^{-1} && \text{(C3)}\\
&= \underline{p_{l,l-1}\, p_{jk}}\, \beta_{j,l-1}\, \underline{p_{jk}^{-1}\, \alpha_{ik}\, p_{l,l-1}^{-1}} && \text{(C1)(C1)(C1)}\\
&= p_{jk}\, p_{l,l-1}\, \beta_{j,l-1}\, p_{l,l-1}^{-1}\, p_{jk}^{-1}\, \alpha_{ik} &&\\
&= \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik}) &&
\end{aligned}$$

Finally, when $q = l$ and $i = l + 1$ we have the following. (Here the (C2)s hold because they always hold for the triples (α, p, p) and (β, p, p).)

$$\begin{aligned}
\Phi_{\sigma_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \underline{p_{il}\, \alpha_{kl}\, p_{il}^{-1}}\, \underline{p_{kj}\, \beta_{ij}\, p_{jk}^{-1}} && \text{(C2)(C2)}\\
&= p_{ik}^{-1}\, \underline{\alpha_{kl}\, \beta_{ij}}\, p_{ik} && \text{(C1)}\\
&= \underline{p_{ik}^{-1}\, \beta_{ij}\, p_{ik}}\, \underline{p_{ik}^{-1}\, \alpha_{kl}\, p_{ik}} && \text{(C2)(C2)}\\
&= p_{jk}\, \beta_{ij}\, p_{jk}^{-1}\, p_{kl}\, \alpha_{kl}\, p_{kl}^{-1} &&\\
&= \Phi_{\sigma_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik}) &&
\end{aligned}$$

Now consider Φ_{τ_q}, there are two non-trivial cases. In the first case $\Phi_{\tau_q}(\alpha_{ik}) = \alpha_{ik}^{-1} p_{ik}$ and we have the following.

$$\begin{aligned}
\Phi_{\tau_q}(\alpha_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \alpha_{ik}^{-1}\, \underline{p_{ik}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}} && \text{(C3)}\\
&= \underline{\alpha_{ik}^{-1}\, p_{jk}\, \beta_{jl}\, p_{jk}^{-1}}\, p_{ik} && \text{(C3)}
\end{aligned}$$

$$
\begin{aligned}
&= p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\, \alpha_{ik}^{-1}\, p_{ik} \\
&= \Phi_{\tau_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\ \alpha_{ik})
\end{aligned}
$$

In the second case $\Phi_{\tau_q}(\beta_{jl}) = \beta_{jl}^{-1} p_{jl}$ and we have the following.

$$
\begin{aligned}
\Phi_{\tau_q}(\alpha_{ik}\ p_{jk}\, \beta_{jl}\, p_{jk}^{-1}) &= \underline{\alpha_{ik}\, p_{jk}\, \beta_{jl}^{-1}}\, p_{jl}\, p_{jk}^{-1} && \text{(C3)} \\
&= p_{jk}\, \beta_{jl}^{-1}\, \underline{p_{jk}^{-1}\, \alpha_{ik}\, p_{jk}\, p_{jl}\, p_{jk}^{-1}} && \text{(C3)} \\
&= p_{jk}\, \beta_{jl}^{-1}\, p_{jl}\, p_{jk}^{-1}\, \alpha_{ik} \\
&= \Phi_{\tau_q}(p_{jk}\, \beta_{jl}\, p_{jk}^{-1}\ \alpha_{ik})
\end{aligned}
$$

(M-x) $$x_{ij}\, p_{ij}\, t_i = p_{ij}\, t_i\, x_{ij} \qquad i < j$$

First consider Φ_{σ_q}. The only non-trivial cases are when $q = i-1$, $q = i$ and $j = i+1$, and $q = j-1$ and $j \neq i+1$.

When $q = i-1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(x_{ij}\, p_{ij}\, t_i) &= p_{i-1,i}\, x_{i-1,j}\, p_{i-1,j}\, \underline{p_{i-1,i}^{-1}\, t_{i-1}} && \text{(C-}pt\text{)} \\
&= p_{i-1,i}\, \underline{x_{i-1,j}\, p_{i-1,j}\, t_{i-1}}\, p_{i-1,i}^{-1} && \text{(M-}x\text{)} \\
&= p_{i-1,i}\, p_{i-1,j}\, \underline{t_{i-1}}\, x_{i-1,j}\, p_{i-1,i}^{-1} && \text{(C-}pt\text{)} \\
&= p_{i-1,i}\, p_{i-1,j}\, p_{i-1,i}^{-1}\, t_{i-1}\, p_{i-1,i}\, x_{i-1,j}\, p_{i-1,i}^{-1} \\
&= \Phi_{\sigma_q}(p_{ij}\, t_i\, x_{ij})
\end{aligned}
$$

When $q = i$ and $j = i+1$ we have the following.

$$
\begin{aligned}
\Phi_{\sigma_q}(x_{ij}\, p_{ij}\, t_i) &= t_j^{-1}\, y_{ij}\, \underline{t_j\, p_{ij}}\, t_j && \text{(C-}pt\text{)} \\
&= t_j^{-1}\, \underline{y_{ij}\, p_{ij}\, t_j}\, t_j && \text{(M-}y\text{)} \\
&= \underline{t_j^{-1}\, p_{ij}\, t_j}\, y_{ij}\, t_j && \text{(C-}pt\text{)} \\
&= p_{ij}\, y_{ij}\, t_j
\end{aligned}
$$

$$= \Phi_{\sigma_q}(p_{ij}\, t_i\, x_{ij})$$

When $q = j-1$ and $j \neq i+1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(x_{ij}\, p_{ij}\, t_i) &= p_{j-1,j}\, x_{i,j-1}\, p_{i,j-1}\, \underline{p_{j-1,j}^{-1}\, t_i} && \text{(C-}pt\text{)}\\
&= p_{j-1,j}\, \underline{x_{i,j-1}\, p_{i,j-1}\, t_i}\, p_{j-1,j}^{-1} && \text{(M-}x\text{)}\\
&= p_{j-1,j}\, p_{i,j-1}\, \underline{t_i}\, x_{i,j-1}\, p_{j-1,j}^{-1} && \text{(C-}pt\text{)}\\
&= p_{j-1,j}\, p_{i,j-1}\, p_{j-1,j}^{-1}\, t_i\, p_{j-1,j}\, x_{i,j-1}\, p_{j-1,j}^{-1}\\
&= \Phi_{\sigma_q}(p_{ij}\, t_i\, x_{ij})
\end{aligned}$$

Now consider Φ_{τ_q}, the only non-trivial case is when $q = i$.

$$\begin{aligned}
\Phi_{\tau_q}(x_{ij}\, p_{ij}\, t_i) &= x_{ij}^{-1}\, p_{ij}\, \underline{p_{ij}\, t_i} && \text{(C-}pt\text{)}\\
&= \underline{x_{ij}^{-1}\, p_{ij}\, t_i}\, p_{ij} && \text{(M-}y\text{)}\\
&= p_{ij}\, t_i\, x_{ij}^{-1}\, p_{ij}\\
&= \Phi_{\tau_q}(p_{ij}\, t_i\, x_{ij})
\end{aligned}$$

(M-y) $$y_{ij}\, p_{ij}\, t_j = p_{ij}\, t_j\, y_{ij} \qquad i < j$$

First consider Φ_{σ_q}. The only non-trivial cases are when $q = i-1$, $q = i$ and $j = i+1$, and $q = j-1$ and $j \neq i+1$.

When $q = i-1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(y_{ij}\, p_{ij}\, t_j) &= p_{i-1,i}\, y_{i-1,j}\, p_{i-1,j}\, \underline{p_{i-1,i}^{-1}\, t_j} && \text{(C-}pt\text{)}\\
&= p_{i-1,i}\, \underline{y_{i-1,j}\, p_{i-1,j}\, t_j}\, p_{i-1,i}^{-1} && \text{(M-}y\text{)}\\
&= p_{i-1,i}\, p_{i-1,j}\, \underline{t_j}\, y_{i-1,j}\, p_{i-1,i}^{-1} && \text{(C-}pt\text{)}\\
&= p_{i-1,i}\, p_{i-1,j}\, p_{i-1,i}^{-1}\, t_j\, p_{i-1,i}\, y_{i-1,j}\, p_{i-1,i}^{-1}\\
&= \Phi_{\sigma_q}(p_{ij}\, t_j\, y_{ij})
\end{aligned}$$

When $q = i$ and $j = i + 1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(y_{ij}\, p_{ij}\, t_j) &= \underline{x_{ij}\, p_{ij}\, t_i} && \text{(M-}x\text{)}\\
&= p_{ij}\, t_i\, x_{ij}\\
&= \Phi_{\sigma_q}(p_{ij}\, t_j\, y_{ij})
\end{aligned}$$

When $q = j - 1$ and $j \neq i + 1$ we have the following.

$$\begin{aligned}
\Phi_{\sigma_q}(y_{ij}\, p_{ij}\, t_j) &= p_{j-1,j}\, y_{i,j-1}\, p_{i,j-1}\, \underline{p_{j-1,j}^{-1}\, t_{j-1}} && \text{(C-}pt\text{)}\\
&= p_{j-1,j}\, \underline{y_{i,j-1}\, p_{i,j-1}\, t_{j-1}}\, p_{j-1,j}^{-1} && \text{(M-}y\text{)}\\
&= p_{j-1,j}\, p_{i,j-1}\, \underline{t_{j-1}}\, y_{i,j-1}\, p_{j-1,j}^{-1} && \text{(C-}pt\text{)}\\
&= p_{j-1,j}\, p_{i,j-1}\, p_{j-1,j}^{-1}\, t_{j-1}\, p_{j-1,j}\, y_{i,j-1}\, p_{j-1,j}^{-1}\\
&= \Phi_{\sigma_q}(p_{ij}\, t_j\, y_{ij})
\end{aligned}$$

Now consider Φ_{τ_q}, the only non-trivial case is when $q = j$.

$$\begin{aligned}
\Phi_{\tau_q}(y_{ij}\, p_{ij}\, t_j) &= y_{ij}^{-1}\, p_{ij}\, \underline{p_{ij}\, t_j} && \text{(C-}pt\text{)}\\
&= \underline{y_{ij}^{-1}\, p_{ij}\, t_j}\, p_{ij} && \text{(M-}y\text{)}\\
&= p_{ij}\, t_j\, y_{ij}^{-1}\, p_{ij}\\
&= \Phi_{\sigma_q}(p_{ij}\, t_j\, y_{ij})
\end{aligned}$$

All that remains is to check that R is closed under $\Phi_{\tau_q^{-1}}$ and $\Phi_{\sigma_q^{-1}}$. From the expressions for the inverses given in the proof of Proposition 6 it follows that for every word x we have $\Phi_{\sigma_q^{-2}}(x) =_R p_{q,q+1}^{-1}\, x\, p_{q,q+1}$ and $\Phi_{\tau_q^{-2}}(x) =_R t_q^{-1}\, x\, t_q$. Therefore whenever $x =_R y$ we have

$$\Phi_{\sigma_q^{-2}}(x) \;=_R\; p_{q,q+1}^{-1}\, x\, p_{q,q+1} \;=_R\; p_{q,q+1}^{-1}\, y\, p_{q,q+1} \;=_R\; \Phi_{\sigma_q^{-2}}(y)$$

and

$$\Phi_{\tau_q^{-2}}(x) \;=_R\; t_q^{-1}\, x\, t_q \;=_R\; t_q^{-1}\, y\, t_q \;=_R\; \Phi_{\tau_q^{-2}}(y)$$

hence $\Phi_{\sigma_q^{-1}}(x) =_R \Phi_{\sigma_q^{-1}}(y)$ and $\Phi_{\tau_q^{-1}}(x) =_R \Phi_{\tau_q^{-1}}(y)$. $\square$

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