POST-SALE COST MODELING AND OPTIMIZATION LINKING WARRANTY AND PREVENTIVE MAINTENANCE

WU JUN
(B. ENG, TSINGHUA UNIVERSITY)

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SUMMARY

This thesis investigates several important issues in post-sale cost modeling and optimization. The costing of a maintenance program is analyzed under the warranty context from both the manufacturer's and consumer's point of view. In particular, for the manufacturer, we study the issue of warranty cost analysis where preventive maintenance (PM) is an important planning tool in terms of service improvement and warranty cost reduction. For the consumer, we investigate the PM scheduling problem during the product life cycle where warranty is an important factor in influencing the maintenance decisions. In addition to costing analysis, warranty as an effective marketing instrument for enhancing the revenue is also discussed.

In the modeling of warranty expense for complex systems, the majority of researchers presume the system as a "black box" which does not utilize the information of inner structure. Chapter 3 studies two basic system structures, i.e. series and parallel, and derives the respective warranty cost functions under renewable warranty policies. Unique in this study is the incorporation of failure dependence factor between each two of the system components. We investigate the impact of such factor on the total warranty expense and the risk of ignoring it.

Manufacturers usually rely on the age information of a product as the single metric for maintenance design under warranty. Such policy may result in unnecessary maintenance operations for products with inadequate deteriorations. In Chapter 4, we propose such a condition-based warranty policy that counts on the product state information for executing maintenance decisions. We derive the warranty cost

functions under both renewable and non-renewable warranty policies, based on which the optimal scheduling of inspection services is further analyzed.

For the owners of industrial equipments, investing in maintenance is widely modeled from the costing perspective while the value or return of maintenance investments is seldom emphasized. Motivated by this, Chapter 5 investigates the optimal design of maintenance servicing on revenue-generating equipments by integrating both the cost and value aspects of maintenance. The study assumes imperfect maintenance operations and generalizes the existing periodic PM models. The influence of warranty as well as many other models parameters on the optimal PM decisions is illustrated.

In the design of maintenance policies, two major assumptions are commonly adopted: binary system (i.e. functioning or failed) and infinite planning horizon. In contrast, almost all systems are operating under finite life times and many of them exhibit multiple performance levels. Therefore, in Chapter 6, we investigate the repair-replacement policies for multi-state systems (MSS) under finite life cycles. Corrective and preventive replacement decisions are modeled and compared via two control parameters – a threshold on the current system state and a threshold on the residual life cycle. Value of time is taken into account for the maintenance cost modeling. Extension is further made to generalize the cost functions under the warranty context.

Warranty as an effective product marketing tool has been extensively studied in the literature. Majority of the researchers focus on the joint determination of selling price and warranty length that maximizes the seller's profit (rate) function. In Chapter 7, we further extend this topic by integrating the factor of the buyer's age-replacement decisions into the design phase of the seller's product marketing strategy. For such

integration to be viable, a game theoretic model is formulated that allows the seller to foresee the buyer's maintenance decisions and subsequently make his own moves.

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LIST OF SYMBOLS

ABAO	as bad as old
AGAN	as good as new
cdf	cumulative distribution function
CM	corrective maintenance
CSW	cost-sharing warranty
CTMC	continuous-time Markov chain
DWC	discounted warranty cost
EDMC	expected discounted maintenance cost
EWC	expected warranty cost
FDPM	failure dependence probability matrix
FIPM	failure independence probability matrix
FRW	free-replacement warranty
i.i.d.	identically and independently distributed
IFR	increasing failure rate
IM	imperfect maintenance
LST	Laplace transform
NHPP	non-homogeneous Poisson process
pdf	probability density function
pmf	probability mass function
PM	preventive maintenance
PRW	pro-rata warranty
RFRW	renewing free-replacement warranty
RPRW	renewing pro-rata warranty
SF	survival function
vs.	versus

w	warranty period
L	system life cycle
$l_{\bar{w}}$	$\max\{0, L - w\}$, post-warranty period
u	system effective age
CW	warranty cost
p_{ij}	probability that a natural failure of component i causes an induced failure of component j
T_S	total time under renewing warranty, or warranty cycle for short
N_i	number of natural failures of component i per warranty cycle T_S
N_{ij}	number of induced failures of component j caused by the natural failures of component i per warranty cycle T_S; $N_{ii} = N_{ij}$
$p_i(w)$	probability that a natural failure of component i occurs within warranty
$\alpha_i(w)$	probability that the breakdown of a series system within warranty is incurred by a natural failure of component i
com_i	average replacement cost for component i
c_M	system maintenance cost
R_S	system reliability function
X	time to the defect arrival
Y	delay time of a failure since the defect arrival
Z	$X + Y$, time to failure without PM intervention
K	time to failure considering PM intervention
F_X, F_Y, F_Z	cdf of X, Y, Z
M_Z	renewal function for Z
T	time to first inspection or inspection interval
C_I	average cost for inspection
C_P	average cost for preventive replacement
C_F	average cost for failure or corrective replacement
c_r	average cost for minimal repair
$c_p(m)$	imperfect PM cost as a function of PM degree m

$\varphi(m)$	age-reduction factor as a function of PM degree m
$\hat{S}(u)$	ageing loss rate as a function of the system effective age u
C_L	life-cycle maintenance cost to the buyer excluding the ageing losses
C_T	life-cycle cost to the buyer including the ageing losses
δ	continuous discounted factor
m_i, d_i	minimal repair and downtime cost for Possion failures during the ith-stage degradation for MSS
α_i	transition rate during the ith-stage degradation for MSS
λ_i	failure rate during the ith-stage degradation for MSS
η_i	$\alpha_i + \lambda_i + \delta$
r_i	replacement cost for MSS during the ith-stage degradation
(J, τ)	maintenance thresholds on the system state and the residual life cycle
$C_i^{(O)}(t)$	EDMC for Policy O when the system is in operational state $2i - 1$ and the residual life cycle is t
$C_i^{(A)}(t\|J, \tau)$	EDMC for Policy A under maintenance thresholds (J, τ) when the system is in operational state $2i - 1$ and the residual life cycle is t
$C_i^{(B)}(t\|J, \tau)$	EDMC for Policy B under maintenance thresholds (J, τ) when the system is in operational state $2i - 1$ and the residual life cycle is t
C_p	purchase price or cost to the buyer
C_d	downtime cost per failure to the buyer
$Q(C_p, w)$	product demand as a function of purchase price and warranty length
$r(.)$	failure rate function
CR	long-run average cost rate to the buyer per item bought
PR	long-run average profit rate to the seller per time sold
π_1	total sales profit to the seller for all product sold
ρ	warranty cost-sharing ratio
$\phi(\rho)$	discounted factor of warranty cost-sharing ratio ρ on the product demand

CHAPTER 1 INTRODUCTION

In this thesis, we investigates several important issues in the post-sale cost modeling and optimization from both the manufacturer's (seller) and consumer's perspective (buyer) by considering a comprehensive set of cost factors, such as characteristics of warranty and preventive maintenance policies, maintenance cost structure, system structure and dependence, product ageing mechanism, value of time, product demand function, and different maintenance planning horizons. Warranty and preventive maintenance (PM) are modeled interactively to either enhance the seller's warranty servicing strategy or reduce the buyer's life-cycle maintenance cost.

Chapter 1 introduces the roles and various types of product warranty, presents the classification of maintenance, illustrates the maintenance cost factors and elaborates the scope and objective of this research. An outline of the thesis is provided thereafter.

1.1 Product Warranty

Almost all products are released in the market with certain forms of warranty. A warranty is a contractual agreement that requires the manufacturer to rectify all the failures incurred within the warranty period. A validated warranty contract should contain at least three attributes: the length of warranty coverage (fixed or random), the methods for compensation, and the conditions under which such compensations would be materialized. The last attribute is closely related to the legal aspect of warranty execution, while the first two are often used to differentiate or categorize the warranties.

1.1.1 Role of Warranty

Warranty serves different roles for manufacturers and consumers. From the manufacturer's perspective, the main role of a warranty is to promote sales. In other words, warranties are mainly used by manufacturers for marketing purposes. Consumers usually view better warranty terms as signals for better product quality and this will have positive influence on their purchase decisions. On the other hand, warranty also serves as an important protection tool for manufacturers in terms of reducing disputes with consumers upon product failures. It elucidates the consumer's obligations for care and maintenance of the product and therefore insures against excessive warranty claims caused by inappropriate use of the product.

From the consumer's perspective, the main role of a warranty is to provide protection against premature failures. By clearly defining the consumer's rights for warranty execution, a warranty provides a means of rectification for failures under normal usage conditions. Also warranty is informative in the sense that it serves as an indicator for product performance and reliability. This typically makes sense in those highly competitive markets where products have become much more complex and less easily evaluated by the users.

Some researchers also report the role of warranty from an investing perspective (Priest, 1981). In general, warranty as an integral part of the sale is factored into the product price. Insightful customers can often view such additional payments at purchase as investments confining future financial losses due to failures. On the other hand, through offering warranty services, manufacturers may successfully establish a long-term relationship with customers, and thereby retain their business even after the warranty has expired (such as by selling extended warranties).

1.1.2 Warranty Policies

Numerous types of warranty policies have been proposed within the last few decades. A simple but relatively complete taxonomy of warranty policies can be found in Blischke and Murthy (1992). To serve our purpose, we categorize warranty policies according to three attributes: renewability, methods for warranty compensation and the dimensionality.

Renewable and Non-Renewable Warranties

Warranty policies can be classified as either renewable or non-renewable. For renewable warranties, whenever a product fails within the warranty period under normal usage conditions, compensation is made to the buyer with an identical warranty provided. The same process is repeated until no failure has incurred in the prescribed warranty length. Thus, the total time under warranty (or warranty cycle for short) is a random variable and jointly determined by the warranty length and product failure mechanism. On the other hand, majority of warranties are non-renewable in the sense that it will automatically expire after a fixed length of usage. One should notice that offering renewable warranties could be costly to the sellers. Therefore, it is mainly provided to certain inexpensive and non-repairable products such as microwaves, coffee grinders and so forth. However, as product features among comparable models of competing brands become more and more indistinguishable, one would expect to observe a growing interest in renewable warranties for marketing high-priced products.

Free-Replacement and Pro-Rata Warranties

Warranty policies can be grouped by their methods of compensation specified in the warranty contracts. A free-replacement warranty (FRW) requires the manufacturer to be fully responsible for the product failures within the warranty period, and fix the

problems (either by repair or replacement) without extra charge to customers. FRW has been offered on a wide range of repairable and irreparable products such as automobiles, home appliances and expensive electronic components. It is considered as favorable to customers since manufacturers have to bear all the warranty cost risk. A pro-rata warranty (PRW), on the other hand, indicates a prorated cost-sharing scheme between the manufacturers and customers. Upon premature failures, the customer can choose either renew the product at a reduced price or receive a cash rebate from the manufacturer. The renewing cost or the amount of rebate is not fixed and usually depends on the age of the product. Many relatively inexpensive products are sold under this policy, such as automobile tires, batteries and so forth. Also, FRW and PRW can be combined. For example, under a one-year warranty contract, the manufacturer can provide free-replacement services upon failure during the first two months and apply the pro-rata policies during the remaining period.

Both FRW and PRW (and combination FRW/PRW) can be further extended to renewable warranty settings. They are mostly known as renewing free-replacement warranties (RFRW) and renewing pro-rata warranties (RPRW) in the literature. These four types of warranty policies serve as a foundation for many variants of post-sale service plans offered in the market, which subsequently provide abundant marketing solutions for the seller.

One-Dimensional and Two-Dimensional Warranties

Warranty policies can be grouped into either one-dimensional or two-dimensional. Most warranties are one-dimensional for which the warranty terms are based on the product age or usage, but not both. In comparison, two-dimensional warranties, which rely on both the product age and usage, are mainly offered in auto industries. For

example, typical warranty terms for a new warranted car include a 36-month age limit and a 36000-mile mileage limit, with certain parts (such as power train) under prolonged warranty protection. Since two-dimensional warranties would expire when either the age or the usage exceeds the limit, it is considered to be more advantageous to manufacturers compared with one-dimensional warranties. One should notice that for many products, measuring product usage (or other replaceable attribute) is difficult to the sellers, and the administration cost for executing such warranties could be very high. Therefore, two-dimensional warranties are not favored in many industries. This thesis will only cover the study of one-dimensional warranties.

1.2 Maintenance

1.2.1 Classification on Maintenance

Maintenance is the set of all technical and administrative actions intended to maintain a system in or restore it to a state in which it can perform its required functions (Dekker, 1996). According to Wang (2002), a maintenance activity can be categorized along three dimensions: the type of maintenance, the degree of maintenance and the type of system to be maintained.

Corrective and Preventive Maintenance

Maintenance can be categorized into two major types: corrective maintenance (CM) and preventive maintenance (PM). Corrective maintenance (CM) actions are unscheduled actions intended to restore a system from a failed state to an operational state. This involves either repair or replacement of failed components. In contrast, preventive maintenance (PM) actions are scheduled actions carried out to either reduce the likelihood of a failure or prolong the life of the system. PM actions are not always

planned by the buyers. Instead, since most products are sold with warranty, it is important for manufacturers to design a good PM program in order to reduce the total cost of warranty service.

Perfect, Minimal and Imperfect Maintenance

Maintenance can be classified according to the degree of improvement on the system operating conditions right after the maintenance. Perfect maintenance restores a system operating condition to "as good as new" (AGAN). That is, upon a perfect maintenance, a system has the same failure mechanism as a new one. Typically, replacement of a failed system by a new one is considered as a perfect maintenance operation, where the number of replacements over a fixed period can be analyzed by the renewal theory (Barlow and Proschan, 1965).

Minimal repair/maintenance, on the other hand, restores a system to the same operating condition right before it fails. It was first proposed in Barlow and Hunter (1960) and also referred as "as bad as old" (ABAO) maintenance in the literature.

For a minimally repaired system, its failure mechanism can be modeled by a non-homogeneous Poisson process (NHPP). Specifically, its failure rate (or hazard rate) function $h(.)$ coincides with the intensity function of NHPP, which can be defined as

$$h(t) = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t[1 - F(t)]} \tag{1.1}$$

where $F(.)$ is the failure time distribution of the system.

As an attempt to generalize perfect and minimal maintenances, imperfect maintenance restores a system operating condition to somewhere between AGAN and ABAO. It is a more realistic model because, for most systems in real-world situations, the working

condition after the maintenance (or repair) does not fall into the two extreme states but rather somewhere in between.

Various methods have been proposed in the literature to model imperfect maintenance operations. Some of them are proposed for the CM and some are designed initially for the PM. However, most of these methods can be well applied to both CM and PM activities and have been thereby used interchangeably in the recent literature (Wu and Zuo, 2010).

Pham and Wang (1996) provided a taxonomy for these models. They emphasized on the modeling methods utilizing (p,q) rule, improvement factor, virtual age, shock model, quasi-renewal process and several other methods. Nakagawa (2005) further classified these methods into four major types, dealing with probability, age, failure rate and cost respectively, and with age and rate being the most popular modeling subjects in the literature.

Maintenance on Single-Component and Multi-Component System

Maintenance can be applied to either a single-component system or a multi-component system. For multi-component systems, the maintenance plan is developed by utilizing both the system structure information and the dependence among its subcomponents. In contrast, the single-component maintenance is mainly modeled via a "black-box" approach. Typically, a complex system may be modeled as a single component if its inner structure cannot be identified or the reliability of the individual subcomponent does not directly affect that of the system (Valdez-Flores and Feldman, 1989).

There are two major types of dependences for multi-component system maintenance: economic dependence and failure dependence. Economic dependence describes the strategy for replacing/repairing a group of components simultaneously at a cost (or

time) less than the separate maintenance of each subcomponent (Tsokos and Shimi, 1977). It is also referred as the opportunistic maintenance strategies. This type of strategy is usually applied to continuous operating systems, such as aircrafts, telecommunication systems, production lines, where the downtime cost of the system is much higher than the maintenance cost. In contrast, failure dependence characterizes the properties of component-level degradations that cannot be stochastically isolated. Failure dependence will normally accelerate the ageing process of a system, and ignoring such factor may lead to either insufficient maintenance or underestimation of the maintenance cost during the planning stage.

1.2.2 Preventive Maintenance Policies

A preventive maintenance (PM) policy represents the set of rules that a PM program should follow in order to achieve certain pre-specified goal(s). The set of rules normally includes the frequency of PM actions, the degree of each PM, and the conditions under which a PM is to be implemented. The goal(s) of a PM policy can be either cost-centered or reliability-centered, or the combination of these two. To be specific, a PM policy can be developed to minimize the maintenance cost, to achieve certain reliability target, or to minimize the maintenance cost subject to reliability constraint. In comparison, most PM policies are developed for cost minimization purposes.

Based on whether the operating condition of a system is monitored or not, preventive maintenance (PM) policies can be grouped into either time-based maintenance (TBM) policies or condition-based maintenance (CBM) policies. The former refers to the planned maintenance actions carried out at specific calendar times regardless of system condition, while the latter is based on observing and collecting information concerning

the condition of system for executing maintenance actions. TBM is easy to implement in practice which assumes the underlying life time distribution has been statistically or experimentally made known (Gertsbakh, 1977). CBM on the other hand is more dependent on the availability of monitoring technique and facilities. Some systems are subject to failures only if it deteriorates beyond certain threshold level. In this sense, CBM can be very effective in preventing excessive maintenance for systems with inadequate deteriorations (Grall et al., 2002a).

Wang (2002) provided a thorough review of the time-based PM policies in the literature. The author addressed the age-dependent PM policy, periodic PM policy, failure limit policy, sequential PM policy, repair limit policy and several other policies. Among these, the age-dependent PM policy and periodic PM policy are most embraced in practice.

1.3 Maintenance Cost Factor

The costing of a maintenance program can be generally studied from two perspectives: the manufacturer's perspective and the consumer's perspective. The former is to evaluate the maintenance cost over the warranty period whereas the latter is to conduct cost analysis over the life time of a product. Each perspective of studies has to be related to many cost factors. The following is a list of common factors which we believe should be contained in any maintenance cost models.

1. Warranty and preventive maintenance policies
2. Maintenance cost structure
3. Product ageing mechanism
4. Impact of maintenance on the product reliability
5. Criterion for cost measurement

1.3.1 Warranty and Preventive Maintenance Policies

Interpretation of the first cost factor is critical for understanding the motivation of this research.

The design of warranty and PM policies is of special importance to the cost analysis of a maintenance program. From the manufacturer's perspective, the maintenance costs under warranty are mainly determined by the attributes of warranty policies such as the length of warranty coverage and the methods for compensation. For example, longer warranty coverage or better compensation to the consumer upon product failures (such as a FRW) will normally incur a higher service cost to the manufacturer. From the consumer's perspective, the maintenance costs will be influenced by the PM policies such as the frequency and degree of PM actions carried out during the product life cycle. For example, more frequent PMs or deeper degree of maintenance (such as perfect maintenance) will reduce the likelihood of unexpected system breakdowns as well as the corrective maintenance (CM) cost upon failures.

On the other hand, it is important to notice that, for many cases the design of warranty and PM policies are not separate processes but has to be integrated. Such argument can be justified from three aspects, and each of them can substantially complicate the procedure of maintenance cost modeling.

1) Almost all products are released with warranty. Implementing a PM program from the consumer's perspective is inevitably affected by the design of warranty policies, which usually cover the initial period of product life cycle. Typically, the presence of warranty protection reduces the maintenance duties of the consumer and may subsequently influence his PM decisions.

2) For the manufacturer, designing a good maintenance program under warranty is an important issue. Majority of warranty policies only specify the types of maintenance actions (repair or replacement) carried out upon product failures. However, such CM policies are sometimes costly if a failure can lead to a much higher repair cost or result in a negative image to the product quality. Therefore, it is justifiable for the manufacturer to further incorporate the concept of "preventive maintenance" into the warranty design.
3) For the manufacturer, the consumer's PM efforts within the warranty period will also have impact on the manufacturer's warranty related decisions. For example, maintenance efforts from the consumer's side reduce the warranty burden for the manufacturer, and this may enable the manufacturer to provide longer warranty coverage as responses to the consumer's maintenance contributions (Pascual and Ortega, 2006; Huang and Yen, 2009). In particular, a common reason why the consumer implements PMs under warranty is to reduce the downtime cost of failures. Practically speaking, warranty and maintenance contracts as such may appear in a negotiable scheme (Jackson and Pascual, 2008; Wang, 2010).

Overall, PM modeling plays an important role in the warranty design and vice versa. The interactions between these two should be carefully investigated in order to enhance the related post-sale decisions.

1.3.2 Maintenance Cost Structure

In addition to the post-sale service design, maintenance cost structure is another key factor to be modeled. It refers to the set of maintenance costs associated with different degrees and types of maintenance actions, which usually includes minimal repair,

imperfect and perfect maintenance (either corrective or preventive), replacement, inspection, testing and so forth. Strictly speaking, these costs are random in nature and need to be modeled by suitable distribution functions. However, due to the scarcity of data, in practice they are often averaged and modeled as constant values.

1.3.3 Product Ageing Mechanism

The product ageing mechanism can be characterized by the joint behavior of product (or system) degradation and failure process. This involves the consideration of system structure and dependence, reliability of components, system state space, and maintenance impact on system functionality. System structure is critical in determining the overall reliability. In particular, for multi-component systems, component failures may not be stochastically independent. The existence of failure dependence accelerates the ageing process, and may cause premature failures and result in higher maintenance cost.

Majority of warranty and maintenance policies are developed for binary systems, which are functioning or failed subject to a single stage of degradation. A multi-state system (MSS) generalizes the binary state assumption by considering a set of system states and involves several stages of degradation before it reaches the complete (or degradation) failure. Maintenance cost modeling for MSS often resorts to the Markov decision processes, with both repair and replacement costs being considered as functions of degradation stages.

1.3.4 Impact of Maintenance on the Product Reliability

The impact of maintenance on product reliability is also crucial to the maintenance cost modeling. For a non-repairable product, given perfect replacement, the time to

first failure and those of the subsequent failures generally follow the same distributions. However, if the product is repairable and the repair is imperfect, then a distinct failure distribution function should be adopted for modeling the maintenance cost after each failure. Similar logic can apply to imperfect PM activities. In particular, the degree of imperfect PM is often negatively correlated with the frequency of PM, both of which have to be carefully designed in the maintenance practice.

1.3.5 Criterion for Cost Measurement

Maintenance cost can be measured based on different criterions. The choice of any will have direct impact on the maintenance costing as well as the design of warranty and maintenance policies. In the following we illustrate three major types of cost criterions.

1) *Expected maintenance cost* (EMC) vs. *expected discounted maintenance cost* (EDMC). The difference between these two is that the EDMC incorporates the value of time and discounts all the future maintenance expenditures into their respective present values. In correspondence, the warranty costs are measured as *expected warranty cost* (EWC) and *expected discounted warranty cost* (EDWC) respectively.
2) *Cost per item sold* vs. *cost for all items sold*. The former calculates the maintenance expenditure on a unit sale and can be analyzed within either warranty period (the seller's perspective) or product life cycle (the buyer's perspective). The latter on the other hand measures the total maintenance cost incurred by a population of identical products which is usually governed by the demand or sales of the product. In practice, instead of analyzing the total maintenance cost under warranty, manufacturers are more interested in measuring the total profit generated from selling these products.

3) *Cost over a finite planning horizon* vs. *cost per unit time*. Most products have finite life cycles in the sense that they will be disposed after certain periods of usage. However, if the assumption of repeated purchases hold for the buyer (mainly to serve for long-term missions), then maintenance actions can be modeled by renewal reward processes (Ross, 1970) which measure the cost on a unit time basis over an infinite planning horizon.

   Let $R(t)$ denote the total reward earned by time t, R_n denote the reward during the nth renewal cycle and Z_n denote the length of nth renewal cycle. The long-run average return is given by the following

$$\lim_{t\to\infty} E\left[\frac{R(t)}{t}\right] = \frac{E[R_n]}{E[Z_n]} \tag{1.2}$$

   where R_n and Z_n (n = 1,2,…) are i.i.d. random variables. Such approach usually leads to the simplification of maintenance cost formulation but only provides an approximation to the reality.

1.4 Problems and Research Objectives

Due to the interdisciplinary nature, warranty has been investigated by researchers in many different ways. A list of over 13 areas of study is given in Murthy and Djamaludin (2002). To serve our purpose, we categorize them into the study of three basic problems.

I. Determining the actual cost of a warranty program for a particular product or maintenance service.

II. Designing a proper warranty service strategy to either reduce the warranty cost or enhance the revenue.

III. Investigating warranty as a factor in studying other disciplines, such as engineering (e.g. maintenance, logistics), accounting, legislation, marketing, economic and societal, etc.

As mentioned earlier, PM modeling plays an important role in the warranty service design and vice versa. In particular, for Problems I and II, PM is an important planning tool in terms of enhancing the warranty servicing, reducing the warranty cost, and promoting the sales revenue. In contrast, for Problem III, warranty design becomes an important factor in influencing the buyer's (preventive) maintenance decisions over the product life cycle.

By realizing this, the primary goal of this research is to model warranty and preventive maintenance (PM) interactively in order to enhance the existing studies on post-sale cost analysis and optimization. As far as we see, such a joint consideration hasn't been sufficiently addressed in the warranty and maintenance literature.

In addition, by putting PM modeling under the warranty context, this research attempts to study or enhance the study of the following research issues.

1. Warranty cost modeling for complex system with failure dependence

   Compared to the "black-box" approach, an improved way for modeling warranty cost of complex system is to incorporate the internal system structure. However, by ignoring failure dependence among system components, warranty cost models may no longer be accurate and mostly the manufacturer risks affording higher warranty expense due to the increase in unprecedented failures. As such it seems necessary to model the warranty cost of complex systems by incorporating the failure dependence among components.

2. Warranty service design considering condition-based maintenance

   Most researchers rely on the age information of a product for the design of maintenance services under warranty, which may result in unnecessary maintenance operations for products with inadequate deteriorations. In order to resolve this problem, condition-based PM strategy may be explored which maintains the product based on the level of actual deterioration.

3. Maintenance service design for multi-state systems (MSS) under finite planning horizon

   Due to the complexity of ageing mechanism, majority of maintenance policies for MSS are proposed under infinite planning horizons, which measure the maintenance cost on a unit time basis and subsequently bring technical convenience for formulating and resolving the maintenance problem. Formulating a MSS maintenance policy under a finite horizon remains to be a challenge research issue despite of the fact that almost all systems are functioning with finite life cycles. It is therefore important to study a finite-horizon maintenance policy dedicated for the MSS.

4. Maintenance service design including the value aspect of maintenance

   Optimal maintenance service design is usually selected from a set of the maintenance policies that leads to the minimum maintenance expenditure. However, such cost-centric maintenance design often ignores the value of maintenance, and may lead to sub-optimal maintenance strategies (Marais and Saleh, 2009). On the other hand, an industrial or service system usually creates more output (or revenue) to the operators if it receives better maintenance services that help retain its productivity. This justifies the value aspect of

maintenance and should be appropriately reflected in the maintenance cost models in order to enhance the maintenance decisions.

5. Integrated consideration of the buyer's and the seller's post-sale service designs

   Post-sale service designs for the manufacturer and the consumer are often made separately. This enables one to focus on the study from a single perspective and helps simplify the decision making processes. However, such arrangement may not be appropriate sometimes, in particular when concerning the interaction between the designs of warranty and PM policies (as highlighted in Section 1.3.1). For such situation, it is important to explore the feasibility of unifying both the buyer's and the seller's post-sale decisions for which knowing the decision of one perspective could potentially enhance that of the other.

6. Maintenance cost modeling incorporating the value of time

   Generally speaking, maintenance cost is not incurred at the stage of maintenance planning but rather spent in future and allocated over the system life cycle. For the purpose of budget allocation or balance-sheet reporting, it seems necessary to incorporate the value of time because the accuracy of cost estimation could be crucial to decision makers. So far, little attention has been paid to studying this issue.

For better understanding the objectives of this research and the limitations of existing literature, a separate review will be provided in the next chapter.

1.5 Organization of the Thesis

Warranty and maintenance literature are vast and disjoint. This thesis does not attempt to cover a wide range of topics in these two areas. Instead, we focus on the costing

perspective of the study and explore the research issues that require an integrated consideration of warranty and preventive maintenance.

In Chapter 2, we present a thorough review of existing literature linking warranty and PM modeling. Reviews of warranty and PM cost models are presented as well.

Chapters 3 and 4 focus on the warranty cost analysis for the manufacturer. In Chapter 3, we study a comprehensive warranty policy for multi-component systems for which component failures are not stochastically independent. In order to improve the system reliability, we assume that opportunistic PM services are implemented upon system failures. Chapter 4 proposes a condition-based warranty servicing strategy that relies on the system state information for executing PM decisions. We derive warranty cost functions under both renewable and non-renewable warranty settings, based on which the optimal scheduling of condition monitoring services is further analyzed.

Chapters 5 and 6 focus on the life-cycle maintenance cost analysis for the buyer. Chapter 5 investigates the optimal design of maintenance servicing on revenue-generating equipments by integrating both the cost and value aspects of maintenance. The influence of warranty as well as many other models parameters on the buyer's optimal PM decisions is discussed. Chapter 6 investigates the repair-replacement policies for multi-state systems (MSS) that operate under finite life cycles. The formulations of maintenance cost models take into account both the value of time and initial warranty coverage during the system life cycle.

Chapter 7 focuses on the enhancement of sales revenue for a specific product by properly integrating the seller's warranty design with the buyer's age-dependent PM decisions. The demand of the product is assumed to be governed by the selling price, the warranty length as well as the buyer's PM decisions. In order to link the buyer's

PM decisions with the seller's warranty design, a game theoretic model is formulated that allows the seller to foresee the buyer's decisions and subsequently make his own decisions.

Chapter 8 presents the summary and future works of this research. A skeleton of this thesis is shown in Figure 1.1.

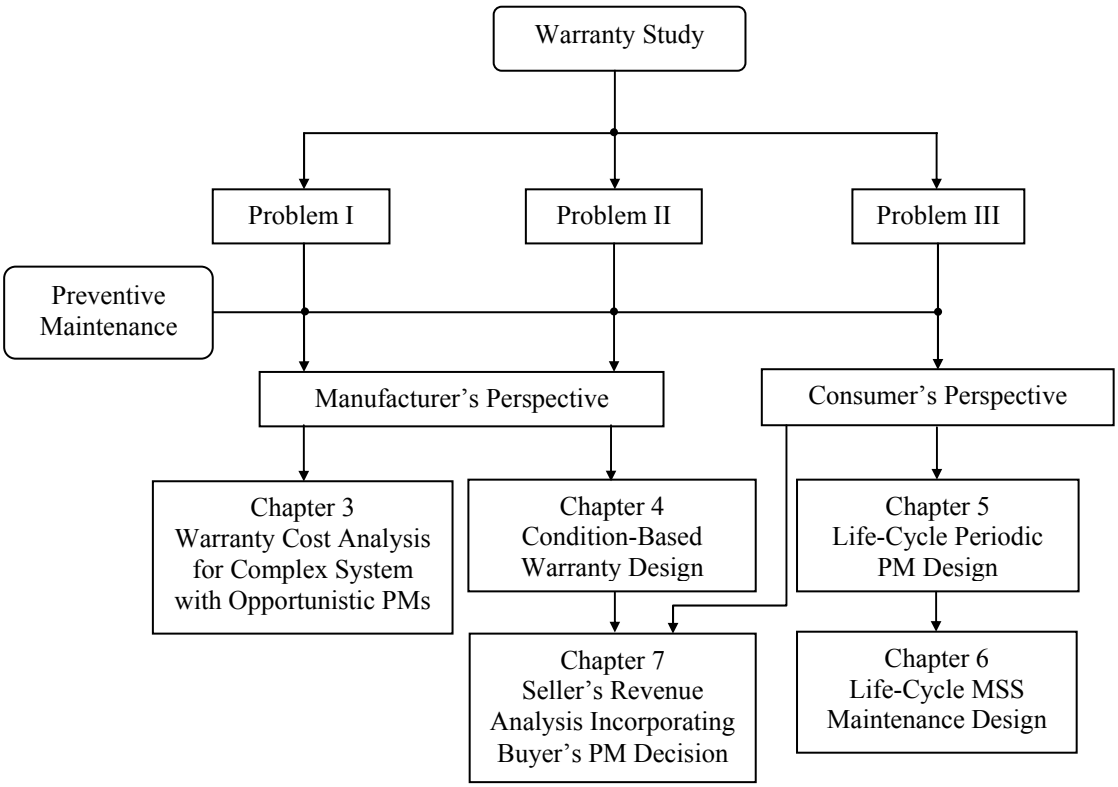

Figure 1.1 Map of study linking warranty and preventive maintenance modeling

CHAPTER 2 LITERATURE REVIEW

In this chapter, we provide some background information for warranty and preventive maintenance (PM) cost modeling and optimization. Reviews of existing literature linking warranty and PM modeling are presented in detail.

2.1 Methods for Modeling Imperfect Maintenance

Post-sale cost modeling and service design depend crucially on the selection of imperfect maintenance (IM) models. An IM model is used to measure the effectiveness of an IM activity on the system reliability, which can be either a CM activity or a PM activity. We refer them in the following as "CM model" and "PM model" respectively.

According to Nakagawa (1988), majority of the IM models deal with the age or the failure rate of the maintained system. These models can be generally categorized into three groups: age reduction models, hazard-rate reduction models, and a hybrid of both.

For the ease of illustration, the following notations are used in this section:

t_k	time between $(k-1)^{\text{th}}$ IM and k^{th} IM
$\sum_{i=1}^{k} t_i$	calendar time of the kth IM
$h_k(.)$	$k = 1,2, \ldots$, hazard rate function of the system after the kth IM
$h_0(.)$	hazard rate function of the system without IM

We assume that the time is reset to zero at each IM. In other words, $h_k(t)$ in fact reflects the hazard level of the system at calendar time $t + \sum_{i=1}^{k} t_i$. In addition, $t_1, t_2, \ldots$

is a series of random variables for imperfect CM activities whereas they are pre-specified for PM activities.

Age Reduction Models

Age reduction models are the most commonly used IM models. Nakagawa (1988) developed two imperfect PM models. One is an age reduction model (NAKI), while another is a hazard rate reduction model (NAKII). In NAKI model, the age of the system after the kth PM reduce to $\alpha_k t$ when it was t before PM. Therefore the hazard rate function $h_k(t)$ after the kth PM is given by

$$h_k(t) = h_0\left(t + \sum_{i=1}^{k} t_i \prod_{j=i}^{k} \alpha_j\right), 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k < 1 \qquad (2.1)$$

Kijima et al. (1988) and Kijima (1989) introduced two types of CM models, Type I and Type II, using the concept of virtual age (also referred as effective age). The idea is to distinguish between the system's calendar age, which is the time elapsed since the system is put into use, with the virtual age of the system, which describes its present functionality when compared to a new system. However, applications of these two models are not constrained to CM activities. They have been recently used for modeling PM activities (Kahle, 2007; Bartholomew-Biggs et al., 2009). When modeling PM activities, Type II model in Kijima (1989) coincides with the NAKI model mentioned previously; therefore, we only present Type I model here. Let v_k denote the virtual age of the system immediately after the kth repair (or PM). Type I model assumes that $v_k = v_{k-1} + \alpha_k t_k$ for $\alpha_k \in (0,1)$, and therefore hazard rate function $h_k(t)$ after the kth repair is given by

$$h_k(t) = h_0\left(t + \sum_{i=1}^{k} \alpha_i t_i\right), \quad 0 < \alpha_1 \leq \alpha_2 \leq \cdots < \alpha_k < 1 \tag{2.2}$$

In other words, the effect of kth repair only works on the ageing of the system taken place since the $(k-1)^{\text{th}}$ repair. It is different from NAKI or Type II model in Kijima (1989) where each PM is assumed to cause an effective decrease in all the aging that has taken place since time zero. Unlike Type II model, repeated repairs under Type I model generally have no cumulative age-reduction effects. Dagpunar (1997) further considered the case in which the virtual age after the kth repair can be expressed as $v_k = \varphi(v_{k-1} + t_k)$ where $\varphi(.)$ is an arbitrary scaling function that models the effectiveness of repair. The model was further generalized in Seo and Bai (2004) where v_k is modeled as $v_k = \varphi(v_{k-1}, t_k)$.

Canfield (1986) considered the following periodic PM model with PM interval τ:

$$h_k(t) = h_0\big(t + k(\tau - a)\big) + \sum_{i=1}^{k}\big[h_0\big((i-1)(\tau - a)\big) - h_0(i\tau - ia)\big], 0 < a < \tau \tag{2.3}$$

where a represents the level of PM restoration or the quality of PM. The main feature of this model is that the PM restores the shape of the hazard function at effect age t to the level at effective age $t - a$ whereas the hazard level remains unchanged after the PM. The model depicts those minor PM activities, such as lubrication, adjustment, cleaning, etc, which slow the system deterioration but fail to bring the system to a younger condition. Parameter a in Canfield's model is assumed to be fixed. Wu and Clements-Croome (2005) further considered a as a random variable and develop PM policies.

Brown and Proschan (1983) proposed the (p, q) model to describe the repair impact on the system reliability. In their model, the repair restores the system operating condition to a brand-new condition with probability p and to a minimally-repaired condition with probability $q = 1 - p$. In this sense, (p, q) model is essentially a type of age reduction model where the amount of age reduction is either zero or the entire operating life since last perfect repair. Again, it includes AGAN and ABAO as special cases when $p = 1$ and $p = 0$ respectively. Block et al. (1985) generalized the (p, q) model to the $(p(t), q(t))$ model by assuming the probabilities to be time-dependent. Their model was later extended to $(p(t, n), q(t, n), s(t, n))$ model in Makis and Jardine (1992) by making the probabilities depend on both time *t* and the number of failures *n* and also by considering the case when the repair is unsuccessful (with probability $s(t, n) = 1 - p(t, n) - q(t, n)$). Compared to other age reduction models, these models assumed either AGAN or ABAO, and therefore failed to describe any kind of system aging.

Hazard-Rate Reduction and Hybrid Models

Another branch of IM model are the hazard-rate reduction models. Nguyen and Murthy (1981) assumed that the system deteriorates with time as well as with the number of CMs carried out. After each CM, the failure rate of the system increases by following a distinct hazard rate function, i.e. $h_{k+1}(t) \geq h_k(t)$ with $h_{k+1}(0) = h_k(0)$. Nakagawa (1988) (NAKII model) considered a special case of Nguyen and Murthy (1981) by assuming $h_{k+1}(t) = \beta_k h_k(t), \beta_k > 1$; and therefore the hazard rate function after the kth CM is simply given by

$$h_{k+1}(t) = \left(\prod_{i=0}^{k} \beta_i\right) h_0(t), \quad 1 < \beta_0 \leq \beta_1 \leq \cdots < \beta_k \tag{2.4}$$

By integrating the NAKI and NAKII models, a hybrid IM model was proposed in Lin et al. (2000, 2001):

$$h_k(t) = \prod_{i=1}^{k} \beta_i \, h_0\left(t + \sum_{i=1}^{k} t_i \prod_{j=i}^{k} \alpha_k\right), 0 < \alpha_1 \leq \cdots \leq \alpha_k < 1 < \beta_1 \leq \cdots < \beta_k \quad (2.5)$$

The effectiveness of IM is modeled from two aspects: one for its immediate reduction of the system effective age and the other for the alteration of the shape of the hazard rate function. NAKI and NAKII become the special cases of the hybrid model when $\beta_1 = \beta_2 = \cdots = 1$ and $\alpha_1 = \alpha_2 = \cdots = 0$ respectively.

Another interesting hybrid model is based on the geometric process and mainly applied for CM activities. Lam (1988) defined the geometric process as an alternative to the non-homogeneous Poisson process (NHPP); that is, a sequence of random variables $t_1, t_2, \ldots$ is a geometric process if the distribution function of t_k is given by $F(\alpha^k t)$for $k = 1,2, \ldots$ and α is a positive constant. The hazard rate changes from $h_{k-1}(t)$ before the kth CM activity to $\alpha h_{k-1}(\alpha t)$ after the CM. The model can be used to describe imperfect CM activities if $0 < \alpha < 1$. Wang and Pham (1996) later referred to a process similar to the geometric process as a quasi-renewal process. Finkelstein (1993) developed a very similar model where he defined a general deteriorating renewal process such that $F_{k+1}(t) \leq F_k(t)$. So far, however, very few works in the literature consider geometric processes in the modeling of PM activities.

2.2 Warranty Cost and Revenue Analysis

In this section, reviews of existing literature on warranty cost modeling per unit sale from the manufacturer's perspective are presented. We focus on one-dimensional warranties offered to both single-component systems and multi-component systems.

For studies covering two-dimensional warranties, we refer to Moskowitz and Chun (1994), Murthy et al. (1995), Iskandar and Murthy (2003, 2005).

In the case of multiple sales, the warranty costs are mainly analyzed as part of the revenue structure. An important branch of warranty literature is to model the warranty, selling price and product demand jointly in order to maximize the total profit for the manufacturer. This review will cover this topic as well.

Warranty Cost Analysis for Single-Component Systems

Majority of warranty cost models are derived for single-component systems via a "black-box" approach. This usually leads to technical convenience for formulating various warranty cost functions.

Blischke and Scheuer (1975) first derived the expected warranty cost (EWC) functions for a free-replacement warranty (FRW) policy. Consider a non-repairable product having failure distribution F and sold under a FRW with warranty period w. Let c_1 be the average cost to a seller for replacing a failed product and CW be the total warranty cost. The EWC per item sold to the seller is given as

$$E[CW] = c_1 M_F(w) \tag{2.6}$$

where $M_F(.)$ is the renewal function associated with F.

More frequently, non-repairable products are sold under pro-rata warranties (PRW) which require the seller to refund part of the purchase price to the buyer upon product failures under warranty. The ways of refund are vast and generally determined by the rebate functions. Let c_2 be the average purchase cost to a buyer and $\rho(.)$ be the rebate function associated with PRW. If a proportional linear rebate function as below is used,

$$\rho(t) = \begin{cases} \alpha(1 - t/w)c_2, & 0 \le t < w \\ 0, & \text{otherwise} \end{cases} \tag{2.7}$$

then the EWC can be expressed as

$$E[CW] = \alpha c_2 \frac{\int_0^w F(t)dt}{w} \tag{2.8}$$

On the other hand, for repairable products, it is natural that the product may not always be replaced upon failures. Many researchers assume that the product is minimally repaired upon failures and therefore the product ageing process can be described by a NHPP. This result has been very useful in analyzing many warranty and maintenance policies involving minimal repair. The related references are Park (1979), Phelps (1983), Sheu (1990), Murthy (1991), Aven and Jensen (2000), Ja et al. (2001). Among these studies, the minimal repair cost can be fixed, deterministic functions of time or random in nature, with the former two as special cases of the latter. For example, consider a minimally repaired product subject to a NHPP ageing process $\{N(t), t \ge 0\}$ with intensity function $\lambda(.)$. Suppose that the repair cost $c_m^{(i)}(t)$ at time t is random and depends on the number of minimal repairs i before time t. By applying the results in Sheu (1990), the EWC for FRW with warranty period w can be given by

$$E[CW] = \int_0^w h(t)\lambda(t)dt \tag{2.9}$$

where $h(t) = E\left[c_m^{(N(t)+1)}(t)\right]$. The special cases of (2.9) when the minimal repair cost is a deterministic function of time, i.e. $c_m^{(i)}(t) \equiv c_m(t), i = 1,2, \ldots$ (Boland, 1982), and a constant value, i.e. $c_m(t) \equiv c_m$, are given separately as

$$E[CW] = \int_0^w c_m(t)\lambda(t)dt \tag{2.10}$$

$$E[CW] = c_m \Lambda(w) \tag{2.11}$$

where $\Lambda(t) = \int_0^t \lambda(y)dy$ represents the cumulative hazard function.

For renewable warranties, the most commonly used ones are renewing free-replacement warranties (RFRW) and renewing pro-rata warranties (RPRW). It is important to notice that almost all cost models for renewable warranties are derived under the assumption that replacements or repairs rectify the failed product to AGAN. Based on such assumption, the number of replacements N until the expiration of warranty simply follows a geometric distribution with parameter $F(w)$. Therefore, we verify that $E[N] = F(w)/[1 - F(w)]$, and the EWCs per item sold under RFPW and RPRW with rebate function as (2.2) are given separately by

$$E[CW] = c_1 \frac{F(w)}{1 - F(w)} \tag{2.12}$$

$$E[CW] = \left[(c_1 - c_2)F(w) - \alpha c_2 \frac{\int_0^w F(t)dt}{w}\right] \frac{F(w)}{1 - F(w)} \tag{2.13}$$

Equation (2.13) is obtained by assuming that the buyer will continue purchasing the product when it fails within warranty. On the other hand, if the buyer will purchase the product whenever it fails (with discounted price within warranty), then it is better to measure the warranty cost based on an infinite planning horizon. Mamer (1982) conducted such study for both FRW and PRW under linear rebate functions. We modify their result under proportional linear rebate functions and present it as

$$E[CW] = \frac{(c_1 - c_2)}{\int_0^w \big(1 - F(t)\big)dt} + \frac{c_2 \alpha \int_0^w F(t)dt}{w \int_0^w \big(1 - F(t)\big)dt} \tag{2.14}$$

On the other hand, combination warranties (CMW), which contain a free replacement period w followed by a pro-rata period T, are also studied by several researchers as a trade-off between FRW and PRW. Although such policies are considered more general than FRW ($T = 0$) and PRW ($w = 0$), the costing procedures are essentially the same as previous discussions. We refer to Blischke (1990) and Nguyen and Murthy (1984) for detailed studies.

Warranty Cost Analysis for Multi-Component Systems

As mentioned before, most researchers models the warranty cost for (complex) systems via a "black-box" approach. However, for multi-component systems, system structure is the key maintenance cost factor to be incorporated and ignoring this may compromise the accuracy of warranty cost estimation.

Warranty cost modeling for multi-component systems is not a new topic, but until now, only limited studies have been contributed to this area. Balachandran et al. (1981) proposed a Markovian approach for modeling warranty cost for a three-component system. Chukova and Dimitrov (1996) analyzed the warranty cost for several series systems and parallel systems under FRW. Hussain and Murthy (1998) estimated the warranty cost for newly-launched parallel systems with uncertain quality. Monga and Zuo (1998) and Lin et al. (2000) focused on the design aspect of the series-parallel systems by incorporating the warranty and PM cost estimation.

More recently, Bai and Pham (2004) derived some distribution properties of the expected discounted warranty cost (EDWC) for minimally repaired series systems under both FRW and PRW warranties. Their results were used to support the decisions on the optimal warranty reserve level and the optimal warranty durations for lot sales. Consider a series system that consists of q stochastically independent components and

is sold under FRW. Let $\lambda_i(.)$ be the failure rate function of the ith component and c_i be its minimal repair cost. The expectation and variance of the warranty cost under a general discounted function $H(.)$ are given by

$$E[CW] = \sum_{i=1}^{q} c_i \int_0^W H(u)\lambda_i(u)du \tag{2.15}$$

$$V[CW] = \sum_{i=1}^{q} c_i^2 \int_0^W H^2(u)\lambda_i(u)du \tag{2.16}$$

Using the Gaussian approximation, the minimal warranty reserve level per sale can be determined by

$$WR_{min} = E[CW] + z_{1-\alpha}\sqrt{V[CW]} \tag{2.17}$$

where $z_{1-\alpha}$ is the $1-\alpha$ quantile of the standard Gaussian distribution and α represents the probability that the actual warranty cost per sale is over the budget level WR_{min}.

Duchesne and Marri (2009) also discussed several risk adjusted (discounted) warranty cost (RAWC) models for minimally repaired systems. Their discussion was more general than Bai and Pham (2004) in the sense that the system structure was not limited to series systems but can be any complex systems subject to competing failure risks. On the other hand, by focusing on the expected warranty costs (EWC), Bai and Pham (2006) further investigated the distribution properties for series, parallel, series-parallel, and parallel-series systems.

Sometimes, component failures may not be stochastically independent. Therefore, it seems necessary to consider failure dependence when modeling the warranty cost for multi-component systems. However, comprehensive studies on this topic are rare under the warranty context (refer to Chukova and Dimitrov (1996) for some simple

case studies) whereas majority of the literature are confined to pure maintenance modeling without warranty incorporation. The interested readers are referred to Murthy and Nguyen (1985a, b), Zequeira and Berenguer (2005), and Lai (2007, 2008), Sun et al. (2009) for some further discussion.

Warranty Revenue Analysis for Multiple Sales

So far we have focused on the warranty cost modeling per unit sale without incorporating the revenue of selling the product. When dealing with a population of products, warranty costs are usually investigated as part of revenue structure and modeled together with the selling price and product demand. Modeling the total warranty costs alone is meaningless because the number of sales is generally determined by the selling price which has to be incorporated into the decision models.

In addition to price, the product demand is also influenced by the seller's warranty strategies. As indicated before, consumers usually view the quality of a product based on its warranty and therefore, a satisfactory warranty policy will certainly enhance their purchase willingness (Menezes and Currim, 1992; Padmanabhan, 1993). On the other hand, a lower price or longer warranty coverage tends to enhance the sales but may lead to a decrease in the unit profit. In this sense, the joint determination of selling price and warranty length is of special importance to the seller in order to maximize his total profit (rate).

Glickman and Berger (1976) presented an early important work of modeling product demand in a static market in which the demand decreases exponentially with respect to price and increases exponentially with warranty length. The objective is to optimally determine the price and warranty length that maximizes a manufacturer's profit. Denote C_p and w as the price and warranty length of product, respectively. The

demand function, $Q(C_p, w)$, in Glickman and Berger's model, is a displaced log-linear function with an exponential form as follows:

$$Q(C_p, w) = k_1 C_p{}^{-a}(k_2 + w)^b, a > 1, 0 < b < 1, K_1 > 0, k_2 \geq 0 \qquad (2.18)$$

where k_1 ($k_1 > 0$) is a constant amplitude factor, and k_2 ($k_2 \geq 0$) is an additive factor that allows for a non-zero demand when $w = 0$. Parameter a ($a > 1$) is the price elasticity, and b ($0 \leq b \leq 1$) is the displaced warranty length elasticity. Applications of this demand function can be widely found in the literature. For example, Mitra and Jayprakash (1990) presented a multi-objective model for warranty estimation based on this demand function; Blischke and Murthy (1992) used this demand function in product warranty management; Mitra and Jayprakash (1997) also applied this demand function to develop a market-share model.

On the other hand, Glickman and Berger's model did not consider the diffusion effect for which the product demand at time t could count on the number of buyers before t. Bass' growth model (Bass, 1969; Robinson and Lakhani, 1975) is an epidemic model that applies to initial purchases. It assumed that the buyers are generally divided into innovators and imitators where the innovators make the purchase decisions independently whereas the decisions of imitators are influenced by that of other individuals in a social system. Teng and Thompson (1996) conducted a further study by applying the maximum principle in analyzing the optimal price and quality policies for introducing a new product. They assumed that the unit cost declines along the learning curve and investigated the dynamics between price and quality, and diffusion process. Lin and Shue (2005) and Wu et al. (2006) modified the Teng-Thompson price-quality model into a price-warranty decision model similar as Glickman and Berger (1976) in which the warranty length represents the quality level. Wu et al.

(2006) derived the marketing strategy for a normal life time distributed product whereas Lin and Shue (2005) investigated numerous basic life time distributions.

Recent studies of this area usually involve the consideration of many other factors. For example, Huang et al. (2007) incorporated the product reliability modeling into the design phase of the product marketing strategy, Matis et al. (2008) and Huang and Fang (2008) considered variants of standard warranties, Wu et al. (2009) and Lin et al. (2009) dealt with the production and inventory problem for a static demand market, Zhou et al. (2009) incorporated the heterogeneous risk attitudes of customers into their decision models, and Fang and Huang (2010) focused on the marketing design for products with uncertain life time distributions due to the scarcity of historical failure data. However, in contrast to the warranty study on a unit sale, the factor of maintenance policies is seldom included in the seller's revenue models. Generally speaking, different maintenance policies lead to different maintenance expenditure, and this will have impact on the seller's profit margin. Ignoring such factor may result in sub-optimal designs of the product marketing strategies.

2.3 Preventive Maintenance Policies

Numerous maintenance policies have been proposed within last few decades in order to model and resolve the maintenance and replacement problems of deteriorating systems. This section reviews both time-based and condition-based PM policies without warranty consideration. In particular, for the time-based PM policies, we focus on two most popular policies: the age-dependent PM policy and periodic PM policy. For the systematic studies of other maintenance policies, we refer to Valdez-Flores and Feldman (1989), Cho and Parlar (1991), Dekker et al. (1997), Wang (2002).

2.3.1 Age-Dependent PM Policy

The most common and popular time-based PM policy is the age-dependent PM policy. Under this policy, a product is preventively maintained at some predetermined age T, or repaired at failure, until a perfect maintenance, preventive or corrective, is received (Wang, 2002). The earliest age-dependent PM policies always assume perfect PM activity, and therefore, the product age is reset to zero whenever a maintenance activity (either CM or PM) is carried out. Such policy is also referred as the age-replacement policy in Barlow and Hunter (1960), for which a product is replaced at age T or failure, whichever occurs first.

The age-replacement policy (and age-dependent policies in general) is developed under infinite planning horizons and therefore the maintenance costs should be evaluated on a unit time basis. Suppose that the life time of a product follows failure distribution $F(.)$. Let c_p be the preventive replacement cost at time T and c_r be the corrective replacement cost if the product fails before T. The long-run average maintenance cost rate $CR(T)$ is given by

$$CR(T) = \frac{c_p + (c_r - c_p)F(T)}{\int_0^T \bar{F}(t)dt} \tag{2.19}$$

where $\bar{F}(t) = 1 - F(t)$ represents the survival function of the product at time t. The optimal PM policy then equals to finding the optimal replacement age T^* so that $CR(T)$ is minimized. To be specific, we set the first derivative of $CR(T)$ equal to zero. T^* should satisfy the following optimal condition:

$$h(T^*)\int_0^{T^*} \bar{F}(u)du - F(T^*) = \frac{c_p}{c_r - c_p} \tag{2.20}$$

where $h(.)$ represents the hazard rate function of the product. Obviously, the existence of the optimal solution is influenced by the cost structure c_r and c_p, and the ageing mechanism of the product. If there is no solution (or $T^* = \infty$), then the product should only be correctively replaced.

Unlike the classic age-replacement policies, many newer policies assume imperfect PM, meaning that the PM operation does not necessarily imply renewal. Nakagawa (1979) conducted a pioneer work by adopting imperfect maintenance in the modeling of age-dependent PM policies. He considered three types of PM models, imperfect, perfect or minimal repair at failure, and analyzed the optimal maintenance policies in terms of PM interval time T. Nakagawa (1984) presented an age-dependent PM policy where the system is replaced at age T or at the Nth failure, whichever occurs first. Any failures before that are minimally repaired. The age and the number of failures are reset to zero after the system is replaced. Note that if $N = 1$, the policy is reduced to the classical age-replacement policy.

Two other extended policies were proposed in Sheu et al. (1993, 1995) by following the (p, q) rule. In Sheu et al (1993), if a product fails at age $y < t$, it is subject to a perfect repair with probability $p(y)$, or undergoes a minimal repair with probability $q(y) = 1 - p(y)$. Otherwise, the product is replaced when the first failure after t occurs or the total operating time reaches age T ($0 \leq t \leq T$), whichever occurs first. The objective is to find the optimal t^* and T^* so that the maintenance cost rate is minimized. Sheu et al. (1995) considered another extension to the age-replacement policy. They assumed that a system is subject to two types of failures and is replaced at the nth Type 1 failure or first Type 2 failure or at age T, whichever occurs first. Type 1 failure occurs with probability $p(y)$ and is rectified by minimal repair. In contrast,

Type 2 failure occurs with probability $q(y) = 1 - p(y)$ and is rectified by perfect repair (replacement). The decision variables for this policy are n and T. Both policies have been further extended recently by considering different IM models (refer to Section 2.1) and more complex system structure and ageing mechanism. For interested readers, we refer to Sheu (1996), Wang and Pham (1999), Frickenstein and Whitaker (2003), Sheu and Chang (2001), and Heidergott and Yuan (2010).

2.3.2 Periodic PM Policy

Another commonly studied time-based PM policy is the periodic PM policy. Under this policy, the system is preventively maintained at fixed time intervals kT ($k = 1,2, \ldots$), and repaired at intervening failures where T is a constant. This policy is easy to implement because it requires no record keeping of the system history. If both PM and CM are perfect, this policy is better known as a block replacement PM policy, for which the product is replaced at per-arranged times kT ($k = 1,2, \ldots$) and at its failures. Boland and Proschan (1982) analyzed such a policy by finding the optimal T^* in order to minimize the expected cost (rate) over both finite and infinite planning horizons. On the other hand, if PM is perfect but CM is minimal, then this policy is referred as the "periodic replacement with minimal repair at failures" policy in Barlow and Hunter (1960), where the product is replaced at per-arranged times kT ($k = 1,2, \ldots$) and minimally repaired at its failures. It is typically suitable for the large or complex systems where minimal repair is plausible at failures. If the failed product remains failed until the next scheduled PM, then the policy is referred as "no failure replacement" policy (Nakagawa, 1980). Such policy can be applied to those maintenance settings where a system is not continuously monitored and its failure is not self-announcing.

Similar as the age-dependent policy, the periodic PM policies can be generalized by considering imperfect PMs. Kijima et al. (1988) developed a block replacement model with general repair at failures. The general repair is modeled by the virtual age method (refer to Section 2.1). Nakagawa (1986) made an extension to the "periodic replacement with minimal repair at failures" policy by considering a constraint on the total PM times. In his model, the imperfect PM operation is performed at times $kT, k = 1,2, \dots, N-1$ with minimal repair at failures. Each PM is imperfect in the sense that it increases the hazard rate of the system, and the system is replaced right before the Nth PM (or at age NT). Liu et al. (1995) developed a policy similar to that in Nakagawa (1986). For both policies, N and T are the two decision variables to be optimized under an infinite planning horizon. Another policy was developed in Sheu et al. (2005) by considering the (p, q) PM model. In other words, after the kth PM, the system is AGAN with probability p_k and ABAO with probability $q_k = 1 - p_k$. p_k is assumed to be a function of the number of previous ABAO PM operations.

So far, the most complicated periodic PM policies might be those combining with age-replacement policies. Recently, Sheu and Chang (2009) developed such a hybrid policy for a system with age-dependent failure type. Similar to Sheu et al. (1995), they considered two types of failures: Type-I failure (minor) and Type-II failure (catastrophic), which is rectified by minimal repair and a major overhaul (equals to a PM) respectively. Unlike previous policies, they assumed that the PM is implemented following a Type-II failure, or at age T, whichever occurs first; and the system is replaced at the Nth PM. They modeled the imperfect maintenance by utilizing the hybrid PM model proposed by Lin et al. (2001) (refer to Section 2.1). The policy turned out to be a generalized form of many existing policies, with the classic age-replacement policy and "periodic replacement with minimal repair at failures" policy

as the special cases. Similar studies can also be found in Sheu and Chang (2010) and Sheu et al. (2010).

2.3.3 Condition-Based PM Policy

Condition-based maintenance (CBM) policies are based on observing and collecting information concerning the condition of system (i.e. system state) to determine maintenance actions and prevent system failures. As mentioned in Section 1.2.2, if a system is subject to failure only if its state deteriorates beyond a given threshold level, then CBM should be more cost-efficient than time-based (preventive) maintenance (TBM) policies which are based solely on the system age and the knowledge of the statistical information on its life time.

Modeling the degradation path of a system is of special importance to the development of CBM policies. In particular, based on the type of system state space, existing CBM policies can be grouped into two major streams: continuous-state CBM policies and discrete-state CBM policies.

The first stream of policies focuses on a system that deteriorates gradually and stochastically, for which it is better to model the degradation path as a continuous-state wear process. The objective is to find the optimal inspection schedule and/or the optimal preventive replacement threshold so that the expected maintenance cost (rate) is minimized. Grall et al. (2002b) and Dieulle et al. (2003) modeled the continuous-state wear process as a gamma process (van Noortwijk, 2009) and studied the inspection-maintenance strategy for a single-component system, where the effect of maintenance operations was assumed to be perfect. Castanier et al. (2003) considered imperfect maintenance for a repairable system monitored by sequential non-periodic inspections. Li and Pham (2005), Liao et al. (2006) and Tai and Chan (2010)

developed the optimal CBM policies for continuously degraded system by maximizing the system average availability. Berenguer et al. (2000) and Castanier et al. (2005) on the other hand investigated the optimal CBM policies for multi-component systems.

The continuous-state models are precise in describing the overall system ageing but are computational demanding. As an approximation, the second stream of CBM policies focuses on the multi-state systems (MSS) and describes the system deterioration as a continuous-time discrete-state (semi-)Markov process. The critical set of states for which the PM is to be carried out is optimally determined such that expected maintenance cost is minimized. Hosseini et al. (2000) and Makis and Jiang (2003) modeled the system deterioration as a continuous-time Markov chain. Hosseini et al. (2000) studied a MSS subject to two types of failures, Poisson failure and degradation failures. Optimal inter-inspection times were derived in order to maximize the system throughput. Makis and Jiang (2003) developed a mathematical framework for the CBM optimization for MSS subject to only degradation failures. Chen and Trivedi (2005), Black et al. (2005) and Moustafa et al. (2004) investigated the CBM policies for MSS with semi-Markov decision process. As a special case, the CBM policies for a two-stage degradation model (i.e. a 3-sate semi-Markov process), also called as the delay time model (DTM), were studied by Okumura et al. (1996), Wang (2007) and Ferreira et al. (2009). Recently, Ghasemi et al. (2007) and Wu and Ryan (2010) developed the optimal inspection-replacement policies for MSS by utilizing the proportional hazards model (Cox, 1972).

So far condition-based PM policies are mainly developed for pure maintenance situations from the user's perspective. Utilizing the CBM concept for enhancing the warranty service design from the manufacturer's perspective is still rare in the literature.

2.4 Literature Review Linking Warranty and Preventive Maintenance

Warranty and preventive maintenance (PM) have been separately studied by researchers from many different ways. However, studies combining these two research areas are still insufficient in the literature. In this section, we present a thorough review of nearly 30 papers that link warranty and PM modeling within the last two decades. The reviews of existing models are carried out for both the manufacturer and the buyer.

2.4.1 Warranty Service Design Incorporating Preventive Maintenance

For the manufacturer, designing a good maintenance program under warranty is an important issue. Existing studies focus on the planning of PM actions over the warranty period as a means of reducing the total warranty service cost.

Chun (1992) first incorporated preventive maintenance actions in designing a product warranty. He considered the case when PM actions are carried out periodically during the warranty period. The number of PM actions N was obtained and minimized over a finite horizon. The model was later generalized by Jack and Dagpunar (1994) by assuming non-periodic PM actions.

Dagpunar and Jack (1994) dealt with a similar model as in Jack and Dagpunar (1994). The difference was that they considered the cost of each PM as a non-decreasing function of the operating age and the effective amount of age reduction. The optimal number of PM actions N^*, operating age s^*, and age reduction factor x^* are obtained jointly so that the manufacturers' expected warranty cost is minimized.

Yeh and Lo (2001) proposed a general “preventive maintenance warranty” (PMW) policy under which the cost of each PM was only related to the degree of age-reduction

factor x. An efficient algorithm was provided to search for the optimal number of PM actions N^*, and the optimal degree of each PM x^*. Their results showed that the system is renewed at each PM.

Wang (2006) presented four warranty cost models all assuming imperfect repair. Among these models, Models B and D consider preventive maintenance during the warranty period. Model B assumes that the item is preventively maintained at times kT $(k = 1,2, \dots, N)$ with warranty period $w \geq NT > 0$. The imperfect PM follows (p, q) rule, i.e., upon PM the item is AGAN with probability p and is restored to ABAO with probability $q = 1 - p$ (Nakagawa, 1979). The warranty cost model is developed based on an infinite horizon and the approximate optimal PM interval T is obtained such that the warranty cost is minimized. Model D considers a k-out-of-n:G (Kuo et al., 2001) system under warranty, and the imperfect PM during the warranty follows quasi-renewal processes (Pham and Wang, 1996). Minimal repairs are performed on failed items before a fixed time τ. CM on the failed items together with PM on all unfailed but deteriorating ones is performed once exactly m items fail between τ and warranty period w. The optimization on (τ^*, m^*) is presented such that expected warranty cost is minimized.

Wu and Li (2007) considered periodic PMs under warranty for repairable products with both dormant and operating states. They assumed that the product deteriorates slower under the dormant state compared to the operating state. They used the PM model proposed in Canfield (1986) and conducted warranty cost analysis under a wide range of model parameters. Preventive replacement policies for non-repairable products with dormant states were further discussed in Wu and Xie (2008).

Huang and Yen (2009) provided an approach for modeling two-dimensional warranty policies with both the age and usage limits under periodic PM operations. They assumed PMs are conducted by the buyer and in exchange, the manufacturer offers the buyer an extended age limit (warranty time limit). They derived the new warranty limit under the condition that the total warranty cost remains unchanged and subsequently used the result to maximize the manufacturer's profit.

On the other hand, it seems necessary to differentiate the above PM policies with the well-known repair-replacement policies under warranty. Both of them attempt to minimize the total warranty cost. However, the repair-replacement policy is part of corrective maintenance (CM) policies in the sense that the decision of repairing (either minimal or imperfect) or replacing the product is only made upon failure; and the products maintained under such policy may be subject to high downtime costs. As a simple example of such policy, a warranted product with warranty length w may be replaced if it fails before T and repaired if it fails between $[T, w]$, where T ($T < w$) is the decision variable to be optimized (Nguyen and Murthy, 1989). For some detailed studies, we refer to Jack and Van der Duyn Schouten (2000), Jack and Murthy (2001) and Jiang et al. (2006) for the case of one-dimensional warranties, Iskandar and Murthy (2003), Iskandar et al. (2005) and Jack et al. (2009) for the case of two-dimensional warranties, and Zuo et al. (2000) and Pan and Thomas (2010) for the case of multi-state products.

2.4.2 Life-Cycle Maintenance Service Design under Warranty Context

Product life cycle is defined as the time span since the product is purchased until the time when it is disposed, and usually includes both warranty and post-warranty period.

Different from the foregoing studies, warranty and PM modeling is conducted from the buyer's perspective under the life-cycle context.

Chun and Lee (1992) first proposed a periodic PM model for a system with an increasing failure rate (IFR) and subjected to periodic PM actions during warranty period and post-warranty period. They used the age-reduction method to model imperfect PM actions. The cost model was developed from the buyer's perspective under the life-cycle context, which consists of portion of the maintenance cost during the warranty, and all the maintenance cost after the warranty. The optimal PM interval and product life cycle were obtained by minimizing the buyer's expected long-run cost rate.

Jung et al. (2000, 2003) considered periodic PM policies after the warranty is expired. They assumed that each PM is modeled by the Canfield's model (Canfield, 1986). Two types of warranty policies, the renewing pro-rata warranty (RPRW) and renewing free-replacement warranty (RFRW), were considered for the model formulation. The optimal number of PMs and the corresponding PM interval were determined jointly such that the expected long-run maintenance cost rate is minimized.

Djamaludin et al. (2001) presented a review of the studies linking warranty and PM modeling and proposed a new continuous PM model for items sold under non-renewing warranties. They considered three maintenance options under a pre-specified warranty period w and product life cycle L. Option A assumes no PM action over the life of the item; Option B assumes continuous maintenance over $[0, L)$; and Option C assume no preventive maintenance over the warranty period and continuous preventive maintenance over $[w, L)$. Cost models for both the manufacturer and buyer were formulated, based on which the optimal maintenance option was further selected such

that the buyer's life-cycle cost is minimized. Kim et al. (2004) further extended the study under the periodic PM operations with discrete and finite number of PM levels.

Pascual and Ortega (2006) described a situation that the system undergoes periodical PM with interval T_s and is replaced by a new system at time $T_1 = nT_s$ after $n - 1$ overhauls (PM). Failures between overhauls are rectified by minimal repair. They assumed that the PM cost during the warranty is borne by the buyer so that the buyer has the option to negotiate an extended warranty period under which the seller's warranty cost remains unchanged. The optimal number of overhauls, their interval T_s, and the negotiated warranty period T_w were obtained to minimize the expected maintenance cost rate over an infinite horizon.

Chen and Chien (2007) considered a repairable system under renewing free-replacement warranty (RFRW) with the failure modes following the (p,q) rule as presented in Sheu et al. (1995). In other words, the system is subject to two types of failures, minor failure (with probability p) and catastrophic failure (with probability 1-p), and is rectified by minimal repair and a major overhaul respectively. They conducted cost analysis under a similar framework as Djamaludin et al. (2001) by assuming continuous maintenance operation. An additional penalty cost was considered for each unplanned catastrophic failure out of warranty.

The above studies mainly dealt with continuous or periodic PM services under the warranty context. On the other hand, the age-replacement policies for warranted products have also been investigated by some researchers.

Ritchken and Fuh (1986) first investigated the age-replacement policy for a non-repairable product under the warranty context. The scheduled replacement was conducted only after the warranty is expired. Sahin and Polatoglu (1996) studied two

replacement policies following the expiration of warranty: 1) replace the item at a pre-scheduled time $T + w$ ($T \geq 0$); and 2) replace it at first failure after $T + w$. For each policy, the optimal replacement age T^* (measured since the expiration of warranty) was derived under both renewing and non-renewing warranties so that the long-run average maintenance cost to the buyer is minimized.

Yeh et al. (2005) analyzed the effects of a renewing free-replacement warranty (RFRW) on the age-replacement policy for a non-repairable product with IFR. Unlike Ritchken and Fuh (1986) and Sahin and Polatoglu (1996), they considered downtime cost to the buyer upon any unplanned failure, and therefore the replacement age could fall into warranty period. They developed cost models for products both with and without warranties, and analytically derived corresponding optimal replacement ages so that the long-run average cost rates are minimized. Their results showed that the optimal replacement age for a warranted product is closer to the end of the warranty period than for a non-warranted product ($w = 0$). Extensions of this study have been made recently by considering different degrees of maintenance (Yeh et al, 2007; Chien, 2008) and variants of warranty policies (Chien, 2010a,b). Yeo and Yuan (2009) further studied age-replacement policies for warranted repairable products where the first failure within warranty is imperfectly repaired with the cost borne by the manufacturer.

CHAPTER 3 WARRANTY COST ANALYSIS FOR COMPLEX SYSTEMS WITH FAILURE INTERACTION

3.1 Introduction

One of the primary goals of warranty study is to analyze the cost of a warranty program. In a multi-component system, the assumption of failure independence among components is seldom valid, especially for those complex systems with complicated failure mechanism. For such systems, warranty cost is subject to all the factors including system structure, quality of each component and the extent of failure dependence among components. In this chapter, we present warranty cost analysis for warranted multi-component systems subject to dependent failure modes. Whenever a component (subsystem) fails, it can induce a failure of one or more of the remaining components (subsystems) of the system. Opportunistic PMs are carried out upon system failures in order to improve the system reliability and reduce the chance of future failure. Results of this chapter can help decision makers better evaluate system reliability and improve the accuracy of warranty cost estimation for complex systems.

Traditionally, warranty costs for complex systems are modeled via a "black-box" approach (Blischke and Murthy, 1996). The disadvantage of such approach lies in the fact that it does not utilize the available information of system structure, and may subsequently compromise the accuracy of warranty cost estimation. Recently, some researchers analyze the warranty costs for multi-component systems by explicitly considering the system structure (refer to Section 2.2). However, none of these studies

have incorporated failure dependence as a factor in the warranty cost modeling (Cho and Parlar, 1991).

Murthy and Nguyen (1985a) formulated three types (Type I-III) of failure dependence models for a two-component system. They referred them as failure interaction models. Type I failure interaction assumes that whenever a component fails, it can induce a simultaneous failure of one or more of the remaining components of the system. They defined this simultaneous failure of the remaining components as "induced failure", compared with so called "natural failure" described by components' life time distribution without failure interaction. The extension work to multi-component system under Type I failure interaction can be further found in Murthy and Nguyen (1985b) which described a special N-component system where one component fails naturally but induces a failure of the remaining $(N-1)$ components with certain probability. Type II failure interaction is known as failure rate interaction, which assumes a modification on the component's failure rate when another component fails. It was discussed in Murthy and Casey (1987), Zequeira and Berenguer (2005), and Lai and Chen (2008). A combination of Type I and Type II is referred as Type III failure interaction. So far, failure interactions were mainly discussed in pure maintenance context. Warranty cost models under failure interaction are seldom explored.

In this chapter, we present warranty cost analysis for both series and parallel systems subject to Type I failure interaction under renewing free-replacement warranty (RFRW) policies. Significance of this research can be justified from three aspects. First, instead of assuming independent failures among components, we derive the EWCs incorporating the factor of failure interaction. Second, we generalize the model in Murthy and Nguyen (1985b) by considering failure interaction between each two components instead of that between one component and the whole remaining system.

Third, the study of warranty costs for basic series and parallel systems provides a foundation for the future study of even more complex system configurations, such as parallel-series, series-parallel, hierarchical, and *k*-out-of-*n* (Kuo et al., 2001).

The rest of this chapter is organized as follows. Section 3.2 presents the model assumptions on failure interaction, maintenance strategy and warranty policy. Sections 3.3 and 3.4 formulate the warranty cost models under series and parallel system configurations. Section 3.5 provides a numerical example for a 3-component parallel system satisfying memoryless property with failure interaction. Conclusions and potential extensions are made in Section 3.6.

3.2 General Model and Assumptions

This section provides consideration of general model and preliminary assumptions for RFRW and failure interaction.

3.2.1 Renewing Free-Replacement Warranty (RFRW)

We follow the same assumptions for RFRW as presented in Bai and Pham (2006). Let w represent the length of warranty period and $T_S = \sum t_i + w$ represent the total time under warranty (i.e. warranty cycle), where t_i $(t_i < w)$ is the system life time between two consecutive failures within warranty.

Assumptions on RFRW for multi-component systems:

1. Failed components upon each system failure within T_S are fully replaced. Simultaneously, warranty terms are renewed by manufacturers.
2. In addition to replacement service, system maintenance is to be conducted in order to reduce the chances of future failures. We assume perfect

maintenance (Pham and Wang, 1996) operation, so that after each warranty service, the restored system is AGAN.

3. Warranty service cost per system failure is decomposed into two parts: the replacement cost for the failed components and the system maintenance cost. The replacement cost per failure is random and depends on the component-level replacement cost and the system failure mechanism.
4. In order to reduce the complexity of the cost model, system maintenance cost per failure is assumed constant (Wang, 2002). It aggregates the cost for maintenance set up, diagnosis, labor, and possible preventive maintenance efforts (only for components subject to IFRs), and averages them throughout all the system failures within warranty cycle T_S.
5. Both replacement and maintenance efforts are free of charge to customers.

3.2.2 Failure Interaction

We consider Type I failure interaction model (Murthy and Nguyen, 1985a,b) as in the following:

1. Denote $\Omega = \{1,2,\dots,n\}$. Consider an n-component system with either series or parallel structure. For each component, there are two types of failures - natural and induced. The natural failure of component i is governed by the life time distribution function $F_i(t)(i \in \Omega)$, whereas the induced failure is described by the failure interaction mechanism.
2. Type I failure interaction assumes that a natural failure of component i can cause the induced failure of component j with probability p_{ij}, and can have no effect on the operating condition of component j with probability $q_{ij} =$

$1 - p_{ij}$, $i, j \in \Omega$. Apparently, $p_{ii} = 1, q_{ii} = 0$, and the failure interaction information is integrated into two matrixes:

Failure dependence probability matrix **(FDPM)** with $P = (p_{ij}), p_{ij} \in [0,1], \forall i, j \in \Omega$,

$$\begin{bmatrix} 1 & p_{12} & \cdots & \cdots & p_{1,n-1} & p_{1,n} \\ p_{21} & 1 & \cdots & \cdots & p_{2,n-1} & p_{2,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ p_{n-1,1} & p_{n-1,2} & \cdots & \cdots & 1 & p_{n-1,n} \\ p_{n,1} & p_{n,2} & \cdots & \cdots & p_{n,n-1} & 1 \end{bmatrix}$$

and ***Failure independence probability matrix*** **(FIPM)** with $Q = (q_{ij}) = (1 - p_{ij}), q_{ij} \in [0,1], \forall i, j \in \Omega$.

Remarks: Failure interactions among components are either due to system ageing mechanisms or design problems. It would decrease the system reliability and increase both the maintenance and warranty cost, neither of which is expected by the manufacturers under RFRW terms. Usually for a well-designed system, FDPM is a sparse matrix and with a small non-zero value for each entry. Throughout this chapter, only Type I failure interaction is considered. The extended study of Type II and III failure interaction is left for future work.

3.2.3 Definition of $N_s(w)$, $p_i(w)$ and $\alpha_i(w)$

Finally, the following definitions are required.

1. Define $N_S(w)$ as the number of warranty renewals within warranty cycle T_S. Let $f_S(t)$ and $F_S(t)$ represent the probability density function (pdf) and the cumulative distribution function (cdf) of the system life times respectively. $R_S(t) = 1 - F_S(t)$ is the system reliability function, and $h_S(t) = \frac{f_S(t)}{R_S(t)}$ is the

system hazard rate function. Similarly, $f_i(t), F_i(t), R_i(t), h_i(t)$ is defined as the pdf, cdf, reliability function, and hazard rate function of component i, $i \in \Omega$.

2. Under perfect maintenance assumption, the system is AGAN after each warranty service. As such the number of warranty renewals simply follows geometric distribution with

$$P[N_S(w) = n] = [F_S(w)]^n R_S(w) \quad \forall n, n = 0,1,2, \ldots$$

and the warranty expires only when the first time the system lifetime exceeds w.

3. Let $T_i (i \in \Omega)$ be the time to nature failure of component i and $Y_i = \min(T_j, \forall j \in \Omega, j \neq i)$ be the shortest life time of the remaining components. For series system structure, define $p_i(w) \equiv \Pr[T_i \leq Y_i, T_i \leq w]$ and $\alpha_i(w) \equiv \frac{p_i(w)}{F_S(w)}$. $p_i(w)$ can be interpreted as the probability that a natural failure of component i occurs within w. Given a system failure before w, $\alpha_i(w)$ describes the probability that it is caused by a natural failure of component i. The exact expressions of $p_i(w)$ and $\alpha_i(w)$ are presented in Lemma 3.1.

Lemma 3.1. For an n-component series system with failure interaction, following the foregoing definitions of T_i, Y_i, $p_i(w)$ and $\alpha_i(w)$, we have

$$p_i(w) = \int_0^w h_i(t) R_S(t)\, \mathrm{d}t$$

$$\alpha_i(w) = \frac{1}{F_S(w)} \int_0^w h_i(t) R_S(t)\, \mathrm{d}t$$

$$\sum_{i=1}^{n} p_i(w) = F_S(w), \qquad \sum_{i=1}^{n} \alpha_i(w) = 1$$

where $h_i(t)$ and $h_S(t)$ represent the component and system hazard rate function, respectively.

Proof: See the Appendix A for the proof.

Remarks: The analytic expressions of $F_S(w), R_S(w), h_S(w)$ depend on the system structure and the degree of failure interaction among components. It is intuitive to say that system reliability will decrease when failure interaction intensifies (series system is an exception as illustrated in Lemma 3.1). Referring to the definition of $p_i(w)$ and $\alpha_i(w)$, as long as the exact expression of $f_i(t)$ is available, $p_i(w)$ and $\alpha_i(w)$ can be obtained numerically. Specifically if the life time of each component satisfies exponential distribution, i.e. $T_i \sim \exp(\lambda_i)$, we have $h_i(w) = \lambda_i$ and $\alpha_i(w) = \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}$.

3.3 RFRW for Series Systems under Failure Interaction

In this section, we follow the definition of $T_S, N_S(w), p_i(w), \alpha_i(w), F_i(w), F_S(w)$ in Section 3.2.3 and derive the warranty cost models for n-component series systems under failure interaction. Apparently, we have

$$F_S(w) = 1 - \prod_{i=1}^{n} \big(1 - F_i(w)\big) = 1 - \prod_{i=1}^{n} R_i(w) \tag{3.1}$$

We further define $N_i (i \in \Omega)$ as the number of natural failures of component i within warranty cycle T_S, and N_{ij} $(i \neq j)$ as the number of induced failures of component j caused by the natural failures of component i per cycle. For consistency of notation, we let $N_{ii} = N_i, \forall i \in \Omega$.

3.3.1 Distribution of N_{ij}

First of all, we identify the independence relationship among several random variables:

(1) $N_{ik}|N_i$ and $N_{jl}|N_j$ are independent, $i \neq j, \forall i, j, k, l \in \Omega$;

(2) N_i and N_{ij} are dependent; $N_{ij}|N_i \sim Binomial(N_i, p_{ij})$; $i, j \in \Omega$;

$N_{ik}|N_i$ can be interpreted as the number of induced failures of component k caused by the natural failures of component i when N_i is known. The conditional distribution of $N_{ik}|N_i$ depends on the failure interaction probability p_{ik} and the number of N_i. As p_{ik} is assumed constant, each natural failure of component i can be treated as a Bernoulli experiment of which $N_{ik}|N_i$ satisfies Binomial Distribution. Besides, although $N_{ik}|N_i$ and $N_{jl}|N_j$ are independent, N_{ik} and N_{jl} are dependent. They all depend on the number of system failures N_S.

Using the above results, the distribution of N_{ij} can be obtained as follow.

Lemma 3.2. For any $i, j \in \Omega, i \neq j$, the probability mass function (pmf) of N_{ij} follows a geometric distribution with parameter $\frac{R_s(w)}{R_s(w)+p_{ij}p_i(w)}$ and

$$\Pr(N_{ij} = n_{ij}) = \left(\frac{p_{ij}p_i(w)}{R_s(w) + p_{ij}p_i(w)}\right)^{n_{ij}} \frac{R_s(w)}{R_s(w) + p_{ij}p_i(w)}, \quad n_{ij} = 0,1,2, \ldots \quad (3.2)$$

$$E[N_{ij}] = \frac{p_{ij}p_i(w)}{R_s(w)} \quad (3.3)$$

Typically, when $i = j$, the number of nature failures of component i satisfies

$$\Pr(N_i = n_i) = \left(\frac{p_i(w)}{R_s(w) + p_i(w)}\right)^{n_i} \frac{R_s(w)}{R_s(w) + p_i(w)}, \quad n_i = 0,1,2, \ldots \quad (3.4)$$

with the expected number of natural failures within w given by

$$E[N_i] = \frac{p_i(w)}{R_s(w)} \tag{3.5}$$

Proof. The proof is based on Lemma 3.1 and the exact expression of $\Pr(N_i|N_S)$. We refer to Appendix B for the detailed proof.

3.3.2 Warranty Cost Analysis

According to the assumptions for RFRW in Section 3.2.1, the warranty cost (CW) can be divided into two parts: component-level replacement cost and system maintenance cost. We define com_i as the replacement cost of component $i(i \in \Omega)$ and c_M as the perfect system maintenance cost. Then

$$CW = \sum_{i,j\in\Omega} com_j N_{ij} + c_M N_S(w) = \sum_{i,j\in\Omega} com_j N_{ij} + \sum_{i\in\Omega} c_M N_i \tag{3.6}$$

$$E[CW] = E\left[\sum_{i,j\in\Omega} com_j N_{ij} + \sum_{i\in\Omega} c_M N_i\right] = \sum_{i\in\Omega} E\left[\sum_{j\in\Omega} com_j N_{ij} + c_M N_i\right]$$

$$= \sum_{i\in\Omega}\left\{\sum_{j\in\Omega} com_j E[N_{ij}] + c_M E[N_i]\right\} \tag{3.7}$$

According to Lemma 3.2, the EWC can be easily obtained.

Corollary 3.1. Under the RFRW policy, the EWC for an *n*-component series system with failure interaction is given by

$$E[CW] = \sum_{i\in\Omega}\left\{\frac{p_i(w)}{R_S(w)}\left(\sum_{j\in\Omega} com_j p_{ij} + c_M\right)\right\} \tag{3.8}$$

Proof. The deviation of $E[CW]$ is straightforward. Substituting (3.3) and (3.5) into (3.7), the results then follow.

Remarks: Under series structure, failure interaction only affects the total warranty cost *CW*, but has no impact on the system failure time distribution $F_S(w)$. Each time a natural failure happens, the system will fail despite of the presence of induced failures. In the following we further derive the warranty cost functions for parallel systems. Since all the components need to failure before the entire system fails, $F_S(w)$ is expected to depend on the degree of failure interaction.

3.4 RFRW for Parallel Systems under Failure Interaction

Under parallel structure, a system is failed given that all components in it are failed. According to the assumptions for RFRW, the joint replacement and system maintenance cost per failure is constant and given by $\sum_{i=1}^{n} com_i + c_M$. System maintenance cost here can purely refer to the cost for maintenance set up and labor, whereas the maintenance cost for series systems may further include the preventive maintenance cost for the survived components in order to restore their operating conditions to AGAN.

On the other hand, the number of warranty renewals $N_S(w)$ satisfies geometric distribution with parameter $R_S(w)$, which is the value of system reliability at time w. Then the EWC for an *n*-component parallel system is straightforward.

Corollary 3.2. Under the RFRW policy, the EWC for an *n*-component parallel system with failure interaction is given by

$$E[CW] = \left(\sum_{i=1}^{n} com_i + c_M\right)\frac{F_S(w)}{R_S(w)} \tag{3.9}$$

Proof. Simply referring to the property of geometrical distribution, the results then follow.

In order to calculate the total warranty cost for a parallel system, we need to calculate either $F_S(w)$ or $R_S(w)$, which may resort to numerical procedures if the number of components is large or if the failure time distribution is arbitrary. In the following we present a simple recursive algorithm for calculating and approximating $F_S(w)$. The availability of analytical expression of $F_S(w)$ is illustrated when each component has an exponentially distributed life time.

3.4.1 Recursive Algorithms for the Calculation of $F_S(w)$

Let S_Ω represent the n-component parallel system with $\Omega = \{1,2,\dots,n\}$, and S_φ represent the subsystem of S_Ω with $\varphi \subseteq \Omega$. For each subsystem S_φ, let $P_\varphi(a,b)$ denote the cdf of failure time distribution before b conditioned on that "no component fails before time a". It is easy to verify that, for $\forall a,b,c \geq 0, a < b, P_\emptyset(a,b) = 1$ and $P_\varphi(c,c) = 0, \varphi \neq \emptyset$.

We assume that two or more natural failures at one time are not allowed (or with zero probability). Then the failure of S_Ω within w can be divided into n conditions, depending on which component fails first. Recall that T_i is the time to natural failure of component i, and $Y_i = \min(T_j, \forall j \in \Omega, j \neq i)$. Define Z_i as the system failure time given that component i fails first. Then we have $\Pr\{T_S \leq w\} = \sum_{i\in\Omega} \Pr\{T_i \leq w, T_i \leq Y_i, T_S \leq w\}$, and thus

$$P_\Omega(a,w) = \sum_{i\in\Omega} \int_a^w G_\Omega(Z_i \leq w, Y_i \geq t) \frac{\prod_{j\neq i, j\in\Omega} R_j(t)}{\prod_{j\in\Omega} R_j(a)} dF_i(t) \qquad (3.10)$$

where $F_i(t)$ is the cdf of failure time distribution of component i, and

$$G_\Omega[Z_i \le w, Y_i \ge t] = \Pr(\text{no induced failure happens})\, P_{\Omega-\{i\}}(t,w)$$

$$= \sum_{j\in\Omega-\{i\}} \Pr\big(\text{only } E_{ij} \text{ happens}\big)\, P_{\Omega-\{i,j\}}(t,w)$$

$$+ \sum_{j,k\in\Omega-\{i\}} \Pr\big(\text{only } E_{ij}, E_{ik} \text{ happens}\big)\, P_{\Omega-\{i,j,k\}}(t,w) + \cdots$$

$$+ \sum_{j\in\Omega-\{i\}} \Pr\big(\text{all but } E_{ij} \text{ happen}\big) P_{\{j\}}(t,w)$$

$$+ \Pr\big(E_{ij} \text{ happens for any } j \in \Omega - \{i\}\big) P_\emptyset(t,w)$$

E_{ij} is defined as the event that component j is failed induced by the natural failure of component i, $\forall i, j \in \Omega$. Following the assumption of Type I failure interaction, we have

$$G_\Omega[Z_i \le w, Y_i \ge t]$$

$$= \left(\prod_{l\in\Omega-\{i\}} q_{il}\right) P_{\Omega-\{i\}}(t,w) + \sum_{j\in\Omega-\{i\}} \left[\left(\prod_{l\in\Omega-\{i,j\}} q_{il}\right) p_{ij} P_{\Omega-\{i,j\}}(t,w)\right]$$

$$+ \sum_{j,k\in\Omega-\{i\}} \left[\left(\prod_{l\in\Omega-\{i,j,k\}} q_{il}\right) p_{ij} p_{ik} P_{\Omega-\{i,j,k\}}(t,w)\right] + \cdots$$

$$+ \sum_{j\in\Omega-\{i\}} \left[q_{ij}\left(\prod_{l\in\Omega-\{i,j\}} p_{il}\right) P_{\{j\}}(t,w)\right] + \prod_{j\in\Omega-\{i\}} p_{ij} \tag{3.11}$$

Specifically, $F_S(w)$ can be interpreted as the cdf of failure time distribution before w conditioned on that "no component fails before time 0". Therefore $P_\Omega(0,w) = F_S(w)$, and we have

$$F_S(w) = \sum_{i\in\Omega} \Pr\,[Z_i \le w, T_i \le Y_i] = \sum_{i\in\Omega} \int_0^w G_\Omega(Z_i \le w, Y_i \ge t) \prod_{j\ne i, j\in\Omega} R_j(t)\, dF_i(t) \tag{3.12}$$

Analytical expression for (3.10) (and (3.12)) is not available in general. However, a simple numerical procedure can help approximate the result. The basic idea is to discretize the integral operator in (3.10). Let $0 = x_0 < x_1 < x_2 < \cdots < x_\theta = w$. Note that $P_\varphi(w, w) = 0, \varphi \subseteq \Omega, \varphi \neq \emptyset$. For any $j = 0,1,2, \ldots \theta - 1$, we have

$$P_\Omega(x_j, w) = \sum_{i\in\Omega}\left(\sum_{k=j}^{\theta-1}\int_{x_k}^{x_{k+1}} G_\Omega(Z_i \le w, Y_i \ge t)\frac{\prod_{j\neq i, j\in\Omega} R_j(t)}{\prod_{j\in\Omega} R_j(x_j)} dF_i(t)\right)$$

$$\approx \sum_{i\in\Omega}\left(\sum_{k=j}^{\theta-1} G_\Omega(Z_i \le w, Y_i \ge x_k)\frac{\prod_{j\neq i, j\in\Omega} R_j(x_k)}{\prod_{j\in\Omega} R_j(x_j)}\left(F_i(x_{k+1}) - F_i(x_k)\right)\right) \tag{3.13}$$

where $G_\Omega(Z_i \le w, Y_i \ge x_k)$ is a linear combination of $P_\varphi(x_k, w)$ for all the subsystems φ, and the initial condition of the numerical procedure satisfies $P_{\{i\}}(x_k, w) = \frac{F_i(w) - F_i(x_k)}{R_i(x_k)}$ for any $i \in \Omega$ and $k = 0,1,2, \ldots \theta - 1$.

Remarks: According to the recursive formula presented in (3.10)–(3.13), we need to calculate the values of $P_\varphi(x_j, w)$ for $j = 0,1,2, \ldots, \theta - 1$ and $\varphi \subseteq \Omega$ in order to obtain $P_\Omega(0, w)$ $(= F_S(w))$, and the number of these specific value points is $(2^n - 1)\theta$. Apparently, it becomes computational demanding when n is large. However, the occurrence of either of the following conditions would significantly simplify the calculation process: (i) identical life time distribution for all the components, which means that the number of subsets is reduced from $2^n - 1$ to $n - 1$, or (ii) exponential life time distribution for each component. In particularly, for condition (ii), the expression of $F_S(w)$ can be analytically derived. To demonstrate this, we present an illustrative example as below.

3.4.2 Illustration Example for Memoryless System

Exponential distribution has been widely adopted in the literature for the system reliability and maintenance modeling. It can also be used to approximate some of the most commonly used distributions, such as Weibll distribution (Xie et al., 2000).

First of all, we show the memoryless property of the parallel system with failure interaction when the life time of each component is exponentially distributed.

Lemma 3.3. For an n-component parallel system S_Ω with failure interaction, if $T_i \sim \exp(\lambda_i)$ for any $i \in \Omega$, we have

$$P_\Omega(x, y) = P_\Omega(x - a, y - a), \;\; 0 \le a \le x \le y \tag{3.14}$$

Proof: The proof is presented in Appendix C by applying the *Mathematical Induction* (MI).

Specifically, if $x = a$, we have

$$P_\Omega(0, y - x) = P_\Omega(x, y) \tag{3.15}$$

In the following we let $P_\Omega(0, y - x) = P_\Omega(y - x)$ for simplicity.

Remarks: Lemma 3.3 is an exhibition of memoryless property generalized from the component level to system level under parallel structure. It indicates that at any time t, the system is considered as a new one as long as no components fail before t.

Given the above results, $P_\Omega(0, w)$ can be calculated by resorting to Laplace transform (LST) technique. Note that $T_i \sim \exp(\lambda_i)$, $i \in \Omega$. Taking the LST of (3.12), we have

$$\mathcal{L}[P_\Omega(0,w)] = \sum_{i\in\Omega} \mathcal{L}\left[\int_0^w G_\Omega(Z_i \le w, Y_i \ge t) \prod_{j\ne i, j\in\Omega} R_j(t)\, dF_i(t)\right]$$

$$= \sum_{i\in\Omega} \mathcal{L}\left[\int_0^w G_\Omega(Z_i \le w-t, Y_i \ge 0)\lambda_i e^{-\sum_{j=1}^n \lambda_j t}\, dt\right]$$

$$= \sum_{i\in\Omega} \mathcal{L}[G_\Omega(Z_i \le w, Y_i \ge 0)] \frac{\lambda_i}{s+\sum_{j=1}^n \lambda_j} \tag{3.16}$$

where $\mathcal{L}[G_\Omega(Z_i \le w, Y_i \ge 0)]$ is expressed as

$$\mathcal{L}[G_\Omega(Z_i \le w, Y_i \ge 0)]$$

$$= \left(\prod_{j\in\Omega-\{i\}} q_{ij}\right) \mathcal{L}\left[P_{\Omega-\{i\}}(0,w)\right]$$

$$+ \sum_{j\in\Omega-\{i\}} \left[\left(\prod_{l\in\Omega-\{i,j\}} q_{il}\right) p_{ij} \mathcal{L}\left[P_{\Omega-\{i,j\}}(0,w)\right]\right]$$

$$+ \sum_{j,k\in\Omega-\{i\}} \left[\left(\prod_{l\in\Omega-\{i,j,k\}} q_{il}\right) p_{ij} p_{ik} \mathcal{L}\left[P_{\Omega-\{i,j,k\}}(0,w)\right]\right] + \cdots$$

$$+ \sum_{j\in\Omega-\{i\}} \left[q_{ij}\left(\prod_{l\in\Omega-\{i,j\}} p_{il}\right) \mathcal{L}\left[P_{\{j\}}(0,w)\right]\right] + \frac{1}{s}\prod_{j\in\Omega-\{i\}} p_{ij} \tag{3.17}$$

In the following we use a 3-component parallel system as an example to explain the calculation procedure.

3.5 Numerical Example and Sensitivity Analysis

Consider a 3-component parallel system shown in Figure 3.1(a) under the RFRW policy. Suppose that system maintenance cost per failure is $c_M = \$220$ and the length of warranty period is $w = 3$ time units. Each time unit is 6 months. All other

parameters needed for the warranty cost analysis are given in Table 3.1. The parameters of components' failure times were chosen such that $R(t) > 0.80$ for $t \leq 10$ time units (5 years). Here we assume that the failure times of all the components follow exponential distribution, which is a special case of Weibull distribution with shape parameter $k = 1$ and also implies that the component has a constant failure rate. For the comparison purpose, the series system composed of the same components is presented in Figure 3.1(b). Note that for both parallel and series structures, system maintenance cost does not involve any preventive maintenance cost because it is generally unnecessary for memoryless system (Barlow and Proschan, 1965).

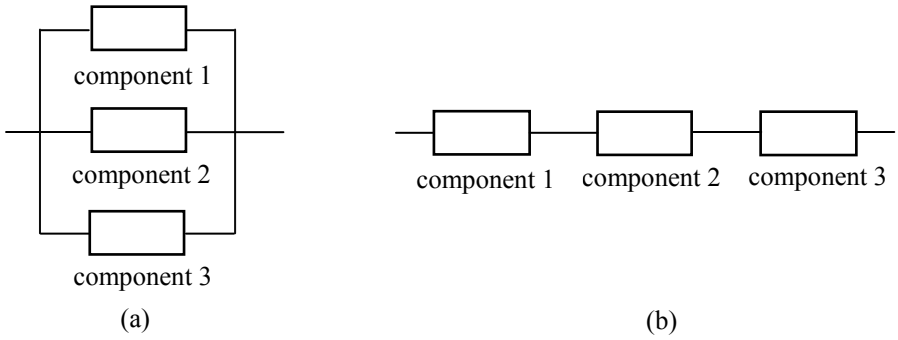

Figure 3.1 Description for three-component systems

Table 3.1 Cost and reliability parameters for system components

i	1	2	3
$\boldsymbol{com_i}$	\$200	\$230	\$180
$\boldsymbol{R_i(t)}$	$e^{-t/50}$	$e^{-t/60}$	$e^{-t/40}$

The failure dependence probability matrix (**FDPM**) $P = (p_{ij})$ $(i, j \in \Omega)$ is given as:

$$P_{3\times3} = \begin{bmatrix} 1 & 0.1 & 0.05 \\ 0.07 & 1 & 0.2 \\ 0.17 & 0.13 & 1 \end{bmatrix}$$

where $\Omega = \{1,2,3\}$ and $q_{ij} = 1 - p_{ij}, \forall i, j \epsilon \Omega$.

The applicability of the recursive algorithm presented in (3.16) and (3.17) can be demonstrated as follow.

Recursive Algorithm:

Step 1: single component sub-system

Subsystem $S_{\{1\}}$: $\mathcal{L}\left[P_{\{1\}}(w)\right] = \frac{1}{s} - \frac{1}{s+\lambda_1} = \frac{1}{s} - \frac{1}{s+1/50}$

Subsystem $S_{\{2\}}$: $\mathcal{L}\left[P_{\{2\}}(w)\right] = \frac{1}{s} - \frac{1}{s+\lambda_2} = \frac{1}{s} - \frac{1}{s+1/60}$

Subsystem $S_{\{3\}}$: $\mathcal{L}\left[P_{\{3\}}(w)\right] = \frac{1}{s} - \frac{1}{s+\lambda_3} = \frac{1}{s} - \frac{1}{s+1/40}$

Step 2: two-component sub-system

Subsystem $S_{\{1,2\}}, S_{\{1,3\}}, S_{\{2,3\}}$:

$$\begin{aligned}
\mathcal{L}\left[P_{\{1,2\}}(w)\right] &= \left[q_{12}\mathcal{L}\left[P_{\{2\}}(w)\right] + \frac{1}{s}p_{12}\right]\frac{\lambda_1}{s+\lambda_1+\lambda_2} \\
&\quad + \left[q_{21}\mathcal{L}\left[P_{\{1\}}(w)\right] + \frac{1}{s}p_{21}\right]\frac{\lambda_2}{s+\lambda_1+\lambda_2} \\
&= \frac{1}{s} - q_{21}\frac{1}{s+\lambda_1} - q_{12}\frac{1}{s+\lambda_2} + (1-p_{12}-p_{21})\frac{1}{s+\lambda_1+\lambda_2} \\
&= \frac{1}{s} - 0.93\frac{1}{s+1/50} - 0.9\frac{1}{s+1/60} + 0.83\frac{1}{s+11/300}
\end{aligned}$$

$$\mathcal{L}\left[P_{\{1,3\}}(w)\right] = \frac{1}{s} - 0.83\frac{1}{s+1/50} - 0.95\frac{1}{s+1/40} + 0.78\frac{1}{s+9/200}$$

$$\mathcal{L}\left[P_{\{2,3\}}(w)\right] = \frac{1}{s} - 0.87\frac{1}{s+1/60} - 0.8\frac{1}{s+1/40} + 0.67\frac{1}{s+5/120}$$

Step 3: 3-component parallel system $S_{\{1,2,3\}}$

$$\mathcal{L}\left[P_{\{1,2,3\}}(w)\right] = \left[q_{12}q_{13}\mathcal{L}\left[P_{\{2,3\}}(w)\right] + p_{12}q_{13}\mathcal{L}\left[P_{\{3\}}(w)\right] + p_{13}q_{12}\mathcal{L}\left[P_{\{2\}}(w)\right]\right.$$

$$\left. + \frac{1}{s}p_{12}p_{13}\right]\frac{\lambda_1}{s+\lambda_1+\lambda_2+\lambda_3}$$

$$+ \left[q_{21}q_{23}\mathcal{L}\left[P_{\{1,3\}}(w)\right] + p_{21}q_{23}\mathcal{L}\left[P_{\{3\}}(w)\right] + p_{23}q_{21}\mathcal{L}\left[P_{\{1\}}(w)\right]\right.$$

$$\left. + \frac{1}{s}p_{21}p_{23}\right]\frac{\lambda_2}{s+\lambda_1+\lambda_2+\lambda_3}$$

$$+ \left[q_{31}q_{32}\mathcal{L}\left[P_{\{1,2\}}(w)\right] + p_{31}q_{32}\mathcal{L}\left[P_{\{2\}}(w)\right] + p_{32}q_{31}\mathcal{L}\left[P_{\{1\}}(w)\right]\right.$$

$$\left. + \frac{1}{s}p_{31}p_{32}\right]\frac{\lambda_3}{s+\lambda_1+\lambda_2+\lambda_3}$$

$$= \frac{1}{s} - \frac{0.789}{s+1/50} - \frac{0.794}{s+1/40} - \frac{0.772}{s+1/60} + \frac{0.599}{s+9/200} + \frac{0.580}{s+11/300}$$

$$+ \frac{0.573}{s+5/120} - \frac{0.398}{s+37/600} \tag{3.18}$$

Step 4: Inverse LST

$$P_{\{1,2,3\}}(w) = 1 - 0.789e^{-w/50} - 0.794e^{-w/40} - 0.772e^{-w/60} + 0.599e^{-(9/200)w}$$

$$+ 0.580e^{-(11/300)w} + 0.573e^{-(5/120)w} - 0.398e^{-(37/600)w} \tag{3.19}$$

Results:

Table 3.2 gives a basic comparison of warranty costs between parallel (Figure 3.1(a)) and series (Figure 3.1(b)) structures with and without failure interaction under $w = 6$ (3 years).

Table 3.2 Cost comparison for 3-component systems with *w*=6

System	Structure	Failure interaction	$F_S(w)$	Expected warranty cost
1	series	No	0.3093	\$188.0
2	series	Yes	0.3093	\$210.3
3	parallel	No	0.0015	\$1.246
4	parallel	Yes	0.0129	\$10.89

From Table 3.2, we can see that the EWC is obviously higher under series structure (Systems 1 and 2) than parallel structure (Systems 3 and 4). It is due to that, under the series structure, the failure of any component (subsystem) will cause the system failure. Consequently, under the series structure, the improvement of reliability for each component (subsystem) is essential; failure interaction is only a minor factor (accounting for $\frac{210.3-188.0}{188.0} \times 100\% \cong 10\%$ of the total warranty cost).

For the parallel system, however, warranty cost is much more sensitive to the existence of failure interaction. Given the same case, failure interaction substantially increases the EWC, which is nearly $\frac{10.89-1.246}{1.246} \cong 8$ times more than the cost for systems without failure interaction. In practice, it is necessary for engineers to examine potential failure interaction in the system and either combine this factor into the warranty cost estimation or design it out from the system in the early stage.

Furthermore, as shown in Table 3.2, for series system, failure interaction only affects the EWC but has no effect on the system failure time distribution; under the parallel structure, however, the system may face severe reliability problems due to failure interaction.

Sensitivity analysis:

Figure 3.2 shows how warranty length w affects the EWC per system. We also illustrate the impact of structure and failure interaction to the total warranty cost.

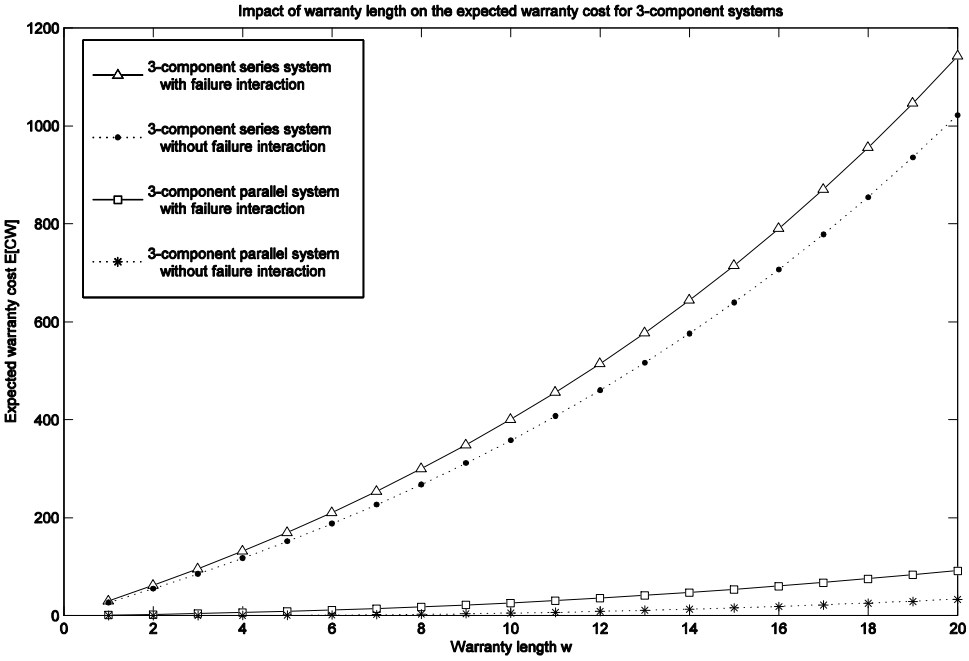

Figure 3.2 Impact of warranty length *w* on the EWC for 3-component systems

Figure 3.3 illustrates how the elements of failure dependence probability matrix (**FDPM**) (take $p(1,2)$ for example) affect the expected warranty cost of the system in Figure 3.1(a):

$$P_{3\times 3} = \begin{bmatrix} 1 & p_{12} & 0.05 \\ 0.07 & 1 & 0.2 \\ 0.17 & 0.13 & 1 \end{bmatrix}$$

It is intuitive to say that the EWC increases while any element of FDPM increases. When $p_{ij} = 1, \forall i, j \in \{1,2,\dots,n\}$, the parallel system transforms to the series structure and the failure time distribution equals to that of an n-component series system. However, the EWC is different as when $p_{ij} = 1, \forall i, j \in \{1,2,\dots,n\}$, the failure of one component will cause other components to fail, which incurs extra cost.

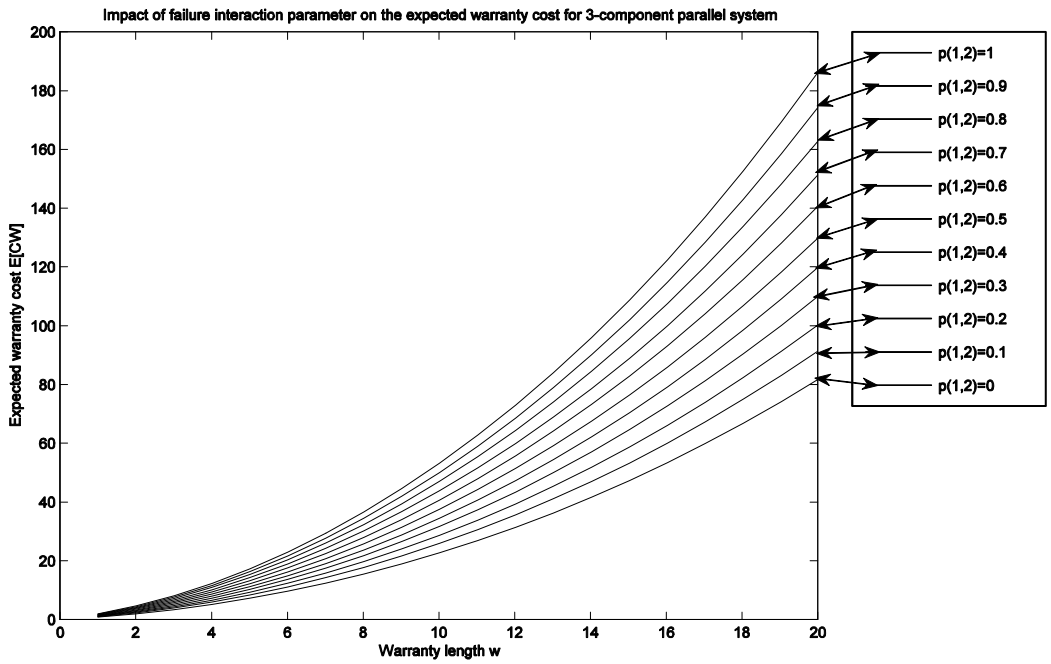

Figure 3.3 Impact of p_{12} on the EWC for 3-component parallel system

3.6 Conclusion

In this chapter, a study of the cost model under RFRW for multi-component systems considering prefect maintenance is presented. Unlike the previous works for complex systems which assume that failure events between components are independent, a new factor of failure interaction is incorporated into the cost modeling. RFRW here assumes free replacements for the failed components and perfect maintenance for the remaining system. Based on these assumptions, we derive the analytical expression of warranty cost functions for both series and parallel structures and propose the recursive algorithm for calculating and approximating the system reliability for parallel system configuration. Specifically, we present the memoryless property of the parallel system with failure interaction when all the components' life times are exponentially distributed. The numerical results reveal that, compared with series systems, failure interaction incurs higher planning uncertainty (risk) of warranty costs for systems with parallel structure. Decision makers are highly recommended to allocate extra warranty

budget to compensate the cost induced by failure interaction. It is also recommended for the engineers to improve product design in order to eliminate the failure interaction among components (subsystems) in the early stage.

Several additional remarks for our models are summarized. First, rather than simply in series or parallel connection among the components, the systems in practice exhibit much more complex structures (Kuo et al., 2001), such as series-parallel, parallel-series, hierarchical series-parallel and *k*-out-of-*n*. Although it is hard to fully apply the methodology derived in this chapter to complex systems, basic studies for series and parallel configurations provide general views of the significance of failure interaction to the system reliability and warranty cost, and show possible ways for evaluating system reliability under complex structure. Secondly, both the assumptions of perfect maintenance and constant maintenance cost in RFRW might be untenable in practice although it is widely adopted in the literature. More practical assumptions such as minimal and imperfect maintenance (Pham and Wang, 1996) and random maintenance cost (Sheu and Liou, 1992) can be adopted in future study. Third, numerical illustration in Section 3.5 is confined to multi-component systems satisfying memoryless property. However, by applying the numerical procedure proposed in (3.13) of Section 3.4.1, the application can be well extended to any arbitrary system. Finally, the parameter estimation for the elements in FDPM is excluded in this research. The reader is referred to the work in Murthy and Wilson (1994) for both Type I and Type II failure interaction.

CHAPTER 4 CONDITION-BASED WARRANTY SERVICE DESIGN

4.1 Introduction

Warranty costs constitute a significant portion of the post-sale servicing cost, typically ranging from 2% to as much as 15% of the net sales (McGuire, 1980). Designing an effective maintenance program under warranty is thereby of special importance to the manufacturer.

It is well acknowledged that the likelihood of a product failure can be reduced by scheduled preventive maintenance (PM) services. From the manufacturer's perspective, providing extra PMs during the warranty not only reduces the corrective maintenance cost of the product but also improves the service level of the company and helps attract more buyers. Two major PM approaches are identified in the literature, namely, time-based maintenance (TBM) and condition-based maintenance (CBM) (Legat et al., 1996). TBM arranges the PM services at pre-determined age or time intervals, while CBM relies on the actual state information of the product for executing the maintenance decisions. Although CBM is widely considered as a superior approach to TBM, a review of the existing literature identifies the unanimous preference of utilizing the TBM approach in warranty-related applications. For example, from the manufacturer's perspective, the age-based PM optimization models during the warranty period are given in: Jack and Dagpunar (1994), Yeh and Lo (2001), Wang (2006), Yun et al. (2008), Huang and Yen (2009) and Jack et al. (2009). From the customer's perspective, the optimal PM strategies under the life-cycle context have

been studied by works such as Chun and Lee (1992), Jung and Park (2003), Kim et al. (2004) and Pascual and Ortega (2006).

The CBM approach, on the other hand, is extensively studied under circumstances that do not include warranty considerations. In particular, two major streams of CBM models are identified in the literature. The first describes the product deterioration as a continuous wear process. Preventive maintenance is carried out when the degree of wear exceeds a threshold value (Grall et al., 2002b; Li and Pham, 2005; Liao et al., 2006; van Noortwijk, 2009). The second stream of work considers the deterioration as a continuous-time (semi-)Markov process. The critical set of states for which the PM is to be carried out is optimally determined such that expected maintenance cost is minimized (Makis and Jiang, 2003; Black et al., 2005; Wu and Ryan, 2010; Zhao et al., 2010). The inspection interval is either pre-determined or to be optimized for both cases. Moreover, as a special case of the (semi-)Markov model, the 3-state semi-Markov system has been extensively studied under the delay-time modeling (DTM) framework (Baker and Christer, 1994). The DTM technique embraces the concept of CBM by assuming that the product with such three states – good, defective and failed, should be preventively replaced only if the product is defective upon inspection. Applications of the DTM technique can be found in Christer et al. (1998), Pillay et al. (2001), Zhao et al. (2007), Scarf et al. (2009) and Wang (2010).

One of the main reasons why the CBM approach is seldom applied to the warranty context is because most of the warranty theories assume binary systems (i.e. either functioning or failed), and do not consider the intermediate states which continuous monitoring or inspection could take place. Although binary state assumption allows technical convenience for the modeling, it does realistically describe the underlying behavior of the deterioration process. An exception is Zuo et al. (2000), who studied

warranty servicing strategies for a multi-state product and proposed condition-based repair-replacement policies in order to minimize the expected warranty cost. Their work was further extended in Pan and Thomas (2010) lately by considering a larger and more general Markov state space. However, both works assume continuous monitoring (instead of periodic monitoring) on the product condition which may not be technically available or cost-efficient in practical situations. Furthermore, a survey of maintenance literature reveals that most of the CBM modeling approaches, or the PM modeling in general, assume infinite planning horizon (Nakagawa and Mizutani, 2009), while warranty cost optimization is essentially a finite-horizon maintenance problem.

This chapter considers a realistic scenario that inspection or periodic monitoring is the only option available to the manufacturer. We study a novel warranty servicing strategy that integrates the basic warranty policies with extra CBM services. For inspection to be viable, we assume that a defect may arise prior to failure, and that these defects are only detectable at inspection. Thus, the product deterioration can be modeled as a two-stage process, namely, from nominal (defect-free) to defective, and from defective (if left unattended) to failure. Preventive maintenance is only implemented when the product is found to be in the defective state. We focus on the inspection scheduling problem for products under both renewing and non-renewing warranty policies. We first consider a simple case that only one inspection service is provided during the warranty period. The objective is to find the optimal inspection time so that the expected warranty cost is minimized. We then extend the simple case to periodic inspection within the warranty and aim at optimally determining the inspection interval. Since many of the warranty servicing plans need to deal with a population of products instead of a single product of interest, we argue that the proposed condition-based servicing strategies can be well applied in the context when

the targeted market of the product is relatively small and the individual selling can be traced. Typical examples include those high-priced commercial and industrial products such as large-scale mechanical facilities, specialized medical devices, IT servicing systems, etc. In contrast, in the context of high sales quantities (typically for consumer products), the manufacturer may instead provide the inspection services as free options to the customer with execution period(s) pre-specified in the warranty contract. Also, the manufacturer may further utilize these additional inspection services to enhance his product marketing strategy for promoting the sales in the marketplace.

The rest of the chapter is organized as follows. Section 4.2 presents the problem and assumptions, and proposes various condition-based warranty servicing policies. Section 4.3 derives warranty cost models under the renewing warranty setting. The cost functions under the non-renewing warranty setting are further discussed in Section 4.4. Section 4.5 provides the numerical illustrations for the proposed models. Conclusion is made in Section 4.6.

4.2 Problem Description

4.2.1 Assumption

1) The product is sold under a free-replacement warranty. Within warranty period w, all the repair and inspection cost is borne by the manufacturer.
2) Product can have such three states: good, defective and failed. At any time, there is at most one (dominant) defect present in the product.
3) Let X represent the time to defective arrival and Y represent the delay time between the defect arrival and the subsequent failure. We assume that Y is independent of X.

4) Product is preventively replaced if a defect is detected upon inspection. Inspection is perfect in the sense that any defect present will be identified.
5) For the renewing warranty, preventive replacement of the product will not renew the warranty. The warranty is renewed only when a failure is incurred during the warranty.
6) Let C_I represent the cost for inspection, C_P represent the cost for preventive replacement and C_F represent the failure or corrective replacement cost; we assume that $C_I < C_P < C_F$.
7) Either corrective or preventive replacement restores the product to AGAN.
8) Let Z represent the time to failure without PM intervention, i.e. $Z = X + Y$. We also let $F_X(.)$, $F_Y(.)$, $F_Z(.)$ and $f_X(.)$, $f_Y(.)$, $f_Z(.)$ represent the cdf and pdf of X, Y and Z.

4.2.2 Condition-based Warranty Servicing Policies

The following condition-based warranty servicing policies are studied in this chapter. Under both renewing and non-renewing warranty settings, two types of inspection policies are further considered: one-time inspection and periodic inspection.

Policy A: The product is sold under a renewing free-replacement warranty.

1) Additional inspection service is to be implemented by the manufacturer at pre-specified time T ($T < w$). The inspection service is renewed upon failure within warranty.
2) Additional inspection services are to be implemented by the manufacturer periodically at time intervals $T, 2T, \ldots, NT$ with $N = \lceil w/T \rceil - 1$. The inspection services are renewed upon failure within warranty.

For the case of non-renewing warranty, we first consider the following inspection policies which are dependent on the failure history of the product.

Policy B: The product is sold under a non-renewing free-replacement warranty.

1) Additional inspection service is to be implemented by the manufacturer at pre-specified time T ($T < w$) given that no failure is incurred before T. Otherwise, no inspection is made at time T.
2) Additional inspection services are to be implemented by the manufacturer at pre-specified times kT for $1 \leq k \leq N = \lceil w/T \rceil - 1$ given that no failure is incurred within $((k-1)T, kT)$. Otherwise, no subsequent inspections are made.

For comparison, we also consider the "block-type" inspection policies which are independent of the failure history of the product.

Policy C: The product is sold under a non-renewing free-replacement warranty.

1) Additional inspection service is to be implemented by the manufacturer at pre-specified time T ($T < w$) irrespective of the failure history before time T.
2) Additional inspection services are to be implemented by the manufacturer at constant time intervals $T, 2T, \ldots, NT$ with $N = \lceil w/T \rceil - 1$ irrespective of the failure history between inspections.

Remarks:

a) A wide range of product types can be covered under the proposed warranty servicing policies. For example, periodic inspection policy is mostly suitable for those high-priced products with relatively long warranty coverage (typically for commercial and industrial products sold under a non-renewing warranty). Failure replacements of

these products may often be costly to the manufacturer. In contrast, for products sold with short warranty coverage or covered by a renewing warranty policy, one-time inspection policy may be appropriate in order to avoid the corrective replacement or/and warranty renewal. Note that not many high-priced products are sold under a renewing warranty. Therefore, periodic inspection Policy A2 could be less applicable compared to other policies.

b) Sometimes failure is also costly to the consumers in terms of the downtime cost per failure. In this sense Policy B can be treated as a "defensive" warranty strategy where the inspection as claimed is only provided for the customer given that no failure replacement has incurred. This requires certain maintenance efforts from the customer and therefore partially protects the benefits of the manufacturer. In contrast, Policy C is more "offensive" or aggressive in the sense that all the inspections are provided irrespective of the failure history of the product. It may be potentially useful for promoting new products in the marketplace with heavy marketing purposes.

c) Note that for Policy A1, B1 and C1, the one-time inspection may be conducted during the early stage of the warranty (e.g. $T < w/2$). Therefore, Policy A1 (B1, C1) is not necessarily a special case of Policy A2 (B2, C2) when $N = 1$. This requires further comparisons between the one-time inspection and periodic inspection policies in order to obtain the global optimal policy.

4.3 Cost Modeling for Renewing Warranty

In this section, the warranty cost functions are derived incorporating inspection services under the renewing warranty setting. An example is shown later to illustrate the application of the model.

We first present some preliminary results for the cost derivation in this section.

4.3.1 Preliminary Results

Consider a product that is sold under a renewing warranty and is subject to CBM service(s) at appropriate time(s) during the warranty period. Let K be the time to product failure since the last warranty renewal and $n-1$ represent the number of replacements until the product survives the warranty. Clearly the random variable n satisfies the following geometric distribution

$$\Pr\{n=i\} = (\Pr\{K \le w\})^{i-1} \Pr\{K > w\}, \;\; i = 1,2,\ldots \tag{4.1}$$

with a finite expectation $E[n] = 1/\Pr\{K > w\}$.

Further let i.i.d. $\{K_1, K_2, \ldots\}$ represent the sequence copies of K and $\{C_1, C_2, \ldots\}$ represent the corresponding cycle cost. The total warranty cost CW can be described by $CW = \sum_{i=1}^{n} C_i$ where n is the stopping time of the sequence $\{C_1, C_2, \ldots\}$. Applying the Wald's Equation (Ross, 1970), the expected warranty cost (EWC) can be given as

$$E[CW] = E\left[\sum_{i=1}^{n} C_i\right] = E[n]E[C_1] = \frac{E[C_1]}{\Pr\{K > w\}} \tag{4.2}$$

Therefore, for the renewing warranty, the problem is simplified to that of evaluating the expected cycle cost $E[C_1]$ and the survival function $\Pr\{K > w\}$ respectively.

As a special case, if no inspection or PM are conducted during the warranty, K reduces to Z where $\Pr\{K \le t\} = F_Z(t) = \int_0^t f_X(u)F_Y(t-u)du$. In this case the expected cycle cost per item is $E[C_1] = F_Z(w)C_F$, and the EWC is further given by

$$E[CW] = \frac{F_Z(w)C_F}{\bar{F}_Z(w)} \tag{4.3}$$

4.3.2 Warranty Cost Model for Policy A1

Here we derive the warranty cost function for Policy A1. Suppose an additional one-time inspection service is conducted at time T ($T < w$). In order to derive the expected cycle cost $E[C_1]$, two cases are considered here: $K \leq w$ and $K > w$.

- $K \leq w$

We first consider three scenarios depending on the time to the defect arrival X and time to the failure Z (without PM intervention).

1) If $Z < T$, the product is correctly replaced with total cost $C_1 = C_F$.
2) If $X < T$ and $Z > T$, the product is preventively replaced at time T with cost $C_I + C_P$; the product fails within $[T, w]$ thereafter and the total cost is $C_1 = C_I + C_P + C_F$.
3) If $T < X < w$ and $Z < T$, the product is inspected at cost C_I but no PM is carried out; the product fails within $[X, w]$ thereafter and the total cost is $C_1 = C_I + C_F$.

- $K > w$

Again, we consider the following three scenarios.

1) If $X < T$ and $Z > T$, the product is preventively replaced at time T with total cost $C_1 = C_I + C_P$; the product does not fail within $[T, w]$.
2) If $T < X < w$ and $Z > w$, the product is inspected at cost C_I but no PM is carried out; the product does not fail within $[X, w]$ and the total cost is $C_1 = C_I$.
3) If $X > w$, the product is inspected at cost $C_1 = C_I$; the product does not fail within $[T, w]$ and the total cost is $C_1 = C_I$.

Based on the above scenarios, the expected cycle cost can be derived as

$$E[C_1] = C_F F_Z(T) + (C_I + C_P + C_F)\left(\int_0^T f_X(t)\bar{F}_Y(T-t)dt\right)F_Z(w-T)$$

$$+ (C_I + C_F)\int_T^W f_X(t)F_Y(w-t)dt$$

$$+ (C_I + C_P)\left(\int_0^T f_X(t)\bar{F}_Y(T-t)dt\right)\bar{F}_Z(w-T)$$

$$+ C_I\int_T^W f_X(t)\bar{F}_Y(w-t)dt + C_I\bar{F}_X(w) \tag{4.4}$$

The survival function $\Pr\{K > w\}$ is given by

$$\Pr\{K > w\} = \left(\int_0^T f_X(t)\bar{F}_Y(T-t)dt\right)\bar{F}_Z(w-T) + \int_T^w f_X(t)\bar{F}_Y(w-t)\,dt + \bar{F}_X(w) \tag{4.5}$$

Referring to (4.2), the EWC as a function of T can be verified as

$$E[CW] = \frac{1}{\bar{F}_Z(T)\bar{F}_Z(w-T) + \bar{F}_X(T)F_Z(w-T) - \int_T^w f_X(t)F_Y(w-t)\,dt}$$

$$\times\left(C_F\left[\int_T^w f_X(t)F_Y(w-t)dt + F_Z(w-T)F_X(T) + \bar{F}_Z(w-T)F_Z(T)\right]\right.$$

$$\left. + C_I\bar{F}_Z(T) + C_P\big(F_X(T) - F_Z(T)\big)\right) \tag{4.6}$$

4.3.3 Warranty Cost Model for Policy A2

Here we further consider the periodic inspection policy for renewing warranty settings. Suppose that the product is inspected at times $T, 2T, \ldots, NT$ and the number of inspections N is given as $\lceil w/T \rceil - 1$. We let $\varepsilon = w - NT$ represent the residual warranty period since the Nth inspection. Note that ε is a constant once T is determined.

As in Policy A1, in the following we derive the expression for $E[C_1]$ and $\Pr\{K > w\}$ respectively. The results can be characterized conveniently in a recursive form presented as follows.

Given inspection at each T time units, define $C^{(J)}(T,\varepsilon)$ as the expected cycle cost for a new system when its warranty length is $JT+\varepsilon$ and $R^{(J)}(T,\varepsilon)$ as its corresponding survive function over this warranty period. The problem then resolves into the evaluation of the expression for $E[C_1] = C^{(N)}(T,\varepsilon)$ and $\Pr\{K > w\} = R^{(N)}(T,\varepsilon)$. Depending on when the product is preventively replaced upon inspection, the following recursive equations on $C^{(J)}(T,\varepsilon)$ and $R^{(J)}(T,\varepsilon)$ can be derived.

$$
\begin{aligned}
C^{(J)}(T,\varepsilon) = &\sum_{i=0}^{J-1}\left[(i+1)C_I + C_P + C^{(J-i-1)}(T,\varepsilon)\right]\int_{iT}^{(i+1)T} f_X(t)\bar{F}_Y[(i+1)T-t]dt \\
&+ JC_I\left(\int_{JT}^{JT+\varepsilon} f_X(t)\bar{F}_Y(JT+\varepsilon-t)dt + \int_{JT+\varepsilon}^{\infty} f_X(t)dt\right) \\
&+ \sum_{i=0}^{J-1}(iC_I + C_F)\int_{iT}^{(i+1)T} f_X(t)F_Y[(i+1)T-t]dt \\
&+ (JC_I + C_F)\int_{JT}^{JT+\varepsilon} f_X(t)F_Y(JT+\varepsilon-t)dt
\end{aligned}
$$

which is further simplified as

$$
\begin{aligned}
C^{(J)}(T,\varepsilon) = &\sum_{i=0}^{J-1}\left[(i+1)C_I + C_P + C^{(J-i-1)}(T,\varepsilon)\right]\int_{iT}^{(i+1)T} f_X(t)\bar{F}_Y[(i+1)T-t]dt \\
&+ JC_I\bar{F}_X(JT) + \sum_{i=0}^{J-1}(iC_I + C_F)\int_{iT}^{(i+1)T} f_X(t)F_Y[(i+1)T-t]dt \\
&+ C_F\int_{JT}^{JT+\varepsilon} f_X(t)F_Y(JT+\varepsilon-t)dt \qquad (4.7)
\end{aligned}
$$

where $C^{(0)}(T,\varepsilon) = C_F F_Z(\varepsilon)$ and similarly,

$$R^{(J)}(T,\varepsilon) = \sum_{i=0}^{J-1} R^{(J-i-1)}(T,\varepsilon) \int_{iT}^{(i+1)T} f_X(t)\bar{F}_Y[(i+1)T-t]dt$$

$$+ \int_{JT}^{JT+\varepsilon} f_X(t)\bar{F}_Y(JT+\varepsilon-t)dt + \int_{JT+\varepsilon}^{\infty} f_X(t)dt$$

$$= \sum_{i=0}^{J-1} R^{(J-i-1)}(T,\varepsilon) \int_{iT}^{(i+1)T} f_X(t)\bar{F}_Y[(i+1)T-t]dt$$

$$- \int_{JT}^{JT+\varepsilon} f_X(t)F_Y(JT+\varepsilon-t)dt + \bar{F}_X(JT) \tag{4.8}$$

where $R^{(0)}(T,\varepsilon) = \bar{F}_Z(\varepsilon)$.

Given (4.2), (4.7) and (4.8), the EWC as a function of T can be expressed as

$$E[CW] = \frac{C^{(N)}(T,\varepsilon)}{R^{(N)}(T,\varepsilon)} \tag{4.9}$$

Remarks: For both Policies A1 and A2, the optimal inspection policy T^* can be obtained numerically by searching over the domain of $T \in (0,w)$ so that the respective warranty cost is minimized. For the case of periodic inspection, it is important to obtain a lower bound of T^* as when T decreases, the number of inspections will increase significantly. To derive this lower bound value, we notice that a meaningful inspection policy T should incur a total cost less than the cost without any inspection (i.e. $C_F F_Z(w)/\bar{F}_Z(w)$). According to (4.7)–(4.9), we also verify that $E[C_1] > NC_I R^{(N)}(T,\varepsilon)$. Therefore it is clear that $\frac{NC_I R^{(J)}(T,\varepsilon)}{R^{(J)}(T,\varepsilon)} < C^{(N)}(T,\varepsilon) < \frac{F_Z(w)}{\bar{F}_Z(w)} C_F \Rightarrow N < \frac{F_Z(w)}{\bar{F}_Z(w)} \frac{C_F}{C_I}$ and the optimal inspection interval T^* can then be searched within $\left[\frac{w}{\hat{N}+1}, w\right)$ where $\hat{N} = \left\lfloor \frac{F_Z(w)}{\bar{F}_Z(w)} \frac{C_F}{C_I} \right\rfloor$.

4.4 Cost Modeling for Non-Renewing Warranty

In this section, warranty cost functions are derived under non-renewing warranty setting. We first consider the case when the inspection service is dependent on the failure history of the product (Policy B). For comparison, the "block-type" inspection policy (Policy C) is subsequently analyzed by considering additional assumption. An example is shown later to illustrate the application of the model.

We first define the renewal functions for two counting processes that will be met in the following model derivation. Let $M_Z(t)$ and $M_D(t)$ represent the number of renewals for the ordinary renewal process $Z_1, Z_2, \ldots$ and the delayed renewal process $Y_1, Z_2, Z_3, \ldots$ over the time period t respectively. According to the classical renewal theory (Ross, 1970), the expressions of $M_Z(t)$ and $M_D(t)$ can be given by

$$M_Z(t) = F_Z(t) + \int_0^t M_Z(t-u)\, dF_Z(u) \tag{4.10}$$

$$M_D(t) = F_Y(t) + \int_0^t M_Z(t-u)\, dF_Y(u) \tag{4.11}$$

In particular, for the case of no inspection or PM over warranty, the EWC is simply given as $C_F M_Z(w)$.

Unfortunately, analytical expressions of renewal functions are not available in general which is the key to our problem of interest. But certain numerical methods (e.g. RS-integration; Xie, 1989) have been able to provide both robust and accurate results for the approximation. We will illustrate this in the numerical session.

4.4.1 Warranty Cost Model for Policy B1

Here we derive the warranty cost function for Policy B1 with one-time inspection at time T. Depending on whether a failure occurred before T, we consider the following four scenarios.

1) If $Z < T$, the product is correctly replaced with cost C_F and no inspection is conducted at time T; the total warranty cost is $CW = C_F + C_F M_Z(w - Z)$.
2) If $X < T$ and $Z > T$, the product is preventively replaced at time T with cost $C_I + C_P$; the total warranty cost is $CW = C_I + C_P + C_F M_Z(w - T)$.
3) If $T < X < w$, the product is inspected at cost C_I but no PM is carried out; the total warranty cost is $CW = C_I + C_F M_D(w - X)$.
4) If $X > w$, the product is inspected at cost C_I but no PM is carried out; the total warranty cost is $CW = C_I$.

Based on the above scenarios, the EWC as a function of T is given by

$$
\begin{aligned}
E[CW] = &\int_0^T C_F[1 + M_Z(w - t)]f_Z(t)dt \\
&+ \int_0^T [C_I + C_P + C_F M_Z(w - T)]f_X(t)\bar{F}_Y(T - t)dt \\
&+ \int_T^w [C_I + C_F M_D(w - t)]f_X(t)dt + C_I \bar{F}_X(w) \\
&= C_F \left(\int_0^T [1 + M_Z(w - t)]f_Z(t)dt + M_Z(w - T)\big(F_X(T) - F_Z(T)\big) \right. \\
&\left. + \int_T^w M_D(w - t)f_X(t)dt \right) + C_I \bar{F}_Z(T) + C_P\big(F_X(T) - F_Z(T)\big) \qquad (4.12)
\end{aligned}
$$

As a special case, when the time to defect arrival is exponentially distributed, we have

$$\int_T^w M_D(w-t)f_X(t)dt = \bar{F}_X(T)[M_D(w-T)F_X(w-T)] = \bar{F}_X(T)M_Z(w-T) \quad (4.13)$$

and therefore (4.12) can be simplified as

$$E[CW] = C_I\bar{F}_Z(T) + C_P\big(F_X(T) - F_Z(T)\big)$$
$$+ C_F\left(\int_0^T [1 + M_Z(w-t)]f_Z(t)dt + M_Z(w-T)\bar{F}_Z(T)\right) \quad (4.14)$$

4.4.2 Warranty Cost Model for Policy B2

Here we consider the case of periodic inspection under non-renewing warranty. Suppose that the product is inspected at times $T, 2T, \dots, NT$ with $N = \lceil w/T \rceil - 1$ and $\varepsilon = w - NT$. Note that once the product fails during warranty, the subsequent planned inspections will not be made.

Similar as Policy A2, we let $CW^{(J)}(T,\varepsilon)$ represent the EWC for a new system when the warranty length is $JT + \varepsilon$. The following recursive function is applied.

$$CW^{(J)}(T,\varepsilon) = \sum_{i=0}^{J-1} \int_{iT}^{(i+1)T} f_X(t) \int_t^{(i+1)T} [iC_I + C_F(1 + M_Z(JT + \varepsilon - u))]f_Y(u-t)dudt$$
$$+ \int_{JT}^{JT+\varepsilon} f_X(t) \int_t^{JT+\varepsilon} [JC_I + C_F(1 + M_Z(JT + \varepsilon - u))]f_Y(u-t)dudt$$
$$+ JC_I\left(\int_{JT}^{JT+\varepsilon} f_X(t)\bar{F}_Y(JT + \varepsilon - t)dt + \bar{F}_X(JT + \varepsilon)\right)$$
$$+ \sum_{i=0}^{J-1}\Bigg(\big[(i+1)C_I + C_P + CW^{(J-i-1)}(T,\varepsilon)\big]$$
$$\times \int_{iT}^{(i+1)T} f_X(t)\bar{F}_Y[(i+1)T - t]dt\Bigg) \quad (4.15)$$

where $CW^{(0)}(T,\varepsilon) = C_F M_Z(\varepsilon)$.

The EWC as a function of T is therefore given by

$$E[CW] = CW^{(N)}(T,\varepsilon) \tag{4.16}$$

Again, a meaningful inspection interval should incur a total warranty cost less than $C_F M_Z(w)$. A relatively less stringent lower bound of T is to be given by $\frac{w}{\widehat{N}+1}$ where $\widehat{N}$ can be derived from $NC_I\bar{F}_X(w) < CW^{(N)}(T,\varepsilon) < C_F M_Z(w) \Rightarrow \widehat{N} = \left\lfloor \frac{C_F M_Z(w)}{C_I(1-F_X(w))} \right\rfloor$.

As a special case, when the time to defect arrival satisfies exponential distribution, each inspection point becomes the renewal point for the product. Therefore (4.15) can be simplified as

$$CW^{(J)}(T,\varepsilon) = C_F \int_0^T f_Z(t)[1 + M_Z(JT + \varepsilon - t)]dt + C_P \int_0^T f_X(t)\bar{F}_Y(T-t)dt$$
$$+ \left(C_I + CW^{(J-1)}(T,\varepsilon)\right)\bar{F}_Z(T) \tag{4.17}$$

where $CW^{(0)}(T,\varepsilon) = C_F M_Z(\varepsilon)$.

Further solve (4.17) iteratively and let $J = N$. We have

$$CW^{(N)}(T,\varepsilon) = \left(C_P \int_0^T f_X(t)\bar{F}_Y(T-t)dt + C_I\bar{F}_Z(T)\right)\frac{1-\left(\bar{F}_Z(T)\right)^N}{F_Z(T)}$$
$$+ C_F\left(\sum_{i=1}^{N}\left(\int_0^T f_Z(t)[1 + M_Z(iT + \varepsilon - t)]dt\right)\left(\bar{F}_Z(T)\right)^{N-i}\right.$$
$$\left. + M_Z(\varepsilon)\left(\bar{F}_Z(T)\right)^N\right) \tag{4.18}$$

Remarks: In the foregoing analysis, we have derived the warranty cost functions for Policy B which is dependent on the product failure history. For comparison, in the

following we further consider a "block-type" inspection policy for the product with delayed failures. However, without further assumptions, it is technically difficult to formulate the cost model. As a result, for Policy C we assume that the time to defect arrival follows exponential distribution. Under this assumption, the product is renewed at each inspection point during the warranty.

4.4.3 Warranty Cost Model for Policy C1

Suppose that the inspection is to be made at time T irrespective of the failure history of the product. Let $P_1(T)$ be the probability that the product is in the defective state at time T. Then the EWC as a function of T can be presented as

$$E[CW] = C_I + C_F M_Z(T) + C_F M_Z(w - T) + C_P P_1(T) \tag{4.19}$$

In order to derive $P_1(T)$, we define $F_Z^{(n)}(t)$ as the distribution function of $S_n = \sum_{i=1}^{n} Z_i = \sum_{i=1}^{n}(X_i + Y_i)$, with $F_Z^{(0)}(t) = 1$ and $F_Z^{(1)}(t) = F_Z(t)$. Then $F_Z^{(n)}(t)$ can be expressed as

$$F_Z^{(n)}(t) = \Pr\{S_n < t\} = \int_0^\infty F_Z^{(n-1)}(t-u)dF_Z(u) \tag{4.20}$$

Since $P_1(T) = \sum_{n=0}^{\infty} \Pr\{S_n + X_{n+1} \le T < S_{n+1}\}$, we have

$$\begin{aligned} P_1(T) &= \sum_{n=0}^{\infty} (\Pr\{S_n + X_{n+1} \le T\} - \Pr\{S_{n+1} \le T\}) \\ &= \sum_{n=0}^{\infty} \left(F_Z^{(n)}(T) * f_X(T) - F_Z^{(n+1)}(T)\right) \end{aligned} \tag{4.21}$$

By further taking the LST of $P_1(T)$, we have

$$\mathcal{L}[P_1(T)] = \mathcal{L}\left[\sum_{n=0}^{\infty}\left(F_Z^{(n)}(T) * f_X(T) - F_Z^{(n+1)}(T)\right)\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{s}(\mathcal{L}[f_Z(T)])^n \mathcal{L}[f_X(T)] - \frac{1}{s}(\mathcal{L}[f_Z(T)])^{n+1}\right)$$

$$= \frac{\mathcal{L}[f_X(T)]}{s} + \mathcal{L}[f_X(T)]\frac{\mathcal{L}[f_Z(T)]}{s(1-\mathcal{L}[f_Z(T)])} - \frac{\mathcal{L}[f_Z(T)]}{s(1-\mathcal{L}[f_Z(T)])} \quad (4.22)$$

According to the renewal theory, $\mathcal{L}[M_Z(T)] = \frac{\mathcal{L}[f_Z(T)]}{s(1-\mathcal{L}[f_Z(T)])}$. Therefore we have

$$P_1(T) = F_X(T) - \int_0^T [1 - F_X(T-u)]\, dM_Z(u) \quad (4.23)$$

The corresponding warranty cost function is further given by

$$E[CW] = C_I + C_F M_Z(T)$$

$$+ C_F M_Z(w-T) + C_P\left(F_X(T) - \int_0^T [1 - F_X(T-u)]\, dM_Z(u)\right) \quad (4.24)$$

4.4.4 Warranty Cost Model for Policy C2

For Policy C2, suppose that the inspections are to be made at times $T, 2T, \ldots, NT$ with $N = \lceil w/T \rceil - 1$ and $\varepsilon = w - NT$. Again, by assuming that the time to defect arrival is exponentially distributed, each inspection point becomes the renewal point for the product. Therefore, the EWC can be easily generalized from the results of Policy C1. We present it below without proof.

$$E[CW] = N\left[C_I + C_F M_Z(T) + C_P\left(F_X(T) - \int_0^T [1 - F_X(T-u)] dM_Z(u)\right)\right] + C_F M_Z(\varepsilon) \quad (4.25)$$

The inspection interval can be searched within $\left[\frac{w}{\widehat{N}+1}, w\right)$ where $\widehat{N}$ is derived from $NC_I < E[CW] < C_F M_Z(w) \Rightarrow \widehat{N} = \lfloor C_F M_Z(w)/C_I \rfloor$.

Alternatively, if the time to defect follows a non-exponential distribution, we may need some simulation methods (e.g. Monte Carlo) for approximating the EWC. However, such treatment is typically time-consuming and less accurate. In the following we will compare Policy B and Policy C using numerical illustrations and investigate the possibility of approximating the cost of Policy C using the results from Policy B.

4.5 Numerical Examples

In this section, numerical examples are proposed separately under the renewing and non-renewing warranty settings in order to demonstrate the effect of inspection and PM services on the total warranty cost minimization.

Weibull models have been widely applied in the reliability engineering and applications (Murthy et al., 2003). Consider the case in the following that $F_X(.)$ and $F_Y(.)$ satisfy Weibull distributions with scale parameter α_X, α_Y and shape parameter β_X, β_Y respectively, where

$$F_X(t) = 1 - e^{-(\alpha_X t)^{\beta_X}},\ \ F_Y(t) = 1 - e^{-(\alpha_Y t)^{\beta_Y}},\ \ F_Z(t) = \int_0^t F_X(t-u)dF_Y(u) \quad (4.26)$$

and μ_X, μ_Y and σ_X^2, σ_Y^2 represent the mean and variance of X and Y respectively.

For non-renewing warranty, we approximate $M_Z(.)$ by utilizing the RS-integration method (Xie, 1989). In the following warranty length w is partitioned into θ intervals where $0 = t_0 < t_1 < t_2 < \cdots < t_\theta = w$. The following recursive equation can be used

$$M_Z(t_j) = \frac{F_Z(t_j) + S_j - F_Z(t_j - t_{j-0.5})M_Z(t_{j-1})}{1 - F_Z(t_j - t_{j-0.5})}, 1 \le j \le \theta \quad (4.27)$$

with $M_Z(0) = 0$ and

$$S_j = \sum_{i=1}^{j-1} F_Z(t_j - t_{i-0.5})(M_Z(t_i) - M_Z(t_{i-1})), 2 \leq j \leq \theta \tag{4.28}$$

where $t_{j-0.5} = 0.5(t_j + t_{j-1})$.

Similarly, $P_1(t)$ can be approximated using the following:

$$P_1(t_j) \approx F_X(t_j) - \sum_{i=1}^{j} \left[1 - F_X\left(t_j - 0.5(t_i + t_{i-1})\right)\right] (M_Z(t_i) - M_Z(t_{i-1})), 1 \leq j \leq \theta \tag{4.29}$$

Without loss of generality, we assume that the warranty coverage is $w = 1$ time unit and the preventive replacement cost is $C_P = 1$. Optimal inspection policies are obtained separately under the renewing and non-renewing warranty settings over a wide range of model parameters. Note that for the one-time inspection policy, T^* is equivalent to the optimal inspection time, while for periodic inspection, it is the optimal the inspection interval. In addition, for comparisons between Policy B and Policy C to be viable, we assume that the time to defect arrival is exponential distributed for non-renewing warranty setting although Policy B can be generally applied with any arbitrary distribution. Table 4.1 and Table 4.2 below show respective optimal inspection policies (in shaded background) under renewing and non-renewing warranty settings. Here $\mu_Z = \mu_X + \mu_Y$ represent the mean life time of the product and CR% represents the percentage of cost reductions by applying the optimal inspection policies in contrast to no inspection within warranty.

Based on these computational results, the following observations and insights are made.

Table 4.1 Optimal inspection scheduling for renewing warranty

No.	w	α_X	β_X	α_Y	β_Y	C_I	C_P	C_F	μ_Z	$\frac{C_F F_Z(w)}{\bar{F}_Z(w)}$	Policy A1		Policy A2		CR%
											T^*	$E^*[CW]$	T^*	$E^*[CW]$	
1	1	1	1	1	2	0.5	1	2	1.89	0.49	0.34	1.02	0.5	1.05	-
2	1	1	1	2	2	0.5	1	2	1.44	1.41	0.53	1.45	0.53	1.45	-
3	1	2	1	2	2	0.5	1	2	0.94	3.49	0.55	2.37	0.55	2.37	32%
4	1	2	2	2	2	0.5	1	2	0.89	3.85	0.58	2.14	0.58	2.14	44%
5	1	2	3	2	2	0.5	1	2	0.89	4.06	0.59	2.01	0.59	2.01	50%
6	1	2	2	1	2	0.5	1	4	1.33	1.59	0.50	1.60	0.50	1.60	-
7	1	2	2	2	2	0.5	1	4	0.89	7.69	0.58	2.93	0.58	2.93	62%
8	1	2	2	3	2	0.5	1	4	0.74	19.06	0.59	5.16	0.25	4.11	78%
9	1	2	2	2	1	0.1	1	4	0.94	6.98	0.59	3.97	0.17	3.18	54%
10	1	2	2	2	3	0.1	1	4	0.89	8.12	0.56	1.87	0.34	1.60	80%
11	1	2	2	2	4	0.1	1	4	0.90	8.23	0.55	1.56	0.34	1.40	83%
12	1	2	2	2	2	0.1	1	4	0.89	7.69	0.57	2.47	0.25	1.99	74%
13	1	1	2	2	2	0.1	1	4	1.33	1.60	0.59	0.96	0.34	0.92	43%
14	1	3	2	2	2	0.1	1	4	0.74	19.06	0.54	3.83	0.25	2.84	85%
15	1	3	2.5	2	2	0.1	1	4	0.74	20.59	0.54	3.52	0.25	2.81	86%
16	1	3	3	2	2	0.1	1	4	0.74	21.38	0.53	3.28	0.25	2.78	87%
17	1	2	2	2	2	0.05	1	4	0.89	7.69	0.57	2.41	0.20	1.82	76%
18	1	2	2	2	3	0.05	1	4	0.89	8.12	0.56	1.81	0.25	1.48	82%
19	1	2	2	2	4	0.05	1	4	0.90	8.23	0.55	1.50	0.34	1.29	84%
20	1	2	2	2	2	0.05	1	8	0.89	15.38	0.57	3.99	0.17	2.12	86%

Table 4.2 Optimal inspection scheduling for non-renewing warranty

No.	w	α_X	β_X	α_Y	β_Y	C_I	C_P	C_F	μ_Z	$C_F F_Z(w)$	Policy B1		Policy B2		Policy C1		Policy C2		CR%
											T^*	$E^*[CW]$	T^*	$E^*[CW]$	T^*	$E^*[CW]$	T^*	$E^*[CW]$	
1	1	1	1	1.5	3	0.5	1	2	1.60	0.65	0.44	1.00	0.50	1.00	0.40	1.01	0.50	1.03	-
2	1	1	1	2	3	0.5	1	2	1.45	0.87	0.53	1.12	0.53	1.12	0.47	1.16	0.50	1.16	-
3	1	2	1	2	2	0.5	1	2	0.94	1.46	0.59	1.67	0.59	1.67	0.47	1.78	0.50	1.78	-
4	1	1	1	1	3	0.5	1	4	1.89	0.63	0.35	0.94	0.50	0.98	0.35	0.95	0.50	0.99	-
5	1	1	1	1.5	3	0.5	1	4	1.60	1.30	0.48	1.15	0.50	1.19	0.46	1.20	0.50	1.20	12%
6	1	1	1	2	3	0.5	1	4	1.44	1.75	0.50	1.48	0.50	1.48	0.49	1.52	0.50	1.52	15%
7	1	1	1	2.5	3	0.5	1	4	1.36	2.05	0.52	1.83	0.52	1.83	0.50	1.87	0.50	1.87	11%
8	1	2	1	2.5	3	0.5	1	4	0.86	3.39	0.50	2.88	0.34	2.69	0.51	2.90	0.34	2.73	21%
9	1	1	1	2	3	0.25	1	4	1.44	1.75	0.49	1.23	0.50	1.23	0.49	1.29	0.50	1.29	30%
10	1	1.5	1	2	3	0.25	1	4	1.11	2.36	0.49	1.66	0.34	1.60	0.50	1.67	0.34	1.65	32%
11	1	2	1	2	3	0.25	1	4	0.95	2.86	0.49	2.02	0.25	1.88	0.50	2.01	0.25	1.91	34%
12	1	2.5	1	2	3	0.25	1	4	0.85	3.27	0.49	2.34	0.25	2.16	0.50	2.32	0.25	2.14	35%
13	1	2	1	2.5	3	0.25	1	4	0.86	3.39	0.47	2.69	0.25	2.28	0.51	2.66	0.25	2.27	33%
14	1	2	1	2	3	0.10	1	4	0.95	2.86	0.49	1.89	0.25	1.62	0.50	1.85	0.25	1.59	44%
15	1	2	1	3	2	0.10	1	4	0.80	3.89	0.40	3.26	0.20	2.43	0.51	3.19	0.20	2.36	39%
16	1	2	1	4	2	0.10	1	4	0.72	4.56	0.31	4.01	0.13	2.84	0.61	3.96	0.17	2.75	40%
17	1	2	1	3	2	0.05	1	4	0.80	3.90	0.39	3.22	0.17	2.22	0.51	3.13	0.17	2.15	45%
18	1	2	1	3	3	0.05	1	4	0.80	3.83	0.36	3.08	0.20	1.88	0.60	3.02	0.20	1.83	52%
19	1	2	1	3	4	0.05	1	4	0.80	3.78	0.34	2.97	0.20	1.70	0.64	2.92	0.20	1.67	56%
20	1	2	1	3	3	0.05	1	8	0.80	7.67	0.35	5.74	0.13	2.18	0.60	5.62	0.13	2.12	72%

Discussion of computational results:

1) No inspection is necessary when the inspection cost C_I is relatively expensive compared to both the preventive replacement cost C_P and corrective replacement cost C_F. It can be observed in Table 4.1 and Table 4.2 when $C_I/C_P = 0.5$ and $C_I/C_F = 0.25$.
2) Given a fixed C_P, the frequency of inspection increases when either C_I decreases or C_F increases. For our case, the one-time inspection is preferred typically when $C_I/C_P = 0.5$ and $C_I/C_F = 0.125$. When the inspection cost further decreases, multiple inspections become economically viable. In particular, results reveal that periodic inspection should be arranged evenly during the warranty period with $N = w/T - 1$. In other words, given a fixed number of inspections, it is always better to implement the inspections with the shortest intervals. In this sense we may instead search over the domain of $N\ (= 1,2, \dots)$ rather than $T\ (\in (0, w))$ in order to reduce the computational complexity.
3) Given a fixed cost structure, the frequency of inspection increases (or non-decreasing) with the scale parameters α_X and α_Y. In other words, when the mean life time $\mu_Z = \mu_X + \mu_Y$ decreases (α_X, α_Y increase), the product deterioration accelerates and consequently more inspections are required in order to identify hidden defects and prevent failures. Note that the results may also be interpreted from the opposite perspective namely longer warranty (given fixed μ_Z) should require more frequent inspections.
4) Given a fixed cost structure, the frequency of inspection decreases (or non-increasing) with the shape parameters β_X, β_Y when $\beta_X, \beta_Y \geq 1$. It may be partially explained by the characteristics of Weibull distribution. Take the delay

time Y for example: σ_Y decreases significantly when β_Y increases from 1 while μ_Y only changes slightly within $[0.89\alpha_Y, \alpha_Y]$. In other words, the chance of short delay time (in terms of its mean delay time) becomes much lower when β_Y increases and therefore less inspection efforts are required.

5) A cost comparison between Policy B and Policy C is further made based on the observations from Table 4.2. Policy B is better than Policy C typically when the number of inspection is small, which is mainly due to a high inspection cost or a relatively large mean life time. The situation reverses when the inspection becomes relatively cheaper or the product is more prone to failure. For both cases, Policy B shows to be a good approximation of Policy C which does not have an analytical expression for non-exponentially distributed random variable X. Such conclusion may still hold when $\beta_X > 1$ due to the fact that μ_X does not vary significantly as β_X increases.
6) Parameter impacts on the optimal inspection policies T^* shows relatively consistent results between renewing warranty and non-renewing warranty settings. In particular, substantial cost reductions (CR%) can been achieved under low inspection cost or a relatively short μ_Z.

4.6 Conclusion

This chapter presents a framework for applying condition-based maintenance in minimizing warranty costs under various warranty schemes from the manufacturer's perspective. For condition monitoring or inspection to be viable, we assumed that product deterioration follows a basic delay time model and can have such three states: good, defective and failed. Preventive maintenance is implemented only when the

product is in the defective state upon inspection. In this work we focused on the inspection scheduling problem for products under both renewing and non-renewing warranty policies. In particular, for non-renewing warranty, we considered two types of inspections. The first type is dependent on the failure history of the product and the inspection service is suspended once a failure is incurred during the warranty. In contrast, the inspection under the second type is conducted at pre-specified time(s) which is irrespective of the product failure history. For each of the above policies, we further considered one-time inspection and periodic inspection within the warranty period in order to cover a wide range of product types and applications. Numerical approximations are then used to demonstrate the applicability of the proposed models. Results reveal that the implementation of inspections within warranty can be cost-efficient for the manufacturer when either the inspection cost is relatively low or the product has a relatively short mean life time.

CHAPTER 5 PERIODIC PM SERVICE DESIGN INCORPORATING VALUE OF MAINTENANCE

5.1 Introduction

In Chapters 3 and 4, we have investigated two important issues in warranty cost modeling from the manufacturer's perspective, where preventive maintenance is used to either enhance the warranty servicing or reduce the warranty cost. In this chapter, we further investigate the life-cycle maintenance service design from the buyer's perspective and analyze warranty as a factor in influencing the buyer's periodic PM decisions.

Optimization of PM strategies under warranty context has received considerable attention in the literature. From the seller's perspective, (periodic) preventive maintenance is mainly designed to minimize the EWC over the warranty period. References of this area can be found in Chun (1992), Jack and Dagpunar (1994), Yeh and Lo (2001), Wang (2006), and Huang and Yen (2009). In comparison, from the buyer's perspective, PM efforts during both the warranty and/or the post-warranty period can have significant impacts on the maintenance cost after the warranty is expired and that will have to be borne by the buyer. As a result, the optimal PM strategy for the buyer should be determined under the life-cycle context. Early studies on the life-time PM modeling can be found in Chun and Lee (1992), Jung et al. (2000), Djamaludin et al. (2001), and Jung and Park (2003). Recently, Kim et al. (2004) developed a framework for the cost analysis linking warranty and preventive maintenance (PM) under the life-cycle context. Different PM options were proposed

and the optimal strategies were further selected such that the buyer's life-cycle maintenance cost was minimized. Pascual and Ortega (2006) described a periodic PM policy by allowing the buyer's negotiation on the length of warranty period. Jain and Maheshwari (2006) developed a discounted PM cost model after the expiration of renewing pro-rata warranty (RPRW). Chen and Chien (2007) considered continuous PM for a repairable system under renewing free-replacement warranty (RFRW) with two failure modes: a minor failure and a catastrophic failure. Jung et al. (2008) investigated optimal preventive replacement policies following the expiration of both renewing and non-renewing warranty.

Some other research in PM modeling related to our problem of interest are also mentioned as follows. These works however do not include warranty consideration. Pongpech and Murthy (2006) studied a periodic PM policy for the leased products. Sheu et al. (2006) studied an optimal periodic PM policy by maximizing the system availability. El-Ferik and Ben-Daya (2008) proposed a new age-based PM strategy where the system underwent PM actions either at failure or after a pre-specified time interval whichever of them occurred first. Jackson and Pascual (2008) focused on the maintenance service negotiation between the agents and clients. More recently, Castro (2009) studied an optimal PM strategy for systems under two types of failure modes: maintainable and non-maintainable. Zhou et al. (2009) considered a multi-unit series system and investigated an opportunistic PM strategy using dynamic programming. Jin et al. (2009) described an option-based PM model under stochastic demand. Nakagawa and Mizutani (2009) presented a summary of various PM models over a finite horizon.

A common assumption for the periodic PM modeling in the literature is that the calendar time of the first PM action is pre-specified. To illustrate this, denote w as the warranty period, L as the length of the system life cycle ($w \leq L$), τ as the constant

interval between consecutive PMs, and t_0 as the calendar time for the first PM action. It is typically assumed that t_0 either equals to τ or $w + \tau$. The first case describes the situation that PMs are planned throughout the system life time; in contrast the second case implies that the PMs are only carried out during the post-warranty period. While such arrangements are easy to be implemented in practice, they do not necessarily lead to the minimization of the life-time cost. Hence a potential improvement of the model is to consider t_0 as a decision variable to be optimized under the life-cycle context.

Another common assumption for the (periodic) PM modeling is that PM strategies are only analyzed from the aspect of maintenance cost while the benefits of PMs are seldom explicitly elaborated in the model optimization. Dekker (1996) highlighted the reason by stating that the maintenance output, in terms of contribution to company profits, is very difficult to quantify. While it is easy to measure the cost of maintenance, it is difficult to measure its benefits. Most recently, Marais and Saleh (2009) developed a framework for capturing and quantifying the value of maintenance activities on revenue-generating facilities. They argued that existing cost-centric maintenance models ignored the value of maintenance, and may lead to sub-optimal maintenance strategies. However, their valuation mechanism only applied to perfect maintenance. For the valuation of (imperfect) preventive maintenance, new methodologies should be proposed.

In this chapter, we study a general periodic preventive maintenance policy for the buyer considering both maintenance cost and the value of maintenance. The first part of the cost model includes the preventive maintenance (PM) cost and the minimal repair cost upon system failures. We assume that PMs are carried out periodically starting from a certain time instant until the end of the system life cycle. The time to the first PM action is a decision variable chosen by the buyer. The second part of the

cost model – the value of maintenance, is quantified through the reduction of ageing losses, which is defined as the total revenue losses caused by the system deterioration over its life cycle. We argue that preventive maintenance, which slows system deterioration, reduces ageing losses and therefore reflects its value. The optimal PM strategies for the buyer, which consider both the maintenance cost and the value of maintenance, are determined jointly by the calendar time of the first PM action and its corresponding maintenance level.

The rest of this chapter is organized as follows. Section 5.2 presents the proposed periodic preventive maintenance model in more details. Section 5.3 derives the total life-cycle cost model under both the warranty contracts and ageing losses. A numerical case is given in Section 5.4, and sensitivity analysis is carried out over various model parameters. Conclusion is made in Section 5.5.

5.2 Periodic Preventive Maintenance Model

Let L represent the system life cycle and τ represent the pre-specified PM interval. In the model, preventive maintenance (PM) is conducted at discrete time instants $y_1, y_2, \ldots, y_n$ with $y_1 = t_0$, $y_i = y_1 + (i-1)\tau$ for $1 \leq i \leq n = \left\lceil \frac{L-t_0}{\tau} \right\rceil$ and $y_{n+1} = L$. Here the integer n represents the number of PMs during the system life cycle. We assume that t_0 has no pre-specified value (i.e. $t_0 = \tau$ or $w + \tau$) and is a decision variable to be optimized. We further assume that the cost for performing PM actions is borne by the buyer while the warranty servicing cost is covered by the manufacturer.

For the sake of generality, we assume that the effect of each PM is *imperfect* (Pham and Wang, 1996) and is modeled using the *virtual age method* (Kijima, 1989). It defines that each PM reduces the system age by a certain amount, and therefore the

system effective age or virtual age is less than its calendar age. Kim et al. (2004) modified Kijima's model by considering discrete PM levels. Such treatment was further adopted by Huang and Yen (2009) in two-dimensional warranty cost modeling under preventive maintenance. Here we further modify Kim's model by considering t_0 as a decision variable, and the system effective age right after performing the ith PM action is therefore given by

$$u_i(m, t_0) = u_{i-1}(m, t_0) + \varphi(m)(y_i - y_{i-1}) \tag{5.1}$$

where y_i is the calendar age of the system when the ith PM action is performed (with $y_1 = t_0$), $\varphi(m)$ is the age-reduction factor and $m(= 0,1, \dots M)$ s represent all the discrete PM levels. Note that $\varphi(m)$ is a decreasing function of m, i.e. a larger m corresponds to greater maintenance efforts and therefore a smaller $\varphi(m)$. In particular, we let $\varphi(0) = 1$ to represent the case of no preventive maintenance. In our model, in addition to t_0, m is another decision variable to be selected and optimized by the buyer.

Under both m and t_0, the virtual age of the system at calendar time t, i.e. $u(t|m, t_0)$, is given by

$$u(t|m, t_0) = u_{i-1}(m, t_0) + t - y_{i-1}, \qquad y_{i-1} \le t < y_i, i = 2,3, \dots, n+1 \tag{5.2}$$

It is important to investigate the effect of the first PM action on the system effective age. Here we assume that for each PM level $m > 0$, there is a limit $K(m)$ on the reduction of system effective age, which describes the capacity of improvement under that PM level. As a result, the amount of age reduction by the first PM is simply the value of $[1 - \varphi(m)]t_0$ or $K(m)$, whichever is smaller. For a general discussion, we let $K(m) = 0$. The system effective age right after the first PM action is therefore given by

$$u_1(m, t_0) = t_0 - \min\{[1 - \varphi(m)]t_0, K(m)\} \tag{5.3}$$

Such arrangement is reasonable in the practical situation as under a fixed maintenance level, the effect of the maintenance action increases with the time span since last maintenance; however, due to the presence of physical and technical constraints, it should not exceed some finite upper limit. To further simplify the problem, we assume that $K(m) \geq [1 - \varphi(m)]\tau$ for any PM level $m > 0$, or $\tau \leq \min_{m\in\{1,2,\ldots,M\}}\left\{\frac{K(m)}{(1-\varphi(m))}\right\}$, implying that for the safety issue, the maintenance of system should not be lower than certain frequency. As a result, the amount of age reduction by the subsequent PMs will not exceed the control limit.

The modified virtual age model is summarized in the following Proposition.

Proposition 5.1: For a repairable system subject to periodic preventive maintenance (PM), the modified virtual age $u(t|m, t_0)$ at calendar age t under both the PM level m and the calendar time of the first PM action t_0 is given by

$$u(t|m, t_0) = \begin{cases} t, \quad 0 \leq t < y_1 = t_0 \\ t - \min\{[1 - \varphi(m)]t_0, K(m)\} \\ \quad -[1 - \varphi(m)](i - 2)\tau, \;\; y_{i-1} \leq t < y_i, i = 2,3, \ldots, n + 1 \end{cases} \tag{5.4}$$

Proof: When $t < y_1 = t_0$, no PM is carried out; therefore we have $u(t|m, t_0) = t$.

When $y_{i-1} \leq t < y_i$ $(i \geq 2)$, the virtual age $u(t|m, t_0)$ is derived as

$$\begin{aligned} u(t|m, t_0) = u_{i-1}(m, t_0) + t - y_{i-1} &= u_1(m, t_0) + \varphi(m)(y_{i-1} - y_1) + t - y_{i-1} \\ &= t_0 - \min\{[1 - \varphi(m)]t_0, K(m)\} + \varphi(m)(y_{i-1} - y_1) + t - y_{i-1} \\ &= t - \min\{[1 - \varphi(m)]t_0, K(m)\} - [1 - \varphi(m)](y_{i-1} - t_0) \\ &= t - \min\{[1 - \varphi(m)]t_0, K(m)\} - [1 - \varphi(m)](i - 2)\tau, \\ &\qquad y_{i-1} \leq t < y_i, i = 2,3, \ldots, n + 1 \end{aligned} \tag{5.5}$$

The result is then straightforward.

Figure 5.1 shows a comparison of the system effective age with ($m > 0$) and without ($m = 0$) preventive maintenance using the modified virtual age method. The new method proposed here has two features: two decision variables (both m and t_0 rather than merely m), and a control limit $K(m)$ to the first PM action.

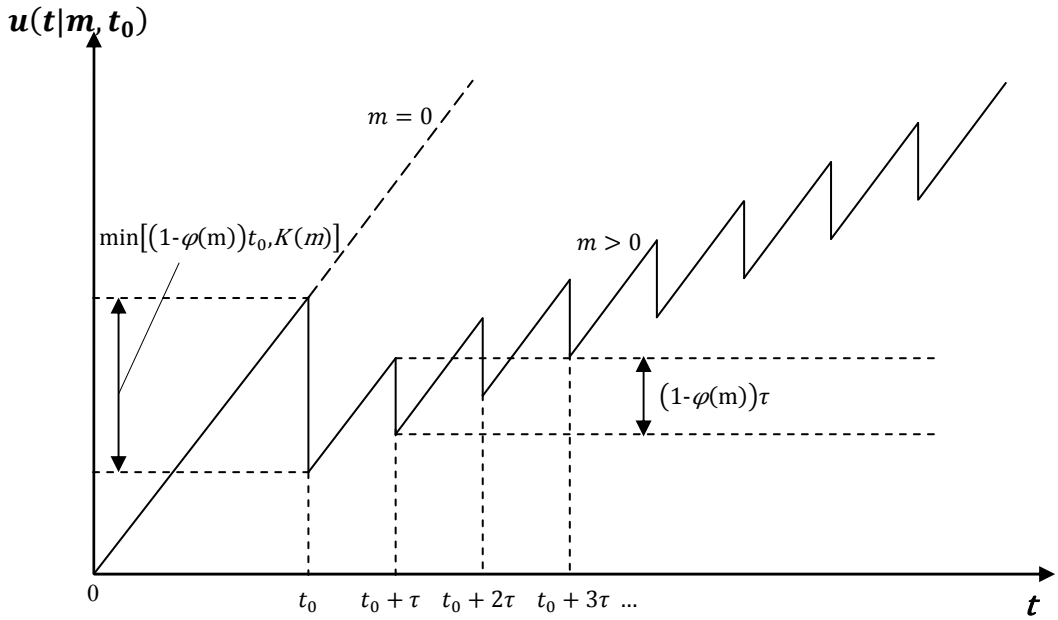

Figure 5.1 System effective age with and without PM

The buyer's life-cycle maintenance cost, which includes both the periodic PM cost throughout the life cycle and minimal repair cost during the post-warranty period, is therefore presented as follows.

Denote $h_0(.)$ as the system failure rate function without PM, $c_p(m)$ as the one-time PM cost under level m, and c_r as the minimal repair cost. Define $C_L(m, t_0)$ as the total maintenance cost under m and c_r. We have

$$C_L(m, t_0) = c_p(m)n + c_r \int_w^L h_0\big(u(t|m, t_0)\big)\, dt \tag{5.6}$$

Note that the three maintenance options studied in Kim et al. (2004) (Options A, B, C) correspond to the special cases of our model when $t_0 = L$ (or $m = 0$), $t_0 = \tau$ and $t_0 = w + \tau$ respectively.

5.3 Life-Cycle Cost Model Incorporating Ageing Losses

In Section 5.2, we have investigated the cost aspect of maintenance based on the modified virtual age method. In this section, we further investigate the value aspect of maintenance by introducing the ageing losses for a revenue-generating system. We first establish the total life-cycle cost model incorporating ageing losses. Then the method for quantifying the ageing losses is explained in detail.

5.3.1 Total Life-Cycle Cost Model

We first assume that the productivity of a revenue-generating system always depends on its effective system age instead of calendar age. Denote $R(u)$ as the revenue generated per unit time when the system works under effective age u. We further assume that $R(u)$ is a non-increasing function of u, implying that the system becomes less productive (or equally productive at best) when it grows old. As a result, the system achieves its maximum productivity when it is new, i.e. $R_{max} = R(0)$. We ignore the learning process by both the workers and machines which could result in $R_{max} > R(0)$.

$\hat{S}(u)$ is defined as the ageing losses per unit time (or loss rate) and is simply the difference between $R(0)$ and $R(u)$. It is given by

$$\hat{S}(u) = R(0) - R(u), \qquad u \geq 0 \tag{5.7}$$

Note that $\hat{S}(u)$ is non-decreasing in u ($u \geq 0$). In order to show the dependence of u with respect to the system calendar age t, the PM level m and the time to the first PM action t_0, we rewrite (5.7) as follow:

$$\hat{S}(u(t|m,t_0)) = R(0) - R(u(t|m,t_0)), \quad 0 \leq t, t_0 \leq L, m \in \{0,1,2,\dots,M\} \tag{5.8}$$

Combining (5.6) and (5.8), the total life-cycle cost model for the buyer is presented as follow.

Proposition 5.2: For a repairable system subject to periodic preventive maintenance (PM) and ageing losses, the total life-cycle cost for the buyer under both the PM level m and the calendar time of the first PM action t_0 is given by

$$C_T(m,t_0) = c_p(m)n + c_r \int_w^L h_0\big(u(t|m,t_0)\big)\,dt + \int_0^L \hat{S}\big(u(t|m,t_0)\big)\,dt \tag{5.9}$$

5.3.2 Modeling of Ageing Losses

A piece-wise function is proposed here in (5.10) to describe the relationship between ageing loss rate $\hat{S}$ and the system effective age u (or $u(t|m,t_0)$).

$$\hat{S}(u(t|m,t_0)) = \begin{cases} 0, & u(t|m,t_0) < \omega_B \\ S_{max}\dfrac{u(t|m,t_0) - \omega_B}{\omega_A - \omega_B}, & \omega_B \leq u(t|m,t_0) \leq \omega_A \leq L \\ S_{max}, & u(t|m,t_0) > \omega_A \end{cases} \tag{5.10}$$

System productivity is assumed to experience three phases before approaching to the end of life cycle. During Phase I, the system effective age u is less than ω_B and ageing loss rate remains at its minimum value (assumed to be zero here). When $\omega_B \leq u \leq \omega_A$, the system enters into Phase II with its ageing loss rate increasing linearly with respect to u. After ω_A, the loss rate reaches its maximum value at S_{max} ($S_{max} > 0$) and remains unchanged during the rest of the time.

Figure 5.2 presents a view of ageing loss rate functions under the system calendar age t. As described below, the ageing structure in (5.10) is flexible enough to model several special cases that can arise from different practical situations.

Without preventive maintenance (i.e. $m = 0$), system effective age $u(t|m, t_0)$ equals to its calendar age t, and $\hat{S}(u(t|m, t_0)) = \hat{S}(t)$. Equation (5.10) is therefore described by Curve O. Curves A, B, C provide the special cases of (5.10) when $\omega_B = 0$, $\omega_A = L$ and $\{\omega_A = L, \omega_B = 0\}$ respecitvely. Curve A describes a system with initially fast decreasing productivity, while Curve B suits for the situation when the ageing loss rate hasn't reach its maximum before the end of life cycle; Curve *C* shows the features of both.

However, when $m \neq 0$, $u(t|m, t_0) \neq t$. The ageing loss rate functions are distorted and transformed to O', A', B', C' under calendar age. The areas between those function-pairs $(O - O', A - A', B - B', C - C')$ are simply the ageing losses that have been successfully reduced by preventive maintenance.

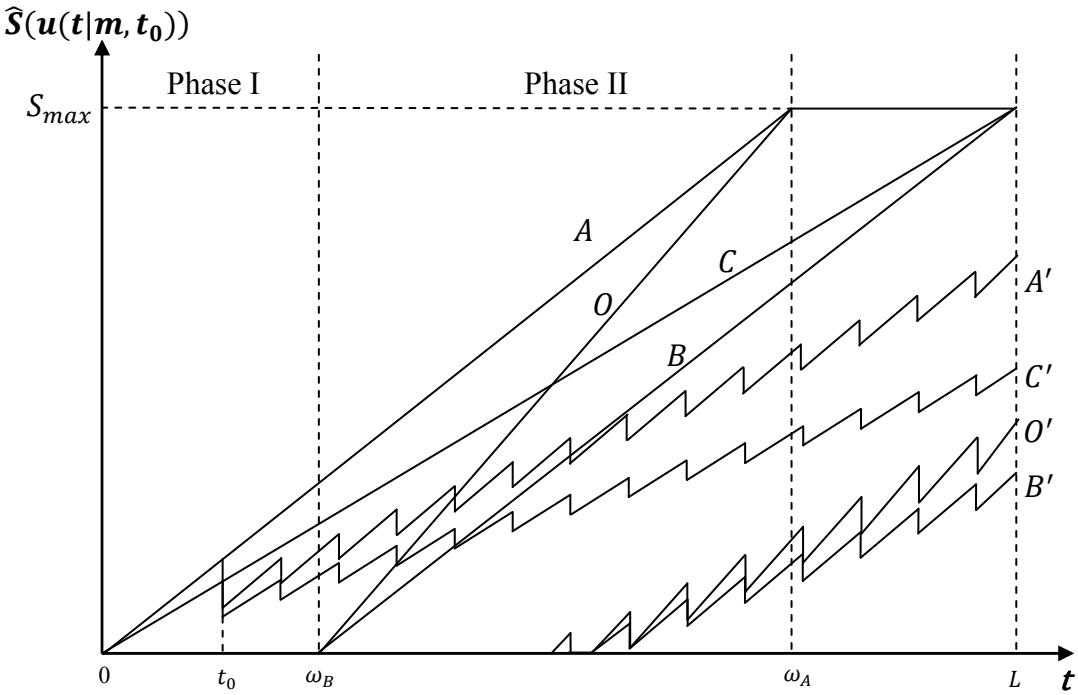

Figure 5.2 Ageing loss rate as functions of system calendar age with and without PM

5.3.3 Discussion on Optimal PM Strategies

The optimal solutions m^* and t_0^* for (5.9) can be obtained numerically by searching over the domains of m ($\in \{0,1,2,\dots,M\}$) and t_0 ($\in [0,L]$) respectively such that $C_T(m,t_0)$ is minimized. It is described as

$$(m^*, t_0^*) = \arg\min_{m\in\{0,1,2,\dots,M\}, t_0\in[0,L]}\{C_T(m,t_0)\} \tag{5.11}$$

Although the analytical forms of the optimal solutions are not available in general, some simple observations of the cost structure can help reduce the complexity of the optimization problem in (5.11). We first note that the cost function $C_T(m,t_0)$ is not continuous in t_0 (as n is not continous in t_0). In the following, define $C_I(m,t_0) = c_p(m)n$ and $C_{II}(m,t_0) = c_r\int_w^L h_0\big(u(t|m,t_0)\big)\,dt + \int_0^L \hat{S}\big(u(t|m,t_0)\big)\,dt$. In other words, $C_T(m,t_0) = C_I(m,t_0) + C_{II}(m,t_0)$. We summarize some properties of the optimal solutions as below.

Proposition 5.3: If $h_0'(.) > 0$ and $\hat{S}(u)$ is non-decreasing in u ($u \geq 0$), then for any given PM level $m > 0$, we have

i) $C_{II}(m,t_0)$ is increasing in t_0 if $t_0 > K(m)/[1-\varphi(m)]$.

ii) For any given $m > 0$, let $t_0^*(m) = \arg\min_{t_0\in[0,L]}\{C_T(m,t_0)\}$. If $t_0^*(m) > K(m)/[1-\varphi(m)]$, then $t_0^*(m)$ must be obtained at the values where $\big(L - t_0^*(m)\big)/\tau$ is an integer.

Proof:

i) When $t_0 > K(m)/[1-\varphi(m)]$, $\min\{[1-\varphi(m)]t_0, K(m)\}$ in (5.4) reduces to $K(m)$. In other words, the PM action under PM level m reaches its maximum capacity. A further study of (5.4) reveals that $u(t|m,t_0)$ is non-decreasing in t_0

(strictly increasing for some intervals of t). This verifies the monotonicity of $C_{II}(m, t_0)$ with respect to t_0 in the interval $(K(m)/[1-\varphi(m)], L)$.

Figure 5.3 illustrates the non-decreasing property of $u(t|m, t_0)$ with respect to t_0 when $t_0 > K(m)/[1-\varphi(m)]$.

ii) Note that for any given m, $C_I(m, t_0)$ remains unchanged when $t_0 \in [L - j\tau, L - (j-1)\tau)$ for $j = 1,2, \ldots, \lfloor L/\tau \rfloor$. Combined with the conclusion in i), $C_T(m, t_0)$ is strictly increasing in t_0 during the intervals $t_0 \in [L - j\tau, L - (j-1)\tau)$ for any integer j that satisfies $L - j\tau > K(m)/[1-\varphi(m)]$. Therefore, $C_T(m, t_0)$ should be minimized at one of the lower bounds of these intervals (i.e. $L - j\tau$). In other words, $\left(L - t_0^*(m)\right)/\tau$ must be an integer.

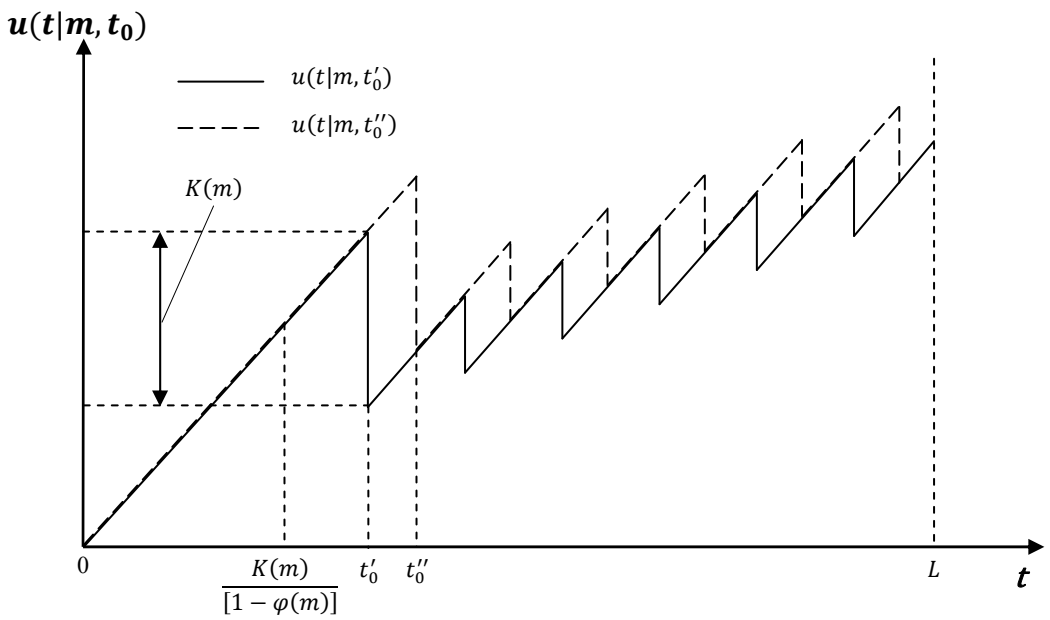

Figure 5.3 Illustration of Proposition 5.3i): $u(t|m,t_0'') \geq u(t|m,t_0')$ when $t_0'' > t_0' > K(m)/[1-\varphi(m)]$

Based on Proposition 5.3ii), the optimization problem in (5.11) can be considerably simplified. To illustrate this, let the integer $j_{max}(m) = \left\lfloor \left(L - \frac{K(m)}{(1-\varphi(m))}\right)/\tau \right\rfloor$. We can rewrite (5.11) as below:

$$(m^*, t_0^*) = \text{argmin}_{m\in\{0,1,2,\ldots,M\}, t_0(m)\in\left[0,\frac{K(m)}{(1-\varphi(m))}\right]\cup\{L-j\tau, j=1,2,\ldots,j_{max}(m)\}}\{C_T(m, t_0)\} \quad (5.12)$$

Here we enforce $\frac{K(m)}{(1-\varphi(m))}|_{m=0} = L$ in order to incorporate the case of no preventive maintenance. In addition, we let $t_0(m)$ replace t_0 in order to show the dependence of the domain of t_0 with respect to the PM level m.

5.4 Numerical Example

In this section, numerical examples are given to show the applicability of the proposed model. It is demonstrated below that, by implementing the proposed framework (i.e. considering both maintenance cost and ageing cost), the cost efficiency of maintenance is improved by at least 4% (see Table 5.2). Parameter effect of the model is further analyzed through extensive sensitivity analysis.

Assume that the failure time distribution of the system satisfies Weibull distribution with shape parameter β and scale parameter α. The failure rate function is therefore given by $h_0(t) = \beta\alpha^\beta t^{\beta-1}$. We assume that $\beta = 2$ and $\alpha = 0.5$, which describes a mean system life time of 1.8 years. Let $\varphi(m) = (1+m)e^{-m}$ for $m = 0,1,2,\ldots,5$ and let the corresponding PM cost $c_p(0) = 0$, $c_p(1) = 10$, $c_p(2) = 30$, $c_p(3) = 60$, $c_p(4) = 100$, $c_p(5) = 160$, which is consistent with Kim et al. (2004). The control limit factor for the first PM is given by $K(m) = k[1-\varphi(m)]\tau$ for $k \geq 1$. A larger k implies a higher capacity of system improvement right after the first PM.

Let $L = 8$, $w = 2$, $\tau = 0.33$, $\omega_A = 3\omega_B = L/2$ and $k = 3$. Table 5.1 shows the total life-cycle cost and the corresponding PM strategies for the buyer given that $c_r = 20\sim400$, $\beta = 2,3$ and $S_{max} = 0, 100, 200$.

Numerical Results:

The optimal solutions are highlighted by shaded background. From Table 5.1, the following results are observed:

(1) A deeper degree of PM action is required when minimal repair cost c_r increases. Typically, the PM levels can vary from $m^* = 0$ ($\varphi(0) = 1$) to $m^* = 4$ ($\varphi(4) = 0.09$) under different value combinations of c_r, S_{max} and β.

(2) A larger β requires a deeper level of preventive maintenance, given that a fixed c_r is applied. It is not surprising since the system deteriorates at a faster speed when β is larger, with both the post-warranty repair cost and ageing losses increasing more rapidly.

(3) The increase of S_{max} has similar effect on the optimal PM level as c_r. However, the effects of S_{max} become almost invisible when c_r is large (m^* is large consequently). For such cases, the system effective age does not exceed ω_B due to heavy preventive maintenance and therefore incurs no ageing losses.

(4) The changes on t_0^* show a clear pattern following the changes on m^*. In other words, m^* is a relatively more dominating factor in determining the optimal PM strategies. To be specific, when m^* remains unchanged, the value of t_0^* either remains unchanged or decreases (given that any of c_r, S_{max}, β increases). The value of t_0^* increases only when m^* jumps from lower level to higher level, implying that an early PM action is not that necessary given that a deeper PM action is conducted.

(5) For those cases when $t_0^* > K(m)/[1 - \varphi(m)] = 0.99$, by applying the results in Proposition 5.3, we obtain $t_0^* = 1.07$ where $(L - t_0^*)/\tau$ is an integer.

Table 5.1 Optimal PM strategies when β = 2,3, S_{max} = 0,100,200, and c_r = 20~400

	$\beta = 2$							$\beta = 3$						
c_r	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	t_0^*	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	t_0^*
	$S_{max} = 0$							$S_{max} = 0$						
20	300.0	309.2	328.1	357.5	397.1	457.0	-	1260.0	914.0	865.7	1258.9	1328.1	1386.4	1.07
50	750.0	757.9	775.3	803.7	842.8	902.4	-	3150.0	1961.4	1219.1	1434.3	2087.3	3044.7	1.07
80	1200.0	1108.5	1152.3	1249.9	1288.5	1347.8	1.07	5040.0	3006.2	1564.3	1539.0	2179.4	3233.4	1.07
110	1650.0	1445.5	1348.2	1653.7	1734.2	1793.3	1.07	6930.0	4051.0	1903.3	1643.6	2223.3	3327.7	1.07
140	2100.0	1782.4	1544.1	1761.1	2179.9	2238.7	1.07	8820.0	5095.8	2242.4	1748.2	2256.9	3384.0	1.07
170	2550.0	2118.3	1739.9	1868.4	2447.7	2684.2	1.07	10710.0	6140.6	2581.5	1852.8	2290.6	3423.4	1.07
200	3000.0	2453.3	1935.8	1975.8	2509.0	3129.6	1.07	12600.0	7185.5	2920.6	1947.8	2324.2	3452.7	0.93
230	3450.0	2788.2	2127.9	2083.2	2570.4	3575.0	1.07	14490.0	8230.3	3259.7	2042.0	2357.8	3466.6	0.93
260	3900.0	3123.2	2319.3	2190.6	2631.7	3702.1	1.07	16380.0	9275.1	3598.8	2136.1	2391.5	3480.5	0.93
290	4350.0	3458.2	2510.8	2297.9	2693.1	3741.5	1.07	18270.0	10320.0	3937.9	2230.3	2425.1	3494.4	0.93
320	4800.0	3793.2	2702.3	2401.9	2754.4	3781.0	0.99	20160.0	11365.0	4277.0	2324.5	2458.7	3508.3	0.93
350	5250.0	4128.2	2893.7	2503.3	2815.8	3820.5	0.99	22050.0	12410.0	4616.1	2418.6	2492.4	3522.3	0.93
380	5700.0	4463.2	3085.2	2604.7	2877.1	3859.9	0.99	23940.0	13454.0	4955.2	2512.8	2526.0	3536.2	0.93
400	6000.0	4686.5	3212.8	2672.4	2918.0	3886.2	0.99	25200.0	14151.0	5181.3	2575.6	2548.4	3545.4	1.07
	$S_{max} = 100$							$S_{max} = 100$						
20	833.3	842.5	861.5	890.8	930.5	990.3	-	1793.3	1360.0	1062.0	1350.1	1861.4	1919.7	1.07
50	1283.3	1217.9	1152.8	1337.0	1376.1	1435.7	1.07	3683.3	2404.9	1412.3	1454.8	2103.3	3128.7	0.93
80	1733.3	1554.9	1348.7	1566.7	1821.8	1881.2	1.07	5573.3	3449.7	1751.4	1559.4	2179.6	3252.2	1.07
110	2183.3	1891.8	1544.6	1674.1	2267.5	2326.6	1.07	7463.3	4494.5	2090.5	1664.0	2223.3	3330.7	1.07
140	2633.3	2226.8	1740.4	1781.5	2386.3	2772.1	1.07	9353.3	5539.3	2429.6	1768.6	2256.9	3384.1	1.07
170	3083.3	2561.8	1932.2	1888.8	2447.7	3217.5	1.07	11243.0	6584.1	2768.7	1868.5	2290.6	3423.5	0.93
200	3533.3	2896.7	2123.6	1996.2	2509.0	3623.1	1.07	13133.0	7628.9	3107.8	1962.7	2324.2	3452.7	0.93
230	3983.3	3231.7	2315.1	2103.6	2570.4	3662.6	1.07	15023.0	8673.8	3446.9	2056.8	2357.8	3466.6	0.93
260	4433.3	3566.7	2506.6	2211.0	2631.7	3702.1	1.07	16913.0	9718.6	3786.0	2151.0	2391.5	3480.5	0.93
290	4883.3	3901.7	2698.0	2315.4	2693.1	3741.5	0.99	18803.0	10763.0	4125.0	2245.2	2425.1	3494.4	0.93
320	5333.3	4236.7	2889.5	2416.9	2754.4	3781.0	0.99	20693.0	11808.0	4464.1	2339.3	2458.7	3508.3	0.93
350	5783.3	4571.7	3080.9	2518.3	2815.8	3820.5	0.99	22583.0	12853.0	4803.2	2433.5	2492.4	3522.3	0.93
380	6233.3	4906.7	3272.4	2619.7	2877.1	3859.9	0.99	24473.0	13898.0	5142.3	2527.7	2526.0	3536.2	1.07
400	6533.3	5130.0	3400.0	2687.3	2918.0	3886.2	0.99	25733.0	14594.0	5368.4	2590.5	2548.4	3545.4	1.07

	$S_{max} = 200$							$S_{max} = 200$						
20	1366.7	1327.4	1153.3	1372.4	1463.8	1523.6	1.07	2326.7	1803.5	1258.4	1370.5	2005.0	2453.1	1.07
50	1816.7	1664.3	1349.2	1479.8	1909.5	1969.1	1.07	4216.7	2848.3	1599.4	1475.2	2112.6	3156.6	1.07
80	2266.7	2000.3	1544.9	1587.1	2236.0	2414.5	0.93	6106.7	3893.2	1938.5	1579.8	2179.9	3255.2	1.07
110	2716.7	2335.3	1736.4	1694.5	2324.3	2859.9	1.07	7996.7	4938.0	2277.6	1684.4	2223.3	3333.7	1.07
140	3166.7	2670.3	1927.9	1801.9	2386.3	3305.4	1.07	9886.7	5982.8	2616.7	1789.0	2256.9	3384.1	1.07
170	3616.7	3005.2	2119.4	1909.3	2447.7	3583.6	1.07	11777.0	7027.6	2955.8	1883.4	2290.6	3423.6	0.93
200	4066.7	3340.2	2310.8	2016.6	2509.0	3623.1	1.07	13667.0	8072.4	3294.9	1977.6	2324.2	3452.7	0.93
230	4516.7	3675.2	2502.3	2124.0	2570.4	3662.6	1.07	15557.0	9117.3	3634.0	2071.7	2357.8	3466.6	0.93
260	4966.7	4010.2	2693.8	2229.0	2631.7	3702.1	0.99	17447.0	10162.0	3973.1	2165.9	2391.5	3480.5	0.93
290	5416.7	4345.2	2885.2	2330.4	2693.1	3741.5	0.99	19337.0	11207.0	4312.2	2260.1	2425.1	3494.4	0.93
320	5866.7	4680.2	3076.7	2431.9	2754.4	3781.0	0.99	21227.0	12252.0	4651.3	2354.2	2458.7	3508.3	0.93
350	6316.7	5015.2	3268.2	2533.3	2815.8	3820.5	0.99	23117.0	13297.0	4990.4	2448.4	2492.4	3522.3	0.93
380	6766.7	5350.2	3459.6	2634.7	2877.1	3859.9	0.99	25007.0	14341.0	5329.5	2542.6	2526.0	3536.2	1.07
400	7066.7	5573.5	3587.3	2702.3	2918.0	3886.2	0.99	26267.0	15038.0	5555.5	2605.3	2548.4	3545.4	1.07

Table 5.2 Cost comparison among different PM strategies given $\beta = 3$ and $S_{max} = 100$

c_r	Minimum Life-cycle Cost under (m^*, t_0^*)	Strategy A ($t_0 = L$ or $m = 0$)		Strategy B ($t_0 = \tau$)		Strategy C ($t_0 = w + \tau$)	
		Life-cycle Cost	Cost Increase (%)	Life-cycle Cost	Cost Increase (%)	Life-cycle Cost	Cost Increase (%)
20	1062.0	1793.3	68.9%	1133.3	6.7%	1320.6	24.4%
50	1412.3	3683.3	160.8%	1472.5	4.3%	1861.0	31.8%
80	1559.4	5573.3	257.4%	1706.2	9.4%	2203.7	41.3%
110	1664.0	7463.3	348.5%	1800.4	8.2%	2546.3	53.0%
140	1768.6	9353.3	428.9%	1894.6	7.1%	2889.0	63.3%
170	1868.5	11243.0	501.7%	1988.8	6.4%	3231.7	73.0%
200	1962.7	13133.0	569.1%	2083.0	6.1%	3493.4	78.0%
230	2056.8	15023.0	630.4%	2177.2	5.9%	3728.5	81.3%
260	2151.0	16913.0	686.3%	2271.5	5.6%	3963.6	84.3%
290	2245.2	18803.0	737.5%	2365.7	5.4%	4198.7	87.0%
320	2339.3	20693.0	784.6%	2459.9	5.2%	4433.8	89.5%
350	2433.5	22583.0	828.0%	2554.1	5.0%	4668.9	91.9%
380	2526.0	24473.0	868.8%	2648.3	4.8%	4904.0	94.1%
400	2548.4	25733.0	909.8%	2711.1	6.4%	5060.8	98.6%

Table 5.2 gives a cost comparison of our proposed (optimal) strategy with other maintenance strategies (A, B, C) that have been widely adopted in the literature. Let $\beta = 3$, $S_{max} = 100$ and c_r vary. Strategy A assumes no PM action during the life cycle; Strategy B considers periodic PMs throughout the life cycle; and Strategy C requires PMs only after the warranty is expired. We observe that, Strategy B is always better than Strategy A and C in total cost minimization. However, compared to the optimal strategy in our model, the cost of Strategy B still turns out to be 4.3%~9.4% higher, depending on the specific values of c_r ($\beta = 3$).

Sensitivity analysis:

So far, we have investigated the effect of c_r, S_{max} and β on the optimal solutions. In the following part, sensitivity analysis is further conducted over a range of other parameters including $L, \tau, w, \omega_a, \omega_b$ and k. By varying one or two parameters each time, it is assumed that the remaining parameters are set at their nominal values (refer to Table 5.3).

Table 5.3 Nominal values of model parameters for sensitivity analysis

(α, β)	L	w	τ	c_r	S_{max}	k	ω_A	ω_B	$\{c_p(.)\}$
(0.5, 2)	8	2	0.33	50	100	3	$\frac{1}{2}L$	$\frac{1}{6}L$	{0, 10, 30, 60, 100, 160}

- **When L, w, and τ vary**

Here we investigate the effect of L (= 6,8,10), τ (= 0.25,0.33,0.5) and w (= 0,2,4) on the optimal PM strategies separately. Let c_r vary from 20 to 400. The rest parameters are set at their nominal values.

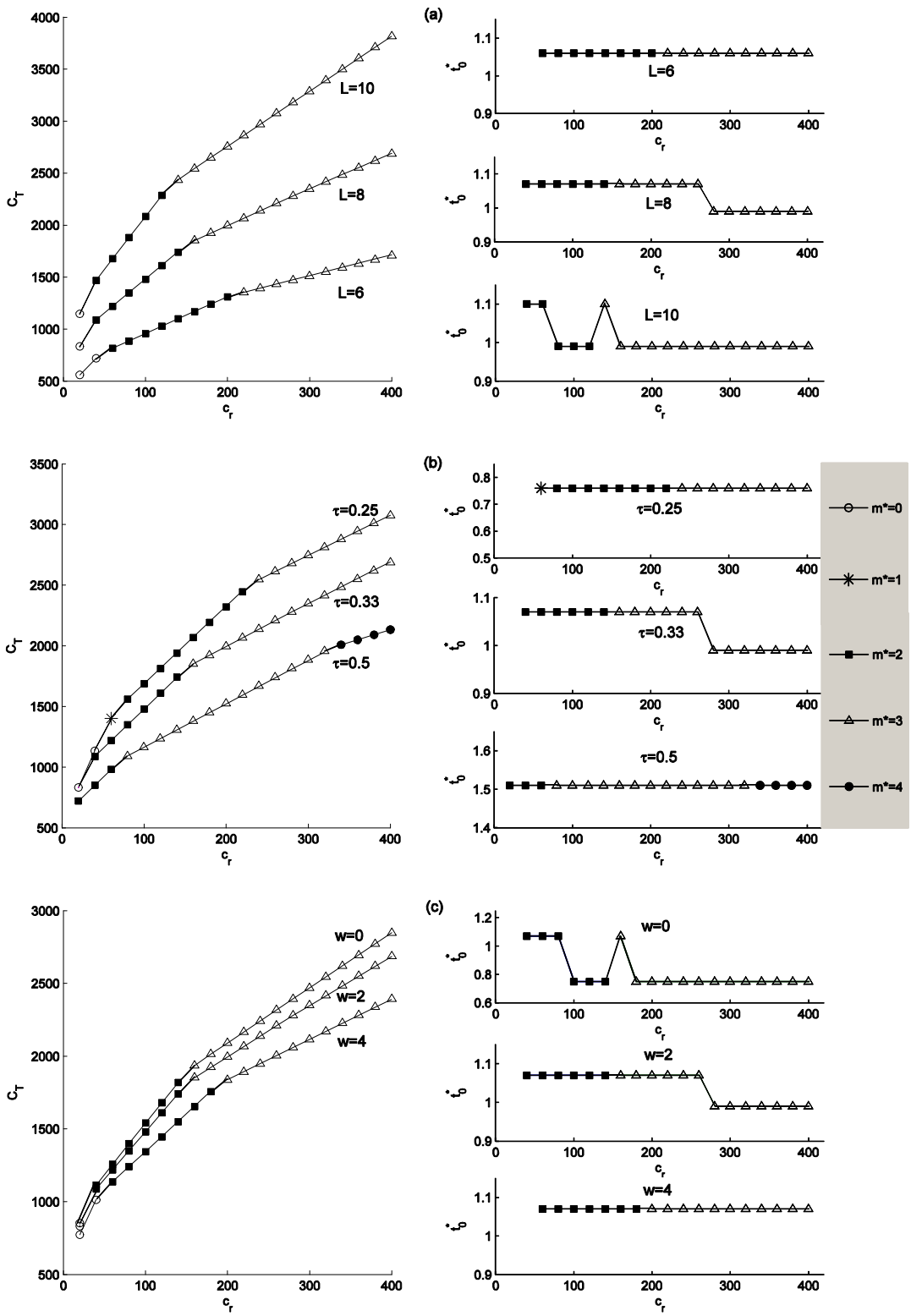

Figure 5.4 Optimal PM strategies under different values of (a) *L*, (b) *τ* and (c) *w*

Results in Figure 5.4(a) show that given a fixed c_r, the system with longer life cycles should require deeper levels (or at least the same levels) of preventive maintenance. However, the impact of L on m^* diminishes when c_r further increases. Typically, if $c_r \geq 220$, PM action with level 3 should be implemented and the results are not affected by the changing values of L.

Results in Figure 5.4(b) show that the more frequently the system is maintained (i.e. a smaller τ), the less effort is required for each PM, but the higher life-cycle cost is incurred to the buyer. In comparison, a large τ always requires a deeper PM each time. Typically, when $\tau = 0.5$, the PMs reach level 4 given $c_r \geq 340$. Such strategy can fully utilize the capacity of each PM given the same PM cost for the same PM level. However, system safety (and consequently a large downtime cost) is often the issue that will require the frequency of PMs above certain level (see Chareonsuk et al., 1997).

Results in Figure 5.4(c) show that more maintenance efforts should be invested when the length of warranty coverage is shorter. On the other hand, the total life-cycle cost is higher to the buyer under shorter warranty coverage since the post-warranty cost is higher.

It is also verified in Figure 5.4 that the changes on t_0^* is mainly dominated by the changes on m^* rather than directly follows the changes on L, τ, and w.

- **When ω_A and ω_B vary**

Two extreme cases are considered in order to investigate the effect of ω_A and ω_B on the optimal PM solutions: 1) $\omega_B = 0$, $\omega_A = 0 \sim L$, and 2) $\omega_A = L$, $\omega_B = 0 \sim L$. It is

noticed that Case 1) and Case 2) correspond to Curves A and B in Figure 5.2 respectively. c_r is set at its nominal value 50.

The results are given in Figure 5.5. For Case 1), we notice that the optimal PM level first increases to level 2 when ω_A increases, and then decreases to level 1 as ω_A approaches to L. The interpretation is that deeper PM actions are not effective in reducing life-cycle cost when ω_A approaches to either 0 or L. When ω_A is small, ageing loss rate increases rapidly and it is hard to be slowed down by conducting deeper levels of PM actions. On the other hand, when ω_A reaches above certain level ($\omega_A \geq 6$ for instance), ageing losses are no longer a main contribution of total cost and therefore deeper levels of PM actions become unnecessary. For Case 2), optimal PMs remains at level 1 when $\omega_B \leq 7.6$ and no PM is needed when ω_B reaches L. Correspondingly, t_0^* increases sharply when ω_B approaches to L and finally increases to $t_0^* = L$ (i.e. $m = 0$).

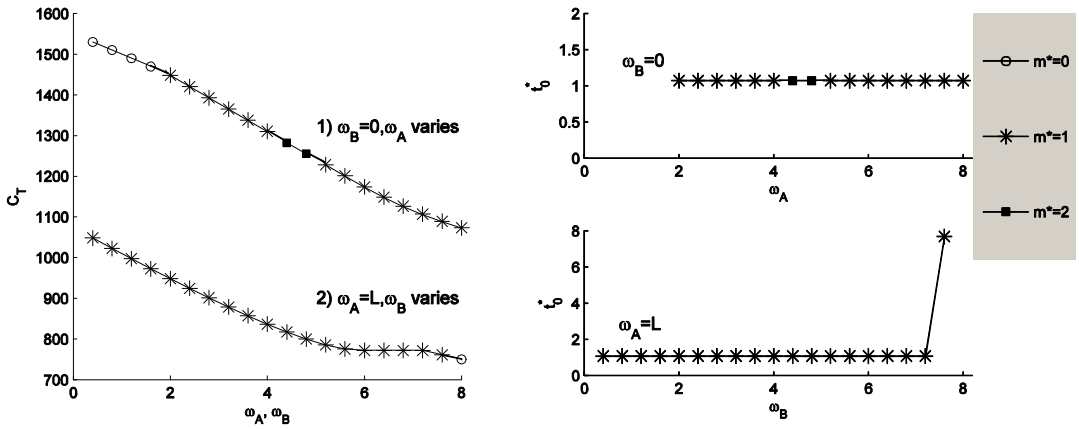

Figure 5.5 Optimal PM strategies following the changes on ω_A and ω_B

- **When k varies**

Here we investigate the effect of control limit factor $K(m)(= k[1 - \varphi(m)]\tau)$ on the optimal solutions. Let k vary from 2 to 40 and other parameters set at their nominal values.

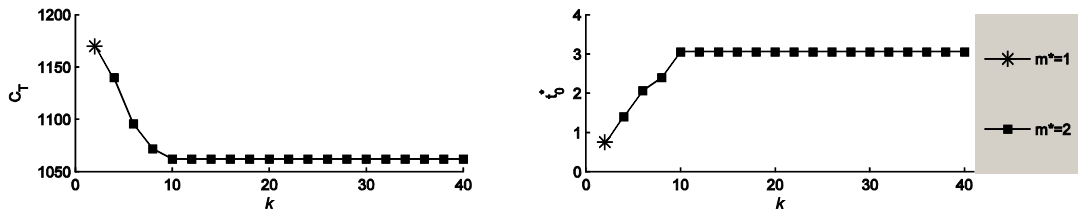

Figure 5.6 Optimal PM strategies following the changes on *k*

Results in Figure 5.6 show that the total life-cycle cost decreases when k increases given $k < 10$ and remains at the same value given $k \geq 10$. When $k \geq 4$, a deeper level of PM is applied implying that a larger capacity of PMs is allowed under a higher control limit. However, the effect of k diminishes after it reaches above certain level (i.e. $k \geq 10$ here).

5.5 Conclusion

In this chapter, a general periodic preventive maintenance (PM) policy is studied for a single buyer under the consideration of warranty contracts and ageing losses. Ageing losses are defined as the total revenue losses due to the decreasing productivity during the system ageing process. Therefore, by slowing down the speed of system deterioration, the implementation of preventive maintenance here has two types of value: 1) the reduction of the life-time maintenance cost, and 2) the reduction of ageing losses. Total life-cycle cost model is developed for the buyer including both the maintenance cost and ageing losses. The proposed model here has two decision variables. The first one is the calendar time of the first PM action. Different from

previous studies, we assume that it is not pre-specified but has to be optimized. The following PM actions are then carried out periodically until the end of the system life cycle. The second decision variable is the degree of each PM action. We follow the treatment in Kim et al. (2004) by assuming that the PM levels are discrete in nature. In order to minimize the buyer's life-cycle cost, we derive some analytical insights of the optimal solutions and then apply the results to simplify the model optimization. A numerical case is presented to demonstrate the applicability of the model, and sensitivity analyses are further conducted to investigate the effect of model parameters on the optimal PM strategies.

CHAPTER 6 MSS MAINTENANCE SERVICE DESIGN

6.1 Introduction

In Chapter 5, we focused on the life-cycle maintenance service design for a binary system whose ageing process is described by a continuous and deterministic function of time (e.g. failure rate function). However, this might not be realistic for some systems of which the ageing processes depend not only on the elapsed operational time, but also on the system status, such as vibration level, efficiency, number of random shocks on the system, etc., any of which causes performance degradation. In this chapter, we further investigate the life-cycle maintenance service design for multi-state systems (MSS) from the buyer's perspective. Maintenance cost models with and without warranty incorporation are derived. The basic concepts of multi-stage degradation models can be found in Barlow and Wu (1978), El-Neweihi et al. (1978) and Ross (1979), which defined the system structure function and its properties. The corresponding performance analysis (e.g. reliability, availability, mean time-to-failure, redundancy etc.) were addressed by Xue and Yang (1995), Pham and Misra (1997), Wu (2005), Zuo and Tian (2006), Tian et al. (2008a, b) and Tai and Chan (2010).

Optimization of maintenance policies for multi-state systems (MSS) is a natural extension of the maintenance studies for the binary systems which utilize many results from the reliability modelling of MSS. The majority of the current literatures assume that maintenance actions for MSS are planned based on an infinite operating horizon and after any replacement or restoration, the system is renewed and the same process is assumed to repeat indefinitely. Characteristics of a system, such as the current state,

the age and the elapsed operating time during each state, are often selected as the optimality criterions and used to minimize the long-run average maintenance cost rate function. Reviews of work in this area can be found in Kao (1973), Sim and Endrenyi (1993), Yeh (1996), Levitin and Lisnianski (2000), Grall et al. (2002a), Moustafa et al. (2004) and Kim and Makis (2009).

In practice, however, the useful life cycle of most systems are finite in nature. For instance, in military applications, a missile launching system is only required to be functioning within the designated mission time. Different from an infinite-horizon maintenance problem, residual life cycle for such system, which is measured from the present time to the end of the mission, is typically finite and decreases over time. When the mission is close to end, replacement of a functioning system becomes less necessary and traditional maintenance strategies, such as those merely relying on the information of the current system state, could turn out to be very costly to the stakeholders. Considering the improper planning horizon, though bringing technical convenience, may not be realistic under these circumstances (Nakagawa and Mizutani, 2009). On the other hand, compared to the vast amount of literature in infinite-horizon maintenance planning, existing works showed very limited options for maintaining MSS with finite life cycles. Su and Chang (2000) proposed a periodic maintenance policy for MSS and derived the optimal number of maintenance activities that minimized the total life-cycle cost. Zuo et al. (2000) investigated the optimal replacement policy for multi-state products under warranty such that the manufacturer's warranty cost was minimized. Ivy and Pollock (2005) and Maillart and Zheltova (2007) analyzed maintenance and inspection policies for a discrete-time Markov system over a finite horizon given that perfect observations of systems states were not available. Ding et al. (2009) studied the optimal corrective maintenance

planning for the MSS subject to availability constraints. Among these studies, very few further considered maintenance optimization with multiple optimality criterions.

In this chapter, we assume that the ageing process of the system is modelled as a continuous-time Markov process that is subject to both degradation and Poisson failures. We assume that the system can fail randomly from any of the operational states (Poisson failures) and can be rectified by minimal repair which returns the system to its previous working state. Any unexpected (Poisson) failure is assumed to result in an extra downtime cost that is borne by the customer. We propose two MSS maintenance policies for controlling the customer's *expected discounted maintenance cost* (EDMC) over a finite system life cycle. The first policy conducts preventive system replacement, i.e. a system may be replaced while still operational. In contrast the second policy allows only corrective replacements, i.e. system replacements are only made when the system suffers a random failure. For both policies, the EDMC is derived as a function of two control parameters, namely, a threshold level on the current state of the system, and a threshold level on the residual life cycle (measured from present time to the end of life cycle). We further propose two different methodologies for the optimization of maintenance thresholds. The first method utilizes the LST and inverse LST techniques, while the second method directly approximates the EDMC and optimizes the maintenance thresholds on the time domain. The applications of both methods are illustrated using a numerical case, and the impact of warranty incorporation is also discussed. Through computational examples, we demonstrate that preventive replacement outperforms corrective replacement when the downtime cost per failure is relatively high compared to the repair cost. Unlike past works, this research incorporates many realistic factors, i.e. multiple system states, discounted economic values, finite planning horizon, warranty coverage, and easy-to-

implement maintenance policies. As such, it should be of interest to both theoreticians and practitioners.

The rest of this chapter is organized as follows. In Section 6.2, we present the system descriptions and propose the maintenance policies for the MSS. Section 6.3 derives the EDMC for the customer under both Policy A and B with and without warranty incorporation. Methodologies for analyzing the optimal maintenance policies are further proposed in Section 6.4. Section 6.5 demonstrates the applicability of the foregoing analysis with numerical examples. Conclusion is made in Section 6.6.

6.2 Model Formulation

6.2.1 System Description

Consider a multi-state system (MSS) that initially works under a perfect condition. The system can have N stages of degradation before reaching a complete failure and let $\Omega = \{1,2,3,\dots,N\}$ represent the set of all these stages. We define three disjoint sets of the states which fully characterize the MSS – the operational states $S_1 = \{2i-1, i \in \Omega\}$, the (Poisson) failure states $S_2 = \{2i, i \in \Omega\}$ and the complete failure state $\{2N+1\}$. State 1 represents the perfect functioning state and the degree of deterioration increases with each subsequent operational state. In particular, once the system degrades to State $2N+1$, it is considered as completely failed and can only be rectified by a replacement. Here the ith stage degradation is defined as the transition period from State $2i-1$ to $2i+1$ ($i \in \Omega$) and is characterized by a degradation rate α_i. In addition to the degradation process, the system is also subject to random (Poisson) failure process from any operational state $2i-1$ (i.e. $2i-1 \rightarrow 2i, i \in \Omega$) and can be rectified by repair. The failure from operational state $2i-1$ ($i \in \Omega$) to failure state $2i$ is characterized by a failure rate λ_i.

Let r_i ($i \in \Omega$) represent the replacement cost for the system when the system is either operating in State $2i-1$ or failed from State $2i-1$, and let m_i ($i \in \Omega$) represent the corresponding minimal repair cost during this stage. For each (unexpected) Poisson failure, we assume that there is an additional downtime cost d_i associated with m_i which is borne by the customer. Also, let r_{N+1} and d_{N+1} represent the replacement and downtime cost for the system when it reaches a complete failure (i.e. State $2N+1$).

A graphical description of the above system is given in Figure 6.1(a). In order to complete our model formulation, the following assumptions are made.

1) The system is replaced with a new one once it degrades to the complete failure state (2*N*+1).
2) The system is minimally repaired after Poisson failures. The repair returns the system to the operational state right before the failure.
3) All the transition rates α_is and λ_is ($i \in \Omega$) are constant but state-dependent. In particular, we assume $\lambda_1 < \lambda_2 < \cdots < \lambda_{N-1} < \lambda_N$ to describe the ageing of the system.
4) The average repair and replacement time is very small compared to mean time between failures and therefore is negligible.
5) The system becomes more costly to repair and replace when it ages, i.e. $m_1 < m_2 < \cdots < m_{N-1} < m_N$ and $r_1 < r_2 < \cdots < r_{N-1} < r_N < r_{N+1}$.
6) No downtime cost is incurred or associated with preventive replacement when the system is still functioning.
7) The current state of the system is always known (observed) for certain by continuous monitoring.

6.2.2 System Replacement Policies

We denote L as the finite planning horizon, or the system life cycle, and δ as the continuous discounted factor over the cycle. It is important to notice that maintenance cost is not incurred at the stage of maintenance planning but rather spent in future and allocated over the system life cycle. Therefore, incorporating δ in the cost forecasting will have practical meanings, in particular for those managerial circumstances such as budget allocation and balance-sheet reporting where the accuracy of cost estimation is crucial to the decision makers. As an endeavour to minimize the EDMC for the customer, we propose the following two maintenance policies (A and B), both of which rely on two threshold parameters (J, τ), where $1 \leq J < N$ and $0 \leq \tau \leq L$.

Policy A (Preventive Replacement): If the system operates in State $2i - 1$ ($i \in \Omega$) and the residual life cycle is t ($0 < t \leq L$), it is then replaced by a new one if and only if $J + 1 \leq i \leq N$ and $t \geq \tau$; otherwise, no replacement is made.

Policy B (Corrective Replacement): If the system fails from State $2i - 1$ ($i \in \Omega$) and the residual life cycle is t ($0 < t \leq L$), it is then replaced by a new one if and only if $J + 1 \leq i \leq N$ and $t \geq \tau$; otherwise, it is minimally repaired.

Both policies utilize the information of the current system state and the residual life cycle. The system is replaced only when its deterioration level is heavier than the threshold parameter J and the residual life cycle is longer than τ. Such policies can avoid expensive replacements when the system is still relatively healthy or when the system is close to retirement. Policy A requires *preventive replacement* for the system when it is still functioning. On the other hand, Policy B implements *corrective replacement* for the system only upon (Poisson) failures.

For comparison purpose, the base case of no corrective or preventive replacement is also defined (i.e. Policy O). Note that when $\tau = L$, both Policy A and B reduce to Policy O.

Policy O: No corrective or preventive replacement.

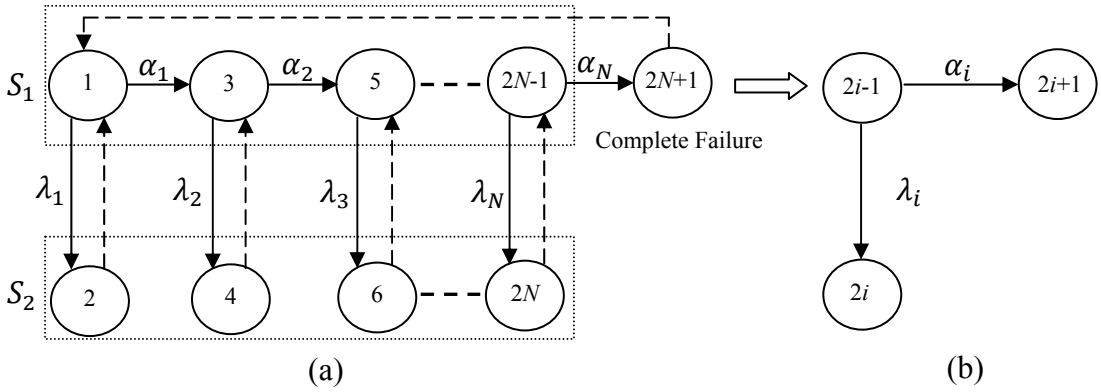

Figure 6.1 Description for (a) an *N*-stage degradation MSS, (b) an isolated 3-state MSS

6.3 Model Development

In this section, we derive the close-to-explicit forms of the EDMC for the customer under Policies O, A and B. The discounted cost models are presented in recursive forms and solved iteratively.

We present some preliminary results for a 1-stage degradation system before proceeding to the analysis of *N*-stage degradation system.

6.3.1 Preliminary Results

Consider a 3-state Markov system ($i \in \Omega$) in Figure 6.1(b) that is isolated from Figure 6.1(a). In contrast to Figure 6.1(a), we assume that both State $2i$ and $2i + 1$ are absorbing states. The objective is to derive the system state transition (degradation) and time-to-failure distributions that are useful in subsequent analysis.

Let $I(y)$ represent the system state after an elapsed life time y. We assume that the system initially operates at State $2i-1$, i.e. $I(0)=2i-1$. Define $Q_i(y)=\Pr\{I(y)=2i-1|I(0)=2i-1\}$, $P_i(y)=\Pr\{I(y)=2i|I(0)=2i-1\}$ and $G_i(y)=\Pr[I(y)=2i+1|I(0)=2i-1]$. Also, define $p_i(y)=dP_i(y)/dy$ and $g_i(y)=dG_i(y)/dy$ as the corresponding probability densities of system failure and degradation at time y. The Chapman-Kolmogorov equations for such a simple Markov system can be written as

$$\begin{cases} \dfrac{dQ_i(y)}{dy}=-(\alpha_i+\lambda_i)Q_i(y) \\ \dfrac{dP_i(y)}{dy}=\lambda_i Q_i(y) \\ \dfrac{dG_i(y)}{dy}=\alpha_i Q_i(y) \end{cases} \tag{6.1}$$

with the initial conditions satisfying $Q_i(0)=1$, $P_i(0)=0$ and $G_i(0)=0$ $(i\in\Omega)$. Solutions for (6.1) are explicitly given as $Q_i(y)=e^{-(\alpha_i+\lambda_i)y}$, $P_i(y)=\lambda_i\left[1-e^{-(\alpha_i+\lambda_i)y}\right]/(\alpha_i+\lambda_i)$ and $G_i(y)=\alpha_i\left[1-e^{-(\alpha_i+\lambda_i)y}\right]/(\alpha_i+\lambda_i)$. Therefore, $p_i(y)=\lambda_i e^{-(\alpha_i+\lambda_i)y}$ and $g_i(y)=\alpha_i e^{-(\alpha_i+\lambda_i)y}$.

6.3.2 The EDMC Model for Policy O

Here we investigate the EDMC for the *N*-stage degradation system under Policy O.

Let $C_i^{(O)}(t)$ represent the EDMC under Policy O when the system is in State $2i-1$ $(i\in\Omega)$ and the residual life cycle is t. The objective is therefore to obtain $C_1^{(O)}(L)$. Note that without preventive or corrective replacement, the system is automatically replaced by a new unit when it degrades to State $2N+1$, until the end of its life cycle. Using the expressions for $p_i(t)$ and $g_i(t)$ and incorporating the discounted factor δ, the recursive form of the cost model is presented as follow:

$$\begin{cases} C_i^{(O)}(t) = \int_0^t \left[m_i + d_i + C_i^{(O)}(x)\right] e^{-\delta(t-x)} p_i(t-x)dx \\ \qquad + \int_0^t C_{i+1}^{(O)}(x) e^{-\delta(t-x)} g_i(t-x)dx = \int_0^t \left[m_i + d_i + C_i^{(O)}(x)\right] \lambda_i e^{-\eta_i(t-x)}\, dx \\ \qquad + \int_0^t C_{i+1}^{(O)}(x) \alpha_i e^{-\eta_i(t-x)} dx \quad \text{for } i = 1,2,\dots,N-1 \\ C_N^{(O)}(t) = \int_0^t \left[m_N + d_N + C_N^{(O)}(x)\right] \lambda_N e^{-\eta_N(t-x)}\, dx \\ \qquad + \int_0^t \left[r_{N+1} + d_{N+1} + C_1^{(O)}(x)\right] \alpha_N e^{-\eta_N(t-x)}\, dx \end{cases} \tag{6.2}$$

The analytical form of $C_1^{(O)}(L)$ can be obtained by solving (6.2) iteratively using LST technique. We present the results in the following proposition. Define $\alpha_0 = 1$, and

$$e_i = \begin{cases} \lambda_i(m_i + d_i) & \text{for } i = 1,2,\dots N-1 \\ \lambda_N(m_N + d_N) + (r_{N+1} + d_{N+1})\alpha_N & \text{for } i = N \end{cases} \tag{6.3}$$

Proposition 6.1: For an *N*-stage degradation system that initially works under a perfect condition, the close-to-explicit form of the EDMC for Policy O is given as

$$C_1^{(O)}(L) = \left(\mathcal{L}^{-1} \left[\frac{\sum_{j=1}^{N} \left[\prod_{k=0}^{j-1} \alpha_k \times e_j \times \prod_{k=j+1}^{N} (s + \alpha_k + \delta) \right]}{s\left[\prod_{j=1}^{N}(s + \alpha_j + \delta) - \prod_{j=1}^{N} \alpha_j\right]} \right] \right) \Bigg|_{t=L} \tag{6.4}$$

Proof: Define $C_i^{(O)}(s) = \mathcal{L}[C_i^{(O)}(t)]$ as the LST of $C_i^{(O)}(t)$ ($i \in \Omega$). We then have

$$\begin{cases} C_i^{(O)}(s) = \dfrac{e_i}{s(s + \alpha_i + \delta)} + C_{i+1}^{(O)}(s) \dfrac{\alpha_i}{s + \alpha_i + \delta} & \text{for } i = 1,2,\dots,N-1 \\ C_N^{(O)}(s) = \dfrac{e_N}{s(s + \alpha_N + \delta)} + C_1^{(O)}(s) \dfrac{\alpha_N}{s + \alpha_N + \delta} \end{cases} \tag{6.5}$$

After simplification,

$$C_1^{(O)}(s) = \frac{\sum_{j=1}^{N} \left[\prod_{k=0}^{j-1} \alpha_k \times e_j \times \prod_{k=j+1}^{N} (s + \alpha_k + \delta) \right]}{s\left[\prod_{j=1}^{N}(s + \alpha_j + \delta) - \prod_{j=1}^{N} \alpha_j\right]} \tag{6.6}$$

From (6.6), Proposition 6.1 is easily obtained.

Remarks: Equation (6.6) is considered as close-to-explicit because obtaining the inverse transform for $C_1^{(O)}(t)$ requires a numerical solver (e.g. Matlab) except for some simple cases. The application of such inversion techniques for the maintenance optimization will be illustrated shortly. Note that when $C_1^{(O)}(t)$ is obtained, we can further obtain $C_i^{(O)}(t)$ for a system starting at any degraded state (i.e. $i > 1$) using the following:

$$C_i^{(O)}(t) = \frac{1}{\alpha_{i-1}}\left[\frac{dC_{i-1}^{(O)}(t)}{dt} + (\alpha_{i-1} + \delta)C_{i-1}^{(O)}(t) - e_{i-1}\right] \text{ for } i = 2,3,\dots,N \quad (6.7)$$

6.3.3 The EDMC Model for Policy A

In this section, we investigate the EDMC model for Policy A when the maintenance thresholds (J, τ) are given. We assume that when the system is preventively replaced, no downtime cost is incurred. One of the justifications for this is that a warm-standby may be initiated before shutting down the old unit for replacement. Let $C_i^{(A)}(t|J, \tau)$ represent the EDMC under (J, τ) when the system is working in State $2i - 1$ ($i \in \Omega$) and the residual life cycle is t. Again, $p_i(t) = \lambda_i e^{-(\alpha_i+\lambda_i)t}$ and $g_i(t) = \alpha_i e^{-(\alpha_i+\lambda_i)t}$. Two cases are further analyzed: $t \leq \tau$ and $t > \tau$.

- When $t \leq \tau$

In this case, no preventive replacement is required under Policy A. $C_i^{(A)}(t|J, \tau)$ is calculated in the same way as Policy O, i.e.

$$C_i^{(A)}(t|J, \tau) = C_i^{(O)}(t), \quad t \leq \tau, i \in \Omega \quad (6.8)$$

- When $t > \tau$

Note that for this case, the deterioration of the system is no heavier than $2J - 1$ ($1 \leq J < N$); otherwise, the system should have been preventively replaced under Policy A. Consequently, we have

$$\begin{cases} C_i^{(A)}(t|J,\tau) = e^{-\eta_i(t-\tau)}C_i^{(O)}(\tau) + \int_\tau^t \left[m_i + d_i + C_i^{(A)}(x|J,\tau)\right]\lambda_i e^{-\eta_i(t-x)}dx \\ \qquad + \int_\tau^t C_{i+1}^{(A)}(x|J,\tau)\alpha_i e^{-\eta_i(t-x)}dx \quad \text{for } i = 1,2\ldots,J-1 \\ C_J^{(A)}(t|J,\tau) = e^{-\eta_J(t-\tau)}C_J^{(O)}(\tau) + \int_\tau^t \left[m_J + d_J + C_J^{(A)}(x|J,\tau)\right]\lambda_J e^{-\eta_J(t-x)}dx \\ \qquad + \int_\tau^t \left[r_{J+1} + C_1^{(A)}(x|J,\tau)\right]\alpha_J e^{-\eta_J(t-x)}dx \end{cases} \tag{6.9}$$

Since $\tau \leq L$, the analytical form of $C_1^{(A)}(L|J,\tau)$ is determined by (6.4) and (6.9). We present the results in the following Proposition.

Proposition 6.2: For an *N*-stage degradation system that initially works under a perfect condition, the close-to-explicit form of the EDMC for Policy A under maintenance thresholds (J,τ) is given as

$$C_1^{(A)}(L|J,\tau) = \left(\mathcal{L}^{-1}\left[\frac{\begin{array}{c}\sum_{j=1}^{J-1}\left[\prod_{k=0}^{j-1}\alpha_k\left(e_j + sC_j^{(O)}(\tau)\right)\prod_{k=j+1}^{J}(s+\alpha_k+\delta)\right] \\ +\prod_{k=0}^{J-1}\alpha_k\left(e_J + r_{J+1}\alpha_J + sC_J^{(O)}(\tau)\right)\end{array}}{s\left[\prod_{j=1}^{J}(s+\alpha_j+\delta) - \prod_{j=1}^{J}\alpha_j\right]}\right]\right)\Bigg|_{t=L-\tau} \tag{6.10}$$

Proof: Let $u = t - \tau$ ($u \geq 0$) and define $H_i^{(A)}(u|J,\tau) = C_i^{(A)}(t|J,\tau)$ for any $t \geq \tau$. Equation (6.9) can be rewritten as

$$\begin{cases} H_i^{(A)}(u|J,\tau) = e^{-\eta_i u} C_i^{(O)}(\tau) + \int_0^u \left[m_i + d_i + H_i^{(A)}(x|J,\tau)\right] \lambda_i e^{-(\alpha_i+\lambda_i+\delta)(u-x)} dx \\ \qquad + \int_0^u H_{i+1}^{(A)}(x|J,\tau) \alpha_i e^{-\eta_i(u-x)} dx \quad \text{for } i = 1,2 \dots, J-1 \\ H_J^{(A)}(u|J,\tau) = e^{-\eta_J u} C_J^{(O)}(\tau) + \int_0^u \left[m_J + d_J + H_J^{(A)}(x|J,\tau)\right] \lambda_J e^{-\eta_J(u-x)} dx \\ \qquad + \int_0^u \left[r_{J+1} + H_1^{(A)}(x|J,\tau)\right] \alpha_J e^{-\eta_J(u-x)} dx \end{cases} \tag{6.11}$$

Define $H_i^{(A)}(s|J,\tau) = \mathcal{L}[H_i^{(A)}(u|J,\tau)]$ as the LST of $H_i^{(A)}(u|J,\tau)$. We then have

$$\begin{cases} H_i^{(A)}(s|J,\tau) = \dfrac{e_i + sC_i^{(O)}(\tau)}{s(s+\alpha_i+\delta)} + H_{i+1}^{(A)}(s|J,\tau) \dfrac{\alpha_i}{s+\alpha_i+\delta} \quad \text{for } i = 1,2,\dots,J-1 \\ H_J^{(A)}(s|J,\tau) = \dfrac{e_J + r_{J+1}\alpha_J + sC_J^{(O)}(\tau)}{s\left(s+\alpha_J+\delta\right)} + H_1^{(A)}(s|J,\tau) \dfrac{\alpha_J}{s+\alpha_J+\delta} \end{cases} \tag{6.12}$$

After some simplification, $H_1^{(A)}(s|J,\tau)$ is given as

$$H_1^{(A)}(s|J,\tau) = \frac{\begin{pmatrix} \sum_{j=1}^{J-1} \left[\prod_{k=0}^{j-1} \alpha_k \left(e_j + sC_j^{(O)}(\tau)\right) \prod_{k=j+1}^{J} (s+\alpha_k+\delta)\right] \\ + \prod_{k=0}^{J-1} \alpha_k \left(e_J + r_{J+1}\alpha_J + sC_J^{(O)}(\tau)\right) \end{pmatrix}}{s\left[\prod_{j=1}^{J}(s+\alpha_j+\delta) - \prod_{j=1}^{J} \alpha_j\right]} \tag{6.13}$$

Since $C_1^{(A)}(L|J,\tau) = H_1^{(A)}(L-\tau|J,\tau)$, from (6.13), Proposition 6.2 is thus obtained.

6.3.4 The EDMC Model for Policy B

In this section, we investigate the EDMC model for the customer under Policy B, i.e. corrective replacements. Similarly, let $C_i^{(B)}(t|J,\tau)$ represent the EDMC under (J,τ) when the system is working in State $2i-1$ ($i \in \Omega$) and the residual life cycle is t. Again, two cases are further considered: $t \le \tau$ and $t > \tau$.

- When $t \leq \tau$

For this case, no corrective replacement is conducted. We simply have

$$C_i^{(B)}(t|J,\tau) = C_i^{(O)}(t), \quad t \leq \tau, i \in \Omega \tag{6.14}$$

- When $t > \tau$

Under Policy B, the system may deteriorate to any of the operational states. Therefore we have

$$\begin{cases} C_i^{(B)}(t|J,\tau) = e^{-\eta_i(t-\tau)}C_i^{(O)}(\tau) + \int_\tau^t \left[m_i + d_i + C_i^{(B)}(x|J,\tau)\right]\lambda_i e^{-\eta_i(t-x)}dx \\ \qquad + \int_\tau^t C_{i+1}^{(B)}(x|J,\tau)\alpha_i e^{-\eta_i(t-x)}dx \quad \text{for } i = 1,2,..,J \\ C_i^{(B)}(t|J,\tau) = e^{-\eta_i(t-\tau)}C_i^{(O)}(\tau) + \int_\tau^t \left[r_i + d_i + C_1^{(B)}(x|J,\tau)\right]\lambda_i e^{-\eta_i(t-x)}\,dx \\ \qquad + \int_\tau^t C_{i+1}^{(B)}(x|J,\tau)\alpha_i e^{-\eta_i(t-x)}dx \quad \text{for } i = J+1, J+2, \dots, N-1 \\ C_N^{(B)}(t|J,\tau) = e^{-\eta_N(t-\tau)}C_N^{(O)}(\tau) + \int_\tau^t \left[r_N + d_N + C_1^{(B)}(x|J,\tau)\right]\lambda_N e^{-\eta_N(t-x)}dx \\ \qquad + \int_\tau^t \left[r_{N+1} + d_{N+1} + C_1^{(B)}(x|J,\tau)\right]\alpha_N e^{-\eta_N(t-x)}dx \end{cases} \tag{6.15}$$

To further derive the analytical form of $C_i^{(B)}(L|J,\tau)$, we follow similar procedures as Proposition 6.2.

Proposition 6.3: For an *N*-stage degradation system that initially works under a perfect condition, the close-to-explicit form of the EDMC for Policy B under maintenance threshold (J,τ) is given as

$$C_1^{(B)}(L|J,\tau) = \left(\mathcal{L}^{-1}\left[\frac{U(s|J,\tau)}{V(s|J,\tau)}\right]\right)\Bigg|_{t=L-\tau} \tag{6.16}$$

where $f_i = \lambda_i(r_i - m_i)$, $u_i(s) = e_i + sC_i^{(O)}(\tau)$ $(i \in \Omega)$ and

$$U(s|J,\tau) = \sum_{j=1}^{J}\left[\left(\prod_{k=0}^{j-1}\alpha_k\right) \times u_j(s) \times \left(\prod_{k=j+1}^{J}(s+\alpha_k+\delta)\right)\left(\prod_{k=J+1}^{N}(s+\eta_k)\right)\right]$$
$$+ \sum_{j=J+1}^{N}\left[\left(\prod_{k=0}^{j-1}\alpha_k\right) \times \left(u_j(s)+f_j\right) \times \left(\prod_{k=j+1}^{N}(s+\eta_k)\right)\right] \tag{6.17}$$

$$V(s|J,\tau) = s\left[\left(\prod_{j=1}^{J}(s+\alpha_j+\delta)\right)\left(\prod_{j=J+1}^{N}(s+\eta_j)\right)\right.$$
$$\left. - \sum_{j=J+1}^{N}\left[\left(\prod_{k=0}^{j-1}\alpha_k\right) \times \lambda_j \times \left(\prod_{k=j+1}^{N}(s+\eta_k)\right)\right] - \prod_{j=0}^{N}\alpha_j\right] \tag{6.18}$$

Proof: Let $u = t - \tau$ $(u \geq 0)$ and define $H_i^{(B)}(u|J,\tau) = C_i^{(B)}(t|J,\tau)$ for any $t \geq \tau$. The remaining procedure is identical to (6.11)–(6.13) in Proposition 6.2.

6.3.5 The EDMC Model with Warranty Incorporation

We further consider the case when the system is released under warranty. Note that the extent of warranty protection depends on the detailed terms specified in the warranty contracts and is usually determined by the price of the product or the price of the warranty if it can be detached. Here we make a realistic assumption that all the system breakdowns (Poisson failures) within the warranty are rectified by the manufacturer (or vendor, seller, third party, etc) for free, while the remaining costs are afforded by the customer. Note that the degradation failure is not a breakdown of the system but

indicates an inferior level of performance which is unacceptable to the customer. Therefore it is not covered under warranty.

Let $l_{\overline{w}} = L - w$ represent the length of the post-warranty period and let $l_{\overline{w}} = 0$ if $L < w$. For Policy O, the EDMC needs to be derived separately for $t \leq l_{\overline{w}}$ and $t > l_{\overline{w}}$. In contrast, for Policy A and B, the EDMC as a function of τ varies when $\tau \leq l_{\overline{w}}$ and $\tau > l_{\overline{w}}$. The procedure of the cost derivation for Policies O, A and B can be easily modified from the case without warranty incorporation. We therefore present the results below without proof.

For Policy O:

1) When $t \leq l_{\overline{w}}$, the EDMC is given by (6.2).
2) When $t > l_{\overline{w}}$, the EDMC is modified from (6.2) by deducting the minimal repair cost and is given as below:

$$\begin{cases} C_i^{(O)}(t) = C_i^{(O)}(l_{\overline{w}})e^{-\eta_i(t-l_{\overline{w}})} + \int_{l_{\overline{w}}}^{t} \left(d_i + C_i^{(O)}(x)\right)\lambda_i e^{-\eta_i(t-x)}\,dx \\ \qquad + \int_{l_{\overline{w}}}^{t} C_{i+1}^{(O)}(x)\alpha_i e^{-\eta_i(t-x)}\,dx \quad \text{for } i = 1,2,\dots,N-1 \\ C_N^{(O)}(t) = C_N^{(O)}(l_{\overline{w}})e^{-(\alpha_N+\lambda_N+\delta)(t-l_{\overline{w}})} + \int_{l_{\overline{w}}}^{t} \left(d_N + C_N^{(O)}(x)\right)\lambda_N e^{-\eta_N(t-x)}\,dx \\ \qquad + \int_{l_{\overline{w}}}^{t} \left[r_{N+1} + d_{N+1} + C_1^{(O)}(x)\right]\alpha_N e^{-\eta_N(t-x)}\,dx \end{cases} \tag{6.19}$$

For Policy A:

1) $\tau \leq l_{\overline{w}}$

- When $t \leq \tau$, the EDMC is given by (6.8).
- When $\tau < t \leq l_{\overline{w}}$, the EDMC is given by (6.9).

- When $t > l_{\overline{w}}$, the EDMC is modified from (6.9) and is given below:

$$\begin{cases} C_i^{(A)}(t|J,\tau) = e^{-\eta_i(t-l_{\overline{w}})}C_i^{(A)}(l_{\overline{w}}) + \int_{l_{\overline{w}}}^{t} \left(d_i + C_i^{(A)}(x|J,\tau)\right)\lambda_i e^{-\eta_i(t-x)}\,dx \\ \qquad + \int_{l_{\overline{w}}}^{t} C_{i+1}^{(A)}(x|J,\tau)\alpha_i e^{-\eta_i(t-x)}dx \quad \text{for } i = 1,2\ldots,J-1 \\ C_J^{(A)}(t|J,\tau) = e^{-\eta_J(t-l_{\overline{w}})}C_J^{(A)}(l_{\overline{w}}) + \int_{l_{\overline{w}}}^{t} \left(d_J + C_J^{(A)}(x|J,\tau)\right)\lambda_J e^{-\eta_J(t-x)}\,dx \\ \qquad + \int_{l_{\overline{w}}}^{t} \left[r_{J+1} + C_1^{(A)}(x|J,\tau)\right]\alpha_J e^{-\eta_J(t-x)}dx \end{cases} \tag{6.20}$$

2) When $\tau > l_{\overline{w}}$

- When $t \leq \tau$, the EDMC is given by (6.8).
- When $t > \tau$, the EDMC is given by (6.20).

For Policy B:

1) When $\tau \leq l_{\overline{w}}$

- When $t \leq \tau$, the EDMC is given by (6.14).
- When $\tau < t \leq l_{\overline{w}}$, the EDMC is given by (6.15).
- When $t > l_{\overline{w}}$, the EDMC is modified from (6.15) and is given below:

$$
\begin{cases}
C_i^{(B)}(t|J,\tau) = e^{-\eta_i(t-l_{\bar{w}})}C_i^{(B)}(l_{\bar{w}}) + \int_{l_{\bar{w}}}^{t}\left(d_i + C_i^{(B)}(x|J,\tau)\right)\lambda_i e^{-\eta_i(t-x)}dx \\
\qquad + \int_{l_{\bar{w}}}^{t} C_{i+1}^{(B)}(x|J,\tau)\alpha_i e^{-\eta_i(t-x)}dx \quad \text{for } i = 1,2,..,J \\
C_i^{(B)}(t|J,\tau) = e^{-\eta_i(t-l_{\bar{w}})}C_i^{(B)}(l_{\bar{w}}) + \int_{l_{\bar{w}}}^{t}\left[r_i + d_i + C_1^{(B)}(x|J,\tau)\right]\lambda_i e^{-\eta_i(t-x)}dx \\
\qquad + \int_{l_{\bar{w}}}^{t} C_{i+1}^{(B)}(x|J,\tau)\alpha_i e^{-\eta_i(t-x)}dx \quad \text{for } i = J+1, J+2, \dots, N-1 \\
C_N^{(B)}(t|J,\tau) = e^{-\eta_N(t-l_{\bar{w}})}C_N^{(B)}(l_{\bar{w}}) + \int_{l_{\bar{w}}}^{t}\left[r_N + d_N + C_1^{(B)}(x|J,\tau)\right]\lambda_N e^{-\eta_N(t-x)}dx \\
\qquad + \int_{l_{\bar{w}}}^{t}\left[r_{N+1} + d_{N+1} + C_1^{(B)}(x|J,\tau)\right]\alpha_N e^{-\eta_N(t-x)}dx
\end{cases}
\tag{6.21}
$$

2) When $\tau > l_{\bar{w}}$

- When $t \leq \tau$, the EDMC is given by (6.14).
- When $t > \tau$, the EDMC is given by (6.21).

Remarks: For Policy O, the inverse LST of the EDMC under warranty can be easily derived by following similar procedures as the derivations for Policies A and B without warranty incorporation. However, it becomes complicated when dealing with Polices A and B under warranty since it requires three-stage derivations for the cases $t \leq \tau$, $\tau < t \leq l_{\bar{w}}$ and $t > l_{\bar{w}}$ respectively which, to our opinion, is workable but turns out to be very cumbersome. Therefore we do not consider inverse LST when warranty is incorporated; instead, we resort to numerical procedures for analyzing EDMC and optimizing the maintenance thresholds.

6.4 Optimization of the Maintenance Thresholds

In the foregoing analysis, we have derived the close-to-explicit forms of the EDMC for Policy A and B when the maintenance thresholds (J, τ) are given without warranty incorporation. Here we further consider the methodologies for optimizing (J, τ) under each of the policies.

6.4.1 *Method 1*: Optimizing (J,τ) Using Inverse LST

A straightforward way for optimize (J, τ) is to obtain the time-domain functions of the EDMC using the inverse LST technique and repeat the same process over the domain of J and τ. Note that the frequency-domain functions in the brackets of (6.4), (6.10) and (6.16) have a simple form that both the numerator and the denominator are rational polynomial functions of s and the degree of the numerator (in terms of s) is smaller than that of the denominator. For such functions, the fundamental theory for conducting the inversion is to apply the *Heaviside's expansion theorem*. Details of the theorem can be found in any textbook of complex analysis. Note that manually implementing the expansion technique is often cumbersome. Alternatively, scientific computing software, such as Matlab, has the embedded function for implementing such technique and is very easy to use. On the other hand, as we will see in the following numerical session, the number of inversions for each policy is entirely determined by the domain of J (i.e. $J = 1,2, \dots, N-1$) ($C_i^{(O)}(\tau), i \in \Omega$ are considered as symbolic values during the inversion). In other words, both Policy A and B requires merely $N-1$ times of inversion. By further optimizing these $N-1$ time-domain functions for each of the policy, optimal maintenance thresholds (J^*, τ^*) can be easily obtained.

Note that nowadays almost all types of the numerical solvers are able to conduct calculations for any pre-specified number of significant digits required. Consequently, the advantage of the above method is that it provides the de facto "analytical" form of the EDMC in a very efficient way, which subsequently guarantees the accuracy of the optimization process. On the other hand, however, the issue of numerical stability associated with existing (Laplace) inversion techniques (Kwok and Barthez, 1989) may surface when the probability of the root-overlapping in the frequency-domain function becomes significantly high, which for our case may only be observed in the MSS with large number of states (reflected by the degree of s in the denominator). Theoretically, such issue can still be addressed by increasing the computational efforts. But clearly this will compromise the efficiency of the numerical inversion. In addition, for systems that are released under initial warranty coverage, formulating the EDMC in the form of inverse LST turns out to be cumbersome (although still workable). Therefore, in the following we consider an alternative method that can approximate the EDMC and optimize (J, τ) directly on the time domain.

6.4.2 *Method 2*: Optimizing (J,τ) Using Discretization in the Time Domain

The following methodology applies to the general case when $w \neq 0$. We first discretize the integral operator in (6.2), (6.9), (6.15) and (6.19)–(6.21) before approximating the EDMC in the time domain.

Set $t_j = jh\ (j = 0,1,2, \dots, \theta)$, $L = t_\theta = \theta h$ and $l_{\overline{w}} = L - w = \theta_{\overline{w}} h$ where h is the minimal step of the approximation. Further set $\tau = lh_\tau\ (l = 0,1, \dots, L/h_\tau)$ as the threshold of the residual life cycle. Note that h_τ may not necessarily equal to h. It could be multiples of h and depend on the accuracy requirements of the optimization.

Let $\tilde{C}_i^{(O)}(t)$, $\tilde{C}_i^{(A)}(t|J,\tau)$ and $\tilde{C}_i^{(B)}(t|J,\tau)$ represent the numerical approximations of $C_i^{(O)}(t)$, $C_i^{(A)}(t|J,\tau)$ and $C_i^{(B)}(t|J,\tau)$ respectively.

For Policy O:

For $j = 0,1,2,\dots,\theta_{\overline{w}}$, we have

$$\begin{cases} \tilde{C}_i^{(O)}(t_j) = \sum_{k=0}^{j-1}\left[\int_{t_k}^{t_{k+1}}\left[m_i+d_i+C_i^{(O)}(x)\right]\lambda_i e^{-\eta_i(t_j-x)}dx + \int_{t_k}^{t_{k+1}} C_{i+1}^{(O)}(x)\alpha_i e^{-\eta_i(t_j-x)}dx\right] \\ \qquad \approx \sum_{k=0}^{j-1} h \times e^{-\eta_i(t_j-t_k)}\left[e_i + \lambda_i\tilde{C}_i^{(O)}(t_k) + \alpha_i\tilde{C}_{i+1}^{(O)}(t_k)\right] \text{ for } i = 1,2,\dots,N-1 \\ \tilde{C}_N^{(O)}(t_j) \approx \sum_{k=0}^{j-1} h \times e^{-\eta_N(t_j-t_k)}\left[e_N + \lambda_N\tilde{C}_N^{(O)}(t_k) + \alpha_N\tilde{C}_1^{(O)}(t_k)\right] \end{cases} \quad (6.22)$$

To further reduce the computational complexity, Equation (6.22) is rewritten in such a linear form that the EDMC with residual time $t = t_j$ only relies on the EDMC at $t = t_{j-1}$:

$$\begin{cases} \tilde{C}_i^{(O)}(t_j) \approx e^{-\eta_i h}\left[\tilde{C}_i^{(O)}(t_{j-1})(1+h\lambda_i) + \alpha_i\tilde{C}_{i+1}^{(O)}(t_{j-1})h + e_i h\right] \text{ for } i = 1,2,\dots,N-1 \\ \tilde{C}_N^{(O)}(t_j) \approx e^{-\eta_N h}\left[\tilde{C}_N^{(O)}(t_{j-1})(1+h\lambda_N) + \alpha_N\tilde{C}_1^{(O)}(t_{j-1})h + e_N h\right] \end{cases} \quad (6.23)$$

For $j = \theta_{\overline{w}}+1, \theta_{\overline{w}}+2,\dots,\theta$, similarly we have

$$\begin{cases} \tilde{C}_i^{(O)}(t_j) \approx e^{-\eta_i h}\left[\tilde{C}_i^{(O)}(t_{j-1})(1+h\lambda_i) + \alpha_i\tilde{C}_{i+1}^{(O)}(t_{j-1})h + (e_i - \lambda_i m_i)h\right] \\ \qquad \text{for } i = 1,2,\dots,N-1 \\ \tilde{C}_N^{(O)}(t_j) \approx e^{-\eta_N h}\left[\tilde{C}_N^{(O)}(t_{j-1})(1+h\lambda_N) + \alpha_N\tilde{C}_1^{(O)}(t_{j-1})h + (e_N - \lambda_N m_N)h\right] \end{cases} \quad (6.24)$$

For Policy A:

- When $\tau \le l_{\overline{w}}$

For $j = 0,1,2,\dots,lh_\tau/h$ and any $i \in \Omega$, we have

$$\tilde{C}_i^{(A)}(t_j|J,\tau) = \tilde{C}_i^{(O)}(t_j) \tag{6.25}$$

For $j = lh_\tau/h + 1, lh_\tau/h + 2, \dots, \theta_{\overline{w}}$, we have

$$\begin{cases} \tilde{C}_i^{(A)}(t_j|J,\tau) \approx e^{-\eta_i h}\left[\tilde{C}_i^{(A)}(t_{j-1}|J,\tau)(1+h\lambda_i)+\alpha_i\tilde{C}_{i+1}^{(A)}(t_{j-1}|J,\tau)h + e_i h\right] \\ \qquad\qquad \text{for } i = 1,2,\dots,J-1 \\ \tilde{C}_J^{(A)}(t_j|J,\tau) \approx e^{-\eta_J h}\left[\tilde{C}_J^{(A)}(t_{j-1}|J,\tau)(1+h\lambda_J)+\alpha_J\tilde{C}_1^{(A)}(t_{j-1}|J,\tau)h + (e_J+r_{J+1}\alpha_J)h\right] \end{cases} \tag{6.26}$$

For $j = \theta_{\overline{w}} + 1, \theta_{\overline{w}} + 2, \dots, \theta$, we have

$$\begin{cases} \tilde{C}_i^{(A)}(t_j|J,\tau) \approx e^{-\eta_i h}\left[\tilde{C}_i^{(A)}(t_{j-1}|J,\tau)(1+h\lambda_i) + \alpha_i\tilde{C}_{i+1}^{(A)}(t_{j-1}|J,\tau)h + \lambda_i d_i h\right] \\ \qquad\qquad \text{for } i = 1,2,\dots,J-1 \\ \tilde{C}_J^{(A)}(t_j|J,\tau) \approx e^{-\eta_J h}\begin{bmatrix} \tilde{C}_J^{(A)}(t_{j-1}|J,\tau)(1+h\lambda_J) + \alpha_J\tilde{C}_1^{(A)}(t_{j-1}|J,\tau)h \\ +(\lambda_J d_J + r_{J+1}\alpha_J)h \end{bmatrix} \end{cases} \tag{6.27}$$

- When $\tau > l_{\overline{w}}$

For $j = 0,1,2,\dots, lh_\tau/h$ and any $i \in \Omega$, we have

$$\tilde{C}_i^{(A)}(t_j|J,\tau) = \tilde{C}_i^{(O)}(t_j) \tag{6.28}$$

For $j = lh_\tau/h + 1, lh_\tau/h + 2, \dots, \theta$, the cost approximation follows (6.27).

For Policy B:

- When $\tau \le l_{\overline{w}}$

For $j = 0,1,2,\dots, lh_\tau/h$ and any $i \in \Omega$

$$\tilde{C}_i^{(B)}(t_j|J,\tau) = \tilde{C}_i^{(O)}(t_j) \tag{6.29}$$

For $j = lh_\tau/h + 1, lh_\tau/h + 2, \dots, \theta_{\overline{w}}$, we have

$$\begin{cases} \tilde{C}_i^{(B)}(t_j|J,\tau) \approx e^{-\eta_i h}\left[\tilde{C}_i^{(B)}(t_{j-1}|J,\tau)(1+h\lambda_i) + \alpha_i \tilde{C}_{i+1}^{(B)}(t_{j-1}|J,\tau)h + e_i h\right] \\ \qquad\qquad \text{for } i = 1,2,\dots,J \\ \tilde{C}_i^{(B)}(t_j|J,\tau) \approx e^{-\eta_i h}\left[\tilde{C}_i^{(B)}(t_{j-1}|J,\tau)(1+h\lambda_i) + \alpha_i \tilde{C}_{i+1}^{(B)}(t_{j-1}|J,\tau)h + (e_i+f_i)h\right] \\ \qquad\qquad \text{for } i = J+1, J+2,\dots,N-1 \\ \tilde{C}_N^{(B)}(t_j|J,\tau) \approx e^{-\eta_N h}\left[\tilde{C}_N^{(B)}(t_{j-1}|J,\tau) + (\lambda_N+\alpha_N)\tilde{C}_1^{(B)}(t_{j-1}|J,\tau)h + (e_N+f_N)h\right] \end{cases} \tag{6.30}$$

For $j = \theta_{\overline{w}} + 1, \theta_{\overline{w}} + 2, \dots, \theta$, we have

$$\begin{cases} \tilde{C}_i^{(B)}(t_j|J,\tau) \approx e^{-\eta_i h}\left[\tilde{C}_i^{(B)}(t_{j-1}|J,\tau)(1+h\lambda_i) + \alpha_i \tilde{C}_{i+1}^{(B)}(t_{j-1}|J,\tau)h + \lambda_i d_i h\right] \\ \qquad\qquad \text{for } i = 1,2,\dots,J \\ \tilde{C}_i^{(B)}(t_j|J,\tau) \approx e^{-\eta_i h}\left[\tilde{C}_i^{(B)}(t_{j-1}|J,\tau)(1+h\lambda_i) + \alpha_i \tilde{C}_{i+1}^{(B)}(t_{j-1}|J,\tau)h + (e_i+f_i)h\right] \\ \qquad\qquad \text{for } i = J+1, J+2,\dots,N-1 \\ \tilde{C}_N^{(B)}(t_j|J,\tau) \approx e^{-\eta_N h}\left[\tilde{C}_N^{(B)}(t_{j-1}|J,\tau) + (\lambda_N+\alpha_N)\tilde{C}_1^{(B)}(t_{j-1}|J,\tau)h + (e_N+f_N)h\right] \end{cases} \tag{6.31}$$

- When $\tau > l_{\overline{w}}$

For $j = 0,1,2,\dots, lh_\tau/h$ and any $i \in \Omega$, we have

$$\tilde{C}_i^{(B)}(t_j|J,\tau) = \tilde{C}_i^{(O)}(t_j) \tag{6.32}$$

For $j = lh_\tau/h + 1, lh_\tau/h + 2, \dots, \theta$, the cost approximation follows (6.31).

An algorithm for optimizing $(\boldsymbol{J}, \boldsymbol{\tau})$

Step 1: Select h and let $L = \theta h$. Set $\tilde{C}_i^{(O)}(t_0) = \tilde{C}_i^{(O)}(0) = 0$ for any $i \in \Omega$.

Step 2: Calculate $\tilde{C}_i^{(O)}(t_j)$ for $j = 1,2,\dots,\theta$ and $\forall i \in \Omega$ following (6.23) and (6.24).

Select h_τ. For each $\tau = kh_\tau$ $(k = 0,1,\dots,L/h_\tau)$ and $J = 1,2,\dots,N-1$

Step 3: Set $\tilde{C}_i^{(A)}(t_j|J,\tau) = \tilde{C}_i^{(B)}(t_j|J,\tau) = \tilde{C}_i^{(O)}(t_j)$ for $j = 0,1,\dots,kh_\tau/h$ and $\forall i \in \Omega$.

Step 4: If $\tau \leq l_{\bar{w}}$, for any $i \in \Omega$, calculate $\tilde{C}_i^{(A)}(t_j|J,\tau)$ and $\tilde{C}_i^{(B)}(t_j|J,\tau)$ for $j = kh_\tau/h + 1, kh_\tau/h + 2, \ldots, \theta_{\bar{w}}$ following (6.26) and (6.30) respectively, and for $j = \theta_{\bar{w}} + 1, \theta_{\bar{w}} + 2, \ldots, \theta$ following (6.27) and (6.31) respectively.

Step 5: If $\tau > l_{\bar{w}}$, calculate $\tilde{C}_i^{(A)}(t_j|J,\tau)$ and $\tilde{C}_i^{(B)}(t_j|J,\tau)$ for $j = kh_\tau/h + 1, kh_\tau/h + 2, \ldots, \theta$ and $\forall i \in \Omega$ following (6.27) and (6.31) respectively.

Step 6: Repeat Steps 3–5.

Step 7: Select the optimal (J^*, τ^*) that minimize $\min\{\tilde{C}_1^{(A)}(L|J,\tau), \tilde{C}_1^{(B)}(L|J,\tau)\}$.

Remarks: (1) The main advantage of the above method is that it is stable and can deal with a wide range of system configurations, which can be subsequently used for the sensitive analysis of the maintenance optimization. In addition, it can be directly applied (without any change) for maintaining a MSS that is not perfect functioning initially since both $\tilde{C}_i^{(A)}(L|J,\tau)$ and $\tilde{C}_i^{(B)}(L|J,\tau)$ $(i \neq 1)$ are automatically calculated in the algorithm. Furthermore, the impact of warranty incorporation can be easily analyzed using the above algorithm. (2) The main drawback of the method is that the optimization process is relatively time-consuming and less accurate. Reducing h, though enhancing the accuracy of the optimization, will further increase the computational efforts significantly. Therefore, h needs to be properly selected to balance the accuracy and efficiency of the algorithm. (3) The EDMC is approximated at the lower limit of the integration and therefore is always smaller than the exact result.

6.5 Numerical Example

In this section we illustrate the optimization process with an example using both methods. In order to compare these two methods, we first consider the case when $w = 0$. The case of warranty incorporation ($w \neq 0$) is presented subsequently.

Consider the following parameters for a MSS with 4-stage degradation ($N = 4$). Let $L = 5$ year and $\delta = 0.05/\text{year}$. The transition rates for degradation are $\alpha_1 = 0.9/\text{year}$, $\alpha_2 = 0.8/\text{year}$, $\alpha_3 = 0.9/\text{year}$ and $\alpha_4 = 1.1/\text{year}$. The Poisson failure rates are $\lambda_1 = 0.4/\text{year}$, $\lambda_2 = 0.6/\text{year}$, $\lambda_3 = 1.0/\text{year}$ and $\lambda_4 = 1.2/\text{year}$. The replacement costs are $r_1 = 200$, $r_2 = 240$, $r_3 = 360$, $r_4 = 520$ and $r_5 = 720$. To further investigate the impact of different cost structures on the optimal policies, we consider the following 3 scenarios.

Scenario 1 $r_i/m_i = 4$ $(i = 1,2,3,4)$ and $d_i = 20$ $(i = 1,2,3,4,5)$, i.e. low repair and downtime cost.

Scenario 2 $r_i/m_i = 4$ $(i = 1,2,3,4)$ and $d_i = 80$ $(i = 1,2,3,4,5)$, i.e. low repair cost and high downtime cost.

Scenario 3 $r_i/m_i = 2$ $(i = 1,2,3,4)$ and $d_i = 80$ $(i = 1,2,3,4,5)$, i.e. high repair and downtime cost.

Results:

- **When $w = 0$**

First of all, the de facto "analytical" forms of the EDMC as functions of (J, τ) are derived and listed in the Appendix D using inverse LST techniques (Method 1). Note that we only show the results for Scenario 1 because it is sufficient for the

demonstration purpose. Here $C_i^{(O)}(\tau)$ $(i \in \Omega)$ are treated as symbolic values during the inversion for Policy A and B and the number of inversions under each of the policies is simply 3 $(N-1)$. For Method 2, we first select a proper minimal interval h to guarantee the accuracy of the approximation. Table 6.1 illustrates that $h = 0.001$ is potentially a good choice since the error between $\tilde{C}_1^{(O)}(L)$ (using Method 2) and $C_1^{(O)}(L)$ (using Method 1) is less than 1%. From the efficiency point of view, further reducing h, say, from 0.001 to 0.0001, barely improves the accuracy of the approximation; for our case, it will increase the system run time from 10 minutes to hours. Therefore, $h = 0.001$ is considered cost-effective here and is applied thereafter.

Table 6.1 Selecting a proper minimal interval *h*

Scenario	$\frac{r_i}{m_i}$	d_i	$C_1^{(O)}(L)$	$h = 0.1$		$h = 0.01$		$h = 0.001$	
				$\tilde{C}_1^{(O)}(L)$	err%	$\tilde{C}_1^{(O)}(L)$	err%	$\tilde{C}_1^{(O)}(L)$	err%
1	4	20	799.5	429.6	46.3%	747.7	6.5%	794.1	0.7%
2	4	80	1023.2	566.3	44.7%	959.5	6.2%	1016.7	0.6%
3	2	80	1279.6	720.6	43.7%	1202.0	6.1%	1271.7	0.6%

Without warranty incorporation, optimal maintenance thresholds (J^*, τ^*) using both methods are presented in Figures 6.2–6.4 under different values of the cost parameters. It is not surprising that the results given by Method 2 when $h = 0.001$ almost overlap with those using Method 1, which always appear to be slightly higher. Based on the results using the inverse LST (Method 1), the following observations are made.

For Scenario 1, Policy B is a better choice for the customer and the minimum EDMC over the 5 years is 655.9 under $J^* = 2$ and $\tau^* = 0.9$ year. In other words, the system is always correctively replaced when it fails during the 3rd stage degradation and the residual life cycle is longer than 0.9 year. For Scenario 2, Policy A (or preventive replacement) should be enforced and the minimum EDMC over the 5 years is 804.8 under $J^* = 2$ and $\tau^* = 1.2$ year. The optimal choice for Scenario 3 is Policy B and the

minimum EDMC over the interval is 909.1 under $J^* = 1$ and $\tau^* = 0.6$ year. From the above results, we conclude that Policy A outperforms Policy B typically when the downtime cost is considerably high compared to the minimal repair cost. In addition, the optimal τ^* under Policy A is larger than the value under Policy B. The interpretation is that for Policy A, more stringent requirements on τ^* are necessary as a balance for more aggressive replacement strategies – preventive replacement, when compared with the corrective replacement strategies under Policy B. Furthermore, the results in Scenario 3 also indicate that when both repair and downtime cost are high, the system should be correctively replaced upon failure even when it is working under a relatively good condition (e.g. the 2nd degradation stage for Scenario 3). Finally, we use Method 2 for the sensitivity analysis of another cost parameter – the discounted factor δ. Results do not show a substantial impact on (J^*, τ^*) (we therefore do not list the results here). Alternatively, it could be important in more practical situations, say, when the cost estimation is also crucial to the decision makers (Note that for the current case, the minimum cost is 10–20% higher if $\delta = 0$).

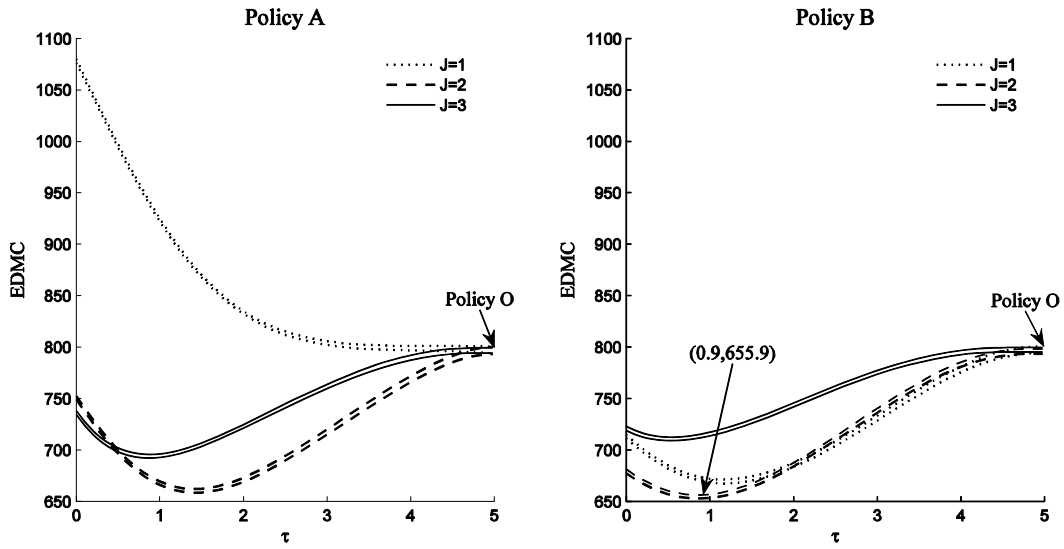

Figure 6.2 The EDMC as a function of (J,τ) for Scenario 1 with w=0

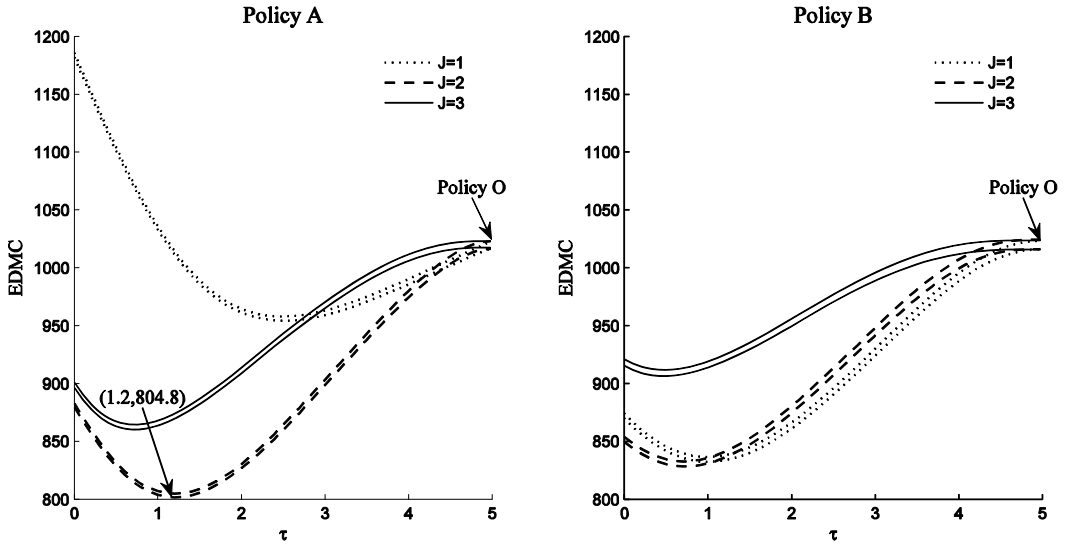

Figure 6.3 The EDMC as a function of (J,τ) for Scenario 2 with w=0

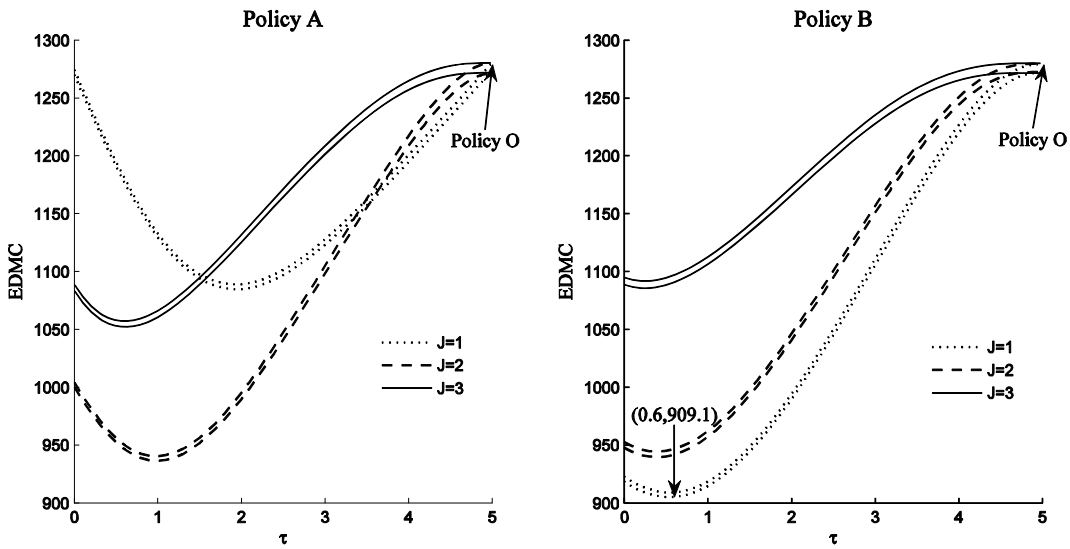

Figure 6.4 The EDMC as a function of (J,τ) for Scenario 3 with w=0

- **When $w \neq 0$**

Here we further investigate the impact of warranty incorporation on the optimal values of maintenance thresholds. The comparison is made between $w = 0$ and $w = 2.5, 5$ by considering Scenarios 1–3 for both Policies A and B. The optimal solutions are highlighted with shaded background.

Table 6.2 Optimal maintenance thresholds (J^*,τ^*) with warranty incorporation

Scenario	$w = 0$		$w = 2.5$		$w = 5$	
	$(J^*, \tau^*, \tilde{C}_1^{(A)*})$	$(J^*, \tau^*, \tilde{C}_1^{(B)*})$	$(J^*, \tau^*, \tilde{C}_1^{(A)*})$	$(J^*, \tau^*, \tilde{C}_1^{(B)*})$	$(J^*, \tau^*, \tilde{C}_1^{(A)*})$	$(J^*, \tau^*, \tilde{C}_1^{(B)*})$
1	(2,1.4,658.7)	(2,0.9,652.4)	(2,1.4,596.7)	(2,0.9,599.1)	(3,1.2,494.0)	(2,1.8,517.6)
2	(2,1.2,801.5)	(2,0.7,827.9)	(2,1.2,739.5)	(2,0.7,774.6)	(2,1.6,658.9)	(2,1.4,701.4)
3	(2,1.0,936.4)	(1,0.6,904.8)	(2,1.0,812.4)	(2,0.4,833.1)	(2,1.6,658.9)	(2,1.4,701.4)

From Table 6.2, we can see that when warranty is incorporated, Policy A becomes a better option for the customer than Policy B. It is not surprising because corrective replacement policy (i.e. Policy B) fails to take advantage of the warranty coverage for which the failed system could have already been repaired by the manufacturer for free. For such case, it is reasonable for the customer to negotiate better warranty terms under which, say, the manufacturer shares part of the replacement cost upon (Poisson) failures within warranty.

On the other hand, for both Policies A and B, when w increases, J^* either remains unchanged or increases. In particular, when J^* remains unchanged, τ^* increases as w increases. This indicates that the system would be less likely to be replaced before degradation failures; τ^* may decrease only when J^* increases.

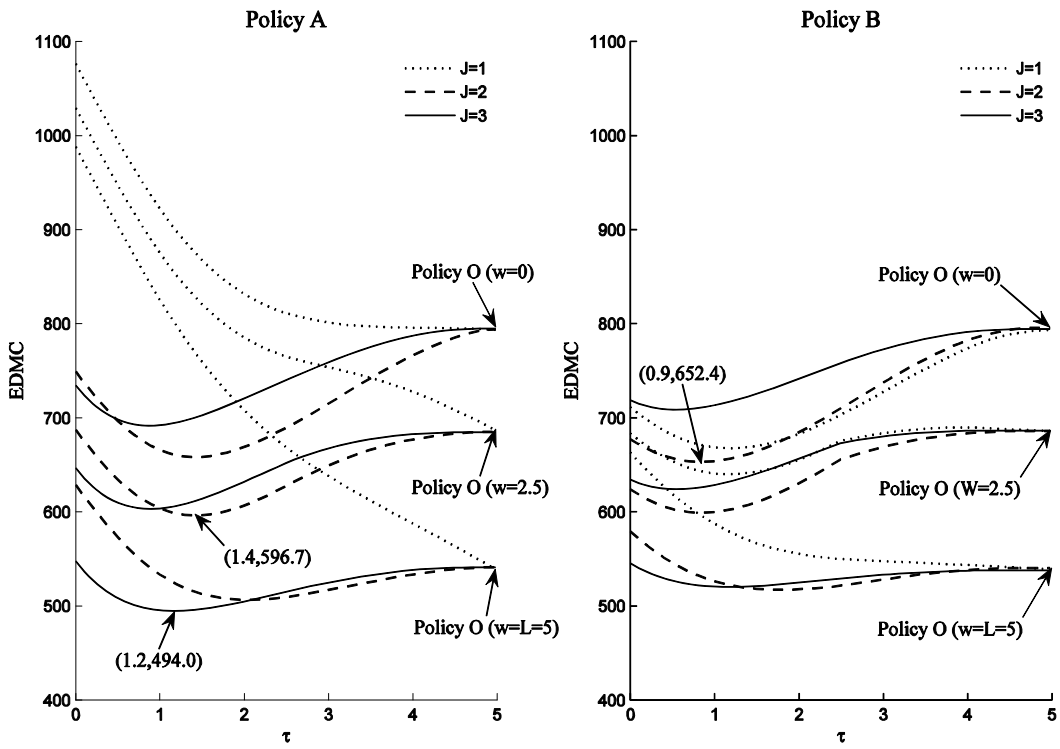

Figure 6.5 The EDMC as a function of (J,τ) for Scenario 1 when w=0,2.5,5

From Table 6.2, we can also estimate the value of warranty perceived by the customer. This is particularly useful when the warranty is sold independently with the product (such as extended warranties). Take Scenario 1 for example. The difference of minimum life-cycle maintenance cost between $w = 2.5$ and $w = 0$ is $652.4 - 596.7 = 55.7$. This implies that the price of 2.5 years warranty the customer is willing to pay should not be higher than 55.7. Again, if the product is sold with 2.5 years embedded warranty (non-detachable), and the customer has the option to purchase another 2.5 years extended warranty, then the price of 2.5 years extended warranty should not be higher than $596.7 - 494.0 = 102.7$. A graphic view of the EDMC as a function of (J, τ) for Scenario 1 is given in Figure 6.5.

6.6 Conclusion

In this chapter, we considered a finite life-cycle MSS that is subject to both degradation and Poisson failures. We study two classes of maintenance policies, that of preventive replacements and corrective replacements. For both policies, the EDMC is derived as a function of two control parameters – a threshold level on the current state of the system, and a threshold level on the residual life cycle both with and without warranty incorporation. In order to obtain the optimal maintenance thresholds to minimize the EDMC, two different methodologies are proposed which utilize (inverse) LST techniques and time-domain numerical approximation respectively. The applications of both methods are illustrated using a numerical case. Through computational examples, we demonstrate that preventive replacements outperform corrective replacements typically when the downtime cost of each failure is relatively high compared to the repair cost. The two proposed replacement policies can effectively detect necessary replacements for the condition when the system has already experienced heavy deterioration and the remaining service time is still long, but can also avoid the excessive replacements for the condition when the system has only experienced minor deterioration.

CHAPTER 7 WARRANTY REVENUE ANALYSIS INTEGRATING USER'S MAINTENANCE DECISIONS

In the warranty and maintenance literature, post-sale decisions for the buyer and the seller are usually considered separately. We followed such logic in the previous discussions: in Chapters 3–4, we presented warranty cost modeling and service design from the seller's perspective, while in Chapters 5–6, we focused on the life-cycle maintenance service design from the buyer's perspective. So far we have yet discussed the feasibility of unifying these two perspectives of decision processes for which knowing the decision of one perspective could help enhance that of the other.

In what follows, we conduct a novel study on the post-sale revenue analysis by jointly considering the seller's product marketing design and the buyer's PM decisions.

7.1 Introduction

The success of launching a new product in the current marketplace requires attractive and efficient product marketing strategies. Recent development in product warranty management has provided managers with very useful instruments in achieving this goal. Better warranty terms are an indicator of higher product quality and manufacturer confidence, and this can have a positive influence on purchasing decisions of consumers. However, offering a competitive warranty such as (renewing) free-replacement warranty (FRW) can be very expensive and risky for the manufacturer and is thus not always an economically attractive option. A middle-ground approach is the cost-sharing warranty, which charges the consumer a pre-specified portion (either fixed or pro-rata) of the cost for each replacement (repair) during warranty. In

application, cost-sharing warranty has become a popular warranty design for both high-priced products such as plant facilities and large-scale machines (Huang et al., 2008), and relatively low-priced products (but with high sales volume) such as tires and batteries (Blischke and Murthy, 1992). It includes the free-replacement warranty (FRW) as a special case and therefore introduces flexibility in promoting different types of products.

Optimizing the seller's marketing strategy by incorporating warranty design is an area of considerable managerial interest. The majority of the literature has focused on the joint determination of selling price and warranty length that maximizes the seller's profit (rate). Glickman and Berger (1976) proposed a demand model to optimally determine the price and warranty length that maximizes a manufacturer's profit. Applications of this demand function can be found in Mitra and Jayprakash (1990, 1997), Lin and Shue (2005), and Wu et al. (2006). Recent studies of this area often considered many other factors. For example, Huang et al. (2007) incorporated the product reliability into the design phase of the product marketing strategy, Matis et al. (2008) and Huang and Fang (2008) considered variants of standard warranties, Wu et al. (2009) dealt with the production and inventory problem for a static demand market, and Lin et al. (2009) and Zhou et al. (2009) incorporated the market dynamics in their decision models.

Though the above studies have considered a wide range of factors that affect the marketing strategies, such as product demand, reliability, production rate, warranty policies, etc, the buyer's maintenance decisions are seldom incorporated into the design phase of the marketing strategy. In practice, a preventive maintenance (PM) effort carried out by the buyer is a critical factor that affects product failures and consequently the seller's servicing cost during the post-sale period. One of the main

incentives for the buyer to implement PMs is to reduce the potential downtime cost that results from unexpected failures as the product ages. Yeh et al. (2005, 2007), Chien (2008) and Yeo and Yuan (2009) have conducted extensive analysis on the optimal age-replacement (PM) strategy for the buyer under the warranty context. In their models, the downtime cost and replacement cost were integrated and minimized on a unit time basis. On the other hand, the necessity of incorporating the buyer's PM actions into the product design was highlighted by Pascual and Ortega (2006) and Huang and Yen (2009). Both of them revealed the impact of the buyer's PM actions on the warranty design (as part of the marketing strategies), and demonstrated that the buyer's PM efforts during warranty enabled the seller to provide a longer warranty coverage for the product while maintain the total warranty cost unchanged.

In this chapter, a model for the integrated analysis of the seller's product marketing strategy and the buyer's maintenance decisions is proposed under a cost-sharing warranty scheme. The motivation of this study is to highlight the buyer's replacement decision as a factor in affecting the seller's marketing decision, which is neglected by most of the previous studies. Part of the existing literature that jointly consider the seller's and buyer's decision making processes under warranty and maintenance context can be found in Singpurwalla and Wilson (1993), Murthy and Ashgarizadeh (1999), DeCroix (1999), Rinsaka and Sandoh (2006), Jack and Murthy (2007) and Jackson and Pascual (2008). Among these studies, the Stackelberg game is a popular approach used to describe the relationship between the seller ("leader") and buyer ("follower"). Similarly, the proposed model in this chapter follows such logic. Given any product price and warranty length, the buyer chooses the optimal age-replacement policy that minimizes his long-run average maintenance cost rate. The seller then derives the buyer's optimal replacement time as a parametric function of product price

and warranty length – both of which are his own design variables, and consequently use that to maximize his long-run average profit rate. The sales model in this chapter uses an extension of the demand function in Glickman and Berger (1976) and is subject to not only product price and warranty length, but also the buyer's age-replacement decision and warranty cost-sharing ratio. Illustrative examples are given to demonstrate the applicability of such game-theoretic model formulation. The feasibility of incorporating the warranty cost-sharing ratio as part of the marketing strategy is further discussed.

The outline of the rest of this chapter is as follows. In Section 7.2, the details of model formulations are presented, including a proposed sales model, and also the buyer's and seller's decision problems. The main results of the buyer's and seller's optimal strategies are presented in Section 7.3, and a special case of the design problem is discussed in Section 7.4. In Section 7.5, two numerical examples are given to demonstrate the applicability of the models and to obtain some insights to the optimal product marketing designs. Conclusions are made in Section 7.6.

7.2 Model Formulation

7.2.1 Product Warranty and Age Replacement

Suppose that a seller adopts a renewing cost-sharing warranty policy to promote a relatively high-priced non-repairable product. The product is offered at price C_p per item and the length of warranty coverage is w. The warranty cost-sharing ratio ρ (< 1) is fixed and pre-specified in the warranty contract. In other words, for given product failure during warranty, the seller charges a discounted price ρC_p for the replacement and an identical new product (with a new warranty) is provided thereafter. Furthermore, pre-market testing indicates that the product has an increasing failure rate (IFR) and its

performance always deteriorates with time. Therefore, the buyer implements an age-replacement strategy to reduce his long-run average maintenance cost per unit time. Let t be the buyer's designed replacement time and C_d be the buyer's downtime cost. We assume that the buyer always conduct the age-replacements for products that survive to time t. If the failure happens before t, a downtime cost C_d is incurred to the buyer and the replacement is carried out immediately. In either case, a purchasing cost is incurred to the buyer with the amount depending on the time the product fails: that is ρC_p for failures during warranty and C_p for failures out of warranty. Such a replacement strategy generates a renewal process and the time span between each two consecutive replacements forms a renewal cycle.

7.2.2 Sales Model

The seller models the sales volume of the new product using a demand function that appropriately accounts for the influence of product price, warranty policy and useful product life. One of the earliest models was proposed in Glickman and Berger (1976), which was given by (2.18) in Section 2.2.

In order to model the product sales under the cost-sharing warranty scheme, we propose a modified demand model based on (2.18). $\phi(\rho)(< 1)$ is defined as a discount factor of the demand due to the application of cost-sharing warranty instead of free-replacement warranty (FRW). Specifically, $\phi(\rho)$ is a non-increasing function of ρ, with $\phi(0) = 1$ (a FRW case), and $\phi(1) = 0$ (no warranty coverage case). In addition, we argue that the marketing power of a warranty should not persist without limit when the useful life of the product (from the buyer's perspective) is limited, which is typically characterized in this chapter by the replacement time of the product. Such proposition mirrors the argument that any warranty coverage beyond the pre-specified

replacement age t brings no extra positive utility to the buyer, and thus cannot influence the buyer's purchasing decision. Taking the above two considerations into account, we propose the following generalized demand model:

$$Q(C_p, w) = k_1 C_p^{-a}[k_2 + \phi(\rho)\min\{w, t\}]^b, a > 1, b < 1, k_1 > 0, k_2 \geq 0, \rho \leq 1 \quad (7.1)$$

Glickman and Berger's demand model in (2.18) therefore coincides with a special case of (7.1) in the case of a free-replacement warranty ($\rho = 0$) and with the buyer's maintenance action being ignored ($t = \infty$). Note that $\phi(1) = 0$ must hold when $\rho = 1$ because the buyer bears the full replacement cost under this case and consequently w is redundant in the demand model.

7.2.3 The Buyer's Cost Model and His Decision Problem

Upon the purchase of the product, the buyer needs to decide a replacement age t that takes into account the product failure characteristic, price C_p and warranty length w. The objective is to minimize his long-run average maintenance cost per unit time.

Let x be the life time of the product and $f(x)$, $F(x)$, $\bar{F}(x) = 1 - F(x)$, $r(x) = \frac{f(x)}{\bar{F}(x)}$ be its p.d.f, c.d.f, survivor function and failure rate function respectively. In particular, $r'(x) > 0$ always holds to describe the IFR property of the product.

To derive the cost models for the buyer, two cases must be considered: $t \geq w$ and $t < w$, conditioned on the seller's decisions of C_p and w.

- When $t \geq w$

Let random variable C_1 be the cost to the buyer during a renewal cycle and x be the product failure time. If $x < w$, the buyer's cost during a cycle includes both warranty renewal cost ρC_p and downtime cost C_d, i.e. $C_1 = \rho C_p + C_d$. If $w \leq x \leq t$, a full

purchasing cost C_p instead of ρC_p is incurred to the buyer and $C_1 = C_p + C_d$. After time t, as no downtime cost is incurred we have $C_1 = C_p$. The expected one-cycle cost to the buyer is then $E[C_1] = (\rho C_p + C_d)F(w) + (C_p + C_d)[F(t) - F(w)] + C_p\bar{F}(t) = C_p\bar{F}(w) + \rho C_p F(w) + C_d F(t)$. Define the random variable L_1 as the length of the replacement cycle. Note that $L_1 = \min\{t, x\}$. Thus we have $E[L_1] = \int_0^t uf(u)dx + t\bar{F}(t) = \int_0^t \bar{F}(u)du$.

According to renewal theory, the long-run average cost rate in this case is given as

$$CR_1(t) = \frac{E[C_1]}{E[L_1]} = \frac{C_p\bar{F}(w) + \rho C_p F(w) + C_d F(t)}{\int_0^t \bar{F}(u)du} \tag{7.2}$$

- When $t < w$

In this case let random variable C_2 be the cost to the buyer during a renewal cycle. If $x \leq t$, the buyer's cost during a cycle includes both warranty renewal cost ρC_p and downtime cost C_d, i.e. $C_2 = \rho C_p + C_d$. Otherwise, no downtime cost is incurred and thus we have $C_2 = C_p$. The expected one-cycle cost to the buyer is then $E[C_2] = (\rho C_p + C_d)F(t) + C_p\bar{F}(t) = C_p\bar{F}(t) + \rho C_p F(t) + C_d F(t)$. Let random variable L_2 be the length of the cycle. Again, we have $E[L_2] = \int_0^t uf(u)dx + t\bar{F}(t) = \int_0^t \bar{F}(u)du$.

The long-run average cost rate in this case is given as

$$CR_2(t) = \frac{E[C_2]}{E[L_2]} = \frac{C_p\bar{F}(t) + \rho C_p F(t) + C_d F(t)}{\int_0^t \bar{F}(u)du} \tag{7.3}$$

The buyer's decision process is determined by first locating the optimal replacement ages t_1^* when $t \geq w$ and t_2^* for the case when $t < w$ so that $CR_1(t)$ and $CR_2(t)$ are minimized respectively, and then looking into the global optimal t^* that minimizes

$CR_1(t_1^*)$ and $CR_2(t_2^*)$. Denote $CR^*(t^*) = \min\{CR_1(t_1^*), CR_2(t_2^*)\}$. The global optimal replacement age t^* is then determined by $\arg CR^*(t^*)$ where

$$CR^*(t^*) = \begin{cases} CR_1(t_1^*) & if\ t^* = t_1^* \\ CR_2(t_2^*) & if\ t^* = t_2^* \end{cases} \tag{7.4}$$

7.2.4 The Seller's Profit Model and His Decision Problem

The seller's decision problem is to locate the optimal C_p^* and w^* to maximize his long-run profit rate for all the products sold taking into account the buyer's replacement age t. Let C_m be the manufacturing cost of the product. Similar to the buyers, the derivation of the seller's profit rate function per item also requires the analysis of two cases, i.e. $t \geq w$ and $t < w$.

- When $t \geq w$

Let random variable P_1 be the profit to the seller during a renewal cycle and x be the product failure time. If $x < w$, the seller's charges ρC_p to the buyer on the warranty renewal and pays C_m for the replacement, which results in a profit that equals to $P_1 = \rho C_p - C_m$. If $x \geq w$, a full purchasing cost C_p instead of ρC_p is charged to the buyer and $P_1 = C_p - C_m$. The expected cycle profit to the seller is thus $E[P_1] = (\rho C_p - C_m)F(w) + (C_p - C_m)\bar{F}(w) = C_p\bar{F}(w) + \rho C_p F(w) - C_m$. In addition, the expected length of the cycle is $E[L_1] = \int_0^t \bar{F}(u)du$.

The long-run average profit rate with respect to product price C_p and warranty length w given $t \geq w$ is then

$$PR_1(C_p, w|t \geq w) = \frac{E[P_1]}{E[L_1]} = \frac{C_p\bar{F}(w) + \rho C_p F(w) - C_m}{\int_0^t \bar{F}(u)du} \tag{7.5}$$

- When $t < w$

Let random variable P_2 be the profit to the seller during a renewal cycle. If $x < t$, the seller charges ρC_p to the buyer on the warranty renewal which results in a profit that equals to $P_2 = \rho C_p - C_m$. If $x \geq t$, we have $P_2 = C_p - C_m$. The expected profit $E[P_2]$ and the cycle length $E[L_2]$ are thus $C_p\bar{F}(t) + \rho C_p F(t) - C_m$ and $\int_0^t \bar{F}(u)du$ respectively.

The long-run average profit rate in this case is

$$PR_2(C_p, w|t < w) = \frac{E[P_2]}{E[L_2]} = \frac{C_p\bar{F}(t) + \rho C_p F(t) - C_m}{\int_0^t \bar{F}(u)du} \tag{7.6}$$

By incorporating the sales volume in (7.1), the total profit rate function for the seller is given as

$$\pi_1(C_p, w|t) = \begin{cases} k_1 C_p^{-a}[k_2 + \phi(\rho)w]^b \dfrac{C_p\bar{F}(w) + \rho C_p F(w) - C_m}{\int_0^t \bar{F}(u)du}, & t \geq w \\ k_1 C_p^{-a}[k_2 + \phi(\rho)t]^b \dfrac{C_p\bar{F}(t) + \rho C_p F(t) - C_m}{\int_0^t \bar{F}(u)du}, & t < w \end{cases} \tag{7.7}$$

7.3 Integrating the Buyer's and Seller's Optimal Strategies

In this section we discuss the buyer's and seller's optimal strategies in an integrated way. As stated in the previous section, the buyer selects the optimal replacement age t^* to minimize his long-run average cost rate based on the price C_p and warranty length w. Given the buyer's strategy the seller seeks to influence the age-replacement decision of the buyer in order to maximize his long-run profits rate. This can be achieved by establishing the relationship between the optimal t^*, C_p and w (i.e. $t^* = t(C_p, w)$). The number of the buyers involved in the game is governed by the sales model

proposed in (7.1) with respect to both the buyer's age-replacement strategy and the product marking strategy. In this sense the buyer-seller relationship is equivalent to a Stackelberg game where the seller is the leader of the game and the buyers are followers. To complete our analysis, the following assumptions are made.

1) The product has an IFR function, i.e. $r'(.) > 0$. It is not desirable to implement preventive replacement for products without an IFR property.
2) The seller is the price maker and warranty designer, and is the leader in the game. The buyers are followers and accept both the price and warranty terms of the product but choose freely the replacement strategy for the product.
3) Both the seller and buyers are rational, risk-neutral and interested in maximizing (minimizing) their own benefits (losses). The buyers know the product failure time distribution and always carry out the age replacements at the time that minimizes the long-run maintenance cost rate. The seller has the proprietary knowledge of the buyers downtime cost and thus can foresee the buyers' action in response to his own marketing strategy.
4) All the buyers bear the same cost structure and thus act unanimously to the seller's decision. Such assumption is typically suitable for a single buyer with a large amount of purchases at once. For multiple buyers, an estimation of the average downtime cost across all the buyers is required from the seller's perspective to represent the single downtime cost mentioned in our model.

7.3.1 The Buyer's Optimal Strategy

We first derive the optimal replacement age of the product as a function of price and warranty length, following the results presented in (7.2)–(7.4). Consider two cases when $t \geq w$ and $t < w$ respectively.

Denote $Y(t) = r(t)\int_0^t \bar{F}(u)\,du - F(t)$. The first derivatives of $CR_1(t)$ and $CR_2(t)$ with respect to replacement age t are

$$\frac{dCR_1(t)}{dt} = \frac{C_d\bar{F}(t)\left(Y(t) - \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}\right)}{\left[\int_0^t \bar{F}(u)du\right]^2} \tag{7.8}$$

$$\frac{dCR_2(t)}{dt} = \frac{\bar{F}(t)\left(\left[C_d - (1-\rho)C_p\right]Y(t) - C_p\right)}{\left[\int_0^t \bar{F}(u)du\right]^2} \tag{7.9}$$

Note that since $r'(t) > 0$, we have $dy/dt = r'(t)\int_0^t \bar{F}(u)du > 0$ and hence $y = Y(t)$ is a non-negative and increasing function in t. Denote $\mu = \int_0^\infty \bar{F}(u)du$ as the mean life time of a product. It is obvious that $Y(0) = 0$ and $\lim_{t\to\infty} Y(t) = Y(\infty) = \mu \lim_{t\to\infty} r(t) - 1$ if $\lim_{t\to\infty} r(t)$ exists. Moreover, the inverse function $t = Y^{-1}(y)$ is also a non-negative and increasing.

In the following development we also define the critical replacement ages $t_0 = Y^{-1}(\frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d})$, $\tau_1 = Y^{-1}(\frac{C_p\bar{F}(\tau_1)+\rho C_p F(\tau_1)}{C_d})$, and $\tau_2 = Y^{-1}(\frac{C_p}{C_d-(1-\rho)C_p})$. Theorem 7.1 below presents the optimal age-replacement strategy for the buyer under a cost-sharing warranty. The proof is given in Appendix E.

Theorem 7.1: Under a renewing cost-sharing warranty policy, given that $r(t)$ is an increasing function in t for a non-repairable product with price C_p, warranty length w, and a fixed cost-sharing ratio ρ ($0 \le \rho < 1$) for each warranty renewal, the following results hold for the optimal replacement age t^*:

1) For $C_p \ge C_d/(1-\rho)$,

   - If $Y(\infty) \le \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$, then $t^* = \infty > w$;

- If $Y(w) < \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} < Y(\infty)$, then $t^* = t_0 > \tau_1 > w$;
- If $Y(w) \geq \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$, then $t^* = w$.

2) For $C_p < C_d/(1-\rho)$,

- If $Y(\infty) \leq \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$, then $t^* = \infty > w$;
- If $Y(w) < \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} < Y(\infty)$, then $t^* = t_0 > \tau_1 > w$;
- If $\frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} \leq Y(w) \leq \frac{C_p}{C_d-(1-\rho)C_p}$, then $t^* = w$;
- If $Y(w) > \frac{C_p}{C_d-(1-\rho)C_p}$, then $t^* = \tau_2 < w$.

where $Y(t) = r(t)\int_0^t \bar{F}(u)\,du - F(t)$.

Remarks:

(a) From Theorem 7.1, with a little reflection we can view τ_1 and τ_2 as typical values of warranty length that demarcate the different regions of the buyer's optimal replacement-age policy (see Figure 7.1). We also note that by definition, τ_1 and τ_2 are functions of C_p but independent of w. In contrast, t_0 is a function of both C_p or w.

(b) The specific characterization of $r(t)$ can determine the boundedness of $Y(.)$, i.e. whether or not $Y(\infty)$ grows without bound, and also the boundedness of t^*. For instance, with the Weibull distribution, $r'(t) > 0$ implies that $\lim_{t\to\infty} r(t) = \infty$ and thus $\lim_{t\to\infty} Y(t) = Y(\infty) = \infty$. Under this case, $\frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} < Y(\infty)$ always holds and t^* never goes to infinite (unless $C_d = 0$). For some other failure processes, such as Gamma distribution with order n (n as an integer), $\lim_{t\to\infty} r(t) < \infty$ and $Y(\infty) = n -$

$1 < \infty$. $t^* = \infty$ holds for those cases when $\frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} \geq n-1$. We will further illustrate this with case studies in Section 7.5.

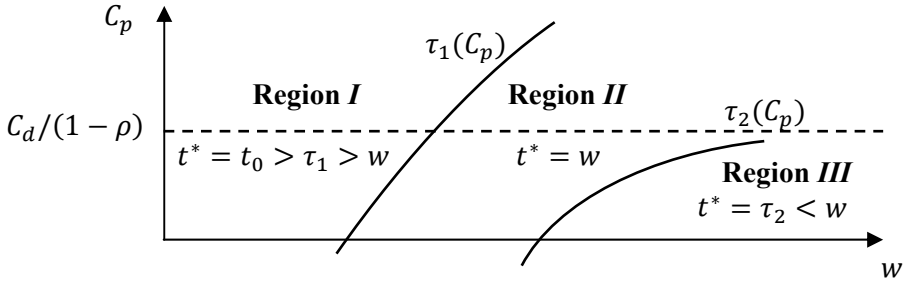

Figure 7.1 Impact of price C_p and warranty length w on optimal replacement age t^*

A graphic view of the buyer's optimal age-replacement strategy with respect to C_p and w is given in Figure 7.1.

For each given $C_p \leq \frac{C_d}{1-\rho}$, when $w < \tau_1(C_p)$, the optimal replacement age is greater than w, specifically $t^* = t_0$; when $\tau_1(C_p) \leq w \leq \tau_2(C_p)$, the product is always replaced by the buyer at the end of warranty period (i.e. $t^* = w$); when $C_p \leq C_d/(1-\rho)$, if $w > \tau_2(C_p)$, the optimal t^* is smaller than w and is given at $t^* = \tau_2 < w$; otherwise, the product is replaced at $t^* = w$. On the other hand, for any given $C_p > \frac{C_d}{1-\rho}$, by assuming $\tau_2(C_p) = \infty$, similar conclusions are obtained.

Note that the above analysis for Figure 7.1 does not included the scenario when $t^* = \infty$. However, by defining t_0, τ_1 and τ_2 to be ∞ when there is no real number solutions for $t_0 = Y^{-1}(\frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d})$, $\tau_1 = Y^{-1}(\frac{C_p\bar{F}(\tau_1)+\rho C_p F(\tau_1)}{C_d})$ and $\tau_2 = Y^{-1}(\frac{C_p}{C_d-(1-\rho)C_p})$ respectively, Figure 7.1 can fully describe Theorem 7.1.

7.3.2 The Seller's Optimal Strategy

To derive the seller's optimal marketing strategy, we assume that all the buyers carry out the optimal age-replacements as represented in Theorem 7.1. Note that for practically meaningful designs we must have $w \geq 0$ and $C_p > C_m$ (a necessary condition that (7.7) never goes to negative). Using (7.7) in conjunction with the results given in Theorem 7.1, the seller's total profit rate function is thus summarized in Theorem 7.2.

Theorem 7.2: Under a renewing cost-sharing warranty policy, given $r(t)$ as an increasing function of t and ρ as the fixed cost-sharing ratio for each warranty renewal, the seller's long-run average profit rate $\pi_1(C_p, w)$ as a function of price $C_p \in (C_m, \infty)$ and warranty length $w \in [0, \infty)$ is as follows:

1) For $C_d \leq (1-\rho)C_p$

$$\pi_1(C_p, w) = \begin{cases} k_1 C_p^{-a} \dfrac{C_p \bar{F}(w) + \rho C_p F(w) - C_m}{\int_0^{t_0} \bar{F}(u)du} [k_2 + \phi(\rho)w]^b, w < \tau_1(C_p) \\ k_1 C_p^{-a} \dfrac{C_p \bar{F}(w) + \rho C_p F(w) - C_m}{\int_0^{w} \bar{F}(u)du} [k_2 + \phi(\rho)w]^b, w \geq \tau_1(C_p) \end{cases} \tag{7.10}$$

2) For $C_d > (1-\rho)C_p$

$$\pi_1(C_p, w) = \begin{cases} k_1 C_p^{-a} \dfrac{C_p \bar{F}(w) + \rho C_p F(w) - C_m}{\int_0^{t_0} \bar{F}(u)du} [k_2 + \phi(\rho)w]^b, w < \tau_1(C_p) \\ k_1 C_p^{-a} \dfrac{C_p \bar{F}(w) + \rho C_p F(w) - C_m}{\int_0^{w} \bar{F}(u)du} [k_2 + \phi(\rho)w]^b, \tau_1(C_p) \leq w \leq \tau_2(C_p) \\ k_1 C_p^{-a} \dfrac{C_p \bar{F}(\tau_2) + \rho C_p F(\tau_2) - C_m}{\int_0^{\tau_2} \bar{F}(u)du} [k_2 + \phi(\rho)\tau_2]^b, w > \tau_2(C_p) \end{cases} \tag{7.11}$$

where $Y(t) = r(t)\int_0^t \bar{F}(u)\,du - F(t)$, $t_0 = Y^{-1}\left(\frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}\right)$, $\tau_1 = \tau_1(C_p) = Y^{-1}(\frac{C_p\bar{F}(\tau_1(C_p))+\rho C_p F(\tau_1(C_p))}{C_d})$ and $\tau_2 = \tau_2(C_p) = Y^{-1}(\frac{C_p}{C_d-(1-\rho)C_p})$.

Proof: Following the rules in Theorem 7.1, Equations (7.10) and (7.11) can be verified. Again, t_0, $\tau_1$and τ_2 go to infinity if there are no real number solutions respectively for $t_0 = Y^{-1}(\frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d})$, $\tau_1 = Y^{-1}(\frac{C_p\bar{F}(\tau_1)+\rho C_p F(\tau_1)}{C_d})$ and $\tau_2 = Y^{-1}(\frac{C_p}{C_d-(1-\rho)C_p})$.

Unfortunately, investigating the first-order derivative conditions of the profit rate function above is extremely cumbersome and does not offer much analytical insight. Since the seller's decision problem involves only two parameters, it is justifiable to use commercially available global search methods to locate the optimal C_p^* and w^*. However, as shown below, it is only necessary to search for the marketing strategy (C_p, w) within the range when $t^*(C_p, w) \geq w$ (Region I and II in Figure 7.1).

Proposition 7.1: Given any product marketing strategy (C_p, w) that satisfies $t^*(C_p, w) = \tau_2(C_p) < w$ (Region III in Figure 7.1), a corresponding strategy $(C_p, \tau_2(C_p))$ that belongs to Region II results in the same long-run average profit rate for the seller, i.e. $\pi_1(C_p, w) = \pi_1(C_p, \tau_2(C_p))$.

Proof: Note that given $t^*(C_p, w) = \tau_2(C_p) < w$, $t^*\left(C_p, \tau_2(C_p)\right) = \tau_2(C_p)$ and thus $\left(C_p, \tau_2(C_p)\right) \in$ Region II. Refer to the Theorem 7.2. The results simply follow.

Proposition 7.1 indicates that any marketing strategy that falls into Region III should never be the unique optimal one to the seller (a corresponding strategy exists in Region II). On the other hand, the numerical tests in Section 7.5 show that no optimal strategy falls into Region III.

7.4 Special Case: $C_d \rightarrow 0$ or $r(t) \equiv r(0)$

The optimization of the seller's marketing strategy is substantially simplified when either $C_d \rightarrow 0$, i.e. insignificant downtime cost or $(t) \equiv r(0)$, i.e. constant failure rate holds. As what is inferred in Theorem 7.1, if the product downtime cost $C_d \rightarrow 0$, $Y(\infty) \leq \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$ always holds. As a result, $t^* = \infty$ for any $C_p \in (C_m, \infty)$ and $w \in [0, \infty)$. The same conclusion can be made when the failure rate of the product is constant. Under this case, $Y(t) \equiv 0$ and $\frac{dCR_1(t)}{dt} < 0$ and $\frac{dCR_2(t)}{dt} < 0$ always hold as represented in (7.8) and (7.9). Thus, $t^* = \infty$. This validates the intuition that no preventive replacement is required when the product has a constant failure rate.

To derive the optimal C_p^* and w^* when $t^* = \infty$, let $\mu = \int_0^\infty \bar{F}(u)du$ be the mean life time of the product, and then (7.10) and (7.11) are reduced to the equation below:

$$\pi_0(C_p, w) = \frac{k_1 C_p^{-a}}{\mu}[k_2 + \phi(\rho)w]^b\left[C_p\bar{F}(w) + \rho C_p F(w) - C_m\right] \qquad (7.12)$$

7.4.1 Stationary Point for $\pi_0(w, C_p)$

It is important to investigate the first partial derivative equations of $\pi_0(C_p, w)$ with respect to C_p and w to obtain the necessary conditions that yield maximum profit. Such two equations are presented below:

$$\frac{\partial \pi_0(C_p, w)}{\partial C_p} = \frac{k_1 C_p^{-a-1}[k_2 + \phi(\rho)w]^b}{\mu}\left(aC_m - (a-1)\left[C_p\bar{F}(w) + \rho C_p F(w)\right]\right) \qquad (7.13)$$

$$\frac{\partial \pi_0(C_p, w)}{\partial w} = \frac{k_1 C_p^{-a}[k_2 + \phi(\rho)w]^{b-1}}{\mu}\left(b\phi(\rho)\left[C_p\bar{F}(w) + \rho C_p F(w) - C_m\right]\right.$$
$$\left. - (1-\rho)C_p f(w)[k_2 + \phi(\rho)w]\right) \qquad (7.14)$$

Both equations are set to zero and solved simultaneously to find the necessary conditions for relative maxima.

$$C_p^* = \frac{aC_m}{(a-1)[\bar{F}(w^*) + \rho F(w^*)]} \tag{7.15}$$

$$w^* = \frac{b}{a(1-\rho)} \frac{[\bar{F}(w^*) + \rho F(w^*)]}{f(w^*)} - \frac{k_2}{\phi(\rho)} \tag{7.16}$$

7.4.2 Second-order Conditions

There may be more than one solution to (7.16) or equivalently (7.13)–(7.14) that satisfy the first-order conditions. In view of this, we further look into the second-order conditions. Let $\pi_{pp} = \frac{\partial^2 \pi_0(C_p,w)}{\partial C_p^2}$, $\pi_{ww} = \frac{\partial^2 \pi_0(C_p,w)}{\partial w^2}$, $\pi_{pw} = \frac{\partial^2 \pi_0(C_p,w)}{\partial C_p \partial w}$ and $D = \pi_{pp}\pi_{ww} - \pi_{pw}^2$. Their values at (C_p^*, w^*) are given as follows:

$$\pi_{pp}^* = -(a-1)[\bar{F}(w^*) + \rho F(w^*)] \frac{k_1 \left(C_p^*\right)^{-a-1} [k_2 + \phi(\rho) w^*]^b}{\mu} \tag{7.17}$$

$$\pi_{ww}^* = -\frac{k_1 \left(C_p^*\right)^{-a+1} [k_2 + \phi(\rho) w^*]^{b-1}}{\mu} (1-\rho)\phi(\rho)$$

$$\times \left((b+1) f(w^*) + f'(w^*) \left[\frac{k_2}{\phi(\rho)} + w^* \right] \right) \tag{7.18}$$

$$\pi_{pw}^* = \frac{k_1 \left(C_p^*\right)^{-a} [k_2 + \phi(\rho) w^*]^b}{\mu} (a-1)(1-\rho) f(w^*) \tag{7.19}$$

$$D^* = \pi_{pp}^* \pi_{ww}^* - \left(\pi_{pw}^*\right)^2 = a(a-1)(1-\rho)^2 \left(C_p^*\right)^{-2a} \frac{k_1^2 [k_2 + \phi(\rho) w^*]^{2b}}{b\mu^2} f(w^*)$$

$$\times \left(f(w^*)(1 + \frac{b}{a}) + f'(w^*) \left(\frac{k_2}{\phi(\rho)} + w^* \right) \right) \tag{7.20}$$

It is easy to verify that $\pi_{pp}^* > 0$ always holds for any $w^* \geq 0$. A necessary condition for (C_p^*, w^*) to be the relative maxima is $\pi_{pp}^* < 0$, $\pi_{ww}^* < 0$ and $D^* > 0$. Or equivalently that $(b+1)f(w^*) + f'(w^*)\left[\frac{k_2}{\phi(\rho)} + w^*\right] > 0$ and $f(w^*)\left(1 + \frac{b}{a}\right) + f'(w^*)\left(\frac{k_2}{\phi(\rho)} + w^*\right) > 0$ must hold. Note that the latter implies the former and hence it suffices to have $f(w^*)\left(1 + \frac{b}{a}\right) + f'(w^*)\left(\frac{k_2}{\phi(\rho)} + w^*\right) > 0$. In all of our numerical examples, (C_p^*, w^*) that falls into the region of $C_p \in (C_m, \infty)$ and $w \in [0, \infty)$ also turns out to be the global maxima.

7.5 Illustrative Examples

In this section, two practical cases are devised to illustrate the applicability of the proposed decision models in this chapter. Sensitivity analysis are then conducted to help managers investigate the impact of the buyer's age-replacement decisions and warranty cost-sharing ratio ρ on the optimal design of the product marketing strategy.

Suppose that two types of large-scale mechanic facilities are developed and planned for release to the market. Cost-sharing warranties are provided to protect consumers against early product failures and balance the high post-sale cost for the manufacturers simultaneously. The in-house testing indicates that the failure rate functions of both products increase with time but follow the Gamma and Weibull distribution respectively. To obtain the optimal product marketing strategies for both products, the seller conducts the following analysis.

7.5.1 Gamma Distribution: When $Y(\infty) < \infty$

The first type of product is assumed to have a life time x that satisfies Gamma distribution of order two:

$$F(x) = 1 - (1 + \lambda x)e^{-\lambda x}, f(x) = \lambda^2 x e^{-\lambda x}, r(x) = \frac{\lambda^2 x}{1 + \lambda x} \tag{7.21}$$

$Y(.)$ is therefore bounded by the value of 1 ($Y(\infty) = \lim_{x\to\infty} r(x) \int_0^\infty \bar{F}(x)\, dx - 1 = 1$), which indicates that under some cases, the optimal replacement age may go to infinity (i.e. $t^* = \infty$). Besides, $r(x)$ is always an increasing function of x.

The warranty contract specifies the warranty cost-sharing ratio $\rho = 0.3$. The market survey and analysis done by the marketing department provides the following parameters of the demand function – $k_1 = 10^6$, $k_2 = 1$ (time unit), $a = 2$, $b = 0.7$ and $\phi(\rho) = 0.7$. The manufacturing cost per item is $C_m = 50$. In addition, engineering department estimates the mean life time of the product to be 20 time units, i.e. $\lambda = 0.1$.

Table 7.1 and Table 7.2 present the seller's optimal marketing strategy and the corresponding buyer's strategy when $C_d = 0$ and $C_d = 100, 200, 500$ respectively. In Table 7.1, given a zero downtime cost, $w^* = 9.74$ is solved iteratively using (7.16) and $C_p^* = 121.71$ is derived from (7.15). The second-order conditions at (121.71,9.74) satisfy $\pi_{pp}^* < 0$, $\pi_{ww}^* < 0$, and $D^* > 0$. In Table 7.2 we notice that when $C_d > 100$, the age-replacement factor t^* decreases with C_d and becomes substantial in the design of (C_p^*, w^*). As C_d increases to 500, the age-replacement (perceived by the seller) will be carried out as soon as the warranty is expired ($t^* = w$).

Figure 7.2 further illustrates the impact of C_d (and t^*) on the seller's optimal strategy in a graphic view. We notice that when C_d increases from 100 to 450, the shortening of the buyer's purchasing cycle t^*enables the seller to decrease price C_p^* to attract more buyers; when C_d increases from 450, the further decrease of t^* reduces the effect of a long warranty on the product demand and we therefore see w^* decrease together with t^* (typically that $w^* = t^*$). Furthermore, when C_d ($\in [0,100]$) is small, t^* goes to

infinity and (7.15)–(7.16) provides a fast approximation of the optimal strategy for the seller.

Table 7.1 Seller's optimal marketing strategy when C_d=0, ρ=0.3 and $\phi(\rho)$=0.7

(C_p^*, w^*)	$\pi_0(C_p^*, w^*)$	π_{pp}^*	π_{ww}^*	D^*	$t^*(C_p^*, w^*)$	$CR^*(t^*)$
(121.71,9.74)	712.13	-0.10	-6.91	0.53	∞	5.00

Table 7.2 Seller's optimal marketing strategy when C_d=100,200,500, ρ=0.3 and $\phi(\rho)$=0.7

C_d	(C_p^*, w^*)	$\pi_1(C_p^*, w^*)$	$t^*(C_p^*, w^*)$	$CR^*(t^*)$	τ_1
100	(121.71,9.74)	712.13	∞	10.00	∞
200	(98.04,9.74)	883.40	21.55	13.66	19.19
500	(96.13,9.01)	1631.71	9.01(= w)	23.70	9.01(= w)

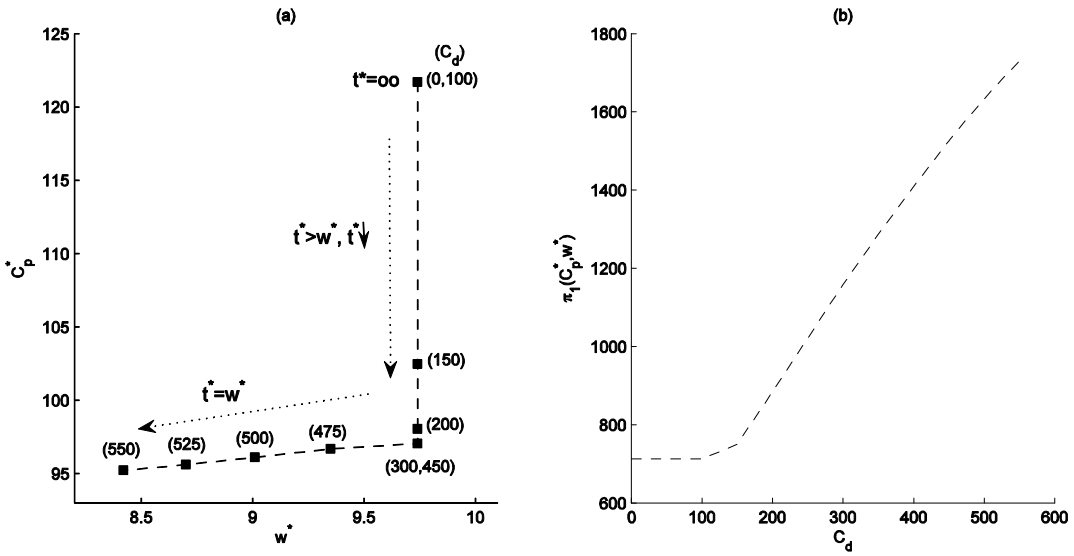

Figure 7.2 Optimal marketing strategies under Gamma distribution given $C_d \in$[0,550]: (a) (C_p^*,w^*) vs. C_d and (b) $\pi_1(C_p^*,w^*)$ vs. C_d

7.5.2 Weibull Distribution: When $Y(\infty) = \infty$

For the second type of product, the life time x is governed by a Weibull distribution with shape parameter β ($\beta > 1$) and scale parameter λ:

$$F(x) = 1 - e^{-(\lambda x)^\beta}, f(x) = \lambda\beta(\lambda x)^{\beta-1}e^{-(\lambda x)^\beta}, r(x) = \lambda\beta(\lambda x)^{\beta-1} \qquad (7.22)$$

Typically for the Weibull distribution, $r'(x) > 0$ implies $\lim_{x\to\infty} r(x) = \infty$ (given $\beta > 1$). Therefore we have $Y(\infty) = \infty$, and $t^* < \infty$.

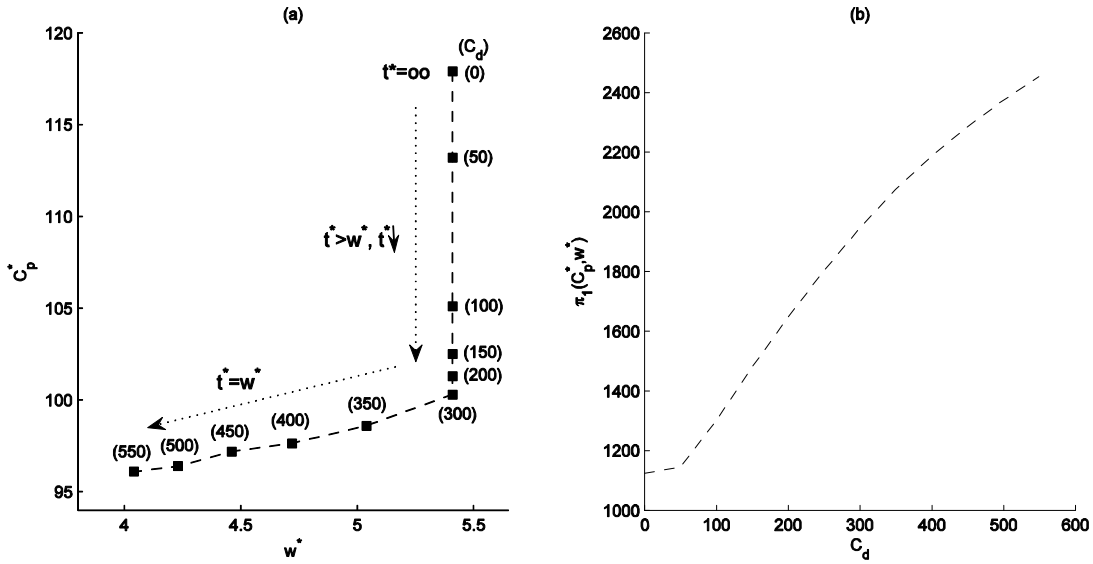

Figure 7.3 Optimal marketing strategies under Weibull distribution given $C_d \in [0,550]$: (a) (C_p^*, w^*) vs. C_d and (b) $\pi_1(C_p^*, w^*)$ vs. C_d

Warranty cost-sharing ratio is designed at $\rho = 0.4$. The data from the marketing department supports the following parameters of the demand function: $k_1 = 10^6$, $k_2 = 0.5$ (time unit), $a = 2$, $b = 0.7$ and $\phi(\rho) = 0.7$. The manufacturing cost per unit is $C_m = 50$. Furthermore, the in-house testing done by the engineering department indicates that the Weibull failure time distribution has a shape parameter $\beta = 2$ and a scale parameter $\lambda = 0.1$, which yields a mean life time of about 9 time units.

Table 7.3 and Table 7.4 present the seller's optimal marketing strategy and its corresponding buyer's strategy when $C_d = 0$ and $C_d = 50, 100, 400$ respectively. The optimal strategy (C_p^*, w^*) under a zero downtime cost is $(117.96, 5.41)$. Again, the impact of C_d and t^* on the design of (C_p^*, w^*) are illustrated in Figure 7.3. The changes of C_p^* and w^* in response to t^* show similar pattern to the previous Gamma case. However we notice that under the Weibull case, any small increase of C_d can result in

a decrease of t^* and a subsequent change on (C_p^*, w^*). Such observation verifies the different boundedness properties of $Y(.)$ between these two distributions.

Table 7.3 Seller's optimal marketing strategy when C_d =0, ρ=0.4 and $\phi(\rho)$=0.7

(C_p^*, w^*)	$\pi_0(C_p^*, w^*)$	π_{pp}^*	π_{ww}^*	D^*	$t^*(C_p^*, w^*)$	$CR^*(t^*)$
(117.96,5.41)	1123.24	-0.16	-45.48	6.16	∞	11.28

Table 7.4 Seller's optimal marketing strategy when C_d=50,100,400, ρ=0.4 and $\phi(\rho)$=0.7

C_d	(C_p^*, w^*)	$\pi_1(C_p^*, w^*)$	$t^*(C_p^*, w^*)$	CR^*	τ_1
50	(113.20,5.41)	1144.41	16.42	16.42	12.23
100	(105.16,5.41)	1300.63	10.20	20.40	8.92
400	(97.65,4.72)	2187.10	4.72(= w)	37.76	4.72(= w)

7.5.3 Impact of Warranty Cost-sharing Ratio ρ on (C_p^*, w^*) and $\pi_1(C_p^*, w^*)$

Conducting a further sensitivity analysis on ρ can help the manager select a better warranty cost-sharing ratio that improves his profit margin. But it relies on more proprietary knowledge about $\phi(\rho)$ that is typically done by the marketing department. In the following we study two types of discount factors $\phi(\rho)$ for $\rho \in [0,0.5]$ that reflects different sensitivity of demand to ρ (described by $\left|\frac{\Delta\phi_1}{\Delta\rho}\right|$ and $\left|\frac{\Delta\phi_2}{\Delta\rho}\right|$). As presented in Table 7.5 and Figure 7.4, $\phi_2(\rho)$ is more sensitive to ρ than $\phi_1(\rho)$ when ρ is small, and the situation reverses when ρ is large.

Table 7.5 Two types of discounted factor $\phi(\rho)$ for $\rho \in [0,0.5]$

ρ	$\phi_1(\rho)$	$\left\|\frac{\Delta\phi_1}{\Delta\rho}\right\|$	$\phi_2(\rho)$	$\left\|\frac{\Delta\phi_2}{\Delta\rho}\right\|$
0	1	---	1	---
0.05	0.985	0.3	0.95	1
0.1	0.965	0.4	0.9	1
0.15	0.94	0.5	0.85	1
0.2	0.91	0.6	0.8	1
0.25	0.87	0.8	0.75	1
0.3	0.82	1	0.7	1
0.35	0.77	1	0.65	1
0.4	0.7	1.4	0.6	1
0.45	0.61	1.8	0.55	1
0.5	0.50	2.2	0.5	1

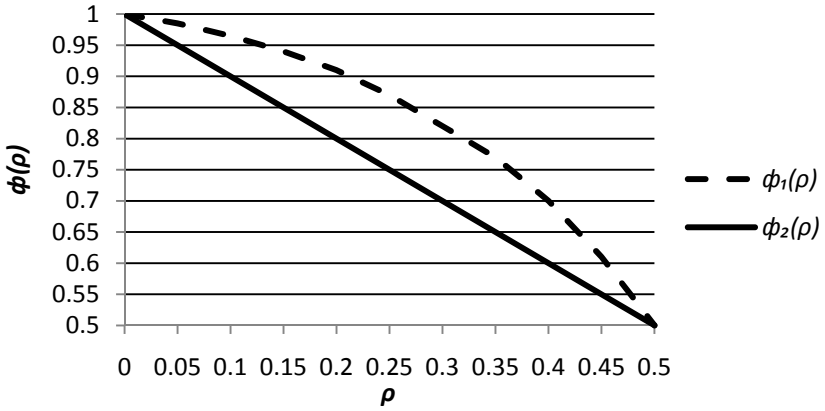

Figure 7.4 Two types of discount factor $\phi(\rho)$ for $\rho \in [0,0.5]$

By applying $\phi_1(\rho)$ and $\phi_2(\rho)$, the impacts of ρ on (C_p^*, w^*) and $\pi_1(C_p^*, w^*)$ are illustrated in Figure 7.5 and Figure 7.6 for Gamma and Weibull distribution respectively. We highlight the following results.

First, the increase of ρ reduces the attractiveness of the warranty program for both products and therefore requires a compensation from the seller's side by increasing the length of the warranty. Second, different discount factor $\phi(.)$s tend to yield different optimal warranty cost-sharing ratios that maximize the seller's profit rate. In our examples, when $\phi = \phi_1$, $\rho^* = 0.25$ is the optimal choice for the Gamma case (Figure 7.5), and $\rho^* = 0.2$ is the optimal choice for the Weibull case (Figure 7.6). However, when $\phi = \phi_2$, $\rho^* = 0$ or a free-replacement warranty (FRW) turns out to be the best choice for both cases.

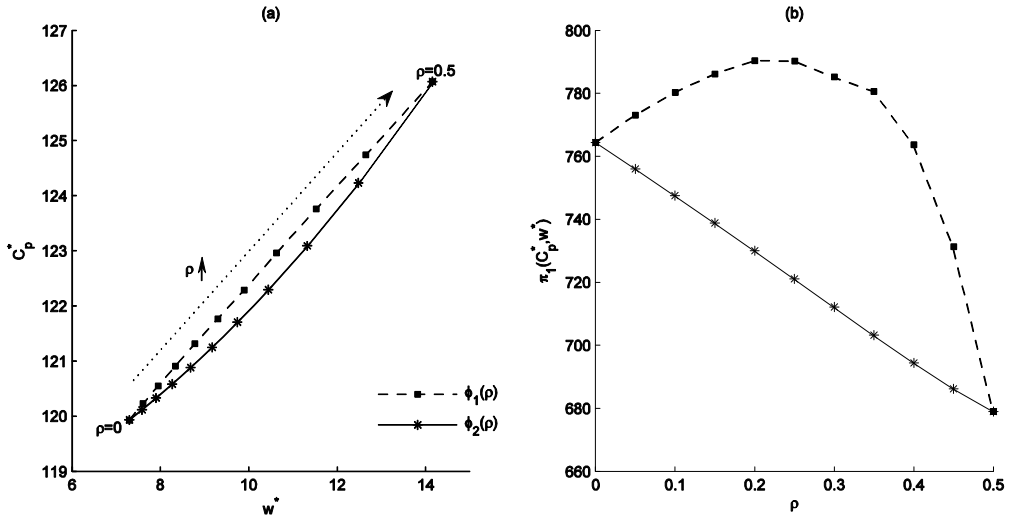

Figure 7.5 (a) (C_p^*,w^*) vs. ρ and (b) $\pi_1(C_p^*,w^*)$ vs. ρ, given $\phi_1(\rho)$, $\phi_2(\rho)$ and $\rho\in[0,0.5]$ for Gamma distribution

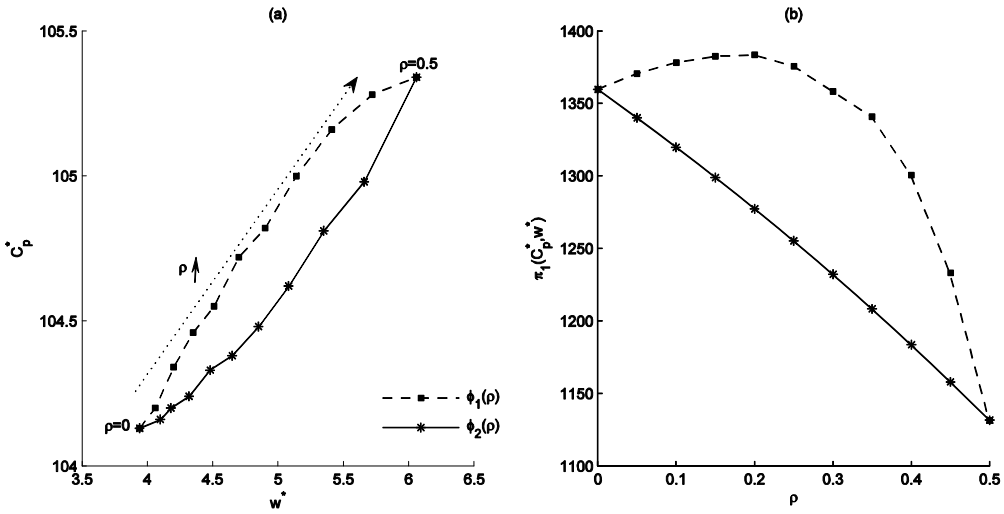

Figure 7.6 (a) (C_p^*,w^*) vs. ρ and (b) $\pi_1(C_p^*,w^*)$ vs. ρ, given $\phi_1(\rho)$, $\phi_2(\rho)$ and $\rho\in[0,0.5]$ for Weibull distribution

7.6 Conclusion

This chapter introduces a model for analyzing optimal product marketing strategy on product price and warranty length that maximizes the seller's long-run average profit rate. The model formulation follows a Stackelberg game where the seller is the leader in the game and the buyer is the follower. In other words, the buyer implements age-replacement for the product to minimize his long-run average maintenance cost rate; the seller derives the buyer's optimal replacement time as a function of product price

and warranty length, and optimally selects these two decision variables to maximize his profit rate function. The sales model in this chapter generalizes the demand function in Glickman and Berger (1976) and is subject to not only product price and warranty length, but also the buyer's age-replacement decision and warranty cost-sharing ratio. A special case without preventive age-replacement is further presented with the optimal seller's strategy being derived and analyzed. Through two illustrative examples, the applicability of the model is demonstrated, and sensitivity analysis are further conducted to investigate the impact of the buyer's age-replacement decisions and warranty cost-sharing ratio on the optimal product marketing strategy.

Based on the computational results done for the two types of products in this chapter, the following insights are provided, which may acquire certain managerial attention. Practically, the buyers always conduct age-replacements in a relatively early period when their downtime cost is high. The shortening of the purchasing cycle helps increase the profit for the seller and therefore enables him to reduce the price of the product to attract more buyers. When the pre-specified purchasing cycle decreases to a certain level, a longer warranty may not be able to increase the demand for the product and as a result, a strategy of reducing the warranty coverage should be appropriate. The seller may further manipulate the warranty cost-sharing ratio to improve his profit margin. The feasibility of this relies on further efforts done by the marketing department in estimating the negative impact of increasing warranty cost-sharing ratio on the product sales. In general, the increase of the ratio may reduce the attractiveness of the warranty program to the buyers and therefore requires certain compensative approach from the seller's side such as providing longer warranty coverage.

CHAPTER 8 CONCLUSION AND FUTURE WORK

This thesis addressed several important issues in the post-sale cost modeling and optimization from both the manufacturer's (seller) and the consumer's (buyer) point of view. Warranty and preventive maintenance (PM) were modeled jointly as a means of either enhancing the seller's warranty service strategy or reducing the buyer's life-cycle maintenance cost.

One of the primary goals of warranty research is to analyze the cost of a warranty program. Chapter 3 conducted warranty cost analysis for multi-component systems under renewing free-replacement warranty (RFRW) policy. Unlike previous works that assumed failure independence among system components, a type of failure dependence models, failure interaction (Murthy and Nguyen, 1985a), was incorporated into the cost modeling. With the presence of failure interaction, the system faces more severe reliability problem during its ageing process. As a result, upon system failures within warranty, opportunistic PMs were carried out for the survived components in order to reduce the chance of future failures. Warranty cost functions for both series and parallel system configurations were derived, followed by a numerical example with sensitivity analysis. The consideration of failure interaction here can help decision makers better evaluate complex system reliability and improve the accuracy of warranty cost estimation.

Designing a good maintenance program under warranty is of great importance to the manufacturer in terms of improving the service quality and reducing the total warranty cost. Chapter 4 applied condition-based maintenance (CBM) in enhancing the existing warranty service design. The implementation of CBM relies on the product state

information, and is typically effective when the product is subject to failures only after it degrades to certain state level. Correspondingly, in this research, we assumed that the deterioration of a product follows a two-stage process, i.e. from nominal to defective and from defective to failed, and PM service is conducted only when the product is defective upon inspection. We derived warranty cost functions under both renewing and non-renewing warranty settings, based on which the optimal scheduling of inspection services was further analyzed. Results revealed that a CBM program within warranty can be cost-efficient to the manufacturer when either the inspection cost is relatively low or the product has a relatively short mean life time.

Chapters 5 and 6 focused on the design of finite life-cycle maintenance policies for the buyer. Chapter 5 studied a general periodic PM policy for warranted revenue-generating systems by integrating both the cost and value aspects of maintenance. We defined ageing losses as the total revenue losses due to the decreasing productivity during the system ageing process. We argued that preventive maintenance, which slows system deterioration, reduces ageing losses and therefore reflects its value. The total life-cycle cost model was developed for the buyer including both the maintenance cost and ageing losses. The optimal PM strategy was further derived as a function of two decision variables, the calendar time of the first PM action and its corresponding maintenance level. In order to investigate the effect of warranty as well as many other model parameters on the buyer's optimal PM decisions, a comprehensive sensitivity analysis was conducted through a numerical case.

Chapter 6 analyzed the repair-replacement policies for multi-state systems (MSS) that operate under finite life cycles. We investigated two classes of policies, that of preventive replacements and corrective replacements. For both policies, the expected discounted maintenance cost (EDMC) was derived as a function of two control

parameters – a threshold level on the current system state, and a threshold level on the residual life cycle, and the cost formulation took into account both the value of time and initial warranty coverage. In order to obtain the optimal maintenance thresholds that minimize the EDMC, two types of optimization methods were proposed. Through computational examples, we demonstrated that preventive replacement policy outperforms corrective replacement policy when either warranty period is long or the downtime cost of each failure is relatively high compared to the repair cost.

Warranty as an efficient marketing instrument for enhancing the sales revenue has been investigated by several researchers. Chapter 7 introduced a model for analyzing optimal marketing decisions on product price and warranty length that maximizes the seller's long-run average profit rate. The decision process was formulated as a Stackelberg game where the seller is the leader in the game and the buyer is the follower. In other words, the buyer implements age-replacement for the product to minimize his long-run average maintenance cost rate; the seller derives the buyer's optimal replacement time as a function of product price and warranty length, and optimally selects these two decision variables to maximize his profit rate function. We proposed a novel sales model that generalizes the demand function in Glickman and Berger (1976) and is subject to not only product price and warranty length, but also the buyer's age-replacement decision and warranty cost-sharing ratio. Two applications were presented to illustrate the applicability of the proposed model and sensitivity analysis was further conducted in order to investigate the impact of warranty and age-dependent PM policy on the optimal product marketing strategy.

As mentioned previously, post-sale cost modeling and optimization linking warranty and PM is an important research area, but has not received enough attention in the

literature. Although we have successfully addressed several issues in this area, there are some other opportunities for future study. Following is a list of potential topics:

- To conduct warranty cost analysis for complex systems with more general system configurations, such as series-parallel, parallel-series, *k*-out-of-*n*, etc (Kuo et al., 2001); to study PM strategy for multi-component systems which is implemented before the entire system fails.
- To design a proper CBM program for products subject to continuous-state degradation processes (e.g. Gamma process); to investigate the CBM policies with imperfect inspections concerning the product condition.
- To develop continuous-time maintenance models (Djamaludin et al., 2001) instead of periodic PM for safety-critical systems.
- To devise repair-replacement policies for MSS subject to semi-Markov degradation processes and imperfect maintenance operations (Soro et al., 2010).
- To study warranty and maintenance policies with non-negligible repair times.
- To formulate the post-sale decisions between the buyer and the seller as a non-cooperative game (Jackson and Pascual, 2008); to involve the customer's negotiation scheme into the seller's post-sale service design.

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APPENDIX A. *Proof* of Lemma 3.1

To prove Lemma 3.1, we have

$$p_i(w) = \int_0^{\infty} \Pr(T_i \le Y_i, T_i \le w | T_i = t)\, \mathrm{d}F_i(t)$$

$$= \int_0^{w} \Pr(Y_i \ge t)\, \mathrm{d}F_i(t) \qquad (Y_i \text{ and } T_i \text{ are independent })$$

$$= \int_0^{w} \prod_{j\in\Omega, j\neq i} \Pr(T_j \ge t)\, \mathrm{d}F_i(t) \qquad (T_j \text{ and } T_i \text{ are independent })$$

$$= \int_0^{w} \left(\prod_{j\in\Omega, j\neq i} R_j(t) \right) f_i(t)\, \mathrm{d}t \quad (\text{since } R_S(t) = \prod_{j\in\Omega} R_j(t) \text{ and } h_i(t)$$

$$= \frac{f_i(t)}{R_i(t)}) = \int_0^{w} h_i(t) R_S(t)\, \mathrm{d}t$$

The definition of $\alpha_i(w)$ is straightforward.

As under series structure, $\Pr[T_S \le w] = \Pr\,[\cup_{i\in\Omega}\{T_i \le Y_i, T_i \le w\}] = \sum_{i\in\Omega} \Pr\,[T_i \le Y_i, T_i \le w]$, we have

$$\sum_{i=1}^{n} p_i(w) = F_S(w) \qquad \sum_{i=1}^{n} \alpha_i(w) = 1$$

APPENDIX B. *Proof* of Lemma 3.2

The proof of (3.4) is presented first. Define $N_S = n_s, n_s = n_i + n_j$. According to the discussion in Section 3.1, the number of natural failures are independent with each other even failure interaction is considered. Using the definition in Lemma 3.1, the probability that the system failure is caused by component i is simply $\alpha_i(w)$. We have $N_i|N_S \sim \text{Binomial}\big(n_i, \alpha_i(w)\big), i \in \Omega$, and thus

$$\Pr(N_i = n_i) = \sum_{n_s=n_i}^{\infty} \Pr(N_i = n_i|N_S = n_s)\big(F_s(w)\big)^{n_s} R_S(w) \text{ (see Section 3.2)}$$

$$= \sum_{n_s=n_i}^{\infty} \binom{n_s}{n_i} \big(\alpha_i(w)\big)^{n_i} \big(1 - \alpha_i(w)\big)^{n_j} \big(F_s(w)\big)^{n_s} R_S(w)$$

$$= \sum_{n_j=0}^{\infty} \binom{n_i + n_j}{n_i} \big(p_i(w)\big)^{n_i} \big(F_s(w) - p_i(w)\big)^{n_j} R_S(w)$$

$$= \left(\frac{p_i(w)}{R_s(w) + p_i(w)}\right)^{n_i} \frac{R_s(w)}{R_s(w) + p_i(w)}$$

The last two steps utilize the result $\sum_{k=0}^{\infty} \binom{k+b}{b} x^k \equiv (1-x)^{-b-1}$. Equation (3.2) can be proved in a similar way, based on which (3.3) and (3.5) simply follow.

$$\Pr\big(N_{ij} = n_{ij}\big) = \sum_{n_i=n_{ij}}^{\infty} \Pr\big(n_{ij}|n_i\big) \Pr(n_i) \quad \text{(using the results in (3.4))}$$

$$= \sum_{n_i=n_{ij}}^{\infty} \binom{n_i}{n_{ij}} p_{ij}^{n_{ij}} (1-p_{ij})^{n_i - n_{ij}} \left(\frac{p_i(w)}{R_s(w) + p_i(w)}\right)^{n_i} \frac{R_s(w)}{R_s(w) + p_i(w)}$$

$$= \left(\frac{p_{ij} p_i(w)}{R_s(w) + p_{ij} p_i(w)}\right)^{n_{ij}} \frac{R_s(w)}{R_s(w) + p_{ij} p_i(w)}$$

APPENDIX C. *Proof* of Lemma 3.3

For any $\varphi \subseteq \Omega = \{0,1,2,\dots,n\}$, let $|\varphi|$ denote the number of elements (i.e. components) in φ.

Using the method of **Mathematical Induction** (MI):

1. Prove that the result holds for any single-component system, i.e. $|\varphi| = 1$

$$P_{\{i\}}(x,y) = \frac{F_i(y) - F_i(x)}{1 - F_i(x)} = \frac{e^{-\lambda_i x} - e^{-\lambda_i y}}{e^{-\lambda_i x}} = 1 - e^{-\lambda_i (y-x)}$$

$$P_{\{i\}}(x-a, y-a) = \frac{e^{-\lambda_i (x-a)} - e^{-\lambda_i (y-a)}}{e^{-\lambda_i (x-a)}} = 1 - e^{-\lambda_i (y-x)}, \forall i \in \Omega$$

   Then for any subsystem with $|\varphi| = 1$, the result holds.

2. Assume that (3.14) is true for any subsystem with $|\varphi| \le k$. Then according to the expression of $G_\varphi[Z_i \le w, Y_i \ge t]$, it is easily verified that, for any subsystem with $|\varphi| = k+1$,

$$G_\varphi[Z_i \le y, Y_i \ge t + a] = G_\varphi[Z_i \le y - a, Y_i \ge t]$$

   According to the definition of $P_\varphi(x,y)$, we have

$$P_\varphi(x,y) = \sum_{i \in \varphi} \int_x^y G_\varphi(Z_i \le y, Y_i \ge t) \lambda_i e^{-\sum_{j=1}^{k+1} \lambda_j (t-x)} \, dt$$

$$= \sum_{i \in \varphi} \int_{x-a}^{y-a} G_\varphi(Z_i \le y, Y_i \ge t + a) \lambda_i e^{-\sum_{j=1}^{k+1} \lambda_j (t+a-x)} \, dt$$

$$= \sum_{i \in \varphi} \int_{x-a}^{y-a} G_\varphi(Z_i \le y - a, Y_i \ge t) \lambda_i e^{-\sum_{j=1}^{n} \lambda_j [t-(x-a)]} \, dt$$

$$= P_\varphi(x-a, y-a)$$

   which exactly means that (3.14) is true for all subsystems of Ω.

APPENDIX D. Expressions of the EDMC for Scenario 1

The de facto "analytical" form of the EDMC under Policies O, A and B for Scenario 1 without warranty incorporation is given as:

Policy O

$$C_1^{(O)}(\tau) =$$

$$5098.8e^{-0.05\tau} + 100.0e^{-1.90\tau} + (226.5cos0.91\tau + 201.6sin0.91\tau)e^{-0.97\tau} - 4772.3$$

$$C_2^{(O)}(\tau) =$$

$$5098.8e^{-0.05\tau} - 105.6e^{-1.90\tau} + (198.2cos0.91\tau - 235.2sin0.91\tau)e^{-0.97\tau} - 5006.3$$

$$C_3^{(O)}(\tau) =$$

$$5098.8e^{-0.05\tau} + 138.7e^{-1.90\tau} - (299.1cos0.91\tau + 189.4sin0.91\tau)e^{-0.97\tau} + 5259.2$$

$$C_4^{(O)}(\tau) =$$

$$5098.8e^{-0.05\tau} - 146.6e^{-1.90\tau} - (183.8cos0.91\tau - 308.4sin0.91\tau)e^{-0.97\tau} - 5429.1$$

Policy A

$$C_1^{(A)}(L|1,\tau) = \left(C_1^{(O)}(\tau) - 4880.0\right)e^{-0.05(L-\tau)} + 4880.0$$

$$C_1^{(A)}(L|2,\tau) = \left(0.47C_1^{(O)}(\tau) + 0.53C_2^{(O)}(\tau) - 3821.2\right)e^{-0.05(L-\tau)} + \left(0.53C_1^{(O)}(\tau) - 0.53C_2^{(O)}(\tau) + 93.2\right)e^{-1.75(L-\tau)} + 3728.0$$

$$C_1^{(A)}(L|3,\tau) = \left(0.32C_1^{(O)}(\tau) + 0.36C_2^{(O)}(\tau) + 0.32C_3^{(O)}(\tau) - 4224\right)e^{-0.05(L-\tau)} + \left[\left(0.68C_1^{(O)}(\tau) - 0.36C_2^{(O)}(\tau) - 0.32C_3^{(O)}(\tau) + 196.2\right)cos0.75(L-\tau) -\right.$$

$\left(0.02C_1^{(O)}(\tau) - 0.58C_2^{(O)}(\tau) + 0.56C_3^{(O)}(\tau) - 109.2\right)sin0.75(L-\tau)\Big]\, e^{-1.35(L-\tau)} +$

4027.8

Policy B

$C_1^{(B)}(L|1,\tau) = \left(0.50C_1^{(O)}(\tau) + 0.32C_2^{(O)}(\tau) + 0.13C_3^{(O)}(\tau) + 0.05C_4^{(O)}(\tau) -\right.$

$\left.3822.8\right)e^{-0.05(L-\tau)} + \left(0.13C_1^{(O)}(\tau) - 0.08C_2^{(O)}(\tau) + 0.07C_3^{(O)}(\tau) - 0.12C_4^{(O)}(\tau) +\right.$

$\left.52.2\right)e^{-2.88(L-\tau)} + \Big[\left(0.38C_1^{(O)}(\tau) - 0.24C_2^{(O)}(\tau) - 0.20C_3^{(O)}(\tau) + 0.06C_4^{(O)}(\tau) +\right.$

$\left.78.7\right)cos1.00(L-\tau) + (0.15C_1^{(O)}(\tau) + 0.24C_2^{(O)}(\tau) - 0.18C_3^{(O)}(\tau) - 0.21C_4^{(O)}(\tau) +$

$135.8)sin1.00(L-\tau)\Big]\, e^{-1.88(L-\tau)} + 3691.8$

$C_1^{(B)}(L|2,\tau) = \left(0.36C_1^{(O)}(\tau) + 0.40C_2^{(O)}(\tau) + 0.17C_3^{(O)}(\tau) + 0.07C_4^{(O)}(\tau) -\right.$

$\left.3828.8\right)e^{-0.05(L-\tau)} + \left(0.10C_1^{(O)}(\tau) - 0.05C_2^{(O)}(\tau) + 0.05C_3^{(O)}(\tau) - 0.10C_4^{(O)}(\tau) +\right.$

$\left.47.6\right)e^{-2.77(L-\tau)} + \Big[(0.23C_1^{(O)}(\tau) + 0.20C_2^{(O)}(\tau) - 0.21C_3^{(O)}(\tau) - 0.22C_4^{(O)}(\tau) +$

$158.4)sin1.02(L-\tau) + (0.54C_1^{(O)}(\tau) - 0.36C_2^{(O)}(\tau) - 0.22C_3^{(O)}(\tau) +$

$0.04C_4^{(O)}(\tau) + 118.2)cos1.02(L-\tau)\Big]e^{-1.64(L-\tau)} + 3663.0$

$C_1^{(B)}(L|3,\tau) = \left(0.28C_1^{(O)}(\tau) + 0.32C_2^{(O)}(\tau) + 0.28C_3^{(O)}(\tau) + 0.11C_4^{(O)}(\tau) -\right.$

$\left.4346.0\right)e^{-0.05(L-\tau)} + \left(0.11C_1^{(O)}(\tau) - 0.06C_2^{(O)}(\tau) + 0.03C_3^{(O)}(\tau) - 0.08C_4^{(O)}(\tau) +\right.$

$\left.45.2\right)e^{-2.64(L-\tau)} + \Big[\left(0.60C_1^{(O)}(\tau) - 0.26C_2^{(O)}(\tau) - 0.31C_3^{(O)}(\tau) - 0.03C_4^{(O)}(\tau) +\right.$

$\left.192.9\right)cos0.96(L-\tau) + (0.09C_1^{(O)}(\tau) + 0.47C_2^{(O)}(\tau) - 0.30C_3^{(O)}(\tau) -$

$0.26C_4^{(O)}(\tau) + 169.8)sin0.96(L-\tau)\Big]\, e^{-1.20(L-\tau)} + 4107.9$

APPENDIX E. *Proof* of Theorem 7.1

Denote t_1^* and t_2^* as the optimal replacement ages for $t \geq w$ and $t < w$ respetively.

1) $C_p \geq C_d/(1-\rho)$

From the assumption, we know that $Y(0) = 0$ and $Y(t)$ is an increasing function of t. Given $C_d \leq (1-\rho)C_p$, it is easy to verify that $CR_2'(t) < 0$ for $t < w$. Thus, the optimal replacement age for $CR_2(t)$ is obtained at $t_2^* = w$.

To derive the optimal replacement age for $t \geq w$, we discuss the following cases.

- When $Y(\infty) \leq \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$

Given $Y'(t) > 0$, we have $Y(w) < Y(\infty) \leq \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$. Thus $CR_1'(t) < 0$ for any $t \geq w$ and this results in $t_1^* = \infty$. Comparing $t_1^* = \infty$ and $t_2^* = w$, we have $CR_1(t_1^*) < CR_2(t_2^*)$. Consequently, $t^* = \infty > w$.

- When $Y(w) < \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} < Y(\infty)$

For $t \geq w$, there is a finite replacement age $t_1^* > w$ such that $CR_1'(t_1^*) = 0$ and $CR_1(t)$ is minimized. t_1^* is thus obtained from $Y(t_1^*) = \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$ and $w < t_1^* = t_0 < \infty$. Again, comparing t_1^* and t_2^* $(= w)$, it is easy to verify that $CR_1(t_1^*) < CR_2(t_2^*)$, and thus $t^* = t_1^* = t_0 > w$.

On the other hand, let $G(t) = Y(t) - \frac{C_p\bar{F}(t)+\rho C_p F(t)}{C_d}$. Since $Y(t)$ is an increasing function of t and $\frac{C_p\bar{F}(t)+\rho C_p F(t)}{C_d}$ is a decreasing function of t, we simply have $G'(t) > 0$.

Furthermore, $G(w) < 0$ and $G(\infty) = Y(\infty) - \frac{\rho C_p}{C_d} > Y(\infty) - \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} > 0$. There is a unique solution $\tau_1 \in (w, \infty)$ that satisfies $G(\tau_1) = 0$, i.e. $\tau_1 = Y^{-1}\left(\frac{C_p\bar{F}(\tau_1)+\rho C_p F(\tau_1)}{C_d}\right) > w$. Combining the definition of t_0, we have $Y(w) < Y(\tau_1) = \frac{C_p\bar{F}(\tau_1)+\rho C_p F(\tau_1)}{C_d} < \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} = Y(t_0)$. Consequently, $w < \tau_1 < t_0$. Note that τ_1 provides a lower bound for t_0 that is tight when $w \to \tau_1$, i.e. $\lim_{w\to\tau_1} t_0 = \tau_1$.

- When $Y(w) \geq \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$

Note that $CR_1'(t) > 0$ for any $t \geq w$, and $t_1^* = w$. Hence, the optimal solution is achieved at $t^* = t_1^* = t_2^* = w$.

2) $C_p < C_d/(1-\rho)$

- When $Y(\infty) \leq \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$

$CR_1(t)$ is minimized at $t^* = \infty > w$, which follows the same logic as 1). Since $Y(t) < Y(w)$ for $t < w$ and $Y(w) < \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} < \frac{C_p}{C_d-(1-\rho)C_p}$, we have $Y(t) < \frac{C_p}{C_d-(1-\rho)C_p}$ for $t < w$. Thus $CR_2'(t) < 0$ and $t_2^* = w$. Again, $CR_1(t_1^*) < CR_2(t_2^*)$. Hence, $t^* = \infty > w$.

- When $Y(w) < \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} < Y(\infty)$

$CR_1(t)$ is minimized at $t_1^* = t_0 > w$ and $t_1^* > \tau_1$, which follows the same logic as 1). Again, $CR_2'(t) < 0$ and $t_2^* = w$. Comparing t_1^* and t_2^*, it is easy to verify that $CR_1(t_1^*) < CR_2(t_2^*)$, and $t^* = t_0 > \tau_1 > w$.

- When $\frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d} \leq Y(w) \leq \frac{C_p}{C_d-(1-\rho)C_p}$

We have $CR_1^{'}(t) > 0$ for $t \geq w$ and $CR_2^{'}(t) < 0$ for $t < w$. Thus $t^* = t_1^* = t_2^* = w$.

- When $Y(w) > \frac{C_p}{C_d-(1-\rho)C_p}$

Note that $Y(w) > \frac{C_p}{C_d-(1-\rho)C_p} > \frac{C_p\bar{F}(w)+\rho C_p F(w)}{C_d}$. Thus, $CR_1'(t) > 0$ for $t \geq w$ and $t_1^* = w$. On the other hand, $CR_2'(t) = 0$ when $Y(t_2^*) = \frac{C_p}{C_d-(1-\rho)C_p}$ and thus $t_2^* = Y^{-1}\left(\frac{C_p}{C_d-(1-\rho)C_p}\right) = \tau_2 < w$. Comparing t_1^* and t_2^*, we have $CR_1(t_1^*) > CR_2(t_2^*)$, and $t^* = \tau_2 < w$.