CONTINUOUS-TIME FINITE-HORIZON OPTIMAL INVESTMENT AND CONSUMPTION PROBLEMS WITH PROPORTIONAL TRANSACTION COSTS

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Contents

Summary

In this thesis, the continuous-time finite-horizon optimal investment and consumption problems with proportional transaction costs are studied. Through probabilistic approach, we investigate the optimal investment problem for a Constant Relative Risk Aversion (CRRA) investor and reveal analytically the connections between the stochastic control problem and an optimal stopping problem, with the existence of optimal stochastic controls and under certain parameter restrictions. Besides, the optimal investment and consumption problem for a Constant Absolute Risk Aversion (CARA) investor is studied through Partial Differential Equation (PDE) approach. Dimensionality reduction and simplification methods are applied to transform the relevant (Hamilton-Jacobi-Bellman) HJB systems to nonlinear parabolic double obstacle problems in different ways and we reveal the equivalence. Important analytical properties of the value function and the free boundaries for the optimal investment and consumption problem are shown through rigorous PDE arguments, while comparison is made between the two cases. In addition, the jump diffusion feature is incorporated into the optimal investment problem for a CARA investor and numerical results are provided.

Chapter 1

Introduction

1.1 Literature review

1.1.1 Optimal investment without transaction costs

The optimal investment problem in the financial markets has usually been modeled as optimizing allocation of wealth among a basket of securities. As a pioneer, Markowitz (1950s) initiated the mean-variance approach for the study of this problem in the single-period settings, which is a natural and illuminating model. In such settings, the investors can only make decisions on their capital allocation at the beginning of the period, and the returns of their portfolio are evaluated until the end. With the risk of the portfolio measured by the variance of its return, Markowitz formulated the problem as minimizing the variance subject to the constraint that the expected return equals to a prescribed level, which turns out to be a quadratic programming problem. As a result, he obtained the well-known Markowitz efficient frontier, which reveals the magnitude of diversification for portfolio management and the optimal tradeoff between risk and expected return. The historical significance of the mean-variance approach is the introduction of quantitative and scientific methods to risk management. This approach provided a fundamental basis for modern portfolio theory, especially the capital asset pricing model (CAPM), and inspired thousands of extensions and applications.

After Markowitz's milestone work, modern portfolio theory has been developed in multi-period discrete-time settings with the whole investment period divided by a sequence of time spots into a series of time intervals. In each time interval between two adjacent time spots, the market is modeled in the same way as in a single-period model. The multi-period model is more than the simple combination of a sequence of single-period models on account of the dynamic evolution of the security prices, which makes the model more practical. The evolution of the prices embeds uncertainty, often depicted by the increments of the price processes, and the information flow that possesses the famous Markov property. Mossin (1968), Samuelson (1969), Hakansson (1971), Grauer and Hakansson (1993), Pliska (1997) et al have developed portfolio selection theory in multi-period discrete-time settings, while Li and Ng (2000) has provided an analytical result for multi-period mean-variance portfolio selection problem.

In more delicate continuous-time models, investors are supposed to be able to make investment decisions at any time during the whole investment period. Often using Bownian Motion to sketch the continuous-time stochastic processes, these models are much more complicated than the discrete-time ones, as they cannot be considered as the limit of the latter by partitioning the investment period into smaller intervals. Louis Bachelier (1900) firstly introduced Brownian Motion to evaluate stock option in his doctorial dissertation "The Theory of Speculation". It was a pioneer work in the study of mathematical finance and stochastic processes, but unfortunately his work did not draw enough attention until the 1960s when stochastic analysis was developed. Subsequently, Black and Scholes (1973) started to adopt the geometric Brownian Motion to model the evolution of stock prices in their seminal work, and using Brownian Motion to model price evolution has since become the standard approach in financial theory. For the optimal investment problem, Merton (1970s) initiated the famous continuous-time stochastic model embedding Brownian Motion in idealized settings, where the market is frictionless, or in other words, no transaction cost exists. One risk-free asset and one risky asset were considered, both of which are infinitely divisible, and the price of the risky asset is driven by the famous It$\hat{o}$ diffusion. Generally, an investor wants to make use of his/her capital as efficiently as possible, and the rules for "efficiency" have to be defined mathematically. In Merton's groundwork (1971), expected utility criteria were employed in Merton's portfolio problem instead of the Markowitz's mean-variance criteria to measure the satisfaction of an individual on the consumption and terminal wealth. Power and logarithm functions were adopted as utility function to represent the preference of Constant Relative Risk Aversion (CRRA) investors. Furthermore, Bellman's principle of dynamic programming, a robust approach to solve optimal control problem, and partial differential equation (PDE) theory were used by Merton to derive and analyze the relevant Hamilton-Jacobi-Bellman (HJB) equation, which is essentially the infinitesimal version of the principle of dynamic programming. In this idealized setting, he obtained a closed-form solution to the stochastic control problem faced by a CRRA investor, and concluded that the optimal investment policy for the investor is to keep a constant fraction of total wealth in the risky asset during the whole investment period, which requires incessant trading. Recent books by Korn (1997) and Karatzas and Shreve (1998) summarized much of this continuous-time optimal investment problem.

1.1.2 Optimal investment with transaction costs

Merton's (1971) idealized model has provided a standard approach to formulate the optimal investment problem for a typical individual investor, and analysis results have been obtained in the absence of transaction costs. However, in real markets, investors have to pay commission fees to their broker when buying or selling a stock. In view of such transaction costs, it has been widely observed that any attempt to apply Merton's strategy would result in immediate penury, since incessant trading is necessary to maintain the proportion on the Merton line. In this case, there must be some "no-transaction" region inside which the portfolio is insufficiently far "out of line" to make transaction worthwhile. In the attempt to understand and explain such phenomenon mathematically, Magil and Constantinides (1976) introduced the proportional transaction costs to Merton's model. They provided a fundamental insight that there exists a no-transaction region in a wedge shape other than the Merton Line, and also expressed hope that their work would "prove useful in determining the impact of trading costs on capital market equilibrium". However, the analysis of transaction cost models has not yet progressed to the point where this hope can be realized since the tools of singular stochastic control were unavailable to these authors. These authors have not given clear prescription as to how to compute the boundaries or what control the investor should take when the process reaches the boundaries, hence their argument is heuristic at best. In terms of rigorous mathematical analysis, Davis and Norman (1990) provided a precise formulation including an algorithm and numerical computations of the optimal policy for the optimal investment problem where the investor maximizes discounted utility of intermediate consumption, and their work became a landmark in the study of transaction cost problems. A key insight suggested by Magil and Constantinides (1976) and exploited by Davis and Norman (1990) is that due to homotheticity of the value function, the dimension of the free boundary problem associated with the original stochastic control problem can be reduced from two to one. In the analysis of the HJB equation for this problem, Davis and Norman (1990) showed that the optimal policies are determined by the solution of the free boundary problem for a nonlinear PDE, and there are two free boundaries indicating separately the optimal purchasing and selling policies. Under a certain parameter condition, they also demonstrated that for an infinite horizon investment and consumption problem with transaction costs, the no-transaction region is a convex cone or a wedge containing the Merton line, and the proportion of total wealth held in the risky asset should be maintained inside some interval without closed-form expression. The results reveal that the optimal transaction policy is an immediate transaction to the closest point in the wedge if the initial endowment is outside the wedge, followed by “minimal trading” to stay within the wedge. The immediate transaction involves “singular control”, and consumption taking place at a finite rate in the interior of the wedge involves “continuous control”. Given the existence of singular control, the problem studied by the authors turns out to be a singular stochastic control problem, which is much more difficult to handle than Merton’s problem. Their work served as a cornerstone to rigorously study the singular stochastic control problem evolved from the optimal investment problem with transaction costs, but it had the deficiency that the results are acquired under restrictive and not fully verifiable assumptions. As a further development, Shreve and Soner (1994) fully characterized the infinite horizon optimal policies under the sole assumption of the finiteness of the value function, relying on the concept of viscosity solutions to HJB equations. The viscosity solution approach uses the principle of dynamic programming to the singular stochastic control problem, assuming only the finiteness of the value function, to show that the equation can be interpreted in the classical sense. In contrast, the classical approach to stochastic control problem involves construction of a function that solves the HJB equation by extraordinary methods, which usually requires considerable ingenuity and sometimes the introduction of extraneous conditions, and verification that the constructed function is indeed the value function using the HJB equation. These characteristics make the classical approach not as powerful as the viscosity approach especially in the case for singular stochastic control problem. The fundamental study on viscosity theory was initiated by Lions (1982), Crandall and Lions (1983), and Crandall, Evans and Lions (1984), all of whose papers deal with first-order equations. As the HJB equation for a controlled diffusion process gives rise to a second-order equation, the extension of the viscosity theory to second-order equations was developed in a series of papers by Lions (1983), Jensen (1988), and Ishii (1989). Furthermore, the use of viscosity solutions in mathematical finance was first studied in the PhD dissertation of Zariphopoulou (1989), and the applications to stochastic control problems were reported in the book by Fleming and Soner (1993). By virtue of the viscosity theory, Shreve and Soner(1994) displayed a comprehensive and robust approach to analyze the singular stochastic problem generated from the optimal investment problem with transaction costs.

Now let us consider the phenomenon that financial consultants typically recommend that younger investors allocate a greater proportion of wealth to stocks than older investors. Malkiel (2000) stated in his popular book A Random Walk Down Wall Street that “The longer period over which you can hold on to your investment, the greater should be the share of common stocks in your portfolio.” In order to be consistent with this clearly horizon-dependent portfolio rule, the model must be considered in finite horizon, where the boundaries of the no-transaction region change as the terminal date approaches. However, it can be seen that the finiteness of the horizon alone is insufficient to justify the horizon-dependent investment policy. Taking Merton’s continuous-time optimal investment problem with idealized settings for example, even though the investor has a finite horizon, his optimal fraction of wealth invested in the stock is still horizon independent. Liu and Loewenstein (2002) focused on the effect of the horizon on an investor's investment policy in the presence of transaction costs, where the optimization problem became more difficult since the two free boundaries also change through time. The authors firstly considered the tractable problem with a stochastic time horizon following Erlang distribution, and derived some analytical properties on the optimal investment policies. They then extended these results to the situation of a deterministic time horizon using the fact that the optimal investment policies of the Erlang distributed case converge to those of the deterministic time case. In order to provide a complete study of the finite-horizon optimal investment problem with proportional transaction costs, Dai and Yi (2009) directly solved the problem faced by a CRRA investor relying on PDE approach. Motivated by the postulation that the spatial partial derivative of the value function might be the solution to some obstacle problem, these authors showed that the resulting equation is linked to a parabolic double obstacle problem, namely, an ordinary parabolic variational inequality problem. The well-developed theory of variational inequality has been very useful in tackling the challenging singular stochastic control problems, since classical compactness arguments that are used for establishing the existence of optimal controls for problems with absolutely continuous control terms do not naturally extend to singular control problems. Using this theory, they successfully obtained regularity of the value function and characterized the optimal investment policies although closed-form solutions are not available. Moreover, Dai et al (2009) took into account investment and consumption together with transaction costs in finite horizon and essentially revealed the connections between singular stochastic control and optimal stopping, while Dai, Xu and Zhou (2010) extended the idea to the continuous-time mean-variance analysis with transaction costs. In another work, Yi and Yang (2008) made use of the approach developed in Dai and Yi (2009) to solve a sub-problem arising from the utility indifference pricing with transaction costs discussed in Davis, Panas and Zariphopoulou (1993). It should be pointed out that this sub-problem is essentially a finite horizon portfolio choice problem for a Constant Absolute Risk Aversion (CARA) investor in no-consumption case, while this thesis studies the consumption case with comparison between the investment strategies of the two cases. The reason for studying the CARA utility case lies in the separability of the utility function by which the multi-asset portfolio choice problem can be reduced to the single risky asset case provided that the assets are uncorrelated, as investigated in Liu (2004).

1.1.3 Connections between singular control and optimal stopping

It has long been observed that there exist connections between singular control problems and certain optimal stopping problems. Such connections were firstly observed by Bather and Chernoff (1966), who posed a specific control problem, introduced a related stopping problem, and argued on heuristic grounds that the optimal risk of the latter ought to be the gradient of the value function of the former. They also stated that the optimal continuation region in the stopping problem ought to be the region of inaction in the control problem. Karatzas and Shreve (1980s) showed by purely probabilistic arguments that, under proper conditions on the cost functions, two typical singular stochastic control problems, the monotone follower problem and the reflected follower control problem, are equivalent to certain optimal stopping problems in the sense described by Bather and Chernoff.

Now that the optimal investment problem with transaction costs has been proven to be a singular stochastic control problem, there seem to be connections between this problem and the optimal stopping problem as well. However, the optimal investment problem with transaction costs is a comparatively more difficult category of singular stochastic control problems, and the connections between optimal investment and optimal stopping in the presence of transaction costs still need to be characterized.

1.2 Scope of this thesis

The optimal investment problem with proportional transaction costs in finite horizon, as well as its connections with optimal stopping, is challenging in theory but interesting in practice. This thesis, for the first time, investigates the continuous-time finite-horizon optimal investment problem with transaction costs for a CRRA investor with logarithm utility function and attempts to reveal its connections with a certain optimal stopping problem through probabilistic approach. Besides, the continuous-time finite-horizon optimal investment problem with transaction costs for a CARA investor with exponential utility function is also studied while jump diffusion feature is incorporated. Another important contribution of this thesis is that analytical and numerical results are obtained for the continuous-time finite-horizon optimal investment and consumption problem with transaction costs for a CARA investor.

In Chapter 2, we attempts to investigate the continuous-time finite-horizon optimal investment problem with transaction costs for a CRRA investor with logarithm utility function by pure probabilistic arguments, and the problem is formulated as a singular stochastic control problem. Properties of the value function for this problem are shown and analytical results are provided for the three transaction regions, which comprises “jump-buy region”, “jump-sell region” and “no-jump-trade region” and prevails for all the problems we study in this thesis. The jumping styles of the singular stochastic controls are further investigated, based on which an equivalent standard stochastic control problem is obtained. This equivalent standard stochastic control problem becomes much simpler than the singular stochastic control problem since jumps of the diffusion processes arising from the singularity of controls have been eliminated. A new diffusion process is further introduced so that the dimensionality of the standard stochastic control problem that innately contains two diffusion processes is reduced based on the result that the CRRA investor should never take short position in the risky asset during the horizon except the initial time and terminal time. Such simplification enables us to seek the relation between this stochastic control problem and a certain optimal stopping problem, especially the connection between the value function of the former and the optimal risk of the latter, with the existence of optimal stochastic controls and under certain parameter restrictions. Our work may shed light on future studies on such optimal investment problem with transaction costs in probabilistic approach.

In Chapter 3, we consider the continuous-time finite-horizon optimal investment and consumption problem with transaction costs for a CARA investor through PDE approach, which constitutes the major contribution of this thesis. It is first observed by probabilistic arguments that the dimensionality of the problem without consumption can be reduced and the optimal investment strategy for the CARA investor is indifferent to the initial endowment in the riskless asset. The relevant HJB systems, in both the no-consumption case and the consumption case, are then transformed and simplified to two nonlinear parabolic double obstacle problems separately, while the equivalence is further revealed. Important properties of the value function and the free boundaries for the optimal investment and consumption problem are revealed analytically by PDE arguments, and comparison is made analytically between the two cases with and without consumption. Besides, the infinite-horizon optimal investment and consumption problem is deduced from the stationary double obstacle problem, which is shown equivalent to the system obtained in Liu (2004). In addition, since the exponential utility function may tolerate negative wealth possibly incurred by the jumping nature, the jump diffusion feature is incorporated in the CARA investor's optimal investment problem and a variational inequality system with gradient constraints is obtained through similar dimensionality reduction. Finite difference methods are implemented to numerically solve the systems, while the impact of the jump diffusion on the optimal investment strategy is explained in the end.

Chapter 2

The CRRA Investor's Optimal Investment Problem with Transaction Costs

2.1 Formulation of the optimal investment problem

2.1.1 The asset market

Throughout this thesis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ denotes a fixed filtered complete probability space on which a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted one-dimensional Brownian Motion $\mathcal{B}(t)$ is defined, with $\mathcal{B}(0) = 0$ almost surely. The formulation of our problem, the continuous-time optimal investment problem with transaction costs in a finite horizon $[0, T]$, is based on this filtered probability space.

Suppose that there are only two assets available in the asset market for investment: one riskless asset (bond) and one risky asset (stock). Their prices, denoted by $S_0(t)$ and $S_1(t)$ separately, evolve as follows:

$$\begin{aligned} dS_0(t) &= rS_0(t)dt, \\ dS_1(t) &= S_1(t)[\alpha dt + \sigma d\mathcal{B}(t)]. \end{aligned}$$

Here $r > 0$ represents the constant riskless interest rate, and $\alpha > r$ and $\sigma > 0$ stand for the constant expected rate of return and the volatility, respectively, of the risky asset. These constitute the simplest standard setting of an asset market, and the investor's problem is derived from such setting.

2.1.2 A singular stochastic control problem

The investor holds a portfolio that consists of $X(t)$ monetary amount in the riskless asset account and $Y(t)$ monetary amount in the risky asset account at any time t in $[0, T]$, hence the investor's position at time t may be referred to as $(X(t), Y(t))$. In the presence of proportional transaction costs, such position satisfies the following diffusion equations:

$$\begin{cases} dX(t) = rX(t-)dt - (1+\lambda)dL(t) + (1-\mu)dM(t), \\ dY(t) = \alpha Y(t-)dt + \sigma Y(t-)d\mathcal{B}(t) + dL(t) - dM(t). \end{cases} \tag{2.1}$$

Here we use $L(\cdot)$ and $M(\cdot)$ to denote cumulative monetary amounts for buying and selling the risky asset separately, both of which are right-continuous, non-negative, and non-decreasing $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted processes with $L(0) = M(0) = 0$. Note that due to possible jumps in $L(\cdot)$ and $M(\cdot)$, we shall use $X(t-)$ and $Y(t-)$ on the right hand side of the stochastic diffusion equations, while the initial endowment is in fact infused at time $0-$. The constants $\lambda \in [0, \infty)$ and $\mu \in [0, 1)$ represent the proportional transaction costs incurred on purchase and sale of the stock separately.

As part of the optimization target, the investor's wealth process is given high concern. Thus if we define

$$w(x, y) := \begin{cases} x + (1-\mu)y, & \text{if} \quad y \geq 0, \\ x + (1+\lambda)y, & \text{if} \quad y < 0, \end{cases}$$

then the net wealth in monetary terms at time t is simply $w(X(t), Y(t))$. Because it is natural to require that the investor's net wealth be positive, we define the solvency region by

$$\mathbb{S} = \left\{(x, y) \in \mathbb{R}^2 : x + (1+\lambda)y > 0, x + (1-\mu)y > 0\right\},$$

inside which $w(x, y) > 0$ holds spontaneously. The following two notations

$$\begin{aligned}\partial_1\mathbb{S} &:= \{(x, y) : x + (1+\lambda)y = 0, x > 0\},\\ \partial_2\mathbb{S} &:= \{(x, y) : x + (1-\mu)y = 0, y > 0\},\end{aligned}$$

refer to the two parts of the solvency region boundary separately.

We define the set of square integrable $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted processes as

$$L^2_{\mathcal{F}} := \left\{\xi \,\middle|\, \{\xi(t)\}_{t\in[0,T]} \text{ is } \{\mathcal{F}_t\}_{t\in[0,T]}\text{-adapted}, \int_0^T \mathbb{E}[\xi^2(t)]dt < \infty\right\},$$

and the set of square integrable random variables as

$$L^2 := \{X \,|X \text{ is a random variable}, \mathbb{E}[X^2] < \infty\}.$$

Assuming that the investor's initial endowment (x_0, y_0) lies in $\mathbb{S}$, we call the investment strategy (L, M) admissible if contained in the following admissible set

$$\mathcal{A} := \left\{ (L, M) \,\middle|\, \begin{array}{l} \{L(t)\}_{t\in[0,T]}, \{M(t)\}_{t\in[0,T]} \text{ are right-continuous, non-negative,} \\ \text{non-decreasing, } \{\mathcal{F}_t\}_{t\in[0,T]} - \text{adapted}, L(0) = M(0) = 0, \\ \text{and its governing processes } (X(\cdot), Y(\cdot)) \in \mathbb{S} \text{ in } [0, T], \\ X \in L^2_{\mathcal{F}}, Y \in L^2_{\mathcal{F}}, \frac{X}{w(X,Y)} \in L^2_{\mathcal{F}}, X(t) \in L^2, Y(t) \in L^2, \forall t \in [0, T] \end{array} \right\}.$$

This admissible set is clearly nonempty, as the investor can always adopt the trading policy that closes out the position in the risky asset at initial time and remains zero position in the risky asset afterwards to satisfy the conditions.

The investor is assumed to be Constant Relative Risk Aversion (CRRA) with logarithm utility function. The associated utility functional J can then be defined as follows:

$$\begin{aligned} J(s,x,y;L,M) &:= \mathbb{E}\left[\log(w(X(T),Y(T)))|X(s-)=x,Y(s-)=y\right] \\ &\quad \text{s.t. } (2.1). \end{aligned}$$

Mathematically, the utility function $\log(\cdot)$ is a real-valued function defined on $(0,\infty)$, strictly increasing, strictly concave, twice continuously differentiable, and satisfies $\lim_{w\downarrow 0}(\log(w))' = \infty$. Based on such cost functional, the investor's problem under expected utility criteria can be formulated as maximizing the cost functional over the admissible set $\mathcal{A}$. Denoting the value function by φ, the problem may be described as follows:

$$\varphi(s,x,y) := \sup_{(L,M)\in\mathcal{A}} J(s,x,y;L,M), \tag{2.2}$$

for any $s \in [0,T]$ and $(x,y) \in \mathbb{S}$. According to the definition of the admissible set $\mathcal{A}$, it is not difficult to show $\varphi(s,x,y) < \infty$ for all $s \in [0,T]$ and $(x,y) \in \mathcal{S}$ by applying Jensen's Inequality.

Problem (2.2) is essentially a singular stochastic control problem, which admits discontinuous controls, or in other words, allows lump-sum investment strategies. Such lump-sum investment strategies will be named as "jump-buy" or "jump-sell" accordingly in most of the cases thereafter.

2.1.3 Properties of the value function

We now introduce several fundamental properties of the value function φ of the singular stochastic control problem (2.2).

Proposition 2.1.1. Given any $s \in [0,T]$, $\varphi(s,\cdot,\cdot)$ is strictly increasing w.r.t. the state arguments x and y.

Proof: It is very easy to obtain this property by investing additional monetary amount in the riskless asset while keeping the investment strategy unchanged afterwards, which makes the value function even larger. □

Proposition 2.1.2. Given any $s \in [0,T]$, $\varphi(s,\cdot)$ is concave in $\mathbb{S}$.

Proof: Let (x_1, y_1) and (x_2, y_2) be in $\mathbb{S}$, and $(X_1(\cdot), Y_1(\cdot))$ and $(X_2(\cdot), Y_2(\cdot))$ be diffusion processes for problem (2.2) with initial states $(X_1(s-), Y_1(s-)) = (x_1, y_1)$ and $(X_2(s-), Y_2(s-)) = (x_2, y_2)$ while subject to investment strategies (L_1, M_1) and (L_2, M_2) respectively. For any $\eta \in (0,1)$, it is easy to see

$$(\eta x_1 + (1-\eta)x_2, \eta y_1 + (1-\eta)y_2) \in \mathbb{S},$$

due to the convexity of $\mathbb{S}$. In view of the linearity of the diffusions, the investment strategy $(\eta L_1 + (1-\eta)L_2, \eta M_1 + (1-\eta)M_2)$ is always admissible for the diffusion processes with initial states $(\eta x_1 + (1-\eta)x_2, \eta y_1 + (1-\eta)y_2)$ at time s.

In order to obtain the convexity of the value function, we need to consider some property possessed by the function $w(x,y)$. Without loss of generality, we take any two points $(\hat{x}_1, \hat{y}_1)$ and $(\hat{x}_2, \hat{y}_2)$ in $\mathbb{S}$ with $\hat{y}_1 \geq \hat{y}_2$. It is not difficult to verify the following results case by case:

$$w(\eta\hat{x}_1 + (1-\eta)\hat{x}_2, \eta\hat{y}_1 + (1-\eta)\hat{y}_2) \begin{cases} = \eta w(\hat{x}_1, \hat{y}_1) + (1-\eta)w(\hat{x}_2, \hat{y}_2), \hat{y}_1 \geq \hat{y}_2 \geq 0, \\ \geq \eta w(\hat{x}_1, \hat{y}_1) + (1-\eta)w(\hat{x}_2, \hat{y}_2), \hat{y}_2 < 0 \leq \hat{y}_1, \\ = \eta w(\hat{x}_1, \hat{y}_1) + (1-\eta)w(\hat{x}_2, \hat{y}_2), \hat{y}_2 \leq \hat{y}_1 < 0, \end{cases}$$

the combination of which leads to

$$w(\eta\hat{x}_1 + (1-\eta)\hat{x}_2, \eta\hat{y}_1 + (1-\eta)\hat{y}_2) \geq \eta w(\hat{x}_1, \hat{y}_1) + (1-\eta)w(\hat{x}_2, \hat{y}_2).$$

Together with the concavity of the utility function $\log(\cdot)$, we have

$$\begin{aligned} & J(s, \eta x_1 + (1-\eta)x_2, \eta y_1 + (1-\eta)y_2; \eta L_1 + (1-\eta)L_2, \eta M_1 + (1-\eta)M_2) \\ = \; & \mathbb{E}\left[\log(w(\eta X_1(T) + (1-\eta)X_2(T), \eta Y_1(T) + (1-\eta)Y_2(T)))\right] \\ \geq \; & \mathbb{E}\left[\log(\eta w(X_1(T), Y_1(T)) + (1-\eta)w(X_2(T), Y_2(T)))\right] \\ \geq \; & \eta\mathbb{E}\left[\log(w(X_1(T), Y_1(T)))\right] + (1-\eta)\mathbb{E}\left[\log(w(X_2(T), Y_2(T)))\right] \\ = \; & \eta J(s, x_1, y_1; L_1, M_1) + (1-\eta)J(s, x_2, y_2; L_2, M_2). \end{aligned}$$

Taking supremum over $(L_1, M_1) \in \mathcal{A}$ and $(L_2, M_2) \in \mathcal{A}$ on the last term of the inequality, we immediately obtain

$$\varphi(s, \eta x_1 + (1-\eta)x_2, \eta y_1 + (1-\eta)y_2) \geq \eta\varphi(s, x_1, y_1) + (1-\eta)\varphi(s, x_2, y_2),$$

which completes the proof. □

Proposition 2.1.3. Given any $s \in [0, T]$, $\varphi(s, \cdot, \cdot)$ has the homotheticity property

$$\varphi(s, \rho x, \rho y) = \varphi(s, x, y) + \log \rho,$$

for any $(x, y) \in \mathbb{S}$ and $\rho > 0$.

Proof: This result follows straightforwardly from the fact that the controls (L, M) for problem (2.2) governing the diffusion processes $(X(\cdot), Y(\cdot))$ with initial states $(X(s-), Y(s-)) = (x, y)$ is admissible if and only if $(\rho L, \rho M)$ governing the diffusion processes $(X_\rho(\cdot), Y_\rho(\cdot))$ with initial states $(X_\rho(s-), Y_\rho(s-)) = (\rho x, \rho y)$ is admissible for all $\rho > 0$. □

Proposition 2.1.4. Given any $(x, y) \in \mathbb{S}$, $\varphi(\cdot, x, y)$ is strictly decreasing with respect to the temporal argument in $[0, T]$.

Proof: Firstly, for any $\delta t \in (0, T]$, we choose the investment strategy as closing out at time $T - \delta t$ and taking no position afterwards, which induces

$$\begin{aligned} \varphi(T - \delta t, x, y) &\geq \varphi(T - \delta t, w(x, y), 0) \geq \varphi(T, w(x, y) \cdot \mathbf{e}^{r\delta t}, 0) \\ &= \varphi(T, w(x, y), 0) + r\delta t = \varphi(T, x, y) + r\delta t > \varphi(T, x, y). \end{aligned}$$

Next, for any $s \in (0, T)$, and $\delta t \in (0, s]$, we denote by $(X_1(\cdot), Y_1(\cdot))$ the diffusion processes with initial states $(X_1((s - \delta t)-), Y_1((s - \delta t)-)) = (x, y)$ and by $(X_2(\cdot), Y_2(\cdot))$ the diffusion processes with initial states $(X_2(s-), Y_2(s-)) = (x, y)$.

Thus it can be deduced that

$$\begin{aligned}\varphi(s-\delta t,x,y) &= \sup_{(L,M)\in\mathcal{A}} \mathbb{E}[\varphi(T-\delta t, X_1(T-\delta t), Y_1(T-\delta t))] \\ &\geq \sup_{(L,M)\in\mathcal{A}} \mathbb{E}[\varphi(T, X_1(T-\delta t), Y_1(T-\delta t))] + r\delta t \\ &= \sup_{(L,M)\in\mathcal{A}} \mathbb{E}[\varphi(T, X_2(T), Y_2(T))] + r\delta t \\ &= \varphi(s,x,y) + r\delta t > \varphi(s,x,y).\end{aligned}$$

Therefore, we conclude that the value function φ is strictly decreasing with respect to the temporal argument. Intuitively, this property reflects the time value of investment. □

Proposition 2.1.5. Given any $s \in [0,T]$, $\varphi(s,\cdot)$ is continuous in $\mathbb{S}$.

Proof: For every $s \in [0,T]$, it is easy to observe that $\varphi(s,\cdot)$ is continuous in $\mathbb{S}$, since a convex function is always continuous on the interior of its domain. ([41], Theorem 10.1) □

2.1.4 Three transaction regions

As aforementioned, the investment strategy $(L,M) \in \mathcal{A}$ may possibly admit jumps, which would make $(X(\cdot),Y(\cdot))$ jump processes. As usual, we define the jumping parts of the diffusion processes by

$$\Delta L(t) := L(t) - L(t-), \Delta M(t) := M(t) - M(t-),$$

for every $t \in [0,T]$ respectively. Thus the continuous parts of the diffusion processes can be expressed by

$$L^c(t) := L(t) - \sum_{s\in[0,t]} \Delta L(s), M^c(t) := M(t) - \sum_{s\in[0,t]} \Delta M(s),$$

both of which remain non-decreasing and $\{\mathcal{F}_t\}_{t\geq 0}$-adapted but are modified to be continuous. We further introduce the following notations for $t \in [0,T]$:

$$\Delta_b(t,x,y) := \sup\{\delta \geq 0 : \varphi(t,x,y) = \varphi(t,x-(1+\lambda)\delta, y+\delta)\},$$
$$\Delta_s(t,x,y) := \sup\{\delta \geq 0 : \varphi(t,x,y) = \varphi(t,x+(1-\mu)\delta, y-\delta)\},$$

which intuitively represent the maximal amount the investor are able to buy and sell at time t without compromising the value function φ. As we trivially have

$$\Delta_b(T,x,y) = \max\{-y,0\}, \Delta_s(T,x,y) = \max\{y,0\},$$

at terminal time T, the characteristics of Δ_b and Δ_s need only to be studied in $[0,T) \times \mathbb{S}$.

For our original problem (2.2), it is apparent to see for any $(x,y) \in \mathbb{S}$ that

$$\begin{aligned} \varphi(s,x,y) \quad = \quad & \sup_{(L,M)\in\mathcal{A}} \mathbb{E}\left[\log(w(X(T),Y(T)))|X(s-)=x, Y(s-)=y\right] \\ & \text{s.t.} \quad dX(t) = rX(t-)dt - (1+\lambda)dL(t) + (1-\mu)dM(t), \\ & \qquad dY(t) = \alpha Y(t-)dt + \sigma Y(t-)d\mathcal{B}(t) + dL(t) - dM(t), \\ & \qquad \Delta L(t) = \Delta_b(t,X(t-),Y(t-)), \Delta M(t) = \Delta_s(t,X(t-),Y(t-)), \end{aligned} \tag{2.3}$$

where the investor is required to adopt the investment strategy with maximal amounts of "jump-buy" and "jump-sell" that would not compromise the value function φ. Such artificial constraint narrows the pool of admissible investment strategies without affecting the value function, thus it facilitates our further analysis of the problem. Moreover, it is natural to distinguish three transaction regions for problem (2.3) as follows:

$$\mathrm{BR}_t := \{(x,y) \in \mathbb{S} : \Delta_b(t,x,y) > 0\},$$
$$\mathrm{SR}_t := \{(x,y) \in \mathbb{S} : \Delta_s(t,x,y) > 0\},$$
$$\overline{\mathrm{NT}_t} := \mathbb{S} \setminus (\mathrm{BR}_t \cup \mathrm{SR}_t),$$

in which the investor should adopt "jump-buy", "jump-sell", or neither at time t respectively. For convenience of analysis, we denote the interior of $\overline{\mathrm{NT}_t}$ by NT_t.

Proposition 2.1.6. Given any $t \in [0, T)$, BR_t, SR_t and NT_t are convex cones if nonempty. Moreover, BR_t and SR_t are open sets.

Proof: Implied by the homotheticity property obtained in Proposition 2.1.3, it is easy to see that $(x, y) \in \mathrm{BR}_t$ if and only if $(\rho x, \rho y) \in \mathrm{BR}_t$, and $(x, y) \in \mathrm{SR}_t$ if and only if $(\rho x, \rho y) \in \mathrm{SR}_t$ for any $\rho > 0$.

Furthermore, if $\mathrm{BR}_t \neq \emptyset$, then for any $(x, y) \in \mathrm{BR}_t$, for any $\kappa > 0$, it is obvious that

$$\varphi(t, x + (1 + \lambda)\kappa, y - \kappa) \geq \varphi(t, x, y),$$

since taking $\Delta L(t) = \kappa$, $\Delta M(t) = 0$ is admissible at $(x + (1 + \lambda)\kappa, y - \kappa)$. Now according to the definition of BR_t, there exists $\delta > 0$ such that

$$\varphi(t, x - (1 + \lambda)\delta, y + \delta) = \varphi(t, x, y).$$

Then for any $\kappa > 0$, using concavity of value function obtained in Proposition 2.1.2, we immediately get $\varphi(t, x + (1 + \lambda)\kappa, y - \kappa) \leq \varphi(t, x, y)$. Hence we have

$$\varphi(t, x + (1 + \lambda)\kappa, y - \kappa) = \varphi(t, x, y).$$

These indicate $(x + (1 + \lambda)\kappa, y - \kappa) \in \mathrm{BR}_t$. Therefore, if $\mathrm{BR}_t \neq \emptyset$, it is a convex cone with $\partial_1\mathbb{S}$ being part of its boundary. Similar argument can be applied to SR_t as well and we conclude that if nonempty it is a convex cone with $\partial_2\mathbb{S}$ being part of its boundary.

In addition, NT_t is also a convex cone between BR_t and SR_t if nonempty based on its definition. Moreover, according to the definition of BR_t, for any $(x, y) \in \mathrm{BR}_t$, it can be easily seen that $(x - \frac{1}{2}(1 + \lambda)\Delta_b(t, x, y), y + \frac{1}{2}\Delta_b(t, x, y)) \in \mathrm{BR}_t$ as well. Applying the same argument for SR_t and together with the convex cone property, we conclude that BR_t and SR_t are open sets. These complete the proof. □

Intuitively, the three transaction regions have the shapes shwon by Figure 2.1 below.

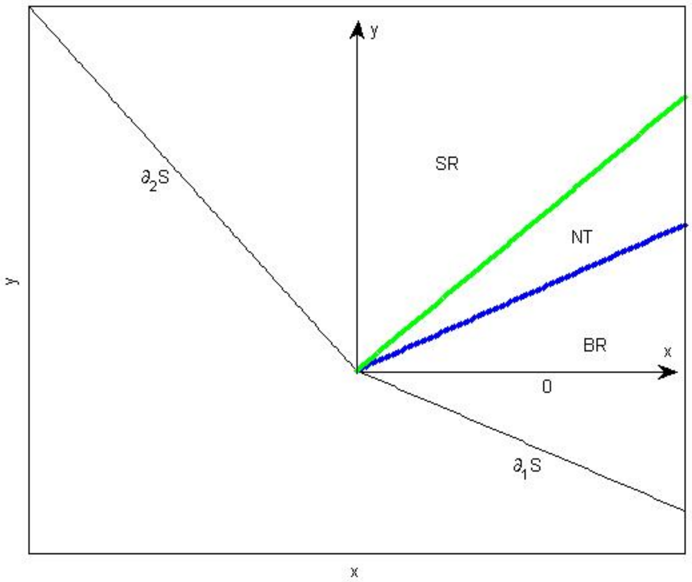

Figure 2.1. Plot of the three transaction regions for the optimal investment problem for a CRRA investor.

Proposition 2.1.7. Given any $t \in [0, T)$, $\varphi(t, \cdot, \cdot)$ is continuously differentiable in arguments x and y respectively in $\mathrm{BR}_t \cup \mathrm{SR}_t$. Moreover, for any $(x_1, y_1) \in \mathrm{BR}_t$ and $(x_2, y_2) \in \mathrm{SR}_t$, we have

$$\begin{array}{ll} \frac{\partial \varphi}{\partial x}(t, x_1, y_1) = \frac{1}{x_1+(1+\lambda)y_1}, & \frac{\partial \varphi}{\partial y}(t, x_1, y_1) = \frac{1+\lambda}{x_1+(1+\lambda)y_1}, \\ \frac{\partial \varphi}{\partial x}(t, x_2, y_2) = \frac{1}{x_2+(1-\mu)y_2}, & \frac{\partial \varphi}{\partial y}(t, x_2, y_2) = \frac{1-\mu}{x_2+(1-\mu)y_2}. \end{array}$$

Proof: Let us consider in the first place the continuous differentiability in x in BR_t, where the direction of contour lines is parallel to $\partial_1\mathbb{S}$, in the following three cases. Firstly, given $(x, y) \in \mathrm{BR}_t \cap \{y < 0\}$, there exists small enough δ_1 such that $(x + \delta_1, y) \in \mathrm{BR}_t \cap \{y < 0\}$ and $(x - \delta_1, y) \in \mathrm{BR}_t \cap \{y < 0\}$ in view of Proposition 2.1.6. For any $\delta \in (0, \delta_1)$, obviously it also holds that $(x + \delta, y) \in \mathrm{BR}_t \cap \{y < 0\}$ and $(x - \delta, y) \in \mathrm{BR}_t \cap \{y < 0\}$ due to Proposition 2.1.6. Furthermore, it can be deduced that

$$\frac{\varphi(t,x+\delta,y)-\varphi(t,x,y)}{\delta} = \frac{\varphi(t,x+c\delta,\frac{y}{x}(x+c\delta))-\varphi(t,x,y)}{\delta} = \frac{\log\left(\frac{x+c\delta}{x}\right)}{\delta},$$
$$\frac{\varphi(t,x,y)-\varphi(t,x-\delta,y)}{\delta} = \frac{\varphi(t,x,y)-\varphi(t,x-c\delta,\frac{y}{x}(x-c\delta))}{\delta} = \frac{\log\left(\frac{x}{x-c\delta}\right)}{\delta},$$

where c is the common edge ratio of certain congruent triangles, which can be shown to be $\frac{x}{x+(1+\lambda)y}$. These indicate the existence of both limits when $\delta \downarrow 0$, and

$$\lim_{\delta\to 0+}\frac{\varphi(t,x+\delta,y)-\varphi(t,x,y)}{\delta} = \lim_{\delta\to 0+}\frac{\varphi(t,x,y)-\varphi(t,x-\delta,y)}{\delta} = \frac{c}{x} = \frac{1}{x+(1+\lambda)y},$$

which appears to be the partial derivative in x in $\mathrm{BR}_t \cap \{y<0\}$. Secondly, for any $(x,y)\in \mathrm{BR}_t\cap\{y=0\}$, for any $\delta\in(0,x)$, it is straightforward to calculate the fractions

$$\frac{\varphi(t,x+\delta,0)-\varphi(t,x,0)}{\delta} = \frac{\log\left(\frac{x+\delta}{x}\right)}{\delta},$$
$$\frac{\varphi(t,x,0)-\varphi(t,x-\delta,0)}{\delta} = \frac{\log\left(\frac{x}{x-\delta}\right)}{\delta},$$

from which we can deduce that

$$\lim_{\delta\to 0+}\frac{\varphi(t,x+\delta,y)-\varphi(t,x,y)}{\delta} = \lim_{\delta\to 0+}\frac{\varphi(t,x,y)-\varphi(t,x-\delta,y)}{\delta} = \frac{1}{x},$$

which appears to be the partial derivative in x on $\mathrm{BR}_t\cap\{y=0\}$. Thirdly, for any $(x,y)\in\mathrm{BR}_t\cap\{y>0\}\cap\{x\neq 0\}$, the argument is the same as in the first case, and we may obtain the partial derivative in x as

$$\lim_{\delta\to 0}\frac{\varphi(t,x+\delta,y)-\varphi(t,x,y)}{\delta} = \frac{1}{x+(1+\lambda)y}.$$

Lastly, for $(x,y)\in\mathrm{BR}_t\cap\{x=0\}$, for any $\delta\in(0,y)$, it is not difficult to calculate the fractions

$$\frac{\varphi(t,\delta,y)-\varphi(t,0,y)}{\delta} = \frac{\varphi(t,0,y+\frac{\delta}{1+\lambda})-\varphi(t,0,y)}{\delta} = \frac{\log\left(\frac{(1+\lambda)y+\delta}{(1+\lambda)y}\right)}{\delta},$$
$$\frac{\varphi(t,0,y)-\varphi(t,-\delta,y)}{\delta} = \frac{\varphi(t,0,y)-\varphi(t,0,y-\frac{\delta}{1+\lambda})}{\delta} = \frac{\log\left(\frac{(1+\lambda)y}{(1+\lambda)y-\delta}\right)}{\delta}.$$

These also indicate the existence of both limits when $\delta\downarrow 0$, and

$$\lim_{\delta\to 0+}\frac{\varphi(t,\delta,y)-\varphi(t,0,y)}{\delta} = \lim_{\delta\to 0+}\frac{\varphi(t,0,y)-\varphi(t,-\delta,y)}{\delta} = \frac{1}{(1+\lambda)y},$$

which accords with the previous formula of the partial derivative.

These results immediately lead to the continuous differentiability of φ in x in BR_t, and we can formally write the general expression as

$$\frac{\partial \varphi}{\partial x}(t,x,y) = \frac{1}{x+(1+\lambda)y},$$

for any $(x,y) \in \mathrm{BR}_t$. Similarly it can be deduced that φ is continuously differentiable in y in BR_t, and

$$\frac{\partial \varphi}{\partial y}(t,x,y) = \frac{1+\lambda}{x+(1+\lambda)y},$$

for any $(x,y) \in \mathrm{BR}_t$. The same argument can be applied in SR_t, where it holds

$$\frac{\partial \varphi}{\partial x}(t,x,y) = \frac{1}{x+(1-\mu)y},$$
$$\frac{\partial \varphi}{\partial y}(t,x,y) = \frac{1-\mu}{x+(1-\mu)y},$$

for $(x,y) \in \mathbb{SR}_t$. Thus we complete the proof. □

Proposition 2.1.8. Given any $t \in [0,T)$, $(x_1,y_1) \in \mathrm{BR}_t$ and $(x_2,y_2) \in \mathrm{SR}_t$, we have

$$(1+\lambda)\frac{\partial \varphi}{\partial x}(t,x_1,y_1) - \frac{\partial \varphi}{\partial y}(t,x_1,y_1) = 0,$$
$$(1-\mu)\frac{\partial \varphi}{\partial x}(t,x_2,y_2) - \frac{\partial \varphi}{\partial y}(t,x_2,y_2) = 0.$$

Proof: For any $t \in [0,T]$, the $C^{1,1}$ regularity of the value function $\varphi(t,\cdot,\cdot)$ obtained in Proposition 2.1.7 guarantees the existence of the first-order partial derivatives. For any $(x_1,y_1) \in \mathrm{BR}_t$, we know from the proof for Proposition 2.1.7 the following expressions of partial derivatives

$$\frac{\partial \varphi}{\partial x}(t,x_1,y_1) = \frac{1}{x_1+(1+\lambda)y_1}, \quad \frac{\partial \varphi}{\partial y}(t,x_1,y_1) = \frac{1+\lambda}{x_1+(1+\lambda)y_1},$$

which immediately leads to

$$(1+\lambda)\frac{\partial \varphi}{\partial x}(t,x_1,y_1) - \frac{\partial \varphi}{\partial y}(t,x_1,y_1) = 0.$$

The other equation for $(x_2,y_2) \in \mathrm{SR}_t$ can be shown in the same manner, hence we complete the proof. □

Corollary 2.1.9. Given any $t \in [0, T)$, $\mathrm{BR}_t \cap \mathrm{SR}_t = \emptyset$, and $\overline{\mathrm{NT}_t} \neq \emptyset$.

Proof: The former conclusion can be directly deduced from Proposition 2.1.7 and Proposition 2.1.8. For the latter one, suppose we have $\overline{\mathrm{NT}_t} = \emptyset$, then either $\mathrm{BR}_t = \mathbb{S}$ or $\mathrm{SR}_t = \mathbb{S}$ holds according to the definitions. Nevertheless, in either case the investor would exercise "jump-transaction" to pull the state $(X(t), Y(t))$ to $\partial_2\mathbb{S}$ or $\partial_1\mathbb{S}$, which is obvious suboptimal since immediate bankruptcy is triggered unnecessarily. Thus we complete the proof. □

Now for any $t \in [0, T)$, we denote the boundary between BR_t and NT_t by $\partial\mathrm{BR}_t$, and the boundary between SR_t and NT_t by $\partial\mathrm{SR}_t$, both of which are radials. Usually, they are also referred to as the free boundaries, which parallel the free boundary we have met in pricing American options.

Proposition 2.1.10. For problem (2.3), for any diffusion processes $(X(\cdot), Y(\cdot))$ with initial states $(X(s-), Y(s-)) = (x, y)$, if the optimal governing controls (L^*, M^*) exist, then such optimal controls are unique almost surely.

Proof: Let us suppose that there exists another pair of controls (L_1^*, M_1^*) that satisfies

$$\mathbb{P}\left[\mathbb{L}\left[t \in [s,T) : \begin{pmatrix} L^*(t) - L^*(s) \\ M^*(t) - M^*(s) \end{pmatrix} \neq \begin{pmatrix} L_1^*(t) - L_1^*(s) \\ M_1^*(t) - M_1^*(s) \end{pmatrix}\right] > 0\right] > 0,$$

with $\mathbb{L}$ being the Lebesgue measure, and $J(s, x, y; L^*, M^*) = J(s, x, y; L_1^*, M_1^*)$. We denote by $(X_1^*(\cdot), Y_1^*(\cdot))$ the corresponding diffusion processes subject to (L_1^*, M_1^*), then the above condition would induce two different distributions of $(X^*(T), Y^*(T))$ and $(X_1^*(T), Y_1^*(T))$, or in other words,

$$\mathbb{P}[(X^*(T), Y^*(T)) \neq (X_1^*(T), Y_1^*(T))] > 0.$$

Hence we may choose the controls as $(\frac{L^*+L_1^*}{2}, \frac{M^*+M_1^*}{2})$, which immediately leads to

$$J(s, x, y; \tfrac{L^*+L_1^*}{2}, \tfrac{M^*+M_1^*}{2}) > \tfrac{1}{2}\left[J(s, x, y; L^*, M^*) + J(s, x, y; L_1^*, M_1^*)\right] = J(s, x, y; L^*, M^*),$$

due to strict concavity of the utility function. This violates the fact that (L^*, M^*) are the optimizer, thus we must have the uniqueness of the optimal controls almost surely if the existence is guaranteed. These complete the proof. □

2.2 Problem transformation and dimensionality reduction

2.2.1 A standard stochastic control problem

So far we have established a partition of the whole spatial domain $\mathbb{S}$ at any time $t \in [0, T)$: The three transaction regions BR_t, SR_t and NT_t, all of which are convex cones, or in other words, wedges if nonempty. The well-known Merton Line in Merton's idealized model is replaced by the "no-jump-transaction region" $\overline{\mathrm{NT}_t}$ in the presence of proportional transaction costs, while the "jump-buy region" BR_t and the "jump-sell region" SR_t are in similar positions as in Merton's model. The artificial investment strategy for problem (2.3) would immediately draw the state inside BR_t and SR_t to $\partial\mathrm{BR}_t$ and $\partial\mathrm{SR}_t$ respectively. In the following, we will reveal a crucial property of the optimal investment strategy.

Proposition 2.2.1. For problem (2.3), the investor should never "jump buy" or "jump sell" during the period (s, T).

Proof: First of all, it has been shown in [10] that the free boundaries $\partial\mathrm{BR}_t$ and $\partial\mathrm{SR}_t$ are continuous via PDE approach, thus it is safe to claim that there are no jump changes of $\partial\mathrm{BR}_t$ or $\partial\mathrm{SR}_t$ across time in $[0, T)$ which may increase BR_t or SR_t abruptly.

For any $s \in [0, T)$, $(x, y) \in \mathbb{S}$, for any coupling diffusion processes $(X(\cdot), Y(\cdot))$ of problem (2.3) with initial states $(X(s-), Y(s-)) = (x, y)$, "jump-buy" or "jump-sell" will be exercised at time s to draw the states (x, y) into $\overline{\mathrm{NT}_s}$. Now for any

$\delta > 0$, suppose for a specific path realization, there exists $t \in (s, T)$ such that $\Delta_b(t, X(t), Y(t)) = \delta$, then we have

$$\varphi(t, X(t), Y(t)) = \varphi(t, X(t) - (1+\lambda)\delta, Y(t) + \delta),$$

and the distance between $(X(t), Y(t))$ and $\partial \mathrm{BR}_t$ is $\sqrt{(1+\lambda)^2+1} \cdot \delta$. However, in view of the claim stated in the beginning of this proof, the constraints in problem (2.3) can force all realized paths not to move into "jump-buy region" exceeding distance $\sqrt{(1+\lambda)^2+1} \cdot \delta/2$ during (s, t) while an abrupt increase of BR_t is impossible, thus the distance between $(X(t), Y(t))$ and $\partial \mathrm{BR}_t$ cannot arrive at $\sqrt{(1+\lambda)^2+1} \cdot \delta$, a contradiction. We may then let δ be arbitrarily small, and it can be seen that $\Delta_b(t, X(t), Y(t)) = 0$ almost surely in (s, T). Similar arguments can be applied to the "jump-sell" region, thus continuous controls dominate the horizon (s, T) while neither "jump buy" nor "jump sell" is possible during (s, T). These complete the proof. $\square$

According to Proposition 2.2.1, we can further strengthen the constraints to the controls, and deduce for any $s \in [0, T]$ and $(x, y) \in \mathbb{S}$ that

$$\begin{aligned} \varphi(s,x,y) \quad = \quad & \sup_{(L,M)\in\mathcal{A}} \mathbb{E}\left[\log(w(X(T),Y(T)))|X(s-)=x, Y(s-)=y\right] \\ & \text{s.t.} \quad dX(t) = rX(t-)dt - (1+\lambda)dL(t) + (1-\mu)dM(t), \\ & \qquad dY(t) = \alpha Y(t-)dt + \sigma Y(t-)d\mathcal{B}(t) + dL(t) - dM(t), \\ & \qquad \Delta L(s) = \Delta_b(s,x,y), \Delta M(s) = \Delta_s(s,x,y), \\ & \qquad \Delta L(t) = 0, \Delta M(t) = 0, \forall t \in (s,T), \end{aligned} \tag{2.4}$$

where the investor only takes continuous controls in (s, T).

Based on problem (2.4), we may consider shifting our target from the singular stochastic control problem to a standard stochastic control problem, which would be much easier to deal with analytically. The new admissible set is defined as

$$\mathcal{A}^c := \{(L, M) \in \mathcal{A} : \{L(t)\}_{t\in[0,T]}, \{M(t)\}_{t\in[0,T]} \text{ are continuous.}\}$$

Thus if we denote the new value function by ψ, the standard stochastic control problem can be described as follows:

$$\begin{aligned}\psi(s,x,y) \quad &:= \quad \sup_{(L,M)\in\mathcal{A}^c} \mathbb{E}\left[\log(w(X(T),Y(T)))|X(s)=x,Y(s)=y\right] \\ &\qquad \text{s.t.} \quad dX(t) = rX(t)dt - (1+\lambda)dL(t) + (1-\mu)dM(t), \\ &\qquad\qquad dY(t) = \alpha Y(t)dt + \sigma Y(t)d\mathcal{B}(t) + dL(t) - dM(t),\end{aligned} \tag{2.5}$$

for any $s \in [0,T]$, $(x,y) \in \mathbb{S}$. Here the continuity of the stochastic controls also leads to the continuity of the diffusion processes, hence it is safe to replace all the $t-$ with t on the right hand side of the diffusion equations.

Since we already know for problem (2.4) that lump-sum trading can only occur at the initial time, it is not difficult to figure out the relation between the value functions φ and ψ:

$$\varphi(s,x,y) = \sup_{\kappa\geq 0}\left\{\psi(s,x-(1+\lambda)\kappa,y+\kappa),\psi(s,x+(1-\mu)\kappa,y-\kappa)\right\}. \tag{2.6}$$

Therefore, once we solve the new value function ψ, the optimal "jump buy" and "jump sell" investment strategies would be explicit and the original value function φ can be obtained immediately.

2.2.2 Properties of the new value function

Similar to the proof aforementioned, we are able to show the following elementary properties for the new value function of the standard stochastic control problem:

1. $\psi(s,\cdot,\cdot)$ is strictly increasing with respect to the state arguments x and y.
2. Given any $s \in [0,T]$, $\psi(s,\cdot)$ is concave in $\mathbb{S}$.
3. Given any $s \in [0,T]$, $\psi(s,\cdot,\cdot)$ has the homotheticity property

$$\psi(s,\rho x,\rho y) = \psi(s,x,y) + \log\rho,$$

for any $(x,y) \in \mathbb{S}$ and $\rho > 0$.

4. Given any $s \in [0,T]$, $\psi(s,\cdot)$ is continuous in $\mathbb{S}$.

Moreover, the following property is now available for the new value function, which is very helpful for our further analysis in the next chapter.

Proposition 2.2.2. Given any $s \in [0,T)$, $(x,y) \in \mathbb{S}$ with $y < 0$, it holds that $\psi(s,x,y) \leq \psi(s,x+(1+\lambda)y,0)$.

Proof: Let $(X(\cdot),Y(\cdot))$ be the coupling diffusion processes of problem (2.5) under $(L,M) \in \mathcal{A}^c$ with initial states $(X(s),Y(s)) = (x,y)$, then the wealth process $W(t) := X(t)+(1+\lambda)Y(t)$ is apparently positive. We define a stopping time as

$$\tau := \inf\{t > s : Y(t) = 0\} \wedge T,$$

then the following inequality holds

$$\psi(s,x,y) = \sup_{(L,M)\in\mathcal{A}^c} \mathbb{E}[\psi(\tau,W(\tau),0)],$$

thanks to the principle of dynamic programming. Furthermore, given the evolutions of $(X(\cdot),Y(\cdot))$, it can be derived that $W(\cdot)$ satisfies the diffusion

$$dW(t) = W(t)\left[rdt + (1+\lambda)(\alpha - r)\tfrac{Y(t)}{W(t)}dt + (1+\lambda)\sigma\tfrac{Y(t)}{W(t)}d\mathcal{B}(t)\right] - (\lambda+\mu)dM(t),$$

and the initial condition $W(s) = x+(1+\lambda)y$. Since the coefficients are all adapted, we denote them by

$$\nu_1(t) := (1+\lambda)(\alpha - r)\tfrac{Y(t)}{W(t)}, \nu_2(t) := (1+\lambda)\sigma\tfrac{Y(t)}{W(t)},$$

both of which are non-positive on $[s,\tau]$. Now we study the SDE

$$\begin{cases} dU(t) &= U(t)\left[rdt + \nu_1(t)dt + \nu_2(t)d\mathcal{B}(t)\right], \\ U(s) &= x+(1+\lambda)y, \end{cases}$$

which naturally makes $U(t) \geq W(t)$ for all $t \in [s,T]$. Directly solving the SDE for $t > s$ gives us

$$\begin{aligned} U(t) &= (x+(1+\lambda)y)\cdot \mathbf{e}^{\int_s^t\left(r+\nu_1(u)-\frac{1}{2}\nu_2^2(u)\right)du+\int_s^t \nu_2(u)d\mathcal{B}(u)} \\ &< (x+(1+\lambda)y)\cdot \mathbf{e}^{r(t-s)}\cdot \mathbf{e}^{\int_s^t \nu_2(u)d\mathcal{B}(u)-\int_s^t \frac{1}{2}\nu_2^2(u)du}. \end{aligned}$$

Letting $\xi(t) := \mathbf{e}^{\int_s^t \nu_2(u)d\mathcal{B}(u) - \int_s^t \frac{1}{2}\nu_2^2(u)du}$, according to the monotonicity Property 1 stated above, we have

$$\begin{aligned}\psi(s,x,y) &= \sup_{(L,M)\in\mathcal{A}^c} \mathbb{E}[\psi(\tau, W(\tau), 0)] \\ &\le \sup_{\nu_1(\cdot),\nu_2(\cdot)\le 0} \mathbb{E}[\psi(\tau, U(\tau), 0)] \\ &\le \sup_{\nu_2(\cdot)\le 0} \mathbb{E}\left[\psi(\tau, (x+(1+\lambda)y)\mathbf{e}^{r(\tau-s)}\xi(\tau), 0)\right].\end{aligned}$$

Furthermore, since the status $(\tau, (x+(1+\lambda)y)\mathbf{e}^{r(\tau-s)}, 0)$ could be arrived from $(s, (x+(1+\lambda)y), 0)$ almost surely by taking null trading strategies during $[s,\tau]$, we must have

$$\psi(\tau, (x+(1+\lambda)y)\mathbf{e}^{r(\tau-s)}, 0) \le \psi(s, (x+(1+\lambda)y), 0).$$

Together with the homotheticity Property 3 as stated above, it consequently holds that

$$\psi(s,x,y) \le \psi(s, x+(1+\lambda)y, 0) + \sup_{\nu_2(\cdot)\le 0} \mathbb{E}\left[\log \xi(\tau)\right].$$

Since the utility function $\log(\cdot)$ is concave, applying Jensen's Inequality would lead to

$$\mathbb{E}\left[\log \xi(\tau)\right] \le \log\left(\mathbb{E}\left[\xi(\tau)\right]\right).$$

It is also worth noting that $\xi(t)$, as the stochastic exponential of a local martingale $\int_s^t \nu_2(u)d\mathcal{B}(u)$, is a local martingale for any $\nu_2(\cdot)$. Moreover, it is obvious $\xi(t) \ge 0$, thus it is a supermartingale and $\mathbb{E}\left[\xi(\tau)\right] \le \xi(s) = 1$. Therefore, combining all of these results, we obtain

$$\psi(s,x,y) \le \psi(s, x+(1+\lambda)y, 0).$$

These complete the proof. $\square$

2.2.3 Evolution behavior of the diffusion processes

In this subsection, we will illustrate the key evolution behavior of the coupling diffusion processes of the problem, as well as some characteristics of the two free boundaries, $\partial\mathrm{BR}_t$ and $\partial\mathrm{SR}_t$, for any $t \in [0, T)$. These characteristics will facilitate further simplification and investigation of the standard stochastic control problem.

Proposition 2.2.3. Given any $s \in [0, T)$, BR_s contains the region $\mathbb{S} \cap \{y < 0\}$.

Proof: According to the relation (2.6) between the value functions φ and ψ and the result obtained in Proposition 2.2.2, for any $(x, y) \in \mathbb{S} \cap \{y < 0\}$, we have

$$\begin{aligned} &\varphi(s,x,y) = \sup_{\kappa\geq 0}\{\psi(s,x-(1+\lambda)\kappa, y+\kappa), \psi(s,x+(1-\mu)\kappa, y-\kappa)\} \\ \leq\ & \max\left\{\sup_{\kappa\geq -y}\{\psi(s,x-(1+\lambda)\kappa,y+\kappa)\}, \sup_{\kappa\geq 0}\{\psi(s,x+(1+\lambda)y-(\lambda+\mu)\kappa,0)\}\right\} \\ \leq\ & \sup_{\kappa\geq 0}\{\psi(s,x+(1+\lambda)y-(1+\lambda)\kappa,\kappa)\} \\ \leq\ & \varphi(s,x+(1+\lambda)y,0). \end{aligned}$$

Since $(x + (1 + \lambda)y, 0)$ is always attainable from (x, y) via lump-sum buying, we must have $\varphi(s, x, y) = \varphi(s, x+(1+\lambda)y, 0)$, which immediately implies $(x, y) \in \mathrm{BR}_s$. Hence we conclude that $\mathbb{S} \cap \{y < 0\} \subset \mathrm{BR}_s$. □

In view of this result, we only need to study the standard stochastic control problem (2.5) with initial states $(x, y) \in \mathbb{S} \cap \{y \geq 0\}$. If for problem (2.5), there exist optimal controls (L^*, M^*) governing the diffusion processes $(X^*(\cdot), Y^*(\cdot))$ with initial states $(X^*(s), Y^*(s)) = (x, y)$, then Proposition 2.2.3 guarantees that $Y^*(\cdot) \geq 0$ almost surely in $[s, T)$. Therefore, we may focus on the following problem

$$\begin{aligned} \psi(s,x,y) \ &= \sup_{(L,M)\in\mathcal{A}^c} \mathbb{E}\left[\log(X(T)+(1-\mu)Y(T))|X(s)=x, Y(s)=y\geq 0\right] \\ &\quad \text{s.t.}\quad dX(t) = rX(t)dt-(1+\lambda)dL(t)+(1-\mu)dM(t), \\ &\qquad\quad\ dY(t) = \alpha Y(t)dt+\sigma Y(t)d\mathcal{B}(t)+dL(t)-dM(t), \\ &\qquad\quad\ Y(\cdot)\geq 0, \end{aligned} \tag{2.7}$$

for any $s \in [0,T)$, $(x,y) \in \mathbb{S} \cap \{y \geq 0\}$. As an immediate corollary to Proposition 2.2.3, $\overline{\mathrm{NT}_s}$ belongs to $\mathbb{S} \cap \{y \geq 0\}$.

Proposition 2.2.4. For any $s \in [0,T)$, $(x,y) \in \mathbb{S} \cap \{y > 0\}$, we have

$$\psi(s,x,y) \geq \psi(s,x+(1-\mu)y,0).$$

Proof: If $(x,y) \in \overline{\mathrm{NT}_s} \cap \{y > 0\}$, it is easily observed that

$$\psi(s,x,y) = \varphi(s,x,y) > \varphi(s,x+(1-\mu)y,0) \geq \psi(s,x+(1-\mu)y,0).$$

If $(x,y) \in \mathrm{BR}_s \cap \{y > 0\}$, then obviously $(x+(1-\mu)y,0) \in \mathrm{BR}_s$ as well, and there exists $\delta > 0$ such that $(x-(1-\mu)\delta, y+\delta) \in \overline{\mathrm{NT}_s} \cap \{y > 0\}$. Because we have

$$\psi(s,x-(1-\mu)\delta,y+\delta) > \psi(s,x+(1-\mu)y,0),$$

thus according to the concavity of $\psi(s,\cdot)$ in $\mathbb{S}$, it holds that

$$\psi(s,x,y) > \psi(s,x+(1-\mu)y,0).$$

If $(x,y) \in \mathrm{SR}_s \cap \{y > 0\}$, we have the following two cases. Firstly let us suppose $\overline{\mathrm{NT}_s} \cap \{y > 0\} \neq \emptyset$, then there exists $\delta > 0$ such that $(x+(1-\mu)\delta, y-\delta) \in \overline{\mathrm{NT}_s} \cap \{y > 0\}$. Utilizing the same reasoning using concavity, we arrive at

$$\psi(s,x,y) > \psi(s,x+(1-\mu)y,0).$$

Otherwise, $\overline{\mathrm{NT}_s} \cap \{y > 0\} = \emptyset$, then we must have $\mathrm{SR}_s = \mathbb{S} \cap \{y > 0\}$. Hence it holds that

$$\psi(s,x,y) = \varphi(s,x,y) = \varphi(s,x+(1-\mu)y,0) = \psi(s,x+(1-\mu)y,0).$$

These complete the proof. □

In order to facilitate our further analysis of the stochastic control problem, we consider a new stochastic control problem for any $s \in [0,T)$, $(x,y) \in \mathbb{S} \cap \{y > 0\}$

at the moment:

$$
\begin{aligned}
\phi(s,x,y) \quad := \quad & \sup_{(L,M)\in\mathcal{A}^c} \mathbb{E}\left[\log(X(T)+(1-\mu)Y(T))|X(s)=x,Y(s)=y>0\right] \\
& \text{s.t.} \quad dX(t) = rX(t)dt-(1+\lambda)dL(t)+(1-\mu)dM(t), \\
& \qquad\;\; dY(t) = \alpha Y(t)dt+\sigma Y(t)d\mathcal{B}(t)+dL(t)-dM(t), \\
& \qquad\;\; Y(\cdot)>0.
\end{aligned} \tag{2.8}
$$

Proposition 2.2.5. Given any $s\in[0,T)$, $(x,y)\in\mathbb{S}\cap\{y>0\}$, we must have $\psi(s,x,y)=\phi(s,x,y)$.

Proof: Let us consider any admissible controls (L,M) for problem (2.7), and denote by $(X(\cdot),Y(\cdot))$ the corresponding diffusion processes with initial states $(X(s),Y(s))=(x,y)$. For a series of numbers $n>0$, we introduce $\mathcal{F}_t$-stopping times

$$
\begin{aligned}
\tau &:= \inf\{t>s: Y(t)=0\}\wedge T, \\
\tau_n &:= \inf\{t>s: M(t)-M(s)\geq \tfrac{1}{n}\}\wedge\tau.
\end{aligned}
$$

Based on these stopping times, a series of controls (L_n,M_n) are chosen as follows:

$$
(dL_n(t),dM_n(t)) = \begin{cases} (dL(t),0), & t\in[s,\tau_n), \\ (dL(t),dM(t)), & t\in[\tau_n,T), \end{cases}
$$

and we denote the diffusion processes subject to (L_n,M_n) with the same initial states (x,y) at time s by $(X_n(\cdot),Y_n(\cdot))$. Then it is not difficult to verify the following inequalities:

$$
\begin{cases} Y_n(t)\geq Y(t)>0, & t\in[s,\tau_n), \\ Y_n(t)>Y(t)\geq 0, & t\in[\tau_n,T), \\ Y_n(T)\geq Y(T)\geq 0, Y_n(T)>0. \end{cases}
$$

Furthermore, we are able to obtain explicit expressions of $X(t)$ and $X_n(t)$ as follows:

$$
\begin{aligned}
X(t) &= x\cdot\mathbf{e}^{r(t-s)}-(1+\lambda)\int_s^t \mathbf{e}^{r(t-u)}dL(u)+(1-\mu)\int_s^t\mathbf{e}^{r(t-u)}dM(u), \\
X_n(t) &= x\cdot\mathbf{e}^{r(t-s)}-(1+\lambda)\int_s^t \mathbf{e}^{r(t-u)}dL_n(u)+(1-\mu)\int_s^t\mathbf{e}^{r(t-u)}dM_n(u),
\end{aligned}
$$

which leads to $X_n(t) \geq X(t) - (1-\mu)\mathbf{e}^{r(t-s)} \cdot \frac{1}{n}$ for any $t \in (s,T]$. In view of these inequalities, for any $\epsilon > 0$, there exists large $N > 0$ such that for any $n > N$, (L_n, M_n) are admissible for problem (2.8) and $X_n(T) \geq X(T) - \epsilon$. Therefore, taking supremum over all $(L,M) \in \mathcal{A}^c$ that are admissible for problem (2.7), we obtain $\psi(s,x,y) = \phi(s,x,y)$. These complete the proof. □

2.2.4 Dimensionality reduction

Previously, we have transformed the original singular stochastic control problem (2.2) into a standard stochastic control problem (2.5), and confined our study to problem (2.7). Moreover, as shown in Proposition 2.2.5, problem (2.7) has the same value function as problem (2.8) in $[0,T) \times \mathbb{S} \cap \{y > 0\}$. In fact, we expect, although unable to show in probabilistic approach within this thesis, that problem (2.7) is equivalent to problem (2.8) for $s \in [0,T)$, $(x,y) \in \mathbb{S} \cap \{y > 0\}$.

In view of the homotheticity property stated in Proposition 2.1.3, dimensionality of the value function could be reduced to cut down the number of arguments, which is similar to the dimensionality reduction discussed in [44], Chapter 8. This motivates us to reduce the dimensionality of the stochastic control problem, which is more fundamental, to obtain a problem associated only with one diffusion process. We will focus on studying problem (2.8) with initial state $(x,y) \in \mathrm{NT}_s$, where we know $\mathrm{NT}_s \subset \mathbb{S} \cap \{y > 0\}$ previously. It is worth mentioning that the two value functions $\varphi(s,\cdot,\cdot)$ and $\psi(s,\cdot,\cdot)$ coincide in NT_s.

Considering problem (2.8), we introduce $(\tilde{L}, \tilde{M})$ be such that

$$\begin{cases} d\tilde{L}(t,X(t),Y(t)) = \frac{1}{Y(t)} dL(t,X(t),Y(t)), \\ d\tilde{M}(t,X(t),Y(t)) = \frac{1}{Y(t)} dM(t,X(t),Y(t)). \end{cases}$$

The new controls $(\tilde{L}, \tilde{M})$ are still continuous, non-negative, non-decreasing, and

$\{\mathcal{F}_t\}_{t\geq 0}$-adapted, but rescaled according to the state in the vertical spatial direction, and we denoted by $\tilde{\mathcal{A}}^c$ the corresponding admissible set. Thus the governing SDE for $Y(\cdot)$ becomes

$$\begin{cases} dY(t) = Y(t)\cdot\left[\alpha dt + \sigma d\mathcal{B}(t) + d\tilde{L}(t) - d\tilde{M}(t)\right], \\ Y(s) = y, \end{cases}$$

where an explicit formula is available for any $t \in [s,T]$:

$$Y(t) = y\cdot \mathbf{e}^{(\alpha-\frac{1}{2}\sigma^2)(t-s)+\sigma(\mathcal{B}(t)-\mathcal{B}(s))+\left(\tilde{L}(t)-\tilde{L}(s)\right)-\left(\tilde{M}(t)-\tilde{M}(s)\right)}.$$

Furthermore, we introduce a new diffusion process as a quotient of the two original diffusion processes

$$Z(t) := \frac{X(t)}{Y(t)},$$

which naturally lies inside $(-(1-\mu),\infty)$. Applying Ito's formula we obtain the diffusion equation of $Z(\cdot)$:

$$dZ(t) = -Z(t)\left[(\alpha - r - \sigma^2)dt + \sigma d\mathcal{B}(t)\right] - (Z(t)+1+\lambda)d\tilde{L}(t) + (Z(t)+1-\mu)d\tilde{M}(t), \tag{2.9}$$

with initial condition $Z(s) = x/y$. It is obvious that $Z(\cdot)$ is still a continuous diffusion process. Problem (2.8) can then be restated in the following form:

$$\begin{aligned} \psi(s,x,y) \quad = \quad & \sup_{(\tilde{L},\tilde{M})\in\tilde{\mathcal{A}}^c} \mathbb{E}\left[\log(Z(T)+1-\mu) + \left(\tilde{L}(T)-\tilde{L}(s)\right) - \left(\tilde{M}(T)-\tilde{M}(s)\right)\Big|\, Z(s)=x/y\right] \\ & + \log y + (\alpha - \tfrac{1}{2}\sigma^2)(T-s) \\ & \text{s.t.(2.9).} \end{aligned}$$

We only focus on the optimized expectation part, where only the status of $Z(T)$ is involved. Taking z as x/y that lies in $(-(1-\mu),\infty)$, we define the value function V of the following problem coupled with only one diffusion process:

$$\begin{aligned} V(s,z) \quad := \quad & \sup_{(\tilde{L},\tilde{M})\in\tilde{\mathcal{A}}^c} \mathbb{E}\left[\log\left(Z(T)+1-\mu\right) + \left(\tilde{L}(T)-\tilde{L}(s)\right) - \left(\tilde{M}(T)-\tilde{M}(s)\right)\Big|\, Z(s)=z\right] \\ & \text{s.t.(2.9).} \end{aligned} \tag{2.10}$$

It is easy to observe the relation between the value functions of the two standard stochastic control problems:

$$\psi(s,x,y) = V(s,\frac{x}{y}) + \log y + (\alpha - \frac{1}{2}\sigma^2)(T-s).$$

It is worth pointing out that it seems difficult for us to obtain a similar simplified problem for the power utility function case, associated with another type of CRRA investor, via the same dimensionality reduction technique, although both the cases can be dealt with similarly via PDE approach. Our attempts lead to the following utility functional for a power utility function with parameter $\gamma < 1$, $\gamma \neq 0$:

$$\mathbb{E}\left[(Z(T)+1-\mu)^{\gamma}\cdot \mathbf{e}^{\gamma\left(\sigma(\mathcal{B}(T)-\mathcal{B}(s))+(\tilde{L}(T)-\tilde{L}(s))-(\tilde{M}(T)-\tilde{M}(s))\right)}\right]\cdot y^{\gamma}\cdot \mathbf{e}^{\gamma(\alpha-\frac{1}{2}\sigma^2)(T-s)},$$

which is not so convenient as the expression in logarithm utility case. Even if we condense the dynamics into a new stochastic process

$$\tilde{Z}(t) := (Z(t)+1-\mu)\cdot \mathbf{e}^{\sigma(\mathcal{B}(t)-\mathcal{B}(s))},$$

the stochastic differential equation of $\tilde{Z}(\cdot)$ would become more complex and our further investigation turns to be formidable. Interested researchers are encouraged to consider this power utility function case and attempt to apply analogous argument as in the next few sections. We reckon that similar connections between the optimal investment problem with proportional transaction costs for the power utility function case and a certain optimal stopping problem still exist.

After such dimensionality reduction is made, the three wedge-shaped transaction regions aforementioned become segments on the z-axis in $(-(1-\mu),\infty)$, and the moving free-boundaries become moving points. We define the corresponding one-dimensional free boundaries as follows:

$$z_s^*(t) := x/y, \qquad (x,y)\in\partial\mathrm{SR}_t,$$
$$z_b^*(t) := \begin{cases} x/y, & \text{if } y>0,\\ +\infty, & \text{if } y=0,\end{cases} \qquad (x,y)\in\partial\mathrm{BR}_t.$$

It is not difficult to observe the corresponding one-dimensional transaction regions

$$\begin{aligned} \mathrm{SR}'_t &= \{z \in \mathbb{R} : -(1-\mu) < z < z_s^*(t)\}, \\ \mathrm{BR}'_t &= \{z \in \mathbb{R} : z > z_b^*(t)\}, \\ \overline{\mathrm{NT}'_t} &= \{z \in \mathbb{R} : z_s^*(t) \le z \le z_b^*(t)\}, \\ \mathrm{NT}'_t &= \{z \in \mathbb{R} : z_s^*(t) < z < z_b^*(t)\}, \end{aligned}$$

respectively, and the optimal investment strategy is the same as in the two-dimension case. As we have emphasized before, we will focus on studying problem (2.10) with initial state $z \in \mathrm{NT}'_s$.

2.2.5 Evolution behavior of the new diffusion process

In order to facilitate our further investigation into the connections between the stochastic control problem and an optimal stopping problem, we confine our assumption to $\alpha - r - \sigma^2 < 0$ in the following and attempt to establish the evolution behavior of the new diffusion process $Z(\cdot)$ in problem (2.10). As a matter of fact, we met insurmountable obstacles in considering the case $\alpha - r - \sigma^2 \ge 0$ using similar probabilistic approach, where the key results for ensuring the positivity of $Z^*(\cdot)$ are not available and the analysis for the connection with optimal stopping cannot be carried on. Results obtained from PDE approach (see [10]) show that part of the region with negative z value belongs to NT in the case of $\alpha - r - \sigma^2 > 0$, while ∂SR coincides with the $z = 0$ radial in the case of $\alpha - r - \sigma^2 = 0$, thus the optimal diffusion process $Z^*(\cdot)$ would not necessarily stay positive in the other two cases.

Letting $U(t) := \log(Z(t) + 1 - \mu) + (\tilde{L}(t) - \tilde{L}(s)) - (\tilde{M}(t) - \tilde{M}(s))$, we may easily obtain the diffusion equation of $U(t)$ by applying Ito's Lemma. Then problem

(2.10) can be converted into the following form:

$$\begin{aligned} V(s,z) &= \sup_{(\tilde{L},\tilde{M})\in\tilde{\mathcal{A}}^c} \mathbb{E}\left[U(T)|U(s)=\log(z+1-\mu)\right] \\ & \quad \text{s.t.} \quad dU(t) = -\left[(\nu_1(t)+\tfrac{1}{2}\nu_2(t)^2)dt + \nu_2(t)d\mathcal{B}(t) + \nu_3(t)d\tilde{L}(t)\right], \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} \nu_1(t) &= (\alpha - r - \sigma^2)\tfrac{Z(t)}{Z(t)+1-\mu}, \\ \nu_2(t) &= \sigma\tfrac{Z(t)}{Z(t)+1-\mu}, \\ \nu_3(t) &= \tfrac{\lambda+\mu}{Z(t)+1-\mu}. \end{aligned}$$

One benefit of such transformation is that the utility functional can be expressed explicitly by $\nu_1(\cdot),\nu_2(\cdot),\nu_3(\cdot)$:

$$\mathbb{E}[U(T)] = \log(z+1-\mu) - \mathbb{E}\left[\int_s^T(\nu_1(t)+\tfrac{1}{2}\nu_2(t)^2)dt\right] - \mathbb{E}\left[\int_s^T \nu_3(t)d\tilde{L}(t)\right],$$

since the Itô integral process $\int_s^{\cdot}\nu_2(t)d\mathcal{B}(t)$ is a square integrable martingale given $\frac{X}{X+(1-\mu)Y}\in L^2_{\mathcal{F}}$.

Proposition 2.2.6. Given any $s\in[0,T)$, $z\in(-(1-\mu),0)$, there do not exist optimal controls $(\tilde{L}^*,\tilde{M}^*)$ governing the diffusion process $Z^*(\cdot)$ with initial state $Z^*(s)=z$ for problem (2.10) if $\alpha - r - \sigma^2 < 0$.

Proof: Suppose such optimal controls $(\tilde{L}^*,\tilde{M}^*)$ exist, we introduce a pair of auxiliary controls $(\tilde{L}_1,\tilde{M}_1)$ as such satisfying

$$(d\tilde{L}_1(t), d\tilde{M}_1(t)) = (d\tilde{L}^*(t), d\tilde{M}^*(t)+dt),$$

for $t\in[s,T)$, and denote by $Z_1(\cdot)$ the corresponding diffusion process with the same initial state $Z_1(s)=z$. It is worth pointing out that $Z_1(t)>Z^*(t)$ almost surely for $t\in(s,T)$. Then two stopping times are defined as follows

$$\begin{aligned} \tau_1 &:= \inf\{t>s : Z_1(t)=0\}\wedge T, \\ \tau_2 &:= \inf\{t>s : Z^*(t)=0\}\wedge T, \end{aligned}$$

which represent the first hitting times of zero. It is also worth mentioning that $\mathbb{P}[s < \tau_1 \leq \tau_2] = 1$. Thus we choose another pair of controls $(\tilde{L}, \tilde{M})$ as

$$(d\tilde{L}(t), d\tilde{M}(t)) = \begin{cases} (d\tilde{L}^*(t), d\tilde{M}^*(t) + dt), & t \in [s, \tau_1), \\ (0, 0), & t \in [\tau_1, \tau_2), \\ (d\tilde{L}^*(t), d\tilde{M}^*(t)), & t \in [\tau_2, T), \end{cases}$$

and we denote by $Z(\cdot)$ the corresponding diffusion process subject to $(\tilde{L}, \tilde{M})$ with the same initial state $Z(s) = z$. Such choice of controls induces

$$\begin{cases} Z^*(t) < Z(t) \leq 0, & t \in (s, \tau_2), \\ Z^*(t) = Z(t), & t \in [\tau_2, T), \end{cases}$$

almost surely. If we use $\nu_1^*(\cdot), \nu_2^*(\cdot), \nu_3^*(\cdot)$ to denote the terms corresponding to $Z^*(\cdot)$, and use $\nu_1(\cdot), \nu_2(\cdot), \nu_3(\cdot)$ to denote the terms corresponding to $Z(\cdot)$, in problem (2.11), respectively, the following relations

$$\begin{cases} \nu_1^*(t) > \nu_1(t) \geq 0, & t \in (s, \tau_2), \\ \nu_1^*(t) = \nu_1(t), & t \in [\tau_2, T), \end{cases}$$
$$\begin{cases} \nu_2^*(t) < \nu_2(t) \leq 0, & t \in (s, \tau_2), \\ \nu_2^*(t) = \nu_2(t), & t \in [\tau_2, T), \end{cases}$$
$$\begin{cases} \nu_3^*(t) > \nu_3(t) > 0, & t \in (s, \tau_2), \\ \nu_3^*(t) = \nu_3(t), & t \in [\tau_2, T), \end{cases}$$

would hold almost surely. These relations directly imply for problem (2.11) that $\mathbb{E}[U(T)] > \mathbb{E}[U^*(T)]$, clearly a contradiction. Therefore, such optimal controls do not exist, and we complete the proof. □

In our point of view, in the case of $\alpha - r - \sigma^2 < 0$, the region $\{z < 0\}$ belongs to SR'_s for any $s \in [0, T)$, which can be inferred from the argument in the proof for Proposition 2.2.6 in the sense that more aggressive selling strategy in this region always produces better outcome. The non-existence of optimal controls for problem (2.10) in this region is due to the fact that the original optimal investment strategy for problem (2.4) is lump-sum selling which is not allowed in the new standard stochastic control problem. Quasi-lump-sum investment strategy will not constitute an optimal choice for the standard stochastic control problem, thus the lack of the non-singular optimal controls in SR'_s and BR'_s. It is also this reason that makes us focus on studying problem (2.10) with initial state $z \in \mathrm{NT}'_s$ in the following analysis.

In view of the results obtained above, let us now consider a new stochastic control problem for any $s \in [0, T)$, $z \geq 0$:

$$\begin{aligned} V_1(s,z) \quad := \quad & \sup_{(\tilde{L},\tilde{M}) \in \tilde{\mathcal{A}}^c} \mathbb{E}\left[\log\left(Z(T)+1-\mu\right)+\left(\tilde{L}(T)-\tilde{L}(s)\right)-\left(\tilde{M}(T)-\tilde{M}(s)\right)\Big|\, Z(s)=z\geq 0\right] \\ & \text{s.t.}(2.9), Z(\cdot) \geq 0. \end{aligned} \tag{2.12}$$

Proposition 2.2.7. Given any $s \in [0, T)$, $z \geq 0$, we must have $V(s, z) = V_1(s, z)$ if $\alpha - r - \sigma^2 < 0$.

Proof: Let us consider any admissible controls $(\tilde{L}, \tilde{M})$ for problem (2.10), and denote by $Z(\cdot)$ the corresponding diffusion process with initial states $Z(s) = z$. We introduce the following $\mathcal{F}_t$-stopping times

$$\begin{aligned} \tau &:= \inf\{t \geq s : Z(t) < 0\} \wedge T, \\ \tau_1 &:= \inf\{t > \tau : Z(t) \geq 0\} \wedge T. \end{aligned}$$

Based on such stopping times, we choose controls $(\tilde{L}_1, \tilde{M}_1)$ as

$$(d\tilde{L}_1(t), d\tilde{M}_1(t)) = \begin{cases} (d\tilde{L}(t), d\tilde{M}(t)), & t \in [s, \tau), \\ (0, 0), & t \in [\tau, \tau_1), \\ (d\tilde{L}(t), d\tilde{M}(t)), & t \in [\tau_1, T), \end{cases}$$

and we denote the diffusion processes subject to $(\tilde{L}_1, \tilde{M}_1)$ with the same initial states z at time s by $Z_1(\cdot)$. Such choice of controls induces

$$\begin{cases} Z(t) = Z_1(t) \geq 0, & t \in (s, \tau), \\ Z(t) < Z_1(t) = 0, & t \in [\tau, \tau_1), \\ Z(t) = Z_1(t), & t \in [\tau_1, T), \end{cases}$$

almost surely. If we use $\nu_1'(\cdot), \nu_2'(\cdot), \nu_3'(\cdot)$ to denote the terms corresponding to $Z_1(\cdot)$, and use $\nu_1(\cdot), \nu_2(\cdot), \nu_3(\cdot)$ to denote the terms corresponding to $Z(\cdot)$, in problem (2.11), respectively, the following relations

$$\begin{cases} \nu_1(t) > \nu_1'(t) = 0, & t \in (\tau, \tau_1), \\ \nu_1(t) = \nu_1'(t), & \text{otherwise}, \end{cases}$$
$$\begin{cases} \nu_2(t) < \nu_2'(t) = 0, & t \in (\tau, \tau_1), \\ \nu_2(t) = \nu_2'(t), & \text{otherwise}, \end{cases}$$
$$\begin{cases} \nu_3^*(t) > \nu_3(t) > 0, & t \in (\tau, \tau_1), \\ \nu_3(t) = \nu_3'(t), & \text{otherwise}, \end{cases}$$

would hold almost surely. These relations directly imply for problem (2.11) that $\mathbb{E}[U'(T)] > \mathbb{E}[U(T)]$ when $\mathbb{P}[\tau_1 > \tau] > 0$. Thus if $Z(\cdot)$ with non-negative initial state goes negative during some period in the horizon, its governing controls $(\tilde{L}, \tilde{M})$ are always suboptimal. Therefore, we may conclude that $V(s,z) = V_1(s,z)$. These complete the proof. □

As a further step, let us consider another stochastic control problem for any $s \in [0, T)$, $z > 0$:

$$\begin{aligned} V_2(s,z) \quad := \quad & \sup_{(\tilde{L},\tilde{M}) \in \tilde{\mathcal{A}}^c} \mathbb{E}\left[\log\left(Z(T) + 1 - \mu\right) + \left(\tilde{L}(T) - \tilde{L}(s)\right) - \left(\tilde{M}(T) - \tilde{M}(s)\right) \Big| Z(s) = z > 0\right] \\ & \text{s.t.}(2.9), Z(\cdot) > 0. \end{aligned} \tag{2.13}$$

Proposition 2.2.8. Given any $s \in [0, T)$, $z > 0$, we must have $V(s,z) = V_2(s,z)$ if $\alpha - r - \sigma^2 < 0$.

Proof: Let us consider any admissible controls $(\tilde{L}, \tilde{M})$ for problem (2.12), and denote by $Z(\cdot)$ the corresponding diffusion process with initial states $Z(s) = z$.

For a series of numbers $n > 0$, we introduce $\mathcal{F}_t$-stopping times

$$\begin{aligned} \tau &:= \inf\{t > s : Z(t) = 0\} \wedge T, \\ \tau_n &:= \inf\{t > s : \tilde{L}(t) - \tilde{L}(s) \geq \tfrac{1}{n}\} \wedge \tau. \end{aligned}$$

Based on these stopping times, a series of controls $(\tilde{L}_n, \tilde{M}_n)$ are chosen as follows:

$$(d\tilde{L}_n(t), d\tilde{M}_n(t)) = \begin{cases} (0, d\tilde{M}(t)), & t \in [s, \tau_n), \\ (d\tilde{L}(t), d\tilde{M}(t)), & t \in [\tau_n, T), \end{cases}$$

and we denote the diffusion processes subject to $(d\tilde{L}_n, d\tilde{M}_n)$ with the same initial states z at time s by $Z_n(\cdot)$. Then it is not difficult to verify the following inequalities:

$$\begin{cases} Z_n(t) \geq Z(t) > 0, & t \in [s, \tau_n), \\ Z_n(t) > Z(t) \geq 0, & t \in [\tau_n, T), \\ Z_n(T) \geq Z(T) \geq 0, Z_n(T) > 0, \end{cases}$$

which imply that $(\tilde{L}_n, \tilde{M}_n)$ are admissible for problem (2.13). Furthermore, according to the construction of $(\tilde{L}_n, \tilde{M}_n)$, we have

$$\begin{aligned} \tilde{L}_n(T) - \tilde{L}_n(s) &\geq \tilde{L}(T) - \tilde{L}(s) - \tfrac{1}{n}, \\ \tilde{M}_n(T) - \tilde{M}_n(s) &= \tilde{M}(T) - \tilde{M}(s). \end{aligned}$$

Therefore, taking supremum over all $(\tilde{L}, \tilde{M}) \in \tilde{\mathcal{A}}^c$ that are admissible for problem (2.12) and over all $(\tilde{L}_n, \tilde{M}_n)$ as constructed above for problem (2.13) corresponding to every pair of $(\tilde{L}, \tilde{M})$, we obtain $V_2(s, x, y) = V_1(s, x, y)$. In view of Proposition 2.2.7, these complete the proof. □

It is worth noting that the three stochastic control problems (2.10), (2.12), (2.13) are expected to be equivalent to each within $[0, T) \times (0, \infty)$, although we don't include in this thesis the rigorous proofs. Interested researchers may use the equalities between the value functions we obtained in this thesis and attempt to reveal the equivalence of the optimal stochastic controls. It is also worth mentioning that we conjecture, although we cannot guarantee the optimal diffusion process

$Z^*(\cdot)$ stay positive across the whole horizon with arbitrary parameter choice, similar simplification of the problem would still be possible for the cases $\alpha - r - \sigma^2 = 0$ and $\alpha - r - \sigma^2 > 0$. According to the results obtained in [10], the optimal diffusion process $Z^*(\cdot)$ with initial endowment $Z^*(s) = z \in \mathrm{NT}'_s$ is expected to stay positive if $z > 0$, stay zero if $z = 0$ and stay negative if $z < 0$, thus the results on the connections with an optimal stopping problem shown in the next section may still be obtained. We also encourage interested researchers to investigate these cases in the future.

2.3 Connections with optimal stopping

In this section, we will reveal the connections between the stochastic control problem (2.13) and a certain optimal stopping problem.

Since the diffusion process $Z(\cdot)$ has been confined to the positive region, we may introduce $(\hat{L}, \hat{M})$ being such that

$$\left(d\hat{L}(t, Z(t)), d\hat{M}(t, Z(t))\right) = \left(\tfrac{Z(t)+1+\lambda}{Z(t)} d\tilde{L}(t, Z(t)), \tfrac{Z(t)+1-\mu}{Z(t)} d\tilde{M}(t, Z(t))\right),$$

which are still continuous, non-negative, non-decreasing, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, and we denote by $\hat{\mathcal{A}}^c$ the corresponding admissible set. Clearly, with such new admissible set, problem (2.13) is equivalent to the following stochastic control problem for any $s \in [0, T)$, $z > 0$:

$$\begin{aligned} V(s,z) \quad &= \max_{(\hat{L},\hat{M})\in\hat{\mathcal{A}}^c} \mathbb{E}\left[\log\left(Z(T)+1-\mu\right) + \int_s^T \tfrac{Z(t)}{Z(t)+1+\lambda} d\hat{L}(t) - \int_s^T \tfrac{Z(t)}{Z(t)+1-\mu} d\hat{M}(t) \Big| Z(s) = z > 0\right] \\ &\text{s.t.} \quad dZ(t) = -Z(t)\left[(\alpha - r - \sigma^2)dt + \sigma d\mathcal{B}(t) + d\hat{L}(t) - d\hat{M}(t)\right]. \end{aligned} \tag{2.14}$$

It is worth mentioning that if for problem (2.14), there exist optimal controls $(\hat{L}^*, \hat{M}^*)$ governing the diffusion process $Z^*(\cdot)$ with initial state $Z^*(s) = z$, then $Z^*(t)$ has an explicit expression

$$Z^*(t) = z \cdot \mathbf{e}^{-(\alpha - r - \frac{1}{2}\sigma^2)(t-s) - \sigma(\mathcal{B}(t) - \mathcal{B}(s)) - \int_s^t d\hat{L}^*(u) + \int_s^t d\hat{M}^*(u)},$$

for any $t \in [s, T)$. For simplicity reasons, all the expectations taken in this section are conditioned on $Z(s) = z$.

Proposition 2.3.1. For problem (2.14), if there exist optimal controls $(\hat{L}^*, \hat{M}^*)$ governing the diffusion process $Z^*(\cdot)$ with initial state $Z^*(s) = z > 0$, then we have the following inequality:

$$\liminf_{\delta \to 0+} \tfrac{V(s,z+\delta)-V(s,z)}{\delta} \geq \inf_{s \leq \tau_\mu \leq T} \sup_{s \leq \tau_\lambda \leq T} \mathbb{E}\left[\tfrac{\frac{1}{z} Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu} \cdot \mathbf{1}_{\{\tau_\mu \leq \tau_\lambda\}} + \tfrac{\frac{1}{z} Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda} \cdot \mathbf{1}_{\{\tau_\lambda < \tau_\mu\}} \right].$$

Proof: Given the optimal controls $(\hat{L}^*, \hat{M}^*)$, it holds for the value function of problem (2.14) that

$$V(s, z) = \mathbb{E}\left[\log\left(Z^*(T) + 1 - \mu\right) + \int_s^T \tfrac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t) - \int_s^T \tfrac{Z^*(t)}{Z^*(t)+1-\mu} d\hat{M}^*(t) \right].$$

For any $\delta > 0$, we define the following $\mathcal{F}_t$-stopping times

$$\tau_\mu^\delta := \inf\left\{ t > s : \int_s^t d\hat{M}^*(u) = \log\left(\tfrac{z+\delta}{z}\right) \right\} \wedge T,$$
$$\tau_\mu^* := \inf\left\{ t > s : \int_s^t d\hat{M}^*(u) > 0 \right\} \wedge T.$$

Obviously $\tau_\mu^\delta \downarrow \tau_\mu^*$ almost surely, as $\delta \downarrow 0$. Now for any $\mathcal{F}_t$-stopping time τ_λ with $\mathbb{P}[s \leq \tau_\lambda \leq T] = 1$, we define

$$\tau_{\min}^\delta := \min\{\tau_\mu^\delta, \tau_\lambda\}, \tau_{\min} := \min\{\tau_\mu^*, \tau_\lambda\},$$

both of which are also $\mathcal{F}_t$-stopping times, and $\tau_{\min}^\delta \downarrow \tau_{\min}$ almost surely, as $\delta \downarrow 0$.

The diffusion process starting from $(s, z + \delta)$ is denoted by $Z^\delta(\cdot)$, while its controls are denoted by $(\hat{L}^\delta, \hat{M}^\delta)$. We choose controls as $d\hat{L}^\delta(t) = d\hat{L}^*(t)$, $d\hat{M}^\delta(t) = 0$ for all $t \in [s, \tau_{\min}^\delta]$, which will induce

$$Z^*(\tau_{\min}^\delta) \leq Z^\delta(\tau_{\min}^\delta) \leq \frac{z+\delta}{z} Z^*(\tau_{\min}^\delta).$$

The choice of controls after time $\tau_{\min}^\delta$ depends on the event $\{\tau_{\min}^\delta = \tau_\mu^\delta\}$ that belongs to $\mathcal{F}_{\tau_{\min}^\delta}$:

(I) If $\tau^\delta_{\min} = \tau^\delta_\mu$, which implies $\tau^\delta_\mu \le \tau_\lambda$. In such a case, we choose controls as $d\hat{L}^\delta(t) = d\hat{L}^*(t)$, $d\hat{M}^\delta(t) = d\hat{M}^*(t)$ for all $t \in (\tau^\delta_\mu, T]$, which would ensure

$$Z^*(T) \le Z^\delta(T) \le \frac{z+\delta}{z} Z^*(T).$$

(II) If $\tau^\delta_{\min} \neq \tau^\delta_\mu$, which implies $\tau_\lambda < \tau^\delta_\mu$. In such a case, we consider an auxiliary process $W_n(\cdot)$ as follows

$$\begin{cases} dW_n(t) = -W_n(t) \cdot [(\alpha - r - \sigma^2)dt + \sigma d\mathcal{B}(t) + dL_n(t)], \\ W_n(s) = z + \delta, \end{cases}$$

with control

$$dL_n(t) = \begin{cases} d\hat{L}^*(t), & t \in [s, \tau_\lambda], \\ d\hat{L}^*(t) + n \cdot dt, & t \in (\tau_\lambda, T]. \end{cases}$$

An auxiliary $\mathcal{F}_t$-stopping time σ_n is defined as

$$\sigma_n := \inf\{t > \tau_\lambda : W_n(t) = Z^*(t)\} \wedge T.$$

Clearly $\sigma_n \downarrow \tau_\lambda$ almost surely as $n \uparrow \infty$. Given $\tau_\lambda < T$, this convergence indicates that there exists $N > 0$ such that for any $n \ge N$, we have $\sigma_n < T$ almost surely. Using such $\mathcal{F}_t$-stopping time with $n \ge N$, we choose controls in $(\tau_\lambda, T]$ as

$$d\hat{L}^\delta_n(t) = \begin{cases} dL_n(t), & \forall t \in (\tau_\lambda, \sigma_n], \\ d\hat{L}^*(t), & \forall t \in (\sigma_n, T], \end{cases} \quad d\hat{M}^\delta_n(t) = \begin{cases} 0, & \forall t \in (\tau_\lambda, \sigma_n], \\ d\hat{M}^*(t), & \forall t \in (\sigma_n, T]. \end{cases}$$

Under such choice of controls, the following relation holds

$$Z^\delta(\sigma_n) = \frac{z+\delta}{z} Z^*(\sigma_n) \mathrm{e}^{-\int_s^{\sigma_n} d\hat{M}^*(u)} \mathrm{e}^{-\int_{\tau_\lambda}^{\sigma_n} n dt} = Z^*(\sigma_n), a.s.,$$

which induces $n(\sigma_n - \tau_\lambda) = \log\left(\frac{z+\delta}{z}\right) - \int_s^{\sigma_n} d\hat{M}^*(u)$ almost surely.

Thus, under such controls $(\hat{L}_n^\delta, \hat{M}_n^\delta)$ with $n \geq N$, we have

$$\begin{aligned} V(s, z+\delta) \ \geq \ & \lim_{n\to\infty} \mathbb{E}\Big[\log(Z^*(T)+1-\mu)\cdot \mathbf{1}_{\{\tau_{\min}^\delta<T\}} \\ & + \log\left(\tfrac{z+\delta}{z} Z^*(T)\mathbf{e}^{-\int_s^T d\hat{M}^*(u)}+1-\mu\right)\cdot \mathbf{1}_{\{\tau_{\min}^\delta=T\}} \\ & + \int_s^{\tau_{\min}^\delta} \tfrac{\frac{z+\delta}{z} Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}}{\frac{z+\delta}{z} Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}+1+\lambda} d\hat{L}^*(t) \\ & + \left(\int_{\tau_\mu^\delta}^T \tfrac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t) - \int_{\tau_\mu^\delta}^T \tfrac{Z^*(t)}{Z^*(t)+1-\mu} d\hat{M}^*(t)\right)\cdot \mathbf{1}_{\{\tau_\mu^\delta\leq\tau_\lambda\}} \\ & + \left(\int_{\tau_\lambda}^{\sigma_n} \tfrac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t) + n\int_{\tau_\lambda}^{\sigma_n} \tfrac{Z^*(t)}{Z^*(t)+1+\lambda} dt\right)\cdot \mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}} \\ & + \left(\int_{\sigma_n}^T \tfrac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t) - \int_{\sigma_n}^T \tfrac{Z^*(t)}{Z^*(t)+1-\mu} d\hat{M}^*(t)\right)\cdot \mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}}\Big]. \end{aligned}$$

Notice the term $\mathbb{E}\left[n\int_{\tau_\lambda}^{\sigma_n} \frac{Z^*(t)}{Z^*(t)+1+\lambda} dt \cdot \mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}}\right]$ records the result of drawing the process $Z^\delta(\cdot)$ to $Z^*(\cdot)$ in a short time. To simplify the limit of this term, we notice for any $n \geq N$ that

$$\left|n\int_{\tau_\lambda}^{\sigma_n} \tfrac{Z^*(t)}{Z^*(t)+1+\lambda} dt \cdot \mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}}\right| \leq n(\sigma_n-\tau_\lambda) \leq \log\left(\tfrac{z+\delta}{z}\right).$$

Thus we apply Bounded Convergence Theorem to obtain

$$\begin{aligned} & \lim_{n\to\infty} \mathbb{E}\left[n\int_{\tau_\lambda}^{\sigma_n} \tfrac{Z^*(t)}{Z^*(t)+1+\lambda} dt \cdot \mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}}\right] \\ = \ & \mathbb{E}\left[\lim_{n\to\infty} n(\sigma_n-\tau_\lambda)\tfrac{Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot \mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}}\right] \\ = \ & \mathbb{E}\left[\left(\log\left(\tfrac{z+\delta}{z}\right) - \int_s^{\tau_\lambda} d\hat{M}^*(u)\right)\tfrac{Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot \mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}}\right]. \end{aligned}$$

Combining the above inequality with the expression of $V(s,z)$, we have

$$\begin{aligned} \liminf_{\delta\to 0+} \tfrac{V(s,z+\delta)-V(s,z)}{\delta} \ \geq \ & \lim_{\delta\to 0+} \tfrac{1}{\delta}\cdot \mathbb{E}\Bigg[\log\left(\tfrac{\frac{z+\delta}{z} Z^*(T)\mathbf{e}^{-\int_s^T d\hat{M}^*(u)}+1-\mu}{Z^*(T)+1-\mu}\right)\cdot \mathbf{1}_{\{\tau_{\min}^\delta=T\}} \\ & + \int_s^{\tau_{\min}^\delta}\left(\tfrac{\frac{z+\delta}{z} Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}}{\frac{z+\delta}{z} Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}+1+\lambda} - \tfrac{Z^*(t)}{Z^*(t)+1+\lambda}\right) d\hat{L}^*(t) \\ & + \int_s^{\tau_{\min}^\delta} \tfrac{Z^*(t)}{Z^*(t)+1-\mu} d\hat{M}^*(t) \\ & + \left(\log\left(\tfrac{z+\delta}{z}\right) - \int_s^{\tau_\lambda} d\hat{M}^*(u)\right)\tfrac{Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot \mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}}\Bigg]. \end{aligned} \quad (*)$$

For the first term on the RHS of the inequality $(*)$, we notice that

$$\begin{aligned}
& \frac{1}{\delta}\cdot\log\left(\frac{\frac{z+\delta}{z}Z^*(T)\mathbf{e}^{-\int_s^T d\hat{M}^*(u)}+1-\mu}{Z^*(T)+1-\mu}\right)\cdot\mathbf{1}_{\{\tau^\delta_{\min}=T\}} \\
= \; & \frac{1}{\delta}\cdot\log\left(\frac{\frac{z+\delta}{z}Z^*(T)\mathbf{e}^{-\int_s^T d\hat{M}^*(u)}+1-\mu}{Z^*(T)+1-\mu}\right)\cdot\mathbf{1}_{\left\{\int_s^T d\hat{M}^*(u)\le\log\left(\frac{z+\delta}{z}\right)\right\}} \\
\ge \; & \frac{1}{\delta}\cdot\log\left(\frac{\frac{z+\delta}{z}Z^*(T)\mathbf{e}^{-\log\left(\frac{z+\delta}{z}\right)}+1-\mu}{Z^*(T)+1-\mu}\right)\cdot\mathbf{1}_{\left\{\int_s^T d\hat{M}^*(u)\le\log\left(\frac{z+\delta}{z}\right)\right\}}=0,
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{\delta\to0+}\frac{1}{\delta}\cdot\log\left(\frac{\frac{z+\delta}{z}Z^*(T)\mathbf{e}^{-\int_s^T d\hat{M}^*(u)}+1-\mu}{Z^*(T)+1-\mu}\right)\cdot\mathbf{1}_{\{\tau^\delta_{\min}=T\}} \\
\le \; & \lim_{\delta\to0+}\frac{1}{\delta}\cdot\log\left(1+\frac{\frac{\delta}{z}Z^*(T)}{Z^*(T)+1-\mu}\right)=\frac{\frac{1}{z}Z^*(T)}{Z^*(T)+1-\mu}\le\frac{1}{z}.
\end{aligned}$$

Thus we apply Bounded Convergence Theorem to obtain

$$\begin{aligned}
& \lim_{\delta\to0+}\mathbb{E}\left[\frac{1}{\delta}\cdot\log\left(\frac{\frac{z+\delta}{z}Z^*(T)\mathbf{e}^{-\int_s^T d\hat{M}^*(u)}+1-\mu}{Z^*(T)+1-\mu}\right)\cdot\mathbf{1}_{\{\tau^\delta_{\min}=T\}}\right] \\
= \; & \mathbb{E}\left[\lim_{\delta\to0+}\frac{1}{\delta}\cdot\log\left(\frac{\frac{z+\delta}{z}Z^*(T)\mathbf{e}^{-\int_s^T d\hat{M}^*(u)}+1-\mu}{Z^*(T)+1-\mu}\right)\cdot\mathbf{1}_{\{\tau^\delta_{\min}=T\}}\right] \\
\ge \; & \mathbb{E}\left[\lim_{\delta\to0+}\frac{1}{\delta}\cdot\log\left(1+\frac{\frac{\delta}{z}Z^*(T)}{Z^*(T)+1-\mu}\right)\cdot\mathbf{1}_{\{\tau_{\min}=T\}}\right] \\
= \; & \mathbb{E}\left[\frac{\frac{1}{z}Z^*(T)}{Z^*(T)+1-\mu}\cdot\mathbf{1}_{\{\tau_{\min}=T\}}\right],
\end{aligned}$$

where the inequality is due to the fact $\{\tau^\delta_{\min}=T\}\supset\{\tau_{\min}=T\}$.

For the second term on the RHS of the inequality $(*)$, we notice that

$$\begin{aligned}
& \frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\left(\frac{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}}{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}+1+\lambda}-\frac{Z^*(t)}{Z^*(t)+1+\lambda}\right)d\hat{L}^*(t) \\
\ge \; & \frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\left(\frac{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\log\left(\frac{z+\delta}{z}\right)}}{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\log\left(\frac{z+\delta}{z}\right)}+1+\lambda}-\frac{Z^*(t)}{Z^*(t)+1+\lambda}\right)d\hat{L}^*(t)=0,
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{\delta\to0+}\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\left(\frac{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}}{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}+1+\lambda}-\frac{Z^*(t)}{Z^*(t)+1+\lambda}\right)d\hat{L}^*(t) \\
\le \; & \lim_{\delta\to0+}\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\left(\frac{\frac{z+\delta}{z}Z^*(t)}{\frac{z+\delta}{z}Z^*(t)+1+\lambda}-\frac{Z^*(t)}{Z^*(t)+1+\lambda}\right)d\hat{L}^*(t) \\
= \; & \int_s^{\tau_{\min}}(1+\lambda)\frac{\frac{1}{z}Z^*(t)}{(Z^*(t)+1+\lambda)^2}d\hat{L}^*(t)\le\frac{1}{2z}\int_s^{\tau_{\min}}d\hat{L}^*(t),
\end{aligned}$$

where $\int_s^{\tau_{\min}}d\hat{L}^*(t)$ is obviously integrable. Thus we apply Dominated Convergence

Theorem to obtain

$$\begin{aligned}
&\lim_{\delta\to 0+}\mathbb{E}\left[\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\left(\frac{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}}{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}+1+\lambda}-\frac{Z^*(t)}{Z^*(t)+1+\lambda}\right)d\hat{L}^*(t)\right]\\
=\ &\mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\left(\frac{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}}{\frac{z+\delta}{z}Z^*(t)\mathbf{e}^{-\int_s^t d\hat{M}^*(u)}+1+\lambda}-\frac{Z^*(t)}{Z^*(t)+1+\lambda}\right)d\hat{L}^*(t)\right]\\
\geq\ &\mathbb{E}\left[\int_s^{\tau_{\min}}\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\left(\frac{\frac{z+\delta}{z}Z^*(t)}{\frac{z+\delta}{z}Z^*(t)+1+\lambda}-\frac{Z^*(t)}{Z^*(t)+1+\lambda}\right)d\hat{L}^*(t)\right]\\
=\ &\mathbb{E}\left[\int_s^{\tau_{\min}}(1+\lambda)\frac{\frac{1}{z}Z^*(t)}{(Z^*(t)+1+\lambda)^2}d\hat{L}^*(t)\right],
\end{aligned}$$

where the inequality is due to the fact $\tau^\delta_{\min}\geq\tau^*_{\min}$.

For the third term on the RHS of the inequality $(*)$, we notice that

$$\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\frac{Z^*(t)}{Z^*(t)+1-\mu}d\hat{M}^*(t)\geq 0,$$

and

$$\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\frac{Z^*(t)}{Z^*(t)+1-\mu}d\hat{M}^*(t)\leq\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\log\left(\frac{z+\delta}{z}\right)=\frac{1}{z}.$$

Thus we apply Bounded Convergence Theorem to obtain

$$\begin{aligned}
&\lim_{\delta\to 0+}\mathbb{E}\left[\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\frac{Z^*(t)}{Z^*(t)+1-\mu}d\hat{M}^*(t)\right]\\
=\ &\mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\frac{Z^*(t)}{Z^*(t)+1-\mu}d\hat{M}^*(t)\right]\\
=\ &\mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\int_{\tau^*_\mu}^{\tau^\delta_\mu}\frac{Z^*(t)}{Z^*(t)+1-\mu}d\hat{M}^*(t)\cdot\mathbf{1}_{\{\tau^*_\mu<T,\tau^\delta_\mu\leq\tau_\lambda\}}\right]\\
\geq\ &\mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\int_{\tau^*_\mu}^{\tau^\delta_\mu}\frac{Z^*(t)}{Z^*(t)+1-\mu}d\hat{M}^*(t)\cdot\mathbf{1}_{\{\tau^\delta_\mu<T,\tau^\delta_\mu\leq\tau_\lambda\}}\right]\\
=\ &\mathbb{E}\left[\frac{Z^*(\tau^*_\mu)}{Z^*(\tau^*_\mu)+1-\mu}\cdot\lim_{\delta\to 0+}\left(\frac{1}{\delta}\cdot\log\left(\frac{z+\delta}{z}\right)\right)\cdot\mathbf{1}_{\{\tau^\delta_\mu<T,\tau^\delta_\mu\leq\tau_\lambda\}}\right]\\
=\ &\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau^*_\mu)}{Z^*(\tau^*_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau^*_\mu<T,\tau^*_\mu\leq\tau_\lambda\}}\right],
\end{aligned}$$

where the inequality is due to the fact $\{\tau^*_\mu<T\}\supset\{\tau^\delta_\mu<T\}$.

For the fourth term on the RHS of the inequality $(*)$, we notice that

$$\frac{1}{\delta}\cdot\left(\log\left(\frac{z+\delta}{z}\right)-\int_s^{\tau_\lambda}d\hat{M}^*(u)\right)\frac{Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau^\delta_\mu\}}\geq 0,$$

and

$$\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\left(\log\left(\frac{z+\delta}{z}\right)-\int_s^{\tau_\lambda}d\hat{M}^*(u)\right)\frac{Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau^\delta_\mu\}}\leq\frac{1}{z}.$$

Thus we apply Bounded Convergence Theorem to obtain

$$\begin{aligned}
&\lim_{\delta\to 0+}\mathbb{E}\left[\frac{1}{\delta}\cdot\left(\log\left(\frac{z+\delta}{z}\right)-\int_s^{\tau_\lambda}d\hat{M}^*(u)\right)\frac{Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}}\right]\\
=\ &\mathbb{E}\left[\frac{Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\left(\log\left(\frac{z+\delta}{z}\right)-\int_s^{\tau_\lambda}d\hat{M}^*(u)\right)\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu^\delta\}}\right]\\
\geq\ &\mathbb{E}\left[\frac{Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\log\left(\frac{z+\delta}{z}\right)\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu^*\}}\right]\\
=\ &\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu^*\}}\right],
\end{aligned}$$

where the inequality is due to the fact $\{\tau_\lambda<\tau_\mu^\delta\}\supset\{\tau_\lambda<\tau_\mu^*\}$.

Therefore, it can be deduced as follows

$$\begin{aligned}
\liminf_{\delta\to 0+}\frac{V(s,z+\delta)-V(s,z)}{\delta}\ \geq\ &\mathbb{E}\Bigg[\frac{\frac{1}{z}Z^*(T)}{Z^*(T)+1-\mu}\cdot\mathbf{1}_{\{\tau_{\min}=T\}}\\
&+\int_s^{\tau_{\min}}(1+\lambda)\frac{\frac{1}{z}Z^*(t)}{(Z^*(t)+1+\lambda)^2}d\hat{L}^*(t)\\
&+\frac{\frac{1}{z}Z^*(\tau_\mu^*)}{Z^*(\tau_\mu^*)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu^*<T,\tau_\mu^*\leq\tau_\lambda\}}\\
&+\frac{\frac{1}{z}Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu^*\}}\Bigg]\\
\geq\ &\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu^*)}{Z^*(\tau_\mu^*)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu^*\leq\tau_\lambda\}}+\frac{\frac{1}{z}Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu^*\}}\right].
\end{aligned}$$

Hence, we may conclude that

$$\liminf_{\delta\to 0+}\frac{V(s,z+\delta)-V(s,z)}{\delta}\ \geq\ \inf_{s\leq\tau_\mu\leq T}\ \sup_{s\leq\tau_\lambda\leq T}\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda\}}+\frac{\frac{1}{z}Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu\}}\right].\ \square$$

Proposition 2.3.2. For problem (2.14), if there exist optimal controls $(\hat{L}^*,\hat{M}^*)$ governing the diffusion process $Z^*(\cdot)$ with initial state $Z^*(s)=z>0$, then we have the following inequality:

$$\limsup_{\delta\to 0+}\frac{V(s,z)-V(s,z-\delta)}{\delta}\leq\inf_{s\leq\tau_\mu\leq T}\ \sup_{s\leq\tau_\lambda\leq T}\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda\}}+\frac{\frac{1}{z}Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu\}}\right].$$

Proof: Similarly, given the optimal controls $(\hat{L}^*,\hat{M}^*)$, it holds for the value function of problem (2.14) that

$$V(s,z)=\mathbb{E}\left[\log\left(Z^*(T)+1-\mu\right)+\int_s^T\frac{Z^*(t)}{Z^*(t)+1+\lambda}d\hat{L}^*(t)-\int_s^T\frac{Z^*(t)}{Z^*(t)+1-\mu}d\hat{M}^*(t)\right].$$

For any $\delta > 0$, we define the following $\mathcal{F}_t$-stopping times

$$\tau_\lambda^\delta := \inf\left\{t > s : \int_s^t d\hat{L}^*(u) = \log\left(\tfrac{z}{z-\delta}\right)\right\} \wedge T,$$
$$\tau_\lambda^* := \inf\left\{t > s : \int_s^t d\hat{L}^*(u) > 0\right\} \wedge T.$$

Obviously $\tau_\lambda^\delta \downarrow \tau_\lambda^*$ almost surely, as $\delta \downarrow 0$. Now for any $\mathcal{F}_t$-stopping time τ_μ with $\mathbb{P}[s \le \tau_\mu \le T] = 1$, we define

$$\tau_{\min}^\delta := \min\{\tau_\lambda^\delta, \tau_\mu\}, \tau_{\min} := \min\{\tau_\lambda^*, \tau_\mu\},$$

both of which are also $\mathcal{F}_t$-stopping times, and $\tau_{\min}^\delta \downarrow \tau_{\min}$ almost surely, as $\delta \downarrow 0$.

The diffusion process starting from $(s, z - \delta)$ is denoted by $Z^\delta(\cdot)$, while its controls are denoted by $(\hat{L}^\delta, \hat{M}^\delta)$. We choose controls as $d\hat{L}^\delta(t) = 0$, $d\hat{M}^\delta(t) = d\hat{M}^*(t)$ for all $t \in [s, \tau_{\min}^\delta]$, which will induce

$$\frac{z-\delta}{z} Z^*(\tau_{\min}^\delta) \le Z^\delta(\tau_{\min}^\delta) \le Z^*(\tau_{\min}^\delta).$$

The choice of controls after time $\tau_{\min}^\delta$ depends on the event $\{\tau_{\min}^\delta = \tau_\lambda^\delta\}$ that belongs to $\mathcal{F}_{\tau_{\min}^\delta}$:

(I) If $\tau_{\min}^\delta = \tau_\lambda^\delta$, which implies $\tau_\lambda^\delta \le \tau_\mu$. In such a case, we choose controls as $d\hat{L}^\delta(t) = d\hat{L}^*(t)$, $d\hat{M}^\delta(t) = d\hat{M}^*(t)$ for all $t \in (\tau_\lambda^\delta, T]$, which would ensure

$$\frac{z-\delta}{z} Z^*(T) \le Z^\delta(T) \le Z^*(T).$$

(II) If $\tau_{\min}^\delta \neq \tau_\lambda^\delta$, which implies $\tau_\mu < \tau_\lambda^\delta$. In such a case, we consider an auxiliary process $W_n(\cdot)$ as follows

$$\begin{cases} dW_n(t) = -W_n(t) \cdot [(\alpha - r - \sigma^2)dt + \sigma d\mathcal{B}(t) - dM_n(t)], \\ W_n(s) = z - \delta, \end{cases}$$

with control

$$dM_n(t) = \begin{cases} d\hat{M}^*(t), & t \in [s, \tau_\mu], \\ d\hat{M}^*(t) + n \cdot dt, & t \in (\tau_\mu, T]. \end{cases}$$

An auxiliary $\mathcal{F}_t$-stopping time σ_n is defined as

$$\sigma_n := \inf\{t > \tau_\lambda : W_n(t) = Z^*(t)\} \wedge T.$$

Clearly $\sigma_n \downarrow \tau_\mu$ almost surely as $n \uparrow \infty$. Given $\tau_\mu < T$, this convergence indicates that there exists $N > 0$ such that for any $n \geq N$, we have $\sigma_n < T$ almost surely. Using such $\mathcal{F}_t$-stopping time with $n \geq N$, we choose controls in $(\tau_\mu, T]$ as

$$d\hat{L}^\delta_n(t) = \begin{cases} 0, & \forall t \in (\tau_\mu, \sigma_n], \\ d\hat{L}^*(t), & \forall t \in (\sigma_n, T], \end{cases} \quad d\hat{M}^\delta_n(t) = \begin{cases} dM_n(t), & \forall t \in (\tau_\mu, \sigma_n], \\ d\hat{M}^*(t), & \forall t \in (\sigma_n, T]. \end{cases}$$

Under such choice of controls, the following relation holds

$$Z^\delta(\sigma_n) = \frac{z-\delta}{z} Z^*(\sigma_n) \mathbf{e}^{\int_s^{\sigma_n} d\hat{L}^*(u)} \mathbf{e}^{\int_{\tau_\mu}^{\sigma_n} n dt} = Z^*(\sigma_n), a.s.,$$

which induces $n(\sigma_n - \tau_\mu) = \log\left(\frac{z}{z-\delta}\right) - \int_s^{\sigma_n} d\hat{L}^*(u)$ almost surely.

Thus, under such controls $(\hat{L}^\delta_n, \hat{M}^\delta_n)$ with $n \geq N$, we have

$$\begin{aligned} V(s, z-\delta) \ \geq \ & \lim_{n\to\infty} \mathbb{E}\Big[\log(Z^*(T) + 1 - \mu) \cdot \mathbf{1}_{\{\tau^\delta_{\min} < T\}} \\ & + \log\left(\tfrac{z-\delta}{z} Z^*(T) \mathbf{e}^{\int_s^T d\hat{L}^*(u)} + 1 - \mu\right) \cdot \mathbf{1}_{\{\tau^\delta_{\min} = T\}} \\ & - \int_s^{\tau^\delta_{\min}} \frac{\frac{z-\delta}{z} Z^*(t) \mathbf{e}^{\int_s^t d\hat{L}^*(u)}}{\frac{z-\delta}{z} Z^*(t) \mathbf{e}^{\int_s^t d\hat{L}^*(u)} + 1 - \mu} d\hat{M}^*(t) \\ & + \left(\int_{\tau^\delta_\lambda}^T \tfrac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t) - \int_{\tau^\delta_\lambda}^T \tfrac{Z^*(t)}{Z^*(t)+1-\mu} d\hat{M}^*(t)\right) \cdot \mathbf{1}_{\{\tau^\delta_\lambda \leq \tau_\mu\}} \\ & - \left(\int_{\tau_\mu}^{\sigma_n} \tfrac{Z^*(t)}{Z^*(t)+1-\mu} d\hat{M}^*(t) + n \int_{\tau_\mu}^{\sigma_n} \tfrac{Z^*(t)}{Z^*(t)+1-\mu} dt\right) \cdot \mathbf{1}_{\{\tau_\mu < \tau^\delta_\lambda\}} \\ & + \left(\int_{\sigma_n}^T \tfrac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t) - \int_{\sigma_n}^T \tfrac{Z^*(t)}{Z^*(t)+1-\mu} d\hat{M}^*(t)\right) \cdot \mathbf{1}_{\{\tau_\mu < \tau^\delta_\lambda\}}\Big]. \end{aligned}$$

Notice the term $\mathbb{E}\left[n \int_{\tau_\mu}^{\sigma_n} \frac{Z^*(t)}{Z^*(t)+1-\mu} dt \cdot \mathbf{1}_{\{\tau_\mu < \tau^\delta_\lambda\}}\right]$ records the result of drawing the process $Z^\delta(\cdot)$ to $Z^*(\cdot)$ in a short time. To simplify the limit of this term, we notice for any $n \geq N$ that

$$\left| n \int_{\tau_\mu}^{\sigma_n} \tfrac{Z^*(t)}{Z^*(t)+1-\mu} dt \cdot \mathbf{1}_{\{\tau_\mu < \tau^\delta_\lambda\}} \right| \leq n(\sigma_n - \tau_\mu) \leq \log\left(\tfrac{z}{z-\delta}\right).$$

Thus we apply Bounded Convergence Theorem to obtain

$$\begin{aligned} & \lim_{n\to\infty} \mathbb{E}\left[n \int_{\tau_\mu}^{\sigma_n} \tfrac{Z^*(t)}{Z^*(t)+1-\mu} dt \cdot \mathbf{1}_{\{\tau_\mu < \tau^\delta_\lambda\}}\right] \\ = \ & \mathbb{E}\left[\lim_{n\to\infty} n(\sigma_n - \tau_\mu) \tfrac{Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu} \cdot \mathbf{1}_{\{\tau_\mu < \tau^\delta_\lambda\}}\right] \\ = \ & \mathbb{E}\left[\left(\log\left(\tfrac{z}{z-\delta}\right) - \int_s^{\tau_\mu} d\hat{L}^*(u)\right) \tfrac{Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu} \cdot \mathbf{1}_{\{\tau_\mu < \tau^\delta_\lambda\}}\right]. \end{aligned}$$

Combining the above inequality with the expression of $V(s,z)$, we have

$$\begin{aligned}\limsup_{\delta\to 0+}\frac{V(s,z)-V(s,z-\delta)}{\delta} \leq\ & \lim_{\delta\to 0+}\frac{1}{\delta}\cdot\mathbb{E}\Bigg[\log\left(\frac{Z^*(T)+1-\mu}{\frac{z-\delta}{z}Z^*(T)\mathbf{e}^{\int_s^T d\hat{L}^*(u)}+1-\mu}\right)\cdot\mathbf{1}_{\{\tau^\delta_{\min}=T\}}\\ & -\int_s^{\tau^\delta_{\min}}\left(\frac{Z^*(t)}{Z^*(t)+1-\mu}-\frac{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}}{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}+1-\mu}\right)d\hat{M}^*(t)\\ & +\int_s^{\tau^\delta_{\min}}\frac{Z^*(t)}{Z^*(t)+1+\lambda}d\hat{L}^*(t)\\ & +\left(\log\left(\frac{z}{z-\delta}\right)-\int_s^{\tau_\mu}d\hat{L}^*(u)\right)\frac{Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu<\tau^\delta_\lambda\}}\Bigg].\end{aligned} \quad (**)$$

For the first term on the RHS of the inequality $(**)$, we notice that

$$\begin{aligned}& \frac{1}{\delta}\cdot\log\left(\frac{Z^*(T)+1-\mu}{\frac{z-\delta}{z}Z^*(T)\mathbf{e}^{\int_s^T d\hat{L}^*(u)}+1-\mu}\right)\cdot\mathbf{1}_{\{\tau^\delta_{\min}=T\}}\\ =\ & \frac{1}{\delta}\cdot\log\left(\frac{Z^*(T)+1-\mu}{\frac{z-\delta}{z}Z^*(T)\mathbf{e}^{\int_s^T d\hat{L}^*(u)}+1-\mu}\right)\cdot\mathbf{1}_{\left\{\int_s^T d\hat{L}^*(u)\leq\log\left(\frac{z}{z-\delta}\right)\right\}}\\ \geq\ & \frac{1}{\delta}\cdot\log\left(\frac{Z^*(T)+1-\mu}{\frac{z-\delta}{z}Z^*(T)\mathbf{e}^{\log\left(\frac{z}{z-\delta}\right)}+1-\mu}\right)\cdot\mathbf{1}_{\left\{\int_s^T d\hat{L}^*(u)\leq\log\left(\frac{z}{z-\delta}\right)\right\}}=0,\end{aligned}$$

and

$$\begin{aligned}& \lim_{\delta\to 0+}\frac{1}{\delta}\cdot\log\left(\frac{Z^*(T)+1-\mu}{\frac{z-\delta}{z}Z^*(T)\mathbf{e}^{\int_s^T d\hat{L}^*(u)}+1-\mu}\right)\cdot\mathbf{1}_{\{\tau^\delta_{\min}=T\}}\\ \leq\ & \lim_{\delta\to 0+}\frac{1}{\delta}\cdot\log\left(1+\frac{\frac{\delta}{z}Z^*(T)}{\frac{z-\delta}{z}Z^*(T)+1-\mu}\right)=\frac{\frac{1}{z}Z^*(T)}{Z^*(T)+1-\mu}\leq\frac{1}{z}.\end{aligned}$$

Thus we apply Bounded Convergence Theorem to obtain

$$\begin{aligned}& \lim_{\delta\to 0+}\mathbb{E}\left[\frac{1}{\delta}\cdot\log\left(\frac{Z^*(T)+1-\mu}{\frac{z-\delta}{z}Z^*(T)\mathbf{e}^{\int_s^T d\hat{L}^*(u)}+1-\mu}\right)\cdot\mathbf{1}_{\{\tau^\delta_{\min}=T\}}\right]\\ =\ & \mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\log\left(\frac{Z^*(T)+1-\mu}{\frac{z-\delta}{z}Z^*(T)\mathbf{e}^{\int_s^T d\hat{L}^*(u)}+1-\mu}\right)\cdot\mathbf{1}_{\{\tau^\delta_{\min}=T\}}\right]\\ =\ & \mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\log\left(1+\frac{\frac{\delta}{z}Z^*(T)}{\frac{z-\delta}{z}Z^*(T)+1-\mu}\right)\cdot\left(\mathbf{1}_{\{\tau_{\min}=T\}}+\mathbf{1}_{\{\tau_{\min}<\tau^\delta_{\min}=T\}}\right)\right]\\ =\ & \mathbb{E}\left[\frac{\frac{1}{z}Z^*(T)}{Z^*(T)+1-\mu}\cdot\mathbf{1}_{\{\tau_{\min}=T\}}\right],\end{aligned}$$

where the last equality is due to the fact $\lim_{\delta\to 0+}\mathbb{P}\left[\tau_{\min}<\tau^\delta_{\min}=T\right]=0$.

For the second term on the RHS of the inequality $(**)$, we notice that

$$\begin{aligned}& \frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\left(\frac{Z^*(t)}{Z^*(t)+1-\mu}-\frac{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}}{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}+1-\mu}\right)d\hat{M}^*(t)\\ \geq\ & \frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}}\left(\frac{Z^*(t)}{Z^*(t)+1-\mu}-\frac{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\log\left(\frac{z}{z-\delta}\right)}}{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\log\left(\frac{z}{z-\delta}\right)}+1-\mu}\right)d\hat{M}^*(t)=0,\end{aligned}$$

and

$$\begin{aligned} & \lim_{\delta\to 0+} \frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}} \left(\frac{Z^*(t)}{Z^*(t)+1-\mu} - \frac{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}}{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}+1-\mu}\right) d\hat{M}^*(t) \\ \le \; & \lim_{\delta\to 0+} \frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}} \left(\frac{Z^*(t)}{Z^*(t)+1-\mu} - \frac{\frac{z-\delta}{z}Z^*(t)}{\frac{z-\delta}{z}Z^*(t)+1-\mu}\right) d\hat{M}^*(t) \\ = \; & \int_s^{\tau_{\min}} (1-\mu)\frac{\frac{1}{z}Z^*(t)}{(Z^*(t)+1-\mu)^2} d\hat{M}^*(t) \le \frac{1}{2z}\int_s^{\tau_{\min}} d\hat{M}^*(t), \end{aligned}$$

where $\int_s^{\tau_{\min}} d\hat{M}^*(t)$ is obviously integrable. Thus we apply Dominated Convergence Theorem to obtain

$$\begin{aligned} & -\lim_{\delta\to 0+} \mathbb{E}\left[\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}} \left(\frac{Z^*(t)}{Z^*(t)+1-\mu} - \frac{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}}{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}+1-\mu}\right) d\hat{M}^*(t)\right] \\ = \; & -\mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}} \left(\frac{Z^*(t)}{Z^*(t)+1-\mu} - \frac{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}}{\frac{z-\delta}{z}Z^*(t)\mathbf{e}^{\int_s^t d\hat{L}^*(u)}+1-\mu}\right) d\hat{M}^*(t)\right] \\ \le \; & -\mathbb{E}\left[\int_s^{\tau_{\min}} \lim_{\delta\to 0+}\frac{1}{\delta}\cdot \left(\frac{Z^*(t)}{Z^*(t)+1-\mu} - \frac{\frac{z-\delta}{z}Z^*(t)}{\frac{z-\delta}{z}Z^*(t)+1-\mu}\right) d\hat{M}^*(t)\right] \\ = \; & -\mathbb{E}\left[\int_s^{\tau_{\min}} (1-\mu)\frac{\frac{1}{z}Z^*(t)}{(Z^*(t)+1-\mu)^2} d\hat{M}^*(t)\right], \end{aligned}$$

where the inequality is due to the fact $\tau^\delta_{\min} \ge \tau^*_{\min}$.

For the third term on the RHS of the inequality $(**)$, we notice that

$$\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}} \frac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t) \ge 0,$$

and

$$\lim_{\delta\to 0+} \frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}} \frac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t) \le \lim_{\delta\to 0+}\frac{1}{\delta}\cdot\log\left(\frac{z}{z-\delta}\right) = \frac{1}{z}.$$

Thus we apply Bounded Convergence Theorem to obtain

$$\begin{aligned} & \lim_{\delta\to 0+} \mathbb{E}\left[\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}} \frac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t)\right] \\ = \; & \mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\int_s^{\tau^\delta_{\min}} \frac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t)\right] \\ = \; & \mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\int_{\tau^*_\lambda}^{\tau^\delta_\lambda} \frac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t)\cdot \mathbf{1}_{\{\tau^*_\lambda<T,\tau^\delta_\lambda\le\tau_\mu\}}\right] \\ = \; & \mathbb{E}\left[\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\int_{\tau^*_\lambda}^{\tau^\delta_\lambda} \frac{Z^*(t)}{Z^*(t)+1+\lambda} d\hat{L}^*(t)\cdot \left(\mathbf{1}_{\{\tau^\delta_\lambda<T,\tau^\delta_\lambda\le\tau_\mu\}} + \mathbf{1}_{\{\tau^*_\lambda<\tau^\delta_\lambda=T,\tau^\delta_\lambda\le\tau_\mu\}}\right)\right] \\ = \; & \mathbb{E}\left[\frac{Z^*(\tau^*_\lambda)}{Z^*(\tau^*_\lambda)+1+\lambda}\cdot\lim_{\delta\to 0+}\left(\frac{1}{\delta}\cdot\log\left(\frac{z}{z-\delta}\right)\right)\cdot\mathbf{1}_{\{\tau^\delta_\lambda<T,\tau^\delta_\lambda\le\tau_\mu\}}\right] \\ = \; & \mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau^*_\lambda)}{Z^*(\tau^*_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau^*_\lambda<T,\tau^*_\lambda\le\tau_\mu\}}\right], \end{aligned}$$

where the fourth equality is due to the fact $\lim\limits_{\delta\to 0+}\mathbb{P}\left[\tau_\lambda^* < \tau_\lambda^\delta = T\right] = 0$.

For the fourth term on the RHS of the inequality $(**)$, we notice that

$$\frac{1}{\delta}\cdot\left(\log\left(\frac{z}{z-\delta}\right) - \int_s^{\tau_\mu} d\hat{L}^*(u)\right)\frac{Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu<\tau_\lambda^\delta\}} \geq 0,$$

and

$$\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\left(\log\left(\frac{z}{z-\delta}\right) - \int_s^{\tau_\mu} d\hat{L}^*(u)\right)\frac{Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu<\tau_\lambda^\delta\}} \leq \frac{1}{z}.$$

Thus we apply Bounded Convergence Theorem to obtain

$$\begin{aligned}
&\lim_{\delta\to 0+}\mathbb{E}\left[\frac{1}{\delta}\cdot\left(\log\left(\frac{z}{z-\delta}\right) - \int_s^{\tau_\mu} d\hat{L}^*(u)\right)\frac{Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu<\tau_\lambda^\delta\}}\right]\\
= \ &\mathbb{E}\left[\frac{Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\lim_{\delta\to 0+}\frac{1}{\delta}\cdot\left(\log\left(\frac{z}{z-\delta}\right) - \int_s^{\tau_\mu} d\hat{L}^*(u)\right)\cdot\left(\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda^*\}} + \mathbf{1}_{\{\tau_\lambda^*<\tau_\mu<\tau_\lambda^\delta\}}\right)\right]\\
= \ &\mathbb{E}\left[\frac{Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\lim_{\delta\to 0+}\left(\frac{1}{\delta}\cdot\log\left(\frac{z}{z-\delta}\right)\right)\cdot\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda^*\}}\right]\\
= \ &\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda^*\}}\right],
\end{aligned}$$

where the second equality is due to the fact $\lim\limits_{\delta\to 0+}\mathbb{P}\left[\tau_\lambda^* < \tau_\mu < \tau_\lambda^\delta\right] = 0$.

Therefore, it can be deduced as follows

$$\begin{aligned}
\limsup_{\delta\to 0+}\frac{V(s,z)-V(s,z-\delta)}{\delta} \ \leq \ &\mathbb{E}\Bigg[\frac{\frac{1}{z}Z^*(T)}{Z^*(T)+1-\mu}\cdot\mathbf{1}_{\{\tau_{\min}=T\}}\\
&-\int_s^{\tau_{\min}}(1-\mu)\frac{\frac{1}{z}Z^*(t)}{(Z^*(t)+1-\mu)^2}d\hat{M}^*(t)\\
&+\frac{\frac{1}{z}Z^*(\tau_\lambda^*)}{Z^*(\tau_\lambda^*)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda^*<T,\tau_\lambda^*\leq\tau_\mu\}}\\
&+\frac{\frac{1}{z}Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda^*\}}\Bigg]\\
\leq \ &\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda^*\}} + \frac{\frac{1}{z}Z^*(\tau_\lambda^*)}{Z^*(\tau_\lambda^*)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda^*<\tau_\mu\}}\right].
\end{aligned}$$

Hence, we may conclude that

$$\limsup_{\delta\to 0+}\frac{V(s,z)-V(s,z-\delta)}{\delta} \ \leq \ \inf_{s\leq\tau_\mu\leq T}\sup_{s\leq\tau_\lambda\leq T}\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda\}} + \frac{\frac{1}{z}Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu\}}\right]. \ \square$$

Proposition 2.3.3. For any $s\in[0,T)$, $V(s,\cdot)$ is concave in $(-(1-\mu),\infty)$.

Proof: For any $z_1, z_2\in(-(1-\mu),\infty)$, for any $\eta\in(0,1)$, it is easy to observe $(\eta z_1+(1-\eta)z_2)\in(-(1-\mu),\infty)$. Given the relation between V and ψ, and the concavity of $\psi(s,\cdot)$ stated in Proposition 2.2.3, we have

$$\begin{aligned} V(s,\eta z_1+(1-\eta)z_2) &= \psi(s,\eta z_1+(1-\eta)z_2,1)-(\alpha-\tfrac{1}{2}\sigma^2)(T-s) \\ &\geq \eta\psi(s,z_1,1)+(1-\eta)\psi(s,z_2,1)-(\alpha-\tfrac{1}{2}\sigma^2)(T-s) \\ &= \eta V(s,z_1)+(1-\eta)V(s,z_2). \end{aligned}$$

Hence we conclude $V(s,\cdot)$ is concave in $(-(1-\mu),\infty)$. $\square$

Theorem 2.3.4. For problem (2.14), if there exist optimal controls $(\hat{L}^*,\hat{M}^*)$ governing the diffusion process $Z^*(\cdot)$ with initial state $Z^*(s)=z>0$, then $\frac{\partial V}{\partial z}(s,z)$ exists, and we have

$$\begin{aligned} \frac{\partial V}{\partial z}(s,z) &= \inf_{s\leq\tau_\mu\leq T}\sup_{s\leq\tau_\lambda\leq T}\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda\}}+\frac{\frac{1}{z}Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu\}}\right] \\ &= \mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu^*)}{Z^*(\tau_\mu^*)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu^*\leq\tau_\lambda^*\}}+\frac{\frac{1}{z}Z^*(\tau_\lambda^*)}{Z^*(\tau_\lambda^*)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda^*<\tau_\mu^*\}}\right], \end{aligned}$$

where

$$\begin{aligned} \tau_\lambda^* &:= \inf\left\{t>s:\int_s^t d\hat{L}^*(u)>0\right\}\wedge T, \\ \tau_\mu^* &:= \inf\left\{t>s:\int_s^t d\hat{M}^*(u)>0\right\}\wedge T. \end{aligned}$$

Proof: We denote $J_0(s,z;\tau_\lambda,\tau_\mu)$ as follows

$$J_0(s,z;\tau_\lambda,\tau_\mu):=\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\leq\tau_\lambda\}}+\frac{\frac{1}{z}Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu\}}\right].$$

According to the results obtained in Proposition 2.3.1 and Proposition 2.3.2, it holds that

$$\begin{aligned} \liminf_{\delta\to0+}\frac{V(s,z)-V(s,z-\delta)}{\delta} &\leq \limsup_{\delta\to0+}\frac{V(s,z)-V(s,z-\delta)}{\delta}\leq\inf_{s\leq\tau_\mu\leq T}J_0(s,z;\tau_\lambda^*,\tau_\mu) \\ &\leq \inf_{s\leq\tau_\mu\leq T}\sup_{s\leq\tau_\lambda\leq T}J_0(s,z;\tau_\lambda,\tau_\mu)\leq\sup_{s\leq\tau_\lambda\leq T}J_0(s,z;\tau_\lambda,\tau_\mu^*) \\ &\leq \liminf_{\delta\to0+}\frac{V(s,z+\delta)-V(s,z)}{\delta}\leq\limsup_{\delta\to0+}\frac{V(s,z+\delta)-V(s,z)}{\delta}. \end{aligned}$$

Moreover, for any positive numbers δ_1,δ_2, we have from concavity of $V(s,\cdot)$ obtained in Proposition 2.3.3 that

$$V(s,z)\geq\frac{\delta_2}{\delta_1+\delta_2}V(s,z+\delta_1)+\frac{\delta_1}{\delta_1+\delta_2}V(s,z-\delta_2).$$

This inequality would lead to

$$\frac{V(s,z+\delta_1)-V(s,z)}{\delta_1} \le \frac{V(s,z)-V(s,z-\delta_2)}{\delta_2},$$

whence

$$\limsup_{\delta\to 0+} \frac{V(s,z+\delta)-V(s,z)}{\delta} \le \liminf_{\delta\to 0+} \frac{V(s,z)-V(s,z-\delta)}{\delta}.$$

Therefore, we obtain the existence of the gradient of $V(s,\cdot)$ and

$$\frac{\partial V}{\partial z}(s,z) = \inf_{s\le\tau_\mu\le T} \sup_{s\le\tau_\lambda\le T} J_0(s,z;\tau_\lambda,\tau_\mu).$$

In addition, since

$$\inf_{s\le\tau_\mu\le T} J_0(s,z;\tau_\lambda^*,\tau_\mu) \le J_0(s,z;\tau_\lambda^*,\tau_\mu^*) \le \sup_{s\le\tau_\lambda\le T} J_0(s,z;\tau_\lambda,\tau_\mu^*),$$

we also have

$$\frac{\partial V}{\partial z}(s,z) = J_0(s,z;\tau_\lambda^*,\tau_\mu^*).\Box$$

Theorem 2.3.5. Let $u(s,z)$ be the optimal risk of the following optimal stopping problem

$$u(s,z) := \inf_{s\le\tau_\mu\le T} \sup_{s\le\tau_\lambda\le T} \mathbb{E}\left[\frac{\frac{1}{z}\bar{Z}(\tau_\mu)}{\bar{Z}(\tau_\mu)+1-\mu}\cdot \mathbf{1}_{\{\tau_\mu\le\tau_\lambda\}} + \frac{\frac{1}{z}\bar{Z}(\tau_\lambda)}{\bar{Z}(\tau_\lambda)+1+\lambda}\cdot \mathbf{1}_{\{\tau_\lambda<\tau_\mu\}}\right], \qquad (2.15)$$

where $\bar{Z}(t) := z\cdot \mathbf{e}^{-(\alpha-r-\frac{1}{2}\sigma^2)(t-s)-\sigma(\mathcal{B}(t)-\mathcal{B}(s))}$. For problem (2.14), if there exist optimal controls $(\hat{L}^*,\hat{M}^*)$ governing the diffusion process $Z^*(\cdot)$ with initial state $Z^*(s)=z$, then we have

$$\frac{\partial V}{\partial z}(s,z) = u(s,z),$$

and the stopping times

$$\tau_\lambda^* := \inf\left\{t>s : \int_s^t d\hat{L}^*(u)>0\right\}\wedge T,$$
$$\tau_\mu^* := \inf\left\{t>s : \int_s^t d\hat{M}^*(u)>0\right\}\wedge T,$$

are optimal for the stopping problem (2.15).

Proof: On the one hand, as we have shown before,

$$\frac{\partial V}{\partial z}(s,z) = \sup_{s\le\tau_\lambda\le T}\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau^*_\mu)}{Z^*(\tau^*_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau^*_\mu\le\tau_\lambda\}} + \frac{\frac{1}{z}Z^*(\tau_\lambda)}{Z^*(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau^*_\mu\}}\right]$$

$$= \sup_{s\le\tau_\lambda\le T}\mathbb{E}\left[\frac{\frac{1}{z}\bar{Z}(\tau^*_\mu)\cdot\exp\left\{-\int_s^{\tau^*_\mu}d\hat{L}^*(t)\right\}}{\bar{Z}(\tau^*_\mu)\cdot\exp\left\{-\int_s^{\tau^*_\mu}d\hat{L}^*(t)\right\}+1-\mu}\cdot\mathbf{1}_{\{\tau^*_\mu\le\tau_\lambda\}} + \frac{\frac{1}{z}\bar{Z}(\tau_\lambda)\cdot\exp\left\{-\int_s^{\tau_\lambda}d\hat{L}^*(t)\right\}}{\bar{Z}(\tau_\lambda)\cdot\exp\left\{-\int_s^{\tau_\lambda}d\hat{L}^*(t)\right\}+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau^*_\mu\}}\right].$$

Now since equality holds in Proposition 2.3.1, we must have

$$\mathbb{E}\left[\int_s^{\tau_{\min}}(1+\lambda)\frac{\frac{1}{z}Z^*(t)}{(Z^*(t)+1+\lambda)^2}d\hat{L}^*(t)\right] = 0,$$

which implies $\mathbb{E}\left[\int_s^{\tau_{\min}}d\hat{L}^*(t)\right] = 0$ since $Z^*(t) > 0$ for all $t\in[s,T]$, where $\tau_{\min} := \min\{\tau^*_\mu,\tau_\lambda\}$. Because $\hat{L}^*(\cdot)$ is non-decreasing, $\int_s^{\tau_{\min}}d\hat{L}^*(t) = 0$ almost surely. Hence we have

$$\begin{aligned}\frac{\partial V}{\partial z}(s,z) &= \sup_{s\le\tau_\lambda\le T}\mathbb{E}\left[\frac{\frac{1}{z}\bar{Z}(\tau^*_\mu)}{\bar{Z}(\tau^*_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau^*_\mu\le\tau_\lambda\}} + \frac{\frac{1}{z}\bar{Z}(\tau_\lambda)}{\bar{Z}(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau^*_\mu\}}\right]\\ &\ge \inf_{s\le\tau_\mu\le T}\sup_{s\le\tau_\lambda\le T}\mathbb{E}\left[\frac{\frac{1}{z}\bar{Z}(\tau_\mu)}{\bar{Z}(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\le\tau_\lambda\}} + \frac{\frac{1}{z}\bar{Z}(\tau_\lambda)}{\bar{Z}(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu\}}\right].\end{aligned}$$

On the other hand, as we have shown before,

$$\frac{\partial V}{\partial z}(s,z) = \inf_{s\le\tau_\mu\le T}\mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu)}{Z^*(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\le\tau^*_\lambda\}} + \frac{\frac{1}{z}Z^*(\tau^*_\lambda)}{Z^*(\tau^*_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau^*_\lambda<\tau_\mu\}}\right]$$

$$= \inf_{s\le\tau_\mu\le T}\mathbb{E}\left[\frac{\frac{1}{z}\bar{Z}(\tau_\mu)\cdot\exp\left\{\int_s^{\tau_\mu}d\hat{M}^*(t)\right\}}{\bar{Z}(\tau_\mu)\cdot\exp\left\{\int_s^{\tau_\mu}d\hat{M}^*(t)\right\}+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\le\tau^*_\lambda\}} + \frac{\frac{1}{z}\bar{Z}(\tau^*_\lambda)\cdot\exp\left\{\int_s^{\tau^*_\lambda}d\hat{M}^*(t)\right\}}{\bar{Z}(\tau^*_\lambda)\cdot\exp\left\{\int_s^{\tau^*_\lambda}d\hat{M}^*(t)\right\}+1+\lambda}\cdot\mathbf{1}_{\{\tau^*_\lambda<\tau_\mu\}}\right].$$

Now since equality also holds in Proposition 2.3.2, we must have

$$\mathbb{E}\left[\int_s^{\tau_{\min}}(1-\mu)\frac{\frac{1}{z}Z^*(t)}{(Z^*(t)+1-\mu)^2}d\hat{M}^*(t)\right] = 0,$$

which implies $\mathbb{E}\left[\int_s^{\tau_{\min}}d\hat{M}^*(t)\right] = 0$ since $Z^*(t) > 0$ for all $t\in[s,T]$, where $\tau_{\min} := \min\{\tau_\mu,\tau^*_\lambda\}$. Because $\hat{M}^*(\cdot)$ is non-decreasing, $\int_s^{\tau_{\min}}d\hat{M}^*(t) = 0$ almost surely. Hence we have

$$\begin{aligned}\frac{\partial V}{\partial z}(s,z) &= \inf_{s\le\tau_\mu\le T}\mathbb{E}\left[\frac{\frac{1}{z}\bar{Z}(\tau_\mu)}{\bar{Z}(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\le\tau^*_\lambda\}} + \frac{\frac{1}{z}\bar{Z}(\tau^*_\lambda)}{\bar{Z}(\tau^*_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau^*_\lambda<\tau_\mu\}}\right]\\ &\le \inf_{s\le\tau_\mu\le T}\sup_{s\le\tau_\lambda\le T}\mathbb{E}\left[\frac{\frac{1}{z}\bar{Z}(\tau_\mu)}{\bar{Z}(\tau_\mu)+1-\mu}\cdot\mathbf{1}_{\{\tau_\mu\le\tau_\lambda\}} + \frac{\frac{1}{z}\bar{Z}(\tau_\lambda)}{\bar{Z}(\tau_\lambda)+1+\lambda}\cdot\mathbf{1}_{\{\tau_\lambda<\tau_\mu\}}\right].\end{aligned}$$

Therefore, we may conclude that $\frac{\partial V}{\partial z}(s,z) = u(s,z)$. In addition, as we have obtained in Theorem 2.3.4,

$$\frac{\partial V}{\partial z}(s,z) = \mathbb{E}\left[\frac{\frac{1}{z}Z^*(\tau_\mu^*)}{Z^*(\tau_\mu^*)+1-\mu}\cdot \mathbf{1}_{\{\tau_\mu^*\le\tau_\lambda^*\}} + \frac{\frac{1}{z}Z^*(\tau_\lambda^*)}{Z^*(\tau_\lambda^*)+1+\lambda}\cdot \mathbf{1}_{\{\tau_\lambda^*<\tau_\mu^*\}}\right],$$

$(\tau_\lambda^*, \tau_\mu^*)$ are clearly optimal stopping times for the stopping problem (2.15). □

It is worth noting that, although it seems difficult for us to rigorously verify the existence of the optimal controls for problem (2.14), we reckon this is true. Based on these, since we are focusing on studying problem (2.10) with initial state $z \in \mathrm{NT}_s'$ according to our earlier emphasis, the value function V serves as a simplified version of the original value function φ only when $x/y \in \mathrm{NT}_s'$:

$$\varphi(s,x,y) = V(s,\frac{x}{y}) + \log y + (\alpha - \frac{1}{2}\sigma^2)(T-s).$$

Thus Theorem 2.3.5 indicates the connections between the value function of the original singular stochastic control problem (2.2) and the optimal risk of the optimal stopping problem (2.15) in NT_s. Up to now, it becomes natural for us to consider the connections in the trivial cases of the buying region and the selling region, within which we have obtained in Proposition 2.1.8 that

$$\begin{aligned} \frac{\partial\varphi}{\partial x}(s,x_1,y_1) &= \frac{1}{x_1+(1+\lambda)y_1}, & \frac{\partial\varphi}{\partial y}(s,x_1,y_1) &= \frac{1+\lambda}{x_1+(1+\lambda)y_1},\\ \frac{\partial\varphi}{\partial x}(s,x_2,y_2) &= \frac{1}{x_2+(1-\mu)y_2}, & \frac{\partial\varphi}{\partial y}(s,x_2,y_2) &= \frac{1-\mu}{x_2+(1-\mu)y_2}, \end{aligned}$$

for $(x_1,y_1) \in \mathrm{BR}_s$ and $(x_2,y_2) \in \mathrm{SR}_s$ respectively. In view of the definition of V, we define $v(s,z) := \varphi(s,z,1) - (\alpha - \frac{1}{2}\sigma^2)(T-s)$, which constitutes an extension of the relation between V and φ to the whole domain. Then it can be immediately deduced that

$$\frac{\partial v}{\partial z}(s,z_1) = \frac{1}{z_1+1+\lambda},\quad \frac{\partial v}{\partial z}(s,z_2) = \frac{1}{z_2+1-\mu},$$

for $z_1 \in \mathrm{BR}_s'$ and $z_2 \in \mathrm{SR}_s'$ respectively. Meanwhile, for the optimal stopping problem (2.15), the cases of buying region and selling region are also trivial, since

$\tau_\lambda^* = s$ and $\tau_\mu^* = s$ almost surely respectively. Simple calculation directly leads to the same equality

$$\frac{\partial v}{\partial z}(s,z) = u(s,z),$$

in both the buying region and the selling region.

Combining all these results, we have the following relation between the value function φ of the singular stochastic control problem (2.2) and the optimal risk u of the optimal stopping problem (2.15):

$$\varphi_x(s,z,1) \quad = \quad u(s,z), \tag{2.16}$$

for any $z > -(1-\mu)$ based on the existence of the optimal controls for problem (2.14). We expect further research may reveal analytically such existence, and the connections between the original singular stochastic control problem (2.2) and the optimal stopping problem (2.15) can be completely established, especially the relation between the value function φ and the optimal risk u as well as the connection between the optimal stochastic controls and optimal stopping times.

2.4 Numerical results

In this section, we present some numerical results related to the original singular stochastic control problem (2.2) via numerical PDE approach. We have to point out that this numerical problem has been studied in [11] using standard penalty methods, while convergence analysis has also been presented in this paper. The general idea for dealing with this problem is to simplify the relevant HJB system of the original problem and implement finite difference method to numerically solve the PDE system. The same technique can also be applied for a CRRA investor associated with power utility function.

For instance, let us consider the problem with the following parameter setting:

$r = 0.07$, $\alpha = 0.12$, $\sigma = 0.3$, $\mu = 0.01$, $\lambda = 0.01$, $T = 3$. As we have pointed out in Proposition 2.2.3, the region with negative position in the risky asset is fully contained in the buying region, thus we exclude the consideration of the case $y < 0$. Moreover, we have also shown in Proposition 2.2.8 that the region with negative position in the riskless asset is fully contained in the selling region if $\alpha - r - \sigma^2 < 0$, hence we may exclude the consideration of the case $x < 0$ as well. The dimensionality of the HJB system is reduced by introducing $z := \frac{y}{x+y}$, so the domain becomes a bounded region $[0, T] \times [0, 1]$. The free boundaries solved by finite difference method in the domain are as shown in Figure 2.2 below.

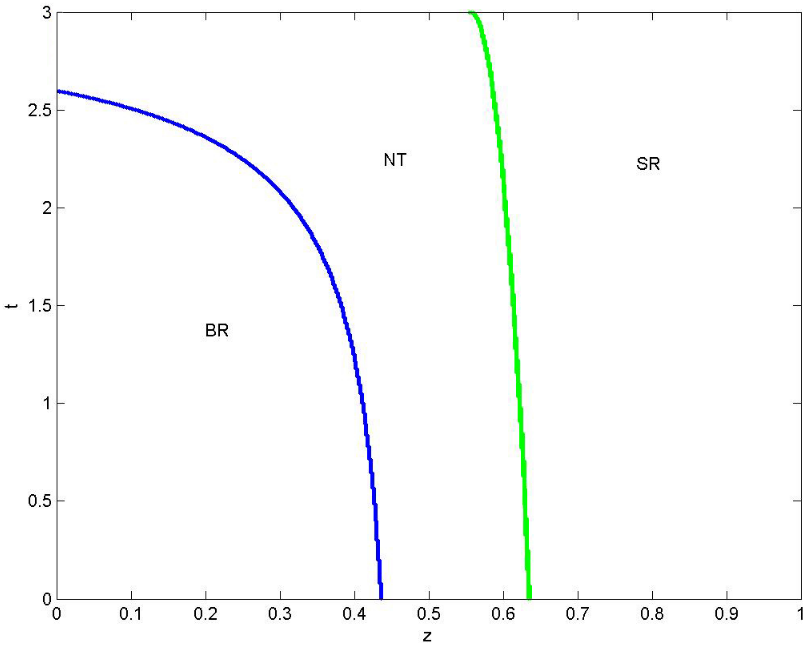

Figure 2.2. Plot of the optimal buying and selling boundaries across the finite horizon for the CRRA investor. The parameter values used are: $r = 0.07$, $\alpha = 0.12$, $\sigma = 0.3$, $\mu = 0.01$, $\lambda = 0.01$, $T = 3$. Note that $z = \frac{y}{x+y}$.

Chapter 3

The CARA Investor's Optimal Investment and Consumption Problem with Transaction Costs

3.1 Formulation of the optimal investment and consumption problem

3.1.1 A generalized optimal investment and consumption problem

As the major difference compared to the case discussed in Chapter 2, the investor we consider in this chapter is assumed to be Constant Absolute Risk Aversion (CARA) associated with exponential utility function $U(w) = -\mathbf{e}^{-\gamma w}$, with $\gamma > 0$ being a constant. Unlike the general settings of the logarithm or power utility functions, we expect the exponential utility function to tolerate negative wealth, which will facilitate our further incorporation of the jump diffusion feature. We also consider the involvement of the consumption term, for which the controlled diffusion processes that describe the underlying dynamics can be modeled as

$$\begin{cases} dX(t) &= rX(t-)dt - (1+\lambda)dL(t) + (1-\mu)dM(t) - \kappa C(t)dt, \\ dY(t) &= \alpha Y(t-)dt + \sigma Y(t-)d\mathcal{B}(t) + dL(t) - dM(t), \end{cases} \tag{3.1}$$

where κ represents an indicator of the involvement of consumption, $C(t) \geq 0$ is the consumption rate at time t, while other settings are the same as before.

Now if we define the admissible set as

$$\mathcal{A}_C := \left\{ (L, M, C) \,\middle|\, \begin{array}{l} \{L(t)\}_{t\in[0,T]}, \{M(t)\}_{t\in[0,T]}, \{C(t)\}_{t\in[0,T]} \text{ are right-continuous,} \\ \text{non-negative, non-decreasing, } \{\mathcal{F}_t\}_{t\geq 0} - \text{adapted}, L(0) = M(0) = 0, \\ X \in L^2_{\mathcal{F}}, Y \in L^2_{\mathcal{F}}, X(t) \in L^2, Y(t) \in L^2, \forall t \in [0,T] \end{array} \right\},$$

the generalized optimal investment and consumption problem with transaction costs can then be formulated as the following singular stochastic control problem:

$$\begin{aligned} &\varphi(s, x, y; \kappa) \\ := &\inf_{(L,M,C)\in\mathcal{A}_C} \mathbb{E}\left[\kappa \int_s^T \mathbf{e}^{-\delta(t-s)} \cdot \mathbf{e}^{-\gamma C(t)} dt + \mathbf{e}^{-\delta(T-s)} \cdot \mathbf{e}^{-\gamma w(X(T),Y(T))} \,\middle|\, X(s-) = x, Y(s-) = y\right] \\ &\text{s.t. } (3.1), \end{aligned} \tag{3.2}$$

where

$$w(x,y) = \begin{cases} x + (1-\mu)y, & y \geq 0, \\ x + (1+\lambda)y, & y < 0. \end{cases}$$

It is not difficult to observe that the problem will be reduced to the optimal investment problem without consumption when κ is set to be 0, and will become the standard optimal investment and consumption problem when κ is set to be 1.

Applying the principle of dynamic programming, the relevant HJB system for problem (3.2) can then be derived as follows:

$$\begin{cases} \max\{-\varphi_t - \mathcal{L}\varphi, \varphi_y - (1-\mu)\varphi_x, -\varphi_y + (1+\lambda)\varphi_x\} = 0, \\ \varphi(T, x, y; \kappa) = \mathbf{e}^{-\gamma w(x,y)}, \end{cases} \tag{3.3}$$

where $\mathcal{L}\varphi = \frac{1}{2}\sigma^2 y^2 \varphi_{yy} + \alpha y \varphi_y + r x \varphi_x - \kappa \left[\frac{1}{\gamma}\left(1 - \log\left(-\frac{\varphi_x}{\gamma}\right)\right)\varphi_x + \delta\varphi\right]$.

3.1.2 Observations in no-consumption case and dimensionality reduction

Similar to the arguments presented in Chapter 2, we are able to produce the following analogous properties related to the CARA investor's optimal investment problem as stated in the following:

1. For any $t \in [0,T]$, $\varphi(t,\cdot,\cdot;0)$ is strictly decreasing w.r.t. the state arguments x and y.

2. For any $t \in [0,T]$, $\varphi(t,\cdot;0)$ is convex in $\mathbb{R}^2$.

3. For any $(x,y) \in \mathbb{R}^2$, $\varphi(\cdot,x,y;0)$ is strictly increasing with respect to the temporal argument t in $[0,T]$.

4. For any $t \in [0,T]$, $\varphi(t,\cdot;0)$ is continuous in $\mathbb{R}^2$; for any $(x,y) \in \mathbb{R}^2$, $\varphi(\cdot,x,y;0)$ is continuous in $[0,T]$.

Note that the homotheticity property does not hold any more in the CARA investor case. As a matter of fact, we will see later that the state variable x can be separated from the problem and the optimal investment strategy is independent of x.

Moreover, if we define

$$g(y) \quad := \quad \begin{cases} (1-\mu)y, & y \geq 0, \\ (1+\lambda)y, & y < 0, \end{cases}$$

problem (3.2) with $\kappa = 0$ can then be rewritten as

$$\begin{aligned} \varphi(s,x,y;0) \quad &= \quad \mathbf{e}^{-\delta(T-s)} \cdot \inf_{(L,M)\in\mathcal{A}} \mathbb{E}\left[\mathbf{e}^{-\gamma X(T)} \cdot \mathbf{e}^{-\gamma g(Y(T))} \middle| X(s-) = x, Y(s-) = y\right] \\ &\quad \text{s.t. } (2.1), \end{aligned}$$

where $X(T)$ has explicit expression

$$X(T) \quad = \quad x\mathbf{e}^{r(T-s)} - \int_s^T (1+\lambda)\mathbf{e}^{r(T-t)}dL(t) + \int_s^T (1-\mu)\mathbf{e}^{r(T-t)}dM(t).$$

At this moment, it is possible to fully draw the deterministic term involving x out of expectation and infimum as follows

$$\begin{aligned}\varphi(s,x,y;0) &= \mathbf{e}^{-\delta(T-s)}\cdot\mathbf{e}^{-\gamma x\exp\{r(T-s)\}}.\\ &\quad \inf_{(L,M)\in\mathcal{A}}\mathbb{E}\left[\mathbf{e}^{-\gamma g(Y(T))}\cdot\mathbf{e}^{\gamma\int_s^T(1+\lambda)\exp\{r(T-t)\}dL(t)}\cdot\mathbf{e}^{-\gamma\int_s^T(1-\mu)\exp\{r(T-t)\}dM(t)}\middle|\,Y(s)=y\right]\\ &\quad \text{s.t.}\quad dY(t)=\alpha Y(t-)dt+\sigma Y(t-)d\mathcal{B}_t+dL(t)-dM(t).\end{aligned}$$

Then it suffices for us to study a new singular stochastic control problem as follows:

$$\begin{aligned}\phi(s,y;0) &:= \inf_{(L,M)\in\mathcal{A}}\mathbb{E}\left[\mathbf{e}^{-\gamma g(Y(T))}\cdot\mathbf{e}^{\gamma\int_s^T(1+\lambda)\exp\{r(T-t)\}dL(t)}\cdot\mathbf{e}^{-\gamma\int_s^T(1-\mu)\exp\{r(T-t)\}dM(t)}\middle|\,Y(s)=y\right]\\ &\quad \text{s.t.}\quad dY(t)=\alpha Y(t-)dt+\sigma Y(t-)d\mathcal{B}_t+dL(t)-dM(t),\end{aligned}$$

which has one less dimension in the spatial direction. Obviously we have the relation $\varphi(s,x,y;0) = \mathbf{e}^{-\gamma x\exp\{r(T-s)\}}\phi(s,y;0)$, and it is clear that the optimal investment strategy for the CARA investor is indifferent to the initial endowment in the riskless asset. It only depends on the absolute value of the initial endowment in the risky asset instead of the relative ratio of the two assets, hence we have the names of Constant Relative Risk Aversion and Constant Absolute Risk Aversion. An collateral result for the value function is as follows:

5. For any $t\in[0,T]$ and any $(x,y)\in\mathbb{R}^2$, for any $x'\in\mathbb{R}$, we have

$$\varphi(t,x',y;0) \;=\; \varphi(t,x,y;0)\cdot\mathbf{e}^{-\gamma(x'-x)\exp\{r(T-t)\}}.$$

Moreover, by considering the corresponding standard stochastic control problem of the CARA investor's optimal investment problem and applying similar argument as provided in Proposition 2.2.3 and Proposition 2.2.4, we are able to show that the "jump-buy" region BR_t for the problem should contain $\{y<0\}$ for any $t\in[0,T)$ via the probabilistic approach. The utility function $\log(w)$ used in the proofs may be replaced with $-\mathbf{e}^{\gamma w}$, which is still concave. However, it is worth noting that the wealth process $W(t) := X(t)+(1+\lambda)Y(t)$ is no longer necessarily positive, which would tentatively be a blocking issue for the rest of the argument. Our solution is to show for a large enough $x_M>0$ that $(x_M,y)\in\mathrm{BR}_t$ instead given the optimal investment strategy would be indifferent of initial endowment x. Because of the restrictions set in $\mathcal{A}_C$, a large enough x_M can always be found to ensure the wealth process $W(\cdot)$ is positive. This result can also be observed in the PDE approach presented below, which suggests that we may confine our study within $\{y \geq 0\}$ in the following analysis.

In order to exploit the dimensionality reduction feature as described above, we attempt to reveal the governing system for the new singular stochastic control problem. In view of the relation $\varphi(t, x, y; 0) = \mathbf{e}^{-\gamma x \exp\{r(T-t)\}}\phi(t, y; 0)$ as well as the HJB system (3.3) when $\kappa = 0$, the system that ϕ satisfies in the viscosity sense can be obtained as follows:

$$\begin{cases} \max\left\{-\phi_t - \frac{1}{2}\sigma^2 y^2 \phi_{yy} - \alpha y \phi_y, \phi_y + (1-\mu)\gamma \mathbf{e}^{r(T-t)}\phi, -\phi_y - (1+\lambda)\gamma \mathbf{e}^{r(T-t)}\phi\right\} = 0, \\ \phi(T, y; 0) = \mathbf{e}^{-\gamma g(y)}. \end{cases} \tag{3.4}$$

Taking note of the simple fact that ϕ is strictly positive, system (3.4) is equivalent to

$$\begin{cases} \phi_t + \frac{1}{2}\sigma^2 y^2 \phi_{yy} + \alpha y \phi_y = 0, & \text{if} \quad -(1+\lambda)\gamma \mathbf{e}^{r(T-t)} < \frac{\phi_y}{\phi} < -(1-\mu)\gamma \mathbf{e}^{r(T-t)}, \\ \phi_t + \frac{1}{2}\sigma^2 y^2 \phi_{yy} + \alpha y \phi_y \geq 0, & \text{if} \quad \frac{\phi_y}{\phi} = -(1+\lambda)\gamma \mathbf{e}^{r(T-t)}, \text{ or } \frac{\phi_y}{\phi} = -(1-\mu)\gamma \mathbf{e}^{r(T-t)}, \\ \phi(T, y; 0) = \mathbf{e}^{-\gamma g(y)}. \end{cases}$$

Now we try to do a series of transformations to further simplify the governing PDE system. Let $\tau := T - t$, $z := \mathbf{e}^{r\tau} y$, $\zeta(\tau, z; 0) := \log \phi(t, y; 0)$. Then the following system is obtained:

$$\begin{cases} \zeta_\tau - (\alpha - r) z \zeta_z - \frac{1}{2}\sigma^2 z^2 (\zeta_{zz} + \zeta_z^2) = 0, & \text{if} \quad -(1+\lambda)\gamma < \zeta_z < -(1-\mu)\gamma, \\ \zeta_\tau - (\alpha - r) z \zeta_z - \frac{1}{2}\sigma^2 z^2 (\zeta_{zz} + \zeta_z^2) \leq 0, & \text{if} \quad \zeta_z = -(1+\lambda)\gamma, \text{ or } \zeta_z = -(1-\mu)\gamma, \\ \zeta(0, z; 0) = -\gamma g(z). \end{cases} \tag{3.5}$$

This gives rise to two free boundaries, but it is difficult to investigate their behaviors by directly studying this problem. Base on system (3.5), we will follow [10] to adopt an indirect approach. Formally, let us define

$$V(\tau, z; 0) = -\frac{1}{\gamma} \cdot \zeta_z(\tau, z; 0),$$

and take the partial derivative with respect to the state variable in the system above, then the following system is obtained:

$$\begin{cases} V_\tau - \tilde{\mathcal{L}}_z V = 0, & \text{if } 1-\mu < V < 1+\lambda, \\ V_\tau - \tilde{\mathcal{L}}_z V \leq 0, & \text{if } V = 1+\lambda, \\ V_\tau - \tilde{\mathcal{L}}_z V \geq 0, & \text{if } V = 1-\mu, \\ V(0,z;0) = \begin{cases} 1-\mu, & z \geq 0, \\ 1+\lambda, & z < 0, \end{cases} \end{cases} \tag{3.6}$$

in $(0,T] \times [0,+\infty)$, where

$$\tilde{\mathcal{L}}_z V = \tfrac{1}{2}\sigma^2 z^2 V_{zz} + (\alpha - r + \sigma^2) z V_z + (\alpha - r)V - \gamma\sigma^2 z V(zV_z + V).$$

This is indeed a nonlinear parabolic double obstacle problem, with $1-\mu$ and $1+\lambda$ being the lower and upper obstacles respectively. Since the Neumann boundary conditions for the PDE system have been transferred to Dirichlet boundary conditions, while the equivalence between the two problems can be achieved, the numerical approach can be much more straightforward and more stable.

It is clear that the buying region BR, selling region SR, and no transaction region NT for problem (3.6) satisfy

$$\begin{aligned} \overline{\text{BR}} &= \{(\tau,z) : V(\tau,z;0) = 1+\lambda\}, \\ \overline{\text{SR}} &= \{(\tau,z) : V(\tau,z;0) = 1-\mu\}, \\ \text{NT} &= \{(\tau,z) : 1-\mu < V(\tau,z;0) < 1+\lambda\}. \end{aligned}$$

Some collateral results regarding the free boundaries can be obtained immediately. For instance, for any $(\tau_1, z_1) \in \text{SR}$, and any $(\tau_2, z_2) \in \text{BR}$, it holds that

$$\begin{aligned} (\partial_\tau - \tilde{\mathcal{L}}_z)V(\tau_1, z_1; 0) &= -(1-\mu)\left[\alpha - r - (1-\mu)\gamma\sigma^2 z_1\right] \geq 0, \\ (\partial_\tau - \tilde{\mathcal{L}}_z)V(\tau_2, z_2; 0) &= -(1+\lambda)\left[\alpha - r - (1+\lambda)\gamma\sigma^2 z_2\right] \leq 0 \end{aligned}$$

which are equivalent to $z_1 \geq \frac{\alpha - r}{(1-\mu)\gamma\sigma^2}$ and $z_2 \leq \frac{\alpha - r}{(1+\lambda)\gamma\sigma^2}$. Thus we must have $\text{SR} \subset \left\{z \geq \frac{\alpha-r}{(1-\mu)\gamma\sigma^2}\right\}$ and $\text{BR} \subset \left\{z \leq \frac{\alpha-r}{(1+\lambda)\gamma\sigma^2}\right\}$. In addition, since the operator

$\tilde{\mathcal{L}}_z$ is degenerate at $z = 0$, we can study the double obstacle problem in $\{z < 0\}$ and $\{z > 0\}$ independently. It is not difficult to verify that

$$V(\tau, z; 0) \quad = \quad 1 + \lambda,$$

is the solution in $\{z < 0\}$, or in other words, $\{z < 0\} \subset$ BR, thus the numerical methods can only be applied within $(0, T] \times (0, \infty)$ in the following. Other related results with this problem can be found in [10] and [45], and we will pay more attention to the characteristics of the CARA investor's optimal investment and consumption problem in the following.

3.1.3 Dimensionality reduction in consumption case

In order to reduce the dimensionality of the standard optimal investment and consumption problem, we define

$$\xi(t) := \tfrac{r}{1-(1-r)\mathbf{e}^{-r(T-t)}},$$

$$g(y) := \begin{cases} (1-\mu)y, & y \geq 0, \\ (1+\lambda)y, & y < 0, \end{cases}$$

and ϕ satisfying $\varphi(t, x, y; 1) = \mathbf{e}^{-\gamma x \xi(t)} \phi(t, y; 1)$. Then in view of the fact that

$$\begin{aligned} &\xi'(t) - \xi(t)^2 + r\xi(t) = 0, \\ &\xi(T) = 1, \end{aligned}$$

as well as the HJB system (3.3) when $\kappa = 1$, we have

$$\begin{cases} \max\{-\phi_t - \mathcal{L}_1\phi, \phi_y + (1-\mu)\gamma\xi(t)\phi, -\phi_y - (1+\lambda)\gamma\xi(t)\phi\} = 0, \\ \phi(T, y; 1) = \mathbf{e}^{-\gamma g(y)}, \end{cases} \tag{3.7}$$

where

$$\mathcal{L}_1\phi \quad = \quad \tfrac{1}{2}\sigma^2 y^2 \phi_{yy} + \alpha y \phi_y + \xi(t)\phi\left(1 - \log(\xi(t)) - \log(\phi)\right) - \delta\phi.$$

Note that ϕ is strictly positive, the system above is thus equivalent to

$$\begin{cases} \phi_t + \mathcal{L}_1\phi = 0, & \text{if } \; -(1+\lambda)\gamma\xi(t) < \frac{\phi_y}{\phi} < -(1-\mu)\gamma\xi(t), \\ \phi_t + \mathcal{L}_1\phi \geq 0, & \text{if } \; \frac{\phi_y}{\phi} = -(1+\lambda)\gamma\xi(t), \text{ or } \frac{\phi_y}{\phi} = -(1-\mu)\gamma\xi(t), \\ \phi(T, y; 1) = \mathbf{e}^{-\gamma g(y)}. \end{cases}$$

Similarly, we try to do a series of transformations to further simplify the governing PDE system. By denoting $\tau := T - t$, and

$$\xi_1(\tau) := \xi(T-\tau) = \tfrac{r}{1-(1-r)\mathbf{e}^{-r\tau}},$$

we introduce $z := \xi_1(\tau)y$, $\zeta(\tau, z; 1) := \log\phi(t, y; 1)$. Note that $\xi_1(\cdot)$ is a decreasing function when $r < 1$ and a constant 1 when $r = 1$, while it becomes increasing when $r > 1$. Based on these, the following system is then obtained:

$$\begin{cases} \zeta_\tau - \mathcal{L}_2\zeta = 0, & \text{if } \; -(1+\lambda)\gamma < \zeta_z < -(1-\mu)\gamma, \\ \zeta_\tau - \mathcal{L}_2\zeta \leq 0, & \text{if } \; \zeta_z = -(1+\lambda)\gamma, \text{ or } \zeta_z = -(1-\mu)\gamma, \\ \zeta(0, z; 1) = -\gamma g(z), \end{cases} \tag{3.8}$$

where

$$\mathcal{L}_2\zeta \;\; = \;\; \tfrac{1}{2}\sigma^2 z^2\left(\zeta_{zz} + \zeta_z^2\right) + (\alpha - r + \xi_1(\tau))z\zeta_z - \xi_1(\tau)\zeta + \xi_1(\tau)(1 - \log(\xi_1(\tau))) - \delta.$$

Forthermore, we formally define

$$V(\tau, z; 1) = -\tfrac{1}{\gamma}\cdot\zeta_z(\tau, z; 1),$$

and take the partial derivative with respect to the state variable in the system above, then the following parabolic double obstacle problem is obtained:

$$\begin{cases} V_\tau - \mathcal{L}_z V = 0, & \text{if } 1-\mu < V < 1+\lambda, \\ V_\tau - \mathcal{L}_z V \leq 0, & \text{if } V = 1+\lambda, \\ V_\tau - \mathcal{L}_z V \geq 0, & \text{if } V = 1-\mu, \\ V(0, z; 1) = (1-\mu)\cdot\mathbf{1}_{z\geq 0} + (1+\lambda)\cdot\mathbf{1}_{z<0}, \end{cases} \tag{3.9}$$

where

$$\mathcal{L}_z V \;=\; \tfrac{1}{2}\sigma^2 z^2 V_{zz} + (\alpha - r + \sigma^2 + \xi_1(\tau)) z V_z + (\alpha - r)V - \gamma\sigma^2 z V (z V_z + V),$$

with $1-\mu$ and $1+\lambda$ being the lower and upper obstacles respectively. Similarly, the buying region BR, selling region SR, and no transaction region NT for problem (3.9) also satisfy

$$\begin{aligned}
&\overline{\mathrm{BR}} = \{(\tau, z) : V(\tau, z; 1) = 1+\lambda\}\,,\\
&\overline{\mathrm{SR}} = \{(\tau, z) : V(\tau, z; 1) = 1-\mu\}\,,\\
&\mathrm{NT} = \{(\tau, z) : 1-\mu < V(\tau, z; 1) < 1+\lambda\}\,.
\end{aligned}$$

The equivalence between the double obstacle problem (3.9) and the original HJB system (3.3) when $\kappa = 1$ can be obtained, while the proof is deferred to Section 3.2.3. It is worth noting that the double obstacle problem (3.9) is indifferent of the discounting factor δ, which is present up to system (3.8). Given the equivalence that we are about to establish, this actually reveals that the optimal investment strategy, which may be characterized by the free boundaries, is independent of the discounting factor δ. Nevertheless, the optimal consumption strategy, which is not characterized by the free boundaries, is still affected by this factor.

3.2 Characteristics of the optimal investment and consumption problem

3.2.1 The existence of $W_p^{1,2}$ solution and properties of the value function

In this section, we will focus on investigating analytically the characteristics of problem (3.9). Since the operator $\mathcal{L}_z$ in the system is degenerate at $z = 0$, then similar to the no consumption case, we can study the problem in $\{z > 0\}$ and $\{z < 0\}$ independently. Moreover, it is not difficult to verify that

$$V(\tau, z; 1) \;=\; 1+\lambda$$

is the solution to the double obstacle problem in $(0,T]\times(-\infty,0)$, or in other words, $\{z<0\}\subset \mathrm{BR}$. Thus we may only focus on the problem in $(0,T]\times(0,\infty)$ in the following. Let $x := \log(z)$, $v(\tau,x;1) := V(\tau,z;1)$, then the system is mapped onto $(0,T]\times(-\infty,\infty)$ as follows:

$$\begin{cases} v_\tau - \mathcal{L}_x v = 0, & \text{if } 1-\mu < v < 1+\lambda, \\ v_\tau - \mathcal{L}_x v \le 0, & \text{if } v = 1+\lambda, \\ v_\tau - \mathcal{L}_x v \ge 0, & \text{if } v = 1-\mu, \\ v(0,x;1) = 1-\mu, & \end{cases} \tag{3.10}$$

where

$$\mathcal{L}_x v \;=\; \tfrac{1}{2}\sigma^2 v_{xx} + \left(\alpha - r + \tfrac{1}{2}\sigma^2 + \xi_1(\tau)\right) v_x + (\alpha - r)v - \gamma\sigma^2 \mathbf{e}^x v(v_x + v).$$

Furthermore, for $n > 0$, we define

$$\begin{aligned} \Omega_T &= (0,T]\times\mathbb{R}, \\ \Omega_T^n &= (0,T]\times(-n,n), \end{aligned}$$

and consider the bounded problem within Ω_T^n:

$$\begin{cases} \partial_\tau v_n - \mathcal{L}_x v_n = 0, & \text{if } 1-\mu < v_n < 1+\lambda, (\tau,x)\in\Omega_T^n, \\ \partial_\tau v_n - \mathcal{L}_x v_n \le 0, & \text{if } v_n = 1+\lambda, (\tau,x)\in\Omega_T^n, \\ \partial_\tau v_n - \mathcal{L}_x v_n \ge 0, & \text{if } v_n = 1-\mu, (\tau,x)\in\Omega_T^n, \\ \partial_x v_n(\tau,x;1) = 0, & \text{if } x = \pm n, \tau\in(0,T], \\ v_n(0,x;1) = 1-\mu, & x\in(-n,n). \end{cases} \tag{3.11}$$

Proposition 3.2.1. Problem (3.10) has a solution $v \in \mathcal{C}(\overline{\Omega}_T)\cap W_p^{1,2}(\Omega_T^n)$ for any $n > 0$, $p > 1$, and $\partial_x v \le 0$ in Ω_T. When $r \le 1$, it holds that $\partial_\tau v \ge 0$ in Ω_T.

Moreover, for any $n > 0$ and any $\alpha \in (0,1)$, we have

$$|v|_{\mathcal{C}^{\alpha/2,\alpha}\overline{((0,T]\times(-\infty,n))}} \le C_n,$$

where C_n is a constant that only depends on n.

Proof: We define a penalty function $\beta_\epsilon(\cdot)$ that satisfies

$$\begin{aligned}
&\beta_\epsilon(\cdot) \in \mathcal{C}^2(-\infty,+\infty), \quad \beta_\epsilon(\cdot) \le 0,\\
&\beta_\epsilon(0) = -C_0, \quad C_0 := \max\{\gamma\sigma^2(1-\mu)^2\mathbf{e}^n, (\alpha-3r)(1+\lambda)\},\\
&\beta_\epsilon'(\cdot) \ge 0, \quad \beta_\epsilon''(\cdot) \le 0,\\
&\lim_{\epsilon\to 0+} \beta_\epsilon(t) = \begin{cases} 0, & t > 0,\\ -\infty, & t < 0.\end{cases}
\end{aligned}$$

Then an approximate problem is constructed as follows:

$$\begin{cases}
\partial_\tau v_{\epsilon,n} - \mathcal{L}_x v_{\epsilon,n} + \beta_\epsilon(v_{\epsilon,n} - (1-\mu)) - \beta_\epsilon(-v_{\epsilon,n} + (1+\lambda)) = 0, \text{ in } \Omega_T^n,\\
\partial_\tau v_{\epsilon,n}(\tau,x;1) = 0, \quad \text{if } x = \pm n, \tau \in (0,T],\\
v_{\epsilon,n}(0,x;1) = 1-\mu, \quad x \in (-n,n).
\end{cases}$$

Applying Leray-Schauder fixed point theorem to the system above, it is not difficult to show that there exists a solution $v_{\epsilon,n} \in W_p^{1,2}(\Omega_T)$ for $1 < p < \infty$. It can be further deduced that, by letting $\epsilon \to 0+$, we have $v_{\epsilon,n} \rightharpoonup v_n$ in $W_p^{1,2}(\Omega_T^n)$ weakly and $v_n \to v$ in $\mathcal{C}(\overline{\Omega}_T)$.

Now we define $u_1 := \partial_x v_{\epsilon,n}$, then u_1 satisfies the following system:

$$\begin{cases}
\partial_\tau u_1 - \frac{1}{2}\sigma^2\partial_{xx}u_1 - \left(\alpha - r + \frac{1}{2}\sigma^2 + \xi_1(\tau) - \gamma\sigma^2\mathbf{e}^x v_{\epsilon,n}\right)\partial_x u_1 - (\alpha - r - 3\gamma\sigma^2\mathbf{e}^x v_{\epsilon,n})u_1\\
\qquad +\beta_\epsilon'(\cdot)u_1 + \beta_\epsilon'(\cdot\cdot)u_1 = -\gamma\sigma^2\mathbf{e}^x(u_1^2 + v_{\epsilon,n}^2) \le 0, \quad \text{in } \Omega_T^n,\\
u_1(\tau,x;1) = 0, \quad \text{if } x = \pm n, \tau \in (0,T],\\
u_1(0,x;1) = 0, \quad x \in (-n,n).
\end{cases}$$

Applying the maximum principle, we obtain $\partial_x v_{\epsilon,n} \le 0$. Furthermore, we define

$u_2 := \partial_\tau v_{\epsilon,n}$, then u_2 satisfies the following system:

$$\begin{cases} \partial_\tau u_2 - \frac{1}{2}\sigma^2\partial_{xx}u_2 - \left(\alpha - r + \frac{1}{2}\sigma^2 + \xi_1(\tau) - \gamma\sigma^2\mathbf{e}^x v_{\epsilon,n}\right)\partial_x u_2 - (\alpha - r - 2\gamma\sigma^2\mathbf{e}^x v_{\epsilon,n})u_2 \\ \qquad\qquad +\gamma\sigma^2\mathbf{e}^x u_2\partial_x v_{\epsilon,n} + \beta_\epsilon'(\cdot)u_2 + \beta_\epsilon'(\cdot\cdot)u_1 = \xi_1'(\tau)\partial_x v_{\epsilon,n} \geq 0, \quad \text{in } \Omega_T^n, \\ \partial_x u_2(\tau, x; 1) = 0, \quad \text{if } x = \pm n, \tau \in (0, T], \\ u_2(0, x; 1) = (\alpha - r)(1-\mu) - [\gamma\sigma^2(1-\mu)^2\mathbf{e}^x + \beta_\epsilon(0)] \geq 0, \quad x \in (-n, n), \end{cases}$$

where $\xi_1'(\cdot) \leq 0$ given $r \leq 1$. Applying the minimum principle, we obtain $\partial_\tau v_{\epsilon,n} \geq 0$. Consequently, we may conclude that $\partial_\tau v \geq 0$, $\partial_x v \leq 0$ due to the convergence.

Moveover, in view of the fact that

$$\begin{aligned} (\partial_\tau - \mathcal{L}_x)(1+\lambda) &= -(\alpha - r)(1+\lambda) + \gamma\sigma^2\mathbf{e}^x(1+\lambda)^2, \\ (\partial_\tau - \mathcal{L}_x)(1-\mu) &= -(\alpha - r)(1-\mu) + \gamma\sigma^2\mathbf{e}^x(1-\mu)^2, \end{aligned}$$

problem (3.11) may be rewritten as

$$\begin{cases} \partial_\tau v_n - \mathcal{L}_x v_n = f(\tau, x), & \text{in } \Omega_T^n, \\ \partial_x v_n(\tau, x; 1) = 0, & \text{if } x = \pm n, \tau \in (0, T], \\ v_n(0, x; 1) = 1 - \mu, & x \in (-n, n), \end{cases}$$

where

$$\begin{aligned} f(\tau, x) &= \mathbf{1}_{\{v_n = 1+\lambda\}}\left[-(\alpha - r)(1+\lambda) + \gamma\sigma^2\mathbf{e}^x(1+\lambda)^2\right] \\ &\quad + \mathbf{1}_{\{v_n = 1+\lambda\}}\left[-(\alpha - r)(1-\mu) + \gamma\sigma^2\mathbf{e}^x(1-\mu)^2\right]. \end{aligned}$$

It is obvious that $|f(\tau, x)| \leq C_n$ for a constant C_n that only depends on n. Thus standard $\mathcal{C}^\alpha$ theorem of parabolic equation may lead to the result. These complete the proof. □

Remark: When $r > 1$, the monotonicity of v in the temporal direction does not necessarily hold true although the monotonicity in the spatial direction is intact. In view of the fact that the monotonicity in both the temporal direction and the spatial direction holds true for the no consumption case, this constitutes one of the differences between the two case.

Corollary 3.2.2. For problem (3.9), we have $V \in \mathcal{C}((0, T] \times (0, \infty))$ and $V_z \leq 0$ in $\{z > 0\}$. When $r \leq 1$, it holds $V_\tau \geq 0$ in $\{z > 0\}$.

Proof: Based on the results attained in Proposition 3.2.1, we may easily deduce that

$$\begin{aligned} V_\tau &= v_\tau \geq 0, \\ V_z &= \mathbf{e}^{-x} v_x \leq 0. \end{aligned}$$

Thus the monotonicity of the value function for problem (3.9) is obtained. □

Remark: Similar remarks as above apply for the value function V.

3.2.2 Characterization of the free boundaries

Utilizing the analytical properties for the value function we obtained in Section 3.2.1, characteristics of the free boundaries for the standard optimal investment and consumption problem for a CARA investor with transaction costs can be revealed. In view of the monotonicity property of the value function in the state variable direction as well as the property $\{z < 0\} \subset \mathrm{BR}$, the buying boundary and the selling boundary for the double obstacle problem (3.9) can be defined as

$$\begin{aligned} z_b(\tau) &:= \sup\{z \geq 0 | V(\tau, z; 1) = 1 + \lambda\}, \quad \tau \in (0, T], \\ z_s(\tau) &:= \inf\{z \geq 0 | V(\tau, z; 1) = 1 - \mu\}, \quad \tau \in (0, T], \end{aligned}$$

and it is clear that

$$\begin{aligned} \overline{\mathrm{BR}} &= \{(\tau, z) \in \Omega : z \leq z_b(\tau)\}, \\ \overline{\mathrm{SR}} &= \{(\tau, z) \in \Omega : z \geq z_s(\tau)\}. \end{aligned}$$

Proposition 3.2.3. $\overline{\mathrm{SR}} \subset \left\{ z \geq \frac{\alpha - r}{(1-\mu)\gamma\sigma^2} \right\}$, and $\overline{\mathrm{BR}} \subset \left\{ z \leq \frac{\alpha - r}{(1+\lambda)\gamma\sigma^2} \right\}$.

Proof: For any $(\tau_1, z_1) \in \overline{\mathrm{SR}}$ and any $(\tau_2, z_2) \in \overline{\mathrm{BR}}$, it holds that

$$\begin{aligned} (\partial_\tau - \mathcal{L}_z) V(\tau_1, z_1; 1) &= -(1 - \mu) \left[\alpha - r - (1 - \mu)\gamma\sigma^2 z_1\right] \geq 0, \\ (\partial_\tau - \mathcal{L}_z) V(\tau_2, z_2; 1) &= -(1 + \lambda) \left[\alpha - r - (1 + \lambda)\gamma\sigma^2 z_2\right] \leq 0, \end{aligned}$$

which are equivalent to $z_1 \geq \frac{\alpha - r}{(1-\mu)\gamma\sigma^2}$ and $z_2 \leq \frac{\alpha - r}{(1+\lambda)\gamma\sigma^2}$. □

Proposition 3.2.4. There exists $z_0 > 0$, $\tau_0 > 0$ such that

$$(0, \tau_0) \times (0, z_0) \quad \subset \quad \mathrm{NT},$$

and all partial derivatives of V are bounded in $(0,\tau_0)\times(0,z_0)$.

Proof: In view of Proposition 3.2.3, we may choose $z_0 \in \left(0, \frac{\alpha-r}{\gamma\sigma^2(1-\mu)}\right)$ such that $V > 1-\mu$ in $(0,T]\times(0,z_0)$. On the other hand, it can be inferred that there exists $\tau_0 \in (0,T)$ such that $V < 1+\lambda$ in $(0,\tau_0)\times(0,z_0)$ according to Corollary 3.2.2. Thus we may conclude that $(0,\tau_0)\times(0,z_0)\subset \mathrm{NT}$, and the following system can be obtained:

$$\begin{cases} \partial_\tau V - \mathcal{L}_z V = 0, & \text{in } (0,\tau_0]\times(0,z_0),\\ V(\tau,z_0;1)\in\mathcal{C}^\infty[0,\tau_0], & \\ V(0,z;1) = 1-\mu, & z\in(0,z_0). \end{cases}$$

Recalling system (3.10) and denoting $\log(z_0)$ by x_0, we have

$$\begin{cases} \partial_\tau v - \mathcal{L}_x v = 0, & \text{in } (0,\tau_0]\times(-\infty,x_0),\\ v(\tau,x_0;1)\in\mathcal{C}^\infty[0,\tau_0], & \\ v(0,x;1) = 1-\mu, & x\in(-\infty,x_0). \end{cases}$$

Applying the Schauder theory of parabolic equation, we have

$$|v|_{\mathcal{C}^{1+\alpha/2,2+\alpha}(\overline{(0,\tau_0)\times(-\infty,x_0)})} \;\leq\; C_{x_0},$$

where C_{x_0} depends on x_0. The bootstrap argument can be further used to obtain the boundedness of all partial derivatives of $v(\tau,x;1)$ on $(0,\tau_0)\times(-\infty,x_0)$.

Now we let $u(\tau,x;1) := \mathbf{e}^{-x}\partial_x v(\tau,x;1)$, which satisfies

$$\begin{cases} \partial_\tau u - \frac{1}{2}\sigma^2\partial_{xx}u - \left(\alpha - r + \frac{3}{2}\sigma^2 + \xi_1(\tau)\right)\partial_x u - (2\alpha-2r+\sigma^2+\xi_1(\tau))u \\ \qquad\qquad = -\gamma\sigma^2\mathbf{e}^x\left(v^2 + (\partial_x v)^2 + 3v\partial_x v + v\partial_{xx}v\right), \qquad \text{in } (0,\tau_0]\times(-\infty,x_0),\\ u(\tau,x_0;1)\in\mathcal{C}^\infty[0,\tau_0],\\ u(0,x;1) = 0, x\in(-\infty,x_0). \end{cases}$$

Since the right hand side of the equation is bounded, hence u is bounded, and so is $\partial_z V$. Applying the same argument, we are able to show $\partial_{zz}V = \mathbf{e}^{-2x}(\partial_{xx}v - \partial_x v)$ is also bounded, while the boundedness of all partial derivatives of $V(\tau,z;1)$ on

$(0, \tau_0) \times (0, z_0)$ can be obtained by bootstrap argument. These complete the proof. $\square$

Proposition 3.2.5. The free boundaries $z_b(\cdot)$ and $z_s(\cdot)$ are both non-decreasing in $(0, T]$, and when $r \leq 1$, there exists a positive constant M such that

$$\begin{aligned}
&\lim_{z \to 0+} V(\tau, z; 1) = V_0(\tau), && \\
&z_b(\tau) \leq \tfrac{\alpha - r}{\gamma\sigma^2(1+\lambda)}, && \forall \tau \in (0, T], \\
&z_b(\tau) = 0, && \forall \tau \in (0, \tau^*], \\
&z_s(\tau) \geq z_s(0) = \tfrac{\alpha - r}{\gamma\sigma^2(1-\mu)}, && \forall \tau \in (0, T], \\
&z_s(\tau) < M, && \forall \tau \in (0, T],
\end{aligned}$$

where $V_0(\tau) = \min\left\{(1-\mu)\mathbf{e}^{(\alpha-r)\tau}, 1+\lambda\right\}$ and $\tau^* = \frac{1}{\alpha - r}\log\left(\frac{1+\lambda}{1-\mu}\right)$.

Proof: In the first place, the monotonicity can be implied by Corollary 3.2.2 straightforwardly. In the next place, by virtue of Proposition 3.2.4, we can let $z \to 0+$ so that the following system is obtained:

$$\begin{cases}
V_0'(\tau) - (\alpha - r)V_0(\tau) = 0, & 1 - \mu < V_0(\tau) < 1 + \lambda, \\
V_0'(\tau) - (\alpha - r)V_0(\tau) \leq 0, & V_0(\tau) = 1 + \lambda, \\
V_0'(\tau) - (\alpha - r)V_0(\tau) \geq 0, & V_0(\tau) = 1 - \mu, \\
V_0(0) = 1 - \mu. &
\end{cases}$$

Let us solve the following auxiliary ODE

$$\begin{cases}
V_1'(\tau) - (\alpha - r)V_1(\tau) = 0, \\
V_1(0) = 1 - \mu,
\end{cases}$$

which gives $\log(V_1(\tau)) = (\alpha - r)\tau + C_1$ with C_1 being a constant, thus it holds that

$$V_1(\tau) = (1-\mu)\mathbf{e}^{(\alpha-r)\tau}.$$

In addition, it is easy to obseve that it is impossible that $V_0(\tau) = 1 - \mu$ for $\tau > 0$ since otherwise we must have $V_0'(\tau) \geq (\alpha - r)(1 - \mu) > 0$, which contradicts

Corollary 3.2.2. Therefore, it is not difficult to deduce that

$$V_0(\tau) \;=\; \min\left\{(1-\mu)\mathbf{e}^{(\alpha-r)\tau}, 1+\lambda\right\}.$$

Based on this, directly solving the equation $(1-\mu)\mathbf{e}^{(\alpha-r)\tau} = 1+\lambda$ would lead to

$$\tau^* \;=\; \tfrac{1}{\alpha-r}\log\left(\tfrac{1+\lambda}{1-\mu}\right).$$

Moreover, according to Proposition 3.2.3, it must hold that $z_b(\cdot) \leq \frac{\alpha-r}{\gamma\sigma^2(1+\lambda)}$ and $z_s(0) \geq \frac{\alpha-r}{\gamma\sigma^2(1-\mu)}$. Now suppose $z_s(0) > \frac{\alpha-r}{\gamma\sigma^2(1-\mu)}$, in view of the fact that $\mathrm{BR} \subset \left\{z \leq \frac{\alpha-r}{\gamma\sigma^2(1+\lambda)}\right\}$, then the following system must hold in the region $(0,T] \times \left(\frac{\alpha-r}{\gamma\sigma^2(1-\mu)}, z_s(0)\right)$:

$$\begin{cases} \partial_\tau V - \mathcal{L}_z V = 0, \\ V(0,z;1) = 1-\mu. \end{cases}$$

Hence it can be deduced that $\partial_\tau V(0,z;1) = (\alpha-r)(1-\mu) - \gamma\sigma^2 z(1-\mu)^2 < 0$, which contradicts Corollary 3.2.2. Therefore, we must have

$$z_s(0) \;=\; \tfrac{\alpha-r}{\gamma\sigma^2(1-\mu)}.$$

Lastly, let us suppose that for any $M > 1$, there exists $\tau' \in (0,T]$, such that $z_s(\tau') > M$, then it holds that

$$1-\mu < V(\tau', z; 1) < 1+\lambda, \quad \text{for } z \in (z_b(\tau'), M).$$

Let z_0 be as defined by Proposition 3.2.4, while we define

$$\begin{aligned} A \;&:=\; \log(M) > 0, \\ B \;&:=\; \begin{cases} \log(z_b(\tau')), & \text{if } z_b(\tau') > 0, \\ \log\left(\frac{z_0}{2}\right), & \text{if } z_b(\tau') = 0. \end{cases} \end{aligned}$$

Then recalling system (3.10), we have

$$\partial_\tau v(\tau', x; 1) = \mathcal{L}_x v(\tau', x; 1), \quad \text{for } x \in (B, A).$$

In view of the properties that $\partial_x v(\tau', B; 1)$ is bounded according to Proposition 3.2.4, $\partial_\tau v \geq 0$ and $\partial_x v \leq 0$ according to Proposition 3.2.1, integrating the above equation from B to A with respect to x results in

$$\begin{aligned}
0 &\leq \int_B^A \left[\tfrac{1}{2}\sigma^2\partial_{xx}v + \left(\alpha - r + \tfrac{\sigma^2}{2} + \xi_1(\tau')\right)\partial_x v + (\alpha - r)v - \gamma\sigma^2 \mathbf{e}^x v(\partial_x v + v)\right] dx \\
&\leq \tfrac{1}{2}\sigma^2\partial_x v(\tau', A; 1) - \tfrac{1}{2}\sigma^2\partial_x v(\tau', B; 1) + (\alpha - r)(1+\lambda)(A - B) - \gamma\sigma^2\left(\int_B^A \mathbf{e}^x v\partial_x v dx + \int_B^A \mathbf{e}^x v^2 dx\right) \\
&\leq -\tfrac{1}{2}\sigma^2\partial_x v(\tau', B; 1) + (\alpha - r)(1+\lambda)(A - B) - \gamma\sigma^2\left(\int_B^A \mathbf{e}^x v dv + \int_B^A \mathbf{e}^x v^2 dx\right).
\end{aligned}$$

By virtue of integration by parts, it can be deduced that

$$\int_B^A \mathbf{e}^x v dv \;\geq\; \tfrac{1}{2}\mathbf{e}^A(1-\mu)^2 - \tfrac{1}{2}\mathbf{e}^B(1+\lambda)^2 - \tfrac{1}{2}\int_B^A v^2\mathbf{e}^x dx.$$

This further leads to

$$\begin{aligned}
0 &\leq -\tfrac{1}{2}\sigma^2\partial_x v(\tau', B; 1) + (\alpha - r)(1+\lambda)(A - B) \\
&\quad -\tfrac{\gamma\sigma^2}{2}\mathbf{e}^A(1-\mu)^2 + \tfrac{\gamma\sigma^2}{2}\mathbf{e}^B(1+\lambda)^2 - \tfrac{\gamma\sigma^2}{2}\int_B^A v^2\mathbf{e}^x dx \\
&\to -\infty, \qquad \text{as } M \to \infty,
\end{aligned}$$

which is obviously a contradiction. Hence there exist a positive constant M such that

$$z_s(\tau) < M, \quad \forall \tau \in (0, T].$$

These complete the proof. □

Proposition 3.2.6. When $r \leq 1$, $z_b(\cdot)$ and $z_s(\cdot)$ are both continuous in $(0, T]$.

Proof: In the first place, we prove $z_b(\cdot) \in \mathcal{C}(0, T]$. Otherwise, there should exist a region $(0, \tau_1) \times (z_1, z_2)$ with $0 \leq z_1 < z_2 \leq \frac{\alpha - r}{\gamma\sigma^2(1+\lambda)}$ such that the following system holds in $(0, \tau_1) \times (z_1, z_2)$:

$$\begin{cases} \partial_\tau V - \mathcal{L}_z V = 0, \\ V(\tau_1, z; 1) = 1 + \lambda. \end{cases}$$

Then, if we define $W(\tau, z; 1) := \partial_z V(\tau, z; 1)$, W would satisfy the following system in $(0,\tau_1)\times(z_1,z_2)$:

$$\begin{cases} \partial_\tau W - \frac{1}{2}\sigma^2 z^2 \partial_{zz} W - (\alpha - r + 2\sigma^2 + \xi_1(\tau)) z \partial_z W - (2(\alpha - r) + \sigma^2 + \xi_1(\tau)) W \\ \qquad + \gamma\sigma^2 z^2 V \partial_z W + \gamma\sigma^2 z (zW + 4V) W = -\gamma\sigma^2 V^2 \le 0, \\ W(\tau_1, z; 1) = 0. \end{cases}$$

Since $W = \partial_z V \le 0$, W achieves its non-negative maximum on $\tau = \tau_1$. Applying the maximum principle, we have $\partial_z V = W \equiv 0$ in the region $(0,\tau_1)\times(z_1,z_2)$, which is clearly a contradiction given $V > 0$. Thus we may conclude $z_b(\cdot) \in \mathcal{C}(0,T]$.

In the next place, we prove $z_s(\cdot) \in \mathcal{C}(0,T]$. Otherwise, there should exist a region $(\tau_1, T)\times(z_1,z_2)$ with $\frac{\alpha-r}{\gamma\sigma^2(1-\mu)} \le z_1 < z_2$ such that the following system holds in $(\tau_1,T)\times(z_1,z_2)$:

$$\begin{cases} \partial_\tau V - \mathcal{L}_z V = 0, \\ V(\tau_1, z; 1) = 1 - \mu. \end{cases}$$

Hence it can be deduced that $\partial_\tau V(\tau_1, z; 1) = (\alpha - r)(1-\mu) - \gamma\sigma^2 z (1-\mu)^2 < 0$, which contradicts Corollary 3.2.2. Thus we may conclude $z_s(\cdot) \in \mathcal{C}(0,T]$. These complete the proof. □

Intuitively, we may have the separation of the three transaction regions as shown in the following graph:

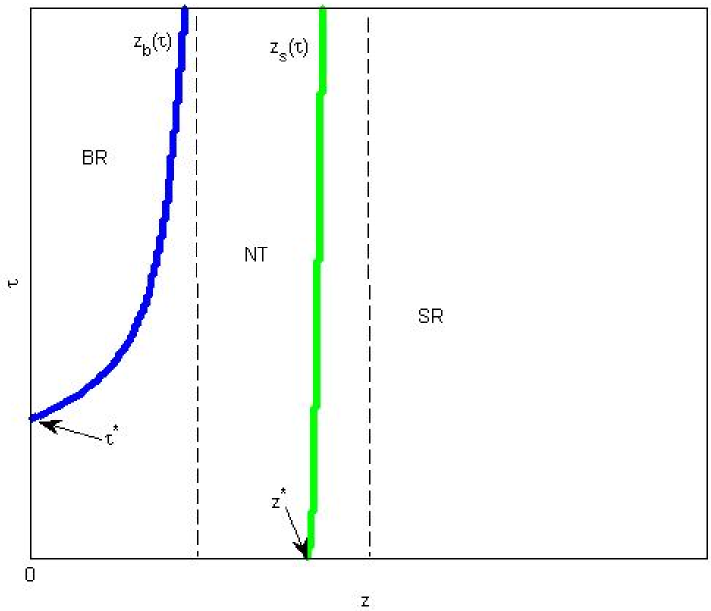

Figure 3.1. Plot of the three transaction regions for the optimal investment and consumption problem for a CARA investor.

3.2.3 Equivalence between HJB system and double obstacle problem

In this section, we attempt to rigorously establish the equivalence between the original HJB system (3.3) when $\kappa = 1$ and the double obstacle problem (3.9). Since the equivalence between problem (3.3) and problem (3.8) is obvious, it suffices for us to show the following results:

Proposition 3.2.7. Let $V(\tau, z; 1)$ be the solution to the double obstacle problem (3.9). Define

$$\zeta(\tau, z; 1) \;\; := \;\; A(\tau) - \gamma(1-\mu)z_s(\tau) - \gamma \int_{z_s(\tau)^z} V(\tau, u)du,$$

where $A(\tau)$ satisfies the following ODE system:

$$\begin{aligned} A'(\tau) &= -\xi_1(\tau)A(\tau) + \tfrac{1}{2}\gamma^2\sigma^2(1-\mu)^2 z_s^2(\tau) - (\alpha - r)\gamma(1-\mu)z_s(\tau) \\ &\quad +\xi_1(\tau)(1 - \log(\xi_1(\tau))) - \delta, \\ A(0) &= 0. \end{aligned}$$

Then $\zeta(\tau, z; 1)$ is the solution to problem (3.8).

Proof: Given $z_s(0) > 0$ as it has been shown in Proposition 3.2.5, simple calculation would verify the initial condition as follows:

$$\zeta(0, z; 1) = \begin{cases} -\gamma(1-\mu)z, & z \geq 0, \\ -\gamma(1+\lambda)z, & z < 0. \end{cases}$$

It is worth noting that $\partial_z\zeta(\tau, z; 1) = -\gamma V(\tau, z; 1)$, which implies ζ, $\partial_z\zeta$ and $\partial_{zz}\zeta$ are continuous across $z_s(\tau)$ in view of Proposition 3.2.1. Moreover, $z_s(\tau) \in \mathcal{C}^\infty(0, T]$ can be shown in the same method as provided in [16] in view of Corollary 3.2.2, thus we have $\partial_\tau\zeta$ is continuous across $z_s(\tau)$ as well. Now given

$$\zeta(\tau, z; 1) = A(\tau) - \gamma(1-\mu)z, \quad z \geq z_s(\tau),$$

we then have

$$\begin{aligned} \mathcal{L}_2\zeta|_{z=z_s(\tau)} &= \tfrac{1}{2}\gamma^2\sigma^2(1-\mu)^2 z_s^2(\tau) - (\alpha - r)\gamma(1-\mu)z_s(\tau) - \xi_1(\tau)A(\tau) \\ &\quad +\xi_1(\tau)(1 - \log(\xi_1(\tau))) - \delta \\ &= A'(\tau) = \partial_\tau\zeta, \end{aligned}$$

where the second last equality is due to the definition of $A(\tau)$.

Note that

$$\partial_z(\partial_\tau - \mathcal{L}_2\zeta) = -\gamma(\partial_\tau V - \mathcal{L}_z V),$$

where V is the solution to the double obstacle problem (3.9), then we have

$$\begin{cases} \partial_z(\partial_\tau - \mathcal{L}_2\zeta) \leq 0, & \text{if } z \geq z_s(\tau), \\ \partial_z(\partial_\tau - \mathcal{L}_2\zeta) = 0, & \text{if } z_b(\tau) < z < z_s(\tau), \\ \partial_z(\partial_\tau - \mathcal{L}_2\zeta) \geq 0, & \text{if } z \leq z_b(\tau). \end{cases}$$

Combining with the result we have shown above

$$\partial_\tau \zeta - \mathcal{L}_2 \zeta = 0, \quad \text{at } z = z_s(\tau),$$

it can be deduced that

$$\begin{cases} \partial_\tau \zeta - \mathcal{L}_2 \zeta = 0, & \text{if } z_b(\tau) < z < z_s(\tau), \\ \partial_\tau \zeta - \mathcal{L}_2 \zeta \leq 0, & \text{if } z \leq z_b(\tau) \text{ or } z \geq z_s(\tau), \end{cases}$$

which completes the proof. □

3.2.4 Comparison between the problems with or without consumption

In this section, we would make comparison between the optimal investment problem and the optimal investment and consumption problem, or specifically, the double obstacle problem (3.6) without consumption and the double obstacle problem (3.9) with consumption.

It is worth noting that the variable y within the HJB system (3.3) represents the same absolute amount of investment in the risky asset for both the no-consumption case and the consumption case, while different transformations for the new variable z are used in both of the cases where $z := \mathbf{e}^{r\tau} y$ in no-consumption case and $z := \xi_1(\tau) y$ in consumption case. In order to reveal the ordering relation of the free boundaries between the two cases, we impose a transformation $w := \frac{\mathbf{e}^{r\tau}}{\xi_1(\tau)} z$, $\overline{V}(\tau, w; 1) := V(\tau, z; 1)$ on problem (3.9) to make it consistent with problem (3.6), then the following system is induced:

$$\begin{cases} \overline{V}_\tau - \mathcal{L}_w \overline{V} = 0, & \text{if } 1 - \mu < \overline{V} < 1 + \lambda, \\ \overline{V}_\tau - \mathcal{L}_w \overline{V} \leq 0, & \text{if } \overline{V} = 1 + \lambda, \\ \overline{V}_\tau - \mathcal{L}_w \overline{V} \geq 0, & \text{if } \overline{V} = 1 - \mu, \\ \overline{V}(0, w; 1) = (1 - \mu) \cdot \mathbf{1}_{w \geq 0} + (1 + \lambda) \cdot \mathbf{1}_{w < 0}, \end{cases} \tag{3.12}$$

where

$$\mathcal{L}_w \overline{V} \;=\; \tfrac{1}{2}\sigma^2 w^2 \overline{V}_{ww} + (\alpha - r + \sigma^2) w \overline{V}_w + (\alpha - r)\overline{V} - \tfrac{r}{\mathrm{e}^{r\tau}-1+r}\gamma\sigma^2 w \overline{V}(w\overline{V}_w + \overline{V}).$$

Correspondingly, we may define the buying boundary and the selling boundary for this problem as

$$\begin{aligned} \overline{z_b}(\tau) &:= \sup\{z \geq 0 | \overline{V}(\tau, z; 1) = 1 + \lambda\}, \quad \tau \in (0, T], \\ \overline{z_s}(\tau) &:= \inf\{z \geq 0 | \overline{V}(\tau, z; 1) = 1 - \mu\}, \quad \tau \in (0, T]. \end{aligned}$$

Recalling system (3.6), the system that the value function $V(\tau, z; 0)$ satisfies can be rewritten as follows:

$$\begin{cases} V_\tau - \mathcal{L}_w V = \left(\frac{r}{\mathrm{e}^{r\tau}-1+r} - 1\right)\gamma\sigma^2 z V(zV_z + V), & \text{if } 1-\mu < V < 1+\lambda, \\ V_\tau - \mathcal{L}_w V \leq \left(\frac{r}{\mathrm{e}^{r\tau}-1+r} - 1\right)\gamma\sigma^2 z V(zV_z + V), & \text{if } V = 1+\lambda, \\ V_\tau - \mathcal{L}_w V \geq \left(\frac{r}{\mathrm{e}^{r\tau}-1+r} - 1\right)\gamma\sigma^2 z V(zV_z + V), & \text{if } V = 1-\mu, \\ V(0, z; 0) = (1-\mu)\cdot \mathbf{1}_{z\geq 0} + (1+\lambda)\cdot \mathbf{1}_{z<0}, \end{cases} \tag{3.13}$$

where it has been shown that $zV_z + V \geq 0$ in [45]. We may also define the buying boundary and the selling boundary for this problem as

$$\begin{aligned} \tilde{z}_b(\tau) &:= \sup\{z \geq 0 | V(\tau, z; 0) = 1 + \lambda\}, \quad \tau \in (0, T], \\ \tilde{z}_s(\tau) &:= \inf\{z \geq 0 | V(\tau, z; 0) = 1 - \mu\}, \quad \tau \in (0, T], \end{aligned}$$

whose behaviors have essentially been characterized in [45]. Note that these free boundaries are consistent with the $\overline{z_b}$ and $\overline{z_s}$ as defined above. We will reveal the ordering relations of the free boundaries between these two cases in the following proposition:

Proposition 3.2.8. The free boundaries $\tilde{z}_b(\tau)$ and $\tilde{z}_s(\tau)$ in the no consumption case and $\overline{z_b}(\tau)$ and $\overline{z_s}(\tau)$ in the consumption case have the following relations for any $\tau \in (0, T]$:

$$\begin{aligned} \tilde{z}_b(\tau) &\leq \overline{z_b}(\tau), \\ \tilde{z}_s(\tau) &\leq \overline{z_s}(\tau). \end{aligned}$$

Proof: On the one hand, it is trivial to observe in $\{z < 0\}$ that

$$V(\tau, z; 0) = \overline{V}(\tau, z; 1) = 1 + \lambda,$$

as argued respectively before. On the other hand, given $zV_z + V \geq 0$ in problem (3.13), applying the maximum principle to system (3.12) and system (3.13) in $\{z > 0\}$ leads to

$$V(\tau, z; 0) \leq \overline{V}(\tau, z; 1).$$

These further imply that

$$\begin{aligned} \{(\tau, z) : V(\tau, z; 0) = 1 + \lambda\} \subset \{(\tau, z) : \overline{V}(\tau, z; 1) = 1 + \lambda\}, \\ \{(\tau, z) : \overline{V}(\tau, z; 1) = 1 - \mu\} \subset \{(\tau, z) : V(\tau, z; 0) = 1 - \mu\}, \end{aligned}$$

which may directly result in

$$\begin{aligned} \tilde{z}_b(\tau) &\leq \overline{z_b}(\tau), \\ \tilde{z}_s(\tau) &\leq \overline{z_s}(\tau). \end{aligned}$$

These complete the proof. □

Essentially, this result reveals intuitively that optimal investment strategy of a CARA investor is more conservative in the no-consumption case compared against the consumption case.

3.2.5 Comparison between the problems with or without consumption in the case without transaction costs

As a special case of the optimal investment problems with or without consumption, the idealized setting in the absence of transaction costs is of particular interest. By setting the proportional transaction cost rates $\lambda = \mu = 0$ and introduce the wealth process $W(t) := X(t) + Y(t)$, problem (3.2) can be readdressed as follows:

$$\begin{aligned} & \varphi_0(s,w;\kappa) \\ := \ & \inf_{(Y,C)} \mathbb{E}\left[\kappa \int_s^T \mathbf{e}^{-\delta(t-s)} \cdot \mathbf{e}^{-\gamma C(t)} dt + \mathbf{e}^{-\delta(T-s)} \cdot \mathbf{e}^{-\gamma W(T)} \Big| W(s-) = w\right] \\ & \text{s.t. } dW(t) \ = \ (rW(t-) - C(t))dt + (\alpha - r)Y(t-)dt + \sigma Y(t-)d\mathcal{B}(t). \end{aligned} \tag{3.14}$$

In the case with consumption, the relevant HJB equation for problem (3.14) is

$$\begin{cases} \partial_t\varphi_0 + rw\partial_w\varphi_0 + (\alpha - r)y_1^*(t)\partial_w\varphi_0 + \frac{\partial_w\varphi_0}{\gamma}\left(\log\left(-\frac{\partial_w\varphi_0}{\gamma}\right) - 1\right) - \delta\varphi = 0, \\ \varphi_0(T,w;1) = \mathbf{e}^{-\gamma w}, \end{cases} \tag{3.15}$$

where

$$y_1^*(t) \ = \ -\frac{(\alpha-r)\partial_w\varphi_0}{\sigma^2\partial_{ww}\varphi_0} = \frac{\alpha-r}{\gamma\sigma^2}\frac{1-(1-r)\mathbf{e}^{-r(T-t)}}{r},$$

represents the optimal investment amount in the risky asset at time t. In the other case without consumption, the relevant HJB equation for problem (3.14) is

$$\begin{cases} \partial_t\varphi_0 + rw\partial_w\varphi_0 + (\alpha - r)y_0^*(t)\partial_w\varphi_0 - \delta\varphi = 0, \\ \varphi_0(T,w;0) = \mathbf{e}^{-\gamma w}, \end{cases} \tag{3.16}$$

where

$$y_0^*(t) \ = \ -\frac{(\alpha-r)\partial_w\varphi_0}{\sigma^2\partial_{ww}\varphi_0} = \frac{\alpha-r}{\gamma\sigma^2}\mathbf{e}^{-r(T-t)},$$

also represents the optimal investment amount in the risky asset at time t.

Comparison can thus be easily made between $y_0^*(t)$ and $y_1^*(t)$ as

$$y_1^*(t) = \frac{\alpha-r}{\gamma\sigma^2}\frac{1-(1-r)\mathbf{e}^{-r(T-t)}}{r} = \frac{\alpha-r}{\gamma\sigma^2}\left(\mathbf{e}^{-r(T-t)} + \frac{1-\mathbf{e}^{-r(T-t)}}{r}\right) > \frac{\alpha-r}{\gamma\sigma^2}\mathbf{e}^{-r(T-t)} = y_0^*(t).$$

This is essentially consistent with the results obtained in Section 3.2.4.

3.2.6 The infinite-horizon optimal investment and consumption problem

In [31], the infinite-horizon optimal consumption and investment policy of a CARA investor was studied, where the optimal investment strategy is characterized by the following ODE system by specifying two critical values ξ_b and ξ_s with $\xi_b < \xi_s$:

$$\begin{cases} \frac{1}{2}\sigma^2\xi^2\left(\psi'' - (\psi')^2\right) + \alpha\xi\psi' - r\psi + \delta - r, & \text{if } \xi_b < \xi < \xi_s, \\ \psi(\xi_b) = (1+\lambda)\xi_b + \tilde{C}_1, \\ \psi'(\xi_b) = 1+\lambda, \\ \psi''(\xi_b) = 0, \\ \psi(\xi_s) = (1-\mu)\xi_s + \tilde{C}_2, \\ \psi'(\xi_s) = 1-\mu, \\ \psi''(\xi_s) = 0, \end{cases} \tag{3.17}$$

where $\tilde{C}_1$ and $\tilde{C}_x$ are two constants to be determined.

Meanwhile, when letting $\tau \to \infty$, which implies $\xi_1(\cdot) \equiv r$, problem (3.9) becomes

$$\begin{cases} \mathcal{L}_\infty V_\infty = 0, & \text{if } 1-\mu < V_\infty < 1+\lambda, \\ \mathcal{L}_\infty V_\infty \geq 0, & \text{if } V_\infty = 1+\lambda, \\ \mathcal{L}_\infty V_\infty \leq 0, & \text{if } V_\infty = 1-\mu, \end{cases} \tag{3.18}$$

where

$$\mathcal{L}_\infty V_\infty \;=\; \tfrac{1}{2}\sigma^2 z^2 V_\infty'' + (\alpha+\sigma^2) z V_\infty' + (\alpha - r)V_\infty - \gamma\sigma^2 z V_\infty (zV_\infty' + V_\infty).$$

In the following, we show that we can deduce the ODE system (3.17) from the stationary double obstacle problem (3.18). If we define

$$\begin{aligned} z_{b,\infty} &:= \lim_{\tau\to\infty} z_b(\tau), \\ z_{s,\infty} &:= \lim_{\tau\to\infty} z_s(\tau), \end{aligned}$$

where it obviously holds $0 < z_{b,\infty} < z_{s,\infty}$, then problem (3.18) can be rewritten as the following stationary free boundary problem:

$$\begin{cases} \mathcal{L}_\infty V_\infty = 0, & \text{if } z_{b,\infty} < z < z_{s,\infty}, \\ V_\infty(z_{b,\infty}) = 1+\lambda, \\ V_\infty'(z_{b,\infty}) = 0, \\ V_\infty(z_{s,\infty}) = 1-\mu, \\ V_\infty'(z_{s,\infty}) = 0. \end{cases} \tag{3.19}$$

Proposition 3.2.9. If V_∞ is the solution to problem (3.19) and we define

$$\psi(\xi) \; := \; \gamma \int_0^{\frac{\xi}{\gamma}} V_\infty(\eta) d\eta + C_1,$$

where

$$C_1 \; := \; \frac{1}{r}\left(-\frac{1}{2}\sigma^2\gamma^2(1+\lambda)^2 z_{b,\infty}^2 + (\alpha - r)\gamma(1+\lambda)z_{b,\infty} + \delta - r, \right),$$

then $\psi(\xi)$ is a solution to problem (3.17) and $\xi_b = \gamma z_{b,\infty}$, $\xi_s = \gamma z_{s,\infty}$.

Proof: It can be easily verified that

$$\psi(\xi) \; = \; \begin{cases} (1+\lambda)\xi + C_1, & \text{if } \xi < \gamma z_{b,\infty}, \\ (1-\mu)\xi + C_2, & \text{if } \xi > \gamma z_{s,\infty}, \end{cases}$$

where $C_2 := \gamma \int_0^{z_{s,\infty}} V_\infty(\eta) d\eta + C_1 - \gamma(1-\mu)z_{s,\infty}$. Moreover, given that V_∞ is the solution to problem (3.19), it can be shown that $V_\infty(z)$ and $V_\infty'(z)$ are continuous, which implies $\psi(\xi)$, $\psi'(\xi)$ and $\psi''(\xi)$ are continuous. Thus, applying a similar argument as in Proposition 3.2.7, we can show that $\psi(\xi)$ satisfies

$$\begin{cases} \frac{1}{2}\sigma^2\xi^2\left(\psi'' - (\psi')^2\right) + \alpha\xi\psi' - r\psi + \delta - r, & \text{if } \gamma z_{b,\infty} < \xi < \gamma z_{s,\infty}, \\ \psi(\gamma z_{b,\infty}) = (1+\lambda)\gamma z_{b,\infty} + C_1, \\ \psi'(\gamma z_{b,\infty}) = 1 + \lambda, \\ \psi''(\gamma z_{b,\infty}) = 0, \\ \psi(\gamma z_{s,\infty}) = (1-\mu)\gamma z_{s,\infty} + C_2, \\ \psi'(\gamma z_{s,\infty}) = 1 - \mu, \\ \psi''(\gamma z_{s,\infty}) = 0. \end{cases}$$

This problem is obviously equivalent to problem (3.17), with $\tilde{C}_1 = C_1$ and $\tilde{C}_2 = C_2$. Note that this is a standard free boundary elliptic problem, whose solution is unique, we may then conclude that $\xi_b = \gamma z_{b,\infty}$ and $\xi_s = \gamma z_{s,\infty}$. These complete the proof. □

3.3 The optimal investment problem with jump diffusion

3.3.1 Formulation of the optimal investment problem with jump diffusion

There is plenty of evidence that every now and then there are sudden unexpected rises or falls in the real financial markets. On all but the shortest timescales the sudden movements appear discontinuous. This striking feature gives rise to the jump diffusion model that was initiated by Merton ([36]), where the Poisson process is added into the building blocks of the geometric Brownian Motion. By virtue of the negative wealth tolerance for the CARA investor's problem, we will incorporate the jump diffusion feature into the optimal investment problem with transaction costs.

In Merton's jump diffusion model, the SDE for the risky asset price becomes

$$dS_1(t) = S_1(t-)[\alpha dt + \sigma d\mathcal{B}(t) + (J-1)dN(t)],$$

where $N(t)$ represents a Poisson process with intensity rate parameter β, and J can be drawn from a pre-specified nonnegative probability distribution which induces the proportional jump magnitude. Thus the investor's position processes turn to the following diffusion equations:

$$\begin{cases} dX(t) = rX(t-)dt - (1+\lambda)dL(t) + (1-\mu)dM(t), \\ dY(t) = \alpha Y(t-)dt + \sigma Y(t-)d\mathcal{B}(t) + (J-1)Y(t-)dN(t) + dL(t) - dM(t). \end{cases} \tag{3.20}$$

In the presence of the jumping term, the optimal investment problem for a

CARA investor now becomes the following problem:

$$\begin{aligned} \varphi(s,x,y;0) \quad &:= \quad \inf_{(L,M)\in\mathcal{A}_C} \mathbb{E}\left[\mathbf{e}^{-\gamma w(X(T),Y(T))}|X(s-)=x, Y(s-)=y\right] \\ & \quad\quad \text{s.t. } (3.20), \end{aligned} \tag{3.21}$$

for any $s \in [0,T]$ and $(x,y) \in \mathbb{R}^2$, where $\mathcal{A}_C$ is as defined in subsection 3.1.1. Note that negative wealth is possible when jump occurs, but it is tolerated by the exponential utility function, which makes the modeling of the problem with jump diffusion robust.

3.3.2 The HJB system and problem simplification

Applying the same principle of dynamic programming, the relevant HJB system for problem (3.21) can then be derived as follows:

$$\begin{cases} \max\left\{-\varphi_t - \frac{1}{2}\sigma^2 y^2 \varphi_{yy} - \alpha y \varphi_y - rx\varphi_x - \beta\cdot\mathbb{E}[\varphi(t,x,Jy)-\varphi(t,x,y)],\right. \\ \qquad \left.\varphi_y - (1-\mu)\varphi_x, -\varphi_y + (1+\lambda)\varphi_x\right\} = 0, \\ \varphi(T,x,y;0) = \mathbf{e}^{-\gamma w(x,y)}. \end{cases} \tag{3.22}$$

Bringing in ϕ that satisfies $\varphi(t,x,y;0) = \mathbf{e}^{-\gamma x\exp\{r(T-t)\}}\phi(t,y;0)$, we obtain the following system:

$$\begin{cases} \max\left\{-\phi_t - \frac{1}{2}\sigma^2 y^2 \phi_{yy} - \alpha y \phi_y - \beta\cdot\mathbb{E}[\phi(t,Jy)-\phi(t,y)],\right. \\ \qquad \left.\phi_y + (1-\mu)\gamma\mathbf{e}^{r(T-t)}\phi, -\phi_y - (1+\lambda)\gamma\mathbf{e}^{r(T-t)}\phi\right\} = 0, \\ \phi(T,y;0) = \mathbf{e}^{-\gamma g(y)}, \end{cases} \tag{3.23}$$

where

$$g(y) \quad := \quad \begin{cases} (1-\mu)y, & y \geq 0, \\ (1+\lambda)y, & y < 0. \end{cases}$$

Here ϕ is also strictly positive, and the it is clearly equivalent to the following system:

$$\begin{cases} \phi_t + \mathcal{L}_y\phi = 0, & \text{if} \quad -(1+\lambda)\gamma\mathbf{e}^{r(T-t)} < \frac{\phi_y}{\phi} < -(1-\mu)\gamma\mathbf{e}^{r(T-t)}, \\ \phi_t + \mathcal{L}_y\phi \geq 0, & \text{if} \quad \frac{\phi_y}{\phi} = -(1+\lambda)\gamma\mathbf{e}^{r(T-t)}, \text{ or } \frac{\phi_y}{\phi} = -(1-\mu)\gamma\mathbf{e}^{r(T-t)}, \\ \phi(T,y;0) = \mathbf{e}^{-\gamma g(y)}, \end{cases}$$

where $\mathcal{L}_y\phi = \frac{1}{2}\sigma^2y^2\phi_{yy} + \alpha y\phi_y + \beta\cdot\mathbb{E}[\phi(t,Jy)-\phi(t,y)]$.

Now the same transformation can be made to further simplify the system by introducing $\tau := T-t$, $z := \mathbf{e}^{r\tau}y$, $\zeta(\tau,z;0) := \log\phi(t,y;0)$. The system could be rescheduled as follows:

$$\begin{cases} \zeta_\tau - \mathcal{L}'_z\zeta = 0, & \text{if} \quad -(1+\lambda)\gamma < \zeta_z < -(1-\mu)\gamma, \\ \zeta_\tau - \mathcal{L}'_z\zeta \leq 0, & \text{if} \quad \zeta_z = -(1+\lambda)\gamma, \text{ or } \zeta_z = -(1-\mu)\gamma, \\ \zeta(0,z;0) = -\gamma g(z), \end{cases} \tag{3.24}$$

where $\mathcal{L}'_z\zeta = \frac{1}{2}\sigma^2z^2(\zeta_{zz}+\zeta_z^2) + (\alpha-r)z\zeta_z + \beta\cdot\mathbb{E}[\mathbf{e}^{\zeta(\tau,Jz)-\zeta(\tau,z)}-1]$. However, due to the existence of the jump diffusion term in the system, we are unable to arrive at the parabolic double obstacle problem (3.6) as we addressed above. We can only obtain the following system by letting $V(\tau,z;0) = -\frac{1}{\gamma}\cdot\zeta(\tau,z;0)$:

$$\begin{cases} V_\tau - \mathcal{L}_zV = 0, & \text{if} \quad 1-\mu < V_z < 1+\lambda, \\ V_\tau - \mathcal{L}_zV \geq 0, & \text{if} \quad V_z = 1+\lambda, \text{ or } V_z = 1-\mu, \\ V(0,z;0) = g(z), \end{cases} \tag{3.25}$$

where $\mathcal{L}_zV = \frac{1}{2}\sigma^2z^2(V_{zz}-\gamma V_z^2) + (\alpha-r)zV_z - \frac{\beta}{\gamma}\cdot\mathbb{E}\left[\mathbf{e}^{-\gamma V(\tau,Jz)+\gamma V(\tau,z)}-1\right]$.

3.4 Numerical methods

3.4.1 The optimal investment problem

Now we provide some numerical methods on the CARA investor's problems with transaction costs. Let us focus on the parabolic double obstacle problem (3.6) within $(0,T]\times(0,\infty)$ given $\{z<0\}\subset$ BR in the first place. Moreover, although the explicit analytical solution is not available, we could consider applying penalty method to the double obstacle problem to attain numerical solutions.

Let us first transform problem (3.6) to convert the domain $(0,T]\times(0,\infty)$ into a bounded region. By denoting a new variable $x:=\frac{z}{z+1}$, and $v(\tau,x;0):=V(\tau,z;0)$, we may then obtain a new bounded problem:

$$\begin{cases} v_\tau-\mathcal{L}_x v=0, 1-\mu<v<1+\lambda, \\ v_\tau-\mathcal{L}_x v\le 0, v=1+\lambda, \\ v_\tau-\mathcal{L}_x v\ge 0, v=1-\mu, \\ v(0,x;0)=1-\mu, x\in[0,1], \\ v(\tau,0;0)=\min\left\{1+\lambda,(1-\mu)\mathbf{e}^{(\alpha-r)\tau}\right\}, \\ v(\tau,1;0)=1-\mu, \tau\in(0,T], \end{cases} \tag{3.26}$$

where

$$\mathcal{L}_x v=\frac{1}{2}\sigma^2x^2(1-x)^2v_{xx}+(\alpha-r+(1-x)\sigma^2)x(1-x)v_x+(\alpha-r)v-\gamma\sigma^2x(xv_x+\frac{1}{1-x}v)v,$$

where the domain becomes $(0,T]\times(0,1)$. Note that the boundary condition at $x=0$ is obtained by solving the corresponding ODE system

$$\begin{cases} v_\tau-(\alpha-r)v=0, \quad 1-\mu<v<1+\lambda, \\ v_\tau-(\alpha-r)v\le 0, \quad v=1+\lambda, \\ v_\tau-(\alpha-r)v\ge 0, \quad v=1-\mu, \\ v|_{\tau=0}=1-\mu. \end{cases}$$

Now we attempt to reconstruct the variational inequality into equality using penalty methods, similar to that studied in [11]. We define C as a positive controlling number and $\epsilon\ll\frac{1}{C}$ as a small regularization parameter depending on the choice of C. Thus problem (3.26) can be approximated by the following problem:

$$\begin{cases} v_\tau - \mathcal{L}_x v + \frac{\epsilon C}{(1+\lambda-v)^+ + \epsilon} - \frac{\epsilon C}{(v-1+\mu)^+ + \epsilon} = 0, \\ v(0,x;0) = 1-\mu, \\ v(\tau,0;0) = \min\left\{1+\lambda, (1-\mu)\mathbf{e}^{(\alpha-r)\tau}\right\}, \\ v(\tau,1;0) = 1-\mu. \end{cases} \tag{3.27}$$

It can be easily seen that $v_\tau - \mathcal{L}_x v = 0$ dominates in NT $(1+\lambda < v < 1-\mu)$; $v_\tau - \mathcal{L}_x v \leq 0$ dominates in BR $(v = 1+\lambda)$; and $v_\tau - \mathcal{L}_x v \geq 0$ dominates in SR $(v = 1-\mu)$ as ϵ approaches to 0. The proper choice of C and ϵ would make the solution to problem (3.27) a good approximation to problem (3.26).

As a standard procedure, we apply finite difference method to the above system and make discretization on the domain $(0,T] \times (0,1)$. Fully implicit scheme is adopted for linear terms, while the nonlinear terms, including the penalty terms, are treated explicitly for simplicity reasons (although Newton-Raphson iteration method can be used to treat such non-linear terms implicitly instead):

$$\begin{cases} \begin{aligned} \frac{v_j^{n+1}-v_j^n}{\Delta t} & - \frac{1}{2}\sigma^2 x_j^2 (1-x_j)^2 \cdot \left[\frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{\Delta x^2}\right] \\ & - (\alpha - r + (1-x_j)\sigma^2) x_j (1-x_j) \cdot \left[\frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2\Delta x}\right] \\ & - (\alpha - r) \cdot v_j^{n+1} \\ & + \gamma\sigma^2 x_j^2 v_j^n \cdot \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} \\ & + \gamma\sigma^2 \frac{x_j}{1-x_j} \cdot (v_j^n)^2 \\ & + \frac{\epsilon C}{\left(1+\lambda-v_j^n\right)^+ + \epsilon} - \frac{\epsilon C}{\left(v_j^n - 1 + \mu\right)^+ + \epsilon} = 0, \\ v_j^0 & = 1-\mu, j = 0..M, \\ v_0^n & = \min\left\{1+\lambda, (1-\mu)\mathbf{e}^{(\alpha-r)n\Delta t}\right\}, n = 1..N, \\ v_M^n & = 1-\mu, n = 1..N, \end{aligned} \end{cases} \tag{3.28}$$

where there are M and N grids in spatial dimension and temporal dimension respectively. The step length in spatial dimension is denoted by $\Delta x \equiv \frac{1}{M}$ and that in temporal dimension is denoted by $\Delta t \equiv \frac{T}{N}$, then the truncation error of the system is $O(\Delta t + \Delta z^2)$. Note that we have to set $\frac{\Delta t}{\Delta x^2} \ll 1$ to guarantee the convergence of this scheme. LU decomposition can be employed in the matrix computation and we obtain each v_i^n iteratively.

As an example, the parameters are set as follows: $r = 0.01$, $\alpha = 0.035$, $\sigma = 0.3$, $\mu = 0.01$, $\lambda = 0.005$, $T = 1$, with $\gamma = 0.5$. The free boundaries solved by the above scheme are as shown in Figure 3.2 below.

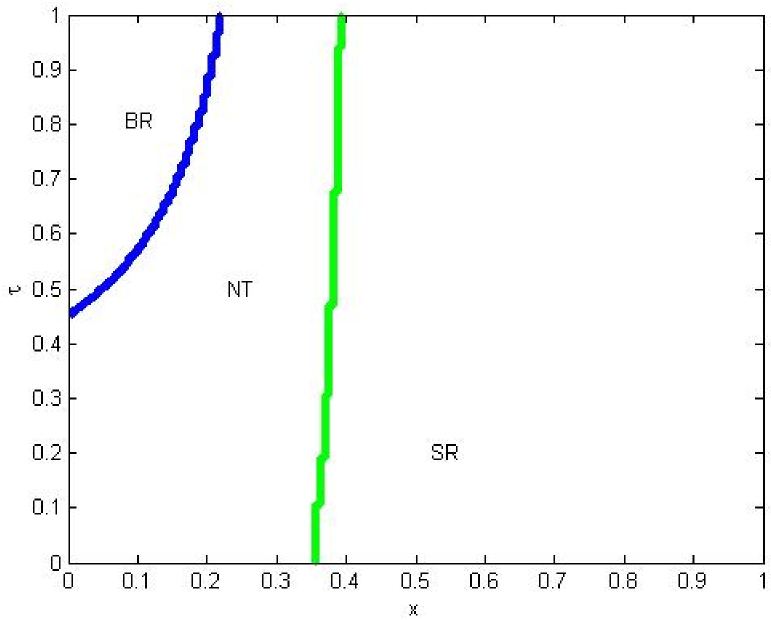

Figure 3.2. Plot of the optimal buying and selling boundaries across the finite horizon for the CARA investor. The parameter values used are: $r = 0.01$, $\alpha = 0.035$, $\sigma = 0.3$, $\mu = 0.01$, $\lambda = 0.005$, $T = 1$, $\gamma = 0.5$. Note that $x = \frac{z}{z+1}$, $z = \mathbf{e}^{r(T-t)}y$, $\tau = T - t$.

3.4.2 The optimal investment and consumption problem

Given the operator L_z in the parabolic double obstacle problem (3.9) is degenerate at $z = 0$ and $(0, T] \times (-\infty, 0] \subset \mathrm{BR}$, we confine our study only in $(0, T] \times (0, \infty)$. Furthermore, if we let $x := \frac{z}{z+1}$, and $v(\tau, x; 1) := V(\tau, z; 1)$, then the system is transformed into

$$\begin{cases} v_\tau - \mathcal{L}_x v = 0, 1-\mu < v < 1+\lambda, \\ v_\tau - \mathcal{L}_x v \leq 0, v = 1+\lambda, \\ v_\tau - \mathcal{L}_x v \geq 0, v = 1-\mu, \\ v(0,x;1) = 1-\mu, x \in [0,1], \\ v(\tau,0;1) = \min\left\{(1-\mu)\mathbf{e}^{(\alpha-r)\tau}, 1+\lambda\right\}, \tau \in (0,T], \\ v(\tau,1;1) = 1-\mu, \tau \in (0,T], \end{cases} \tag{3.29}$$

where

$$\begin{aligned} \mathcal{L}_x v &= \tfrac{1}{2}\sigma^2 x^2 (1-x)^2 v_{xx} + (\alpha - r + (1-x)\sigma^2 + \xi_1(\tau)) x(1-x) v_x + (\alpha - r) v \\ &\quad -\gamma\sigma^2 x \left(x v_x + \tfrac{1}{1-x} v \right) v, \end{aligned}$$

and the domain becomes $(0,T] \times (0,1)$.

Similarly, we attempt to reconstruct the variational inequality into equality using penalty methods by introducing C as a positive controlling number and $\epsilon \ll \frac{1}{C}$ as a small regularization parameter depending on the choice of C. Thus the system can be approximated by the following problem:

$$\begin{cases} v_\tau - \mathcal{L}_x v + \frac{\epsilon C}{(1+\lambda-v)^+ + \epsilon} - \frac{\epsilon C}{(v-1+\mu)^+ + \epsilon} = 0, \\ v(0,x;1) = 1-\mu, \\ v(\tau,0;1) = \min\left\{(1-\mu)\mathbf{e}^{(\alpha-r)\tau}, 1+\lambda\right\}, \tau \in (0,T], \\ v(\tau,1;1) = 1-\mu. \end{cases} \tag{3.30}$$

It can be easily seen that $v_\tau - \mathcal{L}_x v = 0$ dominates in NT $(1+\lambda < v < 1-\mu)$; $v_\tau - \mathcal{L}_x v \leq 0$ dominates in BR $(v = 1+\lambda)$; and $v_\tau - \mathcal{L}_x v \geq 0$ dominates in SR $(v = 1-\mu)$ as ϵ approaches to 0. The proper choice of C and ϵ would make the solution a good approximation.

For the same example as before with the parameter settings being $r = 0.01$, $\alpha = 0.035$, $\sigma = 0.3$, $\mu = 0.01$, $\lambda = 0.005$, $T = 1$, $\gamma = 0.5$, the free boundaries solved by the above scheme are as shown in Figure 3.3 below.

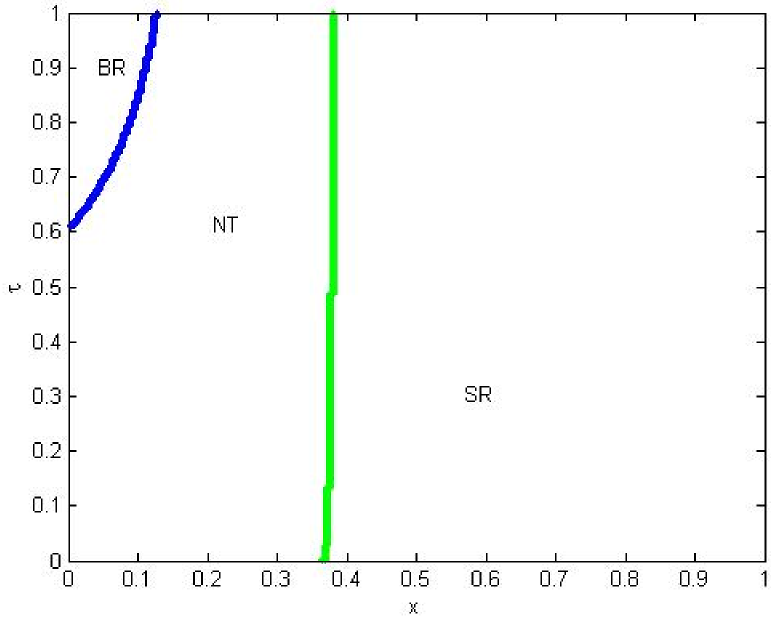

Figure 3.3. Plot of the optimal buying and selling boundaries across the finite horizon for the CARA investor with consumption. The parameter values used are: $r = 0.01$, $\alpha = 0.035$, $\sigma = 0.3$, $\mu = 0.01$, $\lambda = 0.005$, $T = 1$, $\gamma = 0.5$. Note that $x = \frac{z}{z+1}$, $z = \mathbf{e}^{r(T-t)}y$, $\tau = T - t$.

3.4.3 The optimal investment problem with jump diffusion

For the optimal investment problem with jump diffusion feature, we attempt to apply finite difference method to solve the PDE system (3.25) as well. For simplicity reasons, we only model downside jump risk, which is often observed in financial markets, by fixing the proportional jump magnitude random variable $J = j$ almost surely for some $j \in (0, 1)$.

The system (3.25), with gradient constraints, can also be considered as a bounded PDE system by manually imposing two boundaries at $z = 0$ and $z = l^*$. It is expected that the region $\{z < 0\}$ is fully contained in the buying region and the region $\{z > l^*\}$ is fully contained in the selling region, thus we exclude the consideration of these cases. The following PDE system is then obtained:

$$\begin{cases} V_\tau - \mathcal{L}_z V = 0, & \text{if } z_b^\star(\tau) < z < z_s^\star(\tau), \\ V_z = 1+\lambda, & \text{if } 0 \le z \le z_b^\star(\tau), \\ V_z = 1-\mu, & \text{if } z_s^\star(\tau) \le z \le l^*, \\ V(0,z;0) = g(z), \end{cases} \tag{3.31}$$

in the finite domain $(0,T] \times (0,l^*)$. Moreover, it is worth noting that once $z_b^\star(\tau)$ and $z_s^\star(\tau)$ are obtained, the value function expression in $[0, z_b^\star(\tau)]$ and $[z_s^\star(\tau), l^*]$ can be simplified as follows:

$$V(\tau,z;0) = \begin{cases} V(\tau, z_b^\star(\tau);0) - (1+\lambda)(z_b^\star(\tau) - z), & \text{if } z \in [0, z_b^\star(\tau)], \\ V(\tau, z_s^\star(\tau);0) + (1-\mu)(z - z_s^\star(\tau)), & \text{if } z \in [z_s^\star(\tau), l^*]. \end{cases}$$

An N-by-M grid is set up over the domain $(0,T] \times (0,l^*)$, and we let $\Delta\tau := \frac{T}{N}$ and $\Delta z := \frac{L_2 - L_1}{M}$. The mesh points are

$$\{(\tau_n, z_i) : \tau_n = n\Delta\tau, z_i = i\Delta z, n = 0,1,\ldots,N, i = 0,1,\ldots,M\},$$

and we denote $V(\tau_n, z_i)$ by V_i^n. For each time step $n+1$, knowing all V_i^n values, we use the discrete version of $V_\tau - \mathcal{L}_z V = 0$ to obtain V_i^{n+1}. Applying finite difference method with implicit scheme and upwind scheme to $V_\tau - \mathcal{L}_z V = 0$, while treating the nonlinear terms explicitly, we have

$$\begin{aligned} 0 \;=\; & \tfrac{V_i^{n+1} - V_i^n}{\Delta\tau} - \tfrac{1}{2}\sigma^2 z_i^2 \cdot \tfrac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{\Delta z^2} - (\alpha - r) z_i \cdot \tfrac{V_{i+1}^{n+1} - V_i^{n+1}}{\Delta z} \\ & + \tfrac{1}{2}\sigma^2 z_i^2 \gamma \left(\tfrac{V_{i+1}^n - V_{i-1}^n}{2\Delta z} \right)^2 + \tfrac{\beta}{\gamma} \cdot \mathbf{e}^{\gamma(V_i^n - V(\tau_n, j \cdot z_i))}. \end{aligned} \tag{3.32}$$

Note that we can also use Newton-Raphson iteration method to produce an implicit scheme to deal with the nonlinear terms. This bounded PDE system is solved with Neumann boundary conditions, and the truncation error is $O(\Delta t + \Delta z)$. Note that $(\tau_n, j \cdot z_i)$ may not fall on a specific node of the grid, so we need to adopt certain interpolation method to estimate $V(\tau_n, j \cdot z_i)$. LU decomposition could be employed in the following matrix computation and we may obtain each V_i^{n+1}.

The partial derivatives of V at time step $n+1$ can then be approximated and used to determine the positions of $z_b^\star(\tau_{n+1})$ and $z_s^\star(\tau_{n+1})$, the approximated free boundaries. Utilizing such information, we need to update V_i^{n+1} over the intervals $[0, z_b^\star(\tau_{n+1})]$ and $[z_s^\star(\tau_{n+1}), 1]$ respectively before we move on to the next time step $n+2$.

For the same example as before with the parameter settings being $r = 0.01$, $\alpha = 0.035$, $\sigma = 0.3$, $\mu = 0.01$, $\lambda = 0.005$, $T = 1$, $\gamma = 0.5$, while we impose $l^* = 1$. Firstly, we consider the case $j = 0.8$, and the three sets of free boundaries obtained according to the above scheme with different β values are as shown in Figure 3.4 below.

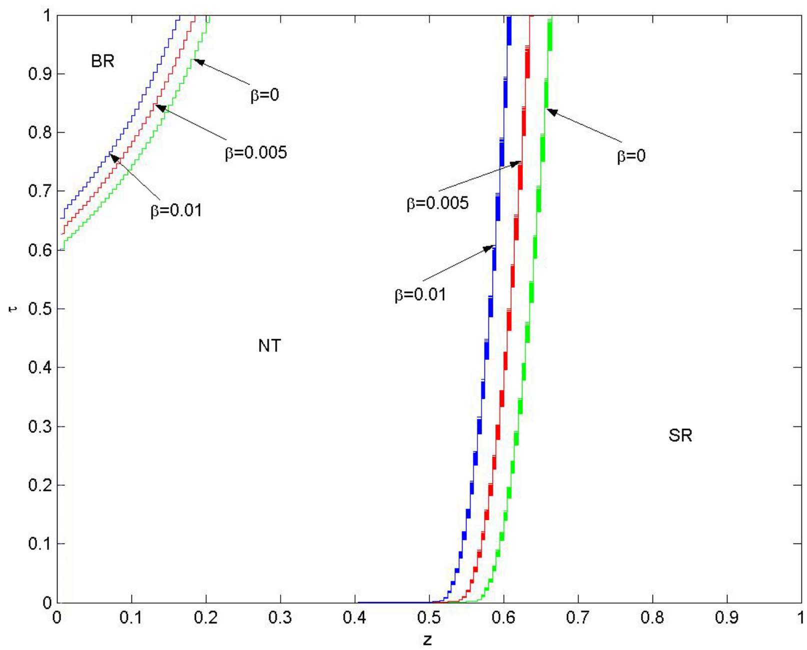

Figure 3.4. Plot of the optimal buying and selling boundaries with different jump intensity rates across the finite horizon. The parameter values used are: $r = 0.01$, $\alpha = 0.035$, $\sigma = 0.3$, $\mu = 0.01$, $\lambda = 0.005$, $T = 1$, $\gamma = 0.5$, $j = 0.8$. Note that $z = \mathbf{e}^{r(T-t)}y$, $\tau = T - t$.

Secondly, another case $j = 0.6$ is considered with different β values and the three sets of free boundaries obtained in the same manner are as shown in Figure 3.5 below.

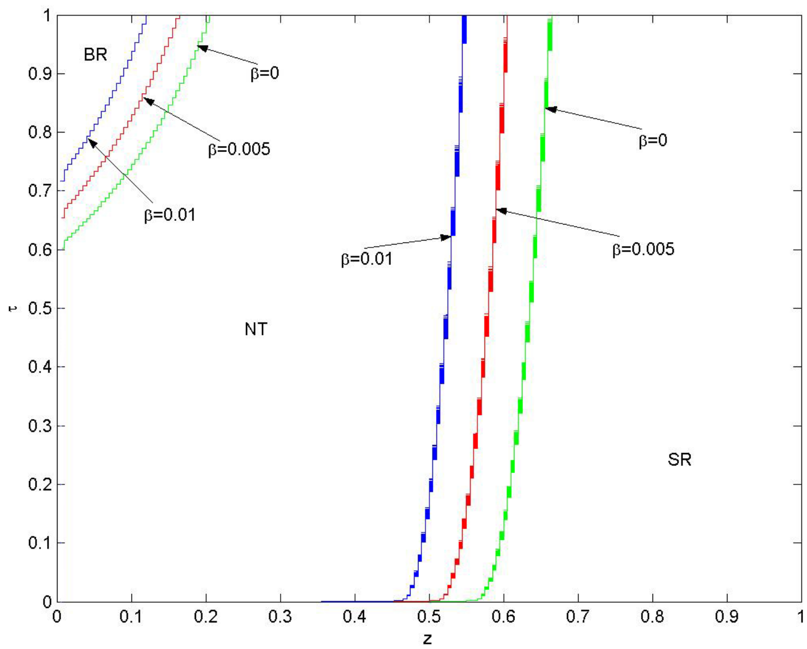

Figure 3.5. Plot of the optimal buying and selling boundaries with different jump intensity rates across the finite horizon. The parameter values used are: $r = 0.01$, $\alpha = 0.035$, $\sigma = 0.3$, $\mu = 0.01$, $\lambda = 0.005$, $T = 1$, $\gamma = 0.5$, $j = 0.6$. Note that $z = \mathbf{e}^{r(T-t)}y$, $\tau = T - t$.

An interesting observation from these graphs is that the buying region shrinks as β, the intensity rate parameter, increases, while the selling region grows as β increases. One natural explanation is that the investor should be more conservative in managing his investment portfolio when the downside jump risk increases.

Chapter 4

Conclusion

In this thesis, the continuous-time finite-horizon optimal investment (and consumption) problems with proportional transaction costs were studied in probabilistic and PDE approaches. Since the problems were all investigated in a finite-horizon setting, the three transaction regions, known as "jump-buy region", "jump-sell region" and "no-transaction region", as well as the optimal investment strategies are horizon-dependent, and the regions are no longer fixed but are varying through time, which make them more difficult than those with infinite-horizon setting.

The continuous-time finite-horizon optimal investment problem with transaction costs for a CRRA investor with logarithm utility function was investigated in the first part of this thesis, and the problem was formulated as singular stochastic control problem. Monotonicity, concavity, homotheticity, and continuity of the value function were proved, and the three transaction regions were shown to be convex cones. A relevant standard stochastic control problem was then constructed based on the result that it is never optimal to exercise "jump-buy" or "jump-sell" during the whole horizon except the initial time and terminal time. This technique is important, as the jumps of the diffusion processes arising from the singularity of controls are eliminated heuristically. By studying this standard stochastic control problem, it was shown in a probabilistic approach that the region with negative states of the risky asset should always be contained in the “jump-buy region”, or in other words, the CRRA investor that applies an optimal investment strategy should never take short positions in the risky asset during the whole time horizon. Utilizing such characteristic, a new diffusion process was brought in as the quotient of the original two diffusion processes in order to reduce the dimensionality of the standard stochastic control problem from two to one. This is inspired by the similarity reduction in Davis and Norman (1990) towards the value function, but it is comparatively more fundamental since both the value function and the problem have been simplified. It is worth pointing out, however, that the dimensionality reduction for the problem with a power utility function, associated with another type of CRRA investor, cannot be achieved using the same approach, although the dimensionality reduction of the value function can be done via PDE approach. Based on the new stochastic control problem, the connections between this problem and an optimal stopping problem were established in the “no-transaction region” with the existence of optimal stochastic controls and under certain parameter restrictions. It is discussed in Section 2.2.5 the difficulties we have encountered in other parameter settings and our intuitive conjecture that may inspire future research in these cases. Shown by rigorous analysis, the connections under such parameter restrictions present that the optimal risk of the optimal stopping problem is in fact the gradient of the value function of the stochastic control problem, and the optimal stopping times are the first times when the optimal stochastic controls, if exist, become non-zero separately. We expect that the existence of the optimal controls can be guaranteed for the stochastic control problems and such connections may apply for the original singular stochastic control problem and the optimal stopping problem across the whole solvency region. Future researches are encouraged to verify these analytically and establish the connections completely.

In the second part of this thesis, the continuous-time finite-horizon optimal investment and consumption problem with transaction costs for a CARA investor, who has an exponential utility function instead, was investigated through PDE approach, which constitutes the major contribution of this thesis. A probabilistic argument was presented for the problem without consumption to separate the state variable of the riskless asset and hence the optimal investment strategy only depends on the absolute value of the endowment in the risky asset instead of the relative ratio of the two assets. The relevant HJB systems, in both the no-consumption case and the consumption case, were then transformed to two nonlinear parabolic double obstacle problems in different ways respectively, while the equivalence for the consumption case was revealed analytically. Important properties of the value function and the free boundaries for the optimal investment and consumption problem were shown analytically through rigorous PDE arguments. It was revealed that the problem is degenerate at zero and the regularity and monotonicity of the value function were obtained. Based on these, monotonicity, continuity, shapes and ranges of the free boundaries for the optimal investment and consumption problem were obtained analytically. Comparison between the two cases with and without consumption was further provided, which reveals the ordering relations of the free boundaries for the two problems and the investor's optimal investment strategy is more conservative in the no-consumption case. Besides, the infinite-horizon optimal investment and consumption problem was deduced from the stationary double obstacle problem, which was shown equivalent to the system obtained in Liu (2004). In addition, since the exponential utility function may tolerate negative wealth possibly incurred by the jumping nature, the jump diffusion feature was able to be incorporated in the CARA investor's optimal investment problem and a variational inequality system with gradient constraints was obtained through similar dimensionality reduction. Finite difference methods were implemented to numerically solve the systems, while it was revealed that the CARA investor should be more conservative in managing his investment portfolio when the downside jump risk increases in the end.

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CONTINUOUS-TIME FINITE-HORIZON OPTIMAL INVESTMENT AND CONSUMPTION PROBLEMS WITH PROPORTIONAL TRANSACTION COSTS

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