SWITCHED DYNAMICAL SYSTEMS: TRANSITION MODEL, QUALITATIVE THEORY, AND ADVANCED CONTROL

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Abstract

Switched Dynamical Systems: Transition Model, Qualitative Theory, and Advanced Control

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This thesis presents a qualitative theory for switched systems and control methods for uncertain switched systems. A transition model of dynamical systems is introduced to obtain a framework for developing qualitative theories. Deriving from the general rule of transition, we obtain a transition model for switched systems carrying the nature of a collection of continuous signals whose evolutions undergo effects of discrete events. The transition mappings are introduced as mathematical description of the continuous motion under interaction with discrete dynamics. Accordingly, results are obtained in terms of the timing properties of discrete advents instead of dynamical properties of the discrete dynamics.

Through the formulation of limiting switching sequences and the quasi-invariance properties of limit sets of trajectories of continuous states, invariance principles are presented for locating attractors in continuous spaces of switched non-autonomous, switched autonomous and switched time-delay systems. The principle of small-variation small-state is introduced for removal of certain limitations of the approach using Lyapunov functions in hybrid space of both continuous and discrete states and the approach imposing the switching decreasing condition on multiple Lyapunov functions on continuous space. The basic observation is that the dwell-time switching events drive the converging behavior and the boundedness of the periods of persistence ensures the boundedness of the diverging behavior of the overall trajectory.

Compactness and attractivity properties of limit sets of trajectories are established for a qualitative theory of switched time-delay systems. It turns out that delay time and time intervals between two dwell-time switching events play the same role of causing instability; furthermore, the Razumikhin condition at switching times is equivalent to the usual switching condition in the sense that they provide the same information on diverging behavior. Accordingly, an invariance principle is obtained for switched time-delay systems and, at the same time, a time-delay approach to stability of delay-free switched systems is introduced.

The gauge design method is introduced for control of a class of switched systems

with unmeasured state and unknown time-varying parameters. The control objective is achieved uniformly with respect to the class of persistent dwell-time switching sequences. Considering the unmeasured dynamics and the controlled dynamics as gauges of each others, we design an adaptive control making the closed-loop system interchangeably driven by the stable modes of these dynamics. In this approach, the unknown time-varying parameter is considered as disturbance whose effect is attenuated through an asymptotic gain. Introducing a condition in terms of observer's poles and gain variations, the gauge design framework is further presented for adaptive output feedback control of the same class of uncertain switched systems.

Adaptive neural control is introduced for a class of uncertain switched nonlinear systems in which the sources of discontinuities making neural networks approximation difficult are uncontrolled switching jumps and the discrepancy between control gains of constituent systems. Neural networks approximations are presented for dealing with unknown functions and a parameter adaptive paradigm is presented for dealing with unknown constant bounds of approximation errors. A condition in terms of design parameters and timing properties of switching sequences is considered for verifying stability conditions.

Thesis' Supervisors: Professor Shuzhi Sam Ge

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Table of Contents

List of Figures

List of Symbols

Chapter 1

Introduction

Dynamical system is a collection of signals evolving under a fixed rule. Signals in real systems are typically of either discrete or continuous nature. Labeling the behavior of the system by the behavior of different groups of signals leads to different classes of systems. Treating discrete signals and continuous signals of a system on an equal footing, we have the concept of *hybrid dynamical system* [3, 6]. Taking the behavior of continuous signals as system behavior and passing the role of event-driving input to discrete signals, we arrive at the notion of *switched dynamical system* [95, 142].

In this thesis, we studies dynamical properties and control of continuous dynamics in dynamical systems consisting of both discrete and continuous signals. The driving question is to make conclusion on ultimate behavior of continuous signals using dwell-time properties [62] of discrete signals. It turns out that richer results can be achieved in the framework of switched dynamical systems.

The presentation is sketched as follows. Looking toward a theory amenable to studying dynamical properties of switched systems under relaxed conditions, we introduce transition models for dynamical systems from which switched systems arise naturally as a special realization of *rule of transition*. We then present various stability theories based on which advanced controls are further developed.

1.1 Motivating Study

At a glance, switched systems are usually described by equations of the form

$$\dot{x}(t) = f_{\sigma(t)}(t, x_t), \tag{1.1}$$

where t is the time variable, $x(t)$ is the state at time t of the system, x_t is the function determined by a trace attached to t of the system trajectory $x(t)$, $\sigma : \mathbb{R}^+ \to \mathbb{Q}$, given the discrete set $\mathbb{Q}$, is the signal describing the dynamics of the discrete state and is usually termed the switching signal, and $f_q, q \in \mathbb{Q}$ are functionals. The discontinuities of σ are termed switching times [95, 142].

We have few notations for discussion. In (1.1), x_t means that the system is delay-dependent. If for all time t, x_t is determined by the single point $x(t)$ of the system trajectory, we then write $x(t)$ instead of x_t to clarify that the system is delay-free. An equation (1.1) with $\sigma(t)$ replaced by a fixed $q \in \mathbb{Q}$ is said to be a switching-free system of (1.1). Given a Lyapunov function V_q for each vector field f_q, the *switching decreasing condition* is: for every $q \in \mathbb{Q}, V_q(x(t))$ is decreasing on the sequence of switching times at which σ either turns to or jumps away q [120, 22]. Dynamical systems described by ordinary differential equations, i.e., equations of the form (1.1) with subscript $\sigma(t)$ dropped, are temporarily called ordinary dynamical systems.

To draw the primary source of the explosion of the area and the current limitation in switched systems, let us consider the simple case of delay-free systems. At the first place, it is worth mentioning that during the long history of the field of ordinary dynamical systems, the celebrated Lyapunov stability theory and LaSalle's invariance principle have always played the principal roles in studying asymptotic behavior of switching-free dynamical systems, i.e., converging properties of the paths of the system state in the state-space. While applications of LaSalle's invariance principle range over a variety of control problems [132, 18, 77], invariant motion is primitive in

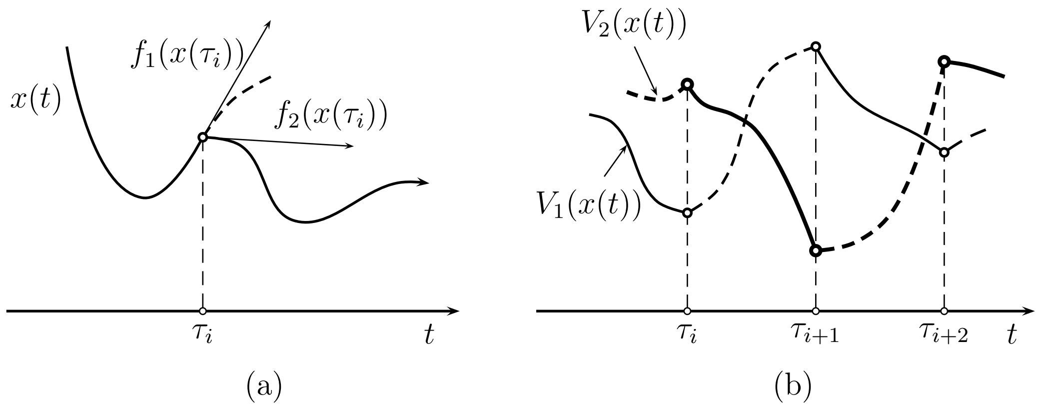

Figure 1.1: Trajectory and Lyapunov functions in switched systems

contemporary control and communication systems [140, 15].

Taking the point of view that evolutions of switched systems and switching-free systems equally draw paths in the state space, it turns out that asymptotic behavior of switched systems can be studied using the framework of ordinary dynamical systems. To this end, we need to establish counterparts in switched systems of well-behaved elements in ordinary dynamical systems such as semi-group property and decreasing condition on Lyapunov function. However, the elegant semi-group property of trajectories of ordinary dynamical systems is lost in switched systems. The behind rationale is: trajectories of switched systems are concatenations of short pieces of trajectories of ordinary dynamical systems, which means that *the behaviors of switched systems are merely determined by transient behaviors of ordinary dynamical systems*. This lays the primary obstacle makes a principal distinction of switched systems.

On the contrary, the mild decreasing condition on Lyapunov function in ordinary dynamical systems has a direct counterpart in switched systems, that is the switching decreasing condition. As witnessed over decades, the switching decreasing condition provides a great convenience in developing stability theories for switched systems following the classical framework dynamical systems [120, 22, 13, 63, 107]. However, unlike the case of ordinary dynamical systems, the switching decreasing condition appears to be restrictive when impose on switched systems.

We have Figure 1.1 to illustrate the loss of the semi-group property and the restrictiveness of the switching decreasing condition in switched systems. There, the set $\mathbb{Q}$ is $\{1, 2\}$ and $\tau_i, \ldots, \tau_{i+2}$ are switching times, V_1 and V_2 are Lyapunov functions of ordinary dynamical systems whose vector fields are f_1 and f_2, and $x(t)$ is the trajectory of the overall system. While the semi-group property states that from the state at a time, we can determine the state at any other time by merely the time interval between these times, Figure 1.1(a) shows that it is impossible to determine the state at a time after τ_i from a state at a time before τ_i without involving the time τ_i at which the vector field of the system is changed. The semi-group property is thus broken in switched systems. Furthermore, in view of Figure 1.1(b), the switching decreasing condition might be desired for convergence. Unfortunately, as the vector fields f_1 and f_2 are independent of each other, the Lyapunov functions are short-time cross-independent along vector fields as well, i.e., in short time periods, behaviors of Lyapunov functions along different vector fields are independent of each other. Thus, in short time periods, decreasing in a Lyapunov function does not prevent the remaining one from increasing. As illustrated in Figure 1.1(b), this may result in diverging behaviors of all Lyapunov functions.

In another consideration, the achieved results in qualitative theory of hybrid systems, which of course applicable to switched systems, usually impose an appropriate semi-group property on system trajectories by either combining the discrete and continuous states into a hybrid state in a merged space or directly making an assumption so that the results can be carried out using the framework of classical theory [154, 30, 102, 126]. It was pointed out that discrete dynamics represented by switching signals have time-varying and hence nonautonomous nature [62, 80]. According to [9], imposing semi-group property in nonautonomous systems leads to conservative results. In light of [58, 9, 62, 80], improvements to qualitative theories for general hybrid systems [154, 30, 102, 126] are well motivated.

Under closer scrutiny, it turns out that models of ordinary differential equations are often only first approximations to the far complex models of the real systems which would include some of the past states [59]. In many applications, time-delay is a source of destabilizing and cannot be ignored [59, 81, 113]. This couples with the important role of switching and delay effect in contemporary systems [5] well motivates studies on switched systems in which the ordinary dynamical systems are delay-dependent. In this merit, a stability theory for this class of switched systems, which can be appropriately termed switched time-delay systems, is not only of theoretical advance but also of practical importance.

In light of the above considerations, a comprehensive stability theory for switched systems might address the destabilizing behavior made by either switching events or time-delay. Nevertheless, while the switching decreasing condition has still played an important role in contemporary stability theories of both (delay-free) hybrid automata and switched systems, stability theory for switched time-delay systems remains open.

By virtue of the vast achievements in control of ordinary dynamical systems [111, 128, 115, 148, 103, 85, 127, 49, 77, 19, 25, 12], well-studied control constraints such as underactuated, unmodeled dynamics, unmeasurable state, and uncertain system models are of either practical or theoretical interests for control of switched systems. However, these constraints often make the widely imposed switching decreasing condition unsatisfiable. Firstly, while this condition requires large decreasing rates, classical adaptive control for handling parameter uncertainties typically exhibits slow parameter convergence rates – an undesired performance in classical adaptive control as well [12]. Though this contradiction can be overcome by means of logic-based switching [65], the problem remains unsolved for systems in which switching signal is not the control variable. Secondly, even if switching logic can be used for control, the unmeasured dynamics makes computation of switching variables based on verification of Lyapunov functions unfeasible. Finally, when only system output is available for control design, state observer is then naturally involved. However, designing an observer fulfilling switching decreasing condition seems to be impossible under fast switching.

Thus, it is not surprising that despite rich achievements in stability analysis, advanced control of switched systems remains in its early stage.

1.2 Early Achievements in the Area

1.2.1 Qualitative Theory

Stability theory of dynamical systems emerged from the foundation works of H. Poincaré, A. M. Lyapunov, and G. D. Birkhoff [17, 56, 131]. In correspondence with significant achievements in the qualitative theory of dynamical systems in Euclidean spaces [83, 90], general dynamical systems in Banach spaces came to interest [58]. It turned out that elegant qualitative properties of dynamical systems in Euclidean spaces such as compactness, invariance, and attractability of limit-sets of trajectories become expensive and are much topology-dependent in the general setting of dynamical systems in Banach spaces [58].

The early efforts to bring out the field of hybrid systems were made through the series of Lecture Notes in Computer Science on Hybrid Systems started with [52, 3, 2]. Since the special issue [4], rich results on qualitative theory of switched/hybrid systems have been actively carried out [22, 154, 102, 30, 13, 63, 107, 126]. Under the primitive assumption on invariant motions of constituent systems/agents, high level control of hybrid systems with operational goals has been addressed [140, 150, 46, 45, 105, 15].

Using hybrid state combined from the discrete and continuous signals to define generalized dynamical systems in merged spaces with axiomatic semi-group property, qualitative theories have been developed for hybrid systems in the framework of classical theory of dynamical systems [108, 102, 126]. It is observed that the topology for convergence in discrete subspace were not appropriately considered.

In [13], a notion of weak invariance in the continuous space was introduced for an invariance principle of switched systems without imposing semi-group property on discrete dynamics. The achieved result is therefore non-conservative. Due to the use of arcs cut-off of one single trajectory, the result gives loose estimates of attractors and is limited to dwell-time switching signals. In [107], another notion of weak invariance is defined via the space of translates of switching signals for a further improvement of LaSalle's invariance principle for switched systems. Though improved estimates of attractors were obtained for the larger class of average dwell-time switched systems, refinement of invariant sets in terms of level sets and hence the structure of attractors has not been studied in [107]. Considering nonlinear norm-observability properties for deriving convergence of a trajectory from its converging segments, LaSalle-like theorems were obtained for asymptotic stability of a more general class of switched systems undergoing regular switching signals [63].

1.2.2 Nonlinear Control

The introduction of the differential geometric approach to nonlinear control made a theoretical clearance for formulating control problems in terms of systems in triangular forms [69]. The emerged facts include: i) under an appropriate transformation, the original model of the interested system can be transformed into a triangular form [69, 70, 28, 34]; and ii) control of systems in triangular forms can be designed in a systematic manner [70,85]. Moreover, if a local transformation was made, then control performance can be specified for preserving the validation of the transformation.

A seminal achievement in control of systems in triangular forms is the backstepping design method undergoing the principle of propagating a desired property through a sequence of augmentations [85]. Stability in backstepping designs is built upon the notion of input-to-state stability and its Lyapunov characterization [133, 136].

During the formulation progress of the notion of input-to-state stability (ISS) as a unification of the notions of Lyapunov stability [56] and input-output stability [159,38], it has turned out that a variety of control problems can be formulated in the framework of input-to-state stability [133,136,135,139]. Particularly, viewing a system in triangular forms with appended dynamics as an interconnection of two separated systems, the superposition property of ISS–Lyapunov functions can be exploited to design a control stabilizing the overall system without measuring the state as well as the Lyapunov function of the appended dynamics [74, 7, 84, 32, 48].

Efforts in dealing with situations of which the control depends on functions whose existence is guaranteed but whose determination is failed gave rise to the field of adaptive neural control [112, 132, 96, 49, 44]. The primary principle is to bring out linear forms of estimation errors amenable to the use of traditional Lyapunov-based adaptive control. Then, parameter estimates can be updated on-line based on the measured regulation error [94,49]. Though the effectiveness of either adaptive control and adaptive neural control ranges over a variety of classes of systems, the parameter update laws usually suffer from discontinuities which switching tends to introduce.

The observation problem arises when there is a need for internal information but only external measurements are available. In nonlinear systems, the notions of controllability and observability were formulated in [60]. Existence of observers for nonlinear systems was studied in [147] through the introduction of the notion of detectability. For nonlinear systems containing a linear part, high-gain observers combined with Lipschitzian condition and singular perturbation were proposed for output feedback controls of nonlinear systems in [20] and [42], respectively.

In summary, the feasibility of nonlinear control is strongly dependent on the problem context. Under certain practical constraints such as unmodelled dynamics, desired information for making switching-logic is not available. In the reversed direction, switching events tend to break feasibility of the control obtained by nonlinear control methods. Control under the combined effects of switching events and practical restrictions therefore deserves study.

1.3 Contribution of the Thesis

The main contributions of the thesis are:

Transition model for dynamical systems. By introducing the notion of rule of transition, we provide a model of dynamical systems amenable to developing qualitative theories. The model generalizes the classical description of dynamical systems as evolution mappings [131] by dropping the particular time transition properties and topological structure of the state space. The behind rationale is to follow the fundamental principle of classical qualitative theory of dynamical systems which states that long-term behavior of a dynamical system is governed by the time transition properties of its motion rather than the specific mechanism generating such motion. In this manner, we expose the facts that i) in order to develop a non-conservative qualitative theory for a dynamical system, the primary step is to identify the defining time transition property of the system; and ii) developing a semi-group property to study long-term behavior shall specialize the class of systems and might give rise to conservative results. For example, in switched systems, including discrete states as part of the limit sets is not meaningful since the discrete parts of these limit sets are usually the whole discrete space.

The notions of switching sequence, transition indicator, and transition mappings for switched systems. With the goal of exposing timing properties of the transition mappings of switched systems, we consider the notion of switching sequence to quantize the evolution of switched systems into running times of constituent dynamical systems. The underlying observation is that though the whole motion of switched systems does not enjoy the semi-group property, the property holds on finite running times of constituent dynamical systems, and hence the transition mapping can be fully determined by the switching sequence, transition indicator, and transition mappings of constituent dynamical systems. The corresponding transition model of switched systems is therefore amenable to the utilization of the achieved results in switched systems and to the development of qualitative theory. By switching sequences, it reflects the observation that stability in switched systems is governed by the timing properties of switching advents rather than the specific mechanism tailoring the advents of switching events [62]. In addition, by switching sequences, there is no preclusion for switching events of zero running times which may occur in limiting behaviors – the main interest in qualitative theories.

A qualitative theory for switched systems. We address the problem of locating attractors of switched systems using auxiliary functions. Instead of merging spaces to bring out a switched autonomous system, we study primitive groups of trajectories generated under fixed switching sequences and develop an invariance principle for the class of switched non-autonomous systems to which switched autonomous systems belong. From these primitives, stronger results can be obtained in terms of uniformity. The results hold over a class of persistent dwell-time switching sequences to which dwell-time and average dwell-time switching sequences are special cases.

We follow the spirit of the original LaSalle's invariance principle that uses decreasing properties of Lyapunov functions to derive the first estimates of attractors, and then uses the characterizing properties of limit sets of trajectories to refine these first estimates. In this spirit, it reveals the fact that the invariance property of limit sets of trajectories of classical dynamical systems is one of the properties amenable to refining the first estimates of attractors, and hence it is more natural to make refinement using typical property of the interested system rather than boiling down to the semi-group and invariance property of classical dynamical systems.

The quasi-invariance property of limit sets of trajectories of switched systems are then revealed through limiting switching sequences. Using this property, we introduce the principle of small-variation small-attractor for an invariance principe of general switched non-autonomous systems without imposing the usual switching decreasing condition. We present the observation that bounded variations are possible via bounded periods of persistence and their compensations can be made in dwell-time intervals. Further invariance principles for switched autonomous systems are then obtained as consequences.

A qualitative theory for switched time-delay systems. We introduce a transition model of switched time-delay systems. Converging behavior of trajectories of switched time-delay systems is studied on the Banach space of continuous functions. We show that bounded trajectories in the Euclidean space give rise to compact and attractive limit sets in the function space. The notion of limiting switching sequence is further utilized to characterize the quasi-invariance property of limit sets.

Treating the delay time and the period of persistence on an equal footing, we show that the decreasing condition on composite Lyapunov function also provides estimates of increments on periods of persistence. Then, we develop a further relaxed invariance principle for switched time-delay systems removing switching decreasing condition. A time-delay approach to delay-free switched systems is presented accordingly.

Small-variation small-state principle for asymptotic gains of switched systems. Looking towards tools for control design of switched systems, we study positive Lyapunov functions for asymptotic gains in switched systems. The principle of small-variation small-state is further studied for relaxed results that do not impose the usual switching decreasing condition. Again, the behind rationale is that small state can be observed from small ultimate variations of auxiliary functions, which can be achieved with dwell-time switching events, while small variations of continuous functions do not impose consistent decrements. For switched time-delay systems, we derive the asymptotic gain via Lyapunov-Razumikhin approach. Upon satisfaction of Razumikhin condition, estimates of past states in terms of current state are available and hence delay-free control is possible.

Gauge design method for uniform-switching adaptive control of switched nonlinear systems. We address the problem of achieving a control objective uniformly with respect to the class of persistent dwell-time switching sequences. Constituent systems whose models contain unknown time-varying parameters and unmeasured dynamics are interested. It is observed that due to unmeasured states, verifying switching conditions using full state feedback is impossible. To overcome this obstacle, we examine the stability characterizations of the appended dynamics such as growth rate, decreasing rate and the timing characterizations of switching sequences such as persistent dwell-time and period of persistence. It turns out that whenever the state of the controlled dynamics is dominated by the unmeasured state, then the desired behavior of the overall system is guaranteed by the stabilizing mode of the unmeasured dynamics, and in the remaining case, i.e., the unmeasured state is dominated by the measured state, estimates of functions of the unmeasured state in terms of the measured state are available and a measured-state dependent control can be designed to make the controlled dynamics the driving dynamics of the overall system. Thus, the gauge design method is introduced undergoing the principle of making the unmeasured dynamics and the controlled dynamics act as gauging dynamics of each others. An important advantage of this method is the allowance of considering unknown time-varying parameters as input disturbance to address the disturbance attenuation problem for switched systems without considering parameter estimates as part of system state and hence increasing difficulty in verifying switching conditions is avoided.

Switching-uniform adaptive output feedback control for switched nonlin-

ear systems. We address the problem of stabilizing the continuous state of uncertain switched systems using only output measurements. The primary difficulty lies in the discrepancy between control gains of constituent systems. The gauge design framework is thus called for an adaptive high-gain observer, in which the dynamics of the whole system is interchangeably driven by the stable modes of the unmeasured dynamics and the coupled dynamics of error variables and state estimates. The resulting output feedback cotnrol scheme is hence of non-separation-principle. It turns out that the observer's poles are no longer arbitrarily assigned as in nonlinear control of continuous dynamical systems and destabilizing terms raised by non-identical control gains might be addressed for non-conservative results. Considering variations in control gains, full-state dependent control gains are allowed.

Switching-uniform adaptive neural control. Adaptive neural control is presented for a class of switched nonlinear systems in which the sources of discontinuities making neural networks approximation difficult are uncontrolled switching jumps and the discrepancy between control gains of constituent systems. Due to switching jumps, neural networks approximations are presented for dealing with unknown functions and a parameter adaptive paradigm is called for dealing with unknown constant bounds of approximation errors. In this way, the orders of functions of signals with discontinuity do not increase as in classical use of adaptive neural control. To deal with discrepancy between control gains, we introduce a discontinuous adaptive neural control and then present its smooth approximation for recursive design. A condition in terms of design parameters and timing properties of switching sequences is considered for verifying stability conditions on the resulting closed-loop system. It is observed that when there is no switching jump, the obtained control achieves the control objective under arbitrary switching.

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Part I

Qualitative Theory

Chapter 2

Transition Model of Dynamical Systems

The main purpose of this chapter is to bring out a model of switched systems from the transition model of general dynamical systems. Among the existing models of switched systems which often involve differential equations for describing subsystems, the transition model of switched systems in this chapter intends to quantize the transition in the continuous space into switching events for studying limiting behavior of switched systems. Due to the fact that zero running time switching events are precluded in the usual description of the discrete dynamics of the underlying hybrid system using piecewise constant right continuous functions, the notions of sequence of switching events and transition indicator are presented for improvement.

2.1 Basic Notations

The notations $\mathbb{N}, \mathbb{R}$ and $\mathbb{R}^+$ denote the sets of nonnegative integers, real numbers, and nonnegative real numbers, respectively. For a $n \in \mathbb{N}$, $\mathbb{R}^n$ is the usual n–dimensional Euclidean space. The notation $|\cdot|$ is used for absolute value of scalars and essential supremum norm of scalar valued functions. We use $\|\cdot\|$ for the Euclidean norm of vectors and essential supremum Euclidean norm of vector valued functions. The notation $\|\cdot\|_F$ is the usual Frobenius norm of matrices. For a subset A of $\mathbb{R}^n$, the distance between a point $x \in \mathbb{R}^n$ and A is $\|x\|_A \stackrel{\text{def}}{=} \inf\{\|x-y\| : y \in A\}$.

We often use $\{\bullet_i\}_i$ to denote infinite sequences $\{\bullet_i\}_{i=0}^{\infty}$. The central dot $\cdot$ represents arguments of functions. For a product set $A = A_1 \times \ldots \times A_n \times \ldots$, $\mathrm{Pr}_i : A \to A_i$ is the i-th coordinate projection mapping, i.e., $\forall i \in \mathbb{N}\backslash\{0\}$, $\mathrm{Pr}_i((a_1, \ldots, a_n, \ldots)) = a_i$. By a time sequence, we means a divergent infinite sequence in $\mathbb{R}^+$.

We shall use the standard notions of comparison functions in [56,133,88]. Consider the continuous functions $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ and $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. The function α is said to be of class-$\mathcal{K}$ if it is strictly increasing and is zero at zero. It is of class–$\mathcal{K}_\infty$ if it is of class–$\mathcal{K}$ and unbounded. The function β is said to be of class–$\mathcal{KL}$ if for each fixed t, the function $\beta(\cdot, t)$ is of class–$\mathcal{K}$ and for each fixed r, $\beta(r, t) \to 0$ as $t \to \infty$. Finally, the function β is said to be of class–$\mathcal{KK}$ if for each fixed r, both functions $\beta(\cdot, r)$ and $\beta(r, \cdot)$ are of class–$\mathcal{K}$.

2.2 Dynamical Systems

The concept of dynamical system has its origin in Newtonian mechanics through the foundation works of H. Poincaré, A. M. Lyapunov, and G. D. Birkhoff [131]. It is a primitive concept whose understanding should be left intuitive in general and precise descriptions of the dynamical system can be postulated in specific applications.

In systems and control, the qualitative properties of dynamical systems are of primary concern and hence models accessible for determining all possible behaviors of the interested dynamical system are of primitive interest.

For the time being, collections of time diagrams of system state and mechanisms for generating such collections are usually called for modeling dynamical systems [151].

In this thesis, we are interested in the transition model of dynamical systems which basically consists of a space and a rule of change of position.

2.2.1 Transition Model

Definition 2.2.1 (dynamical system) *The transition model of dynamical systems is the triple*

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{R}), \tag{2.1}$$

where $\mathbb{T}$ *is a set of real numbers termed time space,* $\mathbb{W}$ *is a set termed the signal space, and* $\mathfrak{R}$ *is the rule of transition that is a map from* $\mathbb{T} \times (\mathbb{T} \times \mathbb{W})$ *to* $\mathbb{W}$*.*

Throughout this thesis, a state of the system is an instant of the signals involved in the system. We shall use the terms *system variables*, *variables in/of the system*, and *state variables* equally in indicating the variables representing instants in time of the signals involved in the system.

Intuitively, by an action of the rule of transition on a point $(t, (s, w)) \in \mathbb{T} \times (\mathbb{T} \times \mathbb{W})$, it means a guided movement in $\mathbb{W}$ from the location w attached to some time s in a time t. We would clarify that s needs not to carry the meaning of the initial time as usual. In the coming model of switched system in Section 2.4.3, it is the time interval since starting for which the system has run to reach the state x.

The above transition model of dynamical systems is equivalent to the behavior model of dynamical systems $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$, where $\mathbb{T}$ and $\mathbb{W}$ are as above and $\mathcal{B}$ is the behavior which is a subset of the set of all maps from $\mathbb{T}$ to $\mathbb{W}$ [151]. In fact, given a rule of transition $\mathfrak{R}$, the set $\mathcal{B} = \{\mathfrak{R}(\cdot, t, w) : (t, w) \in \mathbb{T} \times \mathbb{W}\}$ is a behavior. Conversely, given a behavior $\mathcal{B}$, it is a rule of transition the map $\mathfrak{R}$ defined by $\mathfrak{R}(t, s, w) = \beta(t+s, w), \forall t \in \mathbb{T}, t+s \in \mathbb{T}$ if there is some $\beta \in \mathcal{B}$ such that $\beta(s) = w$ and, for a $t \in \mathbb{T}$, $\mathfrak{R}(t, s, w) = w$ if either $t + s \notin \mathbb{T}$ or no such β exists.

Though the transition and behavior models are equivalent, we are interested in the former one as the rule of transition naturally describes the time transition property along the trajectory of the system which is necessary for accessing the invariance properties of limit sets of trajectories in making conclusions on the long-term behaviors of the systems. This observation has its well root in the classical qualitative theory of dynamical systems [56, 130, 91, 137, 8, 108, 29].

Finally, it is worth mentioning that the above notion of rule of transition does not impose $\mathfrak{R}(t,t,w) = w$ as in the classical notion of motion [56]. As will be annotated in the next sections, this makes (2.1) capable of modeling a large class of real systems.

2.2.2 Equivalence in Classical Models

As well analyzed in [151], the behavior model of dynamical systems respects the nature and hence gives a closer description of the real system. As a result, any model of dynamical system introduced so far including hybrid automata and switched systems ought to have an equivalent behavior model and hence an equivalent transition model. Here, we make manifest the realization of the rule of transition in the classical models of autonomous and non-autonomous dynamical systems.

Definition 2.2.2 ([129]) *Let $\mathcal{X}$ be a topological space, a dynamical system on $\mathcal{X}$ is a continuous mapping $\pi : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ that satisfies the following properties:*

i) $\pi(0,x) = x, \forall x \in \mathcal{X}$*; and*

ii) $\pi(t_2, \pi(t_1,x)) = \pi(t_1 + t_2, x), \forall t_1, t_2 \in \mathbb{R}$.

Definition 2.2.3 ([29]) *Let $\mathcal{X}$ and $\mathcal{W}$ be topological spaces. A non-autonomous dynamical system on $\mathcal{X}$ with base space $\mathcal{W}$ is a couple (π, φ) in which*

i) π is a dynamical system on $\mathcal{W}$ in the sense of Definition 2.2.2; and

ii) $\varphi : \mathbb{R}^+ \times \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ is a cocycle mapping on $\mathcal{X}$, i.e., φ is continuous, $\varphi(0,x,w) = x, \forall x \in \mathcal{X}, w \in \mathcal{W}$, and for all $t_1, t_2 \in \mathbb{R}^+, x \in \mathcal{X}, w \in \mathcal{W}$, we have

$$\varphi(t_1 + t_2, x, w) = \varphi(t_2, \varphi(t_1, x, w), \pi(t_1, w)).$$

The mappings π and φ in the above definitions are usually called the evolution/transition mappings. We shall alternatively call φ the non-autonomous dynamical system without embarrassment. In the context of the general dynamical system in Definition 2.2.1, we call the systems in Definitions 2.2.2 and 2.2.3 the ordinary autonomous dynamical system (OADS) and ordinary non-autonomous dynamical system (ONADS), respectively.

The immediate equivalent transition model of the OADS is the one whose time and signal spaces are $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathcal{X}$, and whose rule of transition is $\mathfrak{R}(t, s, x) = \pi(t, x), \forall t, s \in \mathbb{T}, \forall x \in \mathbb{W}$. Also, an equivalent transition model of the ONADS is the one whose time and signal spaces are $\mathbb{T} = \mathbb{R}^+, \mathbb{W} = \mathcal{W} \times \mathcal{X}$, and whose rule of transition is $\mathfrak{R}(t, s, (w, x)) = (\pi(t, w), \varphi(t, x, w)), \forall s, t \in \mathbb{T}, \forall (w, x) \in \mathbb{W}$.

2.2.3 Trajectory, Motion, Attractor, and Limit Set

The transition model in Definition 2.2.1 tends to a model applicable to all possible dynamical systems by calling for three basic elements any dynamical system ought to have. To classify dynamical systems, more properties on the rule of transition are considered. The qualitative theory of dynamical systems classifies the systems by their limiting behavior such as stability, instability, periodicity, and chaos. In this aspect, the primitive element is trajectory and the primitive qualitative notions are motion, attractor, and limit set [56, 130].

Likewise, as $\mathbb{W}$ models all signals involved in the system, it is natural to divide $\mathbb{W}$ into subspaces when classification of signals is desired. By virtue of the behavioral theory of dynamical systems [151], the variables representing instants of signals in a system can be classified into manifest and latent variables. Continuing this idea, we shall use $\mathbb{W}_{\mathcal{M}} \times \mathbb{W}_{\mathcal{L}}$ to denote $\mathbb{W}$ when it is desired the clarification between the space of manifest variables $\mathbb{W}_{\mathcal{M}}$ and the space of latent variables $\mathbb{W}_{\mathcal{L}}$. Let us begin with the primitive notion of trajectory for systems described by the transition models.

Definition 2.2.4 (trajectory) *Let $\Sigma = (\mathbb{T}, \mathbb{W}_{\mathcal{M}} \times \mathbb{W}_{\mathcal{L}}, \mathfrak{R})$ be a dynamical system. Let $(x_{\mathcal{M}}, x_{\mathcal{L}}) \in \mathbb{W}_{\mathcal{M}} \times \mathbb{W}_{\mathcal{L}}$ and $s \in \mathbb{T}$ fixed. The $(s, x_{\mathcal{L}})$–interacting trajectory through the point $x_{\mathcal{M}} \in \mathbb{W}_{\mathcal{M}}$ in the manifest space of the system is the set $\mathscr{O}_{s,x_{\mathcal{L}}}(x_{\mathcal{M}}) = \{y_{\mathcal{M}} \in \mathbb{W}_{\mathcal{M}} : \exists (t, y_{\mathcal{L}}) \in \mathbb{T} \times \mathbb{W}_{\mathcal{L}}, (y_{\mathcal{M}}, y_{\mathcal{L}}) \in \mathfrak{R}(t, s, (x_{\mathcal{M}}, x_{\mathcal{L}}))\}$.*

Let $t_{\mathbb{T}} = \inf\{t : t \in \mathbb{T}\}$ and $t^{\mathbb{T}} = \sup\{t : t \in \mathbb{T}\}$. Adopting the classical notion of motion [129], we have the following notions of motion, attractor, and limit set for dynamical systems described by transition models.

Definition 2.2.5 (motion) *Let $\Sigma = (\mathbb{T}, \mathbb{W}_{\mathcal{M}} \times \mathbb{W}_{\mathcal{L}}, \mathfrak{R})$ be a dynamical system, in which $\mathbb{W}_{\mathcal{M}}$ is a topological space. Let $(x_{\mathcal{M}}, x_{\mathcal{L}}) \in \mathbb{W}_{\mathcal{M}} \times \mathbb{W}_{\mathcal{L}}$ and $s \in \mathbb{T}$ fixed. For each $t \in \mathbb{T}$, the $(t, s, x_{\mathcal{L}})$–motion through $x_{\mathcal{M}} \in \mathbb{W}_{\mathcal{M}}$ is the set $\mathfrak{R}_{s,x_{\mathcal{L}}}(x_{\mathcal{M}})(t) = \{y_{\mathcal{M}} \in \mathbb{W}_{\mathcal{M}} : \exists y_{\mathcal{L}} \in \mathbb{W}_{\mathcal{L}}, (y_{\mathcal{M}}, y_{\mathcal{L}}) \in \mathfrak{R}(t, s, (x_{\mathcal{M}}, x_{\mathcal{L}}))\}$.*

Definition 2.2.6 (attractor) *Let $\Sigma = (\mathbb{T}, \mathbb{W}_{\mathcal{M}} \times \mathbb{W}_{\mathcal{L}}, \mathfrak{R})$ be a dynamical system, in which $\mathbb{W}_{\mathcal{M}}$ is a topological space. Let $\mathcal{A} \subset \mathcal{D} \subset \mathbb{W}_{\mathcal{M}}$ and $x_{\mathcal{L}} \in \mathbb{W}_{\mathcal{L}}$ fixed. Then, the set $\mathcal{A}$ is said to be an $(s, x_{\mathcal{L}})$–interacting attractor of Σ with basin of attraction $\mathcal{D}$ if for all $x_{\mathcal{M}} \in \mathcal{D}$, the motion $\mathfrak{R}_{s,x_{\mathcal{L}}}(x_{\mathcal{M}})(t)$ topologically converges to $\mathcal{D}$ as $t \to t^{\mathbb{T}}$.*

Definition 2.2.7 (limit set) *Let $\Sigma = (\mathbb{T}, \mathbb{W}_{\mathcal{M}} \times \mathbb{W}_{\mathcal{L}}, \mathfrak{R})$ be a dynamical system, in which $\mathbb{W}_{\mathcal{M}}$ is a topological space. Let $(x_{\mathcal{M}}, x_{\mathcal{L}}) \in \mathbb{W}_{\mathcal{M}} \times \mathbb{W}_{\mathcal{L}}$ and $s \in \mathbb{T}$ fixed. The ω–limit set of the $(s, x_{\mathcal{L}})$–interacting trajectory $\mathscr{O}_{s,x_{\mathcal{L}}}(x_{\mathcal{M}})$ is the set*

$$\omega_{s,x_{\mathcal{L}}}(x_{\mathcal{M}}) = \bigcap_{T \geq t_{\mathbb{T}}} \overline{\bigcup_{t \geq T} \mathfrak{R}_{s,x_{\mathcal{L}}}(x_{\mathcal{M}})(t)}. \tag{2.2}$$

We would mention that the above notions of attractor and limit set have their very primary root in the theory of pullback attractor of ordinary non-autonomous dynamical systems [79, 29]. This theory carried out the fact that though the limit sets of trajectories of non-autonomous dynamical systems are not invariant in general, their non-autonomous limit sets – defined in terms of the interaction with the backward motion of the time-varying parameters – are invariant. Inspired by this fact, we introduce (2.2) with the following observation.

The invariance property of trajectories of ordinary autonomous systems is due to the semi-group property of their transition mappings. As for the general model (2.2.1), there is no restriction on the transition mapping $\mathfrak{R}$, there is no conclusion on invariance of the limit sets can be made. However, by dividing the signal space $\mathbb{W}$ into manifest and latent spaces to bring out the role of the latent variables in tailoring the trajectory of the manifest variables, it suggests that the dynamics of the latent variables can drive the limit sets for invariance. In this thesis, attaching switching sequences to the backward motion of the time-varying parameters for a rule of transition of latent variables consisting of switching sequences and the time-varying parameters, an invariance property is proven for the corresponding non-autonomous ω-limit sets of switched non-autonomous systems.

Finally, when the manifest space $\mathbb{W}_{\mathcal{M}}$ is the whole space $\mathbb{W}$, the prefix "$(s, x_{\mathcal{L}})$-interacting" in the above definitions shall be dropped accordingly.

2.3 Hybrid Systems

The transition model of dynamical systems (2.2.1) at a high level of generality calls for the basic elements that a dynamical system ought to have. While the time and signal spaces usually available from the designation of the interested variables, specification of the rule of transition commits an important role to analyzing mutual effects between signals in the systems.

In most applications, there are two types of signals: discrete signals taking values in discrete sets and continuous signals taking values in continuums. While the rule of transition of the continuous variables and the influence of discrete variables on their dynamics can be sufficiently described by a set of transition mappings of ordinary dynamical systems labeled by discrete values of the discrete variables, rule of transition of the discrete variables are usually of logical description which may be far more complicate for treatability. For the purpose of studying switched systems, we introduce in this section the notion of sequence of switching events for describing the discrete dynamics at the lowered level of abstraction.

2.3.1 Hybrid Transition Model

In the following, $\mathbb{Q}$ is the usual discrete set and $\mathbb{E} = \mathbb{Q} \times \mathbb{Q}$. In the formal language of hybrid automata, we call elements of $\mathbb{E}$ the edges.

Definition 2.3.1 *A transition in the discrete set* $\mathbb{Q}$ *is a sequence* $\sigma = \{(e_i, \Delta\tau_i)\}_i \subset \mathbb{E} \times \mathbb{R}^+$ *satisfying* $\mathrm{Pr}_1(e_0) = \mathrm{Pr}_2(e_0)$ *and* $\mathrm{Pr}_1(e_i) = \mathrm{Pr}_2(e_{i-1}), i \geq 1$. *For each* $i \in \mathbb{N}$, *the pair* $(\mathrm{Pr}_2(e_i), \Delta\tau_i)$ *is called the* i*-th switching event of* σ.

Definition 2.3.2 *A rule of transition in the discrete set* $\mathbb{Q}$ *is a collection* $\mathfrak{R}_{\mathbb{Q}} = \{\sigma_\gamma : \gamma \in \mathcal{I}\}$ *of transitions in* $\mathbb{Q}$, *where* $\mathcal{I}$ *is an index set.*

Intuitively, the discrete dynamics in $\mathbb{Q}$ can be described as follows. The discrete state is initiated at $q_0 = \mathrm{Pr}_1(e_0)$ at some time t_0. Then, at the time $t_1 = t_0 + \Delta\tau_0$, it is transferred to $q_1 = \mathrm{Pr}_2(e_1)$ at which the process continues. We have the following notion of hybrid system.

Definition 2.3.3 (hybrid system) *A hybrid dynamical system is a hexad*

$$\Sigma_{\mathscr{H}} = \left(\mathbb{R}^+, \mathbb{Q}, \mathbb{X}, \{\psi_q\}_{q\in\mathbb{Q}}, \mathfrak{R}_{\mathbb{Q}}, \quad \right), \tag{2.3}$$

where $\mathbb{Q}$ *is a discrete set which is the space of the discrete signals,* $\mathbb{X}$ *is a topological* *space which is the space of the continuous signals,* $\psi_q : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}, q \in \mathbb{Q}$ *are OADS on* $\mathbb{X}$, $\quad : \mathbb{R}^+ \times \mathbb{Q} \times \mathbb{X} \to \mathbb{X}$ *is the transition map of the continuous state, and* $\mathfrak{R}_{\mathbb{Q}} = \{\sigma_\gamma : \gamma \in \mathcal{I}\}$ *is the rule of transition in* $\mathbb{Q}$ *in which each switching events of any transition* σ_γ *is a map from* $\mathbb{R}^+ \times \mathbb{Q} \times \mathbb{X}$ *to* $\mathbb{Q} \times \mathbb{R}^+$*, and the first coordinate of the first switching event of any transition is a constant function.*

The evolution of the hybrid system (2.3.3) can be logically described as follows. Given a transition $\sigma \in \mathfrak{R}_{\mathbb{Q}}$ whose sequence of switching events is $\{(e_{\sigma,i}(\cdot), \Delta\tau_{\sigma,i}(\cdot))\}_i$. Let $q_{\sigma,i} = \mathrm{Pr}_1(e_{\sigma,i})$. The continuous state of the system evolves from the initial state $x_{\sigma,0} \in \mathbb{X}$ at the initial time $t_{\sigma,0}$ under the transition mapping $\psi_{q_{\sigma,0}}$ until the time $t_{\sigma,1} = t_{\sigma,0} + \Delta\tau_{\sigma,0}(t_{\sigma,0}, q_{\sigma,0}, x_{\sigma,0})$. At the time $t_{\sigma,1}$ the continuous state is transferred from $x_1^- = \psi_{q_{\sigma,0}}(\Delta\tau_{\sigma,0}(\cdot), x_{\sigma,0})$ to $x_{\sigma,1} = \quad (t_{\sigma,1}, q_{\sigma,0}, x_{\sigma,1}^-)$ according to the map $\quad$, and the discrete state is transferred to $q_{\sigma,1} = \mathrm{Pr}_2(e_{\sigma,1}(t_{\sigma,1}, q_{\sigma,0}, x_{\sigma,1}^-))$, and a new running time $\Delta\tau_{\sigma,1} = \Delta\tau_{\sigma,1}(t_{\sigma,1}, q_{\sigma,0}, x_{\sigma,1}^-)$ is computed. Then, the process continues.

We now specify for the hybrid system $\Sigma_{\mathscr{H}}$ its basic elements in the general framework of the transition model of dynamical systems. The time and signal spaces are $\mathbb{T} = \mathbb{R}^+$ and $\mathbb{W} = \mathbb{E} \times \mathbb{X}$. In hybrid systems, all variables are manifest. To specify the rule of transition, it is observed that the rule of transition include two parts: rule for transition of continuous variables determined by $\{\psi_q\}_{q \in \mathbb{Q}}$ and $\quad$, and rule for transition of discrete variables determined by $\mathfrak{R}_{\mathbb{Q}}$.

From the above analysis, for each σ, the sequences $\{t_{\sigma,i}\}_i$ and $\{x_{\sigma,i}\}_i$ are well-defined. For a time $t \in \mathbb{R}^+$, let $i_{\mathcal{J}}(t)$ the largest integer satisfying $t_{\sigma, i_{\mathcal{J}}(t)} \leq t$. Let $\quad$ be a fictitious element of $\mathbb{E}$, and define the map $\quad_\sigma : \mathbb{R}^+ \to \mathbb{E}$ defined by $\quad_\sigma(t_{\sigma,i}) = e_{\sigma,i}$ and $\quad_\sigma(t) = \quad$ for $t \notin \{t_{\sigma,i}\}$. Let $\mathfrak{R}_\sigma$ be the map $\mathfrak{R}_\sigma : \mathbb{T} \to \mathbb{W}$ defined by $\mathfrak{R}_\sigma(t; t_{\sigma,0}, e_{\sigma,0}, x_{\sigma,0}) = (\quad_\sigma(t), \psi_{q_{i_{\mathcal{J}}(t)}}(t - t_{i_{\mathcal{J}}(t)}, x_{i_{\mathcal{J}}(t)}))$. Then, $\mathfrak{R}(t, s, (e, x)) \stackrel{\text{def}}{=} \{\mathfrak{R}_\sigma(t; s, e, x) : \sigma \in \mathfrak{R}_{\mathbb{Q}}\}$ is the rule of transition of $\Sigma_{\mathscr{H}}$.

2.3.2 A Comparison

To make manifest the right of the above transition model of hybrid systems, let us consider the following well-known model of hybrid systems reformulated in terms of the above notions.

Definition 2.3.4 ([23]) *A hybrid dynamical system is a hexad*

$$\Sigma_{\mathcal{H},\mathrm{B}} = \big(\mathbb{R}^+, \mathbb{Q}, \mathbb{X}, \{\psi_q\}_{q\in Q}, \mathbf{A}, \mathbf{G}\big), \tag{2.4}$$

where $\mathbb{Q}$ *is the space of the discrete state,* $\mathbb{X}$ *is a topological space which is the space of the continuous state,* $\psi_q : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}, q \in \mathbb{Q}$ *are OADSs,* $\mathbf{A} = \{A_q\}_{q\in\mathbb{Q}}, A_q \subset \mathbb{X}$ *is the set of the jump sets, and* $\mathbf{G} = \{G_q\}_{q\in\mathbb{Q}}, G_q : A_q \to \mathbb{Q}\times\mathbb{X}$ *is the set of the jump transition maps.*

According to [21], the dynamics of $\Sigma_{\mathcal{H},\mathrm{B}}$ is as follows. Starting from a hybrid state $(q_0, x_0) \in \mathbb{Q}\times\mathbb{X}$ at a time t_0, the system evolves according to $x(t) = \psi_{q_0}(t-t_0, x_0), t \in \mathbb{R}^+$ until $x(t)$ enters (if ever) A_{q_0} at the point $x_1^- = \psi_{q_0}(t_1-t_0, x_0)$ at a time t_1. At the time t_1, the transfer $(q_1, x_1) = G_{q_0}(x_1^-)$ is made. Then, from the hybrid state (q_1, x_1) at time t_1, the process continues.

From the above analysis, the time sequence $\{t_i\}_i$ is well-defined. A time $t_i, i \in \mathbb{N}, i > 0$ is determined by the event "$x(t)$ *enters the set* $A_{q_{i-1}}$." As for each $i \in \mathbb{N}\backslash\{0\}$ the autonomous system $\psi_{q_{i-1}}$ is deterministic, the set $A_{q_{i-1}}$ is given *a priori*, and the state x_{i-1} was determined from the previous transition event, the time t_i at which $x(t)$ enters $A_{q_{i-1}}$ is computable from x_{i-1}. As such, for each $i \in \mathbb{N}, i \geq 1$, $\Delta\tau_i \stackrel{\text{def}}{=} t_i - t_{i-1}$ is a function of q_{i-1} and x_{i-1}. Hence, the transition $\sigma = \{((q_{i-1}, q_i), \Delta\tau_i)\}_i$ is well-defined and is an element of the rule of transition $\mathfrak{R}_{\mathbb{Q}}$. Furthermore, let $(q, x) = \mathrm{Pr}_2(G_q(x))$ if $x \in A_q$ and $(q, x) = x$, otherwise. Thus, the transition is well defined, and hence $\Sigma_{\mathcal{H},\mathrm{B}}$ well induces $\Sigma_{\mathcal{H}}$.

Finally, it is worth mentioning that the rule of transition with the last two arguments fixed is a motion defined in [108]. As such, in some aspect, the hybrid transition model in Definition 2.3.3 and the motion-based hybrid model in [108] are equivalent. However, it is a trade-off for its very high level of abstraction the interacting dynamics is hidden in the motion-based hybrid model in [108]. This leads to the fact that the general results achieved in [108] implicitly impose the switching decreasing condition when realizing to switched systems. As the model of hybrid systems in Definition 2.3.3 separates the rule of transition of continuous dynamics and the rule of transition of discrete dynamics, the theory in thesis accepts more relaxed condition, in particular, the switching decreasing condition is no longer used. On the other hand, the model in Definition 2.3.3 is capable of modeling hybrid systems in which discrete variable may exhibit random dynamics, i.e., a transition of both continuous and discrete states can come on the scene at any time. While systems of this property are of normal interest in studying switched systems, the model in [21] does not describe this class of systems. As such, in the context of switched systems, the model in Definition 2.3.3 can be considered as an improvement of the models in [108] and [21].

2.4 Switched Systems

2.4.1 Transition Model

Classifying the continuous and discrete dynamics of hybrid systems into manifest and latent dynamics, respectively, and then studying the continuous dynamics under the influence of the discrete dynamics gives rise to another model of hybrid systems termed switched system. It turns out that the converging behavior of the continuous state is usually governed by the time properties, particularly the dwell-time property, of the rule of discrete transition rather than the specific model of the discrete dynamics [62]. Thus, it is convenience to describe the rule of discrete transition as behaviors in a timing space. In this way, either state-dependent discrete transition or state-independent discrete transition can be dealt with, and hence, in some aspect, the resulting model of switched systems might be more general than the existing models of hybrid systems. In light of this merit, the rule of transition of the discrete (latent) variables can be formulated in terms of the following notion of switching sequence.

Definition 2.4.1 *Let $\mathbb{Q}$ be a discrete set. A switching sequence in $\mathbb{Q}$ is a sequence $\sigma = \{(q_i, \Delta\tau_i)\}_i \subset \mathbb{Q} \times \mathbb{R}^+$. For each $i \in \mathbb{N}$, the pair $(q_i, \Delta\tau_i)$ is called the i-th switching event of σ, and the number $\Delta\tau_i$ is called the i-th running time of σ and the running time of the i-th switching event of σ.*

We have the following notion of switched system.

Definition 2.4.2 (switched autonomous system) *A switched autonomous system is a hexad*

$$\Sigma_{\mathcal{A}} = \left(\mathbb{R}^+, \mathbb{Q}, \mathbb{X}, \{\psi_q\}_{q\in\mathbb{Q}}, \mathbb{S}, \quad \right), \tag{2.5}$$

where $\mathbb{Q}$ is a discrete set which is the space of the discrete signals, $\mathbb{X}$ is a topological space which is the space of the continuous signals, $\psi_q : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}, q \in \mathbb{Q}$ are OADS on $\mathbb{X}$, $\mathbb{S}$ is a collection of switching sequences, and $\quad : \mathbb{R}^+ \times \mathbb{Q} \times \mathbb{X} \to \mathbb{X}$ is the discrete transition map of the continuous state.

In comparison to the transition model of hybrid systems in Definition 2.3.3, the influence of the continuous variables on the dynamics of the discrete variable has been hidden in the set of switching sequences $\mathbb{S}$. The manifest space is now $\mathbb{W}_{\mathcal{M}} = \mathbb{X}$ and the latent space is $\mathbb{W}_{\mathcal{L}} = \mathbb{S}$. In switched systems, the rule of transition $\mathfrak{R}$ is referred to the transition in the space $\mathbb{X}$ of manifest continuous variables.

In the following, we shall call $\psi_q, q \in \mathbb{Q}$ the constituent systems or subsystems of $\Sigma_{\mathcal{A}}$ and the variable q taking values in $\mathbb{Q}$ the switching index. For a $\sigma \in \mathbb{S}$ and for the i-th switching event of σ, $(q_{\sigma,i}, \Delta\tau_{\sigma,i})$, the number $\Delta\tau_{\sigma,i}$ is also called a running time of the respective component system $\psi_{q_{\sigma,i}}$.

Similar to hybrid systems, the evolution of switched system $\Sigma_{\mathscr{A}}$ is as follows. Given a switching sequence $\sigma \in \mathbb{S}$ whose sequence of switching events is $\{(q_{\sigma,i}, \Delta\tau_{\sigma,i})\}_i$. From an initial state $x_0 \in \mathbb{X}$ at some initial time t_0, the system evolves under the transition mapping $\psi_{q_{\sigma,0}}$ until the time $t_1 = t_0 + \Delta\tau_{\sigma,0}$ is reached. At the time t_1 the system state is transferred from $x_1^- = \psi_{q_{\sigma,0}}(\Delta\tau_{\sigma,0}, x_0)$ to $x_1 = \quad (t_1, q_{\sigma,0}, x_{\sigma,1}^-)$ according to the map , and the transition mapping is switched to $\psi_{q_{\sigma,1}}$. Then, the process continues.

At this place, it is worth comparing the notion of switched systems in Definition 2.4.2 to the usual yet simple way for modeling switched systems, i.e., using piecewise constant right-continuous signals σ to model switched systems by equations of the from (1.1) (see Chapter 1) [95, 142]. In the following, by a change of switching index from q_1 to q_2, it means the change of the transition mapping from ψ_{q_1} to ψ_{q_2}.

In hybrid and switched systems $\Sigma_{\mathscr{H}}$ and $\Sigma_{\mathscr{A}}$, a switching interval of the length zero is meaningful either logically or physically. Let $[t_{i-1}, t_i]$ be a such interval, i.e., $t_{i-1} = t_i$. The simple description of the dynamics on this interval is: right at the time $t_i = t_{i-1}$ the continuous state of the system and the switching index are transferred to x_{i-1} and q_{i-1}, respectively, they are transferred further to another state x_i and another index q_i, respectively. Furthermore, if we consider $t_i - t_{i-1}$ as the running time of the system, then a zero running time of the system physically can be: we lock the system and switch its structure around before starting the system again. Unfortunately, the right-continuous convention on switching signals do not describe this important behavior as the mathematical object $[t, t)$ is undefined. Furthermore, as shall be clear, though it can be assumed that the switching intervals are all non-zero, the limiting behavior may exhibit zero length switching intervals. As such, the above notion of switching sequences gives an obvious improvement.

Though the mechanism of evolution of switched systems has described, we need some further following notations for specifying the rule of transition in the transition model of switched systems $\Sigma_{\mathscr{A}}$.

2.4.2 Notations on Switching Sequences

In this subsection, we make some notations on switching sequences that will be used throughout the thesis.

The set of all switching sequences is denoted by $\mathbb{S}$. For a switching sequence $\sigma \in \mathbb{S}$, the notations $e_{\sigma,i}, q_{\sigma,i}$, and $\Delta\tau_{\sigma,i}$ express that $e_{\sigma,i} = (q_{\sigma,i}, \Delta\tau_{\sigma,i})$ is the i-th switching event of σ. For $i \in \mathbb{N}$, the number $\tau_{\sigma,i} = 0$ if $i = 0$ and $\tau_{\sigma,i} = \sum_{j=0}^{i-1} \Delta\tau_{\sigma,j}$, otherwise, is called the starting time of the i-th switching event of σ. We shall call $\Delta\tau_{\sigma,i}$ both the i-th running time of σ and the running time of the i-th switching event.

Associated to each switching sequence $\sigma \in \mathbb{S}$, we have the following useful operator $i_\sigma^-(\cdot)$ which shall be used throughout the thesis. Let $t \in \mathbb{R}^+$ be a nonnegative number, $i_\sigma^-(t)$ is the largest integer satisfying $\tau_{\sigma,i_\sigma^-(t)} \leq t$, i.e., $i_\sigma^-(t) = \max\{i \in \mathbb{N} : \tau_{\sigma,i} \leq t\}$. We shall call i_σ^- the transition indicator.

Definition 2.4.3 *A switching sequence is said to be non-blocking at a time $t \in \mathbb{R}^+$ if the number of its switching events of zero running time at t is finite. It is said to be non-blocking if it is non-blocking at every time $t \in \mathbb{R}^+$*

According to [62], switching sequences can be further classified based on dwell-time properties as follows.

Definition 2.4.4 *A switching sequence $\sigma \in \mathbb{S}$ is said to have*

i) a dwell-time $\tau_{\mathrm{d}} > 0$ if $\Delta\tau_{\sigma,i} \geq \tau_{\mathrm{d}}, \forall i \in \mathbb{N}$;

ii) a persistent dwell-time τ_{p} with chatter bound of persistence N_{p} if it has an infinite number of running times of the length no smaller than τ_{p} and the number of switching events between every two consecutive switching events of the running times no smaller than τ_{p} is bounded by N_{p}; and

iii) a persistent dwell-time τ_{p} with period of persistence T_{p} if it has an infinite number of switching events whose running times are not smaller than τ_{p} and *for every two consecutive switching events $e_{\sigma,i}$ and $e_{\sigma,j}$ of this property, we have $\tau_{\sigma,j} - \tau_{\sigma,i+1} \le T_{\mathrm{p}}$;*

We shall denote by $\mathbb{S}_{\mathscr{D}}[\tau_{\mathrm{d}}]$ the set of all switching sequences having the same dwell-time τ_{d}, by $\mathbb{S}_{\mathscr{A}}[\tau_{\mathrm{p}}, N_{\mathrm{p}}]$ the set of all switching sequences having the same persistent dwell-time τ_{p} with the same chatter bound of persistence N_{p}, and by $\mathbb{S}_{\mathscr{P}}[\tau_{\mathrm{p}}, T_{\mathrm{p}}]$ the set of all switching sequences having the same persistent dwell-time τ_{p} with the same period of persistence T_{p}.

To close this subsection, let us mention that the purpose of introducing $\mathbb{S}_{\mathscr{A}}[\tau_{\mathrm{p}}, N_{\mathrm{p}}]$ is to include switching sequence possessing zero running times. While the notions of dwell-time and persistent dwell-time with a period of persistence switching sequences have been well-recognized [95,62,142], the notion of persistent dwell-time with chatter bound of persistence switching sequence is a modification of the notion of average dwell-time switching signal in the literature [62]. It can be verified that the switching sequences in $\mathbb{S}_{\mathscr{A}}[\tau_{\mathrm{p}}, N_{\mathrm{p}}]$ with no zero running time are equivalent to switching signals of average dwell-time $\tau_{\mathrm{a}} = \tau_{\mathrm{p}}/(N_{\mathrm{p}} + 2)$ and chatter bound $N_{\mathrm{a}} = N_{\mathrm{p}}$.

2.4.3 Continuous Transition Mappings

In this subsection, we introduce the notions of continuous transition mappings for realizing the rule of transition of switched systems in terms of transition mappings of the component systems. In this merit, the continuous transition mappings of switched systems play the important role of carrying semi-group properties of the component systems in their respective running times.

Autonomous Mappings

Consider the general switched system $\Sigma_{\mathscr{A}}$ in Definition 2.4.2. We first consider the case that the discrete dynamics causes no jump in continuous state, i.e., the following condition holds.

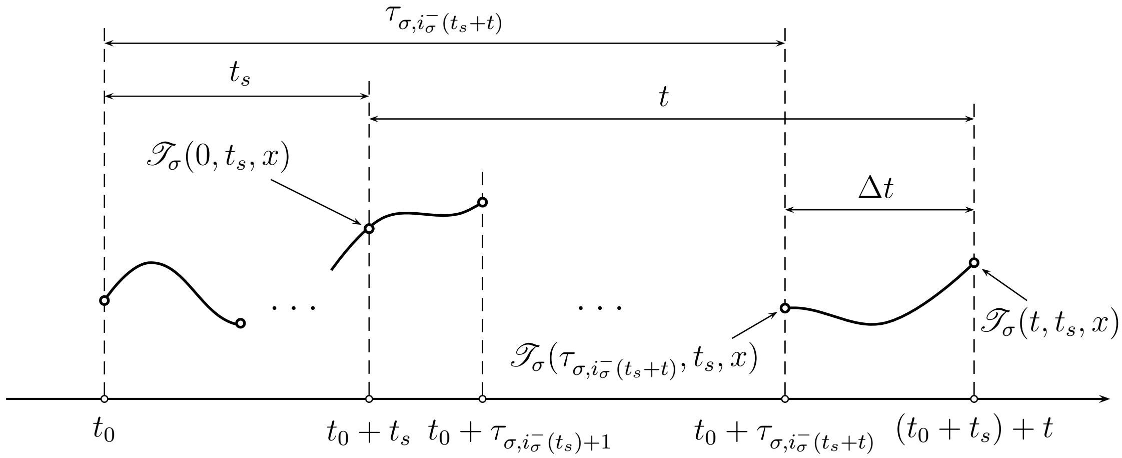

Figure 2.1: Trajectory of switched system: t_0 – the real starting time, t_s – the time elapsed from t_0 to the real time $t_0 + t_s$ at which the system state was read as x, t – the time elapsed from $t_0 + t_s$ to the current real time $(t_0 + t_s) + t$, $\tau_{\sigma,i_\sigma^-(t_s+t)}$ – the time elapsed from t_0 to the real time at which the most recent switch occurred, and Δt – the time elapsed from $t_0 + \tau_{\sigma,i_\sigma^-(t_s+t)}$ to the current real time.

Assumption 2.4.1 *There is no jump in system state, i.e., the discrete transition mapping   is the identity mapping in its third argument:*

$$(t, q, x) = x, \forall (t, q, x) \in \mathbb{R}^+ \times \mathbb{Q} \times \mathbb{X}. \tag{2.6}$$

For a switching sequence $\sigma \in \mathbb{S}$ and for a number t_s, let $(t_s)^b_\sigma$ be the largest number greater than t_s to which there is no blocking time of σ in the interval $[t_s, (t_s)^b_\sigma)$. Let $\mathbb{x}$ be a fictitious element for $\mathbb{X}$. We have the mappings $\mathscr{T}_{\sigma,\mathscr{A}} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{X}, \sigma \in \mathbb{S}, t_s \in \mathbb{R}^+$ defined as $\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x) = \mathbb{x}, \forall t \geq (t_s)^b_\sigma - t_s, t_s \in \mathbb{R}^+, x \in \mathbb{X}$, and

$$\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x) = \begin{cases} \psi_{q_{i_\sigma^-(t_s)}}(t, x) \text{ **if** } t \in [0, \tau_{\sigma,i_\sigma^-(t_s)+1} - t_s], \\ \psi_{q_{i_\sigma^-(t_s+t)}}\big(t_s + t - \tau_{\sigma,i_\sigma^-(t_s+t)}, \mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_\sigma^-(t_s+t)} - t_s, t_s, x)\big) \\ \qquad \qquad \textbf{if } t \geq \tau_{\sigma,i_\sigma^-(t_s)+1} - t_s, \end{cases} \tag{2.7}$$

for all $t \in [0, (t_s)^b_\sigma - t_s), x \in \mathbb{X}$.

Though the mappings $\mathscr{T}_{\sigma,\mathscr{A}}, \sigma \in \mathbb{S}$ defined above are a bit mysterious, they actually play a role no less important than the evolution mappings π and φ in ordinary dynamical systems defined in Definitions 2.2.2 and 2.2.3. The mappings specify the transition in the space $\mathbb{X}$ of the continuous variables under the influence of the dynamics of switching sequence in the space $\mathbb{S}$. With the help of Figure 2.1, this can be made manifest as follows. The problem in question is to determine the future state in the course of time from a visited state x. Like ordinary non-autonomous systems which involve the time at which the system state visited x for determination, in switched systems, the desired additional information is the time t_s elapsed from the time t_0 at which the system started running. The time t_s in fact carries the information on the rule of transition for the determination.

Let us consider a number $t \in [0, (t_s)^b_\sigma - t_s)$. As illustrated by Figure 2.1, if at the time instant $t_0 + t_s$ at which the system state visited x, it is known that the system had run for a time of amount t_s, then the component system driving the dynamics is $\psi_{q_{i^-_\sigma(t_s)}}$. Then, it is also determined that $\psi_{q_{i^-_\sigma(t_s)}}$ was driving the system until the time $t_0 + \tau_{\sigma, i^-_\sigma(t_s)+1}$ at which the driving component system is changed to $\psi_{q_{i^-_\sigma(t_s)+1}}$.

Note that the switching event associated with $q_{i^-_\sigma(t_s)+1}$ may has zero running time. However, for $\tilde{\tau} \stackrel{\text{def}}{=} \tau_{\sigma, i^-_\sigma(t_s)+1}$, by definition of $i^-_\sigma(\cdot)$ and by non-blocking property of σ, $e_{\sigma, i^-_\sigma(\tilde{\tau})}$ is the first non-zero running time switching event at $t_0 + \tau_{\sigma, i^-_\sigma(t_s)+1}$. Thus, it is ready to evolve further since $t_0 + \tau_{\sigma, i^-_\sigma(t_s)+1}$ under $\psi_{q_{i^-_\sigma(\tilde{\tau})}}$.

At a time instant, apart from $t_0 + t_s$ by an amount of t, exceeding the running time $\tau_{\sigma, i^-_\sigma(t_s)+1} - t_s$ (since $t_0 + t_s$) of $\psi_{q_{i^-_\sigma(t_s)}}$, the component system taking the control is $\psi_{q_{i^-_\sigma(t_s+t)}}$. However, in order to determine the state at $t_0 + t_s$ via this transition mapping, it is necessary to know the state $x(t_0 + \tau_{\sigma, i^-_\sigma(t_s+t)})$ at which $\psi_{q_{i^-_\sigma(t_s+t)}}$ started running. Fortunately, as shown in Figure 2.1, this desired state is $\mathscr{T}_{\sigma, \mathscr{A}}(\tau_{\sigma, i^-_\sigma(t_s+t)}, t_s, x)$ which can be determined by recursively applying (2.7) from t_s.

We shall call the family $\mathscr{T}_{\mathscr{A}} = \{\mathscr{T}_{\sigma, \mathscr{A}}\}_{\sigma \in \mathbb{S}}$ defined by (2.7) the autonomous continuous transition mapping (ACTM) of $\Sigma_{\mathscr{A}}$. The following property of this mapping is straightforward.

Proposition 2.4.1 *Under Assumption 2.4.1, the autonomous continuous transition mapping $\mathscr{T}_{\mathscr{A}}$ defined by (2.7) is continuous and satisfies*

$$\mathscr{T}_{\sigma,\mathscr{A}}(t_1 + t_2, t_s, x) = \mathscr{T}_{\sigma,\mathscr{A}}(t_2, t_s + t_1, \mathscr{T}_{\sigma,\mathscr{A}}(t_1, t_s, x)), \tag{2.8}$$

for all $t_1, t_2 \in \mathbb{R}^+, t_1 + t_2 < (t_s)^b_\sigma - t_s, x \in \mathbb{X}$, and $\sigma \in \mathbb{S}$.

We now indicate the basic elements of the transition model of $\Sigma_{\mathscr{A}}$. As we are interested in the dynamics of the continuous variables, the manifest space is $\mathbb{W}_{\mathscr{M}} = \mathbb{X}$ and the latent space is $\mathbb{W}_{\mathscr{L}} = \mathbb{S}$. It is observed that at the time t_s the discrete variable has run for a time amount of t_s, which means that the switching sequence is at the state σ_{t_s} – the t_s-translate of σ. Thus, an obvious choice of rule of transition $\mathfrak{R}_{\mathscr{L}}$ of the latent variable is $\mathfrak{R}(t, t_s, \sigma) \stackrel{\text{def}}{=} \sigma_{t_s+t}$ – the $(t_s + t)$–translate of σ. Accordingly, the rule of transition of $\Sigma_{\mathscr{A}}$ can be specified as

$$\mathfrak{R}(t, t_s, (x, \sigma)) = (\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x), \sigma_{t_s+t}). \tag{2.9}$$

In light of the above consideration, a similar mapping $\mathscr{T}_{\mathscr{A}}$ for the case of non-identity discrete transition mapping can be defined by a similar manner. In this case, though the driving component system at any time apart from t_s by an amount of t is directly determined as $\psi_{q_{i^-_\sigma(t_s+t)}}$, determining the state at the time this component system started running involves the transition mapping   as follows.

For each $\sigma \in \mathbb{S}$, let $\ _\sigma : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ be the mapping defined as

$$_\sigma(t, x) = \quad (t, q_{\sigma,n}(t), \ldots \quad (t, q_{\sigma,1}(t), \quad (t, q_{\sigma,0}(t), x)) \ldots), \tag{2.10}$$

in which, for each $t \in [t_s, (t_s)^b_\sigma - t_s)$, $(q_{\sigma,1}(t), 0), \ldots, (q_{\sigma,n(t)}, 0)$ is the sequence of consecutive switching events of zero running time of σ at t, and $q_{\sigma,0}(t)$ be the index of the switching event right before $(q_{\sigma,1}(t), 0)$.

Let $\mathscr{T}_{\sigma,\mathscr{A}} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ be the mapping defined as $\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x) = \mathbb{x}$ for $t \geq (t_s)^b_\sigma - t_s, t_s \in \mathbb{R}^+, x \in \mathbb{X}$, and, for $t \in [0, (t_s)^b_\sigma - t_s), x \in \mathbb{X}$,

$$\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x) = \begin{cases} \psi_{q_{i^-_\sigma(t_s)}}(t, x) \ \textbf{if} \ t \in [0, \tau_{\sigma,i^+_\sigma(t_s)} - t_s), \\ \quad \sigma(\tau_{\sigma,i^-_\sigma(t_s)+1}, q_{\sigma,i^-_\sigma(t_s)+1}, \psi_{q_{i^-_\sigma(t_s)}}(\tau_{\sigma,i^-_\sigma(t_s)+1} - t_s, x)) \text{ if } t = \tau_{\sigma,i^-_\sigma(t_s)+1} - t_s, \\ \psi_{q_{i^-_\sigma(t_s+t)}}\big(t_s + t - \tau_{\sigma,i^-_\sigma(t_s+t)}, \mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i^-_\sigma(t_s+t)} - t_s, t_s, x)\big) \\ \qquad\qquad \textbf{if} \ t \in (\tau_{\sigma,i^-_\sigma(t_s)+1} - t_s, \tau_{\sigma,i} - t_s), i > i^-_\sigma(t_s + \tau_{\sigma,i^-_\sigma(t_s)+1}), \\ \quad \sigma\Big(\tau_{\sigma,i}, q_{\sigma,i}, \psi_{q_{i^-_\sigma(t_s+t)}}\big(\Delta\tau_{\sigma,i^-_\sigma(t_s+t)}, \mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i^-_\sigma(t_s+t)} - t_s, t_s, x)\big)\Big) \\ \qquad\qquad \textbf{if} \ t = \tau_{\sigma,i} - t_s, i > i^-_\sigma(t_s + \tau_{\sigma,i^-_\sigma(t_s)+1}). \end{cases} \tag{2.11}$$

The family $\mathscr{T}_{\mathscr{A}} = \{\mathscr{T}_{\sigma,\mathscr{A}}\}_{\sigma\in\mathbb{S}}$ defined by (2.11) is also called the autonomous continuous transition mapping of the switched system $\Sigma_{\mathscr{A}}$. Without repeating the above argument, we have the following property of this mapping.

Proposition 2.4.2 *If the transition mapping of the switched system $\Sigma_{\mathscr{A}}$ is deterministic, then the mapping $\mathscr{T}_{\mathscr{A}}$ defined by (2.11) is right-continuous and satisfies*

$$\mathscr{T}_{\sigma,\mathscr{A}}(t_1 + t_2, t_s, x) = \mathscr{T}_{\sigma,\mathscr{A}}(t_2, t_s + t_1, \mathscr{T}_{\sigma,\mathscr{A}}(t_1, t_s, x)), \tag{2.12}$$

for all $t_1, t_2 \in \mathbb{R}^+, t_1 + t_2 < (t_s)^b_\sigma - t_s, x \in \mathbb{X}$, and $\sigma \in \mathbb{S}$.

Likewise, the rule of transition in the signal space $\mathbb{W}_{\mathscr{M}} \times \mathbb{W}_{\mathscr{L}} \overset{\text{def}}{=} \mathbb{X} \times \mathbb{S}$ of the transition model of $\Sigma_{\mathscr{A}}$ with $\mathscr{T}_{\sigma,\mathscr{A}}$ defined by (2.11) is $\mathfrak{R}(t, t_s, (x, \sigma)) = (\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x), \sigma_{t_s+t})$.

Non-autonomous mappings

The transition model of switched systems introduced in Definition 2.4.2 treats all the continuous variables on an equal footing in the interaction with the discrete state. In applications, there is also a large class of systems in which part of the continuous variables plays the role of either driving input or disturbance whose influence on the remaining continuous variables needs to be suppressed. Motivated by this consideration, we introduce the following notion of switched non-autonomous systems.

Definition 2.4.5 *A switched non-autonomous system is an octuple*

$$\Sigma_{\mathcal{NA}} = \big(\mathbb{R}^+, \mathbb{Q}, \mathcal{X}, \mathcal{W}, \{\varphi_q\}_{q\in\mathbb{Q}}, \mathbb{S}, \pi, \quad \big), \tag{2.13}$$

where $\mathbb{Q}$ *is a discrete set which is the space of the discrete signals,* $\mathcal{X}$ *and* $\mathcal{W}$ *are topological spaces which are the space of the continuous signals and the space of the continuous disturbance signals, respectively,* $\varphi_q : \mathbb{R}^+ \times \mathcal{X} \times \mathcal{W} \to \mathcal{X}, q \in \mathbb{Q}$ *are ONADS,* $\mathbb{S}$ *is a collection of switching sequences,* π *is an OADS on* $\mathcal{W}$*, and* $\quad : \mathbb{R}^+ \times \mathbb{Q} \times \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ *is the discrete transition map of the continuous state in* $\mathcal{X}$*.*

To specify the continuous transition mappings for system $\Sigma_{\mathcal{NA}}$, we suppose that   is the identity mapping in its third argument and also use $\mathbb{x}$ as a fictitious element of $\mathcal{X}$. The time $(t_s)^b_\sigma$ is as defined in Page 30. We have the mappings $\mathscr{T}_{\sigma,\mathcal{NA}} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{X} \times \mathcal{W} \to \mathcal{X}, \sigma \in \mathbb{S}$ defined as follows.

$$\mathscr{T}_{\sigma,\mathcal{NA}}(t, t_s, x, w) = \begin{cases} \varphi_{q_{\sigma,i^-_\sigma(t_s)}}(t, x, w) \textbf{ if } t \in [0, \tau_{\sigma,i^-_\sigma(t_s)+1} - t_s], \\ \varphi_{q_{i^-_\sigma(t_s+t)}}\Big(t_s + t - \tau_{\sigma,i^-_\sigma(t_s+t)}, \mathscr{T}_{\sigma,\mathcal{NA}}\big(\tau_{\sigma,i^-_\sigma(t_s+t)} - t_s, x, w\big), \\ \qquad \pi\big(\tau_{\sigma,i^-_\sigma(t_s+t)} - t_s, w\big)\Big) \textbf{ if } t \geq \tau_{\sigma,i^-_\sigma(t_s)+1} - t_s, \end{cases} \tag{2.14}$$

for all $t \in [0, (t_s)^b_\sigma - t_s), x \in \mathcal{X}, w \in \mathcal{W}$, and $\mathscr{T}_{\sigma,\mathcal{NA}}(t, t_s, x, w) = \mathbb{x}, \forall t \geq (t_s)^b_\sigma - t_s$.

It is obvious that the mappings $\mathscr{T}_{\sigma,\mathcal{NA}}$ play the same role as the mapping $\mathscr{T}_{\sigma,\mathcal{A}}$ in the autonomous case. The mapping specifies the transition in the space $\mathcal{X}$ under the influence of the combined dynamics of switching sequence and the autonomous dynamics in $\mathcal{W}$ of the time-varying parameter w.

As we are interested in the dynamics in $\mathcal{X}$, the manifest space is $\mathbb{W}_{\mathcal{M}} = \mathcal{X}$ and the latent space is now $\mathbb{W}_{\mathcal{L}} = \mathbb{S} \times \mathcal{W}$. Based on $\mathscr{T}_{\sigma,\mathcal{NA}}$, the rule of transition for a transition model of $\Sigma_{\mathcal{NA}}$ can be specified once the rule of transition in the latent space – denoted as $\mathfrak{R}_{\mathcal{L}}$ – is specified. Inheriting from theory of pullback attractor of ordinary non-autonomous dynamical systems [79, 29], we have two specifications for $\mathfrak{R}_{\mathcal{L}}$ according to the motion of w: the *rule of forward transition*

$$\overrightarrow{\mathfrak{R}}_{\mathcal{L}}\,(t, t_s, (\sigma, w)) \stackrel{\text{def}}{=} (\sigma_{t_s+t}, \pi(t, w)), \tag{2.15}$$

and the *rule of pullback transition*

$$\overline{\overrightarrow{\mathfrak{R}}}_{\mathcal{L}}\,(t, t_s, (\sigma, w)) \stackrel{\text{def}}{=} (\sigma_{t_s+t}, \pi(-t, w)). \tag{2.16}$$

Accordingly, for the transition model of $\Sigma_{\mathcal{NA}}$, we have the rule of transition

$$\overrightarrow{\mathfrak{R}}\,(t, t_s, (x, (\sigma, w))) \stackrel{\text{def}}{=} (\mathscr{T}_{\sigma,\mathcal{NA}}(t, t_s, x, w), (\sigma_{t_s+t}, \pi(t, w))), \tag{2.17}$$

which is appropriately termed *rule of forward transition*, and the rule of pullback transition

$$\overline{\overrightarrow{\mathfrak{R}}}\,(t, t_s, (x, (\sigma, w))) \stackrel{\text{def}}{=} (\mathscr{T}_{\sigma,\mathcal{NA}}(t, t_s, x, w), (\sigma_{t_s+t}, \pi(-t, w))), \tag{2.18}$$

which is appropriately termed *rule of pullback transition*.

We shall call the family $\mathscr{T}_{\mathcal{NA}} = \{\mathscr{T}_{\sigma,\mathcal{NA}}\}_{\sigma\in\mathbb{S}}$ the non-autonomous continuous transition mapping (NACTM) of the switched non-autonomous system $\Sigma_{\mathcal{NA}}$. From the transition mechanism (2.14), the following property of $\mathscr{T}_{\mathcal{NA}}$ is straightforward.

Proposition 2.4.3 *Under Assumption 2.4.1, the non-autonomous continuous transition mapping $\mathscr{T}_{\mathcal{NA}}$ defined by (2.14) is continuous and satisfies*

$$\mathscr{T}_{\sigma,\mathcal{NA}}(t_1 + t_2, t_s, x, w) = \mathscr{T}_{\sigma,\mathcal{NA}}(t_2, t_s + t_1, \mathscr{T}_{\sigma,\mathcal{NA}}(t_1, t_s, x, w), \pi(t_1, w)). \tag{2.19}$$

for all $t_s, t_1, t_2 \in \mathbb{R}^+, t_1 + t_2 < (t_s)^b_\sigma - t_s, x \in \mathcal{X}, w \in \mathcal{W}$, and $\sigma \in \mathbb{S}$.

In the case of non-identity but deterministic , a similar mapping $\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}$, and hence the rules of forward and pullback transition, for $\Sigma_{\mathscr{N}\mathscr{A}}$ can be specified by the same manner. The specification is straightforward though a bit lengthy and is omitted.

To close this preliminary chapter, we would mention that, in comparison to the cocycle property – second property in ii), Definition 2.2.3 – of the ONADS φ [8, 29], the properties (2.8), (2.12), and (2.19) of the mappings $\mathscr{T}_{\sigma,\mathscr{A}}$ and $\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}$ defined by (2.7), (2.11), and (2.14), respectively, preserves the cocycle property of φ. In light of this merit, the mappings $\mathscr{T}_{\sigma,\mathscr{A}}$ and $\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}$ can be termed *switched cocycle processes*.

Chapter 3

Invariance Theory

The purpose of this chapter is to introduce a qualitative theory for switched systems. Further notions of limiting switching sequence, autonomous and non-autonomous attractors, autonomous and non-autonomous ω-limit sets, and quasi-invariance property are introduced. The principle of small-variation small-attractor is introduced for invariance principles of switched systems. Instead of imposing the usual switching decreasing condition, boundedness of ultimate variations of auxiliary functions is considered for convergence.

3.1 Motivation

In systems and control, the foundation of the qualitative theory – the theory makes conclusions on long-time behavior of a dynamical system without solving for the trajectories – is built on the Lyapunov's second method and its generalization – the LaSalle's invariance principle, in which the central notion is the *Lyapunov function.* Due to its energy embrace, Lyapunov function is not limited in ordinary dynamical systems. Making no exception, LaSalle's invariance principle framework for attractability and stability of sets in switched systems is of natural interest.

The practical motivation for invariance theory of switched systems lies in the use of invariant motions of constituent systems/agents in contemporary control and communication systems [18,140,5,46,45,105,15,160]. Nevertheless, while invariance theories of switched/hybrid systems often impose the semi-group property on system trajectories and the switching decreasing condition which are practically restrictive [108,30], advanced designs of highly complex systems suppose that invariance control of constituent systems has been done *a priori* [140,15]. This obviously lays a significant gap in the field of hybrid systems.

Invariance principles for switched systems have been actively investigated using both differential equations model and hybrid systems' framework [108,102,62,13, 107,126]. Similar to non-autonomous systems, due to the loss of the semi-group property, the ω-limit sets of trajectories of switched systems are generally not invariant. The common approach to overcome this obstacle is to embed the state space into a topological space on which a generalized dynamical system is defined. Then, asymptotic behavior of the original system is investigated through the generalized system [129,9,108,107,80].

In ordinary non-autonomous systems, using the functional space of translates and examining limiting equations, it was shown that the ω-limit set of trajectories of the original system is *semi-quasi-invariant* and a LaSalle-like invariance principle was obtained [9]. Considering switching variables as part of the hybrid state and then imposing a semi-group property on system trajectories, invariance in hybrid spaces and invariance principles for general hybrid systems have been studied [108, 102,126]. Examining the convergence of a sequence of arcs cut off of one single trajectory, a notion of *weak invariance* and an invariance principle was presented for switched systems with dwell-time [13]. Introducing a novel notion of ω-limit set in the product space, another notion of *weak invariance property* was considered and an invariance principle were obtained for a class of switched systems with average dwell-time [107]. In [63], the elegant notions of nonlinear norm-observability were introduced and invariance principle for uniform stability of switched nonlinear systems was carried out.

In summary, the aforementioned results applying to the switched autonomous systems either directly or implicitly make use of the switching decreasing condition. As a result, their applicability does not agree with the generality of the systems. Invariance theory without imposing switching decreasing condition and semi-group property remains open for switched non-autonomous systems and switched systems.

3.2 Limiting Switched Systems

Switched system returns, for each single switching sequence, a non-autonomous system [80]. Therefore, the asymptotic behavior of switched systems can be investigated through the behavior of their limiting systems [9]. The purpose of this section is to bring out such useful limiting systems. For this purpose, we have the following notion of limiting switching sequence.

Throughout this chapter, we suppose that all switching sequences in $\mathbb{S}$ is non-Zeno and nontrivial, i.e., the following assumption holds.

Assumption 3.2.1 *In any finite interval, the number of switching events of any switching sequence in $\mathbb{S}$ is finite. There are infinitely many switching events in each switching sequence and the running times of switching events are all bounded by Δ_T.*

3.2.1 Limiting Switching Sequence

Definition 3.2.1 *Let $\sigma \in \mathbb{S}$ be a switching sequence and let $t \in \mathbb{R}^+$ be a positive number. The t-translate of σ is the switching sequence $\sigma_t \in \mathbb{S}$ whose switching events $e_{\sigma_t,i} = (q_{\sigma_t,i}, \Delta\tau_{\sigma_t,i})$ are determined by*

i) $q_{\sigma_t,i} = q_{\sigma,i_\sigma^-(t)+i}, \forall i \in \mathbb{N}$*; and* *ii)* $\Delta\tau_{\sigma_t,0} = \tau_{\sigma,i_\sigma^-(t)+1} - t,$ *and* $\Delta\tau_{\sigma_t,i} = \Delta\tau_{\sigma,i_\sigma^-(t)+i}, \forall i \in \mathbb{N}\backslash\{0\}.$

It is worth noting that by translating in index variable i instead of translating in time variable t as in [107], the t-translate of a switching sequence preserves the zero running times of σ as well.

Definition 3.2.2 *Let $\{\sigma_n\}_n$ be a sequence of switching sequences in $\mathbb{S}$ and let $\sigma \in \mathbb{S}$. Then, $\{\sigma_n\}_n$ is said to converge to σ as n goes to infinity, denoted as $\sigma_n \to \sigma, n \to \infty$, if the following properties hold:*

i) $\lim_{n\to\infty} q_{\sigma_n,i} = q_{\sigma,i}, \forall i \in \mathbb{N},$ *and*

ii) $\lim_{n\to\infty} \Delta\tau_{\sigma_n,i} = \Delta\tau_{\sigma,i}, \forall i \in \mathbb{N}.$

Definition 3.2.3 (limiting switching sequence) *Let $\sigma \in \mathbb{S}$ be a switching sequence. A switching sequence $\sigma^* \in \mathbb{S}$ is said to be a* limiting switching sequence *of σ if there is a time sequence $\{t_n\}_n$ such that $\sigma_{t_n} \to \sigma^*, n \to \infty$.*

Hereafter, for each switching sequence $\sigma \in \mathbb{S}$, we shall denote by $\mathbb{S}_\sigma^*$ the set of all limiting switching sequences of σ.

3.2.2 Existence and Properties

The following proposition asserts the existence of limiting switching sequence.

Proposition 3.2.1 *Let $\sigma \in \mathbb{S}$ be a switching sequence. Suppose that all the running times of σ are bounded by $\ell > 0$. Then, the set $\mathbb{S}_\sigma^*$ is nonempty.*

Proof: Let $\{t_n\}_n$ be a time sequence. We shall show that there is a subsequence $\{t_{n_m}\}_m$ of $\{t_n\}_n$ such that $\{\sigma_{t_{n_m}}\}_m$ converges.

Let us equip the interval $[0,\ell]$ the usual Euclidean metric d_ℓ defined as $d_\ell(x,y) = |x-y|, \forall x,y \in [0,\ell]$ and equip the discrete set $\mathbb{Q}$ the usual discrete metric defined as $d_{\mathbb{Q}}(q_1,q_2) = 0$ if $q_1 = q_2$ and $d_{\mathbb{Q}}(q_1,q_2) = 1$ if $q_1 \neq q_2$, for every $q_1, q_2 \in \mathbb{Q}$.

Obviously, the space of switching events $\mathcal{Q} = \mathbb{Q} \times [0, \ell]$ and the countable product $\mathcal{Q}^{\mathbb{N}} = \mathcal{Q} \times \ldots \times \mathcal{Q} \times \ldots$ are metrizable topological spaces with the usual product metric [1, Theorem 3.24, page 84]. Furthermore, as $\mathbb{Q}$ and $[0, \ell]$ are compact with respect to their respective metrics, $\mathcal{Q}$ is sequentially compact [72, Definition 10.15, page 170]. Since $\mathcal{Q}^{\mathbb{N}}$ is a metrizable topological space and $\mathcal{Q}$ is sequentially compact, applying [72, Proposition 10.18, page 171], we conclude that $\mathcal{Q}^{\mathbb{N}}$ is sequentially compact.

Consider the sequence of switching sequences $\{\sigma_{t_n}\}_n$. For each $n \in \mathbb{N}$, we have the point $\eta_n \in \mathcal{Q}^{\mathbb{N}}$ defined by

$$\eta_n = (e_{\sigma_{t_n},0}, \ldots, e_{\sigma_{t_n},i}, \ldots), \tag{3.1}$$

where $e_{\sigma_{t_n},i}, i \in \mathbb{N}$ are switching events of σ_{t_n}. Since $\mathcal{Q}^{\mathbb{N}}$ is sequentially compact, there is $\eta^* \in \mathcal{Q}^{\mathbb{N}}$ and a subsequence $\{\eta_{n_m}\}_m$ of $\{\eta_n\}_n$ such that $\{\eta_{n_m}\} \to \eta^*, m \to \infty$. Equivalently, $\sigma_{t_{n_m}} \to \sigma^*, m \to \infty$, where $\sigma^* \in \mathbb{S}$ is the switching sequence whose switching events $e_{\sigma^*,i}, i \in \mathbb{N}$ are defined by

$$q_{\sigma^*,i} = \mathrm{Pr}_1(\mathrm{Pr}_i(\eta^*)), \text{ and } \Delta\tau_{\sigma^*,i} = \mathrm{Pr}_2(\mathrm{Pr}_i(\eta^*)), \tag{3.2}$$

where Pr_i is the usual i-th coordinate projection mapping. Therefore, the set of limiting switching sequence of σ is nonempty. ■

It is important to examine the dwell-time properties of limiting switching sequences. In the following, we show that for certain classes of switching sequences, their limiting switching sequences preserve their dwell-time properties. Note that, by Assumption 3.2.1, we have assumed that the switching sequences are non-Zeno.

Proposition 3.2.2 *Let $\sigma \in \mathbb{S}$ be a switching sequence whose running times are all bounded and let $\sigma^* \in \mathbb{S}^*_\sigma$ be a limiting switching sequence of σ. Then, the following assertions hold:*

i) if $\sigma \in \mathbb{S}_{\mathscr{D}}[\tau_{\rm d}]$*, then so is* σ^* *except the first switching event of* σ^**;*

ii) if $\sigma \in \mathbb{S}_{\mathscr{A}}[\tau_{\rm p}, N_{\rm p}]$*, then so is* σ^**; and*

iii) if $\sigma \in \mathbb{S}_{\mathscr{P}}[\tau_{\rm p}, T_{\rm p}]$ *and* σ^* *is non-Zeno, then* $\sigma^* \in \mathbb{S}_{\mathscr{P}}[\tau_{\rm p}, T_{\rm p}]$*.*

Proof: By Proposition 3.2.1, the set of limiting switching sequences of σ is nonempty. Let σ^* be a limiting switching sequence of σ. By Definition 3.2.3, there is a time sequence $\{t_n\}_n$ such that $\sigma_{t_n} \to \sigma^*, n \to \infty$. Since, for each n, σ_{t_n} is the t_n–translate of σ, for each $n \in \mathbb{N}$, σ_{t_n} have the same dwell-time properties as σ.

The first assertion is obvious as, for each $i \in \mathbb{N}$, $\{\Delta\tau_{\sigma_n,i}\}_n$ is lower bounded by $\tau_{\rm d}$. We prove the second assertion by contradiction. Suppose that ii) is not true, i.e., $\sigma^* \notin \mathbb{S}_{\mathscr{A}}[\tau_{\rm p}, N_{\rm p}]$. From definition of $\mathbb{S}_{\mathscr{A}}[\tau_{\rm p}, N_{\rm p}]$, this implies that σ^* has a sequence of $N_{\rm p}+1$ consecutive switching events whose running times are all strictly less than $\tau_{\rm p}$, i.e., there is $i \in \mathbb{N}$ such that $\Delta\tau_{\sigma^*,i+j} < \tau_{\rm p}, \forall j \in \mathscr{N} \stackrel{\rm def}{=} \{0, 1, \ldots, N_{\rm p}\}$. Since $N_{\rm p}$ is finite, this implies that there is a $\epsilon > 0$ such that $\Delta\tau_{\sigma^*,i+j} < \tau_{\rm p} - \epsilon, \forall j \in \mathscr{N}$. On the other hand, as $\sigma_{t_n} \to \sigma^*, n \to \infty$, for each $j \in \mathscr{N}$, there is a N_j such that $|\Delta\tau_{\sigma_{t_n},i+j} - \Delta\tau_{\sigma^*,i+j}| \le \epsilon/2, \forall n \ge N_j$. This implies that

$$\Delta\tau_{\sigma_{t_n},i+j} \le \Delta\tau_{\sigma^*,i+j} + \epsilon/2 < \tau_{\rm p} - \epsilon/2, \forall n \ge \max\{N_j : j \in \mathscr{N}\}, \forall j \in \mathscr{N}. \qquad (3.3)$$

However, (3.3) implies that, for sufficiently large n, the t_n–translate of σ, that is σ_{t_n}, has a sequence of $N_{\rm p}+1$ consecutive switching events of the running times strictly less than $\tau_{\rm p}$. This contradicts to the dwell-time properties of σ_{t_n}.

We also prove the third assertion by contradiction. Suppose that this assertion is not true. Then, σ^* has two switching events $e_{\sigma^*,i}$ and $e_{\sigma^*,j}$ to which $\tau_{\sigma^*,j} - \tau_{\sigma^*,i+1} > T_{\rm p}$ and the running times of all switching events of σ^* between $e_{\sigma^*,i}$ and $e_{\sigma^*,j}$ are strictly less than $\tau_{\rm p}$. Let $\epsilon > 0$ be the positive number satisfying $(j-i-1)\epsilon < \tau_{\sigma^*,j} - \tau_{\sigma^*,i+1} - T_{\rm p}$ and $\Delta\tau_{\sigma^*,k} < \tau_{\rm p} - \epsilon, \forall k = i+1, \ldots j-1$. Such ϵ exists due to the finiteness of i and j

and the contradiction assumption. Since $\sigma_{t_n} \to \sigma^*, n \to \infty$ and i and j are finite, there is a number $N_0 \in \mathbb{N}$ such that $|\Delta\tau_{\sigma_{t_n},k} - \Delta\tau_{\sigma^*,k}| < \epsilon/2, \forall k = i+1, \dots, j-1, \forall n \geq N_0$. Therefore, for all $n \geq N_0$, we have $\Delta\tau_{\sigma_{t_n},k} \leq \Delta\tau_{\sigma^*} + \epsilon/2 < \tau_{\mathrm{p}} - \epsilon/2$, and

$$\begin{aligned} \tau_{\sigma_{t_n},j} - \tau_{\sigma_{t_n},i+1} &= \sum_{k=i+1}^{j-1} \Delta\tau_{\sigma_{t_n},k} \geq \sum_{k=i+1}^{j-1} (\Delta\tau_{\sigma^*,k} - \epsilon/2) \\ &\geq \tau_{\sigma^*,j-1} - \tau_{\sigma^*,i} - (j-i-1)\epsilon/2 > T_{\mathrm{p}}. \end{aligned} \tag{3.4}$$

This means that for every $n \geq N_0$, the translate σ_{t_n} has two switching events $e_{\sigma_{t_n},i}$ and $e_{\sigma_{t_n},j}$ to which $\tau_{\sigma_{t_n},j} - \tau_{\sigma_{t_n},i+1} > T_{\mathrm{p}}$ and the running times of all switching events between $e_{\sigma_{t_n},i}$ and $e_{\sigma_{t_n},j}$ are all larger than τ_{p}. This is a contradiction since all $\sigma_{t_n}, n \in \mathbb{N}$ belong to $\mathbb{S}_{\mathcal{P}}[\tau_{\mathrm{p}}, T_{\mathrm{p}}]$. ■

Proposition 3.2.3 *Consider a switching sequence $\sigma \in \mathbb{S}$ and its limiting switching sequence $\sigma^* \in \mathbb{S}^*_\sigma$. Let $\{t_n\}_n$ be the sequence to which $\sigma_{t_n} \to \sigma^*, n \to \infty$. Then,*

$$\lim_{n\to\infty} (\tau_{\sigma,i^-_\sigma(t_n)+j} - t_n) = \tau_{\sigma^*,j}, \forall j \in \mathbb{N}\backslash\{0\}, \tag{3.5}$$

and

$$\sigma_{t_n+t} \to \sigma^*_t, n \to \infty, \forall t \in \mathbb{R}^+. \tag{3.6}$$

Proof: We prove (3.5) by contradiction. Suppose that the converse holds, i.e., there are $j_0 \in \mathbb{N}\backslash\{0\}$ and $\epsilon > 0$, such that for every $M \in \mathbb{N}$, there is $n_M \geq M$ such that

$$|\tau_{\sigma,i^-_\sigma(t_{n_M})+j_0} - t_{n_M} - \tau_{\sigma^*,j_0}| > \epsilon. \tag{3.7}$$

Since σ_{t_n}'s are translates of σ, by Definition 3.2.1, we have

$$\tau_{\sigma_{t_n},j_0} \overset{\text{def}}{=} \sum_{k=0}^{j_0-1} \Delta\tau_{\sigma_{t_n},k} = \tau_{\sigma,i^-_\sigma(t_n)+1} - t_n + \sum_{k=1}^{j_0-1} \Delta\tau_{\sigma,i^-_\sigma(t_n)+k} = \tau_{\sigma,i^-_\sigma(t_n)+j_0} - t_n. \tag{3.8}$$

In addition, as $\Delta\tau_{\sigma_{t_n},k} \to \Delta\tau_{\sigma^*,k}, n \to \infty$, we have $|\Delta\tau_{\sigma_{t_n},k} - \Delta\tau_{\sigma^*,k}| < 0.5\epsilon/(j_0 + 1), \forall k = 0, \dots, j_0 - 1$ for sufficiently large n. This coupled with (3.8) yields

$$|\tau_{\sigma,i_\sigma^-(t_n)+j_0} - t_n - \tau_{\sigma^*,j_0}| = \Big|\sum_{k=0}^{j_0-1}(\Delta\tau_{\sigma_{t_n},k} - \Delta\tau_{\sigma^*,k})\Big| < \epsilon/2, \tag{3.9}$$

for sufficiently large n, which obviously contradicts to (3.7).

To prove (3.6), let us consider the number $i_{\sigma^*}^-(t)$, and for each $n \in \mathbb{N}$, let $N_\sigma[t_n, t] = i_\sigma^-(t_n + t) - i_\sigma^-(t_n)$. By definition of $(\cdot)_{\sigma^*}^-$ (see Page 28), we have

$$\begin{aligned} t \geq \tau_{\sigma^*,i_{\sigma^*}^-(t)} &= \sum_{k=0}^{i_{\sigma^*}^-(t)-1} \Delta\tau_{\sigma^*,k} = \lim_{n\to\infty} \sum_{k=0}^{i_{\sigma^*}^-(t)-1} \Delta\tau_{\sigma_{t_n},k} \\ &= \lim_{n\to\infty} \left(\tau_{\sigma,i_\sigma^-(t_n)+1} - t_n + \sum_{k=1}^{i_{\sigma^*}^-(t)-1} \Delta\tau_{\sigma,i_\sigma^-(t_n)+k}\right) \end{aligned} \tag{3.10}$$

On the other hand, by Definition 3.2.1, we have

$$\tau_{\sigma,i_\sigma^-(t_n+t)} = \sum_{k=0}^{i_\sigma^-(t_n+t)} \Delta\tau_{\sigma,k} = \tau_{\sigma,i_\sigma^-(t_n)+1} + \sum_{k=1}^{N_\sigma[t_n,t]-1} \Delta\tau_{\sigma,i_\sigma^-(t_n)+k}. \tag{3.11}$$

As $\tau_{\sigma,i_\sigma^-(t_n+t)} - (t_n + t) \leq t, \forall n \in \mathbb{N}$ by definition, (3.11) implies that

$$t \geq \tau_{\sigma,i_\sigma^-(t_n+t)} - (t_n + t) + t = \tau_{\sigma,i_\sigma^-(t_n)+1} - t_n + \sum_{k=1}^{N_\sigma[t_n,t]-1} \Delta\tau_{\sigma,i_\sigma^-(t_n)+k}. \tag{3.12}$$

Taking limits of both sides of (3.12) as $n \to \infty$, we have

$$t \geq \lim_{n\to\infty} \left(\tau_{\sigma,i_\sigma^-(t_n)+1} - t_n + \sum_{k=1}^{N_\sigma[t_n,t]-1} \Delta\tau_{\sigma,i_\sigma^-(t_n)+k}\right) \tag{3.13}$$

Comparing (3.10) and (3.13) and noting that $\tau_{\sigma^*,i_{\sigma^*}^-(t)+1} > t$ by definition, we obtain $\lim_{n\to\infty} N_\sigma[t_n, t] = i_{\sigma^*}^-(t)$ from which (3.6) follows. ∎

3.2.3 Limiting Switched Systems

In this subsection, we introduce the notion of limiting switched systems for defining quasi-invariance of trajectories of switched systems. Let $\Sigma_{\mathscr{S}}$ be a switched system defined by either Definition 2.4.2 or Definition 2.4.5, and let $\mathbb{S}_1$ be a subset of $\mathbb{S}$. Then, we use $\Sigma_{\mathscr{S}}[\mathbb{S}_1]$ to denote the switched system

$$\Sigma_{\mathscr{S}}[\mathbb{S}_1] \stackrel{\text{def}}{=} (\mathbb{R}^+, \mathbb{Q}, \mathbb{X}, \{\psi_q\}_{q\in\mathbb{Q}}, \mathbb{S}_1, \quad), \tag{3.14}$$

if $\Sigma_{\mathscr{S}}$ is autonomous, and the system

$$\Sigma_{\mathscr{S}}[\mathbb{S}_1] \stackrel{\text{def}}{=} (\mathbb{R}^+, \mathbb{Q}, \mathcal{X}, \mathcal{W}, \{\varphi_q\}_{q\in\mathbb{Q}}, \mathbb{S}_1, \pi, \quad), \tag{3.15}$$

if $\Sigma_{\mathscr{S}}$ is non-autonomous. When $\mathbb{S}_1 = \{\sigma\}$ is a single element set, we then write $\Sigma_{\mathscr{S}}[\sigma]$ for simplicity.

Definition 3.2.4 *Let $\Sigma_{\mathscr{S}}$ be a switched system defined by either Definition 2.4.2 or Definition 2.4.5. Then, for each $\sigma \in \mathbb{S}$, the switched system $\Sigma_{\mathscr{S}}[\mathbb{S}^*_\sigma]$ is said to be the* limiting switched system *of the switched system $\Sigma_{\mathscr{S}}[\sigma]$.*

3.3 Qualitative Notions and Quasi-Invariance

The basic problem in studying asymptotic behavior of dynamical system is to locate attractors of trajectories without solving the system equations. In ordinary dynamical systems, an efficient approach to solve this problem is to combine invariance properties of the limit sets of trajectories and the convergence of auxiliary functions [56, 59]. As shown in Section 2.4.3, when there is no jump in the continuous state, the transition mappings $\mathscr{T}_{\sigma,\mathcal{A}}(t, t_s, x)$ and $\mathscr{T}_{\sigma,\mathcal{NA}}(t, t_s, x, w)$ are continuous functions of the transition time t. Hence, the limit sets of trajectories of switched systems are well-defined in the classical setting. However, due to the dependence on varying quantities σ, t_s, and

w, these limit sets are no longer invariant. As such, a primary step is to bring out another qualitative property for limit sets of trajectories of switched systems.

In this section, using limiting switching signals, we introduce the notion of quasi-invariance for switched systems. For this purpose, let us realize the notions of attractor and limit set from the general framework of dynamical systems in Section 2.2.3 in switched systems as follows.

3.3.1 Qualitative Notions

Autonomous

Let $\mathscr{T}_{\mathscr{A}} = \{\mathscr{T}_{\sigma,\mathscr{A}}\}_{\sigma\in\mathbb{S}}$ be the transition mapping of the switched autonomous system $\Sigma_{\mathscr{A}} = (\mathbb{R}^+, \mathbb{Q}, \mathbb{X}, \{\psi_q\}_{q\in\mathbb{Q}}, \mathbb{S}, \quad)$ with $\mathbb{X} = \mathbb{R}^n$. The latent variable is σ. From the rule of transition (2.9) and Definitions 2.2.4–2.2.7, the following realization of the notions in Section 2.2.3 are obvious.

Definition 3.3.1 (trajectory) *Let $x \in \mathbb{R}^n$, $t_s \in \mathbb{R}^+$ and $\sigma \in \mathbb{S}$ fixed. The (t_s, σ)–interacting trajectory through the point x is the set $\mathscr{O}_{t_s,\sigma}(x) = \{\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x) : t \in \mathbb{R}^+\}$.*

Definition 3.3.2 (motion) *Let $x \in \mathbb{R}^n$. The (t, t_s, σ)–motion through x of $\Sigma_{\mathscr{A}}$ is $\mathfrak{R}_{t_s,\sigma}(x)(t) = \mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x)$.*

Definition 3.3.3 (attractor) *Let $\mathcal{A}$ and $\mathcal{D}$ be closed sets in $\mathbb{R}^n$. The set $\mathcal{A}$ is said to be the (t_s, σ)–forward attractor of $\Sigma_{\mathscr{A}}$ with basin of attraction $\mathcal{D}$ if*

$$\lim_{t\to\infty} \|\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x)\|_{\mathcal{A}} = 0, \forall x \in \mathcal{D}. \tag{3.16}$$

In addition, if this property holds for all $\sigma \in \mathbb{S}$, then $\mathcal{A}$ is said to be the switching-uniform forward attractor with basin of attraction $\mathcal{D}$ of $\Sigma_{\mathscr{A}}$.

Definition 3.3.4 (ω-limit set) *Let $x \in \mathbb{R}^n$ and $t_s \in \mathbb{R}^+$ fixed. The ω–limit set of the (t_s, σ)–interacting trajectory $\mathscr{O}_{t_s,\sigma}(x)$ through x of the system $\Sigma_{\mathscr{A}}$ is the set*

$$\omega_{t_s,\sigma}(x) = \bigcap_{T \geq t_s} \overline{\bigcup_{t \geq T} \mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x)}. \tag{3.17}$$

Non-autonomous

As discussed in Section 2.4.3 – Page 33, switched non-autonomous systems arise when we are interested in the interaction between the dynamics of a part of continuous variables and the combined dynamics of the discrete variables and the remaining continuous variables. From the general framework presented in Section 2.2.3 and the definition of switched non-autonomous systems in Definition 2.4.5, realizing the notions in Section 2.2.3 in terms of the non-autonomous continuous transition mapping $\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}$ gives rise to trajectories, attractors, and limit sets parameterized by part of the continuous variables and switching sequences as follows.

Let $\mathscr{T}_{\mathscr{N}\mathscr{A}} = \{\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}\}_{\sigma\in\mathbb{S}}$ be the continuous transition mapping of the switched non-autonomous system $\Sigma_{\mathscr{N}\mathscr{A}} = (\mathbb{R}^+, \mathbb{Q}, \mathcal{X}, \mathcal{W}, \{\varphi_q\}_{q\in\mathbb{Q}}, \mathbb{S}, \pi, \quad)$ with $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{W} = \mathbb{R}^d$. From the rules of forward transition (2.15) and pullback transition (2.16), we have the following realizations in switched non-autonomous systems.

Definition 3.3.5 (forward trajectory) *Let $x \in \mathbb{R}^n$, $t_s \in \mathbb{R}^+$, $w \in \mathbb{R}^d$, and $\sigma \in \mathbb{S}$ fixed. The (t_s, w, σ)–forward trajectory through the point x is the set $\overrightarrow{\mathscr{O}}_{t_s,w,\sigma}(x) = \{\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}(t, t_s, x, w) : t \in \mathbb{R}^+\}$. In addition, the family $\overrightarrow{\mathscr{O}}_{t_s,\sigma} = \{\overrightarrow{\mathscr{O}}_{t_s,w,\sigma} \colon w \in \mathcal{W}\}$ is termed the* non-autonomous forward trajectory *through x of $\Sigma_{\mathscr{N}\mathscr{A}}$.*

Definition 3.3.6 (pullback trajectory) *Let $x \in \mathbb{R}^n$, $t_s \in \mathbb{R}^+$, $w \in \mathbb{R}^d$, and $\sigma \in \mathbb{S}$ fixed. The (t_s, w, σ)–pullback trajectory through the point x is the set $\overleftrightarrows{\mathscr{O}}_{t_s,w,\sigma}(x) = \{\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}(t, t_s, x, \pi(-t, w)) : t \in \mathbb{R}^+\}$. In addition, the family $\overleftrightarrows{\mathscr{O}}_{t_s,\sigma} = \{\overleftrightarrows{\mathscr{O}}_{t_s,w,\sigma} \colon w \in \mathcal{W}\}$ is termed the* non-autonomous pullback trajectory *through x of $\Sigma_{\mathscr{N}\mathscr{A}}$.*

Definition 3.3.7 (forward motion) *Let $x \in \mathbb{R}^n$, $t_s \in \mathbb{R}^+$, $w \in \mathbb{R}^d$, and $\sigma \in \mathbb{S}$ fixed. The (t, t_s, σ)–forward motion through x of $\Sigma_{\mathscr{NA}}$ is $\overrightarrow{\mathfrak{R}}_{t_s,\sigma,w}(x)(t) = \mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w)$.*

Definition 3.3.8 (pullback motion) *Let $x \in \mathbb{R}^n$, $t_s \in \mathbb{R}^+$, $w \in \mathbb{R}^d$, and $\sigma \in \mathbb{S}$ fixed. The (t, t_s, σ)–pullback motion through x of $\Sigma_{\mathscr{NA}}$ is $\overleftarrow{\overrightarrow{\mathfrak{R}}}_{t_s,\sigma,w}(x)(t) = \mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, \pi(-t, w))$.*

Definition 3.3.9 (forward attractor) *Let $\mathcal{A}$ and $\mathcal{D}$ be closed sets in $\mathbb{R}^n$. The set $\mathcal{A}$ is said to be the (t_s, σ, w)–forward attractor of $\Sigma_{\mathscr{NA}}$ with basin of attraction $\mathcal{D}$ if*

$$\lim_{t\to\infty} \|\mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w)\|_{\mathcal{A}} = 0, \forall x \in \mathcal{D}. \tag{3.18}$$

In addition, if this property holds for all $\sigma \in \mathbb{S}$, then $\mathcal{A}$ is said to be the switching-uniform forward attractor with basin of attraction $\mathcal{D}$ of $\Sigma_{\mathscr{NA}}$.

Definition 3.3.10 (pullback attractor) *Let $\mathcal{A}$ and $\mathcal{D}$ be closed sets in $\mathbb{R}^n$. The set $\mathcal{A}$ is said to be the (t_s, σ, w)–pullback attractor of $\Sigma_{\mathscr{NA}}$ with basin of attraction $\mathcal{D}$ if*

$$\lim_{t\to\infty} \|\mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, \pi(-t, w))\|_{\mathcal{A}} = 0, \forall x \in \mathcal{D}. \tag{3.19}$$

In addition, if this property holds for all $\sigma \in \mathbb{S}$, then $\mathcal{A}$ is said to be the switching-uniform pullback attractor with basin of attraction $\mathcal{D}$ of $\Sigma_{\mathscr{NA}}$.

Definition 3.3.11 ($\boldsymbol{\omega}$-limit set) *Let $x \in \mathbb{R}^n$ fixed. The ω–limit set of the (t_s, w, σ)–forward trajectory $\overrightarrow{\mathscr{O}}_{t_s,w,\sigma}$ through x of the system $\Sigma_{\mathscr{NA}}$ is the set*

$$\omega_{t_s,w,\sigma}(x) = \bigcap_{T \geq t_s} \overline{\bigcup_{t \geq T} \mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w)}. \tag{3.20}$$

In addition, the family $\Omega_{t_s,\sigma}(x) = \{\omega_{t_s,w,\sigma}(x) : w \in \mathcal{W}\}$ is termed the non-autonomous forward ω–limit set *of the non-autonomous trajectory $\overrightarrow{\mathscr{O}}_{t_s,\sigma}$.*

Definition 3.3.12 (pullback ω-limit set) *Let $x \in \mathbb{R}^n$ fixed. The pullback ω–limit set of the (t_s, w, σ)–pullback trajectory $\overleftrightarrow{\mathscr{O}}_{t_s,w,\sigma}$ through x of the system $\Sigma_{\mathscr{NA}}$ is the set*

$$\omega_{t_s,w,\sigma}(x) = \bigcap_{T \geq t_s} \overline{\bigcup_{t \geq T} \mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, \pi(-t, w))}. \tag{3.21}$$

In addition, the family $\Omega_{t_s,\sigma}(x) = \{\omega_{t_s,w,\sigma}(x) : w \in \mathcal{W}\}$ is termed the non-autonomous pullback ω–limit set *of the non-autonomous trajectory $\overleftrightarrow{\mathscr{O}}_{t_s,\sigma}$.*

3.3.2 Quasi-invariance

As analyzed in Section 1.1 and Section 2.4.3, the semi-group property is lost in the transition mappings of switched systems. Thus, the invariance of limit sets of trajectories is lost in switched systems as well. However, as in the framework of LaSalle's invariance principle, invariance property is for refining first estimate of attractors of the system, it is possible to consider other properties of the limit sets for refinement.

In the next section, we shall show that, for a switched system $\Sigma_{\mathscr{S}}$ and a switching sequence $\sigma \in \mathbb{S}$, the limit set of any trajectory of the switched system $\Sigma_{\mathscr{S}}[\sigma]$ is forward invariant under the rule of transition of the limiting switched systems $\Sigma_{\mathscr{S}}[\mathbb{S}^*_\sigma]$. For this reason, we have the following notion of quasi-invariance.

For a set $\mathcal{A}$ and a motion $\mathfrak{R}$, let $\mathfrak{R}(\mathcal{A})(t)$ be the set $\{\mathfrak{R}(x)(t) : x \in \mathcal{A}\}$.

Definition 3.3.13 *Let $\Sigma_{\mathscr{A}}$ be a switched autonomous system. For a fixed switching sequence $\sigma \in \mathbb{S}$, a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be* σ-quasi-invariant *if there is a $\sigma^* \in \mathbb{S}^*_\sigma$ such that $\mathfrak{R}_{0,\sigma^*}(\mathcal{A})(t) \subset \mathcal{A}, \forall t \geq 0$.*

Definition 3.3.14 *Let $\Sigma_{\mathscr{NA}}$ be a switched non-autonomous system. For a fixed switching sequence $\sigma \in \mathbb{S}$, a family of sets $\{\mathcal{A}_w\}_{w \in \mathcal{W}}$ in $\mathbb{R}^n$ is said to be* σ-quasi-invariant *if if there is a $\sigma^* \in \mathbb{S}^*_\sigma$ such that $\overrightarrow{\mathfrak{R}}_{0,\sigma^*,w}(\mathcal{A}_w)(t) \subset \mathcal{A}_{\pi(t,w)}, \forall t \geq 0, w \in \mathcal{W}$.*

3.4 Limit Sets: Existence and Quasi-invariance

The main purpose of this section is to prove that when there is no switching jump, i.e., Assumption 2.4.1 holds, the limit sets of trajectories of switched system exist and are quasi-invariant. In the following we use w_{-t} to denote $\pi(-t, w)$ without embarrassment.

3.4.1 Continuity of Transition Mappings

Lemma 3.4.1 *Let $\Sigma_{\mathscr{NA}}$ be a switched non-autonomous system without switching jump, i.e., Assumption 2.4.1 hold. Suppose that $\mathbb{Q}$ is finite and, for each switching sequence $\sigma \in \mathbb{S}$, $x \in \mathbb{R}^n, w \in \mathbb{R}^d$ and $t_s \geq 0$, the mapping $\mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w_{-t})$ is bounded. Then, $\mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w_{-t})$ is continuous with respect to t.*

Proof: Since t_s and x are fixed, in this proof, we shall suppose that $t_s = 0$ without loss of generality and use $\mathscr{T}_{\sigma,\mathscr{NA}}(t, w)$ to denote $\mathscr{T}_{\sigma,\mathscr{NA}}(t, 0, x, w)$ for short.

Since $\mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w_{-t})$ is bounded, there is a constant $H > 0$ such that

$$\|\mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w_{-t})\| \leq H, \forall t \in \mathbb{R}^+. \tag{3.22}$$

Let $\mathcal{B}_H = \{\zeta \in \mathbb{R}^n : \|\zeta\| \leq H\}$. We prove the theorem by contradiction. Suppose that $\mathscr{T}_{\sigma,\mathscr{NA}}(t, w_{-t})$ is not continuous, i.e., there is $t^* \geq t_s = 0$ such that $\mathscr{T}_{\sigma,\mathscr{NA}}(t, w_{-t})$ is not continuous at t^*. Without loss of generality, suppose that $t^* \geq \tau_{\sigma,1}$. Then, there is a number $\epsilon^* > 0$ such that for every $\delta > 0$, there is $t_\delta > 0$ such that

$$|t_\delta - t^*| \leq \delta \text{ and } \|\mathscr{T}_{\sigma,\mathscr{NA}}(t_\delta, w_{-t_\delta}) - \mathscr{T}_{\sigma,\mathscr{NA}}(t^*, w_{-t^*})\| \geq \epsilon^*. \tag{3.23}$$

By definition of the transition indicator $i_\sigma^-(\cdot)$ in Section 2.4.2, $i_\sigma^-(t^*)$ is the largest number satisfying $\tau_{\sigma,i_\sigma^-(t^*)} \leq t^*$, and since σ is non-Zeno, $i^* \overset{\text{def}}{=} i_\sigma^-(t^*)$ is finite and

$t^* \in [\tau_{\sigma,i^*-1}, \tau_{\sigma,i^*})$. From definition of $\mathscr{T}_{\sigma,\mathcal{NA}}$ in (2.14) (see Page 34), we have

$$\begin{aligned}\mathscr{T}_{\sigma,\mathcal{NA}}(t^*, w_{-t^*-\varsigma}) &= \varphi_{q_{\sigma,i^*}}(t^* - \tau_{\sigma,i^*}, \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,i^*}, 0, x, w_{-t^*-\varsigma}), \pi(\tau_{\sigma,i^*}, w_{-t^*-\varsigma})) \\ &\stackrel{\text{def}}{=} \varphi_{q_{\sigma,i^*}}(t^* - \tau_{\sigma,i^*}, \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,i^*}, w_{-t^*-\varsigma}), w_{-t^*-\varsigma+\tau_{\sigma,i^*}}), \qquad (3.24)\end{aligned}$$

and, for each $i \in \mathcal{N} \stackrel{\text{def}}{=} \{0, \dots, i^* - 1\}$ and for every ς, we have

$$\mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,i+1}, w_{-t^*-\varsigma}) = \varphi_{q_{\sigma,i}}(\Delta\tau_{\sigma,i}, \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,i}, x, w_{-t^*-\varsigma}), w_{(-t^*-\varsigma+\tau_{\sigma,i})}). \qquad (3.25)$$

Let $\epsilon_i > 0, i \in \mathcal{N}$ be arbitrary numbers. From Assumption 3.2.1, the running times of all switching events are bounded by Δ_T. Since $\varphi_{q_{\sigma,i}}$'s are transition mappings of ONADS, they are uniformly continuous on the compact set $\Delta_T \times \mathcal{B}_H \times \mathcal{W}$. Thus, for each $i \in \mathcal{N}$, there is $r_i = r_i(\epsilon_i) > 0$ such that

$$\begin{cases} |t - t'| + \|x - x'\| + \|w - w'\| \le r_i \\ (t, x, w), (t', x', w') \in [0, \Delta_T] \times \mathcal{B}_H \times \mathcal{W} \end{cases}$$
$$\Rightarrow \|\varphi_{q_{\sigma,i}}(t, x, w) - \varphi_{q_{\sigma,i}}(t', x', w')\| < \epsilon_i. \qquad (3.26)$$

Also, since $\mathcal{W}$ is compact, π is uniformly continuous on $\mathcal{W}$. As such, for each $i \in \mathcal{N}$, there is a number $\delta_{r_i} > 0$ such that the following inequality holds for all $\varsigma \in [-\delta_{r_i}, \delta_{r_i}]$:

$$\|w_{-t^*+\tau_{\sigma,i}} - w_{-t^*-\varsigma+\tau_{\sigma,i}}\| = \|\pi(-t^* + \tau_{\sigma,i}, w) - \pi(-t^* - \varsigma + \tau_{\sigma,i}, w)\| < \frac{r_i}{2}. \qquad (3.27)$$

Based on the continuity of the transition mappings $\varphi_q, q \in \mathbb{Q}$, combining (3.26) and (3.27), it follows that for each $i \in \mathcal{N}$ and for every $\epsilon_i > 0$, there are $r_i > 0$ and $\delta_{r_i} > 0$ such that, for all $t \in [0, \Delta_T], x, x' \in \mathcal{B}_H$, and $\varsigma \in [-\delta_{r_i}, \delta_{r_i}]$, we have

$$\|x - x'\| < \frac{r_i}{2} \Rightarrow \|\varphi_{q_{\sigma,i}}(t, x, w_{-t^*+\tau_{\sigma,i}}) - \varphi_{q_{\sigma,i}}(t, x', w_{-t^*-\varsigma+\tau_{\sigma,i}})\| < \epsilon_i. \qquad (3.28)$$

Let r_{i^*} be a design constant to be specified later. For each $i \in \mathcal{N}$, let $\epsilon_i > 0$ and $r_i(\epsilon_i)$ be successively defined in such a way that $\epsilon_i \leq r_{i+1}/2$. Let $\delta_{\min} = \min\{\delta_{r_i} : i \in \mathcal{N}\}$. Note that $\tau_{\sigma,0} = 0$. Using (3.27) with $i = 0$, we have

$$|\Delta\tau_{\sigma,0} - \Delta\tau_{\sigma,0}\| + \|x - x\| + \|w_{-t^*} - w_{-t^*+\varsigma}\| \leq \frac{r_0}{2}, \forall\varsigma \in [-\delta_{\min}, \delta_{\min}]. \tag{3.29}$$

Thus, applying (3.26) yields

$$\begin{aligned}\|\mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,1}, w_{-t^*}) - \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,1}, w_{-t^*-\varsigma})\| = \|\varphi_{q_{\sigma,0}}(\Delta\tau_{\sigma,0}, x, w_{-t^*})\\ - \varphi_{q_{\sigma,0}}(\Delta\tau_{\sigma,0}, x, w_{-t^*-\varsigma})\| \leq \epsilon_0 \leq \frac{r_1}{2}, \forall\varsigma \in [-\delta_{\min}, \delta_{\min}].\end{aligned} \tag{3.30}$$

This coupled with (3.27) applying for $i = 1$ implies that

$$\begin{aligned}|\tau_{\sigma,1} - \tau_{\sigma,1}| + \|\mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,1}, w_{-t^*}) - \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,1}, w_{-t^*-\varsigma})\|\\ + \|w_{-t^*+\tau_{\sigma,1}} - w_{-t^*-\varsigma+\tau_{\sigma,1}}\| \leq r_1, \forall\varsigma \in [-\delta_{\min}, \delta_{\min}].\end{aligned} \tag{3.31}$$

Then, applying (3.25) for $i = 1$, we obtain

$$\begin{aligned}\|\mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,2}, w_{-t^*}) - \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,2}, w_{-t^*-\varsigma})\| = \|\varphi_{q_{\sigma,1}}(\Delta\tau_{\sigma,1}, \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,1}, w_{-t^*}), w_{-t^*+\Delta\tau_{\sigma,1}})\\ - \varphi_{q_{\sigma,1}}(\Delta\tau_{\sigma,1}, \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,1}, w_{-t^*-\varsigma}), w_{-t^*-\varsigma+\Delta\tau_{\sigma,1}})\| \leq \epsilon_1, \forall\varsigma \in [-\delta_{\min}, \delta_{\min}].\end{aligned} \tag{3.32}$$

Since $\epsilon_i \leq r_{i+1}/2, \forall i \in \mathcal{N}$, continuing this procedure, we arrive at

$$\|\mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,i^*}, w_{-t^*}) - \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,i^*}, w_{-t^*-\varsigma})\| \leq \epsilon_{i^*-1}, \forall\varsigma \in [-\delta_{\min}, \delta_{\min}]. \tag{3.33}$$

Due to the continuity of $\varphi_{q_{\sigma,i^*}}$, there is a number r_{i^*} such that in the compact set $[0, \Delta_T] \times \mathcal{B}_H \times \mathcal{W}$, we have

$$|t-t'| + \|x-x'\| + \|w-w'\| \leq r_{i^*} \Rightarrow \|\varphi_{q_{\sigma,i^*}}(t, x, w) - \varphi_{q_{\sigma,i^*}}(t', x', w')\| < \epsilon^*/2. \tag{3.34}$$

As $\epsilon_{i^*-1} \leq r_{i^*}/2$ by construction, combining (3.24), (3.33), and (3.34), we have

$$\|\mathscr{T}_{\sigma,\mathcal{NA}}(t^*, w_{-t^*}) - \mathscr{T}_{\sigma,\mathcal{NA}}(t^*, w_{-t^*-\varsigma})\| \leq \epsilon^*/2, \forall \varsigma \in [-\delta_{\min}, \delta_{\min}]. \tag{3.35}$$

Furthermore, as $\mathbb{Q}$ is finite and $\varphi_q, q \in \mathbb{Q}$ is continuous, there is a r_ϵ such that in the compact set $[0, \Delta_T] \times \mathcal{B}_H \times \mathcal{W}$, we have

$$|t+t'|+\|x+x'\|+\|w-w'\| \leq r_\epsilon \Rightarrow \|\varphi_q(t,x,w)-\varphi_q(t',x',w')\| \leq \epsilon^*/2, \forall q \in \mathbb{Q}. \tag{3.36}$$

This coupled with the condition that there is no switching jump leads to

$$\|\mathscr{T}_{\sigma,\mathcal{NA}}(t^* + \varsigma, w_{-t^*-\varsigma}) - \mathscr{T}_{\sigma,\mathcal{NA}}(t^*, w_{-t^*-\varsigma})\| < \epsilon^*/2 \tag{3.37}$$

for sufficiently small ς. Combining (3.37) and (3.35), we obtain

$$\|\mathscr{T}_{\sigma,\mathcal{NA}}(t^* + \varsigma, w_{-t^*-\varsigma}) - \mathscr{T}_{\sigma,\mathcal{NA}}(t^*, w_{-t^*})\| < \epsilon^* \tag{3.38}$$

for all sufficiently small ς. This contradicts to (3.23). Thus, we conclude that $\mathscr{T}_{\sigma,\mathcal{NA}}(t, w_{-t})$ is continuous with respect to t. ■

3.4.2 Existence and Quasi-invariance

Theorem 3.4.1 *Let $\Sigma_{\mathcal{NA}}$ be a switched non-autonomous system satisfying condition of Lemma 3.4.1. For each $\sigma \in \mathbb{S}$, $x \in \mathcal{X} \subset \mathbb{R}^n$, and $t_s \geq 0$, suppose that the pullback trajectories $\overset{\rightleftarrows}{\mathscr{O}}_{t_s,w,\sigma}(x)$ are bounded for all $w \in \mathcal{W}$. Then, for every $w \in \mathcal{W}$, the pullback ω-limit set $\omega_{t_s,w,\sigma}(x)$ is nonempty and compact. In addition, $\mathscr{T}_{\sigma,\mathcal{NA}}(t, t_s, x, w)$ approaches $\omega_{t_s,w,\sigma}(x)$ as $t \to \infty$.*

Proof: In this proof, we also use $\mathscr{T}_{\sigma,\mathcal{NA}}(t, w)$ to denote $\mathscr{T}_{\sigma,\mathcal{NA}}(t, t_s, x, w)$ for short.

Since $\overrightleftharpoons{\mathscr{O}}_{t_s,w,\sigma}(x) \subset \mathbb{R}^n$ are bounded for all $w \in \mathcal{W}, t_s \in \mathbb{R}^+$, for every time sequence $\{t_n\}_n$, the sequence $\{\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}(t_n, w_{-t_n})\}_n \subset \mathbb{R}^n$ is bounded. By the Bolzano-Weierstrass's lemma there exists a subsequence $\{n_m\}_m$ of $\{n\}_n$ such that the sequence $\{\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}(t_{n_m}, w_{-t_{n_m}})\}_m$ converges to some point $x^* \in \mathbb{R}^n$ which obviously belongs to $\omega_{t_s,w,\sigma}(x)$ by Definition 3.3.12. Thus, $\omega_{t_s,w,\sigma}(x)$ is non-empty for all $w \in \mathcal{W}$.

As $\omega_{t_s,w,\sigma}(x) \subset \mathbb{R}^n$, we shall prove its compactness by showing that it is bounded and closed. Firstly, since $\omega_{t_s,w,\sigma}(x)$ consists of limits of points in bounded trajectories $\overrightleftharpoons{\mathscr{O}}_{t_s,w,\sigma}(x), w \in \mathcal{W}, t_s \in \mathbb{R}^+$, it is bounded. We proceed to prove the closedness of $\omega_{t_s,w,\sigma}(x)$ by considering a limit point y of $\omega_{t_s,w,\sigma}(x)$. By definition, there is a sequence $\{y_n\}_n \subset \omega_{t_s,w,\sigma}(x)$ such that $y_n \to y, n \to \infty$.

Let $\tau > 0$ be any finite number and let $\{\varepsilon_n\}_n$ be any sequence satisfying $\varepsilon_n \to 0, n \to \infty$. For each $n \in \mathbb{N}$, let $k_n \in \mathbb{N}$ be the integer such that $\|y_{k_n} - y\| < \varepsilon_n/2$. Such an integer exists as $y_n \to y, n \to \infty$. For an index $k_n, n \in \mathbb{N}$, as $y_{k_n} \in \omega_{t_s,w,\sigma}(x)$, there is a time sequence $\{t_m^{(k_n)}\}_m$ such that

$$\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}\big(t_m^{(k_n)}, w_{-t_m^{(k_n)}}\big) \to y_{k_n}, m \to \infty. \tag{3.39}$$

From the sequences $\{t_m^{(k_n)}\}_m$, $n \in \mathbb{N}$, let us define the time sequence $\{t_{k_n}\}_n$ as follows. Applying (3.39) for $n = 0$, there is a time $t_{k_0} \in \{t_m^{(k_0)}\}_m$ such that $\|\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}(t_{k_0}, w_{-t_{k_0}}) - y_{k_0}\| < \varepsilon_0/2$. From t_{k_0}, applying (3.39) for each $n \in \mathbb{N}\backslash\{0\}$, we obtain the times $t_{k_n} \in \{t_m^{(k_n)}\}_m$ that satisfies $\|\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}(t_{k_n}, w_{-t_{k_n}}) - y_{k_n}\| < \varepsilon_n/2$ and $t_{k_n} > t_{k_{n-1}} + \tau$.

Obviously, $\{t_{k_n}\}_n$ is a time sequence as its elements are separated by τ. Thus, by construction, we have

$$\|\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}(t_{k_n}, w_{-t_{k_n}}) - y\| \leq \|\mathscr{T}_{\sigma,\mathscr{N}\mathscr{A}}(t_{k_n}, w_{-t_{k_n}}) - y_{k_n}\| + \|y_{k_n} - y\| < \varepsilon_n, \forall n \in \mathbb{N}. \tag{3.40}$$

As $\varepsilon_n \to 0, n \to \infty$, (3.40) implies that $\mathscr{T}_{\sigma,\mathcal{NA}}(t_{k_n}, w_{-t_{k_n}}) \to y, n \to \infty$, i.e., $y \in \omega_{t_s,w,\sigma}(x)$. Thus, $\omega_{t_s,w,\sigma}(x)$ is closed and hence, with its boundedness, is compact.

We prove the last assertion of the theorem by a standard contradiction argument. Suppose that the converse holds, i.e., there is a time sequence $\{t_n\}_n$ and a number $\epsilon > 0$ such that $\|\mathscr{T}_{\sigma,\mathcal{NA}}(t_n, t_s, x, w) - y\| > \epsilon, \forall n \in \mathbb{N}, y \in \omega_{t_s,w,\sigma}(x)$. Since $\{\mathscr{T}_{\sigma,\mathcal{NA}}(t_n, t_s, x, w)\}_n$ is bounded, it has a subsequence converging to some point $x^* \in \mathbb{R}^n$ which is obviously an element of $\omega_{t_s,w,\sigma}(x)$ by definition. This is a contradiction and hence the assertion holds. ■

Theorem 3.4.2 (quasi-invariance) *Let $\Sigma_{\mathcal{NA}}$ be a switched non-autonomous system in which the discrete set $\mathbb{Q}$ is finite and there is no switching jump. Let $t_s \in \mathbb{R}^+, x \in \mathcal{X} \subset \mathbb{R}^n$, and $\sigma \in \mathbb{S}$ fixed. Suppose that the non-autonomous pullback ω-limit set $\Omega_{t_s,\sigma}(x)$ exists and all limiting switching sequences of σ are non-Zeno. Then, $\Omega_{t_s,\sigma}(x)$ is σ-quasi-invariant.*

Proof: We provide a constructive proof of the theorem. Consider a point $y \in \omega_{t_s,w,\sigma}(x)$. By Definition 3.3.11, there is a time sequence $\{t_n\}_n$ such that

$$y = \lim_{n\to\infty} \mathscr{T}_{\sigma,\mathcal{NA}}(t_n, 0, x, w_{-t_n}). \tag{3.41}$$

It follows from Proposition 3.2.1 that there is a subsequence $\{t_{n_m}\}_m$ of $\{t_n\}_n$ such that $\sigma_{t_{n_m}} \to \sigma^*$ as $m \to \infty$. Thus, Proposition 3.2.3 implies that $\sigma_{t_{n_m}+t} \to \sigma^*_t, \forall t \in \mathbb{R}^+$. Hence, by virtue of Definition 3.2.1, we have

$$q_{\sigma,i^-_\sigma(t_{n_m}+t)+j} \overset{\text{def}}{=} q_{\sigma_{t_{n_m}+t},j} \to q_{\sigma^*_t,j} \overset{\text{def}}{=} q_{\sigma^*,i^-_{\sigma^*}(t)+j}, \forall t \in \mathbb{R}^+, \forall j \in \mathbb{N}. \tag{3.42}$$

By definition of the transition indicator $(\cdot)^-_{\sigma^*}$ (see Page 28), $i^-_{\sigma^*}(0)$ is the last index satisfying $\tau_{\sigma^*,i^-_{\sigma^*}(0)} \leq 0$, i.e, $\tau_{\sigma^*,j} = 0, \forall j = 0, \ldots, i^-_{\sigma^*}(0)$ and $\tau_{\sigma^*,i^-_{\sigma^*}(0)+1} > \tau_{\sigma^*,i^-_{\sigma^*}(0)}$. In view of Definition 3.2.1, this also implies that

$$\Delta\tau_{\sigma,i^-_\sigma(t_{n_m})+j} \overset{\text{def}}{=} \Delta\tau_{\sigma_{t_{n_m}},j} \to \Delta\tau_{\sigma^*,j} = 0, m\to\infty, \forall j = 1,\dots,i^-_{\sigma^*}(0)-1, \text{ and}$$
$$\tau_{\sigma,i^-_\sigma(t_{n_m})+1} - t_{n_m} \overset{\text{def}}{=} \Delta\tau_{\sigma_{t_{n_m}},0} \to \Delta\tau_{\sigma^*,0} = 0, m\to\infty. \tag{3.43}$$

Let $\tilde{\tau}_{\sigma,n_m}(0) = 0$. By virtue of (3.43), for each $k = 1,\dots,i^-_{\sigma^*}(0)$, the following number converges to zero as $m \to \infty$:

$$\tilde{\tau}_{\sigma,n_m}(k) = \sum_{j=0}^{k-1} \Delta\tau_{\sigma_{t_{n_m}},j}. \tag{3.44}$$

Using Definition 3.2.1, it is computed that $i^-_\sigma(t_{n_m}+\tilde{\tau}_{\sigma,n_m}(k)) = i^-_\sigma(t_{n_m})+k$. Hence, as $\sigma_{t_{n_m}} \to \sigma^*$ and the functions $\varphi_q, q\in\mathbb{Q}$ are continuous, we obtain the following equality for $k = i^-_{\sigma^*}(0)$:

$$\begin{aligned}
&\lim_{m\to\infty} \mathscr{T}_{\sigma,\mathscr{NA}}(t_{n_m}+\tilde{\tau}_{\sigma,n_m}(k),0,x,w_{-t_{n_m}}) \\
&\quad= \lim_{m\to\infty} \varphi_{q_{\sigma,i^-_\sigma(t_{n_m})+k-1}}(\Delta\tau_{\sigma,i^-_\sigma(t_{n_m})+k-1}, \mathscr{T}_{\sigma,\mathscr{NA}}(t_{n_m}+\tilde{\tau}_{\sigma,n_m}(k-1),0,x,w_{-t_{n_m}}),w) \\
&\quad= \lim_{m\to\infty} \mathscr{T}_{\sigma,\mathscr{NA}}(t_{n_m}+\tilde{\tau}_{\sigma,n_m}(k-1),0,x,w_{-t_{n_m}}).
\end{aligned} \tag{3.45}$$

As $i_{\sigma^*}(0)$ is finite, repeating the above equality for $k = i^-_{\sigma^*}(0)-1$ until $k=1$, we eventually obtain

$$\lim_{m\to\infty} \mathscr{T}_{\sigma,\mathscr{NA}}(t_{n_m}+\tilde{\tau}_{\sigma,n_m}(i^-_{\sigma^*}(0)),0,x,w_{-t_{n_m}}) = y. \tag{3.46}$$

Since $\Delta\tau_{\sigma_{t_{n_m}},i^-_{\sigma^*}(0)} \to \Delta\tau_{\sigma^*,i^-_{\sigma^*}(0)}$ we have $t+\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})} \le \tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})}+\Delta\tau_{\sigma_{t_{n_m}},i^-_{\sigma^*}(0)}$ for sufficiently large m, and hence $i^-_\sigma(t+\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})}) = i_{\sigma,\sigma^*}(t_{n_m}) = i^-_\sigma(t_{n_m})+i^-_{\sigma^*}(0)$ for sufficiently large m. In addition, by virtue of (3.44), we have $i_{\sigma,\sigma^*}(t_{n_m}) \overset{\text{def}}{=} i^-_\sigma(t_{n_m}+\tilde{\tau}_{\sigma,n_m}(i^-_{\sigma^*}(0))) = i^-_\sigma(t_{n_m})+i^-_{\sigma^*}(0)$ and $t_{n_m}+\tilde{\tau}_{\sigma,n_m}(i^-_{\sigma^*}(0)) = \tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})}$. Hence, for a time $t\in[\tau_{\sigma^*,i^-_{\sigma^*}(0)},\tau_{\sigma^*,i^-_{\sigma^*}(0)+1}] = [0,\Delta\tau_{\sigma^*,i^-_{\sigma^*}(0)}]$, applying (3.42) for $t=j=0$ and using the construction (2.14), we obtain

$$\begin{aligned}
\mathscr{T}_{\sigma^*,\mathscr{NA}}(t,0,y,w) &= \varphi_{q_{\sigma^*,i^-_{\sigma^*}(0)}}(t, \lim_{m\to\infty} \mathscr{T}_{\sigma,\mathscr{NA}}(t_{n_m}+\tilde{\tau}_{\sigma,n_m}(i^-_{\sigma^*}(0)),0,x,w_{-t_{n_m}}),w) \\
&= \lim_{m\to\infty} \varphi_{q_{\sigma^*,i^-_{\sigma^*}(0)}}(t, \mathscr{T}_{\sigma,\mathscr{NA}}(\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},0,x,w_{-t_{n_m}-\tilde{\tau}_{\sigma,n_m}(i^-_{\sigma^*}(0))}),w) \\
&= \lim_{m\to\infty} \varphi_{q_{\sigma,i^-_{\sigma}(t_{n_m})+i^-_{\sigma^*}(0)}}(t, \mathscr{T}_{\sigma,\mathscr{NA}}(\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},0,x,w_{-\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})}}),w) \\
&= \lim_{m\to\infty} \mathscr{T}_{\sigma,\mathscr{NA}}(t+\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},0,x,w_{-\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})}}) \\
&= \lim_{m\to\infty} \mathscr{T}_{\sigma,\mathscr{NA}}(t+\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},0,x,w_{-(t+\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})})+t}). \qquad (3.47)
\end{aligned}$$

As $t+\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})} \to \infty, m \to \infty$, the last equality of (3.47) shows that $\mathscr{T}_{\sigma^*,\mathscr{NA}}(t,0,y,w) \in \omega_{t_s,\pi(t,w),\sigma}$. Therefore, $\Omega_{t_s,\sigma}(x)$ is σ-quasi-invariant on $[0, \Delta\tau_{\sigma^*,0}]$. Repeating the above argument for subsequent time stages of σ^*, the conclusion of the theorem follows. ■

By virtue of Theorem 3.4.2, the quasi-invariance property is independent of the "past time" t_s of the switching sequence. Since autonomous systems are non-autonomous systems with constant transition mappings. The following results are straightforward from Theorems 3.4.1 and 3.4.2.

Corollary 3.4.1 *Let $\Sigma_{\mathscr{A}}$ be a switched autonomous system in which the discrete set $\mathbb{Q}$ is finite and there is no switching jump. For each $\sigma \in \mathbb{S}$, $x \in \mathcal{X} \subset \mathbb{R}^n$, and $t_s \geq 0$, suppose that the trajectory $\mathscr{O}_{t_s,\sigma}(x)$ is bounded. Then, the ω-limit set $\omega_{t_s,\sigma}(x)$ is nonempty and compact.*

Corollary 3.4.2 *Let $\Sigma_{\mathscr{A}}$ be a switched autonomous system in which the discrete set $\mathbb{Q}$ is finite and there is no switching jump. Let $t_s \in \mathbb{R}^+, x \in \mathcal{X} \subset \mathbb{R}^n$, and $\sigma \in \mathbb{S}$ fixed. Suppose that the ω-limit set $\omega_{t_s,\sigma}(x)$ exists and all limiting switching sequences of σ are non-Zeno. Then, $\omega_{t_s,\sigma}(x)$ is σ-quasi-invariant.*

To close this section, let us mention that except non-Zeno requirement, Theorem 3.4.2 applies to switching sequences possessing zero running times. In switched systems without dwell-time, arbitrarily short running times are possible and hence the limiting switching sequences tend to exhibit switching events of zero running times. As such, the generality of Theorem 3.4.2 is obvious.

3.5 Invariance Principles for Switched Systems

Since the work [90], the invariance principle has taken a central role in the qualitative theories of dynamical systems. Though the most lucid result is dedicated to autonomous dynamical systems and the invariance property of the limit sets of system trajectories is normally lost in more general classes of dynamical systems, the framework of [90] remains of increasing impact [58, 9, 10, 11, 55, 101, 26, 62, 63, 13, 107, 126]. This is due to the fact that, in locating attractors of the system, the invariance of limit sets plays the role of refining the first estimate of the attractor obtained from examining behavior of the auxiliary Lyapunov function. Therefore, though invariance is the defining property of attractors, there is no restriction in choosing other properties of the limit sets for refinement.

In this section, from the quasi-invariance property of trajectory of switched systems proven in the previous section, we develop further invariance principles for switched systems. The main improvement in comparison to the existing results lies in the relaxation of the switching decreasing condition. We first prove the result for switched non-autonomous systems, and then present the results for switched non-autonomous systems as consequences.

3.5.1 General Result

Notations

In the following $\Sigma_{\mathcal{NA}}$ is the switched non-autonomous system defined in Definition 2.4.5, where the discrete set $\mathbb{Q} = \{1, \ldots, q^\natural\}$ is finite, $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^d$ are topological spaces, and is the identity mapping with respect to its third argument, i.e., there is no switching jump. In this merit, we shall drop the discrete transition mapping and denote the switched non-autonomous system by the hexad $\Sigma_{\mathscr{NA}} = \{\mathbb{R}^+, \mathbb{Q}, \mathcal{X}, \mathcal{W}, \{\varphi_q\}_{q\in\mathbb{Q}}, \mathbb{S}, \pi)$ without embarrassment.

We shall use the term *non-autonomous set* to refer to a mapping $\mathscr{A} : \mathcal{W} \to \mathscr{P}(\mathcal{X})$ taking values in the set $\mathscr{P}(\mathcal{X})$ of subsets of $\mathcal{X}$ [53]. For convenience, we also denote the non-autonomous sets as a family of subsets of $\mathcal{X}$ parameterized by $\mathcal{W}$, i.e., $\mathscr{A}_{\mathcal{W}} = \{A_w : w \in \mathcal{W}\}$. For two non-autonomous sets $\mathscr{A}_{\mathcal{W}} = \{A_w\}_{w\in\mathcal{W}}$ and $\mathscr{B}_{\mathcal{W}} = \{B_w\}_{w\in\mathcal{W}}$ and a $w \in \mathcal{W}$, the set $\mathscr{A}_{\mathcal{W}}$ is said to be contained in $\mathscr{B}_{\mathcal{W}}$ at w and denoted $\mathscr{A}_{\mathcal{W}} \subset_w \mathscr{B}_{\mathcal{W}}$ if $A_w \subset B_w$. If $A_w \subset B_w, \forall w \in \mathcal{W}$, then $\mathscr{A}_{\mathcal{W}}$ is said to be contained in $\mathscr{B}_{\mathcal{W}}$ and denoted $\mathscr{A}_{\mathcal{W}} \subset \mathscr{B}_{\mathcal{W}}$.

For each $w \in \mathcal{W}$, we shall call $\mathfrak{R}_{\varphi,w}(\cdot)(t) : \mathcal{X} \to \mathcal{X}$ the w-motion in $\mathcal{X}$ of the ordinary non-autonomous dynamical system with transition mapping φ described in Definition 2.2.3. As usual $\mathfrak{R}_{\varphi,w}(D)(t) = \{\mathfrak{R}_{\varphi,w}(x)(t), x \in D\}$, and $\mathfrak{R}_{\varphi} \stackrel{\text{def}}{=} \{\mathfrak{R}_{\varphi,w} : w \in \mathcal{W}\}$ is called a non-autonomous motion.

The non-autonomous set $\mathscr{D}_{\mathcal{W}} = \{D_w\}_{w\in\mathcal{W}}, D_w \subset \mathcal{X}$ is said to be forward invariant under the non-autonomous motion $\mathfrak{R}_{\varphi}$ if $\mathfrak{R}_{\varphi,w}(D_w)(t) \subset D_{\pi(t,w)}, \forall t \in \mathbb{R}^+, \forall w \in \mathcal{W}$. The set $\mathscr{D}_{\mathcal{W}}$ is said to be a common forward invariant set for the switched non-autonomous system $\Sigma_{\mathscr{NA}}$ if it is forward invariant under the motions of all constituent systems $\varphi_q, q \in \mathbb{Q}$. If $\wp$ is a property of sets, then the non-autonomous set $\mathscr{A}_{\mathcal{W}} = \{A_w\}_{w\in\mathcal{W}}$ is said to have the property $\wp$ if all $A_w, w \in \mathcal{W}$ has this property. We say that the set $\mathscr{A}$ is the largest set satisfying property $\wp$ contained in the set $\mathscr{B}_{\mathcal{W}}$ at $w \in \mathcal{W}$ if for every set $\mathscr{A}'_{\mathcal{W}}$ satisfying $\wp$ contained in $\mathscr{B}_{\mathcal{W}}$ at w, we have $\mathscr{A}'{}_{\mathcal{W}} \subset_w \mathscr{A}_{\mathcal{W}}$. If for each $w \in \mathcal{W}$, C_w is the largest set satisfying $\wp$ contained in A_w, then the set $\mathscr{C}_{\mathcal{W}} \stackrel{\text{def}}{=} \{C_w\}_{w\in\mathcal{W}}$ is said to be the largest set satisfying $\wp$ contained in $\mathscr{A}_{\mathcal{W}}$.

When σ and t_s are fixed *a priori*, we shall use the notations w_t and $\mathscr{T}_w(t,x)$ to denote $\pi(t,w)$ and $\mathscr{T}_{\sigma,\mathscr{NA}}(t,t_s,x,w_{-t})$, respectively. In the context of the modern theory of non-autonomous systems [8, 29], $\mathscr{T}_w$ can be interpreted as the pullback motion toward the limit set at w.

Let V_1 and V_2 be functions from $\mathbb{R}^n$ to $\mathbb{R}$, the relative variation between V_1 and V_2 at t_1 and t_2 along the pullback motion $\mathscr{T}_w(t,x)$ is

$$\mathrm{Var}_{t_1}^{t_2}[V_1, V_2](\mathscr{T}_w, x) = |V_1(\mathscr{T}_w(t_1, x)) - V_2(\mathscr{T}_w(t_2, x))|. \tag{3.48}$$

Finally, a switching sequence $\sigma \in \mathbb{S}$ is said to have a persistent dwell-time τ_{p} if it has infinitely many switching events of running times no less than τ_{p}. Such switching events are called dwell-time switching events of σ.

General Invariance Principle

Theorem 3.5.1 *Let $\Sigma_{\mathcal{NA}}$ be a switched non-autonomous system in which every switching sequence in $\mathbb{S}$ has a persistent dwell-time. Suppose that $\mathscr{D}_{\mathcal{W}} = \{D_w\}_{w\in\mathcal{W}}, D_w \subset \mathcal{X}$ is a common forward invariant compact non-autonomous set of $\Sigma_{\mathcal{NA}}$ and $D = \bigcap_{w\in\mathcal{W}} D_w \neq \emptyset$.*

Let $\mathcal{G}[\mathcal{X}, \mathbb{R}; \mathcal{W}] = \{g_w\}_{w\in\mathcal{W}}$ and $\mathcal{V}_q[\mathcal{X}, \mathbb{R}; \mathcal{W}] = \{V_{q,w}\}_{w\in\mathcal{W}}, q \in \mathbb{Q}$ be families of functions in which $g_w, V_{q,w}, w \in \mathcal{W}, q \in \mathbb{Q}$ are continuous functions from D_w to $\mathbb{R}$. For each $q \in \mathbb{Q}$, let $r_{q,w} = \sup\{V_{q,w}(x) : x \in D, g_w(x) < 0\}$ if $\{x \in D : g_w(x) < 0\} \neq \emptyset$ and $r_{q,w} = -\infty$, otherwise.

Suppose further that there are nonnegative constants δ_1 and δ_2 such that for any fixed initial state $x \in D$, any fixed time t_s, and any fixed switching sequence $\sigma \in \mathbb{S}$, the following properties hold along the trajectory $\mathscr{T}_w(t,x), t \in \mathbb{R}^+$:

i) in any switching event $(q_{\sigma,i}, \Delta\tau_{\sigma,i}), i \in \mathbb{N}$, if $\varsigma_1, \varsigma_2 \in [0, \Delta\tau_{\sigma,i}]$ are such that $\varsigma_2 > \varsigma_1$ and $g_w(\mathscr{T}_w(\tau_{\sigma,i} + \varsigma_j, x)) \geq 0, j = 1, 2$, then

$$V_{q_{\sigma,i},w}(\mathscr{T}_w(\tau_{\sigma,i} + \varsigma_1, x)) \geq V_{q_{\sigma,i},w}(\mathscr{T}_w(\tau_{\sigma,i} + \varsigma_2, x)); \tag{3.49}$$

ii) for the sequence of dwell-time switching events $\{(q_{\sigma,i_j^{\mathcal{D}}}, \Delta\tau_{\sigma,i_j^{\mathcal{D}}})\}_j$ of σ satisfying $g_w(\mathscr{T}_w(\tau_{\sigma,i_j^{\mathcal{D}}}, x)) \geq 0$, we have

$$\limsup_{j\to\infty} \sup_{k>j} \mathrm{Var}_{\tau_{\sigma,i_j^{\mathcal{D}}}}^{\tau_{\sigma,i_k^{\mathcal{D}}}}[V_{q_{\sigma,i_j^{\mathcal{D}}},w}, V_{q_{\sigma,i_k^{\mathcal{D}}},w}](\mathscr{T}_w, x) < \delta_1, \text{and} \tag{3.50}$$

$$\limsup_{j\to\infty} \max_{t\in[\tau_{\sigma,i_{j-1}^{\mathcal{D}}+1}, \tau_{\sigma,i_j^{\mathcal{D}}}]} \mathrm{Var}_{\tau_{\sigma,i_{j-1}^{\mathcal{D}}+1}}^{t}[V_{q_{\sigma,i_{j-1}^{\mathcal{D}}},w}, V_{q_{\sigma,i_\sigma^-(t)},w}](\mathscr{T}_w, x) \leq \delta_2. \tag{3.51}$$

Let $r_w = \max\{r_{q,w} : q \in \mathbb{Q}\}$, $\mathscr{L}_\gamma = \{L_{w,\gamma}\}_{w\in\mathcal{W}}, \gamma \in \mathbb{R}$ be the non-autonomous level sets defined as $L_{w,\gamma} = \big\{\zeta \in D_w : \exists q \in \mathbb{Q}, V_{q,w}(\zeta) \leq r_w + 2\delta_2 + \delta_1$ or $V_{q,w}(\zeta) \in [\gamma - \delta_2, \gamma + \delta_1 + \delta_2]\big\}$, and $\mathscr{M}_\gamma \overset{\text{def}}{=} \{M_{\gamma,w}\}_{w\in\mathcal{W}}$ be the largest (t_s, σ)-quasi-invariant set contained in $\mathscr{L}_\gamma$ at w.

Then, $\mathscr{T}_w(t, x)$ converges to the set $M_w \overset{\text{def}}{=} \bigcup_{\gamma\in\mathbb{R}} M_{w,\gamma}$ as $t \to \infty$.

Proof: Let us first prove the boundedness of the pullback trajectories $\overset{\rightleftharpoons}{\mathscr{O}}_{t_s,\sigma}(x)$. Consider a time $t \in [0, \tau_{\sigma,i_\sigma^-(t_s)+1} - t_s]$. As $\mathscr{D}_{\mathcal{W}}$ is forward invariant for all $\varphi_q, q \in \mathbb{Q}$ and $D \subset D_w, \forall w \in \mathcal{W}$, from the construction (2.14) (Page 34), we have

$$\begin{aligned}\mathscr{T}_{\sigma,\mathcal{NA}}(t, t_s, x, w_{-t}) &= \varphi_{q_{\sigma,i_\sigma^-(t_s)}}(t, x, w_{-t}) = \mathfrak{R}_{\varphi_{q_{\sigma,i_\sigma^-(t_s)}},w_{-t}}(x)(t) \\ &\in \mathfrak{R}_{\varphi_{q_{\sigma,i_\sigma^-(t_s)}},w_{-t}}(D_{w_{-t}})(t) \subset D_w, \forall t \in [0, \tau_{\sigma,i_\sigma^-(t_s)+1} - t_s].\end{aligned} \tag{3.52}$$

To continue, let $i_{\sigma,1}^*$ denote the number $i_\sigma^-(\tau_{\sigma,i_\sigma^-(t_s)+1})$. We note that $\tau_{\sigma,i_{\sigma,1}^*} = \tau_{\sigma,i_\sigma^-(t_s)+1}$ but $i_{\sigma,1}^* \neq i_\sigma^-(t_s) + 1$ in general due to possible zero running time switching events at $\tau_{\sigma,i_\sigma^-(t_s)+1}$. As σ is non-Zeno, $\tau_{\sigma,i_{\sigma,1}^*+1} > \tau_{\sigma,i_{\sigma,1}^*}$ by definition.

We now consider a time $t \in [\tau_{\sigma,i_{\sigma,1}^*} - t_s, \tau_{\sigma,i_{\sigma,1}^*+1} - t_s] = [\tau_{\sigma,i_\sigma^-(t_s)+1} - t_s, \tau_{\sigma,i_{\sigma,1}^*+1} - t_s]$. Clearly, $i_\sigma^-(t - t_s) = i_{\sigma,1}^*$. From the construction (2.14), we have

$$\mathscr{T}_{\sigma,\mathcal{NA}}(t, t_s, x, w_{-t}) = \varphi_{\sigma,i_{\sigma,1}^*}(t_s + t - \tau_{\sigma,i_{\sigma,1}^*}, \mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,i_{\sigma,1}^*} - t_s, x, w_{-t}), w_{-t+\tau_{\sigma,i_{\sigma,1}^*}-t_s}). \tag{3.53}$$

By expressing $w_{-t} = w_{-(\tau_{\sigma,i_{\sigma,1}^*}-t_s)-(t_s+t-\tau_{\sigma,i_{\sigma,1}^*})}$ and using (3.52), we have $\mathscr{T}_{\sigma,\mathcal{NA}}(\tau_{\sigma,i_{\sigma,1}^*} -$

$t_s, x, w_{-t+\tau_{\sigma,i^*_{\sigma,1}}-t_s}) \in D_{w_{-(t_s+t-\tau_{\sigma,i^*_{\sigma,1}})}}$. This coupled with (3.53) and the forward invariance property of $\mathscr{D}_{\mathcal{W}}$ shows that for all $t \in [\tau_{\sigma,i^-_\sigma(t_s)+1} - t_s, \tau_{\sigma,i^*_{\sigma,1}+1} - t_s]$, we have

$$\mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w_{-t}) \in \varphi_{\sigma,i^*_{\sigma,1}}(t_s + t - \tau_{\sigma,i^*_{\sigma,1}}, D_{w_{-(t_s+t-\tau_{\sigma,i^*_{\sigma,1}})}}, w_{-t+\tau_{\sigma,i^*_{\sigma,1}}-t_s}) \in D_w. \quad (3.54)$$

From $\tau^*_{\sigma,1}$, repeating the above procedure, we conclude that $\mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w_{-t}) \in D_w, \forall t \geq \tau^*_{\sigma,1}$. In summary, $\mathscr{T}_{\sigma,\mathscr{NA}}(t, t_s, x, w_{-t}) \in D_w, \forall t \geq 0$. Thus, $\overset{\rightleftharpoons}{\mathscr{O}}_{t_s,w,\sigma} \subset D_w$ and hence is bounded. Applying Theorems 3.4.1 and 3.4.2, the pullback ω limit set $\Omega_{t_s,\sigma}(x)$ is nonempty, compact, and σ-quasi-invariant.

We proceed to show that if there is a sequence $\{V_{q_{\sigma,i^{\mathscr{D}}_{j_n}}},w(\mathscr{T}_w(\tau_{\sigma,i^{\mathscr{D}}_{j_n}}, x))\}_n$ satisfying $V_{q_{\sigma,i^{\mathscr{D}}_{j_n}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathscr{D}}_{j_n}}, x)) \geq r_{q_{\sigma,i^{\mathscr{D}}_{j_n}},w}, \forall n \in \mathbb{N}$, then there is a number $\gamma \in \mathbb{R}$ such that

$$\gamma + \delta_1 \geq \limsup_{n\to\infty} V_{q_{\sigma,i^{\mathscr{D}}_{j_n}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathscr{D}}_{j_n}}, x)) \text{ and } \liminf_{n\to\infty} V_{q_{\sigma,i^{\mathscr{D}}_{j_n}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathscr{D}}_{j_n}}, x)) \geq \gamma. \quad (3.55)$$

Indeed, from (3.50), there is a number $N \in \mathbb{N}$ such that

$$\mathrm{Var}_{\tau_{\sigma,i^{\mathscr{D}}_{j_N}}}^{\tau_{\sigma,i^{\mathscr{D}}_{j_m}}}[V_{q_{\sigma,i^{\mathscr{D}}_{j_N}},w}, V_{q_{\sigma,i^{\mathscr{D}}_{j_m}},w}](\mathscr{T}_w, x) < \delta_1, \forall m > N, \quad (3.56)$$

which implies that

$$V_{q_{\sigma,i^{\mathscr{D}}_{j_m}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathscr{D}}_{j_m}}, x)) > V_{q_{\sigma,i^{\mathscr{D}}_{j_N}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathscr{D}}_{j_N}}, x)) - \delta_1, \forall m > N. \quad (3.57)$$

As N and x are fixed, (3.57) shows that the sequence $\{V_{q_{\sigma,i^{\mathscr{D}}_{j_m}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathscr{D}}_{j_m}}, x))\}_{m=N}^{\infty}$ is lower bounded. Thus,

$$\gamma = \liminf_{m\to\infty} V_{q_{\sigma,i^{\mathscr{D}}_{j_m}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathscr{D}}_{j_m}}, x)) \quad (3.58)$$

exists and hence the last inequality in (3.55) holds true.

To prove the first inequality in (3.55), we suppose that its converse holds, i.e., there is $\epsilon > 0$ and a sequence $\{j'_n\}_n$ satisfying

$$V_{q_{\sigma,i^{\mathcal{D}}_{j'_n}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_{j'_n}},x)) \geq \gamma + \delta_1 + \epsilon, \forall n \in \mathbb{N}. \tag{3.59}$$

On the other hand, by definition of γ in (3.58), there is a sequence $\{j''_n\}_n$ satisfying $j''_n > j'_n, \forall n \in \mathbb{N}$ and

$$V_{q_{\sigma,i^{\mathcal{D}}_{j''_n}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_{j''_n}},x)) \leq \gamma + \frac{\epsilon}{2}, \forall n \in \mathbb{N}. \tag{3.60}$$

Combining (3.59) and (3.60) yields,

$$\sup_{j_m > j'_n} \mathrm{Var}_{\tau_{\sigma,i^{\mathcal{D}}_{j'_n}}}^{\tau_{\sigma,i^{\mathcal{D}}_{j_m}}}[V_{q_{\sigma,i^{\mathcal{D}}_{j'_n}},w}, V_{q_{\sigma,i^{\mathcal{D}}_{j_m}},w}](\mathscr{T}_w, x) \geq \delta_1 + \frac{\epsilon}{2}. \tag{3.61}$$

Taking limit of the left hand side of (3.61) as $j'_n \to \infty$, we obtain a contradiction to (3.50). Thus the first inequality in (3.55) also holds true.

We now estimate the converging region of the following composite function

$$V_{\mathscr{C},w}(t) = V_{q_{\sigma,i^-_\sigma(t)},w}(\mathscr{T}_w(t,x)), t \geq 0. \tag{3.62}$$

For $t \geq 0$, let $j^-(t) = \max\{j \in \mathbb{N} : \tau_{\sigma,i^{\mathcal{D}}_j} \leq t\}$, i.e., $i^{\mathcal{D}}_{j^-(t)}$ is the index of the switching even closest and before t. Let $\{j_{l,n}\}_{n=0}^{\tilde{N}}$ be the sequence of all indices $j \in \mathbb{N}$ satisfying $V_{q_{\sigma,i^{\mathcal{D}}_j},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_j},x)) \leq r_{q_{\sigma,i^{\mathcal{D}}_j},w}$. We have the following cases.

Case 1: $\tilde{N} = \infty$. In this case, we initially show that, for each $n \in \mathbb{N}$, we have

$$V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}(\mathscr{T}_w(t,x)) \leq r_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}, \forall t \in [\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}}, \tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}+1}]. \tag{3.63}$$

Suppose that the converse holds, i.e., there is a time $t \in [\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}}, \tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}+1}]$ and a number $\epsilon > 0$ such that $V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}(\mathscr{T}_w(t,x)) \geq r_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w} + \epsilon$. Then, as $V_q, q \in \mathbb{Q}$ and $\mathscr{T}_w$ are continuous (proven in Lemma 3.4.1), the time $t_0 = \inf\{t \in [\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}}, \tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}+1}] :$

$V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}(\mathscr{T}_w(t,x)) \geq r_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w} + \epsilon\}$ exists and satisfies $t_0 > \tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}}$. By condition ii), we have $V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}(\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}}) \geq V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}(t_0) > r_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}$ which is a contradiction.

Using the above proven property, we have the following expression

$$\begin{aligned} V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},w}(\mathscr{T}(\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},x)) &= V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}(\mathscr{T}(\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}+1},x)) \\ &\quad + V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},w}(\mathscr{T}(\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},x)) - V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}(\mathscr{T}(\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}+1},x)) \\ &\leq V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}(\mathscr{T}(\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},x)) + \mathrm{Var}[V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}}},w}, V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},w}]^{\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}}}_{\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}}+1}}(\mathscr{T}_w,x). \end{aligned} \tag{3.64}$$

Taking limits of both sides of (3.64) and using (3.51) yields

$$\liminf_{n\to\infty} V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},x)) \quad \leq \limsup_{n\to\infty} V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},x)) < r_w + \delta_2. \tag{3.65}$$

Let $\{j_{u,n}\}_n$ be the sequence of all indices $j \in \mathbb{N}$ such that $V_{q_{\sigma,i^{\mathcal{D}}_j},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_j},x)) > r_{q_{\sigma,i^{\mathcal{D}}_j},w}$. Clearly, $\{j_{l,n}+1\}_n \subset \{j_{u,n}\}_n$. Combining (3.55) and (3.65), we arrive at

$$\begin{aligned} \limsup_{j\to\infty} V_{q_{\sigma,i^{\mathcal{D}}_j},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_j},x)) &\leq \max\{r_w, \liminf_{n\to\infty} V_{q_{\sigma,i^{\mathcal{D}}_{j_{u,n}}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_{j_{u,n}}},x)) + \delta_1\} \\ &\leq \max\{r_w, \liminf_{n\to\infty} V_{q_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_{j_{l,n}+1}},x)) + \delta_1\} \leq r_w + \delta_2 + \delta_1. \end{aligned} \tag{3.66}$$

From condition i) and definition of r_w, we have

$$\begin{aligned} V_{\mathscr{C},w}(t) = V_{q_{\sigma,i^-_\sigma(t)},w}(\mathscr{T}_w(t,x)) &= V_{q_{\sigma,i^{\mathcal{D}}_{j^-(t)}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_{j^-(t)}+1},x)) \\ &\quad + V_{q_{\sigma,i^-_\sigma(t)},w}(\mathscr{T}_w(t,x)) - V_{q_{\sigma,i^{\mathcal{D}}_{j^-(t)}},w}(\mathscr{T}_w(\tau_{\sigma,i^{\mathcal{D}}_{j^-(t)}+1},x)) \\ &\leq \max\{r_w, V_{q_{\sigma,i^{\mathcal{D}}_{j^-(t)}}}(\mathscr{T}(\tau_{\sigma,i^{\mathcal{D}}_{j^-(t)}}))\} \\ &\quad + \max_{s\in[\tau_{\sigma,i^{\mathcal{D}}_{j^-(t)}+1},\tau_{\sigma,i^{\mathcal{D}}_{j^-(t)+1}}]} \mathrm{Var}^s_{\tau_{\sigma,i^{\mathcal{D}}_{j^-(t)}+1}}[V_{q_{\sigma,i^{\mathcal{D}}_{j^-(t)}+1},w}, V_{q_{\sigma,i^-_\sigma(s)},w}](\mathscr{T}_w,x). \end{aligned} \tag{3.67}$$

Taking limits of both sides of (3.67) and using (3.66) and (3.51), we obtain

$$\limsup_{t\to\infty} V_{\mathscr{C},w}(t) \le r_w + 2\delta_2 + \delta_1. \tag{3.68}$$

Case 2: $\tilde{N} < \infty$. In this case, there is a number M such that $V_{q_{\sigma,i_j^{\mathscr{D}}},w}(\mathscr{T}_w(\tau_{\sigma,i_j^{\mathscr{D}}},x)) > r_{q_{\sigma,i_j^{\mathscr{D}}},w}, \forall j \ge M$. Thus, according to (3.55), there is a number γ such that

$$\gamma + \delta_1 \ge \limsup_{j\to\infty} V_{q_{\sigma,i_j^{\mathscr{D}}},w}(\mathscr{T}_w(\tau_{\sigma,i_j^{\mathscr{D}}},x)) \text{ and } \liminf_{j\to\infty} V_{q_{\sigma,i_j^{\mathscr{D}}},w}(\mathscr{T}_w(\tau_{\sigma,i_j^{\mathscr{D}}},x)) \ge \gamma. \tag{3.69}$$

Using (3.51), (3.67), and the first inequality of (3.69), we have

$$\limsup_{t\to\infty} V_{\mathscr{C},w}(t) \le \gamma + \delta_1 + \delta_2. \tag{3.70}$$

In addition, from the first equality of (3.67), it follows that

$$\begin{aligned} V_{\mathscr{C},w}(t) \ge\ & V_{q_{\sigma,i^{\mathscr{D}}_{j^-(t)}}}(\mathscr{T}(\tau_{\sigma,i^{\mathscr{D}}_{j^-(t)}})) \\ & - \max_{s\in[\tau_{\sigma,i^{\mathscr{D}}_{j^-(t)}+1},\tau_{\sigma,i^{\mathscr{D}}_{j^-(t)+1}}]} \mathrm{Var}^s_{\tau_{\sigma,i^{\mathscr{D}}_{j^-(t)}+1}}[V_{q_{\sigma,i^{\mathscr{D}}_{j^-(t)}+1},w}, V_{q_{\sigma,i^-_\sigma(s)},w}](\mathscr{T}_w,x), \end{aligned} \tag{3.71}$$

from which, using (3.51) and the last inequality of (3.69), we have

$$\liminf_{t\to\infty} V_{\mathscr{C},w}(t) \ge \gamma - \delta_2. \tag{3.72}$$

We now consider a limit point of the trajectory $y \in \omega_{t_s,\sigma,w}(x)$. By definition, there is a time sequence $\{t_n\}_n$ such that

$$y = \lim_{n\to\infty} \mathscr{T}_{\sigma,\mathscr{N}\!\mathscr{A}}(t_n, t_s, x, w_{-t_n}). \tag{3.73}$$

Since $\mathbb{Q}$ is finite, there is an index q^* and a subsequence $\{t_{n_m}\}_m$ of $\{t_n\}_n$ such that $q_{\sigma,i^-_\sigma(t_{n_m})} = q^*, \forall m \in \mathbb{N}$. Since the function V_{q^*} is continuous, using (3.68) in the case

$\tilde{N} = \infty$, we obtain

$$V_{q^*}(y) = \lim_{m\to\infty} V_{q_{\sigma,i_\sigma^-(t_{n_m})}}(\mathscr{T}_w(t_{n_m}, x)) \in (V_{-\infty,w}, r_w + 2\delta_2 + \delta_1], \tag{3.74}$$

and using (3.70) and (3.70) in the case $\tilde{N} < \infty$, we have

$$V_{q^*}(y) = \lim_{m\to\infty} V_{q_{\sigma,i_\sigma^-(t_{n_m})}}(\mathscr{T}_w(t_{n_m}, x)) \in [\gamma - \delta_2, \gamma + \delta_1 + \delta_2] \tag{3.75}$$

for some $\gamma \in \mathbb{R}$, where $V_{-\infty,w} = \inf\{V_{q,w}(\zeta) : \zeta \in D_w, q \in \mathbb{Q}\}$.

Let us define the following level sets

$$L_{w,\gamma} = \big\{\zeta \in D_w : \exists q \in \mathbb{Q}, V_{q,w}(\zeta) \le r_w + 2\delta_2 + \delta_1 \text{ or } V_q(\zeta) \in [\gamma - \delta_2, \gamma + \delta_1 + \delta_2]\big\}. \tag{3.76}$$

From (3.74) and (3.75), we have $y \in L_{\gamma,w}, \forall y \in \omega_{t_s,\sigma,w}(x)$, i.e., $\Omega_{t_s,\sigma}(x) \subset_w \mathscr{L}_\gamma$. Since $\Omega_{t_s,\sigma}(x)$ is σ-quasi-invariant, we also have $\Omega_{t_s,\sigma}(x) \subset \mathscr{M}_\gamma$. Therefore, $\omega_{t_s,\sigma,w} \subset M_w$. Since $\omega_{t_s,\sigma,w}(x)$ is compact, we have $\mathscr{T}_w(t,x) \to \omega_{t_s,\sigma,w}(x)$ and hence the conclusion of the theorem follows. ■

Discussion

Invariance principles in qualitative theories of dynamical systems aim at locating attractors of the systems by firstly estimating region of attraction achieved through convergence of Lyapunov functions and then further refining this estimate in terms of invariance properties of limit sets. In this framework, the smaller the convergence region of Lyapunov functions is, the more precise estimate of attractor is.

As discussed in Chapter 1, the existing results on invariance principles for switched systems impose the switching decreasing condition on Lyapunov functions, i.e., the Lyapunov functions are decreasing along the active time of their respective constituent systems [63, 13, 107]. Exploiting this condition, the Lyapunov functions are all uniformly decreasing and hence good first estimates of attractors are guaranteed. However, as discussed, the trade-off for the theoretical interest of the switching decreasing condition is expensive: applicability of the corresponding results is considerably restricted.

Under the above practical consideration, the general result in Theorem 3.5.1 makes no decreasing requirement on periods of persistence. As presented, the natural approach for dealing with the respective difficulty is to estimate the diverging behavior in terms of the converging behavior achievable on dwell-time intervals. The underlying observation in developing the principle is: as Lyapunov functions are continuous functions of state variables, their variations on bounded time intervals are bounded if systems have no finite escape time. As shown in Figure 3.1, at the starting time of a dwell-time switching event, it is always possible to determine the decrements needed for maintaining convergence. For persistent dwell-time switching sequences, as the desired minimum length of the running time of dwell-time switching events is guaranteed by τ_{p}, it is possible to estimate the growth in persistent periods and then design an appropriate control for achieving the desired decrements as long as diverging periods remain bounded. Without involving control design, this observation is formulated in terms of bounded ultimate variations in condition ii) of Theorem 3.5.1. The generality of condition ii) lies in the fact that in the setting of classical dynamical systems, i.e., $V_{q,w}$ are identical with respect to $q \in \mathbb{Q}$, $g_w(x) \equiv 0$, and i) is satisfied, this condition automatically holds with $\delta_1 = \delta_2 = 0$.

We note that, similar to the notion of *small-time norm observability* of switched autonomous systems [63] stated for norms of system state and output, we study convergence in terms of *small-variations small state* to which condition ii) of Theorem 3.5.1 is presented. In general, if a trajectory converges to some region (neighborhood of origin in [63]), then the limit of diverging segments of the trajectory must stay

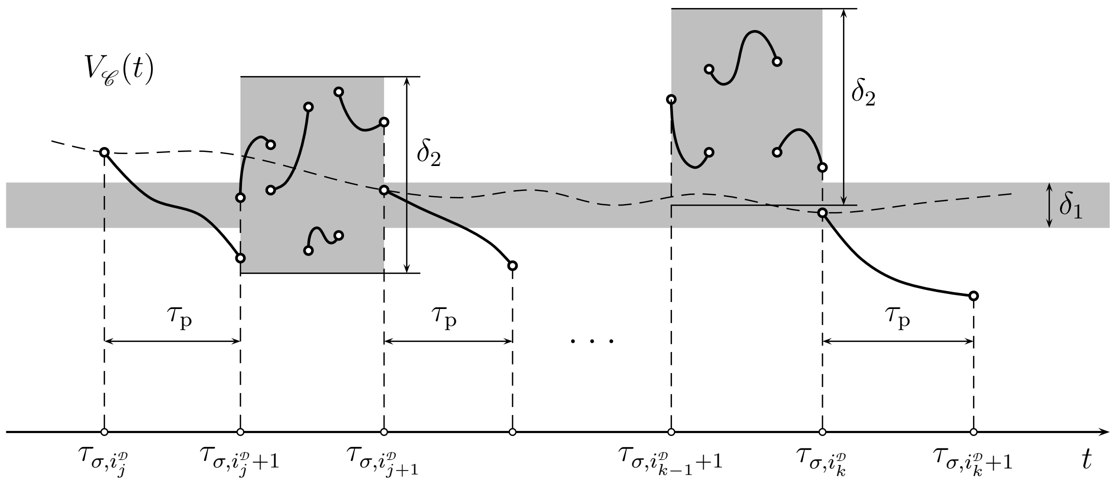

Figure 3.1: Composite Lyapunov function

around this region. The consideration on ultimate variations in condition ii) is to reflect the fact that this desired *detectability* property is satisfiable for systems without finite escape time plus with bounded destabilizing periods.

We note that the condition (3.50) is also of practical relevance. This condition seems to be necessary for converging behavior of any continuous dynamical systems. It is obvious that this condition automatically holds for $\delta_1 = 0$ for ordinary dynamical systems possessing Lyapunov functions and switched systems satisfying the switching decreasing condition which guaranteeing the convergence of all Lyapunov functions. On the other hand, at its high level of generality, this condition seems to be difficult to verify. However, as it is imposed on dwell-time intervals, it is satisfiable by control design. This shall be illustrated in Part II of the thesis - Advanced Control. Also, the consideration of the functions $g_w, w \in \mathcal{W}$ is of practical interest. In control systems with inherent uncertainties, converging to small neighborhoods of an equilibrium – described by $g_w, w \in \mathcal{W}$ – is more realistic than achieving asymptotic convergence to this single equilibrium.

The expense for the generality and the relaxation in Theorem 3.5.1 is the set estimate of attractor (3.76). Due to the allowance of destabilizing behavior on persistent periods, set estimates were achieved instead of single curves/manifolds as in the classical qualitative theories of ordinary dynamical systems [56, 90]. However, rich information on the structure of the attractor is carried on the level sets (3.76). As shall be presented in the next subsection, under less general conditions, (3.76) can be exploited to obtain stronger results.

To close the presentation of the general invariance principle, let us mention that the function $V_{\mathscr{C},w}$ in the above proof of Theorem 3.5.1 is a function of time. As the trajectory of $V_{\mathscr{C},w}(t)$ is the concatenation of the Lyapunov functions $V_{q,w}$'s of constituent systems, we appropriately call $V_{\mathscr{C},w}$ the composite Lyapunov function for the ease of reference.

3.5.2 Case Studies

In this subsection, we demonstrate that using further properties in specific situations, stronger results can be obtained from Theorem 3.5.1. As switched autonomous systems are switched non-autonomous systems with one-element base space $\mathcal{W} = \{w\}$, the first result is a direct application of Theorem 3.5.1 for a version of invariance principle of switched autonomous systems. The last two results are to show that once the dwell-time property of limiting switching sequence gives stronger results while preserving applicability to general switching sequences.

Theorem 3.5.2 *Let $\Sigma_{\mathscr{A}}$ be a switched autonomous system in which every switching sequence in $\mathbb{S}$ has a persistent dwell-time. Suppose that $D \subset \mathcal{X}$ is a nonempty common forward invariant compact set of $\Sigma_{\mathscr{A}}$. Consider the continuous functions $V_q : D \to \mathbb{R}, q \in \mathbb{Q}$ and $g : D \to \mathbb{R}$. Let r_q be $\sup\{V_q(x) : x \in D, g(x) < 0\}$ if $\{x \in D : g(x) < 0\} \neq \emptyset$ and be $-\infty$, otherwise.*

Suppose further that there are nonnegative constants δ_1 and δ_2 such that for any fixed initial state $x \in D$, any fixed time t_s, and any fixed switching sequence $\sigma \in \mathbb{S}$, the following properties hold along the trajectory $x(t) \stackrel{\text{def}}{=} \mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x), t \in \mathbb{R}^+$:

i) in any switching event $(q_{\sigma,i}, \Delta\tau_{\sigma,i}), i \in \mathbb{N}$, if $\varsigma_1, \varsigma_2 \in [0, \Delta\tau_{\sigma,i}]$ are such that

$\varsigma_2 > \varsigma_1$ and $g(x(\tau_{\sigma,i} + \varsigma_j)) \geq 0, j = 1, 2,$ then

$$V_{q_{\sigma,i}}(x(\tau_{\sigma,i} + \varsigma_1)) \geq V_{q_{\sigma,i}}(x(\tau_{\sigma,i} + \varsigma_2)); \tag{3.77}$$

ii) for the sequence of dwell-time switching events $\{(q_{\sigma,i_j^{\mathscr{D}}}, \Delta\tau_{\sigma,i_j^{\mathscr{D}}})\}_j$ of σ satisfying $g(x(\tau_{\sigma,i_j^{\mathscr{D}}})) \geq 0, \forall j \in \mathbb{N}$, we have

$$\limsup_{j\to\infty} \sup_{k>j} \left| V_{q_{\sigma,i_j^{\mathscr{D}}}}(x(\tau_{\sigma,i_j^{\mathscr{D}}})) - V_{q_{\sigma,i_k^{\mathscr{D}}}}(x(\tau_{\sigma,i_k^{\mathscr{D}}})) \right| < \delta_1, \text{and} \tag{3.78}$$

$$\limsup_{j\to\infty} \max_{t\in[\tau_{\sigma,i_{j-1}^{\mathscr{D}}+1}, \tau_{\sigma,i_j^{\mathscr{D}}}]} \left| V_{q_{\sigma,i_{j-1}^{\mathscr{D}}}}(x(\tau_{\sigma,i_{j-1}^{\mathscr{D}}+1})) - V_{q_{\sigma,i_\sigma^-(t)}}(x(t)) \right| \leq \delta_2. \tag{3.79}$$

Let $r = \max\{r_q : q \in \mathbb{Q}\}$, $L_\gamma, \gamma \in \mathbb{R}$ be the level sets defined as $L_\gamma = \big\{\zeta \in D : \exists q \in \mathbb{Q}, V_q(\zeta) \leq r + 2\delta_2 + \delta_1$ or $V_q(\zeta) \in [\gamma - \delta_2, \gamma + \delta_1 + \delta_2]\big\}$, and M_γ be the largest (t_s, σ)-quasi-invariant set contained in L_γ.

Then, $x(t) = \mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x)$ converges to the set $M \stackrel{\text{def}}{=} \bigcup_{\gamma\in\mathbb{R}} M_\gamma$ as $t \to \infty$.

Proof: As $\Sigma_{\mathscr{A}}$ is a switched non-autonomous system with one-element base space $\mathcal{W} = \{w\}$ in which the rule of autonomous transition is $\pi(t, w) = w, \forall t \in \mathbb{R}$. It is obvious that under conditions i) and ii), this non-autonomous system satisfies conditions of Theorem 3.5.1. Thus, the result is obvious. ■

In the following invariance principle, dwell-time property of the limiting switching sequences gives stronger results.

Theorem 3.5.3 *Let $\Sigma_{\mathscr{A}}$ be a switched autonomous system satisfying conditions of Theorem 3.5.2. In addition, suppose that the set $\mathbb{S}_\sigma^*$ of limiting switching sequences of σ is contained in $\mathbb{S}_{\mathscr{D}}[\tau_{\mathrm{p}}]$. Let $r = \max_q \sup\{V_q(x) : x \in D, g(x) < 0\}$, and $L_\gamma, \gamma \in \mathbb{R}$ be the level sets defined as $L_\gamma = \big\{\zeta \in D : \exists q \in \mathbb{Q}, V_q(\zeta) \leq r + \delta_2$ or $V_q(\zeta) \in [\gamma - \delta_2, \gamma + \delta_1]\big\}$, and M_γ be the largest (t_s, σ)-quasi-invariant set contained in L_γ.*

Then, $\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x)$ converges to the set $M \stackrel{\text{def}}{=} \bigcup_{\gamma\in\mathbb{R}} M_\gamma$ as $t \to \infty$.

Proof: Let us consider a limit point $y \in \omega_{t_s,\sigma}(x)$. By Definition 3.3.4, there is a sequence $\{t_n\}_n$ such that

$$y = \lim_{n\to\infty} \mathscr{T}_\sigma(t_n, t_s, x). \tag{3.80}$$

As known from the proof of Proposition 3.2.1, there is a subsequence $\{t_{n_m}\}_m$ of $\{t_n\}_n$ and a limiting switching signal σ^* of σ such that $\sigma_{t_{n_m}} \to \sigma^*, m \to \infty$. Without loss of generality, we suppose that $\sigma_{t_n} \to \sigma^*, n \to \infty$. By condition of the theorem, σ^* is a dwell-time signal in $\mathbb{S}_{\mathscr{D}}[\tau_\mathrm{p}]$.

Consider the case $\{t_{n_m}\}_m$ has infinitely many elements contained in non-dwell-time switching intervals, i.e., $\Delta\tau_{\sigma,i_\sigma^-(t_{n_m})} < \tau_\mathrm{p}$ for infinitely many $m \in \mathbb{N}$. Let us label the sequence of all such elements as $\{t^a_{n_m}\}_m$. As $\sigma_{t_{n_m}} \to \sigma^* \in \mathbb{S}_\sigma[\tau_\mathrm{p}]$, we have $\Delta\tau_{\sigma,i_\sigma^-(t^a_{n_m})} \to 0, m \to \infty$.

For each $t \in \mathbb{R}^+$, let $i_{\mathscr{D}}^-(t)$ be the index of the dwell-time switching event in σ before and closest to t, i.e., $i_{\mathscr{D}}^-(t) = \max\{i \in \mathbb{N} : \Delta\tau_{\sigma,i} \geq \tau_\mathrm{p}, \tau_{\sigma,i} \leq t\}$. By definition of $t^a_{n_m}$, we have $t^a_{n_m} > \tau_{\sigma,i_{\mathscr{D}}^-(t^a_{n_m})}, \forall m \in \mathbb{N}$.

Since all switching events between $i_{\mathscr{D}}^-(t^a_{n_m})$ and $i_\sigma^-(t^a_{n_m})$ are non-dwell-time and $\mathbb{S}_\sigma^* \subset \mathbb{S}_{\mathscr{D}}[\tau_\mathrm{p}]$, we have

$$\lim_{m\to\infty} (t^a_{n_m} - \tau_{\sigma,i_{\mathscr{D}}^-(t^a_{n_m})}) = 0. \tag{3.81}$$

Therefore, by the continuity of the mapping $\mathscr{T}_{\sigma,\mathscr{A}}$ we have

$$y = \lim_{m\to\infty} \mathscr{T}_{\sigma,\mathscr{A}}(t^a_{n_m}, t_s, x) = \lim_{m\to\infty} \mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_{\mathscr{D}}^-(t^a_{n_m})}, t_s, x). \tag{3.82}$$

In view of (3.82), we suppose that all elements of $\{t_{n_m}\}_m$ belong to dwell-time switching intervals of σ without loss of generality.

As $\mathbb{Q}$ is finite, there is a $q^* \in \mathbb{Q}$ and infinitely many numbers $t^b_{n_m}$ such that $q_{\sigma,i_\sigma^-(t^b_{n_m})} = q^*$. According to the proof of Theorem 3.5.1, we have either, in view of

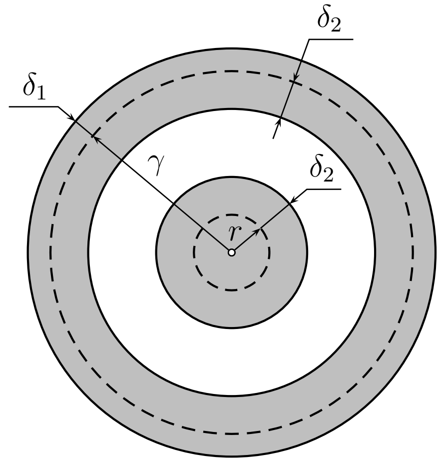

Figure 3.2: Level set L_γ

(3.64) and (3.65),

$$\begin{aligned} V_{-\infty} &= \inf\{V_q(\zeta) : \zeta \in D, q \in \mathbb{Q}\} \leq \liminf_{m\to\infty} V_{q_{\sigma,i_\sigma^-(t_{n_m})}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_\sigma^-(t_{n_m})}, t_s, x)) \\ &\leq \lim_{m\to\infty} V_{q_{\sigma,i_\sigma^-(t^b_{n_m})}}(\mathscr{T}_{\sigma,\mathscr{A}}(t^b_{n_m}, t_s, x)) = V_{q^*}(y) \\ &\leq \limsup_{m\to\infty} V_{q_{\sigma,i_\sigma^-(t_{n_m})}}(\mathscr{T}_{\sigma,\mathscr{A}}(t_{n_m}, t_s, x)) \leq r + \delta_2 \end{aligned} \tag{3.83}$$

or, in view of (3.55) and (3.79),

$$\begin{aligned} \gamma + \delta_1 &\geq \limsup_{m\to\infty} V_{q_{\sigma,i_\sigma^-(t_{n_m})}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_\sigma^-(t_{n_m})}, t_s, x)) \\ &\geq \lim_{m\to\infty} V_{q_{\sigma,i_\sigma^-(t^b_{n_m})}}(\mathscr{T}_{\sigma,\mathscr{A}}(t^b_{n_m}, t_s, x)) = V_{q^*}(y) \\ &\geq \liminf_{m\to\infty} V_{q_{\sigma,i_\sigma^-(t_{n_m})}}(\mathscr{T}_{\sigma,\mathscr{A}}(t_{n_m}, t_s, x)) \geq \liminf_{m\to\infty} V_{q_{\sigma,i_\sigma^-(t_{n_m})}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_\sigma^-(t_{n_m})+1}, t_s, x)) \\ &\geq \liminf_{m\to\infty} \Big(\Big(V_{q_{\sigma,i_\sigma^-(t_{n_m})}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_\sigma^-(t_{n_m})+1}, t_s, x)) \\ &\quad - V_{q_{\sigma,i_\sigma^-(t_{n_m}+1)}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_\sigma^-(t_{n_m})+1}, t_s, x))\Big) \\ &\quad + V_{q_{\sigma,i_\sigma^-(t_{n_m}+1)}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_\sigma^-(t_{n_m})+1}, t_s, x))\Big) \geq \gamma - \delta_2. \end{aligned} \tag{3.84}$$

Combining (3.83) and (3.84), it follows that $y \in L_\gamma$ and hence the conclusion of the theorem follows. ■

The level set L_γ in Theorem 3.5.3 is depicted in Figure 3.2. As can be predicted from Figure 3.1, due to vanishing separation between dwell-time switching intervals, the behavior of the system is eventually governed by the limiting switching sequence and hence good estimates of attractors can be achieved while general switching sequences are permitted. By the following theorem, we show that under the usual conditions on Lyapunov functions, classical analogues of estimates of attractors are obtained.

Theorem 3.5.4 *Let $\Sigma_{\mathscr{A}}$ be a switched autonomous system in which every switching sequence in $\mathbb{S}$ has a persistent dwell-time. Suppose that $D \subset \mathcal{X}$ is a nonempty common forward invariant compact set of $\Sigma_{\mathscr{A}}$ and there are continuous functions $V_q : D \to \mathbb{R}, q \in \mathbb{Q}$ such that for a fixed initial state $x \in D$, a fixed time t_s, and a fixed switching sequence $\sigma \in \mathbb{S}_{\mathscr{P}}[\tau_{\mathrm{p}}, T_{\mathrm{p}}] \subset \mathbb{S}$, the following properties hold along the trajectory $\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x), t \in \mathbb{R}^+$:*

i) $V_{q_{\sigma,i}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i} + \varsigma_1, t_s, x)) \leq V_{q_{\sigma,i}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i} + \varsigma_2, t_s, x)), \forall \varsigma_1, \varsigma_2 \in [0, \Delta\tau_{\sigma,i}], \varsigma_1 < \varsigma_2, i \in \mathbb{N}$;

ii) for the sequence of dwell-time switching events $\{(q_{\sigma,i_j^{\mathscr{P}}}, \Delta\tau_{\sigma,i_j^{\mathscr{P}}})\}_j$ of σ, we have

$$V_{q_{\sigma,i_j^{\mathscr{P}}}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_j^{\mathscr{P}}}, t_s, x)) \geq V_{q_{\sigma,i_k^{\mathscr{P}}}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_k^{\mathscr{P}}}, t_s, x)), \forall k, j \in \mathbb{N}, k > j, and \tag{3.85}$$

$$\limsup_{j\to\infty} \max_{t \in [\tau_{\sigma,i_{j-1}^{\mathscr{P}}}, \tau_{\sigma,i_j^{\mathscr{P}}}]} \left| V_{q_{\sigma,i_{j-1}^{\mathscr{P}}}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_{j-1}^{\mathscr{P}}}, t_s, x)) - V_{q_{\sigma,i_\sigma^-(t)}}(\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x)) \right| = 0. \tag{3.86}$$

Let $L_\gamma, \gamma \in \mathbb{R}$ be the level sets defined as $L_\gamma = \{\zeta \in D : \exists q \in \mathbb{Q}, V_q(\zeta) = \gamma\}$, and M_γ be the largest (t_s, σ)-quasi-invariant set contained in L_γ.

Then, $\mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, x)$ converges to the set $M \stackrel{\text{def}}{=} \cup_{\gamma\in\mathbb{R}} M_\gamma$ as $t \to \infty$.

Proof: From (3.85), $\{V_{q_{\sigma,i_j^{\mathscr{P}}}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_j^{\mathscr{P}}}, t_s, x))\}_j$ is a decreasing sequence. In addition, as $V_q, q \in \mathbb{Q}$ are continuous, and D is compact and common forward invariant compact set for $\Sigma_{\mathscr{A}}$, this sequence is lower bounded. Thus, there is a number $\gamma \in \mathbb{R}$ such that

$$\gamma = \lim_{j\to\infty} V_{q_{\sigma,i_j^{\mathscr{D}}}}(\mathscr{T}_{\sigma,\mathscr{A}}(\tau_{\sigma,i_j^{\mathscr{D}}}, t_s, x)), \tag{3.87}$$

and hence condition (3.78) of Theorem 3.5.2 is satisfied with $\delta_1 = 0$.

Let $g(x) \equiv 0$, then all conditions of Theorem 3.5.1 are satisfied. Applying Theorem 3.5.1 are satisfied and substituting $\delta_1 = \delta_2 = 0$ and $r = -\infty$, we have the conclusion of the theorem. ■

3.6 Examples

In this section, we provides examples to demonstrate applications of the introduced invariance principles in locating attractors of switched systems using auxiliary functions. It shall be demonstrated that by examining non-autonomous attractors, converging to the origin behavior can be achieved. We also show in Example 3.2 the existence of limit cycles in switched systems.

Example 3.6.1 (The Non-autonomous Case)

In this example, we consider the switched non-autonomous system $\Sigma_{\mathscr{N}\mathscr{A}}$ whose constituent systems are described by the following differential equations.

$$\begin{aligned}
\varphi_1: \quad & \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1(3x_1^2 + x_2^2 - w_1) \\ -3x_1 - 2x_2(3x_1^2 + x_2^2 - w_1) \end{bmatrix} \\
\varphi_2: \quad & \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 4x_2 - x_1(x_1^2 + 4x_2^2 - w_2) \\ -x_1 - x_2(x_1^2 + 4x_2^2 - w_2) \end{bmatrix}
\end{aligned} \tag{3.88}$$

where $w = [w_1, w_2]^T$ is the time-varying parameter generated by an autonomous system π on some compact set $\mathcal{W} \subset \mathbb{R}^2$. Let $\lambda(\pi(t,w)) \overset{\text{def}}{=} \partial\pi(t,w)/\partial t$, and let V_w be the non-autonomous Lyapunov function whose components are

$$V_{1,w}(x) = (3x_1^2 + x_2^2 - w_1)^2,\ V_{2,w}(x) = (x_1^2 + 4x_2^2 - w_2)^2, w \in \mathcal{W}. \tag{3.89}$$

Let $\varphi_i(t,x,v) = [\varphi_{i,1}(t,x,v), \varphi_{i,2}(t,x,v)]^T, i = 1,2$ be the component transition mappings of $\Sigma_{\mathcal{NA}}$. For a fixed $w \in \mathcal{W}$, the time derivatives of $V_{i,w}, i = 1,2$ along the pull-back trajectories at w of the constituent systems φ_j, which are $\varphi_j(t,x,w_{-t}), w_{-t} = \pi(-t,w)$, can be computed as

$$D_jV_{i,w}(\varphi_j(t,x,w_{-t})) = \frac{\partial V_{i,w}(x)}{\partial x}\Big|_{x=\varphi_j(t,x,w_{-t})}\Big(\Big(\frac{\partial \varphi_j}{\partial v}\lambda(v)\Big)\Big|_{v=w_{-t}} + \frac{\partial \varphi_j}{\partial t}\Big). \tag{3.90}$$

Let $[\xi_1,\xi_2]^T \overset{\text{def}}{=} \varphi_1(t,x,w_{-t})$ and $[\zeta_1,\zeta_2]^T \overset{\text{def}}{=} \varphi_2(t,x,w_{-t})$ for short. Since π is an autonomous system on compact set $\mathcal{W}$, there is a function $g_w(x_1,x_2) \geq 0$ satisfying

$$g_{0,w}(x_1,x_2) \geq \max_i \Big\{\Big\|\Big(\frac{\partial \varphi_i}{\partial v}\lambda(v)\Big)\Big|_{v=\pi(-t,w)}\Big\|, t \in \mathbb{R}\Big\}. \tag{3.91}$$

A direct calculation according to (3.90) yields

$$\begin{aligned} D_1V_{1,w} &\leq -2(6\xi_1^2 + 4\xi_2^2)(3\xi_1^2 + \xi_2^2 - w_1)^2 + 2g_{0,w}(\xi_1,\xi_2)|3\xi_1^2 + \xi_2^2 - w_1|, \\ D_2V_{2,w} &\leq -2(2\zeta_1^2 + 8\zeta_2^2)(\zeta_1^2 + 4\zeta^2 - w_2)^2 + 2g_{0,w}(\zeta_1,\zeta_2)|\zeta_1^2 + 4\zeta_2^2 - w_2|. \end{aligned} \tag{3.92}$$

In view of (3.92), let us consider the function $g_w(x_1,x_2)$ satisfying

$$g_w(x_1,x_2) \leq \min_i\{g_{i,w}(x_1,x_2)\}, \forall (x_1,x_2) \in \mathbb{R}^2, \tag{3.93}$$

where

$$
\begin{aligned}
g_{1,w}(x_1,x_2) &= 3x_1^2 + x_2^2 - |w_1| - \sqrt{g_{0,w}(x_1,x_2)} \\
g_{2,w}(x_1,x_2) &= x_1^2 + 4x_2^2 - |w_2| - \sqrt{g_{0,w}(x_1,x_2)}.
\end{aligned} \tag{3.94}
$$

Obviously, if $g_w(x_1,x_2) \geq 0$, then $6x_1^2 + 4x_2^2 - \sqrt{g_{0,w}(x_1,x_2)} \geq 0$ and $2x_1^2 + 8x_2^2 - \sqrt{g_{0,w}(x_1,x_2)} \geq 0$, which lead to

$$
\begin{aligned}
D_1V_{1,w} &\leq -2\big(6\xi_1^2 + 4\xi_2^2 - 2\sqrt{g_{0,w}(x_1,x_2)}\big)(3\xi_1^2 + \xi_2^2 - w_1)^2 \leq 0, \\
D_2V_{2,w} &\leq -2(2\zeta_1^2 + 24\zeta_2^4 - 2\sqrt{g_{0,w}(x_1,x_2)})(\zeta_1^2 + 2\zeta_2^4 - w_2)^2 \leq 0.
\end{aligned} \tag{3.95}
$$

As such, for $g_w(x_1,x_2) \geq 0$, $V_{1,w}$ and $V_{2,w}$ are decreasing on running time of their respective constituent systems. Therefore, the switched system $\Sigma_{\mathcal{NA}}$ satisfies condition i) of Theorem 3.5.1.

On the other hand, using (3.90) and Young's inequality, we have

$$
\begin{aligned}
D_2V_{1,w} \leq{}& 2(3\zeta_1^2 + \zeta_2^2 - w_1)\Big(22\zeta_1\zeta_2 - (6\zeta_1^2 + 4\zeta_2^2)(\zeta_1^2 + 4\zeta_2 - w_2)\Big) \\
&+ 2g_{0,w}(\zeta_1,\zeta_2)|3\zeta_1^2 + \zeta_2^2 - w_1| \\
\leq{}& -2(3\zeta_1^2 + \zeta_2^2 - w_1)(6\zeta_1^2 + 4\zeta_2^2)(\zeta_1^2 + 4\zeta_2^2 - 3 - w_2) \\
&+ 2g_{0,w}(\zeta_1,\zeta_2)|3\zeta_1^2 + \zeta_2^2 - w_1| \\
\leq{}& -2(3\zeta_1^2 + \zeta_2^2 - w_1)(6\zeta_1^2 + 4\zeta_2^2)(\zeta_1^2 + 4\zeta_2^2 - w_2) + 6(3\zeta_1^2 + \zeta_2^2 - w_1) \\
&\times (6\zeta_1^2 + 4\zeta_2^2 - w_1) + 6w_1(3\zeta_1^2 + \zeta_2^2 - w_1) + 2g_{0,w}(\zeta_1,\zeta_2)|3\zeta_1^2 + \zeta_2^2 - w_1|.
\end{aligned} \tag{3.96}
$$

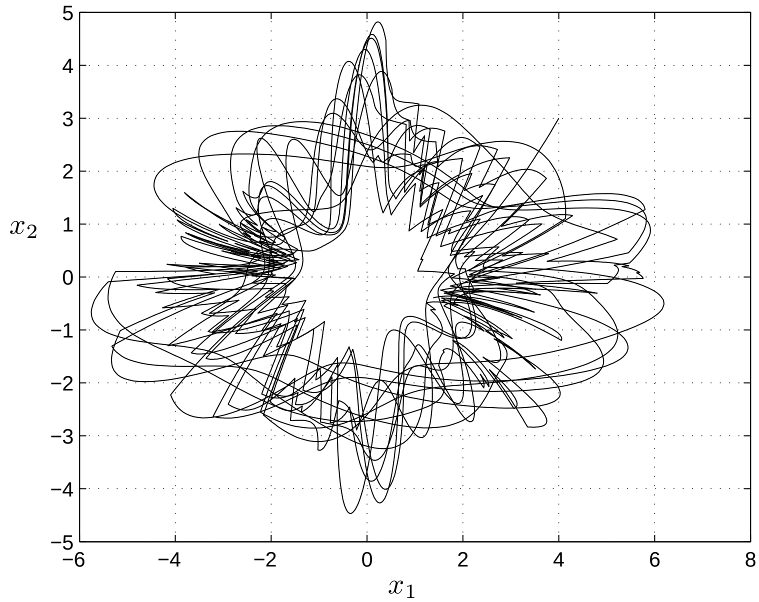

Figure 3.3: Forward attractor.

Since $\mathcal{W}$ is compact and $\sqrt{g_{0,w}(\zeta_1, \zeta_2)} \leq (3\zeta_1^2 + \zeta_2^2 - w_1)$ when $g_w(\zeta_1, \zeta_2) \geq 0$, (3.96) further leads to

$$\begin{aligned} D_2 V_{1,w} &\leq 6(3\zeta_1^2 + \zeta_2^2 - w_1)(6\zeta_1^2 + 4\zeta_2^2 - w_1) + 6w_1(3\zeta_1^2 + \zeta_2^2 - w_1) \\ &\leq c_{11} V_{1,w} + c_{12}\sqrt{V_{2,w}}, \end{aligned} \tag{3.97}$$

where c_{11} and c_{12} are positive constants. Similarly, there are constants $c_{21} > 0$ and $c_{22} > 0$ such that

$$D_1 V_{2,w} \leq c_{21} V_{2,w} + c_{22}\sqrt{V_{1,w}}, \tag{3.98}$$

provided that $g_w(\xi_1, \xi_2) \geq 0$. Directly solving (3.97) and (3.98) on persistent dwell-time intervals and, in view of (3.97), (3.98), and (3.95), making the persistent dwell-time τ_{p} sufficiently large with respect to the period of persistence T_{p}, it is obvious that condition ii) of Theorem 3.5.1 is satisfied.

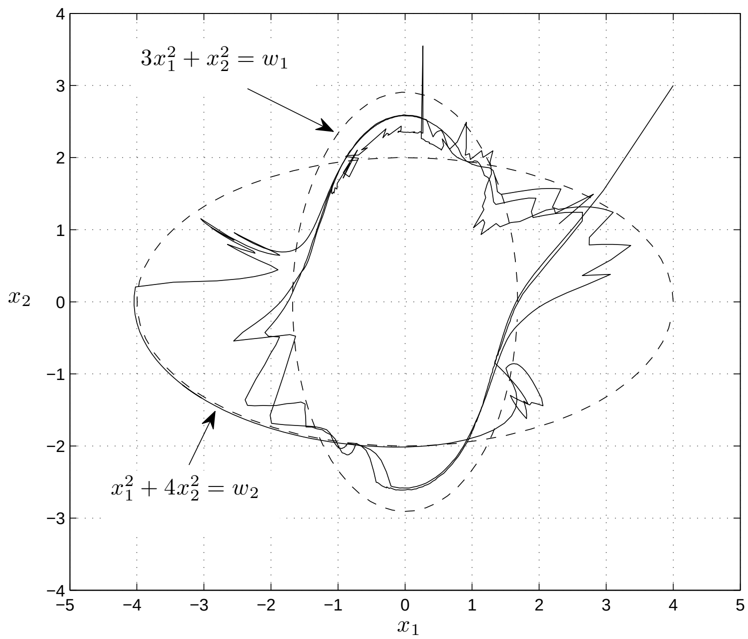

Figure 3.4: Pullback attractor of $\Sigma_{\mathcal{NA}}$ at $w = [w_1, w_2]^T = [8.5, 16]^T$.

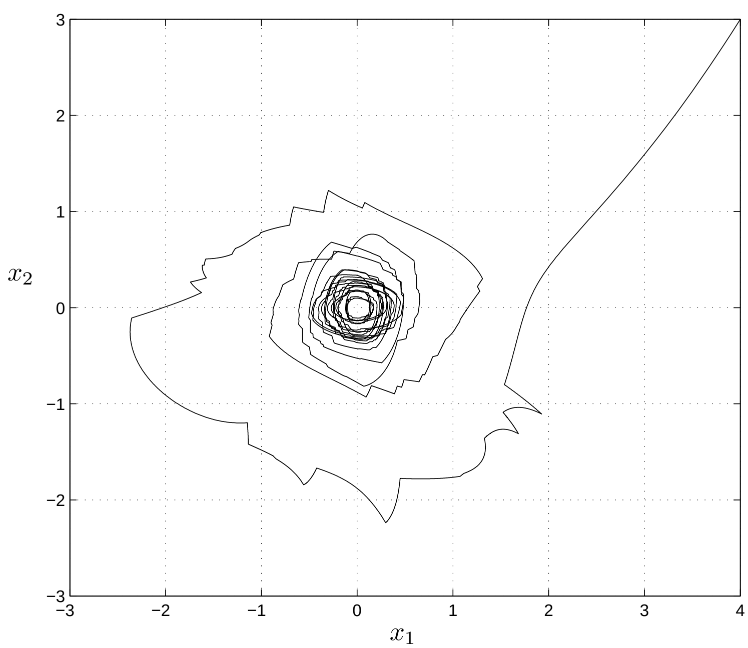

Figure 3.5: Convergence via non-autonomous attractors

In simulation we choose $T_\mathrm{p} = 0.4s$ and $\tau_\mathrm{p} = 1s$. The time-varying parameters w_1 and w_2 are taken as the output of the chaotic system

$$\begin{aligned}
\dot{\eta}_1 &= \sigma(\eta_2 - \eta_1) \\
\dot{\eta}_2 &= \eta_1(\rho - \eta_3) - \eta_2 \\
\dot{\eta}_3 &= \eta_1\eta_2 - \beta\eta_3, \\
w &= \left[\sqrt{\eta_2^2 + 1}, \sqrt{\eta_3^2 + 1}\right]^T,
\end{aligned} \tag{3.99}$$

where $\sigma = 10, \beta = 8/3$, and $\rho = 28$ are constants.

As shown in Figure 3.3, the switched system $\Sigma_{\mathcal{NA}}$ exhibits a chaotic behavior and, as the forward attractor is determined by the specific dynamics of w, no conclusion on the influence of w on the behavior of system state x can be made using forward attractor. However, as shown in Figure 3.4, the above limitation of forward attractor is removed by using pullback attractors. By pushing the initial value $w_0 = \pi(-t, w)$ backward from $w = [w_1, w_2]^T$, the limit set of the pullback trajectory is well-estimate by the level sets shaped by limit cycles $3x_1^2 + x_2^2 = w_1$ and $x_1^2 + 4x_2^2 = w_2$ of the constituent systems.

In addition, since the non-autonomous ω-limit set at a $w \in \mathcal{W}$ can be interpreted as the container of the system state at the current time if the system has run sufficiently long. It is well concluded that for under any parameter w that converge to zero as $t \to \infty$, the system state $x(t)$ shall converge to the region $\{\zeta \in \mathbb{R}^n : g_w(\zeta)|_{w=0} < 0\}$ as $t \to \infty$. This fact is shown in Figure 3.5, where w was generated by the system

$$\dot{\xi} = \begin{bmatrix} 0 & 0.2 \\ -0.3 & -0.7 \end{bmatrix} \xi, \qquad w = [\xi_1^2, \xi_2^2]^T. \tag{3.100}$$

Example 3.6.2 (Autonomous Case)

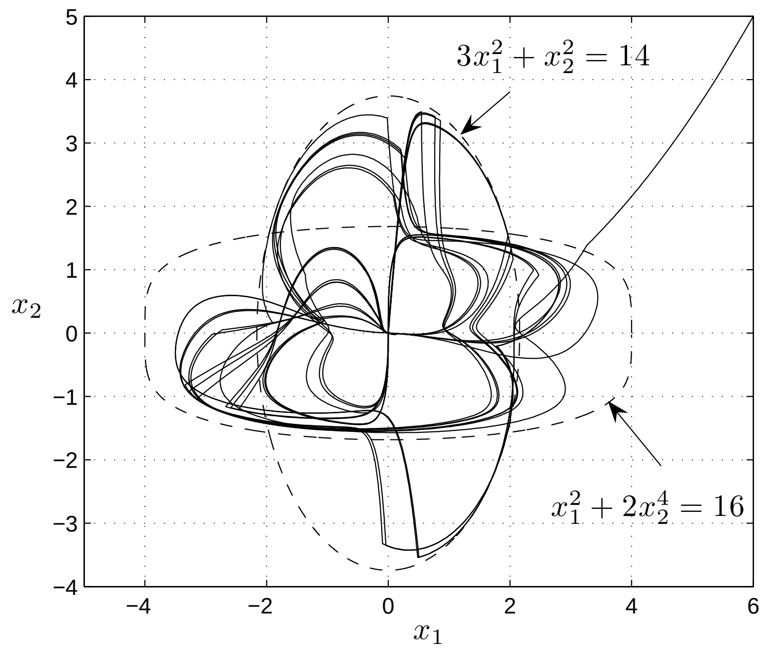

Figure 3.6: Limiting behavior of switched autonomous system $\Sigma_{\mathcal{A}}$

In this example, we consider the switched autonomous system $\Sigma_{\mathcal{A}}$ whose constituent systems are described by the following differential equations.

$$\begin{aligned}
\pi_1 : \quad & \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1(3x_1^2 + x_2^2 - 14) \\ -3x_1 - 2x_2(3x_1^2 + x_2^2 - 14) \end{bmatrix} \\
\pi_2 : \quad & \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 4x_2^3 - x_1(x_1^2 + 2x_2^4 - 16) \\ -x_1 - 3x_2(x_1^2 + 2x_2^4 - 16) \end{bmatrix}
\end{aligned} \tag{3.101}$$

Using the Lyapunov functions $V_1 = (3x_1^2 + x_2^2 - 14)^2$ and $V_2 = (x_1^2 + 2x_2^4 - 16)^2$, it is verified that the sets $\mathcal{O}_1 = \{(x_1, x_2) \in \mathbb{R}^2 : 3x_1^2 + x_2^2 - 14 = 0\}$ and $\mathcal{O}_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + 2x_2^4 - 16 = 0\}$ are attractive invariant sets of π_1 and π_2, respectively. They are limit cycles of the corresponding constituent systems. A direct computation shows that $\dot{V}_1 \leq 0$ and $\dot{V}_2 \leq 0$ along the trajectories of their respective systems and hence condition ii) of Theorem 3.5.2 is satisfied. In simulation a dwell-time switching sequence was used so that condition i) and ii) of Theorem 3.5.2 can be satisfied as well. The function g was selected as $g(x_1, x_2) = \min\{V_1(x), V_2(x)\}$ and the running times of π_1 and π_2 are generated randomly in intervals $[0.8, 0.9]$ and $[1.2, 1.4]$, respectively. As shown in Figure 3.6, the attractor of the system is well estimated by the set $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \{(x_1, x_2) : g(x_1, x_2) \leq 0\}$.

$\bullet$

Chapter 4

Invariance: Time-delay

This chapter aims at a qualitative theory of switched time-delay systems. The delay-dependent systems are modeled as switched autonomous systems without switching jump on the Banach space of continuous functions. The qualitative notions are defined in the Banach space and the converging conditions are stated in the Euclidean space. Treating period of persistence and delay-time on an equal footing, decreasing condition is imposed in the Banach space without conservativeness. Invariance principles are presented for both delay-dependent and delay-free switched systems. The general class of persistent dwell-time switching sequences is again of primary interest.

4.1 Motivation

Infinite dimensional dynamical systems arise in applications where any finite collection of parameters is not sufficient to describe the system dynamics. Practical examples of such systems range over biological, physical, and engineering systems [36, 59, 145, 81, 113,68,54,31]. Systems in which infinite dimensions are called upon the past state are usually termed time-delay systems or retarded systems. Bearing in mind the hybrid nature of contemporary dynamical systems, one might develop a qualitative theory for switched systems with time-delay as a natural advance.

The stability theory of time-delay systems has a long history [59, 89, 113, 54]. Plentiful achievements in this area include Lyapunov-Krasovskii functional method [83], Lyapunov-Razumikhin function method [59], and invariance principles [57,55,67].

In the qualitative theory of time-delay systems, the long-term behavior is usually studied by the functional approach in the Banach space of continuous functions for a richer theory, and is studied by the function approach in the Euclidean spaces for applicability [58, 59]. While the functional approach addresses the issues on compactness of limit sets, the Lyapunov-Razumikhin function approach aims at a relaxed decreasing condition on Lyapunov function for less conservative results.

We would mention that switched systems with time-delay have been studied recently [108,158,100,153]. While [153] restricts to switched linear systems with dwell-time, [158] and [100] adopt the approach of [154] with the inherent conservativeness in the context of switched systems. In particular, the satisfaction of the difference inequality at discrete times in [158,100] requires the knowledge of the discrete dynamics and hence these results become restrictive in the context of switched systems.

Though the general framework of qualitative theory for hybrid systems in [108] applies to switched time-delay systems, the semi-group condition on trajectory becomes restrictive in the context of switched systems. A qualitative theory of switched time-delay systems exploiting basic observations of the original LaSalle's invariance principle and addressing the loss of the semi-group property of trajectories in the continuous space remains open.

Motivated by the above consideration, we develop in this chapter a qualitative theory for switched time-delay systems. The qualitative notions are defined in the Banach space and the converging conditions are stated in the Euclidean space. It turns out that, in switched time-delay systems, relation between time-delay and period of persistence can be exploited for further converging conditions.

4.2 Preliminaries

We shall adopt some standard notations from [59]. Let $[a,b]$ be an interval in $\mathbb{R}$. Then, $\mathscr{C}([a,b],\mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence, i.e., a sequence of function $\{f_n\}_n$ in $\mathscr{C}([a,b],\mathbb{R}^n)$ is said to converge to a function $f \in \mathscr{C}([a,b],\mathbb{R}^n)$ if

$$\lim_{n\to\infty} \sup_{x\in[a,b]} \|f_n(x) - f(x)\| = 0. \tag{4.1}$$

The norm for elements in $\mathscr{C}([a,b],\mathbb{R}^n)$ is designated as

$$\|f\| = \sup\{\|f(x)\| : x \in [a,b]\}, \forall f \in \mathscr{C}([a,b],\mathbb{R}^n). \tag{4.2}$$

Let $T_r \in \mathbb{R}^+$ be a retarded parameter. The space $\mathscr{C}([-T_r,0],\mathbb{R}^n)$ shall be denoted by $\mathscr{C}_r$. Suppose that $\psi \in \mathscr{C}([t_0-T_r,t_0+T],\mathbb{R}^n)$ for some $t_0 \in \mathbb{R}$ and $T \in \mathbb{R}^+$. Then, for a $t \in [t_0,t_0+T]$, ψ_t is the function in $\mathscr{C}_r$ defined by

$$\psi_t(\varsigma) = \psi(t+\varsigma), \forall \varsigma \in [-T_r,0]. \tag{4.3}$$

The distance from a point $\phi \in \mathscr{C}_r$ to a set $\mathcal{A} \subset \mathscr{C}_r$ is $\mathrm{dist}(\phi,\mathcal{A}) \overset{\mathrm{def}}{=} \|\phi\|_{\mathcal{A}} = \inf\{\|\phi-\psi\| : \psi \in \mathcal{A}\}$.

For the ease of reference, let us adopt the following notions and result in topology from [108] for the space $\mathscr{C}([a,b],\mathbb{R}^n)$.

Definition 4.2.1 ([108]) *A subset $\mathcal{F}$ of $\mathscr{C}([a,b],\mathbb{R}^n)$ is said to be equicontinuous if for every $\epsilon > 0$, there is a number $\delta > 0$ such that*

$$x,y \in [a,b], |x-y| < \delta \Rightarrow \|f(x)-f(y)\| < \epsilon, \forall f \in \mathcal{F}. \tag{4.4}$$

In addition, $\mathcal{F}$ is said to be uniformly bounded if there is a constant $H > 0$ such that

$\|f\| \leq H, \forall f \in \mathcal{F}$.

Theorem 4.2.1 (Arzela-Ascoli – [41]) *Let $\mathcal{F}$ be a subset of $\mathscr{C}([a,b],\mathbb{R}^n)$. Suppose that $\mathcal{F}$ is equicontinuous and uniformly bounded. Then, $\mathcal{F}$ is precompact, i.e., the closure* $\mathrm{Cl}(\mathcal{F})$ *is compact in $\mathscr{C}([a,b],\mathbb{R}^n)$.*

We shall say that a subset $\mathcal{A}$ of some topological space $\mathcal{X}$ is compact covered if there is a compact subset $\mathcal{B}$ of $\mathcal{X}$ that contains $\mathcal{A}$.

For a function V mapping $\mathbb{R}^n$ into $\mathbb{R}$, we shall attach the superscript $\natural$ to V to indicate the function $V^\natural$ mapping $\mathscr{C}_r$ into $\mathbb{R}$ defined as

$$V^\natural(\phi) = \sup_{\varsigma \in [-T_r, 0]} V(\phi(\sigma)), \phi \in \mathscr{C}_r. \tag{4.5}$$

Finally, throughout the chapter, the discrete set $\mathbb{Q}$ is supposed to be finite.

4.3 Switched Time-delay Systems

We shall restrict ourselves to the autonomous case for simplicity of exposition as, by virtue of the general results in Chapter 3, the more general result is obtainable. Thus, no issue on the autonomy should arise, and we shall call switched time-delay systems without embarrassment.

4.3.1 The Model

Let $T_r \in \mathbb{R}^+$ be a fixed number. Adopting Definition 2.4.2, we have the following notion of switched time-delay systems as switched autonomous systems without switching jump whose manifest space is $\mathbb{X} = \mathscr{C}_r$.

Definition 4.3.1 (switched time-delay system) *A switched time-delay system is a hexad*

$$\Sigma_{\mathcal{D}} = \big(\mathbb{R}^+, \mathbb{Q}, \mathscr{C}_r, \{\psi_q\}_{q \in \mathbb{Q}}, \mathbb{S}\big), \tag{4.6}$$

where $\mathbb{Q}$ is a discrete set which is the space of the discrete signals, $\mathscr{C}_r$ is the space of continuous functions mapping $[-T_r, 0]$ into $\mathbb{R}^n$ with topology of uniform convergence, $\psi_q : \mathbb{R}^+ \times \mathscr{C}_r \to \mathscr{C}_r, q \in \mathbb{Q}$ are ordinary autonomous dynamical systems on $\mathscr{C}_r$, and $\mathbb{S}$ is a collection of switching sequences.

The transition mapping $\psi_q, q \in \mathbb{Q}$ in the above model of switched time-delay systems can arise in systems described by retarded functional differential equations (RFDEs) of the form

$$\mathfrak{R}_q : \dot{x}(t) = f_q(x_t), q \in \mathbb{Q}, \tag{4.7}$$

where $f_q : \mathscr{C}_r \to \mathbb{R}^n, q \in \mathbb{Q}$ are continuous functions. In fact, consider an index $q \in \mathbb{Q}$. It is well-known from [59] that if f_q takes bounded sets of $\mathscr{C}_r$ into bounded sets of $\mathbb{R}^n$, then for any initial condition $\phi \in \mathscr{C}_r$, the solution $x(\phi) \in \mathscr{C}([-r, \infty), \mathbb{R}^n)$ of (4.7) is well-defined, unique, and continuously dependent on initial condition, and the transition mapping $\psi_q : \mathbb{R}^+ \times \mathscr{C}_r \to \mathscr{C}_r, \phi \mapsto x_t(\phi)$ is well-defined and satisfy the semi-group property

$$\psi_q(t + s, \phi) = \psi_q(t, \psi_q(s, \phi)), \forall t, s \in \mathbb{R}^+, \phi \in \mathscr{C}_r. \tag{4.8}$$

Suppose that the transition mappings $\psi_q, q \in \mathbb{Q}$ of the switched time-delay system $\Sigma_{\mathscr{D}}$ are generated by equations (4.7). The evolution of $\Sigma_{\mathscr{D}}$ can be described as follows. Given a switching sequence $\sigma \in \mathbb{S}$ whose sequence of switching events is $\{(q_{\sigma,i}, \Delta\tau_{\sigma,i})\}_i$. From the initial condition $\phi_0 \in \mathscr{C}_r$ at some initial time t_0, the systems evolves under the law $\mathfrak{R}_{q_{\sigma,0}}$ given by (4.7) with $q = q_{\sigma,0}$ until the time $t_1 = t_0 + \Delta\tau_{\sigma,0}$ is reached. Since $\mathfrak{R}_{q_{\sigma,0}}$ is autonomous, according to [59], the trajectory $x_{q_{\sigma,0}}(\phi_0) : [-T_r, \Delta\tau_{\sigma,0}] \to \mathbb{R}^n$ generated by $\mathfrak{R}_{q_{\sigma,0}}$ on $[t_0, t_1]$ is well-defined, unique, continuous, and independent of t_0. Thus, the transition mapping $\psi_{q_{\sigma,0}}(t) \stackrel{\text{def}}{=} (x_{q_{\sigma,0}})_t(\phi_0), t \in [0, \Delta\tau_{\sigma,0}]$ is well-defined and continuous on $[0, \Delta\tau_{\sigma,0}]$.

Let us recall that $\tau_{\sigma,i} = \sum_{j=0}^{i-1} \Delta\tau_{\sigma,j}$. At the time t_1, the system $\Sigma_{\mathscr{D}}$ changes the rule of transition from $\mathfrak{R}_{q_{\sigma,0}}$ to $\mathfrak{R}_{q_{\sigma,1}}$. Since $x_{q_{\sigma,0}}(\phi_0)$ is continuous on $[-T_r, \tau_{\sigma,1}]$, the initial condition $\phi_1 = (x_{q_{\sigma,0}})_{\Delta\tau_{\sigma,0}}(\phi_0)$ for the new transition rule $\mathfrak{R}_{q_{\sigma,1}}$ is well-defined and belongs to $\mathscr{C}_r$. As such, the trajectory $x_{q_{\sigma,1}}(\phi_1) : [\tau_{\sigma,1} - T_r, \tau_{\sigma,1} + \Delta\tau_{\sigma,1}] \to \mathbb{R}^n$ generated by $\mathfrak{R}_{q_{\sigma,1}}$ on $[t_1, t_2], t_2 = t_1 + \Delta\tau_{\sigma,1}$ is well-defined, continuous, and is uniquely determined the by the initial condition ϕ_0 through ϕ_1. From t_2, the process continues, and we obtain the continuous mapping $x(\cdot, x_0) : [-T_r, \infty) \to \mathbb{R}^b$ defined by

$$x(t;\phi_0) = x_{q_{\sigma,i}}(\phi_i)(t - \tau_{\sigma,i}), t \in [t_i - T_r, t_{i+1}], i \in \mathbb{N}. \tag{4.9}$$

Since $x(t, \phi_0)$ is continuous, by [59, Chapter 2, Lemma 2.1], $x_t(t, \phi_0) = \phi_{q_{\sigma,i_\sigma^-(t)}}(t - \tau_{\sigma,i_\sigma^-(t)})$ is continuous on $\mathscr{C}_r$.

In summary, the model (4.6) well describes the switched time-delay systems whose laws of motions are given by (4.7). Hereafter, we shall deal with switched time-delay systems using the model (4.6).

4.3.2 Transition Mappings

Suppose that the switching sequence is non-blocking. From the general setting of switched autonomous systems in Definition 2.4.2, for each switching sequence σ and initial time $t_s \in \mathbb{R}^+$, the following transition mapping $\mathscr{T}_{t_s,\sigma} : \mathbb{R}^+ \times \mathscr{C}_r \to \mathscr{C}_r$ of switched time-delay system (4.6) is well-defined from (2.7) and is continuous.

$$\mathscr{T}_{t_s,\sigma}(t,\phi) = \begin{cases} \psi_{q_{i_\sigma^-(t_s)}}(t,x) \ \textbf{if } t \in [0, \tau_{\sigma,i_\sigma^-(t_s)+1} - t_s] \\ \psi_{q_{i_\sigma^-(t_s+t)}}\big(t_s + t - \tau_{\sigma,i_\sigma^-(t_s+t)}, \mathscr{T}_\sigma(\tau_{\sigma,i_\sigma^-(t_s+t)} - t_s, t_s, \phi)\big) \\ \qquad\qquad \textbf{if } t \geq \tau_{\sigma,i_\sigma^-(t_s)+1} - t_s \end{cases}. \tag{4.10}$$

We shall alternatively call $\mathscr{T}_{t_s,\sigma}(t,\phi)$ the trajectory in the space $\mathscr{C}_r$ of $\Sigma_{\mathscr{D}}$. Clearly, under the transition (4.10), the following trajectory is continuous and uniquely defined in the state space $\mathbb{R}^n$ of the system:

$$\mathscr{T}^{\phi}_{t_s,\sigma} : \mathbb{R}^+ \to \mathbb{R}^n, t \mapsto \mathscr{T}_{t_s,\sigma}(t, \phi)(0). \tag{4.11}$$

We shall use $\mathscr{T}_{t_s,\sigma}$ for studying qualitative notions and use $\mathscr{T}^{\phi}_{t_s,\sigma}$ for studying converging conditions of $\Sigma_{\mathscr{D}}$.

4.3.3 Derivatives along Trajectories

Let $a \in \mathbb{R}$ be a fixed number. Consider a continuous function $x : [a - T_r, \infty) \to \mathbb{R}^n$, which we call a trajectory in $\mathbb{R}^n$, and continuous functions $V : \mathbb{R}^n \to \mathbb{R}$ and $V^\natural : \mathscr{C}_r \to \mathbb{R}$. For each $t \in [a, \infty)$, we have the function $x_t : [-T_r, 0] \to \mathbb{R}^n$, which we call a trajectory in $\mathscr{C}_r$, defined as $x_t(\varsigma) = x(t + \varsigma), \varsigma \in [-T_r, 0]$. For a time $t \in [a, \infty)$, we have the following Dini derivatives [59, 156]:

$$D^+V(x(t)) = \limsup_{h\to 0^+} \frac{V(x(t+h)) - V(x(t))}{h}; \tag{4.12}$$

$$D^-V(x(t)) = \limsup_{h\to 0^-} \frac{V(x(t+h)) - V(x(t))}{h}; \tag{4.13}$$

$$D^+V^\natural(x_t) = \limsup_{h\to 0^+} \frac{V^\natural(x_{t+h}) - V^\natural(x_t)}{h}; \tag{4.14}$$

$$D^-V^\natural(x_t) = \limsup_{h\to 0^-} \frac{V^\natural(x_{t+h}) - V^\natural(x_t)}{h}. \tag{4.15}$$

We shall call $D^+V(x(t))$ and $D^-V(x(t))$ respectively the upper-right and upper-left Dini derivatives at t of V along the trajectory $x(t)$, and $D^+V^\natural(x_t)$ and $D^-V^\natural(x_t)$ respectively the upper-right and upper-left Dini derivatives at t of $V^\natural$ along the trajectory x_t. For a switching sequence $\sigma \in \mathbb{S}$, we are interested in the following notion of derivative at t of V along the trajectory $x(t)$:

$$D_\sigma V(x(t)) = \begin{cases} D^+V(x(t)) \textbf{ if } t \in \{\tau_{\sigma,i}\}_i \\ \max\{D^+V(x(t)), D^-V(x(t))\} \textbf{ if } t \in \mathbb{R}^+ \backslash \{\tau_{\sigma,i}\}_i. \end{cases} \tag{4.16}$$

4.3.4 Qualitative Notions

Adopting the qualitative notions of the general dynamical systems in Section 2.2.3, we shall bring out the qualitative notions for switched time-delay system $\Sigma_{\mathscr{D}}$ using the transition mapping $\mathscr{T}_{t_s,\sigma}$ defined by (4.10). We first clarify that, in the general framework of the transition model, the manifest space in $\Sigma_{\mathscr{D}}$ is $\mathbb{W}_{\mathscr{M}} = \mathscr{C}_r$ and the latent space is $\mathbb{W}_{\mathscr{L}} = \mathbb{S}$.

Definition 4.3.2 (trajectory) *Let $\phi \in \mathscr{C}_r$, $t_s \in \mathbb{R}^+$ and $\sigma \in \mathbb{S}$ fixed. The (t_s, σ)–interacting trajectory through the point ϕ of $\Sigma_{\mathscr{D}}$ is the set $\mathscr{O}_{t_s,\sigma}(\phi) = \{\mathscr{T}_{t_s,\sigma}(t, \phi) : t \in \mathbb{R}^+\}$.*

Definition 4.3.3 (motion) *Let $\phi \in \mathscr{C}_r$. The (t, t_s, σ)–motion through ϕ of $\Sigma_{\mathscr{D}}$ is $\mathfrak{R}_{t_s,\sigma}(\phi)(t) = \mathscr{T}_{t_s,\sigma}(t, \phi)$.*

Definition 4.3.4 (attractor) *Let $\mathcal{A}$ and $\mathcal{D}$ be closed sets in $\mathscr{C}_r$. The set $\mathcal{A}$ is said to be the (t_s, σ)–forward attractor of $\Sigma_{\mathscr{D}}$ with basin of attraction $\mathcal{D}$ if*

$$\lim_{t\to\infty} \|\mathscr{T}_{t_s,\sigma}(t, \phi)\|_{\mathcal{A}} = 0, \forall \phi \in \mathcal{D}. \tag{4.17}$$

In addition, if this property holds for all $\sigma \in \mathbb{S}$, then $\mathcal{A}$ is said to be the switching-uniform forward attractor with basin of attraction $\mathcal{D}$ of $\Sigma_{\mathscr{D}}$.

Definition 4.3.5 (ω-limit set) *Let $\phi \in \mathscr{C}_r$ and $t_s \in \mathbb{R}^+$ fixed. The ω–limit set of the (t_s, σ)–interacting trajectory $\mathscr{O}_{t_s,\sigma}(\phi)$ through ϕ of the system $\Sigma_{\mathscr{D}}$ is the set*

$$\omega_{t_s,\sigma}(\phi) = \bigcap_{T\geq t_s} \overline{\bigcup_{t\geq T} \mathscr{T}_{t_s,\sigma}(t, \phi)}. \tag{4.18}$$

4.4 Compactness and Quasi-invariance

Compactness of the limit sets of trajectories is an important issue in the general qualitative theory of dynamical systems [57, 58]. Different from dynamical systems on finite dimensional spaces, boundedness of trajectories generally does not ensure compactness for the limit sets in Banach spaces. In addition, similar to delay-free systems, switching events causes the loss of the semi-group property and hence, the limit sets of switched time-delay systems are not invariant as in the classical time-delay systems [57, 59].

In this section, we shall adopt the compact covering condition in [58] for the compactness of limit sets of trajectories of switched time-delay systems. Then, limiting switching sequences introduced in the previous Chapter 3 are further applied to study invariance property of these limit sets of trajectories.

4.4.1 Compactness

Theorem 4.4.1 *Let $\Sigma_{\mathscr{D}}$ be a switched time-delay system. Suppose that for fixed $\phi \in \mathscr{C}_r, \sigma \in \mathbb{S}$, and $t_s \in \mathbb{R}^+$ the interacting trajectory $\mathscr{O}_{t_s,\sigma}(\phi)$ is covered by a compact subset of $\mathscr{C}_r$. Then, the limit set $\omega_{t_s,\sigma}(\phi)$ is non-empty and compact. In addition, $\mathscr{T}_{t_s,\sigma}(t, \phi)$ approaches $\omega_{t_s,\sigma}(\phi)$ as $t \to \infty$.*

Proof: Since $\mathscr{O}_{t_s,\sigma}(\phi)$ is covered by a compact subset of $\mathscr{C}_r$, for any time sequence $\{t_n\}_n$, the sequence $\{\mathscr{T}_{t_s,\sigma}(t_n, \phi)\}_n$ has a subsequence $\{\mathscr{T}_{t_s,\sigma}(t_{n_m}, \phi)\}_m$ that converges. Therefore $\omega_{t_s,\sigma}(\phi)$ is nonempty.

We proceed to prove the compactness of $\omega_{t_s,\sigma}(\phi)$ by showing that $\omega_{t_s,\sigma}(\phi)$ is closed. Suppose that $\{\phi_n\}_n$ is a sequence of functions in $\omega_{t_s,\sigma}(\phi)$ that converges to $\phi^* \in \mathscr{C}_r$ as $n \to \infty$. Let $\tau > 0$ be any finite number and let $\{\varepsilon_n\}_n$ be any sequence converging to zero as $n \to \infty$. As $\phi_n \to \phi^*, n \to \infty$, for each $n \in \mathbb{N}$, there is an integer $k_n \in \mathbb{N}$ such that $\|\phi_{k_n} - \phi^*\| < \varepsilon_n/2$. For each $k_n, n \in \mathbb{N}$, as $\phi_{k_n} \in \omega_{t_s,\sigma}(\phi)$, there is a time sequence $\{t_m^{(k_n)}\}_m$ such that

$$\lim_{m\to\infty} \|\mathscr{T}_{t_s,\sigma}(t_m^{(k_n)}, \phi) - \phi_{k_n}\| = 0. \tag{4.19}$$

From the sequence $\{t_m^{(k_n)}\}_m, n \in \mathbb{N}$, let us define the time sequence $\{t_{k_n}\}_n$ as follows. Applying (4.19) for $n = 0$, there is a time $t_{k_0} \in \{t_m^{(k_0)}\}_m$ satisfying $\|\mathscr{T}_{t_s,\sigma}(t_{k_0}, \phi) - \phi_{k_0}\| < \varepsilon_0/2$. From t_{k_0}, applying (4.19) for each $n \in \mathbb{N}\backslash\{0\}$, we obtain the times $t_{k_n} \in \{t_m^{(k_n)}\}_m$ that satisfies $\|\mathscr{T}_{t_s,\sigma}(t_{k_n}, \phi) - \phi_{k_n}\| < \varepsilon_n/2$ and $t_{k_n} > t_{k_{n-1}} + \tau$.

Obviously, $\{t_{k_n}\}_n$ is a time sequence as its elements are separated by τ. Thus, by construction, we have

$$\|\mathscr{T}_{t_s,\sigma}(t_{k_n}, \phi) - \phi^*\| \leq \|\mathscr{T}_{t_s,\sigma}(t_{k_n}, \phi) - \phi_{k_n}\| + \|\phi_{k_n} - \phi^*\| < \varepsilon_n, \forall n \in \mathbb{N}. \tag{4.20}$$

As $\varepsilon_n \to 0, n \to \infty$, (4.20) implies that $\mathscr{T}_{t_s,\sigma}(t_{k_n}, \phi) \to \phi^*, n \to \infty$. Hence, $\phi^* \in \omega_{t_s,\sigma}(\phi)$ and $\omega_{t_s,\sigma}(\phi)$ is closed accordingly.

Since $\mathscr{O}_{t_s,\sigma}(\phi)$ is contained in a compact subset of $\mathscr{C}_r$, the set of limit points of $\mathscr{O}_{t_s,\sigma}(\phi)$ including those in $\omega_{t_s,\sigma}(\phi)$ is contained in this compact subset. Thus, $\omega_{t_s,\sigma}(\phi)$ is contained in a compact set of $\mathscr{C}_r$. This coupled with the closedness of $\omega_{t_s,\sigma}(\phi)$ implies that $\omega_{t_s,\sigma}(\phi)$ is compact.

We prove the last assertion of the theorem by a contradiction argument similar to [57]. Suppose that the converse holds, i.e., there is a time sequence $\{t_n\}_n$ and a number $\epsilon > 0$ such that $\|\mathscr{T}_{t_s,\sigma}(t_n, \phi) - \phi^*\| > \epsilon, \forall \phi^* \in \omega_{t_s,\sigma}(\phi)$. Since $\{\mathscr{T}_{t_s,\sigma}(t_n, \phi)\}_n \subset \mathscr{O}_{t_s,\sigma}(\phi)$ belonging to a compact subset of $\mathscr{C}_r$, there is a subsequence $\{\mathscr{T}_{t_s,\sigma}(t_{n_m}, \phi)\}_m$ that converges to some $\phi^* \in \mathscr{C}_r$ which is obviously an element of $\omega_{t_s,\sigma}(\phi)$. This is a contradiction, and hence the assertion holds. ■

In Theorem 4.4.1, the technical condition on compact covering of the trajectory $\mathscr{O}_{t_s,\sigma}(\phi)$ is imposed. This condition is adopted from the qualitative theory of dynamical systems on Banach spaces [58]. The following theorem asserts that this condition can be satisfied if the transition mapping in the state space $\mathbb{R}^n$ of the system is bounded and uniformly continuous. We refer to [41] for topological concepts.

Theorem 4.4.2 *Let $\Sigma_{\mathscr{D}}$ be a switched time-delay system whose component transition* *mappings $\psi_q, q \in \mathbb{Q}$ are generated by RFDEs (4.7) with $f_q, q \in \mathbb{Q}$ mapping bounded sets in $\mathscr{C}_r$ into bounded set in $\mathbb{R}^n$. Suppose that, for fixed $\phi \in \mathscr{C}_r, \sigma \in \mathbb{S}$, and $t_s \in \mathbb{R}^+$, where ϕ being uniform continuous on $[-T_r, 0]$, the transition mapping $\mathscr{T}^{\phi}_{t_s,\sigma}$ defined in (4.11) is bounded. Then, the trajectory $\mathscr{O}_{t_s,\sigma}(\phi)$ is compact covered.*

Proof: Since $\{\mathscr{T}^{\phi}_{t_s,\sigma}(t) \in \mathbb{R}^n : t \in [-T_r, \infty)\}$ is bounded, there is a constant H such that $H > \|\mathscr{T}^{\phi}_{t_s,\sigma}(t)\|, \forall t \in \mathbb{R}^+$. Thus, the trajectory $\mathscr{O}_{t_s,\sigma}(\phi) = \{\mathscr{T}_{t_s,\sigma}(t, \phi) \in \mathscr{C}_r : t \in \mathbb{R}^+\}$ is uniformly bounded by H. Since the functions $f_q, q \in \mathbb{Q}$ are continuous and take bounded sets in $\mathscr{C}_r$ into bounded sets in $\mathbb{R}^n$, the right hand sides of (4.7) are bounded for all time and for all constituent systems. This implies that $\mathscr{T}^{\phi}_{t_s,\sigma}(t)$ is uniformly continuous on individual running times of switching events. However, as there is no switching jump and the number of elements of $\mathbb{Q}$ is finite, this further implies that $\mathscr{T}^{\phi}_{t_s,\sigma}$ is uniformly continuous on $[-T_r, \infty)$. Thus, it is obvious that the trajectory $\mathscr{O}_{t_s,\sigma}(\phi)$ is equicontinuous on $[-T_r, 0]$. As $\mathscr{T}^{\phi}_{t_s,\sigma}$ is bounded, applying Arzela-Ascoli theorem, we conclude that $\mathscr{O}_{t_s,\sigma}(\phi)$ is precompact and hence the conclusion of the theorem follows. ■

4.4.2 Quasi-invariance

In this section, we shall introduce a notion of quasi-invariance for switched time-delay systems using limiting switching sequences introduced in Chapter 3. We shall show that the limit sets of trajectories of switched time-delay systems are quasi-invariant. The main complexity lies in the arguments for uniform convergence in function space $\mathscr{C}_r$. In the following, for each $t_s \in \mathbb{R}^+$ and $\sigma \in \mathbb{S}$, the notation $\mathscr{T}_{t_s,\sigma}$ has the obvious meaning from (4.10). Thus, we have the following definition without involving the notion of motion.

Definition 4.4.1 *Let $\Sigma_{\mathscr{D}}$ be a switched time-delay system. For a fixed switching sequence $\sigma \in \mathbb{S}$, a subset $\mathcal{A} \subset \mathscr{C}_r$ is said to be σ-quasi-invariant if there is a limiting switching sequence $\sigma^* \in \mathbb{S}^*_\sigma$ of σ such that for each $\phi \in \mathcal{A}$, $\mathscr{T}_{0,\sigma^*}(t, \phi) \in \mathcal{A}, \forall t \geq 0$.*

In the following theorem, the argument using Arzela-Ascoli theorem for proving uniform convergence of continuous functions in the space $\mathscr{C}_r$ is adopted from [58]. The main difference from the classic results lies in the use of the transition mapping $\mathscr{T}_{t_s,\sigma}$ in dealing with the loss of the semi-group property of the system trajectories.

Theorem 4.4.3 *Let $\Sigma_{\mathscr{D}}$ be a switched time-delay system. Let $t_s \in \mathbb{R}^+, \phi \in \mathscr{C}_r$, and $\sigma \in \mathbb{S}$ fixed. Suppose that the trajectory $\mathscr{O}_{t_s,\sigma}(\phi)$ is compact covered and all limiting sequences of σ are non-Zeno. Then, the limit set $\omega_{t_s,\sigma}(\phi)$ is σ-quasi-invariant.*

Proof: Consider a point $\phi^* \in \omega_{t_s,\sigma}(\phi)$. By Definition (3.3.4), there is a time sequence $\{t_n\}_n$ such that

$$\lim_{n\to\infty} \|\mathscr{T}_{t_s,\sigma}(t_n, \phi) - \phi^*\| = 0. \tag{4.21}$$

It follows from Proposition 3.2.1 that there is a subsequence $\{t_{n_m}\}_m$ of $\{t_n\}_n$ such that $\sigma_{t_{n_m}} \to \sigma^*$ as $m \to \infty$. For switching sequences $\sigma_{t_{n_m}}, m \in \mathbb{N}$ and σ^*, we have the following notations and facts recalled from the proof of Theorem 3.4.1.

i) $\sigma_{t_{n_m}+t} \to \sigma^*_t, \forall t \in \mathbb{R}^+$,

ii) $q_{\sigma,i^-_\sigma(t_{n_m}+t)+j} \overset{\text{def}}{=} q_{\sigma_{t_{n_m}+t},j} \to q_{\sigma^*_t,j} \overset{\text{def}}{=} q_{\sigma^*,i^-_{\sigma^*}(t)+j}, \forall t \in \mathbb{R}^+, \forall j \in \mathbb{N}$,

iii) $i^-_{\sigma^*}(0)$ is the last index satisfying $\tau_{\sigma^*,i^-_{\sigma^*}(0)} \leq 0$, i.e, $\tau_{\sigma^*,j} = 0, \forall j = 0, \ldots, i^-_{\sigma^*}(0)$ and $\tau_{\sigma^*,i^-_{\sigma^*}(0)+1} > \tau_{\sigma^*,i^-_{\sigma^*}(0)}$,

iv) $\Delta\tau_{\sigma,i^-_\sigma(t_{n_m})+j} \overset{\text{def}}{=} \Delta\tau_{\sigma_{t_{n_m}},j} \to \Delta\tau_{\sigma^*,j} = 0, m \to \infty, \forall j = 1, \ldots, i^-_{\sigma^*}(0) - 1$,

v) $\tau_{\sigma,i^-_\sigma(t_{n_m})+1} - t_{n_m} \overset{\text{def}}{=} \Delta\tau_{\sigma_{t_{n_m}},0} \to \Delta\tau_{\sigma^*,0} = 0, m \to \infty$,

vi) $\tilde{\tau}_{\sigma,n_m}(0)$ is designated the value 0 and

$$\tilde{\tau}_{\sigma,n_m}(k) \overset{\text{def}}{=} \sum_{j=0}^{k-1} \Delta\tau_{\sigma_{t_{n_m}},j} \to 0, m \to \infty, \forall k = 1, \ldots, i^-_{\sigma^*}(0), \text{ and}$$

vii) $i^-_\sigma(t_{n_m} + \tilde{\tau}_{\sigma,n_m}(k)) = i^-_\sigma(t_{n_m}) + k$.

We shall show that

$$\lim_{m\to\infty} \|\mathscr{T}_{t_s,\sigma}(t_{n_m} + \tilde{\tau}_{\sigma,n_m}(i^-_{\sigma^*}(0)), \phi) - \phi^*\| = \lim_{m\to\infty} \|\mathscr{T}_{t_s,\sigma}(\tau_{\sigma,i^-_{\sigma,\sigma^*}(t_{n_m})}, \phi) - \phi^*\| = 0, \tag{4.22}$$

where $i_{\sigma,\sigma^*}(t_{n_m}) \stackrel{\text{def}}{=} i^-_\sigma(t_{n_m} + \tilde{\tau}_{\sigma,n_m}(i^-_{\sigma^*}(0))) = i^-_\sigma(t_{n_m}) + i^-_{\sigma^*}(0)$.

In fact, suppose that for each $k \in \{0, \dots, i^-_{\sigma^*}(0) - 1\}$, there exist $\phi^*_k \in \mathscr{C}_r$ such that

$$\lim_{m\to\infty} \|\mathscr{T}_{t_s,\sigma}(t_{n_m} + \tilde{\tau}_{\sigma,n_m}(k), \phi) - \phi^*_k\| = 0. \tag{4.23}$$

From the construction of the transition mapping $\mathscr{T}_{t_s,\sigma}$ in (4.10) and the designation of $\tilde{\tau}_{\sigma,n_m}$ in the fact vi) and vii) above, we have

$$\mathscr{T}_{t_s,\sigma}(t_{n_m} + \tilde{\tau}_{\sigma,n_m}(k+1), \phi) = \psi_{q_{\sigma,i^-_\sigma(t_{n_m})+k}}(\Delta\tau_{\sigma,i^-_\sigma(t_{n_m})+k}, \mathscr{T}_{t_s,\sigma}(t_{n_m} + \tilde{\tau}_{\sigma,n_m}(k), \phi)). \tag{4.24}$$

Since the trajectory $\mathscr{O}_{t_s,\sigma}(\phi)$ is contained in a compact subset of $\mathscr{C}_r$, $\mathbb{Q}$ is finite, and $\Delta\tau_{\sigma,i^-_\sigma(t_{n_m})+k}$ is arbitrarily small for sufficiently large m, (4.24) implies that for each $\epsilon > 0$, there is a $M \in \mathbb{N}$ such that

$$\|\mathscr{T}_{t_s,\sigma}(t_{n_m} + \tilde{\tau}_{\sigma,n_m}(k+1), \phi) - \mathscr{T}_{t_s,\sigma}(t_{n_m} + \tilde{\tau}_{\sigma,n_m}(k), \phi)\| < \epsilon, \forall m \geq M. \tag{4.25}$$

This together with (4.23) show that

$$\lim_{m\to\infty} \|\mathscr{T}_{t_s,\sigma}(t_{n_m} + \tilde{\tau}_{\sigma,n_m}(k+1), \phi) - \phi^*_k\| = 0. \tag{4.26}$$

Starting at $k = 0$ with $\phi^*_0 = \phi^*$, applying (4.26) successively for $k \in \{0, \dots, i^-_{\sigma^*}(0) - 1\}$, we obtain (4.22). Then, we proceed by considering a time $t \in [\tau_{\sigma^*,i^-_{\sigma^*}(0)}, \tau_{\sigma^*,i^-_{\sigma^*}(0)+1}) = [0, \Delta\tau_{\sigma^*,i^-_{\sigma^*}(0)})$ and computing the following limit

$$\lim_{m\to\infty} \mathscr{T}_{t_s,\sigma}(t + \tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})}, \phi). \tag{4.27}$$

As $\Delta\tau_{\sigma_{t_{n_m}},i^-_{\sigma^*}(0)} \to \Delta\tau_{\sigma^*,i^-_{\sigma^*}(0)}$, we have, for sufficiently large m, $t+\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})} \leq \tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})}+\Delta\tau_{\sigma_{t_{n_m}},i^-_{\sigma^*}(0)}$ and hence, $i^-_\sigma(t+\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})}) = i_{\sigma,\sigma^*}(t_{n_m}) = i^-_\sigma(t_{n_m})+i^-_{\sigma^*}(0)$. Thus, from the construction of the transition mapping $\mathscr{T}_{t_s,\sigma}$ in (4.10), we have

$$\begin{aligned}\mathscr{T}_{t_s,\sigma}(t+\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},\phi) &= \psi_{q_{\sigma,i^-_\sigma(t_{n_m})+i^-_{\sigma^*}(0)}}(t,\mathscr{T}_{t_s,\sigma}(\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},\phi))\\ &= \psi_{q_{\sigma^*,i^-_{\sigma^*}(0)}}(t,\mathscr{T}_{t_s,\sigma}(\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},\phi)) \overset{\text{def}}{=} \psi^\natural_{n_m}(t),\end{aligned} \tag{4.28}$$

for all $m \geq M$ for some $M \in \mathbb{N}$.

Since $t \in [\tau_{\sigma^*,i^-_{\sigma^*}(0)},\tau_{\sigma^*,i^-_{\sigma^*}(0)+1})$ is bounded by Δ_T, the functions $\psi_q, q \in \mathbb{Q}$ are continuous, and the trajectory $\mathscr{O}_{t_s,\sigma}(\phi)$ is bounded by a compact subset of $\mathscr{C}_r$, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\begin{aligned}&\|\psi_{q_{\sigma^*,i^-_{\sigma^*}(0)}}(t+\varsigma,\mathscr{T}_{t_s,\sigma}(\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},\phi)) - \psi_{q_{\sigma^*,i^-_{\sigma^*}(0)}}(t,\mathscr{T}_{t_s,\sigma}(\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},\phi))\|\\ &\quad= \|\psi^\natural_{n_m}(t+\varsigma)-\psi^\natural_{n_m}(t)\|\\ &\quad= \|\psi_{q_{\sigma^*,i^-_{\sigma^*}(0)}}(\varsigma,\psi^\natural_{n_m}(t))-\psi^\natural_{n_m}(t)\| < \epsilon, \forall\varsigma\in[0,\delta], t\in[0,\Delta_T],\end{aligned} \tag{4.29}$$

which clearly implies that the family $\{\psi^\natural_{n_m}(t)\}_m$ is equicontinuous.

In addition, as $\mathscr{O}_{t_s,\sigma}(\phi)$ is compact covered, $\{\psi^\natural_{n_m}(t)\}_m$ is uniformly bounded as well. Thus, according to Arzela-Ascoli theorem, there is a subsequence of $\{t_{n_m}\}_m$, which is again labeled by $\{t_{n_m}\}_m$, and a continuous function $\phi^\natural(t)$ such that $\psi^\natural_{n_m}(t) \to \phi^\natural(t), m \to \infty$ uniformly with respect to $t \in [0,\Delta_T]$.

Since $\phi_q, q \in \mathbb{Q}$ are continuous and $t \in [0,\Delta_T]$, from (4.22), we have

$$\begin{aligned}&\lim_{m\to\infty}\|\psi_{q_{\sigma^*,i^-_{\sigma^*}(0)}}(t,\mathscr{T}_{t_s,\sigma}(\tau_{\sigma,i_{\sigma,\sigma^*}(t_{n_m})},\phi)) - \psi_{q_{\sigma^*,i^-_{\sigma^*}(0)}}(t,\phi^*)\|\\ &\quad= \lim_{m\to\infty}\|\psi^\natural_{n_m}(t)-\psi_{q_{\sigma^*,i^-_{\sigma^*}(0)}}(t,\phi^*)\| = 0.\end{aligned} \tag{4.30}$$

Thus, for any $t \in [\tau_{\sigma^*, i^-_{\sigma^*}(0)}, \tau_{\sigma^*, i^-_{\sigma^*}(0)+1})$, we have

$$\|\psi_{\sigma^*, i^-_{\sigma^*}(0)}(t, \phi^*) - \phi^\natural(t)\| = \|\psi_{\sigma^*, i^-_{\sigma^*}(0)}(t, \phi^*) - \psi^\natural_{n_m}(t)\| + \|\psi^\natural_{n_m}(t) - \phi^\natural(t)\|. \quad (4.31)$$

In view of (4.30) and the fact that $\psi^\natural_{n_m}(t) \to \psi^\natural(t)$, the right hand side of (4.31) converges to zero as $m \to \infty$, and hence $\psi_{\sigma^*, i^-_{\sigma^*}(0)}(t, \phi^*) = \phi^\natural(t)$. Clearly, by virtue of (4.28), $\phi^\natural(t) \in \omega_{t_s,\sigma}(\phi)$. As such, $\psi_{\sigma^*, i^-_{\sigma^*}(0)}(t, \phi^*) \in \omega_{t_s,\sigma}(\phi), \forall t \in [\tau_{\sigma^*, i^-_{\sigma^*}(0)}, \tau_{\sigma^*, i^-_{\sigma^*}(0)+1})$.

Repeating the above argument for subsequent time stages of σ^*, the conclusion of the theorem follows. ■

4.5 Invariance Principles

The purpose of this section is to present an invariance principle for switched time-delay systems and its application for further converging criteria of delay-free switched systems. By virtue of the qualitative theory presented in the previous chapter, it is possible to develop a qualitative theory for switched time-delay systems in the Banach space $\mathscr{C}_r$ using Lyapunov functional approach as in the classical theory of time-delay systems [86, 59]. However, we are interested in the Lyapunov-Razumikhin function approach for practical relevance. The main issue thus lies in the relaxation of the decreasing condition on Lyapunov functions, e.g., condition ii) of Theorem 3.5.2, which is restrictive in the context of time-delay systems [57].

For a trajectory $x(t, \phi) \stackrel{\text{def}}{=} \mathscr{T}^{\phi}_{t_s,\sigma}(t)$ in the space $\mathbb{R}^n$ of a switched time-delay system, let us define the functions $V^\natural_q : \mathscr{C}_r \to \mathbb{R}, q \in \mathbb{Q}$ as follows.

$$V^\natural_q(x_t) \stackrel{\text{def}}{=} \sup_{\varsigma \in [-T_r, 0]} V_q(x(t + \varsigma, \phi)). \quad (4.32)$$

Throughout the chapter, whenever the period of persistence T_{p} is involved, we suppose that the delay time T_r is no smaller than T_{p}.

4.5.1 Main Result

Theorem 4.5.1 *Let $\Sigma_{\mathcal{D}}$ be a switched time-delay system. Consider compact subsets D and G of $\mathbb{R}^n$, continuous functions $V_q : G \to \mathbb{R}, q \in \mathbb{Q}$, a class-$\mathcal{K}$ function β, a constant $\delta_V \geq 0$, a function $\phi \in \mathscr{C}_r$ satisfying $\phi(\varsigma) \in D, \forall \varsigma \in [-T_r, 0]$, a time $t_s \in \mathbb{R}^+$ and a switching sequence $\sigma \in \mathbb{S}_{\mathcal{P}}[\tau_{\mathrm{p}}, T_{\mathrm{p}}]$, where $T_{\mathrm{p}} \leq T_r$. Let $x(t,\phi) = \mathscr{T}^{\phi}_{t_s,\sigma}(t), t \in \mathbb{R}^+$ be the trajectory in the space $\mathbb{R}^n$. Suppose that the following conditions holds*

i) *$x(t,\phi)$ is uniformly continuous with respect to t and $x(t,\phi) \in G, \forall t \in \mathbb{R}^+$;*

ii) *$V_{q_1}(x) \leq \beta(V_{q_2}(x)), \forall q_1, q_2 \in \mathbb{Q}, x \in \mathbb{R}^n$;*

iii) *along the trajectory $x(t,\phi)$, the functions $V_q(x(t,\phi)), q \in \mathbb{Q}$ are everywhere Dini differentiable with respect to time;*

iv) *along the trajectory $x(t,\phi), t \in \mathbb{R}^+$, we have*

$$\sup_{\varsigma \in [-T_r,0]} V_{q_{\sigma,i^-_\sigma(t)}}(x(t+\varsigma,\phi)) \leq V_{q_{\sigma,i^-_\sigma(t)}}(x(t,\phi)) \Rightarrow D_\sigma V_{q_{\sigma,i^-_\sigma(t)}}(x(t,\phi)) \leq 0; \quad (4.33)$$

v) *for the sequence of dwell-time switching events $\{(q_{\sigma,i^{\mathcal{D}}_j}, \Delta\tau_{\sigma,i^{\mathcal{D}}_j})\}_j$ of σ, we have*

$$\sup_{\varsigma \in [-T_r,0]} V_{q_{\sigma,i^{\mathcal{D}}_k}}(x(\tau_{\sigma,i^{\mathcal{D}}_k} + \varsigma, \phi)) \leq \sup_{\varsigma \in [-T_r,0]} V_{q_{\sigma,i^{\mathcal{D}}_j}}(x(\tau_{\sigma,i^{\mathcal{D}}_j} + \varsigma, \phi)), \forall k, j \in \mathbb{N}, k > j, \quad (4.34)$$

and

$$\limsup_{j\to\infty} \left(\sup_{\varsigma \in [-T_r,0]} V_{q_{\sigma,i^{\mathcal{D}}_j}}(x(\tau_{\sigma,i^{\mathcal{D}}_j} + \varsigma, \phi)) - \sup_{\varsigma \in [-T_r,0]} V_{q_{\sigma,i^{\mathcal{D}}_j}}(x(\tau_{\sigma,i^{\mathcal{D}}_j+1} + \varsigma, \phi)) \right) \leq \delta_V. \quad (4.35)$$

Let $V_{-\infty} = \inf\{V_q(\zeta) : \zeta \in G, q \in \mathbb{Q}\}$, and $L_\gamma, \gamma \in \mathbb{R}$ be the level sets in $\mathscr{C}_r$ defined as $L_\gamma = \big\{\phi : \phi(\varsigma) \in G, \forall \varsigma \in [-T_r, 0]$ and $\exists q \in \mathbb{Q}, \min\{V_{-\infty}, \gamma - \delta_V\} \leq V^{\natural}_q(\phi) \leq \beta(\gamma)\big\}$.

Then, $\mathscr{T}_{t_s,\sigma}(t,\phi)$ approaches to the set $M \overset{\mathrm{def}}{=} \bigcup_{\gamma \in \mathbb{R}} M_\gamma$ as $t \to \infty$, where M_γ is the largest (t_s,σ)-quasi-invariant set contained in L_γ.

Proof: Since $x(t,\phi)$ is bounded and uniformly continuous, applying the argument in Theorem 4.4.2, it follows that $\mathscr{T}_{t_s,\sigma}(t,\phi)$ is compact covered. Thus, by Theorem 4.4.1, the limit set $\omega_{t_s,\sigma}(\phi)$ in the space $\mathscr{C}_r$ of the trajectory $\mathscr{T}_{t_s,\sigma}(t,\phi)$ attracts this trajectory and is nonempty and compact.

We proceed to estimate $\omega_{t_s,\sigma}(\phi)$ using auxiliary functions $V_q, q \in \mathbb{Q}$. Consider the composite Lyapunov function

$$V_{\mathscr{C}}(t,\phi) = \begin{cases} V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t,\phi)) \textbf{ if } t \in [\tau_{\sigma,i_j^{\mathcal{D}}}, \tau_{\sigma,i_j^{\mathcal{D}}+1}) \\ \sup_{\varsigma\in[-T_r,0]} \beta(V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tau_{\sigma,i_j^{\mathcal{D}}} + \varsigma, \phi)) \textbf{ if } t \in [\tau_{\sigma,i_j^{D}+1}, \tau_{\sigma,i_{j+1}^{\mathcal{D}}}) \end{cases}, \tag{4.36}$$

where, without loss of generality, we have supposed that the first switching event of σ is of dwell-time running time, i.e., $i_0^{\mathcal{D}} = 0$.

Along the trajectory $\mathscr{T}_{t_s,\sigma}(t,\phi)$, let us define the function $V_{\mathscr{C}}^{\natural} : \mathscr{C}_r \to \mathbb{R}$ as follows.

$$V_{\mathscr{C}}^{\natural}(x_t) \stackrel{\text{def}}{=} \sup_{\varsigma\in[-T_r,0]} V_{\mathscr{C}}(x(t+\varsigma,\phi)) \tag{4.37}$$

Clearly $V_{\mathscr{C}}^{\natural}(x_t) = V_{q_{\sigma,i_j^{\mathcal{D}}}}^{\natural}(x_t)$ on dwell-time intervals. Hence, by condition v), the sequence $\{V_{\mathscr{C}}^{\natural}(x_{\tau_{\sigma,i_j^{\mathcal{D}}}})\}_j$ is non-increasing. This coupled with the boundedness of x_t implies that there is a number $\gamma \in \mathbb{R}$ such that

$$\lim_{j\to\infty} V_{\mathscr{C}}^{\natural}(x_{\tau_{\sigma,i_j^{\mathcal{D}}}}) = \gamma, \tag{4.38}$$

which combined with condition v) gives rise to

$$\delta_V \geq \limsup_{j\to\infty} \left(V_{q_{\sigma,i_j^{\mathcal{D}}}}^{\natural}(x_{\tau_{\sigma,i_j^{\mathcal{D}}}}) - V_{q_{\sigma,i_j^{\mathcal{D}}}}^{\natural}(x_{\tau_{\sigma,i_j^{\mathcal{D}}+1}}) \right) = \gamma - \liminf_{j\to\infty} V_{q_{\sigma,i_j^{\mathcal{D}}}}^{\natural}(x_{\tau_{\sigma,i_j^{\mathcal{D}}+1}}). \tag{4.39}$$

From condition iv), applying the argument of [59, Chapter 5, Theorem 4.1] for functions $V_{q_{\sigma,i_j^{\mathcal{D}}}}^{\natural}(x_t), j \in \mathbb{N}$ on dwell-time intervals $[\tau_{\sigma,i_j^{\mathcal{D}}}, \tau_{\sigma,i_j^{\mathcal{D}}+1}], j \in \mathbb{N}$, it follows that $V_{\mathscr{C}}^{\natural}(x_t)$ is non-increasing on each of these intervals. Thus, using (4.39) and the condition (4.35), we arrive at

$$\gamma = \lim_{j\to\infty} V^{\natural}_{q_{\sigma,i^{\mathscr{D}}_j}}(x_{\tau_{\sigma,i^{\mathscr{D}}_j}}) \geq \limsup_{\substack{j\to\infty \\ t\in[\tau_{\sigma,i^{\mathscr{D}}_j},\tau_{\sigma,i^{\mathscr{D}}_j+1}]}} V^{\natural}_{q_{\sigma,i^{\mathscr{D}}_j}}(x_t) \geq \liminf_{j\to\infty} V^{\natural}_{q_{\sigma,i^{\mathscr{D}}_j}}(x_{\tau_{\sigma,i^{\mathscr{D}}_j+1}}) \geq \gamma - \delta_V. \tag{4.40}$$

We now consider a limit point $\phi^* \in \omega_{t_s,\sigma}(\phi)$. By definition, there is a time sequence $\{t_n\}_n$ such that

$$\lim_{n\to\infty} x_{t_n} \overset{\text{def}}{=} \lim_{n\to\infty} \mathscr{T}_{t_s,\sigma}(t_n,\phi) = \phi^*. \tag{4.41}$$

Since $\mathbb{Q}$ is finite, there is an index q^* and a subsequence $\{t_{n_m}\}_m$ of $\{t_n\}_n$ such that $q_{\sigma,i^-_\sigma(t_{n_m})} = q^*, \forall m \in \mathbb{N}$. We have the following cases.

Case 1: There are infinitely many number n_m such that $[\tau_{\sigma,i^-_\sigma(t_{n_m})}, \tau_{\sigma,i^-_\sigma(t_{n_m})+1}]$ are dwell-time intervals. In this case, as $V^{\natural}_{q^*}$ is continuous, according to (4.40), we have

$$V^{\natural}_{q^*}(\phi^*) = \lim_{\substack{m\to\infty \\ \Delta\tau_{\sigma,i^-_\sigma(t_{n_m})}\geq\tau_{\mathrm{p}}}} V^{\natural}_{q^*}(x_{t_n}) = \lim_{\substack{m\to\infty \\ \Delta\tau_{\sigma,i^-_\sigma(t_{n_m})}\geq\tau_{\mathrm{p}}}} V^{\natural}_{q_{\sigma,i^-_\sigma(t_{n_m})}}(x_{t_n}) \in [\gamma - \delta_V, \gamma]. \tag{4.42}$$

Case 2: There is a $M \in \mathbb{N}$ such that the running times of switching events $e_{\sigma,i^-_\sigma(t_{n_m})}, m \geq M$ are all less than τ_{p}. Suppose that all the times t_{n_m} are of this property. Let us recall that, for each $m \in \mathbb{N}$, $i^-_{\mathscr{D}}(t_{n_m}) = \max\{i \in \mathbb{N} : \Delta\tau_{\sigma,i} \geq \tau_{\mathrm{p}}, \tau_{\sigma,i} \leq t_{n_m}\}$ is the index of the dwell-time switching event before and closest to t_{n_m}. We also define $i^+_{\mathscr{D}}(t_{n_m})$ as the index of the dwell-time switching event following $e_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}$.

Since $T_{\mathrm{p}} \leq T_r$ and $\tau_{\sigma,i^+_{\mathscr{D}}(t_{n_m})} - \tau_{\sigma,i^-_{\mathscr{D}}(t_{n_m})+1} \leq T_{\mathrm{p}}$ by specification, using condition ii) and definition of functions $V^{\natural}_q$, we have

$$\begin{aligned} &V_{q_{\sigma,i^-_\sigma(t_{n_m})}}(x(t_{n_m}+\varsigma,\phi)) \leq \beta(V_{q_{\sigma,i^+_{\mathscr{D}}(t_{n_m})}}(x(t+\varsigma,\phi))) \\ &\quad\leq \sup_{\upsilon\in[-T_r,0]} \beta(V_{q_{\sigma,i^+_{\mathscr{D}}(t_{n_m})}}(x(\tau_{\sigma,i^+_{\mathscr{D}}(t_{n_m})}+\upsilon,\phi))) \leq \beta(V^{\natural}_{q_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}}(x_{\tau_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}})) \end{aligned} \tag{4.43}$$

for $\varsigma \in [-T_r, 0]$ and $t_{n_m} + \varsigma \geq \tau_{\sigma, i^-_{\mathscr{D}}(t_{n_m})+1}$, and

$$\begin{aligned}
V_{q_{\sigma,i^-_\sigma(t_{n_m})}}(x(t_{n_m}+\varsigma,\phi)) &\leq \beta(V_{q_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}}(x(t+\varsigma,\phi))) \\
&\leq \sup_{\upsilon\in[-T_r,0]} \beta(V_{q_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}}(x(t_{n_m}+\varsigma+\upsilon,\phi))) \\
&\leq \beta(V^\natural_{q_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}}(x_{t_{n_m}+\varsigma})) \leq \beta(V^\natural_{q_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}}(x_{\tau_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}})) \qquad (4.44)
\end{aligned}$$

for $\varsigma \in [-T_r, 0]$ and $t_{n_m} + \varsigma \leq \tau_{\sigma, i^-_{\mathscr{D}}(t_{n_m})+1}$.

Combining (4.43) and (4.44) yields

$$V^\natural_{q_{\sigma,i^-_\sigma(t_{n_m})}}(x_{t_{n_m}}) \leq \beta(V^\natural_{q_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}}(x_{\tau_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}})). \qquad (4.45)$$

Taking the limits of both sides of (4.45) as $m \to \infty$ using (4.40), we arrive at

$$V^\natural_{q^*}(\phi^*) = \lim_{m\to\infty} V^\natural_{q^*}(x_{t_{n_m}}) = \lim_{m\to\infty} V^\natural_{q_{\sigma,i^-_\sigma(t_{n_m})}}(x_{t_{n_m}}) \leq \beta(V^\natural_{q_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}}(x_{\tau_{\sigma,i^-_{\mathscr{D}}(t_{n_m})}})) \leq \beta(\gamma). \qquad (4.46)$$

By virtue of (4.38), the number γ is independent of the limit point $\phi^* \in \omega_{t_s,\sigma}(\phi)$. Thus, from (4.42) and (4.46), the limit point $\phi^* \in \omega_{t_s,\sigma}(\phi)$ belongs to the level set

$$L_\gamma = \{\phi \in \mathscr{C}_r : \exists q \in \mathbb{Q} : \min\{V_{-\infty}, \gamma - \delta_V\} \leq V^\natural_q(\phi) \leq \beta(\gamma)\}. \qquad (4.47)$$

Since $\omega_{t_s,\sigma}(\phi)$ is (t_s, σ)-quasi-invariant according to Theorem 4.4.3, we also have $\omega_{t_s,\sigma} \subset M_\gamma \subset M$. Finally, as $\omega_{t_s,\sigma}$ is compact and attracts $\mathscr{T}_{t_s,\sigma}(t,\phi)$, this implies that $\mathscr{T}_{t_s,\sigma}(t,\phi) \to M, t \to \infty$. ■

The conversing behavior of Lyapunov functions in Theorem 4.5.1 is illustrated in Figure 4.1. It is observed that the values of Lyapunov functions $V_{q_{\sigma,i^{\mathscr{D}}_j}}$ at starting time $\tau_{\sigma,i^{\mathscr{D}}_j}$ of dwell-time switching events need not to be decreasing. As observed, since the persistence of period T_{p} is smaller than the time-delay T_r, the value of the Lyapunov functions on persistence period is guaranteed to be bounded by

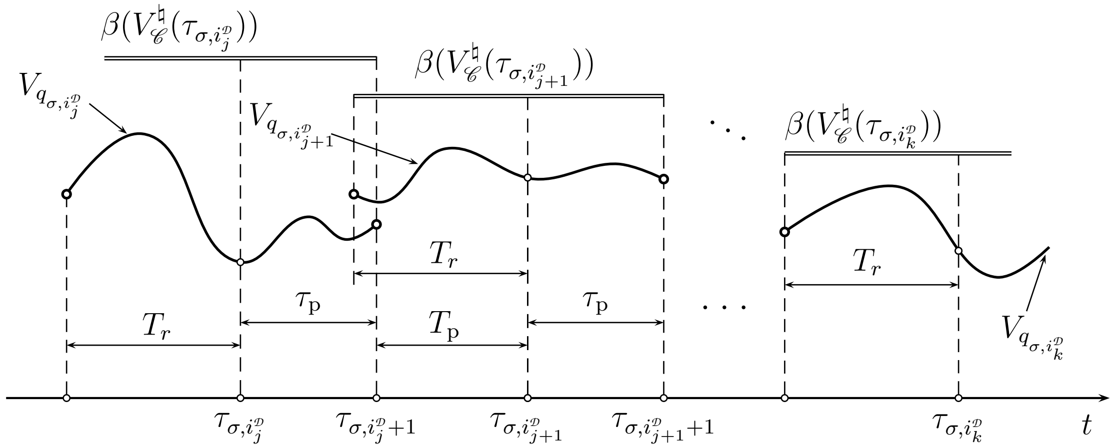

Figure 4.1: Behavior of Lyapunov functions

$\beta(V^\natural_{\mathscr{C}}(\tau_{\sigma,i^{\mathscr{D}}_j})) = \beta(V^\natural_{q_{\sigma,i^{\mathscr{D}}_j}}(x_{\tau_{\sigma,i^{\mathscr{D}}_j}}))$. Thus, the trajectories of Lyapunov functions stay below and hence their convergence is governed by the behavior of the composite function $V^\natural_{\mathscr{C}}$ represented by the double-lines. Since the sequence $\{\beta(V^\natural_{\mathscr{C}}(\tau_{\sigma,i^{\mathscr{D}}_j}))\}_j$ is non-increasing, the convergence of the Lyapunov functions $V_q, q \in \mathbb{Q}$ along the trajectory of the system is guaranteed.

Again, it can be seen that the condition v) in Theorem 4.5.1 is of practical relevance. Since the decreasing condition on Lyapunov functions is imposed only at starting times of dwell-time switching events and the delay time T_r and period of persistence T_{p} are bounded, it is always possible to achieve sufficient decrements on dwell-time switching events.

4.5.2 Application to Delay-free Systems

The results achieved so far show that in qualitative theory of switched systems with persistent dwell-time, estimates of increments of Lyapunov functions on periods of persistence are of natural use in achieving non-conservative results. As illustrated in Figure 4.1, the decreasing condition on Lyapunov functionals $V^\natural_q$ in the space $\mathscr{C}_r$ does not impose decreasing on Lyapunov functions V_q in the space $\mathbb{R}^n$. Furthermore, every trajectory in $\mathbb{R}^n$ induces a trajectory in $\mathscr{C}_r$. Thus, treating the persistence time

T_p and the delay time T_r as the same stimulus of diverging behavior, we have the following further non-conservative result in qualitative theory of delay-free switched systems via time-delay approach.

Let $\mathscr{C}_\mathrm{p}$ be the space $\mathscr{C}([-T_\mathrm{p},0],\mathbb{R}^n)$ of continuous functions from $[-T_\mathrm{p},0]$ to $\mathbb{R}^n$. Consider a delay-free switched autonomous system $\Sigma_{\mathscr{A}}$ with transition mapping $\mathscr{T}_{t_s,\sigma}(t,x)$ defined by (2.7). For a trajectory $x(t,\phi) \stackrel{\mathrm{def}}{=} \mathscr{T}_{t_s,\sigma}(t,\phi(0))$ through $\phi \in \mathscr{C}_\mathrm{p}$, i.e., $x(\varsigma,\phi)=\phi(\varsigma), \forall\varsigma \in [-T_\mathrm{p},0]$, we also denote by $x_t(\phi)$ the continuous function in $\mathscr{C}_\mathrm{p}$ defined as $x_t(\phi)(\varsigma)=x(t+\varsigma,x_0), \varsigma \in [-T_\mathrm{p},0]$. Again, for a continuous function $V:\mathbb{R}^n \to \mathbb{R}$, the function $V^\natural:\mathscr{C}_\mathrm{p} \to \mathbb{R}$ is defined as

$$V^\natural(\phi) = \sup_{\varsigma\in[-T_\mathrm{p},0]} V(\phi(\varsigma)). \tag{4.48}$$

The set $\{x_t(\phi) \in \mathscr{C}_\mathrm{p} : t \in \mathbb{R}^+\}$ is thus termed the trajectory in the space $\mathscr{C}_\mathrm{p}$ of the delay-free switched system $\Sigma_{\mathscr{A}}$.

Definition 4.5.1 *Let $\Sigma_{\mathscr{A}}$ be a switched autonomous system with the transition mapping $\mathscr{T}_{t_s,\sigma}$. A subset $\mathcal{A} \subset \mathscr{C}_r$ is said to be σ-quasi-invariant if there is a limiting switching sequence $\sigma^* \in \mathbb{S}_\sigma^*$ of σ such that for each $\phi \in \mathcal{A}$, $x_t(\phi) = (\mathscr{T}_{0,\sigma^*}(t,\phi(0)))_t \in \mathcal{A}, \forall t \geq 0$.*

Theorem 4.5.2 *Let $\Sigma_{\mathscr{A}}$ be a switched autonomous system. Consider compact subsets D and G of $\mathbb{R}^n$, continuous functions $V_q : G \to \mathbb{R}, q \in \mathbb{Q}$, a class-$\mathcal{K}$ function β, a constant $\delta_V \geq 0$, a function $\phi \in \mathscr{C}_\mathrm{p}$ satisfying $\phi(\varsigma) \in D, \forall\varsigma \in [-T_\mathrm{p},0]$, a time $t_s \in \mathbb{R}^+$, and a switching sequence $\sigma \in \mathbb{S}_{\mathscr{P}}[\tau_\mathrm{p},T_\mathrm{p}]$. Let $x(t,\phi)=\mathscr{T}_{t_s,\sigma}(t,\phi(0)), t \in \mathbb{R}^+$ be the trajectory through $\phi \in D$ in the space $\mathbb{R}^n$ of the system. Suppose that the following conditions hold.*

i) $x(t,\phi)$ is uniformly continuous with respect to t and $x(t,\phi) \in G, \forall t \in \mathbb{R}^+$;

ii) $V_{q_1}(x) \leq \beta(V_{q_2}(x)), \forall q_1,q_2 \in \mathbb{Q}, x \in \mathbb{R}^n$;

iii) *along the trajectory $x(t, x_0)$, the functions $V_q(x(t,\phi)), q \in \mathbb{Q}$ are everywhere Dini differentiable with respect to time;*

iv) *in any switching event $(q_{\sigma,i}, \Delta\tau_{\sigma,i}), i \in \mathbb{N}$, we have*

$$V_{q_{\sigma,i}}(x(\tau_{\sigma,i} + \varsigma_1, \phi)) \geq V_{q_{\sigma,i}}(x(\tau_{\sigma,i} + \varsigma_2, \phi)), \forall \varsigma_1, \varsigma_2 \in [0, \Delta\tau_{\sigma,i}], \varsigma_1 \leq \varsigma_2; \quad (4.49)$$

v) *for the sequence of dwell-time switching events $\{(q_{\sigma,i_j^{\mathcal{D}}}, \Delta\tau_{\sigma,i_j^{\mathcal{D}}})\}_j$ of σ, we have*

$$\sup_{\varsigma \in [-T_{\mathrm{p}},0]} V_{q_{\sigma,i_k^{\mathcal{D}}}}(x(\tau_{\sigma,i_k^{\mathcal{D}}} + \varsigma, \phi)) \leq \sup_{\varsigma \in [-T_{\mathrm{p}},0]} V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(\tau_{\sigma,i_j^{\mathcal{D}}} + \varsigma, \phi)), \forall k, j \in \mathbb{N}, k > j, \quad (4.50)$$

and

$$\limsup_{j\to\infty} \left(\sup_{\varsigma \in [-T_{\mathrm{p}},0]} V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(\tau_{\sigma,i_j^{\mathcal{D}}} + \varsigma, \phi)) - \sup_{\varsigma \in [-T_{\mathrm{p}},0]} V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(\tau_{\sigma,i_j^{\mathcal{D}}+1} + \varsigma, \phi)) \right) \leq \delta_V. \quad (4.51)$$

Let $V_{-\infty} = \inf\{V_q(\zeta) : \zeta \in G, q \in \mathbb{Q}\}$, and $L_\gamma, \gamma \in \mathbb{R}$ be the level sets in $\mathscr{C}_r$ defined as $L_\gamma = \big\{\phi : \phi(\varsigma) \in G, \forall \varsigma \in [-T_{\mathrm{p}}, 0]$ and $\exists q \in \mathbb{Q}, \min\{V_{-\infty}, \gamma - \delta_V\} \leq V_q^\natural(\phi) \leq \beta(\gamma)\big\}$.

Then, $x_t(\phi)$ approaches to the set $M \overset{\text{def}}{=} \bigcup_{\gamma\in\mathbb{R}} M_\gamma$ as $t \to \infty$, where M_γ is the largest (t_s, σ)-quasi-invariant set contained in L_γ.

Proof: Adopting the proofs of Theorems 4.4.2 and 4.4.3, it follows that the limit set

$$\omega(\phi) = \bigcap_{T\geq t_s} \overline{\bigcup_{t\geq T} x_t(\phi)} \quad (4.52)$$

is nonempty, compact and quasi-invariant. As condition ii) guarantees the decreasing behavior of Lyapunov functions, the theorem is a direct consequence of Theorem (4.5.1) and hence the conclusion of the theorem is straightforward. ■

By Theorem 4.5.2, the time-delay nature of switched systems has been revealed. Similar to time-delay systems, the behavior of switched systems in periods of persistence tends to destabilize the systems.

Due to arbitrarily fast switching, it is not realistic to impose and verify stability conditions in the periods of persistence. As a result, a non-conservative approach to deal with the difficulty at hand is to consider the after-effects of switching in periods of persistence, and then study the compensating behavior on the dwell-time intervals. From this point of view, condition v) in Theorem 4.5.2 is of fundamental interest.

$\bullet$

Chapter 5

Asymptotic Gains

In this chapter, positive definite auxiliary functions are studied for stronger converging behavior of switched systems. Under the principle of small-variation small state, we consider ultimate variations of auxiliary functions for asymptotic gain of switched systems. The results are obtained for both delay-free and delay-dependent systems. In the case of delay-dependence, asymptotic gain is achieved via a Lyapunov-Razumikhin function approach.

5.1 Motivation

The attraction of invariance principles presented in the previous chapters lies in their generality in making conclusion on various limiting behaviors of dynamical systems. Of such behaviors, asymptotic convergence under disturbance to a compact set in the state space is of fundamental interest in control theory.

Clearly, such problem can be dealt with by treating the disturbance as a time-varying parameter and then applying Theorem 3.5.1 to determine the non-autonomous attractor of the corresponding switched system. Once the non-autonomous attractor of the corresponding system has been determined, further properties of the time-varying parameter can be exploited to make further conclusion on the "behavior" of the non-autonomous attractor. The expense for the generality of this approach is the complexity in computing with pullback trajectories.

In nonlinear systems, an efficient approach to deal with the problem in question is to study asymptotic gain in the framework of the notion of input-to-state stability [138]. Clearly, if a system has an asymptotic gain χ, it is possible to consider the function $g(x) = \min_{q\in\mathbb{Q}} V_q(x) - \chi(\|w\|)$ and apply Theorem 3.5.2 to make conclusion, where χ is a class–$\mathcal{K}_\infty$ function and $\|w\|$ is the norm of disturbance. It is observed that by addressing convergence to a compact set, we have dropped the need for the structure of the attractor and hence it is not necessary to study the time transition properties. In this context, it is possible to describe the change of state of constituent systems by vector fields for computing time derivatives of auxiliary functions.

Motivated by the above considerations and the motivating study presented in Chapter 1, we shall adopt vector fields to characterize the system evolution and then develop a Lyapunov stability theory for switched systems with persistent dwell-time switching. Bearing in mind the constructibility of positive Lyapunov functions in control design [85], we make use of the approach of combining Lyapunov stability [56] and input-output stability [159] introduced in [133] and summarized in [138, 139]. In this manner, the switching decreasing condition is completely removed.

5.2 Stability of Delay-free Switched System

5.2.1 System with Input and Asymptotic Gain

Consider dynamical systems described by the following differential equations:

$$\dot{x}(t) = f_q(x(t), w(t)), q \in \mathbb{Q}, \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^d$ is the input, $f_q : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n, q \in \mathbb{Q}$ are locally Lipschitz continuous functions, and $\mathbb{Q}$ is a finite discrete set. Suppose that w belongs to the space $\mathcal{L}_\infty^d$ of measurable locally essentially bounded functions mapping $\mathbb{R}$ into $\mathbb{R}^d$. Then, for each $q \in \mathbb{Q}$ and for any $x_0 \in \mathbb{R}^n$ there is a unique maximally extended solution $x(t; x_0, w, q)$ of the following initial value problem [134]:

$$\begin{cases} \dot{x}(t) &= f_q(x(t), w(t)), \\ x(0) &= x_0. \end{cases} \tag{5.2}$$

Thus, if the vector fields $f_q, q \in \mathbb{Q}$ are forward complete [134], then the mappings $\varphi_q : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{L}_\infty^d, (t, t_0, x_0, w) \mapsto x(t; t_0, x_0, w, q)$ are uniquely defined. Consider the autonomous motion in $\mathcal{L}_\infty^d$ defined as $\pi(t, w)(\cdot) = w(\cdot + t), w \in \mathcal{L}_\infty^d$. By virtue of Definition 2.4.5, the following switched non-autonomous system is well-defined

$$\Sigma_{\mathcal{NA}} = \{\mathbb{R}^+, \mathbb{Q}, \mathbb{R}^n, \mathcal{L}_\infty^d, \{\varphi_q\}_{q \in \mathbb{Q}}, \mathbb{S}, \pi\}, \tag{5.3}$$

where we have supposed that there is no switching jump so that the discrete transition mapping was dropped.

In light of the above consideration, it is obvious that the switched systems (5.3) can be equivalently described by

$$\Sigma_{\mathcal{U}} = \{\mathbb{R}^+, \mathbb{Q}, \mathbb{R}^n, \mathcal{L}_\infty^d, \{f_q\}_{q \in \mathbb{Q}}, \mathbb{S}\}. \tag{5.4}$$

We shall call the switched system $\Sigma_{\mathcal{U}}$ described by (5.4) the switched system with input. We denote by $\Sigma_{\mathcal{U}}[\tau_\text{p}, T_\text{p}]$ the switched system $\Sigma_{\mathcal{U}}$ with $\mathbb{S} = \mathbb{S}_{\mathcal{P}}[\tau_\text{p}, T_\text{p}]$.

Note that $x(t; t_0, x_0, w, q)$ is the state at the real time t of the system $\dot{x} = f_q(x, w)$ having started to evolve at the real time t_0. As there is no switching jump, according to (2.14) in Section 2.4.3, if the vector fields $f_q, q \in \mathbb{Q}$ are complete, then for each switching sequence $\sigma \in \mathbb{S}$ and each initial condition $x(t_0) = x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}, w \in \mathcal{L}_\infty^d$, the following mapping is uniquely defined:

$$\mathscr{T}_{\sigma,w,t_s}(t, x_0) = \begin{cases} x(t + t_0; x_0, t_0, w, q_{\sigma,i_\sigma^-(t_s)}) \ \textbf{if} \ t \in [0, \tau_{\sigma,i_\sigma^-(t_s)+1} - t_s] \\ x(t + \tau_{\sigma,i_\sigma^-(t+t_s)} - t_s + t_0; x(\tau_{\sigma,i_\sigma^-(t+t_s)} - t_s + t_0; x_0, w, q_{\sigma,i_\sigma^-(t_s)}), \\ \qquad \tau_{\sigma,i_\sigma^-(t+t_s)} - t_s + t_0, w(\cdot + \tau_{\sigma,i_\sigma^-(t+t_s)} - t_s), q_{\sigma,i_\sigma^-(t+t_s)}) \\ \qquad\qquad\qquad \textbf{if} \ t \geq \tau_{\sigma,i_\sigma^-(t_s)+1} - t_s, \end{cases} \tag{5.5}$$

where $w(\cdot + \tau_{\sigma,i_\sigma^-(t+t_s)}) : \mathbb{R}^+ \to \mathbb{R}^d$ is defined as $w(\cdot + \tau_{\sigma,i_\sigma^-(t+t_s)})(t) = w(t + \tau_{\sigma,i_\sigma^-(t+t_s)})$. Hereafter, we suppose that all switched systems are forward complete, i.e., the mapping $\mathscr{T}_{\sigma,w,t_s}(t, x_0)$ are well-defined for all $t \geq 0$.

Adopting the notion of asymptotic gain for nonlinear systems [138], we have the following analogous notion for switched systems.

Definition 5.2.1 *The switched system Σ_u is said to have a switching-uniform asymptotic gain (SUAG) χ if χ is a class–$\mathcal{K}_\infty$ function and*

$$\limsup_{t\to\infty} \|\mathscr{T}_{\sigma,w,t_s}(t, x_0)\| \leq \chi(\|w\|), \forall w \in \mathcal{L}_\infty^d, x_0 \in \mathbb{R}^n, t_s \in \mathbb{R}^+, \sigma \in \mathbb{S}. \tag{5.6}$$

We would mention that the term "*switching-uniform*" in Definition 5.2.1 is based on the consideration on uniformity over a class of switching signals in [66, 62].

5.2.2 Lyapunov Functions for SUAG

Let us refer to Section 2.4.2 (Page 28) for definition of the transition indicator $i_\sigma^-(\cdot)$. Hereafter, we suppose that $t_s = 0$ without loss of generality. Let $V : \mathbb{R}^n \to \mathbb{R}^+$ be a continuous function. The derivative of V along a trajectory $x(t)$ is [56]

$$DV(x(t)) = \limsup_{h\to 0} \frac{1}{h}(V(x(t+h)) - V(x(t))), \tag{5.7}$$

and the variation of V between t_1 and t_2 along x is

$$\mathrm{Var}_{t_1}^{t_2} V(x) = |V(x(t_1)) - V(x(t_2))|. \tag{5.8}$$

Variation $\mathrm{Var}_a^b V(x(t))$ indicates the deviation of $V(x(t))$ achieved for a duration of $b - a$ of evolution from the time a. This notion is different from the notion of total variation in real analysis [156]. Let V_1 and V_2 be functions from $\mathbb{R}^n$ to $\mathbb{R}^+$. The relative variation between V_1 and V_2 at t_1 and t_2 along a trajectory $x(t)$ is

$$\mathrm{Var}_{t_1}^{t_2}[V_1, V_2](x) = |V_1(x(t_1)) - V_2(x(t_2))|. \tag{5.9}$$

The sequence of dwell-time switching events of σ is labeled as $\{(q_{\sigma,i_j^{\mathscr{D}}}, \Delta\tau_{\sigma,i_j^{\mathscr{D}}})\}_j$. For brevity, let $\tilde{t} = t - t_0$ and $q_\sigma^-(\tilde{t}) = q_{\sigma,i_\sigma^-(\tilde{t})}$, $t_{\sigma,i} \overset{\text{def}}{=} t_0 + \tau_{\sigma,i}$, and $t_{\sigma,i_j^{\mathscr{D}}} \overset{\text{def}}{=} t_0 + \tau_{\sigma,i_j^{\mathscr{D}}}$.

Theorem 5.2.1 *Consider the switched system with input $\Sigma_u[\tau_\mathrm{p}, T_\mathrm{p}]$, the class-$\mathcal{K}_\infty$ functions $\underline{\alpha}, \bar{\alpha}, \alpha$, γ_1, and γ_2, and the continuous functions $V_q : \mathbb{R}^n \to \mathbb{R}^+, q \in \mathbb{Q}$. Suppose that*

$$\underline{\alpha}(\|x\|) \leq V_q(x) \leq \bar{\alpha}(\|x\|), \forall x \in \mathbb{R}^n, q \in \mathbb{Q}, \tag{5.10}$$

and for every (essentially) bounded input $u \in \mathcal{U}$, switching sequence $\sigma \in \mathbb{S}$, and starting point $(t_s, x_0) \in \mathbb{R} \times X$, the following properties hold along the corresponding trajectory $x(t) = \mathscr{T}_{\sigma,u,t_s}(t - t_0, x_0)$:

i) for each $t \in [t_{\sigma,i}, t_{\sigma,i+1}], i \in \mathbb{N}$, if $V_{q_{\sigma,i}}(x(t)) \geq \gamma_1(\|u\|)$, then $DV_{q_{\sigma,i}}(x(t)) \leq -\alpha(V_{q_{\sigma,i}}(x(t)))$;

ii) the relative variations among V_q's on periods of persistence $\mathscr{I}_j^\mathrm{p} = [t_{\sigma,i_j^{\mathscr{D}}+1}, t_{\sigma,i_{j+1}^{\mathscr{D}}}]$ satisfy

$$\limsup_{j\to\infty} \max_{t\in\mathscr{I}_j^\mathrm{p}} \mathrm{Var}_{t_{\sigma,i_j^{\mathscr{D}}+1}}^{t}[V_{q_{\sigma,i_j^{\mathscr{D}}}}, V_{q_{\sigma,i_\sigma^-(\tilde{t})}}](x) \leq \gamma_2(\|u\|). \tag{5.11}$$

Then, the switched system $\Sigma_u[\tau_\mathrm{p}, T_\mathrm{p}]$ has an asymptotic gain.

Proof: Let $\gamma_{\mathrm{d}}(s) = \max\{\tau_{\mathrm{p}}\alpha(\gamma_1(s)), \gamma_2(s)\}$. For an $\epsilon \geq 0$, we have the sets $\mathcal{B}_q(\epsilon) = \{\zeta \in \mathbb{R}^n : \tau_{\mathrm{p}}\alpha(V_q(\zeta)) \leq \gamma_{\mathrm{d}}(\|u\|) + \epsilon\}, q \in \mathbb{Q}$. We shall abbreviate $\mathcal{B}_q(0)$ to $\mathcal{B}_q$.

For each $\epsilon \geq 0$, we have the following claim.

Claim $\mathscr{C}(\epsilon)$: for each $j \in \mathbb{N}$, if $x(t_j^\circ) \in \mathcal{B}_{q_{\sigma,i_j^{\mathcal{D}}}}(\epsilon)$ for some $t_j^\circ \in [t_{\sigma,i_j^{\mathcal{D}}}, t_{\sigma,i_j^{\mathcal{D}}+1}]$, then $x(t) \in \mathcal{B}_{q_{\sigma,i_j^{\mathcal{D}}}}(\epsilon), \forall t \in [t_j^\circ, t_{\sigma,i_j^{\mathcal{D}}+1}]$.

We shall prove this claim in the paradigm of [136, Lemma 2.14]. Suppose that the claim is not true. Then, there are $t \in (t_j^\circ, t_{\sigma,i_j^{\mathcal{D}}+1}]$ and $\epsilon' > \epsilon/2$ such that $\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t))) > \gamma_{\mathrm{d}}(\|u\|)+\epsilon'$. Accordingly, $t_j^* = \inf\{t \in (t_j^\circ, t_{\sigma,i_j^{\mathcal{D}}+1}] : \tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t))) > \gamma_{\mathrm{d}}(\|u\|) + \epsilon'\}$ exists and $\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t_j^*))) \geq \gamma_{\mathrm{d}}(\|u\|) + \epsilon' > \tau_{\mathrm{p}}\alpha(\gamma_1(\|u\|))$. Thus, i) applies and $DV_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t_j^*)) \leq -\alpha(V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t_j^*))) < -\epsilon'/\tau_{\mathrm{p}}$. Hence, $V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t_j^*)) \leq V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(s))$ for some $s < t_j^*$. This contradicts to the minimality of t_j^*. Thus, the claim holds true.

Let us define t_j° to be $t_{\sigma,i_j^{\mathcal{D}}+1}$ if there is no $t \in [t_{\sigma,i_j^{\mathcal{D}}}, t_{\sigma,i_j^{\mathcal{D}}+1}]$ at which $\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t))) \leq \gamma_{\mathrm{d}}(\|u\|)$ and to be $\inf\{t \in [t_{\sigma,i_j^{\mathcal{D}}}, t_{\sigma,i_j^{\mathcal{D}}+1}] : \tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t))) \leq \gamma_{\mathrm{d}}(\|u\|)\}$ if such t exists. According to the above claim applied for $\epsilon = 0$, we have

$$\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_j^{\mathcal{D}}}}(x(t_j^\circ))) \geq \gamma_{\mathrm{d}}(\|u\|), \forall t \in [t_{\sigma,i_j^{\mathcal{D}}}, t_j^\circ]. \tag{5.12}$$

We shall show that

$$\limsup_{j\to\infty} \mathrm{Var}_{t_{\sigma,i_j^{\mathcal{D}}}}^{t_j^\circ} V_{q_{\sigma,i_j^{\mathcal{D}}}}(x) \leq \gamma_{\mathrm{d}}(\|u\|). \tag{5.13}$$

Indeed, suppose that (5.13) does not hold. Then, there is a number $\epsilon > 0$ and an infinite sequence $\{t_{\sigma,i_{j_k}^{\mathcal{D}}}\}_k$ such that

$$|V_{q_{\sigma,i_{j_k}^{\mathcal{D}}}}(x(t_{\sigma,i_{j_k}^{\mathcal{D}}})) - V_{q_{\sigma,i_{j_k}^{\mathcal{D}}}}(x(t_{j_k}^\circ))| > \gamma_{\mathrm{d}}(\|u\|) + \epsilon. \tag{5.14}$$

As $\alpha \in \mathcal{K}_\infty$ which is unbounded and continuous, there is a number $\delta = \delta(\epsilon) > 0$

such that

$$\tau_{\mathrm{p}}\alpha(s) \leq \gamma_{\mathrm{d}}(\|u\|) + \delta \Rightarrow s \leq \alpha^{-1}(\gamma_{\mathrm{d}}(\|u\|)/\tau_{\mathrm{p}}) + \epsilon/3. \tag{5.15}$$

As $\gamma_{\mathrm{d}}(s) \geq \gamma_2(s)$ and $\mathrm{Var}_{t_{\sigma,i_j^{\mathscr{D}}+1}}^{t_{\sigma,i_{j+1}^{\mathscr{D}}}}[V_{q_{\sigma,i_j^{\mathscr{D}}}}, V_{q_{\sigma,i_{j+1}^{\mathscr{D}}}}](x) \leq \max\{\mathrm{Var}_{t_{\sigma,i_j^{\mathscr{D}}+1}}^{t}[V_{q_{\sigma,i_j^{\mathscr{D}}}}, V_{q_{\sigma,i_\sigma^-(\tilde{t})}}](x) : t \in [t_{\sigma,i_j^{\mathscr{D}}+1}, t_{\sigma,i_{j+1}^{\mathscr{D}}}]\}$, condition (5.11) implies that

$$\limsup_{j\to\infty} |V_{q_{\sigma,i_j^{\mathscr{D}}}}(x(t_{\sigma,i_j^{\mathscr{D}}+1})) - V_{q_{\sigma,i_{j+1}^{\mathscr{D}}}}(x(t_{\sigma,i_{j+1}^{\mathscr{D}}}))| \leq \gamma_{\mathrm{d}}(\|u\|). \tag{5.16}$$

Hence, there is an $N_\epsilon \in \mathbb{N}$ such that for all $j > N_\epsilon$, we have

$$V_{q_{\sigma,i_{j+1}^{\mathscr{D}}}}(x(t_{\sigma,i_{j+1}^{\mathscr{D}}})) \leq V_{q_{\sigma,i_j^{\mathscr{D}}}}(x(t_{\sigma,i_j^{\mathscr{D}}+1})) + \gamma_{\mathrm{d}}(\|u\|) + \epsilon^*, \tag{5.17}$$

where $\epsilon^* = \min\{\delta(\epsilon), \epsilon/3\}$.

Consider the case where there is a number $p > N_\epsilon$ such that

$$\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_j^{\mathscr{D}}}}(x(t_{\sigma,i_j^{\mathscr{D}}+1}))) \leq \gamma_{\mathrm{d}}(\|u\|) + \epsilon^* \tag{5.18}$$

holds at $j = p$. We shall show that (5.18) also holds at all $j \geq p$. Indeed, from the above claim we have either $x(t_1) \in \mathcal{B}_{q_{\sigma,i_{p+1}^{\mathscr{D}}}}(\epsilon^*)$ for some $t_1 \in \mathscr{I}_{p+1}^{\mathscr{D}} = [t_{\sigma,i_{p+1}^{\mathscr{D}}}, t_{\sigma,i_{p+1}^{\mathscr{D}}+1}]$ and hence $x(t) \in \mathcal{B}_{q_{\sigma,i_{p+1}^{\mathscr{D}}}}(\epsilon^*), \forall t \in [t_1, t_{\sigma,i_{p+1}^{\mathscr{D}}+1}]$ or $\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_{p+1}^{\mathscr{D}}}}(x(t))) > \gamma_{\mathrm{d}}(\|u\|) + \epsilon^*, \forall t \in \mathscr{I}_{p+1}^{\mathscr{D}}$. Clearly, $\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_{p+1}^{\mathscr{D}}}}(x(t_{\sigma,i_{p+1}^{\mathscr{D}}+1}))) \leq \gamma_{\mathrm{d}}(\|u\|) + \epsilon^*$ in the former case. In the latter case, by definition of γ_{d}, we have $\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i_{p+1}^{\mathscr{D}}}}(x(t))) > \gamma_{\mathrm{d}}(\|u\|) + \epsilon^* \geq \tau_{\mathrm{p}}\alpha(\gamma_1(\|u\|)), \forall t \in \mathscr{I}_{p+1}^{\mathscr{D}}$. Thus, from i) and $t_{\sigma,i_{p+1}^{\mathscr{D}}+1} - t_{\sigma,i_{p+1}^{\mathscr{D}}} \geq \tau_{\mathrm{p}}$, we have

$$\begin{aligned} V_{q_{\sigma,i_{p+1}^{\mathscr{D}}}}(x(t_{\sigma,i_{p+1}^{\mathscr{D}}})) - V_{q_{\sigma,i_{p+1}^{\mathscr{D}}}}(x(t_{\sigma,i_{p+1}^{\mathscr{D}}+1})) &\geq \int_{t_{\sigma,i_{p+1}^{\mathscr{D}}}}^{t_{\sigma,i_{p+1}^{\mathscr{D}}+1}} \alpha(V_{q_{\sigma,i_{p+1}^{\mathscr{D}}}}(x(t)))dt \\ &\geq \tau_{\mathrm{p}} \min_{t\in\mathscr{I}_{p+1}^{\mathscr{D}}}\{\alpha(V_{q_{\sigma,i_{p+1}^{\mathscr{D}}}}(x(t)))\} \geq \gamma_{\mathrm{d}}(\|u\|) + \epsilon^*. \end{aligned} \tag{5.19}$$

Combining (5.19) and (5.17) applied at $j = p$, we obtain

$$V_{q_{\sigma,i^{\mathscr{D}}_{p+1}}}(x(t_{\sigma,i^{\mathscr{D}}_{p+1}+1})) \leq V_{q_{\sigma,i^{\mathscr{D}}_{p}}}(x(t_{\sigma,i^{\mathscr{D}}_{p}+1})). \tag{5.20}$$

Substituting (5.20) into (5.18) applied at $j = p$, it follows that (5.18) also holds true for $j = p + 1$.

In combination, (5.18) holds at $j = p+1$. Continuation of this process shows that (5.18) holds for all $j \geq p$. As $\epsilon^* \leq \delta(\epsilon)$, it follows from (5.18) and (5.15) that

$$V_{q_{\sigma,i^{\mathscr{D}}_{j}}}(x(t_{\sigma,i^{\mathscr{D}}_{j}+1})) \leq \alpha^{-1}(\gamma_{\mathscr{D}}(\|u\|)/\tau_{\mathrm{p}}) + \epsilon/3, \forall j > N_{\epsilon}. \tag{5.21}$$

Combining (5.17) and (5.21), we obtain

$$V_{q_{\sigma,i^{\mathscr{D}}_{j+1}}}(x(t_{\sigma,i^{\mathscr{D}}_{j+1}})) \leq \alpha^{-1}(\gamma_{\mathrm{d}}(\|u\|)/\tau_{\mathrm{p}}) + \gamma_{\mathrm{d}}(\|u\|) + 2\epsilon/3, \tag{5.22}$$

for all $j > N_{\epsilon}$. This coupled with (5.12) yields

$$|V_{q_{\sigma,i^{\mathscr{D}}_{j+1}}}(x(t_{\sigma,i^{\mathscr{D}}_{j+1}})) - V_{q_{\sigma,i^{\mathscr{D}}_{j+1}}}(x(t^{\circ}_{i^{\mathscr{D}}_{j+1}}))| \leq \gamma_{\mathrm{d}}(\|u\|) + 2\epsilon/3, \tag{5.23}$$

for all $j > N_{\epsilon}$, which contradicts to (5.14).

In the case (5.18) does not hold for all $j > N_{\epsilon}$, we have

$$\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i^{\mathscr{D}}_{j+1}}}(x(t))) > \gamma_{\mathrm{d}}(\|u\|) + \epsilon^*, \forall t \in \mathscr{I}^{\mathscr{D}}_{j+1}, j > N_{\epsilon}, \tag{5.24}$$

which, through the argument leading to (5.19), leads to

$$V_{q_{\sigma,i^{\mathscr{D}}_{j}}}(x(t_{\sigma,i^{\mathscr{D}}_{j}+1})) \leq V_{q_{\sigma,i^{\mathscr{D}}_{j}}}(x(t_{\sigma,i^{\mathscr{D}}_{j}})) - \gamma_{\mathrm{d}}(\|u\|) - \epsilon^*, \forall j > N_{\epsilon}. \tag{5.25}$$

Combining (5.25) and (5.17), we arrive at

$$V_{q_{\sigma,i^{\mathcal{D}}_{j+1}}}(x(t_{\sigma,i^{\mathcal{D}}_{j+1}})) \leq V_{q_{\sigma,i^{\mathcal{D}}_{j}}}(x(t_{\sigma,i^{\mathcal{D}}_{j}})), \forall j > N_{\epsilon}. \tag{5.26}$$

In addition, as $V_{q_{\sigma,i^{\mathcal{D}}_{j_k}}}(x(t_{\sigma,i^{\mathcal{D}}_{j_k}+1})) \leq V_{q_{\sigma,i^{\mathcal{D}}_{j_k}}}(x(t^{\circ}_{i^{\mathcal{D}}_{j_k}}))$, (5.17) and (5.14) imply that

$$\begin{aligned} V_{q_{\sigma,i^{\mathcal{D}}_{j_k+1}}}(x(t_{\sigma,i^{\mathcal{D}}_{j_k+1}})) &\leq V_{q_{\sigma,i^{\mathrm{d}}_{j_k}}}(x(t_{\sigma,i^{\mathcal{D}}_{j_k}+1})) + \gamma_{\mathrm{d}}(\|u\|) + \epsilon^* \\ &\leq V_{q_{\sigma,i^{\mathcal{D}}_{j_k}}}(x(t^{\circ}_{i^{\mathcal{D}}_{j_k}})) + \gamma_{\mathrm{d}}(\|u\|) + \epsilon^* \\ &\leq V_{q_{\sigma,i^{\mathcal{D}}_{j_k}}}(x(t_{\sigma,i^{\mathcal{D}}_{j_k}})) - (\epsilon - \epsilon^*) \end{aligned} \tag{5.27}$$

for all k satisfying $j_k > N_{\epsilon}$. Using (5.27) and applying (5.26) successively from j_k to j_{k+1}, we obtain

$$V_{q_{\sigma,i^{\mathcal{D}}_{j_{k+1}}}}(x(t_{\sigma,i^{\mathcal{D}}_{j_{k+1}}})) \leq V_{q_{\sigma,i^{\mathcal{D}}_{j_k}}}(x(t_{\sigma,i^{\mathcal{D}}_{j_k}})) - 2\epsilon/3, \forall j_k > N_{\epsilon}. \tag{5.28}$$

Thus, $V_{q_{\sigma,i^{\mathcal{D}}_{j_k}}}(x(t_{\sigma,i^{\mathcal{D}}_{j_k}})) < 0$ for sufficiently large k, which is a contradiction.

As a result, (5.13) holds true.

The proof of the convergence of $x(t)$ is now in order. We first show the convergence of values of x at end times of dwell-time switching events, i.e.,

$$\limsup_{j\to\infty} \tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i^{\mathcal{D}}_{j}}}(x(t_{\sigma,i^{\mathcal{D}}_{j}+1}))) \leq \gamma_{\mathrm{d}}(\|u\|). \tag{5.29}$$

Suppose that (5.29) is not true. Then, there are $\epsilon > 0$ and sequence $\{j_k\}_k$ such that

$$\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i^{\mathcal{D}}_{j_k}}}(x(t_{\sigma,i^{\mathcal{D}}_{j_k}+1}))) > \gamma_{\mathrm{d}}(\|u\|) + \epsilon, \forall k \in \mathbb{N}. \tag{5.30}$$

According to definition of t°_{j} and claim $\mathscr{C}(\epsilon)$, we have $t^{\circ}_{i^{\mathcal{D}}_{j_k}} = t_{\sigma,i^{\mathcal{D}}_{j_k}+1}$ and hence

(5.30) implies that

$$\tau_{\mathrm{p}}\alpha(V_{q_{\sigma,i^{\mathscr{D}}_{j_k}}}(x(t))) > \gamma_{\mathrm{d}}(\|u\|) + \epsilon, \forall t \in \mathscr{I}^{\mathscr{D}}_{j_k}. \tag{5.31}$$

As $\gamma_{\mathrm{d}}(s) \geq \tau_{\mathrm{p}}\alpha(\gamma_1(s))$, (5.31) implies that $V_{q_{\sigma,i^{\mathscr{D}}_{j_k}}}(x(t)) > \gamma_1(\|u\|)$ and hence, by i), we have

$$DV_{q_{\sigma,i^{\mathscr{D}}_{j_k}}}(x(t)) < -\alpha(V_{q_{\sigma,i^{\mathscr{D}}_{j_k}}}(x(t))), \forall t \in \mathscr{I}^{\mathscr{D}}_{j_k}. \tag{5.32}$$

Taking integrals of both sides of (5.32) and using (5.31) yields

$$\mathrm{Var}_{t_{\sigma,i^{\mathscr{D}}_{j_k}}}^{t_{\sigma,i^{\mathscr{D}}_{j_k}+1}} V_{q_{\sigma,i^{\mathscr{D}}_{j_k}}}(x(t)) \geq \int_{t_{\sigma,i^{\mathscr{D}}_{j_k}}}^{t_{\sigma,i^{\mathscr{D}}_{j_k}+1}} \alpha(V_{q_{\sigma,i^{\mathscr{D}}_{j_k}}}(x(t)))dt \geq \gamma_{\mathrm{d}}(\|u\|) + \epsilon. \tag{5.33}$$

Taking limits of both side of (5.33), we obtain

$$\limsup_{k\to\infty} \mathrm{Var}_{t_{\sigma,i^{\mathscr{D}}_{j_k}}}^{t_{\sigma,i^{\mathscr{D}}_{j_k}+1}} V_{q_{\sigma,i^{\mathscr{D}}_{j_k}}}(x(t)) \geq \gamma_{\mathrm{d}}(\|u\|) + \epsilon, \tag{5.34}$$

which contradicts to (5.13). Thus, (5.29) holds true.

We now examine the converging behavior of the sequence $\{x(t_j)\}_j$, where $\{t_j\}_j \subset [t_0, \infty)$ is an arbitrary divergent sequence. Let us divide $\{t_j\}_j$ into two subsequences $\{t^{\mathscr{D}}_j\}_j$ and $\{t^{\mathrm{P}}_j\}_j$, where the first subsequence consists of all elements of $\{t_j\}_j$ belonging to dwell-time switching intervals and the second one consists of the rest in $\{t_j\}_j$.

For a time $t \in [t_0, \infty)$, there is an interval $[t_{\sigma,i^{\mathscr{D}}_j}, t_{\sigma,i^{\mathscr{D}}_{j+1}}]$ between starting times of two consecutive dwell-time switching events that contains t. Then, we define $i_{\mathscr{D}}(t) = i^{\mathscr{D}}_j$, $\mathscr{I}^{\mathscr{D}}(t) = [t_{\sigma,i^{\mathscr{D}}_j}, t_{\sigma,i^{\mathscr{D}}_j+1}]$, and $\mathscr{I}^{\mathrm{P}}(t) = [t_{\sigma,i^{\mathscr{D}}_j+1}, t_{\sigma,i^{\mathscr{D}}_{j+1}}]$. If $t \in \mathscr{I}^{\mathscr{D}}(t)$, then we further define $t^{\circ}_{\mathscr{D}}(t) = t^{\circ}_j$. Recall that $\tilde{t} = t - t_0$. For the sequence $\{t^{\mathrm{P}}_j\}_j$, we have

$$V_{q_{\sigma,i^{-}_{\sigma}(\tilde{t}^{\mathrm{P}}_j)}}(x(t^{\mathrm{P}}_j)) \leq V_{q_{\sigma,i_{\mathscr{D}}(t^{\mathrm{P}}_j)}}(x(t_{\sigma,i_{\mathscr{D}}(t^{\mathrm{P}}_j)+1})) + \max_{t\in\mathscr{I}^{\mathrm{P}}(t^{\mathrm{P}}_j)} \mathrm{Var}^{t}_{t_{\sigma,i_{\mathscr{D}}(t^{\mathrm{P}}_j)}}[V_{q_{\sigma,i_{\mathscr{D}}(t^{\mathrm{P}}_j)}}, V_{q_{\sigma,i^{-}_{\sigma}(\tilde{t})}}](x). \tag{5.35}$$

Taking the limits of both sides of (5.35) and using (5.29) and (5.11), we obtain

$$\limsup_{j\to\infty} V_{q_{\sigma,i_\sigma^-(\tilde{t}_j^{\mathrm{p}})}}(x(t_j^{\mathrm{p}})) \le \gamma_{\mathrm{d}}(\|u\|) + \gamma_2(\|u\|). \tag{5.36}$$

We now consider the sequence $\{t_j^{\mathcal{D}}\}_j$. As we have shown, $V_{q_{\sigma,i_{\mathscr{D}}(t_j^{\mathcal{D}})}}(x(t))$ is decreasing on $[t_{\sigma,i_{\mathscr{D}}(t_j^{\mathcal{D}})}, t_{\mathscr{D}}^{\circ}(t_j^{\mathcal{D}})]$ and is bounded by $\alpha^{-1}(\gamma_{\mathrm{d}}(\|u\|)/\tau_{\mathrm{p}})$ on $[t_{\mathscr{D}}^{\circ}(t_j^{\mathcal{D}}), t_{\sigma,i_{\mathscr{D}}(t_j^{\mathcal{D}})+1}]$. Let $\gamma_3(\|u\|) = \alpha^{-1}(\gamma_{\mathrm{d}}(\|u\|)/\tau_{\mathrm{p}})$. We have

$$V_{q_{\sigma,i_{\mathscr{D}}(t_j^{\mathcal{D}})}}(x(t_j^{\mathcal{D}})) \le \max\{V_{q_{\sigma,i_{\mathscr{D}}(t_j^{\mathcal{D}})}}(x(t_{\sigma,i_{\mathscr{D}}(t_j^{\mathcal{D}})})), \gamma_3(\|u\|)\}. \tag{5.37}$$

Since $t_{\sigma,i_{\mathscr{D}}(t_j^{\mathcal{D}})}$ is the starting time of a dwell-time switching interval which is also the end time of a period of persistence, taking the limits of both sides of (5.37) and using (5.36), we obtain

$$\limsup_{j\to\infty} V_{q_{\sigma,i_{\mathscr{D}}(t_j^{\mathcal{D}})}}(x(t_j^{\mathcal{D}})) \le \gamma^*(\|u\|), \tag{5.38}$$

where $\gamma^*(s) = \max\{\gamma_{\mathrm{d}}(s) + \gamma_2(s), \gamma_3(s)\}$.

On the other hand, from (5.10), we have $\underline{\alpha}(\|x(t)\|) \le V_q(x(t)), \forall t \in [t_0, \infty)$. As $\underline{\alpha}$ is continuous, combining (5.36) and (5.38), we arrive at

$$\begin{aligned}\limsup_{j\to\infty} \|x(t_j)\| \le \max\Big\{&\underline{\alpha}^{-1}\Big(\limsup_{j\to\infty} V_{q_{\sigma,i_\sigma^-(\tilde{t}_j^{\mathrm{p}})}}(x(t_j^{\mathrm{p}}))\Big),\\ &\underline{\alpha}^{-1}\Big(\limsup_{j\to\infty} V_{q_{\sigma,i_{\mathscr{D}}(t_j^{\mathcal{D}})}}(x(t_j^{\mathcal{D}}))\Big)\Big\} \le \underline{\alpha}^{-1}(\gamma^*(\|u\|)).\end{aligned} \tag{5.39}$$

Let γ be the class-$\mathcal{K}_\infty$ function defined as $\gamma(s) = \underline{\alpha}^{-1}(\gamma^*(s))$. Then, (5.39) gives $\limsup_{j\to\infty} \|x(t_j)\| \le \gamma(\|u\|)$ which, as $\{t_j\}_j$ is arbitrary, further implies that $\limsup_{t\to\infty} \|x(t)\| \le \gamma(\|u\|)$. Finally, as γ is independent of x_0 and σ, the conclusion of the theorem follows accordingly. ■

Similar to the case of invariance principles, condition (5.11) imposes the boundedness of ultimate variations of auxiliary functions. Obviously, this is a necessary condition for stability of any dynamical system. In the classical Lyapunov theory of dynamical systems [56] and in switched systems satisfying the switching decreasing condition [22], this condition automatically holds for $\gamma_2 \equiv 0$. On the other hand, this condition is due to our need for estimations of increments of auxiliary functions on destabilizing periods without involving the number of switches in these periods.

5.3 Stability of Switched Time-delay Systems

5.3.1 System with Input

Let us recall some notations from Chapter 4 as follows. The delay time $T_r > 0$ is a fixed number. The notation $\mathscr{C}_r$ stands for the Banach space of continuous functions mapping the interval $[-T_r, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For a time function $x : [-T_r, \infty) \to \mathbb{R}^n$ and for each $t \in \mathbb{R}^+$, x_t is the function in $\mathscr{C}_r$ defined as $x_t(\varsigma) = x(t+\varsigma), \varsigma \in [-T_r, 0]$. For a function $V : \mathbb{R}^n \to \mathbb{R}$, the superscript $\natural$ is to indicate the function $V^\natural : \mathscr{C}_r \to \mathbb{R}$ defined as

$$V^\natural(\phi) = \sup_{\varsigma \in [-T_r, 0]} V(\phi(\sigma)), \phi \in \mathscr{C}_r. \tag{5.40}$$

The Dini and D_σ derivatives of V and $V^\natural$ along a trajectory $x(t), t \in [-T_r, \infty)$ in $\mathbb{R}^n$ are defined as in Section 4.3.3.

Consider dynamical systems described by the following functional differential equations:

$$\dot{x}(t) = f_q(x_t, w(t)), q \in \mathbb{Q}, \tag{5.41}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^d$ is the input, $f_q : \mathscr{C}_r \times \mathbb{R}^d \to \mathbb{R}^n, q \in \mathbb{Q}$ are continuous functions taking bounded sets in $\mathscr{C}_r \times \mathbb{R}^d$ to bounded sets in $\mathbb{R}^n$, and $\mathbb{Q}$ is a finite discrete set.

Suppose that w belongs to the space $\mathcal{L}_{\infty}^{d}$ of measurable locally essentially bounded functions mapping $\mathbb{R}$ into $\mathbb{R}^d$. Then, for each $q \in \mathbb{Q}$ and for any $\phi \in \mathscr{C}_r$, there is a unique maximally extended solution $x(t; \phi, w, q)$ of the following initial value problem [59, 121]:

$$\begin{cases} \dot{x}(t) &= f_q(x_t, w(t)), \\ x(\varsigma) &= \phi(\varsigma), \forall \varsigma \in [-T_r, 0]. \end{cases} \tag{5.42}$$

Suppose that the vector fields $f_q, q \in \mathbb{Q}$ are forward complete, i.e., for every $\phi \in \mathscr{C}_r, q \in \mathbb{Q}, w \in \mathcal{L}_{\infty}^{d}$, the solution $x(t; \phi, w, q)$ to the problem (5.42) exists and is unique for all $t \in \mathbb{R}^+$.

Thus, according to (2.14) in Section 2.4.3, for each function $\phi \in \mathscr{C}_r$, $t_s \in \mathbb{R}^+$, and switching sequence $\sigma \in \mathbb{S}$, the following recursively defined trajectory in $\mathbb{R}^n$ exists, is continuous and uniquely defined:

$$\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi) = \begin{cases} x(t; \phi, w, q_{\sigma,i_\sigma^-(t_s)}) \ \textbf{if}\ t \in [0, \tau_{\sigma,i_\sigma^-(t+t_s)+1} - t_s] \\ x(t + t_s - \tau_{\sigma,i_\sigma^-(t+t_s)}; \phi_{\tau_{\sigma,i_\sigma^-(t+t_s)}}, w(\cdot + \tau_{\sigma,i_\sigma^-(t+t_s)}), q_{\sigma,i_\sigma^-(t+t_s)}) \\ \qquad\qquad \textbf{if}\ t \geq \tau_{\sigma,i_\sigma^-(t+t_s)+1} - t_s, \end{cases} \tag{5.43}$$

where $\phi_{\tau_{\sigma,i_\sigma^-(t+t_s)}} \stackrel{\text{def}}{=} (\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(\tau_{\sigma,i_\sigma^-(t+t_s)}, \phi))_{\tau_{\sigma,i_\sigma^-(t+t_s)}} \in \mathscr{C}_r$ and $w(\cdot + \tau_{\sigma,i_\sigma^-(t+t_s)})(t) = w(t + \tau_{\sigma,i_\sigma^-(t+t_s)}), t \in \mathbb{R}^+$.

In summary, we have the following collection for a model of switched time-delay system with input with the meanings of symbols are obvious:

$$\Sigma_{\mathcal{U},\mathcal{D}} = \{\mathbb{R}^+, \mathbb{Q}, \mathscr{C}_r, \mathcal{L}_{\infty}^{d}, \{f_q\}_{q\in\mathbb{Q}}, \mathbb{S}\}. \tag{5.44}$$

5.3.2 Stability Notions and Lyapunov-Razumikhin Functions

Adopting the stability notions in continuous time-delay systems [59] and continuous delay-free dynamical systems [138]. Hereafter, we shall call $\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi)$ a trajectory of the switched time-delay system with input $\Sigma_{\mathcal{U},\mathcal{D}}$. We have the following analogous notions for switched time-delay systems with input.

Definition 5.3.1 *A trajectory $\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi)$ of a switched time-delay system with input $\Sigma_{\mathcal{U},\mathcal{D}}$ is said to be bounded if there is a number $R(\phi,w,t_s,\sigma)>0$ such that $\|\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi)\|\le R, \forall t\in[-T_r,\infty)$.*

Definition 5.3.2 *The family of trajectories $\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi), \phi\in\mathscr{C}_r, t_s\in\mathbb{R}^+, w\in\mathcal{L}^d_\infty, \sigma\in\mathbb{S}$ of a switched time-delay system with input $\Sigma_{\mathcal{U},\mathcal{D}}$ is said to be switching-uniform bounded if for every number $r>0$, there is a real number $R(r,w)$ such that for any $\phi\in\mathscr{C}_r$ satisfying $\|\phi\|\le r$, we have $\|\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi)\|\le R(r,w), \forall t\in\mathbb{R}^+, \sigma\in\mathbb{S}, t_s\in\mathbb{R}^+$.*

Definition 5.3.3 *The switched time-delay system $\Sigma_{\mathcal{U},\mathcal{D}}$ is said to have a switching-uniform asymptotic gain χ if χ is a class-$\mathcal{K}_\infty$ function and*

$$\limsup_{t\to\infty}\|\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi)\|\le\chi(\|w\|), \tag{5.45}$$

for all $w\in\mathcal{L}^d_\infty, \phi\in\mathscr{C}_r, t_s\in\mathbb{R}^+$, and $\sigma\in\mathbb{S}$.

Consider a trajectory $\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi)$ of $\Sigma_{\mathcal{U},\mathcal{D}}$. We have the following assumption:

Assumption 5.3.1 *There are continuous functions $V_q:\mathbb{R}^n\to\mathbb{R}^+, q\in\mathbb{Q}$, a non-decreasing continuous function $p:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $p(s)>s$ for $s>0$, and class-$\mathcal{K}_\infty$ functions α_1,α_2,α, and γ_1 such that*

$$\alpha_1(\|x\|)\le V_q(x)\le\alpha_2(\|x\|), \forall x\in\mathbb{R}^n, q\in\mathbb{Q}, \tag{5.46}$$

and along the trajectory $x(t)\stackrel{\text{def}}{=}\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi)$, we have:

$$D_\sigma V_{q_{\sigma,i^-_\sigma(t)}}(x(t))\le-\alpha(V_{q_{\sigma,i^-_\sigma(t)}}(x(t)))+\gamma_1(\|w\|) \tag{5.47}$$

whenever $p(V_{q_{\sigma,i^-_\sigma(t)}}(x(t)))\ge V^\natural_{q_{\sigma,i^-_\sigma(t)}}(x_t)$.

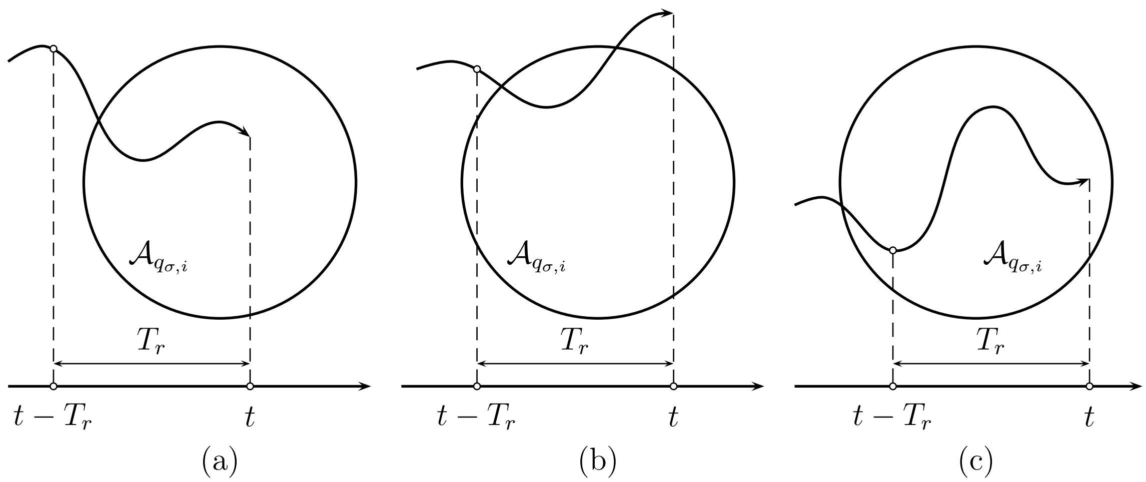

Figure 5.1: Relative positions of the trajectory $\mathscr{T}^{\mathcal{D}}_{\sigma,w,t_s}(t,\phi)$ with respect to $\mathcal{A}_{q_{\sigma,i}}$

We shall call the functions $V_q, q \in \mathbb{Q}$ satisfying Assumption 5.3.1 the Lyapunov-Razumikhin functions for system $\Sigma_{\mathcal{U},\mathcal{D}}$. For a trajectory-based function $V(x(t))$, the Razumikhin condition refers to $p(V(x(t))) \geq \sup_{\varsigma\in[-T_r,0]} V(x(t+\varsigma)) = V^{\natural}(x_t)$.

5.3.3 Boundedness

Theorem 5.3.1 *Suppose that for fixed $\sigma \in \mathbb{S}_{\mathcal{P}}[\tau_{\mathrm{p}}, T_{\mathrm{p}}]$ and $w \in \mathcal{L}^d_\infty$, the switched time-delay system with input $\Sigma_{\mathcal{U},\mathcal{D}}$ satisfies Assumption 5.3.1 for arbitrary $\phi \in \mathscr{C}_r$. In addition, $T_r \geq T_{\mathrm{p}}$ and for the sequence of dwell-time switching events $\{(q_{\sigma,i^{\mathcal{D}}_j}, \Delta\tau_{\sigma,i^{\mathcal{D}}_j})\}_j$ of σ, we have*

$$V^{\natural}_{q_{\sigma,i^{\mathcal{D}}_j}}(x_{\tau_{\sigma,i^{\mathcal{D}}_j}}) \geq V^{\natural}_{q_{\sigma,i^{\mathcal{D}}_k}}(x_{\tau_{\sigma,i^{\mathcal{D}}_k}}), \forall k, j \in \mathbb{N}, k > j. \tag{5.48}$$

Then, the trajectory $x(t) = \mathscr{T}_{\sigma,w,t_s}(t,\phi)$ is bounded.

Proof: We shall prove the boundedness of $x(t)$ by considering the behavior of the following composite Lyapunov function:

$$V^{\natural}_{\mathscr{C}}(t) = \begin{cases} \alpha_2(\alpha_1^{-1}(V^{\natural}_{q_{\sigma,i^{\mathcal{D}}_j}}(x_t)) \textbf{ if } t \in [\tau_{\sigma,i^{\mathcal{D}}_j}, \tau_{\sigma,i^{\mathcal{D}}_j+1}) \\ \alpha_2(\alpha_1^{-1}(V^{\natural}_{q_{\sigma,i^{\mathcal{D}}_j}}(x(\tau_{\sigma,i^{\mathcal{D}}_j})))) \textbf{ if } t \in [\tau_{\sigma,i^{\mathcal{D}}_j+1}, \tau_{\sigma,i^{\mathcal{D}}_{j+1}}) \end{cases}, \tag{5.49}$$

where we have supposed that the first switching event of σ is dwell-time.

Let γ be any class–$\mathcal{K}_\infty$ function satisfying $\gamma(s) > \gamma_1(s), s > 0$. Consider the number $r_\gamma = \alpha^{-1}(\gamma(\|w\|))$, and define, for each $q \in \mathbb{Q}$, the set $\mathcal{A}_q = \{\zeta \in \mathbb{R}^n : V_q(\zeta) \leq r_\gamma\}$.

Consider a non-zero running time switching event $(q_{\sigma,i}, \Delta\tau_{\sigma,i}), i \in \mathbb{N}$ and a time $t \in (\tau_{\sigma,i}, \tau_{\sigma,i+1})$. We have the following cases.

Case 1: $x(t+\varsigma) \in \mathcal{A}_{q_{\sigma,i}}, \forall \varsigma \in [-T_r, 0]$, Figure 5.1(c). It is obvious that, in this case, we have $V^\natural_{q_{\sigma,i}}(x_t) \leq r_\gamma$. We shall show that $x(s) \in \mathcal{A}_{q_{\sigma,i}}, \forall s \in [t, \tau_{\sigma,i}]$. In fact, suppose that the converse holds, i.e., there is $s_1 \in [t, \tau_{\sigma,i}], s_1 > t$ such that $x(s_1) \notin \mathcal{A}_{q_{\sigma,i}}$. Since $V_q(x(t))$ and $x(t)$ are continuous with respect to time t, this implies that there is a number $\epsilon > 0$ such that $V_{q_{\sigma,i}}(x(s_1)) > r_\gamma + \epsilon$. Let $s^* = \inf\{s \in [t, \tau_{\sigma,i}] : V_{q_{\sigma,i}}(x(s)) > r_\gamma + \epsilon\}$. Due to the continuity of solutions, we have $s^* < \tau_{\sigma,i}$ and $V_{q_{\sigma,i}}(x(s^*)) \geq r_\gamma + \epsilon$. Thus, it follows from definition of r_γ that

$$\gamma_1(\|w\|) < \gamma(\|w\|) < \alpha(r_\gamma + \epsilon) \leq \alpha(V_{q_{\sigma,i}}(x(s^*))). \tag{5.50}$$

Moreover, as the current case implies that $r_\gamma \geq V^\natural_{q_{\sigma,i}}(x_t)$, we have $V_{q_{\sigma,i}}(x(s^*)) \geq V^\natural_{q_{\sigma,i}}(x_{s^*})$. Otherwise, the minimality of s^* is violated. Thus, using Assumption 5.3.1 with (5.50) and the fact that $s^* \notin \{\tau_{\sigma,i}\}_i$, we obtain

$$D^- V_{q_{\sigma,i}}(x(s^*)) \leq D_\sigma V_{q_{\sigma,i}}(x(s^*)) \leq -\alpha(V_{q_{\sigma,i}}(x(s^*))) + \gamma_1(\|w\|) < 0. \tag{5.51}$$

However, (5.51) implies that there is a number $s_0 \in (t, s^*)$ such that $V_{q_{\sigma,i}}(x(s_0)) \geq V_{q_{\sigma,i}}(x(s^*)) \geq r_\gamma + \epsilon$, which contradicts to the minimality of s^*. Thus, $x(s) \in \mathcal{A}_{q_{\sigma,i}}, \forall s \in [t, \tau_{\sigma,i+1}]$. Accordingly, $V^\natural_{q_{\sigma,i}}(x_s) \leq r_\gamma, \forall s \in [t, \tau_{\sigma,i+1}]$.

Case 2: $x(t+\varsigma) \notin \mathcal{A}_{q_{\sigma,i}}$ for some $\varsigma \in [-T_r, 0]$, i.e., $V^\natural_{q_{\sigma,i}} > r_\gamma$, Figure 5.1(a)–(b). By definition of $V^\natural_q$, there is a number $\varsigma(t) \in [-T_r, 0]$ such that $V^\natural_{q_{\sigma,i}}(x_t) = V_{q_{\sigma,i}}(x(t+\varsigma(t)))$. We further have the following consideration.

Case 2a: $\varsigma(t) = 0$. In this case, we have $V_{q_{\sigma,i}}(x(t)) > r_\gamma$, and hence $\gamma_1(\|w\|) <$

$\alpha(V_{q_{\sigma,i}}(x(t)))$. Moreover, as $p(s) > s$, $\varsigma(t) = 0$ also implies that $p(V_{q_{\sigma,i}}(x(t))) > V_{q_{\sigma,i}}(x(t)) \geq V^{\natural}_{q_{\sigma,i}}(x_t)$. Hence, using Assumption 5.3.1, we obtain

$$D^+V_{q_{\sigma,i}}(x(t)) \leq D_\sigma V_{q_{\sigma,i}}(x(t)) \leq -\alpha(V_{q_{\sigma,i}}(x(t))) + \gamma_1(\|w\|) < 0. \tag{5.52}$$

As $V_{q_{\sigma,i}}$ and x are continuous, (5.52) implies the existence of a number $h > 0$ such that $V_{q_{\sigma,i}}(x(t+\epsilon)) \leq V_{q_{\sigma,i}}(x(t)), \forall \epsilon \in [0,h)$ and hence $V^{\natural}_{q_{\sigma,i}}(x_{t+\epsilon}) \leq V^{\natural}_{q_{\sigma,i}}(x_t), \forall \epsilon \in [0,h)$. Thus, $D^+V^{\natural}_{q_{\sigma,i}}(x_t) \leq 0$ accordingly.

Case 2b: $\varsigma(t) < 0$. Since $V_{q_{\sigma,i}}$ and x are continuous, there is a number $\epsilon > 0$ such that $V_{q_{\sigma,i}}(x(t+\varsigma(t))) > V_{q_{\sigma,i}}(x(t)) + \epsilon$. Moreover, by the continuity of $V_{q_{\sigma,i}}$ and x again, there also exists a number $h > 0$ such that $|V_{q_{\sigma,i}}(x(t+\varepsilon)) - V_{q_{\sigma,i}}(x(t))| < \epsilon/2, \forall \varepsilon \in [0,h)$. Thus, $V_{q_{\sigma,i}}(x(t+\varepsilon)) + \epsilon/2 \leq V_{q_{\sigma,i}}(x(t+\varsigma(t))), \forall \varepsilon \in [0,h)$ and $V^{\natural}_{q_{\sigma,i}}(x_{t+\epsilon}) = V^{\natural}_{q_{\sigma,i}}(x(t)), \forall \epsilon \in [0,h)$ accordingly. Hence, $D^+V^{\natural}_{q_{\sigma,i}}(x(t)) = 0$.

Combining Cases 2a and 2b, we conclude that $D^+V^{\natural}_{q_{\sigma,i}}(x(t)) \leq 0$ if $x(t+\varsigma) \notin \mathcal{A}_{q_{\sigma,i}}$ for some $\varsigma \in [-T_r, 0]$. Since $V_{\mathscr{C}}(t)^{\natural} = V^{\natural}_{q_{\sigma,i^{\mathscr{D}}_j}}(x_t)$ for $t \in [\tau_{\sigma,i^{\mathscr{D}}_j}, \tau_{\sigma,i^{\mathscr{D}}_j+1})$, combining Cases 1 and 2, it follows that either $V^{\natural}_{\mathscr{C}}(t) \leq r_\gamma$ or $D^+V^{\natural}_{\mathscr{C}}(x_t) \leq 0$ on $\mathbb{T}_{\mathscr{D}} = \cup_j[\tau_{\sigma,i^{\mathscr{D}}_j}, \tau_{\sigma,i^{\mathscr{D}}_j+1}]$. This combined with condition (5.48) shows that $V^{\natural}_{\mathscr{C}}(t)$ is bounded by $\max\{r_\gamma, V^{\natural}_{\mathscr{C}}(0)\}$ on $\mathbb{T}_{\mathscr{D}}$. Clearly, by definition (5.49), $V^{\natural}_{\mathscr{C}}(t)$ is bounded by $\max\{V^{\natural}_{\mathscr{C}}(\tau_{\sigma,i^{\mathscr{D}}_j}) : j \in \mathbb{N}\}$ on $\mathbb{R}^+ \backslash \mathbb{T}_{\mathscr{D}}$. Hence, in conclusion $V^{\natural}_{\mathscr{C}}(t)$ is bounded by $\max\{r_\gamma, V^{\natural}_{\mathscr{C}}(0)\}$.

On the other hand, condition (5.46) implies that

$$V_q(x(t)) \leq \alpha_2(\alpha_1^{-1}(V_{q_{\sigma,i^{\mathscr{D}}_j}}(x(\tau_{\sigma,i^{\mathscr{D}}_j})))), \forall q \in \mathbb{Q}, j \in \mathbb{N}. \tag{5.53}$$

As such, it is straightforward that $V_{q_{\sigma,i^-_\sigma(t)}}(x(t)) \leq V^{\natural}_{\mathscr{C}}(t), \forall t \in \mathbb{R}^+$.

Consider a number $r > 0$ and a function $\phi \in \mathscr{C}_r$ satisfying $\|\phi\| \le r$. From condition (5.46), we have

$$\begin{aligned}\|x(t)\| &\le V_{q_{\sigma,i_\sigma^-(t)}}(x(t)) \le V_{\mathscr{C}}^\natural(t) \le \max\{r_\gamma, V_{\mathscr{C}}^\natural(0)\}\\ &\le \max\{r_\gamma, \alpha_2(\alpha_1^{-1}(V^\natural_{q_{\sigma,i_0^{\mathscr{D}}}}(\phi)))\} \le \max\{r_\gamma, \alpha_2(\alpha_1^{-1}(\alpha_2(r)))\}, \forall t \in \mathbb{R}^+. \quad (5.54)\end{aligned}$$

Thus, $x(t)$ is bounded. ■

5.3.4 Lyapunov-Razumikhin Functions and SUAG

Theorem 5.3.2 *Suppose that for every fixed $\sigma \in \mathbb{S}_{\mathscr{P}}[\tau_{\rm p}, T_{\rm p}]$ and $w \in \mathcal{L}_\infty^d$, the switched time-delay system with input $\Sigma_{\mathscr{U},\mathscr{D}}$ satisfies Assumption 5.3.1 for arbitrary $\phi \in \mathscr{C}_r$. In addition, $T_r \ge T_{\rm p}$ and for the sequence of dwell-time switching events $\{(q_{\sigma,i_j^{\mathscr{D}}}, \Delta\tau_{\sigma,i_j^{\mathscr{D}}})\}_j$ of σ, the following conditions hold:*

i) $V^\natural_{q_{\sigma,i_j^{\mathscr{D}}}}(x_{\tau_{\sigma,i_j^{\mathscr{D}}+1}}) \ge V^\natural_{q_{\sigma,i_{j+1}^{\mathscr{D}}}}(x_{\tau_{\sigma,i_{j+1}^{\mathscr{D}}}}), \forall j \in \mathbb{N}$;

ii) $\limsup_{j\to\infty}\left(V_{q_{\sigma,i_j^{\mathscr{D}}}}(x_{\tau_{\sigma,i_j^{\mathscr{D}}}}) - V_{q_{\sigma,i_j^{\mathscr{D}}}}(x_{\tau_{\sigma,i_j^{\mathscr{D}}+1}})\right) \le \tau_\varepsilon\gamma_2(\|w\|)$; *and*

iii) for each $j \in \mathbb{N}$, there is a $k \in \mathbb{N}, k > j$ satisfying $V_{q_{\sigma,i_j^{\mathscr{D}}+1}}(x(\tau_{\sigma,i_j^{\mathscr{D}}+1})) \ge V_{q_{\sigma,i_k^{\mathscr{D}}}}(x(\tau_{\sigma,i_k^{\mathscr{D}}}))$ and $\Delta\tau_{\sigma,i_k^{\mathscr{D}}} \ge T_r$.

Then, the switched system $\Sigma_{\mathscr{U},\mathscr{D}}[\tau_{\rm p}, T_{\rm p}]$ has a switching-uniform asymptotic gain.

Proof: Let $\gamma(s) = \gamma_1(s) + \gamma_2(s)$ and define $r_\gamma = \alpha^{-1}(\gamma(\|w\|))$. Let $\phi \in \mathscr{C}_r$ be the initial condition of the system and let $x(t)$ denote the trajectory $\mathscr{T}_{\sigma,w,t_s}(t,\phi)$ for short.

Since Assumption 5.3.1 implies that $V^\natural_{q_{\sigma,i_j^{\mathscr{D}}}}(x(t))$ is non-increasing on $[\tau_{\sigma,i_j^{\mathscr{D}}}, \tau_{\sigma,i_j^{\mathscr{D}}+1}]$, condition i) implies condition (5.48) of Theorem 5.3.1. Thus, the system satisfies conditions of Theorem 5.3.1 and hence the trajectory $x(t)$ is bounded accordingly, i.e., there is a number $H = H(\phi, w, \sigma, r_\gamma) > 0$ such that $\|x(t)\| \le H, \forall t \in \mathbb{R}^+$.

As usual, let us assume that the first switching event of σ is dwell-time and prove the convergence on $\mathbb{T}_{\mathscr{D}} = \cup_j [\tau_{\sigma,i_j^{\mathscr{D}}}, \tau_{\sigma,i_j^{\mathscr{D}}+1}]$ of the following composite Lyapunov-Razumikhin function:

$$V_{\mathscr{C}}(t) = \begin{cases} V_{q_{\sigma,i_j^{\mathscr{D}}}}(x(t)) \textbf{ if } t \in [\tau_{\sigma,i_j^{\mathscr{D}}}, \tau_{\sigma,i_j^{\mathscr{D}}+1}) \\ V_{q_{\sigma,i_{j+1}^{\mathscr{D}}}}(x(\tau_{\sigma,i_{j+1}^{\mathscr{D}}})) \textbf{ if } t \in [\tau_{\sigma,i_j^{\mathscr{D}}+1}, \tau_{\sigma,i_{j+1}^{\mathscr{D}}}) \end{cases}, j \in \mathbb{N}. \tag{5.55}$$

By condition (5.46), we have $V_{\mathscr{C}}(t) \le R(\phi, w, \sigma, r_\gamma) \stackrel{\text{def}}{=} \alpha_2(H(\phi, w, \sigma, r_\gamma))$. Suppose that $\epsilon > 0$ is an arbitrary number less than R. Based on the constructive framework of [59, Chapter 5, Theorem 4.2], we shall show that there is a number $t_\epsilon \in \mathbb{T}_{\mathscr{D}}$ such that $V_{\mathscr{C}}(t) \le r_\gamma + \epsilon$ for all $t \in \mathbb{T}_{\mathscr{D}}, t \ge t_\epsilon$.

Since p is continuous and satisfies $p(s) > s, s > 0$, there is a number $a = a(\phi, w, \sigma) > 0$ such that $p(s) - s > a, \forall s \in [r_\gamma + \epsilon, R]$. Let $N_R = N_R(\phi, w, \sigma)$ be the first positive integer satisfying $r_\gamma + \epsilon + N_R a \ge R$.

Our purpose at this point is to show that there is a $t_1 \in \mathbb{T}_{\mathscr{D}}$ such that $V_{\mathscr{C}}(t_1) \le r_\gamma + \epsilon + (N_R - 1)a$. Suppose that the converse holds, i.e., $V_{\mathscr{C}}(t) > r_\gamma + \epsilon + (N_R - 1)a, \forall t \in \mathbb{T}_{\mathscr{D}}$. Since $V_{\mathscr{C}}(t)$ is bounded by R for all $t \in \mathbb{T}_{\mathscr{D}}$, this implies that

$$p(V_{\mathscr{C}}(t)) > V_{\mathscr{C}}(t) + a > r_\gamma + \epsilon + (N_R - 1)a + a \ge R \ge V_{\mathscr{C}}(t + \varsigma), \forall \varsigma \in [-T_r, 0], \tag{5.56}$$

which, by (5.55), further implies that $p(V_{q_{\sigma,i_\sigma^-(t)}}(x(t))) > V^\natural_{q_{\sigma,i_\sigma^-(t)}}(x_t), \forall t \in \mathbb{T}_{\mathscr{D}}$.

Thus, by Assumption 5.3.1, we have

$$D^+ V_{\mathscr{C}}(t) \le D_\sigma V_{\mathscr{C}}(t) \le -\alpha(V_{\mathscr{C}}(t)) + \gamma_1(\|w\|), \forall t \in \mathbb{T}_{\mathscr{D}}. \tag{5.57}$$

Moreover, $V_{\mathscr{C}}(t) > r_\gamma + \epsilon + (N_R - 1)a$ implies that $\alpha(V_{\mathscr{C}}(t)) > \gamma(\|w\|)$ and hence $-\alpha(V_{\mathscr{C}}(t)) + \gamma_1(\|w\|) < -\gamma_2(\|w\|)$. Thus, integrating both side of (5.57) on dwell-time intervals $[\tau_{\sigma,i_j^{\mathscr{D}}},\tau_{\sigma,i_j^{\mathscr{D}}+1})$ yields

$$V_{\mathscr{C}}(\tau_{\sigma,i_j^{\mathscr{D}}})-V_{\mathscr{C}}(\tau_{\sigma,i_j^{\mathscr{D}}+1})\geq\gamma_2(\|w\|)\Delta\tau_{\sigma,i_j^{\mathscr{D}}}\geq\tau_{\mathrm{p}}\gamma_2(\|w\|),\forall j\in\mathbb{N}.\tag{5.58}$$

Taking the limits of both sides of (5.58) as $j\to\infty$, we obtain

$$\limsup_{j\to\infty}(V_{\mathscr{C}}(\tau_{\sigma,i_j^{\mathscr{D}}})-V_{\mathscr{C}}(\tau_{\sigma,i_j^{\mathscr{D}}+1}))\geq\tau_{\mathrm{p}}\gamma_2(\|w\|),\tag{5.59}$$

which, by virtue of $\tau_\varepsilon<\tau_{\mathrm{p}}$, contradicts to condition ii) of the theorem.

Therefore, there is $t_1\in\mathbb{T}_{\mathscr{D}}$ such that the following inequality holds for $t=t_1$:

$$V_{\mathscr{C}}(t)\leq r_\gamma+\epsilon+(N_R-1)a.\tag{5.60}$$

Let us consider the dwell-time interval $[\tau_{\sigma,i_\sigma^-(t_1)},\tau_{\sigma,i_\sigma^-(t_1)+1})$ containing t_1. Let $i_{\mathscr{D},1}$ be the index of the dwell-time switching event of σ satisfying $i_{\mathscr{D},1}>i_\sigma^-(t_1)$, $\Delta\tau_{\sigma,i_{\mathscr{D},1}}\geq T_r$ and $V_{\mathscr{C}}(\tau_{\sigma,i_{\mathscr{D},1}})\leq V_{\mathscr{C}}(\tau_{\sigma,i_\sigma^-(t_1)+1})$. By condition iii) of the theorem and definition of $V_{\mathscr{C}}$, such $i_{\mathscr{D},1}$ exists. The next purpose is to show that (5.60) also holds for all $t\in\mathbb{T}_{\mathscr{D}},t\geq\tau_{\sigma,i_{\mathscr{D},1}}$. We have the following cases at t_1:

Case 1: $V_{\mathscr{C}}(t_1)=r_\gamma+\epsilon+(N_R-1)a$. In this case, we have $p(V_{\mathscr{C}}(t_1))>V_{\mathscr{C}}(t_1)+a\geq R\geq V_{\mathscr{C}}(t_1+\varsigma),\forall\varsigma\in[-T_r,0]$, i.e., the Razumikhin condition holds at t_1. By Assumption 5.3.1 and the fact that $\alpha(V_{\mathscr{C}}(t_1))>\gamma(\|w\|)$, we have $D^+V_{\mathscr{C}}(t_1)<0$. Hence, there is a number $h>0$ such that $V_{\mathscr{C}}(t_1+\varepsilon)<r_\gamma+\epsilon+(N_R-1)a,\forall\varepsilon\in(0,h)$.

Case 2: $V_{\mathscr{C}}(t_1)<r_\gamma+\epsilon+(N_R-1)a$. In this case, due to the continuity of V_q's and $x(t)$, such number $h>0$ in *Case 1* exists.

Combining Cases 1 and 2 above, it follows that there is a number $h>0$ such that $V_{\mathscr{C}}(t_1+\varepsilon)<r_\gamma+\epsilon+(N_R-1)a,\forall\varepsilon\in(0,h)$. Moreover, by virtue of Case 1, if $V_{\mathscr{C}}(t)=r_\gamma+\epsilon+(N_R-1)a$ for some t, then $D^+V_{\mathscr{C}}(t)<0$. As such, $V_{\mathscr{C}}(t)\leq r_\gamma+\epsilon+(N_R-1)a,\forall t\in[t_1,\tau_{\sigma,i_\sigma^-(t_1)+1}]$.

We proceed to consider the interval $[\tau_{\sigma,i_{\mathcal{D},1}}, \tau_{\sigma,i_{\mathcal{D},1}+1})$. By the defining property of $i_{\mathcal{D},1}$, we have $V_{\mathscr{C}}(\tau_{\sigma,i_{\mathcal{D},1}}) \leq V_{\mathscr{C}}(\tau_{\sigma,i_{\sigma}^{-}(t_1)+1}) \leq r_\gamma + \epsilon + (N_R - 1)a$. Thus, applying the above argument for the dwell-time interval $[\tau_{\sigma,i_{\mathcal{D},1}}, \tau_{\sigma,i_{\mathcal{D},1}+1})$, it follows that $V_{\mathscr{C}}(t) \leq r_\gamma + \epsilon + (N_R - 1)a, \forall t \in [\tau_{\sigma,i_{\mathcal{D},1}}, \tau_{\sigma,i_{\mathcal{D},1}+1})$. Furthermore, since $\Delta\tau_{\sigma,i_{\mathcal{D},1}} \geq T_r$, this implies that $V_{\mathscr{C}}^{\natural}(\tau_{\sigma,i_{\mathcal{D},1}+1}) \leq r_\gamma + \epsilon + (N_R - 1)a$.

Consequently, by condition i), we have $V_{\mathscr{C}}(x(\tau_{\sigma,i_j^{\mathcal{D}}})) \leq V_{\mathscr{C}}^{\natural}(\tau_{\sigma,i_j^{\mathcal{D}}}) \leq V_{\mathscr{C}}^{\natural}(\tau_{\sigma,i_{\mathcal{D},1}+1}) \leq r_\gamma + \epsilon + (N_R - 1)a$ for all $i_j^{\mathcal{D}}$ greater than $i_{\mathcal{D},1}$. Applying the above argument for dwell-time intervals after $i_{\mathcal{D},1}$ again, it follows that $V_{\mathscr{C}}^{\natural}(t) \leq r_\gamma + \epsilon + (N_R - 1)a, \forall t \in \mathbb{T}_{\mathcal{D}}, t \geq \tau_{\sigma,i_{\mathcal{D},1}}$.

We now turn to the next levels for function $V_{\mathscr{C}}$. Suppose that for some $k \in \{1, \ldots, N_R - 1\}$, we have derived the existence of a starting time $\tau_{\sigma,i_{\mathcal{D},k}}$ of a dwell-time switching events satisfying $V_{\mathscr{C}}^{\natural}(t) \leq r_\gamma + \epsilon + (N_R - k)a, \forall t \in \mathbb{T}_{\mathcal{D}}, t \geq \tau_{\sigma,i_{\mathcal{D},k}}$. By the defining property of $i_{\mathcal{D},k}$ and p, for a time $t \in \mathbb{T}_{\mathcal{D}}, t \geq \tau_{\sigma,i_{\mathcal{D},k}}$ satisfying $V_{\mathscr{C}}(t) \geq r_\gamma + \epsilon + (N_R - (k+1))a$, we have

$$p(V_{\mathscr{C}}(t)) > V_{\mathscr{C}}(t) + a \geq r_\gamma + \epsilon + (N_R - k)a \geq V_{\mathscr{C}}^{\natural}(t), \tag{5.61}$$

which, by Assumption 5.3.1 and the implication $V_{\mathscr{C}}(t) > r_\gamma + \epsilon \Rightarrow -\alpha(V_{\mathscr{C}}(t)) + \gamma_1(\|w\|) \leq -\gamma_2(\|w\|)$, implies that $D^+V_{\mathscr{C}}(t) \leq -\gamma_2(\|w\|)$. Therefore, repeating the same arguments as above, it follows that there is a time t_{k+1} and a starting time $\tau_{\sigma,i_{\mathcal{D},k+1}}$ of a dwell-time interval of the length no smaller than T_r such that $V_{\mathscr{C}}(t_{k+1}), V_{\mathscr{C}}^{\natural}(t) \leq r_\gamma + \epsilon + (N_R - (k+1))a, \forall t \geq \tau_{\sigma,i_{\mathcal{D},k+1}}$.

At $k = N_R - 1$, setting $t_\epsilon = \tau_{\sigma,i_{\mathcal{D},N_R}}$ and using (5.46), we obtain

$$\alpha_1(\|x(t)\|) \leq V_{\mathscr{C}}^{\natural}(t) \leq r_\gamma + \epsilon, \forall t \in \mathbb{T}_{\mathcal{D}}, t \geq t_\epsilon. \tag{5.62}$$

Furthermore, for a time $t \in [\tau_{\sigma,i_j^{\mathcal{P}}+1}, \tau_{\sigma,i_{j+1}^{\mathcal{P}}})$, using condition i) of the theorem and the property that $T_{\mathrm{p}} \leq T_r$, we have

$$\alpha_1(\|x(t)\|) \leq V_{q_{\sigma,i_{j+1}^{\mathcal{P}}}}(x(t)) \leq V^{\natural}_{q_{\sigma,i_{j+1}^{\mathcal{P}}}}(\tau_{\sigma,i_{j+1}^{\mathcal{P}}}), \forall t \in [\tau_{\sigma,i_j^{\mathcal{P}}+1}, \tau_{\sigma,i_{j+1}^{\mathcal{P}}}), j \in \mathbb{N}. \tag{5.63}$$

Combining (5.62) and (5.63) yields,

$$\alpha_1(\|x(t)\|) \leq V^{\natural}_{\mathscr{C}}(t) \leq r_\gamma + \epsilon, \forall t \geq t_\epsilon. \tag{5.64}$$

Since $\epsilon > 0$ is arbitrary and $r_\gamma = \alpha^{-1}(\gamma(\|w\|))$, (5.64) implies that $x(t)$ approaches the set $\mathcal{O} = \{\zeta \in \mathbb{R}^n : \exists q \in \mathbb{Q}, \|\zeta\| \leq \alpha_1^{-1}(\alpha^{-1}(\gamma(\|w\|)))\}$ as $t \to \infty$. As, $\chi \stackrel{\text{def}}{=} \alpha_1^{-1} \circ \alpha^{-1} \circ \gamma$ is independent of σ, this shows that the system has the switching-uniform asymptotic gain χ. ■

The above result has revealed the important role of relation between delay-time T_r, persistent dwell-time τ_{p}, and period of persistence T_{p} in stability of switched time-delay systems. While the dominance of T_r to T_{p} makes behavior in periods of persistence accessible through behavior in dwell-time intervals, the existence of dwell-time intervals of the lengths no smaller than the delay-time ensures the preservation of the converging behavior through switching events. In comparison to traditional switching decreasing condition, condition i) of Theorem 5.3.2 is much more relaxed as it does not prevent the increasing behavior in period of persistence and is imposed on dwell-time intervals only.

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Part II

Advanced Control

Chapter 6

Gauge Design for Switching-Uniform Adaptive Control

The main purpose of this chapter is to introduce the gauge design method for switching-uniform adaptive control of a class of persistent dwell-time switched systems. Constituent systems possessing unmeasured appended dynamics are interested. The underlying principle is to use the appended dynamics and the controlled dynamics as gauges of each other to make the destabilizing behavior of a constituent system be dominated by the stabilizing behavior of its gauging system. The resulting behavior of the overall switched system is thus ensured to be converging.

6.1 Introduction

Transforming to certain normal forms is usually prerequisite for nonlinear control design [70, 85, 71]. The transformation may result in systems with zero dynamics due to low relative degrees [70, 85] or systems with unmeasured dynamics due to limited modeling capability [144]. Engineering examples of such systems include convey-crane system, robotic systems, hovercraft, surface vessel, and helicopters [43, 39]. In practice, systems with internal/appended dynamics may possess a hybrid nature due to the effects of discrete events such as reconfiguration in system/control structures [24, 109] and impulse effects [150].

Variables driving evolution of switched systems are inputs of constituent systems and switching sequence. As appropriate switching sequences can produce stable switched systems [95,142], extensive research has been carried out for switching strategies [110, 82, 93]. In addition, inappropriate switching sequences may destabilize the system and cause challenges for control design [95, 142]. Of practical relevance, inappropriate switching sequences usually arise in applications where uncertain switches may occur due to failures or where the switching sequences are generated for other purposes instead of stability. Thus, an important problem in switched systems is switching-uniform control, i.e., achieving the control objective uniformly with respect to a class of switching sequences [66,62]. Nevertheless, though stability theories have been developed for general switched systems [154, 152, 22, 102], switching-uniform stabilization of switched systems is still limited to switched linear systems [66] and switched nonlinear systems in Byrnes-Isidori canonical forms [33].

Motivated by these considerations, we study in this chapter control of uncertain switched nonlinear systems whose constituent systems, after suitable changes of coordinates, can be expressed in the following form:

$$\Sigma_q : \begin{cases} \dot{z}(t) & = & Q_q(z(t), x_1(t), \theta(t)) \\ \dot{x}_i(t) & = & g_{q,i}(X_i(t))x_{i+1}(t) + f_{q,i}(X_i(t)), \\ & & i = 1, \ldots, n-1 \\ \dot{x}_n(t) & = & g_{q,n}(X_n(t))u(t) + f_{q,n}(X_n(t)) \end{cases}, \tag{6.1}$$

where $z(t) \in \mathbb{R}^d, x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}$ are unmeasured state, measured state, and system input, respectively, $X_i(t) = [z^T(t), \bar{x}_i^T(t), \theta^T(t)]^T$, q is the system index belonging to the discrete set $\mathbb{Q} = \{1, \ldots, q^\natural\}$, $\theta(t) \in \Omega_\theta \subset \mathbb{R}^{d_\theta}$ is the unknown time-varying parameter, and for each $i = 1, \ldots, n$, $\bar{x}_i(t) = [x_1(t), \ldots, x_i(t)]^T \in \mathbb{R}^i$, and $Q_q, g_{q,i}$, and $f_{q,i}, q \in \mathbb{Q}$ are known and sufficiently smooth functions.

Difficulty in controlling switched systems whose constituent systems are described by (6.1) is manyfold. Firstly, due to low relative degrees, we are able to control a limited number of state variables and might leave the remaining ones evolve autonomously. This raises the challenge that the usual minimum-phase condition for systems with zero-dynamics [71] is not sufficient for stabilization of switched systems. Secondly, unmeasured dynamics make control by switching-logic involving computation based on measurement of all state variables or detectability assumption unfeasible [61, 64, 122, 65, 106]. The third difficulty is due to the inapplicability of traditional adaptive control for handling unknown parameters. The behind rationale is that while the traditional adaptive control results in closed-loop systems with slow dynamics constituent systems [12], finite running times of constituent systems call for fast dynamics for fulfilling switching conditions. Finally, the approach of augmenting Lyapunov functions of the unmeasured dynamics by quadratic functions of error variables [71, 97] is no longer effective. This is due to the fact that the cross-supply rates are positive so that changing supply functions [97, 32] for large decreasing rates on active intervals also causes large growth rates on inactive intervals.

In this chapter, the *gauge design* method is introduced for overcoming the mentioned difficulties. The underlying principle is to use the unmeasured dynamics and the controlled dynamics as gauges of instability of each others. This is possible since whenever the state of the controlled dynamics is dominated by the unmeasured state, then the desired behavior of the overall system is guaranteed by the minimum-phase property of the unmeasured dynamics, and in the remaining case, i.e., the unmeasured state is dominated by the measured state, estimates of functions of the unmeasured state in terms of the measured state are available and a measured-state dependent control can be designed to make the controlled dynamics the driving dynamics of the overall system. The control is responsible for canceling not only known parts of destabilizing terms but also unknown parts of these terms whenever their computable estimates are available.

The advantages of the gauge design lies in i) stability conditions can be verified by the stability properties of the unmeasured dynamics and the dwell-time properties of the switching signal, and ii) the unknown time-varying parameters can be lumped into input disturbance that can be attenuated by tuning control parameters.

6.2 Problem Formulation

In the formal language of transition model of dynamical systems, we have the following collection as a model of switched systems with input, output, disturbance, and appended dynamics:

$$\Sigma_{\mathrm{I/O}} = \{\mathbb{T}, \mathbb{Q}, \mathcal{X}, \mathcal{Z}, \mathcal{L}_\infty^{d_\theta}, \{\Sigma_q\}_{q\in\mathbb{Q}}, \mathbb{S}, \mathcal{U}, \mathcal{Y}\}, \tag{6.2}$$

where $\mathbb{T} = \mathbb{R}^+, \mathbb{Q} = \{1, \dots, q^\natural\}, \mathcal{X} = \mathbb{R}^n, \mathcal{W} = \mathbb{R}^d, \mathcal{U} \subset \mathbb{R}^{n_u}, \mathcal{Y} \subset \mathbb{R}^{n_y}$, and $\mathcal{L}_\infty^{d_\theta}$ are spaces of time, discrete variable, measured variables, unmeasured variables, input, output, and disturbance, respectively, $\mathbb{S}$ is the space of switching sequences, and $\Sigma_q, q \in \mathbb{Q}$ are constituent systems whose evolutions are governed by (6.1). The mechanism of evolution of system $\Sigma_{\mathrm{I/O}}$ is obvious from Section 2.4 of Chapter 2.

Consider the dynamical system described by

$$\Sigma_C : \dot{\zeta} = \Gamma(\zeta, u_C), \tag{6.3}$$

where $\zeta \in \mathbb{R}^{n_C}$ and $u_C \in \mathbb{R}^{m_C}$ are state and input of Σ_C. Let $y_C = h_C(\zeta, u_C)$ and $y_m = h_m(z, x, u)$ be the output of Σ_C and the measured output of Σ_q. For $q \in \mathbb{Q}$,

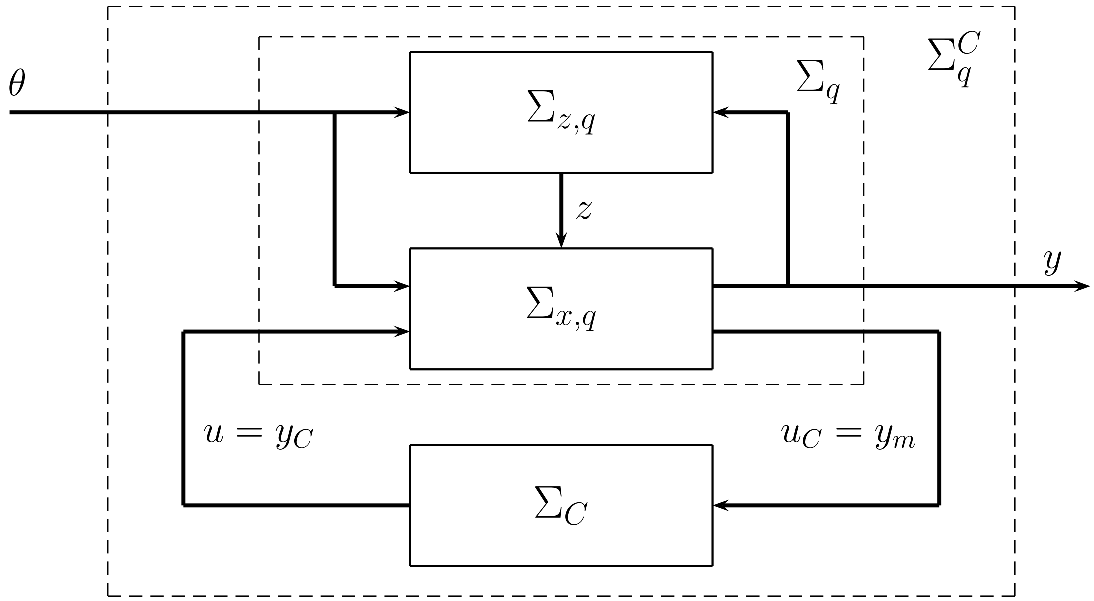


Figure 6.1: q–constituent closed-loop system Σ_q^C: $\Sigma_{z,q}$ – z subsystem of Σ_q, $\Sigma_{x,q}$ – x subsystem of Σ_q

making the interconnection between Σ_q and Σ_C through

$$u_C = y_m, u = y_C, \tag{6.4}$$

we obtain the dynamical systems $\Sigma_q^C, q \in \mathbb{Q}$, see Figure 6.1, from which the following switched system with input are well-defined

$$\Sigma_{\mathscr{C}} = \{\mathbb{R}^+, \mathbb{Q}, \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n_C}, \mathcal{L}_\infty^{d_\theta}, \{\Sigma_q^C\}_{q\in\mathbb{Q}}, \mathbb{S}\}. \tag{6.5}$$

The state of $\Sigma_{\mathscr{C}}$ is $X = [z^T, x^T, \zeta^T]^T$ and its input is θ. The mechanism of evolution of $\Sigma_{\mathscr{C}}$ is obvious from the definition of switched systems in Chapter 2.

For each switching sequence $\sigma \in \mathbb{S}$ and each input $\theta \in \mathcal{L}_\infty^{d_\theta}$, let $X(t) = X(t; \sigma, \theta, X_0)$ denote the trajectory through X_0 of the closed-loop switched system $\Sigma_{\mathscr{C}}$. We have the following control problem for switched system with appended dynamics $\Sigma_{\mathrm{I/O}}$.

Switching-uniform adaptive output regulation: design a dynamical system of the form (6.3) such that, under the interconnection (6.4) with $y_m = x$, the trajectory $X(t) = X(t;\sigma,\theta,X_0)$ of the closed-loop system $\Sigma_{\mathscr{C}}$ generated by any switching sequence $\sigma \in \mathbb{S}[\tau_{\mathrm{p}}, T_{\mathrm{p}}]$, input $\theta \in \mathcal{L}_\infty^{d_\theta}$ and initial condition X_0 satisfies:

i) $X(t)$ is bounded; and

ii) $y(t) = x_1(t)$ approaches to a small neighborhood of zero as $t \to \infty$.

We would mention that by collections (6.2) and (6.5), it is implicitly supposed that there is no switching jump in trajectories. For each $q \in \mathbb{Q}$, we shall call the equation $\dot{z}(t) = Q_q(z(t), x_1(t), \theta(t))$ in (6.1) the z–subsystem of Σ_q and, accordingly, the collection of the remaining equations in (6.1) the x–subsystem of Σ_q. We shall respectively denote these subsystems by $\Sigma_{z,q}$ and $\Sigma_{x,q}$. The interconnection structure of the controlled switched system is shown in Figure 6.1.

As the measured output $y_m = x$ does not contain z, we shall call $\Sigma_{z,q}$ the unmeasured dynamics of Σ_q and we call the dynamics of the continuous state z of system $\Sigma_{\mathrm{I/O}}$, labeled as Σ_Δ, the unmeasured dynamics of $\Sigma_{\mathrm{I/O}}$. In this chapter, we provide solution to the proposed control problem under the following conditions.

Assumption 6.2.1 *There are smooth positive definite and proper functions $U_q : \mathbb{R}^d \to \mathbb{R}, q \in \mathbb{Q}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \bar{\alpha}, \alpha_1, \alpha_2$ and β, and a continuous function $\upsilon : \mathbb{R}^+ \to \mathbb{R}^+, \upsilon(0) = 0$ such that, for all q, q_1, and $q_2 \in \mathbb{Q}, q_1 \neq q_2$, for all $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$ and for all $\theta \in \Omega_\theta$, the following properties hold:*

i) $U_{q_1}(z) \le \beta(U_{q_2}(z))$ *and* $\underline{\alpha}(\|z\|) \le U_q(z) \le \bar{\alpha}(\|z\|)$;

ii) $\dfrac{\partial U_q(z)}{\partial z} Q_q(z, y, \theta) \le -\alpha_1(U_q(z)) + \upsilon(y^2)$; *and*

iii) $\dfrac{\partial U_{q_1}(z)}{\partial z} Q_{q_2}(z, y, \theta) \le \alpha_2(U_{q_1}(z)) + \upsilon(y^2)$.

Let μ be a C^1 class-$\mathcal{K}_\infty$ function and $\omega_k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+, k = 1, 2$ are functions defined as $\omega_k(a, b) = G_k^{-1}(G_k(a) + b)$, where

$$G_1(v) \stackrel{\text{def}}{=} \int_1^v \frac{ds}{-\alpha_1(s) + \mu(s)}, \text{ and } G_2(v) \stackrel{\text{def}}{=} \int_1^v \frac{ds}{\alpha_2(s) + \mu(s)}, v \in \mathbb{R}^+. \tag{6.6}$$

Assumption 6.2.2 $\lim_{s\to 0^+}(\partial\mu(s)/\partial s)\alpha_1(s) = 0$ *and both functions* $\alpha_1 - \mu$ *and* $\alpha_2 + \mu$ *are of class-*$\mathcal{K}_\infty$*. There exist numbers* $R_V > 0$ *and* $\tau_0 > 0$ *and a class–*$\mathcal{KL}$ *function* ω_0 *satisfying* $\omega_0(\omega_0(a, s), t) \leq \omega_0(a, s + t), \forall a, s, t \in \mathbb{R}^+$ *such that* $\omega_2(s, t) < \infty, \forall (s, t) \in [0, R_V) \times [0, T_\mathrm{p}]$ *and*

$$\omega_1(\beta(\omega_2(s, T_\mathrm{p})), \tau_\mathrm{p}) \leq \omega_0(s, \tau_\epsilon), \forall s \in \mathbb{R}^+. \tag{6.7}$$

In system with appended dynamics, the control objective is often to force the system output to follows a prescribed signal [71]. In such context, the system output $y(t)$ plays the role of input disturbance to the z–subsystem. As a result, certain stability property must be imposed on the internal dynamics for solvability [74, 71, 7, 84,32,48]. Without involving the switching signal, Assumption 6.2.1 imposes stability properties of constituent systems Σ_q only.

The first inequality in condition i) of Assumption 6.2.1 is not a restriction as from the second inequality, such a function β can be $\beta(\cdot) = \bar{\alpha}(\underline{\alpha}^{-1}(\cdot))$. However, β is considered as in practice such a function β giving better growing estimates than $\bar{\alpha} \circ \underline{\alpha}^{-1}$ might be utilized for better control.

Condition (6.7) provides a quantitative condition from the well-known but still unutilized fact in switched systems: in comparison to the persistent period and the growth rate, the larger the dwell-time and the decreasing rate are, the better converging behavior is achieved. As ω_1 and ω_2 are class–$\mathcal{KL}$ and class–$\mathcal{KK}$ functions, respectively, condition (6.7) holds if either decreasing rate α_1 or dwell-time τ_p are sufficiently large with respect to the growth rate α_2 and the persistent period T_p.

In the case that the functions β, μ, α_1, and α_2 are linear as in [149], i.e., $\beta(s) = a_0 s, \alpha_1(s) - \mu(s) = a_1 s$, and $\alpha_2(s) + \mu(s) = a_2 s$, the condition (6.7) reduces to $\tau_\epsilon \overset{\text{def}}{=} a_1\tau_\mathrm{p} - a_2 T_\mathrm{p} - \ln a_0 > 0$ and a function β_0 satisfying (6.7) is $\omega_0(s, \tau_\epsilon) = s\exp(-\tau_\epsilon)$. Thus, the generality of (6.7) is obvious.

Assumption 6.2.3 *There are known functions $g_{x,i}, i = 1, \ldots, n$ such that*

$$|g_{q,i}(z, \bar{x}_i, \theta)| \geq g_{x,i}(\bar{x}_i) > 0, \tag{6.8}$$

for all $(z, \bar{x}_i, \theta) \in \mathbb{R}^d \times \mathbb{R}^i \times \Omega_\theta, q \in \mathbb{Q}, i = 1, \ldots, n$.

As the control gains are continuous and bounded away from zero by (6.8), they have unchanged signs. Without loss of generality, we further assume that all the control gains' signs are positive.

Control for systems with changing control gains' signs is an extensive problem [155]. However, control of this class of systems can be developed from controls of systems with known control gains' signs [155, 99]. Therefore, control of systems with known control gains' signs, as assumed in Assumption 6.2.3, also plays an important role in dealing with larger class of systems, whilst state-dependent and un-identical control gains of constituent systems Σ_q in (6.1) present an obvious generalization from systems with constant control gains [85, 71].

Assumption 6.2.3 is instrumental in overcoming the obstacle that the well-known cancelation design for continuous systems, in which the terms $1/g_{k,i}$'s are included in the controls for matching the control to nonlinear functions $f_{k,i}$'s [85, 162], does not apply to switched system (6.2). The behind rationale lies in i) uncomputable unmeasured state dependent control gains $g_{k,i}$'s cannot be included in the controls, and ii) even if z was known, the control and the nonlinear functions of a constituent system Σ_1 are matched only when this system is active.

To close this section, let us recall the following lemmas.

Lemma 6.2.1 ([14]) *Let $v(t)$ be a differentiable function in $J = (a, b)$ such that*

$$\dot{v}(t) \leq g(v(t)), \forall t \in J, \tag{6.9}$$

where g is a nonzero continuous function in $I = (v_1, v_2)$. Let $t_0 \in J$ and $v(t_0) = v_0 \in I$, then

$$v(t) \leq G^{-1}(G(v_0) + t - t_0), \forall t \in [t_0, b_1) = J_1, \tag{6.10}$$

where $G(v) = \int_u^v ds/g(s), u, v \in I$ and $b_1 = \sup\{t \in [t_0, b) : G(v_0)+s-t_0 \in G(I), t_0 \leq s \leq t\}$. In addition, the function $\omega(a,t) \stackrel{\text{def}}{=} G^{-1}(G(a) + t), a \in I, t \in J$ satisfies

$$\omega(\beta(a,s),t) = \omega(a, s+t), \forall a \in I, s, t \in J, s+t \in J. \tag{6.11}$$

Lemma 6.2.2 ([97]) *For any real-valued continuous function $f(x,y)$ where $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, there are smooth scalar-value functions $a(x) \geq, b(y) \geq 0, c(x) \geq 1$ and $d(y) \geq 1$ such that*

$$|f(x,y)| \leq a(x) + b(y) \quad \textit{and} \quad |f(x,y)| \leq c(x)d(y). \tag{6.12}$$

6.3 Switching-Uniform Adaptive Output Regulation

In this section, we present the *gauge design method* for switching-uniform adaptive output regulation of persistent dwell-time switched system with unmeasured dynamics (6.2). The design makes use of the controlled error dynamics as a gauge for instability mode of the unmeasured dynamics to design a control preserving the converging behavior in this mode. To this end, we exploit the fact that in unstable modes of z-system, estimates of z-dependent functions in terms of error variables are available for control design. In this way, auxiliary functions satisfying conditions of Theorem 5.2.1 can be constructed for the convergence of the combined unmeasured dynamics and error dynamics. As the measured state-dependent domination functions of the unmeasured state dependent functions are utilized, we separate the unknown time-varying parameters from known variables and combine them into a lumped input disturbance which can be attenuated by tuning control parameter.

Since $\upsilon(0) = 0$, there is a continuous function υ_0 such that $\upsilon(s) = \upsilon_0(s)s, s \geq 0$. Our design involves the following gauge along the trajectories of z, x:

$$\mu(U_q(z(t))) \leq V_{\mathrm{g}}(\xi(t)) \stackrel{\text{def}}{=} \upsilon_1(\xi_1^2(t))\xi_1^2(t) + \sum_{i=2}^{n} \xi_i^2(t), q \in \mathbb{Q}, \tag{6.13}$$

where U_q's and μ are given in Assumptions 6.2.1 and 6.2.2, $z \in \mathbb{R}^d$ is the state of the unmeasured dynamics Σ_Δ, $\xi_i = x_i - \alpha_{u,i-1} \in \mathbb{R}, i = 1, \ldots, n$ are error variables, $\alpha_{u,0} = 0, \alpha_{u,i}, i = 1, \ldots, n$ are the so-called virtual controls, $\upsilon_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function satisfying $\upsilon_1(s) \geq \max\{\upsilon_0(s), 1\}, \forall s \in \mathbb{R}^+$. Hereafter, we call the differential equations describing the dynamics of the error state $\xi = [\xi_1, \ldots, \xi_n]^T$ the ξ-subsystem or controlled dynamics labeled as $\Sigma_{\mathcal{E}}$.

6.3.1 Control Design

By Assumption 6.2.1, if the inverse of (6.13) holds for all the time, then converging behavior of ξ is guaranteed by the converging behavior of z. Thus, the design purpose is to preserve the converging behavior of z on time stages (6.13) holds. To proceed, let us recall the definition of the transition indicator i_σ^- from Section 2.4.2 of Chapter 2 and the notion of derivative along a trajectory (5.7). For each $i = 1, \ldots, n$, $\bar{\xi}_i$ is the vector $[\xi_1, \ldots, \xi_i]^T$.

First Virtual Control Design

Let $\xi_1 = x_1$ and $\xi_2 = x_2 - \alpha_{u,1}$ where $\alpha_{u,1}$ is the first virtual control to be designed. From the system dynamics (6.1), we have the following evolution rule for ξ_1

$$\dot{\xi}_1 = g_{q,1}(z, x_1, \theta)x_2 + f_{q,1}(z, x_1, \theta), q \in \mathbb{Q}. \tag{6.14}$$

Consider the following function

$$V_{\mathrm{g},1}(\xi_1) = \frac{1}{2}\upsilon_1(\xi_1^2)\xi_1^2, \tag{6.15}$$

which we call the first gauge Lyapunov function candidate as it is a part of V_{g}.

Let $\upsilon_D(s) = (\partial\upsilon_1(s)/\partial s)s + \upsilon_1(s)$. As $V_{\mathrm{g},1}$ is continuously differentiable, the derivative of $V_{\mathrm{g},1}$ along the evolution of ξ_1 is

$$\begin{aligned} DV_{\mathrm{g},1}(\xi_1(t)) \leq \max_{p\in\mathbb{Q}} \Big\{ \upsilon_D(\xi_1^2(t)) \Big(& g_{p,1}(z(t), x_1(t), \theta(t))(\xi_2(t) + \alpha_{u,1}(t)) \\ & + f_{p,1}(z(t), x_1(t), \theta(t)) \Big) \xi_1(t) \Big\}. \end{aligned} \tag{6.16}$$

Bearing in mind that computation are made along evolutions of state variable z and x, we shall often drop the time arguments of evolving variables for short. At this point, our objective is to construct a virtual control $\alpha_{u,1}$ such that if (6.13) holds, then the right-hand-side of (6.16) contains a negative functions of ξ_1^2. To this end, let us estimate the functions $g_{p,1}$'s and $f_{p,1}$'s in terms of known variables ξ_i's upon satisfaction of (6.13). Since $g_{p,1}$'s and $f_{p,1}$'s are continuous and U_p's are radially unbounded by Assumption 6.2.1, for $p \in \mathbb{Q}$, there is a continuous function $\psi_{p,1}$ nondecreasing in its first argument such that

$$g_{p,1}(\cdot)\xi_2 + f_{p,1}(\cdot) \leq \psi_{p,1}(U_q(z), x_1, \xi_2, \theta), \forall (z, x_1, \theta) \in \mathbb{R}^d \times \mathbb{R} \times \Omega_\theta. \tag{6.17}$$

Such a function $\psi_{p,1}$ can be

$$\psi_{p,1}(s, x_1, \theta) = \sup\{g_{p,1}(\zeta, x_1, \theta)\xi_2 + f_{p,1}(\zeta, x_1, \theta) : \zeta \in \mathbb{R}^d, \|\zeta\| \leq \underline{\alpha}^{-1}(s)\}, s \in \mathbb{R}^+, \tag{6.18}$$

where $\underline{\alpha}$ is given by Assumption 6.2.1. Applying Lemma 6.2.2 and Young's inequality, we obtain functions $\psi_{q,1}^a, \psi_{q,1}^b, q \in \mathbb{Q}$ such that, upon satisfaction of (6.13), we have

$$\begin{aligned}
&\upsilon_D(\xi_1^2)(g_{p,1}(\cdot)\xi_2 + f_{p,1}(\cdot))\xi_1 \leq |\upsilon_D(\xi_1^2)|\psi_{p,1}(U_q(z), \xi_1, \xi_2, \theta)|\xi_1| \\
&\quad \leq |\upsilon_D(\xi_1^2)|\psi_{p,1}\Big(\mu^{-1}\Big(\upsilon_1(\xi_1^2)\xi_1^2 + \sum_{i=2}^{n} \xi_i^2\Big), \xi_1, \xi_2, \theta\Big)|\xi_1| \leq \psi_{p,1}^a(\theta)\psi_{p,1}^b(\bar{\xi}_n)|\xi_1|. \quad (6.19)
\end{aligned}$$

where $\xi_i, i = 2, \ldots, n$ are error variables to be defined at the next steps. Since $\psi_{p,1}^a$ is continuous and $\theta \in \mathcal{L}_\infty^{d_\theta}$ is bounded, there is a constant Θ_1 such that $(\psi_{p,1}^a(\theta))^2 \leq 4\Theta_1$. As such, applying Young's inequality, the last term in (6.19) satisfies

$$\psi_{p,1}^a(\theta)\psi_{p,1}^b(\bar{\xi}_n)|\xi_1| \leq K_\Theta(\psi_{p,1}^b(\bar{\xi}_n))^2\xi_1^2 + \frac{(\psi_{p,1}^a(\theta))^2}{4K_\Theta} \leq K_\Theta(\psi_{p,1}^b(\bar{\xi}_n))^2\xi_1^2 + \frac{\Theta_1}{K_\Theta}, \quad (6.20)$$

where $K_\Theta > 0$ is a time-varying design parameter to be updated.

Since $\mathbb{Q}$ is finite, there is a C^1 positive function ψ_1, that is nondecreasing in each individual argument, such that $(\psi_{p,1}^b(\bar{\xi}_n))^2 \leq \psi_1(\bar{\xi}_n^2), \forall p \in \mathbb{Q}$, where $\bar{\xi}_n^2 = [\xi_1^2, \ldots, \xi_n^2]^T$. Such a function ψ_1 can be any C^1 positive function satisfying

$$\psi_1(s) \geq \sup\Big\{(\psi_{p,1}^b(\bar{\xi}_n))^2 : \bar{\xi}_n \in \mathbb{R}^n, \|\bar{\xi}_n\|^2 \leq \sum_{i=1}^{n} s_i, p \in \mathbb{Q}\Big\}, s = [s_1, \ldots, s_n]^T \in (\mathbb{R}^+)^n. \quad (6.21)$$

Applying the following identity [97]

$$\begin{aligned}
f(x_1, \ldots, x_i) &= \Big(\int_0^1 \frac{\partial f(x_1, \ldots, x_{i-1}, s)}{\partial s}\Big|_{s=\beta x_i} d\beta\Big)x_i + f(x_1, \ldots, x_{i-1}, 0) \\
&\stackrel{\text{def}}{=} A(x_1, \ldots, x_i)x_i + f(x_1, \ldots, x_{i-1}, 0), \quad (6.22)
\end{aligned}$$

to the function ψ_1 recursively from $i = n$ to $i = 1$, we obtain positive functions $\varphi_{1,i}$ satisfying

$$\psi_1(\bar{\xi}_n^2) = \sum_{i=1}^{n} \varphi_{1,i}(\bar{\xi}_i^2)\xi_i^2 + \psi_1(0), \bar{\xi}_i^2 = [\xi_1^2, \ldots, \xi_i^2]^T. \quad (6.23)$$

In summary, from (6.19), (6.20) and (6.23), we have

$$\upsilon_D(\xi_1^2)(g_{p,1}(z,x_1,\theta)\xi_2+f_{p,1}(z,x_1,\theta))\xi_1 \leq K_\Theta\Big(\sum_{i=1}^{n}\varphi_{1,i}(\bar{\xi}_i^2)\xi_i^2+\psi_1(0)\Big)\xi_1^2+\frac{\Theta_1}{K_\Theta}, \forall p\in\mathbb{Q} \tag{6.24}$$

whenever (6.13) holds true.

Remark 6.3.1 *As ψ_1 is nondecreasing in each its argument, its partial derivatives are nonnegative. As such, in view of (6.23), the non-negativeness of the functions $\varphi_{1,i}(\cdot)$'s in (6.23) is guaranteed.*

As we are going to design $\alpha_{u,1}$ such that $\alpha_{u,1}\xi_1 \leq 0$, substituting (6.20) and $x_2 = \xi_2 + \alpha_{u,1}$ into (6.16), we obtain

$$\begin{aligned} DV_{\mathrm{g},1}(\xi_1(t)) \leq{}& \upsilon_D(\xi_1^2)\min\{g_{p,1}(z,x_1,\theta) : p\in\mathbb{Q}\}\xi_1\alpha_{u,1} \\ &+ K_\Theta\Big(\sum_{i=1}^{n}\varphi_{1,i}(\bar{\xi}_i^2)\xi_i^2+\psi_1(0)\Big)\xi_1^2+\frac{\Theta_1}{K_\Theta} \end{aligned} \tag{6.25}$$

upon satisfaction of (6.13).

Remark 6.3.2 *The universal design of the functions $\psi_i, i = 1,\ldots,n$ in (6.21) and (6.37) below applies to the generic functions $f_{p,i}$'s and $g_{p,i}$'s. In practice, specific structures of $f_{p,i}$'s and $g_{p,i}$'s can be exploited to improve estimates (6.20) and (6.39) below for better control performance.*

In view of (6.25), let us consider the following first virtual control $\alpha_{u,1}$

$$\alpha_{u,1} = -\frac{1}{g_{x,1}(x_1)}(\varrho_1(\xi_1)+K_\Theta\varphi_1(\xi_1))\xi_1, \tag{6.26}$$

where ϱ_1, and φ_1 are positive smooth functions to be specified. By construction, υ_1 is a nondecreasing function which has nonnegative derivative and $\upsilon_1(s)\geq 1, \forall s\in\mathbb{R}^+$. Therefore,

$$\upsilon_D(s) = \frac{\partial\upsilon_1(s)}{\partial s}s+\upsilon_1(s)\geq\upsilon_1(s)\geq 1, \forall s\in\mathbb{R}^+. \tag{6.27}$$

This together with the property $g_{p,1}(\cdot)/g_{x,1}(\cdot) \geq 1$ from Assumption 6.2.3 implies that

$$\upsilon_D(\xi_1^2)g_{p,1}(\cdot)\xi_1\alpha_{u,1} \leq -\frac{g_{p,1}(\cdot)}{g_{x,1}(\cdot)}(\varrho_1(\xi_1) + K_\Theta\varphi_1(\xi_1))\xi_1^2 \leq -\varrho_1(\xi_1)\xi_1^2 - K_\Theta\varphi_1(\xi_1)\xi_1^2. \tag{6.28}$$

By virtue of (6.25), let φ_1 be the smooth positive function satisfying $\varphi_1(\xi_1) \geq \varphi_{1,1}(\xi_1^2)\xi_1^2 + \psi_1(0)$. Then, substituting φ_1 and (6.28) into (6.25), we arrive at

$$DV_{\mathrm{g},1}(\xi_1) \leq -\varrho_1(\xi_1)\xi_1^2 + K_\Theta \sum_{i=2}^{n} \varphi_{1,i}(\bar{\xi}_i^2)\xi_i^2\xi_1^2 + \frac{\Theta_1}{K_\Theta} \tag{6.29}$$

whenever the condition (6.13) holds. This completes the design of the first virtual control $\alpha_{u,1}$.

Remark 6.3.3 *Different from the traditional backstepping design [85], in gauge design, the inequality (6.13) gives rise to the domination function $\psi_1(\cdot)$ depending on all error variables which cannot be canceled all at once by $\alpha_{u,1}$. The novelty here is the decomposition of ψ_1 into functions of square of error variables (6.23) which can be canceled by the next virtual controls.*

Inductive Virtual Control Design

The purpose of the inductive design is to augment the gauge Lyapunov function candidate $V_{\mathrm{g},1}$ and design the virtual controls $\alpha_{u,i}$'s to propagate (6.29) in such a way that all the positive functions in the derivatives along trajectory are eliminated when the actual control u is reached. Let us state the inductive assumption as follows.

Inductive Assumption: at a step $s \geq 1$, there are

i) gauge Lyapunov function candidates $V_{\mathrm{g},j}, j = 1, \ldots, s$ given by

$$V_{\mathrm{g},j}(\bar{\xi}_j) = V_{\mathrm{g},j-1}(\bar{\xi}_{j-1}) + \frac{1}{2}\xi_j^2, j = 2, \ldots, s \tag{6.30}$$

with $V_{\mathrm{g},1}$ given by (6.15);

ii) virtual controls

$$\alpha_{u,j} \;=\; -\frac{1}{g_{x,j}(\bar{x}_j)}(\varrho_j(\xi_j) + K_\Theta \varphi_i(\bar{\xi}_j))\xi_j, j = 1, \ldots, s, \tag{6.31}$$

where $g_{x,j}$'s are given by Assumption 6.2.3, and $\varrho_j, \varphi_j, j = 1, \ldots, s$ are design positive functions given by (6.43), $i = j$ and (6.45) in the below; and

iii) an unknown constant $\Theta_s > 0$ and nonnegative functions $\varphi_{l,j}, l = 1, \ldots, s, j = s+1, \ldots, n$,

such that whenever (6.13) holds true, the derivatives of $V_{g,s}$ along the trajectory $\bar{\xi}_s(t)$ satisfy

$$DV_{g,s}(\bar{\xi}_s(t)) \;\le\; -\sum_{j=1}^{s} \varrho_j(\xi_j)\xi_j^2 + K_\Theta \sum_{j=s+1}^{n} \sum_{l=1}^{s} \varphi_{l,j}(\bar{\xi}_j^2)\xi_j^2\xi_l^2 + \frac{\Theta_s}{K_\Theta}. \tag{6.32}$$

As shown in the previous subsection, the induction assumption holds for $s = 1$. Suppose that the assumption holds for $s = k-1, k \ge 2$. We will show that it also holds for $s = k$.

From the rule of evolution Σ_q given in (6.1), the rules of evolution for the error $\xi_k = x_k - \alpha_{u,k-1}$ are

$$\dot{\xi}_k = g_{q,k}(z, \bar{x}_k, \theta)x_{k+1} + f_{q,k}(z, \bar{x}_k, \theta) + r_{q,k}(z, \bar{x}_k, \theta), q \in \mathbb{Q} \tag{6.33}$$

where $r_{q,k}$'s are continuous functions given by

$$r_{q,k}(\cdot) = -\sum_{j=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial x_j}(g_{q,j}(\cdot)x_{j+1} + f_{q,j}(\cdot)) - \frac{\partial \alpha_{u,k-1}}{\partial K_\Theta}\dot{K}_\Theta, q \in \mathbb{Q}. \tag{6.34}$$

As the update law for K_Θ shall be designated as a continuous function of ξ_1 (see (6.47) below), the functions $r_{q,k}$'s defined by (6.34) then depends only on $z, \bar{x}_k$ and θ as in (6.33).

Let us define $\xi_{k+1} = x_{k+1} - \alpha_{u,k}$ and consider the following k-th gauge Lyapunov function candidate

$$V_{\mathrm{g},k} = V_{\mathrm{g},k-1} + \frac{1}{2}\xi_k^2. \tag{6.35}$$

From the Inductive Assumption, the derivative of $V_{\mathrm{g},k}$ along the trajectory $\bar{\xi}_k(t)$ satisfy

$$\begin{aligned}
DV_{\mathrm{g},k}(\bar{\xi}_k(t)) &= DV_{\mathrm{g},k-1}(\bar{\xi}_{k-1}(t)) + (g_{q_{\sigma,i_\sigma^-(t)},k}(\cdot)(\xi_{k+1} + \alpha_{u,k}) \\
&\quad + f_{q_{\sigma,i_\sigma^-(t)},k}(\cdot) + r_{q_{\sigma,i_\sigma^-(t)},k}(\cdot))\xi_k \\
&\le -\sum_{j=1}^{k-1} \varrho_j(\xi_j)\xi_j^2 + K_\Theta\Big(\sum_{l=1}^{k-1} \varphi_{l,k}(\bar{\xi}_k^2)\xi_l^2\Big)\xi_k^2 + K_\Theta \sum_{j=k+1}^{n}\sum_{l=1}^{k-1} \varphi_{l,j}(\bar{\xi}_j^2)\xi_j^2\xi_l^2 \\
&\quad + \frac{\Theta_{k-1}}{K_\Theta} + \max_{q\in\mathbb{Q}}\Big\{g_{q,k}(\cdot)\alpha_{u,k}\xi_k + (g_{q,k}(\cdot)\xi_{k+1} + f_{q,k}(\cdot) + r_{q,k}(\cdot))\xi_k\Big\}
\end{aligned} \tag{6.36}$$

whenever (6.13) holds true.

Having (6.36), our next purpose is to design the virtual control $\alpha_{u,k}$ that eliminates part of the positive terms from and add desired negative terms to the right hand side of (6.36). By the same type of reasoning leading to (6.19), we obtain smooth functions $\psi_{p,k}^a, \psi_{p,k}^b, p \in \mathbb{Q}$ such that whenever (6.13) is satisfied, we have

$$(g_{p,k}(\cdot)\xi_{k+1} + f_{p,k}(\cdot) + r_{p,k}(\cdot))\xi_k \le \psi_{p,k}^a(\theta)\psi_{p,k}^b(\bar{\xi}_n)|\xi_k| \le K_\Theta(\psi_{p,p}^b(\bar{\xi}_n))^2\xi_k^2 + \frac{(\psi_{p,k}^a(\theta))^2}{4K_\Theta}. \tag{6.37}$$

Moreover, as $\mathbb{Q}$ is finite, there is a C^1 positive function ψ_k nondecreasing in each individual argument such that $(\psi_{p,k}^b(\bar{\xi}_n))^2 \le \psi_k(\bar{\xi}_n^2), \forall p \in \mathbb{Q}$. Again, applying the identity (6.22) to the function ψ_i recursively from $j = n$ to $j = ki$, we obtain the functions $\varphi_{k,j}, j = k, \dots, n$ satisfying

$$\psi_k(\bar{\xi}_n^2) = \sum_{j=k}^{n} \varphi_{i,j}(\bar{\xi}_j^2)\xi_j^2 + \psi_k(\bar{\xi}_{k-1}^2, 0, \dots, 0). \tag{6.38}$$

As $\theta \in \mathcal{L}_\infty^{d_\theta}$ is bounded and $\psi_{p,p}^a$'s are continuous, there is a constant $\Theta_k > 0$ such that $\max\{\Theta_1, \ldots, \Theta_{k-1}, \|(\psi_{p,k}^a(\theta(\cdot)))^2\|/4, p \in \mathbb{Q}\} \leq \Theta_k$. As such, from (6.37) and (6.38), upon satisfaction of (6.13), we have

$$(g_{p,k}(\cdot)\xi_{k+1} + f_{p,k}(\cdot) + r_{p,k}(\cdot))\xi_k \leq K_\Theta\Big(\sum_{j=k}^{n}\varphi_{k,j}(\bar{\xi}_j^2)\xi_j^2 + \psi_k(\bar{\xi}_{k-1}^2, 0, \ldots, 0)\Big)\xi_k^2 + \frac{\Theta_k}{K_\Theta}. \tag{6.39}$$

Note that we are going to design $\alpha_{u,k}$ such that $\alpha_{u,k}\xi_k \leq 0$. Substituting (6.39) into (6.36), it follows that

$$\begin{aligned} DV_{\mathrm{g},k}(\bar{\xi}_k(t)) \leq &-\sum_{j=1}^{k-1}\varrho_j(\xi_j)\xi_j^2 + K_\Theta\Big(\sum_{l=1}^{k}\varphi_{l,k}(\bar{\xi}_k^2)\xi_l^2 + \psi_k(\bar{\xi}_{k-1}^2, 0, \ldots, 0)\Big)\xi_k^2 \\ &+ K_\Theta\sum_{j=k+1}^{n}\sum_{l=1}^{k}\varphi_{l,j}(\bar{\xi}_j^2)\xi_j^2\xi_l^2 + \min\{g_{p,k}(\cdot) : p \in \mathbb{Q}\}\alpha_{u,k}\xi_k + \frac{\Theta_k}{K_\Theta} \end{aligned} \tag{6.40}$$

whenever (6.13) holds true. Consider the virtual control

$$\alpha_{u,k} = -\frac{1}{g_{x,k}(\bar{x}_k)}\left(\varrho_k(\xi_k) + K_\Theta\varphi_k(\bar{\xi}_k)\right)\xi_k, \tag{6.41}$$

where ϱ_k and φ_k are smooth positive functions to be specified. As $g_{p,k}(\cdot) \geq g_{x,k}(\cdot), \forall p \in \mathbb{Q}$ by Assumption 6.2.3, we have

$$g_{p,k}(\cdot)\xi_k\alpha_{u,k} = -\frac{g_{p,k}(\cdot)}{g_{x,k}(\cdot)}(\varrho_k(\cdot) + K_\Theta\varphi_k(\cdot))\xi_k^2 \leq -\varrho_k(\cdot)\xi_k^2 - K_\Theta\varphi_k(\cdot)\xi_k^2, \forall p \in \mathbb{Q}. \tag{6.42}$$

In view of (6.40), let φ_k be the smooth positive function satisfying

$$\varphi_k(\bar{\xi}_k) \geq \sum_{l=1}^{k}\varphi_{l,k}(\bar{\xi}_k^2)\xi_l^2 + \psi_k(\bar{\xi}_{k-1}^2, 0, \ldots, 0). \tag{6.43}$$

Substituting (6.41) and (6.43) into (6.40), we arrive at

$$DV_{\mathrm{g},k}(\bar{\xi}_k(t)) \le -\sum_{j=1}^{k} \varrho_j(\xi_j)\xi_j^2 + K_\Theta \sum_{j=k+1}^{n} \sum_{l=1}^{k+1} \varphi_{l,j}(\bar{\xi}_j^2)\xi_j^2\xi_l^2 + \frac{\Theta_k}{K_\Theta} \tag{6.44}$$

upon satisfaction of (6.13). Thus, the inductive assumption holds for $s = k$.

Actual Control Design

Following the Inductive Design Step, we obtain at the final step $s = n$ the n-th gauge Lyapunov function candidate $V_\mathrm{g} \stackrel{\text{def}}{=} V_{\mathrm{g},n}$ given by (6.30), $j = n$ and the virtual control $\alpha_{u,n}$ given by (6.31), $j = n$. As $\xi_{n+1} = x_{n+1} - \alpha_{u,n} = u - \alpha_{u,n}$, let us select $u = \alpha_{u,n}$ so that $\xi_{n+1} = 0$. The remaining designs are those for ϱ_j's and the update law for K_Θ.

As $\mu'(s) = \partial\mu(s)/\partial s \ge 0, s > 0$ for $\mu \in \mathcal{K}_\infty$ and $\mu'(s)\alpha_1(s) \to 0$ as $s \to 0^+$ by Assumption 6.2.2, there is a class-$\mathcal{K}$ function α_g chosen to be C^1 such that $\alpha_\mathrm{g}(\mu(s)) \ge \mu'(s)\alpha_1(s)$. We choose ϱ_j's to be C^1 functions satisfying

$$\sum_{j=1}^{n} \varrho_j(\xi_j)\xi_j^2 \ge \alpha_\mathrm{g}(2V_\mathrm{g}(\bar{\xi}_n)). \tag{6.45}$$

Such functions ϱ_j's exist as $\alpha_\mathrm{g}(2V_\mathrm{g}(\bar{\xi}_n))$ can be expressed as

$$\begin{aligned} \alpha_\mathrm{g}(2V_\mathrm{g}(\bar{\xi}_n)) &= \alpha_\mathrm{g}\Big(\sum_{j=1}^{n} \tilde{\upsilon}_j(\xi_j^2)\xi_j^2\Big) \le \sum_{j=1}^{n} \alpha_\mathrm{g}\Big(n\tilde{\upsilon}_j(\xi_j^2)\xi_j^2\Big) \\ &= \sum_{j=1}^{n} \Big(\int_0^1 \frac{\partial \alpha_\mathrm{g}(n\tilde{\upsilon}_j(s)s)}{\partial s}\Big|_{s=\beta\xi_j^2} d\beta\Big)\xi_j^2, \end{aligned} \tag{6.46}$$

where $\tilde{\upsilon}_j(\xi_j^2)$ is $\upsilon_1(\xi_1^2)$ if $j = 1$ and is 1, otherwise.

Let $\epsilon_d > 0$ be a desired accuracy and $k_K > 0$ be a tuning gain. We select the following update law for K_Θ [97]:

$$\dot{K}_\Theta = \begin{cases} k_K(|\xi_1| - \epsilon_d) & \text{if} \quad |\xi_1| \geq \epsilon_d \\ 0 & \text{if} \quad |\xi_1| < \epsilon_d \end{cases}, \; K_\Theta(0) = 1. \tag{6.47}$$

Finally, let $w_\Theta(t) \stackrel{\text{def}}{=} \Theta_n / K_\Theta(t)$. We shall estimate the derivative of $U_{\mathrm{g}}(\xi) \stackrel{\text{def}}{=} \mu^{-1}(2V_{\mathrm{g}}(\xi))$ along the evolution of ξ as follows. As $u = \alpha_n^\circ$ gives $\xi_{n+1} = u - \alpha_n^\circ = 0$ and renders (6.32) hold for $s = n$ under (6.13), using (6.45), we have

$$DV_{\mathrm{g}}(\xi(t)) \leq -\alpha_{\mathrm{g}}(2V_{\mathrm{g}}(\xi(t))) + w_\Theta(t) \tag{6.48}$$

whenever (6.13) holds true. As $\mu \in C^1$ and so is μ^{-1}, we have

$$\begin{aligned} DU_{\mathrm{g}}(\xi(t)) &= 2(\mu'(\mu^{-1}(2V_{\mathrm{g}}(\xi(t)))))^{-1} DV_{\mathrm{g}}(\xi(t)) \\ &\leq -2(\mu'(U_{\mathrm{g}}(\xi(t))))^{-1}(\alpha_{\mathrm{g}}(\mu(U_{\mathrm{g}}(\xi(t)))) - w_\Theta(t)) \end{aligned} \tag{6.49}$$

whenever (6.13) holds true. From (6.49) and the designated property $\alpha_{\mathrm{g}}(\mu(s)) \geq \mu'(s)\alpha_1(s)$, if $\alpha_{\mathrm{g}}(\mu(U_{\mathrm{g}}(\xi(t)))) \geq 2\|w_\Theta\| \geq 2w_\Theta(t)$, then we further have

$$DU_{\mathrm{g}}(\xi(t)) \leq -\alpha_1(U_{\mathrm{g}}(\xi(t))) \tag{6.50}$$

whenever (6.13) holds true. This completes the design procedure.

Remark 6.3.4 *The update law (6.47) is adopted from [97]. Though there is no common Lyapunov function to prove the boundedness of K_Θ as in [97], we will show in the next subsection that, in switched systems (6.2), the update law (6.47) still guarantees the boundedness of K_Θ through the converging-input converging-state property of the closed-loop system.*

Remark 6.3.5 *As (6.50) is guaranteed only when (6.13) holds, stability of the re-*

sulting closed-loop system cannot be obtained from (6.44) with arbitrary ϱ_i's as in the usual Lyapunov-based control design of continuous dynamical systems. As such, condition (6.45) on the design functions ϱ_j's is presented to obtain (6.50) for stability analysis presented in the next subsection.

6.3.2 Stability Analysis

In this section, we prove that the control obtained in the previous subsection achieves the proposed control objective. The main steps are as follows. We first show that the (switched) system of $\tilde{x} = [z^T, \xi^T]^T$ has an asymptotic gain with respect to the input w_Θ. Then, we show that the adaptation of the parameter K_Θ will be stopped when the desired accuracy has been reached. Consider the functions

$$V_q(\tilde{x}) \stackrel{\text{def}}{=} \max\{U_q(z), \mu^{-1}(2V_{\text{g}}(\xi))\}, q \in \mathbb{Q}. \tag{6.51}$$

As $U_q, q \in \mathbb{Q}$, and V_{g} are continuous functions and $z(t)$ and $\xi(t)$ are continuous in t, the functions $V_q(\tilde{x}(t)), q \in \mathbb{Q}$ are continuous in t as well. Let $\rho_\mu(s) = s + \mu(s)$ which is a class-$\mathcal{K}_\infty$ function. Let us verify that

$$V_q(\tilde{x}) \geq \rho_\mu^{-1}(U_q(z) + 2V_{\text{g}}(\xi)), \forall q \in \mathbb{Q}. \tag{6.52}$$

Indeed, if $\mu(U_q(z)) \geq 2V_{\text{g}}(\xi)$ then $V_q(\tilde{x}) = U_q(z)$ and hence $\rho_\mu(V_q(\tilde{x})) = U_q(z) + \mu U_q(z) \geq U_q(z) + 2V_{\text{g}}(\xi)$. In the inverse case of $\mu(U_q(z)) < 2V_{\text{g}}$, we have $V_q(\tilde{x}) = \mu^{-1}(2V_{\text{g}}(\xi))$ and hence $\rho_\mu(V_q(\tilde{x})) = \mu^{-1}(2V_{\text{g}}(\xi)) + 2V_{\text{g}}(\xi) \geq U_q(z) + 2V_{\text{g}}(\xi)$. Combining both cases, we obtain (6.52).

Recall that $\{(q_{\sigma,i_j^{\mathcal{D}}}, \Delta\tau_{\sigma,i_j^{\mathcal{D}}}\}_j$ is the sequence of dwell-time switching events of σ and, given the initial time t_0, $t_{\sigma,i_j^{\mathcal{D}}} = t_0 + \tau_{\sigma,i_j^{\mathcal{D}}}$. We have the following proposition.

Proposition 6.3.1 *Under the input $u = \alpha_n^\circ$, the following properties holds along the* *resulting trajectory* $\tilde{x}(t)$*:*

i) $V_{q_{\sigma,i_j^{\wp}}}(\tilde{x}(t)) \leq \max\{\omega_1(V_{q_{\sigma,i_j^{\wp}}}(\tilde{x}(t_{\sigma,i_j^{\wp}})), t - t_{\sigma,i_j^{\wp}}), \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))\}$, $\forall t \in [t_{\sigma,i_j^{\wp}}, t_{\sigma,i_j^{\wp}+1}]$*; and*

ii) if $V_q(\tilde{x}(t_V)) < R_V$ *for some* $t_V \in [t_{\sigma,i_{j-1}^{\wp}+1}, t_{\sigma,i_j^{\wp}}], q \in \mathbb{Q}$*, then* $V_q(\tilde{x}(t)) \leq \omega_2(V_q^w(t_V), T_{\mathrm{p}}), \forall t \in [t_V, t_{\sigma,i_j^{\wp}}]$*,*

where $V_q^w(s) = \max\{\mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|)), V_q(\tilde{x}(s))\}$*,* ω_1, ω_2*, and* R_V *are given in Assumption 6.2.2, and* $t_{\sigma,i_{-1}^{\wp}+1} \stackrel{\text{def}}{=} t_{\sigma,0}$*.*

Proof: See Section 6.5. ■

Subject to the update law (6.47), $K_\Theta(t)$ is nondecreasing and hence $w_\Theta(t)$ is bounded. Define the constant $V_{ini} = \max\{\chi(\|w_\Theta\|), V_{q_{\sigma,0}}(\tilde{x}(t_0))\}$ and the function $\chi(s) = \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2s)), s > 0$. We have the following theorem.

Theorem 6.3.1 *Consider the switched system (6.2) whose driving dynamics are described by (6.1). Suppose that Assumptions 6.2.1–6.2.3 hold and* $V_{ini} < R_V$*. Then, under the control* $u = \alpha_n^\circ$ *given by (6.31),* $j = n$ *with* ϱ_j*'s satisfying (6.45), the resulting switched system of* $\tilde{x} = [z^T, \xi^T]^T$*, whose driving dynamics are described by* $\dot{z} = Q_q(z, y, \theta)$ *and (6.33) and whose input is* w_Θ*, has an asymptotic gain.*

Proof: We shall prove the theorem by showing that the (switched) system generating $\tilde{x}$ satisfy conditions of Theorem 5.2.1 with the functions $V_q, q \in \mathbb{Q}$ defined in (6.51).

As the driving dynamics of z and x are described by differential equations (6.1) and the resulting control $u = \alpha_n^\circ$ is continuous, from the theory of differential equations [137], we know that the corresponding transition mappings $\mathscr{T}_q$ are continuous in their domains of existence. We shall verify the forward completeness of $\tilde{x}$ through its boundedness in the subsequent verification of condition ii) of Theorem 5.2.1.

From (6.52) and definition of V_q, we have

$$U_q(z) + \mu^{-1}(2V_{\mathrm{g}}(\xi)) \geq V_q(\tilde{x}) \geq \rho_\mu^{-1}(U_q(z) + 2V_{\mathrm{g}}(\xi)). \tag{6.53}$$

As μ and ρ_μ are class-$\mathcal{K}_\infty$ functions, this shows that (5.10) holds.

To verify condition i) of Theorem 5.2.1, we have the following cases at the time $t \in [t_{\sigma,i}, t_{\sigma,i+1}], i \in \mathbb{N}$.

Case 1: Inequality (6.13) does not hold for $q = q_{\sigma,i}$. In this case, we have $V_{q_{\sigma,i}}(\tilde{x}(t)) = U_{q_{\sigma,i}}(z(t))$ and

$$\upsilon(y^2) \leq \upsilon_1(\xi_1^2(t))\xi_1^2(t) \leq \mu(U_{q_{\sigma,i}}(z(t))). \tag{6.54}$$

From Assumption 6.2.1 and (6.54), we have

$$\begin{aligned} DV_{q_{\sigma,i}}(\tilde{x}(t)) &= \frac{\partial U_{q_{\sigma,i}}(z(t))}{\partial z} Q_{q_{\sigma,i}}(z(t), x_1(t), \theta(t)) \\ &\leq -\alpha_1(U_{q_{\sigma,i}}(z(t))) + \mu(U_{q_{\sigma,i}}(z(t))) \leq -\tilde{\alpha}_1(V_{q_{\sigma,i}}(\tilde{x}(t))), \end{aligned} \tag{6.55}$$

where we have defined $\tilde{\alpha}_1(s) = \alpha_1(s) - \mu(s)$.

Case 2: Inequality (6.13) holds for $q = q_{\sigma,i}$. In this case, we have $V_{q_{\sigma,i}}(\tilde{x}(t)) = \mu^{-1}(2V_{\mathrm{g}}(\xi(t))) = U_{\mathrm{g}}(\xi(t))$. By control design, (6.50) holds if $V_{q_{\sigma,i}}(\tilde{x}(t)) = U_{\mathrm{g}}(\xi(t)) \geq \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))$.

Both these cases show that

$$DV_{q_{\sigma,i}}(\tilde{x}(t)) \leq -\tilde{\alpha}_1(V_{q_{\sigma,i}}(\tilde{x}(t))) \text{ if } V_{q_{\sigma,i}}(\tilde{x}(t)) \geq \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|)). \tag{6.56}$$

As μ, α_{g}, and $\tilde{\alpha}_1$ are class-$\mathcal{K}_\infty$ functions, this shows that the condition i) of Theorem 5.2.1 is satisfied.

We now verify condition ii) of Theorem 5.2.1. Suppose that $V_q(\tilde{x}(t_j^*)) < R_V$ for some $t_j^* \in [t_{\sigma,i_{j-1}^{\mathcal{P}}+1}, t_{\sigma,i_j^{\mathcal{P}}}]$. According to ii) of Proposition 6.3.1, $V_q(\tilde{x}(t))$ remains bounded on $[t_j^*, t_{\sigma,i_j^{\mathcal{P}}}]$ and hence, by (6.53), so is $\tilde{x}(t)$. In addition,

$$V_q(\tilde{x}(t_{\sigma,i_j^{\mathcal{P}}})) \leq \max\{\omega_2(V_q(\tilde{x}(t_j^*)), T_{\mathrm{p}}), \omega_2(\mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|)), T_{\mathrm{p}})\}. \tag{6.57}$$

At the time $t_{\sigma,i_j^{\mathcal{D}}}$, we have either *a)* $V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}})) = U_{q_{\sigma,i_j^{\mathcal{D}}}}(z(t_{\sigma,i_j^{\mathcal{D}}}))$ which, by Assumption 6.2.1, is less than $\beta(U_q(z(t_{\sigma,i_j^{\mathcal{D}}})))$ which, in turn, is less than $\beta(V_q(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}})))$ or *b)* $V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}})) = U_{\mathrm{g}}(\xi(t_{\sigma,i_j^{\mathcal{D}}})) \leq V_q(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}}))$. It can be verified that $\beta(s) \geq s, s \geq 0$. These together result in

$$V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}})) \;\leq\; \beta(V_q(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}}))). \tag{6.58}$$

As ω_1 is nonincreasing in its second argument, the boundedness of $V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}}))$ from (6.58) coupled with i) of Proposition 6.3.1 implies that $V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t))$ remains bounded on $[t_{\sigma,i_j^{\mathcal{D}}}, t_{\sigma,i_j^{\mathcal{D}}+1}]$ and hence so does $\tilde{x}(t)$. In conclusion, $\tilde{x}(t)$ is bounded on $[t_j^*, t_{\sigma,i_j^{\mathcal{D}}+1}]$.

Similarly, the boundedness of $\tilde{x}(t)$ on the subsequent time period $[t_{\sigma,i_j^{\mathcal{D}}+1}, t_{\sigma,i_{j+1}^{\mathcal{D}}+1}]$ is obtained if $V_q(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}+1})) < R_V$ for some $q \in \mathbb{Q}$. We shall show that this holds true with $q = q_{\sigma,i_j^{\mathcal{D}}}$. Applying i) of Proposition 6.3.1, we have

$$V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}+1})) \leq \max\{\omega_1(V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}})), \tau_{\mathrm{p}}), \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))\}. \tag{6.59}$$

By stacking (6.57), (6.58), and (6.59) and using Assumption 6.2.2, we obtain

$$\begin{aligned} V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}+1})) &\leq \max\{\mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|)), \omega_1(\beta(\omega_2(\mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|)), T_{\mathrm{p}})), \tau_{\mathrm{p}})), \\ &\qquad \omega_1(\beta(\omega_2(V_q(\tilde{x}(t_j^*)), T_{\mathrm{p}})), \tau_{\mathrm{p}})\} \\ &\leq \max\{\chi(\|w_\Theta\|), \omega_0(V_q(\tilde{x}(t_j^*)), \tau_0)\}, \end{aligned} \tag{6.60}$$

where we have used the property (6.7) from Assumption 6.2.2. As $\chi(\|w_\Theta\|) \leq V_{ini} \leq R_V$ and $\omega_0(s, \tau_0) < s$, $V_q(\tilde{x}(t_j^*) \leq R_V$, (6.60) shows that $V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}+1})) < R_V$ and subsequently $\tilde{x}(t)$ is bounded on $[t_{\sigma,i_j^{\mathcal{D}}+1}, t_{\sigma,i_{j+1}^{\mathcal{D}}+1}]$.

Now, let $t_0^* = t_{\sigma,0} = t_0$ if $t_{\sigma,0} < t_{\sigma,i_0^{\mathcal{D}}}$ and $t_0^* = t_{\sigma,1}$ if $t_{\sigma,0} = t_{\sigma,i_0^{\mathcal{D}}}$. We claim that $\tilde{x}(t)$ is bounded on $[t_0, t_0^*]$ and $V_q(\tilde{x}(t_0^*)) < R_V$ for some $q \in \mathbb{Q}$. This claim is obvious for $t_0^* =$

$t_{\sigma,0}$. In the case $t_0^* = t_{\sigma,1}$, as the first switching event is of dwell-time and t_0^* is its end time, from i) of Proposition 6.3.1 and the property $V_{q_{\sigma,i_0^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_0^{\mathcal{D}}})) = V_{q_{\sigma,0}}(\tilde{x}(t_0)) < R_V$, it follows that $\tilde{x}(t)$ is bounded on $[t_0, t_0^*]$. In addition, at the time t_0^*, as $i_0^{\mathcal{D}} = 0$, using i) of Proposition 6.3.1, we have either $V_{q_{\sigma,0}}(\tilde{x}(t_0^*)) = V_{q_{\sigma,i_0^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_0^{\mathcal{D}}+1})) \leq \chi(\|w_\Theta\|) < R_V$ or $V_{q_{\sigma,i_0^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_0^{\mathcal{D}}+1})) \leq \omega_1(V_{q_{\sigma,i_0^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_0^{\mathcal{D}}})), \tau_{\mathrm{p}}) \leq G_1^{-1}(G_1(V_{q_{\sigma,0}}(\tilde{x}(t_{\sigma,i_0^{\mathcal{D}}}))) + 0) = V_{q_{\sigma,i_0^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_0^{\mathcal{D}}})) = V_{q_{\sigma,0}}(\tilde{x}(t_{\sigma,0})) < R_V$. Thus, $V_{q_{\sigma,0}}(\tilde{x}(t_0^*)) < R_V$, i.e., the claim is true.

By the preceding argument, we conclude that $\tilde{x}(t)$ is bounded on $t \in [t_0, \infty)$ and hence the forward completeness of $\tilde{x}$ follows.

From the properties $\omega_0(s, \tau_0) < s$ and $\omega_0(\omega_0(a, s), t) \leq \omega_0(a, s + t)$ as given in Assumption 6.2.2, applying (6.60) successively from $t_{\sigma,i_j^{\mathcal{D}}+1}$ back to t_0^*, we have

$$\begin{aligned} V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}+1})) &\leq \max\{\chi(\|w_\Theta\|), \omega_0(V_{q_{\sigma,i_{j-1}^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_{j-1}^{\mathcal{D}}+1})), \tau_0)\} \leq \dots \\ &\dots \leq \max\{\chi(\|w_\Theta\|), \omega_0(V_{q_{\sigma,0}}(\tilde{x}(t_0^*)), (j-1)\tau_0)\}. \end{aligned} \tag{6.61}$$

As $\omega_0 \in \mathcal{KL}$, taking the limits of $V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}+1}))$ as $j \to \infty$ we obtain

$$\limsup_{j\to\infty} V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}+1})) \leq \chi(\|w_\Theta\|). \tag{6.62}$$

In addition, since $V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}+1})) < R_V$, from ii) of Proposition 6.3.1, we have $V_{q_{\sigma,i_\sigma^-(\bar{t})}}(x(t)) \leq \beta(V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t))) \leq \beta(\omega_2(V^w_{q_{\sigma,i_j^{\mathcal{D}}}}(t_{\sigma,i_j^{\mathcal{D}}+1}), T_{\mathrm{p}})), \forall t \in [t_{\sigma,i_j^{\mathcal{D}}+1}, t_{\sigma,i_{j+1}^{\mathcal{D}}}]$. Since V_q's are nonnegative, this implies that

$$\begin{aligned} \limsup_{j\to\infty} \max_{t\in\mathscr{I}_j^{\mathrm{P}}} \mathrm{Var}_{t_{\sigma,i_j^{\mathcal{D}}+1}}^{t}[V_{q_{\sigma,i_j^{\mathcal{D}}}}, V_{q_{\sigma,i_\sigma^-(\bar{t})}}](x) &\leq \limsup_{j\to\infty} \max_{t\in\mathscr{I}_j^{\mathrm{P}}} V_{q_{\sigma,i_\sigma^-(\bar{t})}}(x(t)) \\ &\leq \limsup_{j\to\infty} \rho(\omega_2(V^w_{q_{\sigma,i_j^{\mathcal{D}}}}(t_{\sigma,i_j^{\mathcal{D}}+1}), T_{\mathrm{p}})) \\ &\leq \max\left\{\rho(\omega_2(\chi(\|w_\Theta\|), T_{\mathrm{p}})), \limsup_{j\to\infty} V_{q_{\sigma,i_j^{\mathcal{D}}}}(\tilde{x}(t_{\sigma,i_j^{\mathcal{D}}+1}))\right\} \\ &\leq \rho(\omega_2(\chi(\|w_\Theta\|), T_{\mathrm{p}})) \stackrel{\mathrm{def}}{=} \gamma_2(\|w_\Theta\|). \end{aligned} \tag{6.63}$$

Thus, condition ii) of Theorem 5.2.1 is satisfied.

As the satisfaction of the conditions of Theorem 5.2.1 is independent of switching signal, applying Theorem 5.2.1, we have the conclusion of the theorem. ∎

Theorem 6.3.2 *Suppose that the hypotheses of Theorem 6.3.1 hold. Then, under the control $u = \alpha_n$ given by (6.31), $j = n$ with ϱ_i's satisfying (6.45), the output $x_1(t)$ converge to the set $B_{\epsilon_d} = \{s \in \mathbb{R} : |s| \leq \epsilon_d\}$ and the trajectory $\tilde{x}(t)$ remains bounded.*

Proof: As $w_\Theta(t)$ is bounded, by Theorem 6.3.1, there is a class-$\mathcal{K}_\infty$ function γ independent of switching sequence $\sigma \in \mathbb{S}$ such that $\tilde{x}(t) \to \{\zeta \in \mathbb{R}^{d\times n} : \|\zeta\| \leq \gamma(\|w_\Theta\|)\}$. Our first purpose is to show that $K_\Theta(t)$ is bounded. Indeed, suppose that the converse holds. Then there is a divergent sequence $\{t_h\}_j \subset [t_0, \infty)$ such that $\dot{K}_\Theta(t_j) \neq 0, \forall j \in \mathbb{N}$. From the update law (6.47), it must hold that $|\xi_1(t_j)| \geq \epsilon_d, \forall j \in \mathbb{N}$. Let $\epsilon > 0$ and $\delta > 0$ be numbers satisfying $\gamma(\epsilon) + \delta < \epsilon_d$.

As Θ_n is bounded and K_Θ is unbounded and nondecreasing, there is a time $t(\epsilon)$ such that $w_\Theta(t) = \Theta_n/K_\Theta(t) \leq \epsilon, \forall t \geq t(\epsilon)$. Let $t_{\sigma,i(\epsilon)}$ be the switching time of σ that is greater than $t(\epsilon)$. By Theorem 6.3.1, the state $\tilde{x}(t)$ of the switched error system remains bounded for $t \leq t_{\sigma,i(\epsilon)}$. As a result, the trajectory $\tilde{x}(t), t \geq t_{\sigma,i(\epsilon)}$ of the switched error system is the trajectory $\tilde{x}'(t)$ of the same switched error system with initial state $\tilde{x}'(0) = \tilde{x}(t_{\sigma,i(\epsilon)})$, initial value of the time-varying parameter $\theta(0) = \theta(t_{\sigma,i(\epsilon)})$, input $\tilde{w}_\Theta$ defined by $\tilde{w}_\Theta(t) = w_\Theta(t + t_{\sigma,i(\epsilon)}), t \geq 0$, and switching sequence $\sigma(\epsilon)$ defined by $(q_{\sigma(\epsilon),i}, \Delta\tau_{\sigma(\epsilon),i}) = (q_{\sigma,i+i(\epsilon)}, \Delta\tau_{\sigma,i+i(\epsilon)}), i \in \mathbb{N}$. Obviously $\sigma(\epsilon)$ also has the persistent dwell-time τ_{p} with the period of persistent T_{p}. By Theorem 6.3.1, $\limsup_{t\to\infty} \|\tilde{x}'(t)\| \leq \gamma(\|\tilde{w}_\Theta\|) \leq \gamma(\epsilon)$. As $\tilde{x}'(t) = \tilde{x}(t + t(\epsilon)), t \geq 0$ and $\|\tilde{w}_\Theta\| \leq \epsilon$, it follows that there is $T_\delta \in \mathbb{R}^+$ such that $\|\tilde{x}(t+t(\epsilon))\| = \|\tilde{x}'(t)\| \leq \gamma(\epsilon)+\delta < \epsilon_d, \forall t \geq T_\delta$. Thus $|\xi_1(t_j)| \leq \|\tilde{x}(t)\| \leq \gamma(\epsilon)+\delta < \epsilon_d$, for sufficiently large j which is a contradiction. Therefore, $K_\Theta(t)$ is bounded.

Finally, as $\tilde{x}(t)$ is continuous and bounded by Theorem 6.3.1, $\xi_1(t)$ and hence $\dot{K}_\Theta(t)$ are uniformly continuous. Thus, the monotonity from the update law (6.47) and the boundedness of $K_\Theta(t)$ show that $\lim_{t\to\infty} \int_0^t \dot{K}_\Theta(s)ds$ exists and is finite. By Barbalat's lemma, we have $\lim_{t\to\infty} \dot{K}_\Theta(t) = 0$. Therefore, $\xi_1(t) \to B_{\epsilon_d}, t \to \infty$ as, otherwise $\lim_{t\to\infty} K_\Theta(t) \neq 0$, a contradiction. ■

6.4 Design Example

In this section, we present an example to demonstrate the application of the presented theory in switching-uniform output regulation of switched systems with unmeasured dynamics, state dependent control gains, and persistent dwell-time switching.

Consider the switched systems whose constituent systems are

$$\Sigma_1 : \left\{ \begin{array}{lcl} \dot{z}_1 & = & -z_1(1 + z_1^4 + z_2^2) + \frac{1}{4}z_2(1 + \sin x_1) \\ \dot{z}_2 & = & -z_2(1 + z_1^4 + z_2^2) - \frac{1}{4}z_1^3(1 + \sin x_1) \\ \dot{x}_1 & = & \theta_1 x_2 + e^{\theta_2 x_1} z_1^2 \\ \dot{x}_2 & = & (2 + x_2^2 + \theta_1 z_2^2)u + \theta_2 x_1 z_2 \end{array} \right. ,$$

$$\Sigma_2 : \left\{ \begin{array}{lcl} \dot{z}_1 & = & -8z_1 + 2z_2 \\ \dot{z}_2 & = & -z_1 - 12z_2 + x_1 \\ \dot{x}_1 & = & \theta_3 x_2 + 2\theta_4 \sqrt{z_1^2 + 3z_2^2} \\ \dot{x}_2 & = & (\theta_5 + x_2^2)u + \theta_4 x_2(z_1 + z_2) \end{array} \right. , \tag{6.64}$$

where the unknown time-varying parameters are $\theta_1 = 1 + \sin^2 t, \theta_2 = \cos t, \theta_3 = 2 + \cos t, \theta_4 = \sin t$, and $\theta_5 = 2 - \sin t$. The output of the system is $y = x_1$.

Due to high-order terms in Σ_1, a common ISS-Lyapunov function for z-subsystems may not exist. Instead, we have the following Lyapunov functions for z-systems of Σ_1 and Σ_2:

$$U_1(z) = \frac{1}{4}z_1^4 + \frac{1}{2}z_2^2, \text{ and } U_2(z) = z_1^2 + z_1 z_2 + 4z_2^2. \tag{6.65}$$

Let Q_1 and Q_2 denote the vector fields of z-subsystems of Σ_1 and Σ_2, respectively. With the help of Young's inequality, the Lie derivatives of U_1 and U_2 along the vector fields Q_1 and Q_2 are estimated as

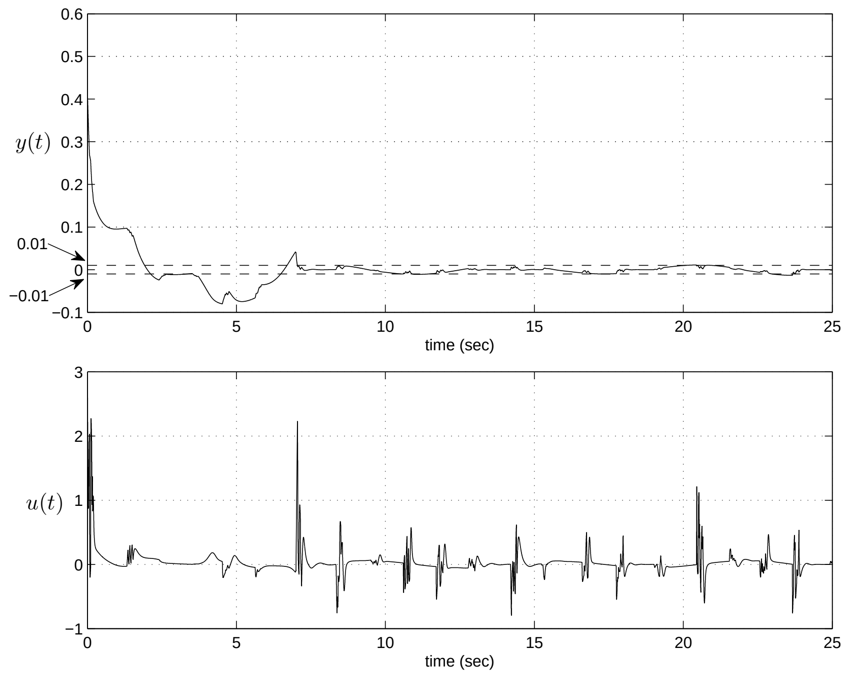
$$\mathcal{L}_{Q_1}U_1(z) \leq -4U_1^2(z) - 2U_1(z),\ \mathcal{L}_{Q_2}U_1(z) \leq U_1^2(z)/150 + U_1(z)/12 + x_1^2/24;$$
$$\mathcal{L}_{Q_1}U_2(z) \leq 0.5U_2(z),\ \mathcal{L}_{Q_2}U_2(z) \leq -4U_2(z) + x_1^2. \tag{6.66}$$

In view of (6.4), as $\mathcal{L}_{Q_2}U_1 \leq U_1^2/150 + U_1/12$, estimate of increment of U_1 has finite escape time. However, if $U_1(z) \leq 150/4$, then we have $\mathcal{L}_{Q_2}U_1 \leq U_1/4 + U_1/12$ so that estimates of increments and decrements of both U_1 and U_2 can be expressed in terms of exponential functions. As such, we will choose small persistent dwell-time T_p and small initial condition $U_1(z(0))$ for simulation to avoid finite escape time.

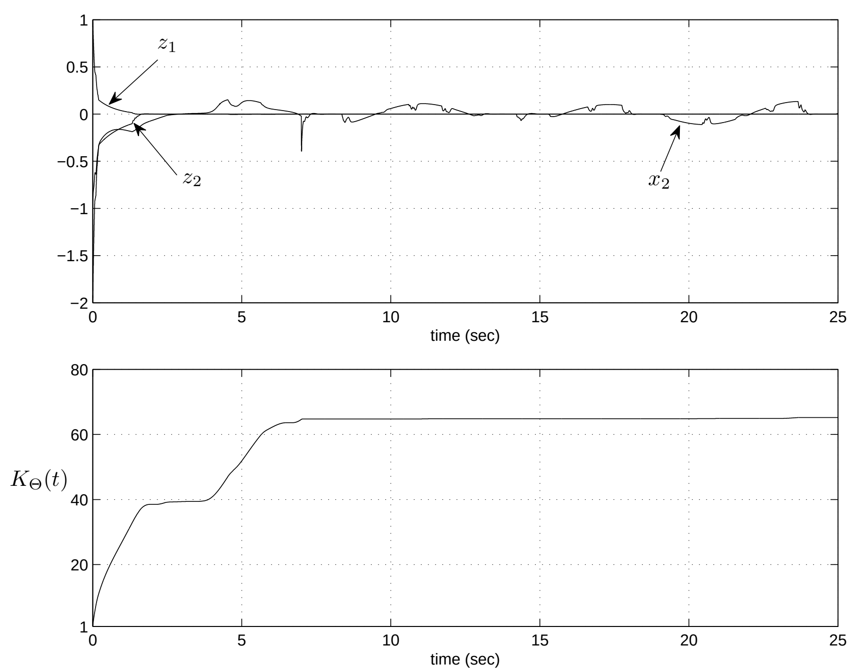
From (6.65), a function β to satisfy Assumption 6.2.1 is $\beta(s) = 3\sqrt{s}+9s+s^2/4$. Let the design gauge be $U_q(z) \leq V_\mathrm{g}$, i.e., $\mu(s) = s$. Then, provided that $U_1(z) \leq 150/4$, the functions β_1 and β_2 in Assumption 6.2.1 can be computed as $\beta_2(s,t) = se^{t/2}$ and $\beta_1(s,t) = se^{-2t}$, and hence, the condition (6.7) turns to impose $\beta(\beta(s))e^{-(2\tau_\mathrm{p} - T_\mathrm{p}/2)}$ to be a class $\mathcal{KL}$ function, which can be satisfied for $2\tau_\mathrm{p} - T_\mathrm{p}/2 > 0$. In addition, the lower bounds for control gains are $g_{x,1} = 1$ and $g_{x,2} = 1 + x_2^2$. As such, the switched system given by (6.64) satisfies conditions of Theorem 6.3.2. Following the design procedure in Section IV, we obtain the following adaptive control

$$\begin{aligned}
\alpha_1 &= -\Big[k_1 + k_2\xi_1^2 + K_\Theta(e^{\xi_1^2} + 2)\xi_1^2\Big]\xi_1 \\
A &= k_1 + 3k_2\xi_1^2 + 3K_\Theta(e^{\xi_1^2} + 2)\xi_1^2 + 2K_\Theta e^{\xi_1^2}\xi_1^4 \\
u &= -\frac{1}{1+x_2^2}\Big(k_3 + k_4\xi_2^2 + K_\Theta A^2 x_2^2 + K_\Theta A^2(e^{\xi_1^2} + 1)(\xi_1^2 + \xi_2^2) \\
&\quad + K_\Theta(\xi_1^2 + \xi_2^2)x_2^2 + K_\Theta(e^{\xi_1^2} + 4)\xi_1^2\Big)\xi_2,
\end{aligned} \tag{6.67}$$

where $k_i, i = 1, \ldots, k_4$ are design parameters, $\xi_1 = x_1$, $\xi_2 = x_2 - \alpha_1$, and K_Θ is updated by (6.47). A value for the unknown constant Θ_2 is $\Theta_2 = 7 > \sup\{\theta_1^2/4 + 2e^{\theta_2^2} + \theta_2^2/2, \theta_3^2/4 + 3\theta_4^2\}$.



(a) Output convergence and control input



(b) $z_1(t), z_2(t), x_2(t)$, and $K_\Theta(t)$

Figure 6.2: Adaptive output regulation

For simulation, we choose $k_1 = k_3 = 1, k_2 = k_4 = 0.5$, the initial state $(z, x) = [1, -2, 0.4, -1]^T$, and the desired accuracy $\epsilon_d = 0.01$. The tuning gain k_K is chosen to be 100 for fast convergence of K_Θ. The switching sequence with persistent period $T_\mathrm{p} = 0.2s$ and dwell-time $\tau_\mathrm{p} = 0.8s$ is generated in such a way that i) on persistent periods, the lengths of switching intervals are generated randomly in $[0, 0.05]$ and ii) the lengths of dwell-time intervals are generated randomly in $[0.8, 1.2]$.

The simulation results for this example are shown in Figure 6.2. It can be seen from Figure 6.2(a) that the output regulation is well obtained. The peak points in control signal are due to the fast transient periods caused by changes of active constituent systems. It is also observed from Figure 6.2(b) that the adaptive gain K_Θ converges to a fixed value and the remaining signals are bounded. Thus, the simulation results well illustrated the presented theory.

6.5 Proof of Proposition 6.3.1

In this proof, a closed interval $[t_1, t_2]$ (an open interval (t_1, t_2), resp.) is said to be $\xi-$domt$[q]$ ($z-$domt$[q]$, resp.) if (6.13) holds (does not hold, resp.) for all t in this interval. An interval of either these properties is said to be maximal in its corresponding property if it has no strict subinterval of the same property. We further denote $q_{\sigma, i_j^\mathcal{D}}$ by $q_j^\mathcal{D}$ for short.

Consider a dwell-time switching event $(q_{\sigma, i_j^\mathcal{D}}, \Delta\tau_{\sigma, i_j^\mathcal{D}}), j \in \mathbb{N}$. We state that if the inequality

$$V_{q_j^\mathcal{D}}(\tilde{x}(t)) \leq \mu^{-1}(\alpha_\mathrm{g}^{-1}(2\|w_\Theta\|)), \tag{6.68}$$

holds for some $t = t_\mathrm{i} \in [t_{\sigma, i_j^\mathcal{D}}, t_{\sigma, i_j^\mathcal{D}+1}]$, then it also holds for all $t \in [t_\mathrm{i}, t_{\sigma, i_j^\mathcal{D}+1}]$.

Indeed, consider the case t_i belongs to a $\xi-$domt$[q_j^\mathcal{D}]$ interval $[t_1, t_2] \subset [t_{\sigma, i_j^\mathcal{D}}, t_{\sigma, i_j^\mathcal{D}+1}]$. As (6.13) holds on $[t_1, t_2]$, we have $V_{q_j^\mathcal{D}}(\tilde{x}(t)) = \mu^{-1}(2V_\mathrm{g}(\xi(t)))$ and (6.48) holds for all $t \in [t_1, t_2]$. As $\mu^{-1} \in \mathcal{K}_\infty$, this coupled with the satisfaction of (6.68) at $t = t_\mathrm{i}$

implies that $2V_{\mathrm{g}}(\xi(t_{\mathrm{i}})) \leq \alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|)$ and (6.48) holds on $[t_1, t_2]$. As such, $2V_{\mathrm{g}}(\xi(t)) \leq \alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|), \forall t \in [t_{\mathrm{i}}, t_2]$. Since $V_{q_j^{\mathcal{D}}}(\tilde{x}(t)) = \mu^{-1}(2V_{\mathrm{g}}(\xi(t))), t \in [t_1, t_2]$, this shows that (6.68) holds on $[t_{\mathrm{i}}, t_2]$.

Let (t_2, t_3) be the $z-$domt$[q_j^{\mathcal{D}}]$ next to $[t_1, t_2]$. On this interval, we have $V_{q_j^{\mathcal{D}}}(\tilde{x}(t)) = U_{q_j^{\mathcal{D}}}(z(t))$ and the inverse of (6.13) holds for $q = q_j^{\mathcal{D}}$. Thus, from Assumption 6.2.1 and the fact that the driving dynamics for z on $[t_{\sigma,i_j^{\mathcal{D}}}, t_{\sigma,i_j^{\mathcal{D}}+1}]$ is that of the index $q_{\sigma,i_j^{\mathcal{D}}}$, for $t \in (t_2, t_3)$, we have

$$\begin{aligned} DU_{q_j^{\mathcal{D}}}(z(t)) &\leq -\alpha_1(U_{q_j^{\mathcal{D}}}(z(t))) + \upsilon(y^2) \\ &\leq -\alpha_1(U_{q_j^{\mathcal{D}}}(z(t))) + \mu(U_{q_j^{\mathcal{D}}}(z(t))) < 0, \end{aligned} \tag{6.69}$$

where we have used the property $\upsilon(\xi_1^2) \leq 2V_{\mathrm{g}}(\xi(t)) \leq \mu(U_{q_j^{\mathcal{D}}}(z(t)))$ held on $z-$domt$[q_j^{\mathcal{D}}]$ intervals. Therefore, $U_{q_j^{\mathcal{D}}}(z(t))$ is decreasing on (t_2, t_3). As we have $U_{q_j^{\mathcal{D}}}(z(t)) = \mu^{-1}(2V_{\mathrm{g}}(\xi(t)))$ at the transition time t_2 and (6.68) holds at $t = t_2$, this coupled with the continuity of $U_{q_j^{\mathcal{D}}}(z(t))$ further implies that $V_{q_j^{\mathcal{D}}}(\tilde{x}(t)) = U_{q_j^{\mathcal{D}}}(z(t))$ is bounded by $\mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))$ on (t_2, t_3) as well. Continuing this process until $t_{\sigma,i_j^{\mathcal{D}}+1}$ is reached, we conclude that the statement is true.

In the case t_{i} belongs to a $z-$domt$[q_j^{\mathcal{D}}]$ interval, it is obvious from the above argument that the statement is true.

We now consider t_{i} to be minimal in the sense that there is no $t_0 \in [t_{\sigma,i_j^{\mathcal{D}}}, t_{\sigma,i_j^{\mathcal{D}}+1}], t_0 < t_{\mathrm{i}}$ such that (6.68) holds for $t = t_0$. For such t_{i}, we have $V_{q_j^{\mathcal{D}}}(\tilde{x}(t)) > \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))$, $\forall t \in [t_{\sigma,i_j^{\mathcal{D}}}, t_{\mathrm{i}})$ and hence (6.50) holds on $\xi-$domt$[q_j^{\mathcal{D}}]$ intervals contained in $[t_{\sigma,i_j^{\mathcal{D}}}, t_{\mathrm{i}})$. Thus, on $\xi-$domt$[q_j^{\mathcal{D}}]$ intervals, as $V_{q_{i_j^{\mathcal{D}}}}(\tilde{x}(t)) = \mu^{-1}(2V_{\mathrm{g}}(\xi(t))) = U_{\mathrm{g}}(\xi(t))$, we have $DV_{q_j^{\mathcal{D}}}(\tilde{x}(t)) \leq -\alpha_1(V_{q_j^{\mathcal{D}}}(\tilde{x}(t)))$. This coupled with (6.69) and the fact that $V_{q_j^{\mathcal{D}}}(\tilde{x}(t)) = U_{q_j^{\mathcal{D}}}(z(t))$ on $z-$domt$[q_j^{\mathcal{D}}]$ intervals show that

$$DV_{q_j^{\mathcal{D}}}(\tilde{x}(t)) \leq -\alpha_1(V_{q_j^{\mathcal{D}}}(\tilde{x}(t))) + \mu(V_{q_j^{\mathcal{D}}}(\tilde{x}(t))), \forall t \in [t_{\sigma,i_j^{\mathcal{D}}}, t_{\mathrm{i}}) \tag{6.70}$$

Since $D^+V_{q_j^p}(\tilde{x}(t)) \le DV_{q_j^p}(\tilde{x}(t))$, using comparison principle [77] in combination with Lemma 6.2.1 for (6.70), we have $V_{q_j^p}(\tilde{x}(t)) \le \omega_1(V_{q_j^p}(\tilde{x}(t_{\sigma,i_j^p})), t - t_{\sigma,i_j^p}), \forall t \in [t_{\sigma,i_j^p}, t_i)$. Combining this estimate with the above estimate (6.68) of $V_{q_j^p}(\tilde{x}(t))$ on $[t_{\mathrm{i}}, t_{\sigma,i_j^p+1}]$, we obtain

$$V_{q_j^p}(\tilde{x}(t)) \le \max\{\omega_1(V_{q_j^p}(\tilde{x}(t_{\sigma,i_j^p})), t - t_{\sigma,i_j^p}), \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))\}, \forall t \in [t_{\sigma,i_j^p}, t_{\sigma,i_j^p+1}], \tag{6.71}$$

i.e., the statement i) of the Proposition is true.

We prove the statement ii) of the Proposition by examining increments of $V_q(\tilde{x}(t))$ on the interval $[t_V, t_{\sigma,i_j^p}]$. Consider the first maximal $z-$domt$[q]$ subinterval (t_1, t_2) of $[t_V, t_{\sigma,i_j^p}]$. As $\upsilon(y_2) \le \mu(U_q(z(t)))$ and $V_q(\tilde{x}(t)) = U_q(z(t))$ on $z-$domt$[q]$ intervals, from Assumption 6.2.1, we have

$$DV_q(\tilde{x}(t)) \le \alpha_2(V_q(\tilde{x}(t))) + \mu(V_q(\tilde{x}(t))), \forall t \in (t_1, t_2). \tag{6.72}$$

Again, applying comparison principle [77] in combination with Lemma 6.2.1 for (6.72), we obtain

$$V_q(\tilde{x}(t)) \le \omega_2(V_q(\tilde{x}(t_1)), t - t_1), t \in (t_1, t_2). \tag{6.73}$$

In addition, as $[t_V, t_1]$ (if not empty) is $\xi-$domt$[q]$, from the above proof of i), we have $V_q(\tilde{x}(t_1)) \le \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))$ if $V_q(\tilde{x}(t_{\mathrm{i}})) \le \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))$ for some $t_{\mathrm{i}} \in [t_V, t_1]$ and $DV_q(\tilde{x}(t)) \le -\alpha_1(V_q(\tilde{x}(t))), \forall t \in [t_V, t_1]$ implying that $V_q(\tilde{x}(t)) \le V_q(\tilde{x}(t_V)), \forall t \in [t_V, t_1]$ if there is no such t_{i}. Therefore, for any $t \in [t_V, t_1]$, we have

$$V_q(\tilde{x}(t)) \le \max\{\mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|)), V_q(\tilde{x}(t_V))\} = V_q^w(t_V). \tag{6.74}$$

Since $\alpha_2(s) + \mu(s) > 0$, ω_2 is nondecreasing in both arguments. Thus, combining

(6.73) and (6.74), we obtain

$$V_q(\tilde{x}(t)) \leq \omega_2(V_q^w(t_V), t - t_V), \forall t \in [t_V, t_2]. \tag{6.75}$$

We now consider the next pair of $\xi-$domt$[q]$ and $z-$domt$[q]$ subintervals of $[t_V, t_{\sigma,i_j^{\mathfrak{p}}}]$, namely $[t_2, t_3]$ and (t_3, t_4), respectively. Since $V_q^w(t_V) \geq \mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))$, ω_2 is nondecreasing, and on $\xi-$domt$[q]$ intervals, $V_q(\tilde{x}(t))$ is decreasing as long as it is not smaller than $\mu^{-1}(\alpha_{\mathrm{g}}^{-1}(2\|w_\Theta\|))$, the inequality in (6.75) holds for $t \in [t_2, t_3]$ as well. Furthermore, as (t_3, t_4) is $z-$domt$[q]$, we also have (6.73) with (t_1, t_2) replaced by (t_3, t_4). Thus, by the additive and nondecreasing properties of ω_2 (see Lemma 6.2.1), we have

$$\begin{aligned} V_q(\tilde{x}(t)) &\leq \omega_2(V_q(\tilde{x}(t_3)), t - t_3) \\ &\leq \omega_2(\omega_2(V_q^w(t_V), t_3 - t_V), t - t_3) \\ &\leq \omega_2(V_q^w(t_V), t - t_V), \forall t \in (t_3, t_4). \end{aligned} \tag{6.76}$$

Continuing this process until $t_{\sigma,i_j^{\mathfrak{p}}}$ is reached, we arrive at

$$V_q(\tilde{x}(t)) \leq \omega_2(V_q^w(t_V), t - t_V), \forall t \in [t_V, t_{\sigma,i_j^{\mathfrak{p}}}]. \tag{6.77}$$

As ω_2 is nondecreasing and $t_{\sigma,i_j^{\mathfrak{p}}} - t_V \leq T_{\mathrm{p}}$, ii) follows (6.77) directly. ■

●

Chapter 7

Switching-Uniform Adaptive Output Feedback Control

In this chapter, adaptive observer is presented for switching-uniform stabilization of switched systems by output feedback. In a gauge design framework, the resulting dynamic output feedback control is of non-separation-principle. The underlying principle is to make the dynamics of the whole system to be interchangeably driven by the stable modes of the unmeasured dynamics and the coupled dynamics of error variables and state estimates. In this way, converging behavior of state estimates of the controlled dynamics are preserved through unstable modes of the unmeasured dynamics which at the same time provides estimates of functions of unmeasured state in terms of errors variables and known variables.

7.1 Introduction

Control by output feedback is a typical problem in feedback control systems [114, 146, 16]. The problem arises in applications in which information available for control design is from only external measurements. For nonlinear systems, the traditional approach is to study conditions under which separation principles apply [114, 16]. Recently, using switching signal as a control variable allowing new switches only when sufficiently accurate state estimates have been reached, [106] introduced a separation principle for switched systems. For switched linear systems, state-dependent switching-logic was developed [49, 50].

The increasing difficulty caused by uncertain and arbitrarily fast switching is twofold. Beside infeasibility of the strategy of switching among the set of observers of constituent systems, achieving sufficiently accurate state estimates on every single switching interval impossible. In comparison to continuous dynamical systems, the discrepancy between control gains of constituent systems in switched systems gives raise to new destabilizing terms in the error dynamics so that the Hurwitz matrix for Luenberger observer is not sufficient for a converging behavior.

In light of the above consideration, output feedback control of switched systems might relax the separation principle and a Hurwitz matrix that is robust/adaptive with respect to the new destabilizing terms is of principal interest.

In this chapter, we use gauge design framework to overcome typical obstacles in output feedback control of switched systems. The main novelty lies in the integration of the presented gauge design method and the method of adaptive high-gain [78, 92] so that control gains dependent on unestimated states are allowed, an enhancement that has not appeared for even continuous systems.

7.2 Problem Formulation

Consider the switched system with input, output, disturbance and appended dynamic $\Sigma_{\mathrm{I/O}}$ modeled by (6.2) in Chapter 6, in which the driving dynamics $\{\Sigma_q\}_{q\in\mathbb{Q}}$ are described by (6.1). Let us refer to Section 6.1 of Chapter 6 for detailed description of the switched system $\Sigma_{\mathrm{I/O}}$ as well as related notations. In this chapter, we are interested in the following output feedback control problem for $\Sigma_{\mathrm{I/O}}$.

Switching-uniform adaptive output feedback stabilization: design a dynamical system of the form (6.3) such that under the interconnection (6.4) with $y_m = x_1$, the trajectory $X(t) = X(t; \sigma, \theta, X_0)$ of the closed-loop system $\Sigma_{\mathscr{C}}$ generated by any switching sequence $\sigma \in \mathbb{S}[\tau_{\mathrm{p}}, T_{\mathrm{p}}]$, input $\theta \in \mathcal{L}_\infty^{d_\theta}$ and initial condition X_0 satisfies the following properties:

i) $X(t)$ is bounded; and

ii) $x(t)$ approaches to a small neighborhood of the origin as $t \to \infty$.

In this chapter, we provide a solution to the proposed output feedback control problem under the following conditions.

Assumption 7.2.1 *There are known constants $g_i, i, = 1, \ldots, n$ and $\Delta_G > 0$ and possibly unknown constant $L_F > 0$ such that for all $z \in \mathbb{R}^d, \bar{x}_i \in \mathbb{R}^i, q \in \mathbb{Q}$, and $i = 1, \ldots, n$, we have*

$$|g_{q,i}(z, \bar{x}_i, \theta) - g_i| \;\leq\; \Delta_G, \text{ and } |f_{q,i}(z, \bar{x}_i, \theta)| \leq L_F(\|z\|^{p/2} + |x_1| + \ldots + |x_i|) \tag{7.1}$$

Assumption 7.2.2 *The system (6.2) satisfies Assumption 6.2.1 for $\beta(s) = a_\beta s, \alpha_1(s) = a_{\alpha,1} s$, and $\alpha_2(s) = a_{\alpha,2} s$, where $\alpha_\beta, a_{\alpha,1}$ and $a_{\alpha,2}$ are positive constants. In addition, $a_{\alpha,1} > a_{\alpha,2}$,*

$$\tilde{a}_1 U_q(\zeta) \geq \|\zeta\|^p, \forall \zeta \in \mathbb{R}^d, q \in \mathbb{Q}, \tag{7.2}$$

and

$$\tilde{a}_1 \tau_{\mathrm{p}} - \tilde{a}_2 T_{\mathrm{p}} > \ln a_\beta, \tag{7.3}$$

where $\tilde{a}_1 = (a_{\alpha,1} - a_{\alpha,2})/2$ and $\tilde{a}_2 = (a_{\alpha,1} + a_{\alpha,2})/2$.

In nonlinear output feedback control, it is often assumed that the systems can be described by models in which control gains are either known constants or known functions of output [104, 124, 98, 73, 84, 92] so that control gains in error equations are vanished. However due to the discrepancy between control gains, non-zero control gains in error equations in switched systems might not be avoided. By Assumption 7.2.1, we shall deal with this typical problem based on consideration of variation Δ_G in control gains. The Lipschitz-like condition for $f_{q,i}$ in (7.1) is instrumental in nonlinear output feedback control via high-gain observer [124, 92].

Let P, Q, Λ, and Π be symmetric matrices satisfying

$$A^T P + PA \leq -2Q, \quad DP + PD \geq 0, \tag{7.4}$$

$$\Gamma^T \Lambda + \Lambda\Gamma \leq -2I, \quad D\Lambda + \Lambda D \geq 0, \text{and} \tag{7.5}$$

$$\Delta_\Lambda I \leq Q, \tag{7.6}$$

where

$$A = \begin{bmatrix} -a_1 & g_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \dots & g_{n-1} \\ -a_n & 0 & \dots & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & g_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_{n-1} \\ -\gamma_1 & -\gamma_2 & \dots & -\gamma_n \end{bmatrix}, \tag{7.7}$$

are design constant matrices, I is the identity matrix, $D = \text{diag}\{1, \dots, n\}$, $\lambda_{\max,P}$ and $\lambda_{\max,\Lambda}$ are the maximal eigenvalues of P and Λ respectively, $\bar{a} = [a_1, \dots, a_n]^T$ and $\bar{\gamma} = [\gamma_1, \dots, \gamma_n]^T$ are design parameters, and

$$\Delta_\Lambda = \frac{\lambda_{\max,P}^2}{2} \Delta_G^2 \Big(1 + \frac{\|\bar{\gamma}\|^2}{g_n}\Big)\Big(1 + \frac{\lambda_{\max,\Lambda}^2}{\lambda_{\max,P}}\|\bar{a}\|^2\Big) + \lambda_{\max,P}\Delta_G \tag{7.8}$$

are constants fixed *a priori*.

Upon the introduction of the gain variation Δ_G for less conservative results, (7.6) is considered for solvability of the problem. While (7.4) and (7.5) are always possible

[92], satisfaction of (7.6) can be made by appropriate matrices A and Q. It is observed from (7.8) that (7.6) automatically holds when $\Delta_G = 0$, i.e., the control gains of constituent systems are identical constants.

For two vectors $a, b \in \mathbb{R}^n$, we have

$$\begin{aligned}\sum_{i=1}^{n}|a_i|\sum_{j=1}^{i}|b_j| &= \sum_{i=1}^{n}\sum_{j=i}^{n}|a_j||b_{j-i+1}| \leq \sum_{i=1}^{n}\Big(\sum_{j=i}^{n}|a_j||b_{j-i+1}| + \sum_{j=1}^{i-1}|a_j||b_{n-i+1+j}|\Big)\\ &\leq \sum_{i=1}^{n}\Big(\sum_{j=1}^{n}a_j^2\Big)^{1/2}\Big(\sum_{j=i}^{n}b_{j-i+1}^2 + \sum_{j=1}^{i-1}b_{n-i+1+j}^2\Big)^{1/2} = n\|a\|\|b\|. \quad (7.9)\end{aligned}$$

7.3 Adaptive Output Feedback Control

In this section, we utilize the adaptive high-gain technique in universal output feedback control design of nonlinear systems [92] to present a gauge design for output feedback control of switched systems. The main advance lies in the use of a gauging inequality to deal with the inherent discrepancy between control gains and the dependence on the unmeasured state of driving systems Σ_q.

7.3.1 Adaptive High-Gain Observer

The purpose of this subsection is to construct an observer providing state estimates for system (6.2). Let $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_n]^T$ be the estimate of x, and $\lambda > 0$ be the time-varying observer's high-gain. We have the following scaled variables

$$e_i = \frac{x_i - \hat{x}_i}{\lambda^i} \text{ and } \zeta_i = \frac{\hat{x}_i}{\lambda^i}, i = 1, \ldots, n. \quad (7.10)$$

In view of (7.4)–(7.6), let us consider the following reduced-order adaptive observer for system (6.2)

$$\dot{\hat{x}}_i(t) = g_i\hat{x}_{i+1}(t) + \lambda^i(t)a_i(x_1(t) - \hat{x}_1(t)), i = 1, \ldots, n \tag{7.11}$$

$$\dot{\lambda}(t) = k_\lambda e_1^2(t), \lambda(0) = 1, \tag{7.12}$$

where $\hat{x}_{n+1} \overset{\text{def}}{=} u$ and $k_\lambda > 0$ is the tuning gain of λ.

From the dynamic equations (6.1) and (7.11), the dynamics of $e_i, i = 1, \ldots, n$ are

$$\begin{aligned}\dot{e}_i(t) = {} & \frac{1}{\lambda^i}\Big(g_i(x_{i+1}(t) - \hat{x}_{i+1}(t)) + (g_{q_{\sigma,i_\sigma^-(t)},i}(z(t), \bar{x}_i(t), \theta(t)) - g_i)x_{i+1}(t) \\ & + f_{q_{\sigma,i_\sigma^-(t)},i}(z(t), \bar{x}_i(t), \theta(t)) - \lambda^i a_i(x_1(t) - \hat{x}_1(t))\Big) - i\frac{\dot{\lambda}}{\lambda}e_i(t) \\ = {} & \lambda g_i e_{i+1}(t) + \tilde{e}_{q_{\sigma,i_\sigma^-(t)},i}(t) - a_i\lambda e_1(t) - i\frac{\dot{\lambda}}{\lambda}e_i(t),\end{aligned} \tag{7.13}$$

where, for each $i = 1, \ldots, n$ and $q \in \mathbb{Q}$, $\tilde{e}_{q,i}$ is

$$\tilde{e}_{q,i} = \frac{1}{\lambda^i}\Big[(g_{q,i}(z, \bar{x}_i, \theta) - g_i)x_{i+1} + f_{q,i}(z, \bar{x}_i, \theta)\Big]. \tag{7.14}$$

Thus, defining $e = [e_1, \ldots, e_n]^T$ and $\tilde{e}_q = [\tilde{e}_{q,1}, \ldots, \tilde{e}_{q,n}]^T, q \in \mathbb{Q}$, (7.13) gives rise to the following compact form describing the dynamics of $e(t)$:

$$\dot{e}(t) = \lambda Ae(t) - \frac{\dot{\lambda}}{\lambda}De(t) + \tilde{e}_{i_\sigma^-(t)}(t). \tag{7.15}$$

In view of (7.14), $\tilde{e}_q$'s are dependent on the unestimated state z and the differences between control gains of subsystems. Thus, the convergence of the estimation error $e(t)$ cannot be derived from (7.15) only as in classical nonlinear output feedback control [78, 124, 92]. For this reason, a gauge function is called for overcoming the obstacle caused by $\tilde{e}_q$'s.

7.3.2 Control Design

Consider the functions

$$V_e = (r_0 + 1)e^T Pe \quad \text{and} \quad V_\zeta = \zeta^T \Lambda \zeta, \tag{7.16}$$

where

$$r_0 = \frac{\lambda^2_{\max,\Lambda}}{\lambda_{\max,P}} \|\bar{a}\|^2. \tag{7.17}$$

Our gauge function is $V_{\mathrm{g}} = V_e + V_\zeta$, where $\zeta = [\zeta_1, \ldots, \zeta_n]^n$. Let $a_\mu = (a_{\alpha,1} - a_{\alpha,2})/2$. Along the evolution of $\mathcal{E} = [e^T, \zeta^T]^T$, we have the following gauges

$$a_\mu U_q(z) \leq V_e + V_\zeta, q \in \mathbb{Q}. \tag{7.18}$$

In the following, we shall show that upon the satisfaction of (7.18), the state estimates provided by the observer (7.11)-(7.12) can be used for designing a control capable of making the dynamics of e and ζ the driving dynamics of the whole system.

By Assumption 7.2.2, whenever (7.18) holds, we have $\|z\|^p \leq V_e + V_\zeta$ and hence,

$$\|z\|^{p/2} = a_\mu^{-1/2}(a_\mu \|z\|^p)^{1/2} \leq a_\mu^{-1/2}(V_e + V_\zeta)^{1/2} \leq a_\mu^{-1/2}(V_e)^{1/2} + a_\mu^{-1/2} V_\zeta^{1/2}. \tag{7.19}$$

The derivative $DV_{\mathrm{g}}(\mathcal{E}(t))$ can be computed through the derivatives of V_e and V_ζ as follows. As $g_{q,i}(\cdot)$'s are positive, using Assumption 7.2.1, (7.19) and replacing x_i by $\hat{x}_i + \lambda^i e_i$, we obtain

$$\begin{aligned} |\tilde{e}_{q,i}| &\leq \frac{|g_{q,i}(z, \bar{x}_i, \theta) - g_i|}{\lambda^i} |\hat{x}_{i+1} + e_{i+1}\lambda^{i+1}| + \frac{L_F}{\lambda^i}\Big(\|z\|^p + \sum_{j=1}^{i} |\hat{x}_j + \lambda^j e_j| \Big) \\ &\leq \frac{\Delta_G}{\lambda^i} |\hat{x}_{i+1}| + \Delta_G |e_{i+1}| \lambda + \frac{L_F}{\lambda^i}\Big(\sqrt{\frac{V_e}{a_\mu}} + \sqrt{\frac{V_\zeta}{a_\mu}} + \sum_{j=1}^{i} |\hat{x}_j + \lambda^j e_j| \Big). \end{aligned} \tag{7.20}$$

As such, upon the satisfaction of (7.18), from (7.4), (7.5) and the designated positiveness of $DP + PD$ and $\dot{\lambda}/\lambda$ (see (7.12)), it follows that

$$\begin{aligned}
DV_e(e(t)) &= \lambda(r_0+1)e^T(A^TP+PA)e - (r_0+1)\frac{\dot{\lambda}}{\lambda}e^T(DP+PD)e + \sum_{i=1}^{n} e^TP_i\tilde{e}_i \\
&\le -2\lambda(r_0+1)e^TQe + (r_0+1)\Big[\sum_{i=1}^{n}\frac{\Delta_G}{\lambda^i}|e^TP_i||\hat{x}_{i+1}| + \lambda\Delta_G\sum_{i=1}^{n}|e^TP_i||e_{i+1}| \\
&\quad + \sum_{i=1}^{n}\frac{L_F|e^TP_i|}{\lambda^i}\Big(\sqrt{\frac{V_e}{a_\mu}} + \sqrt{\frac{V_\zeta}{a_\mu}}\Big) \\
&\quad + \sum_{i=1}^{n}|e^TP_i|\sum_{j=1}^{i}\frac{L_F|e_j|}{\lambda^{i-j}} + \sum_{i=1}^{n}|e^TP_i|\sum_{j=1}^{i}\frac{L_F|\hat{x}_j|}{\lambda^i}\Big]. \qquad (7.21)
\end{aligned}$$

As $\hat{x}_{n+1} = u$ shall be designed in the form (7.28) below, from (7.10), we have

$$\begin{aligned}
\frac{1}{\lambda^i}|\hat{x}_{i+1}| &= \lambda|\zeta_{i+1}|, i = 1, \ldots, n-1, \\
\frac{1}{\lambda^n}|\hat{x}_{n+1}| &= \frac{1}{\lambda^n}|u| = \frac{\lambda}{g_n}|\gamma_1\zeta_1 + \ldots + \gamma_n\zeta_n|. \qquad (7.22)
\end{aligned}$$

Since γ_i's are design constants fixed *a priori*, (7.22) gives rise to

$$\Big\|\Big[\frac{|\hat{x}_2|}{\lambda^2}, \ldots, \frac{|\hat{x}_{n+1}|}{\lambda^{n+1}}\Big]^T\Big\| \le \lambda\sqrt{1 + g_n^{-1}\|\bar{\gamma}\|^2}\|\zeta\|. \qquad (7.23)$$

On the other hand, it is observed that $e_{n+1} = 0$ and $\|e^TP\| \le \lambda_{\max,P}\|e\|$. Thus, using e(7.8), (7.23) and the Cauchy-Schwartz and Young inequalities, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{\Delta_G}{\lambda^i}|e^TP_i||\hat{x}_{i+1}| + \lambda\Delta_G\sum_{i=1}^{n}|e^TP_i||e_{i+1}| \\
&\le \lambda\Delta_G\sqrt{1+\|\bar{\gamma}\|^2}\|e^TP\|\|\zeta\| + \lambda\Delta_G\|e^TP\|\|e\| \\
&\le \lambda\lambda_{\max,P}\Delta_G\sqrt{1+\|\bar{\gamma}\|^2}\|e\|\|\zeta\| + \lambda\lambda_{\max,P}\Delta_G\|e\|^2 \\
&\le \lambda\frac{\lambda_{\max,P}^2}{2}\Delta_G^2(1+\|\bar{\gamma}\|^2)(r_0+1)\|e\|^2 + \frac{\lambda}{2(r_0+1)}\|\zeta\|^2 + \lambda\lambda_{\max,P}\Delta_G\|e\|^2 \\
&\le \lambda\Delta_\Lambda\|e\|^2 + \frac{\lambda}{2(r_0+1)}\|\zeta\|^2. \qquad (7.24)
\end{aligned}$$

Upon satisfaction of (7.18), substituting (7.24) into (7.21) and applying the inequality (7.9) and the Cauchy-Schwartz inequality, we obtain the following estimate

$$
\begin{aligned}
DV_e(e(t)) \leq & -2\lambda(r_0+1)e^TQe + (r_0+1)\Big[\lambda\Delta_\Lambda\|e\|^2 + \frac{\lambda}{2(r_0+1)}\|\zeta\|^2 + \frac{L_F}{\lambda}\sqrt{n}\|e^TP\| \\
& \times\Big(\sqrt{\frac{\lambda_{\max,P}}{a_\mu}}\|e\| + \sqrt{\frac{V_\zeta}{a_\mu}}\Big) + nL_F\|e^TP\|\|e\| + L_Fn\|e^TP\|\|\zeta\|\Big] \\
\leq & -\lambda(r_0+1)(2e^TQe - \Delta_\Lambda\|e\|^2) + \frac{\lambda}{2}\|\zeta\|^2 + (r_0+1)\Big[\frac{L_F\sqrt{n}}{\lambda}\Big(\frac{\lambda_{\max,P}^{3/2}}{\sqrt{a_\mu}}\|e\|^2 \\
& + \frac{\|e^TP\|^2}{2} + \frac{V_\zeta}{2a_\mu}\Big) + nL_F\lambda_{\max,P}\|e\|^2 + \frac{nL_F}{2}\|e^TP\|^2 + \frac{nL_F}{2}\|\zeta\|^2\Big].
\end{aligned}
\tag{7.25}
$$

Further substitution of $V_\zeta \leq \lambda_{\max,\Lambda}$ and $\|e^TP\| \leq \lambda_{\max,P}\|e\|$ into (7.25) with the use of (7.6) yields

$$
\begin{aligned}
DV_e(e(t)) \leq & -\lambda(r_0+1)e^TQe + (r_0+1)L_F\Big(\frac{\lambda_{\max,P}^{3/2}\sqrt{n}}{\lambda\sqrt{a_\mu}} + \frac{\lambda_{\max,P}^2(\sqrt{n}+n)}{2\lambda} \\
& + n\lambda_{\max,P}\Big)\|e\|^2 + \frac{\lambda}{2}\|\zeta\|^2 + (r_0+1)\Big(\frac{L_F\sqrt{n}}{2\lambda\sqrt{a_\mu}}\lambda_{\max,\Lambda} + \frac{nL_F}{2}\Big)\|\zeta\|^2 \\
\leq & -\lambda(r_0+1)e^TQe + \ell(r_0+1)e^TQe + \frac{\lambda}{2}\|\zeta\|^2 + \ell(r_0+1)\|\zeta\|^2
\end{aligned}
\tag{7.26}
$$

upon satisfaction of (7.18), where $\ell > 0$ is time-varying parameter satisfying

$$
\ell \geq \max\Big\{\frac{\lambda_{\max,P}^{3/2}\sqrt{n}}{\lambda\sqrt{a_\mu}} + \frac{\lambda_{\max,P}^2(\sqrt{n}+n)}{2\lambda} + n\lambda_{\max,P}, \frac{\sqrt{n}}{2\lambda\sqrt{a_\mu}}\lambda_{\max,\Lambda} + \frac{n}{2}\Big\}L_F, \tag{7.27}
$$

which is bounded since λ is non-decreasing.

On the other hand, a direct computation from (7.11) and (7.7) shows that under the control

$$
u = -\frac{\lambda^{n+1}}{g_n}(\gamma_1\zeta_1 + \ldots + \gamma_n\zeta_n), \tag{7.28}
$$

the dynamic equation of ζ is

$$\dot{\zeta} = \lambda\Gamma\zeta - \frac{\dot{\lambda}}{\lambda}D\zeta + \lambda\bar{a}e_1. \tag{7.29}$$

Thus, using (7.5) and noting that $\dot{\lambda}/\lambda \geq 0$ and $D\Lambda + \Lambda D \geq 0$, we have

$$\begin{aligned} DV_\zeta(\zeta(t)) &\leq -2\lambda\|\zeta\|^2 - \frac{\dot{\lambda}}{\lambda}\zeta^T(D\Lambda + \Lambda D)\zeta + 2\lambda\|\zeta^T\Lambda\|\|\bar{a}\||e_1| \\ &\leq -2\lambda\|\zeta\|^2 + \lambda\|\zeta\|^2 + \lambda\lambda_{\max,\Lambda}^2\|\bar{a}\|^2\|e\|^2 \\ &\leq -\lambda\|\zeta\|^2 + \lambda\lambda_{\max,\Lambda}^2\|\bar{a}\|^2\frac{\lambda_{\max,P}}{\lambda_{\max,P}}\|e\|^2 \leq -\lambda\|\zeta\|^2 + \lambda r_0 e^T Q e. \end{aligned} \tag{7.30}$$

Combining (7.26) and (7.30), we have the following inequality upon satisfaction of (7.18):

$$DV_{\mathrm{g}}(\mathcal{E}(t)) \leq -(\lambda - \ell(r_0+1))\Big(e^TQe + \frac{1}{2}\|\zeta\|^2\Big). \tag{7.31}$$

7.3.3 Stability Analysis

In this section, we shall show that the adaptive output feedback control given by (7.28), (7.11), (7.12), and (7.10) achieves stabilization uniformly with respect to the class of persistent dwell-time switching sequences $\mathbb{S}_{\mathcal{P}}[\tau_{\mathrm{p}}, T_{\mathrm{p}}]$. In view of (7.31), the uncertainties are lumped into the unknown parameter ℓ to present no separated input disturbance. Hence, asymptotic convergence to the origin can be achieved once λ is sufficiently large. This is slightly different from the case of adaptive output regulation in Chapter 6, where uncertainties is lumped into a separated unknown parameter Θ_n to which disturbance attenuation problem was naturally addressed. We have the following theorem whose proof is carried out in the framework of [92].

Theorem 7.3.1 *Consider the switched system $\Sigma_{\mathrm{I/O}}$ modeled by (6.2). Suppose that Assumptions 6.2.1, 7.2.1, and 7.2.2 are satisfied. Then, under the adaptive output feedback control given by (7.28), (7.11), (7.12), and (7.10), the all signals in the* *resulting closed-loop system $\Sigma_{\mathscr{C}}$ are bounded and the state $x(t)$ converges to the origin uniformly with respect to $\mathbb{S}[\tau_{\mathrm{p}}, T_{\mathrm{p}}]$.*

Proof: From Assumption 7.2.2, let us consider the functions ω_1 and ω_2 defined by

$$\omega_1(s,t) = s\exp(-\tilde{a}_1 t), \omega_2(\upsilon,t) = s\exp(\tilde{a}_2 t), \upsilon, t \in \mathbb{R}^+. \tag{7.32}$$

Using $\beta(s) = a_\beta s$ in Assumption 7.2.2, we have

$$\omega_1(\beta(\omega_2(s,T_{\mathrm{p}})), \tau_{\mathrm{p}})) = a_\rho s\exp(-\tilde{a}_1\tau_{\mathrm{p}} + \tilde{a}_2 T_p) = s\exp(-\tau_\epsilon) \stackrel{\text{def}}{=} \omega_0(s,\tau_\epsilon), \tag{7.33}$$

where τ_ϵ is any positive number satisfying $0 < \tau_\epsilon \le \tilde{a}_1\tau_{\mathrm{p}} - \tilde{a}_2 T_{\mathrm{p}} - \ln a_\beta$. Clearly, these functions satisfy Assumption 6.2.2.

We prove the boundedness of λ by a contradiction argument. Suppose that λ is unbounded. Then, there is a time $t_\lambda > 0$ such that for all $t \ge t_\lambda$, we have $\lambda(t) - \ell(r_0+1) \ge a_{\alpha,1}$, the decreasing rate of z-system. In this case, it follows from (7.31) that $DV_{\mathrm{g}}(\mathcal{E}(t)) \le -a_{\alpha,1}V_{\mathrm{g}}(\mathcal{E}(t))$. Then, applying the argument of the proof of Theorem 6.3.1, it follows that the dynamics of (z,e,ζ) satisfies conditions of Theorem 5.2.1 with auxiliary functions $V_q(z,e,\zeta) = \max\{U_q(z), a_\mu^{-1}V_{\mathrm{g}}(e,\zeta)\}, q \in \mathbb{Q}$, and we obtain $e_1(t) \to 0, t \to \infty$, which coupled with the boundedness and the continuity of $e_1(t)$ further implies that

$$\lambda(\infty) - \lambda(0) = \int_0^\infty e_1^2(t)dt = e_1^2(\infty) - e_1^2(0) < \infty, \tag{7.34}$$

which contradicts to the contradiction hypothesis. Hence, $\lambda(t)$ is bounded.

Our next objective is to prove that $e(t)$ and $\zeta(t)$ converge to 0 as $t \to \infty$.

As designated by (7.12), $\dot{\lambda}(t) = k_\lambda e_1^2(t)$ and $\lambda(t) \geq 1, \forall t$. Hence, from the first inequality in (7.30), the derivative $DV_\zeta(\zeta(t))$ satisfies

$$\begin{aligned} DV_\zeta(\zeta(t)) &\leq -2\lambda\|\zeta\|^2 + 2\lambda\lambda_{\max,\Lambda}\|\zeta\|^2\|\bar{a}\||e_1| \\ &\leq -\lambda\|\zeta\|^2 + \lambda\lambda_{\max,\Lambda}^2\|\bar{a}\|^2 e_1^2 \leq -\|\zeta\|^2 + \lambda_{\max,\Lambda}^2\|\bar{a}\|^2\frac{\lambda\dot{\lambda}}{k_\lambda}, \end{aligned} \tag{7.35}$$

which, by integration, gives rise to

$$V_\zeta(\zeta(t)) + \int_0^t \|\zeta(s)\|^2 ds \leq V_\zeta(\zeta(0)) + \frac{\lambda_{\max,\Lambda}^2\|\bar{a}\|^2}{2k_\lambda}(\lambda^2(t) - \lambda^2(0)), \forall t \geq 0. \tag{7.36}$$

Due to the boundedness of $\lambda(t)$, the right hand side of (7.36) is bounded. Hence, $\zeta(t)$ is well-defined and is bounded for all $t \geq 0$. Applying the Barbalat's lemma, we obtain $\zeta(t) \to 0, t \to \infty$.

We proceed to prove the convergence of $e(t)$. Let $\lambda_0 \geq 1$ be a design constant and consider the following scaled error variables:

$$\varepsilon_i = \frac{x_i - \hat{x}_i}{\lambda_0^i}, i = 1, \ldots, n. \tag{7.37}$$

From (7.11) and (7.37), the time derivatives of ε_i's are computed as

$$\begin{aligned} \dot{\varepsilon}_i = &\frac{1}{\lambda_0^i}(g_i(x_{i+1} - \hat{x}_{i+1}) + (g_{q_{\sigma,i_\sigma^-(t)},i}(z(t), \bar{x}_i(t), \theta(t)) - g_i)x_{i+1} \\ &+ f_{q_{\sigma,i_\sigma^-(t)},i}(z(t), \bar{x}_i(t), \theta(t)) - \lambda^i a_i(x_1 - \hat{x}_1)) \\ = &\lambda_0 g_i \varepsilon_{i+1} + \tilde{\varepsilon}_{q_{\sigma,i_\sigma^-(t)},i} - \lambda\frac{\lambda^i}{\lambda_0^i}a_i e_1, \end{aligned} \tag{7.38}$$

where $e_1 = (x_1 - \hat{x}_1)/\lambda$ and

$$\begin{aligned}\tilde{\varepsilon}_{q,i} &= \frac{1}{\lambda_0^i}\big((g_{q_{\sigma,i_\sigma^-(t)},i}(\cdot) - g_i)x_{i+1} + f_{q_{\sigma,i_\sigma^-(t)},i}(\cdot)\big)\\ &= \frac{1}{\lambda_0^i}\big((g_{q_{\sigma,i_\sigma^-(t)},i}(\cdot) - g_i)\Big(\hat{x}_{i+1} + \lambda_0^{i+1}\varepsilon_{i+1}\Big) + f_{q_{\sigma,i_\sigma^-(t)},i}(\cdot)\big), q \in \mathbb{Q}.\end{aligned} \tag{7.39}$$

Let $\bar{\varepsilon} = [\varepsilon_1, \ldots, \varepsilon_n]^T$ and $\tilde{\varepsilon}_q = [\tilde{\varepsilon}_{q,1}, \ldots, \tilde{\varepsilon}_{q,n}]^T$. Adding $-\lambda_0 a_i \varepsilon_1 + \lambda_0 a_i \varepsilon_1$ to the right hand side of (7.38), we have

$$\dot{\bar{\varepsilon}} = \lambda_0 A\bar{\varepsilon} + \lambda_0 \bar{a}\varepsilon_1 + \tilde{\varepsilon}_{q_{\sigma,i_\sigma^-(t)}} - \lambda B\bar{a}e_1, \tag{7.40}$$

where $B = \mathrm{diag}\{\lambda/\lambda_0, \ldots, (\lambda/\lambda_0)^n\}$. Let us consider the Lyapunov function

$$V_\varepsilon(\bar{\varepsilon}) = \bar{\varepsilon}^T P\bar{\varepsilon}, \tag{7.41}$$

and the following gauges

$$a_\mu U_q(z) \le V_\varepsilon(\bar{\varepsilon}), q \in \mathbb{Q}. \tag{7.42}$$

By virtue of (7.19), upon satisfaction of (7.42), we have

$$\|z\|^{p/2} = a_\mu^{-1/2}(a_\mu\|z\|^p)^{1/2} \le a_\mu^{-1/2}V_\varepsilon^{1/2}. \tag{7.43}$$

Since $\lambda(t)$ is bounded and λ_0 is a constant, let λ_0 is selected such that $\lambda_0 \ge 2\sup\{\lambda(t) : t \in \mathbb{R}^+\}$. As such, in view of (7.10), we have the following expressions

$$\begin{aligned}&\frac{1}{\lambda_0^i}|\hat{x}_{i+1}| = \frac{\lambda^{i+1}}{\lambda_0^i}|\zeta_{i+1}| \le |\zeta_{i+1}|, i = 1, \ldots, n-1,\\ &\frac{1}{\lambda_0^n}|\hat{x}_{n+1}| = \frac{1}{\lambda_0^n}|u| = \frac{\lambda^{n+1}}{g_n\lambda_0^n}|\gamma_1\zeta_1 + \ldots + \gamma_n\zeta_n| \le g_n^{-1}|\gamma_1\zeta_1 + \ldots + \gamma_n\zeta_n|.\end{aligned} \tag{7.44}$$

As a result, we have

$$\left\|\left[\frac{1}{\lambda_0}|\hat{x}_2|,\ldots,\frac{1}{\lambda_0^n}|\hat{x}_{n+1}|\right]^T\right\| \le \sqrt{1+g_n^{-1}\|\bar{\gamma}\|^2}\|\zeta\|. \tag{7.45}$$

In view of (7.39) and (7.43), upon satisfaction of (7.42), we have

$$|\tilde{\varepsilon}_{q,i}| \le \Delta_G\frac{|\hat{x}_{i+1}|}{\lambda_0^i} + \lambda_0\Delta_G|\varepsilon_{i+1}| + \frac{L_F}{\lambda_0^i}\Big(\mu^{-1/2}V_\varepsilon^{1/2} + \sum_{j=1}^{i}|\hat{x}_j + \lambda_0^j\varepsilon_j|\Big), \tag{7.46}$$

which, by a similar use of the Cauchy-Schwartz's inequality as in (7.25), results in

$$\begin{aligned}
\bar{\varepsilon}P\tilde{\varepsilon}_q &\le \Delta_G\sqrt{1+g_n^{-1}\|\bar{\gamma}\|^2}\|\bar{\varepsilon}^TP\|\|\zeta\| + \frac{L_F}{\lambda_0}\sqrt{n}\|\bar{\varepsilon}^TP\|\sqrt{\mu^{-1}\lambda_{\max,P}}\|\bar{\varepsilon}\| \\
&\quad + nL_F\|\bar{\varepsilon}^TP\|\|\bar{\varepsilon}\| + nL_F\|\bar{\varepsilon}^TP\|\|\zeta\| \\
&\le \ell_0(\bar{\varepsilon}^TQ\bar{\varepsilon} + \|\zeta\|^2), q \in \mathbb{Q},
\end{aligned} \tag{7.47}$$

where $\ell_0 > 0$ is a constant that can be selected independent of λ_0.

In addition, using Cauchy-Schwartz's inequality and $\varepsilon_1 = \lambda e_1/\lambda_0$, we also have

$$\begin{aligned}
2\lambda_0\bar{\varepsilon}^TP\bar{a}\varepsilon_1 &\le \lambda_0^2\|P\bar{a}\|^2\varepsilon_1^2 + \|\varepsilon\|^2 = \lambda^2\|P\bar{a}\|^2e_1^2 + \|\bar{\varepsilon}\|^2 \\
2\lambda\bar{\varepsilon}^TPB\bar{a}e_1 &\le \lambda^2\|PB\bar{a}\|^2e_1^2 + \|\bar{\varepsilon}\|^2.
\end{aligned} \tag{7.48}$$

From (7.47), (7.48), and (7.40), the derivative of $DV_\varepsilon(\bar{\varepsilon}(t))$ can be computed as

$$\begin{aligned}
DV_\varepsilon(\bar{\varepsilon}(t)) &\le -2\lambda_0\bar{\varepsilon}^TQ\bar{\varepsilon} + \lambda^2\|P\bar{a}\|^2e_1^2 + \|\bar{\varepsilon}\|^2 + \ell_0(\bar{\varepsilon}^TQ\bar{\varepsilon} + \|\zeta\|^2) + \lambda^2\|PB\bar{a}\|^2e_1^2 + \|\bar{\varepsilon}\|^2 \\
&\le -(2\lambda_0 - \ell_0 - 2\lambda_{\max,Q}^{-1})\bar{\varepsilon}^TQ\bar{\varepsilon} + \ell_0\|\zeta\|^2 + K_ee_1^2
\end{aligned} \tag{7.49}$$

upon satisfaction of (7.42), where $\lambda_{\max,Q}$ is the maximal eigenvalue of Q and K_e is any constant satisfying $K_e \ge \lambda^2\|P\bar{a}\|^2 + \lambda^2\|PB\bar{a}\|^2$ which exists as $\lambda(t)$ is bounded.

In view of (7.49), let λ_0 is such that $\lambda_0 \geq \ell_0 - 2\lambda_{\max,Q}^{-1}$. Then, we have

$$DV_\varepsilon(\bar{\varepsilon}(t)) \leq -\lambda_0 \bar{\varepsilon}^T Q \bar{\varepsilon} + \ell_0 \|\zeta\|^2 + K_e e_1^2 \tag{7.50}$$

upon satisfaction of (7.42). Moreover, for the state $\mathcal{E} = [z^T, \bar{\varepsilon}^T]^T$, let us consider the Lyapunov functions

$$V_q(\mathcal{E}) = \max\{U_q(z), a_\mu^{-1} V_\varepsilon(\bar{\varepsilon})\}. \tag{7.51}$$

At a time instant $t \in \mathbb{R}^+$, we have the following cases:

Case 1: $a_\mu U_{q_{\sigma,i_\sigma^(t)}}(z(t)) \leq V_\varepsilon(\bar{\varepsilon}(t))$. In this case, we have $V_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t)) = a_\mu^{-1} V_\varepsilon(\bar{\varepsilon}(t))$ and hence, it is obvious from (7.50) that

$$DV_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t)) \leq -\lambda_0 \frac{\lambda_{\min,Q}}{\lambda_{\max,Q}} V_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t)) + \ell_0 \|\zeta(t)\|^2 + K_e e_1^2(t), \tag{7.52}$$

where $\lambda_{\min,Q}$ is the smallest eigenvalue of Q.

Case 2: $a_\mu U_{q_{\sigma,i_\sigma^(t)}}(z(t)) > V_\varepsilon(\bar{\varepsilon}(t))$. By Assumption 7.2.2, we have

$$\upsilon(x_1^2) = a_\upsilon x_1^2 = a_\upsilon(x_1 - \hat{x}_1 + \hat{x}_1)^2 = a_\upsilon(\lambda e_1 + \lambda\zeta_1)^2 \leq 2a_\upsilon\lambda^2 e_1^2 + 2a_\upsilon\lambda^2\|\zeta\|^2. \tag{7.53}$$

As $V_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t)) = U_{q_{\sigma,i_\sigma^-(t)}}(z(t))$, using (7.53) and Assumption 7.2.2, we have

$$DV_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t)) \leq -a_{\alpha,1} V_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t)) + 2a_\upsilon\lambda(t)^2 e_1^2(t) + 2a_\upsilon\lambda(t)^2\|\zeta(t)\|^2. \tag{7.54}$$

Letting $a_{\mathcal{E}} = \min\{a_{\alpha,1}, \lambda_0\lambda_{\min,Q}\lambda_{\max,Q}^{-1}\}$ and $\ell_1 = \max\{\ell_0, K_e, 2a_\upsilon\|\lambda(t)\|^2\}$ and combining (7.52) and (7.54), we arrive at

$$DV_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t)) \leq -a_{\mathcal{E}} V_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t)) + \ell_1 e_1^2(t) + \ell_1\|\zeta(t)\|^2, t \in \mathbb{R}^+. \tag{7.55}$$

Integrating both sides of (7.55), we obtain

$$V_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t)) + \int_0^t V_{q_{\sigma,i_\sigma^-(s)}}(\mathcal{E}(s))ds$$
$$\leq V_{q_{\sigma,0}}(\mathcal{E}(0)) + \ell_1 \int_0^t e_1^2(s)ds + \ell_1 \int_0^t \|\zeta(s)\|^2 ds, \forall t \geq 0. \tag{7.56}$$

As $\lambda(t)$ is bounded, (7.34) is satisfied. Thus, in view of (7.34) and (7.36), the right hand side of (7.56) is bounded. As there is no switching jump, $V_{q_{\sigma,i_\sigma^-(t)}}(\mathcal{E}(t))$ is continuous with respect to t. Hence, the boundedness of the RHS of (7.56) implies that $\mathcal{E}(t) \to 0, t \to \infty$, and consequently, $\varepsilon(t)$ and $x(t)$ converge to 0 as $t \to \infty$. ∎

7.4 Design Example

Consider the switched systems whose constituent systems are given by

$$\Sigma_1 : \begin{cases} \dot{z}_1 &=& -z_1 + z_2 z_1 \\ \dot{z}_2 &=& -z_2(1+z_2^2) - z_1^2 + z_2 x_1 \\ \dot{x}_1 &=& \theta_1 x_2 + \ln(1+\|z\|^2) \\ \dot{x}_2 &=& (2+\sin z_2^2)u + \theta_2 x_1 \end{cases}, \quad \Sigma_2 : \begin{cases} \dot{z}_1 &=& -3z_1 + z_2 \\ \dot{z}_2 &=& z_1 - 2z_2 + x_1 \\ \dot{x}_1 &=& \theta_3 x_2 + 2\theta_4 x_1 \\ \dot{x}_2 &=& \theta_5 u + \sqrt{z_1^2 + 3z_2^2} \end{cases}, \tag{7.57}$$

where the unknown time-varying parameters are $\theta_1 = 1 + \sin^2 t, \theta_2 = \cos t, \theta_3 = 1.5 + 0.6\cos t, \theta_4 = \sin t$, and $\theta_5 = 2.5 - 0.5\sin t$. The output of the system is $y = x_1$.

From the given time-varying parameters, we have $g_1 = 1.5, g_2 = 2.5$ and $\Delta_G = 0.6$. Thus, let us choose $a_1 = 30, a_2 = 25, b = 3$, and

$$P = \begin{bmatrix} 2.65 & -3 \\ -3 & 3.7 \end{bmatrix}, \quad Q = \begin{bmatrix} 9 & -1.475 \\ -1.475 & -9 \end{bmatrix},$$
$$\Gamma = \begin{bmatrix} 0 & 1.5 \\ -2 & -5 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2.13 & 0.5 \\ 0.5 & 0.35 \end{bmatrix}, \tag{7.58}$$

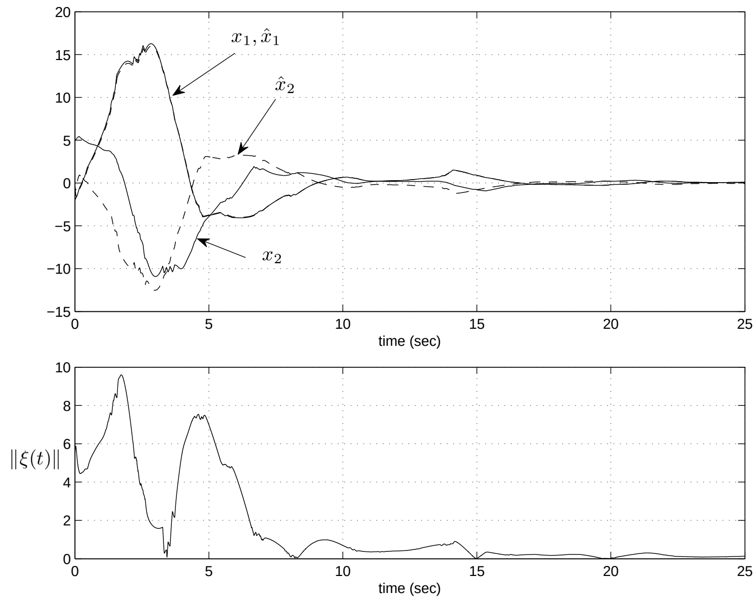


Figure 7.1: State convergence: $\xi = [x_1 - \hat{x}_1, x_2 - \hat{x}_2]^T$

to which a direct calculation shows that (7.4), (7.5), and (7.6) are satisfied.

For z-system of (6.64), we have the Lyapunov function $U_1(z) = U_2(z) = U(z) = z_1^2 + z_2^2$ whose Lie derivatives are

$$\begin{aligned} \mathcal{L}_{Q_1}U(z) &= -2z_1^2 + 2z_2z_1^2 - 2z_2^2 - 2z_2^4 - 2z_2z_1^2 + 2z_2^2x_1 \leq -2U(z) + 0.5x_1^2 \\ \mathcal{L}_{Q_2}U(z) &= -6z_1^2 + 2z_1z_2 + 2z_1z_2 - 4z_2^2 + 2x_1z_2 \leq -2U(z) + x_1^2. \end{aligned} \tag{7.59}$$

Since $U(z)$ is in quadratic form, Assumption 7.2.2 holds for $\mu = 1, p_0 = 2, a_\rho = 1$, and $a_1 = a_2 = 1$. Clearly, the systems (7.57) satisfy the Lipchitz condition in (7.1). As such, conditions of Theorem 7.3.1 are satisfied. Then, by Theorem 7.3.1, the output feedback control $u = -\lambda(2\lambda\hat{x}_1 + 5\lambda\hat{x}_2)$, where $\lambda, \hat{x}_1$, and $\hat{x}_2$ are generated by (7.11) and (7.12) with $n = 2, g_1 = 1.5, g_2 = 2.5, a_1 = 30$, and $a_2 = 25$, stabilizes the system (7.57) for any persistent dwell-time switching sequence satisfying $\tau_{\text{p}} > T_{\text{p}}$.

For simulation, we choose the initial state $(z, x) = [1, -2, -2, 5]^T, \hat{x} = 0, \lambda(0) = 1$, and $k_\lambda = 10$. The switching sequence with persistent period $T_{\text{p}} = 0.4s$ and dwell-

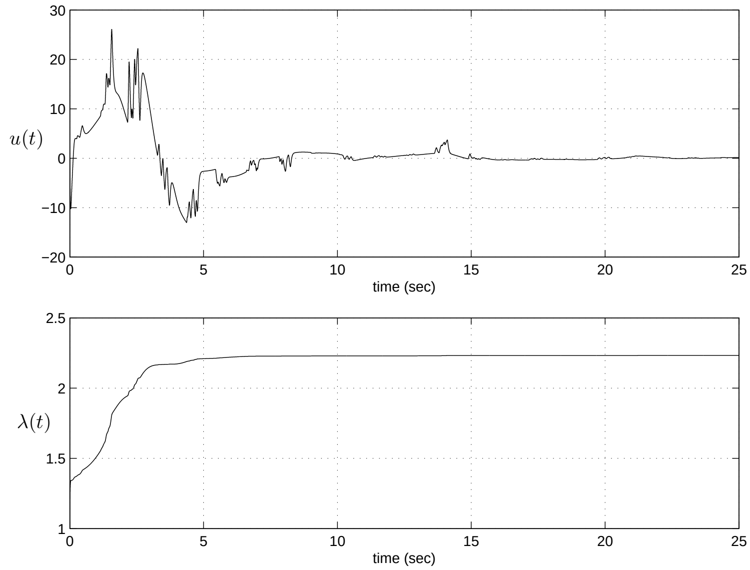


Figure 7.2: Control input $u(t)$ and observer's gain $\lambda(t)$

time $\tau_{\mathrm{p}} = 0.6s$ is generated in such a way that i) on persistent periods, the lengths of switching intervals are generated randomly in $[0, 0.1]$ and ii) the lengths of dwell-time intervals are generated randomly in $[0.6, 0.9]$. The simulation results are shown in Figures 7.1 and 7.2. As observed, Figure 7.1 shows that the stabilization is well obtained, and Figure 7.2 shows that the adaptive observer's gain λ converges to a fixed value. Thus, the simulation results well illustrated the presented theory.

●

Chapter 8

Switching-Uniform Adaptive Neural Control

In this chapter, adaptive neural control is presented for a class of switched nonlinear systems with switching jumps and uncertain system models. Further conditions on limiting variation of the Lyapunov function is given for asymptotic gain of switched systems with switching jumps. The control objective is achieved uniformly with respect to the class of persistent dwell-time switching sequences. The coupled difficulties associated with the discrepancy between control gains and switching jumps are overcome by a discontinuous adaptive neural control combined with the classical adaptive control. Smooth approximations of the discontinuous controls are then presented for a systematic design procedure.

8.1 Introduction

Adaptive neural control is a well-established yet important area in advanced control. It provides an effective tool for dealing with systems of models containing functions whose existence is guaranteed but whose determination is failed [112, 132, 96, 49, 44].

Due to the approximating nature of modeling methods and a variety of sources of uncertainties in practice, switched systems whose constituent systems' models involve unknown functions are obviously of practical relevance and hence are in the scope of control theory. As such, the problem of utilizing the capability of handling unknown functions of adaptive neural networks (NNs) arises naturally.

The challenging obstacle in adaptive neural control of switched systems are due to the contradiction between continuity conditions for validation of neural networks approximation and the discontinuities caused by switching events. In this chapter, we shall deal with this fundamental problem and introduce an adaptive neural control design method for a class of uncertain switched systems in which the sources of discontinuities are uncontrolled switching jumps and discrepancy between control gains of constituent systems.

The destabilizing behavior caused by switching jumps makes the usual decreasing condition on Lyapunov functions in existing stability theories of switched systems [22,63] unsatisfiable, and at the same time, makes the usual use of Young's inequality for decoupling unknown parameters and known functions in adaptive control no longer effective. As well-known in switched systems, due to the destabilizing behavior at switching times, satisfactorily stabilizing performance must be achieved before new switches for stability. However, high-order terms resulted from decoupling operations do not give suitable estimates of variations of Lyapunov functions for tailoring this performance. To overcome this difficulty, we shall adopt Theorem 5.2.1 to present a further Lyapunov-stability theorem addressing switching jumps. We then introduce a control design method that combines advantages of adaptive neural control and classical adaptive control.

Again, we consider constituent systems in triangular form as systems of this form have the advantages i) controls can be obtained via a systematic design procedure, and ii) nonlinear systems can be transformed into triangular forms under appropriate conditions [70, 85]. For this class of systems, the difficulty is also due to uncertain switching making control by switching among a set of controllers predesigned for individual subsystems [106] difficult and the discrepancy between control gains making the usual design of adaptation laws by matching controls to system nonlinearities [85, 49] do not apply. In this chapter, the former difficulty is overcome by exploiting the aforementioned stability theorem to design a control depending on the dwell-time property of the switching sequence only. The later difficulty is overcome by a discontinuous adaptive neural control combined with classical adaptive control. Smooth approximation of this control is then presented for recursive design. A condition in terms of switching sequences' dwell-time property and design parameters is presented for verifying satisfaction of stability conditions of the resulting closed-loop system. It is observed that when there is no switching jump, the obtained control achieves the control objective under arbitrary switching.

8.2 Problem Formulation and Preliminaries

8.2.1 System Model

Consider the collection of dynamical systems that, after suitable changes of coordinates, can be described by the following equations:

$$\Sigma_q : \begin{cases} \dot{x}_i(t) &= g_{q,i}(\bar{x}_i(t))x_{i+1}(t) + f_{q,i}(\bar{x}_i(t)) \\ & \quad i = 1, \dots, n-1 \\ \dot{x}_n(t) &= g_{q,n}(\bar{x}_n(t))u(t) + f_{q,n}(\bar{x}_n(t)) \end{cases}$$
$$q \in \mathbb{Q}, \tag{8.1}$$

where $\mathbb{Q} = \{1, \dots, q^\natural\}$ is a finite discrete set, $x \stackrel{\text{def}}{=} \bar{x}_n = [x_1, \dots, x_n]^T \in \mathbb{R}^n, y \in \mathbb{R}$ and $u \in \mathbb{R}$ are system state, output and input, respectively, $\bar{x}_i = [x_1, \dots, x_i]^T \in \mathbb{R}^i, i \in$

$\{1,\ldots,n\}$, and $g_{q,i}(\cdot), f_{q,i}(\cdot)$, $q \in \mathbb{Q}, i \in \{1,\ldots,n\}$ are unknown smooth functions. Throughout this chapter, $\mathscr{N}$ is the set $\{1,\ldots,n\}$.

In the formal language of transition model of dynamical systems presented in Section 2.4 (Chapter 2) and Section 5.2.1 (Chapter 5), we have the following collection as a model of switched systems with input and switching jump:

$$\Sigma_{\mathcal{J}} = \{\mathbb{T}, \mathbb{Q}, \mathbb{R}^n, \mathcal{L}^1_\infty, \{\Sigma_q\}_{q\in\mathbb{Q}}, \mathbb{S}, \quad \}, \tag{8.2}$$

where $\mathbb{T} = \mathbb{R}^+$ is the time space, $\mathbb{Q}$ and $\mathbb{R}^n$ are spaces of discrete and continuous states, $\mathcal{L}^1_\infty$ is the space of measurable locally essentially bounded functions mapping $\mathbb{R}^+$ to $\mathbb{R}$ and representing the space of one-dimensional input, $\mathbb{S}$ is the space of switching sequences, and $\quad : \mathbb{R}^+ \times \mathbb{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ is the discrete transition mapping. Let $y_m = h_m(x)$ and $y = h(x)$ be the measured output and the controlled output of $\Sigma_{\mathcal{J}}$. Consider the following dynamical system

$$\Sigma_C : \begin{cases} \dot{\zeta} & = & \Gamma(\zeta, u_C) \\ y_C & = & h_C(\zeta, u_C) \end{cases} . \tag{8.3}$$

where $\zeta \in \mathbb{R}^{n_C}$ is the state of Σ_C. For each $q \in \mathbb{Q}$, let Σ_q^C label the dynamical system resulted from the interconnection between Σ_q and Σ_C through $u_C = y_m, u = y_C$. Then, we have the following closed-loop switched systems:

$$\Sigma_{\mathscr{C}} = \{\mathbb{R}^+, \mathbb{Q}, \mathbb{R}^n \times \mathbb{R}^{n_C}, \{\Sigma_q^C\}_{q\in\mathbb{Q}}, \mathbb{S}, \quad \}. \tag{8.4}$$

The output tracking control problem for system $\Sigma_{\mathcal{J}}$ is stated as follows.

Output Tracking Design a dynamical control Σ_C of the form (8.3) such that for every switching sequence $\sigma \in \mathbb{S}$ and initial condition $X(0) \stackrel{\text{def}}{=} [x^T(0), \zeta^T(0)]^T$, the trajectory $X(t) \stackrel{\text{def}}{=} [x^T(t), \zeta^T(t)]^T$ satisfies the following properties:

i) $X(t)$ is bounded; and

ii) the measured output $y(t) = x_1(t)$ follows a prescribed signal $y_d(t)$.

In this chapter, we provide a solution to the proposed control problem under the following conditions.

Assumption 8.2.1 *The the set of switching sequences is* $\mathbb{S} = \mathbb{S}_{\mathcal{A}}[\tau_{\mathrm{p}}, N_{\mathrm{p}}]$ *which consists of all switching sequences having the same persistent dwell-time* $\tau_{\mathrm{p}} > 0$ *and the same chatter bound of persistence* N_{p}*, i.e., i) for every* $T \geq 0$ *there is* $i \in \mathbb{N}$ *such that* $\tau_{\sigma,i} > T$ *and* $\tau_{\sigma,i+1} - \tau_{\sigma,i} \geq \tau_{\mathrm{p}}$*, and ii) the number of switching events of any* $\sigma \in \mathbb{S}_{\mathcal{A}}[\tau_{\mathrm{p}}, N_{\mathrm{p}}]$ *between every two consecutive time intervals of the lengths greater than* τ_{p} *is less than* N_{p}*.*

Assumption 8.2.2 *The measured output is* $y_m = x$*. The desired signal* y_d *is continuously differentiable,* y_d *and its time derivative* $\dot{y}_d$ *are bounded by a constant* y_M*:*

$$\max\{|y_d(t)|, |\dot{y}_d(t)|\} \leq y_M, \forall t \geq 0. \tag{8.5}$$

Assumption 8.2.3 *There is a constant* $\mu \geq 0$ *such that at any switching time* $\tau_{\sigma,i}, i \in \mathbb{N}$*, we have*

$$\| \quad (\tau_{\sigma,i}, q_{\sigma,i}, x^-(\tau_{\sigma,i})) - x^-(\tau_{\sigma,i})\| \leq \mu |e(\tau_{\sigma,i})|, \tag{8.6}$$

where $e(t) = y(t) - y_d(t)$ *is the tracking error and* $x^-(\tau_{\sigma,i})$ *is the departing state of the discrete transition.*

As will be clarified in Remark 8.3.1 below, due to the sudden changes in coefficients of unknown estimation errors, certainty equivalence principle does not apply for adaptive neural control of switched systems, i.e., replacing unknown parameters by their converging estimates for the actual control is not possible. To overcome this obstacle, we consider the scaling functions γ_i's for control gains with the following property.

Assumption 8.2.4 *The control gains $g_{q,i}, q \in \mathbb{Q}, i \in \mathcal{N}$ have known unchanged signs and are bounded. There are known functions $\gamma_i, i \in \mathcal{N}$ and possibly unknown bounded functions $g_{0,i}, i \in \mathcal{N}$ such that*

$$1 \leq \frac{g_{q,i}(\bar{x}_i)}{g_{0,i}(\bar{x}_i)} \leq \gamma_i(\bar{x}_i), \forall \bar{x}_i \in \mathbb{R}^i, \forall q \in \mathbb{Q}, i \in \mathcal{N}, \tag{8.7}$$

and along the trajectory $x(t)$ of the system, the functions $g_{0,i}$'s have finite rates of change, i.e., the time Dini derivatives of $g_{0,i}$'s satisfy

$$|D^+ g_{0,i}(\bar{x}_i(t))| \leq g_d, \forall t \in \mathbb{R}^+, i \in \mathcal{N} \tag{8.8}$$

for some $g_d \geq 0$. In addition, γ_i's and $\partial\gamma_i(\bar{x}_i)/\partial\bar{x}_i, i \in \mathcal{N}$ are bounded.

Without loss of generality, we further assume that the signs of $g_{q,i}$'s are all positive.

Assumption 8.2.3 implies that the jumps in system state at switching instants are governed by the tracking error. This appears to be a necessary condition for convergence of trajectories of systems with uncontrolled state jumps.

The functions γ_i's in Assumption 8.2.4 can be considered as a generalization as in either non-switched systems and switched systems with identical subsystems' control gains, condition (8.7) automatically holds for $\gamma_i = 1, \forall i \in \mathcal{N}$. In addition, as neural networks approximations are implemented over compact sets and the functions $g_{q,i}$'s, $g_{0,i}$'s, and γ_i's are fixed *a priori*, is is well-known that imposing the boundedness on these functions is not restrictive in the context of adaptive neural control [118].

Let us recall the following lemma for smooth approximation of non-smooth functions.

Lemma 8.2.1 ([123]) *The following inequality holds for any $\varepsilon > 0$*

$$0 \leq |\eta| - \eta \tanh(\eta/\varepsilon) \leq k_P \varepsilon, \forall \eta \in \mathbb{R} \tag{8.9}$$

where k_P is a constant that satisfies $k_P = e^{-(k_P+1)}$, i.e., $k_P = 0.2785$.

8.2.2 Switching-Uniform Practical Stability

Consider the general switched autonomous system $\Sigma_{\mathscr{A}}$ defined by Definition 2.4.2 in Chapter 2 with $\mathbb{X} = \mathbb{R}^+ \times \mathbb{R}^n$. Then, for fixed $\sigma \in \mathbb{S}, t_s \in \mathbb{R}^+$ and $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, the transition mapping $\mathscr{T}_{\sigma,\mathscr{A}}$ defined by (2.11) defines a trajectory $(t, x(t)) = \mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, (t_0, x_0))$ of $\Sigma_{\mathscr{A}}$ starting at $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ in the continuous space $\mathbb{R}^+ \times \mathbb{R}^n$. We have the following notion of practical stability for switched systems.

Definition 8.2.1 *The system $\Sigma_{\mathscr{A}}$ is said to be switching-uniformly practically stable (SUpS) if there is a constant $c > 0$ such that for every fixed $\sigma \in \mathbb{S}, t_s \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}^n$, the corresponding trajectory $(t, x(t)) = \mathscr{T}_{\sigma,\mathscr{A}}(t, t_s, (t_0, x_0))$ of $\Sigma_{\mathscr{A}}$ satisfies*

$$\limsup_{t\to\infty} \|x(t)\| \le c. \tag{8.10}$$

The notion of practical stability is useful in describing the behavior of converging to some compact set of fixed size in the state space of system state [87, 75]. As we are interested in converging behavior of the continuous state x of the switched systems with input $\Sigma_{\mathcal{J}}$, which is part of the state of the closed-loop system $\Sigma_{\mathscr{C}}$, and adaptive neural control typically achieves the control objective in the sense of practical stability [49, 48], the above notion of practical stability of switched systems is of instrumental interest.

In the above switched autonomous system $\Sigma_{\mathscr{A}}$, we have considered the time variable t as part of the continuous state. In view of (8.10), there is no confusion should arise. The rule of transition of $\Sigma_{\mathscr{A}}$ consists of the time shift transition of t and the transition mappings $\psi_q : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n, q \in \mathbb{Q}$ of the continuous state $x(t)$. Let $f_q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the time-varying vector fields generating $\psi_q, q \in \mathbb{Q}$. As we are dealing with switching jumps, there are discrete transitions of continuous state at the times of changing rule of transition. As such, in the framework of transition mappings introduced in Section 2.4.2, at each switching time $\tau_{\sigma,i}, i \in \mathbb{N}$, we shall use the notation $x^-(\tau_{\sigma,i})$ to denote the state $\psi_{q_{\sigma,i-1}}(\Delta\tau_{\sigma,i-1}, \mathscr{T}_{\sigma,\mathcal{A}}(\tau_{\sigma,i-1}, t_s, x_0))$–the last state in $(i-1)$-th switching event of σ, and the notation $x(\tau_{\sigma,i})$ to denote the state $(\tau_{\sigma,i}, q_{\sigma,i}, x^-(\tau_{\sigma,i}))$–the starting state of the i-th switching event of σ.

We have the following theorem for SUpS of switched systems.

Theorem 8.2.1 *Suppose that, for switched autonomous system $\Sigma_{\mathcal{A}}$ described above with $\mathbb{S} = \mathbb{S}_{\mathcal{A}}[\tau_{\mathrm{p}}, N_{\mathrm{p}}]$, there exist class–$\mathcal{K}_\infty$ functions $\underline{\alpha}, \bar{\alpha}$, and α, non-negative numbers $\varepsilon_\tau \in [0, \tau_{\mathrm{p}}), p, c_1$ and c_2, and a continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ such that, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ and $q \in \mathbb{Q}$, we have*

$$\underline{\alpha}(\|x\|) \leq V(t, x) \leq \bar{\alpha}(\|x\|), \tag{8.11}$$

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f_q(t, x) \leq -\alpha(\|x\|) + c_1, \tag{8.12}$$

and along the trajectory $(t, x(t)) = \mathscr{T}_{\sigma,\mathcal{A}}(t, t_s, x_0)$, the following properties hold:

i) there is a constant $V_0 = V_0(\sigma, t_s, x_0)$ such that $V(t, x(t)) \leq V_0, \forall t \geq 0$;

ii) for the sequence $\{\tau_{\sigma,i_j}\}_j$ of all switching times satisfying $\Delta\tau_{\sigma,i_j} \geq \varepsilon_\tau$, we have

$$\lim_{j\to\infty} (V(\tau_{\sigma,i_j}, x(\tau_{\sigma,i_j})) - V(\tau_{\sigma,i_j+1}, x^-(\tau_{\sigma,i_j+1}))) \leq \varepsilon_\tau c_2; \text{ and} \tag{8.13}$$

iii) at any switching time $\tau_{\sigma,i}$, we have $V(\tau_{\sigma,i}, x(\tau_{\sigma,i})) \leq pV(\tau_{\sigma,i}, x^-(\tau_{\sigma,i}))$.

Then, the $\Sigma_{\mathcal{A}}$ is switching-uniformly practically stable.

Proof: See Section 8.6.1. ∎

8.3 Direct Adaptive Neural Control Design

The purpose of this section is to develop a systematic design procedure for adaptive neural control of switched system (8.1). The novelty lies in the introduction of a discontinuous adaptive neural control using scaling functions γ_i's combined with classical adaptive control for dealing with discrepancy between subsystems' control gains and switching jumps. Smooth approximations of the discontinuous controls are then presented for desired smoothness in recursive designs. As usual, the design procedure includes n steps as follows.

Initial Step

Let $\xi_1 = x_1 - y_d$ and $\xi_2 = x_2 - \alpha_1$, where α_1 is the virtual control to be designed. From the models (8.1) of constituent systems Σ_q , the dynamic equation for ξ_1 are

$$\dot{\xi}_1(t) = g_{q,1}(x_1)x_2 + f_{q,1}(x_1) - \dot{y}_d(t) \stackrel{\text{def}}{=} Q_{q,1}(x_1, x_2, \dot{y}_d(t)), q \in \mathbb{Q}. \tag{8.14}$$

Consider the following control structure

$$\alpha_1^* = -k_1\xi_1 - \pi_1(x_1)\xi_1, \tag{8.15}$$

where $k_1 > 0$ is a design parameter and $\pi_1(x_1)$ is a positive smooth function to be specified. We have the following Lyapunov function candidate:

$$U_1 = \frac{1}{2g_{0,1}(x_1)}\xi_1^2, \tag{8.16}$$

where $g_{0,1}$ is given in Assumption 8.2.4. From Assumptions 8.2.2 and 8.2.4, using Young's inequality and replacing x_2 by $\xi_2 + \alpha_1$, the time derivatives of U_1 at a time

$t \in \mathbb{R}^+$ along the (time-varying) vector fields $Q_{q,1}, q \in \mathbb{Q}$ satisfy:

$$\begin{aligned}\mathcal{L}_{Q_{q,1}} U_1 &= \frac{-D^+ g_{0,1}(\cdot)}{2g_{0,1}^2(\cdot)}\xi_1^2 + \frac{1}{g_{0,1}(\cdot)}\big(g_{q,1}(\cdot)(\alpha_1 + \xi_2) + f_{q,1}(\cdot) - \dot{y}_d(t)\big)\xi_1 \\ &\le \frac{g_d}{2g_{0,1}^2(\cdot)}\xi_1^2 + \frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}(\alpha_1 - \alpha_1^*)\xi_1 + \frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}\alpha_1^*\xi_1 \\ &\quad + \frac{g_{q,1}^2(\cdot)}{4g_{0,1}^2(\cdot)}\xi_1^2 + \xi_2^2 + \frac{\lambda_1(|f_{q,1}(x_1)| + y_M)^2}{4g_{0,1}^2(\cdot)}\xi_1^2 + \frac{1}{\lambda_1},\end{aligned} \tag{8.17}$$

where λ_1 is a design parameter. In view of (8.17), let π_1 be the smooth function satisfying

$$\pi_1(x_1) \ge \frac{1}{g_{0,1}^2(\cdot)}\Big(\frac{g_d}{2} + \frac{g_{q,1}^2(\cdot)}{4} + \lambda_1\frac{(|f_{q,1}(x_1)| + y_M)^2}{4}\Big), \forall q \in \mathbb{Q}, \tag{8.18}$$

which exists as $f_{q,1}, g_{q,1}, q \in \mathbb{Q}$ are continuous and $\mathbb{Q}$ is finite. As $g_{q,1}(\cdot)/g_{0,1}(\cdot) \ge 1, \forall q \in \mathbb{Q}$ by Assumption 8.2.4, substituting (8.15) and (8.18) into (8.17), we obtain

$$\mathcal{L}_{Q_{q,1}} U_1 \le \frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}(\alpha_1 - \alpha_1^*)\xi_1 - k_1\xi_1^2 + \xi_2^2 + \frac{1}{\lambda_1}, q \in \mathbb{Q}. \tag{8.19}$$

In view of (8.15) and (8.18), α_1^* is dependent on unknown functions $g_{q,1}, f_{q,1}, q \in \mathbb{Q}$. Thus, neural networks (NNs) is called for approximation. From (8.15) and (8.18), we have the following NNs representation of α_1^*.

$$\alpha_1^* = -c_1\xi_1 + W_{1,a}^{*T} S(V_{1,a}^{*T} Z_1) + \varepsilon_{1,a} + \lambda_1(W_{1,a}^{*T} S(V_{1,b}^{*T} Z_1) + \varepsilon_{1,b}), \tag{8.20}$$

where $Z_1 = [x_1, y_d, 1]^T$ and $\varepsilon_{1,a}$ and $\varepsilon_{1,b}$ are approximation errors.

It is observed from (8.20) that we have used two NNs to approximate two unknown smooth functions. The behind rationale is due to the fact that λ_1 is a known parameter and hence the structure of the bounding function π_1 in (8.18) can be utilized for better control performance.

Remark 8.3.1 *In the usual adaptive neural control [49], the virtual control* α_1 *can be designed following certainty equivalence principle (CEP) that simply obtains the control by replacing the unknown NNs parameters* $W^*_{1,a}, V^*_{1,a}, W^*_{1,b}$ *and* $V^*_{1,b}$ *by their estimates* $\hat{W}_{1,a}, \hat{V}_{1,a}, \hat{W}_{1,b}$, *and* $\hat{V}_{1,b}$, *respectively. Then, the parameter update laws are designed such that the terms containing the uncertain difference* $\alpha_1 - \alpha_1^*$ *are eliminated. However, by virtue of (8.19), this method does not apply to the current problem as the term* $\xi_1(\alpha_1 - \alpha_1^*)g_{q,1}(\cdot)/g_{0,1}(\cdot)$ *contains the unknown function* $g_{q,1}(\cdot)$ *which cannot be reduced for matching* $\alpha_1 - \alpha_1^*$ *to the update law as usual. Thus, a non-CEP control is of instrumental interest.*

In (8.20), three-layers NNs are used for function approximations. In comparison to RBF NNs which are also capable of approximating continuous functions [119], multi-layer NNs have the advantage that the basis function set as well as the centers and variations of radial-basis type of activation functions are estimated online and hence, they need not to be specified *a priori* [157, 49]. This means that we need not to fix *a priori* the compact set Ω over which the NNs approximations are employed.

We proceed to define the following variables:

$$\begin{aligned} \eta_1^* &= \hat{W}_{1,a}^{*T} S(V_{1,a}^{*T} Z_1) + \lambda_1 W_{1,b}^{*T} S(V_{1,b}^{*T} Z_1) \\ \hat{\eta}_1 &= \hat{W}_{1,a}^{T} S(\hat{V}_{1,a}^{T} Z_1) + \lambda_1 \hat{W}_{1,b}^{T} S(\hat{V}_{1,b}^{T} Z_1). \end{aligned} \tag{8.21}$$

Consider the following control strategy

$$\alpha_{1,\mathrm{d}} = \begin{cases} -c_1\xi_1 + \hat{\eta}_1 + \alpha_1^s \textbf{ if } \hat{\eta}_1\xi_1 \geq 0, \\ -c_1\xi_1 + \gamma_1(x_1)\hat{\eta}_1 + \alpha_1^s \textbf{ if } \hat{\eta}_1\xi_1 < 0, \end{cases} \tag{8.22}$$

which can be expressed in the following equivalent form for smooth approximation

$$\alpha_{1,\mathrm{d}} = -c_1\xi_1 + \left(1 + \frac{\gamma_1(x_1) - 1}{2}(1 - \mathrm{sgn}(\hat{\eta}_1\xi_1))\right)\hat{\eta}_1 + \alpha_1^s, \tag{8.23}$$

where α_1^s is the leakage term, and $\mathrm{sgn}(s) = 1$ if $s \geq 0$ and $\mathrm{sgn}(s) = -1$, otherwise.

From (8.15), (8.18) and (8.20), we have $(\eta_1^* + \varepsilon_{1,a} + \lambda_1 \varepsilon_{1,b})\xi_1 \leq 0$. For $q \in \mathbb{Q}$, we have the following cases:

- $\hat{\eta}_1\xi_1 \geq 0$: Since $g_{q,1}(\cdot)/g_{0,1}(\cdot) \leq \gamma_1(\cdot)$ by Assumption 8.2.4, from (8.20) and (8.22), we have

$$\begin{aligned}\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}(\alpha_{1,\mathrm{d}} - \alpha_1^*)\xi_1 \leq& -k_1\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}\xi_1^2 + \frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}\hat{\eta}_1\xi_1 + k_1\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}\xi_1^2 \\ &+ \frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}\big(-(\eta_1^* + \varepsilon_{1,a} + \lambda_1\varepsilon_{1,b})\big)\xi_1 + \alpha_1^s\xi_1 \\ \leq& \gamma_1(\cdot)(\hat{\eta}_1 - \eta_1^*)\xi_1 + \alpha_1^s\xi_1 - \gamma_1(\cdot)(\varepsilon_{1,a} + \lambda_1\varepsilon_{1,b})\xi_1. \end{aligned} \tag{8.24}$$

- $\hat{\eta}_1\xi_1 < 0$: using $g_{q,1}(\cdot)/g_{0,1}(\cdot) \geq 1$ from Assumption 8.2.4, we also have

$$\begin{aligned}\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}(\alpha_{1,\mathrm{d}} - \alpha_1^*)\xi_1 \leq& -k_1\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}\xi_1^2 + \gamma_1(\cdot)\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}\hat{\eta}_1\xi_1 + k_1\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}\xi_1^2 \\ &+ \frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}\big(-(\eta_1^* + \varepsilon_{1,a} + \lambda_1\varepsilon_{1,b})\big)\xi_1 + \alpha_1^s\xi_1 \\ \leq& \gamma_1(\cdot)(\hat{\eta}_1 - \eta_1^*)\xi_1 + \alpha_1^s\xi_1 - \gamma_1(\cdot)(\varepsilon_{1,a} + \lambda_1\varepsilon_{1,b}))\xi_1. \end{aligned} \tag{8.25}$$

Combining both cases, we have

$$\begin{aligned}\frac{g_{q,1}(x_1)}{g_{0,1}(x_1)}(\alpha_{1,\mathrm{d}} - \alpha_1^*)\xi_1 \leq& \alpha_1^s\xi_1 + \gamma_1(x_1)\Big(\hat{W}_{1,a}^T S(\hat{V}_{1,a}^T Z_1) - W_{1,a}^{*T} S(V_{1,a}^{*T} Z_1) \\ &+ \lambda_1(\hat{W}_{1,b}^T S(\hat{V}_{1,b}^T Z_1) - W_{1,a}^{*T} S(V_{1,b}^{*T} Z_1))\Big)\xi_1 - \gamma_1(x_1)(\varepsilon_{1,a} + \lambda_1\varepsilon_{1,b})\xi_1. \end{aligned} \tag{8.26}$$

To continue, let us make the following convention.

The notation $\mathbb{w}$ stands for W, V, and θ, and $\mathbb{s}$ stands for the subscripts a and b. An expression containing either $\mathbb{w}$ or $\mathbb{s}$ stands for the group of all expressions obtained by replacing $\mathbb{w}$ and $\mathbb{s}$ by their all possible values, e.g. $\|W_{1,\mathbb{s}}\|^3$ is $\|W_{1,a}\|^3, \|W_{1,b}\|^3$. We shall write $\mathbb{w} \preceq 0$ if all elements of $\mathbb{w}$ are non-positive. Accordingly, we have $\mathbb{w}_1 \preceq \mathbb{w}_2$

if $\mathbb{w}_1 - \mathbb{w}_2 \preceq 0$. The operators $=, \prec, \succeq$, and $\succ$ are defined in the same element-wise manner. In addition, $\|\cdot\|_\sharp$ is $\|\cdot\|$ if $\cdot$ is a vector and is $\|\cdot\|_F$ if $\cdot$ is a matrix.

Let $\tilde{\mathbb{w}}_{1,\mathbb{s}} = \hat{\mathbb{w}}_{1,\mathbb{s}} - \mathbb{w}^*_{1,\mathbb{s}}$ be the estimation errors. From (8.26), using [161, Lemma 3.1], we have

$$\begin{aligned}\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}(\alpha_{1,\mathrm{d}} - \alpha_1^*)\xi_1 \leq{}& \gamma_1(x_1)(\tilde{W}_{1,a}^T(\hat{S}_{1,a} - \hat{S}'_{1,a})\hat{V}_{1,a}^T Z_1 + \hat{W}_{1,a}^T\hat{S}'_{1,a}(\tilde{V}_{1,a})^T Z_1)\xi_1 \\ &+ \gamma_1(x_1)d_{1,a}|\xi_1| + \lambda_1\gamma_1(x_1)(\tilde{W}_{1,b}^T(\hat{S}_{1,b} - \hat{S}'_{1,b})\hat{V}_{1,b}^T Z_1 + \hat{W}_{1,b}^T\hat{S}'_{1,b}(\tilde{V}_{1,b})^T Z_1)\xi_1 \\ &+ \lambda_1\gamma_1(x_1)d_{1,b}|\xi_1| - \gamma_1(x_1)(\varepsilon_{1,a} + \lambda_1\varepsilon_{1,b})\xi_1 + \alpha_1^s\xi_1, q \in \mathbb{Q},\end{aligned} \tag{8.27}$$

where $d_{1,\mathbb{s}} = \|V^*_{1,\mathbb{s}}\|_F\|Z_1\hat{W}^T_{1,\mathbb{s}}\hat{S}'_{1,\mathbb{s}}\|_F + \|W^*_{1,\mathbb{s}}\|\|\hat{S}'_{1,\mathbb{s}}\hat{V}^T_{1,\mathbb{s}}Z_1\| + \|W^*_{1,\mathbb{s}}\|$.

Let $\varepsilon > 0$ be a design parameter fixed *a priori*. By Lemma 8.2.1, we have

$$|\xi_1| \leq ab\Big(\xi_1 \tanh\Big(\frac{\xi_1\sqrt{1+b^2}}{\varepsilon}\Big) + \frac{k_P\varepsilon}{\sqrt{1+b^2}}\Big) \leq a\sqrt{1+b^2}\tanh\Big(\frac{\xi_1\sqrt{1+b^2}}{\varepsilon}\Big)\xi_1 + k_P\varepsilon a, \tag{8.28}$$

for arbitrary $a, b > 0$. By a direct calculation using (8.28), we have

$$\gamma_1(\cdot)d_{1,\mathbb{s}}|\xi_1| \leq \theta^{*T}_{1,a}\Phi_{1,a}\xi_1 + \lambda_1\theta^{*T}_{1,b}\Phi_{1,b}\xi_1 + k_P\varepsilon\Theta_1, \tag{8.29}$$

where $\theta^*_{1,\mathbb{s}}, \Theta_1$, and $\Phi_{1,\mathbb{s}}$ are given by the following recursive formulae with $i = 1$:

$$\begin{aligned}\theta^*_{i,\mathbb{s}} &= \big[\|W^*_{i,\mathbb{s}}\|, \|V^*_{i,\mathbb{s}}\|_F, \varepsilon_{i,\mathbb{s}}\big]^T \\ \Theta_i &= \|V^*_{i,a}\|_F + \|W^*_{i,a}\| + \lambda_i(\|V^*_{i,b}\|_F + \|W^*_{i,b}\|) \\ \Phi_{i,\mathbb{s}} &= \gamma_i(\bar{x}_i)\Big[\sqrt{1+\|Z_i\hat{W}^T_{i,\mathbb{s}}\hat{S}'_{i,\mathbb{s}}\|_F^2}\tanh\Big(\frac{\xi_i\sqrt{1+\|Z_i\hat{W}^T_{i,\mathbb{s}}\hat{S}'_{i,\mathbb{s}}\|_F^2}}{\varepsilon}\Big), \\ &\qquad (1+\sqrt{1+\|\hat{S}'_{i,\mathbb{s}}\hat{V}^T_{i,\mathbb{s}}Z_i\|^2})\tanh\Big(\frac{\xi_i\sqrt{1+2\|\hat{S}'_{i,\mathbb{s}}\hat{V}^T_{i,\mathbb{s}}Z_i\|^2}}{\varepsilon}\Big), -1\Big]^T.\end{aligned} \tag{8.30}$$

The classical adaptive neural control of nonlinear systems decouples the unknown parameters $\|W^*_{1,\mathbb{s}}\|$ and $\|V^*_{1,\mathbb{s}}\|_F$ in $d_{1,\mathbb{s}}$ and then designs the leakage term α_1^s to eliminate the remaining known positive functions [49]. However, this technique leads to functions of high orders of error variables which might be avoided for dealing with switching jumps. By (8.29), classical adaptive control is called for dealing with uncertainties without increasing the orders of error variable ξ_1.

By virtue of (8.29), let us consider the following leakage term

$$\alpha_1^s = -\hat{\theta}_{1,a}^T \Phi_{1,a} - \lambda_1 \hat{\theta}_{1,b}^T \Phi_{1,b}, \tag{8.31}$$

where $\hat{\theta}_{1,\mathbb{s}}$ are estimates of $\theta_{1,\mathbb{s}}^*$. Substituting (8.29) and (8.31) into (8.27) yields

$$\begin{aligned}\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}(\alpha_{1,\mathrm{d}} - \alpha_1^*)\xi_1 \leq \gamma_1(x_1)(&\tilde{W}_{1,a}^T(\hat{S}_{1,a} - \hat{S}'_{1,a})\hat{V}_{1,a}^T Z_1 + \hat{W}_{1,a}^T \hat{S}'_{1,a} \tilde{V}_{1,a}^T Z_1)\xi_1 \\ &+ \lambda_1 \gamma_1(x_1)(\tilde{W}_{1,b}^T(\hat{S}_{1,b} - \hat{S}'_{1,b})\hat{V}_{1,b}^T Z_1 + \hat{W}_{1,b}^T \hat{S}'_{1,b} \tilde{V}_{1,b}^T Z_1)\xi_1 \\ &- \tilde{\theta}_{1,a}^T \Phi_{1,a}\xi_1 - \lambda_1 \tilde{\theta}_{1,b}^T \Phi_{1,b}\xi_1 + k_P \varepsilon \Theta_1, q \in \mathbb{Q}. \end{aligned} \tag{8.32}$$

In the right hand side of (8.32), the unknown variables $\tilde{\mathbb{w}}_{1,\mathbb{s}}$ are in linear forms so that they can be eliminated by appropriate NNs parameter update laws. However, as discontinuous $\alpha_{1,\mathrm{d}}$ given by (8.22) cannot be used for virtual control in recusive design, we further apply Lemma 8.2.1 to obtain the following smooth approximation of $\alpha_{1,\mathrm{d}}$

$$\alpha_{1,\mathrm{sm}} = -k_1\xi_1 + \left(1 + \frac{\gamma_1(x_1) - 1}{2}\left(1 - \tanh\left(\frac{\gamma_1(x_1)(\gamma_1(x_1) - 1)\hat{\eta}_1\xi_1}{2\varepsilon}\right)\right)\right)\hat{\eta}_1 + \alpha_1^s. \tag{8.33}$$

As $1 \leq g_{q,1}(\cdot)/g_{0,1}(\cdot) \leq \gamma_1(\cdot)$ by Assumption 8.2.4, applying Lemma 8.2.1 for $\eta = \hat{\eta}_1\xi_1$ and $\varepsilon = 2\varepsilon/(\gamma_1(x_1)(\gamma_1(x_1) - 1))$, we have

$$\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}(\alpha_{1,\mathrm{sm}} - \alpha_{1,\mathrm{d}})\xi_1 \leq k_P\varepsilon, \forall q \in \mathbb{Q}. \tag{8.34}$$

Consider the first virtual control $\alpha_1 = \alpha_{1,\mathrm{sm}}$ given above. From (8.32) and (8.34), adding $0 = -\alpha_{1,\mathrm{d}} + \alpha_{1,\mathrm{d}}$ to $\alpha_1 - \alpha_1^*$, we obtain

$$\begin{aligned}\frac{g_{q,1}(\cdot)}{g_{0,1}(\cdot)}(\alpha_1 - \alpha_1^*)\xi_1 \le \gamma_1(x_1)(\tilde{W}_{1,a}^T(\hat{S}_{1,a} - \hat{S}'_{1,a})\hat{V}_{1,a}^T Z_1 + \hat{W}_{1,a}^T \hat{S}'_{1,a}\tilde{V}_{1,a}^T Z_1)\xi_1 \\ + \lambda_1\gamma_1(x_1)(\tilde{W}_{1,b}^T(\hat{S}_{1,b} - \hat{S}'_{1,b})\hat{V}_{1,b}^T Z_1 + \hat{W}_{1,b}^T \hat{S}'_{1,b}\tilde{V}_{1,b}^T Z_1)\xi_1 \\ - \tilde{\theta}_{1,a}^T \Phi_{1,a}\xi_1 - \lambda_1 \tilde{\theta}_{1,b}^T \Phi_{1,b}\xi_1 + \delta_{\mathrm{NN},1}, \end{aligned} \tag{8.35}$$

where $\delta_{\mathrm{NN},1} = (\Theta_1 + 1)k_P\varepsilon$ is an unknown constant that can be made arbitrarily small by adjusting ε. Substituting (8.35) into (8.19), we obtain

$$\begin{aligned}\mathcal{L}_{Q_{q,1}} U_1 \le -k_1\xi_1^2 + \xi_2^2 + \gamma_1(x_1)(\tilde{W}_{1,a}^T(\hat{S}_{1,a} - \hat{S}'_{1,a})\hat{V}_{1,a}^T Z_1 + \hat{W}_{1,a}^T \hat{S}'_{1,a}\tilde{V}_{1,a}^T Z_1)\xi_1 \\ + \lambda_1\gamma_1(x_1)(\tilde{W}_{1,b}^T(\hat{S}_{1,b} - \hat{S}'_{1,b})\hat{V}_{1,b}^T Z_1 + \hat{W}_{1,b}^T \hat{S}'_{1,b}\tilde{V}_{1,b}^T Z_1)\xi_1 \\ - \tilde{\theta}_{1,a}^T \Phi_{1,a}\xi_1 - \lambda_1 \tilde{\theta}_{1,b}^T \Phi_{1,b}\xi_1 + \delta_{\mathrm{NN},1} + \frac{1}{\lambda_1}. \end{aligned} \tag{8.36}$$

We are now ready to design the parameter update laws. Let $\Gamma_{\mathbb{W}_{1,\mathbb{S}}}$ be the design adaptation gain matrices of appropriate dimensions and consider the Lyapunov function candidate V_1 given by the following recursive formula with $i = 1$

$$\begin{aligned}V_i = U_i + \frac{1}{2}\tilde{W}_{i,a}^T \Gamma_{W_{i,a}}^{-1}\tilde{W}_{i,a} + \frac{1}{2}\mathrm{tr}\left\{\tilde{V}_{i,a}^T \Gamma_{V_{i,a}}^{-1}\tilde{V}_{i,a}\right\} + \frac{1}{2}\tilde{\theta}_{i,a}^T \Gamma_{\theta_{i,a}}^{-1}\tilde{\theta}_{i,a} \\ + \frac{\lambda_i}{2}\tilde{W}_{i,b}^T \Gamma_{W_{i,b}}^{-1}\tilde{W}_{i,b} + \frac{\lambda_i}{2}\mathrm{tr}\left\{\tilde{V}_{i,b}^T \Gamma_{V_{i,b}}^{-1}\tilde{V}_{i,b}\right\} + \frac{\lambda_i}{2}\tilde{\theta}_{i,b}^T \Gamma_{\theta_{i,b}}^{-1}\tilde{\theta}_{i,b}. \end{aligned} \tag{8.37}$$

Let $\mathbb{W}_{1,\mathbb{S}}^{\min}$ and $\mathbb{W}_{1,\mathbb{S}}^{\max}$ be the lower and upper bounds of the ideal NNs parameters $\mathbb{W}_{1,\mathbb{S}}^*$, i.e., $\mathbb{W}_{1,\mathbb{S}}^{\min} \preceq \mathbb{W}_{1,\mathbb{S}}^* \preceq \mathbb{W}_{1,\mathbb{S}}^{\max}$, and let $\mathbf{Proj}_{\hat{\mathbb{W}}_{1,\mathbb{S}}}$ be the standard projection mapping [51]. We have the following update laws for $\hat{\mathbb{W}}_{1,\mathbb{S}}$

$$\dot{\hat{\mathbb{W}}}_{1,\mathbb{S}} = \tau^{\natural}_{\mathbb{W}_{1,\mathbb{S}}} \stackrel{\mathrm{def}}{=} \mathbf{Proj}_{\hat{\mathbb{W}}_{1,\mathbb{S}}}(\tau_{\mathbb{W}_{1,\mathbb{S}}}), \tag{8.38}$$

where $\tau_{\mathbb{W}_{1,\mathbb{S}}}$ are tuning functions given by the following recursive formulae for $i = 1$

$$\begin{aligned}
\tau_{W_{i,\mathbb{S}}} &= \Gamma_{W_{i,\mathbb{S}}}\big(-\gamma_i(\bar{x}_i)(\hat{S}_{i,\mathbb{S}} - \hat{S}'_{i,\mathbb{S}}\hat{V}^T_{i,\mathbb{S}}Z_i)\xi_i - \sigma_{W_{i,\mathbb{S}}}\hat{W}_{i,\mathbb{S}}\big) \\
\tau_{V_{i,\mathbb{S}}} &= \Gamma_{V_{i,\mathbb{S}}}\big(-\gamma_i(\bar{x}_i)Z_i\hat{W}^T_{i,\mathbb{S}}\hat{S}'_{i,\mathbb{S}}\xi_i - \sigma_{V_{i,\mathbb{S}}}\hat{V}_{i,\mathbb{S}}\big) \\
\tau_{\theta_{i,\mathbb{S}}} &= \Gamma_{\theta_{i,\mathbb{S}}}\big(\Phi_{i,\mathbb{S}}\xi_i - \sigma_{\theta_{i,\mathbb{S}}}\hat{\theta}_{i,\mathbb{S}}\big),
\end{aligned} \tag{8.39}$$

where $\sigma_{\mathbb{W}_{1,\mathbb{S}}}$ are small design constants. For each $q \in \mathbb{Q}$, let us define the vector field:

$$\mathcal{Q}_{q,1} \overset{\text{def}}{=} [Q^T_{q,1}, (\tau^{\natural}_{W_{1,a}})^T, \mathrm{col}(\tau^{\natural}_{V_{1,a}})^T, (\tau^{\natural}_{W_{1,b}})^T, \mathrm{col}(\tau^{\natural}_{V_{1,b}})^T, (\tau^{\natural}_{\theta_{1,a}})^T, (\tau^{\natural}_{\theta_{1,b}})^T]^T. \tag{8.40}$$

From (8.36), by a direct computation using standard completing square and projection computation [128, 49], we obtain

$$\mathcal{L}_{\mathcal{Q}_{q,1}}V_1 \leq -k_1\xi_1^2 - \frac{1}{2}\sum_{\mathbb{W},\mathbb{S}}\sigma_{\mathbb{W}_{1,\mathbb{S}}}\|\tilde{\mathbb{W}}_{1,\mathbb{S}}\|^2_{\sharp} + \xi_2^2 + \delta_1, q \in \mathbb{Q}, \tag{8.41}$$

where

$$\delta_1 = \frac{1}{2}\sum_{\mathbb{W},\mathbb{S}}\sigma_{\mathbb{W}_{1,\mathbb{S}}}\|\mathbb{W}^*_{1,\mathbb{S}}\|^2_{\sharp} + \delta_{\mathrm{NN},1} + \frac{1}{\lambda_1} \tag{8.42}$$

is a constant that can be made arbitrarily small by adjusting design parameters: λ_1, ε, and $\sigma_{\mathbb{W}_{1,\mathbb{S}}}$. This completes the first step.

Inductive Step $(i = 2, \ldots, n)$**:**

Let $\xi_i = x_i - \alpha_{i-1}$ and $\xi_{n+1} = u - \alpha_n$, where α_{i-1} is the $(i-1)$-th virtual control designed at the $(i-1)$-th step.

By a direct computation, we have the following dynamic equations for ξ_i:

$$\dot{\xi}_i = g_{q,i}(\bar{x}_i)x_{i+1} + f^{\circ}_{q,i}(\bar{x}_i, \bar{\psi}_i, \bar{\phi}_i, y_d) - \frac{\partial\alpha_{i-1}}{\partial y_d}\dot{y}_d \overset{\text{def}}{=} Q_{q,i}(\cdot), q \in \mathbb{Q}, \tag{8.43}$$

where

$$
\begin{aligned}
f^{\circ}_{q,i}(\cdot) &= f_{q,i}(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(g_{q,j}(\bar{x}_j)x_{j+1} + f_{q,j}(\bar{x}_j)) - \phi_i, q \in \mathbb{Q} \\
\phi_i &= \sum_{j=1}^{i-1} \sum_{v=a,b} \Big(\frac{\partial \alpha_{i-1}}{\partial \hat{W}_{j,v}} \tau_{W_{j,v}} + \sum_{k=1}^{l_{j,v}} \frac{\partial \alpha_{i-1}}{\partial \hat{V}^{(q)}_{j,v}} \tau^{(k)}_{V_{j,v}} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}_{j,v}} \tau_{\theta_{j,v}} \Big), \bar{\phi}_i = [\phi_1, \ldots, \phi_i]^T \\
\psi_i &= \left[\frac{\partial \alpha_{i-1}}{\partial y_d}, \frac{\partial \alpha_{i-1}}{\partial x_1}, \ldots, \frac{\partial \alpha_{i-1}}{\partial x_{i-1}} \right]^T, \bar{\psi}_i = [\psi_1^T, \ldots, \psi_i^T]^T.
\end{aligned} \tag{8.44}
$$

In (8.44), the variables ϕ_i's and ψ_i's are computable and shall be included in NNs input for reducing computation load, $l_{j,v}$ is the number of columns of $\hat{V}_{j,v}$, and $\hat{V}^{(k)}_{j,v}$ and $\tau^{(k)}_{V_{j,v}}$ are k-th columns of $\hat{V}_{j,v}$ and $\tau_{V_{j,v}}$, respectively.

Consider the following i-th Lyapunov function candidate

$$
U_i = V_{i-1} + \frac{1}{2g_{0,i}(\bar{x}_i)} \xi_i^2. \tag{8.45}
$$

By Young's inequality and Assumption 8.2.4, we have the following estimates for Lie derivatives at time t of U_i along the vector fields $\bar{Q}_{q,i} \overset{\text{def}}{=} [\mathcal{Q}^T_{q,i-1}, Q_{q,i}]^T, q \in \mathbb{Q}$:

$$
\begin{aligned}
\mathcal{L}_{\bar{Q}_{q,i}} U_i &= \mathcal{L}_{\mathcal{Q}_{q,i-1}} V_{i-1} - \frac{D^+ g_{0,i}(\cdot)}{2g^2_{0,i}(\cdot)} \xi_i^2 + \frac{1}{g_{0,i}(\cdot)} \Big(g_{q,i}(\cdot)(\xi_{i+1} + \alpha_i) + f^{\circ}_{q,i}(\cdot) - \frac{\partial \alpha_{i-1}}{\partial y_d} \dot{y}_d \Big) \xi_i \\
&\leq \mathcal{L}_{\mathcal{Q}_{q,i-1}} V_{i-1} + \frac{g_d}{g^2_{0,i}(\cdot)} \xi_i^2 + \frac{g_{q,i}(\cdot)}{g_{0,i}(\cdot)} (\alpha_i - \alpha_i^*) \xi_i + \frac{g_{q,i}(\cdot)}{g_{0,i}(\cdot)} \alpha_i^* \xi_i + \frac{g^2_{q,i}(\cdot)}{4g^2_{0,i}(\cdot)} \xi_i^2 \\
&\quad + \xi_{i+1}^2 + 4\lambda_i \Big(|f^{\circ}_{k,i}(\cdot)| + \Big| \frac{\partial \alpha_{i-1}}{\partial y_d} \Big| y_M \Big)^2 \xi_i^2 + \frac{1}{\lambda_i},
\end{aligned} \tag{8.46}
$$

where $\lambda_i > 0$ is a design parameter. As (8.46) has the same structure as (8.17), following the same design with the same notations $\mathbb{w}$,$\mathbb{s}$, and $\sharp$ of the Initial Step, we obtain

i) the i-th ideal control

$$
\alpha_i^* = -\left(1 + k_i + \pi_i(\bar{x}_i, \bar{\phi}_i, \bar{\psi}_i)\right) \xi_i, \tag{8.47}
$$

where $k_i > 0$ is a design parameter and π_i is the smooth function satisfying

$$\pi_i(\cdot) \geq \frac{1}{g_{0,i}^2(\cdot)}\Big(\frac{g_d}{2} + \frac{g_{q,i}^2(\cdot)}{4} + \frac{\lambda_i}{4}\Big(f_{q,i}^{\circ}(\cdot) + \Big|\frac{\partial \alpha_{i-1}}{\partial y_d}\Big| y_M\Big)^2\Big), \forall q \in \mathbb{Q}; \tag{8.48}$$

ii) the i-th virtual control $\alpha_i = \alpha_{i,\mathrm{sm}}$ given by

$$\begin{aligned}
\alpha_{i,\mathrm{sm}} &= -k_i\xi_i + \Big(1 + \frac{\gamma_i(\bar{x}_i) - 1}{2}\Big(1 - \tanh\Big(\frac{\gamma_i(\bar{x}_i)(\gamma_i(\bar{x}_i) - 1)\hat{\eta}_i\xi_i}{2\varepsilon}\Big)\Big)\Big)\hat{\eta}_i + \alpha_i^s,\\
\hat{\eta}_i &= \hat{W}_{i,a}^T S(\hat{V}_{i,a}^T Z_i) + \lambda_i \hat{W}_{i,b}^T S(\hat{V}_{i,b}^T Z_i), Z_i = [\bar{x}_i^T, y_d, \alpha_{i-1}, \bar{\phi}_i^T, \bar{\psi}_i^T, 1]^T,\\
\alpha_i^s &= -\hat{\theta}_{i,a}^T \Phi_{i,a} - \lambda_i \hat{\theta}_{i,b}^T \Phi_{i,b},
\end{aligned} \tag{8.49}$$

where $\lambda_i > 0$ is the design parameter, $\hat{\theta}_{i,\mathbb{s}}$ are estimates of $\theta_{i,\mathbb{s}}^*$, $\hat{\mathbb{W}}_{i,\mathbb{s}}$ are estimates of the ideal NNs parameters $\mathbb{W}_{i,\mathbb{s}}^*$ of the NNs approximation of α_i^*, and $\theta_{i,\mathbb{s}}^*$, Θ_i, and Φ_i are defined by (8.30) with obvious meanings;

iii) the parameter update laws

$$\dot{\hat{\mathbb{W}}}_{i,\mathbb{s}} = \tau_{\mathbb{W}_{i,\mathbb{s}}}^{\natural} \stackrel{\text{def}}{=} \mathbf{Proj}_{\hat{\mathbb{W}}_{i,\mathbb{s}}}(\tau_{\mathbb{W}_{i,\mathbb{s}}}), \tag{8.50}$$

with the tuning functions $\tau_{\mathbb{W}_{i,\mathbb{s}}}$ given by (8.39); and

iv) the Lyapunov function candidate V_i given by (8.37),

such that, for each $q \in \mathbb{Q}$, we have

$$\mathcal{L}_{\mathcal{Q}_{q,i}} V_i \leq -\sum_{j=1}^{i} c_j\xi_j^2 - \frac{1}{2}\sum_{j=1}^{i}\sum_{\mathbb{s}=a,b} \sigma_{\mathbb{W}_{j,\mathbb{s}}} \|\tilde{\mathbb{W}}_{j,\mathbb{s}}\|_{\sharp}^2 + \xi_{i+1}^2 + \delta_i, \tag{8.51}$$

where $\mathcal{Q}_{q,i}, q \in \mathbb{Q}$ are time-varying vector fields

$$\mathcal{Q}_{q,i} = [\bar{Q}_{q,i}^T, (\tau_{W_{i,a}}^{\natural})^T, \mathrm{col}(\tau_{V_{i,a}}^{\natural})^T, (\tau_{W_{i,b}}^{\natural})^T, \mathrm{col}(\tau_{V_{i,b}}^{\natural})^T, (\tau_{\theta_{i,a}}^{\natural})^T, (\tau_{\theta_{i,b}}^{\natural})^T]^T, \tag{8.52}$$

$\Gamma_{\mathbb{W}_{i,\mathbb{s}}}$ are design adaptation gain matrices, and

$$\delta_i = \delta_{i-1} + \frac{1}{2}\sum_{\mathbb{W},\mathbb{S}} \sigma_{\mathbb{W}_{i,\mathbb{S}}} \|\mathbb{w}^*_{i,\mathbb{s}}\|^2_\sharp + (\Theta_i + 1)k_P\varepsilon + \frac{1}{\lambda_i} \tag{8.53}$$

is a constant which can be made arbitrarily small by adjusting design parameters.

Step n: At this step, $\xi_{n+1} = 0$, we obtain the actual control $u = \alpha_n$ given by (8.49) with $i = n$ and the final Lyapunov function candidate V_n satisfies

$$\mathcal{L}_{\mathcal{Q}_{q,n}} V_n \leq -\sum_{i=1}^{n} k_i\xi_i^2 - \frac{1}{2}\sum_{j=1}^{n}\sum_{\mathbb{s}=a,b} \sigma_{\mathbb{W}_{j,\mathbb{s}}} \|\tilde{\mathbb{w}}_{j,\mathbb{s}}\|^2_\sharp + \delta_n, \forall q \in \mathbb{Q}. \tag{8.54}$$

This completes the design procedure.

8.4 Stability Analysis

In this section, based on Theorem 8.2.1, we show that for appropriate design parameters, the control $u = \alpha_n$ designed above achieves the proposed control objective. To this end, let us define the variables $\mathcal{E}_1 = [\xi_1, \ldots, \xi_n]^T$, $\mathcal{E}_2 = [\tilde{W}^T, \tilde{V}^T]^T$ and $\mathcal{E} = [\mathcal{E}_1^T, \mathcal{E}_2^T]^T$, where

$$\begin{aligned} \tilde{W} &= [\tilde{W}^T_{1,a}, \ldots, \tilde{W}^T_{n,a}, \tilde{W}^T_{1,b}, \ldots, \tilde{W}^T_{n,b}]^T \\ \tilde{V} &= [\mathrm{col}(\tilde{V}_{1,a})^T, \ldots, \mathrm{col}(\tilde{V}_{n,a})^T, \mathrm{col}(\tilde{V}_{1,b})^T, \ldots, \mathrm{col}(\tilde{V}_{n,b})^T]^T. \end{aligned} \tag{8.55}$$

Let $V_{\mathcal{E}_1}$ and $V_{\mathcal{E}_2}$ be Lyapunov functions defined as

$$\begin{aligned} V_{\mathcal{E}_1}(t, \mathcal{E}_1) &= \sum_{i=1}^{n} \frac{1}{g_{0,i}(\bar{x}_i(t))}\xi_1^2, \text{ and} \\ V_{\mathcal{E}_2}(\mathcal{E}_2) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{\mathbb{s}=a,b} \big(\tilde{W}^T_{i,\mathbb{s}}\Gamma^{-1}_{W_{i,\mathbb{s}}}\tilde{W}_{i,\mathbb{s}} + \mathrm{tr}\big\{\tilde{V}^T_{i,\mathbb{s}}\Gamma^{-1}_{V_{i,\mathbb{s}}}\tilde{V}_{i,\mathbb{s}}\big\}\big). \end{aligned} \tag{8.56}$$

In terms of the above notations, let $n_\mathcal{E}$ be the appropriate dimension of $\mathcal{E}$, $V_\mathcal{E} \stackrel{\text{def}}{=} V_n$, and $\mathcal{Q}_q \stackrel{\text{def}}{=} \mathcal{Q}_{q,n}, q \in \mathbb{Q}$. Then, $V_{\mathcal{E}}$ can be expressed as $V_{\mathcal{E}}(t, \mathcal{E}) = V_{\mathcal{E}_1}(t, \mathcal{E}_1) + V_{\mathcal{E}_2}(\mathcal{E}_2)$.

As the functions $g_{0,i}$ are lower bounded by Assumption 8.2.4 and the set $\mathcal{N} = \{1, \ldots, n\}$ is finite, there are positive numbers $g_{\max}$ and $g_{\min}$ such that $g_{\min} \leq g_{0,i}(\zeta) \leq g_{\max}, \forall \zeta \in \mathbb{R}^i, \forall i \in \mathcal{N}$. Let $\lambda_0 > 0$ be the number that is greater than all eigenvalues of $\Gamma^{-1}_{\mathbb{W}_{i,\mathbb{S}}}, i \in N$ and $c_{\mathcal{E}}$ be the positive number defined by

$$c_{\mathcal{E}} = \frac{1}{2} \min\left\{\frac{1}{2\lambda_0} \min\{\sigma_{\mathbb{W}_{i,\mathbb{S}}}\}, g_{\min} \min\{c_1, \ldots, c_n\}\right\}. \tag{8.57}$$

Then, from (8.38), (8.54) and (8.56), we have

$$\mathcal{L}_{\mathcal{Q}_q} V_{\mathcal{E}}(t, \mathcal{E}) \leq -2c_{\mathcal{E}} V_{\mathcal{E}_1}(t, \mathcal{E}_1) - 2c_{\mathcal{E}} V_{\mathcal{E}_2}(\mathcal{E}_2) + \delta_n, \forall t \geq 0, \forall q \in \mathbb{Q}. \tag{8.58}$$

Consider the ball

$$B_{\delta_n} = \left\{\mathcal{E} \in \mathbb{R}^{n\varepsilon} : \frac{c_{\mathcal{E}}}{g_{\max}} \sum_{i=1}^{n} \xi_i^2 + c_{\mathcal{E}} V_{\mathcal{E}_2}(\mathcal{E}_2) \leq \delta_n\right\}. \tag{8.59}$$

Obviously, for $\mathcal{E} \notin B_{\delta_n}$, we have

$$\mathcal{L}_{\mathcal{Q}_q} V_{\mathcal{E}}(t, \mathcal{E}) \leq -c_{\mathcal{E}} V_{\mathcal{E}}(t, \mathcal{E}), \forall t \geq 0, q \in \mathbb{Q}. \tag{8.60}$$

We proceed to consider a switching sequence $\sigma \in \mathbb{S}_{\mathcal{A}}[\tau_{\mathrm{p}}, N_{\mathrm{p}}]$. Let $\{\tau_{\sigma, i_j^{\mathcal{D}}}\}_j$ be the sequence of all starting times of dwell-time switching events of σ, i.e., $\tau_{\sigma, i_j^{\mathcal{D}}+1} - \tau_{\sigma, i_j^{\mathcal{D}}} \geq \tau_{\mathrm{p}}, \forall j \in \mathbb{N}$. As $\sigma \in \mathbb{S}_{\mathcal{A}}[\tau_{\mathrm{p}}, N_{\mathrm{p}}]$, for each $j \in \mathbb{N}$, the number $n_{i_j}^{\mathcal{D}} = i_{j+1}^{\mathcal{D}} - i_j^{\mathcal{D}} - 1$ of switching events between two consecutive dwell-time switching events of σ satisfies $n_{i_j}^{\mathcal{D}} \leq N_{\mathrm{p}}$.

From the boundedness of sigmoid activation functions of NNs and their derivatives, the boundedness of parameter estimates ensured by the adaptation laws (8.38) and (8.50) [161, Lemma 4.1], the boundedness of γ_i's and their gradients from Assumption

8.2.4, and the boundedness of the function $f_{\text{th}}(x,y) = (a+\sqrt{1+x^2})\tanh(y\sqrt{1+bx^2}/\varepsilon)$ and its derivatives, where x, y are independent variables and $a, b \geq 0$ are constants, a direct computation shows that the derivatives of the virtual controls α_i's given by (8.33) and (8.49) are bounded, i.e., there is a constant $q_\alpha > 0$ such that

$$\left\| \frac{\partial \alpha_i}{\partial \bar{x}_i} \right\| \leq q_\alpha, \forall \bar{x}_i \in \mathbb{R}^i, \forall i \in \mathcal{N}. \tag{8.61}$$

We have the following proposition.

Proposition 8.4.1 *At switching times $\tau_{\sigma,i}$, we have*

$$V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}(\tau_{\sigma,i})) \leq q_\mu V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}^-(\tau_{\sigma,i})), \tag{8.62}$$

and for every $j \in \mathbb{N}$, if (8.60) holds for all $t \in [\tau_{\sigma,i_j^{\mathcal{P}}+1}, \tau_{\sigma,i_{j+1}^{\mathcal{P}}})$, then we have

$$V_{\mathcal{E}}(t, \mathcal{E}(t)) \leq q_\mu^{N_{\text{p}}+1} V_{\mathcal{E}}(\tau_{\sigma,i_j^{\mathcal{P}}+1}, \mathcal{E}(\tau_{\sigma,i_j^{\mathcal{P}}+1})) e^{t-\tau_{\sigma,i_j^{\mathcal{P}}+1}}, \forall t \in [\tau_{\sigma,i_j^{\mathcal{P}}+1}, \tau_{\sigma,i_{j+1}^{\mathcal{P}}}], \tag{8.63}$$

where

$$\begin{aligned} q_\mu &= 1 + 4\mu\Big(\delta_G + \frac{g_{\max}}{g_{\min}}(1+\mu(1+q_\alpha))(1+q_\alpha)\Big) \\ \delta_G &= \begin{cases} 0 & \text{if} \quad \mu = 0 \\ \dfrac{g_{\max}}{\mu g_{\min}} & \text{if} \quad \mu \neq 0 \end{cases}. \end{aligned} \tag{8.64}$$

Proof: See Section 8.6.2. ∎

Theorem 8.4.1 *Suppose that Assumptions 8.2.1–8.2.4 hold for system $\Sigma_{\mathcal{J}}$ given in (8.2) and*

$$(N_{\text{p}}+2)\ln q_\mu < c_{\mathcal{E}} \tau_{\text{p}}, \tag{8.65}$$

where q_μ is the constant defined in Proposition 8.4.1. Then, the control $u = \alpha_n$ given *by (8.49), $i = n$ together with parameter update laws (8.50), $i \in \mathcal{N}$ guarantees that all signals in the closed-loop system are bounded and the tracking error converges to a neighborhood of the origin whose size can be made arbitrarily small by adjusting design parameters undergoing (8.65).*

Proof: We first consider the case there is a time t^* such that $\mathcal{E}(t^*) \in B_{\delta_n}$. In this case, we know from the proof of Theorem 8.2.1 that $\mathcal{E}(t)$ escapes from B_{δ_n} only through switching jumps. Let $\tau_{\sigma,j}$ be the first switching time greater than t^* at which $\mathcal{E}(\tau_{\sigma,j}) \notin B_{\delta_n}$ and let $[\tau_{\sigma,i_{\mathcal{D}}^+(j)}, \tau_{\sigma,i_{\mathcal{D}}^+(j)+1}]$ be the first dwell-time interval after $\tau_{\sigma,j}$. By virtue of the proof of Proposition 8.4.1, we have $V_{\mathcal{E}}(\tau_{\sigma,j}, \mathcal{E}(\tau_{\sigma,j})) \leq q_\mu V_{\mathcal{E}}(\tau_{\sigma,j}, \mathcal{E}^-(\tau_{\sigma,j})) \leq q_\mu \delta_n$. Thus, as the number of switches between $\tau_{\sigma,j}$ and $\tau_{\sigma,i_{\mathcal{D}}^+(j)}$ is bounded by N_{p}, using Proposition 8.4.1, we have $V_{\mathcal{E}}(t, \mathcal{E}(t)) \leq q_\mu^{N_{\mathrm{p}}} \delta_n, \forall t \in [\tau_{\sigma,j}, \tau_{\sigma,i_{\mathcal{D}}^+(j)})$ and $V_{\mathcal{E}}(\tau_{\sigma,i_{\mathcal{D}}^+(j)}, \mathcal{E}(\tau_{\sigma,i_{\mathcal{D}}^+(j)})) \leq q_\mu^{N_{\mathrm{p}}+1} \delta_n$. As $\tau_{\sigma,i_{\mathcal{D}}^+(j)+1} - \tau_{\sigma,i_{\mathcal{D}}^+(j)} \geq \tau_{\mathrm{p}}$, this coupled with (8.65) implies that there is $t \in [\tau_{\sigma,i_{\mathcal{D}}^+(j)}, \tau_{\sigma,i_{\mathcal{D}}^+(j)+1}]$ such that $V_{\mathcal{E}}(t, \mathcal{E}(t)) \leq \delta_n$. As such, $V_{\mathcal{E}}(t, \mathcal{E}(t)) \leq q_\mu^{N_{\mathrm{p}}+1} \delta_n, \forall t \geq t^*$ and the conclusion of the theorem follows.

We now complete the proof by considering the case that there is no such t^*, i.e., (8.60) holds for all $t \geq 0$. Let us verify conditions of Theorem 8.2.1 as follows.

Since $g_{0,i}$'s are both upper and lower bounded by Assumption 8.2.4, (8.16), (8.45), and (8.37) imply that $V_{\mathcal{E}}$ is both upper and lower bounded by quadratic forms of $\mathcal{E}$. Hence, condition (8.11) of Theorem 8.2.1 is satisfied. The satisfaction of (8.12) follows directly from (8.60).

To verify condition i) of Theorem 8.2.1, let us consider a dwell-time interval $[\tau_{\sigma,i_j^{\mathcal{D}}}, \tau_{\sigma,i_j^{\mathcal{D}}+1}], j \in \mathbb{N}$. As (8.60) holds for all t, applying comparison principle [88] for (8.60) and using Proposition 8.4.1, we obtain

$$V_{\mathcal{E}}(\tau_{\sigma,i_j^{\mathcal{D}}+1}, \mathcal{E}(\tau_{\sigma,i_j^{\mathcal{D}}+1})) \leq q_\mu V_{\mathcal{E}}(\tau_{\sigma,i_j^{\mathcal{D}}+1}, \mathcal{E}^-(\tau_{\sigma,i_j^{\mathcal{D}}+1})) \leq q_\mu V_{\mathcal{E}}(\tau_{\sigma,i_j^{\mathcal{D}}}, \mathcal{E}(\tau_{\sigma,i_j^{\mathcal{D}}})) e^{-c_{\mathcal{E}} \Delta\tau_{\sigma,i_j^{\mathcal{D}}}}. \tag{8.66}$$

Furthermore, as $\Delta\tau_{\sigma,i_j^{\mathcal{D}}} \geq \tau_{\mathrm{p}}$ by definition, from (8.65) we have

$$q_\mu^{N_{\mathrm{p}}+2} e^{-c_{\mathcal{E}}\Delta\tau_{\sigma,i_j^{\mathcal{D}}}} \leq q_\mu^{N_{\mathrm{p}}+2} e^{-c_{\mathcal{E}}\tau_{\mathrm{p}}} < 1. \tag{8.67}$$

Thus, combining (8.63) and (8.66) and using (8.67) yields

$$V_{\mathcal{E}}(\tau_{\sigma,i_{j+1}^{\mathcal{D}}}, \mathcal{E}(\tau_{\sigma,i_{j+1}^{\mathcal{D}}})) \leq q_\mu^{N_{\mathrm{p}}+2} V_{\mathcal{E}}(\tau_{\sigma,i_j^{\mathcal{D}}}, \mathcal{E}(\tau_{\sigma,i_j^{\mathcal{D}}})) e^{-c_{\mathcal{E}}\Delta\tau_{\sigma,i_{j+1}^{\mathcal{D}}}} < V_{\mathcal{E}}(\tau_{\sigma,i_j^{\mathcal{D}}}, \mathcal{E}(\tau_{\sigma,i_j^{\mathcal{D}}})). \tag{8.68}$$

Since (8.68) holds for all $j \in \mathbb{N}$, it together with Proposition 8.4.1 implies that $V_{\mathcal{E}}(t, \mathcal{E}(t))$ is bounded by $q_\mu^{N_{\mathrm{p}}+2} V_{\mathcal{E}}(0, \mathcal{E}(0))$ for all $t \geq 0$. Thus, condition i) of Theorem 8.2.1 is satisfied.

Moreover, (8.68) together with the condition that $\mathcal{E}(t)$ stays outside the set B_{δ_n} for all $t \geq 0$ implies that the functions $V_{\mathcal{E}}(t, \mathcal{E}(t))$ is decreasing on the time interval $\cup_{i=0}^{\infty}[\tau_{\sigma,i_j^{\mathcal{D}}}, \tau_{\sigma,i_j^{\mathcal{D}}+1}]$. In addition, $V_{\mathcal{E}}$ is lower bounded by construction. Thus, both $\lim_{i\to\infty} V_{\mathcal{E}}(\tau_{\sigma,i_j^{\mathcal{D}}}, \mathcal{E}(\tau_{\sigma,i_j^{\mathcal{D}}}))$ and $\lim_{i\to\infty} V_{\mathcal{E}}(\tau_{\sigma,i_j^{\mathcal{D}}+1}, \mathcal{E}^-(\tau_{\sigma,i_j^{\mathcal{D}}}+1))$ exist and are identical. Therefore, the satisfaction of condition ii) of Theorem 8.2.1 follows accordingly.

Finally, satisfaction of iii) of Theorem 8.2.1 follows (8.62) directly. Thus, applying Theorem 8.2.1, we conclude that $\mathcal{E}(t)$ converges to the set B_{δ_n}, i.e., all the state variable of the closed-loop system are bounded and the tracking error $\xi_1(t) = y(t) - y_d(t)$ satisfying $|\xi_1(t)| \leq \|\mathcal{E}(t)\|$ converges to a small region as desired. ■

The inequality (8.65) in Theorem 8.4.1 can be satisfied for either large dwell-time τ_p or small switching jumps gain μ. In addition, when there is no jump in system state, i.e., $\mu = 0$ implying $q_\mu = 1$, (8.65) is automatically satisfied, and hence, the introduced control achieves the control objective under arbitrary switching.

The inequality (8.65) also reflects the conservativeness of adaptive control in switched systems. For improving accuracy, small $\sigma_{\mathbb{W}_{i,\mathbb{S}}}$'s, leading to small $c_{\mathcal{E}}$, are desired. However, (8.65) shows that the dwell-time property of the switching sequences and the switching jump gain μ must be taken into account in selecting $\sigma_{\mathbb{W}_{i,\mathbb{S}}}$.

Thus, given a switching sequence, arbitrary control accuracy cannot be achieved for switched systems. Instead, the larger $\tau_{\rm p}$ and the smaller μ and $N_{\rm p}$, the better accuracy can be achieved.

8.5 Design Example

In this section, we apply the design procedure presented to design an adaptive neural control for the switched system whose constituent systems are:

$$\begin{aligned}
&\Sigma_1 : \begin{cases} \dot{x}_1 &= x_2\sqrt{1+3x_1^2} + e^{x_1} \\ \dot{x}_2 &= (1+0.5\sin(x_1x_2))u + x_1x_2^2 \end{cases} \\
&\Sigma_2 : \begin{cases} \dot{x}_1 &= (1+5x_1\tanh x_1)x_2 + x_1^2 \\ \dot{x}_2 &= (3+\cos x_2)u + x_2\ln(1+x_1^2x_2^2). \end{cases}
\end{aligned} \tag{8.69}$$

The output is $y = x_1$, the desired signal is $y_d = 0.5(\sin t + 0.3\sin(3t))$, and the switching jump gain is $\mu = 0.1$. Clearly, Assumptions 8.2.2 and 8.2.3 are satisfied. Let us select $\gamma_{0,1} = \sqrt{1+3x_1^2}$ and $\gamma_{0,2} = 0.5$. Thus, Assumption 8.2.4 is satisfied with $\gamma_1 = 3.2$ and $\gamma_2 = 8$.

Following the presented design procedure, we obtain the control $u = \alpha_n$ given by (8.49), $i = 2$, and the parameter update laws given by (8.50), $i = 1, 2$.

For simulation, we choose $\lambda_1 = \lambda_2 = \lambda_1^s = \lambda_2^s = 10, c_1 = c_2 = 5, \varepsilon = 0.01, \sigma_{\mathbb{W}_{1,\mathbb{S}}} = 10^{-3}$. At each step, the neural networks $\hat{W}_{i,\upsilon}^T S(\hat{V}_{i,\upsilon}^T \bar{Z}_i)$, $i = 1, 2, \upsilon = a, b$ contain 10 hidden nodes, i.e., $l_{i,\upsilon} = 10$. The activation function is assigned as $s(z_a) = 1/(1+e^{-\gamma z_a})$ with $\gamma = 5$. $\Gamma_{W_{i,\upsilon}} = 2\times 10^{-4}\times \mathbb{I}_{10}, i = 1, 2, \upsilon = a, b, \Gamma_{V_{1,a}} = \Gamma_{V_{1,b}} = 2\times 10^{-4}\times \mathbb{I}_3, \Gamma_{V_{2,a}} = \Gamma_{V_{2,b}} = 2\times 10^{-4}\times \mathbb{I}_8$ where $\mathbb{I}_d$ is the identity matrix of dimension d. The parameter estimates $\hat{\mathbb{w}}_{i,\mathbb{S}}, i = 1, 2$ are all initialized at 0. The initial values of state variables $[x_1, x_2]^T$ is $[1, -1]^T$. The parameters of the switching sequence are $N_{\rm p} = 10$ and $\tau_{\rm p} = 1.2s$. The lengths of switching intervals between two consecutive

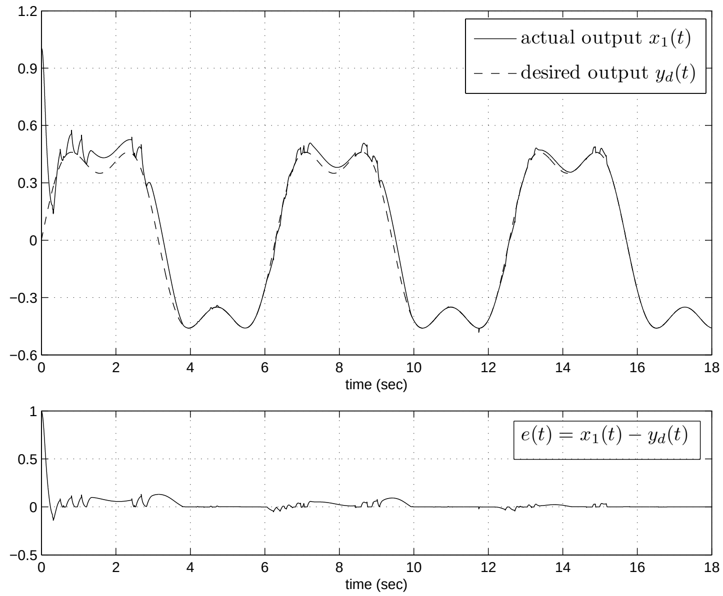


Figure 8.1: Tracking performance

dwell-time intervals are randomly generated between 0s and 0.2s.

The simulation results are shown in Figures 8.1 and 8.2. As observed in Figure 8.1, the tracking objective is well obtained. The state variable x_2, the control signal and the switching history are shown in Figure 8.2. It is observed that the signals are bounded while the switching sequence exhibits arbitrarily fast switching. The highly oscillated control signal is due to the switching in control structure (8.22). Thus, the simulation results well illustrate the theory presented.

8.6 Proofs

8.6.1 Proof of Theorem 8.2.1

To prove the theorem, let $c = c_1 + c_2, b = \bar{\alpha}(\alpha^{-1}(c))$ and, along the trajectory $x(t)$, let $\mathscr{T}_{x,t}, t \in \mathbb{R}^+$ be the sets $\{s \in [\tau_{\sigma,i_\sigma^-(t)}, \tau_{\sigma,i_\sigma^-(t)+1}] : V(s, x(s)) > b\}, t \in \mathbb{R}^+$. Let $(\cdot)_{\mathscr{T}_x}$ be the operator defined as $(t)_{\mathscr{T}_x} = \sup \mathscr{T}_{x,t}$ if $\mathscr{T}_{x,t} \neq \emptyset$ and $(t)_{\mathscr{T}_x} = \tau_{\sigma,i_\sigma^-(t)}$ if $\mathscr{T}_{x,t} = \emptyset$.

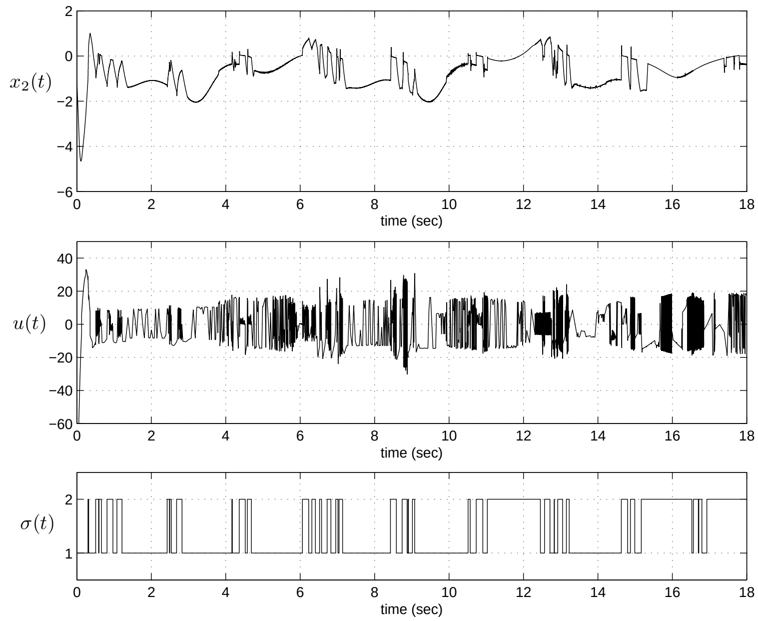


Figure 8.2: State $x_2(t)$, control signal, and switching history

We have the following inequality from condition ii)

$$D_\sigma V(t, x(t)) \le -\alpha(\|x(t)\|) + c_1, \forall t \in \mathbb{R}^+. \tag{8.70}$$

From (8.70), by an argument similar to the proof of Theorem 5.2.1, we know that if $V(t^*, x(t^*)) \le b$ for some $t^* \ge 0$, then $V(s, x(s)) \le b, \forall s \in [t^*, \tau_{\sigma, i_\sigma^-(t^*)+1})$. This also implies that if t is such that $V(t, x(t)) > b$, then $(t)_{\mathscr{T}_x}$ is well defined and $V(s, x(s)) \ge b, \forall s \in [\tau_{\sigma, i_\sigma^-(t)}, (t)_{\mathscr{T}_x})$. Furthermore, if such t^* exists, then $V(t, x(t)) > b$ for some $t > t^*$ is possible only if t is a switching time. As such, we are interested in the case that there are infinitely many switching times $\tau_{\sigma,i}$ at which $V(\tau_{\sigma,i}, x(\tau_{\sigma,i})) > b$ as, by (8.11), the inverse case trivially implies that $\|x(t)\| \le \underline{\alpha}^{-1}(b), \forall t \ge T$ for sufficiently large $T \in \mathbb{R}^+$.

To continue, let $\{\tau_{\sigma,i}^b\}_i$ be the sequence of all switching times of σ satisfying

$$(\tau_{\sigma,i}^b)_{\mathscr{T}_x} - \tau_{\sigma,i}^b > \varepsilon_\tau, \forall i \in \mathbb{N}. \tag{8.71}$$

We now show that for every time sequence $\{t^b_{\sigma,i}\}_i$ satisfying $t^b_{\sigma,i} \in [\tau^b_{\sigma,i}, (\tau^b_{\sigma,i})_{\mathscr{T}_x})$, $V(t^b_{\sigma,i}, x(t^b_{\sigma,i}))$ converges to the set $B = \{s \in \mathbb{R}^+ : s \leq b\}$, as $i \to \infty$. Indeed, assume that the converse holds, i.e., there exist $\varepsilon > 0$ and a subsequence $\{t^b_{\sigma,i_j}\}_j$ of $\{t^b_{\sigma,i}\}_i$ such that

$$V(t^b_{\sigma,i_j}, x(t^b_{\sigma,i_j})) \geq b + \varepsilon = \bar{\alpha}(\alpha^{-1}(c)) + \varepsilon, \forall j \geq 0. \tag{8.72}$$

As noted above, (8.72) also implies that $V(t, x(t)) \geq b, \forall t \in [\tau_{\sigma, i^-_\sigma(t^b_{\sigma,i_j})}, (t^b_{\sigma,i_j})_{\mathscr{T}_x})$. Thus, from (8.11), the nondecreasing property of $\bar{\alpha}$, and the above convention that $\tau_{\sigma, i^-_\sigma(t^b_{\sigma,i_j})} = \tau^b_{\sigma,i)j}$, we have

$$\alpha(\|x(t)\|) \geq c, \forall t \in [\tau^b_{\sigma,i_j}, (t^b_{\sigma,i_j})_{\mathscr{T}_x}), \forall j \geq 0. \tag{8.73}$$

Using (8.73) and the facts that $c > c_1, (\tau^b_{\sigma,i_j})_{\mathscr{T}_x} - \tau^b_{\sigma,i_j} > \varepsilon_\tau$ to integrate both sides of (8.70) over $[\tau^b_{\sigma,i_j}, (\tau^b_{\sigma,i_j})_{\mathscr{T}_x})$, we obtain

$$\begin{aligned} V(\tau^b_{\sigma,i_j}, x(\tau^b_{\sigma,i_j})) - V((\tau^b_{\sigma,i_j})_{\mathscr{T}_x}, x^-((\tau^b_{\sigma,i_j})_{\mathscr{T}_x})) &\geq \int_{\tau^b_{\sigma,i_j}}^{(\tau^b_{\sigma,i_j})_{\mathscr{T}_x}} (\alpha(\|x(s)\|) - c_1) ds \\ &\geq \int_{(t^b_{\sigma,i_j})^-_{\varepsilon_\tau}}^{(t^b_{\sigma,i_j})^+_{\varepsilon_\tau}} (\alpha(\|x(s)\| - c_1)) ds \stackrel{\text{def}}{=} \upsilon_{i_j}, \end{aligned} \tag{8.74}$$

where $[(t^b_{\sigma,i_j})^-_{\varepsilon_\tau}, (t^b_{\sigma,i_j})^+_{\varepsilon_\tau})$ is any subinterval containing t^b_{σ,i_j} of $[\tau^b_{\sigma,i_j}, (\tau^b_{\sigma,i_j})_{\mathscr{T}_x})$, whose length is ε_τ. Since $V(t, x(t))$ is decreasing on $[\tau^b_{\sigma,i_j}, (\tau^b_{\sigma,i_j})_{\mathscr{T}_x})$ and $V(t, x(t))$ is bounded by condition i), the sequence $\{\upsilon_{i_j}\}_j$ is bounded and hence has a subsequence that converges. Without loss of generality, we assume that $\{\upsilon_{i_j}\}_j$ converges. Noting that $(\tau^b_{\sigma,i_j})_{\mathscr{T}_x} \leq \tau_{\sigma, i^-_\sigma(t^b_{i_j})+1}$ and taking the limits of both sides of (8.74) as $t \to \infty$ using condition ii), we have

$$\tau_\varepsilon c_2 \geq \lim_{j\to\infty} (V(\tau^b_{\sigma,i_j}, x(\tau^b_{\sigma,i_j})) - V((\tau^b_{\sigma,i_j})_{\mathscr{T}_x}, x^-((\tau^b_{\sigma,i_j})_{\mathscr{T}_x}))) \geq \int_{(t^b_{\sigma,i_j})^-_{\tau_\varepsilon}}^{(t^b_{\sigma,i_j})^+_{\tau_\varepsilon}} (\alpha(\|x(s)\| - c_1)) ds. \tag{8.75}$$

Since $x(t)$ is uniformly continuous on time intervals $[\tau_{\sigma,i}, \tau_{\sigma,i+1}], i \in \mathbb{N}$, $\mathbb{Q}$ is finite, the norm function $\|\cdot\|$ and $\alpha(\cdot)$ are continuous, and $V(t, x(t))$ is bounded by condition i), (8.72) implies the existence of $\delta \in (0, \varepsilon_\tau]$ such that $\alpha(\|x(t^b_{\sigma,i_j} + s)\|) \geq c + \varepsilon'$ for all $s \in [0, \delta/2]$ if $t^b_{\sigma,i_j} \leq (t^b_{\sigma,i_j})^-_{\varepsilon_\tau} + \varepsilon_\tau/2$ and for all $s \in [-\delta/2, 0]$ if $t^b_{\sigma,i_j} > (t^b_{\sigma,i_j})^-_{\varepsilon_\tau} + \varepsilon_\tau/2$. Let

$$\tilde{t}^b_{\sigma,i_j} = \begin{cases} t^b_{\sigma,i_j} & \text{if} \quad t^b_{\sigma,i_j} \leq (t^b_{\sigma,i_j})^-_{\varepsilon_\tau} + \varepsilon_\tau/2 \\ t^b_{\sigma,i_j} - \delta/2 \;\; \text{if} & \text{if} \quad t^b_{\sigma,i_j} > (t^b_{\sigma,i_j})^-_{\varepsilon_\tau} + \varepsilon_\tau/2 \end{cases}. \tag{8.76}$$

Obviously, $[\tilde{t}^b_{\sigma,i_j}, \tilde{t}^b_{\sigma,i_j} + \delta/2] \subset [(t^b_{\sigma,i_j})^-_{\varepsilon_\tau}, (t^b_{\sigma,i_j})^+_{\varepsilon_\tau}], \forall j \in \mathbb{R}^+$. Thus, using (8.75), we have

$$0 \geq \int_{\tilde{t}^b_{\sigma,i_j}}^{\tilde{t}^b_{\sigma,i_j}+\delta/2} (\alpha(\|x(s)\|) - c))\, ds \geq \int_{\tilde{t}^b_{\sigma,i_j}}^{\tilde{t}^b_{\sigma,i_j}+\delta/2} \frac{\varepsilon'}{2} ds = \frac{\delta\varepsilon'}{4} > 0, \tag{8.77}$$

which is a contradiction. Therefore, $V(t^b_{\sigma,i}, x(t^b_{\sigma,i}))$ converges to B as $i \to \infty$. As $\{t^b_{\sigma,i}\}_i$ is arbitrary, this implies that

$$\lim_{(t)_{\mathscr{T}_x} - \tau_{\sigma,i^-_\sigma(t)} \geq \varepsilon_\tau, t \to \infty} \operatorname{dist}(V(t, x(t)), B) = 0. \tag{8.78}$$

To continue, let us define $\tau^{b+}_{\sigma,i} \stackrel{\text{def}}{=} \tau_{\sigma,i^-_\sigma(\tau^b_{\sigma,i})+1}$ and consider the sequence $\{\tau^{b+}_{\sigma,i}\}_i$. By definition of $\tau^b_{\sigma,i}$ we have either $V(\tau^{b+}_{\sigma,i}, x^-(\tau^{b+}_{\sigma,i})) \leq b$ or $\tau^{b+}_{\sigma,i} = (\tau^b_{\sigma,i})_{\mathscr{T}_x}$ implying that $\tau^{b+}_{\sigma,i} - \delta \in [\tau^b_{\sigma,i}, (\tau^b_{\sigma,i})_{\mathscr{T}_x}), \forall \delta > 0$. Hence, due to the continuity of V and the trajectories of subsystems, we have

$$\lim_{i\to\infty} V(\tau^{b+}_{\sigma,i}, x^-(\tau^{b+}_{\sigma,i})) = \lim_{\delta \to 0^+} \lim_{i\to\infty} V((\tau^{b+}_{\sigma,i} - \delta, x(\tau^{b+}_{\sigma,i} - \delta)) \leq b. \tag{8.79}$$

We proceed to consider the sequence of time intervals $[\tau^{b+}_{\sigma,i}, \tau^b_{\sigma,i+1}), i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $\{\tau^{b,1}_{\sigma,i}, \ldots, \tau^{b,n_i}_{\sigma,i}\}$ be the sequence of switching times in $[\tau^{b+}_{\sigma,i}, \tau^b_{\sigma,i+1})$. As $\sigma \in \mathbb{S}_{\mathscr{A}}[\tau_{\mathrm{p}}, N_{\mathrm{p}}]$, we have $n_i \leq N_{\mathrm{p}}, \forall i \in \mathbb{N}$. Let $\tau^{b,0}_{\sigma,i} = \tau^{b+}_{\sigma,i}$ and $\tau^{b,n_i+1}_{\sigma,i} = \tau^b_{\sigma,i+1}$ for convenience. For each $i \in \mathbb{N}$, let $\Delta V_{\tau_{\sigma,i}} = V(\tau_{\sigma,i}, x(\tau_{\sigma,i})) - V(\tau_{\sigma,i}, x^-(\tau_{\sigma,i}))$.

Again, from (8.70) and condition iii), we have

$$V(t,x(t)) \leq \max\{b, V(\tau_{\sigma,i}, x(\tau_{\sigma,i}))\} \leq \max\{b, \mu V(\tau_{\sigma,i}, x^-(\tau_{\sigma,i}))\}, \forall t \in [\tau_{\sigma,i}, \tau_{\sigma,i+1}). \tag{8.80}$$

Therefore, considering all switches in $[\tau^{b+}_{\sigma,i}, \tau^{b}_{\sigma,i+1})$ using (8.80), we have

$$V(t,x(t)) \leq \max\{b, \mu^{N_{\mathrm{p}}} V(\tau^{b+}_{\sigma,i}, x^-(\tau^{b+}_{\sigma,i}))\}, \forall t \in t \in [\tau^{b+}_{\sigma,i}, \tau^{b}_{\sigma,i+1}). \tag{8.81}$$

This coupled with (8.79) above gives rise to

$$\lim_{t\to\infty, t\in[\tau^{b+}_{\sigma,i}, tau^{b}_{\sigma,i+1})} V(t,x(t)) \leq \max\{b, \mu^{N_{\mathrm{p}}} b\}. \tag{8.82}$$

As $\mathbb{R}^+ = [\tau_{\sigma,0}, \tau^{b}_{\sigma,0}) \cup (\cup_i [\tau^{b}_{\sigma,i}, \tau^{b+}_{\sigma,i})) \cup (\cup_i [\tau^{b+}_{\sigma,i}, \tau^{b}_{\sigma,i+1}))$, the conclusion of the theorem follows (8.78) and (8.82). ■

8.6.2 Proof of Proposition 8.4.1

Since the parameter estimates are controller's state, there is no jump in $\mathcal{E}_2$ at switching times. Therefore, $V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}^-(\tau_{\sigma,i})) = V_{\mathcal{E}_1}(\tau_{\sigma,i}, \mathcal{E}_1^-(\tau_{\sigma,i})) + V_{\mathcal{E}_2}(\mathcal{E}_2(\tau_{\sigma,i})), \forall i \in \mathbb{N}$. Consider an arbitrary switching time $\tau_{\sigma,i}$. Since $g_{0,i}$ are bounded by Assumption 8.2.4, using the mean value theorem, we have

$$\begin{aligned} |V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}(\tau_{\sigma,i})) - V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}^-(\tau_{\sigma,i}))| &= |V_{\mathcal{E}_1}(\tau_{\sigma,i}, \mathcal{E}_1(\tau_{\sigma,i})) - V_{\mathcal{E}_1}(\tau_{\sigma,i}, \mathcal{E}_1^-(\tau_{\sigma,i}))| \\ = \sum_{i=1}^{n} \frac{(\xi_i(\tau_{\sigma,i}))^2 - (\xi_i^-(\tau_{\sigma,i}))^2}{g_{0,s}(\bar{x}_s(\tau_{\sigma,i}))} &+ \sum_{i=1}^{n} \Big(\frac{1}{g_{0,s}(\bar{x}_s(\tau_{\sigma,i}))} - \frac{1}{g_{0,s}(\bar{x}_s^-(\tau_{\sigma,i}))} \Big) \xi_i^-(\tau_{\sigma,i}))^2, \end{aligned} \tag{8.83}$$

where $\mathcal{E}_1' = [\xi_1', \ldots, \xi_n'], \xi_j' \in [\xi_j^-(\tau_{\sigma,i}), \xi_j(\tau_{\sigma,i})], j = 1, \ldots, n$. As the second term in the second equation of (8.83) is zero if $\mu = 0$, defining $q_G = g_{\min}^{-1} g_{\max}$ and $\delta_\mu = 0$ for $\mu = 0$

and $\delta_\mu = q_G/\mu$ for $\mu \neq 0$, we obtain the following inequality from (8.83)

$$\begin{aligned}
&|V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}(\tau_{\sigma,i})) - V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}^-(\tau_{\sigma,i}))| \\
&\quad \leq 2g_{\min}^{-1}\|\mathcal{E}_1'\|\|\mathcal{E}_1(\tau_{\sigma,i}) - \mathcal{E}_1^-(\tau_{\sigma,i})\| + 4\mu\delta_\mu V_{\mathcal{E}_1}(\tau_{\sigma,i}, \mathcal{E}_1^-(\tau_{\sigma,i})). \qquad (8.84)
\end{aligned}$$

By Assumption 8.2.3 and the designed boundedness of derivatives of virtual controls (8.61), we further have

$$\begin{aligned}
\|\mathcal{E}_1(\tau_{\sigma,i}) - \mathcal{E}_1^-(\tau_{\sigma,i})\| &= \|\bar{x}_n(\tau_{\sigma,i}) - \bar{\alpha}_n(\tau_{\sigma,i}) - (\bar{x}_n^-(\tau_{\sigma,i}) - \bar{\alpha}_n^-(\tau_i))\| \\
&\leq \|\bar{x}_n(\tau_{\sigma,i}) - x_n^-(\tau_{\sigma,i})\| + \|\partial\bar{\alpha}_n/\partial\bar{x}_n|_{\bar{x}_n=\bar{x}_n'}\|\|\bar{x}_n(\tau_{\sigma,i}) - \bar{x}_n^-(\tau_{\sigma,i})\| \\
&\leq \mu|e^-(\tau_{\sigma,i})| + q_\alpha\mu|e^-(\tau_{\sigma,i})| = \mu(1+q_\alpha)|e^-(\tau_{\sigma,i})|, \qquad (8.85)
\end{aligned}$$

where $\bar{\alpha}_n = [y_d, \alpha_1, \ldots, \alpha_{n-1}]^T$, $\bar{x}_n' = [x_1', \ldots, x_n']^T$, $x_i' \in [x_i^-(\tau_{\sigma,i}), x_i(\tau_{\sigma,i})]$, $i = 1, \ldots, n$. Since $\|\mathcal{E}_1'\| \leq \|\mathcal{E}_1^-(\tau_{\sigma,i})\| + \|\mathcal{E}_1(\tau_{\sigma,i}) - \mathcal{E}_1^-(\tau_{\sigma,i})\|$ and $|e^-(\tau_{\sigma,i})| = |\xi_1^-(\tau_{\sigma,i})| \leq \|\mathcal{E}_1^-(\tau_{\sigma,i})\|$, using (8.84) and (8.85), we have

$$\begin{aligned}
V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}(\tau_{\sigma,i})) &\leq V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}^-(\tau_{\sigma,i})) + |V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}(\tau_{\sigma,i})) - V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}^-(\tau_{\sigma,i}))| \\
&\leq V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}^-(\tau_{\sigma,i})) + 2g_{\min}^{-1}\big(\|\mathcal{E}_1^-(\tau_{\sigma,i})\| + \mu(1+q_\alpha)|e^-(\tau_{\sigma,i})|\big)\mu(1+q_\alpha)|e^-(\tau_{\sigma,i})| \\
&\quad + 4\mu\delta_\mu V_{\mathcal{E}_1}(\tau_{\sigma,i}, \mathcal{E}_1^-(\tau_{\sigma,i})) \\
&\leq (1 + 4\mu\delta_\mu)V_{\mathcal{E}_1}(\cdot, \mathcal{E}_1^-(\cdot)) + 2g_{\min}^{-1}(1 + \mu(1+q_\alpha))\mu(1+q_\alpha)\|\mathcal{E}_1^-(\cdot)\|^2 + V_{\mathcal{E}_2}(\mathcal{E}_2^-(\cdot)) \\
&\leq (1 + 4\mu\delta_\mu + 4q_G\mu(1 + \mu(1+q_\alpha))(1+q_\alpha))V_{\mathcal{E}_1}(\tau_{\sigma,i}, \mathcal{E}_1^-(\tau_{\sigma,i})) + V_{\mathcal{E}_2}(\mathcal{E}_2^-(\tau_{\sigma,i})) \\
&\leq q_\mu V_{\mathcal{E}_1}(\tau_{\sigma,i}, \mathcal{E}_1^-(\tau_{\sigma,i})) + V_{\mathcal{E}_2}(\mathcal{E}_2^-(\tau_{\sigma,i})) \leq q_\mu V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}^-(\tau_{\sigma,i})). \qquad (8.86)
\end{aligned}$$

This proves (8.62).

We proceed to prove (8.63) as follows. Applying comparison theorem for differential inequality (8.60) with initial condition $\mathcal{E}(\tau_{\sigma,i})$ satisfying (8.86), we obtain

$$\begin{aligned} V_{\mathcal{E}}(t, \mathcal{E}(t)) &\leq V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}(\tau_{\sigma,i}))e^{-c_{\mathcal{E}}(t-\tau_{\sigma,i})} \\ &\leq q_{\mu} V_{\mathcal{E}}(\tau_{\sigma,i}, \mathcal{E}^{-}(\tau_{\sigma,i}))e^{-c_{\mathcal{E}}(t-\tau_{\sigma,i})}, \forall t \in [\tau_{\sigma,i}, \tau_{\sigma,i+1}). \end{aligned} \tag{8.87}$$

Since the number n_i of switches between $\tau_{\sigma,i_j^{\mathcal{D}}}$ and $\tau_{\sigma,i_{j+1}^{\mathcal{D}}}$ is less than N_{p}. Applying consecutively (8.86) and (8.87) through intervals $[\tau_{\sigma,i_j^{\mathcal{D}}}, \tau_{\sigma,i_j^{\mathcal{D}}+1}], \ldots, [\tau_{\sigma,i_{j+1}^{\mathcal{D}}-1}, \tau_{\sigma,i_{j+1}^{\mathcal{D}}}]$, we arrive at (8.63). Hence, the conclusion of the proposition follows. ■

●

Chapter 9

Conclusions

This thesis advances our understanding on qualitative properties of switched systems and our ability to control uncertain switched systems. Transition model of dynamical systems amenable to developing qualitative theories of generic dynamical systems was presented. Elements for studying dynamical properties in the space of continuous state including limiting switching sequence, transition indicator, transition mappings, and quasi-invariance property were introduced. Invariance principles addressing destabilizing behavior were obtained for locating attractors of switched non-autonomous, switched autonomous, and switched time-delay systems. The gauge design method was introduced for control of uncertain switched systems. Adaptive neural control was obtained for output tracking of a class of uncertain switched systems undergoing uncontrolled switching jumps.

9.1 Summary

In Chapter 2, we have introduced the transition model of dynamical systems and its realizations to classical model of dynamical systems, hybrid systems, and switched systems. The model was obtained as a generalization of the classical description of dynamical systems using evolution mappings. By dropping the semi-group hypothesis on transition mapping and the topological structure of the state space, we exposed the rich time-transition property of trajectories of dynamical systems. By decomposing the abstract state space into manifest and latent spaces, we followed the idea that revealing time-transition properties of interacting trajectories of signals in a dynamical system is essential in order to obtain richer results. Accordingly, we have obtained the notions of switching sequence, transition indicator, and transition mappings to bring out transition models of switched dynamical systems amenable for developing qualitative theories.

In Chapter 3, we built up an invariance theory for delay-free switched systems on the time-transition properties of transition mappings obtained in Chapter 2. We have exposed the existence of limiting switching sequences in switched systems. It turned out that the qualitative properties on limiting behavior of trajectories of continuous state in switched systems are governed by the limiting switching sequences. The quasi-invariance property of limit sets of trajectories of switched systems was proven accordingly. Invariance principles with relaxed switching conditions were obtained for switched non-autonomous and switched autonomous systems. Through examples, we have demonstrated that conclusion on the converging-input converging-state property of switched systems can be made by examining the attractors of the systems. By virtue of the results achieved in this chapter, it turned out that other types of motions, to which pullback motion is a special case, can be considered for establishing invariance properties for qualitative theories of dynamical systems.

In Chapter 4, we developed invariance theory for switched time-delay systems. We established the compactness and the attractivity of limit sets of trajectories in the function state space that asserted that asymptotic properties of switched time-delay systems can be studied through these limit sets. In the framework of transition model, the quasi-invariance and invariance principle for switched time-delay systems obtained. The consideration on destabilizing behavior gave rise to the role of the relative sizes of delay-time and periods of persistence on converging behavior of the overall trajectories. It was shown that the Razumikhin condition at switching times can be used to remove the needs for functions estimating state growth in destabilizing periods. A time-delay approach to delay-free switched systems was presented.

In Chapter 5, we presented the principle of small-variation small-state for asymptotic gains of switched systems. The conditions were formulated in terms of comparison functions so that convergence of Lyapunov functions implies convergence of the state via norm estimates. It was shown that the positive definite and radially unbounded properties of Lyapunov functions plus with their bounded ultimate variations gave rise to further relaxation on switching conditions. Stability conditions was also presented for asymptotic gains of switched time-delay systems in the framework of Lyapunov-Razumikhin functions. It was shown that if the dwell-time is larger than the delay-time, then the Razumikhin condition also provides estimates for verifying decreasing behavior.

In Chapter 6, the gauge design method was introduced for switching-uniform adaptive control of uncertain switched systems with unknown time-varying parameters and unmeasured dynamics. Separating the unknown time-varying parameters from state dependent functions, output regulation was achieved in the sense of disturbance attenuation. In this way, parameter estimates were not included in the state of the resulting closed-loop systems and hence the problem of slow parameter convergence in traditional adaptive control as well as the problem of increasing difficulty in verifying switching conditions were not encountered. The method exposed the principle of driving system behavior through converging modes of its component systems. It was also shown that relation between growth and decreasing rates of the appended dynamics and the persistent dwell-time and period of persistence of switching sequence is essential in verifying switching conditions of switched systems undergoing persistent dwell-time switching sequences. The novelty also lies in the recursive design paradigm, where the destabilizing terms were step-by-step eliminated instead of being canceled all at once in each single step.

In Chapter 7, adaptive high-gain observer was designed for switching-uniform output feedback stabilization. It was pointed out that destabilizing terms in estimation error dynamics caused by discrepancy between control gains might not be avoided for non-conservative results. Condition on variation in control gains was introduced for the effectiveness of the proposed observer. Application of the CPLF design method gave rise to an adaptive output feedback control effective in the presence of unknown time-varying parameters and full-state dependent control gains.

The results in Chapters 2–7 were obtained for switched systems undergoing persistent dwell-time switching sequences.

Finally, in Chapter 8, we presented a combined adaptive neural control for output tracking of uncertain switched systems undergoing switching jumps and average dwell-time switching sequences. The underlying principle also lied in the use of dwell-time intervals to compensate the growth raised in destabilizing periods. In achieving this performance, we used parameter adaptive mechanism for dealing with unknown constant bounds of approximation errors without increasing the orders of functions of signals with discontinuity. A condition in terms of design parameters and timing properties of switching sequences was introduced for verifying stability conditions.

9.2 Open Problems

Among the stability conditions presented, there is a question of how to verify the boundedness condition on ultimate variations of auxiliary functions (cf. (3.51), (3.79), (3.86), (4.35), (5.11), and ii) of Theorem 5.3.2). This condition appears to be necessary for converging behaviors. It automatically holds in the classical Lyapunov theorem for ordinary dynamical systems and switched systems satisfying switching decreasing condition. In Chapter 6, this condition was satisfied by utilizing the stabilizing behavior on dwell-time intervals to render the sequences of values of composite Lyapunov function at starting times of dwell-time switching events non-increasing. However, at its high level of relaxation, this condition expresses that once the stationary evolution has been established, the auxiliary functions are still allowed to increase. Therefore, it is obvious that there are possibly further mechanisms for satisfaction of this condition.

With the introduction of the stability conditions on Lyapunov-Razumikhin functions and the introduction of the gauge design method, it opened the possibility of switching-uniform control for switched time-delay systems. It is worth mentioning that the Razumikhin condition provides estimates over a continuum of the past states. Hence, control design for switched systems with distributed delay terms is possible.

Finally, the invariance principles developed in Chapters 3 and 4 can finds their applications in complex systems [141, 35, 27]. In such systems, due to limited interaction range and individuals' independent decision, the connection topology changes frequently and does not follows a specific rule and hence the models of these systems are of switched systems in nature [117, 143]. Understanding the plentiful collective behavior of these systems such as flocking, consensus, and pattern formation [117,47,116,37,125,40,76] call for structures of attractors which are of invariance-principles relevance. In systems such as engineered robot swarms, due to limited capability of sensing units, communication delay [117] arises and hence the resulting systems become relevant to invariance principles of switched time-delay systems presented in Chapter 4.

Author's Publications

Journal papers and submissions

[1] S. S. Ge and T.-T. Han, "Semiglobal ISpS disturbance attenuation with output tracking via direct adaptive design," *IEEE Trans. Neural Netw.*, vol. 18, no. 4, Special Issue on Neural Networks for Feedback Control Systems, pp. 1129-1148, July 2007.

[2] T.-T. Han, S. S. Ge, and T. H. Lee, "Adaptive neural control for a class of switched nonlinear systems," *Systems & Control Letters*, vol. 58, no. 2, pp. 109-118, 2009.

[3] T.-T. Han, S. S. Ge, and T. H. Lee, "Persistent dwell-time switched nonlinear systems: variation paradigm and gauge design," *IEEE Trans. Automat. Contr.*, to appear, tentatively January 2010.

[4] T.-T. Han, S. S. Ge, and T. H. Lee, "Adaptive NN control for a class of nonlinear distributed time-delay systems by method of Lyapunov-Razumikhin function," *IEEE Trans. Neural Netw.*, rejected.

[5] T.-T. Han, S. S. Ge, and T. H. Lee, "Asymptotic behavior of switched nonautonomous systems," IEEE *Trans. Automat. Contr.*, rejected.

Referred Conference Papers

[1] T.-T. Han, S. S. Ge, and T. H. Lee, "Partial state feedback tracking with ISpS disturbance attenuation via direct adaptive design," in *Proc. IEEE CDC '04*, Atlantis, Paradise Island, Bahamas, December 14–17 2004, pp. 656-661.

[2] T.-T. Han, S. S. Ge, and T. H. Lee, "Uniform adaptive neural control for switched underactuated systems," in *Proc. IEEE MSC-ISIC '08*, Hilton Palacio del Rio in San Antonio, Texas, September 3–5, 2008.

[2] T.-T. Han, S. S. Ge, and T. H. Lee, "Variation paradigm for asymptotic gain of switched time-delay systems," *IEEE CDC & CCC '09*, to appear.

Bibliography

[1] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer-Verlag, Berlin, 1999.

[2] R. Alur, T. A. Henzinger, and E. D. Sontag, editors. *Hybrid Systems III*, volume 1066 of *Lecture Notes in Computer Science*. Springer-Verlag, 1996.

[3] P. Antsaklis, W. Kohn, A. Nerode, and S. Sastry, editors. *Hybrid Systems II*, volume 999 of *Lecture Notes in Computer Science*. Springer-Verlag, New York, 1995.

[4] P. J. Antsaklis and A. Nerode, editors. Special issue on hybrid control systems. *IEEE Trans. Automat. Contr.*, 43(4), April 1998.

[5] P. J. Antsaklis and J. Baillieul, editors. Special issue on networked control systems. *IEEE Trans. Automat. Contr.*, 49(9), September 2004.

[6] P. J. Antsaklis and A. Nerode. Hybrid systems: an introductory discussion to the special issue. *IEEE Trans. Automat. Contr.*, 43(4):457–460, April 1998.

[7] M. Arcak. *Unmodeled Dynamics in Robust Nonlinear Control*. PhD thesis, University of California, Santa Barbara, 2000.

[8] L. Arnold. *Random Dynamical Systems*. Springer-Verlag, Berlin, 1998.

[9] Z. Artstein. Limiting equations and stability of nonautonomous ordinary differential equations. Appendix to J. P. LaSalle, The Stability of Dynamical Systems*, CBMS Regional Conference Series in Applied Mathematics,* SIAM, Philadenphia, 1977.

[10] Z. Artstein. The limiting equations of nonautonomous ordinary differential equations. *J. Diff. Equat.*, 25(2):184–202, 1977.

[11] Z. Artstein. Stability, observability and invariance. *J. Diff. Equat.*, 44(2):224–248, 1982.

[12] A. Astolfi, D. Karagiannis, and R. Ortega. *Nonlinear and Adaptive Control with Applications*. Springer-Verlag, Berlin, 2008.

[13] A. Bacciotti and L. Mazzi. An invariance principle for nonlinear switched systems. *Syst. Control Lett.*, 54(11):1109–1119, 2005.

[14] D. Bainov and P. Simeonov. *Integral Inequalities and Applications*. Kluwer Academic Publishers, Dordrecht, 1992.

[15] C. Belta, A. Bicchi, M. Egerstedt, E. Frazzoli, E. Klavins, and G. J. Pappas. Symbolic planning and control of robot motion: state of the art and grand challenges. *IEEE Robot. Autom. Mag.*, 14(1):61–70, March 2007.

[16] G. Besancon, editor. *Nonlinear Observers and Applications*, volume 363 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, 2007.

[17] G. D. Birkhoff. *Dynamical Systems*. AMS, Providence, Rhode Island, 1966.

[18] F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.

[19] A. M. Bloch, J. Baillieul, P. Crouch, and J. Marsden. *Nonholonomic Mechanics and Control*. Springer Verlag, New York, 2003.

[20] G. Bornard and H. Hammouri. A high gain observer for a class of uniformly observable systems. In *Proc. 30th IEEE Conf. Decision Contr.*, pages 1494–1496, Brighton, England, December 1991.

[21] M. S. Branicky. *Studies in hybrid systems: Modeling, analysis, and control*. PhD thesis, Dept. Elec. Eng. and Computer Sci., Massachusetts Inst. Technol., Cambridge, 1995.

[22] M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Automat. Contr.*, 43(4):475–482, April 1998.

[23] M. S. Branicky, V. S. Borkar, and S. K. Mitter. A unified framework for hybrid control: model and optimal control theory. *IEEE Trans. Automat. Contr.*, 43(1):31–45, April 1998.

[24] R. W. Brockett. Hybrid models for motion control systems. In H. L. Trentelman and J. C. Willems, editors, *Essays on Control: Perspectives in the Theorey and its Applications*. Birkhauser, 1993.

[25] F. Bullo and A. D. Lewis. *Geometric Control of Mechanical Systems*. Springer Verlag, New York, 2004.

[26] C. I. Byrnes and C. F. Martin. An integral-invariance principle for nonlinear systems. *IEEE Trans. Automat. Contr.*, 40(6):983–994, June 1995.

[27] C. Castellano, S. Furtunato, and V. Loreto. Statistical physics of social dynamics. *Rev. Mod. Phys.*, 81(2):591–646, 2009.

[28] S. Celikovsky and H. Nijmeijer. Equivalence of nonlinear systems to triangular form: the singular case. *Syst. Con. Lett.*, 27(3):135–144, 2006.

[29] D. N. Cheban. *Global Attractors of Non-Autonomous Dissipative Dynamical Systems*. World Scientific, New Jersey, 2004.

[30] V. Chellaboina, S. P. Bhat, and W. M. Haddad. An invariance principle for nonlinear hybrid and impulsive dynamical systems. *Nonlinear Analysis*, 53(3):527–550, May 2003.

[31] W.-K. Chen. *Nonlinear and Distributed Circuits*. CRC/Taylor & Francis, Boca Raton, 2006.

[32] Z. Chen and J. Huang. Dissipativity, stablization, and regulation of cascade-connected systems. *IEEE Trans. Automat. Contr.*, 49(5):635–650, May 2004.

[33] D. Cheng, G. Feng, and Z. Xi. Stabilization of a class of switched nonlinear systems. *J. Contr. Theory Appl.*, 4(1), 2006.

[34] D. Cheng and W. Lin. On p–normal forms of nonlinear systems. *IEEE Trans. Automat. Contr.*, 48(7):1242–1248, July 2003.

[35] M. Clerc and J. Kennedy. The particle swarm–explosion, stability, and convergence in multidimentional complex space. *IEEE Trans. Evolutionary Computat.*, 6(1):58–73, February 2002.

[36] L. Crocco. *Theory of Combustion Instability in Liquid Propellant Rocket Motors*. Butterworths Scientific Publications, 1956.

[37] F. Cucker and S. Smale. Emergent behavior in flocks. *IEEE Trans. Automat. Contr.*, 52(5):852–862, May 2007.

[38] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, New York, 1975.

[39] W. E. Dixon, A. Behal, D. M. Dawson, and S. P. Nagarkatti. *Nonlinear Control of Engineering Systems*. Birkhauser, Boston, 2003.

[40] M. R. D'Orsogna, Y. L. Chuang, A. L. Bertozzi, and L. S. Chayes. Self-propelled particles with soft-core interactions: patterns, stability and collapse. *Phys. Rev. Lett.*, 96(104302), 2006.

[41] J. Dugundji. *Topology*. Allyn and Bacon, Inc., Boston, 1966.

[42] F. Esfandiari and H. K. Khalil. Output feedback stabilization of fully linearizable systems. *Int. J. Contr.*, 56(5):1007–1037, 1992.

[43] I. Fantoni and R. Lozano. *Non-linear Control for Underactuated Mechanical Systems*. Springer-Verlag, London, 2001.

[44] J. A. Farrell and M. M. Polycarpou. *Adaptive Approximation Based Control*. John Wiley & Sons, New Jersey, 2006.

[45] E. Frazzoli and A Arsie. Groupoids in control systems and the reachability problem for a class of quantized control systems with nonabelian symmetries. In A. Bemporad, A. Bicchi, and G. Buttazzo, editors, *Hybrid Systems: Computation and Control*, volume 4416 of *Lecture Notes in Computer Science*, pages 18–31. Springer-Verlag, 2007.

[46] E. Frazzoli, M. A. Dahleh, and E. Feron. Maneuver-based motion planning for nonlinear systems with symmetries. *IEEE Trans. Robot.*, 21(6):1077–1091, December 2006.

[47] V. Gazi and K. M. Passino. Stability analysis of social forgaging swarms. *IEEE Trans. Syst., Man, Cybern. B*, 34(1):539–557, February 2004.

[48] S. S. Ge and T.-T. Han. Semi-global ISpS disturbance attenuation with output tracking via direct adaptive design. *IEEE Trans. Neutral Netw.,* Special issue on neural networks for feedback control systems, 18(4):1129–1148, July 2007.

[49] S. S. Ge, C.C. Hang, T.H. Lee, and T. Zhang. *Stable Adaptive Neural Network Control*. Kluwer Academic, Norwell, USA, 2001.

[50] J. C. Geromel, P. Colaneri, and P. Bolzern. Dynamic output feedback control of switched linear systems. *IEEE Trans. Automat. Contr.*, 53(4):720–733, April 2008.

[51] J. Q. Gong and B. Yao. Neural network adaptive robust control of nonlinear systems in semi-strict feedback form. *Automatica*, 37(8):1149–1160, 2001.

[52] Robert L. Grossman, Anil Nerode, Anders P. Ravn, and Hans Rischel, editors. *Hybrid Systems I*, volume 736 of *Lecture Notes in Computer Science*. Springer-Verlag, 1993.

[53] L. Grune, P. E. Kloeden, S. Siegmund, and F. R. Wirth. Lyapunov's second method for nonautonomous differential equations. *Discrete & Continuous Dynamical Systems – Series A*, 18(2–3):375–403, July 2007.

[54] K. Gu, V. L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhauser, Boston, 2003.

[55] J. R. Haddock. Liapunov-Razumikhin functions and an invariance principle for functional differential equations. *J. Diff. Equat.*, 48(1):95–122, 1983.

[56] W. Hahn. *Stability of Motion*. Springer-Verlag, Berlin, 1967.

[57] J. K. Hale. Sufficient conditions for stability and instability of autonomous functional differential equations. *J. Diff. Equat.*, 1(4):452–482, 1965.

[58] J. K. Hale. Dynamical systems and stability. *J. Math. Anal. App.*, 26(1):29–59, 1969.

[59] J. K. Hale and S. M. Verduyn Lunel. *Introduction to Functional Differential Equations*. Springer-Verlag, New York, 1993.

[60] R. Hermann and A. J. Krener. Nonlinear controllability and observability. *IEEE Trans. Automat. Contr.*, 22(5):728–740, October 1977.

[61] J. P. Hespanha. *Logic-based switching algorithms in control*. PhD thesis, Dept. Electrical Engineering, Yale University, 1998.

[62] J. P. Hespanha. Uniform stability of switched linear systems: extensions of LaSalle's invariance principle. *IEEE Trans. Automat. Contr.*, 49(4):470–482, April 2004.

[63] J. P. Hespanha, D. Liberzon, D. Angeli, and E. D. Sontag. Nonlinear norm-observability notions and stability of switched systems. *IEEE Trans. Automat. Contr.*, 50(2):154–168, February 2005.

[64] J. P. Hespanha, D. Liberzon, and A. S. Morse. Supervision of integral-input-to-state stabilizing controllers. *Automatica*, 38(8):1327–1335, 2002.

[65] J. P. Hespanha, D. Liberzon, and A. S. Morse. Overcoming the limitations of adaptive control by means of logic–based switching. *Syst. Con. Lett.*, 49(1):49–65, 2003.

[66] J. P. Hespanha and A. S. Morse. Switching between stabilizing controllers. *Automatica*, 38(11):1905–1917, 2002.

[67] W. E. Hornor. Invariance principles and asymptotic constancy of solutions of precompact functional differential equations. *Tohoku Math. J.*, 42(2):217–229, 1990.

[68] H. Y. Hu and Z. H. Wang. *Dynamics of Controlled Mechanical Systems with Delayed Feedback*. Springer-Verlag, Berlin, 2002.

[69] A. Isidori. *Nonlinear Control Systems: An Introduction*. Springer-Verlag, New York, 1985.

[70] A. Isidori. *Nonlinear Control Systems*. Springer-Verlag, New York, 3rd. edition, 1995.

[71] A. Isidori. *Nonlinear Control Systems II*. Springer-Verlag, New York, 1999.

[72] I. M. James. *Topologies and Uniformities*. Springer-Verlag, London, 1999.

[73] Z.-P. Jiang, I. Mareels, D. J. Hill, and J. Huang. A unifying framework for global regulation via nonlinear output feedback: from ISS to iISS. *IEEE Trans. Automat. Contr.*, 49(4):549–562, April 2004.

[74] Z.-P. Jiang and L. Praly. Design of robust adaptive controllers for nonlinear systems with dynamic uncertainties. *Automatica*, 34(7):825–840, 1998.

[75] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Control Signals Systems*, 7(2):95–120, 1994.

[76] D. E. O. Juanico. Self-organized pattern formation in a diverse attractive-repulsive swarm. *EPL*, 86(48004), 2009.

[77] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ, 3rd edition, 2002.

[78] H. K. Khalil and A. Saberi. Adaptive stabilization of a class of nonlinear systems using high-gain feedback. *IEEE Trans. Automat. Contr.*, 32(11):1031–1035, November 1987.

[79] P. E. Kloeden. Pullback attractors of nonautonomous semidynamical systems. *Stochastics and Dynamics*, 3(1):101–112, March 2003.

[80] P. E. Kloeden. Nonautonomous attractors of switching systems. *Dynamical Systems: An International Journal*, 21(2):209–230, 2006.

[81] V. Kolmanovskii and A. Myshkis. *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer Academic, Dordrecht, 1999.

[82] I. Kolmanovsky and N. H. McClamroch. Hybrid feedback laws for a class of cascade nonlinear control systems. *IEEE Trans. Automat. Contr.*, 41(9):1271–1282, 1996.

[83] N. Krasovskii. *Stability of Motion*. Stanford University Press, Moscow, 1963. Translation.

[84] P. Krishnamurthy and F. Khorrami. Dynamic high-gain scaling: state and output feedback with application to systems with ISS appended dynamics driven by all states. *IEEE Trans. Automat. Contr.*, 49(12):2219–2239, December 2004.

[85] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. John Wiley and Sons, New York, 1995.

[86] V. Lakshmikantham and S. Leela. *Differential and Integral Inequalities*, volume II. Academic Press, New York and London, 1969.

[87] V. Lakshmikantham, S. Leela, and A. A. Martynyuk. *Practical Stability of Nonlinear Systems*. World Scientific, Singapore, 1990.

[88] V. Lakshmikantham and X. Z. Liu. *Stability Analysis in Terms of Two Measures*. World Scientific, Singapore, 1993.

[89] V. Lakshmikantham, L. Wen, and B. Zhang. *Theory of Differential Equations with Unbounded Delay*. Kluwer Academic, Dordrecht, 1994.

[90] J. P. LaSalle. Stability theory for ordinary differential equations. *J. Diff. Equat.*, 4(1):57–65, 1968.

[91] J. P. LaSalle. *The Stability of Dynamical Systems.* SIAM Publications, Philadelphia, 1976.

[92] H. Lei and W. Lin. Universal adaptive control of nonlinear systems with unknown growth rate by output feedback. *Automatica*, 42(10):1783–1789, 2006.

[93] A. Leonessa, W. M. Haddad, and V.-S. Chellaboina. Nonlinear system stabilization via hierarchical switching control. *IEEE Trans. Automat. Contr.*, 46(1):17–28, January 2001.

[94] F. L. Lewis, A. Yesildirek, and K. Liu. Multilayer neural-net robot controller with guaranteed tracking performance. *IEEE Trans. Neural Netw.*, 7(2):388–399, March 1996.

[95] D. Liberzon. *Switching in Systems and Control.* Birkhäuser, Boston, MA, 2003.

[96] Y. H. Lim and F. L. Lewis. *High-Level Feedback Control with Neural Networks.* World Scientific, Singapore, December, 1999.

[97] W. Lin and R. Pongvuthithum. Adaptive output tracking of inherently nonlinear systems with nonlinear parameterization. *IEEE Trans. Automat. Contr.*, 48(10):1737–1749, October 2003.

[98] W. Lin, C. Qian, and X. Huang. Disturbance attenuation for a class of nonlinear systems via output feedback. *Int. J. Robust Nonlin.*, 13(15):1359–1369, 2003.

[99] L. Liu and J. Huang. Global robust output regulation of output feedback systems with unknown high-frequency gain sign. *IEEE Trans. Automat. Contr.*, 51(4):625–631, April 2006.

[100] X. Liu and J. Shen. Stability theory of hybrid dynamical systems with time delay. *IEEE Trans. Automat. Contr.*, 51(4):620–625, April 2006.

[101] H. Logemann and E. P. Ryan. Non-autonomous systems: asymptotic behaviour and weak invariance principles. *J. Diff. Equat.*, 189(2):440–460, 2003.

[102] J. Lygeros, K. H. Johansson, S. N. Simic, J. Zhang, and S. Sastry. Dynamical properties of hybrid automata. *IEEE Trans. Automat. Contr.*, 48(1):2–17, January 2003.

[103] R. Marino and P. Tomei. *Nonlinear Control Design: Geometric, Adaptive and Robust.* Prentice Hall, Englewood Cliffs, NJ, 1995.

[104] R. Marino and P. Tomei. Nonlinear output feedback tracking with almost disturbance decoupling. *IEEE Trans. Automat. Contr.*, 44(1):18–28, January 1999.

[105] S. Martinez, F. Bullo, J. Cortes, and E. Frazzoli. On synchronous roboti networks–Part I: models, tasks, and complexity. *IEEE Trans. Automat. Contr.*, 52(12):2199–2213, December 2007.

[106] N. H. El-Farra, P. Mhaskar, and P. D. Christofides. Output feedback control of switched nonlinear systems using multiple lyapunov functions. *Syst. Contr. Lett.*, 54(12):1163–1182, 2005.

[107] J. L. Mancilla-Aguilar and R. A. Garcia. An extension of LaSalle's invariance principle for switched systems. *Syst. Control Lett.*, 55(5):376–384, 2006.

[108] A. N. Michel, K. Wang, and B. Hu. *Qualitative Theory of Dynamical Systems: the role of stability preserving mappings*. Marcel Dekker, New York, 2001.

[109] J. Monaco, D. Ward, and R. Bird. Implementation and flight test assessment of and adaptive, reconfigurable flight control system. In *Proc. AIAA Guidance Navigation and Control Conf.*, New Orleans, LA, August 1997.

[110] A. S. Morse. Control using logic-based switching. In A. Isidori, editor, *Trends in Control*. Springer-Verlag, 1995.

[111] K. S. Narendra and A. M. Annaswamy. *Stable Adaptive Systems*. Prentice Hall, Englewood Cliffs, NJ, 1989.

[112] K. S. Narendra and K. Parthasarathy. Identification and control of dynamic systems using neural networks. *IEEE Trans. Neural Netw.*, 1(1):4–27, March 1990.

[113] S.-I. Niculescu. *Delay Effects on Stability*. Springer-Verlag, Berlin, 2001.

[114] H. Nijmeijer and T. I. Fossen, editors. *New directions nonlinear observer design*, volume 244 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, 1999.

[115] H. Nijmeijer and A. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer-Verlag, New York, 1990.

[116] R. Olfati-Saber. Flocking for multi-agent dynamic systems: algorithm and theory. *IEEE Trans. Automat. Contr.*, 51(3):401–420, March 2006.

[117] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Automat. Contr.*, 49(9):1520–1533, September 2004.

[118] R. Ordonez and K. M. Passino. Adaptive control for a class of nonlinear systems with a time-varying structure. *IEEE Trans. Automat. Contr.*, 46(1):152–155, January 2001.

[119] J. Park and I. W. Sandberg. Universal approximation using radial-basis-function networks. *Neural Computation*, 3(2):246–257, 1990.

[120] P. Peleties and R. DeCarlo. Asymptotic stablity of m-switched systems using Lyapunov-like functions. In *Proc. American Control Conf.*, pages 1679–1684, Boston, MA, June 1991.

[121] P. Pepe and Z.-P. Jiang. A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Syst. Contr. Lett.*, 55(12):1006–1014, 2006.

[122] C. De Persis, R. De Santis, and A. S. Morse. Switched nonlinear systems with state-dependent dwell-time. *Syst. Contr. Lett.*, 50(4):291–302, 2003.

[123] M. M. Polycarpou and P. A. Ioannou. A robust adaptive nonlinear control design. *Automatica*, 32(3):423–427, 1996.

[124] L. Praly. Asymptotic stabilization via output feedback for lower triangular systems with output dependent incremental rate. *IEEE Trans. Automat. Contr.*, 48(6):1103–1108, June 2003.

[125] E. M. Raucha, M. M. Millonasb, and D. R. Chialvod. Pattern formation and functionality in swarm models. *Phys. Lett. A*, 207:185–193, 1995.

[126] R. G. Sanfelice, R. Goebel, and A. R. Teel. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Trans. Automat. Contr.*, 53(12):2282–2297, December 2007.

[127] S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control*. Springer-Verlag, New York, 1999.

[128] S. Sastry and M. Bodson. *Adaptive Control: Stability, Convergence and Robustness*. Prentice Hall, Englewood Cliffs, NJ, 1989.

[129] G. R. Sell. Nonautonomous differential equations and topological dynamics I and II. *Trans. Amer. Math. Soc.*, 127:241–283, 1967.

[130] G. R. Sell. *Topological Dynamics and Ordinary Differential Equations*. Van Nostrand, New York, 1971.

[131] G. R. Sell and Y. You. *Dynamics of Evolutionary Equations*. Springer-Verlag, New York, 2002.

[132] J. J. Slotine and W. Li. *Applied Nonlinear Control*. Prentice Hall, Englewood Cliffs, NJ, 1991.

[133] E. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Contr.*, 34(4):435–443, April 1989.

[134] E. Sontag. Comments on integral variants of ISS. *Syst. Contr. Lett.*, 34(1–2):93–100, 1998.

[135] E. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Trans. Automat. Contr.*, 41(9):1283–1294, September 1995.

[136] E. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Syst. Control Lett.*, 24(5):351–359, 1995.

[137] E. D. Sontag. *Mathematical Control Theory.* Springer-Verlag, New York, 2nd edition, 1998.

[138] E. D. Sontag. The ISS philosophy as a unifying framework for stability-like behavior. In A. Isidori, F. Lamnabhi-Lagarrigue, and W. Respondek, editors, *Nonlinear Control in the Year 2000*, volume 2, pages 443–468. Springer-Verlag, Berlin, 2000.

[139] E. D. Sontag. Input to state stability: Basic concepts and results. In P. Nistri and G. Stefani, editors, *Nonlinear and Optimal Control Theory*, pages 163–220. Springer-Verlag, Berlin, 2006.

[140] J. A. Stiver, X. D. Koutsoukos, and P. J. Antsaklis. An invariant-based approach to the design of hybrid control systems. *Int. J. Robust Nonlinear Contr.*, 11(5):453–478, 2001.

[141] S. H. Strogatz. Exploring complex networks. *Nature*, 410:268–276, 2001.

[142] Z. Sun and S. S. Ge. *Switched Linear Systems: Control and Design.* Springer-Verlag, London, 2005.

[143] H. G. Tanner, Ali Jadbabaie, and G. J. Pappas. Flocking in fixed and switching networks. *IEEE Trans. Automat. Contr.*, 52(5):863–868, May 2007.

[144] D. G. Taylor, P. V. Kokotovic, R. Marino, and I. Kanellakopoulos. Adaptive regulation of nonlinear systems with unmodelled dynamics. *IEEE Trans. Automat. Contr.*, 34(4):405–412, April 1989.

[145] R. Temam. *Infinite-dimensional Dynamical Systems in Mechanics and Physics.* Springer-Verlag, New York, 1997.

[146] M. Thoma and M. Morari, editors. *Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems*, volume 322 of *Lecture Notes in Control and Information Sciences.* Springer-Verlag, 2005.

[147] M. Vidyasagar. On the stabilization of nonlinear systems using state detection. *IEEE Trans. Automat. Contr.*, 25(3):504–509, June 1980.

[148] M. Vidyasagar. *Nonlinear Systems Analysis.* Prentice Hall, Englewood Cliffs, NJ, 2nd. edition, 1993.

[149] L. Vu, D. Chatterjee, and D. Liberzon. Input-to-state stability of switched systems and switching adaptive control. *Automatica*, 43(4):639–646, 2007.

[150] E. R. Westervelt, J. W. Grizzle, and D. E. Koditschek. Hybrid zero dynamics of planar biped walkers. *IEEE Trans. Automat. Contr.*, 48(1):42–56, January 2003.

[151] J.C. Willems. The behavioral approach to open and interconnected systems. *Control Systems Magazine*, 27(6):46–99, December 2007.

[152] W. Xie, C. Wen, and Z. Li. Input-to-state stabilization of switched nonlinear systems. *IEEE Trans. Automat. Contr.*, 46(7):1111–1116, July 2001.

[153] P. Yan and H. Ozbay. Stability analysis of switched time delay systems. *SIAM J. Control and Optimization*, 47(2):936–949, 2008.

[154] H. Ye, A. N. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Trans. Automat. Contr.*, 43(4):461–474, April 1998.

[155] X. Ye and J. Jiang. Adaptive nonlinear design without a priori knowledge of control directions. *IEEE Trans. Automat. Contr.*, 43(11):1617–1621, November 1998.

[156] J. Yeh. *Real Analysis: Theory of Measure and Integration.* World Scientific, New Jersey, 2006.

[157] A. Yesildirek and F. L. Lewis. Feedback linearization using neural networks. *Automatica*, 31(11):1659–1664, 1995.

[158] R. Yuan, Z. Jing, and L. Chen. Uniform asymptotic stability of hybrid dynamical systems with delay. *IEEE Trans. Automat. Contr.*, 48(2):344–348, February 2003.

[159] G. Zames. On the input-output stability of time-varying nonlinear feedback systems Part I: Conditions derived using concepts of loop gain, conicity, and positivity. *IEEE Trans. Automat. Contr.*, 11(2):228–238, April 1966.

[160] M. M. Zavlanos and G. J. Pappas. Dynamic assigment in distributed motion planning with local coordination. *IEEE Trans. Robot.*, 24(1):232–242, February 2008.

[161] T. Zhang, S. S. Ge, and C. C. Hang. Design and performance analysis of a direct adaptive controller for nonlinear systems. *Automatica*, 35(11):1809–1817, 1999.

[162] T. Zhang, S. S. Ge, and C. C. Hang. Adaptive neural network control for strict-feedback nonlinear systems using backstepping design. *Automatica*, 36(12):1835–1846, 2000.