Stability Analysis of Switched Systems

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Abstract

Switched systems are a particular kind of hybrid systems described by a combination of continuous/discrete subsystems and a logic-based switching signal. Currently, switched systems are employed as useful mathematical models for many physical systems displaying different dynamic behavior in each mode. Among the challenging mathematical problems that have arisen in switched systems, stability is the main issue. It is well known that switching can introduce instability even when all the subsystems are stable while on the other hand proper switching between unstable subsystems can lead to the stability of the overall system. In the last few years, significant progress has been made in establishing stability conditions for switched systems. While major advances have been made, a number of interesting problems are left open, even in the case of switched linear systems. With respect to some of these problems, we present some new results in three chapters as follows:

In Chapter 2, we deal with the stability of switched systems under arbitrary switching. Compared to Lyapunov-function methods which have been widely used in the literature, a novel geometric approach is proposed to develop an easily verifiable, necessary and sufficient stability condition for a pair of second-order linear time invariant (LTI) systems under arbitrary switching. The condition is general since all the possible combinations of subsystem dynamics are analyzed.

In Chapter 3, we apply the geometric approach to the problem of stabilization by switching. Necessary and sufficient conditions for regional asymptotic stabilizability are derived, thereby providing an

effective way to verify whether a switched system with two unstable second-order LTI subsystems can be stabilized by switching.

In Chapter 4, we investigate the stability of switched systems under restricted switching. We derive new frequency-domain conditions for the L_2-stability of feedback systems with periodically switched, linear/nonlinear feedback gains. These conditions, which can be checked by a computational-graphic method, are applicable to higher-order switched systems.

We conclude the thesis with a summary of the main contributions and future direction of research in Chapter 5.

Dedicated to my beloved wife
Lan Li
and my dear daughter
Yixin Huang

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Contents

List of Figures

List of Tables

Nomenclature

BCSS Best case switching signal

BMI Bilinear matrix inequality

CLF Common Lyapunov function

CQLF Common quadratic Lyapunov function

GAS Global asymptotic stabilizability, globally asymptotically stabilizable

LMI Linear matrix inequality

LTI Linear time invariant

MIMO Multi-input-multi-output

RAS Regional asymptotic stabilizability, regionally asymptotically stabilizable

SISO Single-input-single-output

WCSS Worst case switching signal

Chapter 1

Introduction

1.1 Hybrid Systems and Switched Systems

A hybrid system is a dynamical system that contains interacting continuous and discrete dynamics. Many systems encountered in practice are intrinsically hybrid systems. For example, a valve or a power switch opening and closing; a thermostat turning the heat on and off; and the dynamics of a car changing abruptly due to wheels locking and unlocking.

Hybrid systems have attracted the attention of people with diverse backgrounds due to their intrinsic interdisciplinary nature. One approach, favored by researchers in computer science, is to concentrate on studying the discrete behavior of the system, while the continuous dynamics are assumed to take a relative simple form. Many researchers in systems and control theory, on the other hand, tend to regard hybrid systems as continuous systems with switching, and place a greater emphasis on properties of the continuous state.

This thesis is written from a control engineer's perspective which adopts the latter point of view. Thus, we are interested in continuous-time systems with switching. We refer to such systems as *switched systems*. Specifically, a switched system is a hybrid system that consists of a family of subsystems and a switching law that orchestrates switching between these subsystems.

A typical switched system is a multi-controller system shown in Fig. 1.1. A given plant is controlled by switching among a family of stabilizing controllers, each of which is designed for a specific task. A high-level decision maker determines which controller is activated at each instant of time via a switching signal.

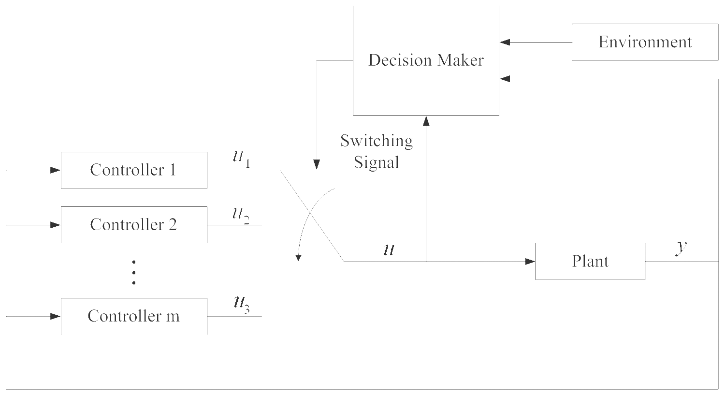


Figure 1.1: A multi-controller switched system.

Mathematically, a switched system can be described by a differential equation of the form

$$\dot{x}(t) = f_\sigma(x(t)), \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the continuous state of the system, $f_p : p \in \mathcal{P}$ is a family of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is parameterized by some index set $\mathcal{P}$, and $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant function of time t or state $x(t)$, called a switching signal.

In particular, if all individual systems are linear, we obtain a *switched linear system*

$$\dot{x}(t) = A_\sigma x(t), \qquad A_\sigma \in \mathbb{R}^{n\times n}. \tag{1.2}$$

Switched systems have been studied for the past fifty years or so, in the course of analysis and synthesis of engineering systems with relays and/or hysteresis. Due not only to their success in applications but also to their importance in theory, the last decade has witnessed burgeoning research activities on their stability [1, 2, 3], controllability [4], observability [5] *etc.*, that aim at designing switched systems with guaranteed stability and performance [6, 7, 8, 9]. Among these research topics, stability and stabilization have attracted most attention.

1.2 Stability of Switched Systems

Stability is a fundamental requirement in any control system, including switched systems which give rise to interesting phenomena. For instance, even when all the subsystems are asymptotically stable, the switched systems may not be stable under all possible switching. Consider two second-order asymptotically stable subsystems whose trajectories are sketched in Fig. 1.2. It is seen that the switched system can be made unstable by a suitable synthesis of trajectories.

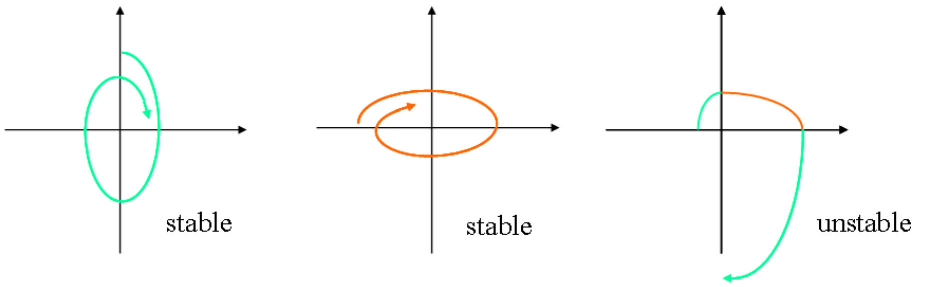


Figure 1.2: Switching between stable systems.

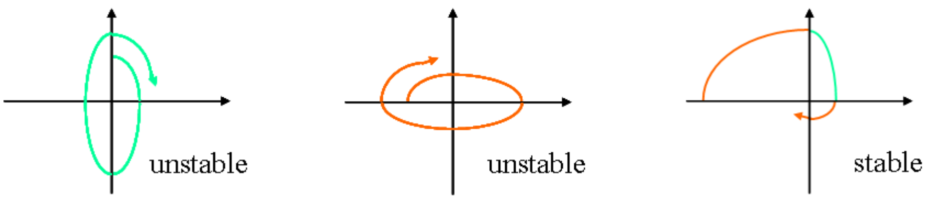


Figure 1.3: Switching between unstable systems.

Similarly, Fig. 1.3 illustrates the fact that, even when all the subsystems are unstable, it is possible to stabilize the system by designing a suitable switching signal.

Such phenomena prompt us to consider three basic problems concerning switched systems.

Problem A: What are the conditions on the subsystems such that a switched system is stable under arbitrary switching?

Problem B: If a switched system is not stable under arbitrary switching, how to identify a class of switching signals under which the switched system is stable?

Problem C: How to design switching signals to stabilize a switched system with unstable subsystems?

1.3 Literature Review on Stability under Arbitrary Switching

In this section, we review some important results in the literature of switched systems, in particular, switched linear systems, under arbitrary switching. See the papers [1, 10, 11] and recent books [12, 13] for an excellent survey.

Consider a switched linear system (1.2)

$$\dot{x} = A_\sigma x, \qquad A_\sigma \in \mathbb{R}^{n\times n}.$$

Clearly, a necessary condition for the switched system to be asymptotically stable under arbitrary switching is that all the subsystems must be asymptotically stable. If one subsystem, say, the p^{th} subsystem is not stable, then the switched system is unstable for $\sigma \equiv p$. However, this condition is not sufficient for the stability under arbitrary switching. Therefore, there is a need to determine the additional conditions on the subsystems for the stability of the complete system.

A simple condition to guarantee stability under arbitrary switching is that the matrices of the subsystems commute [14]. Let us take a switched system with two linear time invariant (LTI) subsystems as an example. Now consider an arbitrary switching signal σ and denote the time intervals on which $\sigma = 1$ and $\sigma = 2$ by t_i and τ_i respectively. The solution of the switched system under this switching signal is

$$x(t) = \cdots e^{A_2\tau_2}e^{A_1t_2}e^{A_2\tau_1}e^{A_1t_1}x(0). \tag{1.3}$$

If $A_1A_2 = A_2A_1$, then we have $e^{A_1t_1}e^{A_2\tau_1} = e^{A_2\tau_1}e^{A_1t_1}$, as can be seen from the definition of a matrix exponential via the series $e^{At} = It + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots$. Hence, we can rewrite (1.3) as

$$x(t) = \cdots e^{A_2\tau_2}e^{A_2\tau_1} \cdots\; e^{A_1\tau_2}e^{A_1t_1}x(0) = e^{A_2(\tau_1+\tau_2+\ldots)}e^{A_1(t_1+t_2+\ldots)}x(0). \tag{1.4}$$

Since both subsystems are stable, it follows that both $e^{A_2(\tau_1+\tau_2+\ldots)}$ and $e^{A_1(t_1+t_2+\ldots)}$ are bounded, and the switched system is stable for all σ.

For the switched systems of the first-order, A_1 and A_2 become scalars, and hence the commutativity condition is always satisfied. However, for higher-order switched systems, the commutativity condition is too restrictive to be satisfied in general. Therefore, more general conditions need to be found.

It is well known that if there exists a common Lyapunov function (CLF) for all subsystems, then the stability of the switched system under arbitrary switching is guaranteed. This has provided, in fact, the motivation to explore the application of quadratic Lyapunov functions (CQLFs) for switched *linear* systems, as found in [10, 15, 16].

1.3.1 Common Quadratic Lyapunov Functions

Consider switched linear systems (1.2). If there exists a positive definite symmetric matrix P satisfying

$$A_p^T P + P A_p < 0 \quad \forall p \in \mathcal{P}, \tag{1.5}$$

where the subscript T denotes transpose, then all subsystems admit a CQLF of the form,

$$V(x) = x^T P x, \tag{1.6}$$

and the switched system is stable under arbitrary switching.

Remark 1.1. The geometrical meaning of the existence of a CQLF is that, in the domain of linearly transformed coordinates, the squared magnitudes of the states of all subsystems decay exponentially.

1.3.1.1 Algebraic Conditions on the Existence of a CQLF

The CQLFs are attractive because the linear matrix inequalities (1.5) in P appear to be numerically solvable. But linear matrix inequalities are inefficient, and offer little insights to stability under arbitrary switching. Therefore, many attempts have been made to derive algebraic conditions on the dynamics of subsystems for the existence of a CQLF.

Shorten and Narendra [17] considered a second-order switched system with two subsystems, and derived the following necessary and sufficient condition for the existence of a CQLF. Let the matrix pencil be denoted by $\gamma_\alpha(A_1, A_2) = \alpha A_1 + (1-\alpha)A_2$ for $\alpha \in [0,1]$. Then,

Theorem 1.1. *[17] A necessary and sufficient condition for the dynamic systems Σ_{A_1} and Σ_{A_2} to have a CQLF is that the pencils $\gamma_\alpha(A_1, A_2)$ and $\gamma_\alpha(A_1, A_2^{-1})$ are both Hurwitz.*

Theorem 1.1 helps to verify the existence of a CQLF based on the state matrix directly, *i.e.*, without the need for solving linear matrix inequalities. It has been extended to switched systems consisting of (a) more than two LTI subsystems in [15], and (b) two third-order as also higher-order subsystems in [18]. However, for general higher-order switched systems and systems with more than two modes, necessary and sufficient conditions for the existence of a CQLF for stability are still not known.

In contrast, for switched systems, Liberzon, Hespanha and Morse [19] propose a Lie algebraic condition, based on the solvability of the Lie algebra generated by the subsystems' state matrices.

Theorem 1.2. *[19] If all the matrices $A_p, p \in \mathcal{P}$ are Hurwitz and the Lie algebra $\{A_p, p \in \mathcal{P}_{LA}\}$ is solvable, then there exists a common quadratic Lyapunov function.*

See [20] for an extension of the above theorem to the local stability of switched nonlinear systems, based on Lyapunov's first method; and [21] for a recent study of global stability properties for switched nonlinear systems and for a Lie algebraic global stability criterion, based on Lie brackets of the nonlinear vector fields.

Note that the systems satisfying Lie algebraic condition are a special case of systems which share a CQLF. Therefore, the Lie algebraic condition is only sufficient but not necessary for the existence of a CQLF (ensuring asymptotic stability of the switched system under arbitrary switching). Further, it is not easy to verify the Lie algebraic condition.

Remark 1.2. The existence of a CQLF is only sufficient for the stability of arbitrary switching systems. See [22] for the counterexample of two (second-order) subsystems which do not have a CQLF, but the switched system is asymptotically stable under arbitrary switching.

It has to be noted that the stability conditions for arbitrarily switched linear systems, based on the existence of a common quadratic Lyapunov function, are sufficient only, except for some special cases. In the next subsection, we discuss these special cases for which (i) quadratic stability is equivalent to asymptotic stability, and (ii) the stability of subsystems guarantees not only the existence of a quadratic Lyapunov function but also the stability of the arbitrarily switched system.

1.3.1.2 Some Special Cases

One special case is that of pairwise commutative subsystems [14], *i.e.*, $A_iA_j = A_jA_i$ for all i, j. As mentioned before, a commutative switched system is stable if and only if all its subsystems are stable. This can be established by a direct inspection of the solution of the switched system, and invoking the commutativity property of the matrices of the subsystems:

$$x(t) = \cdots e^{A_2\tau_2}e^{A_2\tau_1} \cdots\ e^{A_1\tau_2}e^{A_1t_1}x(0) = e^{A_2(\tau_1+\tau_2+\ldots)}e^{A_1(t_1+t_2+\ldots)}x(0).$$

These commutative subsystems share a common quadratic Lyapunov function, which can be obtained by solving a collection of chained Lyapunov equations.

Theorem 1.3. *[14] Let $P_1, \cdots, P_N$ be the unique symmetric positive definite matrices that satisfy the Lyapunov equations*

$$A_1^TP_1 + P_1A_1 = -I,$$

$$A_i^TP_i + P_iA_i = -P_{i-1}, \ \ i = 2, \cdots, N$$

then the function $V(x) = x^TP_Nx$ is a CQLF for the subsystems.

The second special case is when all the subsystems are *symmetric* [23], *i.e.*, $A_i^T = A_i$. In this case, a common quadratic Lyapunov function can be chosen as $V(x) = x^Tx$. Stability of A_i implies that $A_i^T + A_i < 0$, which means that there exists a P which can be chosen as I (the identity matrix) satisfying the inequality $A_i^TP + PA_i < 0$.

The third special case is the *normal* system which is a switched LTI system whose subsystem matrices satisfying $A_iA_i^T = A_i^TA_i$ for every mode i. Notice that the symmetric matrix is always normal. It is shown in [24] that $V(x) = x^Tx$ also serves as a CQLF for such a system.

1.3.2 Converse Lyapunov Theorems

It is known that the existence of a common Lyapunov function implies asymptotic stability of the switched system (1.2) under arbitrary switching. Does the converse hold? Molchanov and Pyatnitskiy [25] provide an affirmative answer to it.

Theorem 1.4. *[25] If the switched linear system is uniformly exponentially stable under arbitrary switching, then it has a strictly convex, homogenous (of second order) common Lyapunov function of a quasi-quadratic form*

$$V(x) = x^T L(x)x,$$

where $L(x) = L^T(x) = L(\tau x)$ *for all nonzero* $x \in \mathbb{R}^n$ *and* $\tau \in \mathbb{R}$*.*

See [22] for a converse theorem concerning the globally uniformly asymptotically stable and locally uniformly exponentially stable (1.2) with arbitrary switching. It is also shown that such a system admits a common Lyapunov function.

Theorem 1.5. *[22] If the switched system is globally uniformly asymptotically stable and in addition uniformly exponentially stable, the family has a common Lyapunov function.*

Even though converse Lyapunov theorems support the use of CQLF for establishing stability conditions for switched systems (1.2), it is evident that a common Lyapunov function need not be quadratic, although most of the available results are on the CQLF. Recently, non-quadratic Lyapunov functions, in particular polyhedral Lyapunov functions, have been explored.

1.3.3 Piecewise Lyapunov Functions

Several methods for automated construction of a common polyhedral (and hence piecewise) Lyapunov function have been proposed. See [26] for the synthesis of a balanced polyhedron satisfying some invariance properties, [25] for an alternative approach in which algebraic stability conditions are derived based on weighted infinity norms, and [27] for a linear programming-based method for deriving the stability conditions; and [28] for a numerical approach (to calculate polyhedral Lyapunov functions) in which the state-space is uniformly gridded in ray directions. However, it has been found that a construction of such piecewise Lyapunov functions is, in general, not simple.

1.3.4 Trajectory Optimization

Another approach to the analysis of stability under arbitrary switching is based on identifying a switching scheme which results in a “most unstable” trajectory. The basic idea is simple: if the worst case trajectory is stable, then the whole system should be stable as well for all the switching schemes. Filippov [29] derives a necessary and sufficient stability condition for a switched system having trajectories rotating around the origin. Pyatnitskiy and Rapoport [30] identify the *most unstable* nonlinearity using variational calculus and derive a necessary and sufficient condition for absolute stability of second- and third-order systems. Unfortunately, this condition is computationally challenging because it requires the solution of a nonlinear equation with three unknowns. In more recent pursuit along this line, Margaliot and Langholz [31], Margaliot and Gitizadeh [32] reduce the number of unknowns of the nonlinear equation from three to one, and derive a verifiable, necessary and sufficient condition for the absolute stability of second-order systems, which is extended to third-order systems in [33]. However, there is still a need to solve a nonlinear equation numerically. Recently, in [34], the relationships between the eigenvectors and eigenvalues of the two subsystems have been exploited to deal with the worst trajectory (which may be chattering) and to derive an easily verifiable, necessary and sufficient condition. However, the stability conditions in the above references are ad hoc, and offer little insight into the actual stability mechanism of switched systems.

1.4 Literature Review on Switching Stabilization

In this section, we review the literature on switching stabilization which is of two types.

1. Feedback stabilization in which the switching signals are assumed to be given or restricted. The problem is to design appropriate feedback control laws, in the form of state or output feedback, to achieve closed-loop system stability [35].

   Several classes of switching signals are considered in the literature, for example arbitrary switching [36], slow switching [37] and restricted switching induced by partitions of the state space [38, 39, 40].

2. Switching stabilization in which it is assumed that there is no external input to the system. The problem is to design a sequence for switching between the two subsystems to achieve system stability.

We consider only the latter mode of stabilizing switched systems.

1.4.1 Quadratic Switching Stabilization

Early research is concerned with quadratic stabilization for certain classes of systems. From the results of the literature [41, 42], it is known that the existence of a stable convex combination state matrix is necessary and sufficient for the quadratic stabilizability of two-mode switched Linear-time-invariant (LTI) systems. However, it should be noted that the existence of a stable convex combination matrix is only sufficient for switched LTI systems with more than two modes. In fact, there are systems for which no stable convex combination state matrix exists, but are quadratic-stabilizable.

Moreover, all the methods that guarantee stability by using a CQLF are conservative in the sense that there are switched systems that can be asymptotically (or exponentially) stabilized without using a CQLF [43].

More recent efforts were based on multiple Lyapunov functions [44], especially piecewise Lyapunov functions [45, 46, 47], to construct stabilizing switching signals. In [46], a probabilistic algorithm was proposed for the synthesis of an asymptotically stabilizing switching law for switched LTI systems along with a piecewise quadratic Lyapunov function.

1.4.2 Switching Stabilizability

Note that the existing stabilizability conditions, which may be expressed as certain linear matrix inequalities and bilinear matrix inequalities, are basically sufficient only, except for certain cases of quadratic stabilization. The more elusive problem is the necessity part. In [48], it is shown that if there exists an asymptotically stabilizing switching signal among a finite number of LTI systems $\dot{x}(t) = A_i x(t)$, where $i = 1, 2, \cdots, N$, then there exists a subsystem, say A_k, such that at least one of the eigenvalues of $A_k + A_k^T$ is a negative real number.

An algebraic necessary and sufficient condition for asymptotic stabilizability of second-order switched LTI systems was derived in [49] by detailed vector field analysis. For more recent results, see [50, 51]. However, the stabilization conditions of the above papers are not general since not all the possible combinations of subsystem dynamics are considered. Recently, Lin and Antsaklis [52] derived a necessary and sufficient condition for the stabilizability of switched linear system affected by parameter variations. However, verification of the necessity of the stabilization condition is not easy in general. This motivates us to derive easily verifiable, necessary and sufficient conditions for the switching stabilizability of switched linear systems.

1.5 Literature Review on Stability under Restricted Switching

Switched systems, which fail to preserve stability under arbitrary switching, may be stable under restricted switching. One may have some knowledge about possible switching signals for a switched system, *e.g.*, certain bound on the time interval between two successive switchings. With a prior knowledge about the switching signals, we can derive a stronger stability condition for a given switched system than the arbitrary switching case which is, by its very nature, the worst case. This knowledge imply restrictions on the switching signals, which may be either time domain restrictions (*e.g.*, dwell-time, average dwell-time, and switching frequency) or state space restrictions (*e.g.*, the state may be trapped in some partitions of the state space). It is shown in [53] that the distinction between time-dependent switching signals and trajectory-dependent switching signals is significant.

Now we proceed to review some important results on two classes of time-dependent constraints: slow switching and periodic switching.

1.5.1 Slow Switching

By studying the divergent trajectory in Fig. 1.2, one may notice that the instability is introduced by the failure to absorb the energy increase caused by the switching. Intuitively, if the switching is sufficiently slow, so as to allow the transient effects to dissipate after each switch, it is possible to attain stability. These ideas are proved to be reasonable and are captured by concepts like dwell time and average dwell time switching in the literature, see for example [54, 53].

Definition 1.1. *τ_d is called the dwell time if the time interval between any two consecutive switchings is no smaller than τ_d.*

Theorem 1.6. *[54] Assume that all subsystems in the switched linear systems are exponentially stable. Then, there exists a scalar $\tau_d > 0$ such that the switched system is exponentially stable if the dwell time is larger than τ_d.*

Definition 1.2. *A positive constant τ_a is called the average dwell time if $N_\sigma(t) \leq N_0 + \frac{t}{\tau_a}$ holds for all $t > 0$ and some scalar $N_0 \geq 0$, where $N_\sigma(t)$ denotes the number of discontinuities of a given switching signal σ over $[0, t)$.*

Here the constant τ_a is called the *average dwell time* and N_0 the *chatter bound.* The reason to call a class of switching signal satisfy $N_\sigma(t) \leq N_0 + \frac{t}{\tau_a}$ have an average dwell time no less than τ_a is that

$$N_\sigma(t) \leq N_0 + \frac{t}{\tau_a} \Leftrightarrow \frac{t}{N_\sigma(t) - N_0} \geq \tau_a,$$

which means that on average the "dwell time" between any two consecutive switchings is no smaller than τ_a.

Theorem 1.7. *[53] Assume that all subsystems in the switched linear systems are exponentially stable. There exists a scalar $\tau_a > 0$ such that the switched system is exponentially stable if the average dwell time is larger than τ_a.*

The stability results for slow switching can be extended to switched systems consisting of both stable and unstable subsystems. When unstable dynamics is considered, slow switching (like long enough dwell/average dwell time) is not sufficient for stability. It has to make sure that the switched system does not spend too much time on the unstable subsystems. We need to consider unstable subsystems in switched systems because there are cases where switching to unstable subsystems is unavoidable once failure occurs. It is interesting to identify conditions under which the stability of the switched systems is still preserved. See [55, 56, 57] for details.

1.5.2 Periodic Switching

Another important class of switched systems is periodically switched systems. For periodically switched linear systems, necessary and sufficient conditions are available from Floquet theory [58, 59]. Since any general system may be thought as a periodic system with an infinite period, it is natural to question as follows.

Consider the system $\dot{x} = A(t)x, A(t) \in \{A_1, \cdots, A_m\}$. Suppose the switching system is exponentially stable for all periodic switching signals σ. Does this imply that the system is exponentially stable for arbitrary switching signals?

The above question has been studied extensively for both discrete- and continuous time switched systems. See [60, 61] and the references therein.

Theorem 1.8. *[10] The switched linear system is asymptotically stable under arbitrary switching if and only if there exists an $\varepsilon > 0$ such that $r(\Phi, \sigma(T, 0)) < 1 - \varepsilon$ for all periodic switching signals σ.*

It is shown that if the switched system is periodically stable with some finite robustness margin ε, then it is exponentially stable for arbitrary switching signals. In principle, Theorem 1.8 gives a practical method for testing the stability of any given switching system.

In addition, the worst case switching signal of a switched linear system with two second-order LTI subsystems is periodic based on our analysis in Chapter 2. The switching period $T = T_A + T_B$, where T_A (2.46) and T_B (2.47) are the time on the subsystems A and B respectively, associated with the worst case switching signal (2.62). We believe that it is true even for higher-order switched systems with more than two subsystems.

In practice, many real-world systems can be modeled as periodically switched systems, *e.g.*, the Buck converter in Fig. 1.4. The Buck converter is widely used in computer power supplies, which converts 12V direct current (DC) voltage to a lower voltage (around 1V) for central processing unit (CPU).

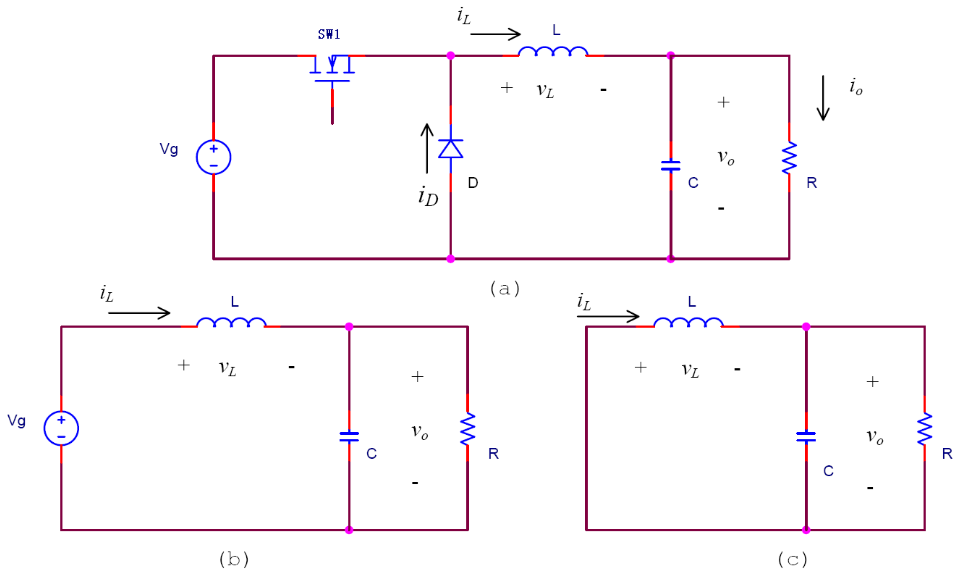


Figure 1.4: A practical example of periodically switched systems - a Buck converter.

Fig. 1.4(a) is the circuit of a Buck converter, where SW1 is switching at a fixed frequency (*e.g.*, 100MHz). When SW1 is on, the equivalent circuit is as Fig. 1.4(b), and when SW1 is off, the equivalent circuit is Fig. 1.4(c).

1.6 Outline of the Thesis

The main aim of the thesis is to present easily verifiable *new* conditions for both the stability and stabilizability of switched systems. To this end, the thesis is organized as follows.

In Chapter 2, we deal with the stability of switched systems under arbitrary switching. Compared to Lyapunov-function methods which have been widely used in the literature, a novel geometric approach is proposed to develop an easily verifiable, necessary and sufficient stability condition for a pair of second-order linear time invariant (LTI) systems under arbitrary switching. The condition is general since all the possible combinations of subsystem dynamics are analyzed.

In Chapter 3, we apply the geometric approach to the problem of stabilization by switching. Necessary and sufficient conditions for regional asymptotic stabilizability are derived, thereby providing an effective way to verify whether a switched system with two unstable second-order LTI subsystems can be stabilized by switching.

In Chapter 4, we investigate the stability of switched systems under restricted switching. We derive new frequency-domain conditions for the L_2-stability of feedback systems with periodically switched, linear/nonlinear feedback gains. These conditions, which can be checked by a computational-graphic method, are applicable to higher-order switched systems.

We conclude the thesis with a summary of the main contributions and future direction of research in Chapter 5.

Chapter 2

Stability Under Arbitrary Switching

In this chapter, we consider the stability of switched systems under arbitrary switching. This problem is important because the switching signal is either unknown or too complicated for some switched systems. Moreover, once the stability of a switched system under arbitrary switching is guaranteed, engineers have more freedom to design a switching signal for better performance, unaffected by stability considerations.

This chapter is organized as follows. In Section 2.1, we show the switched systems, which will be analyzed in this chapter. In Section 2.2, we introduce the concept of constants of integration, which plays a key role in developing the new stability and stabilizability conditions of the thesis. In Section 2.3, we characterize the worst-case switching signal (WCSS) based on the variations of the subsystems' constants of integration. In Section 2.4, we present the main result of this chapter, which is an easily verifiable, necessary and sufficient conditions, under reasonable assumptions, for the stability of switched systems with two continuous-time, second-order linear time invariant (LTI) subsystems, under arbitrary switching. All the possible combinations of the subsystems are analyzed under the WCSS such that no constraint is imposed on the dynamics of the subsystems. Geometrical interpretations of the stability condition are discussed. Examples are given to show its superiority over the stability conditions in the literature. In Section 2.5, we extend the main result to the stability of switched systems with marginally stable subsystems. In Section 2.6, we discuss the relationship between the main result in this chapter and the conditions on the existence of a common quadratic Lyapunov function (CQLF).

2.1 Problem Formulation

Motivated by the limitations of the existing results outlined in Chapter 1, our goal is to derive new and easily verifiable necessary and sufficient stability criterion for switched linear systems under arbitrary switching. In particular, we consider the following switched system with two second-order continuous-time LTI subsystems:

$$S_{ij} : \dot{x} = \sigma(t)x, \quad \sigma(t) \in \{A_i, B_j\}, \tag{2.1}$$

where both $A_i, B_j \in \mathbb{R}^{2\times 2}$ are stable, and $i, j \in \{1, 2, 3\}$ denote the types of A and B. The matrix $A \in \mathbb{R}^{2\times 2}$ is classified into three types according to its eigenvalues and eigenstructure as follows:

- Type 1: A has real eigenvalues and diagonalizable;
- Type 2: A has real eigenvalues but is undiagonalizable;
- Type 3: A has two complex eigenvalues.

In contrast with the existing results, the proposed stability condition has the following features:

1. It is a necessary and sufficient condition for the stability of the switched system (2.1) under arbitrary switching.
2. All combinations of the dynamics of subsystems (*i.e.* all the combinations of i and j in S_{ij}), are analyzed. There is no constraint on the subsystems.
3. It is easily verifiable (even by hand computation) in the sense that no numerical solution of nonlinear equation is required.
4. It is compact, and provides more geometrical insights.

The method to derive the necessary and sufficient condition follows the strategy of finding the worst case trajectory: if the trajectory of (2.1) under the worst case switching signal (WCSS) is stable, then the switched system is stable under arbitrary switching. Distinct from the approaches used in [34, 31, 32], the WCSS is characterized by the variations of the constants of integration of the subsystems.

2.2 Constants of Integration

The concept of *constants of integration* is introduced by analyzing the phase diagrams of switched systems in polar coordinates ($r-\theta$ coordinates). The variation of constants of integration facilitates the construction of an unstable trajectory between two asymptotically stable subsystems. It is interesting that the mathematical results presented in this section can also be applied to study the problem of switching stabilizability in Chapter 3.

2.2.1 Single Second-order LTI System in Polar Coordinates

Consider the second-order LTI system,

$$\dot{x} = Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x. \tag{2.2}$$

Let $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Then (2.2) assumes the form

$$\frac{dr}{dt} = r[a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta], \tag{2.3}$$

$$\frac{d\theta}{dt} = a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta. \tag{2.4}$$

When $\frac{d\theta}{dt} = 0$, it corresponds to the real eigenvector of A. The solutions on the real eigenvectors are

$$r = r_0 e^{\lambda_m t}, \tag{2.5}$$

where r_0 is the magnitude of the initial state and λ_m is the corresponding eigenvalue of the real eigenvector.

Since the worst case switching signal is straightforward on the eigenvectors, we focus on the trajectories not on the eigenvectors.

When $\frac{d\theta}{dt} \neq 0$,

$$\frac{dr}{d\theta} = r\frac{a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta}. \tag{2.6}$$

It follows that

$$\frac{1}{r}dr = f(\theta)d\theta, \tag{2.7}$$

where

$$f(\theta) = \frac{a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22}-a_{11})\sin\theta\cos\theta}. \tag{2.8}$$

2.2.2 Constant of Integration for A Single Subsystem

Lemma 2.1. *The trajectories of the LTI system (2.2) in r-θ coordinates, except the ones along the eigenvectors, can be expressed as*

$$r(\theta) = Cg(\theta), \tag{2.9}$$

where $g(\theta(t)) = e^{\int_{\theta^*}^{\theta(t)} f(\tau)d\tau}$ is positive, and C, the constant of integration, is a positive constant depending on the initial state (r_0, θ_0). Note that θ^* can be chosen as any value except the angle of any real eigenvector of A.

Proof. By integrating both sides of (2.7), we have

$$\int_{r_0}^{r}\frac{1}{r}dr = \int_{\theta_0}^{\theta} f(\tau)d\tau \Longrightarrow \ln r = \int_{\theta_0}^{\theta} f(\tau)d\tau + \ln r_0 \Longrightarrow r(\theta) = r_0 e^{\int_{\theta_0}^{\theta} f(\tau)d\tau}. \tag{2.10}$$

Equation (2.10) can be rewritten as (2.11) by splitting the integral interval,

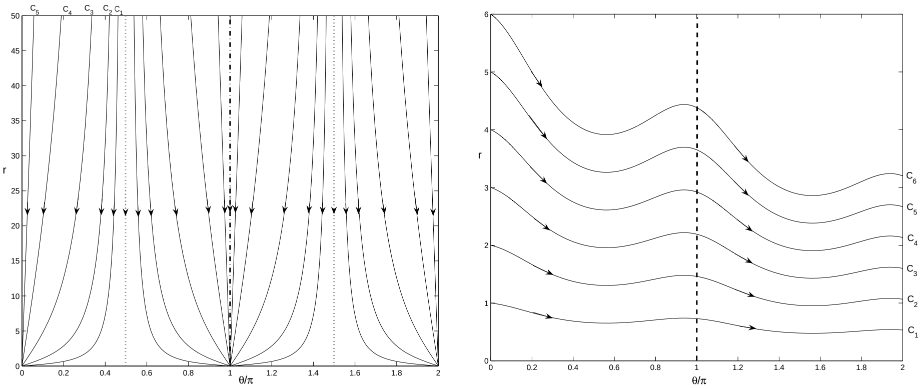
$$r(\theta) = r_0 e^{\int_{\theta_0}^{\theta} f(\tau)d\tau} = r_0 e^{\int_{\theta_0}^{\theta^*} f(\tau)d\tau} e^{\int_{\theta^*}^{\theta} f(\tau)d\tau}. \tag{2.11}$$

Denote the angle of the eigenvector of A as θ_e. Since $\theta^* \neq \theta_e$, $\theta \neq \theta_e$, the integrals $\int_{\theta_0}^{\theta^*} f(\tau)d\tau$ and $\int_{\theta^*}^{\theta} f(\tau)d\tau$ are bounded [1], and (2.11) can be further

[1] If $\theta_e \in (\theta^*, \theta)$, the Cauchy principal value (P.V.) of the improper integral is $\texttt{P.V.}\int_{\theta^*}^{\theta} f(\tau)d\tau = \lim_{\varepsilon\to 0^+}\left(\int_{\theta^*}^{\theta_e-\varepsilon} f(\tau)d\tau + \int_{\theta_e+\varepsilon}^{\theta} f(\tau)d\tau\right)$, which is also bounded because $\lim_{\varepsilon\to 0^+}\int_{\theta_e-\varepsilon}^{\theta_e+\varepsilon} f(\tau)d\tau = 0$.

reduced to (2.9). It can be readily seen that $C = r_0 e^{\int_{\theta_0}^{\theta^*} f(\tau)d\tau}$ is a constant determined by the initial state (r_0, θ_0).

Typical phase trajectories of planar LTI systems in polar coordinates are shown in Fig. 2.1. It can been seen that $f(\theta) = \frac{1}{r}\frac{dr}{d\theta}$, the slope of the trajectories normalized by the magnitude, is a periodic function with a period of π, for both real and complex eigenvalue cases.



(a) The real eigenvalue case. (b) The complex eigenvalue case.

Figure 2.1: The phase diagrams of second-order LTI systems in polar coordinates.

Remark 2.1. It follows from (2.10) that

$$\frac{r(\theta+\pi)}{r(\theta)} = \frac{r_0 e^{\int_{\theta_0}^{\theta+\pi} f(\tau)d\tau}}{r_0 e^{\int_{\theta_0}^{\theta} f(\tau)d\tau}} = e^{\int_0^{\pi} f(\tau)d\tau} \tag{2.12}$$

which is a constant since $f(\theta)$ is a periodic function with a period of π. Therefore, it is sufficient to analyze the stability of the system (2.2), regardless of the types of A, in an interval of θ with the length of π. Without loss of generality, this interval is chosen to be $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$.

Definition 2.1. *The line $\theta = \theta_a$ is said to be an asymptote of A in $r - \theta$ coordinates if the angle of the trajectory of $\dot{x} = Ax$ approaches to θ_a as the time $t \to +\infty$. Similarly, the line $\theta = \theta_{na}$ is said to be a non-asymptote of A if the angle of the trajectory of $\dot{x} = Ax$ approaches θ_{na} as the time $t \to -\infty$.*

For a given $A \in \mathbb{R}^{2\times 2}$ with real eigenvalues, the asymptote θ_a is the angle of the real eigenvector corresponding to a larger eigenvalue of A. Definition 2.1

is applicable to all matrices $A \in \mathbb{R}^{2\times 2}$ with real eigenvalues regardless of the dynamics of A (stable/unstable node, saddle point).

If A is a degenerate node (has only one eigenvector with an angle θ_r), θ_a and θ_{na} are chosen from θ_r^+ or θ_r^- based on the trajectory direction of A.

If A is a counter clockwise/clockwise focus, the asymptote of A in $r - \theta$ coordinates is actually $\theta_a = +\infty/-\infty$.

Remark 2.2. Note that the constant of integration C depends on the initial state. It remains invariant to $r(t)$ and $\theta(t)$ for the whole trajectory. Geometrically, a larger C indicates an outer layer trajectory, as shown in Fig. 2.1, where $C_1 < C_2 < C_3 \cdots < C_n$. Note that $r(\theta(t))$ converges to zero since $g(\theta)$ converges to zero as θ approaches the asymptote of the system associated with a Hurwitz A.

2.2.3 Variation of Constants of Integration for A Switched System

We analyze the switched system with two asymptotically stable subsystems. Using the variations of constants, we show how to construct an unstable trajectory by switching between two asymptotically stable subsystems.

Let the two subsystems be defined by:

$$\Sigma_A : \dot{x} = Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x,$$

$$\Sigma_B : \dot{x} = Bx = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} x. \tag{2.13}$$

To simplify the analysis, two classes of special cases are excluded by the following assumptions. These two special cases will be discussed separately in Section 2.4.1 (and Section 3.3.1 for switching stabilizability problem).

Assumption 2.1. *$A \neq cB$, where $c \in \mathbb{R}$.*

Assumption 2.2. *A and B do not share any real eigenvector.*

Following the definition of $f(\theta)$ in equation (2.8), we define $f_A(\theta)$ and $f_B(\theta)$ for subsystems A and B respectively

$$f_A(\theta) = \frac{a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12} + a_{21})\sin\theta\cos\theta}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22} - a_{11})\sin\theta\cos\theta}, \tag{2.14}$$

$$f_B(\theta) = \frac{b_{11}\cos^2\theta + b_{22}\sin^2\theta + (b_{12}+b_{21})\sin\theta\cos\theta}{b_{21}\cos^2\theta - b_{12}\sin^2\theta + (b_{22}-b_{11})\sin\theta\cos\theta}. \tag{2.15}$$

It follows from Lemma 2.1 that

$$r_A(t) = C_A g_A(\theta(t)), \tag{2.16}$$

$$r_B(t) = C_B g_B(\theta(t)). \tag{2.17}$$

A piecewise solution is obtained by combining (2.16) and (2.17).

$$r(t) = \begin{cases} C_A(t)g_A(\theta(t)), & when \quad \sigma(t) = A \\ C_B(t)g_B(\theta(t)), & when \quad \sigma(t) = B \end{cases}, \tag{2.18}$$

where $C_A(t)$ and $C_B(t)$ are invariant during the period when the states move along their own phase trajectories.

$$\left.\frac{dC_A(t)}{dt}\right|_{\sigma(t)=A} = 0, \quad \left.\frac{dC_B(t)}{dt}\right|_{\sigma(t)=B} = 0. \tag{2.19}$$

From (2.18), a compact solution of the switched system, except the ones along the eigenvectors, can be obtained as

$$r(t) = h_A(\theta(t))g_A(\theta(t)), \tag{2.20}$$

where

$$h_A(\theta(t)) = \begin{cases} C_A(t), & \sigma(t) = A \\ C_B(t)\frac{g_B(\theta(t))}{g_A(\theta(t))}, & \sigma(t) = B \end{cases}, \tag{2.21}$$

or similarly

$$r(t) = h_B(\theta(t))g_B(\theta(t)), \tag{2.22}$$

where

$$h_B(\theta(t)) = \begin{cases} C_A(t)\frac{g_A(\theta(t))}{g_B(\theta(t))}, & \sigma(t) = A \\ C_B(t), & \sigma(t) = B \end{cases}. \tag{2.23}$$

For convenience, we denote

$$H_A(\theta(t)) \triangleq \left.\frac{dh_A(\theta(t))}{dt}\right|_{\sigma(t)=B}, H_B(\theta(t)) \triangleq \left.\frac{dh_B(\theta(t))}{dt}\right|_{\sigma(t)=A}. \tag{2.24}$$

Equation (2.20) indicates that even when the actual trajectory follows Σ_B, it can still be described by the same form as that of the solution of Σ_A with a

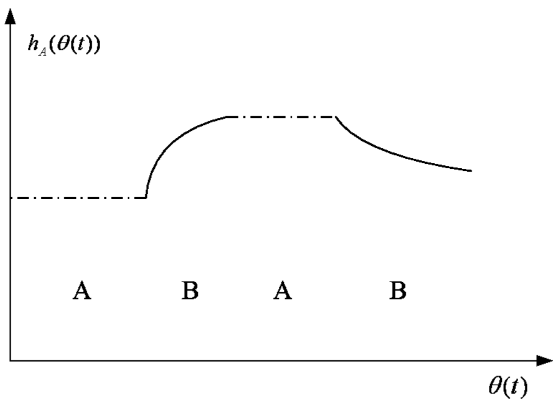


Figure 2.2: The variation of h_A under switching.

varying h_A. Then, we can use the variation of h_A to describe the behavior of the switched system (2.13), as shown in Fig. 2.2.

Geometrically, the positive $H_A(\theta)$, or equivalently the increase of $h_A(\theta)$, means that the vector field of Σ_B points outwards relative to Σ_A. Intuitively, if the increase of h_A can compensate the convergence of g_A for a long term, or in a period of $\theta(t)$, then it is possible to make the switched system unstable. Although the existence of a positive $H_A(\theta)$ or $H_B(\theta)$ is considered to be necessary, it is not sufficient to make the switched system (2.13) unstable. Therefore, there is a need for a comprehensive worst case analysis, which will be given in Section 2.3.

2.3 Worst Case Analysis

In this section, we identify the worst case switching signal (WCSS) for a given switched system, thereby converting the stability problem under arbitrary switching to the stability problem under the WCSS.

2.3.1 Mathematical Preliminaries

To find the WCSS, we need to know which subsystem is more "unstable" for every θ and how θ varies with time t. The former is determined through the signs of $H_A(\theta)$ and $H_B(\theta)$, while the latter is based on the signs of $Q_A(\theta)$ and $Q_B(\theta)$

which are defined as

$$Q_A(\theta(t)) \triangleq \left.\frac{d\theta(t)}{dt}\right|_{\sigma(t)=A}, Q_B(\theta(t)) \triangleq \left.\frac{d\theta(t)}{dt}\right|_{\sigma(t)=B}. \tag{2.25}$$

It follows from equations (2.21) and (2.24) that

$$\begin{aligned} H_A(\theta(t) &= \left.\frac{dh_A(t)}{dt}\right|_{\sigma(t)=B} = C_B(t)\left(\frac{g_B(\theta(t))}{g_A(\theta(t))}\right)' \\ &= -C_B(t)\frac{g_B(\theta(t))}{g_A(\theta(t))}[f_A(\theta(t)) - f_B(\theta(t))]\left.\frac{d\theta(t)}{dt}\right|_{\sigma(t)=B}, \end{aligned} \tag{2.26}$$

where $C_B(t)$ is a constant since $\sigma(t) = B$ in (2.26). Similarly, we have

$$H_B(\theta(t)) = C_A(t)\frac{g_A(\theta(t))}{g_B(\theta(t))}[f_A(\theta(t)) - f_B(\theta(t))]\left.\frac{d\theta(t)}{dt}\right|_{\sigma(t)=A}. \tag{2.27}$$

Equations (2.26) and (2.27) can be rewritten as

$$H_A(\theta(t)) = -K_B(\theta(t))G(\theta(t))Q_B(\theta(t)), \tag{2.28}$$

$$H_B(\theta(t)) = K_A(\theta(t))G(\theta(t))Q_A(\theta(t)), \tag{2.29}$$

where $K_A(\theta(t)) = C_A(t)\frac{g_A(\theta(t))}{g_B(\theta(t))}, K_B(\theta(t)) = C_B(t)\frac{g_B(\theta(t))}{g_A(\theta(t))}$ and

$$G(\theta) = f_A(\theta) - f_B(\theta). \tag{2.30}$$

Remark 2.3. In (2.28) and (2.29), both $K_A(\theta)$ and $K_B(\theta)$ are positive, $G(\theta)$ is the common part, and it can be readily shown that

- If the signs of $Q_A(\theta)$ and $Q_B(\theta)$ are the same, then the signs of $H_A(\theta)$ and $H_B(\theta)$ are opposite.
- If the signs of $Q_A(\theta)$ and $Q_B(\theta)$ are opposite, then the signs of $H_A(\theta)$ and $H_B(\theta)$ are the same.

The geometrical meaning of the signs of $Q_A(\theta)$ and $Q_B(\theta)$ is the trajectory direction. A positive $Q_A(\theta)$ implies a counter clockwise trajectory of Σ_A in $x - y$ coordinates.

Since the interesting interval of θ is $[-\frac{\pi}{2}, \frac{\pi}{2})$, all the functions of θ could be transformed to the functions of k by denoting $k = \tan\theta$. Straightforward algebraic manipulation yields

$$H_A(k) = K_B(k)\frac{N(k)}{D_B(k)}, \tag{2.31}$$

$$H_B(k) = -K_A(k)\frac{N(k)}{D_A(k)}, \tag{2.32}$$

$$Q_A(k) = -\frac{1}{k^2+1}D_A(k), \tag{2.33}$$

$$Q_B(k) = -\frac{1}{k^2+1}D_B(k), \tag{2.34}$$

where

$$D_A(k) = a_{12}k^2 + (a_{11} - a_{22})k - a_{21}, \tag{2.35}$$

$$D_B(k) = b_{12}k^2 + (b_{11} - b_{22})k - b_{21}, \tag{2.36}$$

and

$$N(k) = p_2k^2 + p_1k + p_0, \tag{2.37}$$

where $p_2 = a_{12}b_{22} - a_{22}b_{12}$, $p_1 = a_{12}b_{21} + a_{11}b_{22} - a_{21}b_{12} - a_{22}b_{11}$, and $p_0 = a_{11}b_{21} - a_{21}b_{11}$.

Denote the two distinct real roots of $N(k)$, if exist, by k_1 and k_2, and assume $k_2 < k_1$. Notice that the signs of equations (2.31)-(2.34) depend on the signs of $D_A(k)$, $D_B(k)$ and $N(k)$.

Lemma 2.2. *If A and B do not share any real eigenvector, which was guaranteed by Assumption 2.2, the real roots of $N(k)$ do not overlap the real roots of $D_A(k)$ or $D_B(k)$ when A and B are not singular.*

The proof of Lemma 2.2 is presented in Appendix A.1.

Definition 2.2. *A region of k is a continuous interval where the signs of (2.31)-(2.34) preserve for all k in this interval.*

Remark 2.4. The boundaries of the regions of k, if exist, are the lines whose angles satisfy $D_A(k) = 0$, $D_B(k) = 0$ or $N(k) = 0$.

- If $D_A(k) = 0$, then $\left.\frac{d\theta}{dt}\right|_{\sigma=A} = 0$, they are the real eigenvectors of A.

- If $D_B(k) = 0$, then $\left.\frac{d\theta}{dt}\right|_{\sigma=B} = 0$, they are the real eigenvectors of B.
- Since the real eigenvectors are only located on the boundaries, the solution expressions of (2.20) and (2.22) are always valid inside the regions of k.
- If $N(k) = 0$, then $G(\theta) = f_A(\theta) - f_B(\theta) = 0$, which indicates $\left.\frac{dr}{d\theta}\right|_{\sigma=A} = \left.\frac{dr}{d\theta}\right|_{\sigma=B}$ with reference to (2.6)-(2.8). They are the lines where the trajectories of the subsystems are tangent to each other.
- If $N(k) = (k - k_m)^2$, in the two regions that share the boundary $k = k_m$, the signs of (2.31)-(2.34) are invariant, so the WCSS in these two regions are the same. In addition, the trajectories on the boundary $k = k_m$ are tangent to each other and none of them can stay on this boundary based on Lemma 2.2. As a result, the two regions can be merged to one, which means that the system behavior when $N(k)$ has two multiple roots is entirely similar to the one when $N(k)$ does not have real roots. Therefore, the case when $N(k)$ has two multiple roots will be ignored.
- With reference to Eqn. (2.31)-(2.34), when trajectories cross the boundary k_1 or k_2, the trajectory directions remain unchanged while the signs of $H_A(k)$ and $H_B(k)$ change simultaneously.

These boundaries divide the $x - y$ plane to several conic sectors, *i.e.*, regions of k.

2.3.2 Characterization of the Worst Case Switching Signal (WCSS)

Now we proceed to establish criteria to determine the WCSS for every θ, or k equivalently, based on the signs of H_A and H_B.

1) Both H_A and H_B are positive

Lemma 2.3. *The switched system (2.13) is not stable under arbitrary switching if there is a region of k, $[k_l, k_u]$, where both H_A and H_B are positive.*

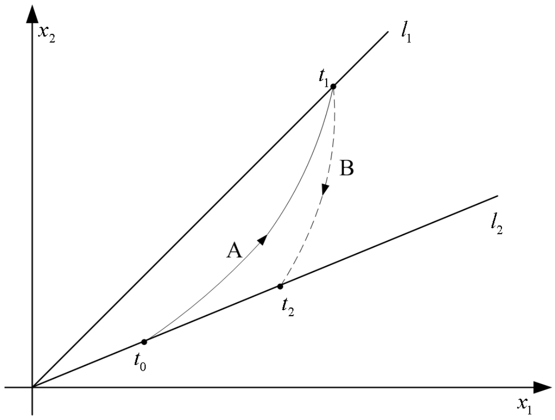


Figure 2.3: The region where both H_A and H_B are positive.

With reference to Fig. 2.3, an unstable trajectory can be easily constructed by switching inside this region. The proof of Lemma 2.3 is given in Appendix A.2.

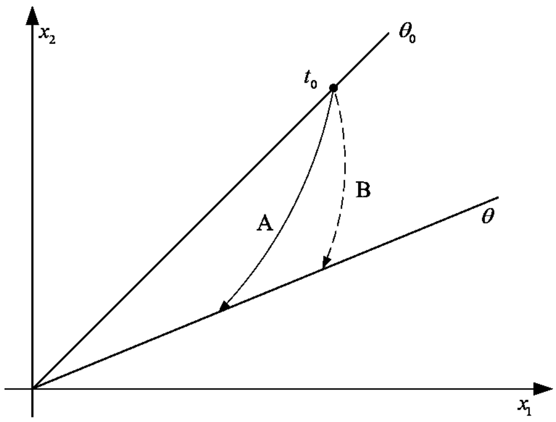


Figure 2.4: The region where H_A is positive and H_B is negative.

2) H_A is positive and H_B is negative

The WCSS is Σ_B. In this case, the trajectories of two subsystems have the same direction, as shown in Remark 2.3. With reference to Fig. 2.4, consider an initial state with an angle θ_0 at t_0. Let $r_B(\theta)$ be the trajectory along Σ_B, and $r_A(\theta)$ be the trajectory along Σ_A. Comparing the magnitudes of the trajectories along different subsystems, we have

$$r_B(\theta) - r_A(\theta) = h_A(\theta)g_A(\theta) - C_A g_A(\theta) = g_A(\theta)\int_{t_0}^{t} H_A(\theta(t))dt > 0 \qquad (2.38)$$

which shows that the trajectory of Σ_B always has a larger magnitude than the corresponding one of Σ_A for all θ in this region.

3) H_A is negative and H_B is positive

Similarly, the WCSS is Σ_A.

4) Both H_A and H_B are negative

First, we show that the switched system is stable in this region if its trajectory does not move out of this region. It follows from Assumption 2.2 that at least one of $g_A(\theta)$ and $g_B(\theta)$ is bounded for any given θ. Since both H_A and H_B are negative, we have $h_A(t) \leq h_A(t_0)$ and $h_B(t) \leq h_B(t_0)$. With reference to (2.20) and (2.22), the magnitude of trajectories r is bounded in this region. Hence the stability of the switched system is determined by other regions.

Next we will discuss the scenarios when the trajectory may move out.

(1) If only the trajectory of one subsystem, assumed to be Σ_A, can go out of this region, then the WCSS in this region is Σ_A. Let r_{σ^*} be the trajectory along Σ_A and let r_σ be the trajectory under any other switching signal. Comparing the magnitudes of the states on the boundary ($\theta = \theta_{bn}$) where the trajectories move out, it can be shown that any switching other than Σ_A in this region will make the switched system more stable since

$$r_{\sigma^*}(\theta_{bn}) = h_A(t_0)g_A(\theta_{bn}) > r_\sigma = h_A(t)g_A(\theta_{bn}). \qquad (2.39)$$

(2) If the trajectories of both subsystems can go out and neither can come back, then no matter which subsystem is chosen, the trajectory will leave this region and the stability of the switched system is determined by other regions.

(3) If the trajectories of both subsystems can go out and at least one of them can come back, then at least one of the boundaries of this region is k_1 or k_2. It was mentioned in Remark 2.4 that $H_A(k)$ and $H_B(k)$ change their signs simultaneously when trajectories cross the boundary k_1 or k_2. In such a case,

there must exist an unstable region, where both H_A and H_B are positive, next to this region. Therefore, the switched system is not stable under arbitrary switching from Lemma 2.3.

5) One of H_A and H_B is zero

If one of H_A and H_B is zero, then it implies that $N(k) = 0$, then both of them are zero. (The case that $N(k)$ has two multiple real roots is ignored based on Remark 2.4.)

(1) If the trajectories of the subsystems cross the line in the same direction, we can choose either subsystem as the WCSS since the trajectories are tangent to each other on this line.

(2) If the trajectories of the subsystems cross the line in the opposite direction, it follows from Remark 2.4 that there exists an unstable region near the line where $N(k) = 0$. Hence the switched system is not stable under arbitrary switching from Lemma 2.3.

6) On real eigenvectors

It can be readily shown that the WCSS is Σ_A on the eigenvectors of B, and vice versa.

We have characterized the WCSS based on the signs of $H_A(k)$, $H_B(k)$, $Q_A(k)$ and $Q_B(k)$, for which we can determine the stability of the switched systems (2.13) under arbitrary switching by the following procedure.

1. Determine all the boundaries: the real eigenvectors of two subsystems and the distinct real roots of $N(k)$. All the boundaries are known since all the entries of the subsystems are known.

2. Determine the signs of $H_A(k)$, $H_B(k)$, $Q_A(k)$ and $Q_B(k)$ for every region of k.

3. Determine the WCSS for every region based on 2 and obtain the WCSS for the given switched system.

4. Determine the stability of the switched system based on the WCSS.

2.4 Necessary and Sufficient Stability Conditions

We now apply the worst case analysis to derive an easily verifiable, necessary and sufficient stability condition for the stability of switched system

$$S_{ij} : \dot{x} = \sigma(t)x, \quad \sigma(t) \in \{A_i, B_j\}, \tag{2.40}$$

where both A_i and $B_j \in \mathbb{R}^{2\times 2}$ are Hurwitz, and $i, j \in \{1, 2, 3\}$ denote the types of A and B respectively. A matrix $A \in \mathbb{R}^{2\times 2}$ is classified into three types according to its eigenvalues and eigenstructure as defined in Section 2.1.

- Type 1: A has real eigenvalues and diagonalizable;
- Type 2: A has real eigenvalues but is undiagonalizable;
- Type 3: A has two complex eigenvalues.

The types of the equilibrium of Type 1-3 are stable nodes, stable degenerate nodes, and stable foci, respectively.

2.4.1 Assumptions

In this subsection, we make some assumptions that are useful for the main results of this chapter. First of all, we recall the two assumptions made in Section 2.2 for the switched system (2.40) and discuss the stability of the special cases when the two assumptions are violated.

Assumption 2.1. $A_i \neq cB_j$, where $c \geq 0$.

When Assumption 2.1 is violated, it is trivial to show that A_i is just scaled B_j, then the switched system is stable under arbitrary switching.

Assumption 2.2. A_i and B_j do not share any real eigenvector.

When Assumption 2.2 is violated, A_i and B_j are simultaneously similar to upper triangular matrices that share a common quadratic Lyapunov function (CQLF) as shown in [62]. In this case, the switched system is stable under arbitrary switching.

In order to reduce the degrees of freedom, we need to employ certain standard forms and standard transformation matrices, as defined below, for different types of second-order matrices.

2.4.1.1 Standard Forms

Without loss of generality, the standard forms (real Jordan forms) for different types of second-order matrices are defined as follows

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \; J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \; J_3 = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}. \tag{2.41}$$

Since the subsystems in (2.40) are Hurwitz, we have

$$\lambda_2 \leq \lambda_1 < 0, \quad \lambda < 0, \quad \mu < 0, \omega > 0. \tag{2.42}$$

Assumption 2.3. *One subsystem of the switched system (2.40) is in its standard form as defined in (2.41),* i.e., $A_i = J_i$.

Note that it is always possible to guarantee one subsystem in its standard form by linear transformation under which the stability of the switched system is preserved.

2.4.1.2 Standard Transformation Matrices

Since one subsystem is in its standard form, the other subsystem can be expressed as $B_j = P_j J_j P_j^{-1}$ with $i \leq j$, where J_j is the standard form of B_j, and P_j is the transformation matrix defined in (2.43).

The standard transformation matrices are defined for different types of B_j as follows.

$$P_1 = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix} \tag{2.43}$$

For any given B_j with its standard form J_j, P_j can be derived from the eigenvectors of B_j.

1. α and β in P_1 can be obtained by calculating the real eigenvectors of B_1. Make sure that the eigenvector $[1, \alpha]^T$ corresponds to λ_1.

2. α in P_2 can be derived by calculating the eigenvector of B_2. And then β can be uniquely determined by the equation $B_2 = P_2 J_2 P_2^{-1}$.

3. α and β in P_3 can be derived from the eigenvector of B_3. If the eigenvector corresponding to the eigenvalue $\mu + j\omega$, is $v = \begin{bmatrix} p_{11} + p_{12}i \\ p_{21} + p_{22}i \end{bmatrix}$, then

$P_3 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$. It is always possible to ensure $p_{11} = 0$ and $p_{12} = 1$ by multiplying v with a factor of $(p_{11} - p_{12}i)i/(p_{11}^2 + p_{12}^2)$.

2.4.1.3 Assumptions on Various Combinations of S_{ij}

In order to further reduce the degrees of freedom such that the final result can be presented in a compact form, certain assumptions have to be made concerning the various parameters in the standard transformation matrices P_j and in the important equation (2.37) $N(k) = p_2k^2 + p_1k + p_0$, where $p_2 = a_{12}b_{22} - a_{22}b_{12}$, $p_1 = a_{12}b_{21} + a_{11}b_{22} - a_{21}b_{12} - a_{22}b_{11}$, and $p_0 = a_{11}b_{21} - a_{21}b_{11}$. These assumptions are listed below.

Assumption 2.4.

1. *if $S_{ij} = S_{11}$, $\beta < 0$;*
2. *if $S_{ij} = S_{12}$, $\alpha < 0$;*
3. *if $S_{ij} = S_{13}$, $k_2 < 0$, where k_2 is the smaller root of $N(k)$;*
4. *if $S_{ij} = S_{33}$, $p_2 \neq 0$, where p_2 is the leading coefficient of $N(k)$;*
5. *if $S_{ij} = S_{33}$, $p_2 < 0$ (if $N(k)$ (2.37) has two distinct real roots).*

Please note that these assumptions do not impose any constraint on the subsystems A_i and B_j as shown by the following lemma.

Lemma 2.4. *Any given switched linear system (2.40) subject to Assumptions 2.1 and 2.2 can be transformed to satisfy Assumption 2.4 by similarity transformations.*

The proof of Lemma 2.4 is given in Appendix A.3.

2.4.2 A Necessary and Sufficient Stability Condition

The principal result of the chapter is the following theorem.

Theorem 2.1. *The switched system (2.40), subject to Assumptions 2.1-2.4, is not stable under arbitrary switching if and only if there exist two independent real-valued vectors w_1, w_2, satisfying the collinear condition*

$$\det([A_iw_1, B_jw_1]) = 0, \ \det([A_iw_2, B_jw_2]) = 0,$$

and the slopes of w_1 *and* w_2*, denoted as* k_1 *and* k_2 *with* $k_2 < k_1$*, satisfy the following inequality:*

$$\begin{cases} L < k_2 < k_1 < M & if\,\det(P_j) < 0 \\ \|\exp(B_j T_B)\exp(A_i T_A) w_2\|_2 > \|w_2\|_2 & if\,\det(P_j) > 0 \end{cases}, \tag{2.44}$$

where M *and* L *correspond to the slopes of the asymptotes (Definition 2.1) of* A_i *and* B_j *respectively*[1]*, such that*

$$M = \begin{cases} 0, & i = 1 \\ +\infty, & i = 2 \\ +\infty, & i = 3 \end{cases}, \qquad L = \begin{cases} \alpha, & j = 1 \\ \alpha, & j = 2 \\ -\infty & j = 3 \end{cases}, \tag{2.45}$$

and

$$T_A = \int_{\theta_2}^{\theta_1} \frac{1}{Q_A(\theta)} d\theta = \int_{\theta_2}^{\theta_1} \frac{1}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22} - a_{11})\sin\theta\cos\theta} d\theta, \tag{2.46}$$

$$T_B = \int_{\theta_2}^{\theta_1} \frac{1}{Q_B(\theta)} d\theta = \int_{\theta_1}^{\theta_2+\pi} \frac{1}{b_{21}\cos^2\theta - b_{12}\sin^2\theta + (b_{22} - b_{11})\sin\theta\cos\theta} d\theta, \tag{2.47}$$

where $\theta_1 = \tan^{-1} k_1$*,* $\theta_2 = \tan^{-1} k_2$*.*

It can be readily seen from above theorem that there are two classes of switched systems (2.40), which are categorized by the sign of $\det(P_j)$ that in some sense indicates the relative trajectory direction of two subsystems, *i.e.*, a negative $\det(P_j)$ implies opposite trajectory direction in certain region. Each class corresponds to a possible instability mechanism as follows.

Class I ($\det(P_j) < 0$): *Unstable chattering* (sliding or sliding-like motion), *i.e.*, when system trajectories can be driven into a conic region where both $H_A(k)$ and $H_B(k)$ are positive. There exists a switching sequence that switches back and forth inside this region to make the system trajectories unstable.

Class II ($\det(P_j) > 0$): *Unstable spiralling*, *i.e.*, when the system trajectory is a spiral around the origin and there exists a switching action to make it unstable.

The above classification is similar to the one in [49] which deals with the stabilization problem.

[1] There is no asymptote for A_3 and B_3. In this case, L is chosen as $-\infty$, and M is chosen as $+\infty$ based on the directions of the subsystems.

Remark 2.5. Theorem 2.1 shows that the existence of two independent vectors w_1, w_2, along which the trajectories of the two subsystems are collinear, is a necessary condition for the switched system (2.40) to be unstable.

Remark 2.6. One of the improvements of Theorem 2.1 compared with the condition proposed in [31] is that the case of $\det(P_j) < 0$ is included. The basic idea in [31] is to find the maximum value of the feedback gain k^* that corresponds to a closed trajectory in phase plane under the WCSS. If $k > k^*$, the worse case trajectory is an unstable spiral; if $k < k^*$, the worse case trajectory is an asymptotically stable spiral. However, it is not clear how this idea can be applied to the case when $\det(P_j) < 0$, where the stability of switched systems depends on the existence of an unstable region rather than the existence of a closed trajectory under the WCSS.

2.4.2.1 Proof of Theorem 2.1 when S_{ij}=S_{11}

Theorem 2.1 is proved in the following fashion. For every possible combination of the subsystems S_{ij}, it will be shown that if the condition (2.44) is satisfied, then there exist switching signals such that the switched system (2.40) is unstable, which constitutes the proof for the sufficiency. It will also be demonstrated that for all the cases when this condition is violated, the switched system is always stable regardless of switching signals, which would establish the necessity.

We prove Theorem 2.1 for the case $S_{ij} = S_{11}$ in the following as an example to show the main idea and process of the proof of Theorem 2.1. The proofs of other cases of S_{ij} are provided in Appendix A.4.

Proof: In the case of $S_{ij} = S_{11}$,

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix}, B_1 = P_1 J_1 P_1^{-1} = \frac{1}{\beta - \alpha} \begin{bmatrix} \beta\lambda_{1b} - \alpha\lambda_{2b} & \lambda_{2b} - \lambda_{1b} \\ \alpha\beta(\lambda_{1b} - \lambda_{2b}) & \beta\lambda_{2b} - \alpha\lambda_{1b} \end{bmatrix}. \tag{2.48}$$

Denote $\lambda_{1a} = k_A\lambda_{2a}$, $\lambda_{1b} = k_B\lambda_{2b}$, we have $0 < k_A, k_B < 1$[1], $\alpha \neq 0$ by Assumption 2.2 and $\beta < 0$ by Assumption 2.4.1. Substituting (2.48) into (2.31)-

[1]If k_A=1, then any vector in the phase plane is the eigenvector of A, which contradicts Assumption 2.2. This is because B have two real eigenvectors.

(2.37), it follows that

$$N(k) = \frac{\lambda_{2a}\lambda_{2b}(k_A - 1)}{\beta - \alpha}\bar{N}(k), \tag{2.49}$$

where

$$\bar{N}(k) = k^2 + \frac{(k_A - k_B)\beta + (1 - k_A k_B)\alpha}{k_B - 1}k + \alpha\beta k_A, \tag{2.50}$$

is a monic polynomial with the same roots as $N(k)$ and

$$H_A(k) = K_B(k)\lambda_{2b}\frac{-\bar{N}(k)}{(\alpha - \beta)k}, \tag{2.51}$$

$$H_B(k) = K_A(k)\frac{\lambda_{2a}(1 - k_A)\bar{N}(k)}{(1 - k_B)(k - \alpha)(k - \beta)}, \tag{2.52}$$

$$Q_A(k) = -\frac{1}{1 + k^2}\lambda_{2a}(k_A - 1)k, \tag{2.53}$$

$$Q_B(k) = \frac{\lambda_{2b}(1 - k_B)}{1 + k^2}\frac{(k - \alpha)(k - \beta)}{\alpha - \beta}. \tag{2.54}$$

It can be readily shown that

$$\operatorname{sgn}(H_A(k)) = \operatorname{sgn}(\alpha - \beta)\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k), \tag{2.55}$$

$$\operatorname{sgn}(H_B(k)) = -\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k - \alpha)\operatorname{sgn}(k - \beta), \tag{2.56}$$

$$\operatorname{sgn}(Q_A(k)) = -\operatorname{sgn}(k), \tag{2.57}$$

$$\operatorname{sgn}(Q_B(k)) = -\operatorname{sgn}(\alpha - \beta)\operatorname{sgn}(k - \alpha)\operatorname{sgn}(k - \beta). \tag{2.58}$$

In order to determine the signs of the equations (2.55)-(2.58) in every region of k, we need the relative position of the boundaries including (i) two eigenvectors of A_1 which are $k = 0$ and $k = \infty$ in S_{11}; (ii) two eigenvectors of B_1 which are $k = \alpha$ and $k = \beta$; and (iii) the two distinct real roots of $N(k)$, which are defined as k_1 and k_2. We analyze all possible sequences of these boundaries with respect to the following three exclusive and exhaustive cases.

Case 1. $\bar{N}(k)$ does not have two distinct real roots.

There are three possibilities: 1) two complex roots; 2) two identical real roots; 3) one root, which are discussed as follows.

1.1) $N(k)$ has two complex roots. Since the complex roots of $N(k)$, denoted as c_1 and c_2, are conjugate, the equation (2.59) below should be positive for any α.

$$(\alpha - c_1)(\alpha - c_2) = \frac{(1 - k_A)k_B\alpha(\alpha - \beta)}{k_B - 1}. \tag{2.59}$$

As a result, the only possible sequence of these boundaries is $\beta < \alpha < 0$. Then the signs of (2.55)-(2.58) could be determined for every region of k, as shown in Fig. 2.5.

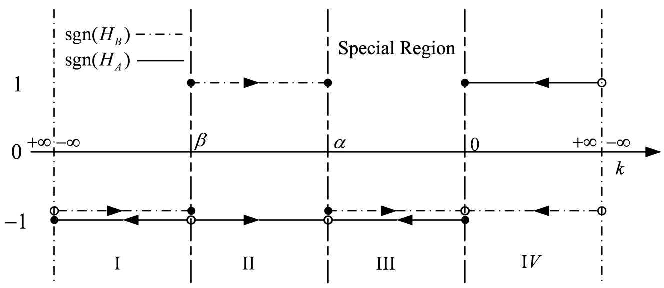


Figure 2.5: S_{11}: $N(k)$ does not have two distinct real roots, the switched system is stable.

Fig. 2.5 is the crucial diagram exhibiting the conditions for the stability of switched systems (2.40). It shows the signs of $H_A(k)$, $H_B(k)$, $Q_A(k)$ and $Q_B(k)$ versus $k \in (-\infty, +\infty)$, corresponding to $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$. The dashed vertical lines are the boundaries of the regions of k. The horizontal lines represent the signs of $H_A(k)$ (the solid) and $H_B(k)$ (the dashed) while the arrows represent the signs of $Q_A(k)$ and $Q_B(k)$ in different regions. If $H_A(k)$ is positive, then the solid line is above the horizontal axis. If $Q_A(k)$ is positive, the arrow on the dashed line points to the right (counter clockwise in $x - y$ plane).

With reference to Fig. 2.5, Regions I and III are stable since both $H_A(k)$ and $H_B(k)$ are negative in these regions. Furthermore, Region III is a special region, where none of the trajectories can go out. Consider all possible initial states in different regions as follows.

1. If the initial state is in Region III, it can not go out of this region.

2. If the initial state is in Region II or IV, it will be brought into Region III by the WCSS, which is Σ_A (H_B is positive and H_A is negative) in Region II and Σ_B in Region IV respectively.

3. If the initial state is in Region I, it must be brought out because Region I is stable. Then the trajectory will go to Region II or Region IV, and go to Region III eventually.

Therefore, when $N(k)$ has two complex roots, the switched system is stable under arbitrary switching.

1.2) $N(k)$ has two identical real roots. Unlike properties of the boundary k_1 or k_2 stated in Remark 2.4, all the signs of (2.31)-(2.34) do not change when system trajectories cross k_m, the identical real root of $N(k)$, because the sign of $N(k)$ does not change. The two regions next to k_m can be merged into one since the signs of $H_A(\theta)$, $H_B(\theta)$, $Q_A(\theta)$, and $Q_B(\theta)$ (2.31)-(2.34) keep the same, and trajectories of both subsystems can cross k_m with the same directions as those in these two regions. It follows that the worst case analysis for this case is similar to the one for Fig. 2.5 regardless of the position of k_m. Since this is true for all S_{ij}, the analysis for the case that $N(k)$ has two identical real roots will be omitted in all other cases.

1.3) $N(k)$ has only one root. In this case, the leading coefficient of $N(k)$, $p_2 = a_{12}b_{22} - a_{22}b_{12} = 0$ based upon (2.37). With reference to (2.48), we have $a_{12} = 0$ and $a_{22} \neq 0$. So $p_2 = 0$ results in $b_{12} = 0$, which implies that B_1 shares a real eigenvector (the y axis) with A_1, which violates Assumption 2.2. Therefore, this case can not happen for S_{11}. It can be readily shown that this is true for all other cases of S_{1j} and S_{2j}. In S_{33}, $p_2 = 0$ was excluded by Assumption 2.4.4. Hence, we will omit the case that $N(k)$ has only one root in the rest of the proof of Theorem 2.1.

Case 2. $\bar{N}(k)$ has two distinct real roots and $\det(P_1) < 0$.

$$\det(P_1) = \beta - \alpha < 0. \tag{2.60}$$

So we have $\alpha > \beta$ in this case. Since $\beta < 0$ is guaranteed by Assumption 2.4.1, there are two possibilities: $\beta < \alpha < 0$ and $\beta < 0 < \alpha$. Then we need to insert k_1 and k_2 into the two possible sequences. Equation (2.61) is useful to determine the relative position between k_1, k_2 and α:

$$(\alpha - k_1)(\alpha - k_2) = \frac{(1 - k_A)k_B\alpha(\alpha - \beta)}{k_B - 1}. \tag{2.61}$$

With reference to equations (2.50) and (2.61), there are only four possible sequences for all the boundaries in this case:

2.1) $\beta < \alpha < k_2 < k_1 < 0$. With reference to Fig. 2.6, both $H_A(k)$ and $H_B(k)$ are positive when $k \in (k_2, k_1)$, the switched system is not stable under arbitrary switching from Lemma 2.3.

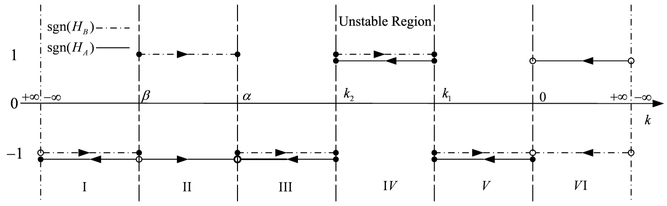


Figure 2.6: S_{11}: $\det(P_1) < 0$, $\beta < \alpha < k_2 < k_1 < 0$, the switched system is not stable for arbitrary switching.

2.2) $\beta < k_2 < k_1 < \alpha < 0$. With reference to Fig. 2.7, the switched system is stable by the similar argument as that for Fig 2.5.

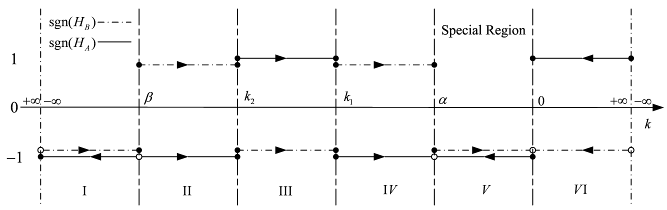


Figure 2.7: S_{11}: $\det(P_1) < 0$, $\beta < k_2 < k_1 < \alpha < 0$, the switched system is stable.

2.3) $\beta < \alpha < 0 < k_2 < k_1$. With reference to Fig. 2.8, the switched system is stable by the similar argument as that for Fig. 2.5.

2.4) $\beta < k_2 < 0 < \alpha < k_1$. With reference to Fig. 2.9, the switched system is stable by the similar argument as that for Fig. 2.5.

In summary, it can be concluded that $\alpha < k_2 < k_1 < 0$ is necessary and sufficient for instability in this case.

Case 3. $\bar{N}(k)$ has two distinct real roots and $\det(P_1) > 0$.

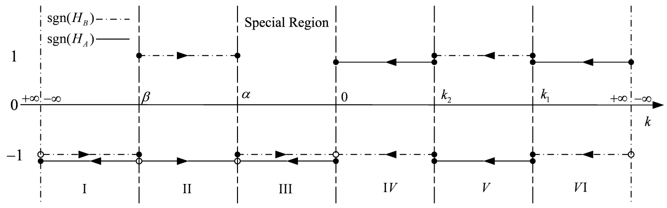


Figure 2.8: S_{11}: $\det(P_1) < 0$, $\beta < \alpha < 0 < k_2 < k_1$, the switched system is stable.

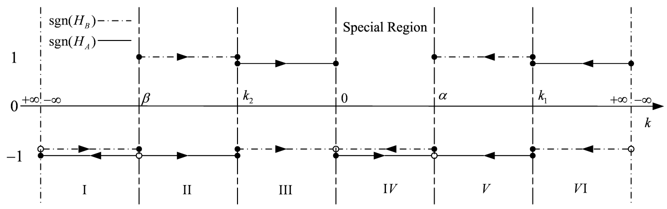


Figure 2.9: S_{11}: $\det(P_1) < 0$, $\beta < k_2 < 0 < \alpha < k_1$, the switched system is stable.

In this case, $\alpha < \beta$ from (2.60). It follows from the equations (2.50) and (2.61) that the only possible sequence of the boundaries is : $k_2 < \alpha < \beta < k_1 < 0$.

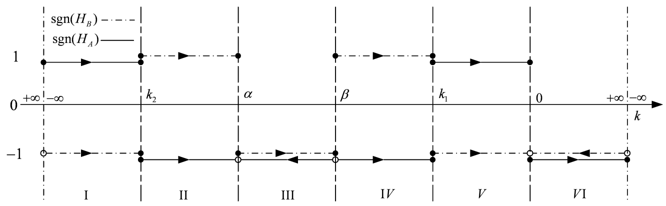


Figure 2.10: S_{11}: $\det(P_1) > 0$, the worst case trajectory rotates around the origin counter clockwise.

With reference to Fig. 2.10, it is straightforward that the WCSS is Σ_B in Regions I and V where H_A is positive and H_B are negative. Similarly, the WCSS is Σ_A in Region II and IV where H_A is positive and H_B is negative. In Region III, both H_A and H_B is negative, but Σ_A is the only subsystem whose trajectory can go out of Region III because the boundaries of Region III, α and β, are the two real eigenvectors of Σ_B. Similarly, the WCSS is Σ_B in Region VI. On k_1 and k_2, the trajectory directions of the two subsystems are the same. Without loss of generality, we choose Σ_B as the WCSS. Based on above analysis, it is concluded that the WCSS σ^* in the whole interval of k is

$$\sigma^* = \begin{cases} A & k_2 < k < k_1, \\ B & \text{otherwise.} \end{cases} \tag{2.62}$$

In this case, the trajectory under the WCSS rotates around the origin counter clockwise. The simplest way to determine stability of the system is to follow a trajectory under the WCSS originating from a line l until it returns to l again, and evaluate its expansion or contraction in the radial direction. Without loss of generality, let $w_2 = [1, k_2]^T$, the switched system is not stable under the WCSS σ^* if and only if $\|\exp(B_1 T_B)\exp(A_1 T_A) w_2\|_2 > \|w_2\|_2$, where T_A and T_B are the time duration on Σ_A and Σ_B, respectively, which could be calculated by

$$T_A = \int_{\theta_2}^{\theta_1} \left.\frac{dt}{d\theta}\right|_{\sigma=A} d\theta = \int_{\theta_2}^{\theta_1} \frac{1}{Q_A(\theta)} d\theta, \tag{2.63}$$

$$T_B = \int_{\theta_1}^{\theta_2+\pi} \left.\frac{dt}{d\theta}\right|_{\sigma=B} d\theta = \int_{\theta_1}^{\theta_2+\pi} \frac{1}{Q_B(\theta)} d\theta, \tag{2.64}$$

where $\theta_1 = \tan^{-1} k_1$ and $\theta_2 = \tan^{-1} k_2$. It corresponds to the second inequality of Theorem 2.1. Hence, the theorem is proved.

2.4.2.2 Application of Theorem 2.1

The condition in Theorem 2.1 can be easily verified by the following procedure:
1. Calculate the eigenvalues and the eigenvectors of two subsystems, and check the following

a) If one of the subsystems is unstable, the switched system (2.40) is not stable under arbitrary switching.

b) If either Assumption 2.1 or 2.2 is violated, the switched system (2.40) is stable under arbitrary switching.

2. Determine S_{ij} with $i \leq j$, where subscript i and j denote the type of A_i and B_j respectively.
3. Check whether A_i is in its standard form J_i. Do similarity transformation for the two subsystems simultaneously to guarantee $A_i = J_i$ if necessary.
4. Calculate P_j, k_1, k_2, and check Assumption 2.4.

a) If Assumption 2.4 is satisfied, go to step 5.

b) Otherwise, do similarity transformation, as stated previously, for two subsystems simultaneously such that Assumption 2.4 is satisfied. Recalculate P_j, k_1 and k_2.

5. If the real roots $k_1 \neq k_2$, go to the next step, otherwise the switched system is stable under arbitrary switching.
6. Calculate $\det(P_j)$.

a) If $\det(P_j) < 0$, determine the values of L and M with reference to (2.45), and check the first inequality of Theorem 2.1.

b) If $\det(P_j) > 0$, calculate the values of T_A and T_B using equations (2.46) and (2.47), which can be easily integrated by changing variable $k = \tan\theta$, and check the second inequality of Theorem 2.1.

We now apply Theorem 2.1 to some examples below.

Example 2.1

Consider a switched linear system with two LTI planar systems

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -10 \\ 1/10 & -1 \end{bmatrix}. \tag{2.65}$$

It has been shown in [22] that the switched system (2.65) does not have a common quadratic Lyapunov function, but is exponentially stable under arbitrary switching. Now we check it using the procedure described in Section 2.4.2.2, based on Theorem 2.1.

1. Both A and B are Hurwitz with a pair of complex eigenvalues: $-1 \pm i$. And Assumptions 2.1 and 2.2 are satisfied. It is the case S_{33} since both subsystems have complex eigenvalues.

2. A_3 is already in its standard form J_3.

3. The eigenvector of B_3 corresponding to the eigenvalue $1+i$ is $[1, 1/10i]^T$, denoted by v_1. Then $P_3 = \begin{bmatrix} 0 & 1 \\ -1/10 & 0 \end{bmatrix}$ is derived from $v_1 * i = \begin{bmatrix} 0+1i \\ -1/10+0i \end{bmatrix}$.

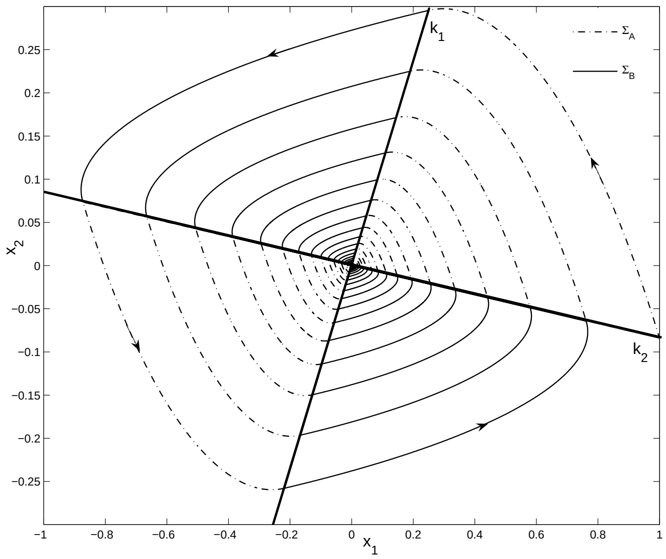


Figure 2.11: The trajectory of the switched system (2.65) under the WCSS.

Substituting the entries of (2.65) into $N(k)$ (2.37), we have $k_1 = 1.184, k_2 = -0.084$. Assumptions 2.4.4-2.4.5 for S_{33} are satisfied due to $p_2 = a_{12}b_{22} - b_{12}a_{22} = -9 < 0$. Hence no further transformation is needed.

4. It follows from $\det(P_3) = 1/10 > 0$ that the second inequality of Theorem 2.1 should be checked. Substituting the entries of (2.65) to equations (2.46) and (2.47), it results in $T_A = T_B = 0.9539$. It follows that

$$\|\exp(BT_B)\exp(AT_A)w_2\|_2 = 0.8758 < \|w_2\|_2 = 1.0036.$$

By Theorem 2.1, the switched system (2.65), with its two subsystems not sharing a CQLF, is stable under arbitrary switching, matching the conclusion of [22]. The trajectory under the WCSS is shown in Fig. 2.11.

Example 2.2

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & -5 \\ 20 & -11 \end{bmatrix}. \tag{2.66}$$

1. Simply checking yields that A has two distinct real eigenvalues: $\lambda_{1a} = -1$ and $\lambda_{2a} = -3$ with corresponding eigenvectors: $[1, 0]^T$ and $[0, 1]^T$, respectively. B has two multiple eigenvalues $\lambda_b = -1$ with a single eigenvector

$[1,2]^T$, which is undiagonalizable. It is the case S_{12} with Hurwitz A_1 and B_2. And it follows that Assumptions 2.1 and 2.2 are satisfied.

2. A_1 is already in its standard form J_1.

3. $P_2 = \begin{bmatrix} 0 & 1 \\ -0.2 & 2 \end{bmatrix}$ is derived from $B_2 = P_2 J_2 P_2^{-1}$. It follows that $\alpha = 2$, which violates Assumption 2.4.2. Therefore, we need to transform A_1 and B_2 simultaneously. By denoting $\bar{x}_1 = -x_1$, we obtain a new switched system

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 9 & 5 \\ -20 & -11 \end{bmatrix}, \tag{2.67}$$

which has the same stability property as the switched system (2.66). Recalculate $\bar{P}_2 = \begin{bmatrix} 0 & 1 \\ 0.2 & -2 \end{bmatrix}$, where $\alpha = -2$ satisfies Assumption 2.4.2. We have $k_1 = -0.7460, k_2 = -1.7873$.

4. The first inequality of Theorem 2.1 should be checked because $\det(\bar{P}_2) = -0.2 < 0$. With reference to (2.45), we have $L = \alpha = -2$ and $M = 0$ for S_{12}, hence the inequality $L < k_2 < k_1 < M$ is satisfied.

It can be concluded that the switched system (2.67), or equivalently the switched system (2.66), is not stable under arbitrary switching. An unstable trajectory of the switched system (2.67) is shown in Fig. 2.12. It is to be noted that the stability condition of [31] can not be applied to this example since its worst case trajectory is chattering rather than spiralling.

2.5 Extension to the Marginally Stable Case

The stability criterion can be extended to the switched system that consists of marginally stable subsystems:

$$S_{ij} : \dot{x} = \sigma x, \quad \sigma \in \{A_i, B_j\}, \tag{2.68}$$

where $A_i, B_j \in \mathbb{R}^{2\times 2}$ are either Hurwitz or marginally stable. The corresponding stability condition for (2.68) is formulated as Theorem 2.2 below.

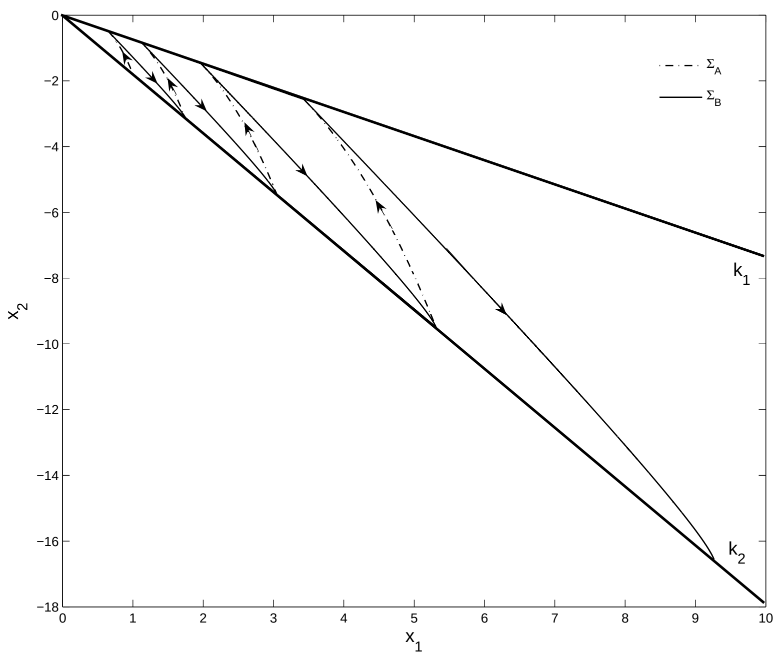


Figure 2.12: A typical unstable trajectory of the switched system (2.67).

Theorem 2.2. *The switched system (2.68), subject to Assumptions 2.1-2.4, is not stable under arbitrary switching if and only if there exist two independent real-valued vectors* w_1, w_2*, satisfying the collinear condition*

$$\det([A_i w_1, B_j w_1]) = 0,\ \det([A_i w_2, B_j w_2]) = 0,$$

and the slopes of w_1 *and* w_2*, denoted as* k_1 *and* k_2 *with* $k_2 < k_1$*, satisfy the following inequality:*

$$\begin{cases} L \le k_2 < k_1 \le M & if\,\det(P_j) < 0 \\ \left\|\exp(B_j T_B)\exp(A_i T_A) w_2\right\|_2 > \left\|w_2\right\|_2 & if\,\det(P_j) > 0 \end{cases}, \tag{2.69}$$

where M*,* L*,* T_A*,* T_B*, and* w *are the same as those defined in Theorem 2.1.*

Remark 2.7. *The only issue caused by the marginally stable subsystem is that the collinear vectors may overlap with an eigenvector of the subsystem. As a result, it takes infinite time for the worst case switching signal* $\sigma^*(\theta(t))$*, which is state-dependent, to bring the trajectory to its eigenvector. However, Theorem 2.2 is still valid by introducing a less worse switching signal* $\sigma(\theta(t))$*, under which the trajectory is close to the worst case trajectory, but associated with a*

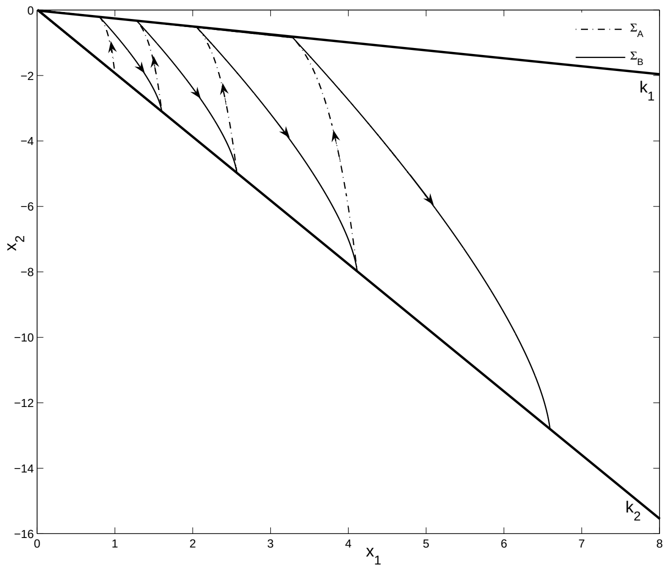


Figure 2.13: A typical unstable trajectory of the switched system (2.70).

finite time. Similar comment applies to the stabilizability condition for marginal unstable cases in Chapter 3.

The proof of Theorem 2.2 is similar to that of Theorem 2.1, and hence is omitted here.

Example 2.3

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}. \tag{2.70}$$

Simple checking yields that A has two distinct eigenvalues $\lambda_1 = -1$, $\lambda_2 = -10$ and B has two complex eigenvalues $\pm i$. So it is the case S_{13}, and Σ_B is marginally stable. Following the steps similar to those of Example 2.2, we find that $L = -\infty < k_2 = -1.9426 < k_1 = -0.2574 < M = 0$. Therefore, the switched system (2.70) is not stable under arbitrary switching. See Fig. 2.13 for its unstable trajectory.

2.6 The connection between Theorem 2.1 and CQLF

In this section, we discuss the relationship between Theorem 2.1 and the existence of a common quadratic Lyapunov function (CQLF). We prove the following result:

Theorem 2.3. *If there do not exist two independent real-valued vectors* w_1, w_2, *satisfying the collinear condition* $\det([Aw_1, Bw_1]) = 0, \det([Aw_2, Bw_2]) = 0$, *or equivalently* $Q = A^{-1}B$ *has two complex eigenvalues, then* A *and* B *share a CQLF.*

Proof: Without loss of generality, it is assumed that the matrix Q with two complex eigenvalues $\mu \pm i\omega$ is in its standard form

$$Q = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}.$$

Denote

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

We have

$$B = AQ = \begin{bmatrix} \mu a_{11} + \omega a_{12} & -\omega a_{11} + \mu a_{12} \\ \mu a_{21} + \omega a_{22} & -\omega a_{21} + \mu a_{22} \end{bmatrix}.$$

The characteristic polynomial of A is given by

$$\det(\lambda I - A) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}.$$

Since A is Hurwitz, we have the conditions

$$a_{11} + a_{22} < 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0. \tag{2.71}$$

Similarly,

$$\det(\lambda I - B) = \lambda^2 - [\mu(a_{11} + a_{22}) + \omega(a_{12} - a_{21})]\lambda + (\mu^2 + \omega^2)(a_{11}a_{22} - a_{12}a_{21}).$$

Since B is also Hurwitz, we obtain another condition

$$\mu(a_{11} + a_{22}) + \omega(a_{12} - a_{21}) < 0. \tag{2.72}$$

We use the matrix pencil condition proposed in [17] to show that A and B share a CQLF.

Let $M_\gamma(A,B) = B + \gamma A, N_\gamma(A,B) = B + \gamma A^{-1}, \gamma > 0$, from which we get

$$\det(\lambda I - M_\gamma) = m_2\lambda^2 + m_1\lambda + m_0,$$

where $m_2 = 1$, $m_1 = -[\mu(a_{11}+a_{22}) + \omega(a_{12}-a_{21}) + \gamma(a_{11}-a_{22})]$, $m_2 = [(\mu+\gamma)^2 + \omega^2](a_{11}a_{22} - a_{12}a_{21})$, and

$$\det(\lambda I - N_\gamma) = n_2\lambda^2 + n_1\lambda + n_0,$$

where $n_2 = 1$, $n_1 = \gamma(a_{11}+a_{22})[\mu(a_{11}+a_{22}) + \omega(a_{12}-a_{21})] + \gamma^2(a_{11}a_{22} - a_{12}a_{21})$, $n_2 = \gamma(a_{11}+a_{22})[\mu(a_{11}+a_{22}) + \omega(a_{12}-a_{21})] + [\omega^2 + (\mu-\gamma)^2](a_{11}a_{22} - a_{12}a_{21})$.

It follows from Conditions (2.71) and (2.72) that $m_1 > 0, m_0 > 0, n_1 > 0, n_0 > 0$, which implies that both $M_\gamma(A,B)$ and $N_\gamma(A,B)$ are Hurwitz for all $\gamma > 0$. Therefore, A and B have a CQLF when $Q = A^{-1}B$ has a pair of complex eigenvalues.

It is concluded from the detailed analysis on each individual case that when the first inequality of Theorem 2.1 is satisfied, the trajectory directions on the collinear vectors are always opposite, which implies that $Q = A^{-1}B$ has two negative eigenvalues. We have known from Theorem 2.1 that the switched system is not stable under arbitrary switching in this case. Here we show that there is no CQLF for A and B is this case.

The case of $Q = A^{-1}B$ having two negative eigenvalues is equivalent to the existence of two real eigenvector x_1 and x_2 with $Ax_1 = \lambda_1 Bx_1$ and $Ax_2 = \lambda_2 Bx_2$ with $\lambda_1 < 0, \lambda_2 < 0$.

For any symmetric positive definite P satisfying

$$x^T(A^TP + PA)x < 0, \forall x \in \mathbb{R}^2/\{0\},$$

we have $x_1^T(A^TP + PA)x_1 < 0$. It follows that $[x_1^T(B^TP + PB)x_1]\lambda_1 < 0$ or $[x_1^T(B^TP + PB)x_1] > 0$. Therefore, none of the quadratic Lyapunov functions of A is valid for B, thereby implying that A and B do not share a CQLF.

2.7 Summary

In this chapter, a necessary and sufficient condition (Theorem 2.1) for the stability of a pair of planar LTI system (2.40) has been derived. The condition is easily verified, even by hand calculation. In contrast with the stability conditions proposed in the literature [31], Theorem 2.1 takes into account all possible combinations of the subsystem dynamics. Moreover, it has been shown (Theorem 2.2) that the result can be generalized to the switched system (2.68) which is made up of marginally stable subsystems. Furthermore, we discussed the relationship between Theorem 2.1 and the existence of a CQLF.

Chapter 3

Switching Stabilizability

In this chapter, we deal with the problem of stabilizability of second-order switched systems with unstable subsystems. In contrast with the earlier use of worst case analysis, we now invoke the idea of best case analysis in order to discover whether the system can be stabilized by switching between the unstable subsystems, and also, at the time, determine the switching sequence for stabilization.

This chapter is organized as follows. Section 3.1 formulates the switching stabilizability problem and defines global asymptotic stabilizability and regional asymptotic stabilizability. Section 3.2 identifies the best-case switching signal (BCSS) to obtain easily verifiable, necessary and sufficient conditions for regional asymptotic stabilizability of switched systems. Section 3.3 presents these conditions for the case of two unstable second-order LTI subsystems. Section 3.4 discusses the connections among the stabilizability conditions in this chapter, the stability condition in Chapter 2, and related results in the literature. Section 3.5 summaries this chapter.

3.1 Problem Formulation

We consider the following switched system with a pair of second-order continuous-time LTI subsystems

$$S_{ij} : \dot{x} = \sigma x, \quad \sigma = \{A_i, B_j\}, \tag{3.1}$$

where A_i and $B_j \in \mathbb{R}^{2\times 2}$ are not asymptotically stable, and $i, j \in \{1, 2, 3\}$ denote the types of A and B, respectively.

For clarity, we define two types of asymptotic stabilizability employed in the derivation of our results.

Definition 3.1. *The switched system (3.1) is said to be* globally asymptotically stabilizable *(GAS), if for any non-zero initial state, there exists a switching signal under which the trajectory will asymptotically converge to zero.*

Definition 3.2. *The switched system (3.1) is said to be* regionally asymptotically stabilizable *(RAS), if there exists at least one region (non-empty, open set) such that for any initial state in that region, there exists a switching signal under which the trajectory will asymptotically converge to zero.*

In addition to global asymptotic stabilizability (GAS), which is the focus of the most of the research in the literature, regional asymptotic stabilizability will also be considered in this thesis. It is due to the fact that there exists a class of switched systems which are not GAS, but still can be stabilized if the initial state is within certain regions. Those switched systems can be stabilized in stark contrast with those that cannot be stabilized irrespective of the initial state. In practice, however, it is likely that the initial state lies inside the stabilizable region.

Note that A_i or B_j in (3.1) can be unstable node, saddle point or even marginally stable subsystem. It is because the existence of a marginally stable A_i or B_j does not guarantee the regional asymptotic stabilizability (RAS) of the switched system (3.1) with reference to Definitions 3.1 and 3.2.

The main technique for stabilizability analysis is based on the characterization of the best case switching signal (BCSS).

3.2 Best Case Analysis

We characterize the *best case switching signal* (BCSS) for a given switched system with a pair of unstable subsystems, thereby converting the switching stabilizability problem to the stability problem under the BCSS.

3.2.1 Mathematical Preliminaries

First of all, we list some equations, which are useful for the characterization of the BCSS, as follows.

$$H_A(\theta(t)) \triangleq \left.\frac{dh_A(\theta(t))}{dt}\right|_{\sigma(t)=B}, H_B(\theta(t)) \triangleq \left.\frac{dh_B(\theta(t))}{dt}\right|_{\sigma(t)=A}, \tag{3.2}$$

and

$$Q_A(\theta(t)) \triangleq \left.\frac{d\theta}{dt}\right|_{\sigma=A}, Q_B(\theta(t)) \triangleq \left.\frac{d\theta}{dt}\right|_{\sigma=B}. \tag{3.3}$$

Similar to the worse case analysis in Chapter 2, with $k = \tan\theta$, we have

$$H_A(k) = K_B(k)\frac{N(k)}{D_B(k)}, \tag{3.4}$$

$$H_B(k) = -K_A(k)\frac{N(k)}{D_A(k)}, \tag{3.5}$$

$$Q_A(k) = -\frac{1}{k^2+1}D_A(k), \tag{3.6}$$

$$Q_B(k) = -\frac{1}{k^2+1}D_B(k), \tag{3.7}$$

where

$$D_A(k) = a_{12}k^2 + (a_{11} - a_{22})k - a_{21}, \tag{3.8}$$

$$D_B(k) = b_{12}k^2 + (b_{11} - b_{22})k - b_{21}, \tag{3.9}$$

and

$$N(k) = p_2k^2 + p_1k + p_0, \tag{3.10}$$

where $p_2 = a_{12}b_{22} - a_{22}b_{12}$, $p_1 = a_{12}b_{21} + a_{11}b_{22} - a_{21}b_{12} - a_{22}b_{11}$, and $p_0 = a_{11}b_{21} - a_{21}b_{11}$.

Let the two distinct real roots of $N(k)$, if they exist, denoted by k_1 and k_2, and assume $k_2 < k_1$. The signs of equations (3.4)-(3.7) depend on the signs of $D_A(k)$, $D_B(k)$, and $N(k)$.

With reference to Definition 2.2, a region of k is a continuous interval where the signs of (3.4)-(3.7) are preserved for all k in this interval.

The boundaries of the regions of k, if they exist, are the lines whose angles satisfy $D_A(k) = 0$, $D_B(k) = 0$, or $N(k) = 0$. These boundaries divide the $x - y$

plane to several conic sectors, *i.e.*, regions of k. Now we proceed to establish criteria to determine the BCSS for every θ, or equivalently k, based on the signs of H_A and H_B.

3.2.2 Characterization of the Best Case Switching Signal (BCSS)

1) Both H_A and H_B are negative

Lemma 3.1. *The switched system (3.1) is regionally asymptotically stabilizable (RAS) if there is a region of k, $[k_l, k_u]$, where both $H_A(k)$ and $H_B(k)$ are negative.*

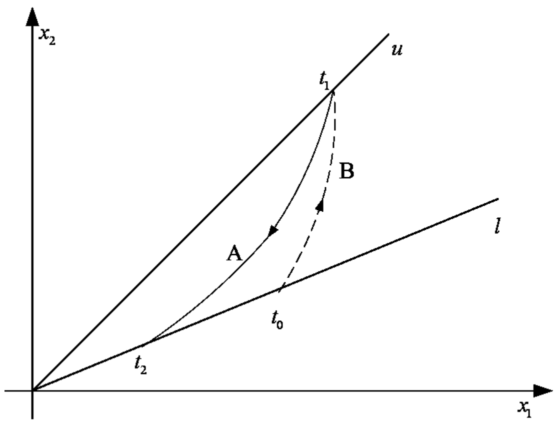


Figure 3.1: The region where both H_A and H_B are negative

With reference to Fig.3.1, a stable trajectory can be easily constructed by switching inside this region. The proof of Lemma 3.1 is similar to the one for Lemma 2.3, hence is omitted here.

2) H_A is positive and H_B is negative

The BCSS is Σ_A. In this case, the trajectories of two subsystems have the same direction based on Remark 2.3. With reference to Fig. 3.2, consider an initial state with an angle θ_0 at t_0. Let $r_B(\theta)$ be the trajectory along Σ_B and let $r_A(\theta)$ be the trajectory along Σ_A. Comparing the magnitudes of the trajectories along different subsystems, we have

$$r_B(\theta) - r_A(\theta) = h_A(\theta)g_A(\theta) - C_A g_A(\theta) = g_A(\theta)\int_{t_0}^{t} H_A(\theta(t))dt > 0, \qquad (3.11)$$

which shows that the trajectory of Σ_A always has a smaller magnitude than the corresponding one of Σ_B for all θ in this region.

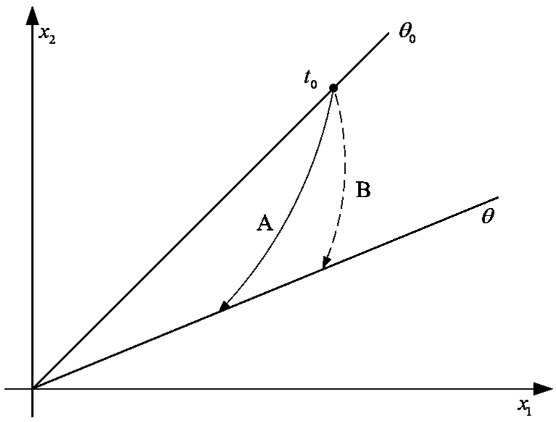


Figure 3.2: The region where H_A is negative and H_B are positive

3) H_A is negative and H_B is positive

Similarly, the BCSS is Σ_B in this case.

4) Both H_A and H_B are positive

First, we will show that the switched system can not be stabilized in this region if its trajectory does not move out. It follows from Assumption 2.2 that at least one of $g_A(\theta)$ and $g_B(\theta)$ is lower-bounded for any given θ. Since both H_A and H_B are positive, we have $h_A(t) \geq h_A(t_0)$ and $h_B(t) \geq h_B(t_0)$. With reference to (2.20) and (2.22), the magnitude r is lower-bounded in this region. Hence the asymptotic stabilizability of the switched system is determined by other regions.

Next we will discuss the scenarios when the trajectory may move out.

1) If only the trajectory of one subsystem, say Σ_A, can go out of this region, then the BCSS in this region is Σ_A. Let r_{σ^*} be the trajectory along Σ_A and let r_σ be the trajectory under any other switching signal. Comparing the magnitudes of the trajectories under different switching on the boundary ($\theta = \theta_{bn}$) where the

trajectories move out, it can be shown that any switching other than Σ_A in this region will make the switched system more unstable since

$$r_{\sigma^*}(\theta_{bn}) = h_A(t_0)g_A(\theta_{bn}) < r_\sigma = h_A(t)g_A(\theta_{bn}). \tag{3.12}$$

2) If the trajectories of both subsystems can go out and neither can come back, then no matter which subsystem is chosen, the trajectory will leave this region and the stabilizability of the switched system is determined by other regions.

3) If the trajectories of both subsystems can go out and at least one of them can come back, then at least one of the boundaries of this region is k_1 or k_2, the root of $N(k)$. It was mentioned in Remark 2.4 that $H_A(k)$ and $H_B(k)$ change their signs simultaneously when trajectories cross the boundary k_1 or k_2, then there must exist a stabilizable region, where both H_A and H_B are negative, next to this region. Therefore, the switched system (3.1) is RAS based on Lemma 3.1.

5) One of H_A and H_B is zero

If one of $H_A(k)$ and $H_B(k)$ is zero, it implies $N(k) = 0$, then both of them are zero at the line k.

1) If the trajectories of the subsystems cross the line with the same direction, we can choose either subsystem as the BCSS since the trajectories are tangent to each other on this line.

2) If the trajectories of the subsystems cross the line with opposite direction, it follows from Remark 2.4 that there exists a stabilizable region near the line where $N(k) = 0$. Hence the switched system is RAS from Lemma 3.1.

6) On real eigenvectors

It can be readily shown that the BCSS is Σ_A on the eigenvectors of B, and vice versa.

In summary, the BCSS is identified based on the signs of $H_A(k)$, $H_B(k)$, $Q_A(k)$, and $Q_B(k)$, which provides an effective way to analyze the problem of regional asymptotical stabilizability of switched systems.

3.3 Necessary and Sufficient Stabilizability Conditions

In this section, we focus on deriving necessary and sufficient conditions for the switched system

$$S_{ij} : \dot{x} = \sigma x, \sigma = \{A_i, B_j\}, A_i, B_j \in \mathbb{R}^{2\times 2}, \mathrm{Re}\{\lambda_{\mathrm{A_i}}\} > 0, \mathrm{Re}\{\lambda_{\mathrm{B_j}}\} > 0, \quad (3.13)$$

where $\mathrm{Re}\{\lambda_{\mathrm{A_i}}\}$ denotes the real parts of the eigenvalues of A_i.

The condition will be extended to the switched system

$$S_{ij} : \dot{x} = \sigma x, \quad \sigma = \{A_i, B_j\}, A_i, B_j \in \mathbb{R}^{2\times 2}, \mathrm{Re}\{\lambda_{\mathrm{A_i}}\} \geq 0, \mathrm{Re}\{\lambda_{\mathrm{B_j}}\} \geq 0 \quad (3.14)$$

in Subsection 3.3.3.

In Subsection 3.3.4, the condition is further extended to the switched system consisting of at least one subsystem (assumed to be A_1) having a negative real eigenvalue

$$S_{ij} : \dot{x} = \sigma x, \quad \sigma = \{A_1, B_j\}, A_1, B_j \in \mathbb{R}^{2\times 2}, \quad (3.15)$$

where $\lambda_{1A}\lambda_{2A} \leq 0$ and B_j is not asymptotically stable. When $\lambda_{1A}\lambda_{2A} < 0$, A_1 is a saddle point. When $\lambda_{1A}\lambda_{2A} = 0$, A_1 is marginally stable but not asymptotically stable.

3.3.1 Assumptions

We need to settle some preliminaries to arrive at the main stabilization results. To this end, we rewrite the two assumptions in Section 2.2 for the switched system (3.1) and discuss the regional asymptotic stabilizability of the special cases when they are violated.

Assumption 3.1. *$A_i \neq cB_j$, where $c \in \mathbb{R}$.*

When Assumption 3.1 is violated, it is trivial to show that A_i is just scaled B_j, and hence the switched system (3.1) is not regionally asymptotically stabilizable (RAS). One difference between Assumption 3.1 and Assumption 2.1 is that the case $c < 0$ is included in Assumption 3.1. This case is possible when both A_i and B_j are saddle points and $A_i = cB_j, c < 0$, where the switched system is not RAS due to Definitions 3.1 and 3.2.

Note that the cases when B_j has two positive real eigenvalues and $A_i = cB_j, c < 0$ is asymptotically stable contradict the condition that neither subsystem of (3.1) is asymptotically stable, hence will not be considered here.

Assumption 3.2. *A_i and B_j do not share any real eigenvector.*

The special cases when Assumption 3.2 is violated will be discussed in Appendix B.1 since they are more complicated than the ones for Assumption 2.2 due to the complexity of the subsystems in (3.1).

3.3.1.1 Standard Forms

Assumption 3.3. *One subsystem of (3.1) is in its standard form as defined in (2.41),* i.e., *$A_i = J_i$,*

where the standard forms (real Jordan forms) J_i are defined as

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \ J_2 = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \ J_3 = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}. \tag{3.16}$$

When the switched system (3.13) is considered, we have

$$\lambda_2 \geq \lambda_1 > 0; \quad \lambda > 0; \quad \mu > 0, \omega < 0. \tag{3.17}$$

Note that it is always possible to guarantee one subsystem in its standard form by linear transformation under which stability of the switched system is preserved.

3.3.1.2 Standard Transformation Matrices

Since one subsystem is in its standard form, the other subsystem can be expressed as $B_j = P_j J_j P_j^{-1}$ with $i \leq j$, where J_j is the standard form of B_j and P_j is the transformation matrix defined as

$$P_1 = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 \\ \beta & \alpha \end{bmatrix}. \tag{3.18}$$

For any given B_j with its standard form J_j, P_j can be derived from the eigenvectors of B_j, which is the same as discussed in Section 2.4.

3.3.1.3 Assumptions on Different Combinations of S_{ij}

In order to further reduce the degrees of freedom, we make the following assumptions concerning the various parameters in the standard transformation matrices P_j and $N(k)$ of (3.10). These assumptions are listed below.

Assumption 3.4. *1. if $S_{ij} = S_{11}$, $\beta < 0$;*

2. if $S_{ij} = S_{12}$, $\alpha < 0$;

3. if $S_{ij} = S_{13}$, $k_2 < 0$, where k_2 is the smaller root of $N(k)$;

4. if $S_{ij} = S_{33}$, $p_2 \neq 0$, where p_2 is the leading coefficient of $N(k)$;

5. if $S_{ij} = S_{33}$, $p_2 < 0$ (if $N(k)$ (3.10) has two distinct real roots).

Please note that Assumptions 3.4 is the same as Assumptions 2.4, thus do not impose any constraint on the subsystems A_i and B_j, as supported by Lemma 2.4.

3.3.2 A Necessary and Sufficient Stabilizability Condition for the Switched System (3.13)

The main result is as follows.

Theorem 3.1. *The switched system (3.13), subject to Assumptions 3.1-3.4, is regionally asymptotically stabilizable if and only if there exist two independent real-valued vectors w_1, w_2, satisfying the collinear condition*

$$\det([A_i w_1, B_j w_1]) = 0,\ \det([A_i w_2, B_j w_2]) = 0, \tag{3.19}$$

and the slopes of w_1 and w_2, denoted as k_1 and k_2 with $k_2 < k_1$, satisfy the following inequality

$$\begin{cases} L < k_2 < k_1 < M & if\ \det(P_j) < 0 \\ \|\exp(B_j T_B)\exp(A_i T_A) w_1\|_2 < \|w_1\|_2 & if\ \det(P_j) > 0 \end{cases}, \tag{3.20}$$

where M and L correspond to the slopes of the non-asymptotes of A_i and B_j [1]

[1] with reference to Definition 2.1

respectively such that

$$M = \begin{cases} 0, & i = 1 \\ +\infty, & i = 2 \\ +\infty, & i = 3 \end{cases}, \qquad L = \begin{cases} \alpha, & j = 1 \\ \alpha, & j = 2 \\ -\infty & j = 3 \end{cases}, \tag{3.21}$$

and

$$T_A = \int_{\theta_2}^{\theta_1} \frac{1}{Q_A(\theta)} d\theta = \int_{\theta_2}^{\theta_1} \frac{1}{a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22} - a_{11})\sin\theta\cos\theta} d\theta, \tag{3.22}$$

$$T_B = \int_{\theta_1}^{\theta_2+\pi} \frac{1}{Q_B(\theta)} d\theta = \int_{\theta_1}^{\theta_2+\pi} \frac{1}{b_{21}\cos^2\theta - b_{12}\sin^2\theta + (b_{22} - b_{11})\sin\theta\cos\theta} d\theta, \tag{3.23}$$

where $\theta_1 = \tan^{-1} k_1$ *and* $\theta_2 = \tan^{-1} k_2$.

Theorem 3.1 shows that the existence of two independent vectors w_1 and w_2, along which the trajectories of the two subsystems are collinear, is a necessary condition for the switched system (3.13) to be stabilizable. It also indicates that there are two classes of switched systems (3.13) categorized by the sign of $\det(P_j)$, which implies the relative trajectory direction of two subsystems in certain regions. For example, when both A_i and B_j are with complex eigenvalues, the positive/negative $\det(P_j)$ implies that the trajectory directions of the two subsystems are the same/opposite for the whole phase plane.

The possible stabilization mechanisms corresponding to the two classes mentioned above are totally different as detailed below.

Class I ($\det(P_j) < 0$): *stable chattering* (sliding or sliding-like motion), *i.e.*, when system trajectories can be driven into a conic region where both $H_A(k)$ and $H_B(k)$ are negative, there exists a switching sequence to stabilize the system inside this region. In Class I, the switched systems are only RAS in the region (L, M), but not GAS unless one of the subsystem is with spiral, which can bring any initial state into the stabilizable region.

Class II ($\det(P_j) > 0$): *stable spiralling*, *i.e.*, when the system trajectory is a spiral around the origin and there exists a switching action to make the magnitude decrease after one or half circle. In Class II, if the condition (3.20) is satisfied, the switched systems are not only RAS, but also GAS.

Remark 3.1. The existence of two distinct stabilization mechanisms was also discussed by [49]. However, no simple algebraic index has been reported in the literature to classify given switched system (3.13) into these two classes. It was shown above that this can be readily done by checking the sign of $\det(P_j)$.

3.3.2.1 Proof of Theorem 3.1 when $S_{ij} = S_{11}$

Theorem 3.1 is proved in the following fashion. For every possible combination of the subsystems S_{ij}, it will be shown that if the condition (3.20) is satisfied, then there exists a switching signal to stabilize the switched system (3.13) if its initial states are in some regions of k, which constitutes the proof for the sufficiency. It will also be demonstrated that for all the cases when this condition is violated, the switched system can not be stabilized by any possible switching, which would establish the necessity.

We prove Theorem 3.1 for the case $S_{ij} = S_{11}$ in the following as an example to show the main idea and process of the proof of Theorem 3.1. The proofs of other combinations of S_{ij} are given in Appendix B.2.

Proof: In the case of $S_{ij} = S_{11}$,

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix}, B_1 = P_2 J_2 P_2^{-1} = \frac{1}{\beta - \alpha} \begin{bmatrix} \beta\lambda_{1b} - \alpha\lambda_{2b} & \lambda_{2b} - \lambda_{1b} \\ \alpha\beta(\lambda_{1b} - \lambda_{2b}) & \beta\lambda_{2b} - \alpha\lambda_{1b} \end{bmatrix}. \tag{3.24}$$

Let

$$\lambda_{1a} = k_A \lambda_{2a}, \lambda_{1b} = k_A \lambda_{2b}. \tag{3.25}$$

Then, we have $0 < k_A, k_B < 1$[1], $\alpha \neq 0$ by Assumption 3.2 and $\beta < 0$ by Assumption 3.4.1. Substituting A_1 and B_1 into (3.4)-(3.10), it follows that

$$N(k) = \frac{\lambda_{2a}\lambda_{2b}(k_A - 1)}{\beta - \alpha} \bar{N}(k), \tag{3.26}$$

where

$$\bar{N}(k) = k^2 + \frac{(k_A - k_B)\beta + (1 - k_A k_B)\alpha}{k_B - 1} k + \alpha\beta k_A, \tag{3.27}$$

is a monic polynomial with the same roots as $N(k)$ and

[1] If k_A=1, any vector in the phase plane is the eigenvector of A, which contradicts Assumption 3.2 since B has two real eigenvectors.

$$H_A(k) = K_B(k)\lambda_{2b}\frac{-\bar{N}(k)}{(\alpha-\beta)k}, \tag{3.28}$$

$$H_B(k) = K_A(k)\frac{\lambda_{2a}(1-k_A)\bar{N}(k)}{(1-k_B)(k-\alpha)(k-\beta)}, \tag{3.29}$$

$$Q_A(k) = -\frac{1}{1+k^2}\lambda_{2a}(k_A-1)k, \tag{3.30}$$

$$Q_B(k) = \frac{\lambda_{2b}(1-k_B)}{1+k^2}\frac{(k-\alpha)(k-\beta)}{\alpha-\beta}. \tag{3.31}$$

It can be readily shown that

$$\operatorname{sgn}(H_A(k)) = -\operatorname{sgn}(\alpha-\beta)\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k), \tag{3.32}$$

$$\operatorname{sgn}(H_B(k)) = \operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k-\alpha)\operatorname{sgn}(k-\beta), \tag{3.33}$$

$$\operatorname{sgn}(Q_A(k)) = \operatorname{sgn}(k), \tag{3.34}$$

$$\operatorname{sgn}(Q_B(k)) = \operatorname{sgn}(\alpha-\beta)\operatorname{sgn}(k-\alpha)\operatorname{sgn}(k-\beta). \tag{3.35}$$

In order to determine the signs of the equations (3.32)-(3.35) in every region of k, we require the relative position of the boundaries including (i) two eigenvectors of A_1, which are $k=0$ and $k=\infty$ in S_{11}; (ii) two eigenvectors of B_1, which are $k=\alpha$ and $k=\beta$; and (iii) the two distinct real roots of $N(k)$ which are defined as k_1 and k_2. We analyze all possible sequences of these boundaries with respect to the following three exclusive and exhaustive cases. Note that the root condition of $\bar{N}(k)$, or $N(k)$, is essentially the same as the one for $\det(Aw, Bw)$ by denoting k as the slope of w. For simplicity, we use the root condition of $\bar{N}(k)$ in the following analysis.

Case 1. $\bar{N}(k)$ does not have two distinct real roots.

There are three possibilities: 1) two complex roots; 2) two identical real roots; 3) one root, which are discussed as follows.

1.1) $\bar{N}(k)$ has two complex roots. Since the complex roots of $N(k)$, denoted as c_1 and c_2, are conjugate, the equation (3.36) below should be positive for any α.

$$(\alpha-c_1)(\alpha-c_2) = \frac{(1-k_A)k_B\alpha(\alpha-\beta)}{k_B-1}. \tag{3.36}$$

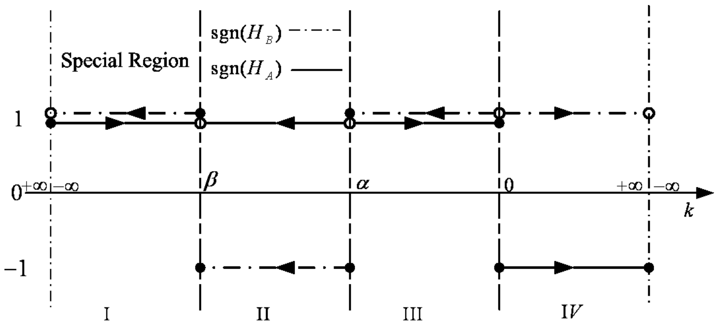


Figure 3.3: S_{11}: $N(k)$ has two complex real roots, the switched system is not RAS.

As a result, the only possible sequence of these boundaries is $\beta < \alpha < 0$. Then the signs of (3.32)-(3.35) could be determined for every region of k, as shown in Fig. 3.3

Fig. 3.3 is the crucial diagram exhibiting the conditions for the stabilizability of switched systems (3.13), as well as switched systems (3.1). It shows the signs of $H_A(k)$, $H_B(k)$, $Q_A(k)$, and $Q_B(k)$ versus $k \in (-\infty, +\infty)$, corresponding to $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$. The dashed vertical lines are the boundaries of the regions of k. The horizontal lines represent the signs of $H_A(k)$ (the solid) and $H_B(k)$ (the dashed) while the arrows represent the signs of $Q_A(k)$ and $Q_B(k)$ in different regions. If $H_A(k)$ is positive, then the solid line is above the horizontal axis. If $Q_A(k)$ is positive, the arrow on the dashed line points to the right (counter clockwise in $x - y$ plane).

With reference to Fig. 3.3, Region I and III are unstabilizable since both $H_A(k)$ and $H_B(k)$ are positive in these regions. Region I is a special region, where none of the trajectories can go out. Consider all possible initial states in different regions as follows.

- If the initial state is in Region I, it can not go out of this region.
- If the initial state is in Region II or IV, it will be brought into Region I by the best case switching signal, which is Σ_A in Region II (H_A is positive and H_B is negative) and Σ_B in Region IV.
- If the initial state is in Region III, it must be brought out because region III is unstabilizable. Then the trajectory will go to Region II or Region IV,

and goes to Region I eventually.

Therefore, when $\bar{N}(k)$ has two complex roots, the switched system (3.13) is not RAS.

1.2) $\bar{N}(k)$ has two identical real roots. Based on Remark 2.4, the best case analysis for this case is similar to the one for Fig. 3.3 regardless of the position of the multiple roots. Since this is true for all S_{ij}, the analysis for the case that $\bar{N}(k)$ having two identical real roots will be omitted in all other cases.

1.3) $\bar{N}(k)$ has only one root. In this case, the leading coefficient of $N(k)$, $p_2 = a_{12}b_{22} - a_{22}b_{12} = 0$ based upon (3.10). With reference to (3.24), we have $a_{12} = 0$ and $a_{22} \neq 0$. So $p_2 = 0$ results in $b_{12} = 0$, which implies that B_1 shares a real eigenvector (the y axis) with A_1, which violates Assumption 3.2. Therefore, this case can not happen for S_{11}. It can be readily shown that this is also true for all other cases of S_{1j} and S_{2j}. In S_{33}, $p_2 = 0$ was excluded by Assumption 3.4.4. Hence, we will omit the case that $N(k)$ has only one root in the rest of the proof of Theorem 3.1.

Case 2. $\bar{N}(k)$ has two distinct real roots and $\det(P_1) < 0$

$\alpha > \beta$, with reference to (3.27) and (3.36), there are totally four possibilities:

2.1) $\beta < \alpha < k_2 < k_1 < 0$

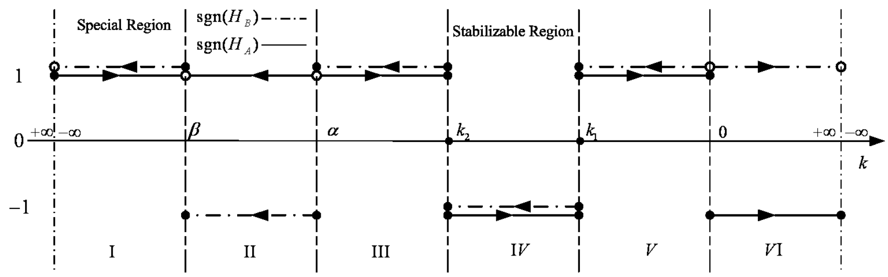


Figure 3.4: S_{11}: $\det(P_1) < 0$, $\beta < \alpha < k_2 < k_1 < 0$, the switched system is RAS.

With reference to Fig. 3.4, if the initial state is in the region of $k \in (-\infty, \alpha]$ or $k \in [0, \infty)$, the trajectory will be driven into the unstabilizable Region I and can not move out no matter which subsystem is chosen. However, if the initial state is in $(\alpha, 0)$, the trajectory can be brought into Region IV, where both $H_A(k)$ and $H_B(k)$ are negative, then the system can be stabilized by switching inside

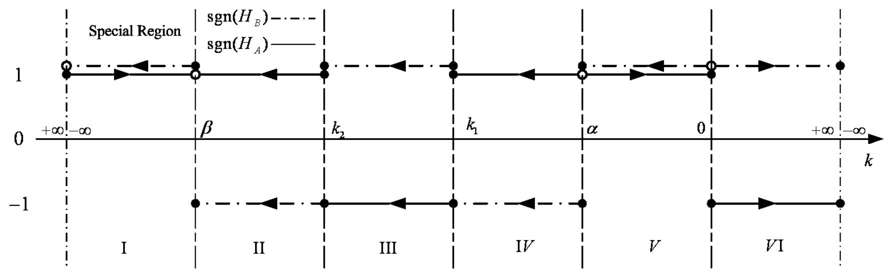


Figure 3.5: S_{11}: $\det(P_1) < 0$, $\beta < k_2 < k_1 < \alpha < 0$, the switched system is not RAS.

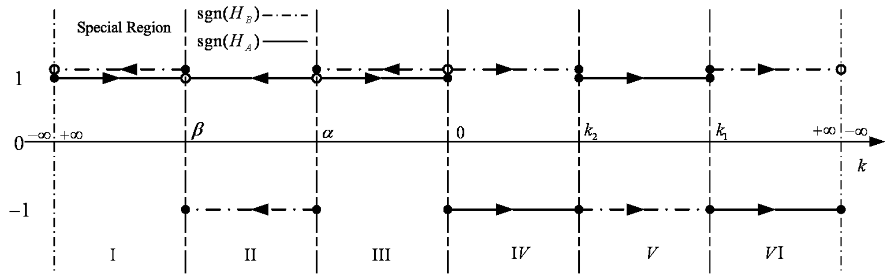


Figure 3.6: S_{11}: $\det(P_1) < 0$, $\beta < \alpha < 0 < k_2 < k_1$, the switched system is not RAS.

the stabilizable Region IV. Therefore, in this case, the switched system is RAS. The stabilizable region is $(\alpha, 0)$.

2.2) $\beta < k_2 < k_1 < \alpha < 0$

The switched system is not RAS with reference to Fig. 3.5.

2.3) $\beta < \alpha < 0 < k_2 < k_1$

The switched system is not RAS with reference to Fig. 3.6.

2.4) $\beta < k_2 < 0 < \alpha < k_1$

The switched system is not RAS with reference to Fig. 3.7.

Case 3. $\bar{N}(k)$ has two distinct real roots and $\det(P_1) > 0$.

$\alpha < \beta$, it follows from (3.27) and (3.36) that $k_2 < \alpha < \beta < k_1 < 0$.

With reference to Fig. 3.8, it is straightforward that the best case switching signal is Σ_B in region I and V because H_A is negative and H_B are positive. Similarly, the BCSS is Σ_A in region II and IV because H_A is positive and H_B are negative. In region III, both of H_A and H_B are positive, but Σ_A is the only subsystem whose trajectory can go out of region III because the boundaries of

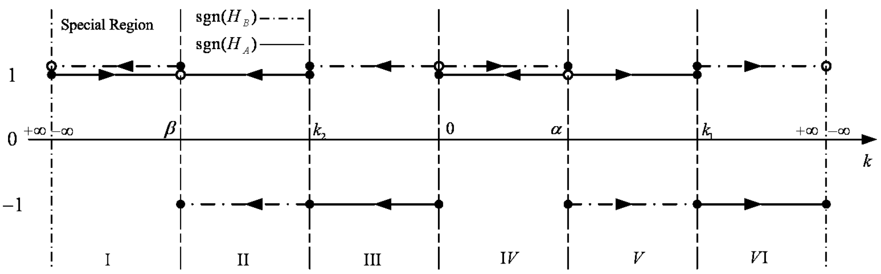


Figure 3.7: S_{11}: $\det(P_1) < 0$, $\beta < k_2 < 0 < \alpha < k_1$, the switched system is not RAS.

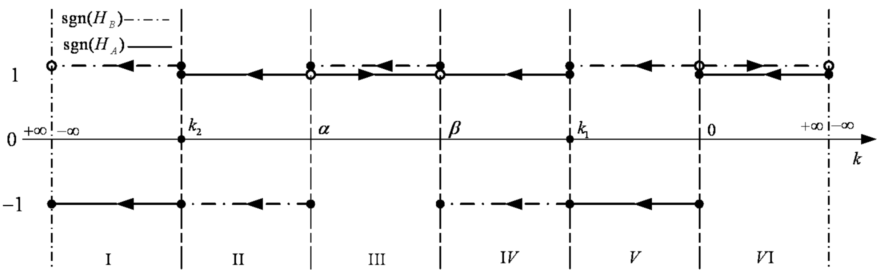


Figure 3.8: S_{11}: $\det(P_1) > 0$, the trajectory under the BCSS rotates around the origin.

region III are α and β that correspond to the eigenvectors of B. Similarly, the BCSS is Σ_B in region VI. On k_1 and k_2, without loss of generality, we choose Σ_B as the BCSS since both H_A and H_B are zeros. It is concluded that the BCSS in the whole interval of k is

$$\begin{cases} \sigma = A & k_2 < k < k_1, \\ \sigma = B & \text{otherwise}. \end{cases} \tag{3.37}$$

In this case, the trajectory under the BCSS rotates around the origin clockwise. The simplest way to determine stabilizability of the system is to follow a trajectory under the BCSS originating from a line l until it returns to l again and evaluate its expansion or contraction in the radial direction. Without loss of generality, let $w_1 = [1, k_1]$, the system is GAS if and only if $\|\exp(B_1 T_B)\exp(A_1 T_A) w_1\|_2 < \|w_1\|_2$. T_A and T_B are the time on Σ_A and Σ_B respectively, which could be calculated by

$$T_A = \int_{\theta_2}^{\theta_1} \frac{dt}{d\theta}\bigg|_{\sigma=A} d\theta = \int_{\theta_2}^{\theta_1} \frac{1}{Q_A(\theta)} d\theta, \tag{3.38}$$

$$T_B = \int_{\theta_1}^{\theta_2+\pi} \frac{dt}{d\theta}\bigg|_{\sigma=B} d\theta = \int_{\theta_1}^{\theta_2+\pi} \frac{1}{Q_B(\theta)} d\theta, \tag{3.39}$$

where $\theta_1 = \tan^{-1} k_1$ and $\theta_2 = \tan^{-1} k_2$. Hence the theorem is proven.

3.3.2.2 Application of Theorem 3.1

Example 3.1

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -9 & 5 \\ -20 & 11 \end{bmatrix}. \tag{3.40}$$

1. Simply checking shows that A has two distinct real eigenvalues $\lambda_{1a} = 1$ and $\lambda_{2a} = 3$ with corresponding eigenvectors $[1, 0]^T$ and $[0, 1]^T$, respectively. B has two multiple eigenvalues $\lambda_b = 1$ with a single eigenvector $[1, 2]^T$ which is undiagonalizable. It is the case S_{12}. It follows that Assumptions 3.1 and 3.2 are satisfied.

2. A is already in its standard form J_1.

3. $P_2 = \begin{bmatrix} 0 & 1 \\ -0.2 & 2 \end{bmatrix}$ is derived from $B = P_2 J_2 P_2^{-1}$. It follows that $\alpha = 2$, which violates Assumption 3.4.2. Therefore, we need to transform A and B simultaneously. By denoting $\bar{x}_1 = -x_1$, we obtain a new switched system

$$\bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} -9 & -5 \\ 20 & 11 \end{bmatrix} \tag{3.41}$$

   which has the same stabilizability property as the switched system (3.40). Recalculate $\bar{P}_2 = \begin{bmatrix} 0 & 1 \\ 0.2 & -2 \end{bmatrix}$, where $\alpha = -1$ satisfies Assumption 3.4.2. And we have $k_1 = -0.7460, k_2 = -1.7873$.

4. The first inequality of Theorem 3.1 should be checked because $\det(\bar{P}_2) = -0.2 < 0$. With reference to (3.21), we have $L = \alpha = -2$ and $M = 0$ for S_{12}, hence the inequality $L < k_2 < k_1 < M$ is satisfied.

It can be concluded that the switched system (3.41), or equivalently, the switched system (3.40), is regionally asymptotically stabilizable. A typical stabilizing trajectory of the switched system (3.41) is shown in Fig. 3.9.

Note that this example corresponds to a class of switched systems which was not considered in [49], [50], or [51].

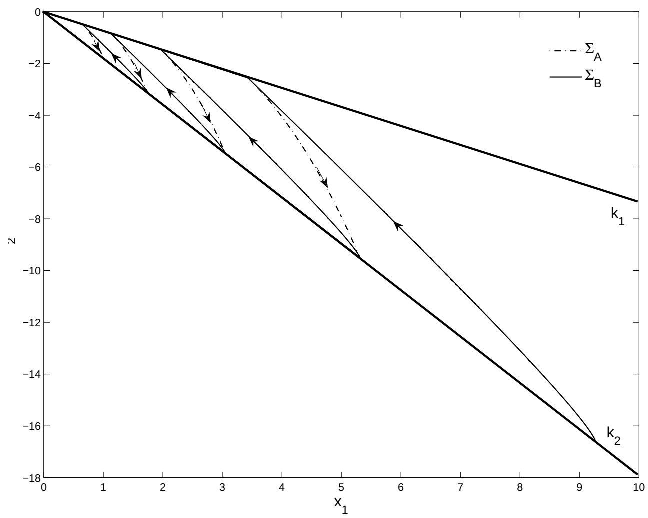


Figure 3.9: A typical stabilizing trajectory of the switched system (3.41).

3.3.3 Extension to the Switched System (3.14)

We now extend Theorem 3.1 to the switched system (3.14), where the real part of the subsystem's state matrix is allowed to be zero here. The standard forms and standard transformation matrices of the switched system (3.14) are the same as those for the switched system (3.13) in (2.41) and (2.43), except that (3.17) is revised as

$$\lambda_2 \geq \lambda_1 \geq 0; \quad \lambda \geq 0; \quad \mu \geq 0, \omega < 0. \tag{3.42}$$

Theorem 3.2. *The switched system (3.14), subject to Assumptions 3.1-3.4, is regionally asymptotically stabilizable if and only if there exist two independent real-valued vectors w_1, w_2, satisfying the collinear condition*

$$\det([A_i w_1, B_j w_1]) = 0, \ \det([A_i w_2, B_j w_2]) = 0, \tag{3.43}$$

and the slopes of w_1 and w_2, denoted as k_1 and k_2 with $k_2 < k_1$, satisfy the *following inequality*

$$\begin{cases} L \leq k_2 < k_1 \leq M & if\,\det(P_j) < 0 \\ \|\exp(B_j T_B)\exp(A_i T_A) w_1\|_2 < \|w_1\|_2 & if\,\det(P_j) > 0 \end{cases}, \tag{3.44}$$

where M, L, T_A, and T_B are the same as those defined in Theorem 3.1.

The proof of Theorem 3.2 is similar to that of Theorem 3.1 by considering the special cases when $k_A = 0$, $k_B = 0$ (3.25), or $\mu = 0$ (3.17). Therefore, it is omitted here.

3.3.4 Extension to the Switched System (3.15)

In this subsection, we analyze the regional asymptotic stabilizability of the switched system (3.15), where at least one of the subsystems has a negative eigenvalue.

It is worth noting that the trajectory staying on the eigenvector with a negative eigenvalue will not be considered as a valid stabilizing trajectory, because it is not possible to bring the trajectory to this eigenvector exactly in practical. Furthermore a small disturbance will divert the trajectory from the eigenvector even if the initial state is on the eigenvector.

Theorem 3.3. *The switched system (3.15) subjected to Assumptions 3.1 and 3.2 is always regionally asymptotically stabilizable.*

Since the proof for Theorem 3.3 is similar to that of Theorem 3.1 by analyzing the cases when k_A and/or k_B are negative, it is omitted here.

Remark 3.2. The switched system (3.15) which is RAS can also be said to GAS

- if $S_{ij} = S_{13}$. In this case, there exists a subsystem along which the trajectories can be driven into the stabilizable region regardless of the initial state.
- if the switched system (3.15) is subject to Assumption 3.1-3.4 and satisfies the condition $\det(P_j) > 0$. In this case, there always exists a trajectory that can rotate around the origin regardless of the type of subsystems.

3.4 Discussion

Now we discuss the connections among (i) the stabilizability conditions of the present chapter, (ii) the stability conditions in Chapter 2, and (iii) related results in literature [49], [34], and [63].

In [49], necessary and sufficient stabilizability conditions for the switched system (3.1) are established first for the cases when both A and B are unstable nodes, unstable spirals, and saddle points.

The stabilizability conditions derived in this chapter extend those found in [49], and have shown to be 1) more general in the sense that all the possible combinations of subsystem dynamics (node, saddle point and focus) and marginally unstable subsystems are taken into account, 2) easily verifiable since the checking algorithm is easy to follow and all the calculations can be done by hand, and 3) in a compact form that facilitates more geometric insights.

In Chapter 2, we analyzed the stability of the switched system (2.40) under arbitrary switching and derived a necessary and sufficient condition (Theorem 2.1) by finding the worst-case switching signals. In the present chapter, we investigated the regional asymptotical stabilizability of the switched system (3.13) and derived necessary and sufficient stabilizability condition (Theorem 3.1), based upon the best-case switching signals. It is interesting to note that, by reversing time, Theorem 3.1 is equivalent to Theorem 2.1. In simpler term, if a switched system (3.13) with a pair of A_i and B_j is not regionally asymptotically stabilizable (RAS), then the corresponding switched system with $-A_i$ and $-B_j$ is stable under arbitrary switching. Similarly, if a switched system (3.13) with A_i and B_j is RAS, then the corresponding switched system with $-A_i$ and $-B_j$ is not stable under arbitrary switching. The equivalence is obvious by comparing of Example 2.2 to Example 3.1.

It is to be noted that the analysis of regional asymptotical stabilizability as proposed in the chapter is non-trivial, although Theorem 3.1 and Theorem 2.1 are found to be equivalent by reversing time. The reasons are listed below

1. When the stabilizability problem is considered, we need to know i) when a switched system is globally asymptotically stabilizable, and ii) where the stabilizable region is if a switched system is only regionally asymptotically stabilizable. In Example 2.1, the initial state has to be inside the region

of k bounded by (L, M) such that its trajectory can go into the stabilizable region (k_2, k_1) where $H_A(k)$ and $H_B(k)$ are negative. The situation is different for the problem of the stability under arbitrary switching. If there exists an unstable region, then the trajectory can be driven into this region regardless of its initial state. This difference can be easily seen by comparing Fig. 2.6 and Fig. 3.4.

2. In Theorem 3.3, the cases when subsystems have eigenvalues with a negative eigenvalue are considered. No corresponding result can be found in the papers by [34, 63].

3. In addition to regional asymptotic stabilizability (RAS), global asymptotic stabilizability (GAS) can also be obtained by similar analysis. The equivalence does not exist anymore when GAS is considered.

3.5 Summary

In this chapter, a necessary and sufficient condition (Theorem 2.1) for regional asymptotic stabilizability of the switched system (3.13) is derived, based on detailed best-case analysis. The condition is easily verifiable without relying on any numerical solution. Furthermore, this stabilizability condition is extended to switched systems (3.14) and (3.15) such that all possible dynamics of the subsystems of (3.1) are covered.

Chapter 4

Stability of Periodically Switched Systems

In this chapter, we investigate the stability of switched systems under periodic switching due to its importance of periodically switched systems in theory and practice. We present frequency-domain L_2- stability conditions for feedback systems with a linear system in the forward path and periodically switched linear and nonlinear gains in the feedback path. These conditions can be easy verified by a computational-graphic method. An interesting phenomenon of the switching feedback systems is discovered: fast switching leading to stability, which is confirmed by our simulation.

This chapter is organized as follows. In Section 4.1, we formulate the problem of L_2-stability of SISO and MIMO systems with a linear/nonlinear, periodically switched single-/matrix-gain described by integral equations. We also introduce the multiplier-function type of stability conditions. In Section 4.2, we present the main results (with proofs) of this chapter, which are frequency-domain stability conditions for single-input-single-output (SISO) systems, and use examples from literature to demonstrate the novelty of the new stability conditions. Moreover, we outline a procedure for synthesizing a multiplier function for linear and a class of nonlinear systems. In Section 4.3, we consider the effect of dwell-time on stability, while in Section 4.4, we derive stability conditions for multi-input-multi-output (MIMO) systems, and illustrate them with an example. In Section 4.5, we compare our results with those found in the recent reference [64]. Section

4.6 summarizes this chapter. Appendix C contains the proofs of some lemmas used in the main theorems.

4.1 Problem Formulation

Concerning SISO systems, we deal with two stability problems (1) L_2-stability of a linear system with a single periodically switching parameter with values in $[0, \overline{K})$ [1], and (2) L_2-stability of a nonlinear system with a nonlinearity in association with a single periodically switching gain, having together a finite gain with values in $[0, \overline{K})$. We also consider the corresponding counterparts for MIMO systems.

It is known that, in general, the standard differential equation description of a system can be converted to an integral form. Conversely, the stability results obtained for integral equations can be specialized to be applicable to differential equations.

4.1.1 SISO Linear Systems

The following $n^{th}-$order differential equation represents the dynamics of a linear system, having y as the dependent variable

$$p(D)y + k(t)q(D)y = f(t),\ t \in [0, \infty), \tag{4.1}$$

where $p(D) = D^n + p_{n-1}D^{n-1} + \cdots + p_0$ and $q(D) = q_m D^m + q_{m-1}D^{m-1} + \cdots + q_0$ are constant coefficient differential operators with the order n of $p(D)$ at least one higher than the order m of $q(D)$.

Let $y = x_1,\ x_2 = dx_1/dt, \cdots, x_n = dx_{n-1}/dt$, and $\underline{x} = [x_1, x_2, \cdots, x_n]'$, with $'$ denoting transpose. Then (4.1) can be converted to vector differential equation

$$\begin{aligned} \frac{d\underline{x}}{dt} &= A\underline{x} + \underline{b}(f(t) - k(t)y(t)), \\ y(t) &= \underline{c}'\underline{x}, \end{aligned} \tag{4.2}$$

where A is a phase-variable canonical form [65] stable matrix with the last row given by $[-p_0, -p_1, \cdots, -p_{n-1}]$, $\underline{b} = [0, 0, \cdots, 1]'$, and $\underline{c} = [-q_0, -q_1, \cdots, -q_m, \cdots, 0]'$. The gain $k(\cdot)$ is a piece-wise continuous switching parameter, assuming values

[1] Applicable to a more general range $[\underline{K}, \overline{K})$, where $\underline{K} \geq 0$

in $[0, \overline{K})$, and having the fundamental switching period $\mathcal{P}$, $f(\cdot)$ is the reference input to the system, $v(\cdot)$ is the error signal, and $y(\cdot)$, in the terminology of system theory, the output of the system.

For simplicity in the proof of stability theorems, it is helpful to enlarge the range $[0, \overline{K})$ to $[0, \infty)$. A conversion scheme, which is standard in the stability theory of feedback systems [66], can be applied in order to arrive at an equivalent system with the gain in $[0, \infty)$. See Fig. 4.1.

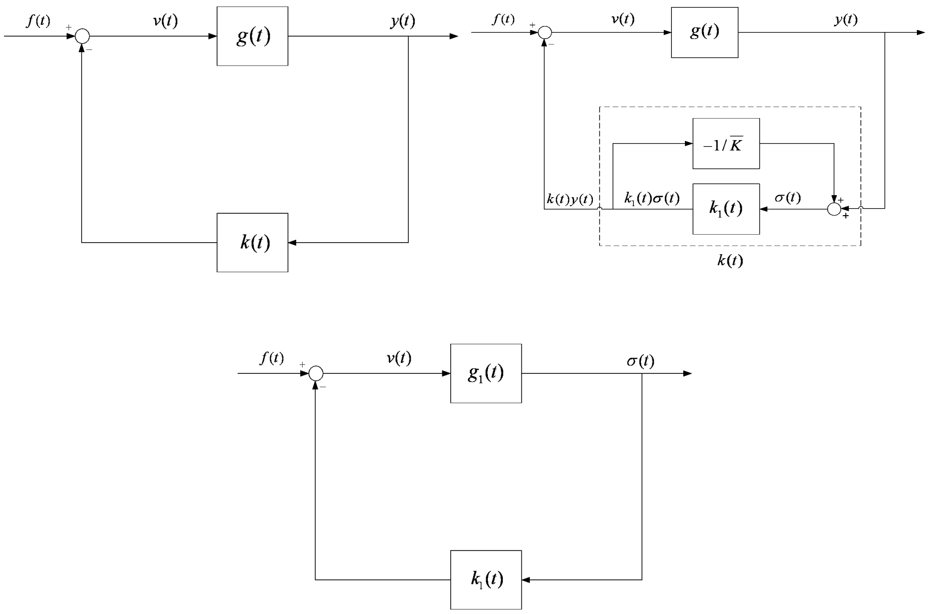


Figure 4.1: Conversion of feedback gain from finite range to infinite range.

Further, (4.2) can be converted to an integral equation which, in a generalized version, assumes the following form

$$
\begin{aligned}
v(t) &= f(t) - k_1(t)\sigma(t), \\
\sigma(t) &= \sum_{m=1}^{\infty} g_{1m}\, v(t-\tau_m) + \int_0^{\infty} g_1(\tau)v(t-\tau)d\tau, \quad t \geq 0,
\end{aligned} \tag{4.3}
$$

where $f(\cdot)$ and $v(\cdot)$ are defined as before, $\sigma(\cdot)$ is the output of the gain-enlargement system, $\{g_{1m}\}$, $\{\tau_m\}$[1] are constant real sequences, with $\tau_m \geq 0$, $\forall m$, $k_1(t) = \frac{\overline{K}k(t)}{\overline{K}-k(t)}$, and assumes values in $[0, \infty)$. Such a gain-enlargement is also applicable, with changes when necessary, to nonlinear systems considered below. See [66] for details as applied to certain classes of feedback systems.

[1] to describe discontinuities.

Let $G_1(j\omega)$ be the Fourier transform of $g_1(t)$, the gain-scaled impulse response of the linear part, and $G(j\omega)$ the Fourier transform of $\{\sum_{m=1}^{\infty} g_{1m}\ \delta(t-\tau_m)+g(t)\}$. Then $G_1(j\omega) = G(j\omega) + 1/\overline{K}$. In this chapter, the *input-output stability* of (4.3) is analyzed, which can be shown to be equivalent to the *Lyapunov-stability* under suitable conditions [67].

4.1.2 SISO Nonlinear Systems

The corresponding nonlinear system in differential is described by

$$\begin{aligned} &\frac{d\underline{x}}{dt} = A\underline{x} + \underline{b}(f(t) - k(t)\varphi(y(t))), \\ &y = \underline{c}'\underline{x}, \end{aligned}$$

or equivalently,

$$\begin{aligned} &v(t) = f(t) - k_1(t)\varphi(\sigma(t)), \\ &\sigma(t) = \textstyle\sum_{m=1}^{\infty} g_{1m}\ v(t-\tau_m)\ +\ \int_0^{\infty} g_1(\tau)v(t-\tau)d\tau, \end{aligned} \tag{4.4}$$

where the switching functions $k(\cdot)$ and $k_1(\cdot)$, and constant real sequences $\{g_{1m}\}$, and $\{\tau_m\}$ are the same as before. $\varphi(\cdot)$, a real-valued function on $(-\infty, \infty)$ is a memoryless, first-and-third-quadrant nonlinearity. It satisfies the following basic properties (1) $\varphi(0) = 0$, (2) there exist positive constants q_1 and q_2 with $q_1 < q_2$ such that $q_1\sigma^2 \leq \varphi(\sigma)\sigma \leq q_2\sigma^2$, $\sigma \neq 0$, and (3) it is monotone, odd-monotone or power-law with additional properties defined in Sec. 4.1.4. When combined with the switching parameter, $k(\cdot)$, the nonlinear gain assumes values in $[0, \overline{K}q_2)$. In this case, with $\overline{K}q_2 = \overline{K^*}$, the scaled transfer function is given by $G_1(j\omega) = G(j\omega) + 1/\overline{K^*}$.

Concerning the systems (4.3) and (4.4), let $L_2[0,\infty)$ be the linear space of real valued functions $x(\cdot)$ on $[0,\infty)$ with the property that $\int_0^{\infty} |x(t)|^2 dt < \infty$, and equipped with the norm, $\|x(\cdot)\| = (\int_0^{\infty} |x(t)|^2 dt)^{\frac{1}{2}}$.

Definition 4.1. *The linear system described by (4.3) and the nonlinear system described by (4.4) are L_2-stable if $v \in L_2[0,\infty)$ for $f \in L_2[0,\infty)$, and an inequality of the type $\|v\| \leq C\|f\|$ holds where C is a constant.*

4.1.3 MIMO Systems

A generalization of (4.4) is the MIMO system with a switching matrix $\mathcal{K}(t)$ of dimension $r \times r$, with a common fundamental period $\mathcal{P}$ for all the elements of $\mathcal{K}(t)$, *i.e.*, $\mathcal{K}(t - m\mathcal{P}) = \mathcal{K}(t)$ for $m = \cdots, -2, -1, 0, 1, 2, \cdots$ and for all t. The elements $k_{mn}(t)$ of $\mathcal{K}(t)$ assume values in $[0, \overline{k_{mn}})$ and the upper-bound matrix of $\mathcal{K}(t)$ is given by $\overline{\mathcal{K}}$. The corresponding system has vector inputs and outputs:

$$\frac{d\underline{x}}{dt} = A\underline{x} + B(\underline{f}(t) - \mathcal{K}(t)\underline{\varphi}(\underline{y}(t))),$$
$$\underline{y} = C\underline{x},$$

or equivalently,

$$\underline{v}(t) = \underline{f}(t) - \mathcal{K}_1(t)\underline{\varphi}(\underline{\sigma}(t)),$$
$$\underline{\sigma}(t) = \sum_{m=1}^{\infty} \mathfrak{S}_{1m} \, \mathtt{Diag}\, [\delta(t-\tau_m) \cdots \delta(t-\tau_m)] + \int_0^{\infty} \mathfrak{S}_1(\tau)\underline{v}(t-\tau)d\tau, \tag{4.5}$$

where all the vectors of reference input $\underline{f}$, error $\underline{v}$, nonlinear gain $\underline{\varphi}$ and output $\underline{\sigma}$ have a dimension of r; $\{\mathfrak{S}_{1m}\}$ is a constant real matrix sequence, and $\{\tau_m\}$ is a real sequence with $\tau_m \geq 0 \; \forall \; m$.

The linear time-invariant block is described by the matrix impulse response $\sum_{m=1}^{\infty} \mathfrak{S}_{1m}$ Diag $[\delta(\mathrm{t}-\tau_\mathrm{m}) \cdots \delta(\mathrm{t}-\tau_\mathrm{m})] + \mathfrak{S}_1(\mathrm{t})$ of size $r \times r$. Its Fourier transform $\Gamma_1(j\omega)$ of is given by $\Gamma_1(j\omega) = I + \overline{\mathcal{K}}\Gamma(j\omega)$, where I is a unit matrix, and $\Gamma(j\omega)$ is the Fourier transform of the unscaled linear forward-block impulse response matrix $\sum_{m=1}^{\infty} \mathfrak{S}_m$ Diag $[\delta(\mathrm{t}-\tau_\mathrm{m}) \cdots \delta(\mathrm{t}-\tau_\mathrm{m})] + \mathfrak{S}(\mathrm{t})$, in the special case of the elements of the constant gain matrix $\mathcal{K}$, assuming values in $[0, \overline{\mathcal{K}})$.

Let $\underline{x}(\cdot)$ denote a real-valued vector function, having elements $x_1, x_2, \cdots, x_r$. If each element of the vector $\underline{x}(\cdot)$ is in L_2, the vector itself is said to be in L_2. Then its L_2-norm is defined by

$$\|\underline{x}(\cdot)\| = \left\{ \sum_{i=1}^{r} \int_0^{\infty} |x_i(t)|^2 dt \right\}^{\frac{1}{2}}.$$

Definition 4.2. *The system described by (4.5) is said to be L_2-stable if $\underline{v} \in L_2[0, \infty)$ for $\underline{f} \in L_2[0, \infty)$, and an inequality of the type $\|\underline{v}\| \leq C\|\underline{f}\|$ holds where C is a constant.*

4.1.4 Classes of Nonlinearity

4.1.4.1 Odd-monotone Nonlinearity

The real-valued function $\varphi(\sigma)$ is monotone nondecreasing, *i.e.*, $\varphi(\cdot) \in \mathcal{M}$, if

$$(\sigma_1 - \sigma_2)(\varphi(\sigma_1) - \varphi(\sigma_2)) \ \geq \ 0, \ \ \forall \sigma_1 \text{ and } \sigma_2. \tag{4.6}$$

If $\varphi(\cdot) \in \mathcal{M}$, then for all σ_1 and σ_2, the following inequality holds [68]:

$$(\sigma_1 \ - \ \sigma_2)\varphi(\sigma_1) \ \geq \ \int_{\sigma_2}^{\sigma_1} \varphi(\sigma)\, d\sigma. \tag{4.7}$$

Further, if $\varphi(\sigma)$ is odd-monotone nondecreasing, then $\varphi(\cdot) \in \mathcal{M}_o$, and has following properties of 1) $\varphi(\cdot) \in \mathcal{M}$, and 2) $\varphi(\sigma) \ = \ -\varphi(-\sigma)$.

When the L_2-stability result derived for the odd-monotone nonlinear system is reduced to the special case of the linear system, there is a stability bound gap between the linear system and the odd-monotone nonlinear system. Therefore, in order to facilitate a smooth transition from the stability results for the odd-monotone nonlinear system to those for the linear system, there is a need to introduce the class $\mathcal{E}$ of power-law nonlinearities as follows.

4.1.4.2 Power-law Nonlinearity

A real-valued function $\varphi(\sigma)$ is a power-law, *i.e.*, $\varphi(\cdot) \in \mathcal{E}$, if its rate of growth is bounded by

$$\frac{1}{\mu} \leq \frac{d}{d\sigma}\{log\ \varphi(\sigma)\} \leq \mu, \sigma > 0, \tag{4.8}$$

where $\mu > 0$ is a constant characterizing the power-law behavior [69, 70].

When $\mu = \infty$, the power-law nonlinearity belongs to class $\mathcal{M}_o$ [70]; and when $\mu = 1$, it becomes a linear function. For the class $\mathcal{E}$ of nonlinearities, the governing inequality is

$$\sigma_1\ \varphi(\sigma_2) - \sigma_2\ \varphi(\sigma_1) \ \leq \ \nu\ \{\sigma_1\ \varphi(\sigma_1) + \sigma_2\ \varphi(\sigma_2)\}, \ \forall \sigma_1 \text{ and } \sigma_2, \tag{4.9}$$

where $\nu > 0$ is associated with the constant μ of the class $\mathcal{E}$ as defined in (4.8), and is given by the following equation [70]

$$\nu = \max_{0<\theta<\infty} \left| \frac{(\theta^{\mu} - \theta)}{(\theta^{\mu+1} + 1)} \right| \tag{4.10}$$

where $\theta = \frac{\sigma_1}{\sigma_2}$. When $\mu = \infty$, the nonlinearity is odd-monotone with $\nu = 1$, but when $\mu = 1$, the nonlinearity reduces to linearity with $\nu = 0$. For a few of the other values of μ, the corresponding values of ν, as found in [70], are as follows: $\mu = 4, \nu = 0.438$; $\mu = 3, \nu = 0.354$; and $\mu = 2, \nu = 0.227$.

4.1.4.3 Relaxed Monotone Nonlinearity

The inequality (4.7) enables us to define a new class $\mathcal{M}_q$ of real-valued functions $\varphi(\cdot)$ with the property of a "relaxed" monotonicity condition. To this end, let $Q(\sigma_1, \sigma_2)$ be a non-negative definite quadratic form in σ_1 and σ_2, as defined below:

$$Q(\sigma_1, \sigma_2) \;=\; q_{11}\sigma_1^2 + q_{12}\sigma_1\sigma_2 + q_{22}\sigma_2^2. \tag{4.11}$$

Then $\varphi(\cdot) \in \mathcal{M}_q$, if

$$(\sigma_1 \,-\, \sigma_2)\varphi(\sigma_1) \;\geq\; \int_{\sigma_2}^{\sigma_1} \varphi(\sigma)\, d\sigma \;+\; Q(\sigma_2, \varphi(\sigma_2)) \;-\; Q(\sigma_1, \varphi(\sigma_1)). \tag{4.12}$$

Note that the actual quadratic form in (4.12) is rather non-standard because the cross-coupling terms $\sigma_1\varphi(\sigma_1)$ and $\sigma_2\varphi(\sigma_2)$ are both always positive for $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$. Further, since $\phi(\sigma)\sigma > 0$ for $\sigma \neq 0$ and $\sigma = \sigma_1$ or σ_2, inequality (4.12) is distinct from the following inequality found in [71] and employed in [64]

$$(\sigma_1 \,-\, \sigma_2)\varphi(\sigma_1) \;\geq\; \int_{\sigma_2}^{\sigma_1} \varphi(\sigma)\, d\sigma \;-\; Q(\sigma_2, \varphi(\sigma_2)) \;-\; Q(\sigma_1, \varphi(\sigma_1)). \tag{4.13}$$

We denote the class of nonlinear functions $\varphi(\cdot)$, satisfying (4.13) as found in [71] and used in [64] by $\mathcal{M}_b$.

For later use in the stability theorems, we define a few characteristic constants of the nonlinear functions belonging to classes $\mathcal{M}, \mathcal{M}_o, \mathcal{M}_q$, and $\mathcal{M}_b$. These constants refer to the upper and lower rates of variation of (i) the nonlinearity $\varphi(\cdot)$, and (ii) the quadratic form that relaxes the restriction of monotonicity on $\varphi(\cdot)$.

1. Classes $\mathcal{M}$ and $\mathcal{M}_o$:

   Let $\Phi(\sigma) \;=\; \int_0^{\sigma} \varphi(\tau)\, d\tau$. Then

$$\nu_s \;=\; \sup_{\sigma\neq 0} \left(\frac{\Phi(\sigma)}{\varphi(\sigma)\sigma}\right),\; \nu_i \;=\; \inf_{\sigma\neq 0} \left(\frac{\Phi(\sigma)}{\varphi(\sigma)\sigma}\right). \tag{4.14}$$

2. Classes $\mathcal{M}_q$ and $\mathcal{M}_b$:

$$\zeta_i = \inf_{\sigma \neq 0} \left(\frac{\varphi(\sigma)}{\sigma} \right), \zeta_s = \sup_{\sigma \neq 0} \left(\frac{\varphi(\sigma)}{\sigma} \right), \nu_s' = \left(\frac{q_{11}}{\zeta_i} + q_{22}\zeta_s + q_{12} \right). \tag{4.15}$$

Note that

$$\zeta_i = \sup_{\phi(\sigma) \neq 0} \left(\frac{\sigma}{\varphi(\sigma)} \right), \ \zeta_s = \inf_{\varphi(\sigma) \neq 0} \left(\frac{\sigma}{\phi(\sigma)} \right).$$

Assumption 4.1. *1. Linear system (4.3): The system is asymptotically stable for all positive constant gains. (Equivalently, the linear system (4.2) or its corresponding integral equation form is asymptotically stable for all constant gains in* $[0, \overline{K})$.*)*

2. Nonlinear system (4.4): The linear system that is obtained from (4.4), by replacing the nonlinearity $\varphi(\sigma)$ *by* $q_2\sigma$*, is asymptotically stable for all positive constant gains.*

3. Nonlinear matrix gain system (4.5): The system is asymptotically stable for a constant gain-matrix, $\overline{\mathcal{K}}$*, which is the matrix* $\mathcal{K}(t)$ *with its elements* $k_{mn}(t)$ *replaced by the constants with values in* $[0, \overline{k_{mn}})$ *and for which the zeros of* $\mid I + \overline{\mathcal{K}}\ \Gamma(s) \mid$ *lie strictly in the left-half (*$\mathrm{Re\ s} < -\delta \ \leq \ 0$*) of the complex plane. The assumptions on* $\underline{\varphi}$ *will be indicated in Section 4.4.*

4.1.5 Objectives and Methodologies

The objectives of this chapter are to find conditions for L_2-stability of the switched feedback systems described by (4.3), (4.4), and (4.5) subject to Assumption 4.1.

The approach for the present results comes from the "multiplier-function" form of the Nyquist criterion for linear time-invariant (feedback) systems [69]. More explicitly, the Nyquist criterion can be rewritten in terms of a multiplier function whose phase angle when added to the phase angle of $G_1(j\omega) = G(j\omega) + 1/\overline{K}$ gives us a composite function with the phase lying in the band $(-\pi/2, \pi/2)$, as follows [74, Theorem 2, page 726]:

The system is asymptotically stable for all constant gains $K \in [0, \overline{K})$ if there exists a frequency function, $Z(j\omega)$ such that $-\pi/2 \leq \arg\{(\mathrm{Z(j}\omega)\} \leq \pi/2$ and

$-\pi/2 < \arg\{\mathrm{Z}(\mathrm{j}\omega)\mathrm{G}_1(\mathrm{j}\omega)\} < \pi/2$, where "arg" denotes "the phase angle of". Alternatively, Re $[\mathrm{Z}(\mathrm{j}\omega)] \geq 0$, and $\mathrm{Re}[\mathrm{Z}(\mathrm{j}\omega)\mathrm{G}_1(\mathrm{j}\omega)] > 0,\ \omega \in (-\infty, \infty)$, where "Re" denotes "the real part of". For simplicity, we call them "Real-Part" conditions.

4.2 Stability Conditions for SISO Systems

Frequency-domain stability conditions for both the linear and nonlinear systems, which are described in an integral equation form, are directly obtained by employing a combination of the Parseval theorem (in Fourier transforms) and certain integral inequalities, originally found in [72, 73], and as developed in a modified form in [74]. The stability conditions involve constraints:

1. In the case of the linear system, the constraints are on (i) the period $\mathcal{P}$ and/or the upper bound $\overline{K}$ of the switching gain $k(t)$, and (ii) $G_1(j\omega)$.

2. In the case of the nonlinear system with monotone, odd-monotone, "relaxed" monotone or power-law function as a gain, the above constraints along with certain additional ones are required.

In order to establish the mathematical result, we need a few more definitions. For any real valued function $x(\cdot)$ on $[0, \infty)$ and any $T \geq 0$, the *truncated function* $x_T(\cdot)$ is defined by:

$$x_T = \begin{cases} x(t) & \text{for } \ 0 \leq t \leq T \\ 0 & \text{for } t < 0 \text{ and } \ t > T \end{cases} .$$

Further, let L_{2e} be the space of those real-valued functions $x(\cdot)$ on $[0, \infty)$ whose truncations $x_T(\cdot)$ belong to $L_2[0, \infty)$ for all $T \geq 0$. Essentially, by assuming infinite *escape time* for the solution of the system with $f \in L_2$, the solution belongs to L_{2e}. Then, it is shown that, under certain conditions on $k(t)$ and on $G_1(j\omega)$, the solution actually belongs to $L_2[0, \infty)$.

We require an operator that generates positive operators in combination with the forward block operator $\mathcal{G}_1(\cdot)$ which in turn is in cascade with the periodic switching gain and the linear/nonlinear gain. In effect, we are looking for an operator $\mathcal{Z}$ whose Fourier transform is such that the Real-Part conditions are satisfied. For the present periodic-switching stability problem, the following linear operator has been found to be appropriate: for given real sequences $\{z_m\}$ and $\{z'_m\}$ in ℓ_1, *i.e.*, $\sum_{m=1}^{\infty} \mid z_m \mid < \infty$ and $\sum_{m=1}^{\infty} \mid z'_m \mid < \infty$,

$$\mathcal{Z}\{\sigma(t)\} = \sigma(t) + \sum_{m=1}^{\infty} z_m \, \sigma(t - m\mathcal{P}) + \sum_{m=1}^{\infty} z'_m \, \sigma(t + m\mathcal{P})). \tag{4.16}$$

Its Fourier transform $Z(j\omega)$ and phase angle $\phi(\omega)$ are given by

$$Z(j\omega) = 1 + \sum_{m=1}^{\infty} z_m (e^{-jm\mathcal{P}\omega}) \; + \; \sum_{m=1}^{\infty} z'_m \, (e^{jm\mathcal{P}\omega})$$

and

$$\tan \, \phi(\omega) \; = \; \frac{\sum_{m=1}^{\infty} (-z_m \; + \; z'_m) \; \sin \, (m\mathcal{P}\omega)}{1 + \sum_{m=1}^{\infty} (z_m \; + \; z'_m) \; \cos \, (m\mathcal{P}\omega)}. \tag{4.17}$$

With the preliminaries settled, we can now state the main results as follows

4.2.1 Stability Conditions for linear and monotone nonlinear systems

Theorem 4.1. *The linear feedback system (4.3) with a periodic switching gain of period* $\mathcal{P}$ *is* L_2*-stable if there exists a multiplier function* $Z(j\omega)$ *of the form (4.17), with* $z_m = -z'_m, m \in [1, \infty]$*, such that*

1) $\mathrm{Re}\,[\mathrm{Z(j\omega)}] \geq 0$*, and*

2) $\mathrm{Re}\,[\mathrm{Z(j\omega)G_1(j\omega)}] \geq \delta > 0,\; \omega \in (-\infty, \infty)$ *for some positive constant* δ*.*

Theorem 4.2. *The nonlinear system (4.4) with* $\varphi(\cdot) \in \mathcal{M}$ *, and a periodic switching gain of period* $\mathcal{P}$ *is* L_2*-stable if there exists a multiplier function* $Z(j\omega)$ *of the form (4.17) with* $z_m < 0,$ and $z'_m < 0, m \in [1, \infty]$ *such that*

1) $\mathrm{Re}\,[\mathrm{Z(j\omega)}] \geq 0,$ *with* $\sum_{m=1}^{\infty}(|z_m| + |z'_m|) < 1,$ *and*

2) $\mathrm{Re}\,[\mathrm{Z(j\omega)G_1(j\omega)}] \geq \delta > 0,\; \omega \in (-\infty, \infty)$ *for some positive constant* δ*.*

Corollary 4.1. *Theorem 4.2 is valid for the nonlinear system (4.4), with* $\varphi(\cdot) \in \mathcal{M}_o$*, and a periodic switching gain of period* $\mathcal{P}$*, with the removal of the constraint of negativity on* z_m *and* z'_m *for all* m*.*

Corollary 4.2. *Theorem 4.2 is valid for the nonlinear system (4.4) with* $\varphi(\cdot) \in \mathcal{E}$ *, and a periodic switching gain of period* $\mathcal{P}$*, if we set* $z_m = -z'_m$ *and* $\sum_{m=1}^{\infty} |z_m| < \frac{1}{2\nu}$*, where* $\nu \in [0, 1]$ *is given by (4.10).*

4.2.2 Stability Conditions for Systems with Relaxed Monotonic Nonlinear Functions

For the new class of relaxed monotone functions $\mathcal{M}_q$ defined by (4.12), we can state the new stability conditions similar to the above in terms of parameters of the quadratic form (4.11) and the constants related to it and $\varphi(\cdot)$, as defined in (4.15).

Theorem 4.3. *The nonlinear system (4.4) with* $\varphi(\cdot) \in \mathcal{M}_q$ *, and a periodic switching gain of period* $\mathcal{P}$ *is* L_2*-stable if there exists a multiplier function* $Z(j\omega)$ *of the form (4.17) with* $z_m < 0$, and $z'_m < 0$, for all $m \in [1, \infty]$ *such that*

1) Re $[\mathrm{Z(j\omega)}] \geq 0$, *with* $\sum_{m=1}^{\infty}(|z_m| + |z'_m|) < 1$ *, and*

2) Re $[\mathrm{Z(j\omega)G_1(j\omega)}] \geq \delta > 0$, $\omega \in (-\infty, \infty)$ *for some positive constant* δ*.*

Corollary 4.3. *Theorem 4.3 is valid for the nonlinear system (4.4) with an odd* $\varphi(\cdot) \in \mathcal{M}_q$ *after removing the negativity restriction on* z_m *and* z'_m*.*

For the class $\mathcal{M}_b$ of monotone functions, Theorem 4.3 and Corollary 4.3 need to be modified only with respect to the second part of condition (1) in Theorem 4.3. That is, replace inequality $\sum_{m=1}^{\infty}(|z_m|+|z'_m|) < 1$ by $\sum_{m=1}^{\infty}(|z_m|+|z'_m|) < (\frac{1}{1+2\nu'_s})$ Then, Theorem 4.3 and Corollary 4.3 become the counterparts of Theorems 4.2 and 4.3 in [64] for $\varphi(\cdot) \in \mathcal{M}_b$.

Theorems 4.2 and 4.3 can be generalized to include slope-restricted nonlinearities by a simple transformation of the $G(j\omega)$ of the function, in the manner of Zames and Falb [68].

4.2.3 Proofs of the Theorems

The proofs of the L_2-stability theorems for the linear system (4.3) and the nonlinear system (4.4) depend on the application of the Parseval theorem (in the theory of Fourier transforms) and establishing positivity conditions for two blocks in cascade. In the case of the linear system (4.3), one block is linear having $Z(j\omega)$, given by (4.17), as its transfer function, and the other is the linear switching gain $k(t)$. In the case of the nonlinear system of (4.4), the first block is linear with the transfer function $Z(j\omega)$ but the second is the linear switching gain $k(t)$ along with the nonlinearity $\varphi(\sigma) \in \mathcal{M},\ \mathcal{M}_o,\ \mathcal{E},\ \mathcal{M}_q$, or $\mathcal{M}_{\mathrm{b}}$.

Parseval's Theorem. Suppose $f_1(\cdot)$ and $f_2(\cdot)$ are real-valued functions defined on $[0, \infty)$ and belong to the class of $L_1 \cap L_2$ functions, then

$$\int_0^{\infty} f_1(t) f_2(t) dt = (1/2\pi) \int_{-\infty}^{\infty} F_1(j\omega) F_2(-j\omega) d\omega.$$

where F_1 and F_2 are Fourier transforms of $f_1(t)$ and $f_2(t)$, respectively.

Lemmas 4.1-4.4 and their corollaries below are concerned with the non-negativity of one of the following integrals in which the operator $\mathcal{Z}$ is the same as the one defined in (4.16).

$$\lambda_1(T) \stackrel{\text{def}}{=} \int_0^{T} \mathcal{Z}\{\sigma_T(t)\} k_1(t) \sigma_T(t) dt \quad \text{or} \quad \lambda_2(\mathrm{T}) \stackrel{\text{def}}{=} \int_0^{\mathrm{T}} \mathcal{Z}\{\sigma_{\mathrm{T}}(\mathrm{t})\} \mathrm{k}_1(\mathrm{t}) \varphi(\sigma_{\mathrm{T}}(\mathrm{t})) \mathrm{dt}. \tag{4.18}$$

Lemma 4.1. *If the operator $\mathcal{Z}$ is constrained by $z_m = -z'_m$ for all $m \in [1, \infty]$, then $\lambda_1(T)$ of (4.18) is non-negative for all σ_T in the domain of $\mathcal{Z}$ and for all $T \geq 0$.*

Lemma 4.2. *With the nonlinearity $\varphi(\cdot) \in \mathcal{M}$, if the operator $\mathcal{Z}$ is constrained by (i) $z_m < 0$ and $z'_m < 0$ for all $m \in [1, \infty]$ and $\sum_{m=1}^{\infty}(|z_m| + |z'_m|) < 1$, then $\lambda_2(T)$ of (4.18) is non-negative for all σ_T in the domain of $\mathcal{Z}$ and for all $T \geq 0$.*

Corollary 4.4. *Lemma 4.2 is valid for an nonlinearity $\varphi(\cdot) \in \mathcal{M}_o$, with the negativity restriction on z_m and z'_m removed.*

Lemma 4.3. *With the nonlinearity $\varphi(\cdot) \in \mathcal{E}$, if the operator $\mathcal{Z}$ is constrained by $z_m = -z'_m$ for all $m \in [1, \infty]$, and $\sum_{m=1}^{\infty} \mid z_m \mid < \frac{1}{2\,\nu}$, where ν is given by (4.10), then $\lambda_2(T)$ of (4.18) is non-negative for all σ_T in the domain of $\mathcal{Z}$ and for all $T \geq 0$.*

Lemma 4.4. *With the nonlinearity $\varphi(\cdot) \in \mathcal{M}_q$, if the operator $\mathcal{Z}$ is constrained by $z_m < 0$ and $z'_m < 0$ for all $m \in [1, \infty]$ and $\sum_{m=1}^{\infty}(|z_m| + |z'_m|) < 1$, then $\lambda_2(T)$ of (4.18) is non-negative for all σ_T in the domain of $\mathcal{Z}$ and for all $T \geq 0$.*

Corollary 4.5. *Lemma 4.4 is valid for an odd nonlinearity $\varphi(\cdot) \in \mathcal{M}_q$, with the restriction of negativity on z_m and z'_m removed.*

Corollary 4.6. *Lemma 4.4 is valid for a nonlinearity* $\varphi(\cdot) \in \mathcal{M}_b$*, if the right hand side of the inequality* $\sum_{m=1}^{\infty}(|z_m| + |z'_m|) < 1$ *is replaced by* $\frac{1}{(1+2\nu'_s)}$.

For proofs of Lemmas 4.1, 4.2 and 4.4, and of Corollary 4.4, see Appendix C.1 and C.2. The proofs of the rest of the corollaries is similar to the proof of Corollary 4.4. The proof of Lemma 4.3 is based on (4.8) in Appendix C.1, see also [70].

The proof of Theorem 4.1 given below is based on Lemma 4.1, and is on the lines of the proof strategy developed in [74].

Proof of Theorem 4.1: Consider the integral, for any $T > 0$,

$$\rho(T) = \int_0^T f_T(t)\mathcal{Z}\{\mathcal{G}_1\{v_T(t)\}\}dt \tag{4.19}$$

where $\mathcal{G}_1\{v_T(t)\} = \int_0^t g_1(\tau)v_T(t-\tau)d\tau$. It follows from $f_T(t) = v_T(t) + k_1(t)\sigma_T(t)$ in (4.3) that

$$\rho(T) = \int_0^T v_T(t)\mathcal{Z}\{\mathcal{G}_1\{v_T(t)\}\}dt + \int_0^T k_1(t)\sigma_T(t)\mathcal{Z}\{\sigma_T(t)\}dt \tag{4.20}$$

Let $V_T(j\omega)$ denote the Fourier transform of $v_T(t)$. Applying the Parseval theorem to the first integral on the right hand side of (4.20), we have

$$\int_0^T v_T(t)\mathcal{Z}\{\mathcal{G}_1\{v_T(t)\}\}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_T(-j\omega)Z(j\omega)G_1(j\omega)V_T(j\omega)d\omega. \tag{4.21}$$

Invoking the condition (2) in Theorem 4.1, Re $[\mathrm{Z(j\omega)G_1(j\omega)}] \geq \delta > 0$ for some $\delta > 0$, the following inequality holds

$$\int_0^T v_T(t)\mathcal{Z}\{\mathcal{G}_1\{v_T(t)\}\}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(j\omega)G_1(j\omega) \mid V_T(j\omega) \mid^2 d\omega \geq \frac{\delta}{2\pi}\|v_T\|^2. \tag{4.22}$$

The second integral on the right hand side of (4.20) is non-negative by virtue of Lemma 4.1. By applying the Parseval theorem to (4.19), and combining the result with (4.21), we get

$$\delta\|v_T\|^2 \leq 2\pi\int_0^T f_T(t)\mathcal{Z}\{\mathcal{G}_1\{v_T(t)\}\}dt = \int_{-\infty}^{\infty} F_T(-j\omega)Z(j\omega)G_1(j\omega)V_T(j\omega)\, d\omega. \tag{4.23}$$

Using Cauchy-Schwarz inequality in (4.23), we get

$$\int_{-\infty}^{\infty} F_T(-j\omega)Z(j\omega)G_1(j\omega)V_T(j\omega)\; d\omega \le \sup_{-\infty<\omega<\infty} |Z(j\omega)G_1(j\omega)|\; \|f_T\|\|v_T\|. \tag{4.24}$$

Note that $\sup_{-\infty<\omega<\infty} |Z(j\omega)G_1(j\omega)|$ is finite by virtue of the assumptions on $Z(\cdot)$ and $G_1(\cdot)$. Let $C = \sup_{-\infty<\omega<\infty} \mid Z(j\omega)G_1(j\omega) \mid$. Then, from (4.23) and (4.24), we get the inequality $\delta\|v_T\| \le C\|f_T\|$ which is valid for all $T > 0$. The theorem is proven.

The proofs Theorems 4.2 and 4.3, and their corollaries, are similar to the proof of Theorem 4.1 except that we now invoke Lemmas 4.2, 4.3 and 4.4, and their corollaries.

4.2.4 Synthesis of a Multiplier Function

For the linear system (4.3), we choose the value of $\overline{K}$ to be greater than the limit obtained from the circle criterion. On the other hand, for the nonlinear system (4.4), this $\overline{K}$ is replaced by $\overline{K^*} = \overline{K}q_2$. Plot the graph of arg $G_1(j\omega)$ and define the following functions

$$\begin{aligned}
\phi_1(\omega) &= \begin{cases} -\frac{\pi}{2} - \arg\{\mathrm{G_1(j\omega)}\} & \text{if } \arg\{\mathrm{G_1(j\omega)}\} < 0 \\ -\frac{\pi}{2} & \text{if } \arg\{\mathrm{G_1(j\omega)}\} \ge 0 \end{cases}, \\
\phi_2(\omega) &= \begin{cases} \frac{\pi}{2} & \text{if } \arg\{\mathrm{G_1(j\omega)}\} < 0 \\ \frac{\pi}{2} - \arg\{\mathrm{G_1(j\omega)}\} & \text{if } \arg\{\mathrm{G_1(j\omega)}\} \ge 0 \end{cases}.
\end{aligned} \tag{4.25}$$

We are now looking for a multiplier function $Z(j\omega) = 1 + j \tan\phi(\omega)$ with the phase angle $\phi(\omega)$ that lies within the band, $(\phi_1(\omega), \phi_2(\omega))$, and is periodic with a fundamental period Ω. By expanding $\tan\phi(\omega)$, which is an odd function of ω, in Fourier series, we get the following representation for the multiplier function

$$Z(j\omega) = 1 + j \sum_{n=-\infty, n\neq 0}^{\infty} z_n e^{jn\frac{2\pi}{\Omega}\omega}, \tag{4.26}$$

where

$$z_n = \frac{1}{\Omega}\int_0^{\Omega} \tan\phi(\omega)\; e^{-jn\frac{2\pi}{\Omega}\omega}\; d\omega.$$

The corresponding multiplier operator $\mathcal{Z}$ is given by

$$\mathcal{Z}\{\sigma(t)\} = \sigma(t) + \; j \sum_{n=-\infty, n\neq 0}^{\infty} \; z_n \; \sigma(t + n\frac{2\pi}{\Omega}), \tag{4.27}$$

where the coefficient z_n is given by (4.26). For the linear system or for the nonlinear system with odd-monotone (or power-law) nonlinearity, comparing (4.27) with (4.16) constrained by $z_m = -z'_m$ for all $m \in [1, \infty]$, we have $\frac{2\pi}{\Omega} = \mathcal{P}$. It indicates that the period Ω of the phase angle $\phi(j\omega)$ is the switching frequency of the gain $k(t)$ for the linear system or for the nonlinear system with odd-monotone (or power-law) nonlinearity.

Remark 4.1. If there exists a periodic frequency function $\phi(\omega)$ in the band $(\phi_1(\omega), \phi_2(\omega))$, defined by (4.25), and $Z(j\omega) = 1 + j \tan \phi(\omega)$ has the Fourier series representation given by (4.26), then the theorems and their corollaries can be cast (at least partly) in terms of the graph of $\arg\{G_1(j\omega)\}$.

4.2.5 Examples

For illustrating the application of Theorems 4.1-4.3, we present a few examples in which the linear forward block is governed by differential equations starting from second order to fifth. For all the examples, the multiplier function has, for simplicity, only one term corresponding to $m = 1$.

Example 4.1. The linear block has the transfer function, $G(s) = \frac{1}{(s^2+s+2)}$. The corresponding *gain-scaled* transfer function, $G_1(s) = G(s) + (1/\overline{K})$. Pyatnitskiy and Skorodinskiy [75] employ a common quadratic Lyapunov function to derive the sufficient condition $\overline{K} < 3.82$ for stability. Zelentsovsky [76] improves this bound, by employing a nonlinear transformation, to $\overline{K} < 5.47$. Xie *et al.* [77] use a piecewise quadratic Lyapunov function to derive the sufficient stability condition $\overline{K} < 5.9$. On the other hand, Margaliot and Langholz [31] apply trajectory optimization to arrive at the necessary and sufficient condition $\overline{K} < 6.89513$ for stability.

By way of applying the present results, with a multiplier function chosen as $Z(j\omega) = 1 - j2 \sum_{m=1}^{\infty} z_m \sin(m\mathcal{P}\omega)$, the conditions to be verified for L_2-stability of the system are 1) Re $[\mathrm{Z(j}\omega)] \geq 0$, and 2) Re $[\mathrm{Z(j}\omega)\mathrm{G_1(j}\omega)] > \epsilon > 0, \omega \in (-\infty, \infty)$. The former condition is satisfied by the chosen multiplier function. As

Table 4.1: Computational results for the second-order system of Example 4.1.

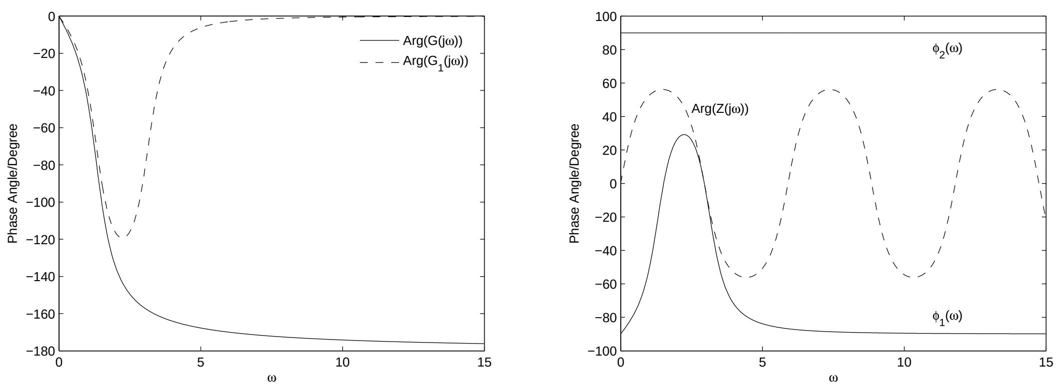
$No.$	$\overline{K}$	$\mathcal{P}_{max}$	z_1	Class of φ	$\mathcal{P}_b$	$\mathcal{P}_b/\mathcal{P}_{max}-1$
1	8	1.0650	−0.746	$\mathcal{E}$	1.2567	18%
2	10	0.953	−1.499	$\mathcal{E}$	1.0864	14%
3	12	0.8759	−0.898	$\mathcal{E}$	0.9634	10%
4	15	0.7886	−1.095	$\mathcal{E}$	0.8517	8%
5	20	0.6872	−1.327	$\mathcal{E}$	0.7284	6%
6	40	0.4909	−1.936	$\mathcal{E}$	0.5105	4%

far as the latter condition is concerned, it can be shown, by standard calculation, to be equivalent to the verification of the following inequality

$$\omega^4 - (\overline{K}+3)\omega^2 + (2\overline{K}+4) - 2\overline{K}\,\omega \left\{ \sum_m z_m \sin(m\mathcal{P}\omega) \right\} > 0,\ \omega \in (-\infty, \infty). \tag{4.28}$$

If we are given any $\overline{K} > 3.828$ (which is the value obtained from the circle criterion), we need to compute the values of $\mathcal{P}$ and z_m, if any, for which (4.28) is satisfied. Or, we can treat the solution to (4.28) as one involving all the parameters, $\overline{K}, \mathcal{P}$, and z_m, and arrive at desired tradeoffs to obey (4.28). In fact, we can cast the L_2-stability problem as an optimization problem[1] for both linear and nonlinear system stability: a) Linear System Stability: Find the maximum value of $\overline{K}$ such that inequality (4.28) is satisfied, for some values of z_m, subject to the constraint that $\mathcal{P}_{min} < \mathcal{P} < \mathcal{P}_{max}$, where $\mathcal{P}_{min}$ and $\mathcal{P}_{max}$ are pre-specified limits for $\mathcal{P}$. b) Nonlinear System Stability: Find the maximum value of $\overline{K}$ such that inequality (4.28) is satisfied subject to the constraints (i) $\sum_m \mid z_m \mid < \frac{1}{2}$, and (ii) $\mathcal{P}_{min} < \mathcal{P} < \mathcal{P}_{max}$, where $\mathcal{P}_{min}$ and $\mathcal{P}_{max}$ are as defined above in item 1. For the case of $\overline{K} = 8$, see Fig. 4.2(a) for the phase plot $G(j\omega)$ and $G_1(j\omega)$, and Fig. 4.2(b) for the multiplier phase angle plot. For computational results of the trade-off between $\overline{K}$ and $\mathcal{P}$, see Table 4.1, where $\mathcal{P}_b$ is the necessary and sufficient boundary of the switching period of $k(t)$ as obtained by simulation.

[1]A similar formulation is applicable to other examples below.



(a) (b)

Figure 4.2: (a) Phase angle plots of $G(j\omega)$ and $G_1(j\omega)$ for $\overline{K} = 8$ (b) a multiplier function of Example 4.1 for $\overline{K} = 8$

Remark 4.2. With reference to Fig. 4.2(b), it is always possible to find other multiplier functions with $\mathcal{P} < \mathcal{P}_{max} = 1.065$ which satisfy Theorem 4.1. It implies that the linear feedback system of Example 4.1 is stable for any $k(t) \in [0, \overline{K})$ if $k(t)$ switches fast enough (small $\mathcal{P}$), which is also evident from Fig. 4.3 and Table 4.1. The simulation shows that when $\mathcal{P} < \mathcal{P}_b$, the system is stable for all $k(t) \in [0, \overline{K}), k(t + \mathcal{P}) = k(t)$. On the other hand, when $\mathcal{P} > \mathcal{P}_b$, there always exists a switching feedback gain $k(t) \in [0, \overline{K}), k(t+\mathcal{P}) = k(t)$ to make the system unstable. A similar observation is applicable to the other examples below.

Remark 4.3. The above-observed phenomenon that fast switching leads to stability appears to be counter-intuitive in view of the well known standard result that sufficiently slow switching *can preserve* the stability of the original time-invariant system. A plausible interpretation of the above phenomenon is that the behavior of a periodically switched system is close to that of its average model if the switching is fast enough. This is also supported by the general results of [78], as well as by applications related to the modeling and control of switched mode power supplies.[1] On the other hand and in contrast, the mathematical fact that fast switching leads to stability is only valid for the switching feedback system governed by (4.1), for which the stability of all possible average models (convex

[1]where a system with high frequency parametric perturbations is modeled and controlled as its average model.

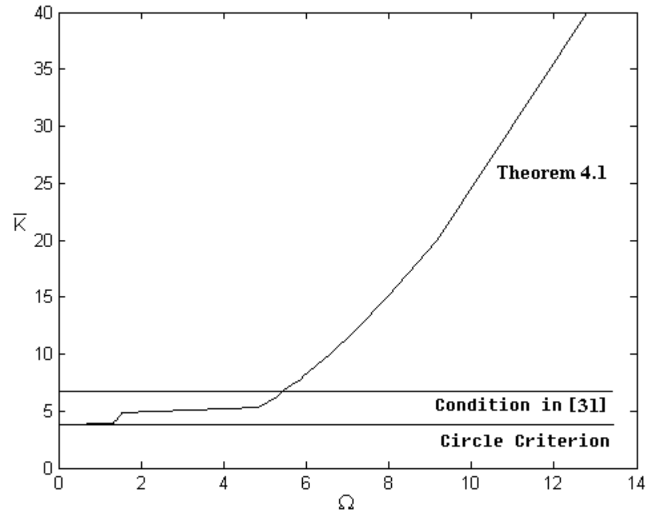
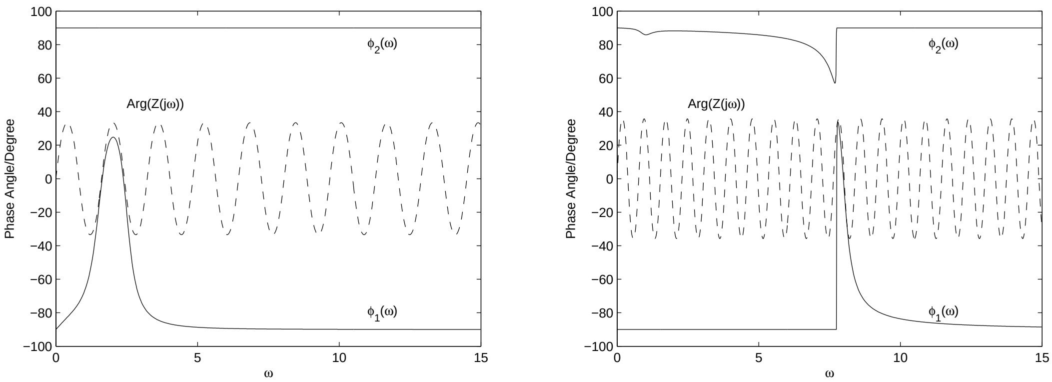


Figure 4.3: Stability Regions of Example 4.1 with respect to $\overline{K}$ and switching frequency Ω

combination) can be guaranteed. It is not generally true for switched systems with a possible unstable average system. In this case, there exists unstable conic sectors (in the case of planar systems, for example), where a divergent trajectory can be constructed by switching back and forth inside those sectors [79].

Remark 4.4. In Tables 4.1 and 4.2, $\mathcal{P}_b/\mathcal{P}_{max} - 1$ (shown in percentage) is a parameter which indicates how close the sufficient-only boundary $\mathcal{P}_{max}$ obtained by Theorem 4.1, is to the actual necessary and sufficient boundary $\mathcal{P}_b$. The simulation result shows that $\mathcal{P}_{max}$ and $\mathcal{P}_b$ are quite close for the general case of $\overline{K}$. It also shows that the difference between $\mathcal{P}_{max}$ and $\mathcal{P}_b$ decreases as $\overline{K}$ increases (when $\overline{K} > 6.982$).

Remark 4.5. When $\overline{K}$ approaches the necessary and sufficient boundary of the switching gain K^* for absolute stability (6.982 for Example 4.1 by Margaliot [31]), the difference between $\mathcal{P}_{max}$ and $\mathcal{P}_b$ will go to infinity. It is because when $\overline{K}$ is close to K^*, $\mathcal{P}_b$ goes to infinity, while $\mathcal{P}_{max}$ obtained from our phase plot does not change dramatically. Nevertheless, as long as there is a reasonable gap between $\overline{K}$ and K^*, the boundary obtained by Theorem 4.1 is considered to be satisfactory. In Example 4.1, when $\overline{K} = 8$, which is quite close to $K^* = 6.982$, the difference is only 18%. In Example 4.2 when $\overline{K} = 5$ (which is not far away



(a) (b)

Figure 4.4: (a) Multiplier phase angle plot for Example 4.2 for $\overline{K} = 3.82$; and (b) Multiplier phase angle plot for Example 4.3 for $\overline{K} = 10$

from $K^* = 3.82$), the difference is only 12.6%.

Remark 4.6. Fig. 4.3 shows that the stability boundary on $\overline{K}$ from Theorem 4.1 is less conservative than the one given by the circle criterion for all Ω because the latter is a special case when the multiplier function is chosen to be merely unity. Fig. 4.3 also shows that the stability bound on $\overline{K}$ from Theorem 4.1 is a tradeoff between $\overline{K}$ and Ω, which is more conservative at low frequencies when compared with the necessary and sufficient condition in [31] but less conservative at high frequencies. More significantly, Theorem 4.1 is applicable to higher-order systems also.

Remark 4.7. It is possible to obtain better (larger) values of $\mathcal{P}$ than the ones listed in Table 4.1 by choosing $m > 1$, especially for a large $\overline{K}$. It is found that when $\overline{K}$ is large, the peak value of $\phi_1(\omega)$ is close to $\phi_2(\omega)$. In this case, a multiplier with $m = 1$ may not give an optimal $\mathcal{P}$, whereas a multiplier with $m > 1$ may.

Example 4.2: In [33], the authors employ an optimization technique to arrive at stability boundaries essentially for a linear third-order system with the linear time-invariant forward block given by $G(s) = \frac{(s+1)}{(s^3+1.5s^2+3s+2)}$. For lack of space, we summarize the L_2-stability conditions for both linear and nonlinear periodic coefficient system in Table 4.2.

Example 4.3. For the fifth-order system with the transfer function

Table 4.2: Computational results for the third-order system of Example 4.2.

$No.$	$\overline{K}$	$\mathcal{P}_{max}$	z_1	Class of φ	$\mathcal{P}_b$	$\mathcal{P}_b/\mathcal{P}_{max} - 1$
1	2	∞	0	Linear and Nonlinear	∞	∞
2	3.82	3.959	-0.33	$\mathcal{M}_o$	∞	∞
3	4	3.89	-0.34	$\mathcal{M}_o$	*Large*	*Large*
4	5	1.191	-1.17	$\mathcal{E}$	1.34	12.6%
5	7	1.045	-1.54	$\mathcal{E}$	1.097	5%
6	9	0.943	-1.70	$\mathcal{E}$	0.9807	4%
7	10	0.903	-1.87	$\mathcal{E}$	0.9391	4%

Table 4.3: Computational results for the fifth-order system of Example 4.3.

$No.$	$\overline{K}$	$\mathcal{P}_{max}$	z_1	Class of φ
1	2	72.97	-0.19	$\mathcal{M}_o$
2	5	19.54	-0.31	$\mathcal{M}_o$
3	10	8.23	-0.36	$\mathcal{M}_o$
4	15	5.01	-0.38	$\mathcal{M}_o$
5	20	3.41	-0.40	$\mathcal{M}_o$
6	25	2.61	-0.41	$\mathcal{M}_o$

$\frac{s(s^2+3s+1)}{(s+14)(s^2+0.5s+1)(s^2+0.01s+60)}$, there seems no comparable result in the literature for the (Lyapunov- or) L_2-stability of the corresponding periodic coefficient monotonic nonlinear feedback system. See Table 4.3 for the stability bounds on $\overline{K}$ and $\mathcal{P}$.

4.3 Dwell-Time and L_2-Stability

It is known that in general, a switched system is stable if the switching between stable subsystems takes place sufficiently slowly[1] [53]. The problem analyzed in this section can be stated as follows. Suppose $k(t)$ is periodic with $\mathcal{P}$, what constraints on the rate of variation of $k(t)$ and on the switching discontinuities in an interval of duration $\mathcal{P}$ are to be satisfied for stability? On the other hand, is it possible to improve the stability regions with respect to $\overline{K}$ and $\mathcal{P}$ derived from Theorem 4.1-4.3 by imposing constraints on $k(t)$ in each period $\mathcal{P}$? When $k(t)$ is not periodic, the same constraints are to be satisfied for the whole (infinite) interval. We consider first the case of the periodic switching gain $k(t)$, and invoke the stability theorem of [74] for which we need the following notation and definitions

Let $\mathcal{C}$ be the class of absolutely continuous [80] functions $k(\cdot)$ on $[0, \infty)$ with $0 < \underline{k} \ \le \ k(t) \ \le \overline{k}, \ t \ \ge \ 0$. An implication is that $k(\cdot)$ is a continuous function of bounded variation whose derivatives are infinite only over a denumerable point set. (This class includes piecewise continuous functions also.)

We now enlarge the class of multiplier functions by considering class of operators satisfying an equation of the type

$$\mathcal{Z}'\{\sigma(t)\} = \sigma(t) + \sum_{m=1}^{\infty} z_m \sigma(t - \tau_m) + \sum_{m=1}^{\infty} z'_m \sigma(t + \tau'_m) + \int_{-\infty}^{\infty} z(\tau)\, \sigma(t - \tau)\, d\tau, \tag{4.29}$$

where the sequences $\{z_m\}$ and $\{z'_m\} \in \ell_1$, *i.e.*, $\sum_{m=1}^{\infty} (|z_m| + |z'_m| < \infty$, $\tau_m, \tau'_m \in (0, \infty)$, and $z(\cdot)$ is a real-valued function on $(-\infty, \infty)$ and is in $L_1(-\infty, \infty)$. Its Fourier transform is given by

$$Z'(j\omega) \ = \ 1 + \sum_{m=1}^{\infty} (z_m\, e^{+j\omega\tau_m} \ + \ z'_m\, e^{-j\omega\tau'_m}) \ + \ \int_{-\infty}^{\infty} z(\tau)\, e^{-j\omega\tau} d\tau. \tag{4.30}$$

Assuming that the switching gain, $k(t)$, is made up of the continuous part, $k_0(t)$, and first-order (jump-) discontinuities at instants $t_m, \ m = 1, 2, \cdots$, with instants t_{m+} corresponding to positive jumps α_m^+, and instants t_{m-} corresponding

[1] A large switching period $\mathcal{P}$ does not necessarily mean slow switching. It is possible that a feedback gain $k(t)$ switches frequently in its large period.

to negative jumps α_m^-. The derivative of $k(t)$ is given by

$$\frac{dk}{dt} = \frac{dk_0}{dt} + \sum_m (\alpha_m^+ \delta(t - \tau_{m+}) + \alpha_m^- \delta(t - \tau_{m-})).$$

Denote $\vartheta(t) = \frac{dk}{dt}/k(t)$, and $\vartheta_0(t) = \frac{dk_0}{dt}/k_0(t)$. At the positive discontinuities t_{m+} of $k(t)$, the value of $k(t)$ is, by convention, taken as $k(t_{m+}^-)$, where t_{m+}^- is the instant just to the left of t_{m+}. Similarly, at the negative discontinuities t_{m-} of $k(t)$, the value of $k(t)$ is taken as $k(t_{m-}^-)$, where t_{m-}^- is the instant just to the left of t_{m-}. Note that based on the assumptions on $k(t)$, $k(t_{m-}^-) \neq 0$, and $k(t_{m+}^-) \neq 0,\ t \geq 0$. Further, let $\vartheta^+(t) = \vartheta(t)$, for $\vartheta(t) > 0$, $\vartheta^+(t) = 0$, for $\vartheta(t) \leq 0$; and $\vartheta^-(t) = \vartheta(t)$, for $\vartheta(t) < 0$, $\vartheta^-(t) = 0$, for $\vartheta(t) \geq 0$. We have $\vartheta(t) = \vartheta^+(t) + \vartheta^-(t)$, where

$$\vartheta^+(t) = \vartheta_0^+(t) + \textstyle\sum_m \frac{\alpha_m^+}{k(t_{m+}^-)} \delta(t - \tau_{m+}),\ \ \vartheta^-(t) = \vartheta_0^-(t) + \textstyle\sum_m \frac{\alpha_m^-}{k(t_{m-}^-)} \delta(t - \tau_{m-}). \tag{4.31}$$

Theorems 4.4 and 4.5 are generalizations of Theorems 4.1 and 4.2, and can be obtained as corollaries of Theorem 1 of [74]. Note that in these generalizations, $\vartheta(t)$ is not restricted because 1) the switching feedback gain, $k(t)$, is assumed to be periodic with period $\mathcal{P}$ and 2) there is a symmetric time-domain condition on the impulse response of the multiplier function $Z'(j\omega)$.

Theorem 4.4. *The linear feedback system described by the pair (4.3), with a periodic switching gain of period $\mathcal{P}$, is L_2-stable, if there exists a frequency function $Z'(j\omega)$ as defined in (4.30), with $z_m = -z'_m$, $\tau_m = \tau'_m$, and $\tau_m = m\mathcal{P}$, $m = 1, 2, \cdots$, such that*

(a) for some positive constant ξ,

$$2\sum_m^{\infty} \mid z_m \mid e^{\xi \tau_m} + \int_0^{\infty} \mid z(\tau) \mid\ e^{\xi\tau} d\tau + \int_{-\infty}^{0} \mid z(\tau) \mid\ e^{-\xi\tau} d\tau < \infty, \text{ and} \tag{4.32}$$

(b) Re $[Z'(j\omega)G_1(j\omega)] \geq \delta > 0,\ \omega \in (-\infty, \infty)$.

Theorem 4.5. *The nonlinear feedback system described by the pair (4.4) with $\varphi(\cdot) \in \mathcal{M}_o$, a periodic switching gain of period $\mathcal{P}$, and with ν_s* and *ν_i defined by (4.14), is L_2-stable, if there exists a frequency function $Z'(j\omega)$ as defined in (4.30), with $z_m = -z'_m$, $\tau_m = \tau'_m$, and $\tau_m = m\mathcal{P}$, $m = 1, 2, \cdots$, such that* *(a) for some positive constant* ξ,

$$2\sum_{m}^{\infty}|z_m|e^{\xi\tau_m}+\int_0^{\infty}|z(\tau)|e^{\xi\tau}d\tau+\int_{-\infty}^{0}|z(\tau)|e^{-\xi\tau}d\tau<\frac{1}{1+\nu_s-\nu_i},\ \textit{and} \tag{4.33}$$

(b) Re $[Z'(j\omega)G_1(j\omega)]\ \geq\ \delta>0,\ \omega\in(-\infty,\infty)$.

In case (4.33) is not satisfied, a more general result involves a constraint on $\vartheta(t)$ as made explicit in the following theorem:

Theorem 4.6. *The nonlinear feedback system described by the pair (4.4) with* $\varphi(\cdot)\in\mathcal{M}_o$, *a periodic switching gain of period* $\mathcal{P}$, *and with* ν_s and ν_i *defined by (4.14), is* L_2*-stable, if there exists a frequency function* $Z(j\omega)$ *as defined in (4.30), with* $z_m\leq 0$, *and* $z'_m\leq 0$, *for all* $m=1,2,\cdots$, *and* τ_m *and* $\tau'_m\in(0,\infty)$ *such that*

(a) for some positive constants ξ *and* ζ

$$\sum_{m}^{\infty}\{|z'_m|e^{\xi\tau'_m}+|z_m|e^{\zeta\tau_m}\}+\int_0^{\infty}|z(\tau)|e^{\xi\tau}d\tau+\int_{-\infty}^{0}|z(\tau)|e^{-\zeta\tau}d\tau<\frac{1}{(1+\nu_s-\nu_i)}, \tag{4.34}$$

(b) Re $[Z'(j\omega)G_1(j\omega)]\ \geq\ \delta>0,\ \omega\in(-\infty,\infty)$, and

(c)

$$\frac{1}{\mathcal{P}}\int_0^{\mathcal{P}}\vartheta^-(t)\,dt\ \geq-\zeta,\ \frac{1}{\mathcal{P}}\int_0^{\mathcal{P}}\vartheta^+(t)\,dt\ \leq\xi, \tag{4.35}$$

where $\vartheta^+(t)$, and $\vartheta^-(\mathrm{t})$ are given by (4.31).

When $k(t)$ is piecewise constant (with respect to time) except at the jump discontinuities, *i.e.*, $\vartheta_0(t)$ is identically zero, (4.35) simplifies to

$$\frac{1}{\mathcal{P}}\sum_{m}\frac{\alpha_m^-}{k(\tau_{m-}^-)}\ \geq-\zeta,\ \frac{1}{\mathcal{P}}\sum_{m}\frac{\alpha_m^+}{k(\tau_{m+}^-)}\ \leq\xi, \tag{4.36}$$

where the summation over the index m applies only to those instants τ_{m+}, and τ_{m-} in the semi-closed interval $(0,\mathcal{P}]$. (This is because a jump can theoretically take place on the boundary of the interval in one period.)

Remark 4.8. When the periodic $k(t)$ is piecewise constant, (4.36) shows that the system is stable as long as the average of the normalized negative and positive switching jumps in the gain at the switching instants, over a period, satisfies a lower and an upper bound, respectively. This is in stark contrast with the results of [81] for switched linear systems in which (1) a minimum dwell time is obtained (by a slight modification of the standard argument of negative definiteness) using a family of quadratic Lyapunov functions, and (2) a stability condition on chattering is expressed by invoking Lyapunov-Metzler inequalities.

It appears, therefore, that the stability results of the literature expressed in terms parameters like the dwell time, average dwell time [53], and chatter bound are conservative. In this context, it can be conjectured that stability in a periodically switched system is not governed by such parameters but by the average of the negative and positive normalized jumps in the gain that take place at the switching instants.

Example 4.4: We apply Theorems 4.4-4.6 to the system considered in Example 4.2. Using these theorems, we can improve the upper-bound of the period $\mathcal{P}$ for given $\overline{K}$ by imposing constraints on the variation of the feedback gain $k(t)$ in its period $\mathcal{P}$ for both linear and odd-monotone nonlinear feedback systems. For illustration, we choose $\overline{K} = 4$. In this case, $\mathcal{P} = 3.89$ for a multiplier defined in (4.17), with $z_m = -z'_m,\ m \in [1, \infty]$. Then with the multiplier function, $Z'(j\omega) = 1 - j2z_1 \sin(\mathcal{P}\omega) + \alpha/(\beta - j\omega)$, where $\beta > 0$, and the added function $\alpha/(\beta - j\omega)$ is defined as the Fourier transform of an impulse response function $\alpha e^{\beta t}$ for $t \leq 0$ and zero for $t > 0$. We need to find for the L_2-stability of the system of Example 4.2, the values, if any, of $z_1,\ \mathcal{P}, \alpha,$ and β such that for all $\omega \in (-\infty, \infty)$ the following inequality is satisfied

$$(\omega^6 - 7.75\omega^4 + 9\omega^2 + 12)\ (1 + \frac{\alpha\beta}{\omega^2 + \beta^2}) - 2z_1\, \omega\ (\omega^2 + 2)\ \sin(\mathcal{P}\omega) + \frac{\alpha\omega^2(\omega^2 + 2)}{\omega^2 + \beta^2} > 0. \tag{4.37}$$

One possible solution to (4.37) is: $\mathcal{P} = 100, z_1 = -0.3, \alpha = 3, \beta = 0.01$. With reference to (4.32), Theorem 4.4 is satisfied if there exists a positive $\xi < \beta = 0.01$, which implies that, for the linear system, $\mathcal{P}$ can be improved dramatically by imposing a constraint on ξ, which relates to the permitted variation of the switching $k(t)$ in a period. However, for the nonlinear system, $\mathcal{P}$ cannot be improved as much as for the linear system: when we invoke Theorem 4.6, another possible solution to (4.37) is found to be $\{\mathcal{P} = 4,\ z_1 = -0.24,\ \alpha = 0.77,\ \beta = 1.71\}$. The time-domain inequality (4.34) simplifies to

$$2 \mid z_1 \mid e^{\zeta \mathcal{P}} + \int_{-\infty}^{0} \mid z(\tau) \mid \ e^{-\zeta\tau} d\tau \ < \ \frac{1}{1+\nu_s-\nu_i}, \tag{4.38}$$

where $\zeta < \beta$, ν_s, and ν_i are defined by (4.14). For demonstrating the existence of a meaningful solution to the problem under consideration, we can choose a) $\zeta = 0.02$, and b) a sub-class of odd-monotone functions having values of ν_s and ν_i such that (4.38) is satisfied. Then, the nonlinear system is L_2-stable if with $\xi = \infty$ and $\zeta = 0.02$, either inequality (4.35) or (4.36) is satisfied, depending on the nature of the (time-varying) switching gain, $k(t)$, and $\varphi(\cdot)$ belongs to a subclass of $\mathcal{M}_o$ with $\nu_s - \nu_i = 0.03$.

4.4 Extension to MIMO Systems

In this section, we extend the frequency-domain stability result to the MIMO switched feedback systems. First of all, we consider the special, limiting cases of the nonlinear MIMO system of (4.5) - the linear constant matrix gain MIMO system:

$$\begin{array}{l} \underline{v}(t) = \underline{f}(t) - \mathcal{K}_0 \underline{\sigma}(t), \\ \underline{\sigma}(t) = \sum_{m=1}^{\infty} \mathfrak{S}_{1m} \mathtt{Diag}[\delta(t-\tau_m)\cdots\delta(t-\tau_m)] + \int_0^\infty \mathfrak{S}_1(\tau)\underline{v}(t-\tau)d\tau, \end{array} \tag{4.39}$$

and the linear periodic matrix-gain MIMO system,

$$\begin{array}{l} \underline{v}(t) = \underline{f}(t) - \mathcal{K}_1(t) \underline{\sigma}(t), \\ \underline{\sigma}(t) = \sum_{m=1}^{\infty} \mathfrak{S}_{1m} \mathtt{Diag}[\delta(t-\tau_m)\cdots\delta(t-\tau_m)] + \int_0^\infty \mathfrak{S}_1(\tau)\underline{v}(t-\tau)d\tau. \end{array} \tag{4.40}$$

As far as the vector nonlinearity of (4.5) is concerned, we make the following assumptions which together constitute a generalization of the assumption (of first-third quadrant function) on the scalar nonlinear gain of (4.4)[1]

$$0 \ \leq \ \underline{\varphi}'(\underline{\sigma})\, \mathcal{K}'(t)\, \underline{\sigma} \ \leq \ \underline{\sigma}'\, \mathcal{K}^{*'}(t)\, \underline{\sigma}, \tag{4.41}$$

[1]Compare with the different bound on the time-invariant vector nonlinearity in [82] which is a special case of the present one.

where the elements $k^*_{mn}(t)$ of $\mathcal{K}^*(t)$ are each bounded by $[0, \overline{k^*_{mn}})$. As its extension, we need another inequality to facilitate establishing a stability condition for (4.5). For an arbitrary, bounded constant matrix Y,

$$0 \;\leq\; \underline{\varphi}'(\underline{\sigma})\, \mathcal{K}'(t)\, Y\, \underline{\sigma} \;\leq\; \|Y\|\, \underline{\sigma}'\, \mathcal{K}^{*'}(t)\, \underline{\sigma}, \tag{4.42}$$

where $\|Y\|$ is the matrix norm[1] of Y.

To generalize the class $\mathcal{M}$ of single variable monotone functions, we define the basic class of monotone vector functions as $\underline{\varphi}(\underline{\sigma})$, for the vector variable $\underline{\sigma}$, which satisfy the following inequality

$$(\underline{\sigma_1} - \underline{\sigma_2})'\underline{\varphi}(\sigma_1) \;\geq\; \int_{\underline{\sigma_2}}^{\underline{\sigma_1}} \underline{\varphi}'(\underline{\sigma})\, d\underline{\sigma}. \tag{4.43}$$

A slight modification of this class (which is needed in proving stability theorems for the systems under consideration), has functions satisfying the following inequality

$$(\underline{\sigma_1} - \underline{\sigma_2})'\, \mathcal{K}\, \underline{\varphi(\sigma_1)} \;\geq\; \int_{\underline{\sigma_2}}^{\underline{\sigma_1}} (\mathcal{K}\, \underline{\varphi}(\underline{\sigma}))'\, d\underline{\sigma}. \tag{4.44}$$

If $\underline{\varphi}(-\underline{\sigma}) = -\underline{\varphi}(\underline{\sigma})$, then the vector nonlinearity is said to be odd. Further, a class of relaxed monotone vector functions can be defined by introducing the quadratic form (4.11) (which will now be in terms of vectors $\underline{\sigma}$ and $\underline{\varphi}(\underline{\sigma})$), on the right hand side of (4.43), as was done in (4.12). This can also be extended to (4.44).

With the upper bound matrix of $\mathcal{K}^*(t)$ defined by $\overline{\mathcal{K}^*}$, the modified (matrix) transfer function of the linear block is given by $\Gamma_1(j\omega) = I + \overline{\mathcal{K}^*}\; \Gamma(j\omega)$. The system described by (4.5) is said to be L_2-stable if $\underline{v} \in L_2[0, \infty)$ for $\underline{f} \in L_2[0, \infty)$, and an inequality of the type $\|\underline{v}\| \leq C\|\underline{f}\|$ holds where C is a constant. As in the scalar case, the problem is to find conditions for the L_2-stability of the feedback system (4.5).

In the literature, the Nyquist criterion for the linear, time-invariant, constant gain-matrix system of (4.39) does not seem to have been formulated in terms of "multiplier" matrix functions and the phase angle characteristics of the Fourier transform $\Gamma_1(j\omega)$ of $\mathfrak{G}_1(t)$. However, for a generalization of the standard encirclement-type of condition, see, for instance, [83, 84].

[1]The matrix norm of Y could be, for instance, $\max_{\sigma \in \mathcal{R}_n} \frac{\|Y\sigma\|}{\|\sigma\|}$.

For our present purposes, we need a matrix operator $\mathcal{Y}$ that has elements which belong to the class of $L_1 \cap L_2$. It turns out that it is possible to formulate the stability condition in terms of the positive definiteness of the real part of the matrix-product of $Y(j\omega)$ and $\Gamma_1(j\omega)$ not only for the linear constant coefficient system of (4.39) but also for (4.40) and (4.5), with additional constraints on $Y(j\omega)$. Note that the constraint on $Y(j\omega)$ is different to positive realness, which is typically the assumption in the literature on results linking Lyapunov-based stability conditions with the frequency domain matrix inequalities (called in the literature as Kalman-Yakubovic-Popov Lemma). See Remark 4.10.

Here we merely state matrix-multiplier conditions that guarantee the L_2-stability of (4.39) for all constant gain matrices $\mathcal{K}_0$ whose elements have values in $[0, \overline{k_{mn}})$. But it is found that the method of its proof does not guarantee necessity of this condition.[1] This result is then extended to cover (4.5). The multiplier (matrix) frequency function $Y(j\omega)$ used in these theorems takes specific forms depending on the system under consideration, and obeys certain constraints. For the linear, constant coefficient system (4.39), the form of the matrix multiplier operator, $\mathcal{Y}^{(l)}$, with its Fourier transform $Y^{(l)}(j\omega)$, is general but obeys some (mild) constraints. Whereas for the linear, periodic coefficient system (4.40), the multiplier operator $\mathcal{Y}^{(p)}$ has a specific form, and is defined by

$$\mathcal{Y}^{(p)}\{\underline{\sigma}(t)\} = \underline{\sigma}(t) + \sum_{m=1}^{\infty}\{Y_{m,1}^{(p)}\,\underline{\sigma}(t - m\mathcal{P}) + Y_{m,2}^{(p)}\,\underline{\sigma}(t + m\mathcal{P})\}, \tag{4.45}$$

where $\{Y_{m,1}^{(p)}\}, \{Y_{m,2}^{(p)}\},\ m = 1, 2, \cdots$, are sequences of constant matrices, $Y_{m,1}$ and $Y_{m,2}$, such that $\sum_{m=1}^{\infty}\{\mid Y_{m,1}^{(p)} \mid + \mid Y_{m,2}^{(p)} \mid\}$ (where $\mid \cdot \mid$ implies that each element of the pair $Y_{m,1}^{(p)}, Y_{m,2}^{(p)}$ is replaced by its magnitude) is a bounded matrix. Its Fourier transform is given by

$$Y^{(p)}(j\omega) = I + \sum_{m=1}^{\infty}\{e^{-jm\omega\mathcal{P}}Y_{m,1}^{(p)} + e^{+jm\omega\mathcal{P}}\,Y_{m,2}^{(p)}\}. \tag{4.46}$$

However, for the nonlinear system of (4.5), the Fourier transform of the multiplier operator $\mathcal{Y}^{(np)}$ has the same form as (4.45), but it has to obey additional constraints as indicated later in the corresponding stability theorem. Note that for a

[1] It would be interesting and valuable to study the relationship between this condition and those found in the literature on multi-variable systems [83, 84].

matrix B, an inequality of the type $B \geq 0$ means that B is positive semi-definite; and $B > 0$ means that B is positive definite. With the above preliminaries, we derive the stability conditions for MIMO systems with a periodic feedback gain as the following theorems.

Theorem 4.7. *The linear constant matrix-gain system described by (4.39) is L_2-stable if there exists a multiplier matrix-operator $\mathcal{Y}^{(l)}$ with the Fourier transform $Y^{(l)}(j\omega)$ such that*

1) $\mathrm{Re}\,[\mathrm{K}_0'\mathrm{Y}^{(l)}(\mathrm{j}\omega)] > 0, \omega \in (-\infty, \infty)$,

2) $\sup_{-\infty<\omega<\infty} \|Y^{(l)}(j\omega) \ \cdot \Gamma_1(j\omega)\| < \infty$, *and*

3) $\mathrm{Re}\,[\mathrm{Y}^{(l)}(\mathrm{j}\omega)\Gamma_1(\mathrm{j}\omega)] \ > \ 0, \ \omega \in (-\infty, \infty)$.

Theorem 4.8. *The linear periodic matrix-gain system described by (4.40) is L_2-stable if there exists a multiplier matrix-operator $\mathcal{Y}^{(p)}$ defined by (4.45), with $\{Y_{m,1}^{(p)}\} = -\{Y_{m,2}^{(p)}\}$, $m = 1, 2, \cdots$, such that*

1) $\mathcal{K}_1(t)$ *is positive definite for all* $t \in [0, \infty)$,

2) $\mathcal{K}_1'(t) Y_m^{(p)} = Y_m^{(p)\prime} \mathcal{K}_1(t), \ t \in [0, \infty), m = 1, 2, \cdots,$

3) $\sup_{-\infty<\omega<\infty} \|Y^{(p)}(j\omega)\Gamma_1(j\omega)\| < \infty$, *and*

4) $\mathrm{Re}\,[\mathrm{Y}^{(\mathrm{p})}(\mathrm{j}\omega)\Gamma_1(\mathrm{j}\omega)] \ > 0, \ \omega \in (-\infty, \infty)$.

Theorem 4.9. *The periodic matrix-gain system (4.5) with monotone vector nonlinearity is L_2-stable, if there exists a multiplier matrix-operator $\mathcal{Y}^{(p)}$, defined by (4.45) and with $Y_{m,1}^{(p)}, Y_{m,2}^{(p)}, m = 1, 2, \cdots$, negative definite, such that*

1) $\mathcal{K}(t) \ \underline{\varphi}(\underline{\sigma})$ *satisfies the assumptions (4.41), (4.42) and (4.44),*

2) $\sum_{m=1}^{\infty} \|Y_{m,1}^{p}\| + \|Y_{m,2}^{p}\| < 1$

3) $\sup_{-\infty<\omega<\infty} \|Y^{(p)}(j\omega)\Gamma_1(j\omega)\| < \infty$, *and*

4) $\mathrm{Re}\,[\mathrm{Y}^{(\mathrm{p})}(\mathrm{j}\omega)\Gamma_1(\mathrm{j}\omega)] \ > 0, \ \omega \in (-\infty, \infty)$.

For odd-monotone vector nonlinearity in the system (4.5), we need to remove the negative definiteness restriction on $Y_{m,1}^{(p)}, Y_{m,2}^{(p)}, \ m = 1, 2, \cdots$, and set $Y_{m,1}^{(p)} = -\, Y_{m,2}^{(p)}, \ m = 1, 2, \cdots$.

Remark 4.9. It is interesting to compare Theorem 4.7 with the result of Davis [85] (Theorem 2, pages 2-3) meant for the linear (continuous-time) constant coefficient MIMO system. In [85], *necessary and sufficient* (bounded-input bounded-output) stability conditions (obtained from the spectral theory of linear operators) are stated in terms of the Nyquist locus of $|\Gamma_1(s)|$ (including the constraint $|\Gamma_1(s)|_{s=j\omega} \neq 0$) or, equivalently, in terms of its zeros in Re [s] ≥ 0. But its proof in [85] cannot be extended to continuous-time linear and nonlinear periodic coefficient MIMO systems. In contrast, Theorem 4.7 (though sufficient for the L_2-stability of a linear, constant coefficient MIMO system), seems to be of interest in its own right. One of the reasons is that it is obtained from a general approach applicable to both linear and nonlinear periodic coefficient MIMO systems. The open question is whether Theorem 4.7 is also necessary for the L_2-stability of the linear constant coefficient MIMO system.

Remark 4.10. Note that the constraint on the multiplier matrix $Y^{(l)}(s)$ is different to the positive realness condition, which is frequently used in deriving Kalman-Yakubovic-Popov Lemma. A rational function $f(s)$ of the complex variable s is said to be positive real if 1) $f(s)$ real for s real, and 2) Re [f(s)] $\geq 0 \; \forall$ Re [s] ≥ 0. See [86] and the references quoted therein. In the condition 1) of Theorem 4.7, the real part of $Y^{(l)}(s)$ is required to be positive. However, $Y^{(l)}(s)$ need not be a positive real matrix (in the network-theoretical sense). With reference to (4.46), $Y^{(l)}(s)$ can be complex even when $s = 0$. Moreover, its elements need not be rational functions of the complex variable s.

Remark 4.11. The condition 2) of Theorem 4.8, which is needed in order to avoid imposition of a restriction on the rate of variation of the elements of $\mathcal{K}_1(t)$, is quite severe. In the special case of a diagonal $\mathcal{K}_1(t)$, the multiplier matrices Y_m can also be chosen to be diagonal to satisfy the condition 2). However, for general $\mathcal{K}_1(t)$, the choice of Y_m for satisfying the condition 2) seems to be impossible since an implication of the condition 2) is that the elements of $Y_r, r = 1, 2, \cdots$, are to be so chosen that $\sum_m \sum_n \alpha_{mn} k_{mn}(t) = 0$, where α_{mn} are linear functions of the elements of Y_r for each value of r, and $k_{mn}(t)$ are the elements of $\mathcal{K}_1(t)$.

Remark 4.12. The elements of $\mathcal{K}_1(t)$ can have different fundamental periods, in which case the value of $\mathcal{P}$, in the multiplier matrix-operator $\mathcal{Y}^{(p)}$ defined by

(4.45), is the least common multiple of the individual periods of the elements of $\mathcal{K}_1(t)$.

The proofs of Theorems 4.7-4.9 can be established on lines similar to those for the single variable case. Due to the lack of space, they are omitted here.

Example 4.5: Since it has turned out to be difficult to illustrate the application of Theorems 4.8 and 4.9 for a general $\mathcal{K}(t)$, we consider only the special case of a diagonal 2×2 matrix $\mathcal{K}(t)$. Let the forward time-invariant linear block transfer function be given by

$$\Gamma(s) \;=\; \begin{bmatrix} \frac{1}{(s^2+2s+0.01)} & \frac{(s+1)}{(s+20)} \\ \frac{1}{(s+100)} & \frac{(s+1)}{(s^3+1.5s^2+3s+2)} \end{bmatrix}, \tag{4.47}$$

and the switching matrix by

$$\mathcal{K}(t) \;=\; \begin{bmatrix} k_{11}(t) & 0 \\ 0 & k_{22}(t) \end{bmatrix}.$$

We first need to find the Routh-Hurwitz limits for a constant gain-matrix, $\overline{\mathcal{K}}$, *i.e.*, the values of $\overline{k_{11}}$ and $\overline{k_{22}}$ for which the zeros of $|I + \overline{\mathcal{K}}\;\Gamma(s)|$ lie strictly in the left-half (Re [s] $< -\delta \le 0$) of the complex plane, where the $\overline{\mathcal{K}}$ has the diagonal elements of $\mathcal{K}(t)$ replaced by constants, $\overline{k_{11}}$ and $\overline{k_{22}}$. This leads to finding $\overline{k_{11}}$ and $\overline{k_{22}}$ for which the following algebraic equation has zeros in the left-half of the complex plane

$$\begin{aligned} &s^5 + 3.5s^4 + (6.01 + \overline{k_{11}} + \overline{k_{22}})s^3 + (6.015 + 2.5\overline{k_{11}} + 3\overline{k_{22}})s^2 \\ &+(0.03 + 5\overline{k_{11}} + 2.01\overline{k_{22}} + \overline{k_{11}k_{22}})s + (0.01\overline{k_{11}} + 0.01\overline{k_{22}} + \overline{k_{11}}^2 + \overline{k_{11}k_{22}}) \;=\; 0. \end{aligned} \tag{4.48}$$

Pairs of two possible solutions of (4.48) are: $\overline{k_{11}} \;= 4,\; \overline{k_{22}} \;> \; -6.29$, and $\overline{k_{11}} \;= 8,\; \overline{k_{22}} \;> \; -47$. Further, with

$$\overline{\mathcal{K}_{ex1}} \;=\; \begin{bmatrix} 4 & 0 \\ 0 & 10 \end{bmatrix}, \tag{4.49}$$

it is found that

$$\text{Re}\;[\text{I} + \overline{\mathcal{K}_{\text{ex1}}}\;\Gamma(\text{j}\omega)] \;=\; \begin{bmatrix} \frac{(0.0401-0.02\omega^2+\omega^4)}{(0.0001+3.98\omega^2+\omega^4)} & \frac{(80+4\omega^2)}{(400+\omega^2)} \\ \frac{1000}{(10000+\omega^2)} & \frac{(24+18\omega^2-13.75\omega^4+\omega^6)}{(4+3\omega^2-3.75\omega^4+\omega^6)} \end{bmatrix}$$

is strictly positive definite for all $\omega \in (-\infty, \infty)$, *i.e.*, the symmetric part of the 2×2 matrix, Re $[\mathrm{I} + \overline{\mathcal{K}_{\mathrm{ex1}}}\Gamma(\mathrm{j}\omega)]$, is positive definite, by checking on the positivity of its first element, and the determinant of its symmetric part, for all $\omega \in (-\infty, \infty)$. The implication is that the switching feedback system is L_2-stable for any switching (periodic or otherwise) feedback linear gain-matrix bounded by (4.49). It is also L_2-stable for any switching (periodic or otherwise) feedback nonlinear gain-matrix satisfying the inequality $0 \le \underline{\varphi}'(\underline{\sigma})\ \mathcal{K}'(t)\ \underline{\sigma} \le \underline{\sigma}'\overline{\mathcal{K}_{ex1}}\underline{\sigma}$.

On the other hand, for a switching gain-matrix bounded by

$$\overline{\mathcal{K}_{ex2}} = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}, \tag{4.50}$$

Re $[\mathrm{I} + \overline{\mathcal{K}_{\mathrm{ex2}}}\ \Gamma(\mathrm{j}\omega)]$ is not positive definite for all $\omega \in (-\infty, \infty)$.

Therefore, we need to check the existence of a multiplier matrix-function of the form (4.46) such that conditions 2) - 4) of Theorem 4.8 for linear system, and Theorem 4.9 for nonlinear system L_2-stability are satisfied. To this end, we explore the following multiplier function: $Y_{ex2}^{(p)}(j\omega) = I - j\ 2\ \sin\ (\mathcal{P}\omega)\ Y_1^{(p)}$, where $Y_1^{(p)}$ is diagonal with unknown elements, y_{11} and y_{22}, which are to be so chosen that Re $[\mathrm{Y}_{\mathrm{ex2}}^{(\mathrm{p})}(\mathrm{j}\omega)\ \Gamma_1(\mathrm{j}\omega)]$ is strictly positive definite for all $\omega \in (-\infty, \infty)$. One solution has been found to be $\mathcal{P} = 1.2, y_{11} = -0.1$, and $y_{22} = -0.3$.

Therefore, the L_2-stability of the system considered in Example 4.5 is as follows:

1) The system with the linear time-invariant block described by (4.47) with a periodically switching gain matrix bounded by (4.50) of period $\mathcal{P} = 1.2$, is L_2-stable.

2) The system with a) the linear time-invariant block described by (4.47), b) odd-monotone nonlinear matrix gain associated with a periodically switching matrix bounded by (4.50) of period $\mathcal{P} = 1.2$, and c) with $\overline{\mathcal{K}*}$ replaced by $\overline{\mathcal{K}_{ex2}}$, the nonlinear matrix gain, $\mathcal{K}(t)\ \underline{\varphi}(\underline{\sigma})$, satisfying the assumptions (4.41) - (4.44) in addition to $\underline{\varphi}(\underline{\sigma})$ being odd, is also L_2-stable, since the norm of $Y_1^{(p)}$ is also bounded by $\frac{1}{2}$.

4.5 Discussion

As mentioned earlier , Altshuller [64] has also derived some interesting conditions for the absolute stability of SISO and MIMO systems with periodic linear and nonlinear feedback gain. However, there exist differences between the L_2-stability conditions presented here and the ones in [64], with respect to the assumptions, definitions, and techniques employed, as explained below.

1. According to [64], the magnitude of the impulse response of the linear block (or its norm in the matrix case) is exponentially bounded. Here we assume that the impulse response of the linear block is in $L_1 \cap L_2$, which is less restrictive.

2. The class $\mathcal{M}_q$ of nonlinear gains defined in (4.12) is distinct from the class $\mathcal{M}_b$ of nonlinear gains (4.13), as found in [71] and employed in [64]. The class $\mathcal{M}_b$ seems to lack symmetry properties. When $\sigma_1 = \sigma_2$, the left hand side of (4.13) is zero and the right hand side is non-positive, the inequality (4.13) being then trivial. And for $\sigma_1 \neq \sigma_2$, the additional terms on right-hand side of (4.13) contribute only negative values. In contrast, in the case of the class $\mathcal{M}_q$, both the left hand and the right hand sides of (4.12) are zero for $\sigma_1 = \sigma_2$. For $\sigma_1 \neq \sigma_2$, the additional terms on right-hand side of (4.12) contribute both positive and negative values. Therefore, (4.12) appears to define a larger class of relaxed monotone nonlinearities.

3. The new stability conditions for (i) $\varphi(\cdot) \in \mathcal{M}_q$ and (ii) $\varphi(\cdot) \in \mathcal{M}_b$, do not have, in the "Real-Part" condition, the frequency term involving the quadratic form $Q(\cdot,\cdot)$ in contrast with Theorems 4.2 and 4.3 of [64] where the quadratic form is denoted by $B(\cdot,\cdot)$. Note further that, in the left hand side of the frequency domain condition (4.3) of [64], we find the term $-2B(W(i\omega))$, which is a consequence of the double negative signs in the quadratic forms of the Definition 4.2 in [64], *i.e.*, class of nonlinear functions $\mathcal{M}_b$ defined above.

4. In [64], there is no results concerning the dwell-time problem. We derive an explicit bound on the sum of the magnitudes of switching discontinuities in a period to guarantee L_2-stability for SISO systems.

5. Our simulation results reveal an interesting phenomenon that fast switching can lead to stability. These can be interpreted as complementing stabilization using vibrational control.

4.6 Summary

In this chapter, we derived frequency-domain stability conditions for two classes of systems: (1) Single-input-single-output (SISO) systems consisting of an LTI part and a periodically switching linear/nonlinear gain. (2) Multi-input-multi-output (MIMO) systems also consisting of an LTI part but with a periodically switching linear/nonlinear matrix gain. The stability conditions are expressed in terms of the magnitude and the period of the gain, and the frequency domain characteristics of the LTI part. With these conditions, we can determine (1) the magnitude boundaries of the gain if the period is fixed or (2) the constraints on the switching period if the magnitude range of the gain is known. These results are believed to be more general than those of the literature on periodic coefficient systems in the sense of their applicability to (a) higher order SISO systems with monotone, odd-monotone and *relaxed* monotone nonlinearities[1] and (b) MIMO systems with vector nonlinearities with similar properties as in the SISO case. For SISO systems with a class of nonlinear gains, the stability conditions can be easily verified because they can be cast in terms of the phase plot of the transfer function of the LTI part. More importantly, we discovered an interesting phenomenon of the switching feedback systems: fast switching leading to stability, and confirmed it by simulation.

[1] When the nonlinearity is odd, it can be treated as a special case of *power-law* nonlinearities. By adopting such a strategy, we arrive at stability conditions for the linear system as a limiting case, which are the same as those obtained from treating the linear system separately.

Chapter 5

Conclusions

Switched systems are dynamical systems that consist of a number of subsystems and a logical rule that orchestrates switching between these subsystems. Switched systems have numerous applications in control of mechanical systems, chemical processes, switching power converters, and many other fields. Due to the importance of switched systems in theory and practice, there have been increasing research activities in this field during the last two decades. Among various topics, the stability issue has attracted most of attention. Interestingly, even when all the subsystems are stable, the switched system may not be stable under arbitrary switching. On the other hand, it is possible to stabilize a switched system with unstable subsystems by an appropriate synthesis of the switching signal. These phenomena lead to three basic problems of switched systems. They are (i) stability under arbitrary switching, (ii) stability under restricted switching, and (iii) switching stabilizability. While many valuable results have been obtained related to these three problems, a number of challenging problems are still open.

Concerning some of them, the thesis proposes new and easily verifiable stability and stabilizability results, which are summarized below.

5.1 A Summary of Contributions

In Chapter 2, we deal with the problem of finding easily verifiable, necessary and sufficient stability criterion for switched systems under arbitrary switching. In contrast with the method of common Lyapunov functions found in the literature, a geometric approach is proposed to attack this problem. The basic idea is that if the trajectory under the worst case switching signal (WCSS) is stable, then the switched system is stable under arbitrary switching.

To facilitate the worst case analysis, the concept of the constant of integration is introduced by analyzing the phase diagrams of switched systems in polar coordinates in Section 2.2. As a result, we are able to use the variations of these constants of integration of subsystems, namely $H_A(k)$ and $H_B(k)$, as the indicators of the "goodness" or "badness" of a trajectory. The worst case switching signals are characterized based on the signs of these indicators, $H_A(k)$ and $H_B(k)$, associated with the signs of the trajectory directions, namely $Q_A(k)$ and $Q_B(k)$. The main result is a necessary and sufficient condition (Theorem 2.1) for the stability of a pair of planar LTI system (2.40) under arbitrary switching. This condition can be easily verified in the sense that it can be checked by hand without the need for any numerical solution. Its generalization is Theorem 2.2 as applied to the switched system (2.68) consisting of marginally stable subsystems.

Compared to the condition obtained by Margaliot and Langholz [31], Theorem 2.1 is more general since it can deal with both chattering and spiralling cases. Furthermore, unlike the conditions derived by Boscain and Balde [34, 63], which are cast case by case, Theorem 2.1 is expressed in a compact form by necessary assumptions, which is believed to offer more geometric insights of switched systems.

It is shown in Theorem 2.1 that the existence of two independent vectors w_1 and w_2, along which the trajectories of the two subsystems are collinear, plays a key role on the stability of switched systems. If the two collinear vectors do not exist, then the subsystems admit a CQLF, and the switched system is asymptotically stable. When the two collinear vectors exist, there are two classes of unstable mechanism. One is unstable chattering, when system trajectories can be driven into a conic region where both $H_A(k)$ and $H_B(k)$ are positive. There exists a switching sequence that switches back and forth inside this region to make the system trajectories unstable. In this case, the collinear vectors are the boundaries of the unstable region. The other mechanism is unstable spiralling, when the system trajectory is a spiral around the origin, and the stability of the switched system depends on the magnitude change of the trajectory under the WCSS. In this case, the collinear vectors act as the switching lines of the WCSS.

In Chapter 3, the problem of switching stabilizability is investigated. In addition to the global asymptotic stabilizability (GAS), which is the focus of the most of the research in the literature, regional asymptotic stabilizability (RAS) is also considered. It is due to the fact that there exists a class of switched systems which are not GAS, but still can be stabilized if the initial state is within certain regions. Similar to the characterization of the worst case switching signals, the best case switching signals (BCSS) are identified based on the signs of the variations of the constant of integration and trajectory directions. We derive easily verifiable, necessary and sufficient conditions for the regional asymptotical stabilizability of switched systems with two second-order LTI systems. These condition are general since all possible combinations of the dynamics of subsystems are taken into account under the assumption that no subsystem is asymptotically stable.

It is worth noting that by reversing time, Theorem 3.1 is equivalent to Theorem 2.1. Simply speaking, if a switched system (3.13) with a pair of A_i and B_j is not regionally asymptotically stabilizable (RAS), then the corresponding switched system with $-A_i$ and $-B_j$ is stable under arbitrary switching. On the contrary, if a switched system (3.13) with A_i and B_j is RAS, then the corresponding switched system with $-A_i$ and $-B_j$ is not stable under arbitrary switching. Therefore, it is the problem of regional asymptotical stabilizability, not the one of global asymptotical stabilizability, that is the dual problem of stability under arbitrary switching.

In Chapter 4, we are concerned with the problem of stability under periodic switching. A non-Lyapunov framework is employed to analyze the L_2-stability of feedback systems with a LTI transfer function in their forward path and a periodically switched linear/nonlinear feedback gain. The new stability conditions are derived based on the construction of multiplier functions in frequency domain. Although these frequency domain stability conditions are sufficient only, they are easily verifiable based on the phase plot of the transfer function $G(j\omega)$ and the upper bound $\overline{K}$.

In distinct contrast with the generally conservative stability conditions on common quadratic Lyapunov-function candidates, the frequency domain L_2-stability conditions derived in Chapter 4 are believed to be more general due to their applicability to a) higher (than two) order systems with linear and also with different classes of nonlinear gains, and b) multi-input-multi-output (MIMO) systems with a matrix of periodically switched linear and nonlinear gains. Compared to the recent results in [64], more general classes of nonlinearities are considered. In addition, an explicit bound on the sum of the magnitudes of switching discontinuities in a period is derived to guarantee L_2-stability of the SISO system.

Furthermore, it is found that fast switching can lead to stability for a class of switched feedback systems. This observation, which seems to be counter-intuitive, has not been proposed in the literature of switched systems. Our simulation has confirmed the phenomenon and also shown that the maximum switching period obtained from our frequency conditions is very close to its necessary and sufficient boundary in general cases.

5.2 Future Research Directions

In this section, we list several future research directions that are related to our work.

Easily verifiable, *even sufficient*, conditions for the stability of switch linear systems of higher (than two) order are sparse. In fact, the conditions found in the literature are for special classes subsystems (commutative, symmetric or normal), as discussed in Section 1.3.1.2. As a starting point, it is desirable to derive an easily verifiable sufficient stability condition for switched systems with two third-order subsystems. The conjecture is that the switched system are stable under arbitrary switching if AB^{-1} has a pair of complex eigenvalues. The proof is based on finding a common quadratic Lyapunov function for A and B. A necessary and sufficient condition for the existence of a CQLF for a switched system with two third-order systems is derived in [18]. This condition is not easy to verify. However, it is conjectured that the case when AB^{-1} has a pair of complex eigenvalues satisfies the sufficient part of this condition.

In Chapter 3, the best case analysis is applied to derive necessary and sufficient conditions for switching stabilizability, which provide a way to verify whether a switched system with unstable subsystems can be stabilized by switching. It is promising to use this idea to design the switching signals to stabilize the switched system. Moreover, for switched systems with external input, it is also possible to apply this idea to design appropriate feedback control laws to achieve closed-loop system stability.

In Chapter 4, we derived frequency-domain stability condition for the L_2-stability of feedback systems with periodical switched linear and nonlinear gain. It is not known how to arrive at instability counterparts for these conditions. Furthermore, it seems to be quite challenging to find stabilization conditions in the frequency-domain for a composite system with stable and unstable subsystems, including linear and nonlinear feedback gains.

In conclusion, the stability of switched systems is importance because switched systems have been employed as useful mathematical models for many practical systems. Easily checkable conditions are needed to verify the stability of switched systems. This thesis represents a further step in that direction.

Appendix A

Appendix of Chapter 2

A.1 Proof of Lemma 2.2

It follows from (2.14), (2.15), and $k = \tan\theta$ that

$$f_A(\theta) - f_B(\theta) = \frac{(k^2+1)N(k)}{D_A(k)D_B(k)}. \tag{A.1}$$

With reference to (2.6) and (2.7), we have

$$f_A(\theta) - f_B(\theta) = \frac{1}{r}\left(\left.\frac{dr}{d\theta}\right|_{\sigma=A} - \left.\frac{dr}{d\theta}\right|_{\sigma=B} \right) \tag{A.2}$$

Combining (A.1) and (A.2), it yields that

$$N(k) = \frac{1}{r(k^2+1)}\left\{ \left.\frac{dr}{d\theta}\right|_{\sigma=A} D_A(k)D_B(k) - \left.\frac{dr}{d\theta}\right|_{\sigma=B} D_A(k)D_B(k) \right\} \tag{A.3}$$

It follows from (2.25), (2.33) and (2.34) that

$$N(k) = \frac{1}{r}\left\{ \left.\frac{dr}{dt}\right|_{\sigma=A}(k)D_B(k) - \left.\frac{dr}{dt}\right|_{\sigma=B}(k)D_A(k) \right\} \tag{A.4}$$

Let $\bar{k}$ be a real root of $D_A(k)$, then $\bar{k}$ is an eigenvector of A, it follows from Assumption 2.2 that $D_B(\bar{k}) \neq 0$. So $N(\bar{k}) = 0$ only if $\left.\frac{dr}{dt}\right|_{\sigma=A}(\bar{k}) = 0$, which implies that the eigenvalue, corresponding to the eigenvector $k = \bar{k}$, is zero. It contradicts the condition that A is Hurwitz.

A.2 Proof of Lemma 2.3

Since $H_A(k)$ and $H_B(k)$ are both positive, the trajectories of two subsystems have opposite directions in this region. With reference to Fig. 2.3, define l_1 and l_2 as the lines where $x_2 = k_u x_1$ and $x_2 = k_l x_1$. Consider an initial state on l_2 at t_0, let trajectory follow Σ_A until it hits l_1 at t_1 and switch to Σ_B until it returns to the line l_2 again at t_2. Define the states at t_0, t_1 and t_2 as (r_0, θ_0), (r_1, θ_1) and (r_2, θ_0) respectively, it yields that

$$\begin{aligned} r_0 &= C_{A0} g_A(\theta_0) = C_{B0} g_B(\theta_0), \\ r_1 &= C_{A0} g_A(\theta_1) = C_{B1} g_B(\theta_1), \\ r_2 &= C_{A1} g_A(\theta_0) = C_{B1} g_B(\theta_0). \end{aligned} \tag{A.5}$$

It follows from (2.26) that $C_{A1} = C_{A0}(1 + \Delta)$, where

$$\Delta = \frac{1}{C_{A0}} \int_{t_1}^{t_2} H_A(\theta(t))dt = \frac{g_A(\theta_1)}{g_B(\theta_1)} \int_{\theta_1}^{\theta_0} \frac{g_B(\theta)}{g_A(\theta)} [f_B(\theta) - f_A(\theta)] d\theta$$

is a positive constant that depends on the known parameters k_l, k_u, and the entries of A and B. An unstable trajectory can be easily constructed by repeating the switching from t_0 to t_2

$$\lim_{n\to\infty} r(t_0 + nT) = \lim_{n\to\infty} C_{A0}(1 + \Delta)^n g(\theta_0) \to \infty,$$

where $T = t_2 - t_0 = \int_{\theta_0}^{\theta_1} \frac{1}{Q_A(\theta)} d\theta + \int_{\theta_1}^{\theta_0} \frac{1}{Q_B(\theta)} d\theta$ and n is the number of switching periods.

A.3 Proof of Lemma 2.4

Assumptions 2.4.1-2.4.3 can be satisfied by the transformation $\bar{x}_1 = -x_1$ when necessary. When $S_{ij} = S_{1j}$, A_1 equals J_1, which is invariant under the transformation $\bar{x}_1 = -x_1$. Therefore, it is reasonable to transform A_1 and B_j simultaneously by $\bar{x}_1 = -x_1$ while the stability of the switched systems S_{1j} preserves. It is assumed that one of the eigenvectors of B is in the fourth quadrant in S_{11} and S_{12} [1]. Similarly, it is assumed that the vector $[1, k_2]^T$ is in the fourth quadrant in S_{13}.

[1]Note that $\beta = 0$ in S_{11} and $\alpha = 0$ in S_{12} have been excluded by Assumption 2.2.

Assumptions 2.4.4-2.4.5 can be satisfied by similarity transformation with a unitary matrix $W = \begin{bmatrix} \gamma & -\eta \\ \eta & \gamma \end{bmatrix}$ when necessary, where $\det(W) = \sqrt{\gamma^2 + \eta^2} = 1$. Geometrically, transformation with W is a coordinate rotation. The phase diagram of $A_3 = J_3$ is a spiral that is invariant under the rotation. Therefore, it is possible to rotate the original coordinate to satisfy Assumptions 2.4.4-2.4.5 while the stability property preserves.

Since W is unitary and real, $W^{-1} = W^T$. In addition, A_3 is in its standard form J_3. It follows that

$$\begin{aligned}
\bar{A}_3 &= W^{-1}A_3W = W^TA_3W = W^TJ_3W \\
&= \begin{bmatrix} \gamma & \eta \\ -\eta & \gamma \end{bmatrix} \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \begin{bmatrix} \gamma & -\eta \\ \eta & \gamma \end{bmatrix} \\
&= \begin{bmatrix} \mu\gamma^2 - (-\omega + \omega)\gamma\eta + \mu\eta^2 & -\omega\gamma^2 + (\mu - \mu)\gamma\eta - \omega\eta^2 \\ \omega\gamma^2 + (\mu - \mu)\gamma\eta - (-\omega)\eta^2 & \mu\gamma^2 + (-\omega + \omega)\gamma\eta + \mu\eta^2 \end{bmatrix} \\
&= J_3
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bar{B}_3 \triangleq \begin{bmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{bmatrix} &= W^{-1}B_3W = W^TB_3W \\
&= \begin{bmatrix} b_{11}\gamma^2 - (b_{12} + b_{21})\gamma\eta + b_{22}\eta^2 & b_{12}\gamma^2 + (b_{11} - b_{22})\gamma\eta - b_{21}\eta^2 \\ b_{21}\gamma^2 + (b_{11} - b_{22})\gamma\eta - b_{12}\eta^2 & b_{22}\gamma^2 + (b_{12} + b_{21})\gamma\eta + b_{11}\eta^2 \end{bmatrix}
\end{aligned}$$

It follows that

$$\bar{p}_2 = \bar{a}_{12}\bar{b}_{22} - \bar{b}_{12}\bar{a}_{22} = a_{12}\bar{b}_{22} - \bar{b}_{12}a_{22} = \eta^2 \left[p_2 \left(\frac{\gamma}{\eta} \right)^2 + p_1\frac{\gamma}{\eta} + p_0 \right] \tag{A.6}$$

The polynomial inside the bracket in (A.6) has the same coefficients as $N(k)$. If $p_2 > 0$ and $N(k)$ has two roots $k_2 < k_1$, it is always possible to get a negative $\bar{p}_2$ by a pair of (γ, η) satisfying $k_2 < \frac{\gamma}{\eta} < k_1$.

Similarly, if $p_2 = 0$ and $p_2^2 + p_1^2 + p_0^2 \neq 0$ that was guaranteed by Assumption 2.1, it is always possible to find a pair of (γ, η) to guarantee $\bar{p}_2 \neq 0$.

A.4 Proof of Theorem 2.1

Proof of $S_{ij} = S_{12}$

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix}, B_2 = \frac{1}{\beta}\begin{bmatrix} \beta\lambda_b - \alpha & 1 \\ -\alpha^2 & \beta\lambda_b + \alpha \end{bmatrix}. \tag{A.7}$$

Denoting $\lambda_{1a} = k_A\lambda_{2a}$, similarly, we have $0 < k_A < 1$ and $\alpha < 0$ by Assumption 2.4.2. Substituting (A.7) into (2.31)-(2.37), it follows that $N(k) = \frac{-\lambda_{2a}}{\beta}\bar{N}(k)$, where

$$\bar{N}(k) = k^2 - [(k_A - 1)\beta\lambda_b + (k_A + 1)\alpha]k + k_A\alpha^2 \tag{A.8}$$

It can be readily shown that

$$\operatorname{sgn}(H_A(k)) = \operatorname{sgn}(\beta)\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k), \tag{A.9}$$

$$\operatorname{sgn}(H_B(k) = -\operatorname{sgn}(\bar{N}(k)), \tag{A.10}$$

$$\operatorname{sgn}(Q_A(k)) = -\operatorname{sgn}(k), \tag{A.11}$$

$$\operatorname{sgn}(Q_B(k)) = -\operatorname{sgn}(\beta). \tag{A.12}$$

Similar with the proof for S_{11}, we go through all possible sequences of the boundaries with respect to the following three catalogs

Case 1. $\bar{N}(k)$ **does not have two distinct real roots**

1.1) $\beta < 0$**.** It follows that the discriminant of equation (A.8)

$$\begin{aligned}\Delta_{12} &= \beta^2\lambda_b^2(k_A - 1)^2 + (k_A + 1)^2\alpha^2 + 2\alpha\beta\lambda_b(k_A - 1)(k_A + 1) - 4k_A\alpha^2 \\ &= \beta^2\lambda_b^2(k_A - 1)^2 + 2\alpha\beta\lambda_b(k_A - 1)(k_A + 1) + (k_A - 1)^2\alpha^2 \\ &= \beta\lambda_b(k_A - 1)[\beta\lambda_b(k_A - 1) + 2\alpha(k_A + 1)] + (k_A - 1)^2\alpha^2 > 0\end{aligned}$$

which contradicts the condition that $N(k)$ does not have two distinct real roots. So $\beta < 0$ is impossible in this case.

1.2) $\beta > 0$**.** With reference to Fig. A.1 and following the similar argument as that for Fig. 2.5, it can be concluded that the switched system is stable.

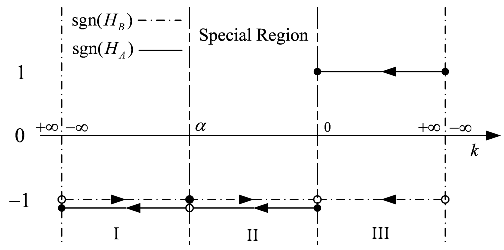


Figure A.1: S_{12}: $N(k)$ does not have two distinct real roots, the switched system is stable.

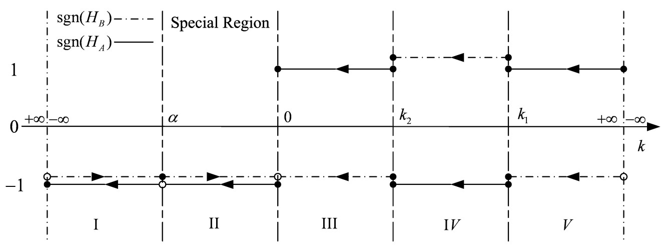


Figure A.2: S_{12}: $\det(P_2) < 0$, $\alpha < 0 < k_2 < k_1$, the switched system is stable.

Case 2. $\bar{N}(k)$ **has two distinct real roots and** $\det(P_2) < 0$

In this case, $\det(P_2) = -\beta < 0$ leads to $\beta > 0$. It follows from $\beta > 0$ and $\alpha < 0$ (Assumption 2.4.2) that equation (A.13) is positive. Thus k_1 and k_2 are in the same side of α. In addition, $|k_1 k_2| = k_A \alpha^2 < \alpha^2$. It results in $\alpha < k_2 < k_1 < 0$ or $\alpha < 0 < k_2 < k_1$.

$$(\alpha - k_1)(\alpha - k_2) = \alpha^2 - \alpha\beta(k_A - 1)\lambda_b - (k_A + 1)\alpha^2 + k_A\alpha^2 = -\alpha\beta(k_A - 1)\lambda_b \quad \text{(A.13)}$$

1.1) $\alpha < k_2 < k_1 < 0$**.** Both (A.9) and (A.10) are positive when $k \in (k_2, k_1)$. Therefore, the switched system is not stable under arbitrary switching based on Lemma 2.3.

1.2) $\alpha < 0 < k_2 < k_1$**.** With reference to Fig. A.2, the switched system is stable by similar argument as that for Fig. 2.5.

It can be concluded that $\alpha < k_2 < k_1 < 0$ is necessary and sufficient for instability in this case.

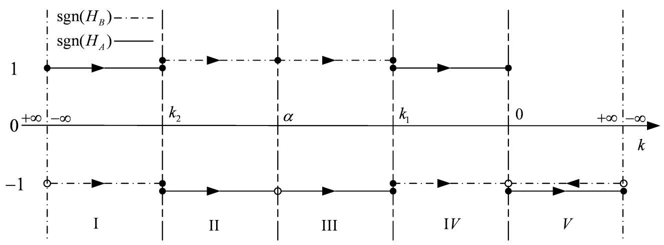


Figure A.3: S_{12}: $\det(P_2) > 0$, the worst case trajectory rotates around the origin counter clockwise.

Case 3. $\bar{N}(k)$ **has two distinct real roots and** $\det(P_2) > 0$

It follows from $\det(P_2) > 0$ that $\beta < 0$. With reference to (A.8) and (A.13), the only possible sequence is $k_2 < \alpha < k_1 < 0$ in this case. With reference to Fig. A.3, the WCSS σ^* can be obtained as follows by similar argument as that for Fig. 2.10.

$$\sigma^* = \begin{cases} A & k_2 < k < k_1, \\ B & \text{otherwise.} \end{cases} \tag{A.14}$$

which is the same as (2.62).

It shows that the second inequality of Theorem 2.1 is necessary and sufficient for instability in this case.

Proof of $S_{ij} = S_{13}$

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix}, B_3 = \frac{\omega}{\beta} \begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix}, \tag{A.15}$$

where $\xi = \frac{\mu}{\omega} < 0$. Following the similar process, we have $N(k) = \frac{-\lambda_{2a}\omega}{\beta}\bar{N}(k)$, where

$$\bar{N}(k) = k^2 - [(k_A - 1)\beta\xi + (k_A + 1)\alpha]k + k_A(\alpha^2 + \beta^2), \tag{A.16}$$

and

$$\operatorname{sgn}(H_A(k)) = \operatorname{sgn}(\beta)\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k), \tag{A.17}$$

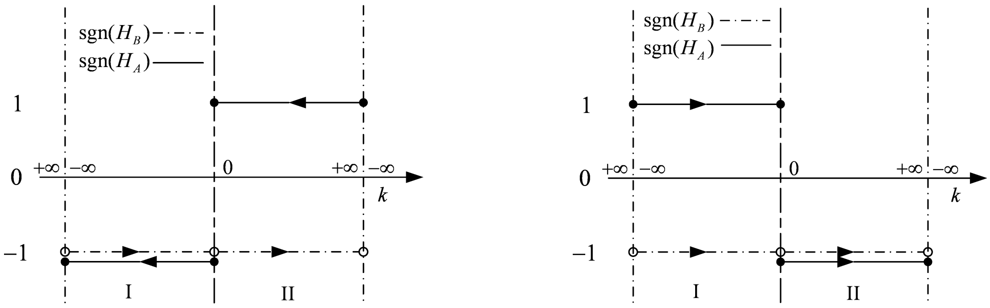
$$\operatorname{sgn}(H_B(k)) = -\operatorname{sgn}(\bar{N}(k)), \tag{A.18}$$

$$\operatorname{sgn}(Q_A(k)) = -\operatorname{sgn}(k), \tag{A.19}$$

$$\operatorname{sgn}(Q_B(k)) = -\operatorname{sgn}(\beta). \tag{A.20}$$

Case 1. $\bar{N}(k)$ **does not have two distinct real roots**

Fig. A.4 shows that the WCSS is Σ_B for all k regardless of the sign of $\det(P_3)$. Hence the switched system is stable under arbitrary switching.



(a) $\det(P_3) < 0$. (b) $\det(P_3) > 0$.

Figure A.4: S_{13}: $N(k)$ does not have two distinct real roots, the switched system is stable.

Case 2. $\bar{N}(k)$ **has two distinct real roots and** $\det(P_3) < 0$

In this case, $\beta > 0$ is obtained from $\det(P_3) = -\beta < 0$. It follows from $k_2 < 0$ (Assumption 2.4.3) and $k_1 k_2 = k_A(\alpha^2 + \beta^2) > 0$ that $k_2 < k_1 < 0$. Hence $H_A(k)$ and $H_B(k)$ are positive when $k \in (k_2, k_1)$, the switched system is not stable under arbitrary switching based on Lemma 2.3.

Case 3. $\bar{N}(k)$ **has two distinct real roots and** $\det(P_3) > 0$

In this case, $\beta < 0$. Similarly, we obtain the WCSS as (2.62) with reference to Fig. A.5.

Proof of $S_{ij} = S_{22}$

$$A_2 = \begin{bmatrix} \lambda_a & 0 \\ 1 & \lambda_a \end{bmatrix}, B_2 = \frac{1}{\beta} \begin{bmatrix} \beta\lambda_b - \alpha & 1 \\ -\alpha^2 & \beta\lambda_b + \alpha \end{bmatrix}, \tag{A.21}$$

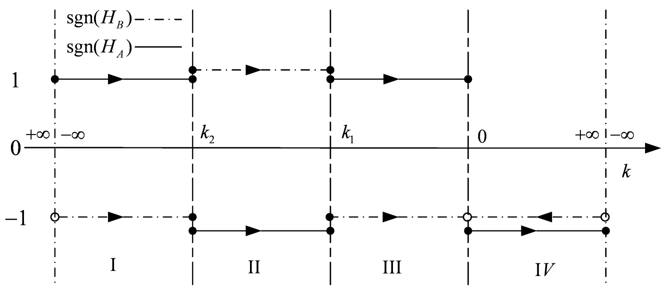


Figure A.5: S_{13}: $\det(P_3) > 0$, the worst case trajectory rotates around the origin counter clockwise.

where $\lambda_a, \lambda_b < 0$. Similarly, we have $N(k) = -\frac{\lambda_a}{\beta}\bar{N}(k)$, where

$$\bar{N}(k) = k^2 - \frac{2\lambda_a\alpha - 1}{\lambda_a}k + \frac{\lambda_a\alpha^2 + (\beta\lambda_b - \alpha)}{\lambda_a} \tag{A.22}$$

and

$$\operatorname{sgn}(H_A(k)) = -\operatorname{sgn}(\beta)\operatorname{sgn}(\bar{N}(k)), \tag{A.23}$$

$$\operatorname{sgn}(H_B(k)) = -\operatorname{sgn}(\bar{N}(k)), \tag{A.24}$$

$$\operatorname{sgn}(Q_A(k)) = 1, \tag{A.25}$$

$$\operatorname{sgn}(Q_B(k)) = -\operatorname{sgn}(\beta). \tag{A.26}$$

Case 1. $\bar{N}(k)$ does not have two distinct real roots

1.1) $\beta < 0$. It follows that

$$\Delta_{22} = \left(\frac{2\lambda_a\alpha - 1}{\lambda_a}\right)^2 - 4\frac{\lambda_a\alpha^2 + (\beta\lambda_b - \alpha)}{\lambda_a}, = \frac{1 - 4\beta\lambda_a\lambda_b}{\lambda_a^2} > 0 \tag{A.27}$$

which contradicts the condition that $N(k)$ does not have two distinct real roots. So $\beta < 0$ is impossible in this case.

1.2) $\beta > 0$. With reference to Fig. A.6, the switched system is stable under arbitrary switching.

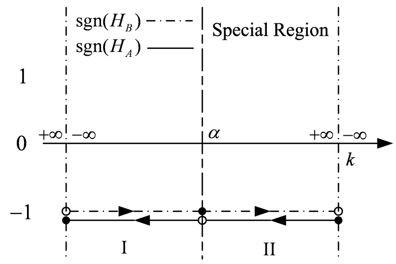


Figure A.6: S_{22}: $N(k)$ does not have two distinct real roots, the switched system is stable.

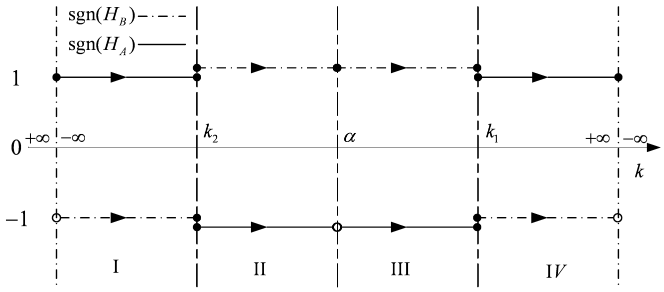


Figure A.7: S_{22}: $\det(P_2) > 0$, the worst case trajectory rotates around the origin counter clockwise.

Case 2. $\bar{N}(k)$ has two distinct real roots and $\det(P_2) < 0$

In this case, $\beta > 0$ is obtained from $\det(P_2) = -\beta < 0$. Then both (A.23) and (A.24) are positive when $k \in (k_2, k_1)$. Based on Lemma 2.3, the switched system is not stable under arbitrary switching as long as k_1 and k_2 exist. In addition, it can be shown that the existence of k_1 and k_2 implies $\alpha < k_2 < k_1$ in S_{22} as follows.

$$k_2 - \alpha = \frac{\frac{2\lambda_a\alpha-1}{\lambda_a} - \sqrt{\Delta_{22}}}{2} - \alpha = \frac{-1-\sqrt{1-4\beta\lambda_a\lambda_b}}{2\lambda_a} > 0. \qquad \text{(A.28)}$$

Hence, it can be concluded that $\alpha < k_2 < k_1$ is necessary and sufficient for instability in this case.

Case 3. $\bar{N}(k)$ has two distinct real roots and $\det(P_2) > 0$

It follows that $\beta < 0$. Similarly, we have the WCSS as (2.62) with reference to Fig. A.7.

Proof of $S_{ij} = S_{23}$

$$A_2 = \begin{bmatrix} \lambda_a & 0 \\ 1 & \lambda_a \end{bmatrix}, B_3 = \frac{\omega}{\beta} \begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix}, \tag{A.29}$$

where $\xi = \frac{\mu}{\omega} < 0$. Similarly, we have $N(k) = -\frac{\lambda_a \omega}{\beta} \bar{N}(k)$, where

$$\bar{N}(k) = k^2 - \frac{2\lambda_a \alpha - 1}{\lambda_a} k + \frac{\lambda_a(\alpha^2 + \beta^2) + (\beta\xi - \alpha)}{\lambda_a}, \tag{A.30}$$

and

$$\operatorname{sgn}(H_A(k)) = -\operatorname{sgn}(\beta)\operatorname{sgn}(\bar{N}(k)), \tag{A.31}$$

$$\operatorname{sgn}(H_B(k)) = -\operatorname{sgn}(\bar{N}(k)), \tag{A.32}$$

$$\operatorname{sgn}(Q_A(k)) = 1, \tag{A.33}$$

$$\operatorname{sgn}(Q_B(k)) = -\operatorname{sgn}(\beta). \tag{A.34}$$

Case 1. $\bar{N}(k)$ does not have two distinct real roots

1.1) $\beta < 0$. (A.31) is positive and (A.32) is negative for all regions. Therefore, Σ_B is the WCSS for all regions. Considering the boundary, which is the eigenvector of σ_A, the WCSS is still Σ_B. Therefore Σ_B is the WCSS for the whole phase plane and the switched system is stable.

1.2) $\beta > 0$. Both (A.31) and (A.32) are negative, since the only boundary is the real eigenvector of A, the trajectory alone A goes to its real eigenvector and can not go out of this region. Hence Σ_B is the WCSS for the whole phase plane and the switched system is stable.

Case 2. $\bar{N}(k)$ has two distinct real roots and $\det(P_3) < 0$

It follows from $\det(P_3) = -\beta < 0$ that $\beta > 0$. Both (A.31) and (A.32) are positive when $k \in (k_2, k_1)$, thus the switched system is not stable under arbitrary switching as long as $k_2 < k_1$ exists, which proves the first inequality of Theorem 2.1 since $M = +\infty$ and $L = -\infty$ for S_{23} with reference to (2.45).

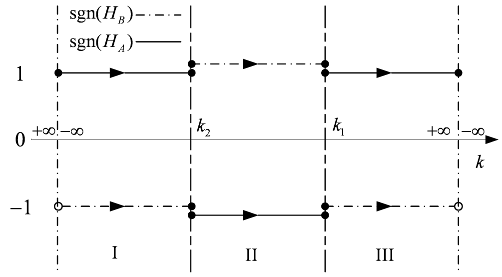


Figure A.8: S_{23}: $\det(P_3) > 0$, the worst case trajectory rotates around the origin counter clockwise.

Case 3. $\bar{N}(k)$ **has two distinct real roots and** $\det(P_3) > 0$

In this case, $\beta < 0$ is derived from $\det(P_3) > 0$. Similarly, we obtain the WCSS that is the same as (2.62) with reference to Fig. A.8.

Proof of $S_{ij} = S_{33}$

$$A_3 = \begin{bmatrix} \mu_a & -\omega_a \\ \omega_a & \mu_a \end{bmatrix}, B_3 = \frac{\omega_b}{\beta} \begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix}, \tag{A.35}$$

where $\omega_a = 1$ is assumed[1] and $\xi = \frac{\mu_b}{\omega_b} < 0$. Similarly, we have

$$\operatorname{sgn}(H_A(k)) = -\operatorname{sgn}(N(k)), \tag{A.36}$$

$$\operatorname{sgn}(H_B(k)) = -\operatorname{sgn}(\beta)\operatorname{sgn}(N(k)), \tag{A.37}$$

$$\operatorname{sgn}(Q_A(k)) = 1, \tag{A.38}$$

$$\operatorname{sgn}(Q_B(k)) = -\operatorname{sgn}(\beta), \tag{A.39}$$

where

$$N(k) = -\tfrac{\omega_b}{\beta}\{[(\beta\xi + \alpha) + \mu_a]k^2 - [(\alpha^2 + \beta^2) + 2\mu_a\alpha - 1]k + \mu_a(\alpha^2 + \beta^2) + (\beta\xi - \alpha)\}$$
$$\triangleq p_2k^2 + p_1k + p_0 \tag{A.40}$$

[1] It can be always satisfied by dividing A_3 and B_3 by ω_a (time scaling) when necessary.

Case 1. $N(k)$ **does not have two distinct real roots**

1.1) $\beta < 0$. One of (A.36) and (A.37) is negative, and the other one is positive for all k. The WCSS is one of the subsystems for the whole phase plane. So the switched system is stable.

1.2) $\beta > 0$ **and** p_2 **is positive.** Both (A.36) and (A.37) are negative for the whole phase plane, then switched system (A.35) is stable under arbitrary switching.

1.3) $\beta > 0$ **and** p_2 **is negative.** With reference to (A.40), we have $p_2 = -\frac{\omega_b}{\beta}[(\beta\xi + \alpha) + \mu_a]$ and $p_0 = -\frac{\omega_b}{\beta}[\mu_a(\alpha^2 + \beta^2) + (\beta\xi - \alpha)]$. If $p_2 < 0$, it follows from $\beta > 0$, $\mu_a < 0$ and $\xi < 0$ that $\alpha > 0$, which leads to $p_0 > 0$, which contradicts the condition that $N(k)$ does not have two distinct real roots. So this case will not happen.

Case 2. $N(k)$ **has two distinct real roots and** $\det(P_3) < 0$

Note that the sign of $N(k)$ is positive when $k \in (k_2, k_1)$ because p_2, the leading coefficient of $N(k)$, was assumed to be negative by Assumption 2.4.5. It follows from $\det(P_3) = -\beta < 0$ that $\beta > 0$. Both (A.36) and (A.37) are positive when $k \in (k_2, k_1)$, thus the switched system is not stable under arbitrary switching as long as the two roots $k_2 < k_1$ exist, which is equivalent to the first inequality of Theorem 2.1.

Case 3. $N(k)$ **has two distinct real roots and** $\det(P_3) > 0$

In this case, we have $\beta < 0$. With reference to Fig. A.9, the WCSS can be derived that is the same as (2.62).

The Theorem 2.1 is proven.

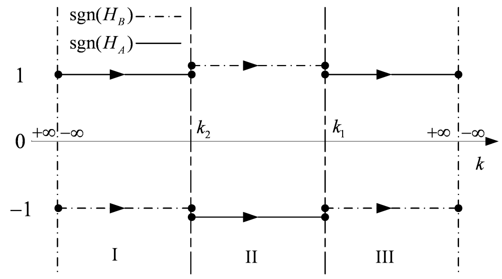


Figure A.9: S_{33}: $\det(P_3) > 0$, the worst case trajectory rotates around the origin counter clockwise.

Appendix B

Appendix of Chapter 3

B.1 Analysis of the special cases when Assumption 3.2 is violated

Case 1) A and B have only one common eigenvector. Without loss of generality, we assume that the eigenvalues of A and B correspond to the common eigenvector are λ_{2A} and λ_{2B}, and the common eigenvector is $[0, 1]^T$, then we have

$$A = \begin{bmatrix} \lambda_{1A} & 0 \\ a_{21} & \lambda_{2A} \end{bmatrix}, B = \begin{bmatrix} \lambda_{1B} & 0 \\ b_{21} & \lambda_{2B} \end{bmatrix},$$

where at least one of a_{21} and b_{21} is not zero. Thus, the dynamic of the switched system can be described as

$$\dot{x} = \begin{bmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & \sigma_{22}(t) \end{bmatrix} x,$$

where $\sigma_{11}(t) \in \{\lambda_{1A}, \lambda_{1B}\}$, $\sigma_{21}(t) \in \{a_{21}, b_{21}\}$, and $\sigma_{22}(t) \in \{\lambda_{2A}, \lambda_{2B}\}$.

For the switched systems (3.13) and (3.14), $\sigma_{11}(t)$ is non-negative because all the eigenvalues of A and B are non-negative. It follows that $|x_1(t)| = e^{\int_0^t \sigma_{11}(\tau)d\tau}|x_1(0)|$ is lower-bounded by $|x_1(0)|$, so the switched system (3.13) or (3.14) is not RAS in this case.

For the switched system (3.15) (if both λ_{1A} and λ_{1B} are non-negative), similarly $|x_1(t)|$ is lower-bounded by $|x_1(0)|$ and the switched system (3.15) is not

RAS. If one of λ_{1A} and λ_{1B} is negative, the switched system (3.15) is RAS, which is proven as follows:

Consider a periodical switching signal $\sigma_T(t)$ with a period of $T = t_A + t_B$

$$\sigma_T(t) = \begin{cases} A & \texttt{if} \quad 0 < t < t_A \\ B & \texttt{if} \quad t_A < t < T \end{cases}.$$

It follows that

$$x(T) = e^{Bt_B} e^{At_A} x(0) \triangleq \Gamma x(0) = \begin{bmatrix} \Gamma_{11} & 0 \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} x(0),$$

where $\Gamma_{11} = e^{\lambda_{1A}t_A + \lambda_{1B}t_B}$, $\Gamma_{22} = e^{\lambda_{2A}t_A + \lambda_{2B}t_B}$, and $\Gamma_{21} = \frac{a_{21}}{(\lambda_{1A} - \lambda_{2A})}\left(e^{\lambda_{1A}t_A} - e^{\lambda_{2A}t_A}\right) e^{\lambda_{1B}t_B} + \frac{b_{21}}{(\lambda_{1B} - \lambda_{2B})}\left(e^{\lambda_{1B}t_B} - e^{\lambda_{2B}t_B}\right) e^{\lambda_{2A}t_A}$.

Let $x(0)$ be on the eigenvector corresponding to the eigenvalue Γ_{11}, *i.e.* $x(0) = \left[1, \ \frac{\Gamma_{21}}{(\Gamma_{11} - \Gamma_{22})}\right]^T$. We have $x(T) = \Gamma_{11}x(0)$. If one of λ_{1A} and λ_{1B} is negative, for every pair $(t_A, \ t_B)$ satisfying $\lambda_{1A}t_A + \lambda_{1B}t_B < 0$, there exists a corresponding vector such that the trajectory starting from this vector is asymptotically stable under the switching signal $\sigma_T(t)$. Since one of a_{21} and b_{21} is nonzero, the collection of these vectors, corresponding to the different pairs $(t_A, \ t_B)$ with $0 < \Gamma_{11} < 1$, is a region instead of a single line. Based on Definition 3.2, the switched system (3.15) is RAS.

Case 2) A and B have two common eigenvectors. In this case, we have

$$\dot{x} = \begin{bmatrix} \sigma_{11}(t) & 0 \\ 0 & \sigma_{22}(t) \end{bmatrix} x.$$

Similarly, the switched system (3.13) or (3.14) is not RAS since both $\sigma_{11}(t)$ and $\sigma_{22}(t)$ are non-negative.

In this case, the switched system (3.15) is RAS if and only if a) one of λ_{1A} and λ_{1B} is negative, b) one of λ_{2A} and λ_{2B} is negative, and c) the product of the two negative eigenvectors is greater than the product of the other two non-negative eigenvectors. These conditions are equivalent to the existence of a pair (t_A, t_B) such that both $\lambda_{1A}t_A + \lambda_{1B}t_B$ and $\lambda_{2A}t_A + \lambda_{2B}t_B$ are negative.

Note that the special cases that Assumption 3.2 is violated can also be solved by direct inspection. They are discussed here just for the completeness of the results.

B.2 Proof of Theorem 3.1

Proof of $S_{ij} = S_{12}$

In this case, the two subsystems are expressed as

$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix}, B_2 = \frac{1}{\beta} \begin{bmatrix} \beta\lambda_b + \alpha & -1 \\ \alpha^2 & \beta\lambda_b - \alpha \end{bmatrix}, \tag{B.1}$$

where $\alpha < 0$ by Assumption 3.2, $\lambda_{2a} > \lambda_{1a} > 0$, and $\lambda_b > 0$. Denote $\lambda_{1a} = k_A\lambda_{2a}$, then $k_A \in (0, 1)$. Substituting the entries of A_1 and B_2 into (3.4)-(3.7), it follows that

$$\operatorname{sgn}(H_A(k) = -\operatorname{sgn}(\beta)\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k), \tag{B.2}$$

$$\operatorname{sgn}(H_B(k) = \operatorname{sgn}(\bar{N}(k)), \tag{B.3}$$

$$\operatorname{sgn}(Q_A(k)) = \operatorname{sgn}(k), \tag{B.4}$$

$$\operatorname{sgn}(Q_B(k)) = \operatorname{sgn}(\beta) \tag{B.5}$$

, where

$$\bar{N}(k) = k^2 + [(k_A - 1)\beta\lambda_b - (k_A + 1)\alpha]k + k_A\alpha^2. \tag{B.6}$$

Similar to the case $S_{ij} = S_{11}$, we need to know the locations of k_1, k_2 relative to α, which is based on

$$\operatorname{sgn}((\alpha - k_1)(\alpha - k_2)) = \operatorname{sgn}(\beta). \tag{B.7}$$

Case 1. $\bar{N}(k)$ does not have two distinct real roots.

1.1) $\beta < 0$: It follows that the discriminant of equation (B.6)

$$\begin{aligned} \Delta_{12} &= \beta^2\lambda_b^2(k_A - 1)^2 + (k_A + 1)^2\alpha^2 - 2\alpha\beta\lambda_b(k_A - 1)(k_A + 1) - 4k_A\alpha^2 \\ &= \beta\lambda_b(k_A - 1)[\beta\lambda_b(k_A - 1) - 2\alpha(k_A + 1)] + (k_A - 1)^2\alpha^2 > 0 \end{aligned}$$

which contradicts the condition that $N(k)$ does not have two distinct real roots. So $\beta < 0$ is impossible in this case.

1.2) $\beta > 0$. With reference to Fig. B.1 and following the similar argument as that for Fig. 3.3, it can be concluded that the switched system is unstabilizable.

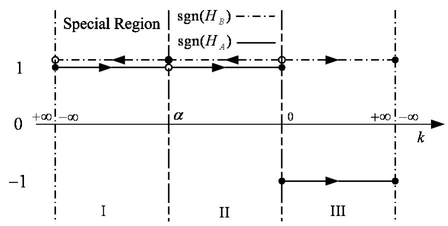


Figure B.1: S_{12}: $N(k)$ does not have two distinct real roots, the switched system is unstabilizable.

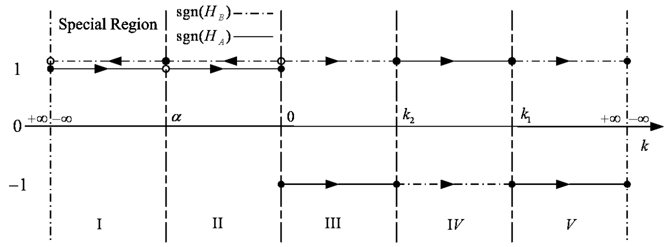


Figure B.2: S_{12}: $\det(P_2) < 0$, $\alpha < 0 < k_2 < k_1$, the switched system is unstabilizable.

Case 2. $\bar{N}(k)$ has two distinct real roots and $\det(P_2) < 0$.

$\det(P_2) = -\beta < 0$ leads to $\beta > 0$. It follows from $\beta > 0$ and $\alpha < 0$ (Assumption 3.2) that Eqn (B.7) is positive. Thus k_1 and k_2 are in the same side of α. In addition, $|k_1 k_2| = k_A \alpha^2 < \alpha^2$. It results in $\alpha < k_2 < k_1 < 0$ or $\alpha < 0 < k_2 < k_1$.

2.1) $\alpha < k_2 < k_1 < 0$. Both (B.2) and (B.3) are negative when $k \in (k_2, k_1)$. Therefore, the switched system is regionally stabilizable based on Lemma 3.

2.2) $\alpha < 0 < k_2 < k_1$. With reference to Fig. B.2, the switched system is stable by similar argument as that for Fig. 3.3. It can be concluded that $\alpha < k_2 < k_1 < 0$ is necessary and sufficient for the stabilizability in Case 2.

Case 3. $\bar{N}(k)$ has two distinct real roots and $\det(P_2) > 0$.

It follows from $\det(P_2) > 0$ that $\beta < 0$. With reference to (B.6) and (B.7), the only possible sequence is $k_2 < \alpha < k_1 < 0$ in this case. With reference to Fig. B.3, the BCSS σ^* for this case is the same as (3.37) by similar argument as that for Fig. 3.8.

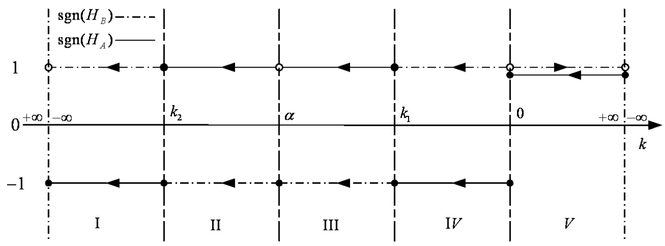


Figure B.3: S_{12}: $\det(P_2) > 0$, the best case trajectory rotates around the origin clockwise.

Proof of $S_{ij} = S_{13}$

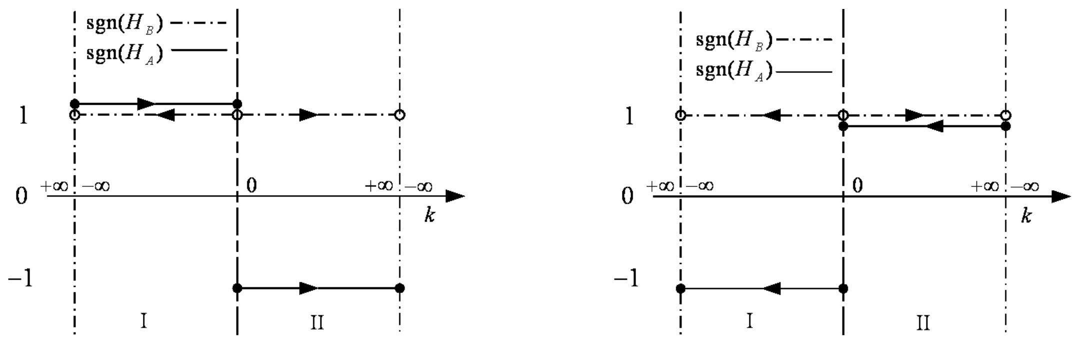
$$A_1 = \begin{bmatrix} \lambda_{1a} & 0 \\ 0 & \lambda_{2a} \end{bmatrix}, B_3 = \frac{\omega}{\beta} \begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix}, \quad \text{(B.8)}$$

where $\mu > 0$, $\omega < 0$, and $\xi = \frac{\mu}{\omega} < 0$. Substituting A_1 and B_3 into (3.4)-(3.7), it follows that $\operatorname{sgn}(H_A(k)) = -\operatorname{sgn}(\beta)\operatorname{sgn}(\bar{N}(k))\operatorname{sgn}(k), \operatorname{sgn}(H_B(k)) = \operatorname{sgn}(\bar{N}(k)), \operatorname{sgn}(Q_A(k)) = \operatorname{sgn}(k)$, and $\operatorname{sgn}(Q_B(k)) = \operatorname{sgn}(\beta)$, where

$$\bar{N}(k) = k^2 - [(k_A - 1)\beta\xi + (k_A + 1)\alpha]k + k_A(\alpha^2 + \beta^2). \quad \text{(B.9)}$$

Case 1. $\bar{N}(k)$ does not have two distinct real roots.

Fig.B.4 shows that the BCSS is Σ_B for all k regardless of the sign of $\det(P_3)$. Hence the switched system is unstabilizable.



(a) $\det(P_3) < 0$.

(b) $\det(P_3) > 0$.

Figure B.4: S_{13}: $N(k)$ does not have two distinct real roots, the switched system is unstabilizable.

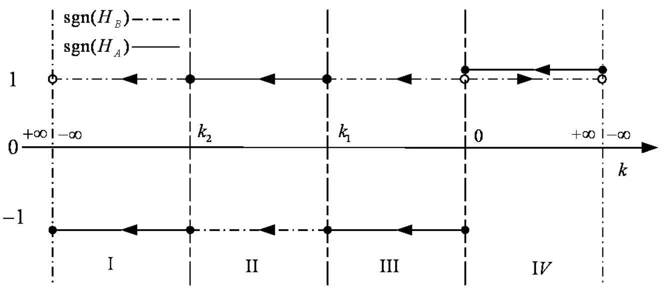


Figure B.5: S_{13}: $\det(P_3) > 0$, the best case trajectory rotates around the origin clockwise.

Case 2. $\bar{N}(k)$ has two distinct real roots and $\det(P_3) < 0$.

In this case, $\beta > 0$. It follows from $k_2 < 0$ (Assumption 3.3) and $k_1 k_2 = k_A(\alpha^2 + \beta^2) > 0$ that $k_2 < k_1 < 0$. Hence $H_A(k)$ and $H_B(k)$ are negative when $k \in (k_2, k_1)$, the switched system is regionally stabilizable based on Lemma 3.

Case 3. $\bar{N}(k)$ has two distinct real roots and $\det(P_3) > 0$.

In this case, we have $\beta < 0$, Similarly, we obtain the BCSS as (3.37) with reference to Fig. B.5.

Proof of $S_{ij} = S_{22}$

$$A_2 = \begin{bmatrix} \lambda_a & 0 \\ -1 & \lambda_a \end{bmatrix}, B_2 = \frac{1}{\beta}\begin{bmatrix} \beta\lambda_b + \alpha & -1 \\ \alpha^2 & \beta\lambda_b - \alpha \end{bmatrix}, \tag{B.10}$$

where $\lambda_a, \lambda_b > 0$. Substituting A_2 and B_2 into (3.4)-(3.7), it follows that $\mathrm{sgn}(H_A(k)) = \mathrm{sgn}(\beta)\,\mathrm{sgn}(\bar{N}(k)), \mathrm{sgn}(H_B(k)) = \mathrm{sgn}(\bar{N}(k)), \mathrm{sgn}(Q_A(k)) = -1$, and $\mathrm{sgn}(Q_B(k)) = \mathrm{sgn}(\beta)$, where

$$\bar{N}(k) = k^2 - \frac{2\lambda_a\alpha + 1}{\lambda_a}k + \frac{\lambda_a\alpha^2 + (\beta\lambda_b + \alpha)}{\lambda_a}. \tag{B.11}$$

Case 1. $\bar{N}(k)$ does not have two distinct real roots.

1.1) $\beta < 0$: It follows that

$$\Delta_{22} = \left(\frac{2\lambda_a\alpha + 1}{\lambda_a}\right)^2 - 4\frac{\lambda_a\alpha^2 + (\beta\lambda_b + \alpha)}{\lambda_a} = \frac{1 - 4\beta\lambda_a\lambda_b}{\lambda_a^2} > 0. \tag{B.12}$$

which contradicts the condition that $N(k)$ does not have two distinct real roots. So $\beta < 0$ is impossible in this case.

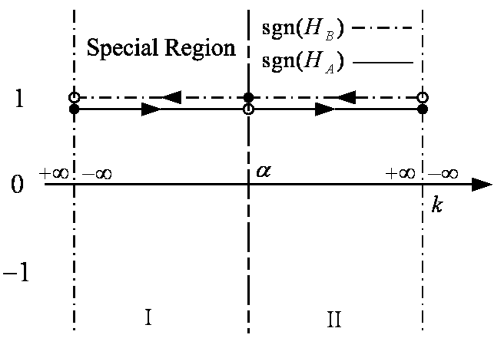


Figure B.6: S_{22}: $N(k)$ does not have two distinct real roots, the switched system is unstabilizable.

1.2) $\beta > 0$: With reference to Fig. B.6, the switched system is unstabilizable.

Case 2. $\bar{N}(k)$ has two distinct real roots and $\det(P_2) < 0$.

In this case ,we have $\beta > 0$. Then both $H_A(k)$ and $H_B(k)$ are negative when $k \in (k_2, k_1)$. Based on Lemma 3, the switched system is regionally stabilizable as long as k_1 and k_2 exist. In addition, it can be shown that the existence of k_1 and k_2 implies $\alpha < k_2 < k_1$ in S_{22} as follows.

$$k_2 - \alpha = \frac{\frac{2\lambda_a\alpha+1}{\lambda_a} - \sqrt{\Delta_{22}}}{2} - \alpha = \frac{1 - \sqrt{1 - 4\beta\lambda_a\lambda_b}}{2\lambda_a} > 0 \tag{B.13}$$

Hence, it can be concluded that $\alpha < k_2 < k_1$ is necessary and sufficient for the stabilizability in Case 2.

Case 3. $\bar{N}(k)$ has two distinct real roots and $\det(P_2) > 0$.

$\beta < 0$, Similarly, we have the BCSS as (3.37) with reference to Fig. B.7.

Proof of $S_{ij} = S_{23}$

$$A_2 = \begin{bmatrix} \lambda_a & 0 \\ -1 & \lambda_a \end{bmatrix}, B_3 = \frac{\omega}{\beta}\begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2+\beta^2) & \beta\xi + \alpha \end{bmatrix}, \tag{B.14}$$

where $\mu > 0$, $\omega < 0$, and $\xi = \dfrac{\mu}{\omega} < 0$. So we have $\operatorname{sgn}(H_A(k)) = \operatorname{sgn}(\beta)\operatorname{sgn}(\bar{N}(k))$, $\operatorname{sgn}(H_B(k)) = \operatorname{sgn}(\bar{N}(k))$, $\operatorname{sgn}(Q_A(k)) = -1$, and $\operatorname{sgn}(Q_B(k)) = \operatorname{sgn}(\beta)$, where $\bar{N}(k) = k^2 - \frac{2\lambda_a\alpha+1}{\lambda_a}k + \frac{\lambda_a(\alpha^2+\beta^2)-(\beta\xi-\alpha)}{\lambda_a}$.

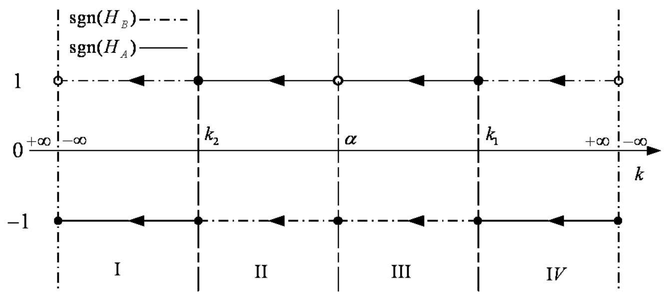


Figure B.7: S_{22}: $\det(P_2) > 0$, the best case trajectory rotates around the origin clockwise.

Case 1. $\bar{N}(k)$ does not have two distinct real roots.

1.1) $\beta < 0$. $H_A(k)$ is negative and $H_B(k)$ is positive for all regions, then Σ_B is the BCSS for all regions. On the boundary, which is the eigenvector of σ_A, the BCSS is still Σ_B. Therefore, Σ_B is the BCSS for the whole phase plane and it is trivial to show that the switched system is unstabilizable.

1.2) $\beta > 0$. Both $H_A(k)$ and $H_B(k)$ are positive, since the only boundary is the real eigenvector of A. The trajectory alone A goes to its real eigenvector and can not go out of this region. Hence, Σ_B is the BCSS for the whole phase plane and the switched system is unstabilizable.

Case 2. $\bar{N}(k)$ has two distinct real roots and $\det(P_3) < 0$.

It follows from $\det(P_3) = -\beta < 0$ that $\beta > 0$. Both $H_A(k)$ and $H_B(k)$ are negative when $k \in (k_2, k_1)$, thus the switched system is regionally stabilizable as long as $k_2 < k_1$ exists. It proves the first inequality of Theorem 3.1 because $M = +\infty$ and $L = -\infty$ for S_{23} with reference to (3.21).

Case 3. $\bar{N}(k)$ has two distinct real roots and $\det(P_3) > 0$.

In this case, $\beta < 0$. Similarly, the BCSS is the same as (3.37) with reference to Fig. B.8.

Proof of $S_{ij} = S_{33}$

$$A_3 = \begin{bmatrix} \mu_a & 1 \\ -1 & \mu_a \end{bmatrix}, B_3 = \frac{\omega_b}{\beta} \begin{bmatrix} \beta\xi - \alpha & 1 \\ -(\alpha^2 + \beta^2) & \beta\xi + \alpha \end{bmatrix},$$

where $\mu_a, \mu_b > 0$, $\omega_b < 0$ and $\xi = \frac{\mu_b}{\omega_b} < 0$. Similarly, we have $\text{sgn}(H_A(k)) =$

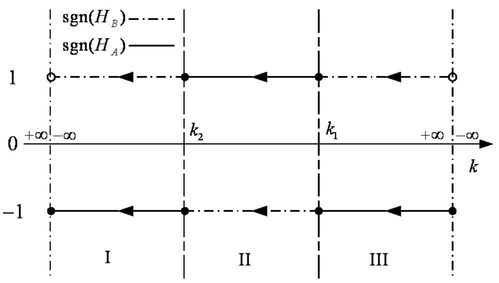


Figure B.8: S_{23}: $\det(P_3) > 0$, the best case trajectory rotates around the origin clockwise.

$\text{sgn}(N(k)), \text{sgn}(H_B(k)) = \text{sgn}(\beta)\,\text{sgn}(N(k)), \text{sgn}(Q_A(k)) = -1$, and $\text{sgn}(Q_B(k)) = \text{sgn}(\beta)$, where

$$N(k) = \tfrac{\omega_b}{\beta}\{[(\beta\xi + \alpha) - \mu_a]k^2 + [1 + 2\mu_a\alpha - (\alpha^2 + \beta^2)]k + (\beta\xi - \alpha) - \mu_a(\alpha^2 + \beta^2)\}$$
$$\triangleq p_2k^2 + p_1k + p_0. \tag{B.15}$$

Case 1. $N(k)$ does not have two distinct real roots.

1.1) $\beta < 0$. One of $H_A(k)$ and $H_B(k)$ is negative, and the other one is positive for all k. The BCSS is one of the subsystems for the whole phase plane. So the switched system is unstabilizable.

1.2) $\beta > 0$ and $p_2 > 0$. Both $H_A(k)$ and $H_B(k)$ are positive for the whole phase plane, then switched system is unstabilizable.

1.3) $\beta > 0$ and $p_2 < 0$. With reference to (B.15), we have $p_2 = \frac{\omega_b}{\beta}[(\beta\xi+\alpha)-\mu_a]$ and $p_0 = \frac{\omega_b}{\beta}[(\beta\xi - \alpha) - \mu_a(\alpha^2 + \beta^2)]$. If $p_2 < 0$, it follows from $\beta > 0$, $\mu_a < 0$ and $\xi < 0$ that $\alpha > 0$, which leads to $p_0 > 0$. This contradicts the condition that $N(k)$ does not have two distinct real roots. So this case will not happen.

1.4) $\beta > 0$ and $p_2 = 0$. The case $p_2 = 0$ has been excluded by Assumption 3.4.

Case 2. $N(k)$ has two distinct real roots and $\det(P_3) < 0$.

Note that the sign of $N(k)$ is positive when $k \in (k_2, k_1)$ because p_2 (the leading coefficient of $N(k)$) was assumed to be negative by Assumption 3.5. It follows from $\det(P_3) = -\beta < 0$ that $\beta > 0$. Both $H_A(k)$ and $H_B(k)$ are negative when

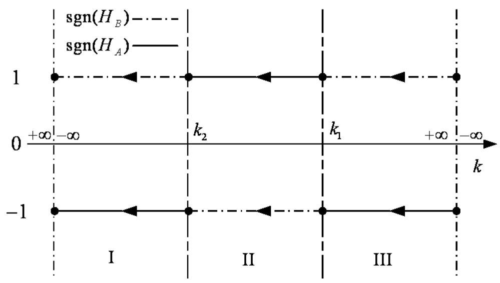


Figure B.9: S_{33}: $\det(P_3) > 0$, the best case trajectory rotates around the origin clockwise.

$k \in (k_2, k_1)$, thus the switched system is regionally stabilizable as long as the two roots $k_2 < k_1$ exists, which is equivalent to the first inequality of Theorem 3.1.

Case 3. $N(k)$ has two distinct real roots and $\det(P_3) > 0$.

In this case, $\beta < 0$. With reference to Fig. B.9, the BCSS can be derived that is the same as (3.37).

The Theorem 3.1 is proven.

Appendix C

Appendix of Chapter 4

C.1 Proof of Lemma 4.1

The integral $\lambda_1(T)$ of (4.18) can be rewritten as

$$\int_0^T \mathcal{Z}\{\sigma_T(t)\}k_1(t)\sigma_T(t)dt = \int_0^T \{\sigma_T(t) + \sum_{m=1}^{\infty} z_m(\sigma_T(t - m\mathcal{P}) - \sigma_T(t + m\mathcal{P}))\}k_1(t)\sigma_T(t)dt, \tag{C.1}$$

for all σ_T in the domain of $\mathcal{Z}$ and for all $T \geq 0$. When the switching gain is actually constant, (C.1) should also be non-negative. This implies, by an application of the Parseval theorem to the left hand side of (C.1), that Re $[\mathrm{Z}(\mathrm{j}\omega)] \geq 0$. For a periodic switching gain, the integral on the right hand side of (C.1) with the first summation can be simplified by a change of variable. To this end, let $t + m\mathcal{P} = \tau$. Then

$$\int_0^T \sigma_T(t + m\mathcal{P})k_1(t)\sigma_T(t)\, dt = \int_{m\mathcal{P}}^{T+m\mathcal{P}} \sigma_T(\tau)k_1(\tau - m\mathcal{P})\sigma_T(\tau - m\mathcal{P})\, d\tau. \tag{C.2}$$

Since $\sigma_T(\tau) = 0$ for $\tau < 0$, and for $\tau > T$, and $k_1(\tau - m\mathcal{P}) = k_1(\tau)$, (C.2) can be reduced to a simpler form

$$\int_0^T \sigma_T(t + m\mathcal{P})k_1(t)\sigma_T(t)dt = \int_0^T \sigma_T(\tau)k_1(\tau)\sigma_T(\tau - m\mathcal{P})d\tau. \tag{C.3}$$

Combining (C.3) and the integral with the first summation in (C.1), we observe that the resulting integrands with the coefficient z_m cancel out. Therefore,

$$\int_0^T \mathcal{Z}\{\sigma_T(t)\}k_1(t)\sigma_T(t)dt = \int_0^T \sigma_T^2(t)k_1(t)dt, \tag{C.4}$$

which is non-negative for all $T \geq 0$ by virtue of the assumption on $k_1(\cdot)$. The lemma is proven.

C.2 Proof of Lemma 4.2

The integral of $\lambda_2(T)$ of (4.18) can be rewritten as

$$\int_0^T \{\sigma_T(t) + \sum_{m=1}^{\infty} (z_m\ \sigma_T(t - m\mathcal{P})\ +\ z'_m\ \sigma_T(t + m\mathcal{P}))\} k_1(t)\varphi(\sigma_T(t))dt \quad \text{(C.5)}$$

for all σ_T in the domain of $\mathcal{Z}$ and for all $T \geq 0$.

Now we establish a couple of inequalities based on the property (4.7) of monotone functions. We have

$$\int_0^T k_1(t)\{\sigma_T(t) - \sigma_T(t - m\mathcal{P})\}\varphi(\sigma_T(t))dt \geq \int_0^T k_1(t)\Phi(\sigma_T(t))dt - \int_0^T k_1(t)\Phi(\sigma_T(t - m\mathcal{P}))dt, \quad \text{(C.6)}$$

where $\Phi(\sigma) = \int_0^\sigma \varphi(\tau)\ d\tau$.

The second integral on the right hand side of (C.6) can be simplified by a change of variable. To this end, let $(t - m\mathcal{P}) = \tau$. Then, invoking the periodicity of $k_1(t)$ and the properties of the integrands with truncation, it can be shown that

$$\int_0^T k_1(t)\Phi(\sigma_T(t - m\mathcal{P}))dt\ =\ \int_0^T k_1(t)\Phi(\sigma_T(t))dt. \quad \text{(C.7)}$$

Therefore, from (C.6) and (C.7), we get the inequality

$$\int_0^T k_1(t)\ \sigma_T(t - m\mathcal{P})\varphi(\sigma_T(t))dt\ \leq\ \int_0^T k_1(t)\sigma_T(t)\varphi(\sigma_T(t))\ dt. \quad \text{(C.8)}$$

On similar lines, we can establish the following inequality

$$\int_0^T k_1(t)\ \sigma_T(t + m\mathcal{P})\varphi(\sigma_T(t))dt\ \leq\ \int_0^T k_1(t)\sigma_T(t)\varphi(\sigma_T(t))\ dt. \quad \text{(C.9)}$$

Combining (C.8) and (C.9), and assuming an interchange of summation and integration to be valid, we conclude that

$$\int_0^T \{\sum_{m=1}^{\infty} (z_m\ \sigma_T(t - m\mathcal{P}) + z'_m\ \sigma_T(t + m\mathcal{P}))\} k_1(t)\varphi(\sigma_T(t))dt \leq \int_0^T \sigma_T(t)k_1(t)\varphi(\sigma_T(t))dt, \quad \text{(C.10)}$$

if (i) $z_m < 0,\ z'_m < 0,\ m = 1, 2, \cdots$, and (ii) $\sum_{m=1}^{\infty} (| z_m | + | z'_m |) < 1$, from which we conclude $\lambda_2(T)$ of (4.18) is nonnegative. Lemma 4.2 is proven.

For the proof of Corollary 4.4, note that when $\varphi(\cdot)$ is odd, $\Phi(\cdot)$ is even. We have

$$\int_0^T k_1(t)\sigma_T(t)\varphi(\sigma_T(t))dt + \int_0^T k_1(t)\sigma_T(t - m\mathcal{P})\varphi(\sigma_T(t))dt =$$
$$\int_0^T k_1(t)\sigma_T(t)\varphi(\sigma_T(t))dt - \int_0^T k_1(t)(-\sigma_T(t - m\mathcal{P}))\varphi(\sigma_T(t))dt \geq$$
$$\int_0^T k_1(t)\Phi(\sigma_T(t))dt - \int_0^T k_1(t)\Phi(-\sigma_T(t - m\mathcal{P}))dt \geq 0. \tag{C.11}$$

Therefore

$$\left|\int_0^T k_1(t)\sigma_T(t - m\mathcal{P})\varphi(\sigma_T(t))dt\right| \leq \int_0^T k_1(t)\sigma_T(t)\varphi(\sigma_T(t))dt.$$

from which, by repeating the remaining part of the proof of Lemma 4.2, Corollary 4.4 follows.

C.3 Proof of Lemma 4.4

For $\varphi(\cdot) \in \mathcal{M}_q$, the defining property is (4.12). In the manner of the proof of Lemma 2, we can establish a couple of inequalities based on (4.12). We have

$$\int_0^T k_1(t)\{\sigma_T(t) - \sigma_T(t - m\mathcal{P})\}\varphi(\sigma_T(t))dt \geq \int_0^T k_1(t)\Phi(\sigma_T(t))dt -$$
$$\int_0^T k_1(t)\Phi(\sigma_T(t-m\mathcal{P}))dt + \int_0^T k_1(t)\{q_{11}\sigma_T^2(t-m\mathcal{P}) + q_{12}\sigma_T(t-m\mathcal{P})\varphi(\sigma_T(t-m\mathcal{P})) +$$
$$q_{22}\varphi^2(\sigma_T(t - m\mathcal{P})) - q_{11}\sigma_T^2(t) - q_{12}\sigma_T(t)\varphi(\sigma_T(t)) - q_{22}\varphi^2(\sigma_T(t))\}dt. \tag{C.12}$$

In the right hand side of (C.12), by changing the variable of integration from $(t - m\mathcal{P})$ to τ, using the periodicity property of $k(t)$ and the truncation properties of the other integrands (and making the necessary changes in the limits of integration), it can shown that the third integral vanishes in the same manner as the first two integrals. As a consequence, (C.8) is valid in this case, too. The rest of the proof of Lemma 4.2 can be applied to complete the proof of the present lemma.

Corollary 4.5 for monotone $\mathcal{M}_q$ can be proved in the same manner as Corollary 4.4.

On the other hand when $\varphi(\cdot) \in \mathcal{M}_b$, the following steps which are a slight modification of the proof of Lemma 4.4 are required.

In the place of (C.12), we now have

$$\begin{aligned}&\textstyle\int_0^T k_1(t)\{\sigma_T(t) - \sigma_T(t-m\mathcal{P})\}\varphi(\sigma_T(t))dt \geq \\ &\textstyle\int_0^T k_1(t)\Phi(\sigma_T(t))dt - \int_0^T k_1(t)\Phi(\sigma_T(t-m\mathcal{P}))dt \\ &\textstyle- \int_0^T k_1(t)\{q_{11}\sigma_T^2(t-m\mathcal{P}) + q_{12}\sigma_T(t-m\mathcal{P})\varphi(\sigma_T(t-m\mathcal{P})) \\ &+q_{22}\varphi^2(\sigma_T(t-m\mathcal{P})) + q_{11}\sigma_T^2(t) + q_{12}\sigma_T(t)\varphi(\sigma_T(t)) + q_{22}\varphi^2(\sigma_T(t))\}dt.\end{aligned} \quad \text{(C.13)}$$

In the right hand side of (C.13), by changing the variable of integration from $(t - m\mathcal{P})$ to τ, it can be shown, as before, that the first two integrals vanish. Therefore, we now have

$$\begin{aligned}&\textstyle\int_0^T k_1(t)\{\sigma_T(t) - \sigma_T(t-m\mathcal{P})\}\varphi(\sigma_T(t))dt \geq \\ &\textstyle- \int_0^T k_1(t)\{q_{11}\sigma_T^2(t-m\mathcal{P}) + q_{12}\sigma_T(t-m\mathcal{P})\varphi(\sigma_T(t-m\mathcal{P})) \\ &+q_{22}\varphi^2(\sigma_T(t-m\mathcal{P})) + q_{11}\sigma_T^2(t) + q_{12}\sigma_T(t)\varphi(\sigma_T(t)) + q_{22}\varphi^2(\sigma_T(t))\}dt,\end{aligned} \quad \text{(C.14)}$$

which, using the characteristic quantities (4.15) of $\varphi(\cdot)$, can be reduced to the following inequality

$$\int_0^T k_1(t)\{\sigma_T(t) - \sigma_T(t-m\mathcal{P})\}\varphi(\sigma_T(t))dt \geq -2\int_0^T k_1(t)\nu_s'\varphi(\sigma_T(t))\sigma_T(t)dt. \quad \text{(C.15)}$$

where $\nu_s' = (\frac{q_{11}}{\zeta_{min}} + q_{22}\zeta_{max} + q_{12})$. We conclude that

$$\int_0^T k_1(t)\sigma_T(t-m\mathcal{P})\varphi(\sigma_T(t))dt \leq (1+2\nu_s')\int_0^T k_1(t)\varphi(\sigma_T(t))\sigma_T(t)\,dt. \quad \text{(C.16)}$$

A similar inequality is valid for $\int_0^T \sigma_T(t+m\mathcal{P})\varphi(\sigma_T(t))dt$. Combining these two, we get

$$\int_0^T \{\sum_{m=1}^{\infty}(z_m\sigma_T(t-m\mathcal{P}) + z_m'\sigma_T(t+m\mathcal{P}))\}k_1(t)\varphi(\sigma_T(t))dt \leq \int_0^T \sigma_T(t)k_1(t)\varphi(\sigma_T(t))dt, \quad \text{(C.17)}$$

if (i) $z_m < 0,\ z_m' < 0,\ m = 1, 2, \cdots$, and (ii) $\sum_{m=1}^{\infty}(|\, z_m \,| + |\, z_m' \,|) < \frac{1}{(1+2\nu_s)}$.

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