MULTIPLE OBJECTIVES SATISFICING UNDER UNCERTAINTY

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"forsan et haec olim meminisse iuvabit."

L. S. W. S.

Table of Contents

List of Figures

List of Tables

Abstract of the Thesis

Multiple Objectives Satisficing Under Uncertainty

We consider a multiple objectives problem in which the objectives are potentially uncertain when the decisions are made. The goal of the decision problem is to select a feasible solution so that all the objectives achieve their specified *targets* as well as possible. The targets may also be potentially uncertain. In order to deal with this multiple objectives problem, we propose a class of objective criteria, called multiple objectives satisficing (MOS) criteria. The MOS criteria provide efficacious evaluations of the level of compliance of the set of objectives in meeting their targets collectively under uncertainty. The MOS criteria include the joint targets achievement probability (or success probability) as a special case and also extend to situations when the probability distribution is not fully characterized (i.e. where only partial distributional information such as the support, mean and covariance are available). We specialize the MOS criteria to a sub-class of *diversification favoring* MOS (DMOS) criteria which has the potential to mitigate severe shortfalls in scenarios when an objective fails to achieve its target. Naturally, the DMOS criteria excludes success probability, which is inherently insensitive to the degree of shortfalls. A specific form of the DMOS criteria that inherits all the desirable characteristics of the MOS criteria and simultaneously incorporates diversification preferences is proposed. The shortfall-aware MOS criterion (S-MOS) is shown to be a lower bound to success probability and allows for the consideration of distributional ambiguity. As the consideration of distributional ambiguity may lead to intractable problems, we show how to construct tractable approximations for the S-MOS criterion. We also propose algorithms for evaluating the S-MOS criterion via solving sequences of convex optimization problems. We report encouraging results via an array of numerical experiments and case studies based on problems in the product development, finance and the oil and gas industry. The numerical studies clearly demonstrate the ability of the S-MOS criterion in overcoming the inherent deficiencies of success probability and expected value based criteria. Specifically, it is shown to mimic probability measures and yet possesses the qualities of sensitivity to the degree of shortfalls, ease of specifications and the ability to deal with decision

problems where only partial distributional information are available.

Keywords: satisficing, targets, multiple objectives, robust optimization

Preamble

Prior to delving into the intricacies of the developments that form the main crux of this thesis on multiple objectives satisficing under uncertainty, we mention some of the research work done through the course of the PhD candidature. These research helped to internalize the inherent challenges and practicality in considering multiple objectives satisficing problems under uncertainty, thereby catalysing the development of the key ideas within this thesis.

Reliability risk management and data modeling [Tang et al., 2010]: Motivated by the fact that the major causes of catastrophic failure in micro hard disk drives are mostly induced by the presence of particles, a new particle-induced failure susceptibility metric, called the Cumulative Particle Counts (CPC), is proposed for managing reliability risk in a fast-paced hard disk drive product development process. This work is thought to represent the first successful attempt to predict particle-induced failure through an accelerated testing framework which leverages on existing streams of research for both particle-injection-based and inherent-particle-generation laboratory experiments to produce a practical reliability prediction framework. In particular, a new testing technique that injects particles into hard disk drives so as to increase the susceptibility of failure is introduced. The experimental results are then analyzed through a proposed framework which comprises the modeling of a CPC-to-failure distribution. The framework also requires the estimation of the growth curve for the CPC in a prime hard disk drive under normal operating conditions without particle injection. Both parametric and non-parametric inferences are presented for the estimation of the CPC growth curve. Statistical inferential procedures are developed in relation to a proposed non-linear CPC growth curve with a change-point. Finally, two applications of the framework to design selection during an actual hard disk drive development project and the subsequent assessment of reliability growth are discussed.

This research won the IIE Transactions best applied paper award, 2011.

Six Sigma methodologies: Fortification of Six Sigma methodologies [Tang et al., 2007]: Six Sigma as a quality improvement framework cannot remain static if it is to sustain its value for businesses beyond the first waves of applications. This research explores the possibilities of enhancing the usefulness and effectiveness of Six Sigma by the integration of established Operations Research/Management Science (OR/MS) techniques. In this research, the needs for OR/MS techniques to enhance Six Sigma deployment in operational and transactional environments are elucidated and a new training roadmap for core Six Sigma professionals (Six Sigma Black Belts) which incorporates these techniques is proposed. A matrix relating the components of the proposed training curriculum to the actual deliverables during implementation for a hybrid of operational and transactional environments is also presented. A practical case study is also presented to demonstrate the usefulness of the OR/MS tools in a typical transactional environment.

Six Sigma methodologies: Statistical experimentation [Goh and Lam, 2010]: In product development, statistical experimentation techniques can help to improve product quality so as to meet or even exceed customers' requirements. To integrate proper statistical techniques effectively and sustainably into new product development and innovation processes, it is important to train product development engineers who are usually experts in their respective engineering domain. In this research, a problem-based learning approach integrated within a Design for Six Sigma (DFSS) training program is described. It is illustrated that instead of relying on ad hoc implementation, statistical experimentation introduced through the DFSS methodology would enable product development engineers to appreciate its usefulness within a broader framework.

Six Sigma methodologies: Six Sigma for healthcare delivery [Tang et al., 2008b]: Due to the numerous success stories of the Six Sigma quality initiative, it has become a subject of intense study and discussion in both the industry and academia over the past fifteen years. A fundamental tenet behind the success of Six Sigma is the use of structured strategies to achieve well-defined business goals. An enhanced Six Sigma framework, which takes into account the distinctly transactional nature of processes found in an extended healthcare delivery system is proposed in this research. Critical differences in the key elements of a Six Sigma program between manufacturing and transactional environments are first identified to motivate the proposed framework. The new framework includes some new systems engineering tools that are deemed more effective than the traditional set of Six Sigma tools. A case study is presented to demonstrate the suitability of these tools within the proposed Six Sigma framework.

Six Sigma methodologies: Statistical tests for normality [Tang and Lam, 2006b]: One of the basic model assumptions in statistical procedures used in Six Sigma applications is that of normality of data. Statistical "goodness-of-fit" (GOF) techniques have been developed in the past decades to assess the adequacy of using the normal distribution to model real-world data so as to limit the risk against severe departures from normality. These GOF techniques have been widely used in Six Sigma projects with the help of statistical software packages such as JMP and MINITAB. This research presents the theoretical concepts together with the operational procedures (coded in Microsoft Excel) for a collection of commonly used GOF tests for normality. The limitations of some of these GOF tests are discussed and compared.

Six Sigma methodologies: Categorical data analysis for Six Sigma [Tang and Lam, 2006a]: Effective statistical analysis hinges upon the use of appropriate techniques for different types of data. As Six Sigma evolves from its original applications in manufacturing environments to new-found applications in transactional operations, categorical responses increasingly become the norm rather than the exception. In this research, two basic schemes, contingency tables and logistic regression, for the analysis of categorical data are presented. These techniques can easily be implemented in Excel or statistical software such as MINITAB. Two case studies, one for each of these methods, are also given to illustrate their application.

Systems engineering methodologies [Ng and Lam, 2010]: The approach of thinking about systems as a whole with the explicit consideration of complex interactions between its constituents is the key conceptual framework underlying systems thinking. Such an approach is particularly useful for unravelling the dynamic interactions that confounds present day maritime systems. In this work, we first undertake a review of existing literature that applies systems thinking to various supply and demand dynamics underpinning the global maritime industry. Thereafter, we distil the DIVER systems inquiry framework that offers a logical and consistent platform to guide the systems thinking process in elucidating the maritime dynamics. We describe the implementation of system dynamics modelling and simulation as a particularly relevant tool within this framework. Its role as a structural modelling and analysis tool for various supply-demand problems is described. Furthermore, as a means to facilitate the systems thinking process in this framework, we propose a set of archetypical system structures that can explain some commonly observed maritime dynamics. Finally, a case study on the evolution of global containership capacity based on DIVER is presented.

Transport scheduling: Multiple response optimization for bus services [Lam et al., 2009]: A multiple response optimization approach based on an integrated simulation model and a service reliability measure is proposed for optimizing the dispatch rules of public bus services. The simulation model is based on two novel Markov models for modeling passenger loads and headway fluctuations. In consideration of the importance of passenger loads for bus services of high frequency, a service reliability measure based on the odds of seeing a full bus plying the route is designed. Such a simulation based multiresponse optimization approach can easily consider sampling variations and is scalable to deal with a larger public bus transit system with other objectives, such as minimizing the cost of operations. Practical data collection, analysis and estimation procedures for the load and headway distributions are also presented. A case study based on an actual public bus route in Singapore is used to demonstrate the usefulness of such an integrated simulation framework.

Transportation policy modeling: Operational efficiencies of Asia Pacific airports [Tang and Lam, 2006a]: This research analyses various dimensions of operational efficiencies in major Asia Pacific airports through Data Envelopment Analysis (DEA) models that simultaneously account for external macroeconomics and price factors. Results from this study show that technical, scale and mix efficiencies are high among the major Asia Pacific airports. In addition, significant disparities in cost efficiencies were detected amongst these airports. These disparities can be attributed to the presence of country-specific effect and differences in allocative efficiencies. A virtual airport is then introduced as a superior performer to rank the efficiency performances of airports.

Transportation policy modeling: Competitive evaluations of hub ports [Low et al., 2009]: This research proposes a novel network-based hub port assessment (NHPA) model through explicit formulations of connectivity and cooperation indices. Such a model is useful for port operators and policy makers to evaluate the relative influences of various quality characteristics from which carriers base their port choices on and identify port partners. Key insights can be obtained for port authorities to improve their port infrastructures and operations to achieve a competitive and sustainable hub port status. Three comprehensive case studies are conducted to assess the current standings and potentials of major Asian ports within networks of major carriers.

Transportation policy modeling: Network based choice evaluation [Low and Lam, 2009]: A novel Network-based Integrated Choice Evaluation (NICE) model is developed to enhance the multinomial logit preference (MNL) model that is widely employed in the existing port choice literature. The NICE model integrates the element of port service network with observational port attributes to identify important quality characteristics on which liner shipping companies base their port choices. An empirical study of the proposed model is conducted through the service schedules of three established liner shipping companies. Results show that port efficiency and scale economies are the more important dimensions influencing liner shipping companies' selection of major Asian ports. Nevertheless, it is important for a competitive port to balance its efforts among all the dimensions.

CHAPTER 1

Introduction

Uncertainty is ever present in real-world decision analytic problems. Examples of these include uncertainties in demand, price, quality and lead-times in supply chain management and chemical process industries; measurement errors and quality defects in engineering and manufacturing industries; and, input prices, quality, service lead times and demand in service industries. Despite the prevalence of uncertainties in real-world decision analytic problems, deterministic frameworks have typically been adopted. In optimization problems, for example, the decision analyst typically assume that data necessary to state the optimization problem is precisely known. The optimization is generally of a minimization (or maximization) of costs (or profits/ payoffs) type over a set of decision variables subjected to some resource constraints. Apart from the presence of uncertainties, real-world decision making may require the evaluation of alternatives across multiple objectives subjected to resource constraints. The most common multiple objectives formulation assumes some reasonable weightings over the individual objectives resulting in a single objective formulation. Such a generic framework includes the multiobjectives optimization and multiattribute utility paradigms. The goal of this optimization problem is to simultaneously optimize two or more conflicting and usually deterministic objectives.

In real-world managerial decision making, decision makers not only have to deal with a ranking of alternatives over numerous and sometimes conflicting objectives under resource constraints but simultaneously, they often have to deal with uncertainty in the problem data. Such problems can often be framed under the multiobjective decision models (MODM) paradigm. The MODM approach encompass the case which may be formulated such that the selection amongst decision alternatives is described by multiple attributes [Keeney and Raiffa, 1976]. In constrained multi-objective optimization problems, the feasible set of solutions may be either small and finite (e.g.

discrete choice problems) or large and perhaps infinite (e.g. continuous optimization problems). Furthermore, due to a lack of complete ordering in multiobjective optimization problems, the solutions are a possibly infinite set of Pareto points (see the reviews by Alves and J. [2007]). Under deterministic assumptions, such infinite set of Pareto points can provide some guide to the selection of desirable alternatives by presenting decision makers with the set of Pareto optimal solutions *a posteriori*, and even facilitate negotiations amongst multiple decision makers [Lam and Tang, 2006]. Nonetheless, when uncertainty is considered under a multiobjective optimization paradigm, it is unlikely (or near impossible) that the decision maker can ever realize the desired optimal objective value. In the presence of uncertainty, constraint violations in implementing the deterministic solution has been found to be the norm, rather than the exception, in many real-world optimization problems [Bertsimas and Sim, 2004]. In these practical situations, the risks of constraint violations may have very adverse implications on the practical applicability of the deterministic solution [Ben-Tal and Nemirovski, 1998] and thus, have to be explicitly considered in the optimization model.

1.1 Uncertainties in decision making: Risk and ambiguity

There is an abundance of management science research that has proposed complex models for deriving optimal managerial decisions in the presence of uncertainty. "Optimality" in decisions has been defined in a variety of ways in the extant literature. Risk and its effects on managerial decision-making has also been extensively studied in these literature. However, most of these models of decision making under uncertainty have considered the implicit assumption that managers can completely assign probabilities to outcomes, and consequently, are able to optimize their actions under full distributional information. On the other hand, researchers working on probability assessments have found an interesting empirical regularity in the sense that managers have difficulty coming up with one single number representing a probability, instead, they often can judge accurately a range for such probabilities, or a family of probability distributions characterized by partial moment information. Such uncertain decision making situations presents the complicating issue of having to deal with both ambiguity and risks. The dichotomization of general uncertainty to risk and ambiguity was first proposed by Knight [1921]. Accordingly, risk is present when an agent does not know the actual state of the world but can however assign subjective priors (probabilities) for each of the possible outcome. Ambiguity on the other hand refers to the case where an agent lacks sufficient knowledge to assign such subjective priors.

Decision problems considering multiple objectives and uncertainty may involve imprecise data which the decision maker do not even have full distributional information. In order to deal with such situations, we introduce the notion of ambiguity through a distinction between risk and ambiguity [Knight, 1921]). Risk considers uncertain outcomes under known probability distributions whereas ambiguity is defined here as uncertainty in the Knightian sense [Knight, 1921], whereby the full distributional information may not be available to make decisions. There are many practical examples of real-world managerial or operations decision problems that have to be made in the presence of both risk and ambiguity. In many cases, engineering or management decisions are frequently made through available data and information that are limited and sometimes "ambiguous" by nature. For example, the decision-making process in the product development process may involve ambiguous data, such as reliability data (see for example Tang et al. [2010]), whose distributional information cannot be fully characterized unless certain rigid assumptions are made. In such product development processes, designers are frequently presented with limited engineering data and tasked to rank design alternatives characterized by multiple product criteria that are important in achieving customers' satisfaction, alternatively known as the Critical-to-Quality characteristics (CTQs) [Goh et al., 2006]. Hence, the design optimization problems often involves multiple objectives. Due to the fact that these are new innovative products, less information is known about them at the design stage, hence, data on the CTQs of the design alternatives can be scarce and ambiguous. The presence of risk and ambiguity in product development processes and a variety of inventory and portfolio management further motivates us to develop a new criterion for multiple objectives problems that can deal with both risk and ambiguity.

Examples of the presence of ambiguity in real-world uncertain decision problems is prevalent. An example in the product development industry is given in the preceding. Another example involves the management of inventory across supply chains. For such problems, classical inventory theory has been overwhelmingly concerned with examining stocking decisions that would reduce the effects of demand variability under the assumption of full distributional knowledge. The demand distribution in making such inventory decisions is likely to be arrived at by examining past demand data. However, irrespective of the statistical procedure used, estimating the exact demand distribution for the optimization of inventory decisions is practically impossible with any finite amount of past data. This difficulty is exacerbated when the demand distributions are not stationary over time. In mathematical finance, much of portfolio theory examines ways and means to balance project portfolios so as to increase returns while mitigating the risks associated with project financing. Many of the large-scale projects may be one-off events that has never been done before, nor likely to be repeated again. For such large scale projects, it may be unlikely that any group of experts is going to arrive at a consensus on the exact distribution of potential returns from a project as budget estimations maybe highly inaccurate [Finnerty, 2007].

Apart from the aforementioned examples in new product development, inventory management and risk management, the practical implications of the presence of ambiguity in risky decision making contexts are well-documented in many other application areas [Natarajan et al., 2010b, Calafiore, 2010, Gilboa and D., 1989, Garlappi et al., 1989, Popescu, 2007]. In fact, the physiological basis that firmly establishes the importance of considering uncertainty in risky decisions have been established recently through functional brain imaging. It was shown that the level of ambiguity in choices correlates positively with activation in the amygdala and orbitofrontal cortex, and negatively with a striatal system [Hsu et al., 2005]. The data suggest a general neural circuit responding to degrees of ambiguity, thereby providing pathological evidence that ambiguity indeed affects human decision behaviors under uncertainty. This implies that the presence of ambiguity cannot be ignored in the most basic human decision processes when risk preferences are to be modelled. Despite these compelling practical and concrete physiological reasons that mandate a model of decision-making to consider ambiguity as one of its core elements, the effects of ambiguity in risky decision making is relatively less well-examined in the analytical stream of management literature.

1.2 Satisficing versus optimizing

It has been postulated that rather than formulating and solving complicated optimization problems, real-world agents often can choose the first available actions which ensure that certain target levels will be achieved [Simon, 1955, 1959]. This can be attributed to the presence of bounded rationality. Specifically, one implication of the bounded rationality concept is that in the presence of resource constraints, information scarcity and data uncertainty, a more sensible (and descriptively accurate) approach may in fact be to view the decision problem of not having an objective to be maximized (or minimized), but rather a constraint relative to some aspiration levels to be "satisficed". The concept of satisficing behaviors explicitly acknowledges the practical impossibility of searching through the feasible space for an optimal solution in many practical decision problems under uncertainty. This is the essence of the satisficing approach defined in Simon [1959]. The basic idea behind Simon's definition of the term satisfice, which is a portmanteau of the concepts encompassed in the terms "satisfy" and "suffice", is reproduced as follows [Simon, 1959]:

> *Individuals have pre-specified targets. They only look for new alternatives when the target isn't met. Stop when you find an option that meets the target.*

One of the most established paradigm in dealing with multiple objectives is the multiattribute utility theory (MAUT) framework. Herein, the optimization model requires the estimation of multiple utility functions and the synthesis of these functions into a single utility measure. In order to implement the MAUT framework, decision makers would have to estimate certain risk tolerance parameters and weighting over the individual utility functions. Such parameters are often highly subjective and difficult to specify (and estimate). An important advantage of the satisficing paradigm is that aspiration levels (or targets) are often more natural for investors to specify, compared to traditional models based on utility functions which depend on these parameters that are often difficult for decision makers to intuitively grasp and accurately assess. Such difficulties are magnified in the case of decision problems considering multiple objectives. Hence a multiple objectives satisficing approach may be a more suitable and practical alternative to the traditional MAUT paradigm.

The consideration of satisficing decisions bears relations to the domain of reference-dependent utility, a framework which arose from the development of prospect theory [Kahneman and Tversky, 1979, Tversky and Kahneman, 1992]. This theory was aimed at correcting perceived inadequacies of expected utility theory from a descriptive perspective. For instance, traditional expected utility theory does not account for the fact that decision makers may be significantly more sensitive (especially locally) to losses around a reference point than they are to gains. These considerations may result in a non-convex value function. Specifically, some recent work, such as [Heath et al., 1999], suggest the interpretation of the inflection point in the S-shaped value function proposed in the context of prospect theory as a target. The relevance of prospect theory in modelling decision behaviors in the presence of uncertainty are often borne out in empirical data of real human decisions [Sudgen, 2003, Kahneman and Tversky, 1979].

A target-oriented approach is intuitively more appealing in a variety of decision making contexts such as in new product development and in some financial decision making such as the design of effective portfolios for managing liabilities over a finite time horizon [Brown and Sim, 2009]. As a simple example, success in competitive new product development is typically determined by the ability of new or improved products to perform across multiple key design parameters relative to competitors, and not on how optimal the design parameters are *per se*. Hence, natural targets can be easily identified in such situations, such as the level of achieving specific critical-to-quality parameters [Goh et al., 2006]. These examples will be used to demonstrate the effective application of the target-based satisficing approach developed in this thesis. The case study in liability management demonstrates the ability of the model developed in this thesis to be superior to results obtained from the classical Markowitz mean-variance model for asset allocation [Markowitz, 1952]. The case study in engineering product development demonstrates the applicability of a target based satisficing approach for design selection in a high quality rapid product development environment. Furthermore, the proposed target-based satisficing model can be easily extended to the deal with situations where design parameters are uncertain and the distributions can only be characterized by some limited distributional information in the form of some descriptive statistics.

1.3 Multiple objectives satisficing under uncertainty

In this thesis, we propose a class of functions, called multiple objective satisficing (MOS) criteria, for evaluating the level of compliance of a set of objectives in meeting their targets collectively under uncertainty. This family of measures can be extended to encompass measures that are consistent with diversification preference. For reasons that would be highlighted later, our proposed family of diversification favoring measures are sensitive to rare events that may result in extreme consequences. This device is technically absent from the traditional measures based on probability and expected value.

Our approach of incorporating targets in multiple objectives decision problems differs markedly from the generic utility approaches as there is no explicit characterization of a utility function. One significant limitation of utility based approaches is that they typically require some calibration of the utility functions, which may not be done easily with reasonable accuracy. Our approach also differs from the usual multiobjectives optimization approach which seek to simultaneously optimizing two or more conflicting and usually deterministic objectives that may evaluate solutions that belongs to a possibly infinite set of Pareto optimal solutions. Such multiobjectives optimization approaches can be differentiated from the multiattribute utility paradigm in that decision preferences may be integrated into the decision process *a posteriori*, in order to identify unique Pareto optimal solutions from the possibly infinite Pareto optimal set [Lam and Tang, 2006]. The proposed family of MOS criteria hence, neither falls under the paradigm of either the multiobjectives optimization nor MAUT approaches. Instead of looking from these usual multiple objectives optimization perspectives, we start from a fundamental target-oriented satisficing framework towards the development of a realistic decision analytic model for multiple objectives decision making problems.

Perhaps the most natural method of incorporating targets in decision making is to evaluate the probability of target achievement or success probability; see for instance Browne [1999], Föllmer and Leukert [1999]. However, one significant drawback in the evaluation of success probability is that it tacitly assumes the decision maker is indifferent to the level of losses when they occur. In real-world situations, decision makers are not completely insensitive to the magnitudes of losses

[Payne et al., 1980, Diecidue and van de Ven, 2005]. Moreover, it is a intractable problem to optimize over success probability and to select the best decision over possibly exponential or infinite alternatives.

Apart from addressing the aforementioned limitations in the maximization of success probability, our proposed model also do not require the problematic calibration of fuzzy and amorphous risk tolerance parameters necessary for implementing the expected utility paradigm. The proposed model further facilitates the consideration of "ambiguity" in risky decision making via the construction of a class of tractable approximations to a specific multiple objectives satisficing criterion that enables the decision maker to only specify partial distributional information such as the support, mean and convariance. These approximations retain the desirable characteristics of the MOS criteria that considers diversification preferences. Distributional ambiguity as defined here is similar to Knightian uncertainty [Knight, 1921]. Simply put, in some situations full distribution information is available for decision making and risks addresses uncertainties under the specific distribution. Sometimes the decision maker may not have the luxury of knowing the full distributional information. What the decision maker may know could be some limited distributional information, such as the support, mean and covariance, and decisions optimal for the family of distributions characterized by these descriptive statistics can be derived through the proposed approach.

The MOS criteria encompass success probability as a special case and can be extended to incorporate a larger scope of uncertainty involving distributional ambiguity. We also propose a class of MOS criteria that favor diversification, which has the potential to mitigate severe shortfalls in scenarios when an objective fails to achieve its target. We also show that the MOS family of criteria have a unique dual relationship to the family of monetary risk measures [Föllmer and Schied, 2004]. This family of monetary risk measures has the well-known Value-at-Risk measure as a member. A specialized form of MOS to consider decision makers' preference for diversification, known as *diversfication favoring multiple objectives satisficing* (DMOS) criteria is also proposed. By being componentwise quasiconcave, the DMOS criteria remains amiable to optimization. The relationship between DMOS and the well known class of monetary risk measures [Föllmer and Schied, 2004] is also established. In the single objective case this dual function is in fact a normalized convex risk measure encompassing well-known risk measures such as the Conditional Value-at-Risk (CVaR) measure. Considering the class of DMOS, a specific functional form, the Shortfall-Aware MOS (S-MOS) is proposed. This is a concrete functional implementation of the DMOS class of satisficing measures, and thus does not suffer from the deficiency of not being able to account for the magnitude of shortfalls, as in success probability criterion. The S-MOS criterion incorporates risk aversion characteristics, through the diversification favouring property, which is a more reasonable behavioral assumption for decision making under uncertainty. We further demonstrate that the S-MOS criterion is a lower bound to success probability.

In many real-world applications that has been studied over the course of this research, it was found that it may not be practical to impose rigid distributional assumptions in quantitative models that supports decision making. Such difficulties were borne out in real-world examples studied in this research for areas ranging from sea cargo operations management [Lam et al., 2007] to the scheduling of public bus services [Lam et al., 2009]. Frequently, such models may require normality assumptions over the model uncertainties that may sometimes not be sufficiently realistic for describing the real-world data. Such deficiencies can be handled by formulating a decision analytic problems that considers the absence of full distributional information [Goh and Sim, 2010], instead of a rigid imposition of unrealistic distributional assumptions. In the presence of distributional ambiguity, we present the techniques for building tractable approximations while preserving the salient properties of MOS criteria. These approximations can be implemented using the distributional robust optimization framework recently proposed by Goh and Sim [2010, 2011].

Distributional robust optimization is an approach where the decision maker is ambiguity averse and the optimal decisions are sought for the worst case probability distributions within a family of possible distributions, defined by certain properties such as their support and moments. In order to deal with distributional robust optimization problems, Goh and Sim [2011] introduced an algebraic modeling toolkit (ROME) and we use it to implement the multiple objectives satisficing framework for situations with ambiguity. As maximizing the S-MOS criterion is not a convex optimization problem, we propose improvement algorithms via solving sequences of convex optimization problems using the ROME toolbox. These algorithms perform remarkably well in our computational studies and we report encouraging results on a refinery blending problem in meeting specification targets even in the absence of full probability distribution description.

Numerical experiments are presented to demonstrate the characteristics of S-MOS in being sensitive to the magnitude of shortfalls and its ability to incorporate the diversification favoring characteristics of decision making under uncertainty. Further computational studies are also conducted to compare the characteristics of the S-MOS measure with existing well-known target based approaches, such as success probability, MAUT and the Markowitz mean-variance model [Markowitz, 1952]. The ability of the tractable robust approximate S-MOS criterion to handle distribution ambiguity will also be demonstrated. In order to show the practical applicability of the S-MOS measure and its distributional robust counterpart in overcoming the aforementioned deficiencies of existing criteria, such as success probability and expected utility, extensive computational studies would be conducted using an array of realistic case studies from the finance, product development and oil and gas industries.

The ability of the tractable robust approximate S-MOS criterion in handling distributional ambiguity would be demonstrated through an extensive case study on a refinery blending problem. The problem was identified after extensive consultations with a global market leader in the oil refinery industry. In this problem, quality uncertainties in the raw material inputs to a refinery blending process were considered critical to the refinery operations, and hence, were explicitly considered. This is similar to existing practical implementations [DeWitt et al., 1989, Rigby et al., 1995, Adhya et al., 1999] published in existing academic literature. The performance of decisions obtained via solving a deterministic model and optimizing the S-MOS and the success probability criteria were compared. The advantage of the S-MOS criterion in its ability to cushion the optimal decision against severe shortfall outcomes is demonstrated. Furthermore, the S-MOS criterion is shown to be capable of producing decisions that is able to account for the absence of full distributional information. The robustness of the decisions derived with the

S-MOS criterion is compared to the decisions derived using success probability measures under known distribution, utilizing different distributions in the simulations. Specifically, the solutions of the model using the S-MOS criterion that considers ambiguity not only yield the highest success probability in the computations, it is also able to handle situations where the true distribution cannot be accurately ascertained. Despite using modest information, the use of the S-MOS criterion is able to produce very good results comparable to the use of existing success probability measures which require large number of samples for solution to solve the optimization problem a sample-average approximation framework. This case study effectively demonstrates the practical applicability of the distributional robust S-MOS criterion developed in this thesis.

1.4 Contributions and structure of thesis

Specific contributions of this thesis are listed concisely as follows:

1. Development of a new family of Multiple Objectives Satisficing (MOS) criteria that encompasses success probability and monetary risk measures as special cases. The family of MOS measure is extended to consider preference for diversification under uncertainty through the *diversification favoring* sub-class of MOS (DMOS) criteria.

2. Development of the Shortfall Aware MOS (S-MOS) criterion as a specific form of the DMOS family of measures. This S-MOS criterion is able to account for the magnitude of shortfalls and presence of ambiguity in decision problems under uncertainty.

3. Development of a tractable robust approximations of the S-MOS criterion that can deal with situations where only partial distributional information is available. Since the consideration of distributional ambiguity leads to an intractable problem, we propose the tractable approximation to S-MOS criterion. The approximate S-MOS criterion retains the desirable characteristics of the class of diversification favoring MOS criteria.

4. Demonstrated the ability of the S-MOS criteria in dealing with rare events with limited predictability that may result in excessive losses. These criteria are able to consider diversi-

fication preferences as a reasonable decision behavior of human agents under uncertainty. In addition, they do not need the specification of subjective risk-tolerance type of parameters that are necessary for many other utility based approaches.

5. Successfully demonstrated the application of the tractable robust approximate S-MOS modeling framework and its ability to deal with distributional ambiguity for practical problems.

Extensive numerical studies were conducted to verify the implementation of the proposed S-MOS (including S-MOS that addresses distributional ambiguity) and improvements made over classical models, such as those based on success probability. Amongst others, these numerical studies demonstrated the ability of the proposed S-MOS criterion to *deal with rare events that may result in excessive losses*, *consider diversification preferences* as a reasonable decision behavior of human agents under uncertainty and *reduce specification complexity* by doing away with the need to estimate subjective risk-tolerance type of parameters that are necessary for utility based approaches. The experiments also demonstrate the application of the tractable robust approximate S-MOS modeling framework and its *ability to deal with distributional ambiguity for practical problems*. These numerical studies are summarized as follows:

1. Experiments are conducted to show the actual numerical manifestations of the prescribed theoretical characteristics of the S-MOS criterion. The performance of the S-MOS criterion are compared with respect to classical measures such as success probability. The sensitivity of the S-MOS criterion to the degree and frequency of shortfall occurrence is demonstrated, thereby lending support to its advantages over the traditional expected value and success probability measures in its ability to consider rare events that may result in extensive losses.

2. An array of numerical experiments is also conducted to show the numerical characteristics of the S-MOS criterion under distributional ambiguity and to compare its performance with respect to success probability. The sensitivity of the S-MOS criterion to the degree and frequency of shortfall occurrence is demonstrated. The ability of S-MOS criterion in mimicking success probability distributions under different correlation structures is also shown. We also show the stability of solutions with respect to the number of samples in

sample-average approximation (SAA) approaches for solving the problem based on success probability by assuming known distributions [Birge and Louveaux, 1997, Ruszczynski and Shapiro, 2003]. These results are compared to those based on tractable robust S-MOS formulations that do not require full distributional assumptions. The SAA based evaluations are shown to vary significantly over different sets of SAA samples whereas the evaluation based on the tractable robust S-MOS on the other hand do not experience such sampling variability.

3. An asset allocation example based on the problem of allocating limited capital amongst risk-free and risky asset classes for covering liabilities over multiple time periods is used to demonstrate the ability of the S-MOS criterion in considering diversification for investment decisions and to cushion against very bad shortfalls against the targets. The results based on the use of the S-MOS criterion are compared against those based on the use of the Markowitz mean-variance model for asset allocation [Markowitz, 1952].

4. A case study in engineering product development processes for design selection amongst competing alternatives of product design in high quality rapid product development processes is presented. This case study demonstrates the applicability of the S-MOS criterion for the ranking of discrete alternatives and its advantage over traditional expected utility models and MAUT models in that the use of a target-based satisficing approach do not require the difficult tasks of having to estimate the risk tolerance parameters in utility models and "weights" necessary in an MAUT approach. In high quality rapid product development processes, the impracticality of obtaining precise risk tolerance parameter estimates and weights to establish the MAUT function may prevent design engineers from clearly differentiating and ranking design alternatives. The advantage of the S-MOS criterion is that it allows the ranking of design alternatives without the need to estimate these problematic risk tolerance parameters and weights.

5. The applicability of the S-MOS criterion under modest distributional information is demonstrated through a case study involving the optimization of a refinery blending problem. The problem have been identified after extensive consultations with a global market leader and

is similar to existing problems published in existing academic literature (see for instance DeWitt et al. [1989], Rigby et al. [1995], Adhya et al. [1999]). Comparisons between the optimization of the S-MOS criterion under distributional ambiguity and the maximization of success probability are made using monte carlo simulation. The advantage of the S-MOS criterion in its ability to cushion the optimal decision against outcomes that may result in severe shortfalls is demonstrated. The ability of the S-MOS criterion based model in producing decisions that is able to account for the absence of full distributional information is also demonstrated. The results show that decisions obtained via the S-MOS criterion under distributional ambiguity outperforms those obtained by assuming a known distribution using either the S-MOS criterion or the probability measures.

The structure of the thesis is as follows:

1. General forms of the multiple objectives satisficing (MOS) criteria are introduced and characterised via some fundamental multiple objectives satisficing properties in **Chapter 3**. The success probability is a special member in this family of MOS criteria, albeit with several significant deficiencies, such as the ability to account for the magnitude of underperformance in the target. The MOS criteria are specialized to a form that incorporates realistic decision behaviors under uncertainty, specifically, the preferences towards diversification in the decisions. This specialized family of MOS is the *diversification favoring MOS* (DMOS). The relationships between the MOS family of criteria and monetary risk measures will also be established.

2. A specific form of the DMOS criteria, the shortfall aware MOS (S-MOS), is proposed in **Chapter 4**. The S-MOS incorporates the consideration of diversification favoring and is constructed based on the idea of approximating a step utility function using a concave approximation. The S-MOS can deal with decision making under ambiguity, when full distributional information is not available for the ranking of alternatives. In order to deal with distributional ambiguity, a family of tractable robust approximation to the S-MOS criteria that retains the desirable characteristics of the family of DMOS criteria is developed. A general algorithm is also presented for the optimization of the S-MOS

criterion under ambiguity. This algorithm is applicable for the more specialized case when full distributional information are available.

3. In **Chapters 4** and **5**, the theoretical characteristics of S-MOS are shown through extensive numerical studies and comparisons are made between the S-MOS (including S-MOS that addresses distributional ambiguity), with success probability measures and Markowitz mean variance model [Markowitz, 1952]. Empirical results in the application of S-MOS for a variety of optimization problems in the aforementioned are also presented in these two chapters. The numerical examples based on examples in the financial portfolio allocation, oil and gas production and engineering product development domains demonstrated the improved characteristics of the proposed S-MOS criterion.

1.5 Some generic notations

For clarity of presentations, we define some generic notations here. Other notations specific to the formulations found in the thesis are the defined prior to the formulations.

In this thesis, matrices and vectors are represented as upper and lower case boldface characters respectively. Sets are denoted by calligraphic notations. If $\boldsymbol{x}$ is a vector, we use the notation x_i to denote the ith component of the vector. Also, given a vector $\boldsymbol{x}$, we define $(y_i, \boldsymbol{x}_{-i})$ to be the vector with all components same as that in $\boldsymbol{x}$ except component i being y_i; i.e. $(y_i, \boldsymbol{x}_{-i}) = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ for a vector of n components. We use $\mathbf{1}$ to denote a vector of all ones and $\mathbf{0}$ to denote the vector of all zeros. For convenience of exposition, we denote $\min\{\min_{i \in N}\{a_i\}, b\}$ where $\boldsymbol{a} \in \Re^N$ and $b \in \Re$ simply by $\min_{i \in N}\{a_i, b\}$.

CHAPTER 2

Literature Review

Perhaps the most fundamental features in modern decision theory are the deeply intertwined problems of uncertainty and human decision behaviors under uncertainty. Coupled with the limitation of resources leading to constrained decision problems, the choice of good criteria to effectively describe the preference of decision makers and the allowance given to cushion against risks in violating any constraints pose fundamental problems in any decision analytic situations. Complex systems, such as those found in financial, operations and engineering decision analysis, require the further consideration of multiple and frequently conflicting objectives. These considerations have resulted in a "trilemma" of sorts, where uncertainties, decision preferences and multiplicity of objectives have to be assessed together, hence, spawning a wide variety of attempts to explicate their complex interactions so as to develop practically relevant and implementable decision analytic models and methodologies. Specifically, the trilemma has been one of being able to design appropriate objectives whilst effectively incorporating realistic decision behaviors, uncertainties (cushioning against constraint violations) and the management of complex interactions between multiple objectives.

This literature review discuss the extant literature in decision analysis across the three dimensions of *uncertain decision making*, *decision behaviours under uncertainty* and the *multiplicity of objectives* in real-world decision making under uncertainty. Consequently, this would provide the underlying motivation for developing the new multiple objectives satisficing decision analytic framework that establishes the foundation of this thesis. The literature review will cover relevant related literature spanning the following dimensions:

- Basic models of decision making under uncertainty,

- Modeling decision preferences under uncertainty, and,

- Multiple objectives decision making under uncertainty.

Important motivations and contributions of this thesis in the development of a novel family of multiple objectives satisficing measures, which forms the central thesis of this research, are discussed in this review.

2.1 Basic models of decision making under uncertainty

In uncertain decision making under constrained resources, we are faced with the prospect of selecting alternatives that best achieves our objectives whilst limiting the risks in the violation of any constraints. In such decision problems, we can be confronted with uncertainties in the objective function, the constraints, or both. One of the simplest way to consider uncertainties for such problems is through expected value approaches [Anderson et al., 2008]. The action to be chosen in such approaches is the one that gives rise to the highest total expected value. Expected value approaches ignore the risks in uncertain decision making and could be useful for deriving decisions which are optimal in the long-term expectation sense. It is often used in situations where decision makers can be reasonably assumed as risk-neutral. Such decision making paradigm may also be more suitable for the the restricted case of recurring decisions under stationary distributions [Sheldon, 2002].

In attempting to mitigate the risks of violating any constraints, a reasonable and prevalent approach is to restrict the probability of constraint violations whilst seeking to achieve the objectives to the largest extent possible. For convenience of exposition, we denote the joint probability of constraint satisfaction in a target based setting (or the joint probability of target achievement) as the *success probability.* Frequently the consideration of success probability would lead to a chance constrained formulation [Charnes et al., 1958, Charnes and Cooper, 1959, 1963]. An example of the chance contrained formulation in terms of a maximization problem is given as follows:

$$\begin{aligned} \max \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & \mathbb{P}(F(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \leq \mathbf{0}) \geq 1 - \alpha \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \Re^n$ is the decision vector, and, $\tilde{\boldsymbol{z}} \in \Re^m$ is a vector of primitive uncertainties in p linear constraints represented by the function $F = (f_1, f_2, \ldots, f_p) : \Re^n \times \Re^m \mapsto \Re^p$. Also, $f : \Re^n \mapsto \Re$ is a real-valued function that can be a profit or utility function. Here, $\mathbb{P}(Y)$ denotes the probability of an event Y. In order to provide a more explicit representation, $\boldsymbol{x} \in \mathcal{X}$ represents the feasible set. The chance constrained approach can be further extended to deal with the objective itself being uncertain. The general chance constraints case requires the set of p linear chance constraints in Problem (1) to be jointly satisfied with probability at least $1 - \alpha$, where $\alpha \in (0, 1)$ represents a desired safety factor that protects against constraint violation. The chance constraints can be modeled with deterministic equivalents for some restricted classes of stochastic models, such as those involving normality assumptions. Under such a distributional assumption, the individual chance-constrained problem can be represented as a second-order cone, thus making it amenable to tractable computational procedures for second order cone optimization problems (SOCP) [Alizadeh and Goldfarb, 2003]. Other tractable distributional assumptions that makes individual chance constraints tractable also typically results when the chance constraints can be second order cone representable (see for example Calafiore and El Ghaoui [2006]). However, under general distributional assumption, Problem (1) is computationally intractable.

Apart from the presence of computational difficulties in optimization, the conceptual framework that underpins expected value and probability measures for decision making under uncertainty have come under severe criticisms in recent years, particularly in the domain when rare and extreme events have to be considered in the decision making process. There are many examples when such events have to be considered, but are not adequately incorporated into the quantitative decision models, especially in diverse applications where the expected value and probability based approaches are used [Taleb, 2001]. Here we utilize a simple example in the engineering discipline related to the design of safety systems for critical infrastructure such as power generation plants to demonstrate the inadequacy of the expected value approach in ranking decisions.

In a typical critical infrastructure [Wright et al., 2006], there usually are a number of safety systems ensuring the safe operations of the entire engineering process. Such safety critical systems can be identified through a Reliability-Centred Maintenance program [Moubray, 1997]. In such a program, the definition of criticality is based on the consequences of failures, whether its failure or malfunction may result in severe consequences, for example, the loss of lives, environmental damage or severe economic loss. These consequences may be estimated in monetary values and are typically assessed based on the expected value (e.g. expected loss). The use of expectation has been found to be severely limiting, particularly so if the failure mode considered could result in a set of different outcomes, or if the underlying uncertainties in the consequences of system failures are large. As an example, assume that the possible consequences of a system failure are 5, 100, or 500, with associated probabilities 0.6, 0.3 and 0.1. Then the expected value is 83. Clearly, 83 could be a poor prediction of the actual outcome, which can have values as extreme as 500. In addition,the probabilities are not objective numbers and the background information for the analysts' estimations could be lacking, thereby further undermining the predictions of the actual outcomes. Hence, by focusing on the expected consequences, a strong element of arbitrariness is introduced. This arbitrariness is due to the possible degree of variations in outcomes coupled with the difficulty of assigning probabilities. To add on, the examples that such rare and extreme events can cause havoc to engineering applications are numerous, stretching from critical infrastructures design to consumer product development.

Apart from the difficulty of expectation and probabilities in considering rare and extreme events and distributional ambiguity, realistic decision behaviors under uncertainty, such as diversification preferences, cannot be adequately incorporated into these basic framework. Hence, more sophisticated models have to be developed to address these deficiencies.

2.2 Modeling decision preferences under uncertainty

2.2.1 Expected utility

In the presence of risky outcomes, a decision maker could use the expected value criterion as a rule of choice. Here, higher expected value decisions are simply the preferred ones, irrespective of the degree of risks such a decision entails. However, such a decision criterion may not be reasonable in real life decision making under uncertainty, given that in some situations, decision makers may not be risk-neutral, instead, they may prefer to take decisions that yield lower expected value in favor of lower risks. In 1738, Daniel Bernoulli [Sommer, 1954] posits that rational behavior can be described as maximizing the expectation of a function that can account for risk aversion. This is the expected utility hypothesis which provides conditions (or axioms) that aims to model the preference characteristics in decision making under uncertainty. For the case when multiple objectives are considered, one of the most pervasive decision analytic methodology is undoubtedly that based on the multiattribute utility theory (MAUT) which engenders the explicit use of utility functions, to incorporate an *a priori* specification of preference information over multiple objectives (or attributes) in the assessment of alternatives [Keeney and Raiffa, 1976, Dyer et al., 1992].

Utility theory, in general, is primarily concerned with the modeling of human preferences in decision making under uncertainty [Friedman and Savage, 1948, Markowitz, 1952]. It proposes the idea of a real-valued functional representation for the utility assigned to each uncertain consequence which can be represented by single or multiple attributes. Through the ranking of the expectation of the utility for each alternative, the best course of action can be determined. The basic premise of the real-valued functional representation is derived from an axiomatic framework that reflects fundamental preference structures of "rational" decision making entities. Examples of such axiomatic frameworks are presented in von Neumann and Morgernstern [1947], Savage [1954], Luce and Raiffa [1957].

Utility theory, despite its widespread appeal, primarily in the domain of economics and decision theory, is not without its detractors. First and foremost, the expected utility hypothesis, as applied to economics, has limited predictive accuracy, simply because in practice, humans may not always behave "rationally" according to the axioms proposed in von Neumann and Morgernstern [1947]. This can be interpreted as evidence that humans are not always rational, or that such rationality characterization is not appropriate to describe human decision making behavior, or some combination of both. Specifically, some of the more famous criticisms against the use of the expected utility theory within the proposed axiomatic framework of rational decision making include the presence of obvious paradoxes arising from simple decision making scenarios, such as the Allais [1953] and Ellsberg [1961] paradoxes. The Allais paradox can be traced to the classical axiomatic framework of rational decision making under uncertainty, in particular, to the independence assumption which states that common outcomes across sets of lotteries are irrelevant to decision choices. The Ellsberg paradox on the other hand, challenges the adequacy of utility theory in addressing issues related to model ambiguity[Knight, 1921]. In contrast to the traditional notion of risk which considers uncertainty on outcomes under known probability distributions, ambiguity refers to uncertainty arising from the possible existence of several different specifications of probabilistic models for the uncertain outcomes.

2.2.2 Target-based utility

Target-based utility approaches have come to the forefront in recent years [Tsetlin and Winkler, 2007, Bordley and Kirkwood, 2004, Castagnoli and LiCalzi, 1996, Bordley and LiCalzi, 2000]. Spurred by recent advances in behavioral economics, Heath et al. [1999] suggests the interpretation of the inflection point in the S-shaped value function proposed in the context of prospect theory by Kahneman and Tversky [1979] as a target. Castagnoli and LiCalzi [1996] demonstrated the equivalence of expected utility and "expected probability" in a target-based model with the utility function appropriately scaled. In the target-based model for the single attribute case, the maximization of expected utility is equivalent to the maximization of the probability of achieving (and exceeding) the target (see Castagnoli and LiCalzi [1996], Bordley and LiCalzi [2000]). Abbas and Matheson [2005] proposed an "aspiration equivalent" measure as a basis for setting targets. The aspiration equivalent measure relates generic normative expected-utility based approaches to a target-oriented perspective where the objective is to meet or exceed the target. Along these lines of research, Tsetlin and Winkler [2007] considered the multiattribute case of using target levels specified across multiple dimensions through the utility functions approach.

A target-oriented approach is intuitively more appealing in a variety of decision making contexts such as in new product development, decision making within regulated environment, the setting of performance standards and resource allocation under uncertain competition [Bordley and Kirkwood, 2004]. As a simple example, success in competitive new product development is typically determined by the ability of new or improved products to perform across multiple key design parameters relative to competitors, and not on how optimal the design parameters are *per se*. This is especially important for highly innovative technological products due to their significantly shorter product development cycles in the range of between 6 to 12 months (see for example Tang et al. [2010]). The speed of such innovative product development cycles limits the amount of information that designers could gather through the design process. Consequently, design decisions may have to be made under bounded rationality. Given the limited time and resource available, the search for a fully optimal design solution may not be feasible. Furthermore, such target-oriented decision making approaches is particularly relevant for products where customers' purchase decisions may depend significantly on how well the product performs amongst its competition. Hence, competitive new product development frequently requires the consideration of uncertainties in both the design parameters and targets, since performance targets may be set relative to those of competing products under similarly uncertain development phase.

Bordley and Kirkwood [2004] argued for the relevance of target orientation in many real-world decision making contexts and proposed a novel approach to assess a preference function for a multiple objectives decision problem under uncertainty. The rationale behind their approach is underpinned by the observation that the expected value of the objective function can be expressed as the probability of the objective function exceeding a random threshold via an expected utility argument. Extending from the single attribute case by Castagnoli and LiCalzi [1996], they proposed a novel multiattribute approach based on the maximization of expected utility where the decision maker's utility depends on the subset of attributes that meet some targets. The approach hinges on a reformulation of the stochastic optimization problem as the maximization of the probability of both the constraints and of the objective function meeting/ exceeding a random threshold. Along these lines of inquiry, Bordley and Pollock [2009] further presents a utility-based probability maximization formulation as an alternative to the chance-constrained programming (CCP) model for structural design analysis and optimization. Such a formulation has also broader practical engineering motivation in its ability to mitigate product liability concerns arising from the use of CCP models in structural design analysis.

2.2.3 Convergence of expected utility and probability maximization

In recent developments related to the use of expected utility in normative decision making, Castagnoli and LiCalzi [1996] and Bordley and LiCalzi [2000] showed that maximizing expected utility is mathematically equivalent to maximizing the probability that the uncertain consequences of a decision are preferable to an uncertain benchmark. As an example of such decision making context in finance, consider the probability that a mutual fund strategy outperforms the S&P 500 or that a firm outperforms the uncertain future performance of a major competitor. This reinterpretation also relates to existing empirical psychological research which suggests that individuals may indeed have implicit aspiration levels in decisions under uncertainty [Lopes and Oden, 1999]. These aspiration levels may be fuzzy and vary amongst different contexts of choice. Hence, the focus has been shifted from utility to individual's uncertain reference point.

From the practical perspective, a target-oriented approach is intuitively more appealing in a variety of decision making contexts[Bordley and Kirkwood, 2004]. Apart from being a more realistic behavioral description in certain settings, it is frequently more natural to think of aspiration levels instead of attempting to articulate a specific utility function or to determine the important risk tolerance parameters which the accurate estimation of utility function requires [Brown and Sim, 2009]. An early research which stimulated the convergence of the work in expected utility theory and success probability under a target-based framework could be attributed to Borch [1968] who, using the concept of ruin probabilities, showed that the maximization of the expected utility is equivalent to choosing the smallest probability of ruin. Castagnoli and

LiCalzi [1996] likewise stated that the expected utility model need not be based on the notion of a cardinal utility function over prizes and can in fact be entirely phrased in the language of probability. Bordley and LiCalzi [2000] further argued that the normative foundation in utility-based decisions likewise imply that one should select the decision which maximizes the probability of meeting the uncertain target. Extending to consideration of multiple attributes in decision making, Bordley and Kirkwood [2004] argued that multiple targets arise in many resource allocation decisions where there are multiple stakeholders involved. For instance, the success of a product may be evaluated by the customer based on different criteria such as price and quality. The authors presented normative target-based formulations for specific conditions of a decision maker's preference. In another related work, Bordley and Pollock [2009] presented a target based probability maximization framework (which is equivalent to the maximization of expected utility) by framing their work in terms of reliability-based optimization, where any violation of a constraint would lead to a failure. They developed a utility-based probability maximization model that may be used in place of chance-constrained programming models. Their formulation once again hinges on the concept of probability maximization. They however take on a product liability perspective such that generated solutions will always satisfy the constraints of the problem.

The aforementioned works revolved around the maximization of target achievement probabilities (or success probabilities). In a related development for decision making under uncertainty, Brown and Sim [2009] generalized the maximization of success probability to a class of satisficing criteria for single objective. These criteria not only accounted for the magnitude of losses and gains but also have quasi-concavity properties that lead to computational tractability. Furthermore, the satisficing criteria are shown to be duals of a corresponding class of risk measures.

2.2.4 Satisficing measures

The idea of aspiration levels is not new in the decision theory literature. As mentioned previously, some recent work focused on the probability of achieving a target level as a way of dealing with decision making under uncertainty, without using utility functions [Castagnoli and LiCalzi, 1996, Bordley and LiCalzi, 2000]. Whilst utility theory proposes a quantitative valuation of alternatives in the context of uncertain decision making with a handle on realistic decision behaviors under uncertainty, scarcity of resources in real-world decision-making requires the imposition of constraints that qualifies solution feasibility for practical implementation of chosen alternatives. Utility theory essentially relies on the concept of rationality in decision analysis. Rationality in decision making postulates that each decision making entity (or agent) behaves in a manner that optimizes his expected utility under full information. In a more general sense, the agent maximizes a value function built upon the basis of rational behavior underpinned by axioms of reasonable behaviors. Despite its widespread appeal, optimizing the expected utility may not be an adequate description of decision making from a behavioral standpoint in the context of bounded rationality (see for example Simon [1959], Lanzillotti [1958], Mao [1970]). Instead, it has been argued that the concept of satisficing may be a more realistic behavioral description in various practical decision scenarios and also from fundamental behavioral perspectives.

Satisfice is a portmanteau term that integrates the concepts of "satisfy" and "suffice". Satisficing decision behaviors explicitly acknowledges the practical impossibility of searching amongst all alternatives for an optimal choice, but postulates the existence of aspiration levels (that may be random) for different objectives which allows an agent to pick the first course of action that meets these aspiration levels (or targets) [Simon, 1955]. More importantly, the satisficing approach recognizes this practical impossibility of obtaining an optimal solution under uncertainty. In many real-world uncertain decision making scenarios, rather than searching among all feasible actions for the optimal one, decision makers often set concrete targets to drive their decisions. Along this line of argument, Simon [1959] mentioned that if business behavior is to be viewed in terms of satisficing, it is to be expected that the firm's target would involve the attainment of a certain level of profit or holding a certain share of the market or a certain level of sales, rather than the maximization of profit. Such a concept of satisficing may be more realistic behavioral description in a significant variety of real-world management decision problems. For example, in the context of risk management, empirical research have also concluded the importance of a target-based perspective [Mao, 1970]. In organizational theory, numerous research have demonstrated the importance of target setting and the multiplicity of targets [Lanzillotti, 1958, Boulding, 1952]. Cyert and March [1963] described the importance of goals (or targets), expectations and choice as the major components of organizational decision processes. In new product development, success is typically determined by the ability of new or improved products to perform across multiple key design parameters relative to competitors, and not on how optimal the design parameters are *per se*. Such target-oriented decision making approaches is particularly relevant for products which customers' purchase decisions depend significantly on how well the product performs amongst its competition.

Apart from being a more realistic description of many managerial decision contexts, the consideration of satisficing behaviors has led to the realization of significant computational benefits under optimization, where they have been defined with respect to quasiconvex risk measures for the single objective case [Brown and Sim, 2009]. In an early work, Charnes and Cooper [1963] first adopted Simon's satisficing approach within the context of optimization problems. They considered this as another set of objectives dependent upon the achievement of target levels. Other than computational attractiveness and natural behavioral insights, the consideration of success probabilities in place of utility functions also serves to eliminate the necessity of estimating imprecise and subjective risk tolerance parameters required in expected utility approaches. Such parameters are often difficult for decision makers to intuitively grasp and even harder to accurately evaluate.

2.3 Multiple objectives decision making under uncertainty

Real-world decision making typically requires the consideration of alternatives assessed across multiple objectives in the presence of uncertainty. In such decision making situations, the setting can be one such that a decision maker has to choose a subset of a set of alternatives evaluated on the basis of two or more objectives. The feasible set of solutions may be either small and finite (e.g. discrete choice problems) or large and perhaps infinite (e.g. continuous optimization problems).

A direct way to adopt a maximization (or minimization) approach under a rational model paradigm for multiple objectives optimization problem is through the framework of multiobjective optimization. The field of multiobjective optimization has witnessed an explosive growth over the last two decades (see the reviews by Alves and J. [2007]). This trend can be attributed primarily to its diverse applicability in real-world problems. However, most of these multiobjective related research dealt with deterministic problems. Such approaches usually attempt to simultaneously optimize two or more conflicting and usually deterministic objectives. In the case without uncertainty, the most usual approach has been to optimize over a single objective function founded on reasonably derived weights over the individual objectives. In the absence of complete ordering in such problems, the solution to a multiobjectives optimization problem is a possibly infinite set of Pareto points; see the reviews by Alves and J. [2007] and Wallenius et al. [2008]. Such conventional methods may also not be appropriate under uncertainty as the concept of an efficient frontier is not readily apparent. Despite the proliferation of research in the area of multiobjective optimization, the consideration of uncertainty in the literature has been noticeably scarce, with most research dealing with the situation when there exists a small number of discrete feasible alternatives [Klein et al., 1990]. Furthermore, most of these research do not have a target-based orientation suitable for addressing decision problems based on the idea of satisficing. In fact, target-based satisficing under uncertainty for the single objective case have only recently come into the spotlight [Chen and Sim, 2009]. Satisficing approaches towards decision optimization under uncertainty have not been convincingly dealt with from a multiple objectives perspective.

In order to facilitate the comparison of alternatives across multiple objectives within the feasible set of solutions, another prevalent framework that has been widely adopted is the multi-attribute utility theory (MAUT). MAUT provides a framework that explicitly models the relative importance of the multiple objectives and the preference structure of decision makers under realistic assumptions. In contrast with multiobjective optimization approaches that frequently generates solutions that belongs to a possibly infinite set of Pareto points, where the best decisions are usually determined *a posteriori* after the generation of the Pareto optimal points, MAUT approaches incorporate decision preference *a priori*. Conceptually, in MAUT, we may assume that a decision maker acts to maximize a utility or value function that depends on the multiple attributes, typically through the maximization of the expected value of a utility function. The generic concept of explicitly modeling the preference behaviors of decision maker *a priori* for multiple attributes have been applied to a wide variety of new domains over the last decade. Some examples of newly minted application over the last decade includes the field of negotiation science [Ehtamo and Hämäläinen, 2001, Ehtamo et al., 1999, Sebenius, 2007, Teich et al., 1994, Wang and Zionts, 2008], e-commerce (auctions and shopping agents) [Chen-Ritzo et al., 2005, Sim and Choi, 2003, Teich et al., 2004], real-world spatial planning and management [Malczewski, 1999] and a multitude of engineering applications ranging from water regulation and energy management to defence and healthcare engineering [Butler et al., 2005, Coello and Lamont, 2004, Hobbs and Meier, 2000, Köksalan and Plante, 2003, Sim and Choi, 2003, Von Winterfeldt and Schweitzer, 1998]. Such sustained and diverse penetration of multiattribute decision models with *a priori* behavioral considerations serves to illustrate its relative importance within the decision theoretic domain. Despite its widespread adoption, the maximization of a multiattribute utility function may not be able to adequately describe many decision contexts. Some of these limitations are legacies of the single attribute utility theory, such as the presence of obvious decision paradoxes [Allais, 1953, Ellsberg, 1961]. In addition, the difficulty in calibrating the risk tolerance parameters necessary for a single attribute utility function is compounded significantly under a MAUT framework [Simon, 1955, Brown and Sim, 2009].

An emerging area of research in MAUT domain is the use of targets in problems involving multiple criteria [Wallenius et al., 2008]. The idea in this emerging field is that the decision maker's utility or value may not depend on the levels of performance on different criteria, but instead on whether the levels meet a target or threshold on one or more criteria. Such a notion has provided a basis for goal programming and has been found promising in overcoming the limitations of a MAUT approach. Specifically, the use of targets have provided an alternative decision paradigm that does not require the tedious and sometimes impractical necessity of deriving the risk tolerance parameters in the MAUT framework [Brown and Sim, 2009]. Many real-world decision making problems require the evaluation of risky alternatives considering multiple objectives under a *target-oriented* perspective. The targets may be fixed, as in the achievement of sales or production quotas, or uncertain, as in the performance against economic indicators. The relevance of targets in managerial decision making is widespread in many business domains [Mao, 1970, Lanzillotti, 1958, Boulding, 1952, Cyert and March, 1963]. The multiplicity of goals can also be seen as a way of hedging under uncertainty [Carter, 1971]. Apart from the multiplicity of targets, it is also important to consider uncertain targets for many real-world decision problems (see for example [Bordley and Kirkwood, 2004]).

In the analysis of decisions under uncertainty, existing approaches often require the assumptions of some plausible models that fully characterises the uncertainties, such as the assumption of normality [Goh et al., 2006, Lam et al., 2007]. Oftentimes, due to a scarcity of data or otherwise (such as the inherent characteristics of the data itself), this endeavour can be very difficult. The resulting assumptions necessary to maintain computational tractability of the resulting models may turn out to be debatable. Frequently, the underlying real-world uncertainties have to be assumed normal, thereby implicitly precluding any possibility of an outlier event from occurring in the analysis [Makridakisa and Taleb, 2009]. In recent years, partly motivated by a post-mortem analysis of the events leading up to the recent sub-prime financial crisis, there is an emergence of interest in the study of rare and unpredictable events in quantitative managerial decision making (see for example Shiller [2008]). Such rare events with significant potential impact have been described as black swan events [Makridakisa and Taleb, 2009, Taleb, 2001]. Black swan events are essentially characterised by the following [Taleb, 2001]:

- The event is an outlier, as it lies outside the realm of regular expectations and nothing in the past can convincingly point to its occurrence.
- The event carries an extreme impact, or that the costs associated with the event is relatively much more malignant than usual.
- The occurrence of such black swan events trigger human nature to concoct explanations for its occurrence *ex post*, after the occurrence. This may make the events "explainable" or predictable *only* a posteriori.

This triplet can be summarized as rarity, extreme impact and limited predictability [Taleb, 2001]. There are many examples that such black swans have occurred in domains ranging from the financial industry [Shiller, 2008, Makridakisa and Taleb, 2009] to engineering where such as in the assessment of vulnerabilities in critical infrastructure protection [Wright et al., 2006]. The recent prolonged subprime financial crisis that started in 2007 [IMF, 2009, Shiller, 2008] have been partially attributed to the limitations of widely adopted quantitative financial models in considering black swan events analytically through the use of archaic risk management paradigms, such as the Value-at-Risk measures[Shiller, 2008]. This financial crisis which started from the US housing markets has resulted in the collapse of large financial institutions, the bailout of banks by national governments, and downturns in stock markets around the world, with top economists claiming it to be comparative in scale to the Great Depression in the 1930s [Roubini et al., 2009].

The characterization of rare events with limited predictability and practical examples of the severe repercussions arising from the neglect of it, have let to an emergence of keen interest in the academic community in re-examining key assumptions underlying existing engineering and financial risk models that may have blatantly disregarded the existence of such events. The deficiencies in basic decision analytic models, such as the expectation and probability *per se*, have been highlighted in the preceding section.

There is often an implicit assumption that the decision maker is in possession of a decision model that accurately describes the distribution of random quantities (such as random returns or demand information) in decision models under uncertainty. In real-life, however, practitioners are typically faced with ambiguity in the knowledge of the distribution. There are practical implications of the presence of ambiguity in a variety risky decision making contexts under ambiguity [Natarajan et al., 2010b, Calafiore, 2010, Gilboa and D., 1989, Garlappi et al., 1989, Popescu, 2007]. As a recapitulation, a significant example in the implication of ambiguity is the idea of ambiguity aversion described in the famous Ellsberg paradox [Ellsberg, 1961]. A simple experiment for the paradox can be demonstrated whereby most people would prefer to bet on an urn with 50 red and 50 blue balls, than in an urn with 100 balls containing an unknown number of red or blue balls. The physiological basis that firmly establishes the importance of considering uncertainty in risky decisions have been established recently whereby functional brain imaging was used to show that the level of ambiguity in choices correlates positively with activation in the amygdala and orbitofrontal cortex, and negatively with a striatal system [Hsu et al., 2005]. The data suggest a general neural circuit responding to degrees of ambiguity, thereby providing pathological evidence that ambiguity indeed affects human decision behaviors under uncertainty. Given concrete pathological evidence, ambiguity can no longer be conveniently ignored in decision models which considers uncertainty when there are *prima facie* evidence of the presence of ambiguity.

Normative literature examining the effects of ambiguity on decision-making is relatively sparse. Shapiro et al. [2002] offers one of the first formulations of decision-making with ambiguity and characterizes the structure of optimal decision-making in a static framework. Iyengar [2005] offers a formulation of dynamic programming under ambiguity. He considers cases where the decision-maker does not know which underlying process is driving the state variables at each stage. Thus in his model it is impossible for the agent to infer at any stage the true state of the world. More recently, Brown and Sim [2009] proposed a satisficing measure based on a target-based paradigm which have an ambiguity interpretation in terms of guarantees on the expected performance when the underlying distribution deviates from the investor's reference distribution. The ability to deal with ambiguity in risky decision making contexts were established through the dual connection between the proposed satisficing measure and the well-established risk measures in the literature of mathematical finance (see for example Föllmer and Schied [2004]).

Distributionally robust optimization or minimax stochastic programming is a general approach to deal with ambiguity. Herein, the decision maker is assumed to be ambiguity averse and the optimal decisions are sought for the worst case probability distributions within a family of possible distributions, defined by certain properties such as their support and moments. This approach was pioneered by Žáčková [1966] and studied in many other works (see, for instance, Dupačová [1987], Shapiro and A. [2002]). Ghaoui et al. [2003] developed worst-case bounds for chance-constraints on uncertainties, when only the bounds on means and covariance matrix were available whilst Delage and Ye [2010] studied distributional robust stochastic programs when the mean and covariance of the primitive uncertainties are themselves subject to uncertainty. Apart from mean and covariance, new distributional properties were proposed under this framework and unified bounds were developed for uncertainties with known support, mean, covariance and new deviation measures such as directional deviations and partitioned covariance [Chen et al., 2003, Chen and Sim, 2009, Goh and Sim, 2010, Natarajan et al., 2010a]. Goh and Sim [2011] introduced an algebraic modeling toolkit (ROME) to facilitate modelling of distributional robust optimization problems and we use it to implement the multiple objectives satisficing framework for situations with ambiguity.

Apart from the need to more adequately address the issues of rare and extreme events and to handle the issue of limited predictability in the context of distribution ambiguity, the inherent decision preferences under uncertainty have to be considered in order to develop a realistic and practical framework for decision making under uncertainty. Utility theory relies on the concept of rationality in decision analysis. Despite its widespread adoption, optimizing the expected utility may not be an adequate description of decision making from a behavioral standpoint (see for example Simon [1959], Lanzillotti [1958], Mao [1970]). This can also be shown convincingly through a number of simple paradoxes Ellsberg [1961], Allais [1953]. Instead, the concept of satisficing has been proposed as a more realistic behavioral description under bounded rationality [Brown and Sim, 2009]. Satisficing can be argued as a more appropriate measure for ranking alternatives when potential rare and extreme events with limited predictability have to be considered, since the satisficing behavior acknowledge the practical impossibility of searching for an optimal decision [Simon, 1955], particularly in an environment of limited predictability and when extreme events can lead to dire consequences.

In this thesis, we consider a real valued multiple objectives decision making problem under uncertainty in which alternatives are evaluated based on the outcomes of two or more objectives in achieving some pre-specified targets that are potentially uncertain when the decisions are made. The goal of this decision problem is to select an alternative amongst a set of feasible decisions characterized by the degree in which the multiple objectives exceed some specified *targets* in all possible outcomes. However, such a utopian solution may not be feasible. In order to evaluate the alternatives against targets in the presence of uncertainty, we introduce a preference structure with reasonably motivated properties to rank the alternatives. The resulting multiple objectives satisficing framework not only allows decision maker to deal with uncertain decision making contexts when risks are present, it also allows the flexibility of considering distributional ambiguity. Distributional ambiguity as defined here is similar to Knightian uncertainty [Knight, 1921]. The MOS criteria will be specialized to deal with the situations whereby the decision maker may know only some limited distributional information, such as the support, mean and covariance, through a specific MOS criterion. Since the consideration of distributional ambiguity in the proposed criterion results in an intractable problem, tractable approximations for the proposed MOS criterion would be proposed.

2.4 Family of multiple objectives satisficing criteria

In this thesis, a class of satisficing criteria for multiple objectives is developed through a set of fundamental properties that describe decision behaviors on the basis of performance relative to targets. The proposed family of multiple objectives satisficing (MOS) criteria is founded on behavioral properties specifically addressing satisficing behaviors in multiple objectives decision problems. Furthermore, the consideration of satisficing behaviors is able to eliminate the need, and hence, the difficulties involved in the estimation of imprecise and subjective risk tolerance parameters and attribute weights that plague existing expected utility and MAUT approaches. The difficulties in estimating such risk tolerance parameters have been well-documented [Keeney and Raiffa, 1976]. A specialization of the MOS to embody realistic diversification favoring characteristics inherent in realistic decision behaviors will be introduced for decision making under uncertainty. A specific member of this sub-class of MOS criteria that incorporates diversification preference is also proposed. This criterion is able to take into account the degree of shortfalls against the targets, thereby producing decisions that are able to mitigate risks in this respect.

Apart from the single objective satisficing criterion proposed in Brown and Sim [2009], to our knowledge, satisficing measures for the more general multiple objectives case (with or without the consideration of ambiguity) have not been formally characterized. In a bid to significantly advance this line of research, a family of MOS criteria would be proposed together with specific forms of these MOS criteria that respect diversification preferences and are able to handle *distributional ambiguity*. These specialized MOS would be able to simultaneously consider realistic decision preferences, risk and ambiguity in the ranking of alternatives, thus overcoming the significant limitations of existing probability and expected value measures. A generic optimization procedure, based on the iterative evaluation of a sequence of convex optimization problems, would be proposed for multiple objectives decision optimization for the specific form of MOS proposed. A number of numerical experiments and case studies are performed to demonstrate the characteristics of the specific MOS criteria and to compare its effectiveness in handling real-world problems relative to existing criteria such as the success probability criterion.

CHAPTER 3

A Framework for Multiple Objectives Satisficing Under Uncertainty

In this chapter, we define a general framework of multiple objectives satisficing (MOS) criteria which encompass the joint probability of target achievement across all objectives (or success probability), $\mathbb{P}(\tilde{\boldsymbol{x}} \geq 0)$ where $\tilde{\boldsymbol{x}} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\}$ as a special case of the MOS criteria. We further specialise the generic MOS family to a specific subclass of diversification favoring MOS (DMOS) by taking into account the characteristic of diversification preference. We also establish the relationship between the MOS and monetary risk measures, with the Value at Risk measure as a special case of the MOS. Furthermore, the dual of the DMOS in the single attribute case is in fact a convex monetary risk measure. The DMOS further allow for the consideration of the extent of shortfalls in the ranking of alternatives and concurrently, preserves computational tractability

3.1 General form of the multiple objective satisficing criteria

In this section, we propose a class of functions for evaluating how well the objectives meet their targets collectively under uncertainty. These functions are called multiple objective satisficing (MOS) criteria which encompass success probability criterion as a special case. We consider uncertainties that are represented by a state-space Ω and a set (sigma-algebra) $\mathcal{F}$ of events. To encompass distributional ambiguity, we do not necessarily specify a probability distribution on $(\Omega, \mathcal{F})$. Instead, we let $\mathbb{F}$ be a set of probability measures on $(\Omega, \mathcal{F})$. For a given distribution, $\mathbb{P} \in \mathbb{F}$, $\mathbb{E}_{\mathbb{P}}(\cdot)$ denotes taking expectation over the distribution $\mathbb{P}$.

We consider n uncertain objectives and represent the outcomes as a multivariate random variable on Ω, i.e, a vector of n functions $\tilde{\boldsymbol{a}} : \Omega \mapsto \Re^n$. We define the index set $N = \{1, \ldots, n\}$. We allow targets to be uncertain and denote this as the random variable $\tilde{\boldsymbol{\tau}}$. For convenience, we suppress the notation of the targets by defining the set of *target excess* as follows

$$\mathcal{X} = \{\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{a}} - \tilde{\boldsymbol{\tau}} \ : \ \tilde{\boldsymbol{a}} \in \mathcal{Y}\},$$

where $\mathcal{Y}$ denotes the set of $\tilde{\boldsymbol{a}}$ random vectors.

For inequalities related to random variables, we use $\tilde{x} \geq 0$ to denote $\mathbb{P}(\tilde{x} \geq 0) = 1$ for all $\mathbb{P} \in \mathbb{F}$ and $\tilde{x} \ngeq 0$ to imply there exists $\mathbb{P} \in \mathbb{F}$ such that $\mathbb{P}(\tilde{x} \geq 0) < 1$. Strict inequality, such as $\tilde{x} < 0$ implies there exists an $\epsilon < 0$ such that $\mathbb{P}(\tilde{x} \leq \epsilon) = 1$ for all $\mathbb{P} \in \mathbb{F}$. Henceforth, we assume that targets are normalized to zeros. We first propose the properties which any MOS criterion should possess.

Definition 1 *A function $\rho : \mathcal{X} \mapsto [0, 1]$, is an MOS criterion if it satisfies the following properties:*

1. ***M**onotonicity: If $\tilde{\boldsymbol{x}} \geq \tilde{\boldsymbol{y}}$, then $\rho(\tilde{\boldsymbol{x}}) \geq \rho(\tilde{\boldsymbol{y}})$.*
2. ***A**ttainment Content: $\rho(\mathbf{0}) = 1$. If there exists $i \in N$, $\tilde{x}_i \geq 0$, then $\rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) = \rho((0, \tilde{\boldsymbol{x}}_{-i}))$.*
3. ***N**on-abandonment: If there exists $i \in N$, $\tilde{x}_i < 0$, then $\rho(\tilde{\boldsymbol{x}}) = 0$.*
4. ***R**ight continuity: $\lim_{a \downarrow 0} \rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) = \rho(\tilde{\boldsymbol{x}})$.*

Monotonicity implies that if an alternative $\tilde{\boldsymbol{x}}$ is almost surely better than $\tilde{\boldsymbol{y}}$ for all distributions in the family, then it would never be less preferred. Attainment content follows from the satisficing principle. Moreover, if there is an objective that always achieves its target, the MOS criterion would be insensitive to magnitude of its over achievement above the target. Non-abandonment states that an alternative is never more preferred if there is an objective that always fails its target. Right continuity implies that if all target excess are augmented with infinitesmally small but positive amounts, the satisficing level cannot be improved in the limit. This also implies that we exclude the consideration of $\mathbb{P}(\tilde{\boldsymbol{x}} > 0)$. Note that the success probability criterion, $\mathbb{P}(\tilde{\boldsymbol{x}} \geq 0)$ is consistent with this definition of the MOS criteria.

The MOS criteria can be characterised from the four basic properties through a representation theorem which shows the relationship in a dual form. This dual form of the MOS has implications as a monetary risk measure. Furthermore, when diversification preference is considered, the dual representation of the single objective satisficing measure is a convex monetary risk measure. A similar result has also been introduced in Brown and Sim [2009]. The consideration of diversification preference in the MOS for the multiple objectives case will be discussed in the next subsection. Extending the result of Brown and Sim [2009], the MOS criteria described in Definition 1 can be equivalently characterized as follows:

Theorem 1 *A function $\rho : \mathcal{X} \mapsto \Re$ is an MOS criterion if and only if*

$$\rho(\tilde{\boldsymbol{x}}) = \begin{cases} \sup\{k \in (0,1) : \eta_k(\tilde{\boldsymbol{x}}) \leq 0\} & \text{if } \exists k \in (0,1) \text{ s.t. } \eta_k(\tilde{\boldsymbol{x}}) \leq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\{\eta_k : k \in (0,1)\}$ is a family of dual functions that are non-decreasing in k for $\tilde{\boldsymbol{x}} \in \mathcal{X}$ and $\eta_k : \mathcal{X} \mapsto \Re$ satisfies the following properties:

1. *If $\tilde{\boldsymbol{x}} \geq \tilde{\boldsymbol{y}}$, then $\eta_k(\tilde{\boldsymbol{x}}) \leq \eta_k(\tilde{\boldsymbol{y}})$*
2. *For all $a \in \Re$, $\eta_k(\tilde{\boldsymbol{x}} + a\mathbf{1}) = \eta_k(\tilde{\boldsymbol{x}}) - a$*
3. *If there exists $i \in N$, $\tilde{x}_i \geq 0$, then $\eta_k((0, \tilde{\boldsymbol{x}}_{-i})) = \max\{0, \eta_k(\tilde{\boldsymbol{x}})\}$*
4. *If there exists $i \in N$, $\tilde{x}_i < 0$, then $\eta_k(\tilde{\boldsymbol{x}}) > 0$*
5. *$\eta_k(\mathbf{0}) = 0$.*

For an MOS criterion, ρ, the underlying dual function for $k \in (0,1)$ is given by

$$\eta_k(\tilde{\boldsymbol{x}}) = \inf\{a : \rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) \geq k\}. \tag{2}$$

Proof : We show that ρ under representation (1) is an MOS criterion if η_k , $k \in (0,1)$, is the family of dual functions with the properties stated in Theorem 1.

Monotonicity: If $\tilde{\boldsymbol{x}} \geq \tilde{\boldsymbol{y}}$, we have by monotonicity $\eta_k(\tilde{\boldsymbol{x}}) \leq \eta_k(\tilde{\boldsymbol{y}})$ for all $k \in (0,1)$. Since η_k is non-decreasing in k, $\rho(\tilde{\boldsymbol{x}}) \geq \rho(\tilde{\boldsymbol{y}})$.

Attainment content: Since $\eta_k(\mathbf{0}) = 0$ for all $k \in (0,1)$, we have $\rho(\mathbf{0}) = 1$. If there exists $i \in N$, $\tilde{x}_i \geq 0$, we will show that $\rho((0, \tilde{\boldsymbol{x}}_{-i})) = \rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}))$ as follows:

$$
\begin{aligned}
\rho((0, \tilde{\boldsymbol{x}}_{-i})) &= \begin{cases} \sup\{k \in (0,1) : \eta_k((0, \tilde{\boldsymbol{x}}_{-i})) \leq 0\} & \text{if feasible} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \sup\{k \in (0,1) : \max\{0, \eta_k((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}))\} \leq 0\} & \text{if feasible} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \sup\{k \in (0,1) : \eta_k((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) \leq 0\} & \text{if feasible} \\ 0 & \text{otherwise} \end{cases} \\
&= \rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})),
\end{aligned}
$$

where the second equality follows from Property 3.

Non-abandonment: If there exists $i \in N$, $\tilde{x}_i < 0$, then $\eta_k(\tilde{\boldsymbol{x}}) > 0$ for all $k > 0$. Hence, $\rho(\tilde{\boldsymbol{x}}) = 0$ under (1).

Right continuity: For right continuity, we need to show that for any $\epsilon > 0$ there exists $a > 0$ such that $\rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) \leq \rho(\tilde{\boldsymbol{x}}) + \epsilon = \bar{\rho}$, so that $\lim_{a \downarrow 0} \rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) = \rho(\tilde{\boldsymbol{x}})$. From (1), we have $\eta_{\bar{\rho}}(\tilde{\boldsymbol{x}}) > 0$ and hence, there exists $a > 0$ such that $\eta_{\bar{\rho}} > a > 0$. By monotonicity and translation invariance of the family of functions $\{\eta_k : k \in (0,1)\}$ we have,

$$
\begin{aligned}
\rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) &= \begin{cases} \sup\{k \in (0,1) : \eta_k(\tilde{\boldsymbol{x}} + a\mathbf{1}) \leq 0\} & \text{if feasible} \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \sup\{k \in (0,1) : \eta_k(\tilde{\boldsymbol{x}}) - a \leq 0\} & \text{if feasible} \\ 0 & \text{otherwise,} \end{cases} \\
&\leq \begin{cases} \sup\{k \in (0,1) : \eta_k(\tilde{\boldsymbol{x}}) < \eta_{\bar{\rho}}(\tilde{\boldsymbol{x}})\} & \text{if feasible} \\ 0 & \text{otherwise,} \end{cases} \\
&\leq \bar{\rho} = \rho(\tilde{\boldsymbol{x}}) + \epsilon.
\end{aligned}
$$

We now show the converse that under representation (2), the family of functions $\{\eta_k : k \in (0,1)\}$ satisfies the properties of Theorem 1.

Property 1: This follows trivially from the monotonicity property of ρ.

Property 2: For any $c \in \Re$,

$$\begin{aligned}\eta_k(\tilde{\boldsymbol{x}} + c\mathbf{1}) &= \inf\{a : \rho(\tilde{\boldsymbol{x}} + (a+c)\mathbf{1}) \geq k\} \\ &= \inf\{a - c : \rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) \geq k\} \\ &= \eta_k(\tilde{\boldsymbol{x}}) - c\end{aligned}$$

Property 3: By definition, $\eta_k((0, \tilde{\boldsymbol{x}}_{-i})) = \inf\{a : \rho((0, \tilde{\boldsymbol{x}}_{-i}) + a\mathbf{1}) \geq k\}$. By attainment content, if $\exists i \in N$ s.t. $\tilde{x}_i \geq 0$, then $\rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) = \rho((0, \tilde{\boldsymbol{x}}_{-i}))$. Also, observe that from the non-abandonment property of ρ,

$$\rho((0, \tilde{\boldsymbol{x}}_{-i}) + a\mathbf{1}) = 0 < k$$

for all $a < 0$, $k \in (0, 1)$. Hence, we have

$$\begin{aligned}\eta_k((0, \tilde{\boldsymbol{x}}_{-i})) &= \inf\{a : \rho((0, \tilde{\boldsymbol{x}}_{-i}) + a\mathbf{1}) \geq k\} \\ &= \inf\{a : \rho((0, \tilde{\boldsymbol{x}}_{-i}) + a\mathbf{1}) \geq k, a \geq 0\} \\ &= \inf\{a : \rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) \geq k, a \geq 0\} \\ &= \max\{0, \eta_k(\tilde{\boldsymbol{x}})\},\end{aligned}$$

where the third equality follows from attainment content and the last equality follows from $\rho(\tilde{\boldsymbol{x}} + a\mathbf{1})$ being a nondecreasing in a.

Property 4: Suppose there exists $i \in N$, $\tilde{x}_i < 0$, we will show that there exists $\epsilon > 0$ such that $\eta_k(\tilde{\boldsymbol{x}}) \geq \epsilon$, or equivalently, by Property 2 shown above, $\eta_k(\tilde{\boldsymbol{x}} + \mathbf{1}\epsilon) \geq 0$. Since $\tilde{x}_i < 0$, it implies that there exists $\epsilon > 0$ such that $\tilde{x}_i + \epsilon < 0$ and consequently, $\rho(\tilde{\boldsymbol{x}} + \mathbf{1}\epsilon) = 0$ from the non-abandonment property of ρ. Therefore, since $k \in (0, 1)$

$$\eta_k(\tilde{\boldsymbol{x}} + \epsilon\mathbf{1}) = \inf\{a : \rho(\tilde{\boldsymbol{x}} + \epsilon\mathbf{1} + a\mathbf{1}) \geq k\} \geq 0.$$

Property 5: For any $k \in (0, 1)$, we observe that $\rho(\mathbf{0} + a\mathbf{1}) = 1 \geq k$ for all $a \geq 0$ and that $\rho(\mathbf{0} + a\mathbf{1}) = 0$ for all $a < 0$. Hence, $\eta_k(\mathbf{0}) = 0$.

We next verify that under representation (2), we can recover ρ from representation (1). For this, we consider two cases. For the first case, there exists $k \in (0, 1)$, such that $\eta_k(\tilde{\boldsymbol{x}}) \leq 0$. Observe that since $\rho(\tilde{\boldsymbol{x}}+a\mathbf{1})$ is right continuous with respect to a, the infimum in (2) is achievable. Hence,

$$\begin{aligned}
& \sup\{k \in (0,1) : \eta_k(\tilde{\boldsymbol{x}}) \leq 0\} \\
= \; & \sup\{k \in (0,1) : \exists a \leq 0 \text{ such that } \rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) \geq k\} \\
= \; & \sup\{\rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) : a \leq 0\} \\
= \; & \rho(\tilde{\boldsymbol{x}}).
\end{aligned}$$

For the second case, there does not exists $k \in (0,1)$ such that $\eta_k(\tilde{\boldsymbol{x}}) \leq 0$, which will lead to $\rho(\tilde{\boldsymbol{x}}) = 0$ under representation (1). Indeed, this is the case, since under representation (2), this condition is the same as the non existence of $a \leq 0$, such that $\rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) \geq k$ for any $k \in (0,1)$. Since $\max\{\rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) : a \leq 0\} = \rho(\tilde{\boldsymbol{x}})$, it also means that $\rho(\tilde{\boldsymbol{x}}) \not> 0$, or equivalently, $\rho(\tilde{\boldsymbol{x}}) = 0$.

■

Given Theorem 1, for ease of exposition, we shall subsequently define the family of functions, $\{\eta_k : k \in (0,1)\}$, as *dual MOS* functions.

3.2 Diversification favoring MOS

Here, we consider a class of MOS that is consistent with diversification preference (or convex preference) and denote them by *diversification favoring* MOS, or DMOS. Diversification preferences is a canonical assumption in economic theory. Although the success probability criterion captures all the properties of the MOS, it does not capture risk aversion which is a desirable property in economic models of risky choice. Specifically, it does not embody the characteristics of *diversification preference* which is a key property for mitigating excessive losses as it is not sensitive to the degree of shortfalls against targets. As a simple example of this lack of diversification preference, consider a single attribute target premium $\tilde{x}$ following a continuous and symmetrical distribution about zero (i.e. $\mathbb{P}(\tilde{x} \geq 0) = 0.5$). Consider another position $\tilde{y} = -\tilde{x} - \delta$ where $\delta > 0$. For δ small enough, $\mathbb{P}(\tilde{y} \geq 0)$ will also be close to 0.5. The diversified position $\tilde{z} = 0.5(\tilde{x} + \tilde{y})$ would then be -0.5δ with probability one, and consequently $\mathbb{P}(\tilde{z} \geq 0) = 0$. This implies that when using probability, we are worse off with diversification [Brown and Sim, 2009]. To further emphasize the importance of diversification preference, it has been noted that diversification is almost always good [Markowitz, 1952]:

> *Diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim.*

In portfolio choice, which is a prominent example of decisions involving risky choice, diversification has been an almost fundamental tenet to investing. Concentration on the other hand, are generally undesirable considering their tendency to expose investors to large positions.

Decision maker's preference for diversification is not only prevalent but fundamentally important for the mitigation of excessive losses arising from uncertainties in decision making. We can observe the impact of diversification via a simple example. Consider two uncertain positions (or scalar-valued target excess), $\tilde{x}$ and $\tilde{y}$. The following inequality shows the impact of diversification over these two uncertain positions:

$$\inf_{\omega\in\Omega}\{\lambda x(\omega)+(1-\lambda)y(\omega)\} \geq \inf_{\omega\in\Omega}\{x(\omega),y(\omega)\} \qquad \forall\lambda\in[0,1].$$

Hence, a diversified position has the effect of cushioning the impact of bad outcomes. We now extend the characterization of MOS to include diversification preference. For the i^{th} objective, diversification preference implies that if both $\tilde{x}_i$ and $\tilde{y}_i$ are preferred over $\tilde{r}_i$, then any convex combination of $\tilde{x}_i$ and $\tilde{y}_i$ is no less preferable to $\tilde{r}_i$. In the multiple objectives setting, we consider each objective is ranked by an autonomous agent that favors diversification. Hence, the DMOS criteria that favors diversification are componentwise quasi-concave functions (see also Brown and Sim [2009]). Formally, we define a sub-class of the MOS criteria in the DMOS criteria as follows:

Definition 2 *A function ρ is a DMOS criterion if it is a componentwise quasi-concave MOS criterion, i.e., for all $i\in N$, $(\tilde{x}_i,\tilde{\boldsymbol{x}}_{-i})$, $(\tilde{y}_i,\tilde{\boldsymbol{x}}_{-i})\in\mathcal{X}$,*

$$\rho((\lambda\tilde{x}_i+(1-\lambda)\tilde{y}_i,\tilde{\boldsymbol{x}}_{-i})) \geq \min\{\rho((\tilde{x}_i,\tilde{\boldsymbol{x}}_{-i})),\rho((\tilde{y}_i,\tilde{\boldsymbol{x}}_{-i}))\} \qquad \forall\lambda\in[0,1].$$

Theorem 2 *A function ρ is a diversification favoring MOS criterion if and only if the underlying function in the dual representation, η_k, $k\in(0,1)$, is also componentwise quasi-convex.*

Proof : Suppose ρ is a diversification favoring MOS criterion, we will show that for any $k \in (0,1)$, the function

$$\eta_k(\tilde{\boldsymbol{x}}) = \inf\{a \ : \ \rho(\tilde{\boldsymbol{x}} + a\mathbf{1}) \geq k\},$$

is componentwise quasi-convex. Since, $\rho(\tilde{\boldsymbol{x}} + a\mathbf{1})$ is right continuous and nondecreasing in a, the infimum in the dual representation is achievable and hence, $\rho(\tilde{\boldsymbol{x}} + \eta_k(\tilde{\boldsymbol{x}})\mathbf{1}) \geq k$. For any objective, $i \in N$, we consider $(\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}), (\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}) \in \boldsymbol{\mathcal{X}}$. Let $a^* = \max\{\eta_k((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})), \eta_k((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))\}$ and we have $\min\{\rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}) + a^*\mathbf{1}), \rho((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}) + a^*\mathbf{1})\} \geq k$. Observe that for any $\lambda \in [0,1]$,

$$\begin{aligned}
\eta_k((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) &= \inf\{a : \rho(\lambda(\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}) + (1-\lambda)(\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}) + a\mathbf{1}) \geq k\} \\
&= \inf\{a : \rho(\lambda((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}) + a\mathbf{1}) + (1-\lambda)((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}) + a\mathbf{1}) \geq k\} \\
&\leq \inf\{a : \min\{\rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}) + a\mathbf{1}), \rho((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}) + a\mathbf{1})\} \geq k\} \\
&\leq a^* = \max\{\eta_k((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})), \eta_k((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))\}.
\end{aligned}$$

where the first inequality follows from quasi-concavity of ρ. Next, we show that if η_k is componentwise quasi-convex and nondecreasing in $k \in (0,1)$, then the function

$$\rho(\tilde{\boldsymbol{x}}) = \begin{cases} \sup\{k \in (0,1) : \eta_k(\tilde{\boldsymbol{x}}) \leq 0\} & \text{if feasible} \\ 0 & \text{otherwise,} \end{cases}$$

is componentwise quasi-concave. For a given objective $i \in N$ and $(\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}), (\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}) \in \boldsymbol{\mathcal{X}}$, let

$$k^* = \min\{\rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})), \rho((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))\}.$$

The result is trivially true if $k^* = 0$. Suppose $k \in (0, k^*)$, we note that $\eta_k((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) \leq 0$ and $\eta_k((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \leq 0$. Since η_k is componentwise quasi-convex and is non-decreasing in k, we have

$$\begin{aligned}
\rho\big(\lambda(\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}) + (1-\lambda)(\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})\big) &= \sup\{k \in (0,1) : \eta_k\left(\lambda(\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}) + (1-\lambda)(\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})\right) \leq 0\} \\
&\geq \sup\{k \in (0,1) : \max\{\eta_k((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})), \eta_k((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))\} \leq 0\} \\
&\geq k^* \\
&= \min\{\rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})), \rho((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))\} > 0.
\end{aligned}$$

■

Apart from being a desirable property in economic models of risky choice, quasi-concavity also leads to the desirable property of solution robustness since it allows the spreading out of risks through splitting up positions, thereby reducing the sensitivity to errors in modeling real-world.

3.2.1 Relationship with monetary risk measures

For the single objective case, Brown and Sim [2009] showed that the satisficing measure can be represented as a dual of monetary risk measures, which originates in mathematical finance community and has witnessed an explosive growth in the last decade; see Artzner et al. [1999], Föllmer and Schied [2004]. Monetary risk measure can be formally defined as follows:

Definition 3 *A function $\mu : \mathcal{X}^1 \mapsto \Re$ is a monetary risk measure if it satisfies:*

1. ***M**onotonicity: If $\tilde{x} \geq \tilde{y}$, then $\mu(\tilde{x}) \leq \mu(\tilde{y})$.*
2. ***T**ranslation invariance: For all $c \in \Re$, $\mu(\tilde{x} + c) = \mu(\tilde{x}) - c$.*

A monetary risk measure can be interpreted as the amount of monetary investment(capital) needed to make a position acceptable by some standard, since, $\mu(\tilde{x} + \mu(\tilde{x})) = \mu(\tilde{x}) - \mu(\tilde{x}) = 0$ implies that adding capital $\mu(\tilde{x})$ to risky position $\tilde{x}$ will result in a position of "zero risk". Moreover, positions with non-positive risk can be considered acceptable. In other words, acceptable positions do not require additional guaranteed capital. A classical example of the monetary risk measure is the value-at-risk (VaR) measure given as follows:

$$\mathrm{VaR}_\alpha(\tilde{x}) = \inf\{c \in \Re : \mathbb{P}(c + \tilde{x} \geq 0) \geq 1 - \alpha\}. \tag{3}$$

In simple terms, the VaR can be interpreted as the smallest amount of capital necessary to augment a risky position $\tilde{x}$ to ensure that the augmented portfolio breaks even with probability at least $1 - \alpha$.

While the dual of single objective satisficing, monetary risk measures, are well studied objects, it is the not the case for dual of MOS. Herein, we extend the dual relationship between the monetary risk measure and the MOS criteria to consider multiple objectives. Here we show an example of how this can be done for a set of objective targets excess (represented by $\tilde{\boldsymbol{x}}$) that are of the same units. Examples of such target excess can be related to the target profit, in monetary terms, to be achieved over a number of period.

In order to show the relationship between the MOS criteria and the monetary risk measure, we provide a simple example of an MOS criteria that can easily be constructed from monetary risk measures for a multiple objectives satisficing problem by redefining a new random variable that is the minimum over the random target excess vector given as follows:

$$\tilde{y} = \min_{j \in \{1,\ldots,n\}} \{\tilde{x}_j\}$$

Other examples may be discovered using appropriate functional constructions. Though it is not necessary, it is natural to assume that monetary risk measures are normalized, i.e., $\mu(0) = 0$. The example of MOS criteria that can be constructed from normalized monetary risk measure is given in the following proposition.

Proposition 1 *Let μ be a normalized monetary risk measure. The following is a dual MOS:*

$$\eta(\tilde{x}) = \mu\left(\min_{j \in \{1,\ldots,n\}} \{\tilde{x}_j\}\right)$$

Proof : Given the properties of monetary risk measure, it is easy to verify properties 1,2 and 5 in Theorem 1 for $\eta(\tilde{x})$ by inspection.

For property 3, suppose $\exists \tilde{x}_j \geq 0$ and note that we have $\mu\left(\min\{0, \tilde{x}_{-j}\}\right) \geq 0$ since,

$$\mu\left(\min\{0, \tilde{x}_{-j}\}\right) \geq \mu(0) = 0.$$

In the case when $\mu(\tilde{\boldsymbol{x}}) \leq 0$, we have $\mu\left(\min\{0, \tilde{x}_{-j}\}\right) \leq 0$. However, from the preceding, $\mu\left(\min\{0, \tilde{x}_{-j}\}\right) \geq 0$, hence, $\mu\left(\min\{0, \tilde{x}_{-j}\}\right) = 0$. For the case when $\mu(\tilde{\boldsymbol{x}}) > 0$, we clearly have $\mu\left(\min\{0, \tilde{x}_{-j}\}\right) = \mu(\tilde{\boldsymbol{x}})$.

For property 4, suppose $\tilde{x}_j < 0$, i.e, there exists $\epsilon < 0$ such that $\tilde{x}_j \leq \epsilon$, then

$$\begin{aligned} \eta(\tilde{x}) &= \mu\left(\min_{j \in \{1,\ldots,n\}} \{\tilde{x}_j\}\right) \\ &\geq \mu(\epsilon) = \mu(0) - \epsilon > 0. \end{aligned}$$

■

An important example of an MOS criterion is the success probability. Intuition on this example can be derived by considering well-known monetary risk measure, the VaR measure, using Proposition 1. Note that the dual relationship between the success probability and VaR measure (in the single objective case) is given by:

$$\mathbb{P}\{\tilde{x} \geq 0\} = \begin{cases} \sup\{k \in (0,1) : \mathrm{VaR}_{1-k}(\tilde{x}) \leq 0\} & \text{if feasible} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we can also establish the success probability, which is an MOS criterion, from the dual MOS of the VaR measure. Indeed, suppose the underlying dual MOS is given by

$$\eta_k(\tilde{y}) = \mathrm{VaR}_{1-k}\left(\min_{i \in \{1,\ldots,n\}} \{\tilde{x}_i\}\right),$$

where,

$$\tilde{y} = \min_{i \in \{1,\ldots,n\}} \{\tilde{x}_i\}$$

then,

$$\begin{aligned} \rho(\tilde{\boldsymbol{x}}) &= \begin{cases} \sup\{k \in (0,1) : \eta_k(\tilde{y}) \leq 0\} & \text{if feasible} \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} \sup\left\{k \in (0,1) : \mathrm{VaR}_{1-k}\left(\min_{i \in \{1,\ldots,n\}} \{\tilde{x}_i\}\right) \leq 0\right\} & \text{if feasible} \\ 0 & \text{otherwise.} \end{cases} \\ &= \mathbb{P}\left\{\min_{i \in \{1,\ldots,n\}} \{\tilde{x}_i\} \geq 0\right\} \\ &= \mathbb{P}\left\{\tilde{\boldsymbol{x}} \geq \mathbf{0}\right\}. \end{aligned}$$

Observe that the success probability does not take into account the severity of violations whenever they occur. Moreover, in general maximizing over success probability is computationally intractable due to the non-convexity of its level sets.

3.2.1.1 Relationship with normalized convex risk measure

For the case of single objective, the function in the dual representation of the DMOS criteria, η_k, $k \in (0,1)$ becomes a well-known normalized convex risk measure [Föllmer and Schied, 2004] with the following properties:

1. **M**onotonicity: If $\tilde{x} \geq \tilde{y}$, then $\eta_k(\tilde{x}) \leq \eta_k(\tilde{y})$.

2. **T**ranslation invariance: For all $a \in \Re$, $\eta_k(\tilde{x} + a) = \eta_k(\tilde{x}) - a$.

3. **C**onvexity: For all $\lambda \in [0,1]$, $\eta_k(\lambda\tilde{x} + (1-\lambda)\tilde{y}) \le \lambda\eta_k(\tilde{x}) + (1-\lambda)\eta_k(\tilde{y})$.

4. **N**ormalization: $\eta_k(0) = 0$.

Observe that for the case of single objective, the function Properties 3 and 4 in Theorem 1 are implied by normalization. Convexity of η_k follows directly from translation invariance and quasi-convexity as follows:

$$\begin{aligned}
& \eta_k(\lambda\tilde{x} + (1-\lambda)\tilde{y}) - \lambda\eta_k(\tilde{x}) - (1-\lambda)\eta_k(\tilde{y}) & \\
= \;& \eta_k\big(\lambda\tilde{x} + (1-\lambda)\tilde{y} + \lambda\eta_k(\tilde{x}) + (1-\lambda)\eta_k(\tilde{y})\big) & : \text{translation invariance} \\
= \;& \eta_k\big(\lambda(\tilde{x} + \eta_k(\tilde{x})) + (1-\lambda)(\tilde{y} + \eta_k(\tilde{y}))\big) & \\
\le \;& \max\{\eta_k(\tilde{x} + \eta_k(\tilde{x})), \eta_k(\tilde{y} + \eta_k(\tilde{y}))\} & : \text{quasi-convexity} \\
= \;& 0, &
\end{aligned}$$

for all $\lambda \in [0,1]$. The dual relationship between the convex risk measure and the satisficing criterion has been established in Brown and Sim [2009] for the single objective case. We extend this relationship to the case of multiple objectives, where the function in dual representation of the DMOS criteria can be viewed as a multivariate extension of the convex risk measure.

3.3 Chapter summary

In this chapter, we have provided detailed characterizations of generic families of MOS and the DMOS family of multiple objectives satisficing criteria. The MOS criteria is the most general family of multiple objectives satisficing measures. The success probability criterion is shown to be a special form of MOS criteria. A diversification favoring sub-class of the MOS criteria, the DMOS criteria, is also proposed. DMOS criteria forms a unique sub-family of MOS criteria that not only possess the fundamental multiple objectives satisficing properties, it is also able to model the characteristic of diversification preference, thereby endowing it with the ability to derive decisions that can cushion against the impact of bad outcomes. The dual of the DMOS criteria is shown to be a monetary risk measures and the relationship between the DMOS criteria and normalized convex risk measures for the single objective case have been established.

The relationship between the specialized class of DMOS criteria, which embodies diversification preference, and convex monetary risk measures [Föllmer and Schied, 2004] is also established.

In the next chapter, we construct a specialized member of the DMOS family of criteria that can be used as a concrete objective measure for comparing alternatives in the presence of uncertainty. A practical iterative solution strategy is also presented for this proposed DMOS criterion. The proposed form of DMOS criterion is also able to handle distribution ambiguity. Since the consideration of distributional ambiguity results in an intractable formulation, we introduce an approximate form that will provide lower bounds to the distributionally robust formulation based on linear decision rules (LDRs) developed in Chen et al. [2003], Goh and Sim [2010] and Goh and Sim [2011]. This will also be presented in the subsequent chapter which introduces the general formulation of a specific DMOS criterion that is able to deal with distributional ambiguity. This criterion would be shown to retain the desirable characteristics of the DMOS criteria.

By proposing the DMOS criterion in a form that can handle distributional ambiguity, we essentially develop a measure that is both sensitive to the magnitude of shortfall and able to handle situations where we do not have full knowledge of the characteristics of the uncertainties underlying the decision problems. In layman terms, the existing measure is able to consider black swan events characterised by its rarity, extreme impact and limited predictability [Taleb, 2001]. The implications of this measures along these lines of enquiry under optimization would be demonstrated in Chapter 5.

CHAPTER 4

Shortfall Aware Multiple Objectives Satisficing Criterion

In the previous chapter, we have provided the general characterizations of the MOS and DMOS families of multiple objectives satisficing criteria. While the probability of achieving targets is an MOS criterion, it is not diversification favoring. Furthermore, it is insensitive to the magnitude of underperformance, or shortfalls, in target achievement. This leads us naturally to the next part of the work. In this chapter, we propose a specific form of a DMOS criteria with many desirable properties that can be implemented to compare and rank alternatives under uncertainty. We consider the more general case of distributional ambiguity in the development of this criterion, since this naturally include the case of a specific distribution in the family of distribution considered.

In this chapter, the shortfall-aware MOS (S-MOS) criterion that is able to address the deficiency of probability measures not being able to account for the magnitude of shortfalls is proposed. This S-MOS criterion also incorporates diversification preference characteristics which is a more reasonable behavioral assumption for decision making under uncertainty and enables the mitigation of risks associated with excessive losses. In Chapter 2, we introduced black swan events as random events that possess the following three characteristics: [Taleb, 2001]:

- The event is an outlier, as it lies outside the realm of regular expectations as nothing in the past can convincingly point to its occurrence.
- The event carries an extreme impact, or that the costs associated with the event is relatively much more malignant than usual.
- The occurrence of such black swan events trigger human nature to concoct explanations for its occurrence *ex post*, after the occurrence. This may make the events "explainable"

or predictable only *a posteriori*.

The S-MOS criterion is able to account for the degree of shortfall in achieving the target in a multiple objectives satisficing framework. This characteristic has been purposefully designed into the S-MOS criterion, hence imbuing the S-MOS criterion with the capability of taking into account the severity of target underachievement which is an important consideration to duly account for the potential occurrence of black swan events.

The multiple objectives satisficing framework based on the family of MOS criteria not only allows decision maker to deal with uncertain decision making when risks are present, it also allows the flexibility of considering *distributional ambiguity*. Ambiguity as defined here is similar to Knightian uncertainty [Knight, 1921]. Instead of adopting an all-encompassing term to describe uncertainties in decision making, Knight [1921] proposed a distinction between ambiguity (or "Knightian" uncertainty) and risk. While risk consists of a future with a known distribution, or a distribution that can be estimated by studying draws over time, "Knightian" uncertainty consists of a future whose distribution is not only cannot be completely characterized, but objectively unknowable [Sarasvathy and Kotha, 2007]. We define ambiguity within the context of Knightian uncertainty. Specifically, ambiguity refers to situations where full distribution information is not available for decision making. What the decision maker may know could be limited distributional information, such as the support, mean and covariance. Hence, the decisions have to be made on the basis of families of distributions characterized by these distributional information. There are practical implications of the presence of ambiguity in a variety risky decision making contexts under ambiguity [Natarajan et al., 2010b, Calafiore, 2010, Gilboa and D., 1989, Garlappi et al., 1989, Popescu, 2007] and the existence of simple examples that convincingly demonstrates the importance of considering ambiguity even within well-established paradigms as discussed in Section 2.3.

Given the practical and scientific evidence supporting the need to consider ambiguity and to sensitize decisions to the degree of shortfalls, the S-MOS criterion that explicitly models distribution ambiguity is presented in this chapter. The resulting S-MOS criterion possesses the characteristics of a DMOS. Since the consideration of distributional ambiguity in the S-MOS criterion results in an intractable problem, we propose some tractable robust approximations to derive valid lower bounds to the S-MOS. The tractable robust bounds are based on the supremal convolution of tractable, positively homogeneous and concave functions. Another approximation approach based on piecewise linear decisions rule proposed in Chen et al. [2003], Goh and Sim [2010] and Goh and Sim [2011], and implemented through the software ROME described in Goh and Sim [2011] would also be proposed. As maximizing the S-MOS criterion is not a convex optimization problem, we propose improvement algorithms via solving sequences of convex optimization problems using the ROME toolbox. The algorithm is described in the context of the distributional robust case and would be applicable in the more specialized situation when full distributional information is available for the optimization of the multiple objectives.

4.1 Shortfall-aware MOS (S-MOS) criterion

In this section we propose a DMOS criteria defined as the shortfall-aware MOS (S-MOS) criterion. As its name implies, the S-MOS criterion is 'shortfall-aware' in the sense that it reflects penalization on the level of target shortfall when a shortfall occurs, in contrast to probability value which is insensitive to the level of shortfall. The shortfall-aware characteristics of the S-MOS criterion is able to take into consideration the severity of underperformance against the targets in the managerial decision making process. The necessity to the motivate such a characteristics has been gaining prominence in recent years particularly with the occurrence of numerous rare events with limited predictability [Taleb, 2001] resulting in severe and crippling consequences [Makridakisa and Taleb, 2009]. The combination of low predictability and large impact have undermined the ability of many basic quantitative decision models (e.g. expected loss, probability, expected utility) to produce accurate and practically useful decisions that are able to adequately deal with decision problems under uncertainty.

One of the key characteristic of black swan events is the presence of rare and unpredictable events that could result in extreme consequences, such as severe monetary losses, the loss of lives or environmental damage. Consequently, the performance measure that is used to rank decisions have to be sensitive to the severity of the consequences. In a target-oriented perspective, this naturally implies sensitivity to the degree of underachievement (or shortfall) of the target. Extending this conceptual framework further to constrained optimization, this implies that the measure should be sensitive to the degree of constraint violations in the ranking of decisions in the presence of uncertainty. As mentioned in the preceding, many traditional measures, such as the expected value approaches or the probability of target achievement, do not possess this capability. Considering this deficiency in traditional performance measures for ranking of alternative under uncertainty, we developed a concrete form of the DMOS criteria that is inherently sensitive to the magnitude of underperformance, or shortfalls, in the targets. To recapitulate, the MOS family of measures defined in the preceding chapter has the success probability, $\mathbb{P}(\tilde{\boldsymbol{x}} \geq 0)$ where $\tilde{\boldsymbol{x}} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\}$ represent the target excess, as a special case. We now specialize the MOS criteria to the shortfall-aware MOS (S-MOS) to allow for the consideration of the extent of shortfall in the ranking of alternatives. The S-MOS criterion is also expected to be more suited in an environment of limited predictability since the satisficing paradigm acknowledges the practical impossibility of searching for an optimal decision [Simon, 1955], particularly in an environment of limited predictability and when extreme events can lead to dire consequences. Furthermore, the proposed S-MOS criterion is able to handle situations where only limited distributional information is available. Note that the case with full distributional information is just a specific instance of such problems.

Before introducing our criterion, we first provide the insights from the perspective of lower bounding the success probability by a family of concave utility functions. The maximization of the probability of target achievement as an objective in optimization has several drawbacks from both the computational and modeling perspectives. The lack of any form of structural convexity leads to computational intractability in optimization problems when this measure is used for decision analysis under uncertainty. Moreover, it is evident from the following that probability does not take into account the level of shortfall from the target and may evaluate a catastrophic event and a mild violation, occurring with the same low probability, equally. Here, we aim to overcome these obvious deficiencies in success probability criterion.

To motivate the development of the S-MOS criterion, first, observe that the success probability criterion can be expressed as an expectation over a step function (or a Heaviside utility function), i.e.,

$$\mathbb{P}(\tilde{\boldsymbol{x}} \geq 0) = \mathbb{E}_{\mathbb{P}}(s(\tilde{\boldsymbol{x}}))$$

where the function $s : \Re^n \mapsto \Re$ is a step-function given by

$$s(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \geq \mathbf{0} \\ 0 & \text{otherwise.} \end{cases}$$

From a utility perspective, such a Heaviside utility function has the undesirable properties of being non-concave and its insensitivity to the degrees of shortfalls. On the other hand, these issues may be addressed by taking expectation over any concave nondecreasing function, $h : \Re^n \mapsto \Re$. Moreover, if the function, h is dominated by the step function, s, i.e., $h(\boldsymbol{x}) \leq s(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Re^n$, then $\mathbb{E}_{\mathbb{P}}(h(\tilde{\boldsymbol{x}}))$ is a lower bound of success probability, i.e.,

$$\mathbb{E}_{\mathbb{P}}(h(\tilde{\boldsymbol{x}})) \leq \mathbb{E}_{\mathbb{P}}(s(\tilde{\boldsymbol{x}})) = \mathbb{P}(\tilde{\boldsymbol{x}} \geq 0).$$

We consider a criterion that selects the best function within the following family of concave functions,

$$\mathcal{H} = \left\{ h : \Re^n \mapsto \Re \mid h \text{ is concave, non-decreasing and } h(\boldsymbol{x}) \leq s(\boldsymbol{x}), \forall \boldsymbol{x} \in \Re^n \right\}.$$

The following result shows how we can optimize over the family of functions, $\mathcal{H}$ to obtain the tightest bound on the success probability criterion.

Proposition 2

$$\sup_{h \in \mathcal{H}} \mathbb{E}_{\mathbb{P}}(h(\tilde{\boldsymbol{x}})) = \sup_{\boldsymbol{u} \geq \mathbf{0}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{u_i \tilde{x}_i, 1\} \right)$$

Proof : Note that for any $\boldsymbol{u} \geq \mathbf{0}$, the function $r(\boldsymbol{x}) = \min_{i \in N}\{u_i x_i, 1\}$ belongs to the family $\mathcal{H}$. Hence, it suffices to show that $\forall h \in \mathcal{H}$, there exists $\boldsymbol{u} \geq \mathbf{0}$ so that:

$$h(x) \leq \min_{i \in N} \{u_i x_i, 1\} \quad \forall \boldsymbol{x} \in \Re^n$$

or equivalently

$$\begin{aligned} h(\boldsymbol{x}) &\leq u_i x_i \quad \forall \boldsymbol{x} \in \Re^n, i \in N \\ h(\boldsymbol{x}) &\leq 1 \qquad \forall \boldsymbol{x} \in \Re^n. \end{aligned} \tag{1}$$

The last inequality is trivially true since, $h(\boldsymbol{x}) \leq s(\boldsymbol{x}) \leq 1$ for all $\boldsymbol{x} \in \Re^n$. Let $g_i(x_i) = \lim_{v\uparrow\infty} h\left((x_i, v\mathbf{1}_{-i})\right)$. Note that g_i is also concave and nondecreasing. Since h is nondecreasing, we have

$$h(\boldsymbol{x}) \leq g_i(x_i).$$

Let u_i be a sub-gradient of g_i at the origin. Since g_i is nondecreasing and concave, we have $u_i \geq 0$ and that

$$u_i\left(x_i - 0\right) \geq g_i(x_i) - g_i(0) \geq g_i(x_i) \quad \forall x_i \in \Re.$$

The last inequality follows from the observation that $g_i(0) \leq 0$. This is true since if $g_i(0) > 0$, then by concavity of g_i on domain $\Re$, there must exist $\epsilon > 0$ small enough so that $0 < g_i(0-\epsilon) = \lim_{v\uparrow\infty} h\left((-\epsilon, v\mathbf{1}_{-i})\right)$. This clearly contradicts $h\left((-\epsilon, v\mathbf{1}_{-i})\right) \leq s\left((-\epsilon, v\mathbf{1}_{-i})\right) = 0\ \forall v \in \Re$. Hence, combining the above, we have $h(\boldsymbol{x}) \leq g_i(x_i) \leq u_i x_i$, and $h(\boldsymbol{x}) \leq 1$ for all $\boldsymbol{x} \in \Re^n$. Applying the argument for each $i \in N$, the desired result follows. ■

Inspired by the bound on success probability via concave functions, we define the shortfall-aware function (SAF) as follows:

Definition 4 *The shortfall aware function (SAF), β, is defined as follows:*

$$\beta(\tilde{\boldsymbol{x}}) = \sup_{\boldsymbol{u}\geq\mathbf{0}} \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{i\in N}\{u_i\tilde{x}_i, 1\}\right)$$

For the case of single objective ($n = 1$), the SAF is the shortfall aspiration measure criterion proposed by Chen and Sim [2009]. When compared to step utility function, which is oblivious to the magnitude of targets' shortfalls, the SAF reflects penalization against the degree of targets' shortfalls. Since, the multipliers u_i are nonnegative, whenever $\tilde{x}_i(\omega) < 0$, the level of target shortfall may contribute to a reduction in the SAF value. The SAF also reflects ambiguity aversion so that for any $\mathbb{P} \in \mathbb{F}$

$$\beta(\tilde{\boldsymbol{x}}) \leq \mathbb{P}(\tilde{\boldsymbol{x}} \geq \mathbf{0}).$$

Observe that since $\beta(\mathbf{0}) = 0$, the SAF is not an MOS criterion. However, we will show that the SAF shares similar properties to the DMOS criteria. For convenience, we henceforth define the index set

$$N(\tilde{\boldsymbol{x}}) \quad = \quad \{i \in N \ : \ \tilde{x}_i \not\geq 0\}.$$

Proposition 3 *The shortfall-aware function (SAF), $\beta : \mathcal{X} \mapsto [0,1]$ satisfies the following properties:*

1. ***M**onotonicity*

2. ***S**trict attainment content: $\beta(\mathbf{1}) = 1$. If there exists $i \in N$, $\tilde{x}_i > 0$, then $\beta((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) = \beta((1, \tilde{\boldsymbol{x}}_{-i}))$.*

3. ***N**on-abandonment*

4. ***C**omponentwise quasi-concave*

5. ***C**omponentwise scale invariant: $\beta(k_1\tilde{x}_1, \dots, k_n\tilde{x}_n) = \beta(\tilde{\boldsymbol{x}})$ for all $\boldsymbol{k} > \mathbf{0}$*

6. ***R**estricted right continuity: $\lim_{a \downarrow 0} \beta((\tilde{x}_i + a, \tilde{\boldsymbol{x}}_{-i})) = \beta(\tilde{\boldsymbol{x}})$, for all $i \in N(\tilde{\boldsymbol{x}})$*

Proof : Note that $\beta(\tilde{\boldsymbol{x}}) \leq 1$ and

$$\beta(\tilde{\boldsymbol{x}}) = \sup_{\boldsymbol{u} \geq \mathbf{0}} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{u_i \tilde{x}_i, 1\} \right) \geq \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{0 \tilde{x}_i, 1\} \right) = 0.$$

Hence, $\beta(\tilde{\boldsymbol{x}}) \in [0,1]$. We next show that the following properties are satisfied. For convenience, denote the index set $N_i = N \setminus \{i\}$.

1. **Monotonicity**: This is straightforward.

2. **Strict attainment content**: Suppose $\tilde{x}_i > 0$, then there exists $u_i > 0$ such that $u_i \tilde{x}_i > 1$. Hence,

$$\begin{aligned} \beta((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) &= \sup_{\boldsymbol{u} \geq 0} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_j \tilde{x}_j, u_i \tilde{x}_i, 1\} \right) \\ &= \sup_{\boldsymbol{u} \geq 0} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_j \tilde{x}_j, 1\} \right) \\ &= \beta((1, \tilde{\boldsymbol{x}}_{-i})). \end{aligned}$$

   For the case when $\tilde{\boldsymbol{x}} > 0$, it is easy to establish that $\beta(\tilde{\boldsymbol{x}}) = 1$.

3. **Non-abandonment:** Suppose there exists $\tilde{x}_i < 0$ for some i, then

$$\beta((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) = \sup_{\boldsymbol{u} \geq 0} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_j \tilde{x}_j, u_i \tilde{x}_i, 1\} \right) \leq 0,$$

   in which the supremum is achieved at $\boldsymbol{u} = \mathbf{0}$. Hence, $\beta(\tilde{\boldsymbol{x}}) = 0$.

4. **Componentwise quasiconcavity**: Let $\beta^* = \min\{\beta((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})), \beta((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))\}$. We will show that for any $\lambda \in [0,1]$,

$$\beta((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \geq \beta^*.$$

The result is trivial if $\lambda \in \{0,1\}$. Hence, we consider $\lambda \in (0,1)$. Observe that for any $\epsilon > 0$, there exists $\boldsymbol{u}, \boldsymbol{v} > 0$ such that

$$\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{j\in N_i}\{u_i\tilde{x}_i, u_j\tilde{x}_j, 1\}\right) \geq \beta^* - \epsilon$$

and

$$\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{j\in N_i}\{v_i\tilde{y}_i, v_j\tilde{x}_j, 1\}\right) \geq \beta^* - \epsilon.$$

Let $\gamma = \lambda v_i/(u_i(1-\lambda)+\lambda v_i)$ and $p_j = \gamma u_j + (1-\gamma)v_j$ for all $j \in N$. Observe that $\gamma \in (0,1)$ and $p_j > 0$ for all $j \in N$. Moreover,

$$p_i(\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i) = \gamma u_i\tilde{x}_i + (1-\gamma)v_i\tilde{y}_i$$

and

$$p_j\tilde{z}_j = \gamma u_j\tilde{x}_j + (1-\gamma)v_j\tilde{x}_j \qquad \forall j \in N_i.$$

Hence, noting that the function $\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_j\{\tilde{x}_j\}\right)$ is concave with respect to $\tilde{\boldsymbol{x}}$, we have

$$\begin{aligned}
& \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{j\in N_i}\{p_i(\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i), p_j\tilde{x}_j, 1\}\right) \\
= \; & \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{j\in N_i}\{\gamma u_i\tilde{x}_i + (1-\gamma)v_i\tilde{y}_i, \gamma u_j\tilde{x}_j + (1-\gamma)v_j\tilde{x}_j, 1\}\right) \\
\geq \; & \gamma \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{j\in N_i}\{u_i\tilde{x}_i, u_j\tilde{x}_j, 1\}\right) + (1-\gamma)\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{j\in N_i}\{v_i\tilde{y}_i, v_j\tilde{x}_j, 1\}\right) \\
\geq \; & \beta^* - \epsilon.
\end{aligned}$$

Hence, $\beta((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \geq \beta^*$.

5. **Componentwise scale invariant**: The result is obvious since the multiplier, u_i lies in a cone.

6. **Restricted right continuity**: For any $i \in N(\tilde{\boldsymbol{x}})$ there exists $\mathbb{P} \in \mathbb{F}$ such that $\mathbb{P}(\tilde{x}_i < 0) > 0$ and so $\mathbb{P}(\tilde{x}_i + \epsilon < 0) > 0$ for some $\epsilon > 0$. Hence, for all $a \in [0, \epsilon]$

$$\begin{aligned}
\lim_{u_i \uparrow \infty} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_i(\tilde{x}_i + a), u_j \tilde{x}_j, 1\} \right) &\leq \lim_{u_i \uparrow \infty} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min\{u_i(\tilde{x}_i + a), 1\} \right) \\
&\leq \lim_{u_i \uparrow \infty} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min\{u_i(\tilde{x}_i + \epsilon), 1\} \right) \\
&\leq \lim_{u_i \uparrow \infty} u_i \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min\{(\tilde{x}_i + \epsilon), 1/u_i\} \right) \\
&\leq -\infty.
\end{aligned}$$

Let $\bar{u}_i$ such that $\inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min\{\bar{u}_i(\tilde{x}_i + \epsilon), 1\} \right) < 0$. Since, the function

$$f(u) = \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min\{u(\tilde{x}_i + \epsilon), 1\} \right)$$

is concave in u with $f(0) = 0$, $f(\infty) = -\infty$, and $f(\bar{u}_i) < 0$, we must have $f(u) < 0$ for all $u \geq \bar{u}_i$. Hence, for all $a \in [0, \epsilon]$, $u_i \geq \bar{u}_i$

$$\inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_i(\tilde{x}_i + a), u_j \tilde{x}_j, 1\} \right) \leq \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min\{u_i(\tilde{x}_i + \epsilon), 1\} \right) < 0.$$

This means that the optimum solution to

$$\sup_{\boldsymbol{u} \geq 0} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_i(\tilde{x}_i + a), u_j \tilde{x}_j, 1\} \right)$$

must require $u_i \leq \bar{u}_i$ for all $a \in [0, \epsilon]$. Hence, for any $i \in N(\tilde{\boldsymbol{x}})$

$$\begin{aligned}
\beta((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) &\leq \lim_{a \downarrow 0} \beta((\tilde{x}_i + a, \tilde{\boldsymbol{x}}_{-i})) \\
&= \lim_{a \downarrow 0} \sup_{\boldsymbol{u} \geq 0} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_i(\tilde{x}_i + a), u_j \tilde{x}_j, 1\} \right) \\
&\leq \lim_{a \downarrow 0} \sup_{\boldsymbol{u} \geq 0} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_i \tilde{x}_i + a\bar{u}_i, u_j \tilde{x}_j, 1\} \right) \\
&\leq \lim_{a \downarrow 0} \left(\sup_{\boldsymbol{u} \geq 0} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_i \tilde{x}_i, u_j \tilde{x}_j, 1\} \right) + a\bar{u}_i \right) \\
&= \beta((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})).
\end{aligned}$$

■

We next show that although SAF is not an MOS criterion, it can be easily transformed to a DMOS criterion in the following result.

Theorem 3 *Suppose a function $\beta : \mathcal{X} \mapsto [0,1]$ that satisfies the following properties in Proposition 3, then the function $\rho : \mathcal{X} \mapsto [0,1]$*

$$\rho(\tilde{\boldsymbol{x}}) = \beta(\hat{\boldsymbol{x}})$$

where we replace the 'tilde' with 'hat' to denote

$$\hat{x}_i = \begin{cases} \tilde{x}_i & \text{if } i \in N(\tilde{\boldsymbol{x}}) \\ 1 & \text{otherwise,} \end{cases}$$

is a diversification favoring MOS criterion.

Proof :

1. **M**onotonicity: Observe that if $\tilde{\boldsymbol{y}} \geq \tilde{\boldsymbol{x}}$, then $N(\tilde{\boldsymbol{y}}) \subseteq N(\tilde{\boldsymbol{x}})$ and the results follows from monotonicity and strict attainment content properties of β.

2. **A**ttainment content: This trivially true from the definition of $N(\tilde{\boldsymbol{x}})$.

3. **N**on-abandonment: If there exists $i \in N$, $\tilde{x}_i < 0$, then $i \in N(\tilde{\boldsymbol{x}})$ and the result follows from the non-abandonment property of β.

4. **R**ight-continuity: Observe that there exists some small $\epsilon > 0$ such that the index set $N(\tilde{\boldsymbol{x}} + a\mathbf{1}) = N(\tilde{\boldsymbol{x}})$ for all $a \in [0, \epsilon]$. Hence, right continuity follows from the restricted right continuity property of β.

5. **C**omponentwise quasi-concavity: We need to show that

$$\rho((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \geq \min\{\rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})), \rho((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))\}$$

for all $\lambda \in [0,1]$, which is trivial for $\lambda \in \{0,1\}$. We hence consider $\lambda \in (0,1)$. Suppose, $\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i \geq 0$, then

$$\rho((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) = \beta((1, \hat{\boldsymbol{x}}_{-i}))$$

which is at least as large as $\beta((\hat{x}_i, \hat{\boldsymbol{x}}_{-i}))$ or $\beta((\hat{y}_i, \hat{\boldsymbol{x}}_{-i}))$. Suppose, $\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i \ngeq 0$, then we consider the two possible cases. The first case being $\tilde{x}_i, \tilde{y}_i \ngeq 0$, we have by quasiconcavity of β

$$\begin{aligned} & \rho((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \\ = & \ \beta((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \hat{\boldsymbol{x}}_{-i})) \\ \geq & \ \min\{\beta((\tilde{x}_i, \hat{\boldsymbol{x}}_{-i})), \beta((\tilde{y}_i, \hat{\boldsymbol{x}}_{-i}))\} \\ = & \ \min\{\rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})), \rho((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))\}. \end{aligned}$$

The next case being $\tilde{x}_i \geq 0$ and $\tilde{y}_i \ngeq 0$, in which case

$$\rho((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) = \beta((1, \hat{\boldsymbol{x}}_{-i})) \geq \beta((\tilde{y}_i, \hat{\boldsymbol{x}}_{-i})) = \rho((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})),$$

and it suffices to show that $\rho((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \geq \rho((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))$. Indeed, since $\tilde{x}_i \geq 0$, by monotonicity of β,

$$\begin{aligned} & \rho((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \\ = & \ \beta((\lambda\tilde{x}_i + (1-\lambda)\tilde{y}_i, \hat{\boldsymbol{x}}_{-i})) \\ \geq & \ \beta(((1-\lambda)\tilde{y}_i, \hat{\boldsymbol{x}}_{-i})) \\ = & \ \beta((\tilde{y}_i, \hat{\boldsymbol{x}}_{-i})) \\ = & \ \rho((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \end{aligned}$$

where the second last equality is due to the componentwise scale invariant property of β.

■

In view of Theorem 3, we define the shortfall-aware MOS (S-MOS) criterion as follows:

Definition 5 *The shortfall aware MOS (S-MOS) criterion, α, is defined as follows:*

$$\alpha(\tilde{\boldsymbol{x}}) = \sup_{\boldsymbol{u} \geq \boldsymbol{0}} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N(\tilde{\boldsymbol{x}})} \{u_i \tilde{x}_i, 1\} \right)$$

The corresponding optimization model utilizing the S-MOS criterion maximizes the shortfall-aware satisficing criterion. We will present some simple computational examples for such an optimization model in Chapter 5 to illustrate this. The practical implications of these qualities present in the S-MOS criterion, but absent in the probability and expected value based measures, would be also be shown in Chapter 5 with some case studies based on realistic examples from the finance, oil and gas, and the product development industries.

In considering ambiguity, computing the SAF, or the S-MOS criterion, requires the ability to evaluate the worst-case expectation over a family of distributions and then optimize it over the parameters, $\boldsymbol{u}$. Even if probability distributions of the uncertainty are known, computing the SAF requires the evaluation of expectation operators, which involves multidimensional integration and is generally computationally intractable; see for instance, Nemirovski and Shapiro [2006]. The tractability problem is further accentuated under distributional ambiguity, where computing the value of the SAF or the S-MOS criterion requires the ability to evaluate the worst-case expectation over a family of distributions and then optimize it over the parameters, $\boldsymbol{u}$. In the next section, we aim to provide a tractable approximation under distributional ambiguity. These tractable approximations can be cast as conic optimization or semi-definite programming problems. We also propose another method based on piecewise linear decisions rules (LDRs) [Chen et al., 2003, Goh and Sim, 2010, 2011] to derive these approximations which can also preserve the desirable characteristics of the DMOS family of criteria.

4.1.1 Tractable robust approximation of S-MOS criterion

If probability distributions of the uncertainty are known, computing the SAF requires the evaluation of expectation operators, which involves multidimensional integration and is generally computationally intractable; see for instance, Nemirovski and Shapiro [2006]. Nevertheless, optimizing the parameters, $\boldsymbol{u}$ in SAF can easily be posed as a standard stochastic optimization problem, which we can approximate using sample average approximation (SAA). However, under distributional ambiguity, computing the value of SAF requires the ability to evaluate the worst-case expectation over a family of distributions and then optimize it over the parameters, $\boldsymbol{u}$. The aim is to provide a tractable approximation of SAF under distributional ambiguity and we focus on the case where the uncertain target excess are affinely dependent on m given random variables or factors, $(\tilde{z}_1, \ldots, \tilde{z}_m)$ on Ω, i.e,

$$\mathcal{X} \subseteq \mathcal{A} = \left\{ \tilde{\boldsymbol{a}} : \exists a_i^j \in \Re, i \in N, j \in M \mid \tilde{a}_i(\omega) = a_i^0 + \sum_{j \in M} a_i^j \tilde{z}_j(\omega) \ \ \forall \omega \in \Omega, i \in N, j \in M \right\},$$

where $M = \{1, \ldots, m\}$. The factor model of uncertainty is common in robust optimization literature in which the descriptive statistics of these factors are specified and form the basis for characterizing the family of distributions, $\mathbb{F}$; see for instance, Ben-Tal and Nemirovski [1998], Chen et al. [2003], Goh and Sim [2010]. In this section, we will focus on $\tilde{\boldsymbol{z}}$ being described by its support $\mathcal{W}$, mean $\boldsymbol{\mu}$ and covariance, $\boldsymbol{\Sigma}$ so that the family of distributions is given as follows:

$$\mathbb{F}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathcal{W}) = \{\mathbb{P} : \mathbb{E}_{\mathbb{P}}(\tilde{\boldsymbol{z}}) = \boldsymbol{\mu}, \mathbb{E}_{\mathbb{P}}(\tilde{\boldsymbol{z}}\tilde{\boldsymbol{z}}') = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}', \mathbb{P}(\tilde{\boldsymbol{z}} \in \mathcal{W}) = 1\}.$$

For computational reasons, we assume that the support $\mathcal{W}$ is a tractable conic representable, closed, convex and bounded set with non empty interior as proposed by Ben-Tal and Nemirovski [1998]. In order to compute the index set $N(\tilde{\boldsymbol{x}})$ as required for S-MOS, we assume the mean $\boldsymbol{\mu}$ and covariance, $\boldsymbol{\Sigma}$ do not further constrain the support set $\mathcal{W}$ so that for every $\boldsymbol{z} \in \mathcal{W}$ and $\delta > 0$ there exists $\mathbb{P} \in \mathbb{F}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathcal{W})$ such that $\mathbb{P}(\tilde{\boldsymbol{z}} \in \{\boldsymbol{z} + \boldsymbol{\epsilon} \ : \ ||\epsilon||_2 \leq \delta\}) > 0$. Hence, $y + \boldsymbol{x}'\tilde{\boldsymbol{z}} \geq 0$ if and only if $y + \min\{\boldsymbol{x}'\boldsymbol{z} \ : \ \boldsymbol{z} \in \mathcal{W}\} \geq 0$, which can easily be computed under our assumption of support set, $\mathcal{W}$.

Under the factor model of uncertainty, the computational tractability for evaluating the worst-case expectation over a family of distributions depends on the specification of distributional ambiguity. Unfortunately, for any $\tilde{\boldsymbol{a}} \in \mathcal{A}$, $b \in \Re$ the problem

$$\inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{\tilde{a}_i, b\} \right) \tag{2}$$

is generally NP-hard, as in the case when the family of distributions $\mathbb{F}$ is given by $\mathbb{F}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathcal{W})$; see Murty and Kabadi [1987]. Nevertheless, there are tractable examples as follows:

Proposition 4 *Let*

$$\mathbb{F}(\boldsymbol{\mu}, \mathcal{W}) = \{\mathbb{P} : \mathbb{E}_{\mathbb{P}}(\tilde{\boldsymbol{z}}) = \boldsymbol{\mu}, \mathbb{P}(\{\tilde{\boldsymbol{z}} \in \mathcal{W}\}) = 1\}$$

and

$$\mathbb{F}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \{\mathbb{P} : \mathbb{E}_{\mathbb{P}}(\tilde{\boldsymbol{z}}) = \boldsymbol{\mu}, \mathbb{E}_{\mathbb{P}}(\tilde{\boldsymbol{z}}\tilde{\boldsymbol{z}}') = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}'\}.$$

Then, for any $\tilde{\boldsymbol{a}} \in \mathcal{A}$, *i.e.,* $\tilde{a}_i = a_i^0 + \boldsymbol{a}_i'\tilde{\boldsymbol{z}}$ *for some* $a_i^0 \in \Re$, $\boldsymbol{a}_i \in \Re^m$, $i \in N$, *we have*

$$\pi_A(\tilde{\boldsymbol{a}}, b) = \inf_{\mathbb{P} \in \mathbb{F}(\boldsymbol{\mu}, \mathcal{W})} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{\tilde{a}_i, b\} \right)$$

$$\pi_B(\tilde{\boldsymbol{a}}, b) = \inf_{\mathbb{P} \in \mathbb{F}(\boldsymbol{\mu}, \boldsymbol{\Sigma})} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{\tilde{a}_i, b\} \right)$$

where the functions $\pi_A : \mathcal{A} \times \Re \mapsto \Re$ *and* $\pi_B : \mathcal{A} \times \Re \mapsto \Re$ *are given by:*

$$\begin{aligned} \pi_A(\tilde{\boldsymbol{a}}, b) = \sup \quad & v + \boldsymbol{\mu}'\boldsymbol{r} \\ s.t. \quad & v + \max_{\boldsymbol{z} \in \mathcal{W}} (\boldsymbol{r} - \boldsymbol{a}_i)'\boldsymbol{z} \leq a_i^0 \quad i \in N \\ & v + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{r}'\boldsymbol{z} \leq b \\ & v \in \Re, \boldsymbol{r} \in \Re^m. \end{aligned}$$

and

$$\begin{aligned} \pi_B(\tilde{\boldsymbol{a}}, b) = \sup \quad & s + \boldsymbol{t}'\boldsymbol{\mu} + tr(\boldsymbol{S}(\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}')) \\ s.t. \quad & \begin{pmatrix} a_i^0 - s & \frac{1}{2}(\boldsymbol{a}_i - \boldsymbol{t})' \\ \frac{1}{2}(\boldsymbol{a}_i - \boldsymbol{t}) & -\boldsymbol{S} \end{pmatrix} \in \mathbb{S}_+^{m+1} \quad \forall i \in N \\ & \begin{pmatrix} b - s & -\frac{1}{2}\boldsymbol{t}' \\ -\frac{1}{2}\boldsymbol{t} & -\boldsymbol{S} \end{pmatrix} \in \mathbb{S}_+^{m+1} \\ & s \in \Re, \boldsymbol{S} \in \mathbb{S}^m, \end{aligned}$$

where $\mathbb{S}^m$ *(respectively,* $\mathbb{S}_+^m$*) denotes the set of symmetric matrices (respectively, symmetric positive semidefinite matrices) in* $\Re^{m \times m}$*.*

Proof :

Applying the strong duality results of Isii [1963], the problem

$$\begin{aligned} \inf_{\mathbb{P}} \quad & \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{a_i^0 + \boldsymbol{a}_i'\tilde{\boldsymbol{z}}, b\} \right) \\ \text{s.t.} \quad & \mathbb{E}_{\mathbb{P}}(\tilde{\boldsymbol{z}}) = \boldsymbol{\mu} \\ & \mathbb{P}(\{\tilde{\boldsymbol{z}} \in \mathcal{W}\}) = 1 \end{aligned}$$

achieves the same optimal objective as

$$\begin{aligned} \sup \quad & v + \boldsymbol{\mu}'\boldsymbol{r} \\ \text{s.t.} \quad & v + \boldsymbol{r}'\boldsymbol{z} \geq \min_{i \in N} \{a_i^0 + \boldsymbol{a}_i'\boldsymbol{z}, b\} \quad \forall \boldsymbol{z} \in \mathcal{W} \\ & v \in \Re, \boldsymbol{r} \in \Re^m. \end{aligned}$$

Hence, the first result follows. Similarly, for the second result, we also apply the strong duality results to obtain the following dual problem

$$
\begin{aligned}
\sup \quad & s + \boldsymbol{t}'\boldsymbol{\mu} + \mathrm{tr}(\boldsymbol{S}(\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}')) \\
\text{s.t.} \quad & s + \boldsymbol{t}'\boldsymbol{z} + \boldsymbol{z}'\boldsymbol{S}\boldsymbol{z} \le a_i^0 + \boldsymbol{a}_i'\boldsymbol{z} \;\; \forall \boldsymbol{z} \in \Re^m \quad i \in N \\
& s + \boldsymbol{t}'\boldsymbol{z} + \boldsymbol{z}'\boldsymbol{S}\boldsymbol{z} \le b \;\; \forall \boldsymbol{z} \in \Re^m \\
& s \in \Re, \boldsymbol{t} \in \Re^m.
\end{aligned}
$$

Observe that each of the $i \in N$ constraint is equivalent to

$$
\begin{pmatrix} 1 \\ \boldsymbol{z} \end{pmatrix}' \begin{pmatrix} s - a_i^0 & \frac{1}{2}(\boldsymbol{t} - \boldsymbol{a}_i)' \\ \frac{1}{2}(\boldsymbol{t} - \boldsymbol{a}_i) & \boldsymbol{S} \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{z} \end{pmatrix} \le 0 \; \forall \boldsymbol{z} \in \Re^m
$$

which can be trivially cast as a positive semi-definite conic constraint. Hence, the result follows.
■

Remark : The optimization problem to evaluate π_A requires the explicit formation of the constraints $v + \max_{\boldsymbol{z} \in \mathcal{W}} (\boldsymbol{r} - \boldsymbol{a}_i)'\boldsymbol{z} \le a_i^0$, $i \in N$ and $v + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{r}'\boldsymbol{z} \le b$ which are known as the *robust counterparts* in the context of robust optimization. Again, under our assumption of support set $\mathcal{W}$ the robust counterparts can be transformed to a small set of tractable conic constraints. We refer interested readers to Ben-Tal and Nemirovski [1998], Ghaoui et al. [2003], Bertsimas and Sim [2004].

Observe that the functions π_A and π_B are concave functions in their inputs and the corresponding SAFs are as follows:

$$
\begin{aligned}
\beta_A\left(x_1^0 + \boldsymbol{x}_1'\tilde{\boldsymbol{z}}, \ldots, x_n^0 + \boldsymbol{x}_n'\tilde{\boldsymbol{z}}\right) = \sup \quad & v + \boldsymbol{\mu}'\boldsymbol{r} \\
\text{s.t.} \quad & v + \max_{\boldsymbol{z} \in \mathcal{W}} (\boldsymbol{r} - u_i \boldsymbol{x}_i)'\boldsymbol{z} \le u_i x_i^0 \quad i \in N \\
& v + \max_{\boldsymbol{z} \in \mathcal{W}} \boldsymbol{r}'\boldsymbol{z} \le 1 \\
& v \in \Re, \boldsymbol{r} \in \Re^m.
\end{aligned}
$$

and

$$\begin{aligned}\beta_B\left(x_1^0+\boldsymbol{x}_1'\tilde{\boldsymbol{z}},\ldots,x_n^0+\boldsymbol{x}_n'\tilde{\boldsymbol{z}}\right)=\sup\quad & s+\boldsymbol{t}'\boldsymbol{\mu}+\operatorname{tr}(\boldsymbol{S}(\boldsymbol{\Sigma}+\boldsymbol{\mu}\boldsymbol{\mu}'))\\ \text{s.t.}\quad & \begin{pmatrix} u_i x_i^0 - s & \frac{1}{2}(u_i\boldsymbol{x}_i-\boldsymbol{t})' \\ \frac{1}{2}(u_i\boldsymbol{x}_i-\boldsymbol{t}) & -\boldsymbol{S}\end{pmatrix}\in\mathbb{S}_+^{m+1}\quad \forall i\in N\\ & \begin{pmatrix} 1-s & -\frac{1}{2}\boldsymbol{t}' \\ -\frac{1}{2}\boldsymbol{t} & -\boldsymbol{S}\end{pmatrix}\in\mathbb{S}_+^{m+1}\\ & \boldsymbol{u}\geq\boldsymbol{0}\\ & \boldsymbol{u}\in\Re^n, s\in\Re, \boldsymbol{S}\in\mathbb{S}^m,\end{aligned}$$

which are tractable conic optimization problems.

Interestingly, although $\mathbb{F}(\boldsymbol{\mu},\boldsymbol{\Sigma},\mathcal{W})$ is the intersection of the families $\mathbb{F}(\boldsymbol{\mu},\mathcal{W})$ and $\mathbb{F}(\boldsymbol{\mu},\boldsymbol{\Sigma})$, the corresponding SAF remains intractable. We therefore propose a robust approximation for this case. Our aim is to develop an approximate SAF, $\bar{\beta}:\mathcal{A}\mapsto[0,1]$ that satisfies the properties of Proposition 3, so that the $\bar{\beta}$ can be transformed to a DMOS criterion via Theorem 3. Indeed, this depends on the properties of the function, π that we use to approximate Problem (2).

Theorem 4 *Let $\pi:\mathcal{A}\times\Re\mapsto\Re$ be a positively homogeneous(with degree 1), tractable and concave function such that*

1. *For all $\tilde{\boldsymbol{a}}\in\mathcal{A}$, $b\in\Re$, $\pi(\tilde{\boldsymbol{a}},b)\leq\inf_{\mathbb{P}\in\mathbb{F}}\mathbb{E}_{\mathbb{P}}\left(\min_{i\in N}\{\tilde{a}_i,b\}\right)$*
2. *For all $(\boldsymbol{a},b)$, $\boldsymbol{a}\in\Re^n$ and $b\in\Re$, $\pi(\boldsymbol{a},b)=\min_{i\in\{1,\ldots,n\}}\{a_i,b\}$.*
3. *For all $\tilde{\boldsymbol{a}}\geq\boldsymbol{0}$, $\tilde{\boldsymbol{a}}\in\mathcal{A}$ and $b\geq 0$, $\pi(\tilde{\boldsymbol{a}},b)\geq 0$*

The following function, $\bar{\beta}:\mathcal{A}\mapsto\Re$

$$\bar{\beta}(\tilde{\boldsymbol{x}})=\sup_{\boldsymbol{u}\geq\boldsymbol{0}}\pi(u_1\tilde{x}_1,\ldots,u_n\tilde{x}_n,1)$$

satisfies the properties of Proposition 3. Moreover, for all $\tilde{\boldsymbol{x}}\in\mathcal{A}$,

$$\bar{\beta}(\tilde{\boldsymbol{x}})\leq\beta(\tilde{\boldsymbol{x}}).$$

Proof : Positive homogeneity with degree 1 for function π implies $\pi(k\tilde{\boldsymbol{a}}) = k\pi(\tilde{\boldsymbol{a}})$ for $k \in \Re$. The first property of the π function, leads to bound $\bar{\beta}(\tilde{\boldsymbol{x}}) \leq \beta(\tilde{\boldsymbol{x}})$ for all $\tilde{\boldsymbol{x}} \in \mathcal{A}$. We go through the properties of Proposition 3 as follows:

1. **Monotonicity**: For any $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}} \in \mathcal{A}$, such that $\tilde{\boldsymbol{x}} \geq \tilde{\boldsymbol{y}}$, we will show that $\bar{\beta}(\tilde{\boldsymbol{x}}) \geq \bar{\beta}(\tilde{\boldsymbol{y}})$. For this, we show that the function π abides by monotonicity, where $\pi(\tilde{\boldsymbol{x}}, a) \geq \pi(\tilde{\boldsymbol{y}}, b)$ for $\tilde{\boldsymbol{x}} \geq \tilde{\boldsymbol{y}}$, $a \geq b$ as follows:

$$\begin{aligned}
\pi(\tilde{\boldsymbol{x}}, a) &= \pi(\tilde{\boldsymbol{y}} + (\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{y}}), b + (a - b)) \\
&\geq \pi(\tilde{\boldsymbol{y}}, b) + \underbrace{\pi((\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{y}}), (a - b))}_{\geq 0} \\
&\geq \pi(\tilde{\boldsymbol{y}}, b).
\end{aligned}$$

The first inequality is due to the fact that π is concave and positively homogeneous, hence, it is also superadditive. The second inequality is due to Property 3.

2. **Strict attainment content**: We first show that for $\tilde{\boldsymbol{x}} \in \mathcal{A}$, if there exists $\tilde{x}_i > 0$, then $\bar{\beta}((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) = \bar{\beta}((1, \tilde{\boldsymbol{x}}_{-i}))$. Suppose $\tilde{x}_i > 0$, $i \in N$, then there exists $\tilde{x}_i \geq \epsilon > 0$ for some ϵ. By monotonicity, we have the following:

$$\sup_{\boldsymbol{u} \geq 0} \lim_{r \uparrow \infty} \pi((u_i r, \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1) \geq \sup_{\boldsymbol{u} \geq 0} \pi((u_i \tilde{x}_i, \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1) \geq \sup_{\boldsymbol{u} \geq 0} \pi((u_i \epsilon, \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1)$$

where "$\circ$" represents the elementwise (or Hadamard) product. Observe in the above that the upper and lower bound for $\sup_{\boldsymbol{u} \geq 0} \pi((u_i \tilde{x}_i, \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1)$ are equivalent for any $\epsilon > 0$, i.e.

$$\sup_{\boldsymbol{u} \geq 0} \lim_{r \uparrow \infty} \pi((u_i r, \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1) = \sup_{\boldsymbol{u} \geq 0} \pi((u_i \epsilon, \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1)$$

This implies that $\bar{\beta}(\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}) = \bar{\beta}(\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})$ for all $\tilde{y}_i > 0$. For the case when $\tilde{\boldsymbol{x}} > 0$, we observe that

$$\sup_{\boldsymbol{u} \geq 0} \lim_{r \uparrow \infty} \pi(\boldsymbol{u} r, 1) \geq \sup_{\boldsymbol{u} \geq 0} \pi(u_1 \tilde{x}_1, \ldots, u_n \tilde{x}_n, 1) \geq \sup_{\boldsymbol{u} \geq 0} \pi(\boldsymbol{u} \epsilon, 1)$$

for some ϵ satisfying $\tilde{\boldsymbol{x}} \geq \epsilon \mathbf{1} > \mathbf{0}$. Since $\pi(\boldsymbol{u} r, 1) = \min_{i \in N} \{u_i r, 1\}$ and $\pi(\boldsymbol{u} \epsilon, 1) = \min_{i \in N} \{u_i \epsilon, 1\}$ and that $\boldsymbol{u} \geq \mathbf{0}$ can be arbitrarily large, we must have

$$\bar{\beta}(\tilde{\boldsymbol{x}}) = \sup_{\boldsymbol{u} \geq 0} \pi(u_1 \tilde{x}_1, \ldots, u_n \tilde{x}_n, 1) = 1.$$

Hence, $\bar{\beta}(\mathbf{1}) = 1$.

3. **Non-abandonment**: For any $\tilde{\boldsymbol{x}} \in \mathcal{A}$ such that $\tilde{x}_i < 0$,, we have from Property 1

$$\bar{\beta}(\tilde{\boldsymbol{x}}) \leq \sup_{\boldsymbol{u} \geq \boldsymbol{0}} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_i \tilde{x}_i, u_j \tilde{x}_j, 1\} \right) \leq \sup_{\boldsymbol{u} \geq \boldsymbol{0}} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_i 0, u_j \tilde{x}_j, 1\} \right) = 0.$$

Moreover,

$$\bar{\beta}(\tilde{\boldsymbol{x}}) = \sup_{\boldsymbol{u} \geq \boldsymbol{0}} \pi(u_1 \tilde{x}_1, \ldots, u_n \tilde{x}_n, 1) \geq \pi(0 \tilde{x}_1, \ldots, 0 \tilde{x}_n, 1) = \min\{0, 1\} = 0.$$

Hence, $\bar{\beta}(\tilde{\boldsymbol{x}}) = 0$.

4. **Componentwise quasiconcavity**: Let $\bar{\beta}^* = \min\{\bar{\beta}((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})), \bar{\beta}((\tilde{y}_i, \tilde{\boldsymbol{x}}_{-i}))\}$. We will show that for any $\lambda \in [0, 1]$,

$$\bar{\beta}((\lambda \tilde{x}_i + (1 - \lambda) \tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \geq \bar{\beta}^*$$

This is clearly the case for $\lambda \in \{0, 1\}$. We will focus on $\lambda \in (0, 1)$. Note that $\bar{\beta}((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) \geq \bar{\beta}^*$ means that for any $\epsilon > 0$, there exists $\boldsymbol{u}, \boldsymbol{v} > 0$ such that

$$\pi((u_i \tilde{x}_i, \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1) \geq \bar{\beta}^* - \epsilon.$$

and

$$\pi((v_i \tilde{y}_i, \boldsymbol{v}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1) \geq \bar{\beta}^* - \epsilon.$$

Let $\gamma = \lambda v_i / (u_i (1 - \lambda) + \lambda v_i)$ and $p_j = \gamma u_j + (1 - \gamma) v_j$ for all $j \in N$. Observe that $\gamma \in (0, 1)$ and $p_j > 0$ for all $j \in N$. Moreover,

$$p_i (\lambda \tilde{x}_i + (1 - \lambda) \tilde{y}_i) = \gamma u_i \tilde{x}_i + (1 - \gamma) v_i \tilde{y}_i$$

and

$$p_j \tilde{z}_j = \gamma u_j \tilde{z}_j + (1 - \gamma) v_j \tilde{z}_j \qquad \forall j \in N_i.$$

Using these transformation of variables and noting that π being a concave function, we have

$$\begin{aligned}
& \pi((p_i (\lambda \tilde{x}_i + (1 - \lambda) v_j \tilde{y}_i, \boldsymbol{p}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1) \\
= \; & \pi((\gamma u_i \tilde{x}_i + (1 - \gamma) v_i \tilde{y}_i, \gamma \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i} + (1 - \gamma) \boldsymbol{v}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1) \\
\geq \; & \gamma \pi((u_i \tilde{x}_i, \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1) + (1 - \gamma) \pi((v_i \tilde{y}_i, \boldsymbol{v}_{-i} \circ \tilde{\boldsymbol{x}}_{-i}), 1) \\
\geq \; & \bar{\beta}^* - \epsilon.
\end{aligned}$$

Hence, $\bar{\beta}((\lambda \tilde{x}_i + (1 - \lambda) \tilde{y}_i, \tilde{\boldsymbol{x}}_{-i})) \geq \bar{\beta}^*$.

5. **Componentwise scale invariant**: The result is obvious since the multiplier, u_i lies in a cone.

6. **Restricted right continuity**: For any $i \in N(\tilde{\boldsymbol{x}})$ there exists $\mathbb{P} \in \mathbb{F}$ such that $\mathbb{P}(\tilde{x}_i < 0) > 0$ and so $\mathbb{P}(\tilde{x}_i + \epsilon < 0) > 0$ for some $\epsilon > 0$. Hence, for all $a \in [0, \epsilon]$
$$\lim_{u_i \uparrow \infty} \pi((u_i(\tilde{x}_i + a), \boldsymbol{u}_{-i} \circ \tilde{\boldsymbol{x}}_{-i})) \leq \lim_{u_i \uparrow \infty} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{j \in N_i} \{u_i(\tilde{x}_i + a), u_j \tilde{x}_j, 1\} \right) = -\infty.$$
Following the proof of right continuity of Proposition 3, there exists $\bar{u}_i > 0$ such that the optimum solution
$$\sup_{\boldsymbol{u} \geq \boldsymbol{0}} \pi(u_1 \tilde{x}_1, \ldots, u_n \tilde{x}_n, 1)$$
must require $u_i \leq \bar{u}_i$ for all $a \in [0, \epsilon]$. Hence, for any $i \in N(\tilde{\boldsymbol{x}})$,
$$\begin{aligned}
\bar{\beta}((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})) &\leq \lim_{a \downarrow 0} \bar{\beta}((\tilde{x}_i + a, \tilde{\boldsymbol{x}}_{-i})) \\
&= \lim_{a \downarrow 0} \sup_{\boldsymbol{u} \geq \boldsymbol{0}} \pi(u_1 \tilde{x}_1, \ldots, u_i(\tilde{x}_i + a), \ldots, u_n \tilde{x}_n) \\
&\leq \lim_{a \downarrow 0} \sup_{\boldsymbol{u} \geq \boldsymbol{0}} \pi(u_1 \tilde{x}_1, \ldots, u_i \tilde{x}_i + \bar{u}_i a, \ldots, u_n \tilde{x}_n) \\
&\leq \lim_{a \downarrow 0} \sup_{\boldsymbol{u} \geq \boldsymbol{0}} \left(\pi(u_1 \tilde{x}_1, \ldots, u_i \tilde{x}_i, \ldots, u_n \tilde{x}_n) - \pi(0, \ldots, -a\bar{u}_i, \ldots, 0) \right) \\
&= \lim_{a \downarrow 0} \sup_{\boldsymbol{u} \geq \boldsymbol{0}} \left(\pi(u_1 \tilde{x}_1, \ldots, u_i \tilde{x}_i, \ldots, u_n \tilde{x}_n) + a\bar{u}_i \right) \\
&= \bar{\beta}((\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i})),
\end{aligned}$$
where the last inequality is due to the superadditivity of the π function.

■

We can easily see that the functions π_A and π_B are positively homogenous, concave and satisfy Properties 1 and 2 of Theorem 4. Moreover, when the family of distribution is given by $\mathbb{F}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathcal{W})$, then if $\tilde{\boldsymbol{a}} \geq 0$, we have $\mathbb{P}(\tilde{\boldsymbol{a}} \geq \boldsymbol{0}) = 1$ for all $\mathbb{P} \in \mathbb{F}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathcal{W})$. Since we assume that $\boldsymbol{\Sigma}$ does not further constrain the support set $\mathcal{W}$, it also implies that $\mathbb{P}(\tilde{\boldsymbol{a}} \geq \boldsymbol{0}) = 1$ for all $\mathbb{P} \in \mathbb{F}(\boldsymbol{\mu}, \mathcal{W})$.Therefore if $b \geq 0$,
$$\pi_A(\tilde{\boldsymbol{a}}, b) = \inf_{\mathbb{P} \in \mathbb{F}(\boldsymbol{\mu}, \mathcal{W})} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{\tilde{a}_i, b\} \right) \geq 0$$
and the function also satisfies Property 3. On the other hand, since π_B does not take into account of the support $\mathcal{W}$, it will not necessarily satisfy Property 3. The following result shows how we

can use these functions to obtain a tighter approximation of Problem (2) while preserving the desirable properties of Theorem 4.

Theorem 5 *Let L be an index set, and the functions $\pi_l : \mathcal{A} \times \Re \mapsto \Re$, $l \in L$ are positively homogeneous, tractable and concave that satisfy Properties 1 and 2 of Theorem 4. Moreover, there exists $l^* \in L$ such that π_{l^*} also satisfy Property 3 of Theorem 4. Let the function $\pi : \mathcal{A} \times \Re \mapsto \Re$ be the supremal convolution of π_l, $l \in L$ given by*

$$\begin{array}{rlr} \pi(a_1^0 + \boldsymbol{a}_1'\tilde{\boldsymbol{z}}, \ldots, a_n^0 + \boldsymbol{a}_n'\tilde{\boldsymbol{z}}, b) = \sup & \displaystyle\sum_{l \in L} \pi_l(a_{1l}^0 + \boldsymbol{a}_{1l}'\tilde{\boldsymbol{z}}, \ldots, a_{nl}^0 + \boldsymbol{a}_{nl}'\tilde{\boldsymbol{z}}, b_l) & \\ s.t. & \displaystyle\sum_{l \in L} \boldsymbol{a}_{il} = \boldsymbol{a}_i, & i \in N \\ & \displaystyle\sum_{l \in L} a_{il}^0 = a_i^0, & i \in N \\ & \displaystyle\sum_{l \in L} b_l = b & \\ & b_l, a_{il}^0 \in \Re, \boldsymbol{a}_{il} \in \Re^m & i \in N, l \in L. \end{array} \tag{3}$$

Then π is also a positively homogeneous, tractable and concave function that satisfies the properties in Theorem 4. Moreover, for all $l \in L$, $\tilde{\boldsymbol{a}} \in \mathcal{A}$, $b \in \Re$.

$$\pi_l(\tilde{\boldsymbol{a}}, b) \leq \pi(\tilde{\boldsymbol{a}}, b).$$

Proof : Since π is the supremal convolution of tractable, positively homogeneous and concave functions, it is also a tractable, positively homogeneous and concave function [Rockafellar, 1970]. To show that π is an improved bound, observe that for any $k \in L$, $z \in \mathcal{W}$,

$$\pi(\tilde{\boldsymbol{a}}, b) \geq \pi_k(a_1^0 + \boldsymbol{a}_1'\tilde{\boldsymbol{z}}, \ldots, a_n^0 + \boldsymbol{a}_n'\tilde{\boldsymbol{z}}, b) + \sum_{l \in L \backslash \{k\}} \underbrace{\pi_l(0 + \boldsymbol{0}'\tilde{\boldsymbol{z}}, \ldots, 0 + \boldsymbol{0}'\tilde{\boldsymbol{z}}, 0)}_{= \min\limits_{i \in \{1, \ldots, n+1\}} \{0\} = 0} = \pi_k(\tilde{\boldsymbol{a}}, b).$$

Let $b_l, a_{il}^0 \in \Re, \boldsymbol{a}_{il} \in \Re^m$, $i \in N, l \in L$ be a feasible solution to Problem (3). Observe that

$$\begin{aligned} \sum_{l \in L} \pi_l(a_{1l}^0 + \boldsymbol{a}_{1l}'\tilde{\boldsymbol{z}}, \ldots, a_{nl}^0 + \boldsymbol{a}_{nl}'\tilde{\boldsymbol{z}}, b_l) &\leq \sum_{l \in L} \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{a_{il}^0 + \boldsymbol{a}_{il}'\tilde{\boldsymbol{z}}, b_l\} \right) \\ &\leq \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \left\{ \sum_{l \in L} (a_{il}^0 + \boldsymbol{a}_{il}'\tilde{\boldsymbol{z}}), \sum_{l \in L} b_l \right\} \right) \\ &= \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N} \{a_i^0 + \boldsymbol{a}_i'\tilde{\boldsymbol{z}}, b\} \right), \end{aligned}$$

where the first inequality is obvious due to Property 1 of π_l in Theorem 4, and the second inequality is valid from noting that $\inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{i\in N}\{\tilde{a}_i, b\}\right)$ is a concave and positively homogeneous function of $(\tilde{\boldsymbol{a}}, b)$ and hence, superadditive. Therefore, for all $\tilde{\boldsymbol{a}} \in \mathcal{A}$, $b \in \Re$, $l \in L$

$$\pi_l(\tilde{\boldsymbol{a}}, b) \leq \pi(\tilde{\boldsymbol{a}}, b) \leq \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{i\in N}\{\tilde{a}_i, b\}\right). \tag{4}$$

Furthermore, for all $\boldsymbol{a} \in \Re^{n+1}$, $l \in L$, we have

$$\min_{i\in\{1,\ldots,n+1\}}\{a_i\} = \pi_l(\boldsymbol{a}) \leq \pi(\boldsymbol{a}) \leq \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{i\in\{1,\ldots,n+1\}}\{a_i\}\right) = \min_{i\in\{1,\ldots,n+1\}}\{a_i\}$$

Hence, $\pi(\boldsymbol{a}) = \min_{i\in\{1,\ldots,n+1\}}\{a_i\}$. Finally, we note that for all $\tilde{\boldsymbol{a}} \geq \boldsymbol{0}$, $\tilde{\boldsymbol{a}} \in \mathcal{A}$ and $b \geq 0$,

$$\pi(\tilde{\boldsymbol{a}}, b) \geq \pi_{l^*}(\tilde{\boldsymbol{a}}, b) \geq 0.$$

■

Using both π_A and π_B in Theorem 3, the improved bound is given explicitly as

$$\begin{array}{rlll}
\pi_{AB}\left(a_1^0 + \boldsymbol{a}_1'\tilde{\boldsymbol{z}}, \ldots, a_n^0 + \boldsymbol{a}_n'\tilde{\boldsymbol{z}}, b\right) = \sup & v + \boldsymbol{\mu}'\boldsymbol{r} + s + \boldsymbol{t}'\boldsymbol{\mu} + \mathrm{tr}(\boldsymbol{S}(\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}')) & & \\
\text{s.t.} & v + \max_{\boldsymbol{z}\in\mathcal{W}}(\boldsymbol{r} - \boldsymbol{a}_{iA})'\boldsymbol{z} \leq a_{iA}^0 & & i \in N \\
& v + \max_{\boldsymbol{z}\in\mathcal{W}} \boldsymbol{r}'\boldsymbol{z} \leq b_A & & \\
& \begin{pmatrix} a_{iB}^0 - s & \frac{1}{2}(\boldsymbol{a}_{iB} - \boldsymbol{t})' \\ \frac{1}{2}(\boldsymbol{a}_{iB} - \boldsymbol{t}) & -\boldsymbol{S} \end{pmatrix} \in \mathbb{S}_+^{m+1} & & \forall i \in N(\tilde{\boldsymbol{x}}) \\
& \begin{pmatrix} b_B - s & -\frac{1}{2}\boldsymbol{t}' \\ -\frac{1}{2}\boldsymbol{t} & -\boldsymbol{S} \end{pmatrix} \in \mathbb{S}_+^{m+1} & & \\
& \boldsymbol{a}_{iA} + \boldsymbol{a}_{iB} = \boldsymbol{a}_i, & & i \in N \\
& a_{iA}^0 + a_{iB}^0 = a_i^0, & & i \in N \\
& b_A + b_B = b & & \\
& s \in \Re, \boldsymbol{S} \in \mathbb{S}^m & & \\
& v \in \Re, \boldsymbol{r} \in \Re^m & & \\
& b_A, b_B, a_{iA}^0, a_{iB}^0 \in \Re, \boldsymbol{a}_{iA}, \boldsymbol{a}_{iB} \in \Re^m & & i \in N
\end{array}$$

which is also a tractable positive semidefinite optimization problem.

4.1.1.1 Robust approximations of Goh and Sim [2010, 2011]

We now look at another way to obtain an approximation of Problem (2) from the perspective of distributional robust optimization described in Goh and Sim [2010, 2011] as follows:

$$
\begin{aligned}
\pi_{\mathcal{M}}\left(a_1^0 + \boldsymbol{a}_1'\tilde{\boldsymbol{z}}, \ldots, a_n^0 + \boldsymbol{a}_n'\tilde{\boldsymbol{z}}, b\right) = \sup \quad & \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}(y(\tilde{\boldsymbol{z}})) \\
\text{s.t.} \quad & y(\tilde{\boldsymbol{z}}) \le b \\
& y(\tilde{\boldsymbol{z}}) \le a_i^0 + \boldsymbol{a}_i'\tilde{\boldsymbol{z}} \quad i \in N \\
& y \in \mathcal{M}
\end{aligned}
\tag{5}
$$

where $\mathcal{M}$ is a space of measurable functions on $\Re^m$. Problem (5) is a well-studied problem in recent progress on robust optimization; see for instance, Ben-Tal et al. [2004], Chen et al. [2003], Goh and Sim [2010] and Kuhn et al. [2010]. Such a formulation using linear decision rules can also be used to model the S-MOS based optimization problem under distributional ambiguity. Although it is typically an intractable problem, it is easy to verify that when y is restricted to affine functions of $\tilde{\boldsymbol{z}}$, i.e.,

$$\mathcal{L} = \{y : \Re^m \mapsto \Re \ : \ \exists v \in \Re, \boldsymbol{r} \in \Re^m \mid y(\boldsymbol{z}) = v + \boldsymbol{r}'\boldsymbol{z}\},$$

then the function $\pi_{\mathcal{L}}$ is exactly the same as π_A. When incorporating covariance of $\tilde{\boldsymbol{z}}$, Goh and Sim [2010] show that we can obtain a tighter approximation by restricting y to a space of piecewise linear functions $\mathcal{P}$, such as deflected linear decisions rule, segregated linear decisions rule or combinations of these rules. Using any of these approaches, we have for all $\tilde{\boldsymbol{a}} \in \mathcal{A}$, $b \in \Re$,

$$\pi_{\mathcal{L}}(\tilde{\boldsymbol{a}}, b) \le \pi_{\mathcal{P}}(\tilde{\boldsymbol{a}}, b) \le \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{i\in N}\{\tilde{a}_i, b\}\right).$$

Furthermore, for all $\boldsymbol{a} \in \Re^{n+1}$

$$\min_{i\in\{1,\ldots,n+1\}}\{a_i\} = \pi_{\mathcal{L}}(\boldsymbol{a}) \le \pi_{\mathcal{P}}(\boldsymbol{a}) \le \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{i\in\{1,\ldots,n+1\}}\{a_i\}\right) = \min_{i\in\{1,\ldots,n+1\}}\{a_i\}.$$

Also, from the above, for all $\tilde{\boldsymbol{a}} \in \mathcal{A}$ and $b \in \Re$, if $\tilde{\boldsymbol{a}} \ge \boldsymbol{0}$, $b \ge 0$, then

$$\pi_{\mathcal{P}}(\tilde{\boldsymbol{a}}, b) \ge \pi_{\mathcal{L}}(\tilde{\boldsymbol{a}}, b) = \pi_A(\tilde{\boldsymbol{a}}, b) \ge 0.$$

Hence, $\pi_{\mathcal{P}}$ satisfies all the desirable properties of Theorem 4. Although, $\pi_{\mathcal{P}}$ may not be as tight as π_{AB}, the resultant model is an attractive second-order conic optimization problem, which has tremendous computational advantage over positive semidefinite optimization problem. Moreover, Goh and Sim [2010] provides a generic framework that we could exploit to extend the description of distributions beyond $\mathbb{F}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathcal{W})$. For instance, it could incorporate stochastically independent factors, directional deviations of Chen et al. [2007] and partitioned statistics of Natarajan et al. [2010a], which are means of incorporating distributional asymmetry that are not captured by mean or covariance. The ROME algebraic toolbox provides a convenient platform for the implementation of these approaches.

4.1.2 Optimizing over S-MOS criterion

In the previous section, we show how to evaluate S-MOS criterion for a given $\tilde{\boldsymbol{x}} \in \mathcal{X}$. Our goal is to maximize the S-MOS criterion by solving the following optimization problem:

$$
\begin{aligned}
Z^* = \quad & \sup \quad \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N(\tilde{\boldsymbol{x}})} \{u_i \tilde{x}_i, 1\} \right) \\
& \text{s.t.} \quad \boldsymbol{u} \geq \mathbf{0} \\
& \qquad\;\; \tilde{\boldsymbol{x}} \in \mathcal{X}.
\end{aligned}
\tag{6}
$$

For the optimization problem to be interesting, we assume that $Z^* \in (0, 1)$. If $Z^* = 0$, then the targets are overly ambitious with respect to the S-MOS criterion. On the other hand if $Z^* = 1$, then there exists $\tilde{\boldsymbol{x}} \in \mathcal{X}$ such that $\tilde{\boldsymbol{x}} \geq \mathbf{0}$, which we can determine its existence without having to solve Problem (6).

Unfortunately, Problem (6) is not a convex optimization problem and the global optimum solution is generally hard to obtain. Our modest goal is to provide a strategy for improving the solutions by solving as a sequence of optimization subproblems. Although we can easily extend this to optimizing the approximate S-MOS based on $\bar{\beta}$ developed in the previous section, for ease of exposition, we will focus on the exact model and assume that we can obtain the optimal solutions to the subproblems as follows:

$$
\begin{aligned}
\alpha(\tilde{\boldsymbol{x}}) = \quad & \max \quad \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N(\tilde{\boldsymbol{x}})} \{u_i \tilde{x}_i, 1\} \right) \\
& \text{s.t.} \quad \boldsymbol{u} \; \geq 0,
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
Z(\boldsymbol{u}, I) = \quad & \max \quad \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i\in I} \{u_i \tilde{x}_i, 1\} \right) \\
& \text{s.t.} \quad \tilde{x}_i \geq 0, \quad i \in N \setminus I \\
& \qquad\;\; \tilde{\boldsymbol{x}} \in \mathcal{X}.
\end{aligned}
\tag{8}
$$

In solving Problem (7), we observe that for any $i \in N(\tilde{\boldsymbol{x}})$, we have $\tilde{x}_i \not\geq 0$. Hence, the corresponding solution, u_i must be finite. We also denote the index set $I \subseteq N$ in (8).

We next present our strategy for solving Problem (6). The algorithm assumes that we can find an initial feasible solution $\tilde{\boldsymbol{x}} \in \mathcal{X}$ such that $\alpha(\tilde{\boldsymbol{x}}) > 0$. In practice, this initial solution can be a benchmark that we would like to improve upon in terms of achieving better satisficing performance with respect to the S-MOS criterion. Alternatively, we can generate random $\boldsymbol{u}$ and solve Problem (8) with $I = N$, until a solution $\tilde{\boldsymbol{x}}$ with $\alpha(\tilde{\boldsymbol{x}}) > 0$ is found.

Algorithm 1

Input: $\tilde{\boldsymbol{x}}$: $\alpha(\tilde{\boldsymbol{x}}) > 0$.

1. Solve Problem (7) with Input $\tilde{\boldsymbol{x}}$. Obtain optimal solution $\boldsymbol{u}^*$. Set $I := N(\tilde{\boldsymbol{x}})$
2. Solve Problem (8) with Input $(\boldsymbol{u}^*, I)$. Obtain optimal solution $\tilde{\boldsymbol{x}}^*$. Set $\tilde{\boldsymbol{x}} := \tilde{\boldsymbol{x}}^*$.
3. Repeat Steps 1 and 2 until a termination criterion is met.

Theorem 6 *Given $\tilde{\boldsymbol{x}}$, let $\boldsymbol{u}^*$ be an optimal solution to Problem (7) and $\boldsymbol{x}^*$ be the optimum solution to Problem (8) in which $I = N(\tilde{\boldsymbol{x}})$ and $\boldsymbol{u} = \boldsymbol{u}^*$. Then*

$$\alpha(\tilde{\boldsymbol{x}}^*) \geq Z(\boldsymbol{u}^*, I) \geq \alpha(\tilde{\boldsymbol{x}}).$$

Proof : Observe that $\tilde{\boldsymbol{x}}$ is a feasible solution to Problem (8) in which $I = N(\tilde{\boldsymbol{x}})$ and $\boldsymbol{u} = \boldsymbol{u}^*$ and the objective is $\alpha(\tilde{\boldsymbol{x}})$. Hence,

$$Z(\boldsymbol{u}^*, I) \geq \alpha(\tilde{\boldsymbol{x}}).$$

Moreover, due to the constraints $\tilde{x}_i \geq 0$ for all $i \in N \setminus I$ in Problem (8), the corresponding optimal solution $\tilde{\boldsymbol{x}}^*$ will have $N(\tilde{\boldsymbol{x}}^*) \subseteq I$. Hence,

$$\alpha(\tilde{\boldsymbol{x}}^*) \geq \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i\in N(\tilde{\boldsymbol{x}}^*)} \{u_i^* \tilde{x}_i^*, 1\} \right) \geq \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i\in I} \{u_i^* \tilde{x}_i^*, 1\} \right) = Z(\boldsymbol{u}^*, I) \geq \alpha(\tilde{\boldsymbol{x}}).$$

$\blacksquare$

Theorem 6 suggests that in Algorithm 1, the sequence of solutions $\{\tilde{\boldsymbol{x}}^k\}$ have non-decreasing objectives, $\alpha(\tilde{\boldsymbol{x}}^k)$. Hence, $\alpha(\tilde{\boldsymbol{x}}^k) \uparrow \delta \leq 1$ in the limit. Suppose the set $\mathcal{X}$ is closed, convex, bounded and has nonempty interior, then we can replace Problem (8) with the following convex optimization problem:

$$
\begin{aligned}
Z_T(\boldsymbol{u}, I) = \quad & \max \quad \inf_{\mathbb{P} \in \mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in I} \{u_i \tilde{y}_i, 1\} \right) \\
& \text{s.t.} \quad \tilde{y}_i \geq 0, \quad i \in N \setminus I \\
& \qquad\;\; (a, \tilde{\boldsymbol{y}}) \in \mathrm{cl}(\bar{\mathcal{X}})
\end{aligned} \tag{9}
$$

where

$$
\bar{\mathcal{X}} = \{(a, \tilde{\boldsymbol{y}}) \ : \ \tilde{\boldsymbol{y}}/a \in \mathcal{X}, a > 0\}
$$

is a also a convex set and cl denotes closure of the set. We then consider the following solution algorithm.

Algorithm 2

Input: $\tilde{\boldsymbol{x}} : \alpha(\tilde{\boldsymbol{x}}) > 0$.

1. Solve Problem (7) with Input $\tilde{\boldsymbol{x}}$. Obtain optimal solution $\boldsymbol{u}^*$. Set $I := N(\tilde{\boldsymbol{x}})$

2. Solve Problem (9) with Input $(\boldsymbol{u}^*, I)$. Obtain optimal solution $(a^*, \tilde{\boldsymbol{y}}^*)$. Set $\tilde{\boldsymbol{x}} := \tilde{\boldsymbol{y}}^*/a^*$.

3. Repeat Steps 1 and 2 until a termination criterion is met.

Our next result suggests that Algorithm 2 has the potential to outperform Algorithm 1.

Theorem 7 *Given $(\boldsymbol{u}, I)$ such that $Z_T(\boldsymbol{u}, I) > 0$. Let $(a^*, \tilde{\boldsymbol{y}}^*)$ be the optimal solutions to Problem (9). Then $a^* > 0$ and*

$$
\alpha(\tilde{\boldsymbol{y}}^*/a^*) \geq Z_T(\boldsymbol{u}, I) \geq Z(\boldsymbol{u}, I).
$$

Moreover, for the case of single objective, the optimum solution of Problem (6) can be obtained directly by solving Problem (9) with inputs $u_1 = 1$ and $I = N = \{1\}$.

Proof : We first observe that Problem (9) can be reformulated as

$$\begin{aligned} Z_T(\boldsymbol{u}, I) = \quad & \max \quad \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{i\in I}\{u_i a\tilde{x}_i, 1\}\right) \\ & \text{s.t.} \quad a\tilde{x}_i \geq 0, \quad i \in N\setminus I \\ & \qquad\ \ a \geq 0 \\ & \qquad\ \ \tilde{\boldsymbol{x}} \in \mathcal{X}, \end{aligned} \tag{10}$$

in which its optimum solution, $\tilde{\boldsymbol{x}}^\dagger = \tilde{\boldsymbol{y}}^*/a^*$, assuming $a^* > 0$. Indeed, since $Z_T(\boldsymbol{u}, I) > 0$, the solution a of Problem (10) cannot approach zero. If a approaches infinity, then we would require $\tilde{\boldsymbol{x}} \geq 0$, which we have assumed is infeasible in the set $\mathcal{X}$. Hence, at optimality, a is finite and strictly positive. Clearly, for the case of single objective, Problem (10) is exactly the same as Problem (6) in which $u_1 = 1$ and $I = N = \{1\}$. Observe that $\tilde{\boldsymbol{x}}^*$, the optimum solution of Problem (9), and $a = 1$ are feasible in Problem (10). Therefore,

$$Z_T(\boldsymbol{u}, I) \geq Z(\boldsymbol{u}, I).$$

Finally, due to the constraints $\tilde{x}_i \geq 0$ for all $i \in N\setminus I$ in Problem (10), the corresponding optimal solution, $\tilde{\boldsymbol{x}}^\dagger$ will be such that $N(\tilde{\boldsymbol{x}}^\dagger) \subseteq I$. Hence,

$$\alpha(\tilde{\boldsymbol{y}}^*/a^*) = \alpha(\tilde{\boldsymbol{x}}^\dagger) \geq \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{i\in N(\tilde{\boldsymbol{x}}^\dagger)}\left\{u_i a^* \tilde{x}_i^\dagger, 1\right\}\right) \geq \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}}\left(\min_{i\in I}\left\{u_i a^* \tilde{x}_i^\dagger, 1\right\}\right) = Z_T(\boldsymbol{u}, I) \geq Z(\boldsymbol{u}, I).$$

■

Remark 1. Theorems 6 and 7 implies that $\alpha(\tilde{\boldsymbol{y}}^*/a^*)$ has a higher lower bound than $\alpha(\tilde{\boldsymbol{x}}^*)$, where $\tilde{\boldsymbol{x}}^*$ is the solution of Problem (8). However, this does not necessarily imply that $\alpha(\tilde{\boldsymbol{y}}^*/a^*) \geq \alpha(\tilde{\boldsymbol{x}}^*)$. Nevertheless, our computational studies in the next section suggests that Algorithm 2 has better performance.

Remark 2. For the case of single objective, Chen and Sim [2009] propose a binary approach to obtain the optimum solution. The result in Theorem 7 shows that this can be solved as a single convex optimization problem.

For the purpose of illustration, we consider a MOS criterion optimization problem in which the feasible solution, $\boldsymbol{x}$ lies in the polytope $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$. Also, for a given $\boldsymbol{x}$, we assume the uncertain objectives are affinely dependent on the uncertainties and given by

$$(\tilde{\boldsymbol{p}}_1'\boldsymbol{x} + \tilde{q}_1, \ldots, \tilde{\boldsymbol{p}}_n'\boldsymbol{x} + \tilde{q}_n).$$

Let $\boldsymbol{\tau}$ be the target. The explicit formulation of Problem (9) is therefore

$$\begin{aligned} Z(\boldsymbol{u}, I) = \quad \max \quad & \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i\in I} \left\{ u_i \left(\tilde{\boldsymbol{p}}_i'\boldsymbol{x} + \tilde{q}_i - \tau_i \right), 1 \right\} \right) \\ \text{s.t.} \quad & \tilde{\boldsymbol{p}}_i'\boldsymbol{x} + \tilde{q}_i \geq \tau_i, \quad i \in N \setminus I \\ & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \end{aligned}$$

which is a convex optimization problem with respect to $\boldsymbol{x}$. Likewise, the explicit formulation of Problem (10) is as follows:

$$\begin{aligned} Z_T(\boldsymbol{u}, I) = \quad \max \quad & \inf_{\mathbb{P}\in\mathbb{F}} \mathbb{E}_{\mathbb{P}} \left(\min_{i\in I} \left\{ u_i \left(\tilde{\boldsymbol{p}}_i'\boldsymbol{y} + a\tilde{q}_i - a\tau_i \right), 1 \right\} \right) \\ \text{s.t.} \quad & \tilde{\boldsymbol{p}}_i'\boldsymbol{y} + a\tilde{q}_i \geq \tau_i a, \quad i \in N \setminus I \\ & \boldsymbol{A}\boldsymbol{y} \leq \boldsymbol{b}a \\ & a \geq 0. \end{aligned}$$

which is also a convex optimization problem with respect to the variables a and $\boldsymbol{y}$. We use these formulations in our computational studies for optimizing the S-MOS criterion. We solve the problem approximately using deflected linear decision rules [Goh and Sim, 2010] which is a straightforward implementation in ROME. Here, the problem is reformulated as a two-stage stochastic programming problem by the definition of a second stage decision variable, $\tilde{d} = \min_{i\in I} \left\{ u_i \left(\tilde{\boldsymbol{p}}_i'\boldsymbol{y} + a\tilde{q}_i - a\tau_i \right), 1 \right\}$. This variable is then restricted to the space of piecewise linear functions with appropriate penalization of constraint violations in the objective function. For known distributions, we can use sampling average approximation (SAA), which is a standard approach in stochastic programming (see for instance, Birge and Louveaux [1997], Ruszczynski and Shapiro [2003]). The SAA approach utilizes a random sample to estimate the decisions with an approximate objective function averaged over these samples.

4.2 Chapter summary

In this chapter, the shortfall-aware MOS (S-MOS) criterion is proposed. S-MOS criterion is able to address the deficiencies of probability measures not being able to account for the magnitude of shortfalls and yet able to incorporate more realistic diversification preferences, for decision making under uncertainty. The S-MOS criterion is a lower bound to success probability and in the presence of distributional ambiguity, we present the techniques for building tractable approximations while preserving the salient properties of MOS criteria. The ability to handle events with extreme consequences by taking into due considerations the magnitude of shortfalls, yet retaining computational tractability, naturally endows the S-MOS criterion to be sensitive to rare events that can potentially result in severe consequences. In many situations, the full characterization of the model uncertainties may not be available, and only limited distributional information, such as the mean, covariance and supports, are known. We describe such situations as distributionally ambiguous and proposed the S-MOS criterion such that it would be able to handle the distributional ambiguity.

These approximations for the S-MOS under distributional ambiguity can be implemented either based on the supremal convolution of tractable, positively homogeneous and concave functions or the distributional robust optimization framework recently proposed by Goh and Sim [2010, 2011]. Some tractable conic optimization or semidefinite problem formulations are shown. The approximations can also be evaluated based on the distributional robust optimization framework recently proposed by Goh and Sim [2010, 2011]. Since maximizing the S-MOS criterion is not a convex optimization, we propose improvement algorithms via solving sequences of convex optimization problems.

CHAPTER 5

Numerical Studies on S-MOS Criterion

In this chapter, we present an array of numerical studies utilizing the shortfall-aware MOS (S-MOS) criterion presented in the Chapter 4. Here,we assume full distributional information, whereas in the next chapter, we extend the numerical studies to S-MOS based problems which considers ambiguity where only limited distributional information are available. We compare the numerical results from implementing the S-MOS criterion with existing target-based measures such as the success probability and the Markowitz mean-variance models [Markowitz, 1952].

We first illustrate the numerical characteristics of the S-MOS criterion and compare it with the characteristics of success probability under different random shortfall assumptions. As mentioned in the preceding, the S-MOS criterion mimics the characteristics of the success probability, yet is able to overcome some of the more glaring deficiencies of the probability criterion, such as its inability to account of the degree of shortfall. Subsequently, we illustrate the implementation of the S-MOS criterion in optimization through the following two practical case studies:

- The case study in liability management demonstrates the ability of the S-MOS criterion in respecting diversification preferences in the portfolio allocation decisions and to cushion against very bad shortfalls against the targets. The ability of the S-MOS criterion in deriving decisions that possess these characteristics is demonstrated through a comparison with results from the Markowitz mean-variance model for asset allocation [Markowitz, 1952].
- The case study in engineering product development for design selection amongst competing alternative of product design under high quality manufacturing environment demonstrates the applicability of the S-MOS criterion for the ranking of discrete alternatives. This case

study also shows the advantage of S-MOS criterion over traditional expected utility models and MAUT models in that the use of a target-based satisficing approach do not require the difficult tasks of having to estimate the risk tolerance parameters necessary when using utility based models. The difficulties in accurately estimating these risk tolerance parameters could severely undermine the ability of the utility based models in ranking potential design or process solutions under high quality manufacturing and product development processes. Since risk tolerance parameters are not required in using the S-MOS criterion, these measures will not encounter such issues posed by these difficulties.

The optimization of these models were performed using the modeling toolbox ROME version 1.0.8 [Goh and Sim, 2011], designed to solve robust optimization problems in the MATLAB environment, together with CPLEX 11.2 solver engine. The computations were performed with an Intel Core 2 Duo CPU, T9600 at 2.80 Ghz processor with 4GB RAM system.

5.1 Numerical characteristics of the S-MOS criterion

To illustrate the sensitivity of the S-MOS criterion measure towards the structure of uncertainties, we first assume a two-point distribution for the target excess such that each target excess $\tilde{x}_i$ has the following distribution:

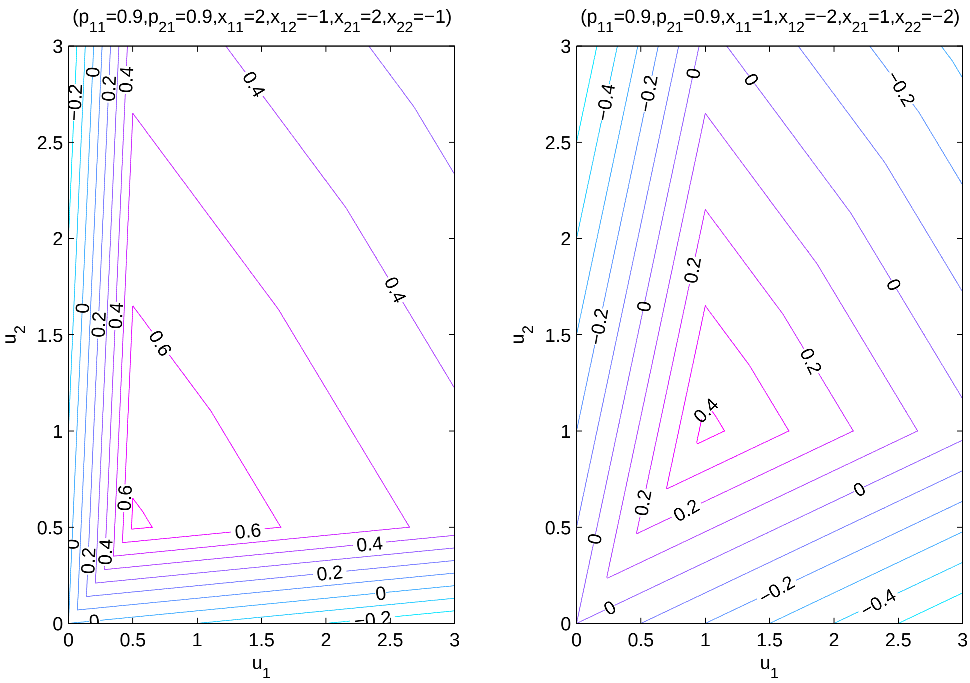
$$\mathbb{P}(\tilde{x}_i = x) = \begin{cases} \kappa_i, & x = a_i \\ 1 - \kappa_i, & x = b_i \end{cases}$$

When $a_i < 0$ the worst-case outcome of target excess i is a shortfall against the target. Also, $\kappa_i \in [0, 1]$ models the corresponding probability of shortfall. For a specified distribution modelling the uncertainties, the S-MOS criterion can be evaluated using a generated set of random samples S via sample average approximation (SAA). Under distributional ambiguity, a tractable robust approximation of S-MOS can be evaluated using deflected linear decision rules in Problem (5) in Chapter 4. In the experiments, unless otherwise specified, a sample size $|S| = 50,000$ is used for the solving the SAA models.

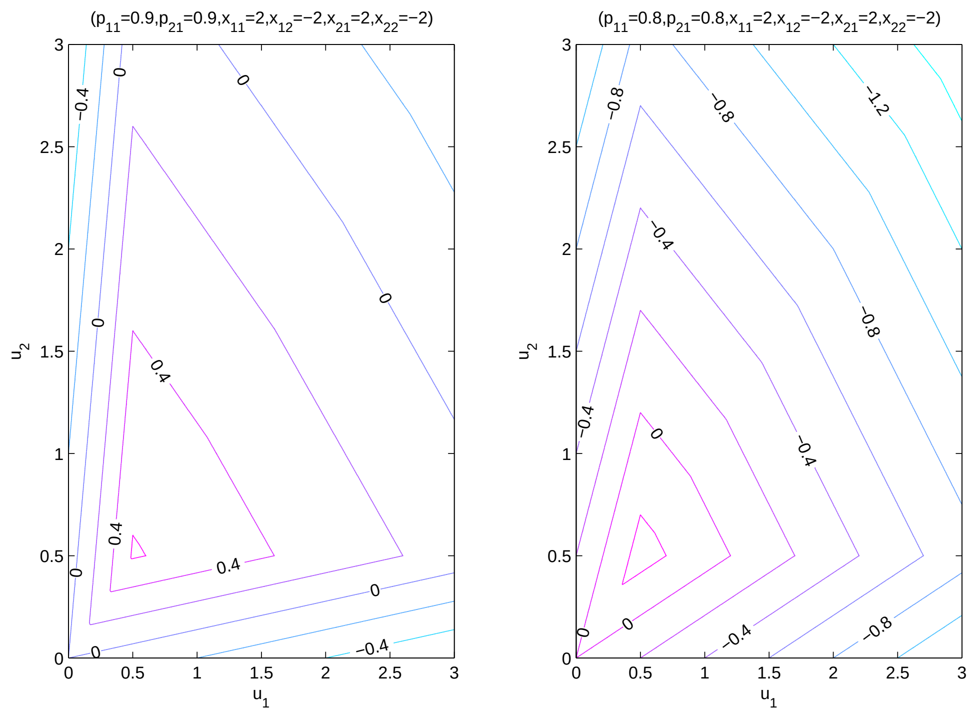
Using the above discrete two-point distribution, we examine the characteristics of S-MOS criterion over different structures of uncertainty where, following Definition 5, the S-MOS criterion for a specific distribution is defined as follows:

$$\alpha(\tilde{\boldsymbol{x}}) = \sup_{\boldsymbol{u} \geq \mathbf{0}} \mathbb{E}\left(\min_{i \in N(\tilde{\boldsymbol{x}})} \{u_i \tilde{x}_i, 1\}\right)$$

We first illustrate the characteristics of S-MOS criterion over the $\boldsymbol{u}$ parameter for target excess, $\tilde{\boldsymbol{x}}$, Note that in the evaluation of S-MOS criterion, the supremum over the parameter vector $\boldsymbol{u}$ of the expectation of a minimum function is evaluated. In order to gain insights over how this supremum is evaluated, variation of the S-MOS criterion against $\boldsymbol{u}$ under different distributional assumptions are plotted for a basic two objectives case via a contour plot as shown in Figure 5.1.



(a) Sensitivity to Degree of Shortfall



(b) Sensitivity to Probability of Shortfall

Figure 5.1: Equi-contour plots of S-MOS under different probabilities of target achievements (p_{11},p_{21}) and degrees of shortfalls (x_{12},x_{22})

From the equi-contour plots, it can be observed that the S-MOS criterion at optimal $\boldsymbol{u}$ is sensitive to both the extremity of the shortfalls and the probability of shortfalls occurring. As the degree of shortfalls and the probability of these shortfalls occurring increases, the optimal S-MOS criterion decreases. The normalization vector, $\boldsymbol{u}$, at the optimal point also tends to change as the optimization tends to allocate higher weights in an attempt to offset the larger potential degree of shortfall in order to achieve the maximum over the minimum weighted shortfalls. In addition, it can be observed that the optimization surface is componentwise concave across the range of values over which $\boldsymbol{u}$ is varied over. In fact, in the general sense, the S-MOS criterion is quasi-concave over each individual objective as described in Chapter 4.

Apart from looking at the optimization surface over the $\boldsymbol{u}$ vector, we examine how the degree of shortfalls and probabilities of these shortfalls occurring impact the S-MOS criterion using a variability chart. A variability chart based on a factorial design of the different factors for such an optimization is plotted. This is shown in Figures 5.2 and 5.3. Here, p_{11} and p_{21} denote the probabilities of achieving the targets whereas p_{12} and p_{22} represent the probabilities of violations. Also, x_{11} and x_{21} represents the extreme values in achieving the targets whereas x_{12} and x_{22} represents the extreme worst case violations of targets. These are the factors that were varied over high and low values described in Table 5.1.

Factor	Low Level	High Level
p_{11}	0.8	0.9
p_{21}	0.8	0.9
x_{11}	1	2
x_{12}	-2	-1
x_{21}	1	2
x_{22}	-2	-1

Table 5.1: Factor levels for the full 2^6 factorial design

For these numerical studies, as it is combinatorially impractical to evaluate the exact expectation of the minimum function over random variables, the evaluation of the S-MOS criterion over the range of $\boldsymbol{u} \geq 0$ is based on solving the following model via sample average approximation over the set of sample S instead:

$$\begin{aligned} \max \quad & \sum_{s \in S} \frac{w_s}{|S|} \\ \text{s.t.} \quad & u_i x_i^s \leq w_s \quad \forall i \in N(\tilde{\boldsymbol{x}}), s \in S \\ & u_i x_i^s \leq 1 \quad\ \ \forall i \in N(\tilde{\boldsymbol{x}}), s \in S \end{aligned} \tag{1}$$

where x_i^s is the s^{th} sample realization of the i^{th} uncertain objective and w_s is an auxiliary variable to obtain a linear SAA formulation.

The variability plots across the parameter setting shown in Table 5.1 are shown in Figures 5.2 and 5.3 for the cases where the probabilities of target achievement in the second objective are 0.8 and 0.9 respectively.

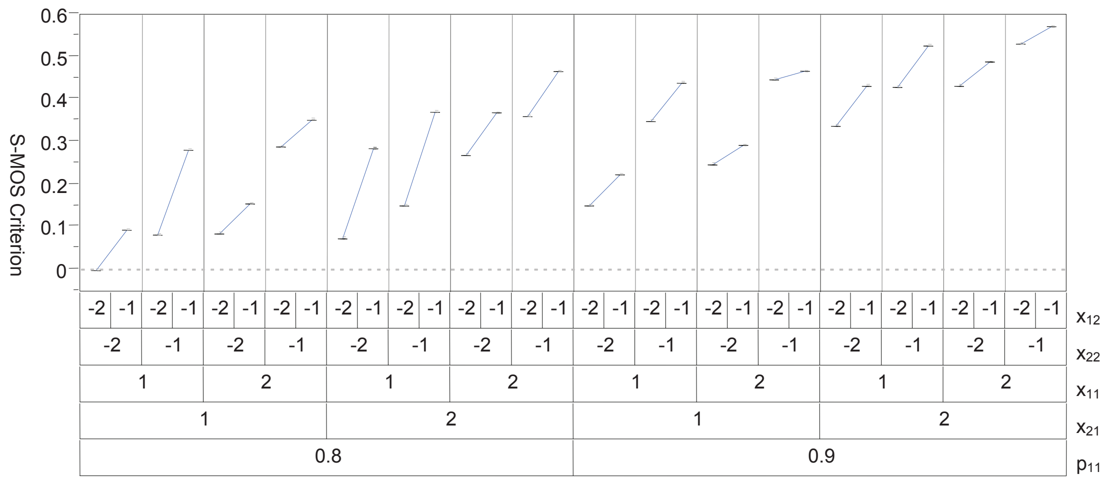


Figure 5.2: Sensitivity of the S-MOS criterion under different probability of target achievements and degree of shortfall ($p_{21} = 0.8$)

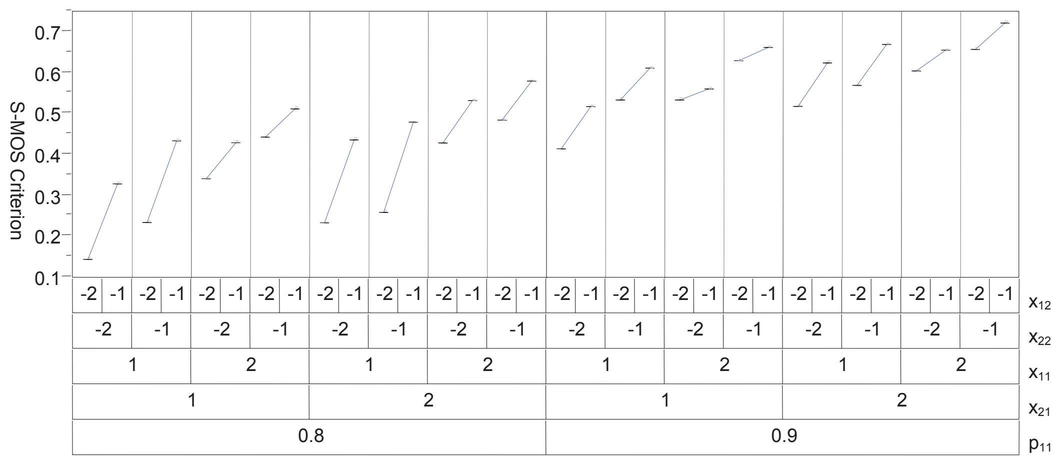


Figure 5.3: Sensitivity of the S-MOS criterion under different probability of target achievements and degree of shortfall ($p_{21} = 0.9$)

The variability charts clearly shows the sensitivity of the S-MOS criterion to levels of shortfalls under different distributions in shortfall probabilities. Both increasing the severity of shortfalls and its probabilities tend to reduce the S-MOS criterion. Also, there appears to be no significant interaction effects in the impact of both the degree and probabilities of shortfalls on the S-MOS criterion. All these are consistent with the theoretical developments underpinning that of a DMOS and overcomes the key deficiency of the probability measure in not being able to take the extremity of shortfalls into consideration. Furthermore, the quasi-concave characteristics imbue the S-MOS criterion with the propensity of being a criterion that do not have to assume risk-neutrality and is able to incorporate diversification preferences as a reasonable decision behavior under uncertainty.

Apart from the earlier basic description of the S-MOS criterion under limited stylized probability distributions of random shortfalls, we look more in-depth into how the S-MOS criterion compares with the success probability across several dimensions, thereby establishing the unique advantages of the S-MOS criterion over the success probability criterion for ranking alternatives. The dimensions that we examine are as follows:

1. Degree of shortfall in target achievement (Figure 5.4), and,

2. Frequency of shortfall in target achievement (Figure 5.5).

In the experiments, unless otherwise specified, we consider four targets to be satisficed. A sample size $|S| = 50,000$ is used for the solving the SAA models.

For the effect of degree of shortfall on the S-MOS criterion, three of the target excess were distributed over a minimum of -0.1 and maximum of 2 and one of the target excess has the worst case shortfall ranging from -2 to 1. Negative values represent positive shortfall and vice versa. Hence, -2 represent 2 units of shortfall and $0-1$ represents no shortfall. There is a 30% chance of shortfall for all the criteria. In assessing the effect of the frequency of shortfalls on the S-MOS criterion, three of the target excess were distributed over a minimum of -0.1 and maximum of 2. The other target excess has a worst case shortfall of -0.5 and -1. For these two cases the probability of achieving the minimum shortfall was varied over the entire range of probabilities. In these cases, we assume all the random target excess are iid uniform.

The S-MOS criterion can be considered a generalization to success probability as it allows the consideration of the level of "shortfall" in target achievement. In the experiments, we demonstrate clearly how the S-MOS criterion compares with success probability across the different dimensions, and where they differ. The success probability, $\mathbb{P}(\tilde{\boldsymbol{x}} \geq 0)$, and the S-MOS criterion are evaluated and compared in the numerical experiments.

In Figure 5.4, we observe that as the degree of violations increases towards the left on the horizontal axis, the S-MOS criterion tends to decrease. In contrast, the success probability criterion remains insensitive to this degree of violation. Both the S-MOS criterion and the success probability criterion are insensitive to the extent of the achievement of targets when targets can be achieved (no violations).

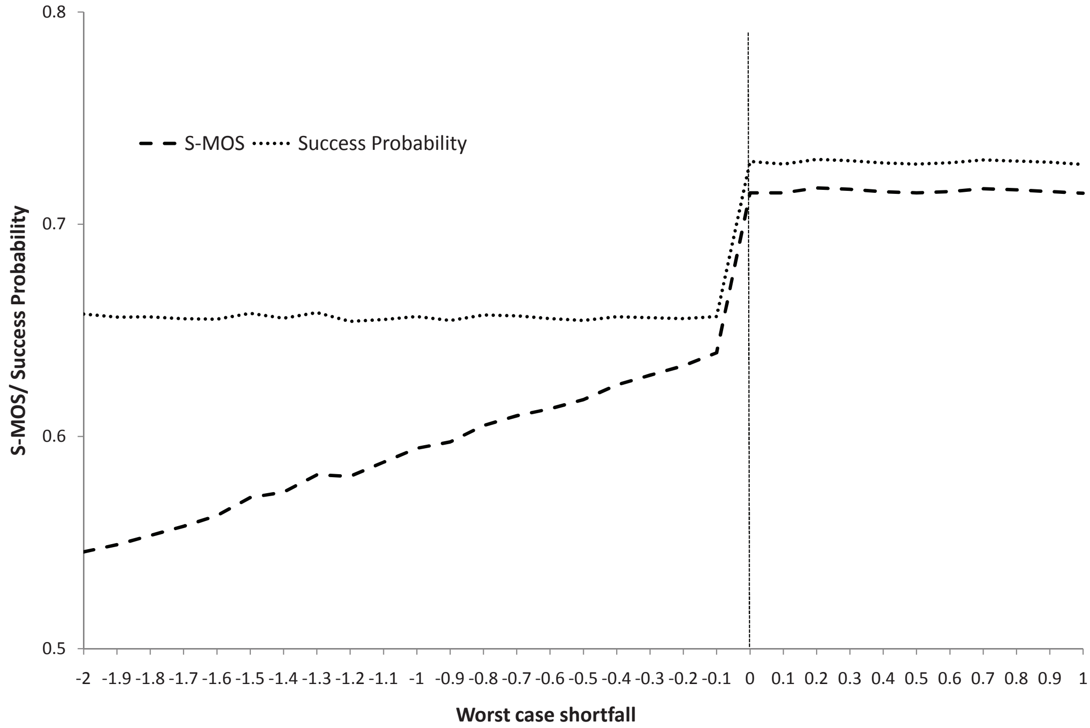


Figure 5.4: Characteristics of S-MOS criterion - Degree of maximum shortfall in target Excess

Figure 5.5 shows the effect on the S-MOS and the success probability criterion when we change the frequency of shortfall occurrence. The success probability criterion effectively tracks the change in the frequency of shortfall occurrence. In addition, the following can be observed here:

1. The S-MOS criterion is effectively a lower bound to the success probability criterion.

2. The rate of change of S-MOS criterion with respect to the change in the frequency of shortfall occurrence increases when the degree of maximum shortfall is higher, whereas the rate of change in the success probability criterion is insensitive to the degree of shortfall

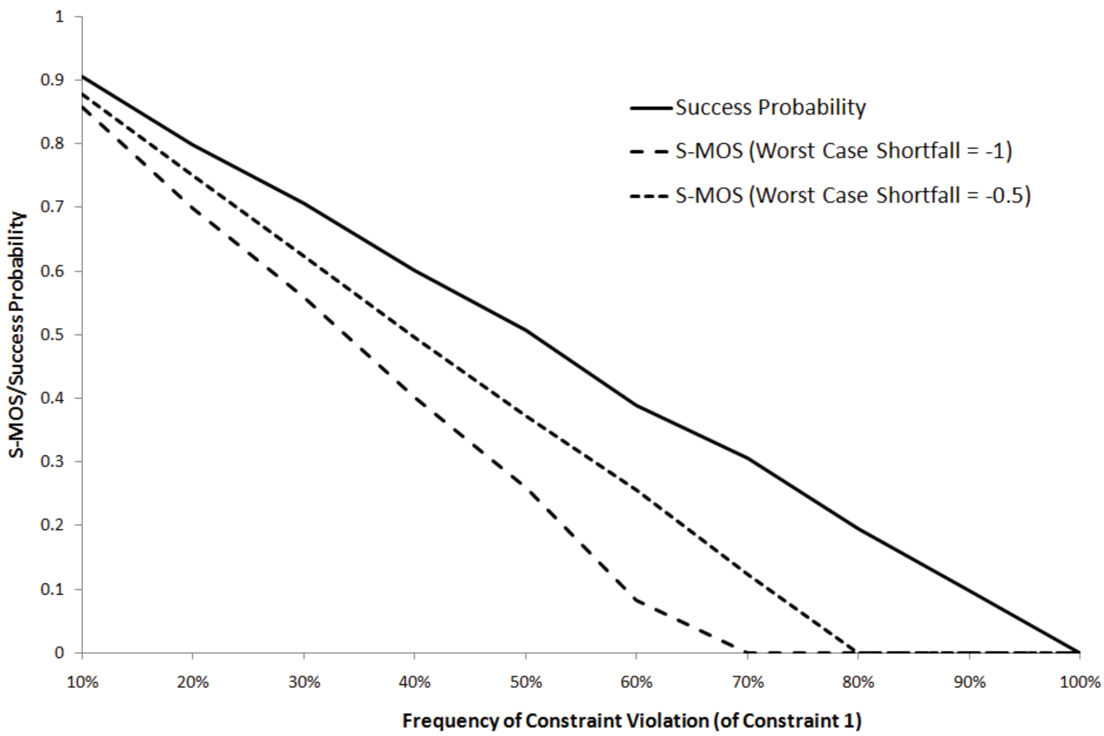


Figure 5.5: Characteristics of S-MOS criterion - Frequency of shortfalls

5.2 Characteristics of S-MOS criterion based optimization: Case study in portfolio liability management

In the previous section, we demonstrated several salient characteristics of the S-MOS criterion and also showed how it compares with success probability. Here we show the behavior and performance of the S-MOS criterion in realistic optimization problems. We undertake this endeavour in a practical setting through the application of the multiple objectives satisficing framework for portfolio optimization against a competing benchmark. In certain portfolio management problems, such as *liability management*, the benchmark can be naturally stated as the long term liabilities that has to be managed.

Through the numerical studies based on the liability management problem, we attempt to demonstrate the diversification characteristics of the decisions and the ability of these optimal decisions in cushioning against bad shortfalls in the achievement of targets. The comparisons would be made against the decisions derived from a traditional Markowitz mean-variance model based on the minimization of the portfolio variance [Markowitz, 1952].

Consider a firm which is looking to create an optimal portfolio mix based on a specific amount of initial capital c for the management of some future liabilities which may be realized over different time horizons in the future. The firm has at their disposal, the option to in a basket of different assets (including possibly some risk-free assets). Hence, the decisions to be made by the firm is to decide how much to allocate to each of the asset in order to effectively manage the future liabilities over the different time horizons.

The single period liability management problem based on a satisficing criterion has been studied in Brown and Sim [2009]. In the liability management problem over multiple time horizons, a multiple objectives satisficing framework (where each objectives relate to one time horizon) is very natural with the targets being the firm's liability amounts that needs to be covered over different time horizons in the future. Furthermore, as the firm may be risk averse, the S-MOS criterion may be a more appropriate measure than a traditional risk-neutral expected value formulation based on the maximization of expected returns. Even though diversification preferences may be taken into consideration in an expected utility paradigm, appropriate risk tolerances necessary to calibrate the utility functions may not be readily available. Hence, we propose the use of the S-MOS criterion which is able to overcome these difficulties and appears to be natural under the portfolio optimization problem for liability management.

In the experiments, for simplicity, we consider two time horizons where liabilities have to be covered by allocating the initial capital into a basket of m asset classes. Before describing the model, we define the following notations:

Table 5.2: Liability management notations

Sets	M	Set of m assets, $M = \{1, \ldots, m\}$
	T_1	Set of m_1 assets that is available to cover liabilities in time horizon 1, $T_1 = \{1, \ldots, m_1\}$
	T_2	Set of assets that is available to cover liabilities in time horizon 2, $T_2 = \{m_1 + 1, \ldots, m\}$
Indices	j	Asset class
	i	Time horizon when liabilities are to be covered
Variables	x_j	Amount of capital to be allocated to asset class j
Coefficients	c	Starting capital
	d_i	Liability (of debt) to be covered over time horizon i
	$\tilde{r}_j$	returns from investing in asset class j. These are uncertain and follows distribution $\mathcal{P}$

Before discussing the S-MOS criterion, we first present the basic expected returns formulation for such a liability management problem similar to the traditional portfolio optimization problem. A risk neutral expected returns maximization formulation for the liability management problem can be stated as follows (where $\mathbb{E}_{\mathbb{P}}$ denotes taking the expectation over the distribution $\mathbb{P}$):

$$\begin{aligned} \max \quad & \sum_{i=1}^{2} \left(\left(\sum_{j=1}^{m} \mathbb{E}_{\mathbb{P}}(1 + \tilde{r}_j) x_j \right) - d_i \right) \\ \text{s.t.} \quad & \boldsymbol{x} \in X \end{aligned}$$

where,

$$X = \left\{ \boldsymbol{x} \in \Re^m \;\middle|\; \begin{array}{l} \displaystyle\sum_{i=1}^{2} \sum_{j \in T_i} x_j \leq c \\ \boldsymbol{x} \geq \mathbf{0} \end{array} \right\}$$

and $\tilde{\boldsymbol{r}}$ follows distribution $\mathbb{P}$.

To cast the problem using a satisficing approach, we note that the targets for each objective of this asset allocation problem is very natural: it is the firm's liability amounts d_i over two time horizons $i \in \{1,2\}$ which they have to cover. As mentioned in the preceding, such a formulation assumes the decision maker is risk neutral. Also, even though MAUT models may be used to model risk aversion, appropriate risk tolerance levels for the firm may not be easily estimated (or reasonable) even for a simplistic single objective case. All these deficiencies can be overcome by a multiple objectives satisficing approach.

In a multiple objectives satisficing setting considering liabilities to be covered in different time horizons, we define the target excess for covering the liability over the i^{th} time horizon as follows:

$$\tilde{p}_i(\boldsymbol{x}) = \sum_{j \in T_i} (1+\tilde{r}_j)x_j - d_i.$$

where $\tilde{p}_i$ can be interpreted as the liability risks over the i^{th} time horizon which the asset allocation plan aims to cover. Herein, there is an implicit assumption of no reinvestments since all capital must be invested at the start of the planning horizon. The S-MOS criterion for a given returns distribution $\mathbb{P}$ is defined as follows:

$$\alpha(\tilde{\boldsymbol{p}}(\boldsymbol{x})) = \sup_{\boldsymbol{u} \geq \boldsymbol{0}} \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N(\tilde{\boldsymbol{p}}(\boldsymbol{x}))} \{u_i \tilde{p}_i(\boldsymbol{x}), 1\} \right) \tag{2}$$

Our goal is to maximize the S-MOS criterion by solving the following optimization model:

$$\begin{aligned} \sup \quad & \mathbb{E}_{\mathbb{P}} \left(\min_{i \in N(\tilde{\boldsymbol{p}}(\boldsymbol{x}))} \{u_i \tilde{p}_i(\boldsymbol{x}), 1\} \right) \\ \text{s.t.} \quad & \boldsymbol{u} \geq \boldsymbol{0} \\ & \boldsymbol{x} \in X. \end{aligned} \tag{3}$$

We consider a simple example where the initial capital to be invested, c, is deterministic and the liabilities, d_i, are assumed to be non-random over both time horizons, $i \in \{1,2\}$. Also, we assume that the covering positions are bonds j with default probabilities q_j. The return of the asset j is θ_j and becomes ϕ_j, $\phi_j \leq \theta_j$ when defaulted. If asset j is a risky bond, we have $q_j > 0$ and $\phi_j = 0$. Otherwise, if it is a risk-free asset, then $q_j = 0$ and $\theta_j = \phi_j$. We also assume capital is protected for all asset classes. Considering independent default probabilities, q_j for all $j \in M$ and subsets, $Q \subseteq M$, representing subsets of asset classes that are not defaulted, we will obtain the following formulation similar to Problem (8) in Chapter 4 for Algorithm (1):

$$
\begin{aligned}
Z(\boldsymbol{u}, I) = \max \quad & \sum_{Q \subseteq M} \left(\prod_{j \in Q} (1 - q_j) \prod_{l \in \bar{Q}} q_l \right) s_Q \\
\text{s.t.} \quad & u_i \left(\left(\sum_{j \in T_i \bigcap Q} (1 + \theta_j) x_j \right) + \left(\sum_{l \in T_i \bigcap \bar{Q}} (1 + \phi_l) x_l \right) - d_i \right) \geq s_Q \quad && \forall i \in \{1, 2\}, Q \subseteq M \\
& s_Q \leq 1 && \forall Q \subseteq M \\
& \sum_{j \in T_i} (1 + \phi_j) x_j \geq d_i && \forall i \in \{1, 2\} \backslash I \\
& \boldsymbol{x} \in X,
\end{aligned}
\tag{4}
$$

where $\bar{Q}$ denotes $M \setminus Q$. Together with Problem (2) in this chapter, we would be able to implement the iterative Algorithm (1) to evaluate the S-MOS criterion.

In the experiments, we assume the existence of a number of risky assets and a risk-free asset for each time horizon. We assume the rate of return for the risk-free assets to be similar for both time horizons at 5%. The number of risky assets available over each of the time horizon studied in the computational experiments are given in Table 5.3, denoted by m_r. Note that $m = m_r + 2$ to include the risk-free assets. The parameters for the computational experiments are also shown in Table 5.3 for risky assets in each covering horizon. These parameters include the probability of default of each asset class in each time horizon and the potential returns from these asset classes. We assume the liabilities to be covered in the over horizon 1 and 2 to be 10 and 15 respectively. These are natural targets for the two criteria considered in the multiple objectives satisficing formulation.

Risky Asset Class	Liability Horizon 1 Default Probability	Liability Horizon 1 Returns	Liability Horizon 2 Default Probability	Liability Horizon 2 Returns
$m_r = 4$				
1	0.01	0.1	0.1	0.3
2	0.1	0.3	0.01	0.1
$m_r = 8$				
1	0.01	0.1	0.1	0.3
2	0.1	0.3	0.01	0.1
3	0.01	0.1	0.1	0.3
4	0.1	0.3	0.01	0.1
$m_r = 12$				
1	0.01	0.1	0.1	0.3
2	0.1	0.3	0.01	0.1
3	0.01	0.1	0.1	0.3
4	0.1	0.3	0.01	0.1
5	0.01	0.1	0.1	0.3
6	0.1	0.3	0.01	0.1
$m_r = 16$				
1	0.01	0.1	0.1	0.3
2	0.1	0.3	0.01	0.1
3	0.01	0.1	0.1	0.3
4	0.1	0.3	0.01	0.1
5	0.01	0.1	0.1	0.3
6	0.1	0.3	0.01	0.1
7	0.01	0.1	0.1	0.3
8	0.1	0.3	0.01	0.1

Table 5.3: Characteristics of risky assets for debt covering

In Figure 5.6, we vary the available initial capital c and compute the optimal S-MOS portfolio and the corresponding satisficing level. It can be observed that the effect of larger amount of investible initial capital tend to increase the satisficing level as it becomes less likely that the firm would default on its payments. There is a distinct level of c beyond which the satisficing level drop to zero. This implies that any level of initial capital that is less than this minimum threshold would be unable to cover either of the liabilities in expectation assuming the given returns distributions. More importantly, it can also be observed that as the number of asset classes increase, the satisficing levels goes up, given a specific initial investible capital and fixed liabilities. Since the probability distributions for the additional asset classes are similar as the number of asset classes are increased, it clearly reflects the diversifying effect of increasing the number of available asset classes, which are mainly positive in the off-diagonal elements. More intuitively, this diversifying effect is exemplified in the decisions when there are more possible asset classes to invest in arises from the ability of the satisficing measure to take into account the risk reducing pooling effect of investing in more asset classes through enabling the thinning effect of the left tail of the distribution.

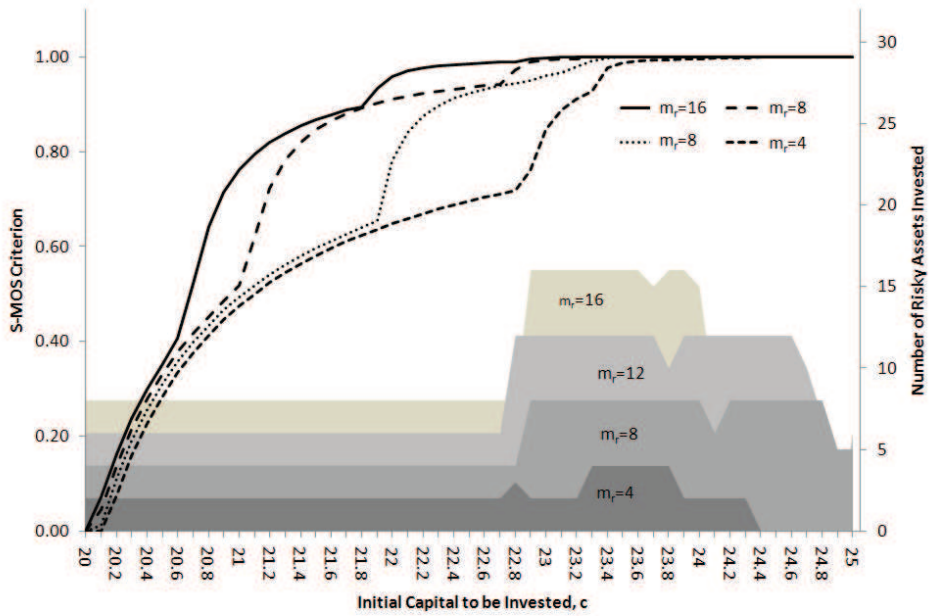


Figure 5.6: Effect of initial capital on satisficing level

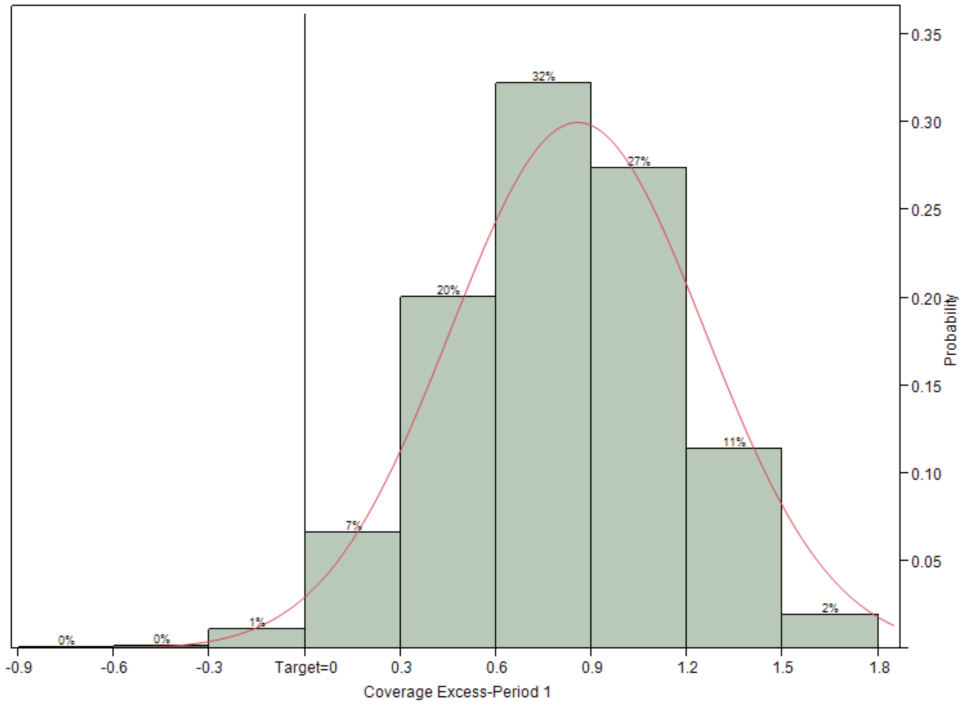
We conduct another set of experiments with different default probabilities and return distributions to illustrate the characteristics of the decisions derived under the multiple objective satisficing framework in comparison with the results obtained from the Markowitz mean-variance model. Under the Markowitz's mean variance model, the portfolio optimization problem can be solved as a quadratic programming problem. Asset selection in this model is only based on mean and covariance information of the uncertain returns. Defining $\boldsymbol{\Sigma} = \text{cov}(\tilde{\boldsymbol{r}})$ as the covariance of the uncertain return vector, $\tilde{\boldsymbol{r}}$, the mean-variance Markowitz model for the liability management example can be formulated as follows:

$$\begin{array}{ll} \min & \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} \\ \text{s.t.} & \displaystyle\sum_{j \in T_i} \mathbb{E}(1+\tilde{r}_j)x_j \geq d_i \quad \forall i \in \{1,2\} \\ & \boldsymbol{x} \in X \end{array}$$

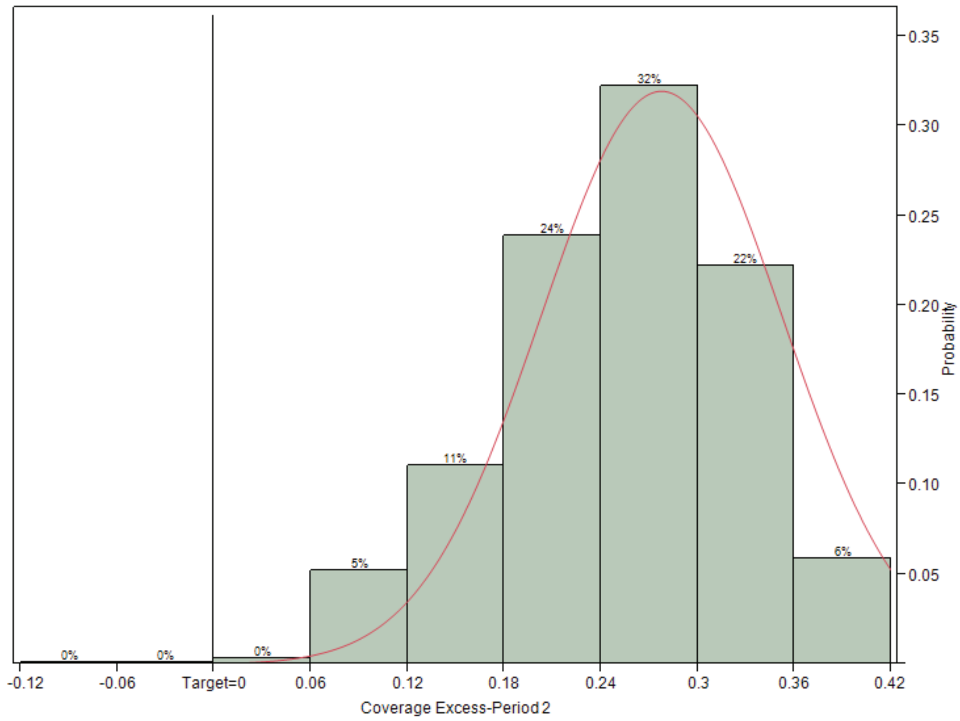
The parameters for this set of experiments are listed in Table 5.4. In this set of experiments, the distributions of the target excess in each covering horizon are studied. The distribution of the target excess for each of the covering horizons are shown in Figure 5.7. The target excess distributions are assessed based on Monte Carlo simulation. After obtaining the optimal decisions based on the Markowitz mean-variance model and the S-MOS criterion, we generate the target excess distributional profiles based on a random sample of size $M = 300,000$ using the assumed distribution given in 5.4. Figure 5.7 shows the excess distributions for the target excess based on $300,000$ random samples based on the decisions obtained using the S-MOS criterion based formulation whereas Figure 5.8 shows the excess distribution for the same $300,000$ random samples based on the decisions obtained using a Markowitz mean-variance model [Markowitz et al., 1987].

Risky Asset Class	Liability Horizon 1		Liability Horizon 2	
	Default Probability	Returns	Default Probability	Returns
1	0.01	0.1	0.001	0.01
2	0.1	0.2	0.05	0.02
3	0.2	0.3	0.1	0.03
4	0.3	0.4	0.2	0.04
5	0.4	0.5	0.3	0.05
6	0.5	0.6	0.4	0.06
7	0.6	0.7	0.5	0.07
8	0.7	0.8	0.6	0.08

Table 5.4: Numerical studies for distribution characteristics of target excess from decisions obtained with the S-MOS criterion and mean variance model

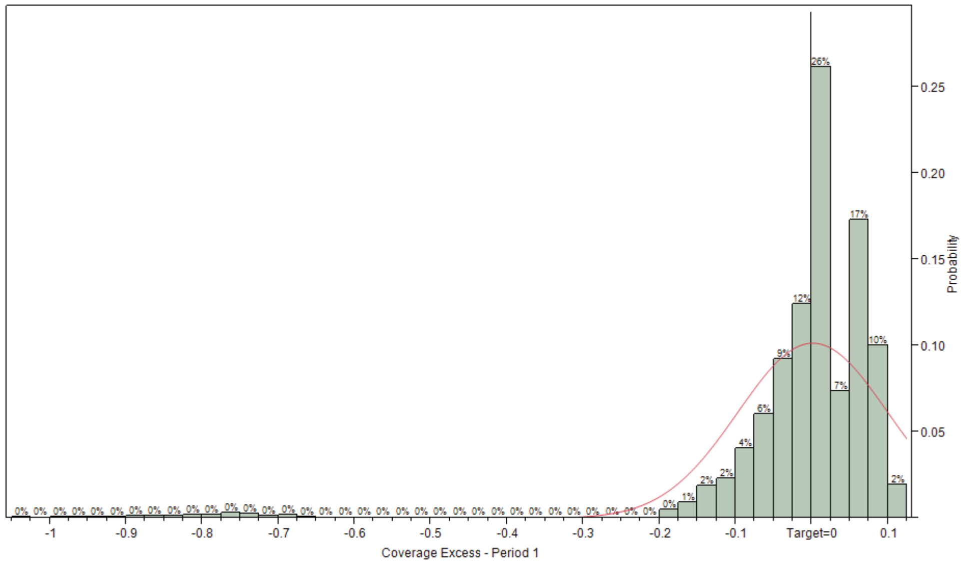


(a) Distribution of target excess for liability horizon 1

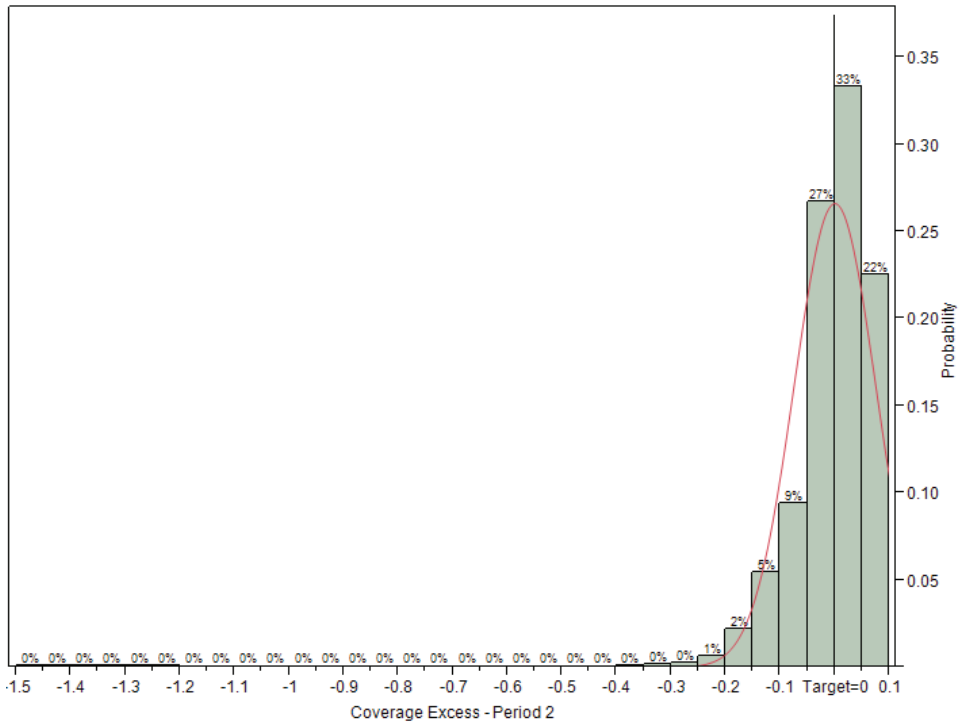


(b) Distribution of target excess for liability horizon 2

Figure 5.7: Distribution of target excess based on S-MOS criterion



(a) Distribution of target excess for liability horizon 1



(b) Distribution of target excess for liability horizon 2

Figure 5.8: Distribution of target excess based on Markowitz mean-variance optimization

From Figure 5.8, it can be observed that the mean-variance optimization was able to derive decisions that has less variability than under that obtained from the S-MOS criterion based objective (see Figure 5.7). Approximately 90% of the target excess falls close to the target liabilities. Over both liability horizons, approximately $30-40\%$ of the random samples resulted in the mean-variance solutions not being able to cover the liabilities. This occurs when the coverage excess becomes negative. Some of these instances resulted in quite extreme losses as can be observed by the long left tail of the target excess distribution. Such a result is expected when using a mean variance formulation since the Markowitz model is similar to a quadratic utility function where the portfolio optimization problem is solved as a quadratic programming problem. The mean-variance optimization model essentially models the risks in uncertain decision making through the variance-covariance matrix of the uncertain parameters. While the resulting model may be simple, such a quadratic utility function is subject to serious limitations in modeling an investor's behavior. Notably, the quadratic utility function is particularly insensitive to asymmetry in returns distribution, meaning that it is insensitive to whether there is an excess or loss relative to the target, as long as the variance is minimized. This is due to the fact that the model penalizes returns upsides and downsides equally. On the other hand, the S-MOS criterion based optimal decisions are able to consider downside risks and ignore the magnitude of upside potential in formulating the decisions. This can be observed from the target excess distribution when the S-MOS criterion based decisions are implemented on the same set of $100,000$ random return samples. Figure 5.7 shows that the S-MOS criterion based decisions were able to cover the liabilities for period 1 for approximately 99% of the time and the liabilities for period 2 for nearly 100% of the samples.

More importantly, from Figures 5.7 and 5.8, it can be observed that the asset allocation decisions based on the mean-variance optimization yielded some returns realizations that are unable to cover the liabilities across the two covering periods to a large extent. Although the possibilities of such events are rare, yet, when they occur, the shortfalls realized can be extremely severe and damaging to any companies implementing managerial decisions without due considerations to the impact of such events. The main idea in behind the proposal of the black swan theory in Taleb [2001] mentioned in Chapter 2 is not to attempt to predict the occurrence of black swan events, but to build robustness against negative ones that can potentially occur and be able to exploit positive ones. The S-MOS measure being shortfall-aware, and hence aware of the potential damage of rare but extreme shortfall events, has been shown to be able to overcome the criticism against expected value and traditional probability measures in not being able to account for the severity of such rare events with limited predictability adequately in its application to managerial decision making.

5.3 Comparison of S-MOS and probability criterion in multicriteria decision analysis: Case study in product design

A computational study for the assessment of decisions characterized by multiple criteria is presented here to demonstrate the advantage of the S-MOS criterion in eliminating the need to estimate risk tolerance parameters necessary of traditional utility-based MAUT models. As mentioned in the preceding, these risk tolerance parameters are often subjective in nature and presents difficulties for decision makers to intuitively grasp and specify with reasonable accuracy. Such difficulties are often exacerbated in the evaluation of product designs in new product development processes of high quality technological products. The problems are accentuated under an MAUT framework that considers multiple utility functions over the attributes.

This case study is based on the evaluation of a proposed new tester for very large scale integrated circuits (VLSI) [Keeney and Lilien, 1987]. The objective of this case study is to demonstrate the advantages of using S-MOS criteria in the selection of product design alternatives over success probability measures typically used in quality assessments. For this example, it is natural to think in terms of performance targets because the explicit purpose of the analysis was to determine whether the new tester designs (OR 9000 designs) were attractive when judged against the incumbent designs (J941 and the Sentry 50). Thus, the best performance of the incumbent testers sets targets against which the OR 9000 designs can be judged. For the evaluation of new product designs and positioning in the market, most methods link product criteria to a preference measure through a functional form that is typically linear [Urban and Hauser, 1980].

In this example, Keeney and Lilien [1987] identified 17 decision criteria from a much larger list of 57 criteria [Healy, 1982]. First, redundancies were eliminated using means-ends relationships. Next, the decisions were grouped into four main categories: technical, economic, software and vendor support. Each of these criteria contains associated quantifiable measures to describe the characteristics of the products. The technical criteria contains six subcriteria, each with a readily available quantitative measure (e.g. pin capacity is measured by the number of pins; vector depth is measured by the memory size in megabits). The economic criteria is evaluated on the basis of three quantifiable characteristics, namely, price, uptime in percent per month and delivery time. The software criteria is similarly described by five quantifiable subcriteria (e.g. development time, existence of networking capabilities). The vendor support characteristics is measured in terms of vendor service level, vendor response time and customer applications of the designs. Some of these criteria are described by logical discrete variables.

Given the characterisation of main and sub-criteria, Keeney and Lilien [1987] proposed the use of a measurable value function approach under the MAUT paradigm [Keeney and Raiffa, 1976] to incorporate the decision preferences of lead users in a primary customer company for the evaluation of different tester designs. The measurable value analysis approach requires three major steps: (1) Specification of product criteria for a given customer; (2) Identification of an evaluation model; (3) Assessment of value judgements to calibrate the value function. The use of lead users proposed in Keeney and Lilien [1987] is in consideration of the highly heterogeneous characteristics of the customer base for the product whose market is characterized by large number of product criteria with small number of high volume buying firms. In such markets, there will usually be a small number of highly influential customers who are consistently early adopters of new technologies - the lead users [von Hippel, 1986]. In Keeney and Lilien [1987] approach, the lead users were asked to provide the minimum acceptability level and maximum desirability level for each of the subcriteria. Keeney and Lilien [1987] then assessed the user preferences and assumed an additive measurable value function based on importance weights for evaluating the VLSI tester. An example of a calibrated value function based on an exponential utility function for criteria “data rate” is as follows:

$$v_{\mathrm{datarate}}(x_{\mathrm{datarate}}) = -0.309(1 - \exp[0.02406(x_{\mathrm{datarate}} - 40)]$$

Other than the calibration of such utility functions, there is a need to estimate the weights associated with each criterion in using a multiattribute utility approach.

For the Keeney and Lilien [1987] decisions, the performance of the new tester design (OR9000) was determined against the performance of two existing incumbent testers (J941 and Sentry 50). Generally, the disadvantages of the multiattribute value analysis based on a utility function approach approach proposed in Keeney and Lilien [1987] are as follows:

1. Difficulty in calibrating the value function for each decision maker.
2. Assessment procedure requires the identification of independence condition required for the form of value function. It is not always practical to rigorously validate this independence condition [Keeney and Raiffa, 1976].
3. Even with the simplest case of an additive utility function approach, the calibration of the attribute weights can already be difficult and imprecise [Keeney and Raiffa, 1976].

Furthermore, the example discussed in Keeney and Lilien [1987] deals with the case without uncertainty about the performance of the three testers, and preferences are monotonic with respect to each evaluation criterion. Hence, the target will be achieved for an evaluation criterion only if performance meets or exceeds a target performance level on that evaluation criterion.

Since the performance of the new tester design (OR9000) was determined against the performance of two existing incumbent testers (J941 and Sentry 50), the performance of the two incumbent testers appears to set targets against which the new tester is evaluated. Bordley and Kirkwood [2004] in recognition of the fact that this decision problem is of a target-oriented type, proposed the use of a multiattribute preference analysis framework that can be extended to deal with uncertainties in the criteria. Here, targets are dependent on the criteria. For example, some criteria may be required to meet certain minimal level whilst other may be required to exceed the performance of the incumbent products by some buffer. Their approach, however, still requires the need to determine the appropriate value function and to calibrate them according to decision makers' preferences. Here, monotonicity of decision preferences were considered for each criterion.

Similar to Keeney and Lilien [1987], targets would be achieved as long as the criterion meets (or exceeds) the target performance level considering monotonicity of preferences. A two-stage process was used to assess the weights for the value function. Relative weights for criteria within each of the four main criteria were first assessed. The weights were then synthesized for each of the main categories based on the assumption of a simple weighted additive function approach. Accordingly, the assumptions underlying the use of an additive value function were validated for this example [Keeney and Lilien, 1987]. In the deterministic case, the overall weight for each design is simply the product of the main category weight and the within category weight since the probability of achieving the target for each criterion is unity if the target is achieved and zero otherwise. The approach here is based on the assumption that the performance target is achieved as long as the criterion meets the *best* performance of either of the two competing incumbents. In the uncertain case, the expected value for the target oriented preference function for each criterion can be easily evaluated based on the cumulative distribution function of the uncertain criteria. Assuming an additive target-oriented preference function defined in Bordley and Kirkwood [2004], the probability analysis can be done one criterion at a time. If the criteria are not probabilistically independent, as in a realistic multiattribute decision problem, the multidimensional integral for the expected preference function may be much more complex [Bordley and Kirkwood, 2004].

For the independent attribute case, Bordley and Kirkwood [2004] provided an example on the evaluation criterion of "data rate" under the technical criteria. Here, we let y_{Sentry50} and y_{J941} denote the data rates for tester models, Sentry 50 and J941, respectively. Since the data rate preference follows positive monotonicity, the new design achieves the target for data rate if y_{OR9000} (representing the data rate for model OR9000) is at least as great as: $\max(y_{\text{J941}}, y_{\text{Sentry50}})$. Hence, defining Y as an indicator function which is equal to unity if the target is achieved and zero otherwise, the conditional probability of achieving the target for the criterion under consideration can be stated as follows:

$$\mathbb{P}(Y|y) = \begin{cases} 1, & y \geq \max(y_{\text{Sentry50}}, y_{\text{J941}}) \\ 0, & \text{otherwise.} \end{cases}$$

The expected value of the preference function for each criterion (v_{datarate}) can then be evaluated as follows:

$$\mathbb{E}[v_{\text{datarate}}|\text{OR9000}] = 1 - F_{\text{OR9000}}(\max(y_{\text{J941}}, y_{\text{Sentry50}}))$$

where F_{OR9000} represent the cumulative distribution function of the "data rate" criterion [Bordley and Kirkwood, 2004]. The expected value of the preference functions for the other 16 criteria can be similarly evaluated based on the distributions of the criteria. The overall preference function can then be evaluated using the weighted approach under a simple additive assumption for the multiattribute value function. Bordley and Kirkwood [2004] highlighted the difficulty of assessing these weights, but advocated that the competitive shortcomings of the designs can be more quickly apparent and easily understood from a target oriented approach than the value function approach proposed in Keeney and Lilien [1987]. The insights they derived from the proposed analysis was that: *"although both approaches are valid for analyzing the decisions, the target-oriented approach seems easier to explain to a nontechnical audience, and it clearly emphasizes the difference among the alternative"* [Bordley and Kirkwood, 2004]. Nonetheless, both approaches still requires the derivation of weights amongst the various criteria impacting on the final decisions. The weights are further subjected to estimation uncertainty, which cannot be taken into account in either approaches. Furthermore, these two approaches require full distributional information in the assessment of decision alternatives when the criteria are uncertain regardless whether the weights can be estimated to full certainty. Realistic multiattribute decision analysis usually have to deal with situations when the distributional information over the random criteria cannot be fully estimated.

In this section, we compare the analysis presented in Bordley and Kirkwood [2004] based on a multiattribute value function approach with the S-MOS criterion. For the computational studies, we assumed the distributions shown in Tables 5.7 and 5.8. Keeney and Lilien [1987] utilized the multiattribute preference analysis based on value preference assumptions of exponential utility function under no uncertainty whilst Bordley and Kirkwood [2004] claimed to have extended the evaluation to consider uncertainties through their proposed approach based on performance targets to assess a preference function for a multiobjective decision under uncertainty. Their approach was founded on the stream of work that resulted in the convergence of the expected utility theory model and the maximization of the probability under a target-based framework Castagnoli and LiCalzi [1996]. Despite the claims, no numerical experiments were presented to validate their model in Bordley and Kirkwood [2004].

In the computational studies, we make some assumptions on the distributions of the product criteria and compare the solution derived using Bordley and Kirkwood [2004] target based utility maximization formulation with that based on the maximization of the proposed S-MOS criterion. Specifically, we compare the performance of Bordley and Kirkwood [2004] approach and the use of S-MOS criterion for the evaluation of new product designs under uncertainty. The S-MOS criterion for the set of 17 product criteria can be solved via the sample average approximation formulation shown in Problem (1) in this chapter over the set of sample S.

Table 5.5 shows the specifications of the incumbent (Sentry 50 and J941) testers and the new testers (OR9000A and OR9000B). In this table, the distribution types of the evaluation criteria (or design parameters) are specified by "Cont" and "Disc", representing continuous type and discrete type distributions respectively.

Evaluation Attribute	Monotonicity	Type	Target	J941	Sentry	OR9000A	OR9000B	Weight
Technical								0.52
Pin capacity	Increasing	Cont	256	96	256	295	268.05	0.15
Vector depth	Increasing	Cont	0.256	0.256	0.064	0.3	0.2685	0.2
Data rate	Increasing	Cont	50	20	50	59	51.9	0.1
Timing accuracy	Decreasing	Cont	600	1000	600	540	575	0.35
Pin capacitance	Decreasing	Cont	40	50	40	36	38.3	0.1
Programmable measurement unit	Increasing	Disc	4	2	4	8	8	0.1
Economic								0.14
Price	Decreasing	Cont	1	1	2.8	0.89	0.958	0.5

continued. . .

Uptime	Increasing	Cont	95	95	95	98	98	0.2
Delivery time	Decreasing	Cont	6	6	6	3	3	0.3
Software								0.32
Software translator (% conversion)	Increasing	Cont	90	90	90	103	94.2	0.15
Networking: Communications	Increasing	Disc	1	1	1	1	1	0.2
Networking: Open	Increasing	Disc	0	0	0	1	1	0.2
Development time	Decreasing	Cont	4	4	4	3	3	0.3
Data analysis software	Increasing	Disc	1	1	1	1	1	0.15
Vendor Support								0.02
Vendor service	Decreasing	Cont	4.75	4.75	6	2	2	0.3
Vendor performance	Decreasing	Cont	4	4	4	3.6	3.83	0.3
Customer applications	Increasing	Disc	1	1	1	1	1	0.4

Table 5.5: Specifications of designs

Table 5.6 shows the deterministic evaluations of the incumbent and new tester designs based on the utility function approach described in Keeney and Lilien [1987]. Herein, exponential utility functions were assumed for each individual criterion and an additive multiattribute utility function was used to derive the final scores of each design. Extending to the case of uncertainty in the evaluation criteria, Tables 5.7 and 5.8 list the assumed distributions used in the numerical experiments. The evaluation results based on probability of target achievement, expected utility and the S-MOS measures under these assumed distributions are shown in the tables. In these tables, “FS” represents the case of the evaluation criteria fully satisfying the targets under the S-MOS evaluation column.

Evaluation Attribute	Target	Weight	OR9000A	OR9000B	J941	Sentry50	OR9000A	OR9000B	J941	Sentry50
Technical		*0.52*					*1*	*1*	*0.2*	*0.8*
Pin capacity	256	0.15	1	1	0	1	0.15	0.15	0	0.15
Vector depth	0.256	0.2	1	1	1	0	0.2	0.2	0.2	0
Data rate	50	0.1	1	1	0	1	0.1	0.1	0	0.1
Timing accuracy	600	0.35	1	1	0	1	0.35	0.35	0	0.35
Pin capacitance	40	0.1	1	1	0	1	0.1	0.1	0	0.1
Programmable measurement unit	4	0.1	1	1	0	1	0.1	0.1	0	0.1
Economic		*0.14*					*1*	*1*	*1*	*0.5*
Price	1	0.5	1	1	1	0	0.5	0.5	0.5	0
Uptime	95	0.2	1	1	1	1	0.2	0.2	0.2	0.2
Delivery time	6	0.3	1	1	1	1	0.3	0.3	0.3	0.3
Software		*0.32*					*1*	*1*	*1*	*1*
Software translator (% conversion)	90	0.15	1	1	1	1	0.15	0.15	0.15	0.15

continued...

Networking: Communications	1	0.2	1	1	1	1	0.2	0.2	0.2	0.2
Networking: Open	0	0.2	1	1	1	1	0.2	0.2	0.2	0.2
Development time	4	0.3	1	1	1	1	0.3	0.3	0.3	0.3
Data analysis software	1	0.15	1	1	1	1	0.15	0.15	0.15	0.15
Vendor Support		*0.02*					*1*	*1*	*1*	*0.7*
Vendor service	4.75	0.3	1	1	1	0	0.3	0.3	0.3	0
Vendor performance	4	0.3	1	1	1	1	0.3	0.3	0.3	0.3
Customer applications	1	0.4	1	1	1	1	0.4	0.4	0.4	0.4
					Overall Weight		***1***	***1***	***0.584***	***0.82***

Table 5.6: Deterministic design evaluations

As mentioned in the preceding, Keeney and Lilien [1987] original example dealt with the case without uncertainty on the performance of the three testers, and preferences are monotonic with respect to each evaluation criteria. In such a simplistic scenario, the target will be achieved for an evaluation criterion only if performance meets or exceeds a target performance level on that evaluation criterion. The criterion score of "1" in Table 5.6 indicates that a particular criterion meets the target and "0" otherwise. From Table 5.6 it can be observed that the OR9000A and OR9000B design both scored the best in the overall assessment. This is in fact rather obvious, by looking at the number of design criteria that are satisfied across each design. The new OR9000 designs are in fact indistinguishable, given that both satisfies all the evaluation criteria under the pre-specified design criteria. Despite these obvious design evaluations proposed by Keeney and Lilien [1987], we still do not have sufficient information to distinguish the quality of the new incumbent designs. Hence, we cannot say that they are equally good. One primary deficiency is that Keeney and Lilien [1987] assumed that all the evaluation criteria can be evaluated without any uncertainty. This assumption is particularly difficult to justify especially in today's highly uncertain and competitive business environment, more so, when it is within the context of a new product development process [Unger and Eppinger, 2009].

According to Bordley and Kirkwood [2004], their target-oriented analysis approach extends the Keeney and Lilien [1987] deterministic approach to the case with uncertainty in a straightforward manner as discussed in the preceding. The stochastic evaluations of the design criteria based on the Bordley and Kirkwood [2004] target oriented expected utility analysis are listed under the "Component Expected Value" column in Tables 5.7 and 5.8. Apart from evaluating the expected utility for each criterion based on Bordley and Kirkwood [2004] target oriented approach, the probability of achieving the criterion's targets for each of the design criteria based on the assumed distributions are also presented in Tables 5.7 and 5.8.

The overall evaluation scores for both OR9000A and OR9000B are indistinguishable at a level of 0.999 in terms of both the joint probability of satisfying all the targets and the target-based utility approach proposed by Bordley and Kirkwood [2004]. This again shows that the use of joint probability or the target-based approach by Bordley and Kirkwood [2004] may be ineffective in dealing with the type of designs with the criteria adhering to the distributional assumptions listed in Tables 5.7 and 5.8. This may be a special example whereby the quality of components used in the product may be of extremely high quality, thereby yielding a product that is able to achieve all the target criteria jointly with a probability exceeding 99.9%. It maybe argued as improbable to achieve such high quality products perhaps two decades ago. However, in today's highly competitive and innovative high technology product development processes, this may instead be a typical case. A prominent example is product development processes that have adopted a Six Sigma quality framework.

Six Sigma makes use of sound statistical methods and quality management principles to improve processes and products via the Define-Measure-Analyze-Improve-Control (DMAIC) quality improvement framework to meet customer needs on a project-by-project basis. With many high-profile adoptions by companies such as General Electric (GE) in the 1990s, Six Sigma has spread like wild fire towards the end of the 20th century [Tang et al., 2007]. The focus of Six Sigma in reducing variability in key product quality characteristics around specified target values to the level at which failure or defects are extremely unlikely is well documented and explained (see for example Tang et al. [2007], Goh et al. [2006]). The Six Sigma concept (see for example Tang et al. [2007] and Tang et al. [2008a]) aims to reduce the variability in the process, so that the specification limits are at least six standard deviations from the target. This will result in about two parts per billion non-conforming to specifications. Under the Six Sigma concept, however, an assumption was made that when the process reached the Six Sigma quality level, the process mean may still be subjected to disturbances that could cause it to shift by as much as 1.5 standard deviations off target. Hence, under this assumption, a Six Sigma process would produce up to 3.4 parts per million (ppm) non-conforming to specifications. Some quality management practitioners have rejected Six Sigma on the grounds that the 3.4 ppm is a "numerical goal", or that the 1.5 sigma shift is arbitrary, and that Six Sigma is simply a"slogan" and supposedly violated some of Deming quality philosophy [Deming, 1986]. However, no process or system is ever truly stable and even in the best of situations, disturbances occur. These disturbances can result in the process mean shifting off-target, an increase in the process standard deviation, or both. The Six Sigma process concept is one way to model this behaviour.

The distributional assumptions for the numerical experiments shown in Tables 5.7 and 5.8 are based on the abovementioned assumptions and follows closely the Six Sigma quality criteria for the evaluation criteria. Practically, it may mean that all the suppliers for the components of the products have to comply with the Six Sigma requirements. As mentioned in the preceding, given such high quality level, the overall evaluation scores for both OR9000A and OR9000B are indistinguishable at a level of 0.999 in terms of both the joint probability of satisfying all the targets and the target-based utility approach proposed by Bordley and Kirkwood [2004]. As an additional note, it can be asserted that it is practically impossible to improve the precision of the estimates for the target based utility approach proposed by Bordley and Kirkwood [2004] to offer a distinction between designs OR9000A and OR9000B, due to the high quality processes coupled with the need to estimate the weights required to use an additive value function approach under the classical MAUT framework [Keeney and Raiffa, 1976]. This may be due to the limited information on customers' risk tolerance parameters, or may be simply due to the highly subjective nature of utility assessments. In using the joint probability of target achievement, this evaluation may be possible, but difficulties exists due to the lack of adequate data or information about the distribution of the evaluation criterion. This is particularly an issue in an environment of information scarcity when we are dealing with new product development.

Both of these deficiencies can be handled by using the S-MOS criterion, which adheres to the natural target-based orientation, yet able to deal with the inherent characteristics of data scarcity in a new product development environment. For a start, the S-MOS criterion do not require the estimation of the weights required for a value function approach under the MAUT paradigm. Furthermore, the S-MOS criterion has the capability to incorporate distributional ambiguity that requires only the estimation of specific partial distributional information. The need to have this capability is particularly relevant in situations where limited product information is available in a new product development process for design evaluations.

Evaluation Attribute	Type of distribution	Mean	Standard deviation	Probability of Violation (ppm)	Probability of Achievement	Component Expected Value	Satisficing Approach, u
Technical						1.00	
Pin capacity	Normal	295	8.85	5.25	0.99999	0.15	0.50230
Vector depth	Normal	0.3	0.009	0.51	1.00000	0.20	FS
Data rate	Normal	59	1.77	0.18	1.00000	0.10	0.84156
Timing accuracy	Normal	540	16.2	106.24	0.99989	0.35	0.25800
Pin capacitance	Normal	36	1.08	106.24	0.99989	0.10	3.26136
Programmable unit	Binomial	7	0.21	8.59	0.99999	0.10	FS
Economic						1.00	
Price	Normal	0.89	0.0267	18.96	0.99998	0.50	FS
Uptime	Normal	109	3.27	9.29	0.99999	0.20	1.10536
Delivery time	Normal	5.3	0.159	5.35	0.99999	0.30	FS
Software						1.00	
Software translator	Normal	103	3.09	12.93	0.99999	0.15	1.48115

continued...

Networking: Communications	Bernoulli	0.999997		3.40	1.00000	0.20	FS
Networking: Open	Bernoulli	0.999997		3.40	1.00000	0.20	FS
Development time	Normal	3.5	0.105	0.96	1.00000	0.30	FS
Data analysis software	Bernoulli	0.999997		3.40	1.00000	0.15	FS
Vendor Support						1.00	
Vendor service	Normal	4.2	0.126	6.35	0.99999	0.30	FS
Vendor performance	Normal	3.6	0.108	106.24	0.99989	0.30	34.03024
Customer applications	Bernoulli	0.999997		3.40	1.00	0.40	10.00

Table 5.7: Stochastic design evaluations for OR9000A

Evaluation Attribute	Type of distribution	Mean	Standard deviation	Probability of Violation (ppm)	Probability of Achievement	Component Expected Value	Satisficing Approach, u
Technical						1.00	
Pin capacity	Normal	268.05	2.6805	3.47	1.00000	0.15	FS
Vector depth	Normal	0.2685	0.002685	1.62	1.00000	0.20	FS
Data rate	Normal	51.9	0.519	125.67	0.99987	0.10	7.32376
Timing accuracy	Normal	575	5.75	6.87	0.99999	0.35	0.43904
Pin capacitance	Normal	38.3	0.383	4.53	1.00000	0.10	25.75296
Programmable unit	Binomial	6	0.06	5.39	0.99999	0.10	FS
Economic						1.00	
Price	Normal	0.958	0.00958	5.82	0.99999	0.50	394.50827
Uptime	Normal	100	1	0.29	1.00000	0.20	FS
Delivery time	Normal	5.75	0.0575	6.87	0.99999	0.30	110.16862
Software						1.00	
Software translator	Normal	94.2	0.942	4.12	1.00000	0.15	FS

continued...

Networking: Communications	Bernoulli	0.999997		3.40	1.00	0.20	10.00
Networking: Open	Bernoulli	0.999997		3.40	1.00000	0.20	FS
Development time	Normal	3.8	0.0383	4.53	1.00000	0.30	FS
Data analysis software	Bernoulli	0.999997		3.40	1.00	0.15	10.00
Vendor Support						1.00	
Vendor service	Normal	4.55	0.0455	5.52	0.99999	0.30	FS
Vendor performance	Normal	3.83	0.0383	4.53	1.00000	0.30	FS
Customer applications	Bernoulli	0.999997		3.40	1.00000	0.40	FS

Table 5.8: Stochastic design evaluations for OR9000B

The S-MOS criterion is evaluated for the OR9000 designs and listed in Tables 5.7 and 5.8. The overall S-MOS criterion value for OR9000A is 0.9870 and that for OR9000B is 0.9998. The normalizing weights estimated in the normalization vector, $\boldsymbol{u}$, for each of the evaluation criterion are also listed in the "*Satisficing Approach*, u" column. For the cases where the uncertain distribution of the evaluation criteria fully satisfies the target, that particular criterion will not be considered in the evaluation of the final satisficing value. The criteria are given the "FS" values under this column. By taking away the considerations of these fully satisfied criteria and focusing on the remaining few important criteria enables one to look at each individual criterion and determine which of these criteria is able to differentiate between the two high quality design. The final satisficing score, being sensitive only to the remaining underperforming objectives, is also able to distinguish the best-in-class from the close runner-ups. More importantly, as a reiteration, the evaluation of the S-MOS criterion do not require the unwieldy need to estimate weights or tolerance parameters necessary in an MAUT approach. Such parameters would unnecessarily degrade the precision of evaluations, which is particularly important in today's high quality product development and manufacturing processes operating at the Six Sigma level of quality.

Furthermore, the generic S-MOS criterion (see Definition 5) is able to handle distributional ambiguity. This makes the proposed satisficing framework particularly relevant for high quality new product development processes, since most of the distributional information on the product criteria are likely to evade accurate and precise estimation, such that the full distributional information is available for the evaluation of these designs. The ability to handle distributional ambiguity would be shown in the case study describe in Chapter 6.

5.4 Chapter summary

In this chapter, an array of numerical experiments were described to numerically show the characteristics of the S-MOS criterion and compare its performance with respect to classical measures such as in terms of the expected value, expected utility and the success probability.

The variations of the S-MOS criterion with respect to the structure of uncertainties are first shown through the use of simple two-point distributions. The sensitivity of the S-MOS criterion to the frequency of shortfall occurrence, which is also the fundamental characteristics of the probability measure, is demonstrated through the numerical experiments. In addition, the sensitivity of S-MOS criterion to the degree of shortfalls in target achievement is also demonstrated via the stylized numerical experiments involving simplistic two-point distributions. As mentioned in the preceding, the absence of this sensitivity to the magnitude of shortfall is the fundamental flaw in success probability and expected value type measures.

Apart from showcasing the characteristics of the S-MOS criterion and its sensitivity to the magnitude of shortfall when shortfalls occur, the numerical experiments also demonstrated the applications of the S-MOS criterion for problems in engineering (in terms of design selection for high quality engineering product development process) and the financial industry (in terms of portfolio allocation for liability management).

The liability management example involves the problem of allocating of capital amongst risk-free and risky asset classes. The case study in liability management demonstrated clearly the ability of the S-MOS criterion to diversify the investment decisions and to cushion against very bad shortfalls against the targets. The ability of the S-MOS criterion to derive decisions that possess these characteristics is demonstrated through a comparison against the use of the Markowitz mean-variance model for asset allocation [Markowitz, 1952].

The case study in engineering product development processes for design selection amongst competing alternative of product design under high quality manufacturing environment demonstrates the applicability of the S-MOS criterion for the ranking of discrete alternatives. This case study also demonstrates the advantage of S-MOS criterion over traditional expected utility models and MAUT models in that the use of a target-based satisficing approach do not require the difficult tasks of having to estimate the risk tolerance parameters in utility models. As mentioned in the preceding, this difficulty is much more pronounced in using MAUT models. In addition, MAUT models require the estimations of "weights" to establish the MAUT objective function, irrespective of whether a simplistic additive model suffices. In a high quality product development processes, the impracticality of obtaining precise risk tolerance parameter estimates and weights to establish the MAUT function may prevent design engineers from clearly differentiating and ranking design alternatives. The advantage of the S-MOS criterion is that it allows the ranking of design alternatives without the need to estimate these problematic risk tolerance parameters and weights.

CHAPTER 6

Numerical Studies on S-MOS Criterion (With Ambiguity)

In this chapter, we look at the characteristics of the S-MOS criterion using simple distributions and present an extensive case study using the shortfall-aware multiple objectives satisficing criterion considering distributional ambiguity. Under distributional ambiguity, the S-MOS criterion is presented in the Definition 5. We also compare the numerical results from implementing this measure with the results obtained via the success probability measure.

The computations in this chapter were performed using the modeling toolbox ROME version 1.0.8 [Goh and Sim, 2011], designed to solve robust optimization problems in the MATLAB environment, together with CPLEX 11.2 solver engine. The computations were performed with an Intel Core 2 Duo CPU, T9600 at 2.80 Ghz processor with 4GB RAM system.

6.1 Numerical characteristics of the S-MOS criterion under ambiguity

In the numerical experiments, we first look at how the S-MOS criterion compares with the success probability across several dimensions under ambiguity. As in the preceding chapter, the dimensions that we examine are as follows:

1. Degree of shortfall in target achievement (Figure 6.1),
2. Frequency of shortfall in target achievement (Figure 6.2), and,
3. Degree of correlations between target premias (Figure 6.3).

In the numerical experiments, unless otherwise specified, we consider four targets to be satisficed. A sample size $|S| = 50,000$ is used for the solving the SAA models. For purpose of illustration, we assume a two-point distribution for the target excess such that each target excess $\tilde{x}_i$ has the following distribution:

$$\mathbb{P}(\tilde{x}_i = x) = \begin{cases} \kappa_i, & x = a_i \\ 1 - \kappa_i, & x = b_i \end{cases}$$

When $a_i < 0$ the worst-case outcome of target excess i is a shortfall against the target. Also, $\kappa_i \in [0, 1]$ models the corresponding probability of shortfall. For a specified distribution modelling the uncertainties, the S-MOS criterion can be evaluated using a generated set of random samples S via sample average approximation (SAA). Under distributional ambiguity, a tractable robust approximation of S-MOS can be evaluated using deflected linear decision rules in Problem (5) in Chapter 4. In the experiments, unless otherwise specified, we consider four targets to be satisficed. A sample size $|S| = 50,000$ is used for the solving the SAA models.

Figure 6.1 compares the S-MOS and success probability criteria at different levels of target shortfall of one of the targets over $a_i \in [-2, 1]$. The rest of the target excess are assumed to follow two-point distributions with lower and upper supports at -0.1 and 2 respectively, and $\kappa_i = 0.1$ for all target excess. It can be observed in Figure 6.1 that the S-MOS criterion value decreases steadily as target shortfall level increases. A similar behavior is observed for the tractable robust S-MOS criterion. In contrast, the success probability value remains insensitive to the level of shortfall even as the target shortfall level increases. On the other hand, when the targets are always achievable, both the S-MOS and success probability criteria values are indifferent to the level of target achievement. Furthermore, we also show the stability of solutions with respect to the number of SAA samples for $|S| = 1000, 5000, 50000$. From Figure 6.1, it is clear that the SAA based evaluations varies significantly over the set of SAA samples generated for evaluating S-MOS and the success probability. Clearly, evaluation based on the tractable robust S-MOS on the other hand will not experience such sampling variability.

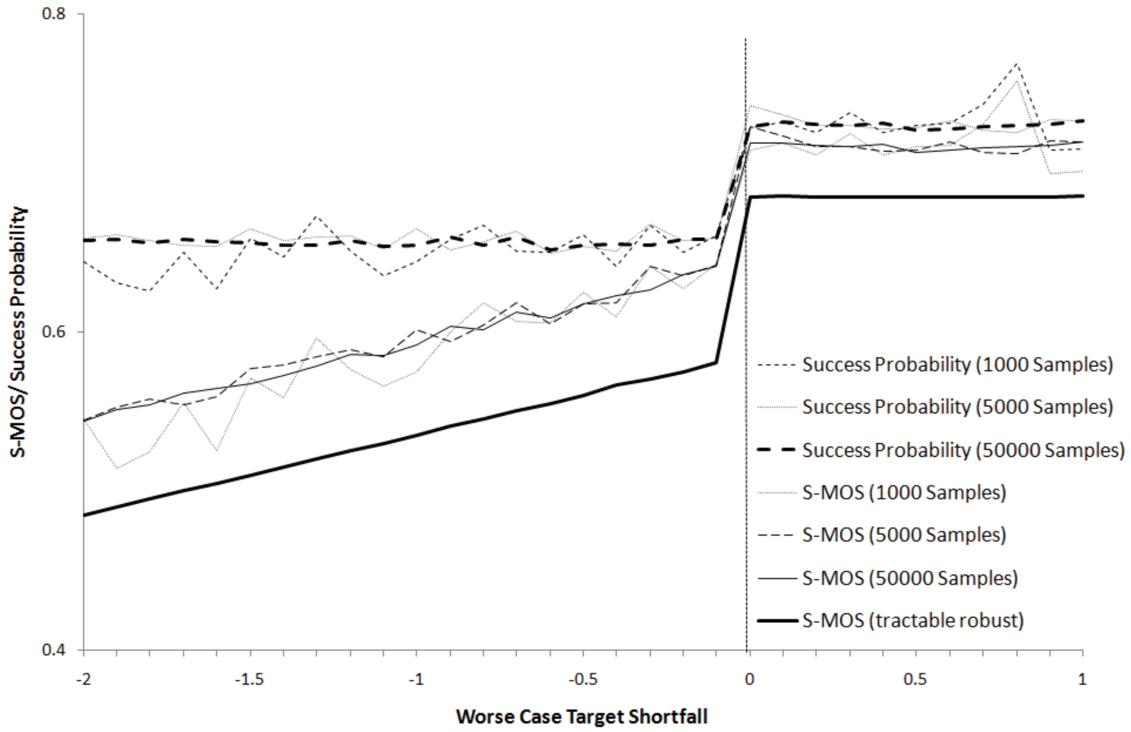


Figure 6.1: Characteristics of S-MOS criterion - Worst-case target shortfall a_i

Figure 6.2 illustrates the effect of varying the frequency of shortfalls on the S-MOS, tractable robust S-MOS and success probability. Here, we assume the worst-case shortfall for the i^{th} target excess to be $a_i = -0.5$ whilst varying κ_i for $\kappa_i \in [0.1, 1]$. The rest of the worst-case shortfall were set at $a_j = -0.1$ and $\kappa_j = 0.1$ for all $j \in \{1, \ldots, 4\} \backslash i$. For evaluating the tractable robust S-MOS, we assumed the mean, and support of the two-point distributions are specified according to the same set of distributions over $\kappa_i \in [0.1, 1]$. The S-MOS criterion values provide lower bounds to the success probability for each level of shortfall frequency, and these bounds are distinguished by the level of shortfalls incurred. Naturally, the tractable robust S-MOS provides the most conservative values since it accounts for distributional ambiguity.

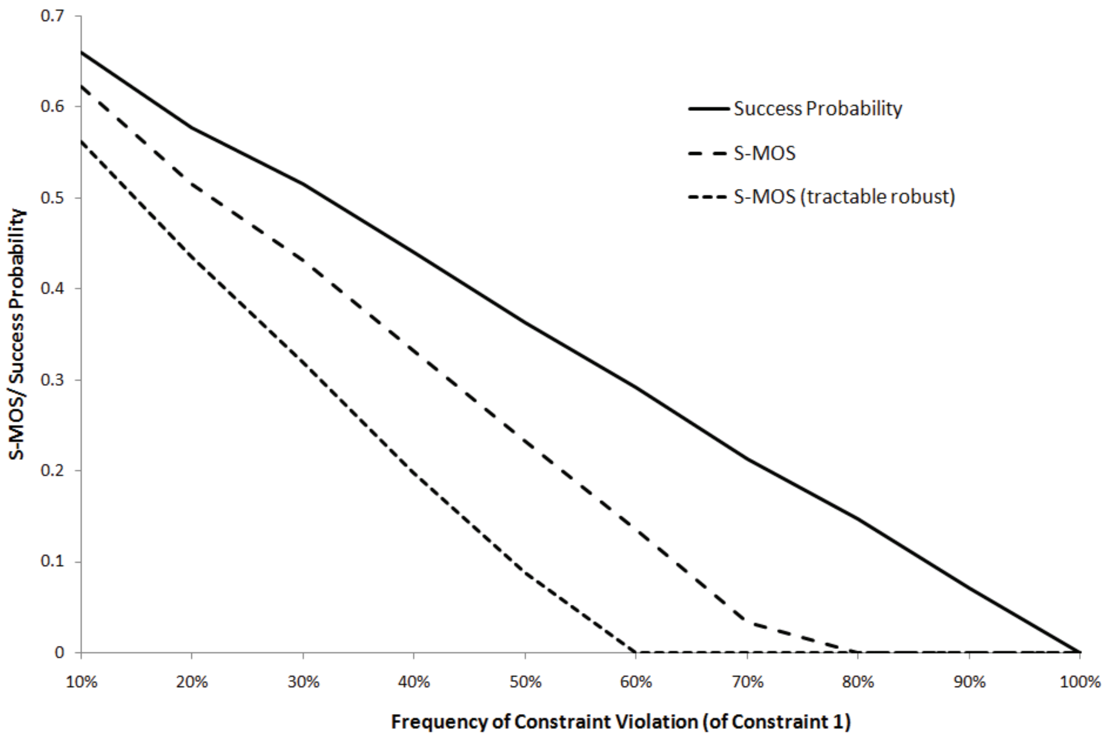


Figure 6.2: Characteristics of S-MOS criterion - Frequency of target shortfalls κ_i

Finally in Figure 6.3, the success probability and S-MOS criteria values are plotted when two of the target excess $\tilde{x}_1$ and $\tilde{x}_2$ are correlated via the scalar parameter $r \in (-1, 1)$. In particular, the two correlated target excess, $\tilde{x}_1$ and $\tilde{x}_2$, are defined as follows:

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\tilde{z}_1$ and $\tilde{z}_2$ are assumed to be uniform iid random variables over $[-\sqrt{3}, \sqrt{3}]$. Hence the degree of correlation between $\tilde{x}_1$ and $\tilde{x}_2$ increases with increasing r. Figure 6.3 verifies that the S-MOS criteria are safe approximations to the success probability criterion. A more interesting and important observation is that both these exhibit a trend that is similar to the success probability. In this example, all three criteria are maximized when $\tilde{x}_1$ and $\tilde{x}_2$ are associated with the highest degree of correlation. These illustrations are consistent with the results established in the previous sections that S-MOS criterion offers a viable and tractable alternative to probability in the selection of decision alternatives. The implementation of S-MOS criterion for an optimization problem is shown in the following subsection.

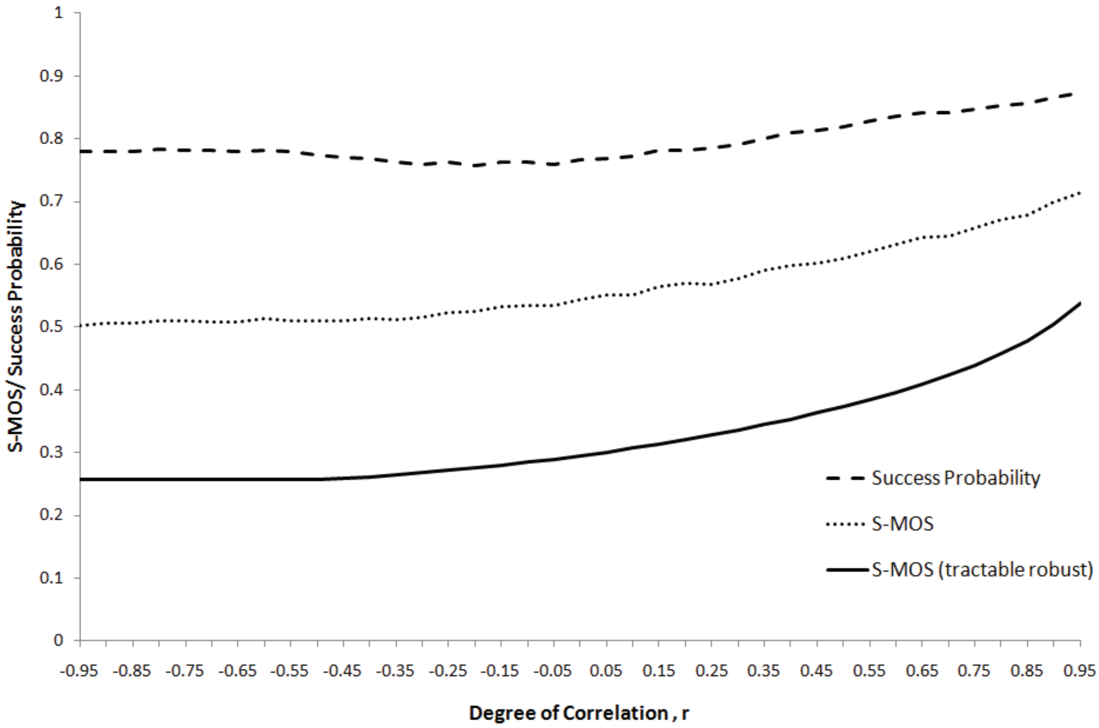


Figure 6.3: Characteristics of S-MOS criterion - Correlation of target excess

6.2 Case study in refinery blending problem

In this section, we present a computational study based on a refinery blending problem with parameter uncertainty, and compare the performance of decisions obtained via solving a deterministic model and optimizing the S-MOS and success probability criteria. In this case study, the performance of decision derived from the deterministic model, optimizaing S-MOS and maximizing success probability are evaluated using Monte Carlo simulation. The ability of the S-MOS based decisions to mimic those derived from the model assuming the maximization of the probability of target achievement objective is shown. In addition, the advantage of the S-MOS criterion in its ability to cushion the optimal decision against severe shortfall outcomes is also demonstrated via the simulation studies. The ability of the S-MOS criterion in producing decisions that is able to account for the absence of full distributional information is also demonstrated. The robustness of the decisions derived with the S-MOS criterion is compared to the decisions derived using success probability measures under known distribution, utilizing different distributions in the simulations.

Blending of raw materials and intermediate products are important steps in the synthesis of blended crude and refined petroleum products (Adhya et al. [1999]). In practice, both the feeds and the end products can have diverse quality parameters. The quality parameters for the feeds into the blending process may not be known with certainty and the end products usually have to satisfy certain pre-specified quality targets. For example, the product quality parameters may consider the levels of certain chemicals in the final blends (e.g. sulfur and benzene) and the blending process is required to produce blends that should not contain levels of such chemicals beyond stipulated thresholds. There are also minimum output levels to be met, since the output products typically form material requirements for the downstream production stages. In addition, there are also constraints on raw material availability. In our computational studies, we implemented a simplified instance of a practical refinery blending problem (see for example DeWitt et al. [1989] and Rigby et al. [1995]) possessing the aforementioned features.

The problem instance used in this case study is derived from an example in Adhya et al. [1999]. In this problem instance, there are eight raw materials that can be used for blending into five output products. The blended products have to satisfy certain pre-specified quality requirements whilst the raw material quality parameters may be uncertain. The total processing costs incurred depends on the product blending options used and the production output have to meet minimum output specifications. Before describing the mathematical model, we first introduce the following notations:

Table 6.1: Blending problem notations

Indices	i	raw material
	j	blended product
	k	quality parameter
Variables	x_{ij}	amount of raw material i blended into product j
Coefficients	$\tilde{a}_{ik}$	k^{th} quality parameter of raw material i which can be potentially uncertain
	c_{ij}	cost of processing a unit of i^{th} raw material for j^{th} product
	b_{jk}	threshold acceptable level of quality parameter k per unit of product j
	r_i	availability of raw material i
	d_j	minimum output levels required demand of product j

For each raw material, Table 6.2 provides the processing cost, raw material availability and the nominal values of the two quality parameters associated with each raw material. Each of these quality parameters can be potentially uncertain over their nominal levels. For each product, Table 6.3 provides the minimum demand that must be satisfied and the threshold acceptable quality level for each product quality parameter. In the example, we assume that the processing cost is the raw material cost, hence $c_{ij} = c_i \ \ \forall j \in \{1, \ldots, 5\}$.

Raw Material Type	Raw Material Cost	Raw Material Availability	Raw Material Quality Parameter	
			Parameter 1	Parameter 2
1	15	85	1.3	1.0
2	7	85	1.7	1.6
3	4	85	1.4	1.4
4	5	85	1.7	1.3
5	6	85	1.6	2.0
6	3	85	1.4	2.0
7	5	85	1.5	1.5
8	5	85	1.2	1.5

Table 6.2: Raw material data

Product	Demand	Product Specifications	
		Parameter 1	Parameter 2
1	15	1.4	1.7
2	25	1.8	1.4
3	10	1.9	1.9
4	20	1.7	1.6
5	15	2.0	2.5

Table 6.3: Product data

Without consideration of uncertainties in the quality parameters, the blending problem can be formulated as the following deterministic linear programming model with an objective of minimizing total processing costs whilst satisfying the quality and demand requirements.

Problem LP

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{5} \sum_{i=1}^{8} c_{ij} x_{ij} \\
\text{s.t.} \quad & b_{jk} \sum_{i=1}^{8} x_{ij} - \sum_{i=1}^{8} a_{ik} x_{ij} \geq 0 \quad j \in \{1, \ldots, 5\}, k \in \{1, 2\} \\
& \boldsymbol{x} \in X
\end{aligned}
\tag{1}
$$

where

$$
X = \left\{ \boldsymbol{x} \in \Re^{8\times 5} \;\middle|\; \begin{array}{l} \sum_{i=1}^{8} x_{ij} \geq d_j \quad j \in \{1, \ldots, 5\} \\ \sum_{j=1}^{5} x_{ij} \leq r_i \quad i \in \{1, \ldots, 8\} \\ \boldsymbol{x} \geq \mathbf{0} \end{array} \right\}
$$

The constraints in (1) defines the quality requirements on the products. The set X ensures that the production levels meet minimum specified demands and raw materials availability.

The quality parameters for the raw materials can be uncertain before the execution of the blending decisions. In this case, we can formulate an S-MOS optimization problem to locate a blending decision that satisfies the target quality requirements as well as possible under uncertainty, given a pre-specified budget τ_b for the total processing costs. For this problem, we define the target excess for the quality constraint on quality parameter k of product j as follows:

$$
\tilde{p}_{jk} = b_{jk} \sum_{i=1}^{8} x_{ij} - \sum_{i=1}^{8} \tilde{a}_{ik} x_{ij}.
$$

The budget τ_b can be specified based on the optimal objective value of the deterministic problem. In the case when the probability distributions of the uncertain quality parameters are known, an SAA model for the S-MOS optimization problem can be solved. For the convenience of discussing the computational results, we define this as the Problem S-MOS.

A corresponding SAA model for the optimization of success probability can be formulated as a mixed integer programming model. For this formulation, let M be a suitably large real number.

Problem SP

$$
\begin{aligned}
\max \quad & \frac{1}{|S|}\sum_{s\in S} w_s \\
\text{s.t.} \quad & b_{jk}\sum_{i=1}^{8} x_{ij} - \sum_{i=1}^{8} a^s_{ik}x_{ij} \geq -M(1-w_s) \quad s\in S, j\in\{1,\dots,5\}, k\in\{1,2\} \\
& \sum_{i=1}^{8}\sum_{j=1}^{5} c_{ij}x_{ij} \leq \tau_b \\
& \mathbf{x}\in X \\
& \boldsymbol{w}\in\{0,1\}^{|S|}
\end{aligned}
\tag{2}
$$

In the above w_s is used to indicate the occurrence of the event of achieving all quality targets in realization $s \in S$. The budget constraint in the formulation ensures that the total processing costs do not exceed budget τ_b. The problem is simply to maximize the number of successes in achieving all quality targets in realization $s \in S$. It should be noted that with increased sample size, Problem SP becomes extremely difficult to solve.

In the case of distributional ambiguity, where only certain distributional information on the random parameters are available, we apply the results of Section 4.1.1 based on Problem (5) in Chapter 4 with deflected linear decisions rule to optimize the tractable robust approximation of S-MOS. The following model of uncertainty is assumed for the uncertain raw material quality coefficients:

$$\tilde{a}_{ik} = \bar{a}_{ik}(1+\tilde{z}_{ik}),$$

where $\bar{a}_{ik}$ represents the nominal k^{th} quality parameter of raw material i. The family of distributions is specified over the mean, support and covariance statistics as follows:

$$\mathbb{F}(\delta,\sigma) = \left\{\mathbb{P} \ : \ \mathbb{E}_{\mathbb{P}}(\boldsymbol{z}) = 0,\ \mathbb{E}_{\mathbb{P}}(\boldsymbol{z}\boldsymbol{z}') = \boldsymbol{I}\sigma^2,\ \mathbb{P}(\tilde{\boldsymbol{z}} \in [-\delta\mathbf{1}, \delta\mathbf{1}]) = 1\right\}.$$

In our computational experiments, we assumed $\tilde{z}_{ik}$ to be uniformly distributed, hence, $\sigma = \delta/\sqrt{3}$. For ease of exposition, we define this problem as the Problem DRS (Distributional Robust S-MOS).

6.3 Computational experiments

In following computational studies, we compare the following characteristics of decisions derived from maximizing the S-MOS and success probability.

1. Performance of optimal solutions of Problems S-MOS, DRS and SP in terms of the probability of jointly achieving all quality specifications (success probability).

2. Ability of these optimal decisions in cushioning against extreme shortfalls.

3. Convergence characteristics of Algorithms 1 and 2.

In the computational studies, the raw material quality levels were assumed to vary over $1-3\%$ away from their nominal values shown in Table 6.2 (or $\delta \in [0.01, 0.03]$). Unless otherwise specified, the computations in this section assumed that the processing cost budget τ_b for the problem with uncertainties was set at 3% above the minimum processing cost achieved by Problem LP.

To solve Problems S-MOS and SP by SAA, 300 simulated samples were used. Both Problems S-MOS and SP were implemented using ROME and solved using the linear and mixed integer optimization routines in CPLEX 11.2. Algorithm 2 was implemented to solve Problem S-MOS. The maximum CPU time for solving the mixed integer Problem SP was 2500 seconds, while Problems S-MOS and DRS were both solved using Algorithm 2 in a few seconds.

The optimal objective values of Problems S-MOS, SP and DRS over the range of variations in the quality parameters are shown in Figure 6.4. The performance of the decisions are evaluated based on $200,000$ out-of-sample realizations of the input data. Figure 6.5 plots the success probability based on these out-of-sample realizations over the range of variations in the quality coefficients using optimal decisions derived from solving Problems S-MOS, SP and DRS. It is clear that, although the use of the success probability criterion through Problem SP achieved the best in-sample objective value (see Figure 6.4), the out-of-sample performance of the other two problems, Problems S-MOS and DRS, in terms of success probability itself, are highly comparable with that achieved using the solutions of Problem SP. In fact at higher variability levels, the solutions from Problem SP yield the lowest success probability, while the solutions of Problem

DRS yield the highest success probability. This indicates that S-MOS and its tractable robust approximation not only improves computational efficiency but also the solution performance. It should also be noted that Problem SP requires much more computational time to solve. Furthermore, similar to Problem S-MOS, its solutions depend on the set of samples utilized in the SAA. Figure 6.1 demonstrates clearly through a simple two-point distribution that the SAA based evaluations are sensitive to the set of SAA samples generated for evaluating S-MOS and the success probability. Evaluation of Problem DRS on the other hand will not experience such sampling variability. If the actual distribution is different from what is being assumed, SAA performance might be further degraded.

Admittedly, the trends shown in Figure 6.5 might be different if we use a larger sample size for the computations, provided we can still solve the model. However, in practice, one rarely knows the exact distribution and such a comparison may not even be meaningful. Nonetheless, the solutions from Problem DRS not only yield the highest success probability in the computations, it is also able to handle situations where the true distribution cannot be accurately ascertained. This is the major advantage of the formulation based on Problem DRS. Specifically, despite using modest information, Problem DRS is able to produce very good results comparable to SAA-based evaluations of Problems SP and S-MOS which require large number of samples. All these were achieved with no increase in computational complexity.

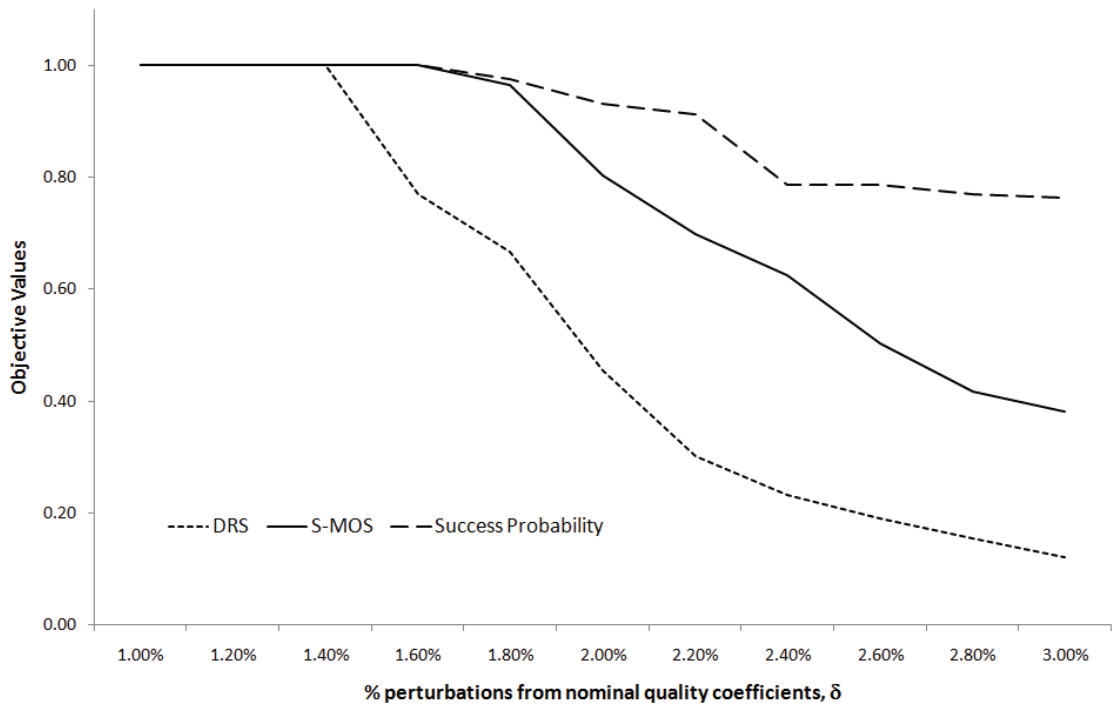


Figure 6.4: Objective values for the maximization of S-MOS and success probability

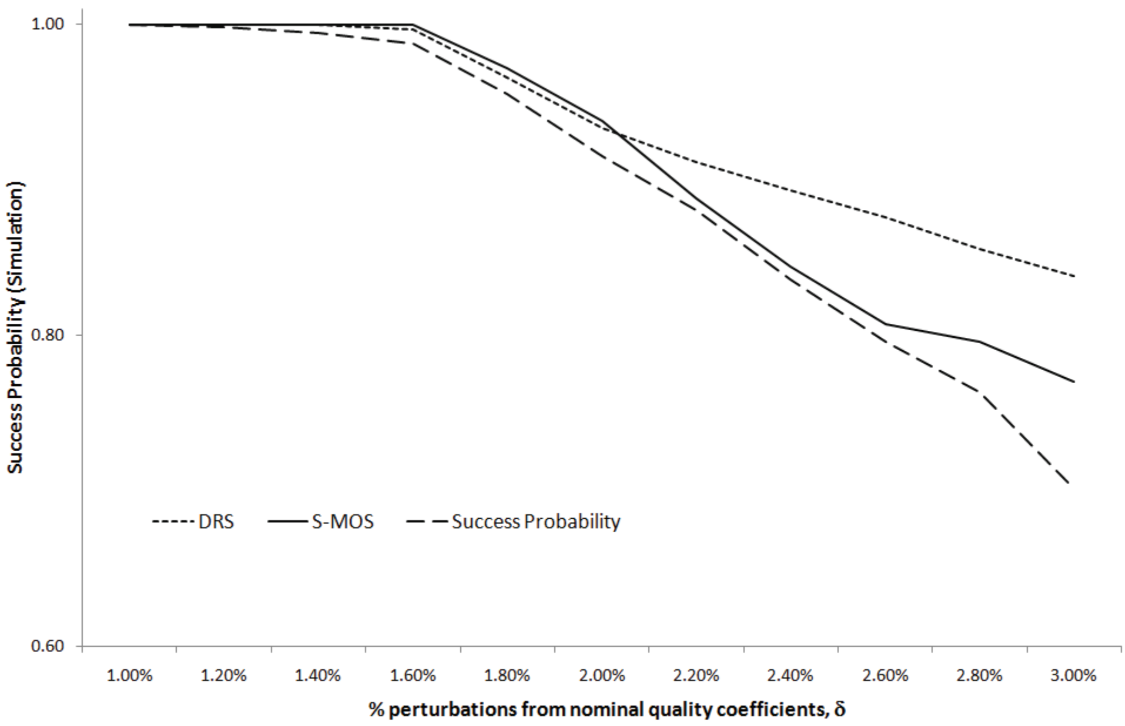


Figure 6.5: Probability of achieving quality specifications (simulation results)

Next, to compare the ability of the solutions in cushioning against extreme outcomes, we evaluated the 90^{th}, 95^{th} and 99^{th} percentile of the percentage violation (or shortfall) of each quality constraint. The percentage violation, $\tilde{w}_{jk}$, of a product j and quality parameter k is defined as the percentage of blend quality exceeding the specified threshold quality levels:

$$\tilde{w}_{jk} = \left(\frac{\sum_{i=1}^{8} \tilde{a}_{ijk} x_{ij}}{b_{jk} \sum_{i=1}^{8} x_{ij}} - 1 \right) \times 100\%$$

The results for Problems LP, S-MOS, SP and DRS are shown in Tables 6.4, 6.5, 6.6 and 6.7 respectively. The first rows of these tables shows the random perturbations away from the nominal values of the quality parameters shown in Table 6.2 over the range of $1-3\%$. The darkened cells in the tables indicate the outcomes that violate the quality threshold levels. It can be observed that the solutions from solving Problem S-MOS outperforms both the solutions of LP and SP, both in the number of violations and the corresponding percentile levels. The solution from DRS yields the smallest number of quality violations, and comparably lower percentile values. In contrast, the solution from Problem LP, while yielding the minimum cost, performed badly under uncertainty. 50% of the quality constraints were violated across the entire range of quality uncertainty considered (Table 6.4). Also, the solution from SP also resulted in higher degree of constraint violations, even though the optimal objective values were the higher than the SMOS criteria. The use of S-MOS and its tractable robust approximation effectively reduced the number of quality constraint violations under the range of uncertainty in the raw materials. Furthermore, the solutions seemed more stable across the range of variability studied. For instance, for the 90^{th} to 99^{th} percentile levels (Table 6.7), the violations always occurred for quality parameter 2 of products 1, 2 and 3. On the other hand, little such inference can be made for the solutions of Problem SP.

Table 6.4: Percentile of quality violations - deterministic solution (simulation results)

90th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	0.6%	0.7%	0.8%	0.9%	1.0%	1.1%	1.2%	1.3%	1.5%	1.6%	1.6%
1	2	0.8%	1.0%	1.1%	1.3%	1.4%	1.6%	1.8%	1.9%	2.1%	2.2%	2.4%
2	1	0.7%	0.8%	1.0%	1.1%	1.3%	1.4%	1.5%	1.7%	1.8%	2.0%	2.1%
2	2	0.6%	0.7%	0.8%	0.9%	1.0%	1.1%	1.2%	1.3%	1.5%	1.6%	1.7%
3	1	-19.4%	-19.2%	-19.1%	-19.0%	-18.8%	-18.7%	-18.6%	-18.5%	-18.3%	-18.2%	-18.1%
3	2	0.6%	0.7%	0.8%	0.9%	1.0%	1.1%	1.2%	1.3%	1.4%	1.5%	1.7%
4	1	-21.6%	-21.5%	-21.3%	-21.2%	-21.1%	-21.0%	-20.9%	-20.7%	-20.6%	-20.5%	-20.4%
4	2	-25.8%	-25.7%	-25.6%	-25.5%	-25.4%	-25.3%	-25.2%	-25.1%	-25.0%	-24.9%	-24.8%
5	1	-17.2%	-17.1%	-17.0%	-16.9%	-16.8%	-16.7%	-16.6%	-16.5%	-16.4%	-16.3%	-16.2%
5	2	-29.4%	-29.3%	-29.2%	-29.1%	-29.0%	-28.9%	-28.8%	-28.6%	-28.5%	-28.4%	-28.3%

95th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	0.7%	0.8%	1.0%	1.1%	1.2%	1.4%	1.5%	1.7%	1.8%	1.9%	2.0%
1	2	0.9%	1.1%	1.3%	1.4%	1.6%	1.8%	2.0%	2.2%	2.3%	2.5%	2.7%
2	1	0.8%	1.0%	1.1%	1.3%	1.4%	1.6%	1.7%	1.9%	2.1%	2.2%	2.4%
2	2	0.7%	0.8%	0.9%	1.1%	1.2%	1.4%	1.5%	1.7%	1.8%	1.9%	2.0%
3	1	-19.3%	-19.1%	-19.0%	-18.9%	-18.7%	-18.6%	-18.4%	-18.3%	-18.1%	-18.0%	-17.8%
3	2	0.7%	0.8%	1.0%	1.1%	1.2%	1.4%	1.5%	1.6%	1.8%	1.9%	2.1%
4	1	-21.5%	-21.4%	-21.2%	-21.1%	-21.0%	-20.8%	-20.7%	-20.5%	-20.4%	-20.3%	-20.1%
4	2	-25.8%	-25.6%	-25.5%	-25.4%	-25.3%	-25.2%	-25.1%	-25.0%	-24.8%	-24.7%	-24.6%
5	1	-17.1%	-16.9%	-16.8%	-16.7%	-16.6%	-16.5%	-16.4%	-16.3%	-16.1%	-16.0%	-15.9%
5	2	-29.4%	-29.2%	-29.1%	-29.0%	-28.9%	-28.7%	-28.6%	-28.5%	-28.4%	-28.2%	-28.1%

99th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	0.9%	1.0%	1.2%	1.4%	1.5%	1.7%	1.9%	2.1%	2.2%	2.4%	2.6%
1	2	1.0%	1.2%	1.4%	1.6%	1.8%	2.0%	2.2%	2.4%	2.5%	2.7%	2.9%
2	1	0.9%	1.1%	1.3%	1.4%	1.6%	1.8%	2.0%	2.2%	2.4%	2.5%	2.7%
2	2	0.9%	1.0%	1.2%	1.4%	1.6%	1.7%	1.9%	2.1%	2.2%	2.4%	2.6%
3	1	-19.2%	-19.1%	-18.9%	-18.7%	-18.6%	-18.4%	-18.3%	-18.1%	-18.0%	-17.8%	-17.6%
3	2	0.9%	1.0%	1.2%	1.4%	1.5%	1.7%	1.9%	2.1%	2.2%	2.4%	2.6%
4	1	-21.5%	-21.3%	-21.2%	-21.0%	-20.8%	-20.7%	-20.5%	-20.4%	-20.2%	-20.1%	-19.9%
4	2	-25.7%	-25.5%	-25.4%	-25.3%	-25.1%	-25.0%	-24.9%	-24.7%	-24.6%	-24.5%	-24.3%
5	1	-16.9%	-16.8%	-16.7%	-16.5%	-16.4%	-16.2%	-16.1%	-15.9%	-15.8%	-15.7%	-15.5%
5	2	-29.3%	-29.2%	-29.0%	-28.9%	-28.8%	-28.6%	-28.5%	-28.4%	-28.2%	-28.1%	-27.9%

Table 6.5: Percentile of quality violations - success probability solution (simulation results)

90th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	-5.7%	-0.1%	-0.2%	-0.4%	-0.5%	-0.5%	-0.1%	-0.5%	0.0%	0.2%	0.1%
1	2	-0.3%	-0.2%	-0.2%	-0.4%	-0.5%	-0.1%	-0.2%	0.1%	0.2%	0.6%	0.9%
2	1	-0.1%	-0.1%	-0.1%	-0.2%	-0.4%	-0.4%	0.0%	-0.3%	-0.4%	-0.1%	-0.1%
2	2	-0.2%	-0.2%	-0.2%	-0.4%	-0.5%	-0.3%	-0.2%	-0.5%	0.1%	0.5%	0.1%
3	1	-19.4%	-19.2%	-19.1%	-19.0%	-18.9%	-18.7%	-18.6%	-18.5%	-18.3%	-18.2%	-18.1%
3	2	-0.2%	-0.5%	-0.3%	-0.6%	-0.1%	-0.7%	-0.3%	-0.1%	0.1%	-0.9%	-0.7%
4	1	-19.5%	-19.7%	-18.8%	-18.2%	-17.5%	-18.0%	-17.5%	-17.7%	-17.3%	-17.9%	-18.0%
4	2	-23.2%	-23.1%	-23.2%	-24.2%	-25.5%	-25.4%	-25.3%	-25.2%	-25.1%	-25.0%	-24.9%
5	1	-17.2%	-17.1%	-17.0%	-16.9%	-16.8%	-16.7%	-16.6%	-16.5%	-16.4%	-16.3%	-16.2%
5	2	-29.4%	-29.3%	-29.2%	-29.1%	-29.0%	-28.9%	-28.8%	-28.7%	-28.5%	-28.4%	-28.3%

95th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	-5.6%	0.0%	0.0%	-0.2%	-0.3%	-0.3%	0.2%	-0.2%	0.3%	0.5%	0.5%
1	2	-0.2%	-0.1%	-0.1%	-0.2%	-0.3%	0.1%	0.1%	0.4%	0.5%	0.9%	1.2%
2	1	0.0%	0.1%	0.1%	-0.1%	-0.2%	-0.2%	0.2%	-0.1%	-0.1%	0.2%	0.2%
2	2	-0.1%	0.0%	0.0%	-0.2%	-0.3%	0.0%	0.1%	-0.2%	0.4%	0.9%	0.5%
3	1	-19.3%	-19.1%	-19.0%	-18.9%	-18.7%	-18.6%	-18.4%	-18.3%	-18.1%	-18.0%	-17.8%
3	2	-0.1%	-0.3%	-0.2%	-0.4%	0.1%	-0.5%	0.0%	0.2%	0.4%	-0.6%	-0.4%
4	1	-19.4%	-19.6%	-18.7%	-18.0%	-17.3%	-17.9%	-17.3%	-17.5%	-17.1%	-17.6%	-17.7%
4	2	-23.2%	-23.0%	-23.1%	-24.1%	-25.3%	-25.2%	-25.1%	-25.0%	-24.9%	-24.8%	-24.7%
5	1	-17.1%	-17.0%	-16.8%	-16.7%	-16.6%	-16.5%	-16.4%	-16.2%	-16.1%	-16.0%	-15.9%
5	2	-29.4%	-29.2%	-29.1%	-29.0%	-28.9%	-28.7%	-28.6%	-28.5%	-28.4%	-28.2%	-28.1%

99th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	-5.5%	0.2%	0.2%	0.0%	0.0%	0.1%	0.5%	0.2%	0.7%	1.0%	1.0%
1	2	0.0%	0.0%	0.1%	0.0%	-0.1%	0.4%	0.4%	0.7%	0.8%	1.2%	1.9%
2	1	0.1%	0.2%	0.2%	0.1%	0.0%	0.0%	0.4%	0.2%	0.2%	0.5%	0.6%
2	2	0.1%	0.2%	0.2%	0.1%	0.0%	0.3%	0.4%	0.2%	0.8%	1.4%	1.0%
3	1	-19.2%	-19.1%	-18.9%	-18.7%	-18.6%	-18.4%	-18.3%	-18.1%	-18.0%	-17.8%	-17.6%
3	2	0.0%	-0.1%	0.1%	-0.1%	0.4%	-0.2%	0.3%	0.6%	0.8%	-0.1%	0.1%
4	1	-19.3%	-19.5%	-18.5%	-17.9%	-17.1%	-17.6%	-17.1%	-17.2%	-16.7%	-17.3%	-17.4%
4	2	-23.0%	-22.9%	-22.9%	-24.0%	-25.1%	-25.0%	-24.9%	-24.8%	-24.6%	-24.5%	-24.4%
5	1	-16.9%	-16.8%	-16.6%	-16.5%	-16.4%	-16.2%	-16.1%	-15.9%	-15.8%	-15.6%	-15.5%
5	2	-29.3%	-29.2%	-29.0%	-28.9%	-28.8%	-28.6%	-28.5%	-28.4%	-28.2%	-28.1%	-27.9%

Table 6.6: Percentile of quality violations - S-MOS solution (simulation results)

90th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	-0.9%	-0.8%	-0.8%	-0.6%	-0.5%	-0.6%	-0.4%	-0.4%	-0.3%	-0.3%	-0.3%
1	2	-0.8%	-0.6%	-0.5%	-0.4%	-0.4%	-0.1%	0.0%	0.2%	0.4%	0.7%	0.8%
2	1	-1.0%	-0.9%	-0.8%	-0.5%	-0.5%	-0.5%	-0.4%	-0.7%	-0.5%	-0.8%	-0.8%
2	2	-0.8%	-0.8%	-0.7%	-0.6%	-0.5%	-0.4%	-0.4%	-0.2%	-0.2%	0.0%	-0.1%
3	1	-19.4%	-19.2%	-19.1%	-19.0%	-18.8%	-18.7%	-18.6%	-18.5%	-18.3%	-18.2%	-18.1%
3	2	-1.1%	-1.0%	-0.8%	-0.7%	-0.4%	-0.5%	-0.4%	-0.4%	-0.4%	-0.5%	-0.2%
4	1	-18.4%	-18.3%	-18.2%	-18.0%	-17.7%	-18.0%	-17.8%	-17.9%	-17.9%	-18.0%	-17.9%
4	2	-25.8%	-25.8%	-25.7%	-25.6%	-25.5%	-25.4%	-25.3%	-25.2%	-25.1%	-25.0%	-25.0%
5	1	-17.2%	-17.1%	-17.0%	-16.9%	-16.8%	-16.7%	-16.6%	-16.5%	-16.4%	-16.3%	-16.2%
5	2	-29.4%	-29.3%	-29.2%	-29.1%	-29.0%	-28.9%	-28.8%	-28.7%	-28.5%	-28.4%	-28.3%

95th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	-0.8%	-0.7%	-0.6%	-0.4%	-0.3%	-0.3%	-0.1%	-0.2%	0.0%	0.0%	0.0%
1	2	-0.7%	-0.5%	-0.4%	-0.3%	-0.2%	0.1%	0.2%	0.5%	0.7%	1.0%	1.1%
2	1	-0.9%	-0.8%	-0.6%	-0.3%	-0.3%	-0.3%	-0.2%	-0.4%	-0.2%	-0.5%	-0.5%
2	2	-0.7%	-0.6%	-0.6%	-0.4%	-0.3%	-0.2%	-0.1%	0.1%	0.1%	0.3%	0.3%
3	1	-19.3%	-19.1%	-19.0%	-18.8%	-18.7%	-18.6%	-18.4%	-18.3%	-18.1%	-18.0%	-17.8%
3	2	-1.0%	-0.9%	-0.6%	-0.5%	-0.2%	-0.3%	-0.1%	-0.1%	-0.1%	-0.2%	0.1%
4	1	-18.4%	-18.2%	-18.1%	-17.9%	-17.6%	-17.8%	-17.6%	-17.7%	-17.7%	-17.8%	-17.6%
4	2	-25.8%	-25.7%	-25.6%	-25.4%	-25.3%	-25.2%	-25.1%	-25.0%	-24.9%	-24.8%	-24.7%
5	1	-17.1%	-17.0%	-16.8%	-16.7%	-16.6%	-16.5%	-16.4%	-16.2%	-16.1%	-16.0%	-15.9%
5	2	-29.4%	-29.2%	-29.1%	-29.0%	-28.9%	-28.7%	-28.6%	-28.5%	-28.4%	-28.2%	-28.1%

99th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	-0.5%	-0.5%	-0.4%	-0.2%	0.0%	0.0%	0.2%	0.2%	0.4%	0.5%	0.5%
1	2	-0.5%	-0.4%	-0.2%	-0.1%	0.0%	0.4%	0.5%	0.8%	1.0%	1.3%	1.5%
2	1	-0.8%	-0.7%	-0.4%	-0.1%	-0.1%	0.0%	0.1%	-0.1%	0.1%	-0.2%	-0.1%
2	2	-0.5%	-0.4%	-0.3%	-0.1%	0.1%	0.1%	0.3%	0.5%	0.6%	0.8%	0.8%
3	1	-19.2%	-19.1%	-18.9%	-18.7%	-18.6%	-18.4%	-18.3%	-18.1%	-18.0%	-17.8%	-17.7%
3	2	-0.9%	-0.7%	-0.4%	-0.2%	0.2%	0.0%	0.2%	0.3%	0.3%	0.3%	0.6%
4	1	-18.2%	-18.1%	-17.9%	-17.7%	-17.4%	-17.5%	-17.4%	-17.4%	-17.4%	-17.5%	-17.3%
4	2	-25.7%	-25.5%	-25.4%	-25.3%	-25.1%	-25.0%	-24.9%	-24.8%	-24.6%	-24.5%	-24.4%
5	1	-16.9%	-16.8%	-16.6%	-16.5%	-16.4%	-16.2%	-16.1%	-15.9%	-15.8%	-15.6%	-15.5%
5	2	-29.3%	-29.2%	-29.0%	-28.9%	-28.8%	-28.6%	-28.5%	-28.4%	-28.2%	-28.1%	-27.9%

Table 6.7: Percentile of quality violations - distributional robust S-MOS solution (simulation results)

90th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	-0.6%	-0.7%	-0.7%	-0.8%	-0.9%	-1.0%	-0.9%	-0.9%	-0.9%	-0.9%	-0.9%
1	2	-0.4%	-0.4%	-0.5%	-0.4%	0.0%	0.1%	0.3%	0.5%	0.7%	0.9%	1.1%
2	1	-0.6%	-0.6%	-0.5%	-0.5%	-0.6%	-0.7%	-0.7%	-0.8%	-0.9%	-1.0%	-1.0%
2	2	-0.6%	-0.6%	-0.6%	-0.7%	-0.8%	-0.8%	-0.6%	-0.6%	-0.5%	-0.3%	-0.2%
3	1	-19.6%	-19.4%	-19.1%	-19.0%	-18.8%	-18.7%	-18.6%	-18.5%	-18.3%	-18.2%	-18.1%
3	2	-0.6%	-0.6%	-0.7%	-0.8%	-0.9%	-0.6%	-0.8%	-0.7%	-0.6%	-0.6%	-0.6%
4	1	-19.1%	-18.7%	-18.2%	-18.2%	-18.4%	-18.3%	-18.4%	-18.4%	-18.4%	-18.4%	-18.4%
4	2	-25.7%	-25.7%	-25.6%	-25.6%	-25.5%	-25.4%	-25.3%	-25.2%	-25.1%	-25.1%	-25.0%
5	1	-17.1%	-17.0%	-17.0%	-16.9%	-16.8%	-16.7%	-16.6%	-16.5%	-16.4%	-16.3%	-16.2%
5	2	-29.4%	-29.3%	-29.2%	-29.1%	-29.0%	-28.9%	-28.8%	-28.7%	-28.5%	-28.4%	-28.3%

95th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	-0.5%	-0.5%	-0.5%	-0.6%	-0.7%	-0.8%	-0.7%	-0.6%	-0.6%	-0.6%	-0.5%
1	2	-0.3%	-0.3%	-0.3%	-0.2%	0.1%	0.3%	0.5%	0.8%	1.0%	1.2%	1.4%
2	1	-0.5%	-0.4%	-0.4%	-0.4%	-0.4%	-0.5%	-0.5%	-0.6%	-0.6%	-0.7%	-0.7%
2	2	-0.4%	-0.5%	-0.5%	-0.5%	-0.5%	-0.6%	-0.4%	-0.3%	-0.2%	0.0%	0.1%
3	1	-19.5%	-19.3%	-19.0%	-18.8%	-18.7%	-18.6%	-18.4%	-18.3%	-18.1%	-18.0%	-17.8%
3	2	-0.4%	-0.5%	-0.5%	-0.6%	-0.7%	-0.4%	-0.5%	-0.4%	-0.3%	-0.3%	-0.3%
4	1	-19.0%	-18.6%	-18.1%	-18.0%	-18.2%	-18.2%	-18.2%	-18.2%	-18.2%	-18.2%	-18.2%
4	2	-25.6%	-25.6%	-25.5%	-25.4%	-25.3%	-25.2%	-25.1%	-25.0%	-24.9%	-24.8%	-24.7%
5	1	-17.0%	-16.9%	-16.8%	-16.7%	-16.6%	-16.5%	-16.4%	-16.2%	-16.1%	-16.0%	-15.9%
5	2	-29.3%	-29.2%	-29.1%	-29.0%	-28.9%	-28.7%	-28.6%	-28.5%	-28.4%	-28.2%	-28.1%

99th Percentile

Product	Quality Attribute	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
1	1	-0.4%	-0.3%	-0.3%	-0.3%	-0.4%	-0.4%	-0.3%	-0.3%	-0.2%	-0.1%	0.0%
1	2	-0.1%	-0.2%	-0.2%	0.0%	0.4%	0.5%	0.8%	1.0%	1.3%	1.5%	1.7%
2	1	-0.4%	-0.3%	-0.2%	-0.2%	-0.2%	-0.2%	-0.2%	-0.2%	-0.3%	-0.3%	-0.3%
2	2	-0.3%	-0.2%	-0.2%	-0.2%	-0.2%	-0.3%	0.0%	0.1%	0.3%	0.5%	0.6%
3	1	-19.5%	-19.2%	-19.0%	-18.7%	-18.6%	-18.4%	-18.3%	-18.1%	-18.0%	-17.8%	-17.6%
3	2	-0.3%	-0.3%	-0.3%	-0.3%	-0.4%	-0.1%	-0.1%	0.0%	0.1%	0.2%	0.2%
4	1	-18.9%	-18.4%	-18.0%	-17.8%	-18.1%	-17.9%	-18.0%	-17.9%	-17.9%	-17.9%	-17.9%
4	2	-25.5%	-25.5%	-25.4%	-25.3%	-25.1%	-25.0%	-24.9%	-24.8%	-24.6%	-24.5%	-24.4%
5	1	-16.9%	-16.7%	-16.6%	-16.5%	-16.4%	-16.2%	-16.1%	-15.9%	-15.8%	-15.6%	-15.5%
5	2	-29.3%	-29.2%	-29.0%	-28.9%	-28.8%	-28.6%	-28.5%	-28.4%	-28.2%	-28.1%	-27.9%

Figure 6.6 plots the S-MOS criterion values under various processing cost budgets τ_b at 3%, 5%, 7% and 10% above the minimum cost solution from Problem LP. While it is obvious that relaxing the cost budget improves the solution performance, the results also indicate that a small increase in operating budget can improve the ability to hedge uncertainty in the target achievement substantially. Over the range of perturbations studied $(2-7\%)$, this effect appears to be most significant when the budget is set at a level of 5% above the deterministic minimum cost solution. This may provide some insights to support managerial decisions and could be used for justification of operating budget levels.

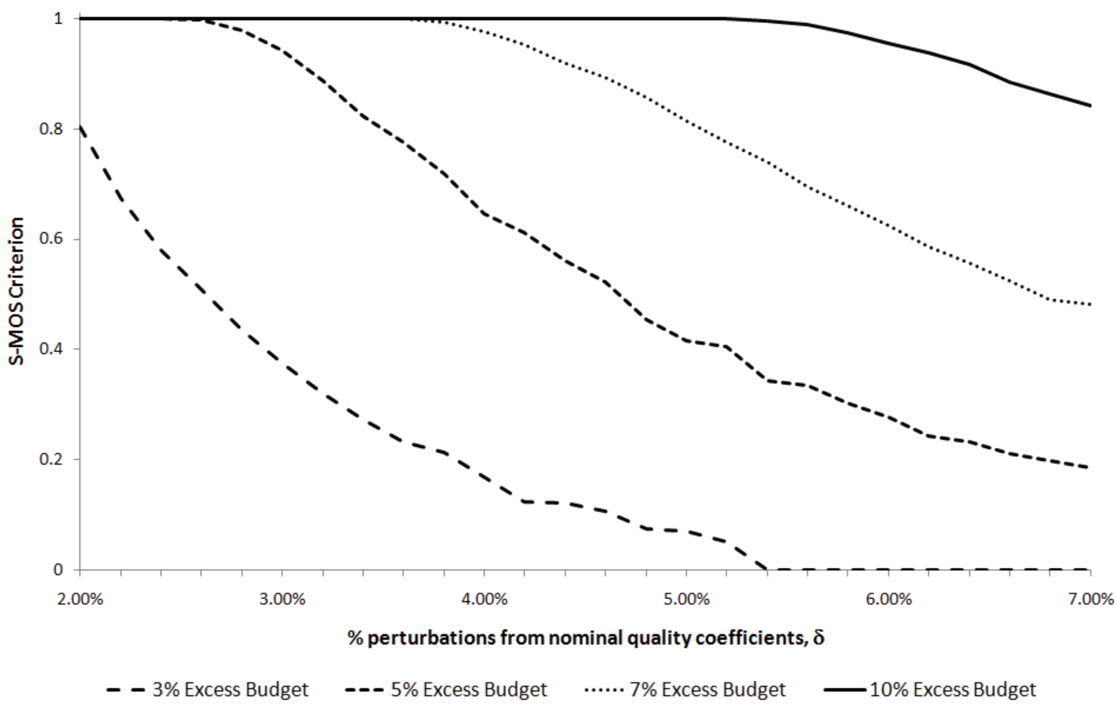


Figure 6.6: Sensitivity to budget requirements (Problem S-MOS)

In order to evaluate the effectiveness of the distributional robust Problem DRS in dealing with distributional ambiguity, we compared the success probability of satisfying the quality constraints based on out-of-sample realizations generated from a distribution that is different to that assumed for solving Problem S-MOS and SP. These distributions however, have the same mean and support as the assumed distribution. We study two different distributions on the out-of-sample realization, independent triangle and two-point distributions. As before, uniform distributions were assumed for solving the S-MOS and success probability based models via SAA. Since Problem DRS evaluates decisions that are supposed to be robust for the family of distributions characterized by these descriptive statistics, solving Problem DRS should produce decisions that will have higher probability of satisfying the quality constraints in the out-of-sample simulations. Indeed, as shown in Figure 6.7, decision based evaluating the success probability model fairs worse than decisions derived from the Problem DRS when the underlying uncertainties follows triangle distributions instead of the assumed uniform distributions. It can also be observed that the superiority of using the distributional robust model over either the S-MOS and success probability based models is more pronounced as the extent of possible perturbations increases.

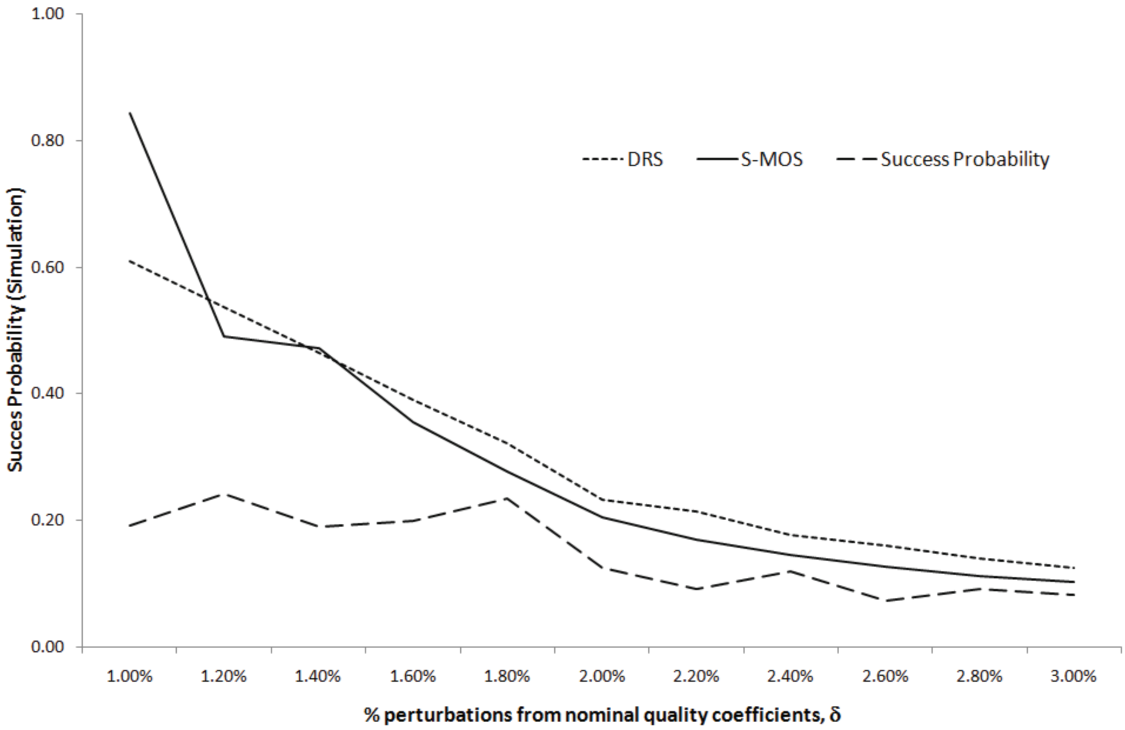


Figure 6.7: Probability of achieving quality specifications (simulation results)

In Figure 6.8, we further provide computational evidence to show the effectiveness of the distributional robust S-MOS model in dealing with distributional ambiguity. Here again, to solve Problem S-MOS and SP using SAA assuming uniformly distributed quality coefficients. The actual distribution used for out-of-sample simulations however were from a family of two-point distributions characterized by the same mean and support as the assumed uniform distribution. It can be observed that the optimal decisions from the distributional robust optimization (based on Problem DRS) outperformed those evaluated under the S-MOS and the success probability criteria (respectively Problems S-MOS and SP when the out-of-sample distributions were different, being that of a two-point distributions (in particular for data perturbations less than 2%). This verifies its ability to hedge against distributional ambiguity effectively. The result is similar to the case when the underlying distributions are that of triangle distributions discussed in the preceding. Furthermore, as shown in Figure 6.8, the objective function values of Problem DRS provide performance gaurantees on the realized joint probability of achieving quality specifications.

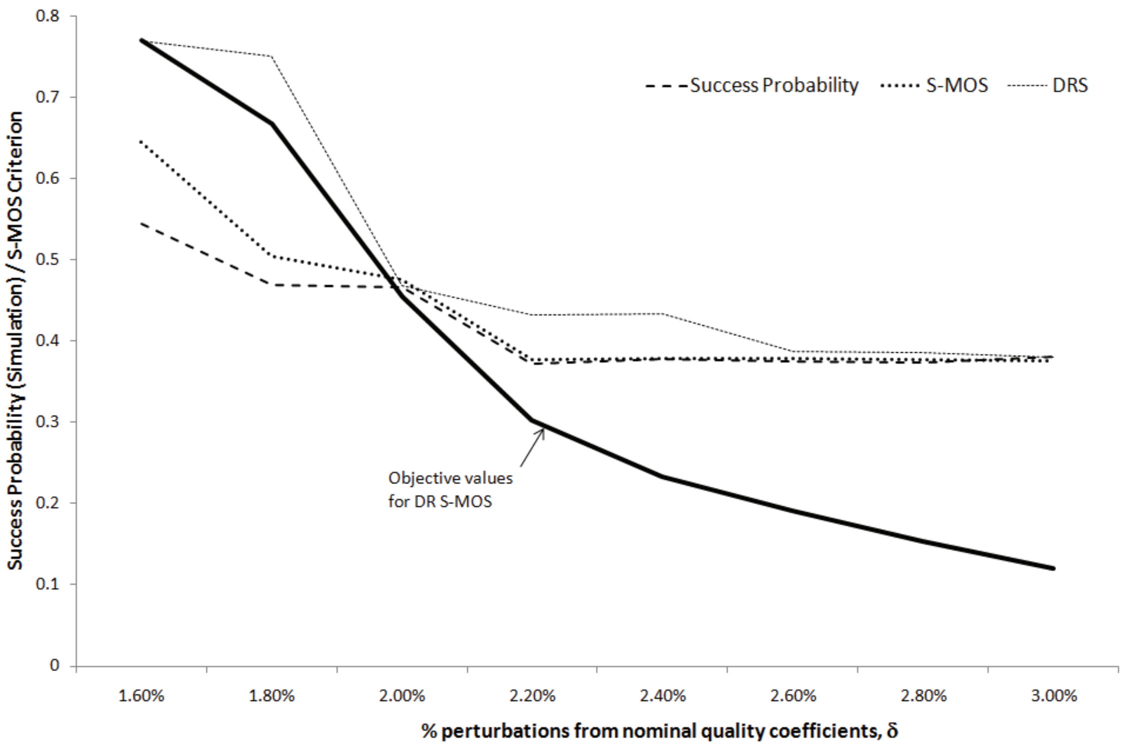


Figure 6.8: Comparison of solution performance derived from Problem DR S-MOS

Finally, we compare the performance of Algorithms 1 and 2, both of which depend on the initial search points. We focus on the case where a starting $\boldsymbol{u}$ vector is specified for solving Problems (8) and (9) in Chapter 4. In the following computations, we are interested in the behavior of the solution paths under different starting $\boldsymbol{u}$. The initial $\boldsymbol{u}$ are generated randomly with components iid and uniform over $[1, 2]$. For each of these starting points, we implement Algorithms 1 and 2 and track the objective values of the sample solutions at every iteration. The maximum allowed perturbations in the uncertain parameters are 2.5% away from the nominal levels. In Table 6.8, we present the distribution of the trajectories for solutions to Problem (7) in Algorithms 1 and 2. The first column of Table 6.8 indicates the range in which the objective values fall within. For example, at the end of the fifth iteration, 74% of the sample path solutions has objective values in $(0.2108, 0.2109]$ using Algorithm 1. It can be observed that all solutions from Algorithm 2 and 95% of the solutions from Algorithm 1 converged to $(0.2108, 0.2109]$. However, Algorithm 2 clearly outperforms Algorithm 1 in the number of iterations used. All the sample paths in Algorithm 2 converged after only seven iterations as compared Algorithm 1, whereas 95% of the objective values converged after fifteen iterations. It can also be observed that all the objective values converged to $(0.2107, 0.2109]$ after only five iterations using Algorithm 2, whereas only 86% of the objective values converged to this range using Algorithm 1 in the same number of iterations. For three sample paths, the objective values converged to a different solution for Algorithm 1, whereas all the sample paths converged to the same solution under Algorithm 2.

6.4 Chapter summary

In this chapter, an array of numerical experiments were described to numerically show the characteristics of the S-MOS criterion under distributional ambiguity and compare its performance with respect to success probability. As in the preceding chapter, the sensitivity of the S-MOS criterion to the frequency of shortfall occurrence is demonstrated through the numerical experiments. The ability of S-MOS criterion to mimic the joint probability distribution under different correlations is also shown for the case of distributional ambiguity. In addition, the sensitivity of S-MOS criterion to the degree of shortfalls in target achievement is also demonstrated. Furthermore, we also show the stability of solutions with respect to the number of SAA samples by assuming known distribution compared to the tractable robust S-MOS formulation, which do not require such assumptions. The SAA based evaluations was shown to vary significantly over the set of SAA samples generated for both the evaluation of the S-MOS criterion and the success probability. The evaluation based on the tractable robust S-MOS on the other hand will not experience such sampling variability.

Apart from the aforementioned comparisons of S-MOS criterion and success probability, the practical applicability of the S-MOS criterion is demonstrated through a case study involving the optimization of a refinery blending problem. In this case study, the decisions derived from the assuming a deterministic problem, optimizing the S-MOS criterion and maximizing success probability are evaluated using monte carlo simulation. The advantage of the S-MOS criterion in its ability to cushion the optimal decision against severe shortfall outcomes was also demonstrated through the out-of-sample simulation studies. The ability of the S-MOS criterion based model in producing decisions that is able to account for the absence of full distributional information was also demonstrated. The robustness for the decisions derived with the S-MOS criterion is compared to the decisions derived using the S-MOS criterion and success probability assuming known distribution. This was done by assuming that the distributions underlying the monte carlo simulation in the performance evaluations have the same descriptive statistics as the assumed uncertainty distributions. Apart from these limited distributional information, the actual form of the distribution differs between the uncertainty models used for optimization and in the performance evaluations. The results demonstrated that the decisions obtained in using the S-MOS under distributional ambiguity outperforms those obtained by assuming a known distribution using either the S-MOS criterion or the probability measures, both evaluated with prior assumed distributions.

	Iteration: 1		Iteration: 2		Iteration: 3		Iteration: 5		Iteration: 7		Iteration: 15	
	Alg 1	Alg 2	Alg 1	Alg 2	Alg 1	Alg 2	Alg 1	Alg 2	Alg 1	Alg 2	Alg 1	Alg 2
(0, 0.2000]	100	100	0	0	0	0	0	0	0	0	0	0
(0.2000, 0.2100]	0	0	12	14	9	0	6	0	5	0	3	0
(0.2100, 0.2101]	0	0	0	2	0	0	0	0	0	0	0	0
(0.2101,0.2102]	0	0	1	0	3	0	0	0	0	0	0	0
(0.2102,0.2103]	0	0	2	3	0	3	0	0	1	0	0	0
(0.2103,0.2104]	0	0	9	0	2	0	3	0	0	0	0	0
(0.2104,0.2105]	0	0	10	10	0	2	3	0	0	0	0	0
(0.2105,0.2106]	0	0	13	10	5	11	0	0	1	0	0	0
(0.2106,0.2107]	0	0	19	14	12	4	2	0	5	0	0	0
(0.2107,0.2108]	0	0	27	33	50	38	12	20	8	0	2	0
(0.2108,0.2109]	0	0	7	14	19	42	74	80	80	100	95	100

Table 6.8: Distribution of objective values

CHAPTER 7

Conclusions

In this thesis, we developed a special family of *multiple objectives satisficing* (MOS) criteria for target-oriented multiple objectives decision making. This family of measure implements the essence of the satisficing concept described by Simon [1959], which may be a more realistic representation of many decision making processes in management science for multiple objectives decision analysis. For example, in the realm of financial risk management, empirical research have concluded the importance of a target-based perspective [Mao, 1970]. In organizational theory, numerous research have also demonstrated the importance of target setting and the multiplicity of targets [Lanzillotti, 1958, Boulding, 1952]. From the target-oriented satisficing perspective, one can view the multiple objectives decision problem in a multi-agent decision making context. An example of such a decision context is in organizational decision problems where multiple performance goals have to be achieve (e.g. financial, operations, marketing, organizational learning goals) under the purview of multiple agents. The use of a target based satisficing approach also eliminates the necessity of having to estimate frequently amorphous and subjective risk tolerance parameters and attribute weights necessary to implement an MAUT based approach for multiattribute decision making.

The MOS criteria that has been developed in the thesis encompass success probability as a special case. We also showed that the MOS family of criteria has a dual relationship with the family of monetary risk measures [Föllmer and Schied, 2004]. This family of monetary risk measures has the well-known Value-at-Risk measure as a special case. Specializing the MOS to consider decision makers' preference for diversification, we introduce an important class of *diversfication favoring multiple objectives satisficing* (DMOS) criteria, which is a subclass of MOS that favors diversification. For the single attribute case, the relationship between DMOS and the well known class of normalized convex risk measures [Föllmer and Schied, 2004] is also established. This class of convex risk measures includes the well-known Conditional Value-at-Risk (CVaR) measure.

Considering the class of DMOS, a specific functional form, the Shortfall-aware MOS (S-MOS) is proposed. This measure is able to address the deficiency of probability measures in not being able to account for the magnitude of shortfalls. The S-MOS criterion also incorporates diversification preference characteristics which is a more reasonable behavioral assumption for decision making under uncertainty. In the computational experiments described, the S-MOS criterion is shown to be capable of deriving decisions that mitigate the impact of events characterised by rarity, extreme impact and limited predictability. In order to achieve this, the S-MOS measure allows decision makers to explicitly consider the consequences of potentially bad outcomes, regardless of how rare these are, by being conscious of the degree of shortfall. This device is technically absent from the traditional probability and expected value based measures. The general formulation of the S-MOS criterion also considers distributional ambiguity. In addition, a solution strategy for the S-MOS is proposed. The proposed strategy is also applicable for the more restricted case of no distributional ambiguity for decision making under uncertainty.

An array of numerical experiments were presented to demonstrate the characteristics of the S-MOS criterion, its sensitivity to the magnitude of shortfalls and its ability to incorporate the diversification favoring characteristics of decision making under uncertainty. We compared the numerical results from implementing these measures with existing target-based measures such as the success probability and the Markowitz mean-variance models [Markowitz, 1952]. Apart from demonstrating the basic characteristics of the S-MOS criterion through numerical studies, three case studies in the domains of portfolio management (on liability management), engineering product development processes and oil refinery blending process management were presented.

The case study in liability management demonstrated the ability of the S-MOS criterion in diversifying asset allocation decisions and in cushioning against very bad shortfalls against the targets. The distribution of target excess in the performance evaluation illustrates how the S-MOS criterion can more appropriately reduce the impact of bad outcomes compared to the use of a Markowitz mean-variance model [Markowitz, 1952] for asset allocation. The case study in engineering product development processes for design selection amongst competing alternative of product design under high quality manufacturing environment demonstrates the advantage of S-MOS criterion over traditional expected utility models and MAUT models in that the use of the S-MOS criterion do not require the difficult tasks of having to estimate the risk tolerance parameters and attribute weights necessary for implementing MAUT models. The need for the estimation of these parameters and weights degrade the precision of the final weighting required for the ranking of design alternatives. In a high quality product development processes, this degradation of precision is shown to have adverse consequences in differentiating design alternatives.

Finally, the case study in the application of the S-MOS criterion to the refinery blending problem demonstrated the practical applicability of the criterion under distributional ambiguity. In this case study, the decisions derived from optimizing deterministic, S-MOS criterion based and success probability based problems were evaluated using monte carlo simulation. The inherent advantage of the S-MOS criterion model in cushioning the optimal decisions against severe shortfall outcomes was demonstrated through out-of-sample simulation studies. The ability of the S-MOS criterion in dealing with the absence of full distributional information was also demonstrated through the case study. The robustness for the decisions derived using the S-MOS criterion under distributional ambiguity is compared to those derived using S-MOS criterion and success probability under known distribution. This was done by assuming the distributions underlying the monte carlo simulation in performance evaluation has the same descriptive statistics as the uncertainty distributions used in the optimization for the S-MOS criterion and success probability based models. The actual form of the distribution, however, differs between the uncertainty models used for optimization and in the performance evaluation other than these specific distributional information. The results demonstrated that the decisions obtained in using the S-MOS criterion in its most generic form, taking into account distributional ambiguity, outperforms those obtained by assuming a known distribution using either the S-MOS criterion or success probability.

7.1 Future research

The implementation of the S-MOS criterion within the MOS framework developed in this thesis is restricted to the case of continuous decision variables and single stage stochastic optimization problems. A natural progression from the single stage decision problem using S-MOS criterion is to develop the framework for using this measure in multi-stage decision problems. These types of decisions occur recurringly across various organizational settings. The decision problems include investment problems where resources may need to be allocated in multiple time periods within a planning horizon. In such scenarios, the decisions may need to be made with respect to multiple targets such as cost, investment returns or some institutional regulations imposed on the organization over several discrete decision epochs. Frequently, there may also be specific targets corresponding to the stability of production and distribution levels over a planning horizon. These problems can be considered from a multistage perspective in which future decision epochs are considered uncertain but depends upon the realizations of uncertainties.

The S-MOS based optimization models can also be extended to the case where integer or binary variables have to be considered in the formulation. These generalizations can enable the MOS framework to deal with stochastic binary integer, integer and mixed integer linear programs that are necessary for a large number of practical problems. Even greater applicability can be derived when the formulations and combinatorial algorithms are developed for the multistage instances of such problems. As an example in the oil and gas industry, the upstream development planning problems requires the evaluation of discrete oil extraction infrastructural investment decisions, such as the construction of new rigs, the expansion of existing rigs and the laying of oil distribution pipelines, together with the production decisions in a multistage decision framework over multiple years, typically for a planning period of up to 30 years time horizon. The basic search algorithm with perspective transformation proposed in Chapter 4 has to be extended to deal with specific formulations for different problems to retain computational tractability when integer decision variables are considered. This is especially important when a multistage decision problem is considered as the search space will become prohibitively huge with this basic search algorithm.

The S-MOS measure is a specific form of the DMOS family of criteria that specifically deploys the properties of DMOS criteria as a criterion that is conscious of the magnitude of adverse outcomes. This endows the S-MOS criterion with the ability to address the specific deficiency of commonly used measures for decision making under uncertainty, such as the probability and the VaR measure, in not being sensitive to the degree of shortfall in the evaluation of alternatives. The S-MOS criteria has been described in a general form that allows it to account for distributional ambiguity when only partial distributional information are available for optimization. Other specific forms of the DMOS criteria can be developed that adheres to the fundamental characteristics of MOS and diversification preference to consider other salient characteristics, such as the joint awareness of the degree of shortfalls and excesses against the targets. Such considerations may still preserve the characteristics of the DMOS within the MOS framework using clever formulations, such as by assuming excesses against the targets using reflection, or it may require the definition of an entirely new family of measure that is beyond the MOS framework. This development may be useful given the existence of target-based decision problems that require the consideration of the magnitude of excess in target achievement in evaluating the decision. All these problem classes fall within the realm of target-based decision making, although not all adheres to the core satisficing concept defining the fundamental scope of this thesis.

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