OPTIMAL POLICIES FOR INVENTORY SYSTEMS WITH DEMAND CANCELLATION

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SUMMARY

An Inventory Network (IN), a logistics network focusing on inventory, comprises a set of inventories located in different regions connected via material flow, information flow, and cash flow. In practice, such network is commonly managed with its retailers to fulfill customers' demand via an advanced sales or reservation system. In practice, customers are often allowed to cancel their orders such as "money back guarantee". The majority of inventory models found in literature do not consider customers' cancellation despite being a commonly observed phenomenon. Ignoring cancellation can lead to the problems of over-estimating demands. Complicated and difficult to manage, such inventory system is becoming increasingly ubiquitous in today's globalized economy. The goal is to model inventory networks where the retailer faces demand uncertainties together with either an unreliable supplier, a capacitated supplier, or two simultaneous suppliers competing for procurement. The possibility of customers' cancellation is captured in these models where novel replenishment policies are analytically developed. The majority of industries appeal to the choice of "order-up-to" policy because of its simplicity. Our results show that such policy need not be optimal depending on suppliers' characteristics. Thus, our research offers a note of caution to guard against complacency in assuming that "order-up-to" is always optimal.

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CHAPTER 1

INTRODUCTION

The purpose of this chapter is to provide a foundational note to the motivation of this thesis. The outline of the thesis and the author's contribution will be presented. Furthermore, the alignment of this work with respect to the vision of Planning and Operations Management of enhancing the three core competency areas of modelling and analysis, operations research techniques, and heuristics techniques will be clarified.

1.1 Motivation

The trend of globalization is one of the key drivers enabling companies to strategically choose their suppliers, locate their manufacturing plants and warehouses that totally decouples from customers' base. According to a survey between July 2008 and July 2010 by comScore, Inc., six in ten consumers in United States feel that the internet has a profound impact on their purchasing decisions. Over the same period, it is found that consumers' loyalties to specific retailers have steadily decrease, while the likelihood to shop for deals online has risen over 8%. The total revenue generated via e-commerce up to Q2 of 2010 has risen by 7% compared to one year ago. According to an industry risk report, Best Buy, Inc. cites that global supply chain as one of its primary risks. "Our 20 largest suppliers account for over three fifths of the merchandize we purchase," the company writes in an annual report filed with the

SEC on May 2, 2007. Amazon.com and Barnes & Noble, Inc are good examples of companies which orchestrate supply chain networks that include internally operated distribution centers, diversely located warehouses and multiple suppliers to satisfy their worldwide customer base. These companies thrive on the basis of being able to provide greater convenience and price transparency for the consumers. This paradigm shift from the traditional "Brick and Mortar" to the "Click and Mortar" retailing is a result of human being's relentless desire for greater efficiencies, ushering in new levels of competition among online businesses never imagined previously. Due to the erosion of entry barriers for online retailers, even traditional "Brick and Mortar" companies are increasingly leveraging on the internet, leading to the prevalent practice of reservation. As more firms are employing web savvy operators to convert cyber-passerby into sales via clicking, allowing customers to cancel is becoming increasingly popular. Advertising campaign such as "money back guarantee" is common among online webshops to promote sales. Customers are usually given a limited amount of time to try a certain product and if they are not satisfied, a full refund can be given. Such risk-free promise on the part of the online retailer has motivated some customers to cancel their orders to try a different product. In the service industries among airline and hotel companies, the majority of bookings is reserved online. Customers are indemnified against the loss of non-refundable deposits when they cancel as a result of purchasing travel insurance such as "24Protect" and "HolidayGuard".

The scope of this thesis centers on modeling and optimizing inventory networks that includes the supplier, retailer, and random demand that allows customers to cancel their orders. One of the goals is to analytically derive optimal replenishment policies given the various suppliers' configurations. Specifically, we focus on three different problems by varying the different environment in which suppliers exists in the supply chain. The first problem we analyze is related suppliers' uncertainty. In the supply network, the deviation from the original order can be costly for the company. Such unreliability can be due to loss of items during transportation or pilferage within the network. The second problem involves the retailer facing two suppliers in which one of them is capacitated. Due to limited supply of raw materials, the retailer has to procure from an alternative but more costly source so as to meet customers' stochastic demand. In order to solve this problem, we extend previous work relating to the retailer entering into a transportation contract with the supplier. Finally, the third problem involves finding the retailer's optimal procurement and replenishment strategy for raw materials in the face of two suppliers competing in parallel. For all the three problems described, we are able to obtain the optimal replenishment policy for the single and multiple-period problem using cost as the objective function. We also try to develop algorithms that can potentially be useful for the industry.

SIMTech is a research institute that primarily engages in research that relates to manufacturing technology. One important role of POM is to encourage small medium enterprises (SME) to move up the value chain and to reap the benefits of knowledge-intensive manufacturing. The Singapore government has other notable and high profile efforts to turn Singapore into a high value manufacturing hub and supply chain nerve center. The IDA (Infocomm Development Authority of Singapore) has an initiative using info-communication technologies using a budget of $10 million RFID initiative was launched in 2004 and aims to build RFID-enabled supply chains by bringing together manufacturers, logistics service providers, retailers, and infrastructure providers. This is a move towards "High Value" manufacturing which involves the complex interplay of manufacturers' production process, inventory stocking strategies, marketing campaigns and service providing. The title "Optimal Policies for Inventory Systems with Demand Cancellation" per se can potentially have an extremely broad scope. In this thesis, the focus is to consider modelling and optimizing inventory networks under different supply environments. Specifically, we concentrate on deriving the optimal replenishment policies for minimizing the cost of managing the supply chain. In the light of our government's strong financial support for growth in knowledge-based high value manufacturing, it is hoped that the work in this thesis can play a role in enhancing SIMTech's capability in helping local enterprises.

1.2 Outline

All the models discussed in the thesis assume that the review policy is periodic and thus, the main tool used is Markov decision process. Furthermore, all customers' demand are stochastic, are reserved and can be canceled via a reservation system. This thesis is organized as follows: Chapter 2 provides an overview of the existing literature relating to inventory modeling. Chapter 3 discusses an inventory model whose supplier is unreliable in a multiple period framework. The focus of this work is to obtain the optimal replenishment policy in the presence of supply uncertainty. The impact of supply uncertainty is discussed so that its supply certain counterpart can be compared. Chapter 4 focuses on a model whose supplier is capacitated but additional procurement of raw materials or items for sale can be done via an alternative source. The optimal replenishment policy is derived for the single period, finite and infinite horizon cases. We will also highlight the technical differences in solving the optimal inventory policy between this model to the case when the supplier is unreliable. Chapter 5 considers the model in which procurement of raw materials is made via two suppliers which compete in parallel. The single and finite horizon models are presented. In addition to finding out the optimal quantity to order, the choice of the supplier is explicitly stated.

1.3 Contribution

Although ubiquitous in practice, demand cancellation and reservation has not been addressed in the vast collection of inventory literature until the pioneering work of Cheung and Zhang (1999) who explicitly model the cancellation phenomenon and evaluate its impact on inventory systems. In their work, they develop results by assuming stationary order-up-to and (s, S) policy. Later Yuan and Cheung (2003) address the fundamental issue of optimality. This thesis is an extension of the work of Yuan and Cheung (2003) by studying three inventory models that are not yet found in the current literature, to the author's best knowledge. In Yuan and Cheung (2003), supply of raw items is unlimited and no ordering costs is incurred. They show that optimal inventory policy is of an order-up-to type for the single, multiple and infinite horizon models.

Chapter 3 is the culmination of the work found in Yeo and Yuan (2011). Inspired by the work of Wang and Gerchak (1996) of using random yield to model uncertainty, Yeo and Yuan (2011) consider the impact of unreliable supplier on the optimal replenishment policy which turns out to be a critical point type. Yuan and Cheung (2003) assume that suppliers are reliable and their model is subsumed in the work of Yeo and Yuan (2011). The optimal inventory policy is of a critical point type. This is a more general form of policy which "collapses" to an order-up-to policy whenever there is no supply uncertainty, thereby, generalizing the result of Yuan and Cheung (2003). The impact of "stochastically" varying demand cancellation on the critical point and ordering quantity is studied. Specifically, if the demand cancellation has a lower expected value, it is always beneficial to order a larger quantity and the critical point is higher. It is also rigorously shown that the cost of managing the firm is always higher when the variance of the supply uncertainty and demand cancellation is higher.

Chapter 4 is adapted from Yeo and Yuan (2010b) extending the work of Yuan and Cheung (2003) to incorporate ordering costs into the inventory model. This work considers the inventory manager entering into a multi-tier supply contract with its supplier. The effect of introducing such a contract creates the tradeoff between ordering to limit stockout and additional cost incurred due to ordering. Such a transportation contract has been first considered in the work of Henig et al (1997) who did not take customers' cancellation into consideration. Mathematically, the model of Yeo and Yuan (2010b) in considering a multi-tier supply contract is also useful in a situation where the inventory manager faces multiple suppliers. In the two-tier scenario, the manager faces one supplier who rations a limited source of items at a lower ordering costs while the other supplier offers an unlimited, but is a more expensive source for procurement. Interestingly, the optimal policy of Yuan and Cheung (2003) with ordering costs can be deduced simply by using the single-tier version of Yeo and Yuan (2010b). The optimal inventory policy is derived for the single period, finite period and infinite period horizon models. Similar to the approach in Chapter 3, the convexity (in the initial inventory level) for optimal cost during each period is proven. However, there are some technical differences in order to establish the optimality for infinite horizon case. This is due to the presence of ordering costs. To overcome this, I appeal to the proof of Theorem 8-14 of Heyman and Sobel (1984). Some modifications are required as their formulation of functional equations developed is single variable and based on maximization, while I consider bivariate equation and this model involves cost minimization.

Chapter 5 is an extension the work of Henig et al (1997) to consider the impact of an additional supplier on the structure of the optimal inventory policy when the other enters into a supply contract. In the presence of two suppliers competing in parallel and offering two different types of supply contracts, my goal is to prove a novel replenishment inventory policy for the multiple period model. Instead of convex cost function in Chapter 3 and Chapter 4, the ordering cost turns out to be concave. Interestingly, the first period cost function exhibits quasi-convexity and its first order derivative is single-crossing in the initial inventory level. The proof of optimality vastly differs from the two previous models as our optimal cost function is quasi-convex. It is well-known that quasi-convexity is not necessarily closed under the sum of two quasi-convex functions. I apply the theory of aggregating single-crossing functions that is recently developed by John and Bruno (2010) to prove the optimal inventory policy.

CHAPTER 2

LITERATURE REVIEW

Inventory theory is viewed as the scientific rationalization of management decisions which falls under broad disciplines such as “operations research” and “management science”. One of the greatest impetuses of inventory theory seems to have arisen during the early twentieth century when manufacturing firms produced items in lots sizes with huge setup costs. Most inventory studies are dedicated to finding out the amount of inventory to stock at the beginning of each period (month, year etc) so as to satisfy future customers’ demand. If the problem is related to production, the inventory problem becomes determining the amount of raw materials to order or to procure so as to meet production schedules requirement. Such practical interests has led to a concentration of combined research efforts of prominent economists and mathematicians leading to the “Stanford Studies ” which is the landmark for the development of inventory theory. Two seminal works that serve as the starting points in that famous “Stanford Studies” are the “Arrow-Harris-Maschak” and “Dvoretzky-Kiefer-Wolfowitz” papers (see Arrow et al (1951) and Dvoretzky et al (1952)). The classical work of Arrow, Harris and Maschak investigates many aspects of inventory theory. Their models take into account of issues under which demand is deterministic, single-period models with random demand, and general dynamic inventory models. The cost is composed of two parts: a set-up cost, which is incurred whenever an order is placed; and a unit cost that is proportional to the size of the order. At that time, the optimality of the (s, S) policy is not known but they restrict their attention to this particular form so as to compute the discounted cost and discuss the selection of the (s, S) pair. An inventory policy of two-bin or (s, S) type is defined as follows: order only when the present level of inventory falls below some given value s and the level of stock is brought up to S after ordering. An inventory policy also known as "order-up-to" is characterized by a sequence of numbers $y_1, ..., y_n$ as follows: if the inventory on hand plus on orders is x_i at the beginning of period i is less than or equal to y_i, then order y_i; otherwise do not order. This policy is the special case when $s = S$. Veinott refers the "order-up-to" policy as the "base-stock policy" and the sequence $\{y_n : n \geq 0\}$ is the base stock level for period n. Arrow et al (1951) popularize the functional equation method in mathematical inventory problem by focusing on a special type of policy where the solution is examined in full generality by Dvoretzky et al (1952). Later Karlin extends the work of Arrow et al (1951) by considering demand density of Polya-type and contribute two chapters in the classical compilation of Arrow, Karlin and Scarf (1958) (see chapter 8 and 9). As for the development of theory for the infinite horizon inventory problem, the work of Bellman, Glicksberg and Gross (1955) is instrumental and most accessible. They show the existence, uniqueness and convergence of its successive approximation of the solution to the infinite stage functional equation. For a good treatment to the origins of modern inventory theory and its connections with the famous "Stanford Studies", one can refer to the work of Girlich and Chikan (2001).

The optimality of the (s, S) policy is first established in the foundational work of Clark and Scarf (1960) who analyze a multi-echelon inventory model. A supply chain consisting of multiple stages with a serial structure is considered. They prove the optimality of order-up-to policies based on the inventory positions (stock on hand plus stock on order, regardless of delivery dates) in the absence of fixed cost and ordering costs. In the presence of fixed costs, (s_i, S_i) is the optimal policy for each period i. For multi-echelon systems up to N stages, the optimal policy is either a vector of re-ordering points $(S_1, ..., S_N)$ or a vector $[(s_{i,1}, ..., s_{i,N}), (S_{i,1}, ..., S_{i,N})]$. The discounted cost criterion is used as a performance measure. In this seminal work, the concept of $K-$ convexity is first introduced to solve the problem. The basic model of Clark-Scarf (1960) has been extended along various directions. The optimality of (s, S) for the infinite horizon model has gathered considerable attention. Inglehart (1963) provides bounds for the pair of critical numbers and discuss the convergence of sequences $\{s_n\}$, $\{S_n\}$. The existence of a limiting (s, S) policy is given in the infinite horizon setting is given as well. The proof of optimality in the work of Clark-Scarf (1960) hinges on the loss function being convex in the initial inventory level. In many practical situations, this assumption may not be appropriate. To overcome this difficulty, Veniott (1966) offers a different yet elegant proof for the optimality of the (s, S) policy by relaxing the loss function to be quasi-convex. Kaplan (1970) considers stochastic lead-time for a periodic review finite horizon problem. By assuming that the orders do not cross over, they are able to apply the state space reduction techniques to prove the optimality of (s, S) policy in the presence of fixed cost. Later, Ehrhardt (1984) extends Kaplan's work to the infinite horizon setting. He provided sufficient conditions under which stationary base stock policies and (s, S) in the presence of lead-time are optimal. Finally, Zheng (1991) provide a simple proof for the optimality of (s, S) policy. The issue of formulating efficient algorithms for the (s, S) policy is also an active research field. Veinott and Wagner (1965), later Bell (1970) develop an efficient method of computing the optimal parameters for finding the (s, S) policy using renewal theory. Archibald and Silver (1978) considers the continuous review inventory problem with compound Poisson arrivals. The optimality of (s, S) replenishment policy is proven for their inventory system. They develop a recursive formulation to compute the cost for any pair of (s, S). Tighter bounds for the quantity $S - s$ than that of Veinott and Wagner (1965) are developed for the periodic review case. Later, Zheng and Federgruen (1991) develop an algorithm that achieves even greater computational efficiencies than that of Veinott and Wagner (1965) and Archibald and Silver (1978). Another notable work involves the investigation of the impact of (s, S) policy on the macroeconomic level by Caplin (1983). The main objective is to describe the economy-wide behavior of inventories by the aggregation of a vast number of individual optimizing decisions. He shows that adopting (s, S) policies increases the variability of demand, with the variance of orders exceeding the variance of sales.

In the wide range of literature surveyed, the scope of inventory problems can be confined to a few main themes according to a classification given by Silver (1981).

- Single vs. Multiple Items
- Deterministic vs. Probabilistic Demand
- Single Period vs. Multiperiod
- Stationary vs. Time-Varying Parameters
- Nature of the Supply Process
- Procurement Cost Structure
- Backorders vs. Lost Sales
- Shelf Life Considerations
- Single vs. Multiple Stocking Points

As it is almost impossible to summarize the enormous literature on inventory models inspired by the pioneering works of Arrow et al (1951) and Dvoretzky (1952), I have chosen to focus on existing works that sought to characterize the form of optimal replenishment strategies for inventory models that have periodic review policy. All works apply dynamic programming and aim to find the tradeoff between costs of holding inventory and stock out possibility via minimization. Unless explicitly stated, all the models surveyed are confined to multiple period and/or infinite period problems.

2.1 Inventory Models with Multiple Class Customers

Veinott (1965a;b) considers a class of multi-period inventory problem in which there are several demand classes for both single and multi-product. He assumes there is a fixed lead-time and the objective is to minimize the discounted cost criterion using discounting factors that varies for each period. One important contribution in Veniott's (1965a;b) works is providing conditions under which a stationary base stock policy is optimal. Topkis (1968) considers an inventory model with several prioritized demand classes. The penalty cost is lower when a relatively lower class customer is being rejected to satisfy the larger class customer. He shows that under certain conditions the optimal ordering policy is characterized as a base stock policy and the optimal rationing policy can be specified by a set of rationing levels. Sobel and Zhang (2001) consider a finite horizon periodic review inventory system, with non-stationary demand arriving simultaneously from a deterministic source and a random source. The deterministic demand has to be satisfied immediately and the stochastic demand can be backlogged. They prove that under certain conditions, a modified (s, S) policy in which s is dependent on the deterministic demand of the current period. Frank, Zhang and Duenyas (2003) study a similar periodic review inventory system in which one source is deterministic while the other is stochastic. However, the units of stochastic demand that are not satisfied during the period when demand occurs are treated as lost sales. At each decision epoch, one has to decide not only whether an order should be placed and how much to order, but also how much demand to fill from the stochastic source. They prove the optimality of the (s, k, S) policy where k is the rationing decision variable for the stochastic demand. Chen and Xu (2010) show that the condition in Sobel and Zhang (2001) can be relaxed and the optimality of the (s, S) policy still holds.

2.2 Inventory Models with Multiple Suppliers

It is commonplace that inventory problems are often concerned with two types of suppliers, a “regular” and “emergency” supplier with different unit prices of ordering and different leadtimes. Barankin (1961) initiated the study of the optimal policy for dual supply sources for the single period problem which is extended to the multiple period case by Fukuda (1964). He prove the existence of two parameters $y^0 < y^1$ such that if the stock on hand is less than y^0, then order up to the base stock level at the emergency mode and $y^1 - y^0$ at the regular mode, otherwise the optimal policy is a base stock policy at the regular delivery mode. The difference between the leadtimes of the expedited and regular source is one. Daniel (1963) and Neuts (1964) show the optimality of order-up-to polices for the case when leadtime for the emergency and regular suppliers are zero and one period, respectively. Porteus (1971) considers a single product, periodic review, stochastic inventory model when the ordering cost function is concave increasing rather than simply linear setup cost. He introduces the concept of quasi-K-convex functions. Such functions are extensions of both K-convex functions and quasi-convex functions of a single variable. For the a finite number of suppliers, he has shown that the optimal policy is of the generalized (s, S) form. Let m be a fixed integer such that there exists a set of numbers $s_m \leq s_{m-1} \leq ... \leq s_1 \leq S_1 \leq S_2 \leq ... \leq S_m$. Let x be the initial inventory level. A decision rule is called generalized (s, S) if

$$\delta(x) = \begin{cases} S_m & \text{if } x < s_m \\ S_i & \text{if } s_{i+1} \leq x < s_i \text{ for } i = 1, 2, ..., m-1 \\ x, & \text{otherwise.} \end{cases}$$

Whittemore and Saunders (1977) study the dual sourcing problem when the difference between regular and emergency leadtime is arbitrary and the holding and shortage cost functions are allowed to be nonlinear. They derive sufficient conditions under which only one supplier is used in the infinite horizon discounted cost problem. But the form of the optimal policy is extremely complicated. Chiang and Gutierrez (1997) analyze an inventory model whose review period is larger than the supply leadtimes of both suppliers. Two types of orders can be placed at the regular review and emergency epochs. They determine the optimal policy for placing orders at the different epochs. Yang et al (2005) consider an inventory model with Markovian in-house production capacity, facing stochastic demand and having the option to outsource. They show that the optimal outsourcing policy is always of the (s, S) type and the optimal production policy is of the modified base-stock type under fairly general assumptions. Frederick (2009) develops a model for multiple sources of supply. He assumes that when the initial inventory exceeds a certain critical level, the manager will return or "order down to" an optimal quantity of inventory at no additional cost. Under the single, finite and infinite horizon period, he prove the optimality of the "finite generalized Base Stock" policy for the discounted cost criterion. By using a vanishing discount approach, he proves the optimality of inventory policy for the average cost criterion as well. The mathematical model considered in his work is a generalization of Henig et al (1997) who study a supply contract embedded in an inventory model. Sethi, Yan and Zhang (2003) analyze a system where there are two delivery modes (fast and slow) with a fixed cost for both the fast and slow orders. The decision variables are the replenishment quantities from the fast and slow mode of deliveries. The information available for making such decisions are based on initial demand forecast, periodical demand forecast updates and the realized customers' demand. They prove the optimality of the (s, S) policy for the finite horizon period when the demand process is non-stationary. Feng et al (2006) analyze a periodic review inventory problem and question the validity of "order-up-to" policy for three or more suppliers when their leadtimes are consecutive integers. For multiple consecutive delivery modes, they have shown that only the fastest two modes have optimal base stocks while the rest do not by means on counter-example. Anshul et al (2010) show that sourcing of two suppliers is a generalization of the "lost sales" models of Karlin and Scarf (1958). They propose and generalize the class of dual index policies by Veeraraghavan and Scheller-Wolf (2008), which has an order-up-to structure for the orders placed on the emergency supplier as well as for the orders placed on the regular supplier. They provide analytical results that are useful for determining optimal or near-optimal policies within the class of policies that have an order-up-to structure for the emergency supplier.

2.3 Markov-Modulated Inventory Models

Most classical inventory models assume demand in each period to be a random variable independent of environmental factors other than time. With the business environment in the manufacturing industry getting more unpredictable, it is more relevant to consider demand being subjected to a fluctuating environment due to changing economic conditions. For such situations, the Markov chain approach provides a natural and flexible alternative for modeling the demand process. Karlin and Fabens (1960) analyze an inventory model where the demand process is modulated by a Markov chain. They postulate the optimality of (s, S) type policy given the Markovian demand structure in their model but did not give a proof explicitly. The work mainly focus on optimizing the two parameters s and S that is independent of the state. Iglehart and Karlin (1962) study an inventory model in which the distribution of demand in a period depends on the state of the environment and it follows a Markov chain. They also assume that the (s, S) policy and develop algorithm to compute the parameters. Kalymon (1971) studies a multiple-period inventory model in which the costs are determined by a Markovian stochastic process. He is the first to prove the optimality of the (s, S) policy where the parameters depends on the price. Parlar et al (1995) consider an inventory where the availability of the supplier forms a Markov chain. In their paper, the supply state takes two values of either "available" or "unavailable". They show the optimality of the (s, S) policy in the presence of a fixed cost. Cheng and Sethi (1997) extends the work of Karlin and Fabens (1960) by proving the optimality of environment-dependent (s, S) policy with a fixed cost and non-stationary demand for the finite and infinite horizon models. Later, Beyer and Sethi (1997) and Beyer et al. (1998) extends the work of Cheng and Sethi (1997) by considering unbounded demand and general costs including lower-semicontinuous surplus cost with polynomial growth. Ozekici and Parlar (1999) consider an infinite horizon inventory control problem whose supply is unreliable and its parameters (such as holding costs, demand and supply) are dependent on the environment. Using dynamic programming, they show that the environment-dependent order-up-to policy is optimal in the absence of fixed costs. However, when there is a fixed cost, the structure of the optimal replenishment policy is of environment dependent (s, S) type. These results hold for both the finite and infinite horizon models. Erdem and Ozekici (2002) extend the work of Ozekici and Parlar (1999) by considering inventory models where supply is always available but with random yield. In their model, yield is the result of supplier's uncertain capacity to fulfill where the supply and the demand processes are modulated by a Markov chain that depicts the state of the environment. The optimal policy is the well-known base-stock policy where the optimal order-up-to level depends on the state of the environment. They compare the result with that of a supplier whose capacity is unconstrained. Arifoglu and Ozekici extend both Ozekici and Parlar (1999) and Erdem and Ozekici (2002) by considering a more general framework in which there is a supplier with random capacity and a transporter with random availability. As a result of their analysis, they show that an environment-dependent base-stock policy is optimal. Srinagesh (2004) considers an inventory model in which the purchasing cost forms a Markov chain, from one period to the next. He shows that the base-stock policy is optimal. Gallego and Hu (2004) extends the work of Parlar et al (1995) by considering random yield (see Section 2.4 for a discussion) and demand that are Markov-driven, with limited capacity. They show that the optimal production and ordering policy is a modified state-dependent "inflated base-stock" policy. This means that the optimal production/ordering quantity for each period is decreasing with respect to the initial level and the optimal order-up-to level is decreasing with respect to the initial level. The term inflated base-stock policy was coined by Zipkin, see (Zipkin, 2000, p. 392). Arifoglu and Ozekici (2010) extend the work of Gallego and Hu (2004) by considering environment that is only partially observable via the use of POMDP or “partially observed Markov decision process”. They show that the optimality of state-dependent modified “inflated base-stock” policy still holds. The work of Yang et al (2005) who consider an inventory model with Markovian in-house production capacity also falls into this subcategory, see Section 2.2. Papacritos and Katsaros (2008) investigate the optimal replenishment of a periodic-review inventory model in a fluctuating environment with a fixed lead-time. They model the environment at the beginning of each period as a homogeneous Markov chain. Furthermore, the model takes into account of supplier’s uncertainty for their capacity level is also modulated by a Markov chain. The ordering, holding, and penalty costs are state-dependent. The results are proven for the finite and infinite horizon and the structure of the optimal replenishment policy is in the form of an environment-dependent order-up-to level policy.

2.4 Inventory Models with Supply Uncertainty

The influence of supply uncertainty has been studied and its impact on the replenishment strategy of stochastic inventory control problem has been considered. “Supply reliability” is a collective term referring to various factors that may contribute to a less reliable supply, including production yield and quality problems, insufficient capacity allocation due to scarce supply, theft, and store execution errors. Any combination of these factors limits the ability of the retailer to put an appropriate amount of stock on store shelves when demand arrives. These common supply chain glitches cause the quantity delivered by the supplier to be deviated from the original order. Henig and Gerchak (1990) introduce the concept of random yield in a production environment and imperfect production process results in some of the processed items becoming defective. The stochastically proportional yield model is used. This means that random yield is the product of the chosen production level and a random multiplier, called the yield rate (independent of the production level). Henig and Gerchak (1990) study a periodic review model in which actual order received is a random size bounded above by the lot size. The optimal policy is the so-called nonorder-up-to" policy defined by a critical inventory level under which an order is given. But, the order quantity does not necessarily bring the inventory position to a fixed base-stock level. Therefore, random yield models do not necessarily lead to nice characterizations on the optimal policy. Henig For a good review of how random yield is considered in the modeling of inventory problem, one can refer to the work of Yano and Lee (1995). Another study is that by Ciarallo et al. (1994) where the problem is similar to Henig and Gerchak (1990), except that the random yield is the consequence of random capacity with a known distribution function. The optimal policy is a base-stock policy where the order-up-to level is a constant as the objective function is quasi-convex. Later, Wang and Gerchak (1996) study and derive the optimal policy for the inventory model under the influence of both variable production capacity and random yield (i.e. processes which caused the manufacturing of unusable items). Variable production capacity and random yields are two main categories of supply uncertainty. They study the optimal policy for the finite and infinite horizon model but the structure is not an order-up-to policy. Erdem and Ozekici (2002) extend the work of Hernig et al (1990) by considering Markov modulated yield of unreliable supplier. From Section 2.2, the work of Yang et al (2005) who consider an inventory model with Markovian in-house production capacity also falls into this subcategory. Chao et al (2009) study a capacity expansion problem of a service firm (subscription-based service) which faces three issues: demand variability, (existing) capacity obsolescence and deterioration, and capacity supply uncertainty. This firm has to decide on the capacity expansion for its customer base in the face of uncertain supplier. The firm has the options to use futures contract to secure delivery. Using futures, the optimal capacity expansion policy for the current period is determined by a base-stock policy. The result is compared when no futures contracts are used.

2.5 Reverse Logistics and Remanufacturing Models

Reverse logistics is defined as the management of returned merchandize whose material flow is opposite to the conventional supply chain so as to re-salvage its value by making it reusable or ensuring proper disposal. Remanufacturing and refurbishing activities also may be included in the definition of reverse logistics. In recent times, customers are getting more environmentally conscious and coupled with enhanced legislation, the roles of manufacturers ensuring proper handling of take-backs has increased significantly. A good review of this growing trend is addressed in the work of Fleischmann et al (1997).

Cohen et al (1980) deal with a periodic review inventory system where a constant proportion of stock issued to meet demand each period feeds back into the inventory after a fixed number of periods. They assume that a fixed share of the products issued in a given period is returned after a fixed leadtime and on hand inventory is subject to proportional decay. Demands in successive periods are assumed to be independent identically distributed random variables. This model is an extension of a simple stochastic inventory model with proportional costs only, but with a consideration for reusable items. The objective is to optimize the trade-off between holding costs and shortage costs. Under certain assumptions, an "order-up-to" policy is optimal. Simpson (1978) proposes a first product recovery model explicitly considering distinct inventories for serviceables and recoverables. The basic solution methodology is a backward dynamic programming technique in two dimensions with the Kuhn-Tucker saddle point theorem applied in every stage. The structure of an optimal policy is based on three dependent parameters: the repair-up-to level, the purchase-up-to level and the scrap-down-to level. However, neither fixed cost nor leadtimes are involved. Inderfurth (1996, 1997) extend the work of Simpson (1978) by considering the effects of non-zero leadtimes for orders and remanufacturing. The activities of procurement, remanufacturing, and disposal are charged with linear costs, but fixed costs are not considered. He shows that a decisive factor for the complexity of the system is the difference between the two leadtimes. The model in the work of Simpson (1978) is a special example when the two leadtimes are identical. In fact, for identical leadtimes, the model is similar to the work of Cohen et al (1980), but has been extended by a disposal option. The optimal policy obtained has a two parameters "order-up-to", "dispose-down-to" policies. In the case where ordering leadtime exceeds the remanufacturing leadtime, the curse of dimensionality of the underlying Markov model prohibits simple optimal control rules. Fleischmann and Kuik (1998) provide another optimality result for a single stock point. They show that a traditional (s, S) policy is optimal if demand and returns are independent, recovery has the shortest lead 3 time of both channels, and there is no disposal option.

DeCroix (2006) extends the work of Clark-Scarf (1960), Simpson (1978), and Inderfurth (1997) by analyzing a multi-echelon inventory system with inventory stages arranged in series. In addition to traditional forward material flows, used products are returned to a recovery facility, where they can be stored, disposed, or remanufactured and shipped to one of the stages to re-enter the forward flow of material. His objective is to determine to what extent can the optimal policy for managing a multi-echelon inventory system that includes the reverse flows due to product recovery and remanufacturing be derived based on Clark-Scarf method of stochastic decomposition. The problem is solved via decomposing it into a sequence of single-stage problems, and the optimal policy for each single-stage problem has a fairly simple structure. The optimal policy for managing the system is simply a combination of the optimal policies for managing a traditional series system without remanufacturing and a single-stage system with remanufacturing. Huang et al (2008) consider the impact of warranty on the optimal replenishment of a single-product inventory. The firm faces demand from two sources: demand for new items and demand to replace failed items under warranty. Demands for new items in different periods are independent and the demands for replacing failed items depend on the number and ages of the items under warranty. Using an appropriate choice for the terminating cost, the optimal replenishment policy is a stationary warranty dependent order-up-to policy. The choice of warranty policy in their model is the free replacement warranty.

The above literatures considers only one core product which is defined as the condition of the returned products, ranging from slightly used up to significantly damaged. Zhou et al (2010) extends the above work by considering a remanufacturing inventory model with possibly of multiple cores. In particular, they show that the optimal manufacturing, remanufacturing, disposal policy has a simple structure and is characterized by a sequence of constant parameters when the holding and disposal costs for all types of cores are the same.

2.6 Inventory Models with Advanced Demand Information

Customers with positive demand leadtimes place orders in advance of their needs results in advance demand information. Such research evolve as a result of risk averse consumers who want to minimize the risks of disappointment that are frequently observed in the service and retailing industries. Examples include airlines selling discount tickets to advance purchase customers, hotels selling discount rooms to advance booking. In Hariharan and Zipkin (1995), perfect demand information is assumed over the demand leadtime, i.e., every single unit of demand reserved will be realized. They study a model of a supplier who uses a continuous-review order-base-stock replenishment policy to meet customer orders that arrive according to a Poisson process. Each customer order is for a single item to be delivered a fixed demand lead-time following the order. DeCroix and Mookerjee (1997) consider a problem in which there is an option of purchasing advance demand information at the beginning of each period. They consider two levels of demand information: Perfect information allows the decision maker to know the exact demand of the coming period, whereas the imperfect one identifies a particular posterior demand distribution. They characterize the optimal policy for the perfect information case. Gallego and Ozer (2001) analyze an inventory system where advanced demand information is known up to some known period in the future. This vector of information is random and only realized some periods later. In the presence of a fixed cost, the structure of the optimal replenishment policy is state-dependent (observable part of the ADI) (s, S) policy. In the absence of the fixed cost, we have a base-stock policy. Gallego and Ozer (2003) extend the work of Gallego and Ozer (2001) to the multi-echelon inventory system. Using the modified inventory position concept introduced in Gallego and Ozer (2001), they obtain state-dependent, echelon base-stock policy for managing the inventory. When demand and cost parameters are stationary, they show that myopic policy is optimal for the finite and infinite horizon models. Wang and Toktay (2008) incorporate flexible delivery into ADI whereby customers are willing to accept orders which comes earlier than expected. They show that the optimal inventory policy is state-dependent (s, S) policy when the leadtimes of all the customers are identical. They also consider the case when customers are differentiated by demand leadtimes. However, they did not solve for an optimal policy but propose a tractable approximation and implementable heuristics.

2.7 Inventory Models with Demand Cancellation

It is a prevalent practice to sell a product through a reservation or advance sales system where cancellation of orders is allowed. During a demand leadtime, there are many reasons why cancellation is legitimate from the consumers' perspective. It is possible to extend the ideas of perfect demand information in Hariharan and Zipkin (1995) to take into account of demand cancellation. The class of inventory models where customers are allowed to cancel their orders received considerably less attention despite being commonly observed in practice. The work of Cheung and Zhang (1999) explicitly model the cancellation phenomenon and evaluate its impact on inventory system based on assuming stationary order-up-to and (s, S) policy. Yuan and Cheung (2003) address the fundamental issue of optimality in the periodic inventory model where the ordering policy is affected by the reservation and customers' cancellation. In their model, demand are reserved by a lead-time of one period and demand are satisfied, but could be canceled at a random fraction. They show that the order-up-to policy is optimal whose re-order point is dependent on the reservation parameter. Tan, Gullu and Erkip (2007) extend the work of Zipkin and Hariharan (1995) by considering the impact of imperfect advanced demand information. Similar to Yuan and Cheung (2003), the information needed to make ordering decisions is based on on-hand inventory and advanced demand information. However, the time to demand realization is greater than one. After one period, there is a fixed probability p that this demand will be realized during demand leadtime. They prove that the optimal policy is an "order-up-to" policy that is dependent on the given size of ADI. Gayon et al (2009) consider a make-to-stock supplier (facing customer of multiple classes) that operates a production facility with limited capacity. Customers share imperfect ADI with the supplier because there is a possibility of order dates not known exactly and orders can be canceled by customers. Assuming Poisson demands, they formulate the problem as a continuous time MDP with finite transition rates. Using uniformization technique of Lippman (1975), they transform the continuous time decision process into an equivalent discrete time decision process. The optimal production policy consists of a base-stock policy with state-dependent base-stock levels, where the state is determined by the inventory level and the number of announced orders from each class. The optimal inventory allocation policy consists of a rationing policy with state-dependent rationing levels such that it is optimal to fulfill orders from a particular class only if the inventory level is above the rationing level corresponding to that class.

CHAPTER 3

OPTIMAL INVENTORY POLICY WITH SUPPLY UNCERTAINTY AND DEMAND CANCELLATION

3.1 Introduction

This chapter considers a single item, periodic review inventory model where demand is reserved and customers are allowed to cancel their orders, at the same time, the supplier is unreliable. Our objective is to derive optimal inventory policy for such a system. In our model, we do not consider penalty on the customers whenever they cancel their orders.

3.2 Literature Review

Generally, there is a dearth of literature considering the impact of demand cancellation on the optimal ordering policy of the inventory model. Cheung and Zhang (1999) study the impact of cancellation of customer orders via assuming an (s, S) policy and Poisson demands. They develop a Bernoulli type cancellation behaviour in which a reservation will be canceled with probability p. In addition, the timing to cancellation is considered. In particular, they show that a stochastically larger elapsed time from reservation to cancellation increases the systems penalty and holding costs. Yuan and Cheung (2003) consider a periodic review inventory model in which all demands are reserved with one-period leadtime, but orders can be canceled during the reservation period. They formulated a dynamic programming model and show that the order-up-to policy is optimal. You and Hsieh (2007) develop a continuous time model to determine the production level and pricing decision by considering constant rate of demand cancellation. They formulate a system of differential equations for inventory level so that holding and penalty costs can be calculated. However, they did not address the impact of cancellation on the optimal cost of managing the system. You and Wu (2007) consider a joint ordering and pricing decision problem where both cancellation and demand (price-dependent) are deterministic. Their aim is to maximize total profit over a finite time planning horizon by determining the optimal advance sales price, spot sales price, order size, and replenishment frequency over a planning horizon.

On the other hand, supply uncertainty is one of the common supply chain glitches whereby the quantity delivered by the supplier may be deviated from the original order. Such loss of items can be due to strikes, misplacement of products, or incorrect shipment quantities on the supplier's side. The topic of supply uncertainty has been included in stochastic inventory models in the following ways. Wang and Gerchak (1996) use the concept of random yield to model supply uncertainty. In their work, random yield is the fraction in which the manufactured quantity turns out to be usable. They derive the optimal policy for the inventory model under the influence of both variable production capacity and random yield. They study the optimal policy for the finite and infinite horizon model and the structure is not an order- up-to policy. G*ü*ll*ü* et al (1999) consider the the supply uncertainty using Bernoulli process in which either the supply arrives or not. In other words, the quantity ordered either arrive or do not arrive. They study a periodic review model and obtain a non-stationary order-up-to policy. However, they assume that the demand in each period is deterministic.

Li and Zheng (2006) characterize the structure of the optimal policy that jointly determines the production quantity and the price for each period to maximize the total discounted profit, in the presence of random yield and stochastic demand. Using price-dependent demand function which is additive, they show that a threshold type policy is optimal. Furthermore, the optimal price decreases in the starting inventory. Following closely to the work of G$\ddot{u}$ll$\ddot{u}$ et al (1999), Serel (2008) develops a single period model to identify the best stocking policy for a retailer with uncertain demand and supply. Finally, Liu et al (2010) consider the impact of supply uncertainty on the firm's performance under joint marketing and inventory decisions. They develop a single period model showing that reducing variance of supply uncertainty improves the firms' profit. Rather than focusing on the structure of the optimal policy, their aim is to derive managerial insights based on firm's willingness to pay for reducing supply uncertainty.

In this chapter, we will consider the effect of supply uncertainty or yield on the optimal inventory policy with demand cancellation. To our best knowledge, no research has been done to address demand cancellation and supplier uncertainty concurrently. One main contribution of our work is to show that the optimal inventory policy with supply uncertainty shares similar structural properties as that with supply certainty for both the finite horizon case and the infinite horizon case. In particular, we show that due to the presence of supply uncertainty, the optimal inventory policy is characterized by a re-order point. Furthermore, we show that this re-order point is independent of the supply uncertainty factor. Gerchak et al (1988) derive a similar policy which they call it the "critical point" policy. Their work features a production model with yield uncertainty and stochastic demand. However, their objective function is the profit function. Wang and Gerchak (1996) also show a similar inventory policy but their critical point is dependent of supply uncertainty ("random yield factor"). We also establish the fact that the expected cost of managing a firm is higher when its supply uncertainty has a relatively larger variance. Interestingly, we show that a more variable yield distribution does not necessarily increase the optimal ordering quantity due to the influence of cancellation. Similarly, we also prove that it is less costly if the firm is to reduce the variance of cancellation behaviour. However, reducing the frequency of demand cancellation does not necessarily translate to cost reduction for the firm. Therefore, we can only develop a bound on the difference between the optimal cost in the presence of differing cancellation behaviour. It turns out that the bound is proportional to the difference between the mean number of items not eventually canceled.

The rest of the chapter is organized as follows. The model and notations are developed in Section 3.3. Section 3.4 presents a model for the single period. The convexity for the optimal cost is established and the optimal ordering level is derived. We show that reducing the variance of either the distribution of yield or the distribution of demand cancellation leads to a lower cost of managing the supply chain. Section 3.5 is similar to Section 3.4, but explores the finite horizon case. In Section 3.6, we discuss the infinite horizon model and solve the optimal policy. We also show that the cost of managing a firm is higher when the distributions of demand cancellation and yield are more variable in the sense of convex ordering. In Section 3.7, we provide numerical evidences to our observations made in earlier sections. We also propose an algorithm to obtain the optimal ordering quantity. An example is given using our proposed algorithm. Finally, we provide a concluding note with some possible extensions to this work in Section 3.8.

3.3 Model

Consider a periodic review inventory system. All demands are made through reservations. Demands reserved in the previous periods are supposed to be fulfilled in the current period. However, due to customers' indecisiveness, demands may be canceled. Suppose $\mathbf{N}$ is the set of non-negative integers. Let D_n be the demand that is reserved during period $n \in \mathbf{N}$, and let R_n be the ratio of the demand reserved during the previous period that is eventually not canceled during period n. Finally, $1-\theta_n$ is the supply uncertainty factor, where θ_n represents the ratio of items that is received after an order has been made during period n. If production is involved, then θ_n can be interpreted as the yield ratio during period n. If $\theta_n = 1$ with probability one, then there is no supply uncertainty. We assume that $\{D_n : n \in \mathbf{N}\}$ is a sequence of i.i.d demand random variables with a common distribution $H(x)$ (with $H(0) = 0$ and $H(\infty) = 1$), density function $h(x)$, and mean ζ. We let $\{R_n : n \in \mathbf{N}\}$ be a sequence of i.i.d ratio random variables whose c.d.f is $G(x)$ (with $G(0) = 0$ and $G(1) = 1$), density function $g(x)$, and mean γ. Similarly, we let $\{1 - \theta_n : n \in \mathbf{N}\}$ be a sequence of i.i.d supply uncertainty in each period. If $\theta_n \stackrel{\mathrm{d}}{=} \theta$ is a random variable, then we write its c.d.f as $F(x)$(with $F(0) = 0$ and $F(1) = 1$) and its p.d.f as $f(x)$.

We also make the assumption that cancellation ratios R_n, demands D_n, and supply uncertainty factors $1-\theta_n$ are independent of each other. All the unfulfilled orders are backordered. The inventory holding cost (h) and penalty cost (p) are both incurred on a per unit per unit time basis. At the beginning of a period, the inventory level is x and the demand reserved in the previous period is $z(> 0)$. Let y be the decision variable representing the order quantity made at the beginning of the current period. Define $[x]^+ = \max\{x, 0\}$ and $[x]^- = \max\{-x, 0\}$. The leadtime is assumed to be zero. Suppose θ is the current period supply uncertainty, then θy is the amount that is available to fulfil the demand, thus the one period cost can be written as

$$\begin{aligned}\varphi(x,y,z) &= hE[x+\theta y - zR]^{+} + pE[x+\theta y - zR]^{-} \\ &= h\int_0^1 \int_0^{\frac{x+sy}{z}} (x+sy-zt)dG(t)dF(s) \\ &\quad + p\int_0^1 \int_{\frac{x+sy}{z}}^1 (zt-x-sy)dG(t)dF(s). \end{aligned} \tag{3.1}$$

Following Yuan and Cheung (2003), we let $C_n(x,z)$ be the optimal total cost from period n to period 1 given that the initial inventory level is x and the demand reserved in period $n+1$ is z. We define the cost when there are no periods left to be $C_0(x,z) \equiv 0$ for all x, z. Suppose D is the demand that arrives during period n, and $\alpha \in [0,1)$ is the discount factor. Then,

$$C_n(x,z) = \min_{y\geq 0}\{\varphi(x,y,z) + \alpha E_D E_{\theta,R} C_{n-1}(x+\theta y - zR, D)\}. \tag{3.2}$$

Set $\Phi_n(x,y,z) = \varphi(x,y,z) + \alpha E_D E_{\theta,R} C_{n-1}(x+\theta y - zR, D)$. From (3.1), we have $C_n(x,z) = \min_{y\geq 0} \Phi_n(x,y,z)$.

3.4 Single Period Analysis

In this section, we shall explore the impact of supply uncertainty on the ordering policy for the single period case. We assume that x denotes the inventory level and z denotes the demand reserved in the previous period. Differentiating (3.1), we obtain

$$\begin{aligned}\frac{\partial}{\partial y}\varphi(x,y,z) &= h\int_0^1\int_0^{\frac{x+sy}{z}} s dG(t)dF(s) - p\int_0^1\int_{\frac{x+sy}{z}}^1 s dG(t)dF(s)\\ &= (h+p)\int_0^1 s\left[G\left(\frac{x+sy}{z}\right) - \frac{p}{h+p}\right]dF(s). \qquad (3.3)\end{aligned}$$

Let $L(x,y,z) = \frac{\partial}{\partial y}\varphi(x,y,z)$. It is easy to see that $L(x,y,z)$ is increasing in y. If a^* is the minimizer of $\varphi(x,y,z)$, we denote $y^*(x,z) = \max\{a^*, 0\}$ to be the optimal ordering quantity.

Lemma 3.4.1 *If $x \geq zG^{-1}(\frac{p}{h+p})$, then $y^*(x,z) = 0$.*

Proof: If $x \geq zG^{-1}(\frac{p}{h+p}) \Rightarrow G(\frac{x}{z}) \geq \frac{p}{h+p}$. In particular, $L(x,0,z) \geq 0$, hence this implies that $y^*(x,z) = 0$. ◇

Lemma 3.4.2 *The optimal ordering quantity $y^*(x,z)$ is increasing in z and decreasing in x. In fact,*

$$y^*(x,z)\begin{cases} > 0 & \text{if } x < zG^{-1}(\frac{p}{h+p}) \\ = 0 & \text{if } x \geq zG^{-1}(\frac{p}{h+p}). \end{cases}$$

Proof: From Lemma 3.4.1, it suffices to show for the case when $x < zG^{-1}(\frac{p}{h+p})$. Note that $\lim_{y\to\infty+} L(x,y,z) = hE(\theta) > 0$. On the other hand, we have $L(x,0,z) < 0$, thus there exists $y^*(x,z)$ such that $L(x, y^*(x,z), z) = 0$. The uniqueness follows since $L(x,y,z)$ is increasing in y. It is easy to see from

$$(h+p)\int_0^1 s\left[G\left(\frac{x+sy^*(x,z)}{z}\right)-\frac{p}{h+p}\right]dF(s)=0,$$

that we have that $y^*(x,z)$ is increasing in z but decreasing in x. $\diamond$

Remark: Let us define the re-order point $x(z) = \inf\{x > 0 : y^*(x,z) = 0\}$. It can be shown that $x(z) \le zG^{-1}(\frac{p}{h+p})$. Suppose not, we assume that $x(z) > zG^{-1}(\frac{p}{p+h})$. Choose $\epsilon = \frac{1}{2}(x(z) - zG^{-1}(\frac{p}{p+h})) > 0$. By definition of $x(z)$, for every $\epsilon > 0$, there exists $x_\epsilon > x(z)$ such that $y^*(x_\epsilon, z) = 0$. But

$$\begin{aligned}
L(x_\epsilon,0,z) &= (h+p)\int_0^1 s\left[G\left(\frac{x_\epsilon}{z}\right)-\frac{p}{p+h}\right]dF(s)\\
&= (h+p)E(\theta)\left[G\left(\frac{x_\epsilon}{z}\right)-\frac{p}{p+h}\right]\\
&\ge (h+p)E(\theta)\left[G\left(\frac{x(z)}{z}\right)-\frac{p}{p+h}\right] > 0.
\end{aligned}$$

The above is a contradiction to $y^*(x_\epsilon, z) = 0$. Thus, $x(z) \le zG^{-1}(\frac{p}{p+h})$.

More interestingly, we have the following result indicating the independence of the re-order point $x(z)$ w.r.t the supply uncertainty.

Lemma 3.4.3 *The re-order point $x(z)$ is independent of the supply uncertainty factor and is equal to $zG^{-1}(\frac{p}{h+p})$.*

Proof: Let z be given. For the function $L(x,0,z)$, we have

$$\frac{\partial}{\partial x}L(x,0,z)=\frac{h+p}{z}\int_0^1 sg\left(\frac{x}{z}\right)dF(s)>0.$$

Thus, $L(x,0,z)$ is strictly monotone in x. Now, $L(0,0,z) < 0$ and

$\lim_{x\to\infty} L(x,0,z) > 0$, there exists a unique $x(z) > 0$ such that $L(x(z),0,z) = 0$. Thus, $x(z) = zG^{-1}\left(\frac{p}{h+p}\right)$ is the unique solution to $L(x,0,z) = 0$. ◇

From Lemma 3.4.3, the re-ordering point is dependent of the demand cancellation of the customers but independent of the supply uncertainty. The derivation of our inventory policy is very similar to the work of Wang and Gerchak (1996). In fact, the yield rate in their work has the same interpretation as our supply uncertainty factor. However, they show that the re-order point is dependent on the yield rate and this is due to the presence of the unit production cost which is paid before imperfect production is carried out. Therefore, one might argue that the assumption of a unit ordering cost in our model will also result in the dependency on the mean of the supply uncertainty factor $(1-\theta)$. This is true only when we assume that the supplier is not liable for any of the delivery losses. In most situations, for every item paid to the supplier, he is liable to make compensation for the amount lost during delivery. This implies that the inventory manager only pays for what he receives. As a result, the cost borned by the inventory manager is directly proportional to the unit ordering cost times $E(\theta)$, causing the re-order point to be independent of θ even when the unit ordering cost is assumed. Since we do not impose a unit ordering cost, there is no need to account for any losses on part of the supplier. This is in contrast to the model of Wang and Gerchak (1996) where the dependency follows after paying a unit cost to produce an item. Let us put things into perspective. Suppose there is a variable cost (say c) for each item ordered, according to Wang and Gerchak's model, the formulation of the one period cost would be $cy + hE[x+\theta y - zR]^+ + pE[x+\theta y - zR^-]$. Upon differentiation w.r.t to y,

$$(h+p)\int_0^1 s\left[G\left(\frac{x+sy^*(x,z)}{z}\right) - \frac{p-c/E\theta}{h+p}\right]f(s)ds = 0.$$

To determine the reorder point, when $y^*(x,z) = 0$, what is the value of x? Now, we see that $x^*(z) = zG^{-1}\left(\frac{p-c/E\theta}{h+p}\right)$, dependent of $E\theta$.

If the inventory manager only pays for what he receives because of supply uncertainty (on the failure of the supplier to deliver everything due to loss, pilferage etc), then the formulation of the one period cost becomes $cE(\theta)y + hE[x+\theta y - zR]^+ + pE[x+\theta y - zR^-]$ given θ. Upon differentiation w.r.t y, the first order condition becomes

$$(h+p)\int_0^1 s\left[G\left(\frac{x+sy^*(x,z)}{z}\right) - \frac{p-c}{h+p}\right] f(s)ds = 0.$$

To determine the reorder point, we ask the question: when $y^*(x,z) = 0$, what is the value of x? Now, it is easy to see that $x^*(z) = zG^{-1}\left(\frac{p-c}{h+p}\right)$, independent of $E\theta$. Finally, we obtain a bound for the optimal ordering level when the distribution function of R is convex.

Lemma 3.4.4 *If $G(x)$ is a convex c.d.f, then for all $x < zG^{-1}(\frac{p}{h+p})$, we have $zG^{-1}(\frac{p}{h+p}) - x \le y^*(x,z) \le \frac{1}{E(\theta)}\left[zG^{-1}(\frac{p}{h+p}) - x\right]$.*

Proof: Let $t(r) = r\left[G\left(\frac{1}{z}(x+ry)\right) - \frac{p}{h+p}\right]$. Since $G(r)$ is convex on $r \in [0,1] = I$, this implies that $t''(r) > 0$ on I. Using Jensen's Inequality, $t(E(\theta)) \le E(t(\theta))$. Then,

$$\begin{aligned} L(x,y,z) &= (h+p)E(t(\theta)) \\ &\ge (h+p)t(E(\theta)) = (h+p)E(\theta)\left\{G\left(\frac{x+yE(\theta)}{z}\right) - \frac{p}{h+p}\right\}. \end{aligned}$$

When $y = \frac{1}{E(\theta)}\left[zG^{-1}(\frac{p}{h+p}) - x\right]$, we have $L(x,y,z) \ge 0$. Thus, by the definition of $y^*(x,z)$, the lemma holds. ◇

There are some continuous random variables whose cumulative distribution functions are convex. The common examples include the uniform distribution on $[0,1]$ and the Beta distribution with $\alpha = 5, \beta = 1$.

3.4.1 Structural Properties of $C_1(x,z)$ and $y^*(x,z)$

In this section, we will discuss the structural properties of the one period optimal cost function in x (the initial inventory level), in the event of supply uncertainty and demand cancellation. Following the notation of Yuan and Cheung (2003), we denote the re-ordering point as $x_1(z) = zG^{-1}(\frac{p}{h+p})$. We denote $y^*(x,z)$ to be the optimal ordering quantity given x, z, θ and R, and $y_c^*(x,z) = y^*(x,z)$ when $\theta = 1$, a.s. After some simplifications, it can be shown that for $x < x_1(z)$,

$$C_1(x,z) = (h+p)\int_0^1 \left[xG\left(\frac{x+sy^*}{z}\right) - \int_0^{\frac{x+sy^*}{z}} ztdG(t)\right] dF(s) + p(zER - x)$$

$$= -py^*E\theta + z(h+p)\int_0^1 \int_0^{\frac{x+sy^*}{z}} G(t)dtdF(s) + p(zER - x). \tag{3.4}$$

And for $x \geq x_1(z)$, we have

$$C_1(x,z) = (h+p)G\left(\frac{x}{z}\right)x - px + \left(p\int_{\frac{x}{z}}^1 tdG(t) - h\int_0^{\frac{x}{z}} tdG(t)\right)z.$$

Lemma 3.4.5 *For $z > 0$, we have*

$$\frac{\partial}{\partial x}C_1(x,z) = \begin{cases} (h+p)\int_0^1 \left[G\left(\frac{x+sy^*(x,z)}{z}\right) - \frac{p}{h+p}\right] dF(s) & \text{if } x < x_1(z), \\ (h+p)G\left(\frac{x}{z}\right) - p & \text{if } x \geq x_1(z). \end{cases}$$

Proof: In particular, we have

$$C_1(x,z) = \begin{cases} \varphi(x, y^*(x,z), z) & \text{if } x < x_1(z) \\ \varphi(x, 0, z) & \text{if } x \geq x_1(z). \end{cases}$$

The case when $x \geq x_1(z)$ is easy. Denote $y^* = y^*(x,z)$ and $(y^*)'_x = \frac{\partial}{\partial x} y^*(x,z)$. For $x < x_1(z)$, we differentiate (3.4) w.r.t x using Leibniz rule for differentiation to obtain

$$\begin{aligned} \frac{\partial}{\partial x} C_1(x,z) &= (h+p) \int_0^1 \left[G\left(\frac{x+sy^*}{z}\right) - y^* sg\left(\frac{x+sy^*}{z}\right)\left(\frac{1+s(y^*)'_x}{z}\right)\right] dF(s) - p \\ &= (h+p)\int_0^1 G\left(\frac{x+sy^*}{z}\right) dF(s) - p. \end{aligned}$$

The last equality holds because $y^* = y^*(x,z)$ satisfies the equation

$$(h+p)\int_0^1 s\left[G\left(\frac{x+sy^*}{z}\right) - \frac{p}{h+p}\right] dF(s) = 0.$$

Differentiating w.r.t x, we obtain $(h+p)\int_0^1 sg\left(\frac{x+sy^*}{z}\right)\left(\frac{1+s(y^*)'_x}{z}\right) dF(s) = 0.$ ◇

Since $y^*(x,z)$ is monotone decreasing in x, then $\frac{\partial}{\partial x} y^*(x,z)$ exists. Thus, $y^*(x,z)$ is continuous in x. In Yuan and Cheung (2003), it is argued that $\frac{\partial}{\partial x} y^*_c(x,z) = -1$. We show that in the presence of supply uncertainty, -1 is the upper bound of $\frac{\partial}{\partial x} y^*(x,z)$.

Lemma 3.4.6 *For any $z > 0$ and $x \in (-\infty, x_1(z)]$,*

$$\frac{\partial}{\partial x} y^*(x,z) \leq -1 = \frac{\partial}{\partial x} y^*_c(x,z).$$

Proof: From proof of Lemma 3.4.5, $(h+p)\int_0^1 sg\left(\frac{x+sy^*}{z}\right)\left(\frac{1+s(y^*)'_x}{z}\right)dF(s)=0$

$$\Rightarrow \frac{\partial}{\partial x}y^*(x,z) = -\frac{\int_0^1 sg\left(\frac{x+sy^*}{z}\right)dF(s)}{\int_0^1 s^2 g\left(\frac{x+sy^*}{z}\right)dF(s)} = -\frac{E(\theta g\left(\frac{x+\theta y^*}{z}\right))}{E(\theta^2 g\left(\frac{x+\theta y^*}{z}\right))}.$$

Define $f_{y^*,z,x}(s) = s(1-s)g\left(\frac{x+sy^*}{z}\right)$ and we have on $s \in [0,1]$,

$$f_{y^*,z,x}(s) \geq s(1-s)\inf_{s\in[0,1]} g\left(\frac{x+sy^*}{z}\right) \geq 0,$$

since $g(\cdot)$ is continuous and thus attains a minimum on the compact interval $[0,1]$. Hence, $f_{y^*,z,x}(\theta) \geq 0$ a.s, and by the linearity of the expectation operator, we have $E(f_{y^*,z,x}(\theta)) \geq 0$. The result readily follows from the simple algebraic rearrangement.$\diamond$

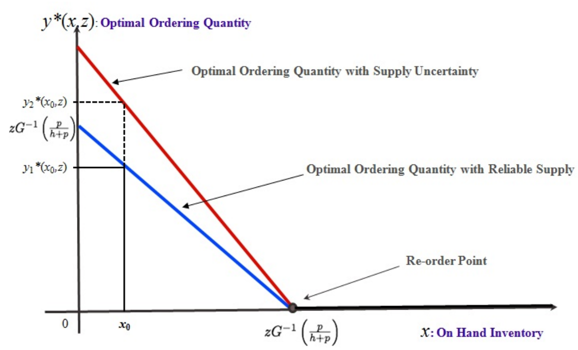

Figure 3.1: Optimal ordering quantities with reliable supply and supply uncertainty, respectively, for the special case where $G(x) = x$ for $x \in [0,1]$.

The interpretation for Lemma 3.4.6 is as follows: $\frac{\partial}{\partial x}y^*(x,z) < -1$ implies that the optimal policy for the single period is not necessarily an order-up-to policy because $\frac{\partial}{\partial x}(x+y^*(x,z)) < 0$ while $\frac{\partial}{\partial x}y^*(x,z) = -1$ implies that $x+y^*(x,z)$ is independent of x. The latter is true when there is no supply uncertainty (Yuan and Cheung (2003)), where the optimal policy is of an order-up-to type. When on-hand inventory is x_0, the quantities $y_1^*(x_0,z)$ and $y_2^*(x_0,z)$ in Figure 3.1 represent optimal ordering levels when the supplier is reliable and uncertain, respectively. In general, the first order derivative of $y^*(x,z)$ w.r.t x need not necessarily linear except in some special cases such as $G(x) = x$ for $x \in [0,1]$.

Theorem 3.1 *$C_1(x,z)$ is convex in $x \in (-\infty,\infty)$ and $z > 0$.*

Proof: From Lemma 3.4.5, if $x < x_1(z)$, then $\frac{\partial}{\partial x}C_1(x,z) = (h+p)\int_0^1 \left[G\left(\frac{x+sy^*(x,z)}{z}\right) - \frac{p}{h+p}\right] dF(s)$. Denote $(y^*)'_x = \frac{\partial}{\partial x}y^*(x,z)$. Thus,

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}C_1(x,z) &= (h+p)\int_0^1 g\left(\frac{x+sy^*_{\theta,R}(x,z)}{z}\right)\left(\frac{1+s(y^*)'_x}{z}\right) dF(s) \\
&= (h+p)\int_0^1 (1+s(y^*)'_x) g\left(\frac{x+sy^*(x,z)}{z}\right)\left(\frac{1+s(y^*)'_x}{z}\right) dF(s) \\
&= \frac{h+p}{z}\int_0^1 (1+s(y^*)'_x)^2 g\left(\frac{x+sy^*(x,z)}{z}\right) dF(s) \geq 0.
\end{aligned}$$

This is because $(h+p)\int_0^1 s(y^*)'_x g\left(\frac{x+sy^*}{z}\right)\left(\frac{1+s(y^*)'_x}{z}\right) dF(s) = 0$ (c.f Lemma 3.4.5). Finally, it is easy to see that for $x \geq x_1(z)$, we have $\frac{\partial^2}{\partial x^2}C_1(x,z) \geq 0$. ◇

Corollary 3.4.1 *For $z > 0$, we have*

$$\frac{\partial}{\partial x}C_1(x,z) = \begin{cases} < 0 & \text{if } x < x_1(z), \\ \geq 0 & \text{if } x \geq x_1(z). \end{cases}$$

Proof: When $x = x_1(z)$, $\frac{\partial}{\partial x}C_1(x,z) = 0$. The result is thus immediate from the convexity of $C_1(x,z)$. ◇

3.4.2 Impact of Supply Uncertainty

Assume that there are two firms, Firm i, where $i = 1,2$. Let R be the common ratio of demand reserved in the previous period that is not eventually canceled for these two firms. It will be shown that there is benefit to reduce volatility of supply uncertainty. Denote $1-\theta_i$ ($F_i(x)$ being the c.d.f of θ_i) to be the supply uncertainty factor of Firm i. Let $C_{1,i}(x,z)$ and $y_i^*(x,z)$ be the optimal cost and optimal ordering quantity of Firm i, respectively, given x, z, θ_i and R. Note that if the initial inventory level satisfies $x \geq x_1(z)$, then $y_1^*(x,z) = y_2^*(x,z) = 0$ and $C_{1,1}(x,z) = C_{1,2}(x,z)$. Therefore, the results in this subsection are proved only for the case when $x < x_1(z)$.

Definition 1 *Let X and Y be two random variables. X is stochastically larger than Y, denoted by $X \geq_{st} Y$ if $P\{X \geq x\} \geq P\{Y \geq x\}$ for all x.*

Lemma 3.4.7 *The following results hold:*

(i). *The optimal replenishment quantity in the presence of supply uncertainty is higher than one with supply certainty,*

(ii). *The optimal cost of managing the supply chain with supply uncertainty is not less than that with supply certainty.*

Fact: If $f(x)$ and $g(x)$ are two differentiable functions that are decreasing convex on $I = [a,b]$ such that $f'(b) = 0 = g'(b)$ and $f(b) = g(b)$, then $f(x) \geq g(x) \Leftrightarrow f^{'}(x) \leq g^{'}(x)$ on I.

Proof: **(i).** Let $y_u^*(x,z)$ ($C_u(x,z)$) and $y_c^*(x,z)$ ($C_c(x,z)$) denote the respective optimal ordering quantities (optimal costs) in the presence and absence of supply uncertainty issues, respectively. Let $\varphi(x,y,z) = hE[x+\theta y - zR]^+ + pE[x+\theta y - zR]^-$. To see why $y_u^*(x,z) \geq y_c^*(x,z) = x_1(z) - x$, we consider

$$\begin{aligned} L_u(x,y,z) = \frac{\partial}{\partial y}\varphi(x,y,z) &= (h+p)\int_0^1 s\left[G\left(\frac{x+sy}{z}\right) - \frac{p}{h+p}\right]dF(s) \\ &\leq (h+p)\int_0^1 s\left[G\left(\frac{x+y}{z}\right) - \frac{p}{h+p}\right]dF(s) \\ &= (h+p)E(\theta)\left[G\left(\frac{x+y}{z}\right) - \frac{p}{h+p}\right]. \end{aligned}$$

In particular, $L_u(x, x_1(z) - x, z) \leq 0 = L_u(x, y_u^*(x,z), z)$. From the above, it is easy to see that $\frac{\partial^2}{\partial y^2}\varphi(x,y,z) \geq 0$, thus $\varphi(x,y,z)$ is convex in $y \geq 0$. Thus, we have $y_u^*(x,z) \geq y_c^*(x,z)$.

(ii). Suppose for some y_0, $C_u(y_0,z) < C_c(y_0,z)$, then we claim that for all $y \in [y_0, x_1(z)]$, we have $C_u(y,z) < C_c(y,z)$. We show by way of contradiction by assuming there exists some $y_1 > y$ such that $C_u(y_1,z) \geq C_c(y_1,z)$. Using Fundamental Theorem of Calculus, we have

$$\begin{aligned} \int_{y_0}^{y_1} \frac{\partial}{\partial y}C_c(y,z)dy &= C_c(y_1,z) - C_c(y_0,z) \\ &\leq C_u(y_1,z) - C_u(y_0,z) \leq \int_{y_0}^{y_1} \frac{\partial}{\partial y}C_u(y,z)dy. \end{aligned}$$

From the work of Yuan and Cheung (2003), $\frac{\partial}{\partial y}C_c(y,z) = 0$. Thus, there exists some $y_0 \leq y' \leq y_1 < x_1(z)$ such that $\frac{\partial}{\partial y}C_u(y,z)|_{y=y'} \geq 0$. This is a contradiction because Corollary 1 states that for all $y < x_1(z)$, $\frac{\partial}{\partial y}C_u(y,z) < 0$. As $C_u(y,z) < C_c(y,z)$ for all $y \in [y_0, x_1(z)]$, $\frac{\partial}{\partial y}C_u(y,z)|_{y=x_1(z)} = 0 = \frac{\partial}{\partial y}C_c(y,z)|_{y=x_1(z)}$, and $C_u(x_1(z),z) = C_c(x_1(z),z)$, applying the above fact, we must have $\frac{\partial}{\partial x}C_u(x,z) > \frac{\partial}{\partial x}C_c(x,z) = 0$ for all $x < x_1(z)$, again contradicting Corollary 3.4.1. $\diamond$

However, we observe that the first order stochastic dominance of the supply uncertainty has no impact on the optimal ordering quantity. This means that $\theta_1 \leq_{st} \theta_2$ can imply either $C_{1,1}(x,z) \leq C_{1,2}(x,z)$ or $C_{1,1}(x,z) \geq C_{1,2}(x,z)$. Example 1 in Section 3.7 illustrates this phenomena. Having a higher expectation of the supply uncertainty random variable does not imply a higher cost of managing the supply chain. This observation leads us to investigate the impact of variability on the optimal costs of managing the reservation system. To compare the performance of two firms based on impact of variability between two random variables, we adopt Definition 4.8 in Song (1994).

Definition 2 *Consider two random variables X and Y having the same mean $EX = EY$, having distributions F and G with densities f and g. Suppose X and Y are either both continuous or both discrete. We say that X is more variable than Y, denoted by $X \geq_{var} Y$, if f crosses g exactly twice, first from above and then from below.*

Definition 3 *Let X and Y be two random variables. X stochastically dominates Y in the convex order, denoted by $X \geq_{cx} Y$ if $E[f(X)] \geq E[f(Y)]$ for all convex functions f.*

Remark: Theorem 4.A.35(a) of Shaked and Shanthikumar (page 197) implies that $\geq_{var} \Rightarrow \geq_{cx}$.

Lemma 3.4.8 *Suppose $\theta_1 \geq_{var} \theta_2$, then $C_{1,1}(x,z) \geq C_{1,2}(x,z)$. In particular, if $G(\cdot)$ is convex, then $y_1^*(x,z) \leq y_2^*(x,z)$.*

Proof: Define

$$g_{x,y,z}(s) = h \int_0^{\frac{x+sy}{z}} (x + sy - zt)dG(t) + p \int_{\frac{x+sy}{z}}^1 (zt - x - sy)dG(t).$$

Then, it is clear that $g''_{x,y,z}(s) = \frac{y}{z}(h+p)g\left(\frac{x+sy}{z}\right) > 0$, implying that $g_{x,y,z}(s)$ is convex in $s \in [0,1]$. Thus, we have

$$\begin{aligned} C_{1,1}(x,z) &= \varphi_1(x, y_1^*, z) = E(g_{x,y_1^*,z}(\theta_1)) \\ &\geq E(g_{x,y_1^*,z}(\theta_2))(\text{since } \theta_1 \geq_{cx} \theta_2) \\ &= \varphi_2(x, y_1^*, z) \geq C_{1,2}(x,z). \end{aligned}$$

Suppose $G(\cdot)$ is convex. Let y_j^* satisfy $L_j(x,y,z) = 0$ for $j = 1,2$ and $u_y(s) = sG\left(\frac{x+sy}{z}\right)$. Since for any $y \geq 0$, $u_y(s)$ is convex on $s \in [0,1]$. Hence, $\theta_1 \geq_{var} \theta_2 \Rightarrow E(u_y(\theta_1)) \geq E(u_y(\theta_2))$ for all $y \geq 0$. In particular when $y = y_2^*$, we have

$$\begin{aligned} & E\left(\theta_1 G\left(\frac{x+\theta_1 y_2^*}{z}\right)\right) \geq E\left(\theta_2 G\left(\frac{x+\theta_2 y_2^*}{z}\right)\right) \\ & \Rightarrow E\left(\theta_1 G\left(\frac{x+\theta_1 y_2^*}{z}\right)\right) - \frac{p}{h+p}E\theta_1 \geq E\left(\theta_2 G\left(\frac{x+\theta_2 y_2^*}{z}\right)\right) - \frac{p}{h+p}E\theta_2 \\ & \Rightarrow L_1(x, y_2^*, z) \geq L_2(x, y_2^*, z). \end{aligned}$$

But $L_2(x, y_2^*, z) = 0 = L_1(x, y_1^*, z)$, and the fact that $L_1(x,y,z)$ is increasing in y implies that $y_2^* \geq y_1^*$. ◇

Now, $\theta G\left(\frac{x+\theta y^*(x,z)}{z}\right)$ can be interpreted as the probability that demand reserved will be satisfied. A greater dispersion in supply uncertainty actually increases the expectation of this probability as convexity of $sG(\cdot)$ is preserved. Intuitively, the manager exercises more caution in his ordering behavior due to greater volatility of θ. If the c.d.f of R is not convex, the comparison of the optimal ordering may be non-trivial. In Section 3.7, we provide Example 2 to illustrate our claim. Lemma 3.4.8 tells us that variability has a greater effect on the cost of managing the firm. In particular, there is always incentive to reduce the variance of supply uncertainty or yield.

3.4.3 Impact of Demand Cancellation

Denote y_i^* to be the optimal ordering level of Firm i whose common supply uncertainty factor is $1-\theta$ for $i = 1, 2$. Let R_i and $C_{1,i}(x, z)$ be the demand cancellation random variable and optimal cost of Firm i, respectively. Let $G_i(x)$ be the respective c.d.f of R_i. Let the one period cost be $\varphi_i(x, y, z) = hE[x+\theta y - zR_i]^+ + pE[x+\theta y - zR_i]^-$. When the customers' demand eventually not canceled becomes stochastically larger, is the cost of managing the system lower? The answer is in fact negative. Thus, it is not true that there is always an incentive to increase the mean of customers' ratio of demand that is eventually not canceled. In section 3.7, we provide an example (Example 3) to illustrate that even when R_2 is stochastically larger than R_1, the optimal cost of system 2 can be higher than that of system 1. We are motivated to develop a bound on the difference between the optimal cost of managing system 1 and system 2. These bounds turn out to be proportional to the difference between the mean number of items not eventually canceled.

Lemma 3.4.9 *For all $x \in (-\infty, \infty), z > 0$, let θ be given, and if $R_1 \leq_{st} R_2$, then*

(i). $y_1^*(x, z) \leq y_2^*(x, z)$,

(ii). $-hz(ER_2 - ER_1) \leq C_{1,2}(x, z) - C_{1,1}(x, z) \leq pz(ER_2 - ER_1)$.

Proof: **(i).** Suppose we have $R_1 \leq_{st} R_2$, and let $G_1(\cdot)$ and $G_2(\cdot)$ be the respective c.d.f's, this implies that $G_1(y) \geq G_2(y)$. But Lemma 3 tells us that the re-ordering point for Firm i is given by $x_i(z) = zG_i^{-1}\left(\frac{p}{h+p}\right)$. Thus, we have

$$G_1\left(\frac{x_1(z)}{z}\right) = G_2\left(\frac{x_2(z)}{z}\right) \leq G_1\left(\frac{x_2(z)}{z}\right).$$

Since $G(y)$ is non-decreasing in y, we have $x_1(z) < x_2(z)$. Let x and z be fixed. There are three cases to consider:

Case 1: If $x < x_1(z)$, then we have $y_1^*(x,z), y_2^*(x,z) > 0$ satisfying $L_{\theta,R_1}(x, y_1^*(x,z), z) = 0 = L_{\theta,R_1}(x, y_2^*(x,z), z)$. Thus,

$$\int_0^1 s\left[G_1\left(\frac{x + sy_1^*(x,z)}{z}\right)\right] dF(s) = \int_0^1 s\left[G_2\left(\frac{x + sy_2^*(x,z)}{z}\right)\right] dF(s).$$

Again, since $G_1(y) \geq G_2(y)$ for all $y \in [0,1]$, then $y_1^*(x,z) \leq y_2^*(x,z)$.

Case 2: If $x_1(z) \leq x \leq x_2(z)$, then $y_2^*(x,z) \geq 0 = y_1^*(x,z)$.

Case 3: If $x > x_2(z)$, then $y_2^*(x,z) = 0 = y_1^*(x,z)$.

(ii). We first prove that for any $y \geq 0$, we have

$$-hz(ER_2 - ER_1) \leq \varphi_2(x,y,z) - \varphi_1(x,y,z) \leq pz(ER_2 - ER_1). \tag{3.5}$$

Using the fact that for any random variable, $X = X^+ - X^-$, we obtain

$$\begin{aligned}\varphi_i(x,y,z) &= hE[x + \theta y - zR_i]^+ + pE[x + \theta y - zR_i]^- \\ &= hE[x + \theta y - zR_i] + (h+p)E[zR_i - x - \theta y]^+.\end{aligned}$$

Thus, we have

$$\varphi_2(x,y,z) - \varphi_1(x,y,z) = -zh(ER_2 - ER_1) \\ + (h+p)\{E[zR_2 - x - \theta y]^+ - E[zR_1 - x - \theta y]^+\}.$$

The lower bound of (3.5) can be obtained by showing that $E[zR_2 - x - \theta y]^+ \geq E[zR_1 - x - \theta y]^+$. This can be done by defining $g(s,r) = [zr - x - sy]^+$. Since $g(s,r)$ is non-decreasing in r and given $R_1 \leq_{st} R_2$, we obtain $E[g(s,R_2)] \geq E[g(s,R_1)]$. Since θ and R are independent random variables, we obtain

$$E[g(\theta,R_1)] = \int_0^1 E[g(s,R_1)]dF(s) \\ \leq \int_0^1 E[g(s,R_2)]dF(s) = E[g(\theta,R_2)].$$

The upper bound of (3.5) can similarly be shown by expressing

$$\varphi_i(x,y,z) = (h+p)E[x + \theta y - zR_i]^+ - pE[zR_i - x - \theta y]$$

and noting that $E[x + \theta y - zR_2]^+ \leq E[x + \theta y - zR_1]^+$. Finally, from the second inequality in (3.5), for all $y \geq 0$,

$$C_{1,2}(x,z) = \varphi_2(x, y_2^*, z) \leq \varphi_1(x,y,z) + pz(ER_2 - ER_1).$$

In particular, $C_{1,2}(x,z) \leq C_{1,1}(x,z) + pz(ER_2 - ER_1)$. The lower bound is proved similarly. $\diamond$

The next result shows that variability of demand cancellation has a greater impact on system performance than the mean.

Proposition 3.4.1 *If $R_1 \leq_{var} R_2$, then $C_{1,1}(x,z) \leq C_{1,2}(x,z)$.*

Proof: We define $f_{x,z,y}(\theta, R) = h[x+\theta y-zR]^+ + p[x+\theta y-zR]^-$. Let $s \in [0,1]$ be given, then $f_{x,z,y}(s,r)$ is convex in $r \in [0,1]$. Since $R_1 \leq_{var} R_2 \Rightarrow R_1 \leq_{cx} R_2$, and we get $E_{R_1}[f_{x,z,y}(s,R_1)] \leq E_{R_2}[f_{x,z,y}(s,R_2)]$, where $E_{R_i}[f_{x,z,y}(s,r)] = \int_0^1 f_{x,z,y}(s,r)dG_i(r)$. Since θ and R are independent random variables, it is easy to see that

$$\begin{aligned} E[f_{x,z,y}(\theta, R_1)] &= \int_0^1 E_{R_1}[f_{x,y,z}(s,R_1)]dF(s) \\ &\leq \int_0^1 E_{R_2}[f_{x,y,z}(s,R_2)]dF(s) = E[f_{x,z,y}(\theta, R_2)]. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} &C_{1,1}(x,z) \leq \varphi_1(x, y_2^*, z) \\ &= E[f_{x,z,y_2^*}(\theta, R_1)] \leq E[f_{x,z,y_2^*}(\theta, R_2)] \\ &= C_{1,2}(x,z). \end{aligned}$$

The result can be interpreted as follows. As the demand eventually not canceled becomes stochastically larger, the optimal ordering quantity increases. Finally, there is an incentive to reduce the variance of the demand cancellation random variable. $\diamond$

3.5 Multiple Period Analysis

This section is devoted to determining the optimal policy for the finite horizon model. Let x be the inventory on hand at the beginning when there are n periods left and z be the demand reserved during period $n+1$. If there are n periods to go, let $C_n(x,z)$ be the cost given that the supply uncertainty and demand not eventually canceled are θ_n and R_n, respectively. Let $C_n'(x,z)$ be the first order derivatives of $C_n(x,z)$ w.r.t x. Throughout the rest of this chapter, let us denote $L_n(x,y,z) = \frac{\partial}{\partial y}\Phi_n(x,y,z)$. The following lemma is crucial.

Lemma 3.5.1 *For all $n \geq 1$, we have*

(i). $C_n'(x,z)$ *is increasing in x.*

(ii). $C_n'(x,z) \leq 0$ *for all $x \leq 0$.*

(iii). $\lim_{x\to\infty} C_n'(x,z) \geq 0$ *for all z.*

Proof: We show the following by induction. To argue that (i) and (ii) hold, we simply note that $C_1'(x_1(z),z) = 0$ and the convexity of $C_1(x,z)$ in x implies that for all $x < x_1(z)$, $C_1'(x,z) \leq 0$ (c.f Lemma 3.4.5). Lemma 3.4.5 implies that $\lim_{x\to\infty} C_1'(x,z) \geq 0$.

Assume that for $n = k$, both (i) and (ii) are true. In the following, we prove that these properties are also true for $n = k+1$.

(i). Given that $L_{k+1}(x,y,z) = \frac{\partial}{\partial y}\Phi_{k+1}(x,y,z)$ and together with (3.2), we have

$$\begin{aligned} L_{k+1}(x,y,z) &= (h+p)\int_0^1 s\left[G\left(\frac{x+sy}{z}\right) - \frac{p}{p+h}\right]dF(s) \\ &\quad + \alpha\frac{\partial}{\partial y}\int_0^\infty\int_0^1\int_0^1 C_k(x+sy-zt,w)dF(s)dG(t)dH(w) \\ &= \int_0^1 s\left[(h+p)G\left(\frac{x+sy}{z}\right) - p + \alpha\int_0^\infty\int_0^1 C_k'(x+sy-zt,w)dG(t)dH(w)\right]dF(s). \end{aligned} \tag{3.6}$$

We want to determine the property of $L_{k+1}(x,y,z)$ w.r.t y. It is easy to see that $\frac{\partial^2}{\partial y^2}\varphi(x,y,z) = (h+p)\int_0^1 s^2 g\left(\frac{x+sy}{z}\right) dF(s) > 0$. Thus, we have

$$\frac{\partial}{\partial y}L_{k+1}(x,y,z) = \frac{\partial^2}{\partial y^2}\varphi(x,y,z) + \alpha \int_0^\infty \int_0^1 \int_0^1 \underbrace{\frac{\partial^2}{\partial y^2}C_k(x+sy-zt,w)}_{\geq 0 \ (\text{since } C_k''(x,z)\geq 0)} dF(s)dG(t)dH(w) > 0.$$

Hence, $L_{k+1}(x,y,z)$ is strictly increasing in y. Moreover, using $\lim_{x\to\infty} C_k'(x,z) \geq 0$, we have

$$\lim_{y\to\infty} L_{k+1}(x,y,z) = hE\theta + \alpha \int_0^\infty \int_0^1 \int_0^1 \lim_{y\to\infty} \frac{\partial}{\partial y}C_k(x+sy-zt,w)dF(s)dG(t)dH(w) > 0.$$

Next, we observe that $L_{k+1}(x,-\infty,z) < 0$ and by the Intermediate Value Theorem, there exists $y_{k+1}(x,z) \in (-\infty,\infty)$ such that $L_{k+1}(x, y_{k+1}(x,z), z) = 0$. It is readily observed that $y_{k+1}(x,z)$ is decreasing in x but increasing in z. Note that $L_{k+1}(x,0,z)$ is increasing in x and $L_{k+1}(0,0,z) < 0$. Thus, there exists a unique $x_{k+1}(z) = \inf\{x > 0 : y_{k+1}(x,z) = 0\}$. Let $y^*_{k+1}(x,z)$ be the optimal ordering quantity. Since $y_{k+1}(x,z)$ is decreasing in x and increasing in z, this implies that $x < x_{k+1}(x,z)$, then $y^*_{k+1}(x,z) = y_{k+1}(x,z)$. Otherwise, $y^*_{k+1}(x,z) = 0$. Now, we have

$$\Phi_{k+1}(x,y,z) = \begin{cases} \Phi_{k+1}(x, y^*_{k+1}(x,z), z) & \text{if } x < x_{k+1}(x,z) \\ \Phi_{k+1}(x,0,z) & \text{if } x \geq x_{k+1}(x,z). \end{cases}$$

Let us denote $(y^*_{k+1})'_x = \frac{\partial}{\partial x} y^*_{k+1}(x,z)$ and $(y^*_{k+1})''_x = \frac{\partial^2}{\partial x^2} y^*_{k+1}(x,z)$. For $x < x_{k+1}(z)$,

$$\begin{aligned}\frac{\partial}{\partial x} C_{k+1}(x,z) &= \int_0^1 (1 + s(y^*_{k+1})'_x)\Bigg[(h+p)G\left(\frac{x + sy^*_{k+1}(x,z)}{z}\right) - p \\ &\quad + \alpha \int_0^\infty \int_0^1 C'_k(x + sy^*_{k+1}(x,z) - zt, w) dG(t) dH(w)\Bigg] dF(s) \\ &= \int_0^1 \Bigg[(h+p)G\left(\frac{x + sy^*_{k+1}(x,z)}{z}\right) - p \\ &\quad + \alpha \int_0^\infty \int_0^1 C'_k(x + sy^*_{k+1}(x,z) - zt, w) dG(t) dH(w)\Bigg] dF(s). \end{aligned} \quad (3.7)$$

Differentiating again w.r.t x and using the fact that $L_{k+1}(x, y^*_{k+1}(x,z), z) = 0$, we have

$$\begin{aligned}\frac{\partial^2}{\partial x^2} C_{k+1}(x,z) &= \int_0^1 (1 + s(y^*_{k+1})'_x)^2 \Bigg\{(h+p)g\left(\frac{x + sy^*_{k+1}(x,z)}{z}\right) \\ &\quad + \alpha \int_0^\infty \int_0^1 C''_k(x + sy^*_{k+1}(x,z) - zt, w) dG(t) dH(w)\Bigg\} dF(s).\end{aligned}$$

Using our induction hypothesis that $C'_k(x,z)$ is increasing in x, implying that that $C''_{k+1}(x,z) \geq 0$ when $x < x_{k+1}(z)$.

For $x \geq x_{k+1}(z)$, we have

$$C'_{k+1}(x,z) = E(\theta)\left[(h+p)G\left(\frac{x}{z}\right) - p + \alpha \int_0^\infty \int_0^1 C'_k(x - zt, w) dG(t) dH(w)\right]. \quad (3.8)$$

Finally, it is easily seen from (3.8) that

$$\begin{aligned}\frac{\partial^2}{\partial x^2} C_{k+1}(x_{k+1}, z) = E(\theta)\Bigg[&\frac{h+p}{z} g\left(\frac{x}{z}\right) \\ &+ \alpha \int_0^\infty \int_0^1 C''_k(x - zt, w) dG(t) dH(w)\Bigg] \geq 0.\end{aligned}$$

Hence, when $x > x_{k+1}(z)$, we have $C''_{k+1}(x,z) \geq 0$. This implies that $C_{k+1}(x,z)$ is convex in x. Thus, (i) holds for $n = k+1$.

(ii). We show that if $x = x_{k+1}(z)$, then $\frac{\partial}{\partial x} C_{k+1}(x,z) = 0$. For notational simplicity, we denote $x_{k+1}(z)$ by x_{k+1}. By definition $y^*_{k+1} = y^*_{k+1}(x_{k+1}, z) = 0$, then $L_{k+1,\theta,R}(x_{k+1}, y^*_{k+1}, z) = 0$. From (3.6), we infer that

$$\left[(h+p)G\left(\frac{x_{k+1}}{z}\right) - p + \alpha \int_0^\infty \int_0^1 C'_k(x_{k+1} - zt, w)dG(t)dH(w)\right] = 0.$$

Thus, $C'_{k+1}(x_{k+1}, z) = 0$. Therefore, by the convexity of $C_{k+1}(x,z)$ in x, we infer that $C'_{k+1}(x,z) \leq 0$ whenever $x < x_{k+1}(z)$. Hence, (ii) holds for $n = k+1$.

(iii). Taking limits in (3.8), we see that $\lim_{x\to\infty} C'_{k+1}(x,z) > 0$ due to our induction hypothesis. $\diamond$

Corollary 3.5.1 *For $z > 0$ and any $n \in \boldsymbol{N}$,*

(i). *$y^*_n(x,z)$ is differentiable w.r.t x.*

(ii). *$C_n(x,z)$ is differentiable w.r.t x.*

Proof: (i). From the proof of Lemma 3.5.1 (i), it is shown that for any k, $y^*_n(x,z)$ is decreasing in x, thus it is a monotone function and is differentiable almost everywhere on an interval, except on a set which has Lebesgue measure zero.

(ii). Lemma 3.4.5 implies that $k = 1$ is true. Suppose $C_{k-1}(x, z)$ is differentiable and convex in x. From Lemma 3.5.1,

$$
\begin{aligned}
C_k(x, z) &= \min_{y \geq 0} \{\varphi(x, y, z) + \alpha E_D E_{\theta,R} C_{k-1}(x + \theta y - zR, D)\} \\
&= \begin{cases}
\varphi(x, y_k^*(x, z), z) & \\
\quad + \alpha E_D E_{\theta,R} C_{k-1}(x + \theta y_k^*(x, z) - zR, D) & \text{if } x < x_k(z) \\
\varphi(x, 0, z) & \\
\quad + \alpha E_D E_{\theta,R} C_{k-1}(x - zR, D) & \text{if } x \geq x_k(z).
\end{cases}
\end{aligned}
$$

From the differentiability of $y_k^*(x, z)$, $C_{k-1}(x, z)$, and $\varphi(x, y, z)$ in x, we have the differentiability of $C_k(x, z)$. Hence, by mathematical induction, the statement is true.

$\diamond$

Theorem 3.2 *For N periods, $1 \leq n \leq N$, the optimal policy is a re-order point policy when the supply is uncertain. The optimal policy can be specified as follows: there exists $x_n^*(z)$ (independent of θ) such that the optimal order quantity (dependent on θ) is*

$$
y_n^*(x, z) = \begin{cases} > 0 & \text{if } x < x_n^*(z) \\ 0 & \text{if } x \geq x_n^*(z), \end{cases}
$$

and the minimal cost is

$$
C_n(x, z) = \begin{cases} \Phi_n(x, y_n^*(x, z), z) & \text{if } x < x_n^*(z) \\ \Phi_n(x, 0, z) & \text{if } x \geq x_n^*(z). \end{cases}
$$

The optimal policy is an order up to policy when the supply is reliable.

Proof: It suffices to show that for each k, $\frac{\partial}{\partial x} y^*_{k+1}(x,z)$ is less than 1 whenever $\theta \leq_{st} 1$. For $x < x_{k+1}(z)$, we have $L_{k+1}(x, y_{k+1}(x,z), z) = 0$. Then differentiating and rearranging, we obtain

$$\frac{\partial}{\partial x} y_{k+1}(x,z) = -\frac{E\left(\theta\left(\frac{h+p}{z} g\left(\frac{x+\theta y_{k+1}}{z}\right) + \alpha C''_k(x+\theta y_{k+1} - zR, D)\right)\right)}{E\left(\theta^2\left(\frac{h+p}{z} g\left(\frac{x+\theta y_{k+1}}{z}\right) + \alpha C''_k(x+\theta y_{k+1} - zR, D)\right)\right)}.$$

Lemma 3.5.1 implies that $C_k(x,z)$ is convex in $x \in (-\infty, \infty)$ and thus, $C''_k(x,y) \geq 0$. Then following the proof of Lemma 3.4.6, we show that $\frac{\partial}{\partial x} y_{k+1}(x,z) < -1$. Thus, whenever $\theta \leq_{st} 1$, we have $\frac{\partial}{\partial x}(x + y_{k+1}(x,z)) < 0$. Thus, $x + y_{k+1}(x,z)$ is a function of x which implies that the optimal inventory policy is not an order up to policy. $\diamond$

3.5.1 Impact of Supply Uncertainty

In this subsection, we will state without proof that managing a firm whose yield distribution being more variable is always more costly. The structure of the proof is exactly the same as Corollary 3.5.4. Assuming there are two firms, Firm i, where $i = 1, 2$. Let R be the common ratio of demand reserved in the previous period that is not eventually canceled for these two firms. Let $\{\theta_{n,i} : n \in \mathbf{N}\}$ be two sequences of i.i.d random variables, such that $\theta_{n,i} \stackrel{\mathrm{d}}{=} \theta_i$ for $i = 1, 2$. We state the next result while its proof will be deferred until Section 3.5.2.

Corollary 3.5.2 *For each period* $1 \leq n \leq N$, *let* $R_n \stackrel{\mathrm{d}}{=} R$ *and suppose* $\theta_{n,1} \leq_{var} \theta_{n,2}$, *then* $C_{n,1}(x,z) \leq C_{n,2}(x,z)$.

Proof: Similar to Corollary 3.5.4. $\diamond$

3.5.2 Impact of Demand Cancellation

Let us consider two firms in this section. For Firm j, $j = 1, 2$, let $\{R_{n,j} : n \in \mathbf{N}\}$ be a sequence of ratio where demand is not eventually canceled. For ease of analysis, we assume that $R_{n,j} \stackrel{\text{d}}{=} R_j$ for $j = 1, 2$. Unless specified, let θ_n be the common supply uncertainty during period n such that $\theta_n \stackrel{\text{d}}{=} \theta$. For Firm j, denote $y^*_{n,j}(x, z)$ to be the optimal ordering quantity given x and z when there are n periods remaining (from period n to period 1), given R_j and θ.

Lemma 3.5.2 *Suppose there is no supply uncertainty, let $x \in (-\infty, \infty), z > 0$ be given. If $R_{n,1} \leq_{st} R_{n,2}$ for all $n \in \boldsymbol{N}$, then*

(i). *the re-order points $x_{n,1}(z) \leq x_{n,2}(z)$.*

(ii). $C'_{n,1}(x, z) \geq C'_{n,2}(x, z)$.

(iii). $y^*_{n,1}(x, z) \leq y^*_{n,2}(x, z)$.

Proof: We argue that $n = 1$ is true. Lemma 3.4.3 implies that $x_{1,i}(z) = zG_i^{-1}\left(\frac{p}{h+p}\right)$ and since $G_1(x) \geq G_2(x)$ for all $x \in [0, 1]$, implying (i) holds. Lemma 9 implies that (iii) is true. Let $G_i(x)$ denote the c.d.f for $R_{n,i}$ for each $n \in \mathbf{N}$ of Firm i. To show that (ii) is true, there are three cases to consider:

Case 1: If $x < x_{1,1}(z)$, then $C'_{1,1}(x, z) = 0 = C'_{1,2}(x, z)$ using Lemma 3.4.5.

Case 2: If $x_{1,1}(z) \leq x \leq x_{1,2}(z)$, then Lemma 3.4.5 implies that $C'_{1,1}(x, z) \geq 0 \geq C'_{1,2}(x, z)$.

Case 3: If $x_{1,2}(z) \leq x$, Lemma 3.4.5 and $G_1(\cdot) \geq G_2(\cdot)$ imply that $C'_{1,1}(x, z) = (h+p)G_1\left(\frac{x}{z}\right) - p \geq (h+p)G_2\left(\frac{x}{z}\right) - p = C'_{1,2}(x, z)$. Hence, (i)–(iii) are true for $n = 1$.

Assume that for $n = k$, (i)$-$(iii) are true. We prove that these properties are also true for $n = k + 1$.

(i). To solve for the re-order points $x^*_{k+1,i} = x_{k+1,i}(z)$, we consider $L_{k+1,i}(x^*_i, 0, z) = 0$. From (3.6), we have

$$(h+p)G_i\left(\frac{x^*_{k+1,i}}{z}\right) - p + \alpha \int_0^\infty \int_0^1 C'_{k,i}(x^*_{k+1,i} - zt, w)dG_i(t)dH(w) = 0.$$

Since $G_1(x) \geq G_2(x)$ for all $x \in [0,1]$ and using the induction hypothesis that $C'_{k,1}(x,z) \geq C'_{k,2}(x,z)$, we have $x^*_{k+1,1} \leq x^*_{k+1,2}$.

(ii). From (3.7) and (3.8) , we have

$$\frac{\partial}{\partial x} C_{k+1,i}(x,z) = \begin{cases} 0 & \text{if } x < x_{k+1,i}(z) \\ (h+p)G_i\left(\frac{x}{z}\right) - p & \\ \quad +\alpha \int_0^\infty \int_0^1 C'_{k,i}(x - zt, w)dG_i(t)dH(w) & \text{if } x \geq x_{k+1,i}(z). \end{cases}$$

Case 1: If $x < x_{k+1,1}(z)$, then $C'_{k+1,1}(x,z) = 0 = C'_{k+1,2}(x,z)$.

Case 2: If $x_{k+1,1}(z) \leq x \leq x_{k+1,2}(z)$, then using the fact that $C'_{k+1,i}(x,z)$ is increasing in x (c.f Lemma 3.5.1 (i)) and $C'_{k+1,i}(x,z) = 0$ for $x = x_{k+1,i}(z)$, we have $C'_{k+1,1}(x,z) \geq 0 \geq C'_{k+1,2}(x,z)$.

Case 3: If $x_{k+1,2}(z) \leq x$, using the induction hypothesis that $C'_{k,1}(x,z) \geq C'_{k,2}(x,z)$ and $G_1(\cdot) \geq G_2(\cdot)$, we have $C'_{k+1,1}(x,z) \geq C'_{k+1,2}(x,z)$. Hence, (ii) is true for $n = k+1$.

(iii). Denote $y^*_{k+1,i} = y^*_{k+1,i}(x,z)$. Again, we consider the same three cases:

Case 1: If $x < x_{k+1,1}(z)$, then

$$L_{k+1,i}(x, y^*_{k+1,i}, z) = (h+p)G_i\left(\frac{x + y^*_{k+1,i}}{z}\right) - p + \alpha \int_0^\infty \int_0^1 C'_{k,i}(x + y^*_{k+1,i} - zt, w)dG_i(t)dH(w).$$

Since for $i = 1, 2$, we have $L_{k+1,i}(x, y^*_{k+1,i}(x,z), z) = 0$, $G_1(x) \geq G_2(x)$ and $C'_{k,1}(x,z) \geq C'_{k,2}(x,z)$, this implies that $y^*_{k+1,1}(x,z) \leq y^*_{k+1,2}(x,z)$.

Case 2: If $x_{k+1,1}(z) \leq x \leq x_{k+1,2}(z)$, then Theorem 3.2 implies that $y^*_{k+1,1}(x,z) = 0 \leq y^*_{k+1,2}(x,z)$.

Case 3: If $x_{k+1,2}(z) \leq x$, then Theorem 3.2 implies that $y^*_{k+1,1}(x,z) = 0 = y^*_{k+1,2}(x,z)$. Hence, (iii) is true for $n = k+1$. Therefore, by induction, the statement is true for all $n \in \mathbf{N}$. $\diamond$

When there is no supply uncertainty issue from the supplier, it is always beneficial to order a larger quantity when the frequency of cancellation is lower.

Corollary 3.5.3 *For any given supply uncertainty, if $R_{n,1} \leq_{st} R_{n,2}$ for all $n \in \boldsymbol{N}$, then $x_{n,1}(z) \leq x_{n,2}(z)$ for all x, z.*

Proof: From Theorem 3.2, the re-order point is independent of θ and combining with Lemma 3.5.2, the result is immediate. $\diamond$

The above result implies that during each period, a stochastically larger fraction of demand not canceled eventually leads to a higher re-order point for the inventory manager. Intuitively, as the expected demand canceled becomes lower, the point that triggers ordering when the inventory on-hand falls below it should be kept higher. This result is true regardless of any form of unreliability from the suppliers' side.

Theorem 3.A.12 (a) of Shaked and Shanthikumar (2007) states that if $X \leq_{var} Y$, then $-X \leq_{var} -Y$. This fact allows us to compare the costs of managing the firms in the presence of relatively more variable demand ratios.

Corollary 3.5.4 *For each period $1 \leq n \leq N$, let $\theta_n \stackrel{\mathrm{d}}{=} \theta$ and suppose $R_{n,1} \leq_{var} R_{n,2}$, then $C_{n,1}(x,z) \leq C_{n,2}(x,z)$.*

Proof: The result is proved by induction. Proposition 1 implies that the statement is true for $n = 1$. Suppose that for some $k < N$, the statement is true. Let y_2^* be the optimal ordering quantity for Firm 2 given x, z, and $\{R_{n,2} : n \in \mathbf{N}\}$. Also, we let $\varphi_i(x,y,z)$ be the cost at the beginning of the planning horizon for Firm i. Define $f_{x,z,y}(s,r) = h[x+sy-zr]^+ + p[x+sy-zr]^-$. Since $R_1 \leq_{var} R_2$, then $E[f_{x,z,y}(s,R_1)] \leq E[f_{x,z,y}(s,R_2)]$ for all $y \geq 0$ since $f_{x,z,y}(s,r)$ is convex in r. Taking expectation w.r.t θ, we obtain $\varphi_1(x,y_2^*,z) = E_{\theta,R_1}[f_{x,z,y_2^*}(\theta,R_1)] \leq E_{\theta,R_2}[f_{x,z,y_2^*}(\theta,R_2)] = \varphi_2(x,y_2^*,z)$. It suffices to show that $E_D E_{\theta,R_1} C_{k,1}(x+\theta y_2^* - zR_1, D) \leq E_D E_{\theta,R_2} C_{k,2}(x+\theta y_2^* - zR_2, D)$, when θ, R_j are independent. Then, we have

$$
\begin{aligned}
C_{k+1,1}(x,z) &= \min_{y \geq 0}\{\varphi_1(x,y,z) + \alpha E_D E_{\theta,R_1} C_{k,1}(x+\theta y - zR_1, D)\} \\
&\leq \varphi_1(x,y_2^*,z) + \alpha E_D E_{\theta,R_1} C_{k,1}(x+\theta y_2^* - zR_1, D) \\
&\leq \varphi_2(x,y_2^*,z) + \alpha E_D E_{\theta,R_2} C_{k,2}(x+\theta y_2^* - zR_2, D) \\
&= \min_{y \geq 0}\{\varphi_2(x,y,z) + \alpha E_D E_{\theta,R_2} C_{k,2}(x+\theta y - zR_2, D)\} = C_{k+1,2}(x,z).
\end{aligned}
$$

Thus, the statement is true for $n = k + 1$ and the corollary is proven. To complete our proof, observe that $R_1 \leq_{var} R_2$ implies $-R_1 \leq_{cx} -R_2$. Then, we have

$$\begin{aligned} E_{R_1} C_{k,1}(x + sy_2^* - zR_1, w) &\leq E_{R_2} C_{k,1}(x + sy_2^* - zR_2, w) \\ &\leq E_{R_2} C_{k,2}(x + sy_2^* - zR_2, w). \end{aligned}$$

The fact that translation has no effect on the relative convex orderings between $x + sy_2^* - zR_j$ (for $j = 1, 2$) and the convexity of $C_{k,i}(x, z)$ in x (c.f Lemma 3.5.1) implies that we have the first inequality. The second inequality is due to our induction hypothesis ($C_{k,1}(x, z) \leq C_{k,2}(x, z)$) and the linearity of expectation operator. Applying Fubini's Theorem, we have

$$\begin{aligned} E_D E_{\theta,R_1} C_{k,1}(x + \theta y_2^* - zR_1, D) &= \int_0^1 \int_0^1 E_{R_1} C_{k,1}(x + sy_2^* - zR_1, w) dH(w) dF(s) \\ &\leq \int_0^1 \int_0^1 E_{R_2} C_{k,2}(x + sy_2^* - zR_2, w) dH(w) dF(s) \\ &= E_D E_{\theta,R_2} C_{k,2}(x + \theta y_2^* - zR_2, D). \end{aligned}$$

The proof is now complete. $\diamond$

The above result implies that when the ratio of demand not eventually canceled is stochastically more variable across a multiperiod time horizon, it has a greater detrimental impact on the profitability of the firm. Both Corollaries 3.5.2 and 3.5.4 imply that it is always more costly to manage a firm when either the distribution of demand cancellation or the distribution of yield is more variable than his competitor. Thus, there is incentive to reduce the variance any one of these factors.

3.6 Infinite Horizon Analysis

The discounted infinite horizon model is a natural extension to the finite horizon case as $n \to \infty$. One of our goals is to establish the Bellman's equation for the infinite horizon model and show that we can drop subscripts n and $n-1$ in (3.2). Throughout the section, we shall assume that $\{R_n : n \in \mathbf{N}\}$ and $\{\theta_n : n \in \mathbf{N}\}$ are the sequences of demand ratios not canceled and supply uncertainty factors. We will follow the proof closely to that of Yuan and Cheung (2003) by showing the existence of the limit of $\{C_n(x, z) : n \geq 0\}$. There are two crucial observations that can be made. It is easy to see that $C_{1,}(x, z) \geq 0 = C_0(x, z)$ and using induction, we see that for x and z, $\{C_n(x, z) : n \geq 0\}$ is an increasing sequence. In fact, it is also not hard to see that there exists a function

$$U(x,z) = \begin{cases} \dfrac{\alpha p \gamma^2}{1-\alpha} + p\gamma z & \text{if } x \leq 0 \\ \dfrac{\alpha p \gamma^2}{1-\alpha} + \dfrac{hx}{1-\alpha} + p\gamma z & \text{if } x > 0, \end{cases}$$

such that $C_n(x, z) \leq U(x, z)$ for all $n \in \mathbf{N}$. The reasoning is as follows. From (3.2),

$$\begin{aligned} C_n(x,z) &= \min_{y \geq 0} \{\varphi(x,y,z) + \alpha E_D E_{\theta,R} C_{n-1}(x + \theta y - zR, D)\} \\ &\leq \varphi(x,0,z) + \alpha E_D E_{\theta,R} C_{n-1}(x - zR, D). \end{aligned}$$

It is clear that when $y = 0$, the cost function in Yuan and Cheung (2003) and $C_n(x, z)$ are equal. Thus, $\lim_{n\to\infty} C_n(x, z)$ exists for all x, z, this limit is denoted by $C(x, z)$.

Lemma 3.6.1 *$C(x,z)$ satisfies the equation:*

$$C(x,z) = \min_{y\geq 0}\{\Phi(x,y,z)\},$$

where $\Phi(x,y,z) = \varphi(x,y,z) + \alpha E_D E_{\theta,R} C(x+\theta y - zR, D)$.

Proof: We apply Theorem 8−14 in Heyman and Sobel (1984) by showing that the four conditions are satisfied. Condition (a) is satisfied because $\lim_{n\to\infty} C_n(x,z)$ exists for all x, z. Condition (b) is satisfied because $\varphi(x,y,z) \geq 0$ for all x, y, and z. Condition (d) is trivially satisfied because both $\varphi(x,y,z)$ and $C_0(x,z)$ are continuous. Using induction, it is clear that $\Phi_n(x,y,z)$ is continuous x and y for all $n \in \mathbf{N}$. ◇

Corollary 3.6.1 *For $z > 0$, we have*

(i). *$C(x,z)$ is differentiable w.r.t x and convex in x.*

(ii). *$\Phi(x,y,z)$ is convex and continuous in x and y, respectively.*

Proof: (i). Since $\lim_{n\to\infty} C_n(x,z) = C(x,z)$ and for each n, $C_n(x,z)$ is convex (c.f Lemma 3.5.1) and differentiable (c.f Corollary 3.5.1). Theorem 10.8 of Rockafellar (1970) guarantees that $C(x,z)$, as a limit of a sequence of convex functions, is both convex and differentiable.

(ii). Theorem 3.1 implies that $\varphi(x,y,z)$ is convex in x. Next, the convexity of $C(x,z)$ in x implies that $E_D E_{\theta,R} C(x+\theta y - zR, D)$ is also convex in x. Thus, $\Phi(x,y,z)$ as a sum of convex functions is convex in x. Next, $C_n(x+sy-zt,w)$ is continuous in y because differentiability of $C(x,z)$ implies continuity in x. Since $\varphi(x,y,z)$ and $E_D E_{\theta,R} C(x+\theta y - zR, D)$ are continuous in y, $\Phi(x,y,z)$ as a sum, is also continuous in y. ◇

Remark: (i). Define $L(x,y,z) = \frac{\partial}{\partial x}\Phi(x,y,z)$. Thus, we have

$$L(x,y,z) = \int_0^1 s\Bigg[(h+p)G\left(\frac{x+sy}{z}\right) - p + \alpha \int_0^\infty \int_0^1 C^{'}(x+sy-zt,w)dG(t)dH(w)\Bigg]dF(s).$$

By the convexity of $C(x,z)$ in x, $\frac{\partial}{\partial y}L(x,y,z) \geq 0$, implying $L(x,y,z)$ is increasing in y. Next, $L(x,0,z) < 0$ and $\lim_{y\to\infty} L(x,y,z) > 0$. Thus, there exists a unique $y^*(x,z)$ such that $L(x,y,z) = 0$. Now, $y^*(x,z)$ is decreasing in x, resulting in the existence of left and right derivatives. Let $(y^*_+)^{'}$ be the right hand derivative of $y^*(x,z)$, we have

$$\int_0^1 s(1+s(y^*_+)^{'})\Bigg[\frac{h+p}{z}g\left(\frac{x+sy}{z}\right) + \alpha \int_0^\infty \int_0^1 C^{''}(x+sy-zt,w)dG(t)dH(w)\Bigg]dF(s) = 0.$$

Clearly, $(y^*_+)^{'}$ exists. A similar expression holds true for the left hand side derivatives and thus, $(y^*_+)^{'} = (y^*_-)^{'}$, concluding differentiability of $y^*(x,z)$ in x.

(ii). Since $C_n(x,z)$ converges pointwise to $C(x,z)$ in x and $\{C_n(x,z)\}$, $C(x,z)$ are differentiable, we have pointwise convergence of $C_n^{'}(x,z)$ to $C^{'}(x,z)$ in x using Lemma 8-5 of Heyman and Sobel (1984).

(iii). Now, $L(x,0,z) = E(\theta)\left[(h+p)G\left(\frac{x}{z}\right) - p + \alpha E_{R,D}(C^{'}(x-zD,R))\right]$. It can be shown that $\frac{\partial}{\partial x}L(x,0,z) > 0$. Furthermore, using Lemma 3.5.1(ii) and the above remark, $E_{R,D}(C^{'}(-zD,R)) \leq 0$. Therefore, $L(0,0,z) < 0$, implying the existence of unique $x^*(z)$ satisfying $L(x,0,z) = 0$. Furthermore, $x^*(z)$ is independent of θ.

(iv). Remark (iii) implies that

$$y^*(x,z) = \begin{cases} > 0 & \text{if } x < x^*(z) \\ 0 & \text{if } x \geq x^*(z), \end{cases}$$

(v). Following the proof of Lemma 3.4.6, we can show that $\frac{\partial}{\partial x}(x + y^*(x, z)) < 0$.

Finally, we aim to characterize the optimal policy for the infinite horizon model. To proceed, we require an additional assumption that the demand that is reserved in each period D is finite random variable, i.e. $D < \infty$, a.s. The following lemma is useful.

Lemma 3.6.2 $\{L_n(x, y, z) : n \in \boldsymbol{N}\}$ *is a sequence of continuous functions that converges pointwise in* x *to* $L(x, y, z)$*, is also continuous.*

Proof: Note that $\{C_n^{'}(x, z)\}$ is a sequence of Riemann integrable functions (because of continuity in x), converging to $C^{'}(x, z)$ (Riemann integrable). Next, the fact that $D < \infty$,a.s implies that we can apply Bounded Convergence Theorem twice so that the order of integration and limit can interchange, resulting in our conclusion. ◇

Using Lemma 3.6.2, one can proceed to show that $y^*(x, z)$ is the limit point of a sequence of $\{y_n^*(x, z) : n \in \mathbf{N}\}$, where each $y_n^*(x, z)$ is the optimal ordering policy of the period n problem. Similarly, one can show that $x^*(z)$ is the limit of $\{x_n^*(z) : n \in \mathbf{N}\}$. We refer the readers to the work of Yuan and Cheung (2003) since the proof is similar.

Theorem 3.3 *In the infinite horizon, the optimal policy is a re-order point policy when the supply is uncertain. The optimal policy can be specified as follows: there exists* $x^*(z)$ *(independent of* θ*) such that the optimal order quantity (dependent on* θ*) is*

$$y^*(x, z) = \begin{cases} > 0 & \text{if } x < x^*(z) \\ 0 & \text{if } x \geq x^*(z), \end{cases}$$

and the minimal cost is

$$C(x,z) = \begin{cases} \Phi(x, y^*(x,z), z) & \text{if } x < x^*(z) \\ \Phi(x, 0, z) & \text{if } x \geq x^*(z). \end{cases}$$

However, if the supply is reliable, then the optimal ordering policy is an order up to policy.

Following the finite horizon model, we proceed to discuss the impact of managing a firm whose distribution of demand cancellation is more variable than his competitor. Again, we suppose that there are two firms, Firm 1 and Firm 2, having the same supply uncertainty factor $1 - \theta$.

Corollary 3.6.2 *For the infinite horizon model, the cost of managing a firm is higher when it sells items to customers whose distribution of demand cancellation is relatively more variable. The statement is true for a relatively more variable yield distribution.*

Proof: We only show that the cost of managing inventory is higher when $R_1 \leq_{var} R_2$ as the proof for the case when $\theta_1 \leq_{var} \theta_2$ is similar. Denote $\varphi_i(x,y,z) = hE[x + \theta y - zR_i]^+ + pE[x + \theta y - zR_i]^-$. The infinite horizon cost for Firm i satisfies $C(x,z) = \min_{y \geq 0} \Phi_i(x,y,z)$, where $\Phi_i(x,y,z) = \varphi_i(x,y,z) + \alpha E_D E_{\theta,R_i} C(x + \theta y - zR_i, D)$. Let $y_i^* = y_i^*(x,z)$ be the optimal ordering level so that the cost for Firm i for the infinite horizon problem is minimized. Now, $R_1 \leq_{var} R_2 \Rightarrow R_1 \leq_{cx} R_2$ and the convexity of $hE[x + sy - zr]^+ + pE[x + sy - zr]^-$ in $r \geq 0$, together with the independence of θ and R_i implies that $\varphi_1(x, y_2^*, z) \leq \varphi_2(x, y_2^*, z)$. Furthermore, $R_1 \leq_{var} R_2 \Rightarrow -R_1 \leq_{cx} -R_2$ and the convexity of $C(x,z)$ in x implies that $E_D E_{\theta,R_1} C(x + sy_2^* - zR_1, w) \leq$

$E_D E_{\theta,R_2} C(x + sy_2^* - zR_2, w)$. As θ, R_i and D are mutually independent, we have $E_D E_{\theta,R_1} C(x + \theta y_2^* - zR_1, D) \leq E_D E_{\theta,R_2} C(x + \theta y_2^* - zR_2, D)$. Thus, we have

$$\begin{aligned}\Phi_1(x, y_2^*, z) &= \varphi_1(x, y_2^*, z) + \alpha E_D E_{\theta,R_1} C(x + \theta y_2^* - zR_1, D) \\ &\leq \varphi_2(x, y_2^*, z) + \alpha E_D E_{\theta,R_2} C(x + \theta y_2^* - zR_2, D) = \Phi_2(x, y_2^*, z).\end{aligned}$$

Finally, the cost for managing inventory in Firm 1 is lower when $R_1 \leq_{var} R_2$ because $\min_{y \geq 0} \Phi_1(x, y, z) \leq \Phi_1(x, y_2^*, z) \leq \Phi_2(x, y_2^*, z) = \min_{y \geq 0} \Phi_2(x, y, z)$. $\diamond$

3.7 Numerical Examples

The motivation of this section is two-fold: we provide numerical evidences to the claims made in the previous sections and more importantly, we illustrate how the results proven can be applied to computing the optimal ordering quantities. The first example shows an instance when the first order stochastic dominance of the supply uncertainty has no impact on the optimal ordering quantity.

Example 1 Let $P\{\theta_1 = 0.3\} = P\{\theta_2 = 0.4\} = P\{\theta_3 = 0.7\} = 0.3$, and $P\{\theta_i = 1\} = 0.7$ for $i = 1, 2, 3$. Thus, $\theta_1 \leq_{st} \theta_2 \leq_{st} \theta_3$. We assume $R \sim Beta(0.5, 1)$, so $G(x) = \sqrt{x}$. For simplicity, let $y_i^* = y_i^*(x, z)$ and $C_{1,i} = C_{1,i}(x, z)$ for $i = 1, 2, 3$. Suppose $h = 100, p = 5000, x = 150$ and $z = 200$. It can be shown that $43.95 = y_1^* \leq y_3^* \leq y_2^* = 46.44$, where $y_3^* = 45.42$. Using (3.4), we have $14626.43 = C_{1,1} \geq C_{1,2} \geq C_{1,3} = 13204.40$, where $C_{1,2} = 14139.29$. If $P\{\theta_4 = 0.2\} = 1$, then the optimal cost in managing the supply chain is $C_{1,4} = 3284.15 \leq C_{1,1}$ but $\theta_4 \leq_{st} \theta_1$.

The following example shows that for non-convex c.d.f, it is not possible to compare the optimal ordering quantity even though one firm has a more variable yield distribution.

Example 2 Let $\theta_1 \sim \text{Beta}(\frac{1}{2}, \frac{1}{2})$, $\theta_2 \sim \text{Beta}(2,2)$, and $R \sim \text{Beta}(1,2)$. Thus, $\theta_1 \geq_{var} \theta_2$. Suppose $x = 0, z = 10, h = 2, p = 3$, then using (3.3), we have $6.86 = y_1^*(0,10) > y_2^*(0,10) = 3.33$. However, if we consider $R' \sim \text{Beta}(\frac{1}{2}, 1)$. Then, we see that $y_1^*(0,10) = 0.99 < y_2^*(0,10) = 1.23$.

Note that both R and R' have c.d.f's which are concave. If the demand cancellation does not have a convex c.d.f, the effect of variability on the relative optimal ordering quantities is not clear. Finally, we provide a counter-example to show that when R_2 is stochastically larger than R_1, the optimal cost of system 2 is higher than system 1.

Example 3 Let $P\{R_1 = 1\} = 0.7$, $P\{R_1 = 0.1\} = 0.3$, and $R_2 = 1$. Thus, $R_1 \leq_{st} R_2$. We assume $\theta \sim U[0,1]$. Suppose $h = 2, p = 3, x = 3$ and $z = 10$. It can be shown that $y_1^* = 8.21$ and $y_2^* = 11.068$. Using (3.4), we have $19.962 = C_{1,1}(3,10) < C_{1,2}(3,10) = 26.514$.

We have conducted extensive studies and found that it is generally not easy to compute the ordering quantities for any initial inventory level. However, for the case $x > x_n(z)$, we can deduce from Theorem 3.2 that $y_n^*(x,z) = 0$. As $x_n(z)$ is independent of supply uncertainty, we can simply apply the results in Yuan and Cheung (2003) to compute $x_n(z)$. For example, suppose the distribution D is exponential with parameter λ, and R being uniformly distributed on $[0,1]$. Yuan and Cheung (2003) show that $x_n(z) = \frac{zp}{p+h}$. For $x < x_n(z)$, the exact ordering quantities can be tedious to compute. However, we provide an example where we can get closed form solution to the optimal ordering quantity.

Example 4 Let $P(R=0)=\frac{1}{4}$, $P(R=1)=\frac{3}{4}$, and $h=\frac{9}{19}p$. If $\theta \sim U[0,1]$ and $D \sim U[50,100]$, then $y_k^*(x,z)=\frac{7}{3}[z-x]^+$ for all $k \geq 1$ and $\frac{7}{4}z-37.5 \leq x \leq z$.

We shall provide an algorithm to illustrate how the optimal quantities are computed in general.

Step 0. Initialization:

Step 0a. Let $C_0(x,z) \equiv 0$ for all x and z.
Step 0c. Given initial inventory level x and reservation level z.
Step 0b. Let $n=1$.

Step 1.

Step 1a. Solve $L_n(x_n(z),0,z)=0$ to obtain $x_n(z)$.
Step 1b. **If** $x \geq x_n(z)$, $y_n^*(x,z)=0$, and go to **Step 3**;
Else, go to **Step 2**.

Step 2.

Step 2a. Solve $L_n(x,y_n^*(x,z),z)=0$ to obtain $y_n^*(x,z)$.
Step 2b. Update $C_n(x,z)=\Phi_n(x,y_n^*(x,z),z)$.
Step 2c. Obtain $C_n'(x,z)=\frac{\partial}{\partial x}C_n(x,z)=\frac{\partial}{\partial x}\Phi_n(x,y_n^*(x,z),z)$.

Step 3.

Let $n \Leftarrow n+1$.
If $n \leq N$, go to **Step 1**;
Otherwise, stop the iteration and complete the computation.

3.8 Concluding Remarks

This chapter has explored the structure of the optimal policy in the presence of demand cancellation and supply uncertainty in the multiple period framework. Extending the work of Yuan and Cheung (2003), this chapter shows that in the presence of supply uncertainty, the optimal replenishment policy has the structure of re-order point type, i.e, in each period, there exists a re-order point such that we order only when the initial inventory level falls below it. It is interesting to note that the re-order point is independent of the supply uncertainty factor. In the presence of supply uncertainty, the convexity of the optimal cost function is preserved, allowing us to derive the optimal replenishment policy using dynamic programming. We also provide some managerial insights into the impacts of both supply uncertainty and demand cancellation on the cost of managing the inventory. For the single period case, we provide an example to show that a stochastically larger demand cancellation does not imply that the optimal cost is reduced. Therefore, we develop a bound on the difference between the cost of managing two firms when they have different cancellation random variables. In our model, the bound turns out to be proportional to the difference between mean number of items not eventually canceled. We also show that if during each period, a stochastically larger fraction of demand not canceled eventually leads to a higher re-order point regardless of how unreliable the supplier is. In particular, if the supply is reliable, it is always more beneficial to order a greater quantity. In terms of managing the inventory cost, we show that variability plays a more significant role. In fact, we show that for the single period, finite horizon and infinite horizon models, it is always more expensive to manage a firm whose distribution of demand cancellation has a relatively higher variability. With a more variable yield distribution, it is also always more expensive to manage the inventory.

This work can be extended in the following directions. The fundamental question of how to optimally handle inventory replenishment in the presence of a setup cost is yet unknown. Incorporating pricing decision into our problem extending the work of Li and Zheng (2006) is another avenue which is interesting. For example, one can determine if a stochastically larger fraction of demand not canceled should lead to higher pricing of the product. The impact of cancellation on the optimal ordering policy in a multi-echelon inventory system such as the Clark-Scarf model is also unknown, providing new arena for future research.

CHAPTER 4

IMPACT OF TRANSPORTATION CONTRACT ON INVENTORY SYSTEMS WITH DEMAND CANCELLATION

4.1 Introduction

This work considers a periodic review inventory system with demand reservation and cancellation under a supply contact. All customer orders need to be reserved one period in advance, and all the reserved orders are allowed to be cancelled before their realization. There is a supply contract where a higher ordering cost is incurred whenever the quantity exceeds a certain number. The optimal inventory policies are analytically derived for single period, finite and infinite horizons, respectively, which are of the type "finite generalized base stock" policy, similar to that in Frederick (2009). The techniques in proving the policy optimality in the infinite horizon scenario in Yuan and Cheung (2003) or Yeo and Yuan (2011) are no longer applicable to the case due to the presence of the ordering cost under the supply contract. We mathematically prove the optimality of these inventory policies, particularly, for the infinite horizon scenario, and analyze the impacts of the supply contract on these optimal inventory policies. The results in the work are an interesting extension to those in Yuan and Cheung (2003).

Our work is motivated by two industrial situations involving retailers obtaining their supply of raw materials to distribute different lines of end products. The first scenario is an online computer retailer connected to an assembly plant, which in turns obtains a certain component from two suppliers: one of them is being capacitated; charges lower ordering cost of delivery via truck while the other is located offshore, incurring higher shipment cost. Ignoring transportation cost differential between the two suppliers, the firm practices an order-up-to policy at an aggregate level for this particular component. Will this ordering policy be optimal? From another perspective, this model is equivalent to the manager facing one supplier offering a single-tier supply contract where a higher ordering cost is charged if the item ordered exceeds a certain quantity. When the supplier offers a multi-tier supply contract, it is equivalent to the scenario of facing two or more capacitated suppliers. According to comScore, Inc., "computer and electronics" category is the greatest outperformer of more than 9% y.o.y (year-on-year) in e-commerce sales growth for Q2 2010. The second scenario involves the gas industry where transportation contracts play a huge role because its reserves are normally quite distant from consumer markets. Natural gas is a commodity with relatively inelastic supply due to recent efforts by countries with proven gas reserves to form a cartel, the Gas Exporting Countries Forum (GECF), to control output. In contrast to supply contract that traditionally fix the volume and price of gas over a specified period, multi-tier contract is often written to provide greater flexibility to reflect the economic value under changing conditions such as winter or output tightening by GEFC. According to NaturalGas.org, a relatively new phenomenon known as "natural gas marketing" has become an integral component of the gas industry. Such marketing activity involves coordinating the business of bringing natural gas from the wellhead to end-users. At AllConnect.com or Whitefence.com, consumers are able to obtain gas via internet marketers and there is a grace period for cancellation without incurring penalty.

Traditionally, a supply contract is a commitment that is established between two parties stretching over a long period of planning horizon. Bassok and Anupindi (1997) analyze a periodic review stochastic inventory model in which the buyer is committed to buying a total minimum quantity over the planning horizon. Henig et al. (1997) study a multi-period inventory-control model under a supply contract that specifies a fixed volume of inventory to deliver. During each period, ordering any quantity exceeding the contracted volume will result in a cost that is proportional to the excess borned by the retailer. They show that the structure of the optimal policy is a three-parameter policy, instead of a base stock policy. The finite and infinite horizon models are solved completely. There have been two works that extend the model considered by Henig et al (1997): the first work is due to Chao and Zipkin (2008) who consider a fixed cost if the order quantity is above the contract volume. They partially characterize the optimal policy for the problem and propose a simple heuristic to compute the parameters of the optimal policy. Xu (2005) considers a periodic review inventory problem with supply contract allowing buyer to cancel his orders. His goal is to choose an ordering and canceling policy so as to minimize the expected cost during the planning horizon. Bassok and Anupindi (2008) consider an important class of supply contract known as the Rolling Horizon Flexibility (RHF) contracts in a multiple period setting. Under such a contract, the buyer is allowed to adjust and update its future commitment in every period. Thus, the contract represents a high level of long term and low level of short term flexibility. They discuss a general model to incorporate adjustment flexibility, and present two heuristics, demonstrating their effectiveness but the structure of the optimal inventory policy is unknown. Lian and Deshmukh (2009) study Rolling Horizon Planning (RHP) supply contracts where the buyer is allowed to increase order amount of future orders on a rolling horizon manner, and has to pay extra cost for any extra quantities of unit ordered. They develop heuristics known as Frozen Ordering Planning (FOP) and second level Frozen Ordering Planning (FOPII). These heuristics are compared against the order-up-to policy using the objective which minimizes the total holding and penalty costs.

Our work can also be viewed as a variant of an inventory problem with multiple suppliers with capacity limit. In the literature on the inventory systems with multiple suppliers, most works focus on dual delivery modes with higher cost and shorter delivery leadtime for the emergency supplier. The pioneering work of Barankin (1961) investigates the optimal policy for dual supply sources for the single period problem. Fukuda (1964) extends his work to the multiple period case. He proves the existence of two parameters $y^0 < y^1$ such that if the stock on hand is less than y^0, then order-up-to the base stock level at the emergency mode and $y^1 - y^0$ at the regular mode, otherwise the optimal policy is a base stock policy at the regular delivery mode. The difference between the leadtimes of the expedited and regular source is one. Whittmore and Saunders (1977) study the multiple period inventory model by allowing the expedited and regular lead times to be of arbitrary length. But the form of the optimal policy is extremely complicated. Chiang and Gutierrez (1996) analyze an inventory model whose review period is larger than the supply leadtimes of both suppliers. Two types of orders can be placed at the regular review and emergency epochs. They determine the optimal policy for placing orders at the different epochs. Yang et al (2005) consider an inventory model with Markovian in-house production capacity, facing stochastic demand and having the option to outsource. They show that the optimal outsourcing policy is always of (s, S) type and the optimal production policy is of the modified base-stock type under fairly general assumptions. Frederick (2009) develops an inventory model with multiple sources of supply. He assumes that when the initial inventory exceeds a certain critical level, the manager will return or "order-down-to" an optimal quantity of inventory at no additional cost. He proves the optimality of the "finite generalized base stock" policy for the discounted cost criterion. The mathematical model considered in his work is a generalization of Henig et al (1997) who study a supply contract embedded in an inventory model.

The positioning of our work with respect to the existing literature is as follows. When the manager orders from a capacitated supplier while simultaneously having a more expensive unlimited supply source, this problem is mathematically equivalent to the retailer engaging in two-tier supply contract. Such equivalence entails us to extend the work of Henig et al. (1997) in an inventory model by embedding it with a two-tier level supply contract. Specifically, if the quantity ordered is greater then v_1, then a cost of c_1 is incurred for delivering the $(v_1 + 1)$-th unit up to $v_2 > v_1$. Furthermore, if the quantity exceeds v_2, then a cost of $c_2 > c_1$ is incurred for delivering the $(v_2 + 1)$-th unit. However, we also take into account the impact of demand cancellation on the optimal replenishment policy. As a result, our analysis of the discounted cost function is bivariate in two information given: initial inventory level and number of items reserved in the previous period. When both v_1 and v_2 goes to infinity, our model collapses to that of Yuan and Cheung (2003). It turns out that our model can easily extend the work of Yuan and Cheung (2003) with non-negative ordering costs. Technically, it is the special case of a one-tier supply contract problem when $v_1 = v_2 = 0$. Our supply contract also differs from Lian and Deshmukh (2009) who assume that unit costs for ordering each unit decreases on the rolling horizon. Furthermore, they did not focus in addressing policy optimality.

Our research yields the following insights. Firstly, much as "order-up-to" policy is popular among industries due to its simple structure, it is in fact suboptimal in the presence of a supply contract and is dominated by "finite generalized base stock" policy. Moreover, the critical numbers are dependent on the reservation parameter. Secondly, we show that the structure of "order-up-to" policy is still preserved when we assume ordering costs in the work of Yuan and Cheung (2003). Without ordering cost, moral hazard on the ordering behavior is induced and optimal quantity in Yuan and Cheung (2003) is always greater than using "generalized base stock policy".

The rest of the work is organized as follows. The model and notations are developed in Section 4.2. Section 4.3 presents a model for the single period. The convexity for the optimal cost is established and the optimal ordering level is derived. Section 4.4 analyzes the finite horizon model. We also compare the differences of the optimal policies among our model, the model of Yuan and Cheung (2003) and the model of Yeo and Yuan (2011). In Section 4.5, we solve the optimal policy for the infinite horizon model. Finally, we provide a concluding note including some possible extensions to this work in Section 4.6.

4.2 Model

We consider a periodic review inventory system. Following the work of Yuan and Cheung (2003) and Yeo and Yuan (2011), all demands are made through reservations. Demands reserved in the previous periods are supposed to be fulfilled in the current period. However, customers' are allowed to cancel their reservation. Denote $\mathbf{N}$ to be the set of non-negative integers. Let D_n be the demand that is reserved during period $n \in \mathbf{N}$, and let R_n be the ratio of the demand reserved during the previous period that is eventually not canceled during period n. We assume that $\{D_n : n \in \mathbf{N}\}$ is a sequence of i.i.d demand random variables with a common distribution $H(x)$ (with

$H(0) = 0$ and $H(\infty) = 1$) and density function $h(x)$. We let $\{R_n : n \in \mathbf{N}\}$ be a sequence of i.i.d ratio random variables whose c.d.f is $G(x)$ (with $G(0) = 0$ and $G(1) = 1$), and p.d.f $g(x)$. Assume that for the transportation contract, a supply of v_1 items is available for the retailer at no additional costs. For orders exceeding v_1 but less than $v_2(> v_1)$, c_1 is charged to retailer for every unit ordered. Finally, $c_2(> c_1)$ is charged for every unit of orders exceeding v_2. Let y be the number of items ordered, it can be seen that the ordering costs for the retailer can be written as $c(y) = c_1[y \wedge v_2 - v_1]^+ + c_2[y - v_2]^+$. We also make the assumption that the cancellation ratios R_n and demands D_n are independent. All the unfulfilled orders are backordered. The inventory holding cost (h) and penalty cost (p) are both incurred on a per unit per unit time basis. At the beginning of the period, the inventory level is x and the demand reserved in the previous period is $z(\geq 0)$. Let y be the decision variable representing the order quantity made at the beginning of the current period. The leadtime is assumed to be zero. As in Yuan and Cheung (2003), the one period cost can be written as $c(y)+\varphi(x, y, z)$, where $\varphi(x, y, z) = hE[x+y-zR]^+ + pE[x+y-zR]^-$. We suppose that $z > 0$. Let $C_n(x, z)$ be the optimal total cost from period n to period 1 given that the initial inventory level is x and the demand reserved in period $n + 1$ is z. We define $C_0(x, z) \equiv 0$ for all x, z. Suppose D is the demand that arrives during period n, and $\alpha \in [0, 1)$ is the discount factor. Denote

$$U_n(x, z) = \min_{x \leq Q \leq x+v_1} \{\varphi(x, Q - x, z) + \alpha E_{R,D} C_{n-1}(Q - zR, D)\}$$

$$V_n(x, z) = -c_1(x + v_1) + \min_{x+v_1 \leq Q \leq x+v_2} \{c_1 Q + \varphi(x, Q - x, z) + \alpha E_{R,D} C_{n-1}(Q - zR, D)\}$$

$$W_n(x, z) = -c_2(x + v_2) + c_1(v_2 - v_1) + \min_{Q \geq x+v_2} \{c_2 Q + \varphi(x, Q - x, z) + \alpha E_{R,D} C_{n-1}(Q - zR, D)\}.$$

One can easily verify that

$$\begin{aligned} C_n(x,z) &= \min_{y\geq 0}\{c(y)+\varphi(x,y,z)+\alpha E_{R,D}C_{n-1}(x+y-zR,D)\} \\ &= \min\{U_n(x,z),V_n(x,z),W_n(x,z)\} \end{aligned} \tag{4.1}$$

Set $\Phi_n(x,y,z) = c(y)+\varphi(x,y,z)+\alpha E_{R,D}C_{n-1}(x+y-zR,D)$, then we can write (4.1) as $C_n(x,z) = \min_{y\geq 0}\Phi_n(x,y,z)$.

For both v_1 and v_2 going to infinity, there is no additional costs for any amount ordered and this model is equivalent to the work of Yuan and Cheung (2003) without ordering costs. If $v_1 = v_2 = v$, then we have a single tier contract where only v is allocated for the supplier at no costs. Furthermore, when $v = 0$ and $c_1 = c_2 = c$, it is the model of Yuan and Cheung (2003) with ordering costs.

4.3 Single Period Analysis

In this section, we will explore the structure of the ordering policy for the single period case. We assume that x denotes the inventory level and z denotes the demand reserved in the previous period. It is easy to see that $S_1(z) = zG^{-1}(\frac{p}{h+p})$, $s_1'(z) = zG^{-1}(\frac{p-c_1}{h+p})$, and $s_1(z) = zG^{-1}(\frac{p-c_2}{h+p})$ are minimizers of $J(Q) = hE(Q-zR)^{+}+pE(Q-zR)^{-}$, $c_1Q+J(Q)$, and $c_2Q+J(Q)$ respectively. Define $J(Q) = \varphi(x,Q-x,z)$.

Theorem 4.1 *Let $y^*(x,z)$ denote the optimal ordering quantity. The optimal policy for the single period problem is of $(s_1(z), s_1'(z), S_1(z), v_1, v_2)$ given by*

$$y^*(x,z) = \begin{cases} (s_1(z) - x) \vee v_2 & \textit{if } x < s_1'(z) - v_2 \\ (s_1'(z) - x) \vee v_1 & \textit{if } s_1'(z) - v_2 \le x < S_1(z) - v_1 \\ (S_1(z) - x) \vee 0 & \textit{if } x \ge S_1(z) - v_1. \end{cases}$$

Proof: (i). For $x < s_1(z) - v_2$, we have $U_1(x,z) = \varphi(x, v_1, z)$. Since $x + v_2 < s_1(z)$, then $V_1(x,z) = -c_1(x+v_1) + \{c_1Q + J(Q)\}|_{Q=x+v_2} \le -c_1(x+v_1) + \{c_1Q + J(Q)\}|_{Q=x+v_1} = U_1(x,z)$. Furthermore, $W_1(x,z) = -c_2(x+v_2) + c_1(v_2 - v_1) + \{c_2Q + J(Q)\}|_{Q=s_1(z)} \le -c_2(x+v_2) + c_1(v_2 - v_1) + \{c_2Q + J(Q)\}|_{Q=x+v_2} = V_1(x,z)$. Thus, $Q^*(x,z) = s_1(z)$ and $y^*(x,z) = s_1(z) - x$.

(ii). For $s_1(z) - v_2 \le x < s_1'(z) - v_2$, then $U_1(x,z) = \varphi(x, v_1, z)$. Since $x + v_2 < s_1'(z)$, then $V_1(x,z) = -c_1(x+v_1) + \{c_1Q + J(Q)\}_{Q=x+v_2} \le -c_1(x+v_1) + \{c_1Q + J(Q)\}_{Q=x+v_1} = U_1(x,z)$. The second inequality is due to $c_1Q + J(Q)$ being convex over $x + v_1 < x + v_2 < s_1'(z)$. Finally, for $Q \ge x + v_2 > s_1(z)$, $W_1(x,z) = -c_2(x + v_2) + c_1(v_2 - v_1) + \{c_2Q + J(Q)\}_{Q=x+v_2} = V_1(x,z)$. Thus, $Q^*(x,z) = x + v_2$ and $y^*(x,z) = v_2$.

(iii). For $s_1'(z) - v_2 \le x < s_1'(z) - v_1$, then $x + v_1 < s_1(z)$ implies that $U_1(x,z) = \varphi(x, v_1, z) \ge -c_1(x+v_1) + \{c_1Q + J(Q)\}|_{Q=s_1'(z)} = V_1(x,z)$. The inequality is due to $x + v_1 < s_1(z) < s_1'(z)$. Next, $x + v_2 \ge s_1'(z) > s_1(z)$, then

$$\begin{aligned} W_1(x,z) &= -c_2(x+v_2) + c_1(v_2 - v_1) + \{c_2Q + J(Q)\}|_{Q=x+v_2} \\ &= -c_1(x+v_1) + \{c_1Q + J(Q)\}|_{Q=x+v_2} \\ &\ge -c_1(x+v_1) + \{c_1Q + J(Q)\}|_{Q=s_1'(z)} = V_1(x,z). \end{aligned}$$

Thus, $Q^*(x,z) = s_1'(z)$ and $y^*(x,z) = s_1'(z) - x$.

(iv). For $s_1(z) - v_1 \le x < S_1(z) - v_1$, then $U_1(x,z) = \varphi(x, v_1, z)$. Next, $x < s'(z)$ implies $V_1(x,z) = -c_1(x+v_1) + \{c_1 Q + J(Q)\}|_{Q=s_1'(z)} \le U_1(x,z)$. Finally, as $x + v_2 > s_1'(z)$, then $W_1(x,z) = -c_2(x+v_2) + c_1(v_2 - v_1) + \{c_2 Q + J(Q)\}|_{Q=x+v_2} \ge V_1(x,z)$. Thus, $Q^*(x,z) = s_1'(z)$ and $y^*(x,z) = s_1'(z) - x$.

(v). For $S_1(z) - v_1 \le x < S_1(z)$, then $x + v_1 \ge S_1(z)$ and $U_1(x,z) = \varphi(z, S_1(z) - x, z)$. As $x + v_1 > s_1'(z)$, then $V_1(x,z) = \varphi(x, v_1, z)$. Finally, $x + v_2 > s_1(z)$ implies $W_1(x,z) = \varphi(x, v_2, z)$. Thus, $Q^*(x,z) = S_1(z)$ and $y^*(x,z) = S_1(z) - x$.

(vi). For $x \ge S_1(z)$, then $U_1(x,z) = \varphi(x, 0, z) \le \varphi(0, v_1, z) = V_1(x,z)$. Finally, $W_1(x,z) = c_1(v_2 - v_1) + \varphi(x, v_2, z) \ge U_1(x,z)$. Thus, $Q^*(x,z) = x$ and $y^*(x,z) = 0$. $\diamond$

From Theorem 4.1, the optimal ordering quantity $y^*(x,z)$ is increasing in z and decreasing in x. Let $C_1(x,z)$ be the given single period optimal cost. Our next goal is to show that $C_1(x,z)$ is convex in $x \in (-\infty, \infty)$. Let x be the given initial inventory level, the optimal cost is given by $\varphi(x, v_i, z)$. It is easy to see that $\frac{\partial}{\partial x}\varphi(x, v_i, z) = (h+p)G\left(\frac{x+v_i}{z}\right) - p$ and $\frac{\partial^2}{\partial x^2}\varphi(x, v_i, z) = \frac{h+p}{z} g\left(\frac{x+v_i}{z}\right) > 0$. Hence, $C_1(x,z)$ is also convex on these intervals.

4.4 Finite Horizon Analysis

This section is devoted to determining the optimal policy for the finite horizon model. Denote $C_n'(x,z)$ be the first order derivatives of $C_n(x,z)$ w.r.t x. To begin our exposition, we need a few technical lemmas in order to make our argument complete. Suppose X and Y are independent random variables with probability spaces given by $(\Omega_X, \mathcal{F}_X, P_X)$ and $(\Omega_Y, \mathcal{F}_Y, P_Y)$ respectively. Define the left hand derivatives and right hand derivatives of $g(x)$ at x to be $g^-(x)$ and $g^+(x)$, respectively.

Lemma 4.4.1 *Let $h : \boldsymbol{R}^2 \to \boldsymbol{R}$ is a Borel measurable function that is convex such that $E[h(a - zX, Y)] < \infty$ for all $a \in \boldsymbol{R}$ and $z > 0$. Then, $E[h(a - zX, Y)]$ is convex for all $a \in \boldsymbol{R}$. Furthermore, the left and right hand derivatives w.r.t a exists of $E[h(a - zX, Y)]$ and is equal to $E[h^-(a - zX, Y)]$ and $E[h^+(a - zX, Y)]$, respectively.*

Proof: Let $z > 0$ be given and $\omega_X \in \Omega_X, \omega_Y \in \Omega_Y$. For notational simplicity, for the realized values of X and Y given by $X \equiv X(\omega_X)$ and $Y \equiv Y(\omega_Y)$. For any real numbers $a_1 < a_2$ and $\lambda \in (0, 1)$, we have $h((1 - \lambda)a_1 + \lambda a_2 - zX, Y) \leq (1 - \lambda)g(a_1 - zX, Y) + \lambda g(a_2 - zX, Y)$ due to the convexity of h in the first variable. Due to the finiteness of $E[h(a - zX, Y)]$ for all a, we can take expectation (where linearity holds) by applying Fubini's Theorem to conclude our result. Next, we prove the existence of $E[h^-(a - zX, Y)]$. The convexity of $E[h(a - zX, Y)]$ in a implies that $\frac{1}{\delta}[h(a - zX, Y)] - [h(a - \delta - zX, Y)]$ is an monotone sequence of increasing reals as $\delta \to 0^+$ that converges to $h^-(a - zX, Y)$. Furthermore, $E[h(a - zX, Y)] < \infty$ implies that $\frac{1}{\delta}[h(a - zX, Y)] - [h(a - \delta - zX, Y)] < \infty$. Thus, we can interchange the limit and expectation via the monotone convergence theorem, together with Fubini's Theorem to conclude that $\frac{1}{\delta}[h(a - zX, Y)] - [h(a - \delta - zX, Y)] \to E[h^-(a - zX, Y)]$ due to the uniqueness of limit. The right hand derivative of $E[h(a - zX, Y)]$ w.r.t a is proven similarly. $\diamond$

Lemma 4.4.2 *For all $n \geq 1$, we have*

(a). *$C_n'(x, z)$ exists and is increasing in x.*

(b). *$C_n'(x, z) \leq 0$ for some $S_n(z)$ and all $x \leq S_n(z)$.*

(c). $\lim_{x\to\infty} C_n'(x,z) \geq 0$ *for all* z.

Proof: Let $z > 0$ be given. For $n = 1$, it is clear that (a)-(c) is true. Suppose for $n = k$, the above properties are also true, we want to show that these statements are true for $n = k + 1$ as well. By induction hypothesis, $C_k(x, z)$ is finite for all $x \in (-\infty, \infty)$ and $E[C_k(x - zR, D)] < \infty$. Furthermore, $C_k(x, z)$ is continuous in x. Thus, Lemma 4.4.1 implies that $E[C_k(x-zR, D)]$ is convex in x and $E[C_k'(x-zR, D)]$ exists. This implies that $U_{k+1}(x, z), V_{k+1}(x, z)$, and $W_{k+1}(x, z)$ are convex in x. Define $F_n(Q) = \varphi(x, Q-x, z)+\alpha E_{R,D}C_{n-1}(Q-zR, D)$. Let us assume that $S_{k+1}(z)$, $s'_{k+1}(z)$, and $s_{k+1}(z)$ be minimizers of $F_{k+1}(Q), c_1Q+F_{k+1}(Q)$, and $c_2Q+F_{k+1}(Q)$ respectively.

Claim 1 *For each* k*, we have* $s_k(z) \leq s_k'(z) \leq S_k(z)$.

Proof [1]: As $S_k(s)$ is the minimizer of $F_k(Q)$, then $c_1S_k(z) + F_k(S_k(z)) \geq c_1s_k'(z) + F_k(s_k'(z)) \geq c_1s_k'(z) + F_k(S_k)$. Thus, $S_k(z) \geq s_k'(z)$. Similarly, $S_k(z) \geq s_k(z)$. Next, we have

$$
\begin{aligned}
c_1s_k(z) + F_k(s_k(z)) &\geq c_1s_k'(z) + F_k(s_k'(z)) \\
&= c_2s_k'(z) + F_k(s_k'(z)) + (c_1 - c_2)s_k'(z) \\
&\geq c_2s_k(z) + F_k(s_k(z)) + (c_1 - c_2)s_k'(z).
\end{aligned}
$$

This implies that $(c_1-c_2)s_k(z) \geq (c_1-c_2)s_k'(z)$, and since $c_1 \leq c_2$, then $s_k(z) \leq s_k'(z)$. $\diamond$

To prove (c), for $x \geq S_{k+1}(z)$, then $U_{k+1}(x, z) = \varphi(x, 0, z) + \alpha E_{R,D}C_k(x - zR, D)$. For $x \in [S_{k+1}(z), \infty)$, $W_{k+1}(x, z) \geq V_{k+1}(x, z) \geq U_{k+1}(x, z)$. Thus, we are left to consider the limit of $C_{k+1}(x, z) = M_{k+1}(x, z) = \varphi(x, 0, z) + \alpha E_{R,D}[C_k(x - zR, D)]$ as x tends to ∞. By induction hypothesis, $\lim_{x\to\infty} C_k(x, z) > 0$, thus, we can conclude that $\lim_{x\to\infty} C_{k+1}(x, z) = \lim_{x\to\infty} M_{k+1}(x, z) > 0$. We now prove (a) and (b).

(i). If $x < s_{k+1}(z) - v_2$, then $U_{k+1}(x,z) = F_{k+1}(x+v_1) \geq F_{k+1}(x+v_2)$ since $x+v_1 < x+v_2 < s_{k+1}(z)$. Then, $V_{k+1}(x,z) = -c_1(x+v_1) + \{c_1(x+v_2) + F_{k+1}(x+v_2)\} \geq U_{k+1}(x,z)$. Finally, $W_{k+1}(x,z) = -c_2(x+v_2) + c_1(v_2 - v_1) + \{c_2 s_{k+1}(z) + F_{k+1}(s_{k+1}(z))\} \leq V_{k+1}(x,z)$. Thus, $y^*_{k+1}(x,z) = s_{k+1}(z) - x$, $C_{k+1}(x,z) = W_{k+1}(x,z)$ and $C'_{k+1}(x,z) = -c_2 < 0$.

(ii). If $s_{k+1}(z) - v_2 \leq x \leq s'_{k+1}(z) - v_2$, then $U_{k+1}(x,z) = F_{k+1}(x+v_1) \geq -c_1(x+v_1) + \{c_1 Q + F_{k+1}(Q)\}|_{Q=x+v_2} = V_{k+1}(x,z)$. Furthermore, we see that $W_{k+1}(x,z) = -c_2(x+v_2) + c_1(v_2 - v_1) + \{c_2 Q + F_{k+1}(Q)\}|_{Q=x+v_2} = V_{k+1}(x,z)$. Hence, $y^*_{k+1}(x,z) = v_2$ and $C_{k+1}(x,z) = c_1(v_2 - v_1) + F_{k+1}(x+v_2)$. Thus, for $x \in [s_{k+1}(z) - v_2, s'_{k+1}(z) - v_2)$, we have

$$\begin{aligned} C'_{k+1}(x,z) &= \frac{\partial}{\partial x} V_{k+1}(x,z)|_{Q=x+v_2} = \frac{\partial}{\partial Q} V_{k+1}(x,z) \times \frac{\partial x}{\partial Q}|_{Q=x+v_2} \\ &= \frac{\partial}{\partial Q} V_{k+1}(x,z)|_{Q \in [s_{k+1}(z), s'_{k+1}(z))} < 0. \end{aligned}$$

The last inequality holds because of the convexity of $F_{k+1}(Q)$ and $s'_{k+1}(z)$ being its minimum.

(iii). If $s'_{k+1}(z) - v_2 \leq x < s'_{k+1}(z) - v_1$, then $U_{k+1}(x,z) = F_{k+1}(x+v_1)$ and

$$\begin{aligned} W_{k+1}(x,z) &= -c_2(x+v_2) + c_1(v_2 - v_1) + \{c_2 Q + F_{k+1}(Q)\}|_{Q=x+v_2} \\ &= -c_1(x+v_1) + \{c_1 Q + F_{k+1}(Q)\}|_{Q=x+v_2} \\ &\geq -c_1(x+v_1) + \{c_1 Q + F_{k+1}(Q)\}|_{Q=s'_{k+1}(z)} = V_{k+1}(x,z). \end{aligned}$$

Clearly, $U_{k+1}(x,z) \geq V_{k+1}(x,z)$ and thus $y^*_{k+1}(x,z) = s'_{k+1}(z) - x$. It is easy to see that $C'_{k+1}(x,z) = -c_1 < 0$.

(iv). If $s'_{k+1}(z) - v_1 \le x < S_{k+1}(z) - v_1$, then

$$\begin{aligned} W_{k+1}(x,z) &= -c_2(x+v_2) + c_1(v_2 - v_1) + \{c_2 Q + F_{k+1}(Q)\}|_{Q=x+v_2} \\ &= -c_1(x+v_1) + \{c_1 Q + F_{k+1}(Q)\}|_{Q=x+v_2} \\ &\ge -c_1(x+v_1) + \{c_1 Q + F_{k+1}(Q)\}|_{Q=x+v_1} = V_{k+1}(x,z) = U_{k+1}(x,z). \end{aligned}$$

The inequality is due to $s'_{k+1}(z) \le x + v_1 < x + v_2$ and $c_1 Q + F_{k+1}(Q)$ is increasing on $[s'_{k+1}(z), \infty)$. Thus, $y^*_{k+1}(x,z) = v_1$ and $C'_{k+1}(x,z) = \frac{\partial}{\partial Q} F_{k+1}(Q)|_{Q=x+v_1}$. Since $s'_{k+1}(z) \le x + v_1 < S_{k+1}(z)$, then $C'_{k+1}(x,z) < 0$ as $S_{k+1}(z)$ is the minimizer for $F_{k+1}(Q)$.

(v). If $S_{k+1}(z) - v_1 \le x < S_{k+1}(z)$, then $x + v_1 \ge S_{k+1}(z)$ implies that $F_{k+1}(S_{k+1}(z))$. Furthermore, as $x + v_1 \ge S_{k+1}(z) > s'_{k+1}(z)$ and $c_1 Q + F_{k+1}(Q)$ is increasing convex on $Q \in [x+v_1, \infty)$, we have $V_{k+1}(x,z) = F_{k+1}(x+v_1) \ge U_{k+1}(x,z)$. Finally, $W_{k+1}(x,z) = -c_2(x+v_2) + c_1(v_2 - v_1) + \{c_2 Q + F_{k+1}(Q)\}|_{Q=x+v_2} = c_2(v_2 - v_1) + F_{k+1}(x+v_2) \ge V_{k+1}(x,z)$. Thus, $y^*_{k+1}(x,z) = S_{k+1}(z) - x$ and $C'_{k+1}(x,z) = 0$. Therefore, from (i)-(v), (b) holds.

Since $C_{k+1}(x,z) = M_{k+1}(x,z)$ is independent of x, we note that the first order derivative is thus 0, i.e. $C'_{k+1}(x,z) = 0$. Define $\rho(\lambda) = (h+p)G\left(\frac{x+\lambda}{z}\right) - p +$

$\alpha E_{R,D}C'_k(x+\lambda-zR,D)$. In summary, we can express the first order derivative of $C_{k+1}(x,z)$ to be

$$C'_{k+1}(x,z)=\begin{cases}-c_2 & \text{if } x<s_{k+1}(z)-v_2,\\ \rho(v_2) & \text{if } s_{k+1}(z)-v_2\le x<s'_{k+1}(z)-v_2,\\ -c_1 & \text{if } s'_{k+1}(z)-v_2\le x<s'_{k+1}(z)-v_1,\\ \rho(v_1) & \text{if } s'_{k+1}(z)-v_1\le x\le S_{k+1}(z)-v_1,\\ 0 & \text{if } S_{k+1}(z)-v_1\le x\le S_{k+1}(z),\\ \rho(0) & \text{if } x\ge S_{k+1}(z).\end{cases}$$

Thus, it is clear that $C'_{k+1}(x,z)$ is non-decreasing in $x\in(-\infty,\infty)$ and (a) is proven. $\diamond$

Part (a) of Lemma 4.4.2 states that the optimal cost function is convex for every period n. This is crucial to establish the structure of the optimal inventory policy. Part (b) of Lemma 4.4.2 states that at every period n, there exists a turning point $S_n(z)$ so that the optimal cost function is increasing in the initial inventory level. We formally state the optimal inventory policy in our next result as follows.

Theorem 4.2 *Let $y_n^*(x,z)$ denote the optimal ordering quantity during period n. The optimal policy for the period n problem is of the form $(s_n(z),s'_n(z),S_n(z),v_1,v_2)$ given by*

$$y_n^*(x,z)=\begin{cases}(s_n(z)-x)\vee v_2 & \textit{if } x<s'_n(z)-v_2\\ (s'_n(z)-x)\vee v_1 & \textit{if } s'_n(z)-v_2\le x<S_n(z)-v_1\\ (S_n(z)-x)\vee 0 & \textit{if } x\ge S_n(z)-v_1.\end{cases}$$

The minimal cost is

$$C_n(x,z) = \begin{cases} \Phi_n(x, (s_n(z) - x) \vee v_2, z) & \text{if } x < s'_n(z) - v_2, \\ \Phi_n(x, (s'_n(z) - x) \vee v_1, z) & \text{if } s'_n(z) - v_2 \le x < S_n(z) - v_1, \\ \Phi_n(x, (S_n(z) - x) \vee 0, z) & \text{if } x \ge S_n(z) - v_1. \end{cases}$$

In particular, the optimal policy is an order-up-to policy with the re-order point being $S_n(z)$ *when* $v_1 = v_2 = 0$.

During k^{th} period, let $\mathcal{T}_{k,1}$ denote the set of critical points such that $\mathcal{T}_{k,1} = \{s_{k,1}, s_{k,2}, s_{k,3}\} \cup \{-\infty, \infty\}$ such that $-\infty = s_{k,0} < s_{k,1} < s_{k,2} < s_{k,3} < s_{k,4} = \infty$. Define $\mathcal{T}_{k,2} = \{r_{k,0}, r_{k,1}, r_{k,2}, r_{k,3}\}$ such that $r_{k,0} = \infty > r_{k,1} > r_{k,2} > r_{k,3}$. The set $\mathcal{T}_{k,1}$ is the base stock level for the policy that is dependent on the initial inventory during period k. Finite generalized base stock policy (see Frederick (2009)) can be described as follows: when $s_{k,i} - r_{k,i} \le x < s_{k,i+1} - r_{k,i}$, we order exactly $r_{k,i}$ units; and when $s_{k,i} - r_{k,i-1} \le x < s_{k,i} - r_{k,i}$, we order-up-to $s_{k,i}$ for $i = 1, 2, 3$. In the context of our model, $\mathcal{T}_{k,1} = \{s_k(z), s'_k(z), S_k(z)\} \cup \{-\infty, \infty\}$ and $\mathcal{T}_{k,2} = \{\infty, v_2, v_1, 0\}$. One key difference is that our critical points in $\mathcal{T}_{k,1}$ depends on z, the reservation quantity. Furthermore, his model assumes that inventory are returnable at no costs. As a result, his optimal ordering rule can become negative when the inventory on hand becomes greater or equal to $s_{k,3}$. In our model, we simply do not order. The form of our optimal policy also generalizes the work of Henig et al (1997).

Let us graphically establish the relationships among the optimal inventory policies for this current model with two other existing models. Yeo and Yuan (2011) prove the optimality of the critical point policy for the unreliable supply problem which generalizes the work of Yuan and Cheung (2003) who establish the optimality of order-up-to policy when the supplier is reliable. In order to represent graphically the relationships among the three models, we need to consider the relative values of $s_k(z) - v_2 < s'_k(z) - v_2 < s'_k(z) - v_1 < S_k(z) - v_1 < S_k(z)$. Figure 4.1 depicts the five possibilities of zero that can occur on the real line.

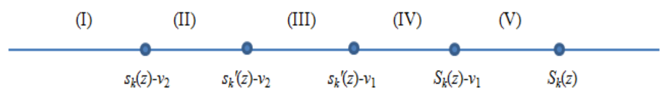

Figure 4.1: The positions that zero can lie in.

Furthermore, in each of these cases, there are a number of further subcases (which can be shown to be a total of ten) which describes relative size of the elements in the set $\mathcal{U}_k = \{v_1, v_2, s_k(z), s'_k(z), S_k(z)\}$, under the constraints that $v_1 < v_2$ and $s_k(z) < s'_k(z) < S_k(z)$. Table 4.1 shows all the different possible arrangements for the elements.

Table 4.1: Possible arrangement for elements in $\mathcal{U}_k$

Cases	
$v_2 < s_k(z)$	$v_1 < v_2 < s_k(z) < s'_k(z) < S_k(z)$
$s_k(z) \le v_2 < s'_k(z)$	$v_1 < s_k(z) < v_2 < s'_k(z) < S_k(z)$
	$s_k(z) < v_1 < v_2 < s'_k(z) < S_k(z)$
$v_1 < s'_k(z) < v_2$	$s_k(z) < v_1 < s'_k(z) < v_2 < S_k(z)$
	$v_1 < s_k(z) < s'_k(z) < v_2 < S_k(z)$
	$s_k(z) < v_1 < s'_k(z) < S_k(z) < v_2$
	$v_1 < s_k(z) < s'_k(z) < S_k(z) < v_2$
$s'_k(z) < v_1 < S_k(z)$	$s_k(z) < s'_k(z) < v_1 < v_2 < S_k(z)$
	$s_k(z) < s'_k(z) < v_1 < S_k(z) < v_2$
$S_k(z) < v_1$	$s_k(z) < s'_k(z) < S_k(z) < v_1 < v_2$

We shall state a result comparing the optimal replenishment quantities between our current model and the model of Yuan and Cheung (2003) for the finite horizon case. For this purpose, denote $y^*_{k,o}(x, z)$ and $y^*_{k,t}(x, z)$ be the k^{th} period replenishment quantities for the model of Yuan and Cheung (2003) and our model, respectively. For convenience, we shall call them M0 and M1. The argument is similar to the case of the single period model.

We shall give the illustrations of the following two cases, while the rest can be developed similarly.

Case (I): $0 < s_k(z) - v_2$. Under this case, $v_1 < v_2 < s_k(z) < s'_k(z) < S_k(z)$ is the only possible subcase. Figure 4.2 illustrates the connections among the three inventory models with demand cancellations.

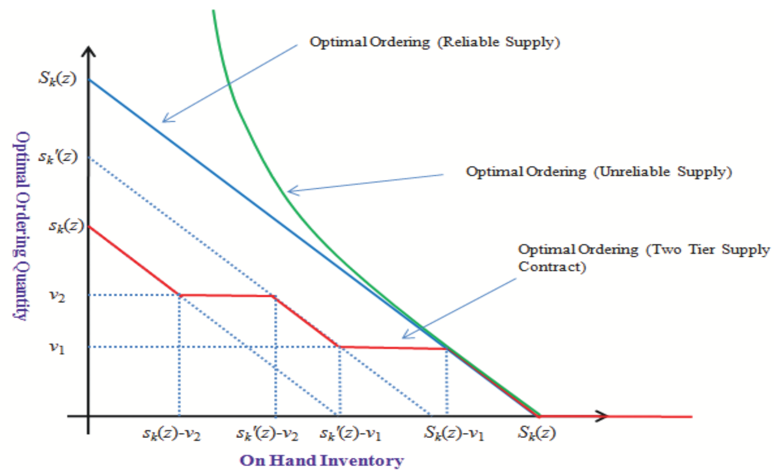

Figure 4.2: Optimal Inventory Policy for three models for Case (I).

Case (II): $s_k(z) - v_2 \leq 0 < s'_k(z) - v_2$. Under this case, we have $s_k(z) < v_2 < s'_k(z) < S_k(z)$. However, under the constraint of $v_1 < v_2$, we have two further subcases: (i). $v_1 < s_k(z) < v_2 < s'_k(z) < S_k(z)$ and (ii). $s_k(z) < v_1 < v_2 < s'_k(z) < S_k(z)$. Figure 4.3 illustrates similar connections for the two subcases.

Corollary 4.4.1 *For period k, the optimal replenishment quantity in M0 is always greater than equal to M1, i.e.* $y^*_{k,o}(x,z) \geq y^*_{k,t}(x,z)$.

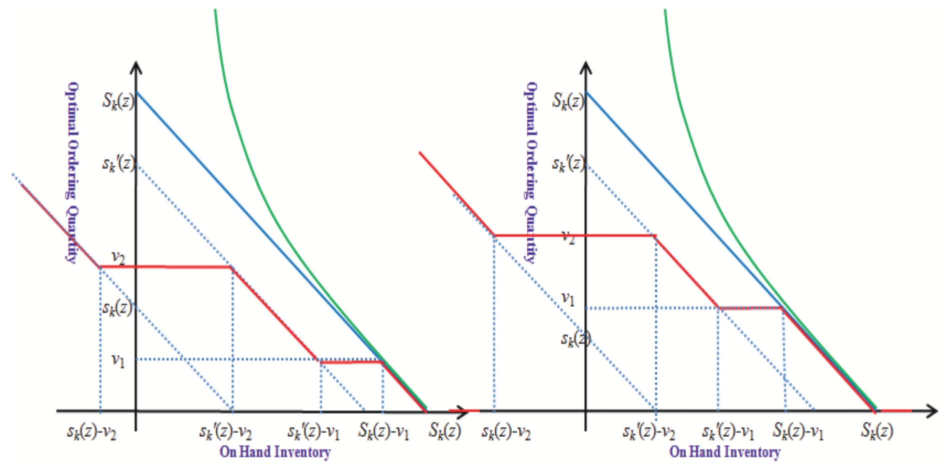

Figure 4.3: Optimal Inventory Policy for three models for Case (II).

Proof: From Claim 1, we have $s_k(z) \leq s_k'(z) \leq S_k(z)$. Furthermore, the optimal replenishment policy given by Yuan and Cheung (2003) is given by

$$y^*_{k,o}(x,z) = \begin{cases} S_k(z) - x & \text{if } x < S_k(z) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have $y^*_{k,o}(x,z) \geq y^*_{k,t}(x,z)$. ◇

Furthermore, we can deduce the exact difference in the optimal replenishment quantity between M0 and M1. Let $d_k(x,z) = y^*_{k,o}(x,z) - y^*_{k,t}(x,z)$ denote the difference. As in the case of the single period, we can easily show that for each period k,

$$d_k(x,z) = \begin{cases} S_k(z) - s_k(z) & \text{if } x < s_k(z) - v_2 \\ S_k(z) - (x + v_2) & \text{if } s_k(z) - v_2 \leq x < s_k'(z) - v_2 \\ S_k(z) - s_k'(z) & \text{if } s_k'(z) - v_2 \leq x < s_k'(z) - v_1 \\ S_k(z) - (x + v_1) & \text{if } s_k'(z) - v_1 \leq x < S_k(z) - v_1 \\ 0 & x \geq S_k(z). \end{cases}$$

We shall extend the model of Yuan and Cheung (2003) by considering the presence of non-negative ordering cost. Using our model, the proof of policy optimality is easily deduced by considering $v_1 = v_2 = 0$. Let c be the per unit ordering cost and $F_n(Q) = \varphi(x, Q - x, z) + \alpha E_{R,D} C_{n-1}(Q - zR, D)$. Furthermore, suppose $s_n(z)$ and $S_n(z)$ are the minimizers of $cQ + F_n(Q)$ and $F_n(Q)$, respectively.

Corollary 4.4.2 (Yuan and Cheung (2003)) *For n periods, $1 \leq n \leq N$, the optimal policy is an order-up-to policy. That is, when $x < s_n(z)$, the optimal policy is to order-up-to $x + y_n(x, z)$, where $y_n(x, z) = s_n(z) - x > 0$, and the optimal total cost is $\Phi_n(x, y_n(x, z), z)$. When $x \geq s_n(z)$, the optimal policy is not to order anything, and the optimal total cost is $\Phi_n(x, 0, z)$.*

4.5 Infinite Horizon Analysis

This section focuses on studying the discounted infinite horizon model. Similar to the work of Yuan and Cheung (2003), our goal is to establish the Bellman's equation for the infinite horizon model and show that the subscript n can be dropped. There are two crucial observations that can be made. It is easy to see that $C_{1,}(x, z) \geq 0 = C_0(x, z)$ and using induction, we see that for x and z, $\{C_n(x, z) : n \geq 0\}$ is an increasing sequence. Following the argument of Yuan and Cheung (2003), there exists a function

$$\Xi(x, z) = \begin{cases} \dfrac{\alpha p \gamma^2}{1 - \alpha} + p\gamma z & \text{if } x \leq 0 \\ \dfrac{\alpha p \gamma^2}{1 - \alpha} + \dfrac{hx}{1 - \alpha} + p\gamma z & \text{if } x > 0, \end{cases}$$

such that $C_n(x,z) \leq \Xi(x,z)$ for all $n \in \mathbf{N}$. Throughout this section, we denote the initial inventory level and reservation level to be x_0 and z_0. We can represent the discounted cost by the extended real-valued objective function

$$J^\alpha(x_0, z_0; U) = \sum_{k=0}^{\infty} \alpha^k E[c(u_k) + \varphi(x_k, u_k, z_k)],$$

where $x_{k+1} = x_k + u_k - z_k R_k$. It is easy to see that if $U = (0, 0, ...)$, then $J^\alpha(x, z; U) < \infty$.

Proposition 4.5.1 *Let x and $z > 0$ be given. Suppose $y_n^*(x,z)$ is the optimal decision for $C_n(x,z)$ for $n \geq 1$. There exists a compact set $I_{x,z} = [q_+, q^+]$ for some $q_+ \equiv q_+(x,z)$ and $q^+ \equiv q^+(x,z)$ such that $y_n^*(x,z) \in I_{x,z}$ for all $n \geq 1$.*

Proof: Suppose we have $\widetilde{u}_k(x,z) = 0$ for all $k \geq 1$. Then the infinite horizon cost under this rule is given by $J^\alpha(x, z; \mathbf{0})$. Define

$$q_+ = \sup\{y \geq Q : \varphi(x, Q - x, z) > J^\alpha(x, z; \mathbf{0})\}$$
$$q^+ = \inf\{y \leq Q : \varphi(x, Q - x, z) > J^\alpha(x, z; \mathbf{0})\}.$$

Since $\lim_{Q\to+\infty} \varphi(x, Q - x, z) > J^\alpha(x, z; \mathbf{0})$ and $\varphi(x, -x, z) < J^\alpha(x, z; \mathbf{0})$, the convexity of $\varphi(x, Q - x, z)$ in Q ensures that $q_+ < q^+$ exists. Let $q \notin [q_+, q^+]$, then $c(q) + \varphi(x, q - x, z) + \alpha E[C_{k-1}(q - zR, D)] > J^\alpha(x, z; \mathbf{0})$. Since $C_k(x,z)$ is the optimal cost, we have

$$C_k(x,z) = \min_{y\geq 0}\{c(y) + \varphi(x, y - x, z) + \alpha E[C_{k-1}(x + y - zR, D)]\} \leq J^\alpha(x, z; \mathbf{0}).$$

Hence, $Q_k^*(x,z) = x + y_k^*(x,z) \in [q_+, q^+]$. ◇

It is established in Lemma 4.4.2 that $C_n'(x,z)$ exists and is non-decreasing in x, thus implying continuity in x. Proposition 4.5.1 allows us to apply Dini's Theorem (see Theorem A.2.1 of Beyer et al (2009)) where pointwise convergence implies uniform convergence of a monotone sequence of continuous function on a compact set. The next result shows that we can remove the subscript n in the dynamic programming formulation of the infinite horizon case. The idea of the proof comes from Theorem 8-14 of Heyman and Sobel (1984). They show that the subscript n can be dropped when the dynamic programming involves the maximization of the functional equation. As our optimal cost formulation for this problem is bivariate and involves cost minimization, we modify our proof accordingly.

Theorem 4.3 *Let x and z be given. Then, $\lim_{n\to\infty} C_n(x,z)$ exists, denoted by $C(x,z)$. Furthermore, $C(x,z)$ is convex in $x \in (-\infty,\infty)$ and satisfies the equation: $C(x,z) = \min_{y\geq 0}\Phi(x,y,z)$, where $\Phi(x,y,z) = c(y) + \varphi(x,y,z) + \alpha E_{R,D}C(x+y-zR,D)$.*

Proof: Note that each $C_n(x,z)$ is non-negative in x and forms a non-decreasing sequence in $n \geq 0$. Hence, the existence of pointwise limit, denoted by $C(x,z)$ is due to the convergence of this monotone sequence. As each $C_n(x,z)$ is convex, the convexity and differentiability of limiting function $C(x,z)$ is guaranteed by Theorem 10.8 of Rockafellar (1970). Next, we show that $C(x,z) = \inf_{y\geq 0}\Phi(x,y,z)$. First of all, we prove by induction that $C_n(x,z) \leq \inf_{y\geq 0}\Phi(x,y,z)$. Clearly, the statement is true for $n = 0$ and suppose for some $n = k-1$, the statement is true. Due to the linearity of expectation operator and induction hypothesis, we have

$$\begin{aligned} C_n(x,z) &= \inf_{y\geq 0}\{c(y) + \varphi(x,y,z) + \alpha E_{R,D}C_{n-1}(x+y-zR,D)\} \\ &\leq \inf_{y\geq 0}\{c(y) + \varphi(x,y,z) + \alpha E_{R,D}C(x+y-zR,D)\}. \end{aligned}$$

Taking limits, we obtain $C(x,z) \leq \inf_{y\geq 0} \Phi(x,y,z)$. Next, we note that $C(x,z) \geq \inf_{y\geq 0} \Phi_n(x,y,z)$ for all $n \geq 1$. Using Proposition 4.5.1, we can always find a compact set I_x such that it contains all the minimizers of $\Phi_n(x,y,z)$ for each $n \geq 1$ given any starting inventory x. By Dini's Theorem, $\Phi_n(x,y,z)$ (which is continuous in y guaranteed by convexity on open set) converges uniformly to $\Phi(x,y,z)$ on I_x. That is, for all $y \in I_x$ and $\epsilon > 0$, there exists N such that $\Phi(x,y,z) - \epsilon < \Phi_n(x,y,z) < \Phi(x,y,z) + \epsilon$. As I_x is compact, the sequence of minimizers $\{y_n : n \geq 1\} \subset I_x$ of $\Phi_n(x,y,z)$ is bounded and thus, by Bolzano-Weierstrass Theorem, there exists a subsequence $\{y_{n_k} : k \geq 1\}$ of minimizers that converges to $\widetilde{y}$. Observe that

$$
\begin{aligned}
&|\Phi_{n_k}(x, y_{n_k}, z) - \Phi(x, \widetilde{y}, z)| \\
&\leq |\Phi_{n_k}(x, y_{n_k}, z) - \Phi_{n_k}(x, \widetilde{y}, z)| + |\Phi_{n_k}(x, \widetilde{y}, z) - \Phi(x, \widetilde{y}, z)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\end{aligned}
$$

The first inequality is due to the continuity of $\Phi_{n_k}(x,y,z)$ in $y \in I_x$ while the second inequality is due to uniform convergence of $\{\Phi_n(x,y,z) : n \geq 0\}$ on I_x. Thus, for large k, we have $\inf_{y\geq 0} \Phi(x,y,z) \leq \Phi(x,\widetilde{y},z) - \epsilon < \inf_{y\geq 0} \Phi_{n_k}(x,y,z) \leq C(x,z)$. As $\epsilon > 0$ can be chosen to be arbitrarily small, we have $\inf_{y\geq 0} \Phi(x,y,z) \leq C(x,z)$. Hence, the conclusion holds. ◇

A consequence of the above result is that $\Phi_k(x,y,z)$ converges pointwise to $\Phi(x,y,z)$ in $y \in [0,\infty)$ using Monotone Convergence Theorem. Since $\Phi_n(x,y,z)$ converges pointwise to $\Phi(x,y,z)$ in y. Using Theorem 10.8 of Rockafellar, the convexity of each $\Phi_n(x,y,z)$ converges to $\Phi(x,y,z)$ which is again convex in y. Let $\widetilde{y}$ is the minimizer of $\Phi(x,y,z)$. Our next goal focuses on proving the convergence in optimal policy for the infinite horizon case. The convergence of $y_k^*(x,z)$ in k cannot be proven by simply appealing to Yuan and Cheung (2003) or Yeo and Yuan (2011). This is because $c_1 = \lim_{y \to v_1^+} c(y) \neq \lim_{y \to v_1^-} c(y) = 0$. Even if $C_n(x,z) \to C(x,z)$ implies $\Phi_n(x,y,z) \to \Phi(x,y,z)$ does not necessarily imply that $\Phi_n^-(x,y,z) \to \Phi^-(x,y,z)$. This is because the limit of the one-sided derivatives is not always the one-sided derivatives of the limiting function. An example is given in Heyman and Sobel (1984) (see Example 8-37, pg. 425). In fact, the sufficient condition that guarantees the convergence in policy is when the convergence of $\Phi_n(x,y,z)$ is monotone. The following two lemmas are useful for proving convergence of ordering policy of the infinite horizon model. The proof is left to the readers to verify.

Lemma 4.5.1 *Let $g(\cdot), g_1(\cdot), ...$ be convex functions on an open convex subset $X \in (-\infty, \infty)$ such that $g_n(x) \to g(x)$ as $n \to \infty$ and $g_n(x) \leq g_{n+1}(x)$ for all n and x. Then, for all $x \in X$, we have*

$$g^-(x) \leq \liminf_{n \to \infty} g_n^-(x) \leq \limsup_{n \to \infty} g_n^+(x) \leq g^+(x).$$

Lemma 4.5.2 *Let $\{a_n : n \in \boldsymbol{N}\}$ and $r \in [0, \infty)$ be given. Then, we have*

(i). $\limsup_{n \to 0} a_n \leq r$ *if and only if* $\forall \epsilon > 0, \exists N$ *such that* $a_n \leq r + \epsilon$ *whenever* $n \geq N$.

(ii). $\liminf_{n \to 0} a_n \geq r$ *if and only if* $\forall \epsilon > 0, \exists N$ *such that* $a_n \geq r + \epsilon$ *whenever* $n \geq N$.

Proof: We only show (i). ($\Rightarrow$) Suppose $\limsup_{n \to \infty} a_n = U$. Let $\epsilon > 0$ be given. By definition, there exists N_1 such that $n > N_1 \Rightarrow a_n \leq \sup_{k \geq n} a_k < U + \epsilon$. ($\Leftarrow$) Suppose not, i.e. $\limsup_{n \to \infty} a_n > r$. Choose $\epsilon = \frac{1}{2}(\limsup_{n \to} a_n - r) > 0$. For any given N, we have $a_n > r + \epsilon$. This is a contradiction. $\diamond$

Theorem 4.4 *Suppose x and z be given. Let $y_n^*(x,z)$ and $\widetilde{y}(x,z)$ be the minimizers of $\Phi_n(x,y,z)$ and $\Phi(x,y,z)$ respectively. Then, $\lim_{n \to \infty} y_n^*(x,z) = \widetilde{y}(x,z)$.*

Proof: Let $\epsilon > 0$ be given. From Lemma 4.5.1 and Lemma 4.5.2, let $\delta > 0$ be given, there exists N_1 such that $n > N_1$ implies that $\Phi_n^+(x, y, z) \leq \Phi^+(x, y, z) + \delta$. Since the sequence of minimizers of $\Phi_n(x, y, z)$ is bounded by $I_{x,z}$, by Bolzano Weierstrass Theorem, there exists a subsequence $\{y_{n_k}^*(x, z) : k \geq 1\}$ such that $y_{n_k}^* \equiv y_{n_k}^*(x, z)$ converges in k. Thus, we have $\Phi_{n_k}^+(x, y_{n_k}^* + \epsilon, z) \leq \Phi^+(x, y_{n_k}^* + \epsilon, z)$ whenever $n_k \geq N_1$. By the convexity of $\Phi_{n_k}(x, y, z)$ in y and $y_{n_k}^*$ being the minimizer, we must have $\Phi_{n_k}^+(x, y_{n_k}^* + \epsilon, z) > 0$. Since $\widetilde{y}(x, z)$ is the minimizer of $\Phi(x, y, z)$ (which is convex in y), we have $\widetilde{y} \leq y_{n_k}^* + \epsilon$. Note that the case for $\widetilde{y} \geq y_{n_k}^* - \epsilon$ for some $n_k \geq N_2$ is proven similarly. Thus, we have $y_{n_k}^* - \epsilon \leq \widetilde{y} \leq y_{n_k}^* + \epsilon$ whenever $n_k \geq \max(N_1, N_2)$. Since $\epsilon > 0$ can be chosen to be small arbitrarily, the result is proven. $\diamond$

Finally, we discuss the optimality of the policy in the infinite horizon. For our purpose, we need to observe that $U_n(x, z) \leq U_{n+1}(x, z)$. This is because $C_n(x, z) \leq C_{n+1}(x, z)$. The same holds true for $V_n(x, z)$ and $W_n(x, z)$. Furthermore, it can be seen that for each $n \geq 0$, $U_n(x, z), V_n(x, z)$ and $W_n(x, z)$ are bounded above by $\Xi(x, z)$. Thus, $\lim_{n\to\infty} U_n(x, z) = U(x, z)$, $\lim_{n\to\infty} V_n(x, z) = V(x, z)$, and $\lim_{n\to\infty} W_n(x, z) = W(x, z)$ are finite. As each $U_n(x, z)$ is convex, the convexity of the limiting function $U(x, z)$ is guaranteed by Theorem 10.8 of Rockafellar. Similarly, $V(x, z)$ and $W(x, z)$ are both convex. Given that for any two functions, $\min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$, the pointwise limit of a sequence of minimum of two functions is the minimum of the two limiting functions, i.e. $\min\{f_n(x), g_n(x)\} \to \min\{f(x), g(x)\}$. Thus,

$$C_n(x, z) = \min\{U_n(x, z), V_n(x, z), W_n(x, z)\} \to \min\{U(x, z), V(x, z), W(x, z)\}.$$

However from Theorem 4.3, $C_n(x, z) \to C(x, z)$ guaranteeing the uniqueness of the pointwise limit so that $C(x, z) = \min\{U(x, z), V(x, z), W(x, z)\}$. To this end, let us assume that $S(z)$, $s'(z)$ and $s(z)$ are the minimizers of $\varphi(x, Q - x, z) + \alpha E_{R,D} C(Q -$

$zR, D)$, $c_1Q + \varphi(x, Q - x, z) + \alpha E_{R,D}C(Q - zR, D)$, and $c_2Q + \varphi(x, Q - x, z) + \alpha E_{R,D}C(Q - zR, D)$.

Theorem 4.5 *Let $y^*(x, z)$ denote the optimal ordering quantity. The optimal policy for the infinite horizon problem is of the form $(s(z), s'(z), S(z), v_1, v_2)$ given by*

$$y^*(x, z) = \begin{cases} (s(z) - x) \vee v_2 & \textit{if } x < s'(z) - v_2 \\ (s'(z) - x) \vee v_1 & \textit{if } s'(z) - v_2 \leq x < S(z) - v_1 \\ (S(z) - x) \vee 0 & \textit{if } x \geq S(z) - v_1. \end{cases}$$

The minimal cost is

$$C(x, z) = \begin{cases} \Phi(x, (s(z) - x) \vee v_2, z) & \textit{if } x < s'(z) - v_2, \\ \Phi(x, (s'(z) - x) \vee v_1, z) & \textit{if } s'(z) - v_2 \leq x < S(z) - v_1, \\ \Phi(x, (S(z) - x) \vee 0, z) & \textit{if } x \geq S(z) - v_1. \end{cases}$$

In particular, the optimal policy is an order-up-to policy with the re-order point being $S(z)$ when $v_1 = v_2 = 0$.

Graphically, the optimal inventory policy can be represented in Figure 4.4.

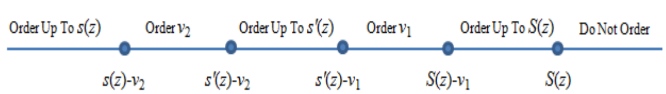

Figure 4.4: Optimal Inventory Policy for the Infinite Horizon Model.

Hence, for the infinite horizon model, the optimal replenishment quantity in Yuan and Cheung (2003) is always greater than that in our model. Overall, the structure of the optimal inventory policy for the infinite horizon model has the similar structure to that of the finite horizon case. Suppose $y_o^*(x, z)$ and $y_t^*(x, z)$ are the limits of $y_{k,o}^*(x, z)$

and $y^*_{k,t}(x,z)$ as $k \to \infty$. The difference in replenishment quantity between the two models is shown in Figure 4.5.

Corollary 4.5.1 *For the infinite horizon model, the optimal replenishment quantity in M0 is always greater than or equal to M1, i.e.* $y^*_o(x,z) \geq y^*_t(x,z)$.

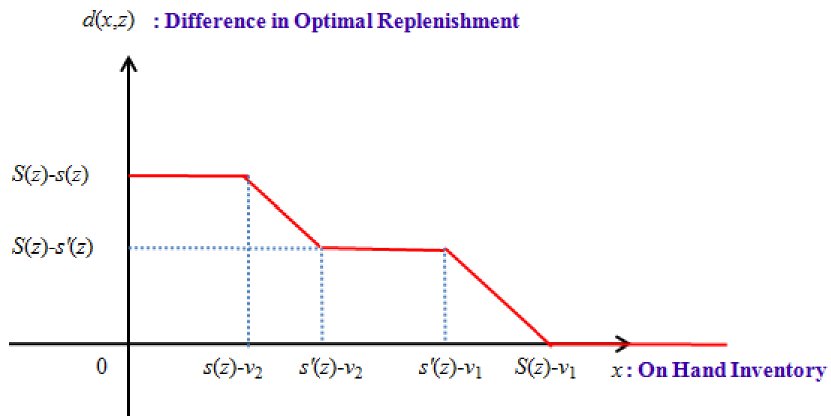

Figure 4.5: Difference in Optimal Replenishment Quantity between M0 and M1.

The intuition is as follows: in M0 without ordering costs, there is a tendency to order a larger quantity in order to minimize the one period cost function that is convex (in the on-hand inventory level). However in M1, the presence of ordering costs induces the tradeoff between ordering a large quantity and cost accumulated as a result of large ordering. For the infinite horizon model, the optimal inventory policy for the model of Yuan and Cheung (2003) in the presence of non-negative ordering cost can be deduced similarly. Let c be the per unit ordering cost and $F(Q) = \varphi(x, Q-x, z) + \alpha E_{R,D}C(Q-zR, D)$. Let $s(z)$ be the minimizer of $cQ+F(Q)$.

Corollary 4.5.2 (Yuan and Cheung (2003)) *In the infinite horizon case with ordering costs, the optimal policy is an order-up-to policy. That is, when* $x < s(z)$*, the optimal policy is to order-up-to* $x + y(x,z)$*, where* $y(x,z) = s(z) - x > 0$*, and the* *optimal total cost is* $\Phi(x, y(x,z), z)$*. When* $x \geq s(z)$*, the optimal policy is not to order anything, and the optimal total cost is* $\Phi(x, 0, z)$*.*

We shall provide a pseudo-code to illustrate how the optimal quantities can be computed in general. For notational simplicity denote $F_n(Q) = \varphi(x, Q-x, z) + \alpha E_{R,D} C_{n-1}(Q - zR, D)$ and $F_n'(Q)$ to be the first order derivatives w.r.t Q.

Step 0. Initialization:
Step 0a. Let $C_0(x,z) \equiv 0$ for all x and z.
Step 0c. Given initial inventory level x and reservation level z.
Step 0b. Let $n = 1$.

Step 1.
Step 1a. Solve $F_n'(Q) = 0$, $F_n'(Q) = -c_1$, and $F_n'(Q) = -c_2$
to obtain $S_n(z), s_n'(z)$, and $s_n(z)$.
Step 1b. **If** $x \geq S_n(z)$: $y_n^*(x,z) = 0$, and go to **Step 3**;
Else go to **Step 2**.

Step 2.
Step 2a. **If** $x < s_n(z) - v_2 : y_n^*(x,z) = s_n(z) - x$.
ElseIf $s_n(z) - v_2 \leq x < s_n'(z) - v_2 : y_n^*(x,z) = v_2$.
ElseIf $s_n'(z) - v_2 \leq x < s_n'(z) - v_1 : y_n^*(x,z) = s_n'(z) - x$.
ElseIf $s_n'(z) - v_1 \leq x < S_n(z) - v_1 : y_n^*(x,z) = v_1$.
ElseIf $S_n(z) - v_1 \leq x < S(z) : y_n^*(x,z) = S(z) - x$.
Step 2b. Update $C_n(x,z) = \Phi_n(x, y_n^*(x,z), z)$.
Step 2c. Obtain $C_n'(x,z) = \frac{\partial}{\partial x} C_n(x,z) = \frac{\partial}{\partial x} \Phi_n(x, y_n^*(x,z), z)$.

Step 3.
Let $n \Leftarrow n+1$.
If $n \leq N$, go to **Step 1**;
Otherwise, stop the iteration and complete the computation.

4.6 Concluding Remarks

The focus of this work is to study an inventory model in which the retailer enters into a "two-tier" supply contract with the supplier. This means that there are two contract values $v_1 < v_2$ and ordering costs $c_1 < c_2$ such that if the quantity ordered is greater than v_1, then c_1 is incurred for ordering every unit exceeding v_1 up to v_2^{th} unit. The cost c_2 is incurred for ordering every unit whenever the ordered quantity exceeds v_2. This model can easily be extended up to n-tier supply contract. Mathematically, the two-tier scenario is equivalent to the model where one of the suppliers is capacitated but is able to provide a cheaper source of supply compared to an alternative and more expensive source without supply constraint. Inspired by the works of Henig et al (1997) and Frederick (2009) but with the different focus, we assume all demands are reserved through a reservation system and can be canceled. To this end, we formulate the discounted cost criterion and prove the optimality of the inventory policy that is similar to the "finite generalized base stock" policy as in Frederick (2009), except that we do not return inventory when the on-hand inventory becomes sufficiently large. Moreover, the critical values in our model depend on the demand reserved in the previous period. Along similar veins of research involving demand cancellation, our model further extends the work of Yuan and Cheung (2003) by showing that even with ordering cost, the "order-up-to" structure of the optimal policy is still preserved. In our investigation, some technical differences are present when we establish policy optimality for the infinite horizon case. This difference is attributed to the presence of a piecewise continuous, non-differentiable, and convex ordering cost when a supply contract is considered. Hence, the techniques in proving the policy optimality for the infinite horizon model in Yuan and Cheung (2003) or

Yeo and Yuan (2011) cannot be applied directly. This model has promising applications in e-commerce, by utilizing internet portals as potential gateway to customers. Such phenomenon is already very prevalent and is even practiced by many traditional "Brick and Mortar" companies that used to rely heavily on consumers' loyalty. In practice, the majority of industries appeal to the choice of "order-up-to" policy because of its simplicity. Our results show that such policy need not be optimal in the presence of transportation contracts. Thus, our research offers a note of caution to guard against complacency in assuming that "order-up-to" is always the best solution when the firm is in a supply contract with its supplier. The model can be extended further by attaching a fixed cost, similar to the work in Chao and Zipkin (2008). To date, there is no work that considers the impact of a fixed cost on the optimal inventory policy for a system whose demands is reserved and can be canceled. In the presence of a supply contract, the proof for optimality may be more technically challenging.

CHAPTER 5

OPTIMAL INVENTORY POLICY FOR COMPETING SUPPLIERS WITH DEMAND CANCELLATION

5.1 Introduction

In practice, it is very common for an inventory manager to have more than one choice of suppliers. One reason is the ability to hedge itself against supply or demand uncertainty via diversification of supply sources. Furthermore, the varying supply contracts provided by different suppliers allows more flexibility for ordering cost management if there is a decision for significantly large or small orders. We analyze a periodic review inventory model where the inventory manager has more than one choice of suppliers. With the first supplier, the inventory manager incurs a higher cost of ordering when the replenishment quantity exceeds a certain level, otherwise there is no ordering costs involved. There is an alternative supplier who charges an ordering cost for every unit ordered but is lower than that of first supplier. Thus, it is seen attractive to order from the first supplier when the ordering quantity is sufficiently small, otherwise we order from the second supplier. The retailer faces stochastic demand that is reserved via a reservation system and they can be canceled within one period. It turns out that the periodic ordering cost function is neither concave nor convex and is piece-wise continuous. In particular, we show that the first period optimal cost function is not necessarily convex and continuous, but quasi-convex. Consequently, we restrict our study to the class of demand function whose distributions is Polya frequency function of order two (PF_2). While restrictive, the class of PF_2 distribution is a very common assumption in the inventory literature which includes strongly unimodal densities. The theory of single-crossing functions developed by John and Bruno (2010) is critical in proving optimality as aggregations of quasi-convex functions is not necessarily quasi-convex.

To justify our models, consider the following examples. During the production process, a component is either produced in-house or outsourced from external vendors. The external supplier may provide a transportation contract that is similar in structure to the model in Henig et al (1997) while in-house manufacturing typically requires a lower cost to produce each component. This scenario depicts the tradeoff between ordering with a transportation contract and variable costs. Even procurement of raw materials sometimes may face ordering cost of similar structure. Suppose there are two choices: a centralized large warehouse and a smaller warehouse located at a less strategic location. Due to the economy of scale, the larger warehouse is able to supply lower ordering cost for every raw material. On the other hand, the smaller warehouse may charge a larger ordering cost for every unit whenever a certain quantity is exceeded due to transportation over a longer distance.

The goal of this work is to consider the impact of suppliers competing in parallel on the optimal replenishment policy. During each period, the replenishment quantity and the choice of the supplier will be optimal. With the first supplier, the decision maker can be seen as entering into a transportation contract inspired by the work of Henig et al (1997). This contract specifies a promise to deliver a volume of items for negligible fixed costs when quantity does not exceed a certain amount. On the other hand, a second supplier provides items with an ordering costs lower than the first one, if any. In our model when there are two suppliers, our cost function becomes concave.

There is a dearth of literature that analyzes multi-period inventory systems with more than one suppliers competing for procurement. Porteus (1971) examines an inventory model with concave increasing ordering costs due to the presence of multiple suppliers competing in parallel. For each ordering decision, a variety of options is available: a low unit cost, high fixed cost option up to a high unit cost, low fixed cost option. Assuming that the class of demand follow a one-sided Polya densities, he demonstrated that the generalized (s, S) policy (with multiple re-order points and target levels) is optimal for the multiple period model. To prove this result, he defines a new class of functions with the property of quasi-K-convexity. Porteus (1972) shows that the same structure of the optimal policy continues to hold when demand has the uniform distribution. As the class of one-sided Polya densities is extremely difficult to characterize and restrictive, Fox et al (2006) derive the optimal policy for two suppliers whose demand follows strongly unimodal densities. In order to accommodate the analysis to a larger class of demand distribution, they assume that one of the suppliers have negligible fixed costs. They prove the optimal choice theorem and characterize the optimal policy for both the finite and infinite horizon case under lost sales and backorder costs. They show that there are three possible optimal inventory policy: (i). (s, S) from the lower variable cost supplier, (ii). an order-up-to policy from the high variable cost supplier, or (iii). a mixed-ordering policy between the two policies from both suppliers. In a similar setting, Hua et al (2009) extend the work of Fox et al (2006) by considering capacitated suppliers. The ordering cost function which is neither convex or concave complicates the analysis. Therefore, the optimal policy derived is significantly different from that of Porteus (1971,1972) and Fox et al (2006).

Entering into a supply contract is a commonly observed phenomena in many industries where risk is transferred to the inventory manager who needs to pay a higher premium for transporting a larger quantity of items. Stochastic inventory models under periodic review policy with supply contract can be found in the works of Bassok and Anupindi (1997), Henig et al. (1997), Xu (2005), Zhao and Katehakis (2006), Chao and Zipkin (2008), Bassok and Anupindi (2008), and Lian and Deshmukh (2009). All of the above works only consider a single source of suppliers and hence, this work is an extension by considering two competing suppliers. In particular, we extend the work of Henig et al (1997) to consider the impact of an additional supplier on the structure of the optimal inventory policy when the other enters into a supply contract.

Our current model is an extension of the work initiated by Yuan and Cheung (2003) who settle the issue of policy optimality where all demands are reserved via a booking system and cancellations are allowed. In Yuan and Cheung (2003) supply of raw items is unlimited and no ordering costs is incurred. They show that optimal inventory policy is of an order-up-to type for the single, multiple and infinite horizon models. To incorporate ordering costs into the inventory model of Yuan and Cheung (2003), Chapter 4 considers the inventory manager entering into a multi-tier supply contract with its supplier. The effect of introducing such a contract creates the trade-off between ordering to limit stockout and additional cost incurred due to ordering. Such a transportation contract has been first considered in the work of Henig et al (1997) who did not take customers' cancellation into consideration. Mathematically, entering into a multi-tier supply contract (see Chapter 4) can be applied to a situation where the inventory manager faces multiple suppliers. Specifically, in the two-tier scenario, the manager faces one supplier who rations a limited source of items at a lower ordering costs while the other supplier offers an unlimited, but is a more expensive source for procurement. Interestingly, the optimal policy of Yuan and Cheung (2003) with ordering cost of the model is subsumed in Chapter 4. To motivate our problem in this chapter, it is necessary to state the main result when the inventory manager faces a supplier who charges an ordering cost whenever the replenishment order exceeds the contracted quantity v.

Theorem 5.1 *Let x and z be initial inventory and reserved quantity, respectively. Suppose $y_n^*(x,z)$ denote the optimal ordering quantity during period n. For each period n, there exists $s_n(z)$ and $S_n(z)$ such that*

$$y_n^*(x,z) = \begin{cases} s_n(z) - x & \text{if } x < s_n(z) - v \\ v & \text{if } s_n(z) - v \leq x < S_n(z) - v \\ S_n(z) - x & \text{if } S_n(z) - v \leq x < S_n(z) \\ 0 & \text{if } x \geq S_n(z) \end{cases}$$

This is the generalized base stock policy of the form $(s_n(z), S_n(z), v)$.

The central theme of this work: if we introduce another supplier who charges a proportional ordering costs (whenever one orders) that is lower, what is the impact on the optimal inventory policy? Thus, the inventory manager is given two choices of suppliers competing with each other depending on the ordering quantity. Such a scenario certainly provides greater flexibility as it is cheaper to order from the supplier (who provides the supply contract) when the quantity decided is less than the contracted quantity v. Otherwise, order can be directed to the supplier who charges a proportional ordering costs when the quantity chosen is sufficiently large. How should one choose which supplier to procure given the information on the customers' reservation and on-hand inventory?

The rest of the work is organized as follows: In section 5.2, we present the model and illustrate graphically the ordering cost for each period. The optimal policy for single period model is discussed in section 5.3. We derive some properties for the optimal cost functions such as single-crossing property which is crucial in proving the policy optimality for the finite horizon model. The impact of an alternative supplier is also discussed. Section 5.4 derives the main result for the optimal inventory policy and choice of suppliers in the multiple period setting. The impact of an additional supplier on the optimal inventory policy and cost savings is discussed in Section 5.5. Section 5.6 concludes.

5.2 Model

Let c_h and $c_l(< c_h)$ be ordering costs charged by supplier 1 (S1) and 2 (S2) respectively. For simplicity, we assume that the supply contract offered by S1 is of the form: for orders exceeding v (the contractual quantity), the inventory manager pays c_h, otherwise, he pays nothing (see Henig et al (1997)). Figure 5.1 shows the concave ordering cost function when the two suppliers are competing for procurement. From S2, c_l is incurred for every unit ordered. Following the work of Yuan and Cheung (2003), all demands are made through reservations. Demands reserved in the previous periods are supposed to be fulfilled in the current period. However, customers' are allowed to cancel their reservation. Suppose $\mathbf{N}$ is the set of non-negative integers. Let D_n be the demand that is reserved during period $n \in \mathbf{N}$, and let R_n be the ratio of the demand reserved during the previous period that is eventually not canceled during period n. Let $\{R_n : n \in \mathbf{N}\}$ be a sequence of i.i.d ratio random variables whose c.d.f is $G(x)$ (with $G(0) = 0$ and $G(1) = 1$), and p.d.f $g(x)$. We assume that $\{D_n : n \in \mathbf{N}\}$

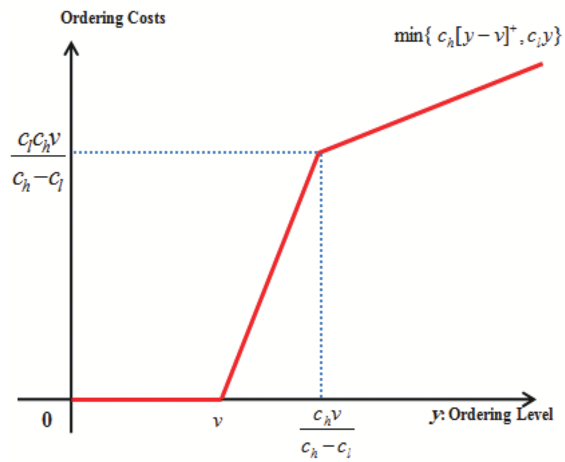

Figure 5.1: Minimal Cost.

is a sequence of i.i.d demand random variables with a common distribution $H(x)$ (with $H(0) = 0$ and $H(\infty) = 1$) and density function $h(x)$. We further assume that $h(x)$ is strongly unimodal. It is well known that strongly unimodal densities can be characterized simply by the fact that $\log(h(x))$ is concave whenever $h(x)$ is a density function. Ross (1983) defines such class of densities as belonging to Polya frequency of order 2, or simply PF_2. As argued in numerous literatures such as Porteus (1971), the class of PF_2 densities is not lacking many significant members. We also make the assumption that the cancellation ratios R_n and demands D_n are independent. All the unfulfilled orders are backordered. The inventory holding cost (h) and penalty cost (p) are both incurred on a per unit per unit time basis. At the beginning of the period, the inventory level is x and the demand reserved in the previous period is $z(\geq 0)$. The leadtime is assumed to be zero. Due to the presence of two suppliers and the structure of the cost functions involved, there is a need to consider splitting and non-splitting of ordering behavior of the inventory manager. In practice, there are many regulations in the procurement industry that sets threshold that prohibits the splitting of orders from different vendors.

For the rest of our work, we shall assume that splitting orders is not allowed. Suppose y is the quantity of items ordered, it can be seen that the ordering costs for the inventory manager becomes $c(y) = \min\{c_h[y-v]^+, c_l y\}$.

5.3 Single Period Analysis

From Section 5.2, the ordering cost is neither concave nor convex that is written as $c(y) = \min\{c_h[y-v]^+, c_l y\}$. Let the optimal cost function when there are n periods left given that the initial inventory level and the demand reserved in period $n+1$ to be denoted by $C_n(x,z)$. Define $\varphi(x,y,z) = hE[x+y-zR]^+ + pE[x+y-zR]^-$. Let $C_0(x,z) = 0$. The following observation $\varphi(x,y,z) \in \{\varphi(a,b,z) : a+b = x+y\}$ is useful and will be used throughout the entire work. Using dynamic programming, we can express our optimal cost function as $C_n(x,z) = \min_{y\geq 0}\{c(y) + \varphi(x,y,z) + \alpha E_{R,D} C_{n-1}(x+y-zR, D)\}$. In order to understand the structure of the optimal replenishment policy for the first period, we need the following definition.

Definition 4 *A function $f : \boldsymbol{R} \to \boldsymbol{R}$ is quasi-convex on a convex set $X \subseteq \boldsymbol{R}$ if for any $x, y \in X$ and $\theta \in [0,1]$, we have $f(\theta x + (1-\theta)y) \leq \max\{f(x), f(y)\}$.*

An alternative way to defining the quasi-convex function $f(x)$ is that every sub-level set $S_\beta(f) = \{x | f(x) \leq \beta)\}$ is convex. To obtain the optimal inventory policy for the case of suppliers competing in parallel, the analysis is somewhat different from the works of Yeo and Yuan (2011) and Yuan and Cheng (2003). This is because the first period optimal cost function is not necessarily convex in the initial inventory level. It turns out that the optimal cost function is quasi-convex and piecewise continuous. One of the issues that we need to contend with is the preservation of quasi-convexity under aggregation which is not true in general. For our purpose, let us define $L(x) = hx1_{\{x>0\}} - px1_{\{x\leq 0\}}$ and $f_n(x) = L(x) + \alpha E_D C_{n-1}(x, D)$. For any function $f(x)$, denote the right-hand derivatives of $f(x)$ to be $f^+(x)$. Using the above notations, we can rewrite our optimal cost function as $C_n(x, z) = \min\{U_n(x, z), V_n(x, z), W_n(x, z)\}$, where

$$\begin{aligned}
U_n(x, z) &= \min_{Q\geq x} E_R f_n(Q - zR) \\
V_n(x, z) &= -c_l x + \min_{Q\geq x}\{c_l Q + E_R f_n(Q - zR)\} \\
W_n(x, z) &= -c_h(x + v) + \min_{Q\geq x+v}\{c_h Q + E_R f_n(Q - zR)\}.
\end{aligned}$$

Let us suppose for now that it is possible obtain the minimizers (we will justify why) for $E_R f_n(Q - zR)$, $c_l Q + E_R f_n(Q - zR)$, and $c_h Q + E_R f_n(Q - zR)$. Let $S_{h,n}(z)$, $s_{l,n}(z)$, and $s_{h,n}(z)$ be the respective minimizers. We show that for each n, we have $s_{h,n}(z) \leq s_{l,n}(z) \leq S_{h,n}(z)$. Since

$$\begin{aligned}
c_l s_{l,n}(z) + E_R f_n(s_{l,n}(z) - zR) &\leq c_l S_{h,n}(z) + E_R f_n(S_{h,n}(z) - zR) \\
&\leq c_l S_{h,n}(z) + E_R f_n(s_{l,n}(z) - zR),
\end{aligned}$$

thus, we have $s_{l,n}(z) \leq S_{h,n}(z)$. The proof that $s_{h,n}(z) \leq S_{h,n}(z)$ is similar. Next, we note that

$$\begin{aligned}
c_h s_{h,n}(z) + E_R f_n(s_{h,n}(z) - zR) &\leq c_h s_{l,n}(z) + E_R f_n(s_{l,n}(z) - zR) \\
&= (c_h - c_l)s_{l,n}(z) + c_l s_{l,n}(z) + E_R f_n(s_{l,n}(z) - zR) \\
&\leq (c_h - c_l)s_{l,n}(z) + c_l s_{h,n}(z) + E_R f_n(s_{h,n}(z) - zR) \\
&\Rightarrow c_h s_{h,n}(z) \leq c_h s_{l,n}(z) + c_l(s_{h,n}(z) - s_{l,n}(z)).
\end{aligned}$$

Rearranging $(c_h - c_l)(s_{l,n}(z) - s_{h,n}(z)) \geq 0$ and thus, $s_{h,n}(z) \leq s_{l,n}(z)$. For the purpose of analyzing the single period ordering policy, let us denote the following notations:

$$s_{l,1}(z) = zG^{-1}\left(\frac{p-c_l}{p+h}\right), s_{h,1}(z) = zG^{-1}\left(\frac{p-c_h}{p+h}\right), S_{h,1}(z) = zG^{-1}\left(\frac{p}{p+h}\right)$$
$$\pi_1(c,z) = z\left(p\int_{G^{-1}\left(\frac{p-c}{p+h}\right)}^{\infty} tdG(t) - h\int_0^{G^{-1}\left(\frac{p-c}{p+h}\right)} tdG(t)\right), s_1^*(z) = \frac{\pi_1(c_h) - \pi_1(c_l,z) - c_h v}{c_h - c_l}.$$

We can interpret $\pi(c,z)$ as the optimal cost when the ordering cost per unit is c and the number of items reserved over the previous period is z. $zG^{-1}\left(\frac{p-c}{p+h}\right)$ is the respective optimal ordering quantity. In Yeo and Yuan (2010b) where the inventory manager enters into a single-tier supply contract, the optimal policy is described by three parameters, namely $(s_1(z), S_1(z), v)$. In the presence of an additional supplier offering a lower ordering cost, the optimal policy is described by two additional parameters. One of which is the point where the manager is indifferent between choosing any supplier while the other is the optimal ordering quantity when the manager faces a single supplier with ordering costs of c_l. Our next task is to state and prove the optimality of replenishment policy for our model with two suppliers under the given contracts. We divide our proofs into two main cases: (I). $s_{l,1}(z) < S_{h,1}(z) - v$ and (II). $s_{l,1}(z) \geq S_{h,1}(z) - v$. Let us define $J_1(x) = c_l x + \varphi(x,v,z) - \pi_1(c_l,z)$. It is easy to check that $J_1(x)$ is convex in x. Throughout our results in this work, x and z represents the initial inventory and demand reserved in the last period. Note that $S_{h,1}(z)$ is the optimal ordering quantity when the inventory model has zero ordering costs (Yuan and Cheung (2003)) while $s_{l,1}(z)$ is the optimal ordering quantity when an ordering cost of c_l is considered in the same inventory model. The quantity $S_{h,1}(z) - s_{l,1}(z)$ is the difference in quantity ordered attributed to the moral hazard on part of the inventory manager's ordering behavior in the absence of ordering costs. The condition

$s_{l,1}(z) < S_{h,1}(z) - v$ is equivalent to the contracted volume, v being less than this difference. In a similar way, $s_{l,1}(z) \geq S_{h,1}(z) - v$ means that the contracted volume for supplier one is greater than the difference in optimal replenishment quantity between the model without ordering cost and the model with ordering cost of c_l.

Theorem 5.2 *Let $y_1^*(x, z)$ denotes the optimal ordering quantity and $s_1^*(z) < s_{h,1}(z) - v$. The optimal policy is characterized by*
(i). If $x < s_1^*(z)$, order up to $s_{l,1}(z)$ from supplier two.
(ii). If $x \geq s_1^*(z)$, order from supplier one using generalized base stock policy of $(s_{h,1}(z), S_{h,1}(z), v)$.

Proof: **Case I:** $s_{l,1}(z) < S_{h,1}(z) - v$. We divide our proof into the following disjoint intervals: (a).$(-\infty, s_1^*(z))$, (b).$[s_1^*(z), s_{h,1}(z) - v)$, (c).$[s_{h,1}(z) - v, S_{h,1}(z) - v)$, (d).$[S_{h,1}(z) - v, S_{h,1}(z))$, and (e).$[S_{h,1}(z), \infty)$.

(a) and (b). If $x < s_{h,1}(z) - v$, then the optimal cost of ordering from supplier one and supplier two are given by $-c_h(x+v) + \pi_1(c_h)$ and $-c_l x + \pi_1(c_l, z)$ respectively. The level of initial inventory under which the manager is indifferent to ordering either from supplier one or supplier two is the point $s_1^*(z)$. There are two subcases to consider: $s_1^*(z) < s_{h,1}(z) - v$ or $s_1^*(z) \geq s_{h,1}(z) - v$. Now, $\hat{s}_1(z) < s_{h,1}(z) - v$ implies that $s_1^*(z) < s_{h,1}(z) - v$ holds, then we further split $(-\infty, s_{h,1}(z) - v)$ into $(-\infty, s_1^*(z)) \cup [s_1^*(z), s_{h,1}(z) - v)$. For $x < s_1^*(z)$, this implies that $-c_l x + \pi_1(c_l, z) \leq -c_h(x+v) + \pi_1(c_h)$ and we order up to $s_{l,1}(z)$ from supplier two. When $x \in [s_1^*(z), s_{h,1}(z) - v)$, we order-up-to $s_{h,1}(z)$ from supplier one.

(c). We divide our argument into two sub-intervals: $[s_{h,1}(z) - v, s_{l,1}(z)) \cup [s_{l,1}(z), S_{h,1}(z) - v)$. If $s_{h,1}(z) - v \leq x < s_{l,1}(z)$, then the optimal ordering cost or ordering from supplier one and supplier two are $\varphi(x, v, z)$ and $-c_l x + \pi_1(c_l, z)$. We claim that there does not exists $x \in [s_{h,1}(z) - v, s_{l,1}(z))$ such that the manager is better off by ordering from the alternative supplier. Differentiate $J_1(x)$ w.r.t x, we obtain $J_1'(x) = c_l+(h+p)G\left(\frac{x+v}{z}\right)-p$ and the turning point is $x = s_{l,1}(z)-v$. Furthermore, $J_1''(x) > 0$ and thus it is convex in x. Using the fact that $\pi_1(c_l, z) = c_l s_{l,1}(z)+\varphi(x, s_{l,1}(z)-x, z)$. Now, $s_1^*(z) < s_{h,1}(z)-v$ implies that $J_1(s_{h,1}(z)-v) < 0$. Next, $J_1(s_{l,1}(z)^-)$ simplifies to $\varphi(s_{l,1}(z), v, z)-\varphi(s_{l,1}(z), 0, z)$. Note that $\varphi_y'(s_{l,1}(z), y, z) = (h+p)G\left(\frac{s_{l,1}(z)+y}{z}\right)-p$. Clearly, $\varphi_y''(s_{l,1}(z), y, z) > 0$ and the turning point is $y^* = S_{h,1}(z)-s_{l,1}(z) > v$. Thus, for $y \in [0, v]$, $\varphi(s_{l,1}(z), y, z)$ is decreasing in y and thus, $J_1(s_{l,1}(z)^-) < 0$. Given that $J_1(x)$ is convex and $J_1(x) < 0$ at the end points of $[s_{h,1}(z)-v, s_{l,1}(z))$, we conclude that $\varphi(x, v, z) < -c_l x + \pi_1(c_l, z)$. Thus, its always optimal to order exactly v units from supplier one when the on-hand inventory lies in this interval.

If $s_{l,1}(z) \leq x < S_{h,1}(z)-v$, then the optimal cost of ordering from supplier one is $\varphi(x, v, z)$. From supplier two, the optimal cost of ordering is $\varphi(x, 0, z)$. As $x < S_{h,1}(z)-v = zG^{-1}\left(\frac{p}{p+h}\right)-v \Rightarrow (h+p)G\left(\frac{x+v}{z}\right)-p < 0$. Therefore, $\varphi_y'(x, y, z)|_{y=0} < \varphi_y'(x, y, z)|_{y=v} < 0$, and as $\varphi(x, y, z)$ is convex in y, this implies that $\varphi(x, 0, z) \geq \varphi(x, v, z)$. Thus, we order up to $v + x$.

(d). If $S_{h,1}(z)-v \leq x < S_{h,1}(z)$, then the optimal cost of ordering from supplier one and supplier two are $\varphi(x, S_{h,1}(z)-x, z) = \pi_1(0)$ and $\varphi(x, 0, z)$. Thus, it is clear that we order up to $S_{h,1}(z)$.

(e). If $x \geq S_{h,1}(z)$, then it is clear that the optimal cost of ordering from both supplier one and supplier two is $\varphi(x, 0, z)$ and thus, we do not order.

Case II: $s_{l,1}(z) \geq S_{h,1}(z)-v$. (a). Over the interval $(-\infty, s_1^*(z)) \cup [s_1^*(z), s_{h,1}(z)-v)$, the proof is omitted as it is similar to the above case when $s_{l,1}(z) < S_{h,1}(z)-v$.

(b). If $s_{h,1}(z)-v \leq x < S_{h,1}(z)-v(\leq s_{l,1}(z))$, then the optimal ordering cost of ordering from supplier one and supplier two are $\varphi(x, v, z)$ and $-c_l x + \pi_1(c_l, z)$. Now, $J_1(x)$ is convex in x and has a turning point $x = s_{l,1}(z)-v$. Again, $s_1^*(z) < s_{h,1}(z)-v$ implies that $J_1(s_{h,1}(z)-v) < 0$. It is easy to check that $J_1(S_{h,1}(z)-v) = c_l(S_{h,1}(z)-$

$v - s_{s,1}(z)) + (\varphi(S_{h,1}(z), v, z) - \varphi(x, s_{l,1}(z) - x, z)) < 0$ since $S_{h,1}(z) - v \leq s_{s,1}(z)$ and $S_{h,1}(z)$ minimizes $\varphi(x, Q - x, z)$. Thus, $J_1(x) < 0$ for $x \in [s_{h,1}(z) - v, S_{h,1}(z) - v)$.

(c). If $S_{h,1}(z) - v \leq x < s_{l,1}(z)$, then the optimal cost of ordering from supplier one and supplier two are $\varphi(x, S_{h,1}(z) - x, z)$ and $-c_l x + \pi_1(c_l, z)$ respectively. As $S_{h,1}(z) - x$ is the minimizer of $\varphi'_y(x, y, z) = 0$, we have $\varphi(x, S_{h,1}(z) - x, z) \leq \varphi(x, s_{l,1}(z) - x, z) \leq -c_l x + c_l s_{l,1}(z) + \varphi(x, s_{l,1}(z) - x, z)$. The last inequality holds as $s_{l,1}(z) > x$. Thus, we order from supplier one and quantity is $S_{h,1}(z) - x$.

(d). If $s_{l,1}(z) \leq x < S_{h,1}(z)$, then the optimal cost of ordering from supplier one and supplier two are $\varphi(x, S_{h,1}(z) - x, z)$ and $\varphi(x, 0, z) \geq \varphi(x, S_{h,1}(z) - x, z)$. Thus, we order from supplier one and quantity is $S_{h,1}(z) - x$. $\diamond$

Theorem 5.3 *Let $y_1^*(x, z)$ denotes the optimal ordering quantity and $s_1^*(z) \geq s_{h,1}(z) - v$. The optimal policy is characterized by some $r_1(z) \in [s_{h,1}(z) - v, s_{l,1}(z) - v)$, such that the optimal policy is characterized by*

(i). If $x < r_1(z)$, order up to $s_{l,1}(z)$ from supplier two.

(ii). If $x \geq r_1(z)$, order from supplier one using generalized base stock policy of $(s_{h,1}(z), S_{h,1}(z), v)$.

Proof: For the case when $s_1^*(z) \geq s_{h,1}(z) - v$, the interval $(s_1^*(z), s_{h,1}(z) - v]$ is vacuous and thus, only invoke supplier two for $x < s_{h,1}(z) - v$.

Case I: $s_{l,1}(z) < S_{h,1}(z) - v$.(a) and (b). If $x < s_{h,1}(z) - v$, we shall omit the proof as it is similar to (a) and (b) of Theorem 5.2.

(c) and (d). It is routine to consider $[s_{h,1}(z) - v, s_{l,1}(z)) \cup [s_{l,1}(z), S_{h,1}(z) - v)$. If $s_{h,1}(z) - v \leq x < s_{l,1}(z)$, then the optimal ordering cost of ordering from supplier one and supplier two are $\varphi(x, v, z)$ and $-c_l x + \pi_1(c_l, z)$. Now, $\hat{s}_1(z) \geq s_{h,1}(z) - h$ implies that $J_1(s_{h,1}(z) - v) \geq 0$. Given that $J_1(x)$ is decreasing in $x \in [s_{h,1}(z) - v, s_{l,1}(z))$ (due to convexity) and that the minimum $J_1(s_{l,1}(z) - v) = -c_l v < 0$, there exists

$r_1(z) \in [s_{h,1}(z) - v, s_{l,1}(z) - v)$ such that $J_1(r_1(z)) = 0$. Following Theorem 5.2 and $s_{l,1}(z) < S_{h,1}(z) - v$, we have $J_1(s_{l,1}(z)^-) < 0$. This implies that $J_1(x) \geq 0$ for $x \in [s_{h,1}(z) - v, r_1(z))$ but $J_1(x) < 0$ for $x \in [r_1(z), s_{l,1}(z))$. The proof that it is optimal to order v quantity in $[s_{l,1}(z), S_{h,1}(z) - v)$ follows from Theorem 5.2.

(e). Over the interval $[S_{h,1}(z) - v, \infty)$, proof is the same as Theorem 5.2 and will be omitted.

Case II: $s_{l,1}(z) \geq S_{h,1}(z) - v$. If $x < s_{h,1}(z) - v$, we shall omit the proof as it is similar to (a) and (b) of Theorem 5.2.

(c) and (d). It is routine to consider $[s_{h,1}(z) - v, s_{l,1}(z)) \cup [s_{l,1}(z), S_{h,1}(z) - v)$. If $s_{h,1}(z) - v \leq x < s_{l,1}(z)$, then the optimal ordering cost of ordering from supplier one and supplier two are $\varphi(x, v, z)$ and $-c_l x + \pi_1(c_l, z)$. Now, $\hat{s}_1(z) \geq s_{h,1}(z) - h$ implies that $J_1(s_{h,1}(z) - v) \geq 0$. Given that $J_1(x)$ is decreasing in $x \in [s_{h,1}(z) - v, s_{l,1}(z))$ (due to convexity) and that the minimum $J_1(s_{l,1}(z) - v) = -c_l v < 0$, there exists $r_1(z) \in [s_{h,1}(z) - v, s_{l,1}(z))$ such that $J_1(r_1(z)) = 0$. Since $J_1(S_{h,1}(z) - v) < 0$, then it is clear that there is at most one root $r_1(z)$ of $J_1(x)$ because of convexity in x. Thus, $J_1(x) \geq 0$ for $x \in [s_{h,1}(z) - v, r_1(z))$ while $J_1(x) < 0$ for $x \in [r_1(z), S_{h,1}(z) - v)$.

(e). If $S_{h,1}(z) - v \leq x < s_{l,1}(z)$, then the optimal cost of ordering from supplier one and supplier two are $\varphi(x, S_{h,1}(z) - x, z)$ and $-c_l x + \pi_1(c_l, z)$ respectively. As $S_{h,1}(z) - x$ is the minimizer of $\varphi'_y(x, y, z) = 0$, we have $\varphi(x, S_{h,1}(z) - x, z) \leq \varphi(x, s_{l,1}(z) - x, z) \leq -c_l x + c_l s_{l,1}(z) + \varphi(x, s_{l,1}(z) - x, z)$. The last inequality holds as $s_{l,1}(z) > x$. Thus, we order from supplier one and quantity is $S_{h,1}(z) - x$.

(f). If $s_{l,1}(z) \leq x < S_{h,1}(z)$, then the optimal cost of ordering from supplier one and supplier two are $\varphi(x, S_{h,1}(z) - x, z)$ and $\varphi(x, 0, z) \geq \varphi(x, S_{h,1}(z) - x, z)$. Thus, we order from supplier one and quantity is $S_{h,1}(z) - x$. $\diamond$

In order to set the stage for seeking the optimal structure of the inventory policy, we shall review some important concepts which can be found in the works of John and Bruno (2010).

Definition 5 *A function $f : \boldsymbol{R} \to \boldsymbol{R}$ is said to be single-crossing if for any $x_2 \geq x_1$, $f(x_1) \geq 0 \Rightarrow f(x_2) \geq 0$.*

An immediate consequence of $f(x)$ being a single-crossing function is that the number of sign changes of $f(x)$ as x traverses from $-\infty$ to ∞ is at most once. It is known that any continuous and piecewise differentiable function $f(x)$ will be quasi-convex if the one sided partial derivatives $f^-(x)$ (or $f^+(x)$) exists and is single-crossing. Recently, John and Bruno (2010) resolve the fundamental issue of aggregating single-crossing functions which is crucial to discussing the optimal policy of our problem. They provide precise conditions under which single-crossing property is preserved under aggregation.

Definition 6 *On the set of functions, we define $h \succeq g$ if whenever $g(s') < 0$ and $h(s') > 0$, then for $s'' > s'$, we have $\frac{g(s')}{h(s')} \leq \frac{g(s'')}{h(s'')}$. We say that two functions h and g are related, denoted by $h \sim g$ if $h \succeq g$ and $g \succeq h$.*

John and Bruno (2010) show that for two single-crossing functions f and g, $f + g$ is again single-crossing whenever $f \sim g$. The next result is also useful (c.f Theorem 2 of John and Bruno (2010)). It is easy to check from Theorem 5.2 and Theorem 5.3 that $C_1^+(x, z)$ always exists.

Lemma 5.3.1 *The family of function $\{C_1^+(x, z) : z > 0\}$ is class of related single-crossing functions in x.*

Proof: It is easy to see that the structure of the right-hand side derivative is

$$C_1^+(x,z)\begin{cases} < 0 & \text{if } x < S_{h,1}(z) - v \\ = 0 & \text{if } S_{h,1}(z) - v \le x < S_{h,1}(z) \\ > 0 & \text{if } x \ge S_{h,1}(z) \end{cases}$$

Let $0 < z_1 < z_2$ be given. We only need to consider two cases: (i). $S_{h,1}(z_1) \ge S_{h,1}(z_2) - v$ and (ii). $S_{h,1}(z_1) < S_{h,1}(z_2) - v$. In case (i), it is not possible to find an interval I such that for $x \in I$, $C_1^+(x,z) < 0$ and $C_2^+(x,z) > 0$ or $C_2^+(x,z) < 0$ and $C_1^+(x,z) > 0$. In case (ii), let $I = [S_{h,1}(z_1), S_{h,1}(z_2) - v]$. Then for all $s' \in I$, $C_1^+(s', z_1) > 0$ and $C_1^+(s', z_2) < 0$. Thus, it remains to show that $-\frac{C_1^+(s,z_2)}{C_1^+(s,z_1)}$ is non-increasing in $s \in I$. However on I, we have

$$C_1^+(s,z_1) = (h+p)G\left(\frac{s}{z_1}\right) - p,$$
$$C_1^+(s,z_2) = (h+p)G\left(\frac{s+v}{z_2}\right) - p.$$

It is sufficient to compute $-\frac{\partial}{\partial s}\left(\frac{C_1^+(s,z_2)}{C_1^+(s,z_1)}\right)$. The numerator of the first order derivative w.r.t s can be calculated to be negative. Thus, we have $C_1^+(s,z_1) \succeq C_1^+(s,z_2)$. As we cannot find an interval I such that $s \in I$, $C_1^+(s,z_1) < 0$ and $C_1^+(s,z_2) > 0$, thus, we have $C_1^+(s,z_2) \succeq C_1^+(s,z_1)$. Thus, $C_1^+(\cdot,z_1) \sim C_1^+(\cdot,z_2)$ for any $z_1, z_2 > 0$. $\diamond$

5.4 Finite Horizon Analysis

For the finite horizon model, the proof of the optimality of the inventory policy for Yuan and Cheung (2003) and Yeo and Yuan (2011) hinges on the fact that the one period cost is convex and continuous in the on-hand inventory level. For the current model, our one period cost function is quasi-convex and piecewise-continuous. Thus, the techniques used in the above models do not work anymore. This section will be devoted to finding the optimal policy for the case when the suppliers are competing in parallel for manager's procurement. Throughout the rest of this work, let us denote $f_n(x) = L(x) + \alpha E_D C_{n-1}(x, D)$ and $f_n^+(x) = L_x^+(x) + \alpha E_D C_{n-1}^+(x, D)$.

Lemma 5.4.1 *Let $\mu : \boldsymbol{R} \to \boldsymbol{R}$ be a function such that $\mu^+(x)$ is single-crossing and let ζ be a non-negative random variable with PF_2 density. Then, $g(x) = \phi x + E\mu(x - z\zeta)$ is a quasi-convex function defined on $\boldsymbol{R}$ for any $z > 0$ and $\phi \geq 0$.*

Proof: Let $\phi \geq 0$ be given. Instead of following Sethi and Cheng (1997) or Porteus (1971), we provide an alternative proof. It is sufficient to show that $g(x) = \phi x + E\mu(x - \zeta')$ is quasi-convex because the density of $\zeta' = z\zeta$ is again PF_2. Now, it is necessary that $\phi x + \mu(x)$ is quasi-convex in x since $\phi + \mu^+(x)$ is single-crossing after translation. Using Lemma 6 of Schoenberg (1951), the convolution of μ^+ with PF_2 density is a proper sign variation diminishing transformation, thus, $g^+(x) = \phi + E\mu^+(x - \zeta')$ changes sign at most once and implies that $g(x)$ is quasi-convex. $\diamond$

Lemma 5.4.2 *For all $n \geq 1$, there exists some ζ_n such that we have*

(a). *$C_n(x, z)$ is piecewise continuous and PF_2-integrable on $I_n = (-\infty, \zeta_n]$,*

(b). *For any constant $\phi \geq 0$, if $\phi + f_n^+(x)$ is single-crossing.*

(c). *The family of function $\{C_n^+(x, z) : z \geq 0\}$ is a related class of single-crossing functions in x.*

(d). *$C_n(x, z)$ is non-increasing on I_n and non-decreasing on $\boldsymbol{R} \setminus I_n$. Furthermore, $\lim_{|x| \to \infty} C_n(x, z) \geq 0$ for all z.*

Proof: It is easy to see that for $n = 1$, (a), (c), and (d) are true. To show that (b) is true for $n = 1$, $f_1(x) = k + h1_{\{x>0\}} - p1_{\{x\leq 0\}}$ is clearly single-crossing, thus, the statement is true. We assume that for $n = k$, statements (a)-(d) are true. Then, we go on to show that these statement are true for $n = k + 1$.

(c). $\Rightarrow$ (b). Assume that $\phi + f_k^+(x) = \phi + L_x^+(x) + \alpha E_D C_{k-1}^+(x, D)$ is single-crossing for any non-negative ϕ. Using Lemma 5.4.1, we conclude that $\phi Q + E_R f_k(Q - zR)$ is a quasi-convex function in Q. From (d), the induction hypothesis states that there exists some I_k such that $C_k(x, z)$ is non-increasing on I_k and non-decreasing on $\mathbf{R}\backslash I_k$. Thus, this implies that $C_k^+(x, z)$ is single-crossing in x. The induction hypothesis from (c) implies that $\{C_k^+(x, z) : z \geq 0\}$ is a related class of single-crossing functions in x. Using Theorem 2 of John and Bruno (2010), $E_D C_k^+(x, D)$ is also a single-crossing function. As $\phi + L_x^+(x)$ is a constant, thus, $\phi + f_{k+1}^+(x) = \phi + L_x^+(x) + \alpha E_D C_k^+(x, D)$ is again a single-crossing function. Hence, (b) is true for $n = k + 1$.

(b). $\Rightarrow$ (d). Given that for any non-negative ϕ, $\phi + f_n^+(x)$ is a single-crossing function. Now Lemma 5.4.1 implies that $\phi Q + E_R f_{k+1}(Q - zR)$ is quasi-convex in Q. Let us denote the following:

$$\text{arginf}_{\,Q\geq x} E_R f_{k+1}(Q - zR) = S_{h,k+1}(z),$$

$$\text{arginf}_{Q\geq x}\{c_l Q + E_R f_{k+1}(Q - zR)\} = s_{l,k+1}(z),$$

$$\text{arginf}_{Q\geq x+v}\{c_h Q + E_R f_{k+1}(Q - zR)\} = s_{h,k+1}(z).$$

It is previously shown that $s_{h,k+1}(z) \leq s_{l,k+1}(z) \leq S_{h,k+1}(z)$. Our next step is to prove the optimal policy for the finite horizon problem. This result in turn, will prove that (d) is true for $n = k + 1$. The argument used is similar to the one in the single period problem and we shall present it for formality sake. Let us further denote the following:

$$\pi_{k+1}(c_h, z) = c_h s_{h,k+1}(z) + E_R f_{k+1}(s_{h,k+1}(z) - zR),$$
$$\pi_{k+1}(c_l, z) = c_l s_{l,k+1}(z) + E_R f_{k+1}(s_{l,k+1}(z) - zR)$$
$$\pi_{k+1}(0, z) = E_R f_{k+1}(S_{h,k+1}(z) - zR).$$

Define $s^*_{k+1}(z) = \frac{\pi_{k+1}(c_h,z)-\pi_{k+1}(c_l,z)-c_h v}{c_h-c_l}$. As in the proof for the single period model, there are two cases to consider: (I). $s^*_{k+1}(z) < s_{h,k+1}(z) - v$, and (II). $s^*_{k+1}(z) \geq s_{h,k+1}(z) - v$. Furthermore, we define $J_{k+1}(x) = c_l x + \varphi(x, v, z) + \alpha E_{R,D} C_k(x + v - zR, D) - \pi_{k+1}(c_l, z)$. Define $r_{k+1}(z)$ such that $J_{k+1}(r_{k+1}(z)) = 0$. However, notice that $\varphi(x, v, z) + \alpha E_{R,D} C_k(x + v - zR, D)$ is symmetrical about x and v. Using the definition of $s_{l,k+1}(z)$, we have $J_{k+1}(s_{l,k+1}(z) - v) = -c_l v < 0$. Next, we show that $s_{l,k+1}(z) - v$ is the minimizer of $J_{k+1}(x)$. Observe that $J_{k+1}(x) + \pi_{k+1}(c_l, z) + c_l v = c_l(x + v) + \varphi(x, v, z) + \alpha E_{R,D} C_k(x + v - zR, D)$. As $s_{l,k+1}(z)$ is the minimizer of $c_l Q + \varphi(x, Q - x, z) + \alpha E_{R,D} C_k(Q - zR, D)$, allowing $x + v = s_{l,k+1}(z)$ minimizes $J_{k+1}(x) + \pi_{k+1}(c_l, z) + c_l v$ implying that $x = s_{l,k+1}(z) - v$. Thus, $J_{k+1}(s_{l,k+1}(z) - v) \leq J_{k+1}(x)$. Note that $J_{k+1}(s_{h,k+1}(z) - v) < 0$ is equivalent to $s^*_{k+1}(z) < s_{h,k+1}(z) - v$.

Case (I): $s^*_{k+1}(z) < s_{h,k+1}(z) - v$.

Subcase (A): $s_{l,k+1}(z) < S_{h,k+1}(z) - v$. We divide our proof into (a). $(-\infty, s^*_{k+1}(z))$, (b). $[s^*_{k+1}(z), s_{h,k+1}(z) - v)$, (c). $[s_{h,k+1}(z) - v, S_{h,k+1}(z) - v)$, (d). $[S_{h,k+1}(z) - v, S_{h,k+1}(z))$, and (e). $[S_{h,k+1}(z), \infty)$.

(a) and (b). If $x < s_{h,k+1}(z) - v$, the optimal cost for the manager becomes $-c_h(x + v) + \pi_{k+1}(c_h, z)$ and $-c_l x + \pi_{k+1}(c_l, z)$ when he orders from supplier one and supplier two respectively. We solve for x such that the manager is indifferent to ordering from any of the suppliers. Letting $-c_h(x+v)+\pi_{k+1}(c_h,z) = -c_l x + \pi_{k+1}(c_l,z)$ and solving for x, we obtain $s^*_{k+1}(z) = \frac{\pi_{k+1}(c_h,z)-\pi_{k+1}(c_l,z)-c_h v}{c_h-c_l}$. Finally, $-c_h(x+v)+\pi_{k+1}(c_h,z) > -c_l x + \pi_{k+1}(c_l,z)$ if and only if $x < s^*_{k+1}(z)$. Thus, if $x < s^*_{k+1}(z)$ then we order up to $s_{l,k+1}(z)$ and if $s^*_{k+1}(z) \le x < s_{h,k+1}(z) - v$, then we order up to $s_{h,k+1}(z)$. Note that there are two further subcases: (a). $s^*_{k+1}(z) \ge s_{h,k+1}(z) - v$ or (b). $s^*_{k+1}(z) < s_{h,k+1}(z) - v$. For (a), always order-up-to $s_{h,k+1}(z)$. For (b). order up to $s_{l,k+1}(z)$ if $x < s^*_{k+1}(z)$ and order up to $s_{h,k+1}(z)$ if $s^*_{k+1}(z) \le x < s_{h,k+1}(z) - v$.

(c). We divide $[s_{h,k+1}(z)-v, S_{h,k+1}(z)-v)$ into $[s_{h,k+1}(z)-v, s_{l,1}(z)) \cup [s_{l,1}(z), S_{h,k+1}(z)-v)$. If $s_{h,k+1}(z) - v \le x < s_{l,k+1}(z)$, then from supplier one, the optimal cost can be written as $C_{k+1}(x,z) = \min\{U_{k+1}(x,z), W_{k+1}(x,z)\}$. As $x < S_{h,k+1}(z)$, then $U_{k+1}(x,z) = \min_{Q \ge x} E_R f_{k+1}(Q - zR)|_{Q=x} = E_R f_{k+1}(x - zR)$ and $W_{k+1}(x,z) = -c_h(x+v) + \min_{Q \ge x+v}\{c_h Q + E_R f_{k+1}(Q - zR)\}|_{Q=x+v} = E_R f_{k+1}(x + v - zR)$. As $f^+_{k+1}(x)$ is single-crossing in x, Lemma 5.4.1 implies that $E_R f_{k+1}(Q - zR)$ is quasi-convex at $S_{h,k+1}(z)$. Furthermore, $x < x + v < S_{h,k+1}(z)$, the quasi-convexity of $E_R f_{k+1}(Q - zR)$ in Q implies that $W_{k+1}(x,z) \le U_{k+1}(x,z)$. Therefore, the optimal cost of ordering from supplier one is $E_R f_{k+1}(x + v - zR)$ or $\varphi(x,v,z) + \alpha E_{R,D} C_k(x + v - zR, D)$. It is easy to see that the optimal cost of the manager when he orders from the supplier two is $-c_l x + \pi_{k+1}(c_l, z)$. We want to determine the level of initial inventory such that the manager is indifferent to ordering from any of the retailers. Due to the quasi-convexity of $J_{k+1}(x)$, together with $J_{k+1}(s_{l,k+1}(z) - v) = -c_l v < 0$, we have $r_{k+1}(z) > s_{l,k+1}(z) - v$. Finally, we establish the sign of $J_{k+1}(s_{l,k+1}(z))$. Now we see that $J_{k+1}(s_{l,k+1}(z)) = c_l s_{l,k+1}(z) + \varphi(s_{l,k+1}(z), v, z) + \alpha E_{R,D} C_k(s_{l,k+1}(z) + v - zR, D) - \pi_{k+1}(c_l, z) = g_{k+1}(v) - g_{k+1}(0)$, where $g_{k+1}(y) = \varphi(s_{l,k+1}(z), y, z) + \alpha E_{R,D} C_k(s_{l,k+1}(z) + y - zR, D)$. Now, $S_{h,k+1}(z)$ is the minimizer of $E_R f_{k+1}(Q - zR)$ and has minimum $g_{k+1}(S_{h,k+1}(z) - s_{l,k+1}(z))$ (by the definition of $s_{l,k+1}(z)$). But $S_{h,k+1}(z) - s_{l,k+1}(z) > v$, and the quasi-convexity of $g_{k+1}(y)$ in y leads us to conclude that $J_{k+1}(s_{l,k+1}(z)) < 0$. Thus, $s_{l,k+1}(z) < r_{k+1}(z)$. Together with $J_{k+1}(s_{h,k+1}(z) - v) < 0$ (implied by $s_1^*(z) < s_{h,k+1}(z) - v$), $J_{k+1}(x) < 0$ for $x \in [s_{h,k+1}(z) - v, s_{l,k+1}(z))$. Hence, when the on-hand inventory lies in this interval, order exactly v units from supplier one. On the other hand, if $s_{l,k+1}(z) \leq x < S_{h,k+1}(z) - v$, then the optimal cost function is given by $\min\{U_{k+1}(x,z), W_{k+1}(x,z)\}$ when the manager orders from supplier one. The analysis is similar and the optimal cost is $E_R f_{k+1}(x + v - zR)$. As $x \geq s_{l,k+1}(z)$, the optimal cost is obtained when the manager orders nothing due to the quasi-convexity of $c_l Q + E_R f_{k+1}(Q - zR)$ and $s_{l,k+1}(z)$ being the minimizer. Thus, the optimal cost of ordering from supplier two is $E_R f_{k+1}(x - zR)$. As $x < S_{h,k+1}(z) - v$, the quasi-convexity of $E_R f_{k+1}(Q - zR)$ in Q and $S_{h,k+1}(z)$ being the minimizer implies that $E_R f_{k+1}(x+v-zR) \leq E_R f_{k+1}(x-zR)$. Thus, it is optimal to order exactly v units from supplier one.

(d). If $S_{h,k+1}(z) - v \leq x < S_{h,k+1}(z)$, then $x + v \geq S_{h,k+1}(z) > s_{h,k+1}(z)$. Now, $W_{k+1}(x,z) = -c_h(x+v) + \min_{Q \geq x+v}\{c_h Q + E_R f_{k+1}(Q - zR)\} = E_R f_{k+1}(x+v-zR)$. As $x < S_{h,k+1}(x,z)$, then $U_{k+1}(x,z) = \min_{Q \geq x} E_R f_{k+1}(Q - zR) = E_R f_{k+1}(S_{h,k+1}(z) - zR)$. By the quasi-convexity of $E_R f_{k+1}(Q - zR)$ in Q and $S_{h,k+1}(z)$ being the minimizer, the manager orders up to $S_{h,k+1}(z)$ from supplier one. The optimal cost of ordering from supplier two is $E_R f_{k+1}(x - zR) \geq E_R f_{k+1}(S_{h,k+1}(z) - zR)$. Thus, it is optimal to order from supplier one and order up to $S_{h,k+1}(z)$.

(e). If $x \geq S_{h,k+1}(z)$, then we do not order anything. This is because $E_R f_{k+1}(Q-zR)$, $c_h Q + E_R f_{k+1}(Q - zR)$, and $c_l Q + E_R f_{k+1}(Q - zR)$ are quasi-convex in Q, together with $x \geq S_{h,k+1}(z) > s_{l,k+1}(z) > s_{h,k+1}(z)$.

For the case when $s_{l,k+1}(z) < S_{h,k+1}(z) - v$, the optimal cost function for the period $k+1$ by the following:

$$C_{k+1}(x,z) = \begin{cases} -c_l x + \pi_{k+1}(c_l, z) & \text{if } x < s^*_{k+1}(z), \\ -c_h(x+v) + \pi_{k+1}(c_h, z) & \text{if } s^*_{k+1}(z) \le x < s_{h,k+1}(z) - v, \\ \varphi(x,v,z) + \alpha E_{R,D} C_k(x+v-zR, D) & \text{if } s_{h,k+1}(z) - v \le x < S_{h,k+1}(z) - v, \\ \pi_{k+1}(0,z) & \text{if } S_{h,k+1}(z) - v \le x < S_{h,k+1}(z), \\ \varphi(x,0,z) + \alpha E_{R,D} C_k(x-zR, D) & \text{if } x \ge S_{h,k+1}(z). \end{cases}$$

Subcase (B): $s_{l,k+1}(z) \ge S_{h,k+1}(z) - v$. In this case, our proof is divided into (a). $(-\infty, s^*_{k+1}(z))$, (b). $[s^*_{k+1}(z), s_{h,k+1}(z) - v)$, (c). $[s_{h,k+1}(z) - v, S_{h,k+1}(z) - v)$, (d). $[S_{h,k+1}(z) - v, s_{l,k+1}(z))$, (e). $[s_{l,k+1}(z), S_{h,k+1}(z))$, and (f). $[S_{h,k+1}(z), \infty)$. We only provide proof for (b) and (c) as the rest are similar.

(b). If $s_{h,k+1}(z) - v \le x < S_{h,k+1}(z) - v (\le s_{l,k+1}(z))$, then the optimal cost ordering from supplier one and two are $E_R f_{k+1}(x+v-zR)$ and $-c_l x + \pi_{k+1}(c_l, z)$. Thus, we use $J_{k+1}(x)$ to determine the optimal policy. Again, $s^*_{k+1}(z) < s_{h,k+1}(z) - v$ implies $J_{k+1}(s_{h,k+1}(z) - v) < 0$ and since $J_{k+1}(S_{h,k+1}(z) - v) < 0$, the quasi-convexity of $J_{k+1}(x)$ in x implies that $J_{k+1}(x) < 0$ for $x \in [s_{h,k+1}(z) - v, S_{h,k+1}(z) - v)$. Thus, it is optimal to order exactly v units from supplier one.

(c). If $S_{h,k+1}(z) - v \le x < s_{l,k+1}(z)$, then the optimal cost from ordering from supplier one and two are $E_R f_{k+1}(S_{h,k+1}(z) - zR)$ and $-c_1 x + \pi_{k+1}(c_l, z)$. Now, it is clear that $E_R f_{k+1}(S_{h,k+1}(z) - zR) \le -c_l x + c_l s_{l,k+1}(z) + E_R f_k(s_{l,k+1}(z) - zR)$. The last inequality holds as $x < s_{l,k+1}(z)$. Thus, it is optimal to order up to $S_{h,k+1}(z)$ from supplier one.

(d). For $s_{l,k+1}(z) \leq x < S_{h,k+1}(z)$, the optimal costs when ordering from supplier one and two are $Ef_{k+1}(S_{h,k+1}(z) - zR)$ and $Ef_{k+1}(x - zR)$. Clearly, ordering up to $S_{h,k+1}(z)$ from supplier one is optimal.

For the case when $s_{l,k+1}(z) \geq S_{h,k+1}(z) - v$, the optimal cost function for the period $k+1$ by the following:

$$C_{k+1}(x,z) = \begin{cases} -c_l x + \pi_{k+1}(c_l, z) & \text{if } x < s^*_{k+1}(z), \\ -c_h(x+v) + \pi_{k+1}(c_h, z) & \text{if } s^*_{k+1}(z) \leq x < s_{h,k+1}(z) - v, \\ \varphi(x,v,z) + \alpha E_{R,D} C_k(x+v-zR, D) & \text{if } s_{h,k+1}(z) - v \leq x < S_{h,k+1}(z) - v, \\ -c_l x + \pi_{k+1}(c_h, z) & \text{if } S_{h,k+1}(z) - v \leq x < s_{l,k+1}(z), \\ \pi_{k+1}(0, z) & \text{if } s_{l,k+1}(z) \leq x < S_{h,k+1}(z), \\ \varphi(x,0,z) + \alpha E_{R,D} C_k(x-zR, D) & \text{if } x \geq S_{h,k+1}(z). \end{cases}$$

Case (II): $s^*_{k+1}(z) \geq s_{h,k+1}(z) - v$.

Subcase (A): $s_{l,k+1}(z) < S_{h,k+1}(z) - v$. We divide our proof into (a). $(-\infty, s_{h,k+1}(z) - v)$, (b). $[s_{h,k+1}(z) - v, S_{h,k+1}(z) - v)$, (c). $[S_{h,k+1}(z) - v, S_{h,k+1}(z))$, and (d). $[S_{h,1}(z), \infty)$.

(a). If $x < s_{h,k+1}(z) - v$, the proof is similar to Subcase (A) of Case (I).

(b). If $s_{h,k+1}(z) - v \leq x < S_{h,k+1}(z) - v$, it is routine to consider $[s_{h,k+1}(z) - v, s_{l,k+1}(z)) \cup [s_{l,k+1}(z), S_{h,k+1}(z))$. For $x \in [s_{h,k+1}(z) - v, s_{l,k+1}(z))$, then the optimal costs by ordering from supplier one and two are $E_R f_k(x+v-zR)$ and $-c_l x + \pi_{k+1}(c_l, z)$. We can invoke the sign of $J_1(x)$ to determine the optimal policy. Note that $s^*_{k+1}(z) \geq s_{h,k+1}(z) - v$ implies that $J_1(s_{h,k+1}(z) - v) \geq 0$. Furthermore, $s_{l,k+1}(z) < S_{h,k+1}(z) - v$ implies that $J_{k+1}(s_{l,k+1}(z)) < 0$ (see Subcase (A) of Case (I)). There exists a root of $J_{k+1}(x)$, say $r_{k+1}(z) \in [s_{h,k+1}(z), s_{l,k+1}(z))$. Thus, $J_{k+1}(x) \geq 0$ for $x \in [s_{h,k+1}(z) - v, r_{k+1}(z))$ while $J_{k+1}(x) < 0$ for $x \in [r_{k+1}(z), s_{l,k+1}(z))$. We order up to $s_{l,k+1}(z)$

from supplier two whenever $x \in [s_{h,k+1}(z) - v, r_{k+1}(z))$ and order exactly v from supplier one whenever $x \in [r_{k+1}(z), s_{l,k+1}(z))$. For $x \in [s_{l,k+1}(z), S_{h,k+1}(z) - v)$, then the optimal costs by ordering from supplier one and two are $E_R f_{k+1}(x + v - zR)$ and $E_R f_{k+1}(x - zR)$, respectively. Given that $x < S_{h,k+1}(z) - v$ and that $S_{h,k+1}(z)$ being the minimizer of the quasi-convex function $E_R f_{k+1}(Q - zR)$ in Q, we have $E_R f_{k+1}(x + v - zR) \leq E_R f_{k+1}(x - zR)$. Thus, it is optimal to order exactly v from supplier one.

The proof of (c) and (d) follows exactly from Subcase (A) of Case (I).

Subcase (B): $s_{l,k+1}(z) \geq S_{h,k+1}(z) - v$. We divide our proof into (a). $(-\infty, s_{h,k+1}(z) - v)$, (b). $[s_{h,k+1}(z) - v, S_{h,k+1}(z) - v)$, (c). $[S_{h,k+1}(z) - v, s_{l,k+1}(z))$, (d). $[s_{l,k+1}(z), S_{h,k+1}(z))$ and (e). $[S_{h,1}(z), \infty)$. We only argue for (b) while the others are similar to those in subcase (B) of Case (I).

(b). Over $[s_{h,k+1}(z) - v, S_{h,k+1}(z) - v)$, we further split our analysis into $[s_{h,k+1}(z) - v, s_{l,k+1}(z)) \cup [s_{l,k+1}(z), S_{h,k+1}(z) - v)$. For $x \in [s_{h,k+1}(z) - v, s_{l,k+1}(z))$, the optimal cost of ordering from supplier one and two are $E_R f_k(x + v - zR)$ and $-c_l x + \pi_{k+1}(c_l, z)$. Next, $s^*_{k+1}(z) \geq s_{h,k+1}(z) - v$ implies that $J_{k+1}(s_{h,k+1}(z) - v) \geq 0$ and since $J_{k+1}(s_{l,k+1}(z) - v) < 0$, there exists $r_{k+1}(z)$ such that $J_{k+1}(r_{k+1}(z)) = 0$. Thus, we have $J_{k+1}(x) \geq 0$ for $x \in [s_{h,k+1}(z) - v, r_{k+1}(z))$ while $J_{k+1}(z) < 0$ for $x \in [r_{k+1}(z), s_{l,k+1}(z))$. For $x \in [s_{l,k+1}(z), S_{h,k+1}(z) - v)$, the optimal cost of ordering from supplier one and two are $E_R f_{k+1}(x + v - zR)$ and $E_R f_{k+1}(x - zR)$ It is optimal to order exactly v from supplier one as the argument is exactly that of (b) in Subcase (A) of Case (II).

Similar to the single period case, the structure of the right-hand side derivative is

$$C^+_{k+1}(x,z)\begin{cases} < 0 & \text{if } x < S_{h,k+1}(z) - v \\ = 0 & \text{if } S_{h,k+1}(z) - v \le x < S_{h,k+1}(z) \\ > 0 & \text{if } x \ge S_{h,k+1}(z) \end{cases}$$

Therefore, there exists $S_{h,k+1}(z)$ such that on $I_{k+1} = (S_{h,k+1}(z), \infty)$, $C_{k+1}(x,z)$ is non-decreasing while on $R \setminus I_{k+1}$, $C_{k+1}(x,z)$ is non-increasing. Using the expression of $C_{k+1}(x,z)$, we can easily conclude that as $|x| \to \infty$, we have $C_{k+1}(x,z) \to \infty$. Hence, the statement (d) is true for $n = k+1$.

(d). $\Rightarrow$ **(c).** The proof is similar to Lemma 5.3.1 and can be adapted to the case when $n = k+1$. Hence, the statement (c) is true for $n = k+1$. $\diamond$

We shall formally state the optimal policy for the model with two suppliers competing in parallel. Define $r_n(z)$ and $J_n(x) = c_l x + \varphi(x,v,z) + \alpha E_{R,D} C_n(x+v-zR, D) - \pi_n(c_l)$ such that $J_n(r_n(z)) = 0$ and $s^*_n(z) = \frac{\pi_n(c_h)-\pi_n(c_l)-c_h v}{c_h - c_l}$.

Theorem 5.4 *Let $y^*_n(x,z)$ denotes the optimal ordering quantity. For $s^*_n(z) < s_{h,n}(z) - v$, the optimal policy is characterized by*

(i). If $x < s^*_n(z)$, order up to $s_{l,n}(z)$ from supplier two.

(ii). If $x \ge s^*_n(z)$, order from supplier one using generalized base stock policy of $(s_{h,n}(z), S_{h,n}(z), v)$.

For $s^*_n(z) \ge s_{h,n}(z) - v$, the optimal policy is characterized by some $r_n(z) \in [s_{h,n}(z) - v, s_{l,n}(z) - v)$, such that

(i). If $x < r_n(z)$, order up to $s_{l,n}(z)$ from supplier two.

(ii). If $x \ge r_n(z)$, order from supplier one using generalized base stock policy of $(s_{h,n}(z), S_{h,n}(z), v)$.

Corollary 5.4.1 *(Optimal Choice Theorem) The optimal choice of the suppliers is described by a simple threshold policy. For every period n, there exists a critical number c_n such that if the inventory level is less than c_n, it is optimal to order from supplier two, otherwise order from supplier one.*

5.5 Impact of Additional Supplier

Let us assume that the inventory manager is in a supply contract with original supplier (supplier one) such that in each period, it pays c_h for every unit ordered in excess of v. This section focuses on the central theme by considering the impact of the alternative supplier (supplier two) who charges $c_l < c_h$ for every unit ordered. Suppose k is the period number of our interest. From Yeo and Yuan (2011), if the manager only orders from supplier one, then the optimal policy is of the type $(s_{h,k}(z), S_{h,k}(z), v)$. If he only orders from supplier two, then the optimal policy is an order-up-to $s_{l,k}(z)$ policy. The goal of this section is two-fold. First, we examine the role played by the alternative supplier in which the order-up-to $s_{l,k}(z)$ policy becomes more attractive option due to increased intensity of competition between the suppliers. Secondly, we want to quantify the impact of the alternative supplier in terms of cost savings. The natural question to ask: how much can the inventory manager save by having an alternative supplier to manage the supply chain?

5.5.1 Impact On Optimal Policy

From Theorem 5.4, the impact on the generalized base stock policy $(s_{h,n}(z), S_{h,n}(z), v)$ when we introduce an alternative supplier is characterized by additional parameters:

$s_n^*(z)$ and $r_n(z)$. In summary, the resulting ordering policy is piecewise-continuous. At the inventory level $s_n^*(z)$, the manager is indifferent between choosing either ordering up to $s_{h,n}(z)$ or ordering up to $s_{l,n}(z)$. For the initial inventory level at $r_n(z)$, the manager is indifferent between ordering exactly v and ordering up to $s_{l,n}(z)$. The dependence on these two extra parameters is intuitive since the optimal policies by ordering from supplier one and two are $(s_{h,n}(z), S_{h,n}(z), v)$ and order-up-to $s_{l,n}(z)$ during period n, given z. Interestingly, Theorem 5.4 implies that for initial inventory level exceeding either $s_n^*(z)$ or $r_n(z)$, it is optimal to retain the use of the original supplier via the generalized base-stock $(s_{h,n}(z), S_{h,n}(z), v)$ policy, otherwise we use the alternative supplier by ordering up to $s_{l,n}(z)$. From Theorem 5.4, the resulting optimal policy with two competing suppliers is affected either by $s_n^*(z)$ or $r_n(z)$, but not both. The primary conditions that govern which parameter affects the policy are $s_{h,n}^*(z) < s_{h,n}(z) - v$ and $s_{h,n}^*(z) \geq s_{h,n}(z) - v$. The reasoning is as follows. By the definition of $s_n^*(z)$, it is necessary that when $x < s_n^*(z)$ ordering up to $s_{l,n}(z)$ from supplier two is optimal, while ordering up to $s_{h,n}(z)$ from supplier one is optimal for $x \in [s_n^*(z), s_{h,n}(z) - v)$. Under $s_n^*(z) < s_{h,n}(z) - v$, the interval $A_n = (s_n^*(z), s_{h,n}(z) - v)$ is non-empty implying that whenever $x \in A_n$, order up to $s_{h,n}(z)$ from supplier one remains attractive. It is easy to see that whenever $s_{h,n}^*(z) \geq s_{h,n}(z) - v$, it is necessary that we always order from the alternative supplier offering a lower ordering cost of c_l whenever $x < s_{h,n}(z) - v$. Beyond inventory level $s_{h,n}(z) - v$, do we order exactly v or order up to $s_{l,n}(z)$? It turns out that the same primary conditions also determine whether or not $r_n(z)$ plays the role in the resultant policy. Our answer is based on the definition of $J_n(x)$ which is the difference in the optimal cost when ordering exactly v from supplier one and ordering up to $s_{l,n}(z)$ from supplier two.

For the finite horizon problem with N periods to go, what happens when the contracted volume is sufficiently large so that $\max_{1 \leq k \leq N}\{S_{h,k}(z)\} < v$? This scenario

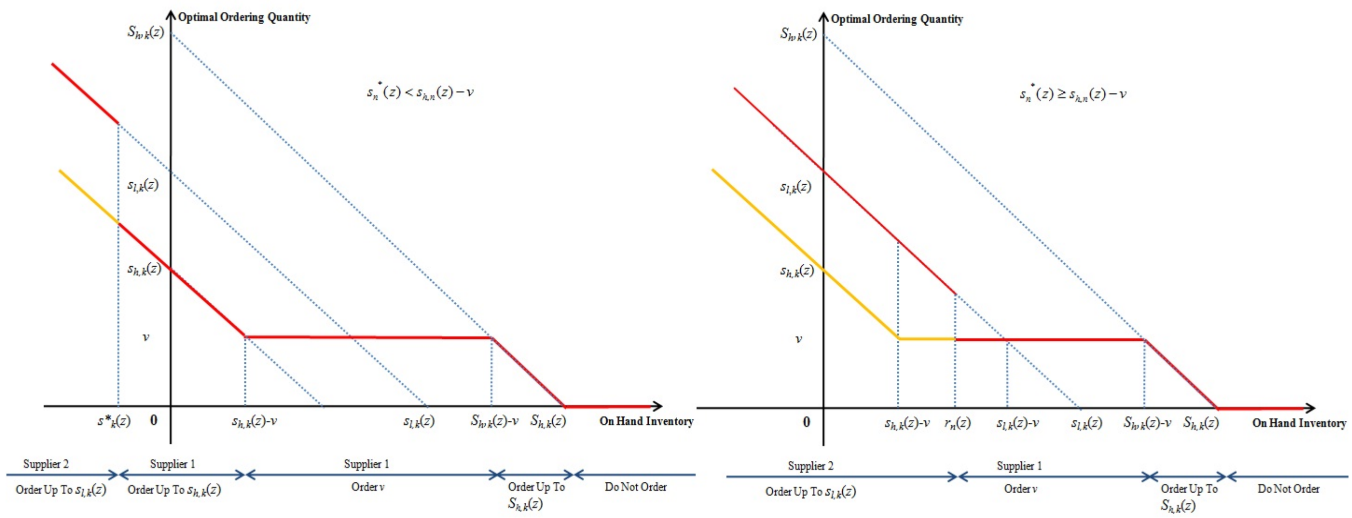

Figure 5.2: Optimal policies for two suppliers.

is similar to an order-up-to policy in the model of Yuan and Cheung (2003). Furthermore, due to the presence of non-zero ordering cost, it is certainly unattractive to use the alternative supplier. This is always true when the on-hand inventory is greater than zero.

5.5.2 Impact On Cost Savings

Suppose x and z be the on-hand inventory and number of items reserved in the previous period. Denote $C_1(x,z)$ to be the optimal cost when the inventory manager faces both supplier one and supplier two while $\widetilde{C}_1(x,z)$ is the optimal cost when he faces only supplier one. For our purpose of illustration, we assume that the distribution function for demand cancellation to be uniformly distributed $G(x) = x$ for $x \in [0,1]$. The analysis can be carried for other distributions. It is seen that $s_{h,1}(z) = z\frac{p-c_h}{h+p}$, $S_{h,1}(z) = z\frac{p}{h+p}$ and $s_{l,1}(z) = z\frac{p-c_l}{h+p}$. Some straightforward computation shows that when the ordering cost is c per unit, the optimal cost given

x and z is given by $\pi_1(c,z) = \frac{z}{2}\left(\frac{hp+2pc-c^2}{h+p}\right)$ and for $c_l < c_h$, $\pi_1(c_h) - \pi_1(c_l,z) = \frac{z}{2(h+p)}(2p-(c_l+c_h))(c_h-c_l)$. Thus, $s_1^*(z) = \frac{z}{2(h+p)}(2p-(c_l+c_h))-\frac{c_h v}{c_h-c_l}$. Finally, we have $\varphi(x,v,z) = \frac{h(x+v)^2}{2z} + p\left(\frac{x+v}{\sqrt{2z}} - \sqrt{\frac{z}{2}}\right)^2$. Let us define the term $\Delta(x) = \frac{\widetilde{C}_1(x,z)-C_1(x,z)}{\widetilde{C}_1(x,z)}$ and thus, $\Delta(x)\times 100\%$ is the percentage cost savings due to the presence of the alternative supplier.

Proposition 5.5.1 *If $x < s_1^*(z)$, then when the backlogging is asymptotically large, the savings has an upper bound of $1-\frac{c_l}{c_h}$. The saving decreases in the initial inventory.*

Proof: If $x < s_1^*(z)$, then it can be shown that $\widetilde{C}_1(x,z) = -c_h(x+v)+\pi_1(c_h)$ and $C_1(x,z) = -c_l x+\pi_1(c_l,z)$. Some algebra shows that $\widetilde{C}_1(x,z)-C_1(x,z) = -c_h v+(c_h-c_l)\left[-x+\frac{z}{2(h+p)}(2p-(c_h+c_l))\right]$. Given that $\widetilde{C}_1(x,z) = -c_h(x+v)+\frac{z}{2(h+p)}(hp+2pc_h-c_h^2)$. As $x \downarrow -\infty$, the asymptotic cost savings is given by applying L'Hospital rule on $\Delta(x)$. Furthermore, by differentiating $\Delta(x)$ w.r.t x, we get $\frac{\partial}{\partial x}\Delta(x) = \frac{1}{\widetilde{C}_1(x,z)^2}(-c_h c_l v+\frac{z}{2(h+p)}(c_h-c_l)(-hp-c_h c_l)) < 0$. Thus, the cost savings is decreasing in the initial inventory over this interval.

To assess the merit of the alternative supplier with a relatively lower c_l using the condition $s_{l,1}(z) \geq S_{h,1}(z) - v$, we let $J_1(x) = c_l x + \varphi(x,v,z) - \pi_1(c_l,z)$ and $J_1(s_1^{\#}(z)) = 0$. In this case, the role played by $s_1^{\#}(z)$ is examined. To this end, $-v+z\frac{p-c_l}{h+p}+\sqrt{\frac{2vc_l z}{h+p}}$ solves $J_1(x) = 0$. We note that the ordering between $s_{h,1}(z)$ and $s_1^{\#}(z)$ depends on either $z\left(\frac{c_h-c_l}{h+p}\right)+\sqrt{\frac{2vc_l z}{h+p}}$ and v. Finally, $J_1(x)$ has minimum point on $[s_1^{\#}(z), S_{h,1}(z)-v)$ which is $s_{l,1}(z)-v$. Thus, the maximum saving that can be attained is at the point $c_l v$. In percentage terms, the saving is at most $\Delta(s_{l,1}(z)-v) = \frac{|J_1(s_{l,1}(z)-v)|}{\tilde{C}_1(x,z)} = \frac{2(h+p)c_l v}{z(hp+c_l^2)}$. It is easy to see that when z increases, $\Delta(s_{l,1}(z)-v)$ decreases, i.e. the role of the alternative supplier is diminished. Suppose z increases, the minimum point of $J_1(x)$, $s_{l,1}(z)-v$ increases as well. Thus, the differences between ordering from the original supplier and ordering with an alternative supplier becomes smaller as the optimal costs shift towards the right, reducing the attractiveness of the alternative supplier. When v increases, $\Delta(s_{l,1}(z)-v)$ increases. The attractiveness of the alternative supplier is highlighted by the fact that the minimum point $s_{l,1}(z)-v$ decreases as v increases. Finally, we look at the impact of increasing holding and penalty costs on the savings. We will only argue for the case of h as p is similar. Using Bernoulli's rule, the limit for $\Delta(s_{l,1}(z)-v) \to \frac{2c_l v}{zp}$ as $h \to \infty$. Furthermore, by differentiating $\Delta(s_{l,1}(z)-v)$ w.r.t h, we obtain $\frac{d}{dh}\Delta(s_{l,1}(z)-v) = 2c_l v\left(1-\frac{h+p}{hp+c_l^2}\right) > 0$. Thus, as either the holding cost or penalty cost increases, the percentage savings increases to a limit.

Proposition 5.5.2 *Suppose $s_{l,1}(z) \geq S_{h,1}(z)-v$, then the maximum savings attained is at $s_{l,1}(z)-v$ and has a supremum at $\frac{2(h+p)c_l v}{z(hp+c_l^2)}$.*

Hence, the maximum savings as a result of introducing an alternative supplier supplier two is given by $\Delta(x) \leq \max\{1-\frac{c_l}{c_h}, \frac{2(h+p)c_l v}{z(hp+c_l^2)}\}$ whenever R is uniformly distributed on $[0,1]$. Similar analysis can be performed on other distributions.

5.6 Concluding Remarks

This work is an extension of Yeo and Yuan (2010b) by considering two suppliers competing in parallel for procurement. Assuming that all demands are reserved and cancellation is allowed within a leadtime of one period, the problem when the manager enters into a single-tier supply contract is solved. We introduce another supplier who charges a lower ordering cost for every item ordered so that the manager has an option. Our aim is to determine the impact of the alternative supplier on the original ordering policy, which is the generalized base-stock policy of type (s, S, v).

We characterize the optimal policy for the single and multiple horizon cases. It turns out that the optimal cost in each case is quasi-convex, the method used in Yuan and Cheung (2003), or Yeo and Yuan (2011,2010b) are not applicable anymore. This is because aggregation of two quasi-convex functions is not necessarily quasi-convex. Fortunately, we are able apply the theory of single-crossing functions developed by John and Bruno (2010) to establish the policy. Unlike the optimal policies in Yuan and Cheung (2003) or Yeo and Yuan (2011,2010b), the optimal policies is not continuous in the on-hand inventory in our model although the random variable for demand not cancelled eventually has a continuous distribution. For each period n, the optimal policy is a hybridized form of $(s_{h,n}(z), S_{h,n}(z), v)$ and order-up-to $s_{l,n}(z)$. Furthermore, we graphically illustrate the impact of the alternative suppliers' ordering cost on the generalized base-stock policy. Our optimal choice theorem states that for every period n, there exists a critical number such that if the inventory level falls below it, the manager will choose the alternative supplier, otherwise he will choose the original supplier offering the transportation contract. Finally, we assess the impact of the alternative supplier using cost savings as a performance measure. Assuming that distribution of demand cancellation being uniform on $[0, 1]$, we derive the upper bound on which the cost saving is achieved. There are numerous ways to extend the this work. One can consider the impact of an alternative supplier offering a fixed setup cost together with a lower ordering cost. Therefore, our model is a special case when the setup cost is zero. Similar analysis can be done by even considering three suppliers. Our model assumes that splitting of orders between the suppliers is not allowed and thus, as an extension, we can consider splitting of orders between the two suppliers. Finally, one can explore the possibility of extending this model to the infinite horizon case.

CHAPTER 6

CONCLUSIONS AND FUTURE WORKS

Exponential growth has been observed in internet retailing seeing scores of industries market or selling their diverse range products online, bringing about the paradigm shift of penetrating the market from the more traditional "brick-and-mortar" to the increasingly popular "click-and-mortar" approach. First movers that failed during the dot.com era neglected the value of supply chain management, but focus on front-end activities such as increasing website appeal. Many businesses that improved the infrastructure of inventory management systems succeeded, while businesses that focused on web development failed (see Tarn et al (2003)). With a dearth of literature investigating periodic review inventory systems involving demand cancellation, I investigate three models of inventory networks useful to internet retailing with various suppliers configurations.

In this thesis, I extend the foundational work of Yuan and Cheung (2003) who consider a periodic review inventory system with demand reservation and cancellation. All models in this work assume that demands are reserved one period in advance and cancellation is possible. One central issue in this thesis is to study the impact of the different types of suppliers on the optimal inventory policy. In practice, many companies still favor the simple strategy of "order-up-to" policy. Using scientific methodology, this research guards against the complacency of using "order-up-to" policy.

Chapter 3 extends Yuan and Cheung (2003) to consider supply uncertainty. It is proven that the "critical-point" policy dominates the "order-up-to" policy. I go beyond by using stochastic ordering to quantify the importance of reducing the variance of either the distribution of yield or the distribution of demand cancellation.

Chapter 4 focuses on the impact of introducing a multi-tier supply contract on the optimal inventory policy. Inspired by Henig et al (1997), we prove that the optimal policy is "finite generalized base stock" which is similar to Frederick (2009). However, our critical points depend on customers' reservation parameter. The analysis of cost function is bivariate in the on-hand inventory and customers' reservation. The presence of a continuous, non-differentiable (at countably many points) ordering cost presents some difficulty to proving the infinite horizon case. However, I overcome that hurdle appealing to Theorem 8-14 of Heyman and Sobel (1984). A comparison to the optimal policies is illustrated between this model and Yuan and Cheung (2003) and Yeo and Yuan (2011). This allows us to quantify the impact of not considering ordering cost (see Yuan and Cheung (2003)) where moral hazard is induced in the ordering behavior. Moreover, the work of Yuan and Cheung (2003) with non-negative ordering cost is easily subsumed in this model.

Chapter 5 extends the work of Yuan and Cheung (2003) and Chapter 4 by considering the presence of two suppliers offering different supply contracts. It turns out that the ordering cost is neither concave nor convex and is non-differentiable (at countably many points) in the on-hand inventory. The optimal policy is derived using a recent theory developed by John and Bruno (2010). This is due to the quasi-convexity in the on-hand inventory of the optimal cost function. I justify the impact on the optimal replenishment policy of introducing an alternative supplier (offering a lower ordering cost).

This thesis is a first step at studying how the choice of cancellation can affect inventory manager's optimal ordering decision in the multiple period setting. Due to the different suppliers types assumed, I have presented three different theoretical developments of the optimal inventory policies. One basic assumption in this thesis is that there is no fixed cost or leadtime. Therefore, one is able to study the system with a fixed cost and the inclusion of leadtime. In the presence of delays, it is important to note that customers' cancellation can occur while items are still in transhipment. To illustrate this, suppose L is the leadtime. If there are $n\,(> L)$ periods left, we should consider $\mathbf{z} = (z_1, z_2, ..., z_{L-1})$ where z_i is the item reserved i periods ago (but not canceled). For simplicity, we assume that R be the ratio of items reserved during the last period but is not canceled eventually in the next period while still under transhipment or delivery. Let x and D be the level of initial inventory and demand during period n. Then, the dynamic evolution for $\mathbf{z}' = (z_1', z_2', ..., z_{L-1}')$, the vector of items reserved when there are $n-1$ periods left can be described by $z_1' = D, z_j' = Rz_{j-1}$ for $2 \leq j \leq L-1$. Let x' be the initial demand during period $n-1$, then $x' = x + \theta y - Rz_{L-1}$. Due to tractability concerns, the study of this problem is deferred.

In all our models, I have assumed system dynamics to be linear. In dealing with more complex network of suppliers and even with the possibility of incorporating remanufacturing, one might need to consider non-linear dynamics. Systems with non-linear dynamics involving manufacturing have appeared in the work of Zhou and Sethi (1994) and Sethi and Zhang (1994). Furthermore, our main concern has been the construction of optimal decisions in observable inventory networks with full information of the on-hand inventory and reservation parameters. However, there are situations in inventory systems where completely observable information can be difficult to achieve. For example if the product comes with a warranty agreement, the on-hand inventory needed to manage the system depends on the returnable through reliability of the products. Another example is the dependence of demand on the environment. Such systems are useful using partially observable stochastic processes such as partially observable Markov decision processes (POMDP). In practice, a customer can have several choices of online retailers such as eBay or Amazon. As such it will be interesting to look at how competition (with demand uncertainty and cancellation) will have an impact on the optimal inventory policy at each retail company. Systems involving several players competing against each other might require stochastic differential game formulations (see Yeo and Lim (2010)).

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