# CALIBRATION TO SWAPTIONS IN THE LIBOR MARKET MODEL

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# Abstract

In this dissertation, the Libor Market Model is presented and its calibration process is derived. We assume the Forward Libor Rates follow log-normal stochastic processes with a $d$-dimensional Brownian motion and build an interest rates model able to price interest rate derivatives. We emphasize how different it is from the usual short-term interest rates models (Hull-White). Nevertheless, this pricing model only makes sense if vanilla products, namely caps and European swaptions, can be well priced with respect to their market value. To check this, we propose different parametric forms of instantaneous volatilities $\sigma_i(t)$ and correlations $\rho_{ij}$ to obtain the best results. Then, we show a method to reduce the dimensionality of the Libor Market model compared to the number of Forward rates involved by using Rebonato Angles and Frobenius norm. Finally, we derive approximations formula for European swaptions and show we can avoid Monte-Carlo simulations for the calculations of the swaptions during the calibration. Some numerical results are given on a 3 factors model.

We discuss then different issues raised and current developments, more specifically the SABR skew form and cross-asset products.

Keywords : Interest Rate Derivatives, Libor Market Model, Calibration, Rank reduction methods, Swaption Approximations.

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# Chapter 1

# Interest Rates Models

At the end of the 70's, after Black and Scholes breakthrough with their formula to value a European option, Black also proposed the alter ego of this formula in the world of interest rates. This was the beginning of the interest rates derivatives.

Since 1976 and Black's formula [2], a lot has been proposed on the interest rates topic. First were presented models that tried to adapt the frameworks coming from the equity world : those used a stochastic equation to describe a short-term rate as it was done for a stock. From this basic idea different evolutions rose by changing the form of this stochastic differential equation to fit the economic behavior of the interest rates generally observed - for instance the mean reversion phenomenon. Finally in 1997, Brace, Gatarek and Musiela proposed a new concept where observable rates were modeled using the work of Heath, Jarrow and Morton in 1992. This completely redefined the vision of pricing and everything needs to be done in this field.

The purpose of this model is undoubtedly to be able to fit the market. Hence, we call calibration the choice of the different assumptions and inputs so that we obtain the best fit to the market.
Calibration is always a huge issue for market operators as they may face severe misprices if the model they use is not well calibrated and I will be

presenting how this can be handled in the second part; before explaining what are the main issues and how some are managed (skew/smile, liquidity..) and what are the next challenges faced by the Libor Market Model (Cross-asset hybrid products).

In this first chapter the main definitions and the models currently used in the world of interest rates are defined and explained.

## 1.1 Important concepts

### 1.1.1 Zero coupon bonds

The first concept we have to define when discussing interest rates products is the Zero coupon bond (Z.C.). In this thesis, the underlying assets are not stocks like in Black-Scholes original framework in 1973 in [1] but bonds. Several bonds can be defined, paying various coupons, depending on some conditions...[1] Hence, it is necessary to define a *simplest* underlying: this one is the set of discount factors for different maturities. We will denote them by $B(t,T)$. This bond represents at time $t$ the price of 1 paid at time $T$, the maturity of the bond. See Figure 1.1 for a more visual explanation.

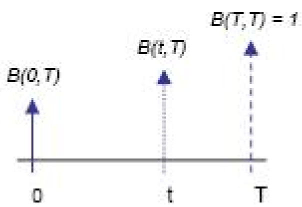


Figure 1.1: Zero-coupon bond mechanism

[1]For instance, a daily range accrual coupon: I pay $X\%\frac{n}{N}$ where $n$ is the number of days 3-months LIBOR rate stays below 6.5% and $N$ the number of days in the accrual period.

One can observe that at any date $t$, those prices are not all quoted on the market but can be obtained from other zero coupons bonds. This bond does not pay any coupon, that is why we generally call the discount factors $B(t,T)$ the Zero coupon bonds (Z.C.).

We introduce very generally the log-normal dynamic for a Zero Coupon bond as:

$$dB(t,T) = m(t,T)tB(t,T)dt + \sigma^B B(t,T)dW_t, \qquad B(T,T) = 1 \qquad (1.1)$$

With $m(t,T)$, the drift, equal to the short term interest rate $r_t$ in a risk-neutral world, $\sigma^B$, the volatility eventually stochastic or time-dependent and $W_t$ a Brownian motion.

### 1.1.2 Short-Term interest rate

We just mentioned the short term interest rate in the previous section. Traditional stochastic interest rates models are based on the exogenous specification of a short-term interest rate and its dynamic. We will denote by $r_t$ the *instantaneous interest rate* or *short-term interest rate* the rate one can borrow in a risk free loan beginning at $t$ over the infinitesimal period $dt$.

In general, we assume that $r_t$ is an adapted process on a filtered probability space. The important thing about short term interest rate is that by consideration over the absence of arbitrage in the market we can create links between $r_t$ and $B(t,T)$.

### 1.1.3 The Arbitrage free assumption

This classic assumption introduces constraints on the payoff of derivatives. Here when we study rate issues, this assumption is made on the Zero coupon bonds as we can link long maturities (more than 1 year) bonds with coupons with Zero coupon bonds by considering the Arbitrage free assumption.

*The price of an asset delivering fixed cash-flows in the future is given by the sum of its cash-flows weighted by the price of the Zero coupon bonds of the settlement dates.*

We make the usual mathematical assumption: all processes are defined on a probability space $(\Omega, \{\mathfrak{F}_t; t \geq 0\}, \mathbb{Q}_0)$. The probability measure $\mathbb{Q}_0$ is any risk neutral probability measure whose existence is given by the no-arbitrage assumption (See The Girsanov transformation in section 1.1.6). The filtration $\{\mathfrak{F}_t; t \geq 0\}$[2] is the filtration generated in $\mathbb{Q}_0$ by a $d$-dimensional Brownian motion $W^{\mathbb{Q}_0} = \{W^{\mathbb{Q}_0}(t); t \geq 0\}$.

Now, we infer that one can invest in a savings account continuously compounded with the stochastic short rate $r_s$ prevailing at time $s$ over the time $[s; s + ds]$. The value of 1 invested at time $t$ at time $T$ is $\beta_T$:

$$\beta_T = \exp \int_t^T r_s ds$$

Therefore, if we invest $B(t,T)$ in a Z.C. of maturity $T$ and the same amount in our saving account, the fundamental theorem of asset pricing (this will be detailled in 1.1.6) ensures that they produce *on average over all the paths* the same amount namely 1. This equality at time $t$ can be written:

$$B(t,T) = \mathbb{E}_t^{\mathbf{Q}_0} \left[ \exp \left( \int_t^T -r_s ds \right) |\mathfrak{F}_t \right]$$

In the case of a deterministic rate $r_s$, as $B(T,T) = 1$:

$$B(t,T) = \exp \left( \int_t^T -r_s ds \right)$$

[2]In a financial point of view, the filtration$\{\mathfrak{F}_t; t \geq 0\}$ represents the structure of all the information known by every market agent.

And in the case of a constant deterministic rate $r$ compound $n$-times per year:

$$B(t,T) = \frac{1}{(1+\frac{r}{n})^{(T-t)}} \tag{1.2}$$

### 1.1.4 Forward Interest rates

We can define Forward Interest Rates for all the previous rates we saw:

- $B_t(T, T+\delta)$ is the forward value at $t$ of a Z.C. invested at $T$ which will pay 1 at $T+\delta$. By arbitrage we know it is worth:

$$B_t(T, T+\delta) = \frac{B(t, T+\delta)}{B(t,T)}$$

- The equivalent rate simply compounded to this Zero Coupon Bond can be computed writing:

$$F_\delta(t,T) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right) \tag{1.3}$$

This rate is named the *Forward Rate* and is the constant rate simply compounded to be paid if you want to borrow money at time $t$ for a future time period between $T$ and $T+\delta$.
We can also define $f(t,T)$ the instantaneous forward interest rate, the forward version of $r_t$. Formally, $f(t,T)$ is the forward rate at $t$ one can borrow in a risk free loan beginning at $T$ over the infinitesimal period $dt$. This concept is rather a mathematical idealization as it can not be observed in the market but is useful to describe bond price models. One can write:

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right), \quad \forall t \in [0,T] \tag{1.4}$$

### 1.1.5 LIBOR interest rate and swaps

#### Libor interest rate

During the 80's, Libor (which stands for London Inter Bank Offered Rates) interest rates have become more and more traded. This rate is declined for different short maturities (inferior to one year) and is a benchmark of the main banks of their loan rate for those maturities. It is fixed everyday at 11h00 am, London Time. It is considered in general as the risk-free interest rate by the investors: even credit default swaps values are given with respect to the LIBOR curve. However, *this is not true*, those financial institutions have a probability of default and hence this default risk is quantified. In the markets, the risk free does not really exist but it can be assumed that the main central banks (More specifically: US Fed, ECB, CBE) have an almost nil probability of default as they can literally print their money and hence the bonds they issue called treasuries have almost no probability of default[3]. The spread between the LIBOR and the treasury rate represents this risk to default. For the USD Market, LIBOR rates trade around 50 basis points above treasury rates.

We call $L_\delta(t,t)$, the LIBOR Interest rate at time $t$ for a maturity of $\delta$:

$$\frac{1}{1+\delta L_\delta(t,t)} = B(t,t+\delta) \tag{1.5}$$

with $\delta$ is three or six months usually.

Using the arbitrage free rule and applying the previous section about Forward Interest rates to Libor Interest Rates and their Forwards $L_\delta(t,T)$ the Libor rate at time $t$ at which one can borrow money at time $T$ for a maturity of $\delta$ we can write:

$$\frac{1}{1+\delta L_\delta(t,T)} = \frac{B(t,T+\delta)}{B(t,T)}$$

[3]It should be emphasized that the sovereign risk is real: in July 1998, Russia defaulted on its bonds causing the fall of the famous hedge-fund LTCM.

That is,

$$L_\delta(t,T) = \frac{B(t,T) - B(t,T+\delta)}{\delta B(t,T+\delta)} \tag{1.6}$$

We will skip the index $\delta$ when there will be no ambiguities about the maturity.

### Swap rate

The first swap contracts were also negotiated in the early 1980s. Since, it has shown an amazing growth becoming more and more important in the exotic derivatives market.

A swap is a contract between two companies to exchange a predefined cash flow in the future. The schedule of the cash flows and the way they are calculated is specified in this agreement. At the beginning, swaps were tailored for companies who wanted to hedge their loans exposure and lock in a good level of interest rate.

Hence one can decide to enter a swap where he will exchange his semi-annual fixed rates cash-flows at $x\%$ against a floating rate, for instance the value of the 6-months LIBOR rate with fixing date at the beginning of the 6-months period (Fixing in advance [4]) The following Figure 1.2 explains how is built the exchange of cash-flows from the customer point of view. This type of

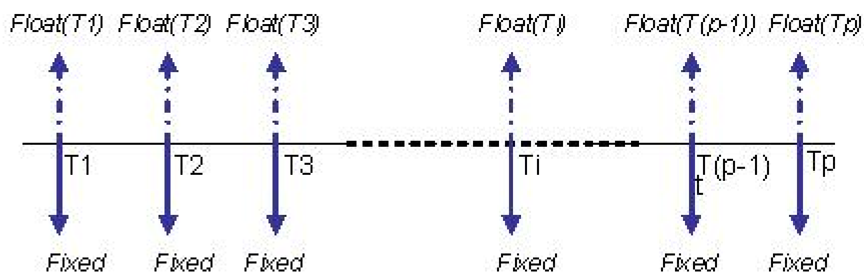


Figure 1.2: Exchange of cash-flows for a Payer Swap

[4]Several issues are not mentioned here about the fixing dates and the convexity adjustment that are necessary when pricing non perfectly scheduled structure or in arrears fixing structures, for instance see [3]

swap is called is called a payer swap. The symmetric version is called receiver swap.

As a matter of fact, from this definition appears the swap rate $S_{p,n}(t)$defined as the rate which gives a net present value of 0 at time $t$ to the swap which exchange this swap rate against a floating one ($\delta$-months Libor $L_\delta(t, T_i)$) on a schedule $T_i, i = p, \ldots, n$. We can compute this swap rate $S_{p,n}(t)$ by arbitrage considerations and, it is worth noticing it, independently of any model assumption.

The fixed leg is:

$$Fixed_{p,n}(t) = \sum_{i=0}^{n-p} S_{p,n}(t) \delta B(t, T_{p+i})$$

And the floating leg is:

$$\begin{aligned} Floating_{p,n}(t) &= \sum_{i=1}^{n-p} B(t, T_{i+p}) \delta L(t, T_{i-1+p}) \\ &= \sum_{i=1}^{n-p} B(t, T_{i+p}) \left( \frac{B(t, T_{i-1+p})}{B(t, T_{i+p})} - 1 \right) \\ &= \sum_{i=1}^{n-p} B(t, T_{i-1+p}) - B(t, T_{i+p}) \\ &= B(t, T_p) - B(t, T_n) \end{aligned}$$

The swap rate is by definition the one that equalize both legs:

$$\begin{aligned} Fixed_{p,n}(t) &= Floating_{p,n}(t) \\ S_{p,n}(t) &= \frac{B(t, T_p) - B(t, T_n)}{\sum_{i=0}^{n-p} \delta B(t, T_{p+i})} \end{aligned}$$

This swap was more precisely a forward start interest rate swap which first settlement date is $T_p$. Once this product was well understood by every one on the markets, it naturally gave rise to its first most natural derivative:

the European swaption [5]. A European swaption is a one-time option on a swap rate. From now, we will always refer to *European swaptions* when we describe swaptions. When one is long a swaption strike $S_{p,n}$, he owns the right and not the obligation to enter a swap of tenor $T_n$ at maturity $T_p$.
A swaption can be computed through different methods but the market in general quotes the implied volatility of the swaption with the generalization of the Black formula (See section 1.3.2). On the mathematical side this arise issues as one can show that swap rates and forward rates can not be log normal at the same time. We will discuss later this point in section 2.4.

### 1.1.6 Stochastic tools

This subsection is going to present a few stochastic tools we need to describe the basics of the Libor Market Model. This subsection does not seek to be exhaustive and *totally rigorous* in stochastic calculus but just to give a general idea about the tools we will be using in the construction of the models in the next section. For further details about stochastic calculus please refer to the excellent [5].

#### Numeraire

A *Numeraire* is a price process $(A(t))_T$ (a process is a sequence of random variables), which is strictly positive for all $t \in [O, T]$.
Numeraires are used to express prices in order to have *relative prices*. The application of this rather abstract concept can be seen in what follows.

#### Change of numeraire

Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent measures[6] with respect to the numeraires $A(T)$ and $B(t)$. The Radon-Nikodym derivative that changes the equivalent mea-

[5] American and Bermudean swaption also exist but are not as liquid and as vanilla than European

[6] $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if : $\mathbb{P}(M) = 0 \leftrightarrow \mathbb{Q}(M) = 0, \quad \forall M \in \mathfrak{F}$

sure $\mathbb{P}$ in $\mathbb{Q}$ is given by:

$$R = \frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{A(T)B(t)}{A(t)B(T)} \tag{1.7}$$

This derivative is very useful: due to the no arbitrage rule the price of an asset $X$ should be independent from the choice of the measure and numeraire:

$$A(t)\mathbb{E}^P\left[\frac{X(T)}{A(T)}|\mathfrak{F}_t\right] = B(t)\mathbb{E}^Q\left[\frac{X(T)}{B(T)}|\mathfrak{F}_t\right]$$

If one introduces: $G(T) = \frac{X(T)}{A(T)}$ and doing some simple manipulation on the previous equation:

$$\begin{aligned}\mathbb{E}^P(G(T)|\mathfrak{F}_t) &= \mathbb{E}^Q\left(G(T)\frac{A(T)B(t)}{A(t)B(T)}|\mathfrak{F}_t\right)\\ &= \mathbb{E}^Q(G(T)R|\mathfrak{F}_t)\end{aligned}$$

We can see that we can change the probability measure just by multiplying the martingale by its Radon-Nikodym derivative.

**Girsanov theorem**

For any adapted stochastic process $k(t)$ which satisfies the following condition:

$$\mathbb{E}\big(e^{\frac{1}{2}\int_0^t k^2(s)ds}\big) < +\infty,$$

Consider the Radon-Nikodym derivative $R = \frac{d\mathbb{P}}{d\mathbb{Q}}$ given by:

$$R = \exp\left(\int_0^t k(s)dW(s) - \frac{1}{2}\int_0^t k^2(s)ds\right),$$

where $W$ is a Brownian motion under the measure $\mathbb{Q}$.
Under the measure $\mathbb{P}$ the process

$$W^{\mathbb{P}}(t) = W(t) - \int_0^t k(s)ds,$$

is a Brownian motion.

The main consequence of the Girsanov theorem is that when one changes measures the drift component is impacted but the volatility component remains unaffected. One can say that switching from one measure to another just changes the relative likelihood of a particular path being chosen. For example the Brownian motion $W(t)$ above might follow a path which drifts downward at a rate of about $-k$ but under the measure $\mathbb{P}$ it is more likely to drift to 0. The general purpose of this theorem is to get rid of the drift. For proof of the previous theorem, please consider [5], page 153-157.

**Equivalent Martingale Measure** An Equivalent Martingale Measure (EMM) $\mathbb{Q}$ is a probability measure on the space $(\Omega, \mathfrak{F})$ such that:

- $\mathbb{Q}$ and $\mathbb{Q}_0$ are equivalent
- The Radon-Nykodym derivative $R = \frac{d\mathbb{Q}_0}{d\mathbb{Q}}$ is positive
- The process $W^{\mathbb{Q}}(t) = W^{\mathbb{Q}_0}(t) - \int_0^t k(s)ds$ is a martingale with respect to $\mathbb{Q}$.

**Fundamental Theorem of Asset Pricing**

All these definitions led us to the fundamental theorem.[7] :

> A market has no-arbitrage opportunity if and only if there exists an EMM.
> A market is complete (All contingent claims can be replicated using admissible portfolio) if and only if there exists a unique EMM.

**Forward measure**

We name *Forward measure*, $\mathbb{P}^i$, the probability measure with as numeraire

[7] This theorem is very well proved and described in [5]

the Zero coupon bond maturing at $T_i$, namely $B(t, T_i)$.

Under this measure,

$$\frac{X(t)}{B(t, T_i)}$$

is a martingale for all contingent claim $X(t)$ and we can price it saying:

$$X(t) = B(t, T_i)\mathbb{E}^i[X(T_i)|\mathfrak{F}_t]$$

**Spot measure**

Using the definition of Jamshidian in [13] we introduce the *spot measure*. Consider a portfolio of Zero coupon bond created by the investment strategy following:

- At $t = 0$, we invest 1 buying $\frac{1}{B(0,T_1)}$Zero coupon maturing at $T_1$
- At $t = T_1$, we receive $\frac{1}{B(0,T_1)}$ and we buy $\frac{1}{B(0,T_1)}\frac{1}{B(0,T_2)}$Zero coupon maturing at $T_2$
- At $t = T_2$, we receive $\frac{1}{B(0,T_1)}\frac{1}{B(0,T_2)}$ and we buy $\frac{1}{B(0,T_1)}\frac{1}{B(0,T_2)}\frac{1}{B(0,T_3)}$ Zero coupon maturing at $T_3$
- ...

Hence, at every $t$, one hold a portfolio of $\frac{1}{\prod_{j=1}^{\lceil t \rceil} B(T_{j-1},T_j)}$ (where $\lceil t \rceil$ is the next date in the tenor). This portfolio can be chosen as a numeraire for a certain measure that we will call the spot measure noted $\mathbb{P}^*$.

## 1.2 Interest Rates Models

Since they have been more and more used several models have been proposed to describe interest rates using different approaches. This part will describe the two models, the most used including at the Royal Bank of Scotland. For further details one can refer to [4] a very detailed review by Rebonato of how these models were built and how did we get there.

### 1.2.1 Short term interest rates

The first generation of models to price Interest Rates structured products were developed in the early 80's. Since, numerous models have been created and we will not describe all of them as the purpose of this part is to show how is built the next generation of models.
For enrichment purpose one can consider other important short term structure models, including Cox, Ingersoll and Ross Model [6], Ho-Lee [7], Black-Karasinski [8], Vasicek [9], Rendleman and Bartter[10].
The most used short-term interest rates model in the financial industry is the one by Hull and White (with one or two factors). Actually, this model is a generalization of the anterior Vasicek model (See [9]). Hull and White are considering a Vasicek model which models the instantaneous short-term interest rate as:

$$dr = a(b - r)dt + \sigma dz, \qquad a, b, \sigma \ constant \tag{1.8}$$

#### Mean Reversion

This model is describing the *mean-reversion* phenomenon: unlike a stock, interest rates appear to be pulled back to some long-run average level over time. Practically, it means that when $r_t$ is high, mean reversion tends to cause it to have a negative drift; when $r_t$ is low, mean reversion tends to cause it to have a positive drift.

This feature can be justified economically; basically, when rates are high, the economy tends to slow down and the demand for fund from borrower decrease. Hence, rates tend to go down, so the demand for fund from borrowers increase and rates tend to increase.
In Vasicek model, the short rate tends to go to $b$ at a rate $a$. The idea of Hull and White is to use the same rate $a$ and the same constant volatility but to add a time dependent feature to the *mean* value: $\frac{\theta(t)}{a}$.

### Hull-White Model

Using these considerations, the Hull-White model consider the instantaneous short term dynamics as:

$$dr = [\theta(t) - ar]dt + \sigma dt \tag{1.9}$$

where the parameters are as explained in the previous section.
The $\theta(t)$ function can be expressed from the initial term structure by using a change of numeraire. We get:

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

Assuming that the last term is very small (which is true in practice), this equation implies that the short term interest rate $r_t$ follows the slope of the initial instantaneous forward rate curve. When it deviates from this curve, it reverts back to $a$, following the *mean-reversion* feature.

Bond prices can be derived using Vasicek [9] idea. First, one can write the partial differential equation verified by any contingent claim and then apply the boundaries conditions to obtain the price of the zero coupon bond. Hence, the price $B(t,T)$ at time $t$ of a Z.C. bond maturing at $T$ can be given using (1.10) in terms of the short rate at time $t$ and the prices of the Z.C. bond today $B(0,T)$ and $B(0,t)$.

$$B(t,T) = C(t,T)\exp^{-D(t,T)r(t)} \tag{1.10}$$

where,

$$D(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

and,

$$\ln C(t,T) = ln\frac{B(0,T)}{B(0,t)} + B(t,T)F(0,t) - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{aT})^2(e^{2at} - 1))$$

With these equations we have defined everything in our model to price any contingent claim.

The issue about this model is that the underlying, namely *the short-term interest rate* is not an *observable* of the market. On the contrary, some zero coupon bonds are traded in a liquid way in the market and hence are *observable* of the market. It would be easier to have a model that describes observable products like Forward rates. This is the purpose of the Libor Market Model.

### 1.2.2 Heath Jarrow and Morton Framework

The previous frameworks we just discussed are easy to implement and give, when used with caution, good prices with respect to actively traded instruments like caps and floors.
However, there are limitations to this approach: the volatility structure is a deterministic function of time and one can not adapt this structure in the time as the volatility structure in the future will probably different from the one observed in the market at $t$.

In 1992, Heath, Jarrow and Morton published an important paper [11] to describe the no-arbitrage condition that must be satisfied by every model of yield curve.
The main idea is to consider the dynamics of instantaneous, continuously compounded forward rates $f(t,T)$ instead of the short-term rate $r$. At time $t$, for a maturity $T+dt$:

$$df(t,T) = a(t,T)dt + \gamma(t,T) \cdot dW_t, \tag{1.11}$$

where $a(t,T)$ and $\gamma(t,T)$ are adapted stochastic processes and $W_t$ is a $d$-dimensional standard Brownian motion with respect to the actual probability $\mathbb{P}$. This rate corresponds to the rate that one contract for at time $t$ on a risk less loan that begins at date $T$ and is returned an instant later.[8]

The assumption of no arbitrage in this market implies a unique relation between the drift $a$ and the volatility $\gamma$. The purpose of this section is to find out what is this relation.

The no instantaneous forward rate in the continuously compound way (same process that for determining (1.3)) is related to the Zero Coupon bond; by arbitrage we have:

$$F_t(T, T+\delta) = \frac{1}{\delta} \ln\left(\frac{B(t,T)}{B(t,T+\delta)}\right)$$

Hence when $\delta$ goes to 0, we can find $f(t,T)$:

$$f(t,T) = -\frac{\partial \ln(B(t,T))}{\partial T} \tag{1.12}$$

Then by applying the Itô lemma to (1.12) with the dynamic given in (1.1) one can get:

$$df(t,T) = \sigma^B(t,T)\frac{\partial \sigma^B(t,T)}{\partial T}dt - \frac{\partial \sigma^B(t,T)}{\partial T}dW_t \tag{1.13}$$

This equation gives the link between the drift and the volatility of the instantaneous forward rate $f(t,T)$. Therefore, integrating between $t$ and $T$, one can obtain:

$$\sigma^B(t,T) - \sigma^B(t,t) = \int_t^T \frac{\partial \sigma^B(t,\tau)}{\partial \tau}d\tau$$

We set $\sigma^B(t,t) = 0$ as it seems obvious that the volatility of a Zero Coupon bond at maturity is nil, and:

$$\sigma^B(t,T) = \int_t^T \frac{\partial \sigma^B(t,\tau)}{\partial \tau}d\tau \tag{1.14}$$

[8] One can notice that this is just the forward version of the instantaneous rate $r_t = f(t,t)$

Using the notation of the preliminaries of this section we can write the fundamental HJM result:

$$a(t,T) = \gamma(t,T)\int_t^T \gamma(t,\tau)d\tau \tag{1.15}$$

**Remark:** This result was proved in a one factor case. It is quiet straight forward to show it with several independent factors, see [11]. If we suppose in a risk neutral world a dynamic for the instantaneous forward rate such that:

$$df(t,T) = a(t,T)dt + \sum_{k=1}^{d} \gamma^k(t,T)dW_k \tag{1.16}$$

with the $\gamma^k(t,T)$ are a family of volatility coefficients for each factor $W_k$ (Independent Brownian motions) left unspecified except on integrability and measurability (quiet weak conditions) then one can get:

$$a(t,T) = \sum_{k=1}^{d} \gamma^k(t,T)\int_t^T \frac{\partial \gamma^k(t,\tau)}{\partial \tau} d\tau \tag{1.17}$$

This new condition is applicable to every interest rates models, including short-term interest rates models like the Hull-White one we reviewed before. But it still gives condition on an *unobservable* of the market, the instantaneous forward rate.

However, this new implied condition gave a new angle of study and Brace, Gatarek and Musiela in [12] have applied it to Forward Libor rate, which are directly observable on the market, developing the so-called *Libor Market Model*

### 1.2.3 The Libor Market Model

This model is very important nowadays in the financial industry and is subject to a lot of research in the banks including the Royal Bank of Scotland as it is harder to implement than the short rate model in term of calibration.

### General Principle

As told previously this model is using as inputs the forward rates and from them build the Zero Coupons curve. The fundamental assumption is that forward rates follow a log-normal dynamic. One can notice that this practice is directly taken from equity markets: operators are looking for a model so that Libor rates and swaps rates follow a log-normal process.
One should highlight the fact that this assumption is not related to the central limit theorem as it is for equity prices but because historically the market quotes Libor rates and swaps rates using Black volatility model in [2]. Hence, the log normal assumption for those rates arises naturally.

### Assumption on the dynamics of the Forward Libor Rates

In 1997, Brace *et al.* proposes a model where the Libor rates follow a log normal process in the forward measure associated. Namely, for a given maturity $\delta$, (the typical maturity are 3, 6, 9 and 12 months), the associated forward Libor rate process $\{L(t,T); t \geq 0\}$ which is defined by

$$1 + \delta L(t,T) = \exp \int_T^{T+\delta} f(t,\nu) d\nu \tag{1.18}$$

follows a log normal process in the spot martingale measure $\mathbb{P}^*$(and a martingale process in its Forward measure $\mathbb{P}^i$):

$$dL(t,T) = (\ldots)dt + L(t,T)\gamma(t,T)dW_t^* \tag{1.19}$$

with $\gamma(t,T)$ a deterministic function bounded and piecewise continuous following the conditions to apply the Girsanov theorem.

### Main results

#### i. Setup of a unique yield curve form the Forward LIBOR Rate

We will use Jamshidian approach [13] to explain how this model is built and how it is related to the Zero coupon bond. We apply the Ito Lemma to the equality shown before and using 1.1:

$$
\begin{aligned}
1 + \delta L(t,T) &= \frac{B(t,T)}{B(t,T+\delta)} \\
\delta dL(t,T) &= \frac{B(t,T)B(t,T+\delta)(m(t,T) - m(t,T+\delta))}{B^2(t,T+\delta)} \cdot dt \\
&+ \frac{B(t,T)B(t,T+\delta)(\sigma^B(t,T) - \sigma^B(t,T+\delta))}{B^2(t,T+\delta)} \cdot dW_t \\
&+ \frac{B(t,T)B(t,T+\delta)(\sigma^B(t,T+\delta))^2 - B(t,T)B(t,T+\delta)\sigma^B(t,T+\delta)\sigma^B(t,T)}{B^2(t,T+\delta)} \cdot dW_t^2 \\
dL(t,T) &= \frac{B(t,T)\Big((m(t,T) - m(t,T+\delta)) - \sigma^B(t,T+\delta)(\sigma^B(t,T) - \sigma^B(t,T+\delta))\Big)}{\delta B(t,T+\delta)} \cdot dt \\
&+ \frac{B(t,T)(\sigma^B(t,T) - \sigma^B(t,T+\delta))}{\delta B(t,T+\delta)} \cdot dW_t
\end{aligned}
$$

Re-organizing this equation, we can find that:

$$
dL(t,T) = \mu(t,T)dt + \gamma(t,T)L(t,T)dW_t \tag{1.20}
$$

where:

$$
\mu(t,T) = \frac{B(t,T)}{\delta B(t,T+\delta)}\big(m(t,T) - m(t,T+\delta)\big) - \gamma(t,T)L(t,T)\sigma^B(t,T+\delta)
$$

and

$$
\gamma(t,T)L(t,T) = \frac{B(t,T)}{\sigma^B(t,T+\delta)}(\sigma^B(t,T) - \sigma^B(t,T+\delta))
$$

which gives the fundamental relation

$$
(\sigma^B(t,T) - \sigma^B(t,T+\delta)) = \frac{\delta L(t,T)\gamma(t,T)}{1 + \delta L(t,T)} \tag{1.21}
$$

Brace, *et al.* (1997) have noticed that the identification equation (1.21) is actually a recurrence relation on $\sigma^B(t,T)$:

$$\sigma^B(t,\lceil t\rceil))-\sigma^B(t,T+(j+1)\delta)=\sum_{k=\lceil\delta^{-1}t\rceil}^{j}\frac{(\delta L(t,t+k\delta))}{1+\delta L(t,t+k\delta)}\gamma(t,t+k\delta) \quad (1.22)$$

where $\lceil\delta^{-1}t\rceil$ is the the next integer.

If we assume the Spot Libor Measure $\mathbb{P}^*$ is equivalent to the market measure $\mathbb{P}$, we can assume the existence of $h_t$, some adapted process, the Radon Nykodym derivative of the two measures such that:

$$dW_t = dW_t^* + h_t dt$$

Using the change of numeraire techniques and the Ito Lemma, we can show that:

$$\frac{m(t,T)-m(t,\lceil t\rceil)}{(\sigma^B(t,T)-\sigma^B(t,\lceil t\rceil))}=\sigma^B(t,\lceil t\rceil)-h_t$$

Combining the previous equation with 1.21 we obtain:

$$\frac{B(t,T)}{\delta B(t,T+\delta)}\big(m(t,T)-m(t,T+\delta)\big)=\gamma(t,T)L(t,T)\cdot\big(\sigma^B(t,\lceil t\rceil)-h_t\big)$$

So we finally get to:

$$dL(t,T)=\gamma(t,T)L(t,T)\Big(\big(\sigma^B(t,\lceil t\rceil)-\sigma^B(t,T+\delta)-h_t\big)dt+dW_t\Big)$$

More exhaustively:

$$dL(t,T)=\Big(\gamma(t,T)L(t,T)\sum_{k=\lceil\delta^{-1}t\rceil}^{j}\frac{(\delta L(t,t+k\delta))}{1+\delta L(t,t+k\delta)}\gamma(t,t+k\delta)\Big)dt+L(t,T)\gamma(t,T)dW_t^* \quad (1.23)$$

This process finishes the setup of the yield curve dynamics as we are given the $\delta-$Libor rate process, the zero coupon volatility in (1.22) and the value of the forward curve today. What should be emphasize is that we have worked the other way that the other short term interest rates model: from the Forward rates known at time 0 (the *observables*) we have defined a unique yield curve dynamic using the arbitrage-free assumption and HJM result described in 1.2.2. Furthermore, the volatility of the zero-coupon is a priori stochastic.

**Remark 1:** Brace *et al.* (1997) have shown with details that the solution to this problem exists and is unique.

**Remark 2:** This model respects the principle of the mean reversion behavior of interest rates in the market as it can be well observed on empirical studies for instance in [12].

**Remark 3:** This expression is very convenient and was proposed by Jamshidian in [13] as it permits to implement numerically the Libor Market Model with only one expression on the opposite of the Forward measure ones. This is the purpose of the Libor Market Model.

**ii. Expression of the LIBOR Forward Rates under different numeraires (Forward measures)** Even if they are less convenient for computation these expressions give sense to what is behind the idea of the Libor Market Model.

Without loss of generality and for simplification purpose, we are going to consider from now a family of $\delta$ Libor forward rates $\{L(t, T_k), t \leq 0\}_n$ which matures at $\{T_k\}_n$. Hence, we will denote by $L_k(t)$ the Libor rate such that:

$$L_k(t) = L(t, T_k - \delta) \tag{1.24}$$

With the new notations for the Forward rates $L_i(t)$ the previous expression in the spot martingale measure becomes:

$$L_k(t) \sum_{j=1}^{T_k} \frac{\delta L_j(t)(\gamma_k(t) \cdot \gamma_j(t))}{1 + \delta L_j(t)} dt + L_k(t)\gamma_k(t) dW_t^* \tag{1.25}$$

Consider the probability measure $\mathbb{P}^k$, the forward measure with maturity $T_k$, associated with numeraire $B(\cdot, T_k)$, the Zero coupon bond maturing at $T_k$. We have seen previously that:

$$L_k(t)B(t,T_k) = \frac{B(t,T_{k-1}) - B(t,T_k)}{\delta} \tag{1.26}$$

One can observe that we can replicate $L_k(t)B(t,T_k)$ by buying and selling the bonds $B(t,T_{k-1})$ and $B(t,T_k)$. Furthermore, the price of the asset $L_k(t)B(t,T_k)$ divided by the numeraire $B(\cdot,T_k)$ is a martingale under $\mathbb{P}^k$ and is as a matter of fact $L_k(t)$. So one can write:

$$dL_k(t) = L_k(t)\gamma_k(t)dW_t^k, \qquad t \leq T_{k-1} \tag{1.27}$$

For the other cases in order to express $L_k(t)$ in the forward measure $\mathbb{P}^i$, we are going to use Girsanov transformation for $\mathbb{P}^k$ to $\mathbb{P}^i$. We can show that case $i < k$ as the case $i > k$ is analogous.
We proceed by recurrence. The Radon Nikodym derivative associated to the change of numeraire from $\mathbb{P}^k$ to $\mathbb{P}^{k-1}$is:

$$R = \frac{\partial \mathbb{P}^{k-1}}{\partial \mathbb{P}^k} = \frac{B(t,T_{k-1})}{B(t,T_k)}\frac{B(T_{k-1},T_k)}{B(T_{k-1},T_{k-1})} \tag{1.28}$$

According to Girsanov theorem we know that $R$ is an exponential martingale under $\mathbb{P}^k$ such that it exists $\phi$ a regular process[9] so:

$$\frac{dR}{R} = \phi \; dW_t^k$$
$$\text{where} \quad dW_t^k = dW_t^{k-1} + \phi \; dt$$

[9]Regular here means several conditions including integrable in $\mathcal{L}^2$

We are going to determine this process $\phi$

$$\frac{dR}{R} = \frac{d(\frac{B(t,T_{k-1})}{B(t,T_k)})}{\frac{B(t,T_{k-1})}{B(t,T_k)}} = \frac{d(1+\delta L_k(t))}{1+\delta L_k(t)}$$

$$= \frac{\delta dL_k(t)}{1+\delta L_k(t)} = \frac{\gamma_k(t)L_k(t)}{1+\delta L_k(t)} dW_t^k$$

Therefore when assembling the two sides,

$$dW_t^k = dW_t^{k-1} + \frac{\gamma_k(t)L_k(t)}{1+\delta L_k(t)} dt \tag{1.29}$$

An important thing to remind is that in a model with $d$ factors, $dW_t$ is a $d$-dimension Brownian motion and $\gamma_k(t)$ is a $d$-dimension vector.

By recurrence, we can exogenously give the dynamic of the $k$-th forward rate under measure $i$. Finally, summing up the different expressions of $L_k(t)$ under the Forward measure $\mathbb{P}^i$:

$$dL_k(t) = \begin{cases} L_k(t)\sum_{j=i+1}^{k} \frac{\delta L_j(t)(\gamma_k(t)\cdot\gamma_j(t))}{1+\delta L_j(t)} dt + L_k(t)\gamma_k(t) dW_t^k, & i < k, t \leq T_i; \\ L_k(t)\gamma_k(t) dW_t^i, & i = k, t \leq T_{k-1}; \\ -L_k(t)\sum_{j=i+1}^{k} \frac{\delta L_j(t)(\gamma_k(t)\cdot\gamma_j(t))}{1+\delta L_j(t)} dt + L_k(t)\gamma_k(t) dW_t^k, & i < k, t \leq T_{k-1}; \end{cases} \tag{1.30}$$

with $W^i$ the standard $d$-dimensional Wiener process under $\mathbb{P}^i$.

All the point with this description of the Libor Forward Rates is that we can see arise the correlation between those Forward rates:

$$\gamma_i(t) \cdot \gamma_j(t) = \sum_{k=1}^{d} (\gamma_i)_k (\gamma_j)_k$$

$$\gamma_i(t) \cdot \gamma_j(t) = \rho_{ij} \|\gamma_i\| \|\gamma_j\|$$

$$\text{what we note } = \rho_{ij}\sigma_i\sigma_j$$

With $\rho_{ij}$ the instantaneous correlation between $i$-th and $j$-th Forward rate. We will study in chapter 2 those two components $\sigma_i$ and $\rho_i$.

### 1.2.4 Libor Market model summary

The Libor market model is an interest rates model whose input are:

- A set of bond maturities $\{T_i\}_n$
- The Libor Forward rates at time zero $L_1(0), \dots, L_n(0)$
- The instantaneous volatilities of the forward rates $\gamma_i(.)$ for $i - 1, \dots, n$

The $\gamma_i(.)$ are the parameters of the BGM model and those need to be calibrated so that our model reflects correctly the prices of assets traded actively in the markets. This calibration procedure will be described in Chapter 2.

## 1.3 Pricing Vanilla Derivatives

Vanilla derivatives are the most liquid which makes them very efficient to track volatility information in interest rate markets. On the contrary of the Swaps and the Forward Rates in 1.1.5 we need in order to price them to use the previous models and assumptions we described before.

### 1.3.1 Interest rate options: cap and floor

Let consider a floating rate note where the interest rate is reset equal to LIBOR periodically (usually using a tenor of 3 months). To protect himself against the rise of LIBOR, the investor can buy an interest rate *cap* so that the floating-rate will not raise above a certain level: *the cap rate.*

In a forward cap, settled in arrears at time $T_j, j = 1 \dots n$, the cash-flows are $(L_j(T_j) - \kappa)^+\delta$ paid at time $T_{j+1}$ with a notional 1. The rule of no arbitrage and the discount factors $B(t, T_{j+1})$ gives the price of the cap:

$$cap_t = \sum_{j=0}^{n-1} B(t, T_{j+1}) \mathbb{E}^{j+1}[(L_j(T_j) - \kappa)^+ \delta] \tag{1.31}$$

where here $\mathbb{E}^{j+1}$ is the expectation under the forward measure $\mathbb{P}^{j+1}$ as we defined it in section 1.1.6. The formula (1.31) permits to consider a cap as a portfolio of $n$ interest rate options also known as caplets: the elementary cash-flow $(L_j(T_j) - \kappa)^+ \delta$ is the pay off of a call option on the LIBOR rate observed in arrears at time $T_j$ and settled at time $T_{j+1}$.

Similarly, one can define a *floor* which provides an insurance that the floating rate will not fall under the floor rate to be defined. The *floorlet* is a put option on the LIBOR rate observed at time $T_j$ and settled at time $T_{j+1}$.

### Pricing caplets with Black Formula

Using Black in [2] a closed formula for the price of a caplet can be derived. We assume that the forward rates are log-normally distributed under some probability measure $\mathbb{Q}$ [10]and have a constant volatility $\sigma > 0$.

$$dL_i(t) = L_i(t) \sigma dW_t \tag{1.32}$$

This stochastic differential equation can easily be solved:

$$L_i(t) = L_i(0) \exp^{(\sigma W_t - \frac{1}{2}\sigma^2 t^2)}, \quad \forall t \in [0, T_i], \tag{1.33}$$

and we know the initial condition by:

$$L_i(0) = \frac{1}{\delta} \left( \frac{B(0,T)}{B(t,T)} - 1 \right)$$

[10]No formal definition is available for this probability, we will refer to $\mathbb{Q}$ as the *market probability*

The payoff of the caplet with strike $\kappa$ at time $T_i$ over the LIBOR rate $L_i(T_i)$ on a notional amount 1 is:

$$1\delta \max\left(L_i(T_i) - \kappa, 0\right),$$

Then, the price of this caplet at time $t$ is:

$$Caplet^{Bl}(t) = \delta B(t, T_{i+1})\mathbb{E}^{\mathbb{Q}}\big((L_i(T_i) - \kappa)^{+}|\mathfrak{F}_t\big)$$

Using Black-Scholes formula we get, $\forall t \in [0, T_i]$,

$$Caplet^{Bl}(t) = 1\delta B(t, T_{i+1})[L_i(t)N(d_1(t, T_i)) - \kappa N(d_2(t, T_i))], \qquad (1.34)$$

$$\text{with,} \qquad d_1 = \frac{ln(L_i(t)/\kappa) + \sigma^2 \frac{(T_i - t)}{2}}{\sigma\sqrt{T_i - t}} \qquad (1.35)$$

$$d_2 = d_1 - \sigma\sqrt{T_i - t} \qquad (1.36)$$

where $N : R \to [0, 1]$ is the standard normal distribution: $N(x) = \frac{1}{\sqrt{2}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$.

For a cap, one can get:

$$Cap^{Bl}(t) = \sum_{j=0}^{n-1} \delta B(t, T_{j+1})\big(L_j(t)N(d_1(t, T_j)) - \kappa N(d_2(t, T_j))\big) \qquad (1.37)$$

The parameter $\sigma$ is usually referred to as the Forward volatility of $L_i$. Caps are quoted for indicative prices by the volatility for a strike equal to the forward rate, they are the famous at the Money *Black implied volatility.*

In order to get the floor price one can use the *cap-floor* parity which can be shown straightforward writing the cap and floor definitions and using the no-arbitrage property:

$$Cap(t) - Floor(t) = \sum_{i=0}^{n} (B(t, T_i)[L_i(t) - \kappa]) \tag{1.38}$$

**Pricing caplets in the Libor Market Model**

We follow the general idea of Miltersen et al. in [15]. As seen before we place ourselves in the forward measure $\mathbb{P}^i$. Under this measure the $i$-th Libor Forward rate is a martingale:

$$dL_i(t) = L_i(t)\gamma_i(t) \centerdot dW_t^i, \quad t \leq T_i \tag{1.39}$$

We recognize an exponential martingale in this stochastic differential equation and we can check by the Ito Lemma that the following is solution of this equation:

$$L_i(t) = L_i(0)e^{\int_t^{T_i} \gamma_i(s) \centerdot dW_s^i - \frac{1}{2}\int_t^{T_i} \|\gamma_i(s)\|^2 ds}, \quad t \leq T_i, \tag{1.40}$$

Hence $L_i(T_i)$ is a martingale under its measure and we can use the no arbitrage rule:

$$\begin{aligned} Caplet^{LMM}(t) &= \delta B(t, T_{i+1})\mathbb{E}^{i+1}\Big[(L_i(T_i) - \kappa)^+ | \mathfrak{F}_t\Big] \\ &= \delta B(t, T_{i+1})\mathbb{E}^{i+1}\big(L_i(T_i)\mathbf{1}_D | \mathfrak{F}_t\big) - \kappa\delta B(t, T_{i+1})Prob(D|\mathfrak{F}_t) \\ &= \delta B(t, T_{i+1})(I_1 - I_2), \end{aligned}$$

where $D = \{L_i(T_i) > \kappa\}$ is the exercise set.

Furthermore, $\gamma_i$ is a deterministic function, hence the probability law under $\mathbb{E}^i$ of its Ito integral is Gaussian with mean 0 and a variance $\zeta_i(t)$:

$$\zeta_i(t) = \int_t^{T_i} \|\gamma_i(s)\|^2 ds$$

So we get:

$$I_2 = \kappa N \left( \frac{\ln(L_i(t)) - \ln \kappa - \frac{1}{2}\zeta_i^2(t)}{\zeta_i(t)} \right) \tag{1.41}$$

The derivation is similar for $I_2$ and we will not reproduce it:

$$I_1 = L_i(t) N \left( \frac{\ln(L_i(t)) - \ln \kappa + \frac{1}{2}\zeta_i^2(t)}{\zeta_i(t)} \right) \tag{1.42}$$

Finally summing everything for the caplets and every caplets to get the cap price:

$$Cap^{LMM}(t) = \sum_{i=0}^{n-1} \delta B(t, T_{i+1}) \big( L_i(t) N(d1(t) - \kappa N(d2(t)) \big)$$

$$\text{with} \quad d_{1,2}(t) = \frac{\ln(L_i(t)) - \ln \kappa \pm \frac{1}{2}\zeta_i^2(t)}{\zeta_i(t)}$$

and

$$\zeta_i^2(t) = \int_t^{T_i} \|\gamma_i(s)\|^2 ds$$

Reminding (1.34) we can define $\sigma_n^{Black,LMM}$ the Black implied volatility of caplet priced by LMM.

$$\sigma_i^{Black,LMM} = \sqrt{\frac{1}{T_n} \int_0^{T_i} \|\gamma_i(s)\|^2 ds} \tag{1.43}$$

Hence, the BGM caplet can also be quoted in terms of its Black implied volatility. That is the way caplets are generally quoted using at the money rate. Using this formula we see well why the Libor Market Model is auto calibrated on the caplets volatilities as we have not done any approximation in this derivation.

Floor prices can be obtained by using the cap-floor parity equation shown previously in 1.38.

### 1.3.2 Swaptions

As previously described, we are now going to derive analytical formula to price a swaption, i.e. a contract where you pay a premium to get the option to enter a swap of a certain tenor at maturity where you pay a pre-negotiated fixed rate (the strike) against a floating one.

#### Black Formula

We have seen previously how to express the swap rate $S_{p,q}$. Thus, we are going to deduce the swaption price the same way as for caplets: that means we assume log-normality of the *forward swap rate* and *constant positive volatility* $\sigma$. Comparing the future cash-flows on a swap rate starting at $T_p$ with fixed rate $S_{p,q}(T_p)$ to those of a swap starting at $T_p$ with fixed rate $\kappa$, we can show the payoff of a payer swaption on a unitary notional as a series of caplet payoffs paid later :

$$\sum_{i=p+1}^{q} [\max(S_{p,q}(T_p) - \kappa) \tag{1.44}$$

Hence using the no-arbitrage assumption and in the market probability measure already mentioned $\mathbb{Q}$ we have:

$$Swaption_{p,q}^{Bl}(t) = \sum_{i=p+1}^{q} B(0,T_i)\mathbb{E}^{\mathbb{Q}}\big((S_{p,q}(T_p) - \kappa)_+ | \mathfrak{F}_t\big) \tag{1.45}$$

Hence we can use Black Formula 1.37 adapted to a delayed payoff (from $T_p$ to $T - i$) and one can get:

$$Swaption_{p,q}^{Bl}(t) = \sum_{i=p+1}^{q} B(0,T_i)[(S_{p,q}(t)N(d_1) - \kappa N(d_2)] \tag{1.46}$$

with

$$d_1 = \frac{ln((S_{p,q}(t)/\kappa) + \sigma^2 \frac{(T_i - t)}{2}}{\sigma\sqrt{(T_p - t)}}$$
$$d_2 = d_1 - \sigma\sqrt{T_p - t}$$

Finally we also obtain here a Black implied volatility which will be used later to give a price to those swaptions. I would like to emphasize the assumption of the log-normality of the forward swap rate *which is not the case in the Libor Market Model.*

**Pricing in the Libor Market Model - Swap Market Model**

In the Libor Market Model the pricing cannot be done using an exact closed formula and this is the purpose of chapter 2. However, one can develop the same model as the Libor Market Model but using the assumption that Forward swap rates are log-normal: this model is called the *Swap Market Model*. See [13] for further details about this model.
Hence, an exact price can be derived as for the caplets in LMM. With straightforward notations:

$$Swaption^{SMM}_{p,q}(t) = \sum_{i=p+1}^{q} B(t, T_i)[S_{p,q}(t)N(d_1) - \kappa N(d_2)] \tag{1.47}$$

with

$$d_{1,2}(t, T_i) = \frac{ln((S_{p,q}(t)/\kappa) \pm \frac{1}{2}\zeta^2(t, T_i)}{\zeta(t, T_i)}$$

and the Black volatility $\zeta$ computed as:

$$\zeta^2(t, T_i) = \int_t^{T_i} \|\nu_i(s)\|^2 ds$$

where $\nu_i$ is the deterministic volatility (well adapted) of the forward swap rate in the corresponding forward swap measure.

# Chapter 2

# Calibration of the Libor Market Model

## 2.1 The settings: Main purpose of the Calibration

Before starting a calibration, a list of calibration objects should be given. A calibration object can be either a caplet price, a forward rate correlation or a swaption price. Each of the entries in this list requires a precise description of the object itself - for instance, for a swaption: which tenor period the swaption is associated with and what the expiry date is - and of course market value of the liquid traded securities we consider.

Note that caplet and swaption prices are quoted here in implied volatilities. Say a calibration has M calibration objects, with market values $x_k^{Traded}, k = 1, \ldots, M$. Given a set of parameters, it is possible to compute the model values of the $M$ calibration objects with the formulas derived in the first part.

This will yield M model values $x_k^{Model}, k = 1, \ldots, M$. This will lead us to highlights $M$ different errors between the $k$-th model value $x_k^{Model}$ and the market value $x_k^{Traded}$.

As a bottom line, we add every errors to obtain how far our parameters for the models are from the market value. The calibration process consists in the minimization of this error over the parameters so as to get the model to resemble the market as close as possible. What we could sum up by:

$$\min_{param} \sum_{k=1}^{M} Error^k(x_k^{Model}(param); x_k^{Traded}) \tag{2.1}$$

In this part, we discuss the main methods of calibration of the Libor Market Model. By calibration we mean the computation of the parameters (the instantaneous volatilities and correlations) of the Libor Market Model so as to match as closely as possible derivative prices computed and observed prices of actively traded securities: caplets and swaptions.

It is very easy to calibrate the BGM model to caplet volatilities as it is almost straight forward because we assumed the log normality of the underlying (The forward rates). But in order to price products involving swap prices, we need to calibrate it also on the swaption market and the swap rates are not log-normal if the forward rates are.

First we have to take care of the volatility of the Forward rates that we defined previously: assuming their log-normality created this volatility. Different parameterizations are possible for this.

To price correctly we have to work using a sole numeraire (The spot measure) which implies a correlation between the different forward rates and changes the drift (which does not impact our study) This will lead us to the debate between historical and implied data. We will show different solutions for the parameterization of the correlation structure and in last section if we should choose historical data or implied data as inputs.

In this chapter we consider a Libor Market Model with $d$-factors described by:

$$dL_i(t) = \mu_i(t)dt + \sum_{k=1}^{d} \gamma_{ik}(t)dW_t^k \tag{2.2}$$

where all the $W_t^k$ are orthogonal and the $\gamma_{ik}$ are the *loadings* of each factors. We know that we have the relation:

$$\gamma_i \cdot dW_t = \sigma_i \sum_{k=1}^{d} b_{ik} dW_t^k \tag{2.3}$$

So we can see the relation with the correlation arises:

$$\begin{aligned}\gamma_i \cdot \gamma_j &= \sigma_i \sigma_j \rho_{ij} \\ &= \sigma_i \sigma_j \sum_{k=1}^{d} b_{ik} b_{kj}\end{aligned}$$

where $b_i$ are correlation vectors in $(\mathbb{R}^+)^d$ and $\gamma_k : [0, T_{k-1}] \to (\mathbb{R}^+)^d$.
On top of this, we have in order to ensure a good pricing of the caplets

$$\sum_{k=1}^{d} b_{ik}^2 = 1 \tag{2.4}$$

This description has the huge advantage to distinguish the volatility and the correlation information. Then a separate calibration is possible where $\sigma_i$ will influence price of the caplets (See [12]) and the choice of $(b_{ik}$ will influence the correlation structure.

## 2.2 Structure of the instantaneous volatility

As previously explained we have to give a shape to the instantaneous volatility of the forward rates. To clarify, we have to fill in the matrix given in 2.1. We are given the choice between several parameterizations for the structure with different advantages.

### 2.2.1 Total parameterized volatility structure

A first simple idea would be to choose a total parameterization considering that each $\sigma_{ij}$ is independent and fit the matrix to both caplets and swap-

| Instant. Vol | $t \in (T_0, T_1]$ | $(T_1, T_2]$ | $(T_2, T_3]$ | $\cdots$ | $(T_{M-1}, T_M]$ |
|---|---|---|---|---|---|
| $L_1(t)$ | $\sigma_{1,1}$ | dead | $\cdots$ | $\cdots$ | dead |
| $L_2(t)$ | $\sigma_{2,1}$ | $\sigma_{2,2}$ | dead | $\cdots$ | $\vdots$ |
| $L_3(t)$ | $\sigma_{3,1}$ | $\sigma_{3,2}$ | $\sigma_{3,3}$ | $\cdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $L_i(t)$ | $\sigma_{i,1}$ | $\sigma_{i,2}$ | $\sigma_{i,3}$ | $\cdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | dead |
| $L_M(t)$ | $\sigma_{M,1}$ | $\sigma_{M,2}$ | $\sigma_{M,3}$ | $\cdots$ | $\sigma_{M,M}$ |

Table 2.1: General volatility structure

tions. However as it is described in [17] this process involves numerous issues including over-parameterization. Though, the system only have a finite number of degree of freedom and cannot be constrained everywhere. That is why we need to consider a semi parameterized structure.

### 2.2.2 General Piecewise-Constant Parameterization

A very used structure is the one that makes the volatility depends only on *the distance to maturity.* For practical purposes, if we force the volatility to be constant on each time bucket, we can write:

$$\sigma_i(t) = \sigma(T_i - t) = \eta_{i-k}, \qquad t = [T_k; T_{k+1}]$$

Finally we can organize instantaneous volatilities in a matrix as follows: We can notice that due to the number of parameters, the main issue with this structure is that it does not allow a simultaneous calibration of both caplets and swaptions volatilities but only for one of them (in LMM, it is on caplets). See [19] for further details about it.

| Instant. Vol | $t \in (T_0, T_1]$ | $(T_1, T_2]$ | $(T_2, T_3]$ | $\cdots$ | $(T_{M-1}, T_M]$ |
|---|---|---|---|---|---|
| $L_1(t)$ | $\eta_1$ | dead | $\cdots$ | $\cdots$ | dead |
| $L_2(t)$ | $\eta_2$ | $\eta_1$ | dead | $\cdots$ | $\vdots$ |
| $L_3(t)$ | $\eta_3$ | $\eta_2$ | $\eta_1$ | $\cdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $L_i(t)$ | $\eta_i$ | $\eta_{i-1}$ | $\eta_{i-2}$ | $\cdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | dead |
| $L_M(t)$ | $\eta_M$ | $\eta_{M-1}$ | $\eta_{M-2}$ | $\cdots$ | $\eta_1$ |

Table 2.2: Piecewise-constant volatility structure

### 2.2.3 Laguerre function linear combination type volatility

Rebonato has proposed a more accurate structure adding one more parameter to the forward rates and keeping the assumption that volatility depends on the distance to maturity. As a matter of fact doing this we enrich the structure and permits a better fit with market prices (on both caplets and swaptions) than the previous one by adding a stationary part $\eta_{i-k}$:

$$\sigma_i(t) = c_i \eta_{i-k}, \qquad t = [T_k; T_{k+1}]$$

Once again we can sum up this structure in a new matrix:

| Instant. Vol | $t \in (T_0, T_1]$ | $(T_1, T_2]$ | $(T_2, T_3]$ | $\cdots$ | $(T_{M-1}, T_M]$ |
|---|---|---|---|---|---|
| $L_1(t)$ | $c_1\eta_1$ | dead | $\cdots$ | $\cdots$ | dead |
| $L_2(t)$ | $c_2\eta_2$ | $c_2\eta_1$ | dead | $\cdots$ | $\vdots$ |
| $L_3(t)$ | $c_3\eta_3$ | $c_3\eta_2$ | $c_3\eta_1$ | $\cdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $L_i(t)$ | $c_i\eta_i$ | $c_i\eta_{i-1}$ | $c_i\eta_{i-2}$ | $\cdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | dead |
| $L_M(t)$ | $c_M\eta_M$ | $c_M\eta_{M-1}$ | $c_M\eta_{M-2}$ | $\cdots$ | $c_M\eta_1$ |

Table 2.3: Laguerre type volatility structure

Of course, one can observe that we have introduced $2N$ parameters instead of $N$ in the previous one. To ease the computation, we are going to

use Rebonato idea about the stationary part of the volatility $\eta_i$.
Most of the time this part is a decreasing exponential with a small hump at the beginning of the curve. Financial justification for this hump can be found in [16]. The idea is to represent it using a linear combination of Laguerre functions, especially the two first.

$$\zeta_1 : \tau \to e^{-\frac{\tau}{2}}$$
$$\zeta_2 : \tau \to \tau e^{-\frac{\tau}{2}}$$

So we obtain for $\eta$:

$$\eta(\tau) = ae^{-\beta\tau} + b\tau e^{-\beta\tau} + c$$
$$\eta(\tau) = e^{-\beta\tau}(a + b\tau) + c$$

Without loss of generality we force:

$$\eta(0) = 1 = a + c$$

and we get with a slight change of notation to reflect what these constants represent :

$$\eta(\tau) = \eta_\infty + (1 - \eta_\infty + b\tau)e^{-\beta\tau}$$

And finally we get :

$$||\gamma_i(t)|| = \sigma_i(t) = c_i\eta(\tau) \tag{2.5}$$

This structure for volatility is a good choice between number of parameters and quality of the fit: compare to the previous structure, we have to propose values for $\eta_\infty, \beta, b$ on the top of the $c_i$ (they are here as normalization factors after the first coefficients have well reproduced the shape of the term-structure volatility) and this gives the best fit to the market as we can use also data from the swaption market (The piece-wise structure only permits to fit the caplet volatilities in each bucket). We can for instance set the $c_i$ using the Black volatility definition for a caplet and fit perfectly the caplet market and optimize the parameters of $\eta(\tau)$ on the swaption volatilities :

$$c_i = \frac{\sigma_i^{BS}\sqrt{T_i}}{\sqrt{\int_0^{T_i} \eta(T_i - s)ds}} \tag{2.6}$$

Hence, we will continue to use this instantaneous volatility term structure for the next parts. An example of such structure is given in 2.1

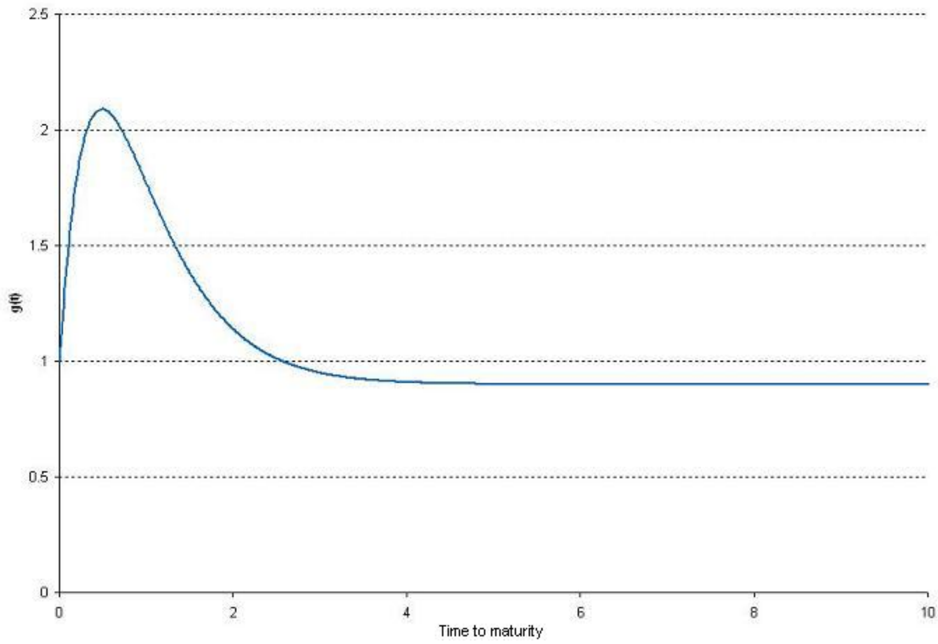


Figure 2.1: Example of a humped Laguerre-type instantaneous volatility for $b = 5.60,\ \beta = 1.75,$ and $\eta_\infty = 0.96$ before normalization by the $c_i$ factor

## 2.3 Structure of the correlation among the Forward Rates

To price an interest rate derivative, it seems pretty clear that we are going to face correlation issues among the state variables. Hence, we have to consider that the forward rates are correlated and to estimate this.

Let consider the family of the forward rates $\{L_i(t)\}$ we can write:

$$\frac{dL_i(t)}{L_i(t)} = \mu_i(\{L_i(t)\}, t)dt + \gamma_i(t) \cdot dW_t$$

where we can recognize the volatility term we defined in the previous chapter and where $W_t$ is the usual d-dimensionnal orthogonal Brownian motion. The correlation very simply appears when taking the inner product of the volatility terms:

**Definition:** The instantaneous correlation between two forward rates $L_i(t)$ and $L_j(t)$ is defined by:

$$\rho_{ij} = \frac{cov(L_i(t), L_j(t))}{\sqrt{Var(L_i(t))Var(L_j(t))}} \tag{2.7}$$

In the BGM case, this definition becomes:

$$\rho_{ij} = \frac{\gamma_i(t) \cdot \gamma_j(t)}{|\gamma_i(t)||\gamma_j(t)|} = b_i \centerdot b_j$$

Finally, the calibration consists in finding a matrix $B \in \mathcal{M}(M, d)$ with $M$ the number of forward rates necessary to build the price of our derivative and $d$ the number of factors of our model which permits the best to approach the correlation matrix using a norm we have to define.
One such distance could be the *Frobenius* norm as we will see in section 2.3.2.

### 2.3.1 Historic correlation vs parametric correlation

The choice of this structure is one of the key of a good BGM calibration. We will see what are the different possibilities and what is the best way to calibrate the correlation.

#### Historical correlation

A rather natural choice would be to consider the historic correlation between forward rates as a good estimation for the present one. In practice, you need to collect during the largest period of time the daily changes in the different forward rates and compute the correlation (Here we assume that the correlation matrix is constant over time as we consider a large period of time (1994-2006) but some operators of the market have observed that duo to market jumps this information is not accurate and propose to use a sliding window of $N$ days that exclude *special days* like FED meetings, CPI announcements....

We remind the formula to estimate the historical correlation $\rho_{ij}$ between the Forward Rates $L(T_i)$ and $L(T_j)$ is given by (2.7)

The results obtained show a clearly visible de-correlation along the columns when moving away from the diagonal. Finally, we can see that those data are very often disturbed as shown in 2.2.

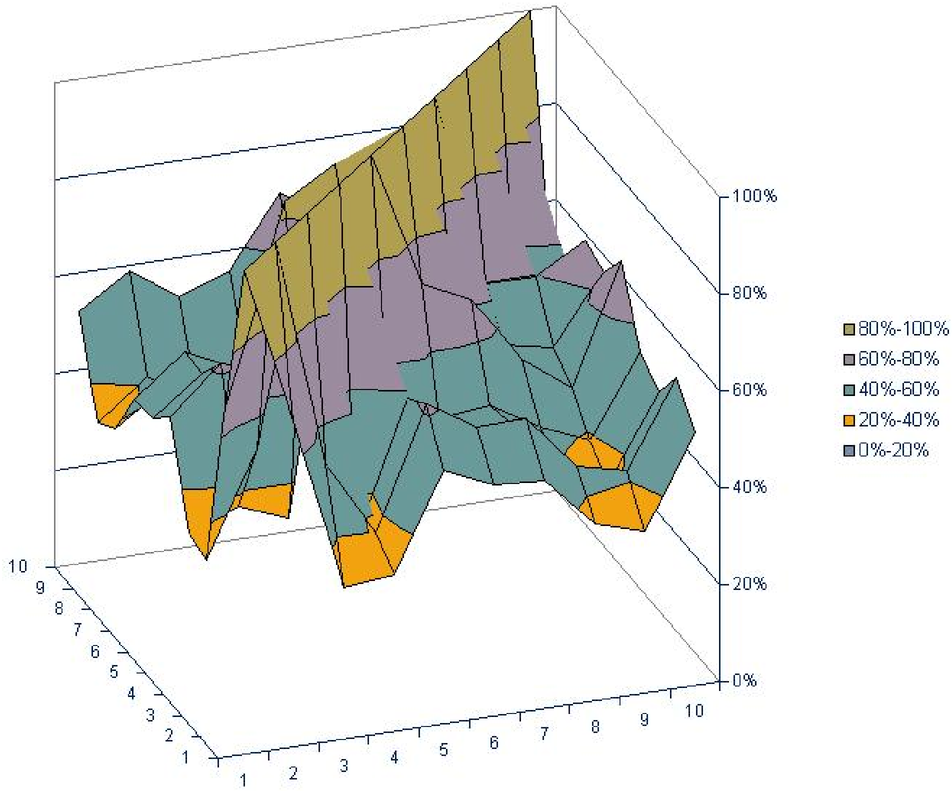


Figure 2.2: Historical correlation among Forward 1Y-Libor rates between 1994 and 2006 with daily observations

For these reasons, several models have been proposed in order to give a more regular shape to historical data and simplify the computation. Furthermore, This is better in terms of consistent pricing and risk management as the greeks will get smoother with a smoother correlation surface.

### Parameterized correlation models

#### Simple exponential correlation function

The simplest functional form for a correlation function is possibly the following:

$$\rho_{ij} = \exp[-\beta|T_i - T_j|], \qquad t \leq \min(T_i, T_j) \tag{2.8}$$

with $T_i$ and $T_j$, the expiring dates of the $i$-th and $j$-th forward rates, and $\beta$ a positive constant.

This form respects several financial requirements:

1. The farther apart two forward rates are, the more de-correlated they are.
2. The condition $\beta \geq 0$ assure that the correlation matrix $[\rho_{ij}]$ is admissible (A real symmetric matrix with positive eigenvalues).
3. However, one may notice that this form is not precise enough as it does not give the possibility to indicate how fast with respect to the time between the expiring dates the forward rates de-correlate. In other words, the 30Y Forward rate and the 10Y Forward rate have the same correlation that the 20Y Forward rate and the 3m Forward Rate. One can refer to the correlation surface given in 2.3.

This can be explained by the fact that this form does not depend on time $t$ explicitly as one can see in equation 2.8. One understands that this feature is also an advantage on a computational point of view (for the integration of the covariance $\int \rho_{ij}\sigma_i(t)\sigma_i(t)dt$) but is this simplification worth it?

Finally, we can generalize this functional form 2.8 by adding a term of asymptotic de-correlation which means that when the distance between the expiring dates goes to $+\infty$ the correlation cannot go to zero but to a finite level $\rho_\infty$. The equation 2.8 is changed into the following one:

$$\rho_{ij} = \rho_\infty + (1 - \rho_\infty)\exp[-\beta|T_i - T_j|] \tag{2.9}$$

One can check that this structure gives a matrix of course real, symmetric and has positive eigenvalues: it is an admissible correlation matrix.

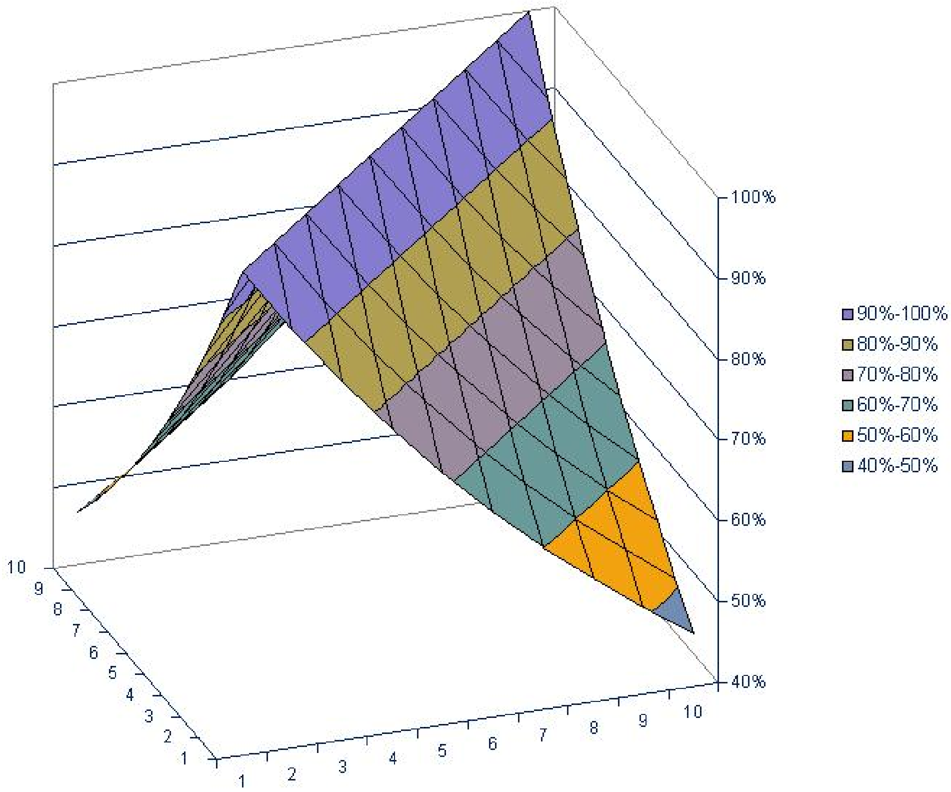


Figure 2.3: Simple Exponential Parameterized correlation among Forward rates with $\beta = 9\%$

### Modified exponential correlation function

Rebonato in [16] has proposed a slight modification which gives better results:

$$\rho_{ij} = \exp[-\beta_{\min(T_i,T_j)}|T_i - T_j|] \tag{2.10}$$

Here $\beta_{\min(T_i,T_j)}$ is not a constant anymore but a function of the earliest expiring forward date.

Nevertheless, Schoenmakers and Coffey in [18] have shown that this type of function does not assure anymore that the eigenvectors of the correlation matrix will remain positive, a necessary condition for a matrix to be correlation admissible.

But, if we choose:

$$\beta_{\min(T_i,T_j)} = \beta_0 \exp(-\gamma \min(T_i, T_j)) \tag{2.11}$$

then the eigenvalues of $\rho_{ij}$ are all positive. This form fits *the rate of decorrelation* feature discussed before while still not depending of $t$ preserving the computational feature.

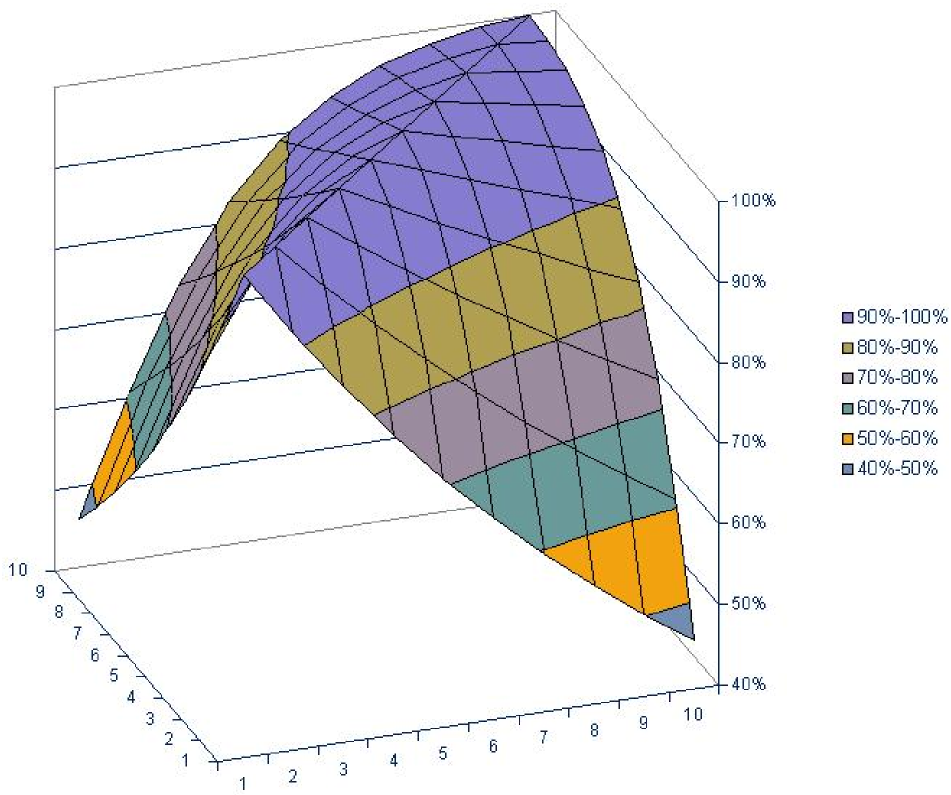


Figure 2.4: Modified Exponential Parameterized correlation among Forward rates with $\beta_0 = 12\%$ and $\gamma = 33\%$

**Schoenmakers-Coffey approach** Schoenmakers-Coffey have proposed in [18] a semi-parametric full rank structure for the correlation matrix. This semi-parametric structure provides a correlation matrix by subjecting a ratio correlation structure which obeys to simple economical principles. They describe the correlation matrix $\rho_{i,i+p}$ with an increasing function of $i$ when $p$ is fixed. This structure is more involved but it has the more robustness and generates admissible correlation matrices.

$$\begin{aligned}\rho_{ij} = \exp\Big( - \frac{|i-j|}{m-1}\big(\ln\rho_\infty \\ &+ \eta_1 \frac{i^2+j^2+ij-3mi-3mj+3i+3j+2m^2-m-4}{(m-2)(m-3)} + \\ &- \eta_2 \frac{i^2+j^2+ij-mi-mj-3i-3j+3m+2}{(m-2)(m-3)}\big)\Big), \\ &(i,j)\in[1,m]^2, 3\eta_1 \le \eta_2 \le 0, 0 \le \eta_1+\eta_2 \le -ln\rho_\infty\end{aligned}$$

This structure enjoys some very interesting properties:

Firstly : the matrices produced are automatically positive semi-definite, as every correlation matrix has to be.

Secondly: the structure produces correlation decreasing as the distance between rates increases.

Finally: the sub-diagonals of the resulting matrix are increasing while moving to longer tenors (South East of the matrix). This property is also visible in the modified exponential form and means that changes in long tenor Forward Rates are more correlated.

Thereafter in 2.5 is given the correlation surface with parameters that RBS is using to book trades.

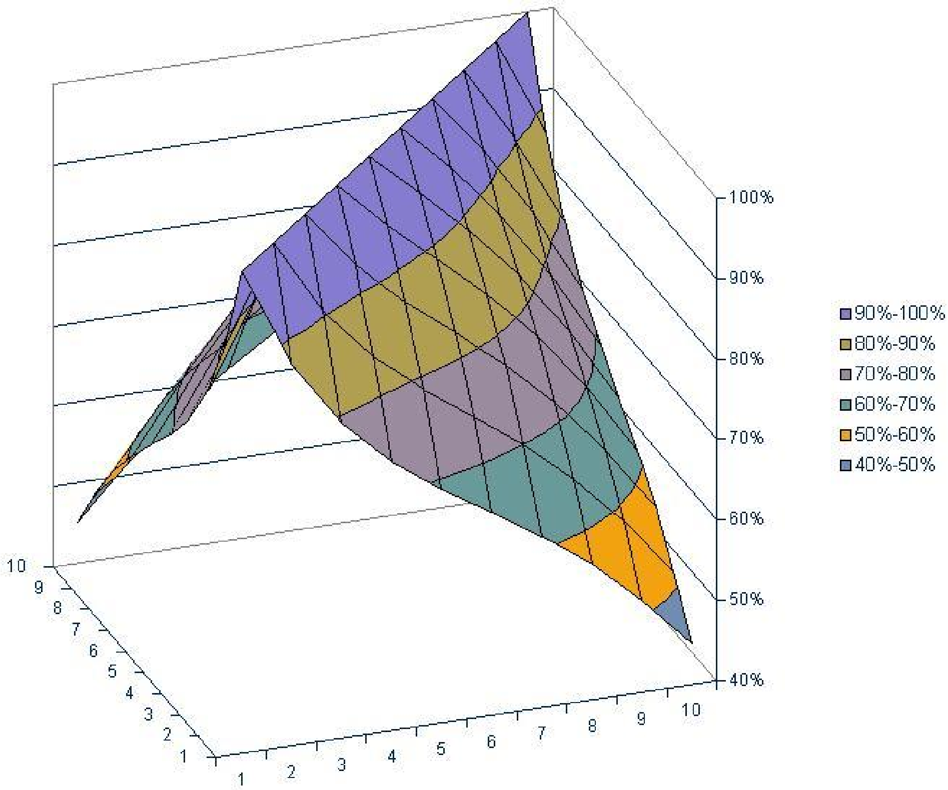


Figure 2.5: Schoenmakers Coffey correlation among Forward Libor rates with $\eta_1 = 19.99\%, \eta_2 = 59.99\%$ and $\rho_\infty = 45\%$

### 2.3.2 Rank Reduction methods

Now that we have obtained a smoother correlation matrix for our Libor Market model giving the inputs, we are going to calibrate our model with a smaller number of factors than the number of Forward rates that is inputted originally as a BGM model with Monte Carlo simulation with 15 factors is not possible.

#### Rebonato parameterization

Rebonato in [16] gives an interesting way to tackle the generation of correlation matrix for the LMM with $d$ factors. Generalizing the BGM model and more specifically 1.30 to $d$ factors we can write in any Forward measure:

$$dL_i(t) = \mu_i(t)dt + \sum_{k=1}^{d} \gamma_{ik}(t)dW_t^k \tag{2.12}$$

Where all the $W_t^k$ are orthogonal and the $\gamma_{ik}$ are the *loadings* of each factors as described in the introduction of this chapter. We know that we have the relation:

$$\gamma_i \cdot dW_t = \sigma_i \sum_{k=1}^{d} b_{ik} dW_t^k \tag{2.13}$$

So we can see the relation with the correlation arises:

$$\begin{aligned}\gamma_i \cdot \gamma_j &= \sigma_i \sigma_j \rho_{ij} \\ &= \sigma_i \sigma_j \sum_{k=1}^{d} b_{ik} b_{kj}\end{aligned}$$

And we have in order to ensure a good pricing of the caplets:

$$\sum_{k=1}^{d} b_{ik}^2 = 1 \tag{2.14}$$

We are going to show that this very general formulation of the BGM model permits us to parameterize the $\gamma_i$.

**Two-factor Case** Let assume that $d = 2$, then in their forward measure (drifts are irrelevant in this discussion):

$$\frac{dL_i(t)}{L_i(t)} = \sigma_i(t)[b_{1i}(t)dW_t^1 + b_{2i}(t)dW_t^2]$$

then the condition 2.4 becomes:

$$b_{1i}^2(t) + b_{2i}^2(t) = 1 \tag{2.15}$$

There we can introduce any coefficient $\theta$ and it is always correct that

$$\cos^2(\theta) + \sin^2(\theta) = 1,$$

which specifies a set of coefficients $b_{1i}, b_{2i}$ and hence a possible distribution of the loadings onto the two Brownian motions compatible with our BGM model. How can we choose among all the possible solutions? We are going to impose the correlation condition to this choice of $\theta$. Using (2.7):

$$\rho_{ik} = \frac{\mathbb{E}\left[\frac{dL_k(t)}{L_k(t)}\frac{dL_i(t)}{L_i(t)}\right]}{\sqrt{\mathbb{E}\left[\frac{dL_k(t)}{L_k(t)}\frac{dL_k(t)}{L_k(t)}\right]\mathbb{E}\left[\frac{dL_i(t)}{L_i(t)}\frac{dL_i(t)}{L_i(t)}\right]}} \tag{2.16}$$

First, the denominator:

$$\begin{aligned}\mathbb{E}\left[\frac{dL_k(t)}{L_k(t)}\frac{dL_k(t)}{L_k(t)}\right] &= \sigma_k^2(t)\mathbb{E}\left[[b_{1k}(t)dW_t^1 + b_{2k}(t)dW_t^2][b_{1k}(t)dW_t^1 + b_{2k}(t)dW_t^2]\right] \\ &= \sigma_k^2(t)(b_{1k}(t)^2 + b_{2k}(t)^2)dt = \sigma_k^2(t)dt\end{aligned}$$

As we have chosen a 2-dimensional Brownian motion with orthogonal Brownian increments. Hence, the denominator simplifies to:

$$\sqrt{\mathbb{E}\left[\frac{dL_k(t)}{L_k(t)}\frac{dL_k(t)}{L_k(t)}\right]\mathbb{E}\left[\frac{dL_i(t)}{L_i(t)}\frac{dL_i(t)}{L_i(t)}\right]} = \sigma_k(t)\sigma_i(t)dt \tag{2.17}$$

For the numerator we derive the same calculus using the orthogonality between the two Brownian motions:

$$\begin{aligned}
\mathbb{E}\left[\frac{dL_k(t)}{L_k(t)}\frac{dL_i(t)}{L_i(t)}\right] &= \mathbb{E}\left[\sigma_k(t)[b_{1k}(t)dW_t^1 + b_{2k}(t)dW_t^2]\sigma_i(t)[b_{1i}(t)dW_t^1 + b_{2i}(t)dW_t^2]\right] \\
&= \mathbb{E}\left[\sigma_k[\sin\theta_k dW_t^1 + \cos\theta_k dW_t^2]\sigma_i(t)[\sin\theta_i dW_t^1 + \cos\theta_i dW_t^2]\right] \\
&= \sigma_k\sigma_j[\sin\theta_k \sin\theta_i + \cos\theta_k \cos\theta_i]dt \\
&= \sigma_k\sigma_j[\cos(\theta_k - \theta_i)]dt
\end{aligned}$$

Finally,

$$\rho_{ik} = [\cos(\theta_k - \theta_i)] \tag{2.18}$$

Hence, this application to a 2-factor case show that the correlation between 2 Forward rates is purely a function of the difference between the "angles" we associated to the loadings $b_{ik}$.

### Generalization to a $d$ factor case

This case is generalizable to a $d$ factors case. Reminding the condition $\sum_{k=1}^{d} b_{ik}^2 = 1$, we recognize the co-ordinates of a point on the surface of hyper-sphere of radius 1. The expression for the polar co-ordinates of a point on the surface of a unit-radius hyper-sphere gives:

$$\begin{aligned}
b_{ik} &= \cos\theta_{ik} \prod_{j=1}^{k-1} \sin\theta_{jk}, \qquad k = 1, 2, \ldots, d-1 \\
b_{ik} &= \prod_{j=1}^{k-1} \sin\theta_{jk}, \qquad k = d
\end{aligned}$$

This parameterization $\{\theta\}$ is very useful on a computational side as we will see it later.

### The Frobenius norm

We explained before we were trying to find $B \in \mathcal{M}(M,d)$ so that $BB^T$ was near $A = [\rho_{ij}]_{Traded} \in \mathcal{M}(M,M)$. This subsection will give a sense to what *near* mean.

In optimization several views can be taken about distance using subordinated norms, penalty function, obstacle function. We will stick to the simplest case of the Frobenius norm.

Formally, we consider a weighted Frobenius inner product $\langle .,. \rangle_W$ on a Hilbert space of real symmetric matrix $M \times M$ defined by:

$$\langle X, Y \rangle_W = trace(XWYW), \quad X, Y \in \mathcal{M}(M,M) \tag{2.19}$$

We use the equally weighted Frobenius norm, hence $W = I$ and we get the norm induced by $\langle .,. \rangle_W$:

$$\|X\|^2 = \langle X, X \rangle_W = trace(X^2), \quad X \in \mathcal{M}(M,M) \tag{2.20}$$

Applying this norm to our optimization problem: we are trying to reduce the distance $[\rho_{ij}]_{model} - [\rho_{ij}]_{traded}$ which can be traduced in:

$$\begin{aligned}\chi^2 = \|[\rho_{ij}]_{model} - [\rho_{ij}]_{traded}\|^2 &= \sum \left( |[\rho_{ij}]_{model} - [\rho_{ij}]_{traded}|^2 \right) \\ &= \sum |\sum_{r=1}^{d} (b_{jr} b_{rk}) - [\rho_{ij}]_{traded}|^2\end{aligned}$$

This norm defines how near is our modeled correlation matrix from the market.

### Principal component analysis - PCA

Back to the Rebonato angle parametrization, we can object that this has only made us go from calibrating $M \times d$ factors to $M \times (d-1)$ factors that integrate the constraints of 2.4.

However, if we use a 3 factors model to simulate the 10Y USD Libor rate (quoted in annually compound) we still have a problem of $10 \times (3-1) = 20$ variables. Hence, we need to find a good start to find out the solution. We will use the principle component analysis.

This technique is the optimal linear transform that transforms the correlation matrix to a new vector basis. This vector system (depending on the correlation matrix) is such that the greatest variance by any projection of the data comes to lie on the first coordinate, the second greatest variance on the second coordinate, and so on.

Practically, we are given a correlation matrix $[\rho_{ij}]$ that we can always diagonalize to find a diagonal matrix $\Lambda = [\lambda_i]$ and an orthonormal diagonal matrix $V$ such that $[\rho] = V\Lambda V^{-1}$. These matrices are easily found using a QR algorithm with Gram-Schmidt method.
Then, you can form a matrix $B \in \mathcal{M}(M, d)$ defined with:

$$B = \sqrt{\Lambda}P = \left(\sqrt{\lambda_1}V_1, \ldots, \sqrt{\lambda_i}V_i, \ldots, \sqrt{\lambda_d}V_d\right)$$

One keeps the $d$ most important eigenvalues $\{\lambda_i\}$ and their eigenvectors $\{V_i\}$. With this choice we have $BB^T \in \mathcal{M}(M, M)$ close in norm to the market input $[\rho_{ij}]$. On top of this, we will use this $B$ to describe the factors $b_{ik}$ as defined in the definition of our Libor Market Model in 2.13.
We also have an indication of the number of factors important to create a good approximation of the original rank $M$ matrix. In our example, we find that the first three eigenvalues account for 93.6% of the sum of the eigenvalues as shown in 2.4. This means that we can explain 93.6% of the variance with the first three factors.

### A PCA Interpretation

We can easily draw a parallel between those eigenvalues and the moves of the curve. The first factor, the most important, explains the parallel shifts

| Eigenvalues | Value | Proportion |
|---|---|---|
| 1st | 7.86 | 78.6% |
| 2nd | 1.07 | 10.7% |
| 3rd | 0.427 | 4.27% |
| Sum of the others | 0.64 | 6.40% |

Table 2.4: Main eigenvalues of the correlation matrix: the PCA arises naturally to explain the moves of the curve

movements of the yield curve. The second one explains the inversion moves of the curve: when the short dated increase while the long dated decrease or the opposite. Finally, the third factor explains the torsion moves of the curve: when long and short rates dated increase and middle dated decreases or the opposite.
Hence, thanks to the PCA, we have a good approximation of the exogeneously given full rank correlation matrix at a relatively low computation cost. Moreover, we know how much of the variance of the correlation matrix we account for when using a 3 factors model.

### Rebonato angles optimized method

Going to a full optimization of the problem $\chi^2$ under a 3 factors model, we obtain very close results to the PCA. Thereafter is given a figure comparing the two methods. The optimization can be done using Broyden-Fletcher-Goldfarb-Shanno algorithm (as detailed in [14]) using parameters for the Schoenmakers-Coffey structure used by RBS for the booking for Interest rate derivatives and based on a sliding window of the last 12 years[1] on USD 12m Libor. The Royal Bank of Scotland is using a slightly modified version of this algorithm that gives better results. Obviously, the norm optimization looks better and the Forward rates are close to the input matrix.

We can see in the next figure that the eigenvectors for both methods are quiet similar although there is no orthogonalization process in the Rebonato

[1] This window can change as one can argue that a shorter window gives a better trend; however this choice is very conditional to trader and risk management opinion

angles optimized method (Fully optimized method). What we have done in these process is just a linear transformation of the original Libor Market Model correlation matrix.

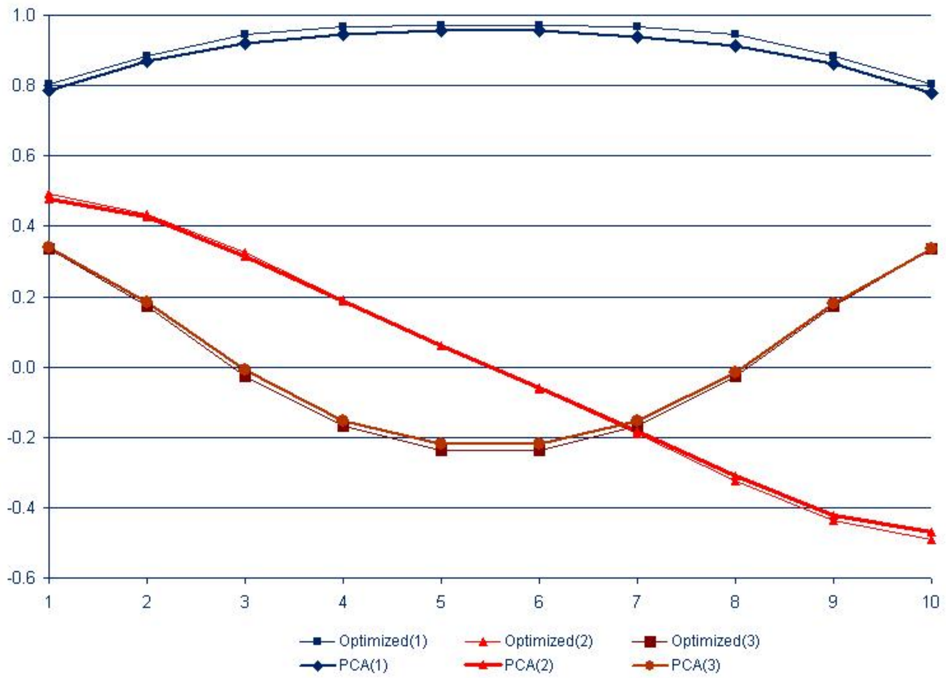


Figure 2.6: Eigenvectors comparison between PCA and Rebonato angles optimized method

### Comparison between rank reduced correlations

The next figures 2.7, 2.8 and 2.9 plot a column (Second, Fifth and Tenth column) of each correlation matrix formed (Market, PCA and Fully optimized) and compare them. What is plotted is the correlation between a Forward Libor rate ($2^{nd} : \rho 2, j, 5^{th} : \rho 5, j, 10^{th} : \rho 10, j$) with the other Forward Libor rates for each matrix formed.

Looking at these figures several remarks can be done. In general, we observe that these rank reduction methods tend to overestimate the correlation between the adjacent Forward rates (thus the terms $\rho_{i,i-1} \ldots$) and lower the correlation between the distant one (the terms $\rho_{i,\epsilon} \ldots$). Hence, this lead to systematical misprice on the swaptions: short maturities swaption will always be too expensive because model correlation will be too high and long maturities swaption will be too cheap because model correlation will be too low. With those reserves in mind results remain acceptable for at-the-money swaptions. Nevertheless, we can see that in our case the low correlation effect is not very well observed, this is due to the rather small size (10 Years) of our matrix.

Increasing the number of factors to 4 does not improve as much as from 2 to 3 as the 4-th eigenvalue is smaller than the first 3 (in our case we would have taken account of 95.7% (vs 93.6% with 3 factors) of the variance with 4 factors); hence depending on the complexity and the accuracy needed we can increase the number of factors but never let it go below 3.

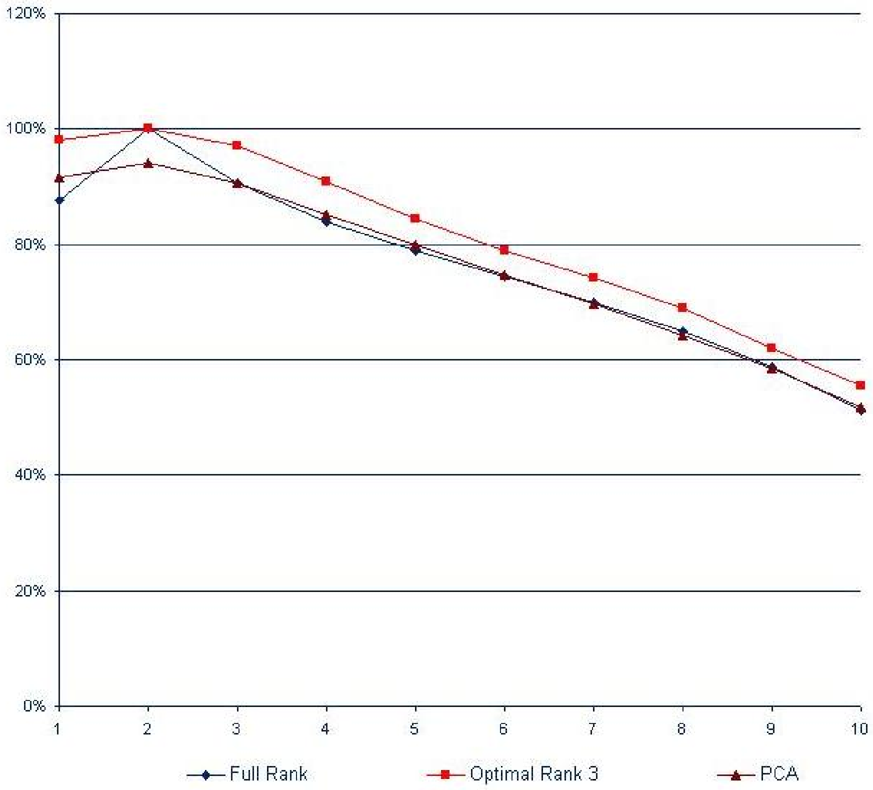


Figure 2.7: Comparison of the 2Y Forward Libor Rates correlation simulated by PCA and complete optimization

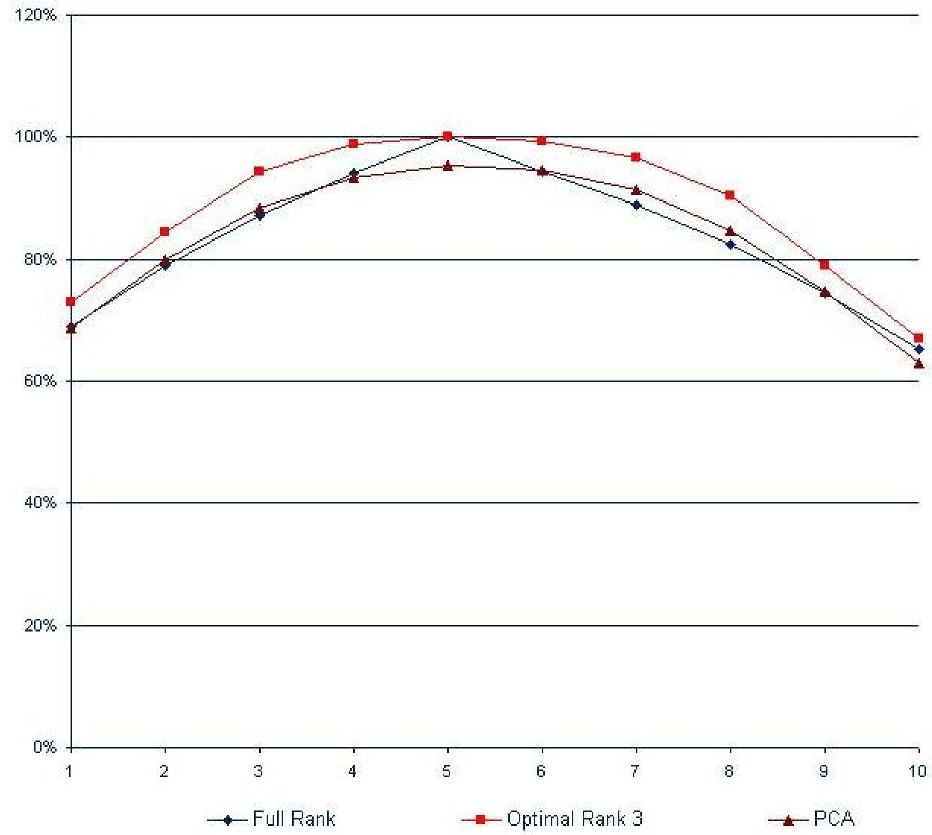


Figure 2.8: Comparison of the 5Y Forward Libor Rates correlation simulated by PCA and complete optimization

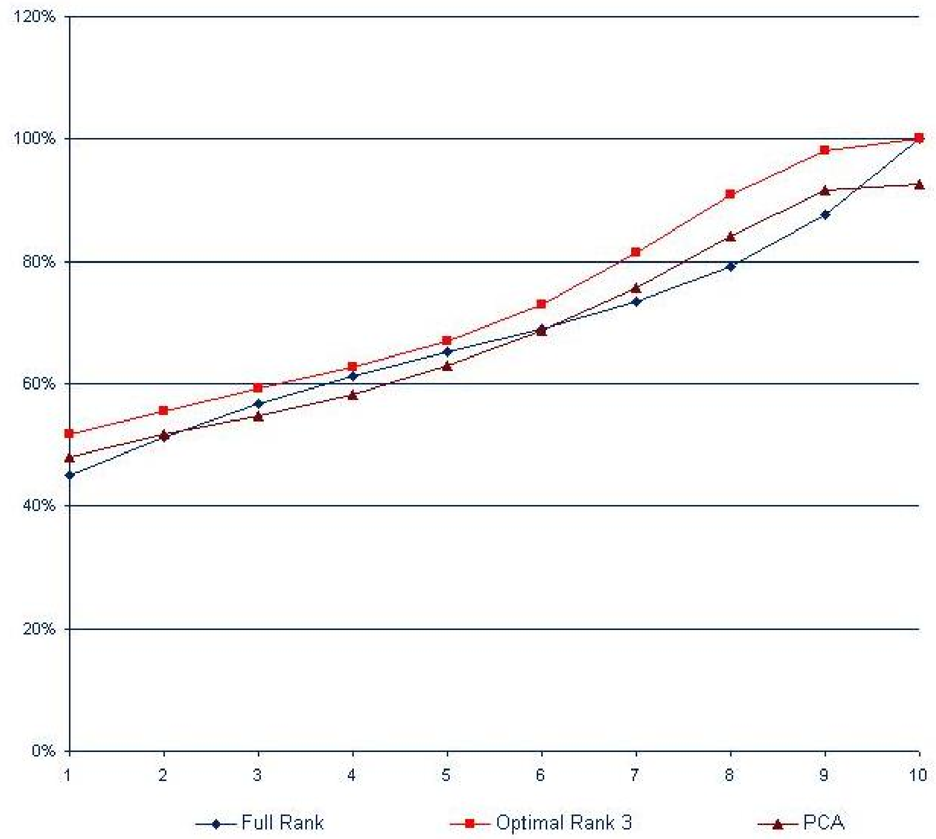


Figure 2.9: Comparison of the 10Y Forward Libor Rates correlation simulated by PCA and complete optimization

## 2.4 Swaption Approximation formulas

We have seen that a financial model is usable by operators only if it reflects prices of the market. This calibration is usually a very time consuming operation.

Theoretically, in order to find the parameters of our problem, we should propose a set of parameters for the instantaneous volatility and correlation, run Monte Carlo simulation on the Forward Rates and from those, derive the volatility of the swaptions. This process needs Monte Carlo simulations at each step which is too much time consuming. Hence, we need to find an approximate closed formula for this price\volatility.

In the market, swaptions at the money are quoted using their implied volatility: the market uses Black Formula to create the relation between the prices of the swaptions and the implied volatility used for the quotation. The use of this Black formula request that one assume the log normality of the Forward rates in their Forward Measure and as a matter of fact no log-normality for the swap rates.

### 2.4.1 Rebonato Formula

Rebonato in [16] proposed an approximation in order to compute the swaption prices. A swap rate $S_{p,q}(t)$ as we saw it before can be written as a linear combination of Forward rates:

$$S_{p,q}(t) = \sum_{k=p}^{q-1} w_{p,q}^k L_k(t) \tag{2.21}$$

where the weights $\{w\}$ are given by:

$$w_{p,q}^k = \frac{\delta B(t, T_k + \delta)}{\sum_{i=1}^{q-p} \delta B(t, T_i + i\delta)}$$

Here, we assume that in the dynamic of the swap rate $dS_{p,q}$ the weighings $\{w\}$ in the linear combination are constant and equal to their value in 0,

$w_{p,q}(0)$. Hence,

$$dS_{p,q} \approx \sum_{k=p}^{q-1} w_{p,q}(0)^k dL_k(t) \tag{2.22}$$

Then we can write the implied volatility $\sigma_{p,q}^{Black}$ using the relation showed in the chapter 1.

$$(\sigma_{p,q}^{Black})^2 T_p = \int_0^{T_p} \|\gamma_{p,q}\|^2 dt = \int_0^{T_p} \left(\frac{dS_{p,q}}{S_{p,q}}\right)^2 \tag{2.23}$$

Therefore,

$$\begin{aligned}
\left(\frac{dS_{p,q}}{S_{p,q}}\right)^2 &= \sum_{j,k=p}^{q-1} \frac{w_{p,q}^k(0) w_{p,q}^j(0) dL_k(t) dL_j(t)}{S_{p,q}^2} \\
&= \sum_{j,k=p}^{q-1} \frac{w_{p,q}^k(0) w_{p,q}^j(0) (\gamma_k \cdot \gamma_j) L_k(t) L_j(t)}{S_{p,q}^2} dt \\
&= \sum_{j,k=p}^{q-1} \frac{w_{p,q}^k(0) w_{p,q}^j(0) \rho_{kj} \sigma_k \sigma_j L_k(t) L_j(t)}{S_{p,q}^2} dt
\end{aligned}$$

And finally,

$$\begin{aligned}
(\sigma_{p,q}^{Black})^2 T_p &\approx \int_0^{T_p} \sum_{j,k=p}^{q-1} \frac{w_{p,q}^k(0) w_{p,q}^j(0) \rho_{kj} \sigma_k \sigma_j L_k(t) L_j(t)}{S_{p,q}^2} dt \\
&\approx \sum_{j,k=p}^{q-1} \frac{w_{p,q}^k(0) w_{p,q}^j(0) L_k(t) L_j(t)}{S_{p,q}^2} \int_0^{T_p} \rho_{kj} \sigma_k \sigma_j dt \\
&\text{Here we assume } L_k(t) = L_k(0) \\
&\approx \sum_{j,k=p}^{q-1} \frac{w_{p,q}^k(0) w_{p,q}^j(0) L_k(t) L_j(t)}{S_{p,q}^2} \rho_{kj}(0) \int_0^{T_p} \sigma_k \sigma_j dt \\
&\text{Here we also assume } \rho_{jk}(t) \approx \rho_{jk}(0)
\end{aligned}$$

So putting things together Rebonato approximation formula is:

$$\sigma_{p,q}^{Black} = \sqrt{\frac{1}{T_p} \sum_{j,k=p}^{q-1} \frac{w_{p,q}^k(0) w_{p,q}^j(0) L_k(t) L_j(t)}{S_{p,q}^2} \rho_{kj}(0) \int_0^{T_p} \sigma_k(t) \sigma_j(t) dt} \tag{2.24}$$

This approximation works quiet well but it can be fine tuned using Hull and White idea in [19].

### 2.4.2 Hull and White Formula

In [19], Hull and White have proposed an improvement of the previous formula using the first order for the coefficient $\{w\}$. We will omit the subscripts $p$ and $q$ to light the notation.

The derivation is for this one:

$$\begin{aligned}
dS_{p,q} &= \sum_{k=p}^{q-1} d(w^k(t)L_k(t)) \\
&= \sum_{k=p}^{q-1} w^k(t)dL_k(t) + L_k(t)dw^k(t) \qquad \text{as the weightings are deterministic functions of} L_k : \\
&= \sum_{k=p}^{q-1} w^k(t)dL_k(t) + \sum_{k=p}^{q-1} L_k(t) \sum_{i=p}^{q-1} \frac{\partial w^k(t)}{\partial L_i} dL_i(t) \\
&= \sum_{k=p}^{q-1} \Big( w^k(t)dL_k(t) + L_k(t) \sum_{i=p}^{q-1} \frac{\partial w^k(t)}{\partial L_i} dL_i(t) \Big)
\end{aligned}$$

The first order derivative can be computed by writing:

$$\begin{aligned}
w^k_{p,q} &= \frac{B(t, T_k + \delta)}{\sum_{i=1}^{d} \delta B(t, T_i + \delta)} \\
&= \frac{\prod_{i=0}^{k} \frac{1}{1+\delta L_i(t)}}{\sum_{i=1}^{q-p} \left(\delta \prod_{i=0}^{p+k-1} \frac{1}{1+\delta L_i(t)}\right)}
\end{aligned}$$

The derivation is straightforward and we will not reproduce it. The reader can refer to [19] for further details. Finally, we obtain:

$$\frac{\partial w^k}{\partial L_i} = \frac{w^k \delta}{1 + \delta L_i} \left( \mathbf{1}_{i>k} - \frac{\sum_{k=1}^{i-p+1} B(t, p+k)}{\sum_{k=1}^{q-p} B(t, p+k)} \right)$$

Finally we can use the convenient expression of Rebonato given in 2.24 where we switch $w^k$ by $\bar{w}^k$ defined by:

$$\bar{w}^k = w^k + \sum_{k=p}^{q-1} L_k(t) \frac{\partial w^k}{\partial L_i}$$

### 2.4.3 Andersen and Andereasen Formula

A third approximation possible is the one given by Andersen et Andreasen in [20]. The idea is to differentiate the swap rate $S_{p,q}$ with respect to the Forward Rates $L_i$.(Basically, using the partial derivatives $\frac{\partial S_{p,q}}{\partial L_i}$ relevant with the maturity we consider that is from $T_p$ to $T_q$)

$$dS_{p,q} = \sum_{k=p}^{q-1} \frac{\partial S_{p,q}}{\partial L_k} dL_k$$

$$\frac{dS_{p,q}}{S_{p,q}} = \frac{1}{S_{p,q}} \sum_{k=p}^{q-1} \left( \frac{\partial S_{p,q}}{\partial L_k} L_k \gamma_k dW_t^k + X_k dt \right)$$

Once again we do not compute the drift $X_k$ as we are interested only in the quadratic variation:

$$\left(\frac{dS_{p,q}}{S_{p,q}}\right)^2 = \frac{1}{S_{p,q}^2} \sum_{j,k=p}^{q-1} \frac{\partial S_{p,q}}{\partial L_k} \frac{\partial S_{p,q}}{\partial L_j} L_k L_j \gamma_k \cdot \gamma_j dt$$

$$\int_0^{T_p} \left(\frac{dS_{p,q}}{S_{p,q}}\right)^2 dt \approx \frac{1}{S_{p,q}^2} \sum_{j,k=p}^{q-1} \frac{\partial S_{p,q}(0)}{\partial L_k} \frac{\partial S_{p,q}(0)}{\partial L_j} L_k(0) L_j(0) \int_0^{T_p} \gamma_k \cdot \gamma_j dt$$

Where we use the same approximation as previously taking for constant the partial derivatives, the forward rates and the instantaneous correlation at 0. So finally we get for the swaption price:

$$\sigma_{p,q}^{Black} = \sqrt{\frac{1}{T_p S_{p,q}^2(0)} \sum_{j,k=p}^{q-1} \frac{\partial S_{p,q}(0)}{\partial L_k} \frac{\partial S_{p,q}(0)}{\partial L_j} L_k(0) L_j(0) \rho_{jk}(0) \int_0^{T_p} \sigma_k \cdot \sigma_j dt} \tag{2.25}$$

where we have derived the term for each $k$:

$$\frac{1}{S_{p,q}}\frac{\partial S_{p,q}}{\partial L_k} = \frac{\delta}{1+\delta L_k}\left(\frac{B(t,T_q)}{B(t,T_p)-B(t,T_q)} + \frac{\sum_{j=k-p+1}^{q-p}\delta B(t,T_{p+j})}{\sum_{j=1}^{q-p}\delta B(t,T_{p+j})}\right)$$

In this approximation, we finally only have changed the $w_{p,q}^{j}$ by $\frac{\partial S_{p,q}}{\partial L_j}$.

## 2.5 Monte Carlo Simulation and Results on 3 Factors BGM

This section is going to compare the different formulas in term of their ability to fit the swaption market simulated by Monte Carlo methods and given the same set of parameters.

### 2.5.1 Monte Carlo Method

The idea of the Monte Carlo method is to compute values of any kind of derivatives instruments from simulated trajectories and evaluate the result as the average of this values.
In general, Monte Carlo computation are used for simulation and optimization problems. In Libor Market model, we have to compute expectations and therefore we can use this process.
In a mathematical point of view, consider a square-integrable function $f \in \mathcal{L}^2(0,1)$ and a uniform distributed random variable $x \in \mathcal{U}[0,1]$. MC permits us to compute expectations as we know that:

$$\mathbb{E}[f(x)] = \int_0^1 f(x)dx,$$

Consider a sequence $x_{in}$ sampled from $\mathcal{U}[0,1]$. An empirical approximation of the expectation is then:

$$\mathbb{E}[f(x)] \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

The justification of this approximation is given by the Strong Law of Large Numbers. This law implies that this approximation is convergent with probability one, i.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \int_0^1 f(x)dx \tag{2.26}$$

The error we make using this approximation hence is:

$$\epsilon_n = \int_0^1 f(x)dx - \frac{1}{n}\sum_{i=1}^{n} f(x_i) \tag{2.27}$$

This error can be described in a statistical point of view using the Central Limit Theorem.

As $n \to \infty$, $\sqrt{n}\epsilon_n(f)$ converges in distribution to $\sigma\nu$ where $\nu$ is a standard normal random variable (with mean nil and variance of 1) and $\sigma$ is the square-root of the variance of $f$:

$$\sigma(f) = \Big[\int_0^1 (f(t) - \int_0^1 f(x)dx)dt\Big]^{1/2}$$

### 2.5.2 Numerical Results

In order to simulate the Forward Libor using Monte Carlo, we need a unique measure. As previously explained in 1.1.6 we will use the spot martingale measure $\mathbb{P}^*$ and its numeraire $B_{spot}(t)$. We have discretised the $\{L_i\}$ under their exponential form using 1.25 on a tenor that coincide with the reset dates $T_i$ for practical reasons:

$$d(\ln L_i(t)) = \left(\sum_{\lfloor\delta^{-1}T\rfloor}^{k} \frac{\delta L_j(t)(\gamma_k(t)\centerdot\gamma_j(t))}{1+\delta L_j(t)} - \frac{\|\gamma_i^2(t)\|^2}{2}\right) dt + L_k(t)\gamma_k(t)\centerdot dW_t^*$$

Therefore, $\forall k \in [0, n-1]$:

$$L_i(T_{k+1}) = L_i(T_k) \exp \Big[ \big( \sum_{\lfloor j=\delta^{-1}T_k \rfloor}^{k} \frac{\delta L_j(T_k)(\gamma_j(T_k) \cdot \gamma_i(T_k))}{1+\delta L_j(T_k)} - \frac{\|\gamma_i^2(T_k)\|^2}{2} \big) \Delta T_k + \|\gamma_i(T_k)\| \sum_{j=1}^{d} b_{ij} \epsilon_{jk} \sqrt{\Delta T_k} \Big]$$

where $\epsilon_{jk} \rightsquigarrow N_d(0,1)$. To compare figures comparable, the same change of numeraire must be done for the swaption payoff:

$$\begin{aligned} Swaption_{p,q}(0) &= B_{spot}(0)\mathbb{E}^* \left( \frac{Swaption_{p,q}(T_p)}{B_{spot}(T_p)} | \mathfrak{F}_0 \right) \\ &= \mathbb{E}^* \left( \sum_{j=p}^{q} \delta B(t, T_{p+j}) \frac{(S_{p,q}(T_p) - \kappa)_+}{B_{spot}(T_p)} ) \right) \\ &\text{As} \quad B_{spot}(0) = 1 \end{aligned}$$

Back to the swaptions, we express the integral of the instantaneous volatility according to section 2.2 (the last term of the generic swaption formula):

$$\begin{aligned} \int_0^{T_p} \sigma_i(s)\sigma_j(s)ds &= \int_0^{T_p} c_i c_j \eta(T_i - s)\eta(T_j - s)ds \\ \text{and as} \quad \gamma_i^2 &= \frac{1}{T_i} \int_0^{T_i} \sigma_i^2(s)ds \\ \gamma_i^2 &= \frac{c_i^2}{T_i} \int_0^{T_i} \eta(T_i - s)^2 ds \\ \int_0^{T_p} \sigma_i(s)\sigma_j(s)ds &= \gamma_i \gamma_j \sqrt{T_i}\sqrt{T_j} \frac{\int_0^{T_p} \eta(T_i - s)\eta(T_j - s)ds}{\sqrt{\int_0^{T_i} \eta(T_i - s)^2 ds}\sqrt{\int_0^{T_j} \eta(T_j - s)^2 ds}} \end{aligned}$$

This permits us to give an explicit generic formula for the swaptions:

$$(\gamma_{p,q})^2 \approx \frac{1}{T_p} \sum_{j,i=1}^{q-1} \frac{w_{p,q}^i w_{p,q}^j L_i L_j}{S_{p,q}^2} \gamma_i \gamma_j \rho_{ij} \varsigma_{i,j,p}^{\beta,b,\eta_\infty} \tag{2.28}$$

with $\varsigma_{i,j,p}^{\beta,b,\eta_\infty}$ the integral term of the previous derivation:

$$\varsigma_{i,j,p}^{\beta,b,\eta_\infty} = \sqrt{T_i}\sqrt{T_j}\frac{\int_0^{T_p}\eta(T_i - s)\eta(T_j - s)ds}{\sqrt{\int_0^{T_i}\eta(T_i - s)^2 ds}\sqrt{\int_0^{T_j}\eta(T_j - s)^2 ds}}$$

Thanks to the Royal Bank of Scotland, I could run tests on a swaption matrix $10 \times 10$ with these formulas on the Libor Market Model with market parameters in date of October $30^{th}$ 2006 and compare them to a Monte Carlo simulation. By swaption matrix, we mean the Black volatilities of the swaptions put in an array with on the x-axis the tenor of the underlying swap and on the y-axis the maturity of the swaption. Hence a $N \times M$ swaption is a swaption of maturity N Years on a M Years swap.

We ran Monte Carlo simulations over 1 million paths on a 10Y tenor swap at maximum (which is a very common tenor for structured products in Asia) with a maximum option maturity of 10Y. (Hence, we had to use the North-West part of a correlation surface $20 \times 20$). We could estimate the average error between the previous formula applied to Hull-White, Rebonato and Andersen and Andreasen and the results obtained by Monte-Carlo simulation by using the expression : $\frac{1}{10\times 10}\sum|\gamma_{p,q}^{Monte-Carlo} - \gamma_{p,q}^{Formula}|$. We can conclude it is non relevant.

| Approximation Accuracy | Maximum Discrepancy | Average Discrepancy |
|---|---|---|
| Rebonato | 0.34% ($1 \times 2$) | 0.18% |
| Hull and White | 0.17% ($5 \times 2$) | 0.10% |
| Andersen and Andreasen | 0.22% ($3 \times 2$) | 0.08% |

Table 2.5: Swaption approximation accuracy for different formulas

Some comments about the general behaviour of each formula. Rebonato and Hull White formula seem to be quiet off on the short maturity and short tenor (First line and first column) and otherwise with a constant discrepancy along the matrix. Andersen and Andreasen formula is behaving the opposite as the approximation quality decrease when the maturity and the tenor increase (Going South East in the matrix).

Hence a good strategy for a calibration would be to use Hull White for the short dated swaption (inferior to 5 years) and then Andersen and Andreasen formula, this is still work in progress as it is very involved to get consistent results with this method all along the swaption matrix.

From a risk management point of view, some products do not depend on some tenors or maturities, we can decide to eliminate these irrelevant swaptions or reduce their influence in the calibration process (For instance by changing the weight matrix in the Frobenius norm). This is very useful for pricing accurately Bermudan swaptions where the co-terminal swaptions[2] are very important. Several procedures have been proposed, see [16] for further details.

Finally to put this in perspective a typical bid-offer spread in USD would be 0.50% highlighting how good are those approximations.

[2] Co-terminal swaptions are the swaption on the diagonal SW-NE of the matrix

# Chapter 3

# Perspectives and issues

## 3.1 Stochastic volatility models applied to Libor Market Model

The work we have produced until now was assuming a deterministic volatility. Like for the equities, volatility mappings suffer from a smile (here a skew) that makes the implied volatility when moving away from at the money point. Several propositions have been worked out to fit the very out or in the money implied volatility and this is still work in progress. Here is the general framework the most used nowadays in the world of rates.

### 3.1.1 Stochastic $\alpha$ $\beta$ $\rho$ model - SABR

Operators have figured out since a long time that interest rates products were not well quoted using deterministic volatility (even the previous piecewise or Laguerre type volatility). Hagan in [21] has introduced a *local volatility model* self-consistent, arbitrage-free and which match observed market skews. We will present its main features and how it is handled in the Libor Market Model.

Main assumption is that the volatility follows a stochastic process correlated to the forward price $L_i(t)$ in its forward measure:

$$dL_i(t) = \Sigma_B L_i^{\beta}(t) dW_1$$
$$d\Sigma_B = \nu \Sigma_B dW_2, \qquad \Sigma_B(0) = \sigma_B$$

where $\nu$ is named the volatility of the volatility, namely *volvol*.
The two processes $W_1$ and $W_2$ are correlated by:

$$dW_1 dW_2 = \rho dt$$

Many other forms have been proposed for the stochastic process for the volatility, with a drift, with a mean reversion *etc* but this original form gives the means to manage the skew risk in markets with only exercise date which is our case with the caplets and the swaptions markets.
In the operator point of view, managing the *vega* risk becomes like delta-hedging as the trader will have to buy and sell options to become vega neutral.

Using singular perturbation techniques we can derive a price for Europeans options, we will let the reader refer to [21] for a complete proof. European prices are given using the Black formula with an other Black volatility $\Sigma_B(L_i(t), \kappa)$. Using the same notations as in 1.3.1:

$$Caplet^{SABR}(t) = 1\delta B(t, T_{i+1})[L_i(t)N(d_1(t, T_i)) - \kappa N(d_2(t, T_i))],$$
$$\text{with,} \qquad d_1 = \frac{ln(L_i(t)/\kappa) + \Sigma_B^2 \frac{(T_i - t)}{2}}{\Sigma_B \sqrt{(T_i - t)}}$$
$$d_2 = d_1 - \Sigma_B \sqrt{(T_i - t)}$$

and where the implied volatility is given exogenously:

$$\Sigma_B(L_i(t),\kappa) = \frac{\sigma_B}{(L_i(0)\kappa)^{\frac{1-\beta}{2}}\left(1+\frac{(1-\beta)^2}{24}\ln^2\frac{L_i(0)}{\kappa}+\frac{(1-\beta)^4}{1920}\ln^4\frac{L_i(0)}{\kappa}+\dots\right)} \cdot \left(\frac{z}{x(z)}\right) \cdot \left(1+\left[\frac{(1-\beta)^2}{24}\frac{\sigma_B^2}{(L_i(0)\kappa)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\sigma_B}{(L_i(0)\kappa)^{\frac{1-\beta}{2}}}+\frac{2-3\rho^2}{24}\nu^2\right](T_i-t)+\dots\right)$$

where we refer to $z$ as:

$$z = \frac{\nu}{\sigma_B}(L_i(0)\kappa)^{\frac{1-\beta}{2}}\ln(L_i(t)/\kappa),$$

and to $x(z)$ as:

$$x(z) = \ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}$$

These formulas give an explicit[1] form for the volatility in the European case and this can be highlighted as it becomes easily implementable in this model, which is generally not the case in the stochastic volatility model.

In order to fit the market, we can play on the parameters of the model.

The $\beta$ controls the backbone of the skew that means the ATM volatility $\Sigma_B(L_i(t), L_i(t))$ estimated with a historical log-log plot of the ATM volatilities. In general we use $\beta = 0.5$ for the USD Interest rate market (like in the CIR Model).

The $\alpha$ parameter is conveniently replaced by the ATM volatility (One can numerically invert the formula) and is changed almost every hours.

$\rho$ and $\nu$ control the skew. $\nu$ is very high for short-dated options, and decrease as the time-to exercise increases, while the correlation $\rho$ starts near 0 and becomes substantially negative along time-to exercise. It should be noticed that there is a weak dependence of the market skew

[1] The omitted terms in ... are much smaller

on the tenor of the underlying swap hence those parameters are fairly constant along market moves for each tenor. In general, they are updated on a monthly basis.

One should notice that the calibration of theses volatility models is made hard by the absence of liquidity of some parts of the skew in the market : very out of the money or deeply in the money swaptions are less likely to be traded and consistency between prices is quiet hard to be found. Extensions can be made with a volatility model that handle market jumps or uses instantaneous stochastic correlation. This is obviously very *work-in-progress*.

## 3.2 Hybrids Products

This section is much more qualitative as this topic is a very new and confidential one and a very few academic paper are available. After discussions and attendance to meetings with market operators, I am going to present some general views over these new derivatives.

A derivative is an hybrid when the whole or part of the trade has risk across two or more asset classes that *cannot be decomposed* into specific asset classes[2]. It can be both considered as a product or an asset class since due to cross convexity one asset class cannot be risk managed without considering other asset classes in a given trade.

In a pricing perspective the main difference with single asset structured products is the important combination of joint distributions, correlation and cross convexity.

**Joint distribution** Two different ways to calculate the expectation of the payoff (in other words the integral and the joint distribution of the two assets) have been proposed using the work done on single asset exotics: Implied distributions (Interest Rates) and Copulas (Credit Derivatives).

[2]Main asset classes are: Equity, Rates, FX, Credit, Commodities, Inflation.

Using the implied distribution means that from the caplet/floorlet prices we build an empirical distribution for each asset class involved in the trade.
A Copula is a real function $C$ such that in a 2 dimensions case is defined on $I^2 = [0,1]^2$ and:

$$C(x,0) = C(0,x) = 0 \quad \text{and} \quad C(x,1) = x,\ C(1,z) = z \tag{3.1}$$

Very basically, using the Sklar theorem that sets that for each Joint distribution $F(X_1, X_2)$ there exist a function $C$ depends on $C(F_1(X_1), F_2(X_2))$ where the $F_i$ are the marginal distribution of our assets, we can determine the Joint distribution of the 2 assets.

**Correlation** This is an issue for risk management and for pricing. We saw in this thesis that pricing was all about correlation and market data are a crucial point for a good calibration. One can understand that when two classes of assets are involved the issue is even bigger than when talking about just two Forward Libor rates. This is still an open problem for many houses: operators are talking about stochastic correlation but most of all refer to the *common sense* before giving a price.

**Cross-convexity** Convexity problems are not new to anyone who already dealt with Constant Maturity Swap and in general interest rates. Basically, in hybrids, managing the risk in terms of delta and gamma is much more involved due to this term of convexity across the asset classes.

**Summary** Hybrids are a *hot* topic and we have seen a growing demand for those kinds of products all around Asia. Pricing is very involved and risk management can be a nightmare: for instance, volatility jumps in one asset class very often brings a jump in other asset classes; then, the market might probably get upset and all assumptions previously made will have to be reconsidered.

## 3.3 Issues raised

### 3.3.1 Choice between Historical and Implied volatility

They are two approaches to the calibration of the swaption. Whether we decide to smooth the historical correlation matrix with a parametric form. Then by using this form in the approximation formula we fit the swaption prices with the parameter $\varsigma$. Or we ignore the historical correlation and we only adapt the parameters of the correlation structure to calibrate the model on the swaption prices.

Indeed, one would say that those two methods should produce similar results. It is not the case as the derivation of the correlation matrix even after smoothing by a parametric form gives different results from the swaption prices quoted in the market. This explains also why the implicit correlation surface obtained in the second approach is different from the one obtained using historical data.

Nevertheless, operators have tried to integrate both historical and implied information. This does not seem to work properly. Hence, as the historical approach does not permit to find the swaption prices and has less value that the implied value (which basically price what is going to be the market) we prefer to choose to use the implied correlation.

### 3.3.2 Interest-rates skew

Except in this section 3.1, we have supposed the volatility to be deterministic and at most time dependent. Great improvements to the calibration of the LMM can be done by using stochastic volatility to model interest rate skew. As described before, SABR Model developed by Hagan in [21] is the most used (and the one used at the Royal Bank of Scotland).

### 3.3.3 Approximation formula

By nature using approximations brings you issues. In our case we have found good approximations to swaption prices. Those are the state-of-the-art of this topic but still they do not permit to price accurately swaptions all along the matrix but still, gives an almost log-normal behaviour to swap rates.

### 3.3.4 Market liquidity

In order to price long trades, we need to calibrate a rather big swaption matrix. After several discussions with traders, I happened to realize that some are very illiquid (Quotes are even worst in non USD or EUR market like emerging currencies: KRW, THB, TWD, SGD, HKD) and therefore the quotes given by brokers can be strange leading to a bad calibration.

# Chapter 4

# General Methodology proposed for calibration

This is a short summary of what we have proposed in this thesis as methodology to calibrate the Libor Market model to the swaption prices.

## 4.1 Assumptions

- Libor Market Model: Lognormality of forward rates
- Volatility: Deterministic
- Correlation: Deterministic

## 4.2 Modeling choices

- Volatility structure: Laguerre type

$$\begin{aligned} \|\gamma_i(t)\| &= \sigma_i(t) = c_i \eta(T_i - t) \\ \eta(s) &= \eta_{a,\beta,\eta_\infty}(s) = \eta_\infty + (1 - \eta_\infty + bs)e^{-\beta s} \\ b, \beta, &\eta_\infty \geq 0 \end{aligned}$$

- Correlation structure: Schoenmakers and Coffey

$$\rho_{ij} = \exp\Big[ -\frac{|i-j|}{m-1}\big( \ln \rho_\infty + \eta_1 \frac{i^2+j^2+ij-3mi-3mj+3i+3j+2m^2-m-4}{(m-2)(m-3)} + - \eta_2 \frac{i^2+j^2+ij-mi-mj-3i-3j+3m+2}{(m-2)(m-3)} \big) \Big],$$

$$(i,j) \in [1,m]^2, 3\eta_1 \leq \eta_2 \leq 0, 0 \leq \eta_1 + \eta_2 \leq -ln\rho_\infty$$

- Approximation formula: Rebonato, Hull White or Andersen & Andreasen

## 4.3 Market data

- The yield curve (Current price $B(0,t)$ of the bonds maturing at time $t$)
- Caplet volatilities:

$$\left(\sigma_i^{Black,LMM}\right)^2 = \frac{c_i^2}{T_i} \int_0^{T_i} \eta^2(T_i - s) ds$$

- At-The-Money Swaptions quotations in volatilities

## 4.4 Calibration process

- Fit roughly the swaption matrix $\gamma_{p,q}$ with *the approximation formula* and the market data,
- Run a Principal Component Analysis on the correlation matrix previously used and keep the most important factors,
- Use the rank reduction method with Rebonato angles to obtain a closer correlation matrix $\rho_{ij}^{Model}$,

- Re-run the first 3 steps with the parameters already found and using only as reference the swaptions useful for the pricing of the derivative,
- Finally, the model is well calibrated on caplets and on the swaptions we need.
- Therefore we can price Interest rates derivatives with this calibrated model: from this correlation matrix, the volatility mapping and the Forward rates at time 0, run a Monte Carlo simulation on the discretized version of the Forward rates in the Libor Market Model to obtain their diffusion through the time.

## 4.5 Conclusion

This thesis has described extensively the Libor Market Model and how it is an important step in Interest Rates model. After this theoretical description, we have proposed different parametric forms for the instantaneous volatility and correlation and chosen a set of parameters: Laguerre type volatility and Schoenmakers-Coffey semi-parametric correlation.
Then, a 3-factor case calibration process of this model was selected according to the results of a Principal component analysis done on the correlation matrix chosen before. From several market inputs and different justified assumptions, we could calibrate the model to caplets and swaptions in a reasonable computation time and with acceptable approximations thanks to closed formula for swaption prices. This formula permitted us to avoid running several Monte-Carlo simulations.
As highlighted, this process is still an open problem especially for skew issues and pricing of cross-asset products.

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