COMPOSITE NONLINEAR FEEDBACK CONTROL FOR SYSTEMS WITH ACTUATOR SATURATION — TOWARDS IMPROVED TRACKING PERFORMANCE

HE YINGJIE
***(B. Eng, Shanghai Jiaotong University, P. R. China*)**

A THESIS SUBMITTED
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
NATIONAL UNIVERSITY OF SINGAPORE

2005

To my family

Acknowledgements

During my years as a postgraduate student at National University of Singapore, I have benefitted from interactions with many people to whom I am deeply grateful. I would like to express my grateful appreciation to those who have guided me during my postgraduate course in National University of Singapore, in one way or another.

First of all, I wish to express my utmost gratitude to my supervisor, Prof. Ben M. Chen for his unfailing guidance and encouragement throughout the course of my research project in both professional and personal aspects of life. Prof. Chen's successive and endless enthusiasm in research arouses my interest in various aspects of control engineering. I have indeed benefitted tremendously from the many discussions I have had with him.

I am also privileged by the close and warm association with my labmates in the Control and Simulation laboratory. I would appreciate the opportunity to interact extensively with Dr Kemao Peng, especially benefiting from his enlightening perspectives. I would like to thank Dr Weiyao Lan, Dr Miaobo Dong and Mr. Guoyang Cheng, Mr. Chao Wu who is now pursing his PhD in US, for their tremendous effort in giving me valuable advice and ideas. I would also like to thank Dr Huajing Tang, Ms. Rui Yan, Mr. Shengqiang Ding, Ms. Yu Sun, Mr. Hanle Zhu and Mr. Wei Wang for their valuable comments and advice, and all the exchange of information in the laboratory. All these have made my postgraduate studies in NUS an unforgettable and enjoyable experience.

I would also like to thank my buddies, friends and other postgraduate students who have in one way or another, rendered their encouragement and helped me greatly enjoy my course of study in NUS. Special thanks go to my family, especially my wife Ma Qin, for their endless love, care and support throughout the years of my study. Naturally I would like to dedicate this work to my dearest wife and my recently born daughter.

Last but not least, I would take this opportunity to thank NUS for its financial support without which I might not have come to Singapore, and my postgraduate study in control engineering might remain a dream for ever.

YINGJIE HE

Kent Ridge, Singapore

August 2005

Contents

Summary

The problem of tracking control for linear systems has been investigated for a fairly long time. When actuator saturates, the controller designed based on ideal assumptions without saturation will cause system performance degrade and even destabilize the whole system. In this thesis, the author aims at proposing a simple control structure yet with improved performance for set-point tracking as in the literature very few works have been done on transient performance improvement. The reason lies in that it is difficult to consider transient performance for more general references tracking. As for set-point tracking, indices like settling time, rise time, overshoot and so on are well defined.

Based on any linear feedback law found using previously proposed methods in the literature which solves the tracking problem under actuator saturation, a so-called Composite Nonlinear Feedback control method is proposed. Both the state feedback case and the measurement feedback case are considered without imposing any restrictive assumption on the given systems, i.e., the systems considered are controllable and also observable for measurement back cases. The composite nonlinear feedback control consists of a linear feedback law and a nonlinear feedback law without any switching element. Typically, the linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time not exceeding the actuator limits for the desired command input levels. This can be done by using any previously developed methods in the literature. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part.

The results for linear continuous-time systems follow some previously reported results where they all consider only certain special cases. Either they consider only some specific class of systems like second-order systems, or only state feedback case for more general systems yet with a restrictive condition imposed on the systems, or although they consider state feedback and measurement feedback cases the systems under investigation are single variable systems. The first objective of my work is to generalize this CNF scheme to its most general form for linear systems. The author considers linear continuous-time and discrete-time systems and all cases of state feedback and measurement feedback. Examples will be given to show the effectiveness of this methodology. A fairly complete theory for CNF control technique has been established.

To go a step further, it is possible to apply this CNF scheme to more general systems. Firstly, it is applied to nonlinear linearizable systems under actuator saturation. Next, the author extends the CNF scheme to be applicable to partially linear composite systems. The partially linear composite system includes two parts, the linear one with actuator saturation and the nonlinear zero dynamics. The output of the linear system is connected to the nonlinear zero dynamics as input. It turns out that by making the output of the saturated linear part decrease faster than a certain exponential rate, the stability of the whole connected system is sustained with improved transient performance.

Finally the author discusses the possible applications of the CNF control scheme and points out some further topics for future research.

Chapter 1

Introduction

Control theory and engineering plays a more and more important role in everyday life nowadays and quite a complete theory has been established in this field. However, in practice, when a controller is implemented, saturation of elements may cause system performance degrade a lot, which has to be investigated carefully in order to obtain satisfactory performance. Due to both its theoretical and practical importance, tracking control, together with tracking control under saturation, has been studied for a fairly long time (Saberi *et al.*, 1999 [63]). From the 1950's many important advancements have been achieved by several researchers, yet the controller structures proposed tend to be rather complex. The author's focus, however, will be exclusively on proposing a simple controller structure while at the same time improving transient performance for set-point tracking or constant reference/signal tracking problem of input constrained linear systems or, linear systems with actuator saturation or constrained input.

I will review some related important results for tracking problem under saturation. Then I will propose my own solution to this classical problem. Especially, I will look into the problem of improving the closed-loop transient response, which is rather important from a practical point of view and rarely considered in the literature. The controller design is based on linear feedback controllers proposed already in other researchers' papers. The reason for using linear controller as a base is obvious as it has a very simply controller structure and thus can be very easily implemented. Based on this linear feedback controller which gives exact tracking under saturation, let one add additional nonlinear law so that by tuning some gains carefully one gets better performance. This idea is not new but we fully explore it and extend previous results to its most general case. Eventually, an easily constructed controller with a simple structure can then be obtained which gives better performance than its linear counterpart. It will also be extended to some classes of nonlinear systems with actuator saturation. I believe that it will contribute to the development of many real application controllers and provide insights into improving transient response for even more general systems.

This chapter serves to give the background and motivation for this research. The research scope and contributions of this research and the organization of this thesis will also be briefly explored.

1.1 Background and Motivation

Control engineering is a fundamental and important field of technology which is applied in almost any man-made systems nowadays. Although many significant achievements (Bennett, 1993 [7]), *e.g.*, spacecraft motion control, satellite status control, high-precision positioning control in micro-electronic manufacturing plant, have been reached in this fascinating area, there are still quite a lot of unsolved problems. For example, as a very central topic in modern as well as classical control theory, tracking control still remains not fully understood (Saberi *et al.*, 1999 [63]). On the other hand, even though we have a good tracking controller design at hand, when it is applied in real applications, system performance usually degrades a lot from what one expects. The presence of saturations, especially actuator saturation is one major reason (Hu and Lin, 2001 [36]). In order to reduce the adverse effect caused by actuator saturation, many efforts have been done on the topic of tracking control of systems with actuator saturation.

Roughly speaking, there are two methods adopted in the literature in order to deal with the adverse effect caused by saturation in tracking problems. One is an indirect approach which, based on controller designed by ignoring saturations at first, modifies this controller by considering saturations. It turns out that the indirect approach tends to produce fairly complex controllers, called anti-windup scheme and typically, these controllers are not easily implemented in practice. The other approach, the direct approach, considers saturations at the onset of controller design and hence provides a straightforward method which can take into consideration of many performance requirements. The controller turns out to be much less complex. Along this latter line, a lot of significant results have been obtained during the last two decades. The author's work, falls in this latter approach too and in fact, this approach dated back to time-optimal control in the 1950's.

Bang-bang control or time-optimal control may be the first attempt to tackle actuator saturation in set-point tracking and naturally this is a direct approach. Although theoretically this scheme can achieve exact point-to-point tracking with shortest time, the controller obtained is a nonlinear one and is non-robust to parameter uncertainty and thus it is rarely implemented in real applications (Athans and Falb, 1966 [4]). Later as a modification to Bang-bang control, PTOS or Proximal Time-Optimal Control was proposed by Workman (1987) [78] in order to get fast and accurate positioning performance in Hard Disk Drives. In order to deal with uncertainty, adaptive PTOS scheme was also proposed. The limitation of PTOS is obvious as it is applicable only to double integrator systems.

As a continual effort to find effective alternatives to Bang-bang control, except the above-mentioned PTOS, many other control schemes dealing with actuator saturation have been proposed, Berstein and Michel (1995) [8]. As a major breakthrough, Gutman and Hagander (1986) [30] presented a systematic (and also direct) method to find stabilizing saturated linear state feedback controllers for linear continuous-time and discrete-time systems. The method is theoretically sound and applicable to tracking not only constant signals and considers general actuator and state saturations whether they be symmetric or not, but it is not easily applied in actual controller design as no explicit and numerically efficient algorithm has been proposed. Trial and error seems inevitable and this can become a tedious job.

Another important result was due to Blanchini and Miani (2000) [11]. Starting from the stabilization problem for linear systems with control and state constraints, the authors proved that any domain of attraction for linear systems with state and actuator constraints is actually also a constant constraint-admissible reference tracking domain of attraction. They showed that the tracking controller can be inferred from the stabilizing (possibly nonlinear) controller associated with the domain of attraction. The main contributions of this paper is that it gives a clear connection between domain of attraction and set-point tracking domain of attraction for linear constrained systems and also gives some relation between the constant constraint-admissible tracking output sets and the tracking domain of attraction (of initial conditions). Again these results are more of theoretical significance and the proposed controller design procedure is quite complex.

Some researchers, however, investigated this sort of tracking problem from other perspectives and offered interesting insights (*e.g.*, Teel, 1992 [71] and Romanchuck, 1995 [61]). Teel (1992) [71] considered nonlinear tracking of an integrator chain of arbitrary order while Romanchuck (1995) [61] examined tracking for linear constrained systems from an input output point of view. Some other literature has been concerned with how a linear feedback can be constructed so that control constraints are not violated, for example Bitsoris (1998 a,b) [9, 10]. The merits of a linear controller are obvious as it can be implemented easily due to its simple structure and thus practically attractive.

It is worth noting that when dealing with set-point tracking, the so-called reference management approach was also proposed in the framework of model predictive control (Bemporad *et al.*, 1997 [5]) and uncertain linear systems (Bemporad and Mosca, 1998 [6]). An improved error governor and a reference governor based on the concept of maximal output admissible sets were adopted to track reference signals inside some constraint set for the output in Gilbert and Tan (1991) [26] and Gilbert *et al.* (1995) [27] respectively. In Graettinger and Krogh (1992) [29], the authors considered the computation of reference signal constraints for guaranteed tracking performance in supervisory control environment. These ideas were also adopted in Blanchini and Miani (2000) [11].

Although there seem to be many schemes proposed for set-point tracking, many are rarely implemented in practice due to either their complicated and computationally expensive structure, or their lack of correspondence to practical engineering systems. So far the only schemes designed to cope with control limits and to be implemented are the retro-fitted anti-windup compensators (Turner *et al.*, 2000) [74]. Thus controllers with simple structure become very appealing in real applications and thus the method proposed by Lin *et al.* (1998) [53], which was later called Composite Nonlinear Feedback (CNF) control, has attracted much attention.

1.2 Composite Nonlinear Feedback (CNF) Control

Rather recently, a new method of achieving accurate tracking in linear systems, while heeding control constraints was suggested by Lin *et al.* (1998) [53], which was built on previous work found in Lin and Saberi (1995) [56]. They proposed a nonlinear state feedback control which was the composition of a nominal linear feedback, superposed with a novel nonlinear feedback (this scheme, was named Composite Nonlinear Feedback (CNF) control by Chen *et al.* (2003) [19]). They showed that for an arbitrary nonnegative nonlinear element in the nonlinear feedback, the system would asymptotically track a constant reference signal, and that the state would be confined to a certain ellipsoidal domain of attraction. Furthermore, they gave a great deal of insight on how to choose the nonlinear parameter in their feedback scheme. Of course, the size of the reference signal which could be tracked was bounded by an a priori determined amount, but simulations on a flight control system indicated excellent results (Lin *et al.*, 1998 [53]).

Indeed, the power of Lin et al's results was only limited by their scope: they were confined to single-input-single-output (SISO) second-order linear systems. Later Turner et al (2000) [74] generalized many of Lin et al's results to higher order and multivariable systems and simulations on a helicopter pitch control and an MIMO missile control showed better performance than conventional linear controllers. And Chen *et al.* (2003) [19] extended it to general linear SISO systems but considered state feedback case as well as measurement feedback cases. However, Chen *et al.* (2003) [19] didn't consider MIMO systems and the extension reported in Turner *et al.* (2000) [74] was made under a pretty odd assumption (Chen *et al.*, 2003 [19]) on the system that excludes many systems including those originally considered in Lin *et al.* (1998) [53]. Also as in Lin *et al.* (1998) [53], only state feedback is considered in Turner *et al.* (2000) [74]. The author's work, will remove all these restrictions, and will extend this CNF control to general linear continuous-time or discrete-time SISO or MIMO systems with state or measurement feedback control and thus make this scheme complete (Lin *et al.*, 1998 [53]).

1.3 Towards Improving Transient Performance

Even though many results have been obtained about how to design a controller for a saturated linear systems, the transient performance is not considered in most of these works. It is a tough task to study the transient performance of the general tracking problem, especially when the reference inputs are time-varying signals. On the other hand, since it is well understood in the literature that certain performance indexes can be established for set-point tracking purposes, for example, settling time, rise time, overshoot, undershoot and so on, let me limit the scope to considering in this work a tracking control problem with a constant (or step) reference. Namely, I will consider the following multivariable linear system Σ with an amplitude-constrained actuator characterized by

$$\begin{cases} \delta(x) = A\ x + B\ \mathrm{sat}(u), \quad x(0) = x_0 \\ y \ \ = C_1\ x \\ h \ \ = C_2\ x + D_2\ \mathrm{sat}(u) \end{cases} \tag{1.1}$$

where $\delta x = \dot{x}$ if Σ is a continuous-time systems, or $\delta x = x(k+1)$ if Σ is a discrete-time systems. As usual, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $h \in \mathbb{R}^\ell$ are respectively the state, control input, measurement output and controlled output of the given system Σ. A, B, C_1 and C_2 are appropriate dimensional constant matrices, and the saturation function is defined by

$$\mathrm{sat}(u) = \begin{pmatrix} \mathrm{sat}(u_1) \\ \mathrm{sat}(u_2) \\ \vdots \\ \mathrm{sat}(u_m) \end{pmatrix}, \tag{1.2}$$

with

$$\text{sat}(u_i) = \text{sign}(u_i)\ \min(|u_i|,\ \bar{u}_i), \tag{1.3}$$

where $\bar{u}_i$ is the maximum amplitude of the i-th control channel. The objective of this work is to design an appropriate control law for (1.1) using the CNF approach such that the resulting controlled output will track some desired step references as fast and as smooth as possible. I will address the CNF control system design for the given system (1.1) for three different situations, namely, the state feedback case, the full order measurement feedback case, and the reduced order measurement feedback case. For tracking purpose, the following assumptions on the given system are required: i) (A, B) is stabilizable; ii) (A, C_1) is detectable; and iii) (A, B, C_2, D_2) is right invertible and has no invariant zeros at $s = 0$ (for continuous-time systems), or $z = 1$ (for continuous-time systems). The objective here is to design control laws that are capable of achieving fast tracking of target references under input saturation. As such, it is well understood in the literature that these assumptions are standard and necessary.

We note that this approach is based on a linear feedback controller found with any previously proposed method in the literature (see, *e.g.*, Blanchini and Miani, 2000 [11]; Gutman and Hagander, 1986 [30]; Bitsoris, 1988a,b [9, 10]), but the resulting controller outperforms these linear controllers by adding additional nonlinear feedback law to the original linear control law which doesn't violate the control constraints. It is noted that when the gains in the nonlinear feedback law vanish, the whole controller reverts to the linear controller. Therefore, one has additional freedom in choosing these gains in order to get better transient performance. The issues regarding domain of attraction, admissible tracking reference signals and other related problems can be explored similarly by using the methods suggested in the literature (see, *e.g.*, Gutman and Hagander, 1986 [30]); Blanchini and Miani, 2000 [11]; Gilbert and Tan, 1991 [26] and the references therein). Of course, the initial conditions should be met and thus must be investigated carefully when one applies this CNF control scheme.

In Blanchini and Miani (2000) [11], the authors suggested also possible nonlinear controllers as their controller was inferred from original stabilizing (possibly nonlinear) controller yet the procedure may not be easily implemented. The CNF controller, however, has a very simple structure and is quite easily constructed.

Finally, it is worth emphasizing that in the literature, much research has been conducted on stabilization problem for systems under actuator saturation or even state saturation, output saturation. It is a common approach when dealing with tracking problem without saturation by transforming it into a stabilization problem. However, when saturation occurs, this approach is not so seemingly available. Rather, people try to solve the tracking problem directly. Although there are many results on stabilization, semi-global and even global stabilization for systems with actuator saturation, their results are mostly limited to the so-called Asymptotical Null Controllable linear systems with Bounded Control (ANCBC), and a recent book Hu and Lin (2001) [36] reflects most updated results achieved during the past years. My focus, is exclusively on a controller with simple structure yet provides one certain freedom to improve closed-loop transient performance and this approach can be applied to general systems, not necessarily ANCBC systems. The simple structure of linear controller is of special interest to practitioners and researchers, which hopefully may be used extensively in practice.

1.4 Contributions of This Research

As a matter of fact, this work will help to complete the theory for CNF control for continuous-time and discrete-time, SISO or MIMO linear systems with state feedback or measurement feedback control. Thus, it is possible for control engineers to adopt this scheme like other practically popular methods, say PID, Model Predictive Control and so on. I believe that this work will benefit them by providing a new choice of design tools in order to obtain improved performance.

The major theoretical contribution of this work is that for the first time, from a rather general perspective, the problem of improving system transient tracking performance under actuator saturation is fully discussed and the CNF controller proves to be effective to reach this target with its simple structure. In fact, by setting the saturation level to very high values, it is easy to see that one can improve transient tracking performance for systems without saturation also. Thus one can explore this possibility when doing normal controller design.

In order to show the effectiveness of the CNF scheme, I will apply it to some real application problems. One is an air-air missile autopilot system which was also considered in Turner *et al.* (2000) [74] but I will apply this method and see whether the simulation results are at least as good as those given by Turner *et al.* (2000) [74] or even better. We will also consider measurement feedback cases which were not covered in Turner *et al.* (2000) [74]. The other example is a Magnetic-Tape-Drive system cited from a standard textbook Franklin *et al.* (1998) [24], which is a discrete-time system application and compare both performances. These simulation examples will serve to verify the theory and also give one certain practical experience about how to tune the parameters for nonlinear feedback law, which, like gains tuning in multivariable control theory, is far from maturity. Rather the tuning method is mainly based on users' experience.

Although I will try to extend the CNF control scheme to its most general form possible, I will study only the set-point tracking problem for linear systems with symmetric actuator saturation. Similar results regarding asymmetric saturations may be sought by shifting the center of the saturation limits. For tracking a group of reference signals not necessarily constant ones, other methods for example, those developed for output regulation (see, *e.g.* Saberi *et al.*, 1999 [63]) or those proposed in the works previously mentioned may be used. Also, it is still too early to expect satisfactory results on improving transient performance for general reference tracking problem.

Finally, it is also of interest for one to apply this control scheme to nonlinear systems. I will extend it to a class of nonlinear linearizable SISO systems and simulation on a pendulum system is given in this thesis. It should also be extended to nonlinear linearizable MIMO systems but the result may be quite restricted. Still further, I will extend this method to partially linear systems where its zero dynamics is nonlinear in nature. It might also be extended to even more general nonlinear systems. However, this is not so easy due to the complex nature of general nonlinear systems. Typically researchers in nonlinear tracking control focus on the so-called output regulation problem without any saturation in the system (Byrnes *et al.*, 1997 [13]). Also they consider only reference signals produced by an exo-system which are neutrally stable, and thus excluding step function signals. For step function signals tracking, people tend to convert this problem to a nonlinear regulation or stabilization problem. When actuator saturation comes into picture, very few works have been done. We hope that the CNF control approach may provide some insights into solving nonlinear tracking problem and improving its tracking transient performance as well.

1.5 Organization of Thesis

This thesis is organized as follows.

In Chapter 2, I will extend the CNF control to linear continuous-time MIMO system, which still renders asymptotic tracking in state feedback case and measurement feedback case. I will also give some guidelines for selecting the key parameter in the proposed controller. An application in an air-air missile autopilot system and a numerical example are included to show the effectiveness of the proposed design methodology.

Parallel to Chapter2, I will extend the CNF control to linear discrete-time MIMO system in Chapter 3. Again, three cases of feedback laws are considered. An application in a Magnetic-Tape-Drive system shows significant transient performance improvement.

Chapter 4 applies the developed CNF control scheme to nonlinear linearizable continuous-time SISO systems. It is applied in a pendulum system. Further extension to nonlinear linearizable continuous-time MIMO systems is possible but the results will be restricted. Similarly, extension to discrete-time systems is quite obvious but not explored in detail in this Chapter.

In the next two chapters, extension of CNF to be applied in partial linear systems is presented. Results for continuous-time systems are reported in Chapter 5 while those for discrete-time systems are presented in Chapter 6. For partial linear systems, since their zero dynamics is nonlinear, the problem of peaking phenomenon in linear part should be examined carefully in order not to drive the zero dynamics to infinity which destabilizes the whole system. Simulation examples will be included to verify the results.

In Chapter 7, I will discuss a so-called asymptotical time-optimal tracking control problem for double integrator systems, which was originally posed in [18] as an open problem. Interestingly, CNF controller can be a good candidate for practically solving this problem. I will give detailed results with rigorous analysis to this problem and propose some suboptimal yet practical controller designs.

Finally, conclusions, discussions and recommendation for future work will be discussed in the last chapter, Chapter 8.

Chapter 2

CNF Control for Continuous-Time Systems with Input Saturation

In this chapter, I will present a design procedure of composite nonlinear feedback control for general multivariable systems with actuator saturation. I will consider both the state feedback case and the measurement feedback case without imposing any restrictive assumption on the given systems. The composite nonlinear feedback control consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time not exceeding the actuator limits for the desired command input levels. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part. The application of this technique to an air-to-air missile autopilot system and a numerical example shows that the proposed design method yields a very satisfactory performance.

2.1 Introduction

Every physical system in our real life has nonlinearities and very little can be done to overcome them. Many practical systems are sufficiently nonlinear so that important features of their performance may be completely overlooked if they are analyzed and designed through linear techniques (see *e.g.*, Hu and Lin [36]). For example, in the computer hard disk drive (HDD) servo systems (see *e.g.*, Chen *et al.* [18]), major nonlinearities are friction, high frequency mechanical resonance and actuator saturation nonlinearities. Among all these, the actuator saturation could be the most significant nonlinearity in designing an HDD servo system. When the actuator is saturated, the performance of the control system designed will seriously deteriorate. As such, the topic of linear and nonlinear control for saturated linear systems has attracted considerable attentions in the past (see *e.g.*, Garcia *et al.* [25], Henrion *et al.* [35], Suarez *et al.* [69], and Wredenhagen and Belanger [79] to name a few). Most of these works are using approaches based on certain parameterized Riccati equations.

Typically, when dealing with "point-and-shoot" fast-targeting for single-input and single-output (SISO) systems with actuator saturation, one would naturally think of using the well known time optimal control (TOC) (known also as the bang-bang control), which uses maximum acceleration and maximum deceleration for a predetermined time period. Unfortunately, it is well known that the classical TOC is not robust with respect to the system uncertainties and measurement noises. It can hardly be used in any real situation. For SISO systems with input saturation, another commonly used controller for target tracking is known as the proximate time-optimal servomechanism (PTOS), which was originally proposed by Workman [78] to overcome the above mentioned drawback of the TOC design.

Inspired by a work of Lin *et al.* [53], which was introduced to improve the tracking performance under state feedback laws for a class of second order systems subject to actuator saturation, Chen *et al.* [19] have recently extended the technique to general SISO systems with measurement feedback. The work of Chen *et al.* [19] has been successfully applied to design an HDD servo system, which outperforms conventional methods by more than 30%. The extension of the results of [53] to multi-input and multi-output (MIMO) systems under state feedback was reported in a nice work by Turner *et al.* [74]. However, the extension was made under a pretty odd assumption on the system that excludes many systems including those originally considered in [53]. The restrictiveness of the assumption of [74] will be discussed later. Also, as in [53], only state feedback is considered in [74].

In this chapter, I will present a design procedure of composite nonlinear feedback (CNF) control for general multivariable systems with actuator saturation. I will consider both the state feedback case and the measurement feedback case without imposing any restrictive assumption on the given systems. As in the earlier works [19, 53, 74], the CNF control consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time not exceeding the actuator limits for the desired command input levels. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part.

This chapter is organized as follows. In Section 2.2, the theory of the composite nonlinear feedback control is developed. Three different cases, *i.e.*, the state feedback, the full order measurement feedback, and the reduced order measurement cases, are considered with all detailed derivations and proofs. I will also address the issue on the selection of nonlinear gain parameter in this section. The application of the CNF technique to an air-to-air missile autopilot system will be presented in Section 2.3, which shows that the proposed design method yields a very satisfactory performance. Finally, some concluding remarks will be drawn in Section 2.4.

2.2 Composite Nonlinear Feedback Control for MIMO Systems

I will present in this section the CNF controller design for the following multivariable linear system Σ with an amplitude-constrained actuator characterized by

$$\begin{cases} \dot{x} = A\ x + B\ \mathrm{sat}(u), \quad x(0) = x_0 \\ y = C_1\ x \\ h = C_2\ x + D_2\ \mathrm{sat}(u) \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $h \in \mathbb{R}^\ell$ are respectively the state, control input, measurement output and controlled output of the given system Σ. A, B, C_1 and C_2 are appropriate dimensional constant matrices, and the saturation function is defined by

$$\mathrm{sat}(u) = \begin{pmatrix} \mathrm{sat}(u_1) \\ \mathrm{sat}(u_2) \\ \vdots \\ \mathrm{sat}(u_m) \end{pmatrix}, \tag{2.2}$$

with

$$\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\ \min(|u_i|,\ \bar{u}_i), \tag{2.3}$$

where $\bar{u}_i$ is the maximum amplitude of the i-th control channel. The objective of this chapter is to design an appropriate control law for (2.1) using the CNF approach such that the resulting controlled output will track some desired step references as fast and as smooth as possible. I will address the CNF control system design for the given system (2.1) for three different situations, namely, the state feedback case, the full order measurement feedback case, and the reduced order measurement feedback case. For tracking purpose, the following assumptions on the given system are required:

i) (A, B) is stabilizable;

ii) (A, C_1) is detectable; and

iii) (A, B, C_2, D_2) is right invertible and has no invariant zeros at $s = 0$.

The objective here is to design control laws that are capable of achieving fast tracking of target references under input saturation. As such, it is well understood in the literature that these assumptions are standard and necessary.

2.2.1 State Feedback Case

Let us first proceed to develop a composite nonlinear feedback control technique for the case when all the state variables of the plant Σ are measurable, *i.e.*, $y = x$. The design will be done in three steps, which is a natural extension of the results of Chen *et al.* [19]. One has the following step-by-step design procedure.

STEP S.1: Design a linear feedback law,

$$u_{\mathrm{L}} = Fx + Gr, \tag{2.4}$$

where $r \in \mathbb{R}^m$ contains a set of step references. The state feedback gain matrix $F \in \mathbb{R}^{m\times n}$ is chosen such that the closed-loop system matrix $A + BF$ is asymptotically stable and the resulting closed-loop system transfer matrix, *i.e.*, $D_2 + (C_2 + D_2F)(sI - A - BF)^{-1}B$, has certain desired properties, *e.g.*, having a small dominating damping ratio in each channel. Note that such an F can be worked out using some well-studied methods such as the LQR, H_∞ and H_2 optimization approaches (see, *e.g.*, Anderson and Moore [1], Chen [17] and Saberi *et al.* [62]). Furthermore, G is an $m \times m$ square constant matrix and is given by

$$G := G_0' \left(G_0 G_0'\right)^{-1}, \tag{2.5}$$

with $G_0 := D_2 - (C_2 + D_2F)(A + BF)^{-1}B$. Here note that both G_0 and G are well defined because $A + BF$ is stable, and (A, B, C_2, D_2) is right invertible and has no invariant zeros at $s = 0$, which implies $(A + BF, B, C + D_2F, D_2)$ is right invertible and has no invariant zeros at $s = 0$ (see *e.g.*, Lemma 2.5.1 of Chen [17]).

STEP S.2: Next, compute

$$H := \left[I - F(A + BF)^{-1}B\right] G \tag{2.6}$$

and

$$x_{\mathrm{e}} := G_{\mathrm{e}}\, r := -(A + BF)^{-1}BG\, r. \tag{2.7}$$

Note that the definitions of H, G_{e} and x_{e} would become transparent later in the derivation. Given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, solve the following Lyapunov equation:

$$(A + BF)'P + P(A + BF) = -W, \tag{2.8}$$

for $P > 0$. Such a P exists since $A + BF$ is asymptotically stable. Then, the nonlinear feedback control law u_{N} is given by

$$u_{\mathrm{N}} = \rho(r, y)B'P(x - x_{\mathrm{e}}), \tag{2.9}$$

where

$$\rho(r, y) = \mathrm{diag}\{\rho_1, \cdots, \rho_m\} = \begin{bmatrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_m \end{bmatrix}, \tag{2.10}$$

and $\rho_i = \rho_i(r, y)$, $i = 1, 2, \cdots, m$, are respectively some nonpositive functions, uniformly bounded and locally Lipschitz in y, which are used to change the closed-loop system damping ratios as the outputs approach the targets. The choice of these nonlinear functions will be discussed at the end of this section.

Step s.3: The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller:

$$u = u_{\mathrm{L}} + u_{\mathrm{N}} = Fx + Gr + \rho(r, y)B'P(x - x_{\mathrm{e}}). \tag{2.11}$$

This completes the design of the CNF controller for the state feedback case.

For further development, partition $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{m\times n}$ and $H \in \mathbb{R}^{m\times m}$ as follows:

$$B = [\, B_1 \quad \cdots \quad B_m \,], \quad F = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ \vdots \\ H_m \end{bmatrix}. \tag{2.12}$$

The following theorem shows that the closed-loop system comprising the given plant in (2.1) and the CNF control law of (2.11) is asymptotically stable. It also determines the magnitudes of the step functions in r that can be tracked by such a control law without exceeding the control limit.

Theorem 2.1. *Consider the given system in (2.1) with $y = x$, which satisfies the assumptions i) and iii), the linear control law of (2.4) and the composite nonlinear feedback* *control law of (2.11). For any $\delta \in (0,1)$, let $c_\delta > 0$ be the largest positive scalar such that for all $x \in \boldsymbol{X}_\delta$, where*

$$\boldsymbol{X}_\delta := \Big\{ x : x' P x \le c_\delta \Big\}, \tag{2.13}$$

the following property holds,

$$| F_i\, x | \le (1-\delta)\bar{u}_i, \quad i = 1, \cdots, m. \tag{2.14}$$

Then, the linear control law of (2.4) is capable of driving the system controlled output $h(t)$ to track asymptotically a set of step references, i.e., r, provided that the initial state x_0 and r satisfy:

$$\tilde{x}_0 := (x_0 - x_\mathrm{e}) \in \boldsymbol{X}_\delta, \quad |H_i\, r| \le \delta \bar{u}_i, \quad i = 1, \cdots, m. \tag{2.15}$$

Furthermore, for any nonpositive function $\rho(r,y)$, uniformly bounded and locally Lipschitz in y, the composite nonlinear feedback law in (2.11) is capable of driving the system controlled output $h(t)$ to track asymptotically the step command input of amplitude r, provided that the initial state x_0 and r satisfy (2.15).

Proof. Let us first define a new state variable $\tilde{x} = x - x_\mathrm{e}$. It is simple to verify that the linear feedback control law of (2.4) can be rewritten as

$$u_\mathrm{L}(t) = F\tilde{x}(t) + [I - F(A+BF)^{-1}B]Gr \tag{2.16}$$

$$= F\tilde{x}(t) + Hr, \tag{2.17}$$

and hence for all $\tilde{x} \in \boldsymbol{X}_\delta$ and, provided that $|H_i\, r| \le \delta \bar{u}_i$, $i = 1, \cdots, m$, the closed-loop system is linear and is given by

$$\dot{\tilde{x}} = (A+BF)\tilde{x} + Ax_\mathrm{e} + BHr. \tag{2.18}$$

Noting that

$$\begin{aligned} Ax_\mathrm{e} + BHr &= \Big\{ B[I - F(A+BF)^{-1}B]G - A(A+BF)^{-1}BG \Big\} r \\ &= \Big\{ [I - BF(A+BF)^{-1}]BG - A(A+BF)^{-1}BG \Big\} r \\ &= \Big\{ I - BF(A+BF)^{-1} - A(A+BF)^{-1} \Big\} BGr \\ &= 0, \end{aligned} \tag{2.19}$$

the closed-loop system in (2.18) can then be simplified as

$$\dot{\tilde{x}} = (A + BF)\tilde{x}. \tag{2.20}$$

Similarly, the closed-loop system comprising the given plant in (2.1) and the CNF control law of (2.11) can be expressed as

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + Bw, \tag{2.21}$$

where

$$w = \text{sat}(F\tilde{x} + Hr + u_{\text{N}}) - F\tilde{x} - Hr. \tag{2.22}$$

Clearly, for the given x_0 satisfying (2.15), one has $\tilde{x}_0 = (x_0 - x_{\text{e}}) \in \boldsymbol{X}_\delta$. Note that (2.21) is reduced to (2.20) if $\rho(r, y) = 0$.

Next, define a Lyapunov function $V = \tilde{x}'P\tilde{x}$ and evaluate the derivative of V along the trajectories of the closed-loop system in (2.21), *i.e.*,

$$\begin{aligned} \dot{V} &= \dot{\tilde{x}}'P\tilde{x} + \tilde{x}'P\dot{\tilde{x}} \\ &= \tilde{x}'(A + BF)'P\tilde{x} + \tilde{x}'P(A + BF)\tilde{x} + 2\tilde{x}'PBw \\ &= -\tilde{x}'W\tilde{x} + 2\tilde{x}'PBw. \end{aligned} \tag{2.23}$$

Note that for all

$$\tilde{x} \in \boldsymbol{X}_\delta = \{\tilde{x} : \tilde{x}'P\tilde{x} \le c_\delta\} \quad \Rightarrow \quad |F_i\,\tilde{x}| \le (1 - \delta)\bar{u}_i, \quad i = 1, \cdots, m. \tag{2.24}$$

In the remainder of this proof, I will adopt similar lines of reasoning as those of Turner *et al.* [74] by considering the following different scenarios. For simplicity, I will drop the dependent variables of the nonlinear function ρ in the rest of this proof.

Case 1. All input channels are unsaturated. It is obvious that one has

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\tilde{x}'PB\rho B'P\tilde{x} \le -\tilde{x}'W\tilde{x}. \tag{2.25}$$

Case 2. All input channels are exceeding their upper limits. In this case, one has

$$F_i\tilde{x} + H_ir + \rho_iB_i'P\tilde{x} \ge \bar{u}_i, \;\; i = 1, \cdots, m. \tag{2.26}$$

For all $\tilde{x} \in \boldsymbol{X}_\delta$, which implies (2.24) holds, and r satisfying (2.15), one has

$$F_i\tilde{x} + H_i r \leq \bar{u}_i, \quad i = 1, \cdots, m, \tag{2.27}$$

and thus

$$w_i = \text{sat}(F_i\tilde{x} + H_i r + \rho_i B_i' P\tilde{x}) - F_i\tilde{x} - H_i r = \bar{u}_i - F_i\tilde{x} - H_i r \geq 0 \tag{2.28}$$

and

$$\rho_i B_i' P\tilde{x} \geq \bar{u}_i - (F_i\tilde{x} + H_i r) \geq 0 \;\; \Rightarrow \;\; B_i' P\tilde{x} = \tilde{x}' P B_i \leq 0. \tag{2.29}$$

Hence,

$$\dot{V} = -\tilde{x}' W\tilde{x} + 2\sum_{i=1}^{m} \tilde{x}' P B_i \bar{w}_i \leq -\tilde{x}' W\tilde{x}. \tag{2.30}$$

Case 3. All input channels are exceeding their lower limits. For this case, one has

$$F_i\tilde{x} + H_i r + \rho_i B_i' P\tilde{x} \leq -\bar{u}_i, \quad i = 1, \cdots, m. \tag{2.31}$$

For all $\tilde{x} \in \boldsymbol{X}_\delta$, which implies (2.24) holds, and r satisfying (2.15), one has

$$F_i\tilde{x} + H_i r \geq -\bar{u}_i, \quad i = 1, \cdots, m, \tag{2.32}$$

and thus

$$w_i = \text{sat}(F_i\tilde{x} + H_i r + \rho_i B_i' P\tilde{x}) - F_i\tilde{x} - H_i r = -u_i - F_i\tilde{x} - H_i r \leq 0 \tag{2.33}$$

and

$$\rho_i B_i' P\tilde{x} \leq -\bar{u}_i - (F_i\tilde{x} + H_i r) \leq 0 \;\; \Rightarrow \;\; B_i' P\tilde{x} = \tilde{x}' P B_i \geq 0. \tag{2.34}$$

Hence,

$$\dot{V} = -\tilde{x}' W\tilde{x} + 2\sum_{i=1}^{m} \tilde{x}' P B_i w_i \leq -\tilde{x}' W\tilde{x}. \tag{2.35}$$

Case 4. Some control channels are saturated and some are unsaturated. In view of Cases 1 to 3, it is simple to note that for those unsaturated channels, one has

$$\tilde{x}' P B_i w_i = \rho_i \tilde{x}' P B_i B_i' P\tilde{x} \leq 0, \tag{2.36}$$

and those input channels whose signals exceeding their upper limits, one has

$$w_i \geq 0, \quad \tilde{x}' P B_i \leq 0 \;\; \Rightarrow \;\; \tilde{x}' P B_i w_i \leq 0, \tag{2.37}$$

and finally for those channels whose signals exceeding their lower limits,

$$w_i \leq 0, \quad \tilde{x}' P B_i \geq 0 \;\Rightarrow\; \tilde{x}' P B_i w_i \leq 0. \tag{2.38}$$

Thus, for this case, again one has

$$\dot{V} = -\tilde{x}' W \tilde{x} + 2 \sum_{i=1}^{m} \tilde{x}' P B_i w_i \leq -\tilde{x}' W \tilde{x}. \tag{2.39}$$

In conclusion, I have shown that

$$\dot{V} \leq -\tilde{x}' W \tilde{x}, \quad \tilde{x} \in \boldsymbol{X}_\delta, \tag{2.40}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (2.21). Noting that $W > 0$, all trajectories of (2.21) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial state x_0 and the step command input r that satisfy (2.15), one has

$$\lim_{t \to \infty} x(t) = x_e, \tag{2.41}$$

which implies

$$\lim_{t \to \infty} u(t) = F \lim_{t \to \infty} x(t) + Gr + \lim_{t \to \infty} \rho B' P [x(t) - x_{\mathrm{e}}] = F x_{\mathrm{e}} + Gr, \tag{2.42}$$

since $\rho(r, y)$ is uniformly bounded. Hence,

$$\begin{aligned}
\lim_{t \to \infty} h(t) &= C_2 \lim_{t \to \infty} x(t) + D_2 \lim_{t \to \infty} u(t) \\
&= C_2 x_{\mathrm{e}} + D_2 (F x_{\mathrm{e}} + Gr) \\
&= (C_2 + D_2 F) x_{\mathrm{e}} + D_2 G r \\
&= -(C_2 + D_2 F)(A + BF)^{-1} B G r + D_2 G r \\
&= [D_2 - (C_2 + D_2 F)(A + BF)^{-1} B] G r \\
&= G_0 G_0' (G_0 G_0')^{-1} r = r.
\end{aligned} \tag{2.43}$$

This completes the proof of Theorem 2.1.

Lastly, assuming that the dynamic equation of the given system is transformed into the following form,

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} \mathrm{sat}(u), \tag{2.44}$$

where $\bar{B}$ is nonsingular, Turner *et al.* [74] have solved the problem under a rather strange condition, *i.e.*, A_{11} is nonsingular. It was suggested in [74] to add some small perturbations to A_{11} if it is singular. Recently, it has been pointed by Turner and Postlethwaite [73] for the case when the system is stabilizable and B is of full rank, there exists nonsingular state transformation that would convert the given system with the form of (2.44) with A_{11} being nonsingular. Nonetheless, it is obvious from the development that such a transformation is totally unnecessary. Please note further that the above approach to the CNF design is much more elegant compared to that given in [74], and it carries over nicely to the measurement feedback cases in the following subsections.

2.2.2 Full Order Measurement Feedback Case

The assumption that all the state variables of the given system Σ are measurable is, in general, not practical. For example, in HDD servo systems (see Chen *et al.* [18]), the velocity of the actuator is usually hard to be measured. As such, in this subsection and the next subsection, I will proceed to develop CNF design using only measurement information. Let us first deal with the full order measurement feedback case, in which the dynamical order of the controller is exactly the same as that of the given plant. The following is a step-by-step procedure for the CNF design using full order measurement feedback.

STEP F.1: First construct a linear full order measurement feedback control law,

$$\begin{cases} \dot{x}_{\rm v} = (A + KC_1)x_{\rm v} - Ky + B\ \text{sat}(u_{\rm L}) \\ u_{\rm L} = F(x_{\rm v} - x_{\rm e}) + Hr, \end{cases} \tag{2.45}$$

where r is the set of step reference signals and $x_{\rm v}$ is the state of the controller. As usual, K, F are gain matrices and are chosen such that $(A + KC_1)$ and $(A + BF)$ are asymptotically stable and the resulting closed loop system having desired properties. Finally, H and $x_{\rm e}$ are as defined in (2.6)–(2.7).

STEP F.2: Given a positive definite matrix $W_{\rm P} \in \mathbb{R}^{n\times n}$, solve the Lyapunov equation

$$(A + BF)'P + P(A + BF) = -W_{\rm P}, \tag{2.46}$$

for $P > 0$. As in the state feedback case, the linear control law of (2.45) obtained in the above step is to be combined with a nonlinear control law to form the following CNF controller:

$$\begin{cases} \dot{x}_{\rm v} = (A + KC_1)x_{\rm v} - Ky + B\ \mathrm{sat}(u) \\ u\ = F(x_{\rm v} - x_{\rm e}) + Hr + \rho(r, y)B'P(x_{\rm v} - x_{\rm e}), \end{cases} \tag{2.47}$$

where $\rho(r, y)$ is as given in (2.10) with all its diagonal elements being respectively a nonpositive function, locally Lipschitz in y, which are to be chosen to improve the performance of the closed-loop system.

It turns out that, for the measurement feedback case, the choice of $\rho_i(r, y)$, $i = 1, \ldots m$, the nonpositive scalar functions, are not totally free. They are subject to certain constraints. One has the following result.

Theorem 2.2. *Consider the given system in (2.1), which satisfies the standard assumptions i) to iii), the full order linear measurement feedback control law of (2.45) and the composite nonlinear measurement feedback control law of (2.47). Given a positive define matrix* $W_{\rm Q} \in \mathbb{R}^{n\times n}$ *with*

$$W_{\rm Q} > F'B'PW_{\rm P}^{-1}PBF, \tag{2.48}$$

let $Q > 0$ *be the solution to the Lyapunov equation,*

$$(A + KC_1)'Q + Q(A + KC_1) = -W_{\rm Q}. \tag{2.49}$$

Note that such a Q *exists as* $A + KC_1$ *is asymptotically stable. For any* $\delta \in (0, 1)$, *let* $c_\delta > 0$ *be the largest positive scalar such that for all* $\begin{pmatrix} x \\ x_v \end{pmatrix} \in \boldsymbol{X}_{F\delta}$, *where*

$$\boldsymbol{X}_{F\delta} := \left\{ \begin{pmatrix} x \\ x_v \end{pmatrix} : \begin{pmatrix} x \\ x_v \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} x \\ x_v \end{pmatrix} \le c_\delta \right\}, \tag{2.50}$$

the following property holds

$$\left| [\,F_i \quad F_i\,] \begin{pmatrix} x \\ x_{\rm v} \end{pmatrix} \right| \le (1 - \delta)\bar{u}_i, \quad i = 1, \cdots, m. \tag{2.51}$$

Then, the linear measurement feedback control law in (2.47) will drive the system's controlled output $h(t)$ *to track asymptotically a set of step references, i.e.,* r, *from an* *initial state x_0, provided that x_0, $x_{v0} = x_v(0)$ and r satisfy:*

$$\begin{pmatrix} x_0 - x_e \\ x_{v0} - x_0 \end{pmatrix} \in \boldsymbol{X}_{F\delta} \quad \text{and} \quad |H_i r| \le \delta \bar{u}_i, \;\; i = 1, \cdots, m. \tag{2.52}$$

Furthermore, there exist positive scalars $\rho_i^* > 0$, $i = 1, \ldots m$, such that for any nonpositive functions $\rho_i(r, y)$, $i = 1, \cdots, m$, locally Lipschitz in y and $|\rho_i(r, y)| \le \rho_i^*$, $i = 1, \cdots, m$, the CNF control law of (2.47) will drive the system controlled output $h(t)$ to track asymptotically the reference r from an initial x_0, provided that x_0, x_{v0} and r satisfy (2.52).

Proof. For simplicity, again I drop r and y in $\rho(r, y)$ throughout the proof of this theorem. Let $\tilde{x} = x - x_e$ and $\tilde{x}_v = x_v - x$. The linear feedback control law of (2.45) can be written as

$$\dot{\tilde{x}}_v = (A + KC_1)\tilde{x}_v, \quad u_L = [\,F \quad F\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr. \tag{2.53}$$

Hence, for all

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \in \boldsymbol{X}_{F\delta} \quad \Rightarrow \quad \left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right| \le (1 - \delta)\bar{u}_i, \;\; i = 1, \cdots, m, \tag{2.54}$$

and for any r satisfying

$$|H_i r| \le \delta \bar{u}_i, \;\; i = 1, \cdots, m, \tag{2.55}$$

each channel of u_L, say $u_{L,i}$, has the following property

$$u_{L,i} = \left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + H_i r \right| \le \left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right| + |H_i r| \le \bar{u}_i. \tag{2.56}$$

Thus, for all $\tilde{x}$ and $\tilde{x}_v$ satisfying the condition as given in (2.54), the closed-loop system comprising the given plant and the linear control law of (2.45) can be rewritten as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_v \end{pmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}. \tag{2.57}$$

Similarly, the closed-loop system with the CNF control law of (2.47) can be expressed as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_v \end{pmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w, \tag{2.58}$$

where

$$w = \operatorname{sat}\left[[\,F \quad F\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr + \rho\,[\,B'P \quad B'P\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right] - [\,F \quad F\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} - Hr. \tag{2.59}$$

Clearly, for x_0 and $x_{\rm v0}$ satisfying (2.52), one has

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_{\rm v0} \end{pmatrix} \in \boldsymbol{X}_{\rm F\delta}, \tag{2.60}$$

where $\tilde{x}_0 = \tilde{x}(0)$ and $\tilde{x}_{\rm v0} = \tilde{x}_{\rm v}(0)$. Note that (2.57) and (2.58) are identical when $\rho = 0$. Again, the results of Theorem 2.2 for both the linear and the nonlinear feedback case can be proved in one shot.

Next, define a Lyapunov function:

$$V = \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}, \tag{2.61}$$

and evaluate the derivative of V along the trajectories of the closed-loop system in (2.58), *i.e.*,

$$\dot{V} = \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}' \begin{bmatrix} -W_{\rm P} & PBF \\ F'B'P & -W_{\rm Q} \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + 2\tilde{x}'PBw. \tag{2.62}$$

Note that for all

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \in \boldsymbol{X}_{\rm F\delta} \;\Rightarrow\; \left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \right| \le (1-\delta)\bar{u}_i, \;\; i = 1, \cdots, m. \tag{2.63}$$

Again, as done in the full state feedback case, let us find the above derivative of V for four different cases.

Case 1. All input channels are unsaturated. For this case, one has

$$\left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + H_i r + \rho_i [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \right| \le \bar{u}_i, \;\; i = 1, \cdots, m, \tag{2.64}$$

which implies

$$w_i = \rho_i [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}, \tag{2.65}$$

and

$$\begin{aligned} \dot{V} &= \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}' \begin{bmatrix} -W_{\rm P} & PB(F+\rho B'P) \\ (F+\rho B'P)'B'P & -W_{\rm Q} \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + 2\rho\tilde{x}'PBB'P\tilde{x} \\ &\le \begin{pmatrix} \hat{x} \\ \tilde{x}_{\rm v} \end{pmatrix}' \begin{bmatrix} -W_{\rm P} & 0 \\ 0 & -\tilde{W}_{\rm Q} \end{bmatrix} \begin{pmatrix} \hat{x} \\ \tilde{x}_{\rm v} \end{pmatrix}, \end{aligned} \tag{2.66}$$

where

$$\hat{x} = \tilde{x} - W_{\rm P}^{-1} PB(F+\rho B'P)\tilde{x}_{\rm v} \tag{2.67}$$

and

$$\tilde{W}_{\rm Q} = W_{\rm Q} - (F + \rho B'P)'B'PW_{\rm P}^{-1}PB(F + \rho B'P). \tag{2.68}$$

Noting (2.48), *i.e.*, $W_{\rm Q} > F'B'PW_{\rm P}^{-1}PBF$, and ρ_i is locally Lipschitz, it is clear that there exist positive scalars $\rho^*_{i,1} > 0$, $i = 1, \cdots, m$, such that for any scalar function satisfying $|\rho_i| \le \rho^*_{i,1}$, $i = 1, \cdots, m$, one has $\tilde{W}_{\rm Q} > 0$ and hence $\dot{V} \le 0$.

Case 2. All input channels are exceeding their upper limits. In such a situation, one has for all $i = 1, \cdots, m$,

$$[\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + H_i\, r + \rho_i\, [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \ge \bar{u}_i. \tag{2.69}$$

For all the trajectories inside $\boldsymbol{X}_{{\rm F}\delta}$,

$$\left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + H_i r \right| \le \bar{u}_i, \tag{2.70}$$

one has for $i = 1, \cdots, m$,

$$0 \le w_i \le \rho_i\, [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}. \tag{2.71}$$

Next, let us express

$$w_i = q_i \rho_i\, [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}, \tag{2.72}$$

for some appropriate positive continuous function matrix $q_i(t)$ bounded by 1 for all t. In this case, the derivative of V becomes

$$\begin{aligned} \dot{V} &= \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}' \begin{bmatrix} -W_{\rm P} & PB(F + q\rho B'P) \\ (F + q\rho B'P)'B'P & -W_{\rm Q} \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + 2q\rho \tilde{x}'PBB'P\tilde{x} \\ &\le \begin{pmatrix} \hat{x}_+ \\ \tilde{x}_{\rm v} \end{pmatrix}' \begin{bmatrix} -W_{\rm P} & 0 \\ 0 & -\tilde{W}_{\rm Q+} \end{bmatrix} \begin{pmatrix} \hat{x}_+ \\ \tilde{x}_{\rm v} \end{pmatrix}, \end{aligned} \tag{2.73}$$

where

$$q = \text{diag}\Big\{ q_1, \cdots, q_m \Big\}, \tag{2.74}$$

$$\hat{x}_+ = \tilde{x} - W_{\rm P}^{-1} PB(F + q\rho B'P)\tilde{x}_{\rm v} \tag{2.75}$$

and

$$\tilde{W}_{\rm Q+} = W_{\rm Q} - (F + q\rho B'P)'B'PW_{\rm P}^{-1}PB(F + q\rho B'P). \tag{2.76}$$

Again, noting (2.48), *i.e.*, $W_{\rm Q} > F'B'PW_{\rm P}^{-1}PBF$, and ρ_i is locally Lipschitz, it is clear that there exist positive scalars $\rho_{i,2}^* > 0$, $i = 1, \cdots, m$, such that for any scalar function satisfying $|\rho_i| \le \rho_{i,2}^*$, $i = 1, \cdots, m$, one has $\tilde{W}_{\rm Q+} > 0$ and hence $\dot{V} \le 0$.

Case 3. All input channels are exceeding their lower limits. In this case, one has for $i = 1, \cdots, m$,

$$[\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + H_i\, r + \rho_i \, [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \le -\bar{u}_i. \tag{2.77}$$

For all the trajectories inside $\boldsymbol{X}_{\rm F\delta}$,

$$\left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + H_i r \right| \le \bar{u}_i, \tag{2.78}$$

one has for $i = 1, \cdots, m$,

$$\rho_i \, [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \le w_i \le 0. \tag{2.79}$$

Next, let us express

$$w_i = q_i \rho_i \, [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}, \tag{2.80}$$

for some appropriate positive continuous function matrix $q_i(t)$ bounded by 1 for all t. Following the similar arguments as in the previous case, one can show that there exist positive scalars $\rho_{i,3}^* > 0$, $i = 1, \cdots, m$, such that for any scalar function satisfying $|\rho_i| \le \rho_{i,3}^*$, $i = 1, \cdots, m$, the corresponding $\dot{V} \le 0$.

Case 4. Some control channels are saturated and some are unsaturated. Following the similar arguments as those in Cases 1 to 3, one can express that for $i = 1, \cdots, m$,

$$w_i = q_i \rho_i \, [\, B_i'P \quad B_i'P \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}, \tag{2.81}$$

for some appropriate positive continuous function matrix $q_i(t)$ bounded by 1 for all t, and show that there exist positive scalars $\rho_{i,4}^* > 0$, $i = 1, \cdots, m$, such that for any scalar function satisfying $|\rho_i| \le \rho_{i,4}^*$, $i = 1, \cdots, m$, the corresponding $\dot{V} \le 0$.

Finally, let $\rho_i^* = \min\{\rho_{i,1}^*, \rho_{i,2}^*, \rho_{i,3}^*, \rho_{i,4}^*\}$. Then, one has for any scalar function ρ_i satisfying $|\rho_i| < \rho_i^*$, $i = 1, \cdots, m$,

$$\dot{V} \le 0, \quad \forall \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \in \boldsymbol{X}_{\rm F\delta}. \tag{2.82}$$

Thus, $\boldsymbol{X}_{\mathrm{F}\delta}$ is an invariant set of the closed-loop system in (2.58), and all trajectories starting from $\boldsymbol{X}_{\mathrm{F}\delta}$ will remain inside and asymptotically converge to the origin. This, in turn, indicates that, for the initial state of the given system x_0, the initial state of the controller $x_{\mathrm{v}0}$, and step command input r that satisfy (2.52),

$$\lim_{t\to\infty} \tilde{x}_{\mathrm{v}}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} x(t) = x_{\mathrm{e}}, \tag{2.83}$$

and then it follows from (2.43) that the controlled output $h(t)$ converges asymptotically to the target reference r. This completes the proof of Theorem 2.2.

2.2.3 Reduced Order Measurement Feedback Case

For the given system in (2.1), it is clear that there are p state variables of the system, which are measurable if C_1 is of maximal rank. Thus, in general, it is not necessary to estimate these measurable state variables in measurement feedback laws. As such, I will proceed in this subsection to design a dynamic controller that has a dynamical order less than that of the given plant. For simplicity of presentation, assume that C_1 is already in the form

$$C_1 = [\, I_p \quad 0\,]. \tag{2.84}$$

Then, the system in (2.1) can be rewritten as

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathrm{sat}(u) \\ y = [\ I_p \quad 0\ ] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ h = C_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_2 \mathrm{sat}(u), \qquad x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \end{cases} \tag{2.85}$$

where the original state x is partitioned into two parts, x_1 and x_2 with $y \equiv x_1$. Thus, one will only need to estimate x_2 in the reduced order measurement feedback design. Next, let F be chosen such that i) $A + BF$ is asymptotically stable, and ii) $(C_2 + D_2F)(sI - A - BF)^{-1}B + D_2$ has desired properties, and let K_{R} be chosen such that $A_{22} + K_{\mathrm{R}}A_{12}$ is asymptotically stable. Here note that it can be shown that (A_{22}, A_{12}) is detectable if and only if (A, C_1) is detectable. Thus, there exists a stabilizing $K_{\rm R}$. Again, such F and $K_{\rm R}$ can be designed using an appropriate control technique. One then partitions F in conformity with x_1 and x_2:

$$F = [\,\mathbb{F}_1 \quad \mathbb{F}_2\,]\,. \tag{2.86}$$

Let us further partition F_2 as follows:

$$\mathbb{F}_2 = \begin{bmatrix} \mathbb{F}_{2,1} \\ \vdots \\ \mathbb{F}_{2,m} \end{bmatrix}. \tag{2.87}$$

Also, let G, H and $x_{\rm e}$ be as given in (2.5)–(2.7). The reduced order CNF controller is given by

$$\dot{x}_{\rm v} = (A_{22}{+}K_{\rm R}A_{12})x_{\rm v} + (B_2 + K_{\rm R}B_1)\,{\rm sat}(u) + \Big[A_{21}{+}K_{\rm R}A_{11}{-}(A_{22}{+}K_{\rm R}A_{12})K_{\rm R}\Big]y \tag{2.88}$$

and

$$u = F\left[\begin{pmatrix} y \\ x_{\rm v} - K_{\rm R}y \end{pmatrix} - x_{\rm e}\right] + Hr + \rho(r,y)B'P\left[\begin{pmatrix} y \\ x_{\rm v} - K_{\rm R}y \end{pmatrix} - x_{\rm e}\right], \tag{2.89}$$

where $\rho(r, y)$ is as given in (2.10).

Next, given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, let $P > 0$ be the solution to the Lyapunov equation

$$(A + BF)'P + P(A + BF) = -W_{\rm P}. \tag{2.90}$$

Given another positive definite matrix $W_R \in \mathbb{R}^{(n-p)\times(n-p)}$ with

$$W_{\rm R} > \mathbb{F}_2' B' P W_{\rm P}^{-1} P B \mathbb{F}_2, \tag{2.91}$$

let $Q_{\rm R} > 0$ be the solution to the Lyapunov equation

$$(A_{22} + K_{\rm R}A_{12})'Q_{\rm R} + Q_{\rm R}(A_{22} + K_{\rm R}A_{12}) = -W_{\rm R}. \tag{2.92}$$

Note that such P and $Q_{\rm R}$ exist as $A + BF$ and $A_{22} + K_{\rm R}A_{12}$ are asymptotically stable. For any $\delta \in (0, 1)$, let c_δ be the largest positive scalar such that for all

$$\begin{pmatrix} x \\ x_{\rm v} \end{pmatrix} \in \boldsymbol{X}_{{\rm R}\delta} := \left\{\begin{pmatrix} x \\ x_{\rm v} \end{pmatrix} : \begin{pmatrix} x \\ x_{\rm v} \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q_{\rm R} \end{bmatrix} \begin{pmatrix} x \\ x_{\rm v} \end{pmatrix} \le c_\delta \right\} \tag{2.93}$$

the following property holds:

$$\left| [\,F_i \quad \mathbb{F}_{2,i}\,] \begin{pmatrix} x \\ x_{\rm v} \end{pmatrix} \right| \le \bar{u}_i(1-\delta), \quad i = 1, \cdots, m. \tag{2.94}$$

One has the following theorem.

Theorem 2.3. *Consider the given system in (2.1), which satisfies the usual assumptions i) to iii). Then, there exist positive scalars* $\rho_i^* \geq 0$, $i = 1, \cdots, m$, *such that for any nonpositive function* $\rho_i(r, y)$, $i = 1, \cdots, m$, *locally Lipschitz in* y_i *and* $|\rho_i(r, y)| \leq \rho_i^*$, *the reduced order CNF law given by (2.88) and (2.89) will drive the system controlled output* $h(t)$ *to asymptotically track the reference* r *from an initial state* x_0, *provided that* x_0, $x_{\rm v0}$ *and* r *satisfy*

$$\begin{pmatrix} x_0 - x_{\rm e} \\ x_{\rm v0} - x_{20} - K_{\rm R} x_{10} \end{pmatrix} \in \boldsymbol{X}_{\rm R\delta}, \quad |H_i r| \leq \delta \bar{u}_i, \quad i = 1, \cdots, m. \tag{2.95}$$

Proof. Let $\tilde{x} = x - x_{\rm e}$ and $\tilde{x}_{\rm v} = x_{\rm v} - x_2 - K_{\rm R} x_1$. Then, the closed-loop system comprising the given plant in (2.1) and the reduced order CNF control law of (2.88) and (2.89) can be expressed as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_{\rm v} \end{pmatrix} = \begin{bmatrix} A + BF & B\mathbb{F}_2 \\ 0 & A_{22} + K_{\rm R} A_{12} \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \tag{2.96}$$

where

$$w = \mathrm{sat}\left\{ [\, F \quad \mathbb{F}_2 \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + Hr + \rho(r, y) B' P \left[\tilde{x} + \begin{pmatrix} 0 \\ \tilde{x}_{\rm v} \end{pmatrix} \right] \right\} - [\, F \quad \mathbb{F}_2 \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} - Hr. \tag{2.97}$$

The rest of the proof follows along similar lines to the reasoning given in the full order measurement feedback case.

2.2.4 Selecting the Nonlinear Gain $\rho(r, y)$

The freedom to choose the function $\rho(r, y)$ is used to tune the control laws so as to improve the performance of the closed-loop system as the controlled output h approaches the set point. Since the main purpose of adding the nonlinear part to the CNF controllers is to speed up the settling time, or equivalently to contribute a significant value to the control input when the tracking error, $r - h$, is small. The nonlinear part, in general, will be in action when the control signal is far away from its saturation level, and thus it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising the given plant in (2.1) and the three different types of control law can be expressed as

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + \rho(r, y) BB' P \tilde{x}. \tag{2.98}$$

Note that the additional term $\rho(r, y)$ does not affect the stability of the estimators. It is now clear that eigenvalues of the closed-loop system in (2.98) can be changed by the function $\rho(r, y)$. There are different types of nonlinear gains that have been suggested in the literature (see *e.g.*, [19, 53, 74]). Assuming that h is available, let us follow the work of [19] to propose the following nonlinear gains,

$$\rho_i(r_i, h_i) = -\frac{\beta_i}{1 - e^{-1}} \left(e^{-|1-[h_i - h_i(0)]/[r_i - h_i(0)]|} - e^{-1} \right), \quad i = 1, \cdots, m, \tag{2.99}$$

when $h_i(0) \neq r_i$ (for the trivial case of $h_i(0) = r_i$, no control input is needed). Or, one may choose

$$\rho_i(r, h) = -\beta_i \left| e^{-\alpha_i \|h(t) - r\|} - e^{-\alpha_i \|h(0) - r\|} \right|, \quad i = 1, \cdots, m, \tag{2.100}$$

which starts from 0 and gradually increases to a final gain of $-\beta_i \left| 1 - e^{-\alpha_i \|h(0)-r\|} \right|$ as h approaches to the target reference r. α_i is used to determine the speed of change in ρ_i. Thus, one could properly select scalar gains β_i, $i = 1, \cdots, m$, to yield a desired performance. Note further that for the case when (A, B, C_2, D_2) is a SISO system, Chen *et al.* [19] have recently shown a nice interconnection on the mechanism of the nonlinear gain ρ with the classical root-locus theory. They have also shown that W can actually be connected to the zero placement for an auxiliary system. Unfortunately, these nice properties generally do not carry over to the MIMO systems.

To examine the behavior of the closed-loop system (2.98) more explicitly, let us define an auxiliary system $G_{\text{aux}}(s)$ as

$$G_{\text{aux}}(s) := C_{\text{aux}}(sI - A_{\text{aux}})^{-1} B_{\text{aux}} := B' P (sI - A - BF)^{-1} B. \tag{2.101}$$

Obviously, $G_{\text{aux}}(s)$ is stable. The closed-loop system (2.98) can then be cast under the framework of the multivariable root locus theory as shown in Figure 2.1 (let us hereafter drop the dependent variables of ρ for simplicity). Note that

$$C_{\text{aux}} B_{\text{aux}} = B' P B > 0, \tag{2.102}$$

which implies $G_{\text{aux}}(s)$ is a square, invertible and uniform rank system with m infinite zeros of order 1 and with $n-m$ invariant zeros. Noting that

$$\det(sI - A_{\text{aux}} - B_{\text{aux}} \cdot \rho \cdot C_{\text{aux}}) = \det(\rho) \cdot \det \begin{bmatrix} sI - A_{\text{aux}} & B_{\text{aux}} \\ C_{\text{aux}} & \rho^{-1} \end{bmatrix}, \tag{2.103}$$

it is clear that for any eigenvalue of the closed-loop system (2.98), *i.e.*, $s \in \lambda(A + BF + B\rho B'P)$,

$$\det \begin{bmatrix} sI - A_{\text{aux}} & B_{\text{aux}} \\ C_{\text{aux}} & \rho^{-1} \end{bmatrix} = 0. \tag{2.104}$$

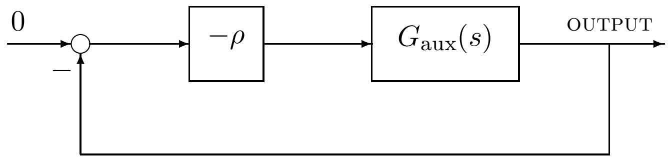


Figure 2.1: Interpretation of the nonlinear function $\rho(r, y)$.

Thus, when all diagonal elements of ρ, *i.e.*, $\rho_i,\ i = 1, 2, \cdots, m$, approach to $-\infty$, the closed-loop eigenvalues of (2.98) approach to the zeros of $G_{\text{aux}}(s)$ including the invariant zeros of $(A_{\text{aux}}, B_{\text{aux}}, C_{\text{aux}})$ and those at infinity. Since it was shown that the closed-loop system remains stable for any ρ whose diagonal elements are nonpositive, the invariant zeros of $G_{\text{aux}}(s)$ have to be stable. Hence, $G_{\text{aux}}(s)$ is of minimum phase.

It should be noted that there is freedom in pre-selecting the locations of these invariant zeros by selecting an appropriate W in (2.8). In general, one should select the invariant zeros of $G_{\text{aux}}(s)$, which are corresponding to the closed-loop poles of (2.98) for large $|\rho|$, such that the dominated ones have a large damping ratio, which in turn will generally yield a smaller overshoot. The following procedure for selecting an appropriate W is adopted from that reported in [19]:

Given the pair $(A_{\text{aux}}, B_{\text{aux}})$ and the desired locations of the invariant zeros of G_{aux}, let us follow the result reported in Chapter 9 of Chen *et al.* [20] on finite and infinite zero assignment to obtain an appropriate matrix C_{aux} such that $(A_{\text{aux}}, B_{\text{aux}}, C_{\text{aux}})$ has the desired relative degree and invariant zeros.

Solve $C_{\rm aux} = B'P$ for a $P = P' > 0$. In general, the solution is non-unique as there are $n(n+1)/2$ elements in P available for selection. However, if the solution does not exist, one goes back to the previous step to re-select the invariant zeros.

Calculate W using (2.8) and check if W is positive definite. If W is not positive definite, one goes back to the previous step to choose another solution of P or go to the first step to re-select the invariant zeros.

Another method for selecting W is based on a trial and error approach by limiting the choice of W to a diagonal matrix and adjusting its diagonal weights through simulation. The software package for realizing the CNF design reported in Cheng *et al.* [22] was implemented based on such an approach. Generally, it will also yield a satisfactory result. I will illustrate such a design approach in two examples in the following section.

2.3 Illustrative Examples

To illustrate the concept of the CNF control, I will present in this section two examples. One is a real application example while the other one is a numerical example.

Example 2.1. The first example is a roll-yaw autopilot system for the Extended Medium Range Air-to-Air Technology (EMRAAT) airframe. I will compare the performance of the CNF design with a corresponding LQR design. The airframe is a generic, non-axisymmetrical airframe and as such, lends itself to highly g coordinated bank-to-turn maneuvers. The linearized roll-yaw state space model for the EMRAAT airframe for the flight conditions of Mach = 2.5, Velocity = 2420 ft/sec, Dynamic Pressure = 1720 lbs/ft^2, and Angle of Attack = 10°, is given by

$$\dot{x} = \begin{bmatrix} -0.501 & -0.985 & 0.174 & 0 & 0.109 & 0.007 \\ 16.83 & -0.575 & 0.0123 & 0 & -132.8 & 27.19 \\ -3227 & 0.321 & -2.10 & 0 & -1620 & -1240 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -179 & 0 \\ 0 & 0 & 0 & 0 & 0 & -179 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 179 & 0 \\ 0 & 179 \end{bmatrix} \mathrm{sat}(u), \quad (2.105)$$

where

$$x = \begin{pmatrix} \beta \\ \alpha \\ p \\ \int p \\ \delta_{\rm r} \\ \delta_{\rm a} \end{pmatrix}, \quad u = \begin{pmatrix} \delta_{\rm rc} \\ \delta_{\rm ac} \end{pmatrix}. \tag{2.106}$$

and where β is sideslip, α is yaw rate, p is roll rate, $\int p$ is roll angle, $\delta_{\rm r}$ is rudder position, $\delta_{\rm a}$ is aileron position, and $\delta_{\rm rc}$ and $\delta_{\rm ac}$ are respectively the controls applied to the rudder and aileron. The measurement of the system is given by

$$y = \begin{pmatrix} \beta \\ \alpha \\ p \\ \int p \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x. \tag{2.107}$$

This air-to-air missile system is taken from the work of Wilson *et al.* [77], in which the authors had designed an autopilot system based on a Lyapunov-constrained eigenstructure assignment approach. Note that in [77], they did not consider any input saturation in their formulation. The same system was adopted by Turner *et al.* [74] for illustration of their work, although they had added a small perturbation in the $(4, 4)$ entry in the system matrix A into order to make A_{11} nonsingular. However, in [74], the authors had assumed that all the state variables of the system are measurable and assumed that both input channels are bounded by $\pm 20^o$. The controlled output of the system is defined as the sideslip and the the roll angle, *i.e.*,

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \beta \\ \int p \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u. \tag{2.108}$$

To demonstrate the results, let us choose a a command reference:

$$r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 50 \end{pmatrix}. \tag{2.109}$$

The aim is to design appropriate CNF controllers with full state feedback, full order measurement feedback and reduced order measurement feedback, which would control the controlled output of the system to track the command reference as fast as possible and as smooth as possible. Following the procedures given in the previous section and with appropriate selections of design parameters, I have obtained the following CNF control laws. Note that the linear parts of the control laws are carried out using the standard LQR design.

1. CNF controller using full state feedback:

$$u = Fx + Gr + \rho(r, y)F_{\mathrm{n}}(x - x_{\mathrm{e}}), \tag{2.110}$$

where

$$F = \begin{bmatrix} -2.573875 & 0.124261 & 0.037199 & 1.891459 & -0.351318 & -0.186503 \\ -0.039226 & -0.131115 & 0.037657 & 1.192637 & -0.186503 & -0.235628 \end{bmatrix},$$

$$G = \begin{bmatrix} 1.675090 & -1.891459 \\ -2.656604 & -1.192637 \end{bmatrix},$$

$$F_{\mathrm{n}} = \begin{bmatrix} 2.573875 & -0.124261 & -0.037199 & -1.891459 & 0.351318 & 0.186503 \\ 0.039226 & 0.131115 & -0.037657 & -1.192637 & 0.186503 & 0.235628 \end{bmatrix},$$

$$x_{\mathrm{e}} = [9 \quad 4.117493 \quad 0 \quad 50 \quad -2.897455 \quad -19.635324]'$$

and

$$\rho(r, y) = \mathrm{diag}\Big\{\rho_1(r_1, h_1),\ \rho_2(r_2, h_2)\Big\},$$

and where

$$\rho_1(r_1, h_1) = -\frac{0.5}{1 - e^{-1}}(e^{-|1-\frac{h_1 - h_1(0)}{r_1 - h_1(0)}|} - e^{-1}), \tag{2.111}$$

$$\rho_2(r_2, h_2) = -\frac{1.5}{1 - e^{-1}}(e^{-|1-\frac{h_2 - h_2(0)}{r_2 - h_2(0)}|} - e^{-1}). \tag{2.112}$$

2. CNF controller using full order measurement feedback:

$$\begin{cases} \dot{x}_{\mathrm{v}} = (A + KC_1)x_{\mathrm{v}} - Ky + B\ \mathrm{sat}(u) \\ u \ = F(x_{\mathrm{v}} - x_{\mathrm{e}}) + Hr + \rho(r, y)F_{\mathrm{n}}(x_{\mathrm{v}} - x_{\mathrm{e}}), \end{cases} \tag{2.113}$$

where F, F_{n}, x_{e}, $\rho(r, y)$ are as given in the state feedback case, and

$$K = \begin{bmatrix} -29.6237 & 0.7142 & -0.1485 & 0 \\ -46.2737 & 119.3702 & -0.6416 & 0 \\ 3495.4107 & 18.1069 & 105.4275 & 0 \\ 0 & 0 & -1 & -60 \\ -20.7195 & 131.5970 & 2.0269 & 0 \\ 56.8973 & -169.5411 & 13.2893 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} -0.321939 & 0 \\ -2.181703 & 0 \end{bmatrix}.$$

3. CNF controller using reduced order measurement feedback:

$$\dot{x}_{\rm v} = A_{\rm cmp} x_{\rm v} + K_{\rm cmp} y + B_{\rm cmp}\ {\rm sat}(u) \tag{2.114}$$

and

$$u = F\left[\begin{pmatrix} y \\ x_{\rm v} - K_{\rm R} y \end{pmatrix} - x_{\rm e}\right] + Hr + \rho(r,y) F_{\rm n}\left[\begin{pmatrix} y \\ x_{\rm v} - K_{\rm R} y \end{pmatrix} - x_{\rm e}\right], \tag{2.115}$$

where

$$A_{\rm cmp} = \begin{bmatrix} -15 & 0 \\ 0 & -20 \end{bmatrix}, \quad K_{\rm cmp} = \begin{bmatrix} 52.553834 & -14.061997 & -0.287191 & 0 \\ 347.215285 & 23.940526 & -1.796177 & 0 \end{bmatrix},$$

$$B_{\rm cmp} = \begin{bmatrix} 179 & 0 \\ 0 & 179 \end{bmatrix}, \quad K_{\rm R} = \begin{bmatrix} 0.000578 & -0.974320 & -0.021364 & 0 \\ -0.000730 & 1.234094 & -0.101165 & 0 \end{bmatrix},$$

and F, H, $x_{\rm e}$, $\rho(r,y)$ and $F_{\rm n}$ are the same as those given in the previous two cases.

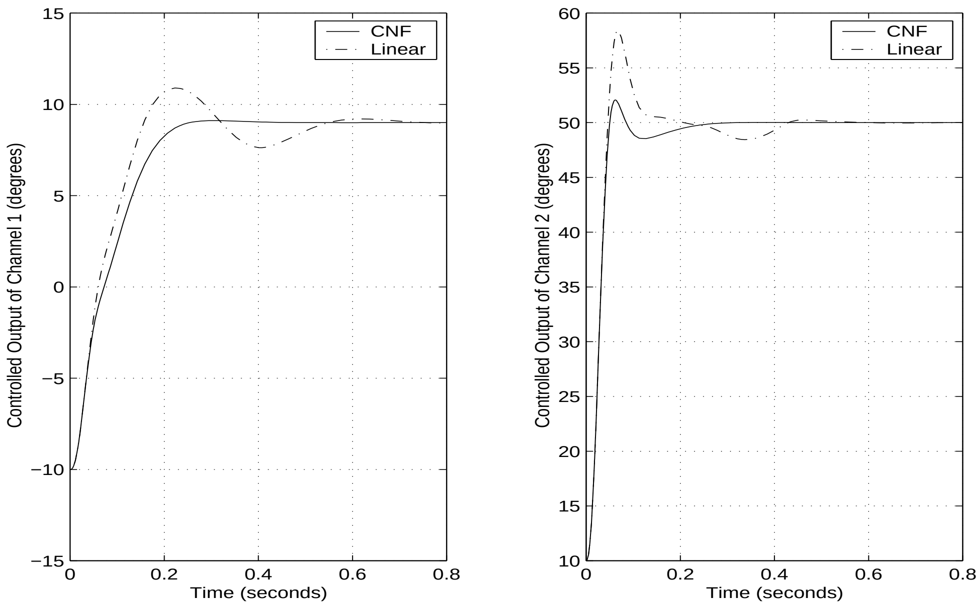
Using Simulink in Matlab, I obtain a set of simulation results in Figures 2.2–2.4, which are done under the following initial condition,

$$x_0 = [\,-10 \quad 0 \quad 0 \quad 10 \quad 0 \quad 0\,]', \tag{2.116}$$

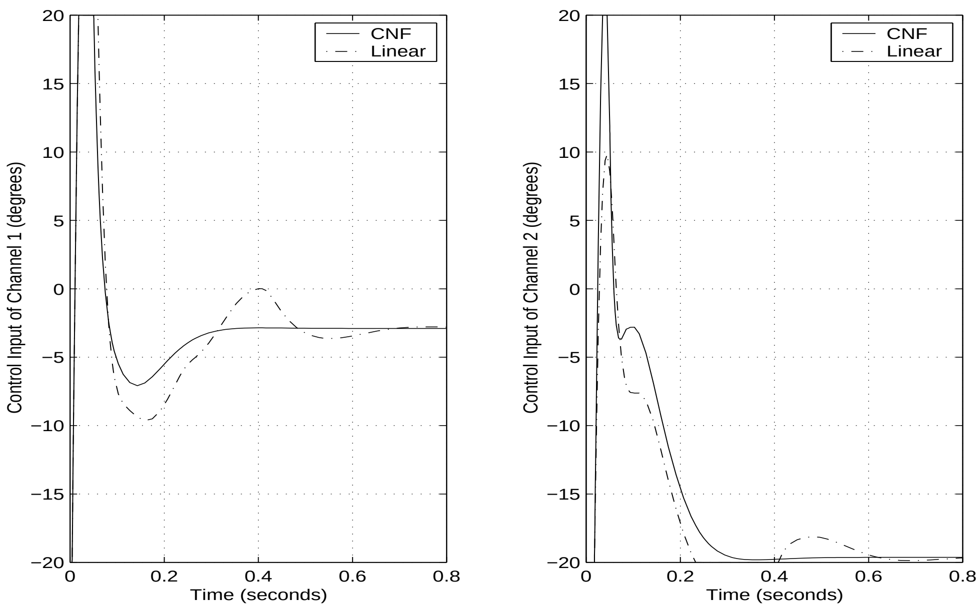
together with initial conditions for both full and reduced order controllers being set to zero. The results clearly show that the control laws with the nonlinear components, *i.e.*, the CNF controllers, outperform their linear counterparts a great deal. It is interesting to note that the results for the CNF state feedback case and the CNF reduced order measurement feedback case are almost identical, and have almost no overshoot at all in their controlled output responses. The controlled output responses in the CNF full order measurement feedback case are, however, having some small overshoot.

Example 2.2. Now let us consider a numerical example. The system considered is a two-input and two-output system characterized by (2.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -2 & -2 & -2 & -1 & -2 \\ 1 & 2 & 2 & 2 & 2 & 3 \\ -1 & -2 & -2 & -2 & -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad x_0 = \begin{pmatrix} -0.6 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \end{pmatrix}, \tag{2.117}$$

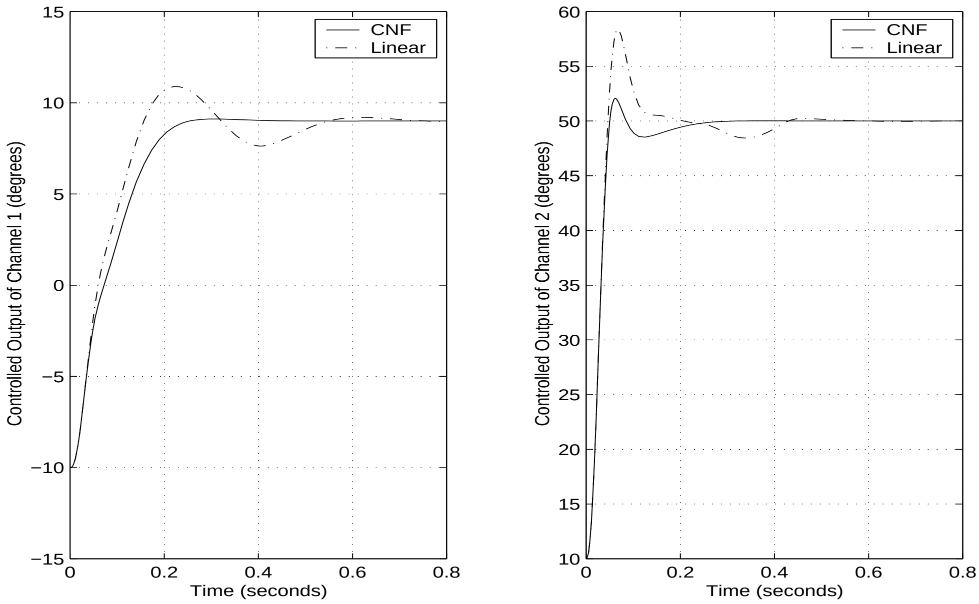


(a) Controlled output

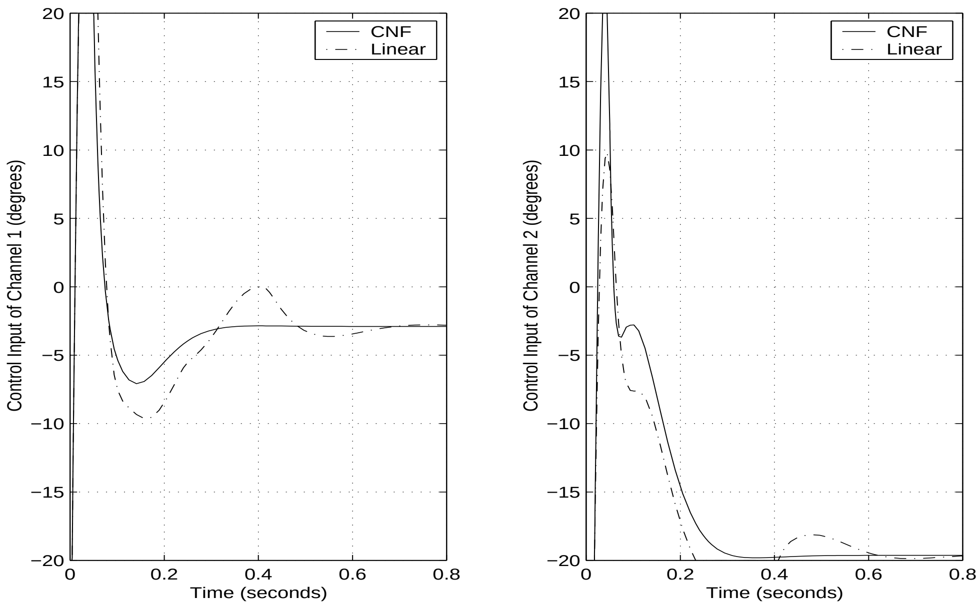


(b) Control input

Figure 2.2: Input and output responses under state feedback.

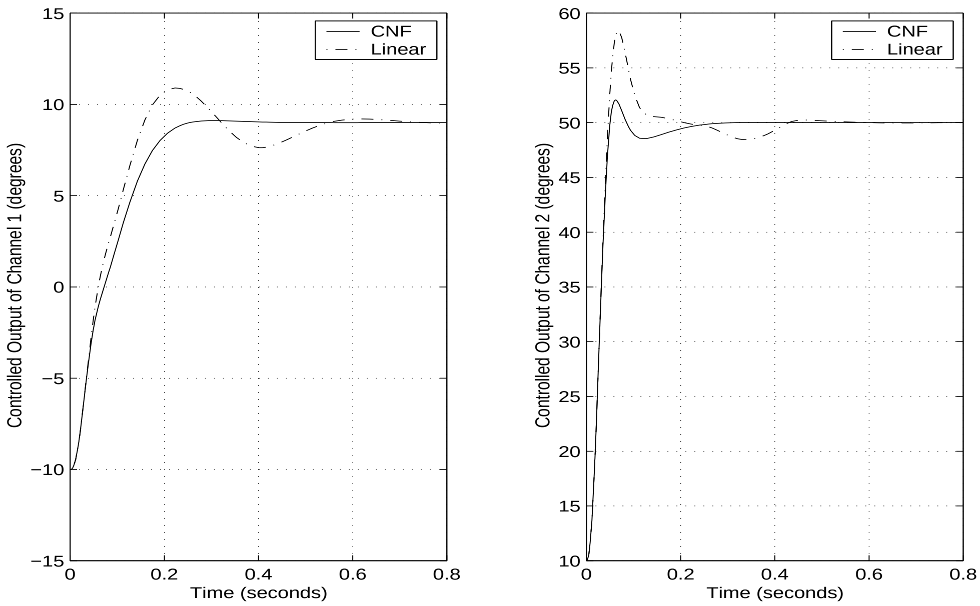


(a) Controlled output

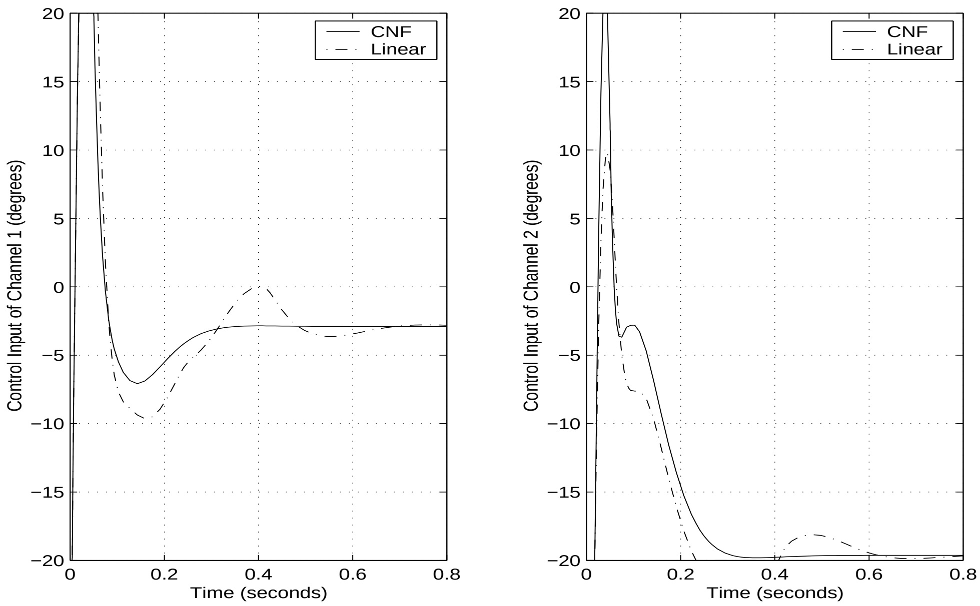


(b) Control input

Figure 2.3: Input and output responses under full order measurement feedback.



(a) Controlled output



(b) Control input

Figure 2.4: Input and output responses under reduced order measurement feedback.

and

$$C_1 = C_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.118}$$

The maximum amplitudes of both control channels are given by $\bar{u}_1 = \bar{u}_2 = 1$. The target references are

$$r = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{2.119}$$

The aim is to design appropriate CNF controllers with full state feedback, full order measurement feedback and reduced order measurement feedback, which would control the controlled output of the system to track the command reference as fast as possible and as smooth as possible. Following the procedures given in the previous section and with appropriate selections of design parameters, I have obtained the following CNF control laws. Note that the state feedback gain F is carried out by carefully examining the structural properties of the given system using the techniques reported in [20] whereas the full order and reduced order observer gain matrices are computed using the H_2 optimization technique given in [62].

1. CNF controller using full state feedback:

$$u = Fx + Gr + \rho(r, y)F_{\rm n}(x - x_{\rm e}), \tag{2.120}$$

where

$$F = \begin{bmatrix} -1 & -1 & -3 & -2 & 2 & 2 \\ 1 & 2 & 2 & 0 & -1 & -3 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

and

$$F_{\rm n} = B'P = \begin{bmatrix} 0.25 & 3.75 & 4.75 & 2.50 & 0.25 & -1.75 \\ -1.75 & -3.75 & -2.75 & 0.25 & 9.00 & 10.75 \end{bmatrix},$$

where P is the solution of the Lyapunov equation (2.8) with $W = I$. Finally,

$$x_{\rm e} = [\,2 \quad -1 \quad 1 \quad -1 \quad 0 \quad 0\,]'$$

and

$$\rho(r, y) = \mathrm{diag}\Big\{\rho_1(r_1, h_1),\ \rho_2(r_2, h_2)\Big\}, \tag{2.121}$$

and where

$$\rho_1(r, h) = -2.8\Big|e^{-||h(t)-r||} - e^{-||h(0)-r||}\Big|, \tag{2.122}$$

$$\rho_2(r, h) = -1.7\Big|e^{-||h(t)-r||} - e^{-||h(0)-r||}\Big|. \tag{2.123}$$

2. CNF controller using full order measurement feedback:

$$\begin{cases} \dot{x}_{\rm v} = (A + KC_1)x_{\rm v} - Ky + B\ {\rm sat}(u) \\ u = F(x_{\rm v} - x_{\rm e}) + Hr + \rho(r,y)F_{\rm n}(x_{\rm v} - x_{\rm e}), \end{cases} \tag{2.124}$$

where F, $F_{\rm n}$, $x_{\rm e}$, $\rho(r,y)$ are as given in the state feedback case, and

$$K = \begin{bmatrix} 65.9921 & -65.9537 \\ -57.5639 & 64.9515 \\ 92.5836 & -73.3967 \\ -27.4805 & 38.5006 \\ 26.4782 & -30.0729 \\ -34.9271 & 65.0887 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and $\rho(r,y)$ is slightly adjusted from that of (2.121) with $\rho_1(r,y)$ being modified as

$$\rho_1(r,h) = -2.5 \left| e^{-||h(t)-r||} - e^{-||h(0)-r||} \right|. \tag{2.125}$$

3. CNF controller using reduced order measurement feedback:

$$\dot{x}_{\rm v} = A_{\rm cmp}x_{\rm v} + K_{\rm cmp}y + B_{\rm cmp}\ {\rm sat}(u) \tag{2.126}$$

and

$$u = F\left[\begin{pmatrix} y \\ x_{\rm v} - K_{\rm R}y \end{pmatrix} - x_{\rm e}\right] + Hr + \rho(r,y)F_{\rm n}\left[\begin{pmatrix} y \\ x_{\rm v} - K_{\rm R}y \end{pmatrix} - x_{\rm e}\right], \tag{2.127}$$

where

$$A_{\rm cmp} = \begin{bmatrix} -15 & 0 \\ 0 & -20 \end{bmatrix}, \quad B_{\rm cmp} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & -2 \\ 0 & 1 \end{bmatrix},$$

$$K_{\rm cmp} = \begin{bmatrix} 52.553834 & -14.061997 & -0.287191 & 0 \\ 347.215285 & 23.940526 & -1.796177 & 0 \end{bmatrix},$$

$$K_{\rm cmp} = 10^3 \times \begin{bmatrix} -1.4088 & 1.3335 \\ -1.1589 & 0.1610 \\ -1.4815 & 2.4787 \\ 0.1749 & -1.2475 \end{bmatrix}, \quad K_{\rm R} = \begin{bmatrix} 99.0046 & -87.7874 \\ 74.1569 & -12.6217 \\ 98.4364 & -160.9248 \\ -13.1539 & 86.8591 \end{bmatrix},$$

and F, H, $x_{\rm e}$ and $F_{\rm n}$ are the same as those given in the previous two cases whereas $\rho(r,y)$ is identical to that given in the full order measurement feedback case.

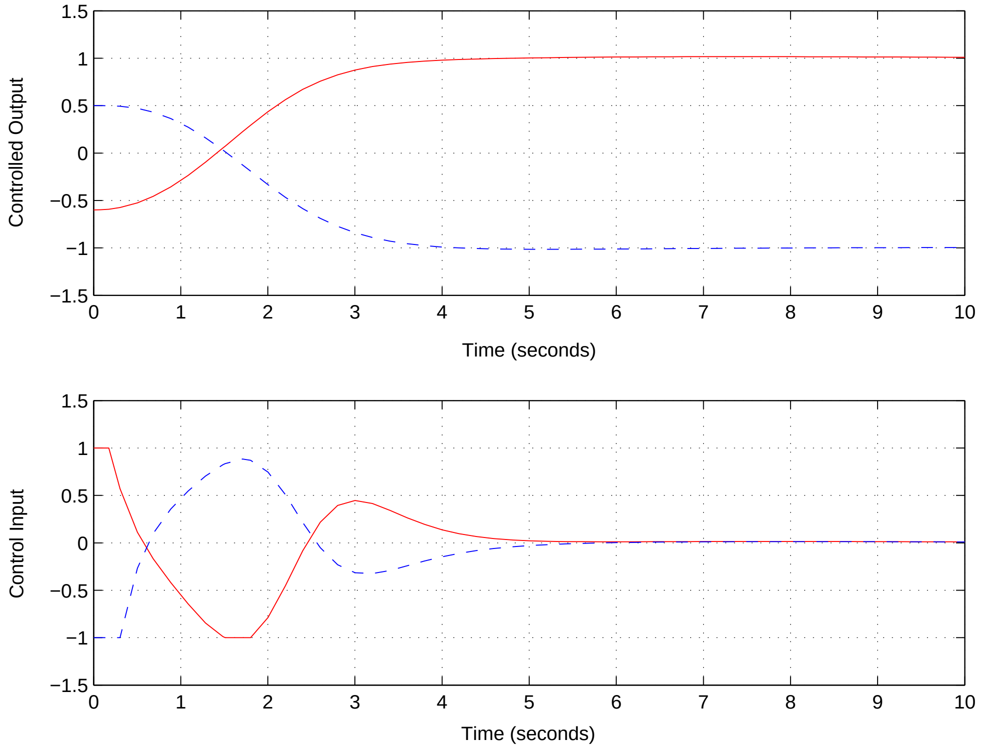


Figure 2.5: Simulation result for the full state CNF case.

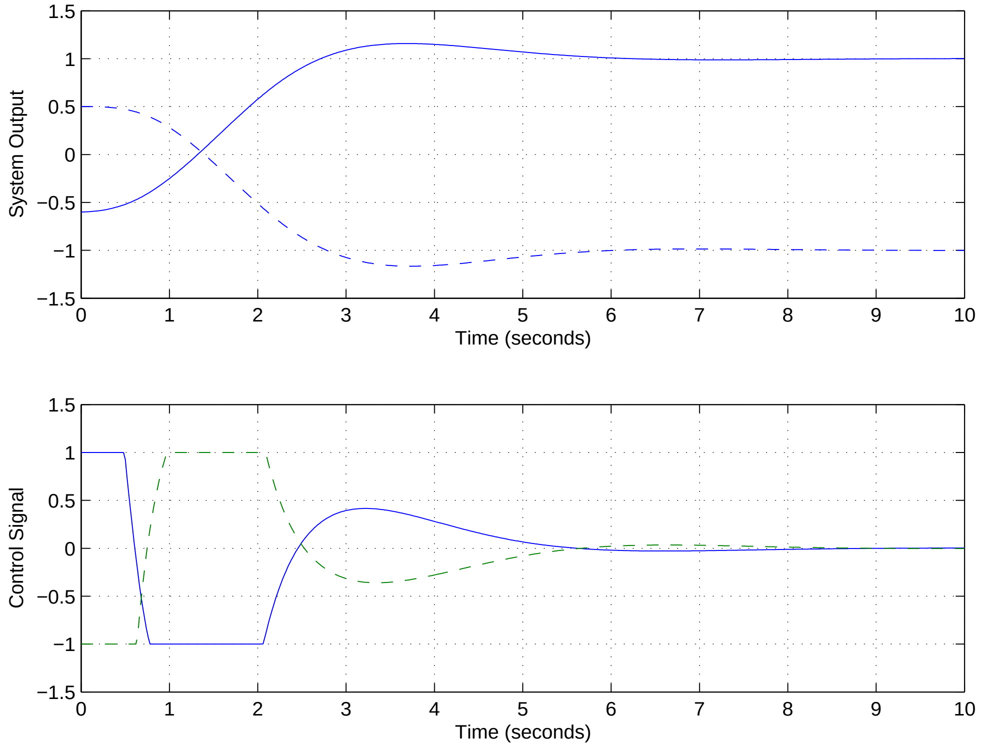


Figure 2.6: Simulation result for the full state H_2 linear feedback case.

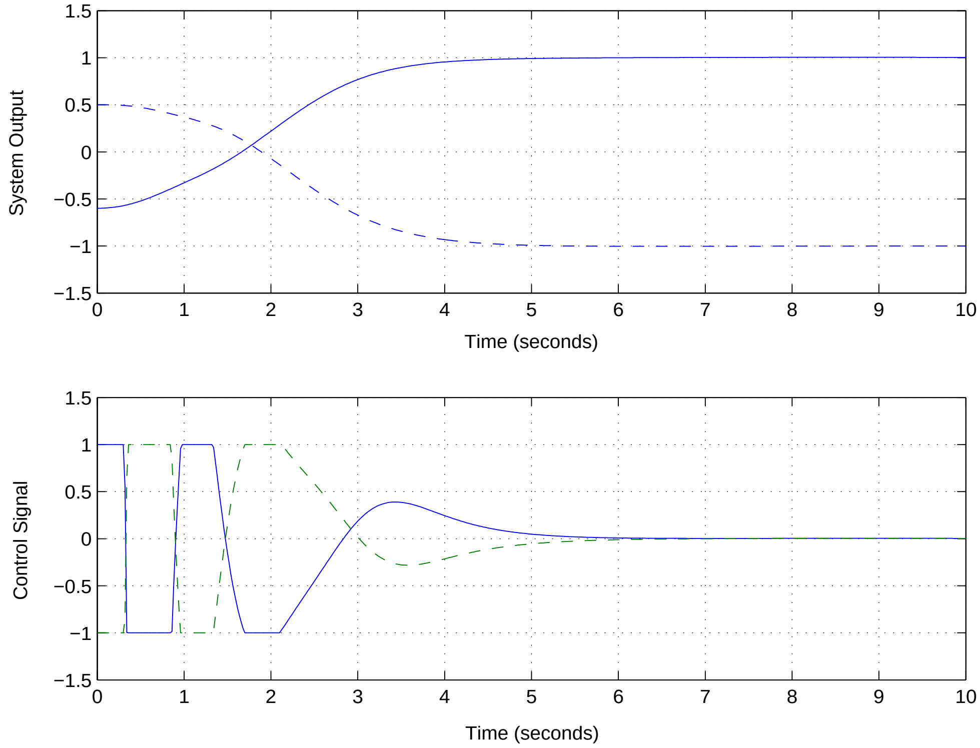


Figure 2.7: Simulation result for the full order measurement CNF case.

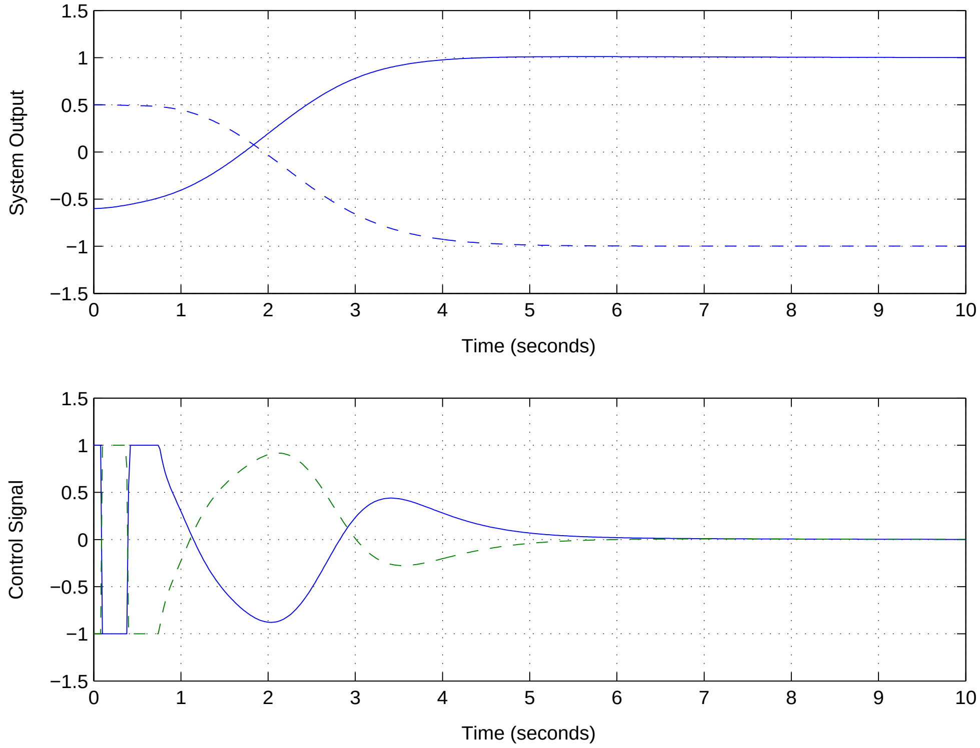


Figure 2.8: Simulation result for the reduced order measurement CNF case.

Using SIMULINK in MATLAB, I obtain a set of simulation results in Figures 2.5-2.8. The initial conditions for both full and reduced order controllers are set to zero. The results are very satisfactory for all three cases. Note that the settling times for the full order and reduced order measurement feedback cases are slightly longer compared to those of the full state feedback case. For comparison, I include in Figure 2.6 the simulation result of a carefully tuned state feedback linear control law using an H_2 optimization approach. Obviously, the CNF controller has a better performance compared to that of a best tuned linear controller.

2.4 Conclusion

I have proposed a nonlinear tracking control technique, *i.e.*, the so-called composite nonlinear feedback (CNF) control design, which consists of two parts, a linear component and a nonlinear component. The former is usually chosen to give fast rising time while the latter is added to smooth out the transient peaks or overshoots when the controlled output is approaching the target reference. The technique is applicable to general multivariable system with some standard assumptions and a natural extension of some recent work in the field. It was successfully demonstrated by a practical example on an air-to-air missile system. Finally, note that unlike the SISO case, the relationship between the physical meaning and the tuning mechanism of the nonlinear gains in the CNF design for MIMO systems is still not clearly captured due to coupling of channels. It requires more investigations and research.

Chapter 3

CNF Control for Discrete-Time Systems with Input Saturation

From previous chapter, one knows that the CNF controller is based on any linear feedback law which solves the tracking problem under actuator saturation. By adding additional nonlinear term, one is able to improve transient performance. The CNF controller has a very simple structure and is easily implemented. It is, of course, natural for one to ask whether this scheme will be extended to linear discrete-time systems. The answer is positive and in this chapter, I will present the composite nonlinear feedback control technique for linear discrete-time multivariable systems with actuator saturation. The goal of this chapter is to complete the theory for general discrete-time systems. Again, I will consider both the state feedback case and the measurement feedback case without imposing any restrictive assumption on the given systems. It will be applied to a Magnetic-Tape-Drive servo system design and yields an improvement of more than 50% in settling time compared to that of standard LQ controller which doesn't violate control constraints.

3.1 Introduction and Problem Formulation

Since the CNF control scheme has been derived for continuous-time systems, it should be natural to extend it to discrete-time systems as in practice more and more controllers are of digital types. Although people may transform continuous-time controllers into digital controllers using zero-order hold, first-order hold or bi-linear transformation methods, it is still of interest for one to explore discrete-time counterpart for CNF scheme as in many cases, when one designs the controller based on discrete-time model, it has special properties which may not be captured by transformed controller.

Note that for designing tracking controller for linear discrete-time systems, certain results can be found in the literature. In fact, some researchers proposed methods which deal with both continuous-time and discrete-time systems either in a single work or in some separate papers. However, with no exception, their proposed controllers are very complex and even hard to be designed as they lack of clear design steps, it is again appealing to find a simple controller like linear controller, as in continuous-time setting.

Unfortunately, along the same line as that of CNF control, very little has been done for linear discrete-time systems besides the work of Venkataramanan *et al.* [75], which is only applicable to linear single-input and single-output (SISO) systems with state feedback. In this chapter, I will present a complete CNF control technique for discrete-time multivariable systems with actuator saturation. Both the state feedback case and the measurement feedback case without imposing any restrictive assumption on the given systems are considered. This work aims to complete the theory for general discrete-time systems. As mentioned earlier in the abstract, the CNF control consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is typically designed to yield a quick response at the initial stage (obviously any method in the literature can be adopted to seek such a linear feedback law which does not violate the control constraints), while the nonlinear feedback law is used to smooth out overshoots in the system output when it approaches the target reference. As such, the resulting closed-loop system generally has very fast transient response and minimal overshoot.

To be specific, let us consider in this chapter the following multi-input and multi-output (MIMO) discrete-time system Σ with an amplitude-constrained actuator characterized by

$$\begin{cases} x(k+1) = Ax(k) + B\,\text{sat}(u(k)), & x(0) = x_0 \\ y(k) = C_1 x(k) \\ h(k) = C_2 x(k) + D_2\,\text{sat}(u(k)) \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $h \in \mathbb{R}^\ell$ are respectively the state, control input, measurement output and controlled output of the given system Σ. A, B, C_1 and C_2 are appropriate dimensional constant matrices, and the saturation function is defined by

$$\text{sat}(u) = \begin{pmatrix} \text{sat}(u_1) \\ \text{sat}(u_2) \\ \vdots \\ \text{sat}(u_m) \end{pmatrix} \tag{3.2}$$

with

$$\text{sat}(u_i) = \text{sign}(u_i)\,\min(|u_i|,\ \bar{u}_i), \tag{3.3}$$

where $\bar{u}_i$ is the maximum amplitude of the i-th control channel. The objective of this chapter is to design an appropriate control law for (3.1) using the CNF approach such that the resulting controlled output will track some desired step references as fast and as smooth as possible. I will address the CNF control system design for the given system (3.1) for three different situations, namely, the state feedback case, the full order measurement feedback case, and the reduced order measurement feedback case. For tracking purpose, the following assumptions on the given system are made:

1. (A, B) is stabilizable.
2. (A, C_1) is detectable.
3. (A, B, C_2, D_2) is right invertible (and hence $m \geq l$) and has no invariant zeros at $z = 1$.

Note that these assumptions are necessary for tracking control of discrete-time systems.

This chapter is organized as follows. Section 3.2 deals with the theory of the composite nonlinear feedback control for the state feedback case, whereas Section 3.3 deals with the detailed development of the CNF design with the full order measurement feedback and the reduced order measurement feedback cases. I will address the issue on the selection of nonlinear gain parameters in Section 3.4. The technique is then illustrated in a Magnetic-Tape-Drive design example in Section 3.5, which shows that the proposed design method yields an improvement of more than 50% in settling time compared to that of conventional linear state feedback design approaches. Finally, I will draw some concluding remarks in Section 3.6.

3.2 State Feedback Case

Let us first proceed to develop a composite nonlinear feedback control technique for the case when all the state variables of the plant Σ are measurable, *i.e.*, $y = x$. The design will be done in three steps. One has the following step-by-step design procedure.

STEP S.1: Design a linear feedback law,

$$u_{\rm L}(k) = Fx(k) + Gr, \tag{3.4}$$

where $r \in \mathbb{R}^m$ contains a set of step references. The state feedback gain matrix $F \in \mathbb{R}^{m\times n}$ is chosen such that the closed-loop system matrix $A+BF$ is asymptotically stable and typically the resulting closed-loop system transfer matrix, *i.e.*, $D_2 + (C_2 + D_2F)(zI - A - BF)^{-1}B$, has certain desired properties, *e.g.*, having a small dominating damping ratio in each channel. Note that such an F can be worked out using some well-studied methods such as the LQR, H_∞ and H_2 optimization approaches (see, *e.g.*, Anderson and Moore [1], Chen [17] and Saberi *et al.* [62]). Furthermore, G is an $m \times l$ constant matrix and is given by

$$G := G_0' \left(G_0 G_0'\right)^{-1}, \tag{3.5}$$

with $G_0 := D_2 + (C_2 + D_2F)(I - A - BF)^{-1}B$. Here note that both G_0 and G are well defined because $A + BF$ is stable, and (A, B, C_2, D_2) is right invertible and has no invariant zeros at $z = 1$, which implies $(A + BF, B, C_2 + D_2F, D_2)$ is right invertible and has no invariant zeros at $z = 1$ (see *e.g.*, Lemma 2.5.1 of Chen [17]).

Step s.2: Next, compute

$$H := \big[I + F(I - A - BF)^{-1}B\big]\, G \tag{3.6}$$

and

$$x_{\rm e} := G_{\rm e}\, r := (I - A - BF)^{-1} BG\, r. \tag{3.7}$$

Note that the definitions of H, $G_{\rm e}$ and $x_{\rm e}$ would become transparent later in the derivation. Given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, solve the following Lyapunov equation:

$$P = (A + BF)' P (A + BF) + W, \tag{3.8}$$

for $P > 0$. Such a P exists since $A + BF$ is asymptotically stable. Then, the nonlinear feedback control law $u_{\rm N}(k)$ is given by

$$u_{\rm N}(k) = \rho(r, y) B' P (A + BF)(x(k) - x_{\rm e}), \tag{3.9}$$

where

$$\rho(r, y) = \mathrm{diag}\{\rho_1, \cdots, \rho_m\} = \begin{bmatrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_m \end{bmatrix}, \tag{3.10}$$

and $\rho_i = \rho_i(r, y)$, $i = 1, 2, \cdots, m$, are some nonpositive functions, locally Lipschitz in y, which are used to change the closed-loop system damping ratios as the outputs approach the targets. The choice of these nonlinear functions will be discussed in Section 3.4.

Step s.3: The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller:

$$u(k) = u_{\rm L}(k) + u_{\rm N}(k) = Fx(k) + Gr + \rho(r, y) B' P (A + BF)(x(k) - x_{\rm e}). \tag{3.11}$$

This completes the design of the CNF controller for the state feedback case.

For further development, let us partition $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{m\times n}$ and $H \in \mathbb{R}^{m\times l}$ as follows:

$$B = [\, B_1 \quad \cdots \quad B_m \,], \quad F = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ \vdots \\ H_m \end{bmatrix}. \tag{3.12}$$

The following theorem shows that the closed-loop system comprising the given plant in (3.1) and the CNF control law of (3.11) is asymptotically stable. It also determines the magnitudes of the step functions in r that can be tracked by such a control law without exceeding the control limit.

Theorem 3.1. *Consider the given system Σ in (3.1) with $y = x$, which satisfies Assumptions 1 and 3, the linear control law of (3.4) and the composite nonlinear feedback control law of (3.11). For any $\delta \in (0,1)$, let $c_\delta > 0$ be the largest positive scalar such that for all $x(k) \in \boldsymbol{X}_\delta$, where*

$$\boldsymbol{X}_\delta := \left\{ x : x'Px \le c_\delta \right\}, \tag{3.13}$$

the following property holds,

$$| F_i\, x(k) | \le (1-\delta)\bar{u}_i, \quad i = 1, \cdots, m. \tag{3.14}$$

Then, the linear control law of (3.4) is capable of driving the system controlled output $h(k)$ to track asymptotically a set of step references, i.e., r, provided that the initial state x_0 and r satisfy:

$$\tilde{x}_0 := (x_0 - x_{\mathrm{e}}) \in \boldsymbol{X}_\delta, \quad |H_i\, r| \le \delta \bar{u}_i, \quad i = 1, \cdots, m. \tag{3.15}$$

Furthermore, for any nonpositive function $\rho(r, y)$, locally Lipschitz in y, which satisfies

$$2\rho + \rho B'PB\rho \le 0, \quad \text{or} \quad \rho^{-1} \le -\frac{1}{2}B'PB \tag{3.16}$$

if ρ is selected to be non-singular, the composite nonlinear feedback law in (3.11) is capable of driving the system controlled output $h(k)$ to track asymptotically the step command input of amplitude r, provided that the initial state x_0 and r satisfy (3.15).

Proof. Let us first define a new state variable $\tilde{x}(k) = x(k) - x_{\mathrm{e}}$. It is simple to verify that the linear feedback control law of (3.4) can be rewritten as

$$u_{\mathrm{L}}(k) = F\tilde{x}(k) + [I + F(I - A - BF)^{-1}B]Gr = F\tilde{x}(k) + Hr, \tag{3.17}$$

and hence for all $\tilde{x}(k) \in \boldsymbol{X}_\delta$ and, provided that $|H_i\, r| \le \delta \bar{u}_i$, $i = 1, \cdots, m$, the closed-loop system is linear and is given by

$$x(k+1) = (A + BF)\tilde{x}(k) + Ax_{\mathrm{e}} + BHr. \tag{3.18}$$

Noting that

$$\begin{aligned} Ax_{\rm e} + BHr &= \Big\{A(I-A-BF)^{-1}BG + B[I+F(I-A-BF)^{-1}B]G\Big\}r \\ &= \Big\{A(I-A-BF)^{-1}BG + [I+BF(I-A-BF)^{-1}]BG\Big\}r \\ &= \Big[A(I-A-BF)^{-1} + I + BF(I-A-BF)^{-1}\Big]BGr \\ &= (I-A-BF)^{-1}BGr = x_{\rm e}, \end{aligned} \tag{3.19}$$

the closed-loop system in (3.18) can then be simplified as

$$\tilde{x}(k+1) = (A+BF)\tilde{x}(k). \tag{3.20}$$

Similarly, the closed-loop system comprising the given plant in (3.1) and the CNF control law of (3.11) can be expressed as

$$\tilde{x}(k+1) = (A+BF)\tilde{x}(k) + Bw(k), \tag{3.21}$$

where

$$w(k) = \mathrm{sat}(F\tilde{x}(k) + Hr + u_{\rm N}(k)) - F\tilde{x}(k) - Hr. \tag{3.22}$$

Clearly, for the given x_0 satisfying (3.15), one has $\tilde{x}_0 = (x_0 - x_{\rm e}) \in \boldsymbol{X}_\delta$. Note that (3.21) is reduced to (3.20) if $\rho(r,y) = 0$.

Next, let us define a Lyapunov function $V(k) = \tilde{x}'(k)P\tilde{x}(k)$ and evaluate the increment of $V(k)$ along the trajectories of the closed-loop system in (3.21), *i.e.*,

$$\begin{aligned} \triangle V(k+1) &= \tilde{x}'(k+1)P\tilde{x}(k+1) - \tilde{x}'(k)P\tilde{x}(k) \\ &= \tilde{x}'(k)(A+BF)'P(A+BF)\tilde{x}(k) - \tilde{x}'(k)P\tilde{x}(k) \\ &\qquad + 2\tilde{x}'(k)(A+BF)'PBw(k) + w'(k)B'PBw(k) \\ &= -\tilde{x}'(k)W\tilde{x}(k) + 2\tilde{x}'(k)(A+BF)'PBw(k) + w'(k)B'PBw(k). \end{aligned} \tag{3.23}$$

Note that for all

$$\tilde{x}(k) \in \boldsymbol{X}_\delta = \{\tilde{x}(k) : \tilde{x}'(k)P\tilde{x}(k) \le c_\delta\} \quad \Rightarrow \quad |F_i\,\tilde{x}(k)| \le (1-\delta)\bar{u}_i, \quad i = 1, \cdots, m. \tag{3.24}$$

In the remainder of this proof, let us consider the following different scenarios. For simplicity, I will drop the dependent variables of the nonlinear function ρ in the rest of this proof.

Case 1. All input channels are unsaturated. It is obvious that one has

$$w(k) = u_{\rm N}(k) = \rho B' P(A + BF)\tilde{x}(k) \tag{3.25}$$

and thus

$$\begin{aligned} \triangle V(k+1) &= -\tilde{x}'(k)W\tilde{x}(k) + 2\tilde{x}'(k)(A+BF)'PB\rho B'P(A+BF)\tilde{x}(k) \\ &\quad + \tilde{x}'(k)(A+BF)'PB\rho B'PB\rho B'P(A+BF)\tilde{x}(k) \\ &= -\tilde{x}'(k)W\tilde{x}(k) \\ &\quad + \tilde{x}'(k)(A+BF)'PB(2\rho + \rho B'PB\rho)B'P(A+BF)\tilde{x}(k) \end{aligned} \tag{3.26}$$

In view of (3.16), one has

$$\triangle V(k+1) \le -\tilde{x}'(k)W\tilde{x}(k) < 0. \tag{3.27}$$

Case 2. All input channels are exceeding their upper limits. In this case, let

$$u_{{\rm N}i}(k) = \rho_i B_i' P(A + BF)\tilde{x}(k). \tag{3.28}$$

Thus, the assumption that all input channels are exceeding their upper limits, *i.e.*,

$$F_i\tilde{x}(k) + H_i r + u_{{\rm N}i}(k) \ge \bar{u}_i, \ \ i = 1, \cdots, m, \tag{3.29}$$

implies that

$$u_{{\rm N}i}(k) \ge \bar{u}_i - F_i\tilde{x}(k) - H_i r, \ \ i = 1, \cdots, m \tag{3.30}$$

and

$$w_i(k) = \bar{u}_i - (F_i\tilde{x}(k) + H_i r). \tag{3.31}$$

For all $\tilde{x}(k) \in \boldsymbol{X}_\delta$, which implies that (3.24) holds, and r satisfies (3.15), one has

$$F_i\tilde{x}(k) + H_i r \le \bar{u}_i, \ \ i = 1, \cdots, m, \tag{3.32}$$

Hence,

$$0 \le w_i(k) \le u_{{\rm N}i}(k). \tag{3.33}$$

$$\begin{aligned}\triangle V(k+1) &= -\tilde{x}'(k)W\tilde{x}(k) + w'(k)[2B'P(A+BF)]\tilde{x}(k) + w'(k)B'PBw(k) \\ &= -\tilde{x}'(k)W\tilde{x}(k) + \sum_{i=1}^{m} w_i(k)[2\rho_i^{-1}u_{\mathrm{N}i}(k)] + w'(k)B'PBw(k) \\ &\le -\tilde{x}'(k)W\tilde{x}(k) + \sum_{i=1}^{m} w_i(k)[2\rho_i^{-1}w_i(k)] + w'(k)B'PBw(k) \\ &= -\tilde{x}'(k)W\tilde{x}(k) + w'(k)(2\rho^{-1})w(k) + w'(k)B'PBw(k) \\ &= -\tilde{x}'(k)W\tilde{x}(k) + w'(k)(2\rho^{-1} + B'PB)w(k) < 0. \end{aligned} \tag{3.34}$$

Case 3. All input channels are exceeding their lower limits. For this case, one has

$$F_i\tilde{x}(k) + H_ir + \rho_iB_i'P(A+BF)\tilde{x}(k) \le -\bar{u}_i, \quad i = 1, \cdots, m. \tag{3.35}$$

Following similar arguments as in the previous case, one can show that

$$\triangle V(k+1) \le -\tilde{x}'(k)W\tilde{x}(k) < 0. \tag{3.36}$$

Case 4. Some control channels are saturated and some are unsaturated. In view of Cases 1 to 3, the increment is just a combination of the above three cases. For those unsaturated channels, one has

$$w_i(k) = u_{\mathrm{N}i}(k) = \rho_iB_i'P(A+BF)\tilde{x}(k) \tag{3.37}$$

and

$$w_i(k)(2\rho_i^{-1})u_{\mathrm{N}i}(k) = w_i(k)(2\rho_i^{-1})w_i(k). \tag{3.38}$$

On the other hand, for those saturated channels, one has either

$$0 \le w_i(k) = \bar{u}_i(k) - (F_i\tilde{x}(k) + H_ir) \le u_{\mathrm{N}i}(k) \tag{3.39}$$

or

$$u_{\mathrm{N}i}(k) \le w_i(k) = -\bar{u}_i(k) - (F_i\tilde{x}(k) + H_ir) \le 0. \tag{3.40}$$

Thus, one has

$$w_i(k)[2\rho_i^{-1}u_{\mathrm{N}i}(k)] \le w_i(k)(2\rho_i^{-1})w_i(k). \tag{3.41}$$

It is then straightforward to verify that for this case, again, one has

$$\triangle V(k+1) \le -\tilde{x}'(k)W\tilde{x}(k) < 0. \tag{3.42}$$

In conclusion, I have shown that

$$\triangle V(k+1) \leq -\tilde{x}'(k)W\tilde{x}(k), \quad \tilde{x}(k) \in \boldsymbol{X}_\delta, \tag{3.43}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (3.21). Noting that $W > 0$, all trajectories of (3.21) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial state x_0 and the step command input r that satisfy (3.15), one has

$$\lim_{k\to\infty} x(k) = x_\mathrm{e}, \tag{3.44}$$

which implies

$$\lim_{k\to\infty} u(k) = F\lim_{k\to\infty} x(k) + Gr + \rho B'P(A+BF)[\lim_{k\to\infty} x(k) - x_\mathrm{e}] = Fx_\mathrm{e} + Gr. \tag{3.45}$$

Hence,

$$\begin{aligned}
\lim_{k\to\infty} h(k) &= C_2 \lim_{k\to\infty} x(k) + D_2 \lim_{k\to\infty} u(k) \\
&= C_2 x_\mathrm{e} + D_2(Fx_\mathrm{e} + Gr) \\
&= (C_2 + D_2F)x_\mathrm{e} + D_2Gr \\
&= (C_2 + D_2F)(I - A - BF)^{-1}BGr + D_2Gr \\
&= [D_2 + (C_2 + D_2F)(I - A - BF)^{-1}B]Gr \\
&= G_0G_0'(G_0G_0')^{-1}r = r.
\end{aligned} \tag{3.46}$$

This completes the proof of Theorem 3.1.

3.3 Measurement Feedback Case

The assumption that all the state variables of the given system Σ are measurable is generally neither feasible nor practical. In this section, let us proceed to design CNF control laws using only measurement information. Both full order and reduced order control laws are considered.

3.3.1 Full Order Measurement Feedback Case

Let us first deal with the full order measurement feedback case, in which the dynamical order of the controller is exactly the same as that of the given plant. The following is a step-by-step procedure for the CNF design using full order measurement feedback.

STEP F.1: First construct a linear full order measurement feedback control law,

$$\begin{cases} x_{\rm v}(k+1) = (A+KC_1)x_{\rm v}(k) - Ky(k) + B\ \mathrm{sat}(u_{\rm L}(k)) \\ \qquad u_{\rm L}(k) = F(x_{\rm v}(k) - x_{\rm e}) + Hr, \end{cases} \tag{3.47}$$

where r is the set of step reference signals and $x_{\rm v}(k)$ is the state of the controller. As usual, K, F are gain matrices and are chosen such that $(A + KC_1)$ and $(A + BF)$ are asymptotically stable and the resulting closed loop system having desired properties. Finally, H and $x_{\rm e}$ are as defined in (3.6)–(3.7).

STEP F.2: Given a positive definite matrix $W_{\rm P} \in \mathbb{R}^{n\times n}$, solve the Lyapunov equation

$$P = (A + BF)'P(A + BF) + W_{\rm P}, \tag{3.48}$$

for $P > 0$. As in the state feedback case, the linear control law of (3.47) obtained in the above step is to be combined with a nonlinear control law to form the following CNF controller:

$$\begin{cases} x_{\rm v}(k+1) = (A+KC_1)x_{\rm v}(k) - Ky(k) + B\ \mathrm{sat}(u(k)) \\ \quad u(k) \quad\ = F(x_{\rm v}(k) - x_{\rm e}) + Hr + \rho(r,y)B'P(A+BF)(x_{\rm v}(k) - x_{\rm e}), \end{cases} \tag{3.49}$$

where $\rho(r, y)$ is as given in (3.10) with all its diagonal elements being respectively a nonpositive function, locally Lipschitz in y, which are to be chosen to improve the performance of the closed-loop system.

One has the following result.

Theorem 3.2. *Consider the given system in (3.1), which satisfies the standard Assumptions 1–3, the full order linear measurement feedback control law of (3.47) and the composite nonlinear measurement feedback control law of (3.49). Given a positive definite matrix* $W_{\rm Q} \in \mathbb{R}^{n\times n}$ *with*

$$W_{\rm Q} > F'[B'PB + B'P(A + BF)W_{\rm P}^{-1}(A + BF)'PB]F, \tag{3.50}$$

let $Q > 0$ be the solution to the following Lyapunov equation:

$$Q = (A + KC_1)' Q (A + KC_1) + W_{\mathrm{Q}}. \tag{3.51}$$

Note that such a Q exists as $A + KC_1$ is asymptotically stable. For any $\delta \in (0,1)$, let $c_\delta > 0$ be the largest positive scalar such that for all

$$\begin{pmatrix} x(k) \\ x_{\mathrm{v}}(k) \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta} := \left\{ \begin{pmatrix} x(k) \\ x_{\mathrm{v}}(k) \end{pmatrix} : \begin{pmatrix} x(k) \\ x_{\mathrm{v}}(k) \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} x(k) \\ x_{\mathrm{v}}(k) \end{pmatrix} \le c_\delta \right\}, \tag{3.52}$$

the following property holds

$$\left| [\, F_i \quad F_i \,] \begin{pmatrix} x(k) \\ x_{\mathrm{v}}(k) \end{pmatrix} \right| \le (1-\delta)\bar{u}_i, \quad i = 1, \cdots, m. \tag{3.53}$$

Then, there exist nonpositive scalars $\rho_i^* \le 0$, $i = 1, \ldots, m$, such that for any nonpositive functions $\rho_i(r, y)$, $i = 1, \cdots, m$, locally Lipschitz in y and $\rho_i^* \le \rho_i(r, y) \le 0$, $i = 1, \cdots, m$, the control law in (3.49) will drive the system's controlled output $h(k)$ to track asymptotically a set of step references, i.e., r, from an initial state x_0, provided that x_0, $x_{\mathrm{v}0} = x_{\mathrm{v}}(0)$ and r satisfy:

$$\begin{pmatrix} x_0 - x_{\mathrm{e}} \\ x_{\mathrm{v}0} - x_0 \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta} \quad \text{and} \quad |H_i r| \le \delta \bar{u}_i, \quad i = 1, \cdots, m. \tag{3.54}$$

Proof. For simplicity, I will again drop r and y in $\rho(r, y)$ throughout the proof of this theorem. Let $\tilde{x} = x - x_{\mathrm{e}}$ and $\tilde{x}_{\mathrm{v}} = x_{\mathrm{v}} - x$. The linear feedback control law of (3.47) can be written as

$$\tilde{x}_{\mathrm{v}}(k+1) = (A + KC_1)\tilde{x}_{\mathrm{v}}(k), \quad u_{\mathrm{L}}(k) = [\, F \quad F \,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{v}}(k) \end{pmatrix} + Hr. \tag{3.55}$$

Hence, for all

$$\begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{v}}(k) \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta} \quad \Rightarrow \quad \left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{v}}(k) \end{pmatrix} \right| \le (1-\delta)\bar{u}_i, \quad i = 1, \cdots, m, \tag{3.56}$$

and for any r satisfying

$$|H_i r| \le \delta \bar{u}_i, \quad i = 1, \cdots, m, \tag{3.57}$$

each channel of u_{L}, say $u_{\mathrm{L},i}$, has the following property

$$u_{\mathrm{L},i}(k) = \left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{v}}(k) \end{pmatrix} + H_i r \right| \le \left| [\, F_i \quad F_i \,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\mathrm{v}}(k) \end{pmatrix} \right| + |H_i r| \le \bar{u}_i. \tag{3.58}$$

Thus, for all $\tilde{x}(k)$ and $\tilde{x}_{\rm v}(k)$ satisfying the condition as given in (3.56), the closed-loop system comprising the given plant and the linear control law of (3.47) can be rewritten as

$$\begin{pmatrix} \tilde{x}(k+1) \\ \tilde{x}_{\rm v}(k+1) \end{pmatrix} = \begin{bmatrix} A+BF & BF \\ 0 & A+KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix}. \tag{3.59}$$

Similarly, the closed-loop system with the CNF control law of (3.49) can be expressed as

$$\begin{pmatrix} \tilde{x}(k+1) \\ \tilde{x}_{\rm v}(k+1) \end{pmatrix} = \begin{bmatrix} A+BF & BF \\ 0 & A+KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w(k), \tag{3.60}$$

where

$$\begin{aligned} w(k) &= \text{sat}\left[[\,F \quad F\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} + Hr + \rho\,[\,B'P \quad B'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \right] \\ &\quad - [\,F \quad F\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} - Hr. \end{aligned} \tag{3.61}$$

Let us consider the following possible situations that could happen to the control input channels.

Case 1. If an input channel, say channel i, is unsaturated, *i.e.*,

$$-\bar{u}_i \le [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} + H_i r + \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \le \bar{u}_i, \tag{3.62}$$

then one has

$$w_i(k) = \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix}. \tag{3.63}$$

Case 2. If an input channel is exceeding its upper limit, *i.e.*,

$$[\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} + H_i r + \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \ge \bar{u}_i, \tag{3.64}$$

then for all trajectories inside $\boldsymbol{X}_{{\rm F}\delta}$, one has

$$\left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} + H_i r \right| \le \left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \right| + |H_i r| \le \bar{u}_i, \tag{3.65}$$

and thus

$$0 \le w_i(k) = \bar{u}_i - [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} - H_i r \le \rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix}. \tag{3.66}$$

Case 3. Similarly, for the case when an input channel is exceeding its lower limits, one has

$$\rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF) \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \le w_i(k) \le 0. \tag{3.67}$$

Clearly, for all the above cases, one can express

$$w_i(k) = q_i\rho_i\,[\,B_i'P \quad B_i'P\,]\,(A+BF)\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix} \tag{3.68}$$

for some scalar function $q_i \in [0,1]$. Defining a diagonal matrix $q := \text{diag}\{q_1, \cdots q_n\}$, one has

$$w(k) = \tilde{\rho}\,[\,B'P \quad B'P\,]\,(A+BF)\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix} \tag{3.69}$$

where $\tilde{\rho} = q\rho$.

Next, note that (3.59) and (3.60) are identical when $\rho = 0$. Again, the results of Theorem 3.2 for both the linear and the nonlinear feedback case can be proved in one shot. Let us proceed to prove the result by defining a Lyapunov function:

$$V(k) = \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}'\begin{bmatrix}P & 0\\ 0 & Q\end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}, \tag{3.70}$$

and evaluating the increment of $V(k)$ along the trajectories of the closed-loop system in (3.60), one obtains

$$\begin{aligned}\triangle V(k+1) &= \begin{pmatrix}\tilde{x}(k+1)\\ \tilde{x}_{\rm v}(k+1)\end{pmatrix}'\begin{bmatrix}P & 0\\ 0 & Q\end{bmatrix}\begin{pmatrix}\tilde{x}(k+1)\\ \tilde{x}_{\rm v}(k+1)\end{pmatrix} - \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}'\begin{bmatrix}P & 0\\ 0 & Q\end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}\\ &= \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}'\begin{bmatrix}-W_{\rm P} & (A+BF)'PBF\\ (BF)'P(A+BF) & -W_{\rm Q}+(BF)'PBF\end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}\\ &\quad + w'(k)\begin{bmatrix}(A+BF)'PB\\ (BF)'PB\end{bmatrix}'\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix} + \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}'\begin{bmatrix}(A+BF)'PB\\ (BF)'PB\end{bmatrix}w(k)\\ &\quad + w'(k)B'PBw(k).\end{aligned} \tag{3.71}$$

Substituting (3.69) into $\triangle V(k+1)$, one has

$$\begin{aligned}\triangle V(k+1) &= \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}'\begin{bmatrix}-W_{\rm P} & (A+BF)'PBF\\ (BF)'P(A+BF) & -W_{\rm Q}+(BF)'PBF\end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}\\ &\quad + \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}'\begin{bmatrix}(A+BF)'PB\\ (A+BF)'PB\end{bmatrix}\tilde{\rho}[B'P(A+BF) \;\; B'PBF]\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}\\ &\quad + \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}'\begin{bmatrix}(A+BF)'PB\\ (BF)'PB\end{bmatrix}\tilde{\rho}[B'P(A+BF) \;\; B'P(A+BF)]\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}\\ &\quad + \begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}'\begin{bmatrix}(A+BF)'PB\\ (A+BF)'PB\end{bmatrix}\tilde{\rho}B'PB\tilde{\rho}\begin{bmatrix}(A+BF)'PB\\ (A+BF)'PB\end{bmatrix}'\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\rm v}(k)\end{pmatrix}\end{aligned} \tag{3.72}$$

Letting $T := B'P(A+BF)$, one gets

$$\begin{aligned}
\triangle V(k+1) &= \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix}' \begin{bmatrix} -W_{\rm P} & T'F \\ F'T & -W_{\rm Q}+(BF)'PBF \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \\
&\quad + \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix}' \begin{bmatrix} 0 & T'\tilde{\rho}B'PBF \\ T'\tilde{\rho}T & T'\tilde{\rho}B'PBF \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \\
&\quad + \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix}' \begin{bmatrix} 0 & T'\tilde{\rho}T \\ F'B'PB\tilde{\rho}T & F'B'PB\tilde{\rho}T \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \\
&\quad + \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix}' \begin{bmatrix} 0 & T'\tilde{\rho}B'PB\tilde{\rho}T \\ T'\tilde{\rho}B'PB\tilde{\rho}T & T'\tilde{\rho}B'PB\tilde{\rho}T \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \\
&\quad + \tilde{x}(k)'T'\tilde{\rho}(2I+B'PB\tilde{\rho})T\tilde{x}(k) \\
&= -\begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix}' \begin{bmatrix} W_{\rm P} & -W_{\rm d} \\ -W_{\rm d}' & W_{\rm m} \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} \\
&\quad + \tilde{x}(k)'T'\tilde{\rho}(2I+B'PB\tilde{\rho})T\tilde{x}(k), \qquad (3.73)
\end{aligned}$$

where

$$W_{\rm d} = T'(I+\tilde{\rho}B'PB)(F+\tilde{\rho}T) \qquad (3.74)$$

and

$$W_{\rm m} = W_{\rm Q} - (F+\tilde{\rho}T)'B'PB(F+\tilde{\rho}T). \qquad (3.75)$$

Defining

$$\hat{x}_{\rm m}(k) := \begin{pmatrix} \tilde{x}(k) - W_{\rm P}^{-1}W_{\rm d}\tilde{x}_{\rm v}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix}, \qquad (3.76)$$

one has

$$\triangle V(k+1) = -\hat{x}_{\rm m}(k)' \begin{bmatrix} W_{\rm P} & 0 \\ 0 & \tilde{W}_{\rm Q} \end{bmatrix} \hat{x}_{\rm m}(k) + \tilde{x}(k)'T'\tilde{\rho}(2\tilde{\rho}^{-1}+B'PB)\tilde{\rho}T\tilde{x}(k), \qquad (3.77)$$

where

$$\begin{aligned}
\tilde{W}_{\rm Q} &:= W_{\rm m} - W_{\rm d}'W_{\rm P}^{-1}W_{\rm d} \\
&= W_{\rm Q} - (F+\tilde{\rho}T)'\Big[B'PB + (I+\tilde{\rho}B'PB)B'P(A+BF)W_{\rm P}^{-1} \\
&\qquad \cdot (A+BF)'PB(I+\tilde{\rho}B'PB)\Big](F+\tilde{\rho}T). \qquad (3.78)
\end{aligned}$$

Noting that (3.50), *i.e.*,

$$W_{\rm Q} > F'[B'PB + B'P(A+BF)W_{\rm P}^{-1}(A+BF)'PB]F,$$

and ρ_i is locally Lipschitz, it is clear that there exist nonpositive scalars $\rho_i^* \le 0$, $i = 1, \cdots, m$, such that for any scalar function satisfying $\rho_i^* \le \rho_i \le \tilde{\rho}_i \le 0$, one has

$$\tilde{W}_{\mathrm{Q}} > 0 \quad \text{and} \quad 2\tilde{\rho}^{-1} + B'PB < 0.$$

and hence $\triangle V(k+1) \le 0$.

Thus, $\boldsymbol{X}_{\mathrm{F}\delta}$ is an invariant set of the closed-loop system in (3.60), and all trajectories starting from $\boldsymbol{X}_{\mathrm{F}\delta}$ will remain inside and asymptotically converge to the origin. This, in turn, indicates that, for the initial state of the given system x_0, the initial state of the controller $x_{\mathrm{v}0}$, and step command input r that satisfy (3.54), one has

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_{\mathrm{v}0} \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta}, \tag{3.79}$$

where $\tilde{x}_0 = \tilde{x}(0)$ and $\tilde{x}_{\mathrm{v}0} = \tilde{x}_{\mathrm{v}}(0)$, and

$$\lim_{k\to\infty} \tilde{x}(k) = 0 \quad \text{and hence} \quad \lim_{k\to\infty} x(k) = x_{\mathrm{e}}, \tag{3.80}$$

and on the other hand,

$$\lim_{k\to\infty} \tilde{x}_{\mathrm{v}}(k) = 0 \quad \text{and hence} \quad \lim_{k\to\infty} x_{\mathrm{v}}(k) = \lim_{k\to\infty} x(k) = x_{\mathrm{e}}, \tag{3.81}$$

which implies

$$\lim_{k\to\infty} u(k) = F[\lim_{k\to\infty} x_{\mathrm{v}}(k) - x_{\mathrm{e}}] + Hr + \rho B'P(A+BF)[\lim_{k\to\infty} x_{\mathrm{v}}(k) - x_{\mathrm{e}}] = Hr = Fx_{\mathrm{e}} + Gr \tag{3.82}$$

and then it follows from (3.46) that the controlled output $h(k)$ converges asymptotically to the reference, r. This completes the proof of Theorem 3.2.

3.3.2 Reduced Order Measurement Feedback Case

For the given system in (3.1), it is clear that there are p state variables of the system, which are measurable if C_1 is of maximal rank. Thus, in general, it is not necessary to estimate these measurable state variables in measurement feedback laws. As such, I will proceed in this subsection to design a dynamic controller that has a dynamical order less than that of the given plant. For simplicity of presentation, let us assume that C_1 is already in the form

$$C_1 = [\, I_p \quad 0 \,]. \tag{3.83}$$

Then, the system in (3.1) can be rewritten as

$$\begin{cases} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathrm{sat}(u(k)) \\ y(k) = [\ I_p \quad 0\ ] \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \\ h(k) = C_2 \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{bmatrix} D_{21} \\ D_{22} \end{bmatrix} \mathrm{sat}(u(k)), \quad x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \end{cases} \tag{3.84}$$

where the original state x is partitioned into two parts, x_1 and x_2 with $y \equiv x_1$. Thus, one will only need to estimate x_2 in the reduced order measurement feedback design. Next, let F be chosen such that i) $A+BF$ is asymptotically stable, and ii) $D_2+(C_2+D_2F)(zI-A-BF)^{-1}B$ has desired properties, and let K_R be chosen such that $A_{22}+K_\mathrm{R}A_{12}$ is asymptotically stable. Here note that it can be shown that (A_{22}, A_{12}) is detectable if and only if (A, C_1) is detectable. Thus, there exists a stabilizing K_R. Again, such F and K_R can be designed using an appropriate control technique. One then partitions F in conformity with x_1 and x_2:

$$F = [\,\mathbb{F}_1 \quad \mathbb{F}_2\,]\,. \tag{3.85}$$

Let us further partition $\mathbb{F}_2$ as follows:

$$\mathbb{F}_2 = \begin{bmatrix} \mathbb{F}_{2,1} \\ \vdots \\ \mathbb{F}_{2,m} \end{bmatrix}. \tag{3.86}$$

Also, let G, H and x_e be as given in (3.5)–(3.7). The reduced order CNF controller is given by

$$x_\mathrm{v}(k+1) = (A_{22}+K_\mathrm{R}A_{12})x_\mathrm{v}(k) + (B_2+K_\mathrm{R}B_1)\,\mathrm{sat}(u(k)) + \Big[A_{21}+K_\mathrm{R}A_{11}-(A_{22}+K_\mathrm{R}A_{12})K_\mathrm{R}\Big]y(k) \tag{3.87}$$

and

$$u(k) = F\left[\begin{pmatrix} y(k) \\ x_\mathrm{v}(k)-K_\mathrm{R}y(k) \end{pmatrix} - x_\mathrm{e}(k)\right] + Hr + \rho(r,y)B'P(A+BF)\left[\begin{pmatrix} y \\ x_\mathrm{v}(k)-K_\mathrm{R}y(k) \end{pmatrix} - x_\mathrm{e}(k)\right], \tag{3.88}$$

where $\rho(r, y)$ is as given in (3.10).

Next, given a positive definite matrix $W_{\rm P} \in \mathbb{R}^{n\times n}$, let $P > 0$ be the solution to the Lyapunov equation

$$P = (A + BF)'P(A + BF) + W_{\rm P}. \tag{3.89}$$

Given a positive definite matrix $W_{\rm R} \in \mathbb{R}^{n\times n}$ with

$$W_{\rm Q} > \mathbb{F}_2'[B'PB + B'P(A + BF)W_{\rm P}^{-1}(A + BF)'PB]\mathbb{F}_2, \tag{3.90}$$

let $Q_{\rm R} > 0$ be the solution to the Lyapunov equation

$$Q_{\rm R} = (A_{22} + K_{\rm R}A_{12})'Q_{\rm R}(A_{22} + K_{\rm R}A_{12}) + W_{\rm R}. \tag{3.91}$$

Note that such P and $Q_{\rm R}$ exist as $A + BF$ and $A_{22} + K_{\rm R}A_{12}$ are asymptotically stable. For any $\delta \in (0, 1)$, let c_δ be the largest positive scalar such that for all

$$\begin{pmatrix} x(k) \\ x_{\rm v}(k) \end{pmatrix} \in \boldsymbol{X}_{{\rm R}\delta} := \left\{ \begin{pmatrix} x(k) \\ x_{\rm v}(k) \end{pmatrix} : \begin{pmatrix} x(k) \\ x_{\rm v}(k) \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q_{\rm R} \end{bmatrix} \begin{pmatrix} x(k) \\ x_{\rm v}(k) \end{pmatrix} \le c_\delta \right\} \tag{3.92}$$

the following property holds:

$$\left| [\,F_i \quad \mathbb{F}_{2,i}\,] \begin{pmatrix} x(k) \\ x_{\rm v}(k) \end{pmatrix} \right| \le \bar{u}_i(1 - \delta), \quad i = 1, \cdots, m. \tag{3.93}$$

One has the following theorem.

Theorem 3.3. *Consider the given system in (3.1), which satisfies the standard Assumptions 1–3. Then, there exist nonpositive scalars $\rho_i^* \le 0$, $i = 1, \ldots m$, such that for any nonpositive functions $\rho_i(r, y)$, $i = 1, \cdots, m$, locally Lipschitz in y and $\rho_i^* \le \rho_i(r, y) \le 0$, $i = 1, \cdots, m$, the reduced order CNF law given by (3.87) and (3.88) will drive the system controlled output $h(k)$ to asymptotically track the reference r from an initial state x_0, provided that x_0, $x_{\rm v0}$ and r satisfy*

$$\begin{pmatrix} x_0 - x_{\rm e} \\ x_{\rm v0} - x_{20} - K_{\rm R}x_{10} \end{pmatrix} \in \boldsymbol{X}_{{\rm R}\delta}, \quad |H_i\, r| \le \delta\bar{u}_i, \quad i = 1, \cdots, m. \tag{3.94}$$

Proof. Let $\tilde{x}(k) = x(k) - x_{\rm e}$ and $\tilde{x}_{\rm v}(k) = x_{\rm v}(k) - x_2(k) - K_{\rm R}x_1(k)$. Then, the closed-loop system comprising the given plant in (3.1) and the reduced order CNF control law of (3.87) and (3.88) can be expressed as

$$\begin{pmatrix} \tilde{x}(k+1) \\ \tilde{x}_{\rm v}(k+1) \end{pmatrix} = \begin{bmatrix} A + BF & B\mathbb{F}_2 \\ 0 & A_{22} + K_{\rm R}A_{12} \end{bmatrix} \begin{pmatrix} \tilde{x}(k) \\ \tilde{x}_{\rm v}(k) \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \tag{3.95}$$

where

$$w = \mathrm{sat}\left\{[\,F \quad \mathbb{F}_2\,]\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{v}}(k)\end{pmatrix} + Hr + \rho(r,y)B'P(A+BF)\left[\tilde{x}(k) + \begin{pmatrix}0\\ \tilde{x}_{\mathrm{v}}(k)\end{pmatrix}\right]\right\}$$
$$- [\,F \quad \mathbb{F}_2\,]\begin{pmatrix}\tilde{x}(k)\\ \tilde{x}_{\mathrm{v}}(k)\end{pmatrix} - Hr. \tag{3.96}$$

The rest of the proof follows along similar lines to the reasoning given in the full order measurement feedback case.

3.4 Selecting the Nonlinear Gain $\rho(r, y)$

The key component in designing the CNF controllers is the selection of ρ and W. The freedom to choose the function $\rho(r, y)$ is used to tune the control laws so as to improve the performance of the closed-loop system as the controlled output h approaches the set point. Since the main purpose of adding the nonlinear part to the CNF controllers is to speed up the settling time, or equivalently to contribute a significant value to the control input when the tracking error, $r - h$, is small, it is appropriate for one to select a nonlinear gain matrix such that the nonlinear part will be in action when the control signal is far away from its saturation level, and thus it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising the given plant in (3.1) and the three different types of control law can be expressed as

$$\tilde{x}(k+1) = (A+BF)\tilde{x}(k) + B\rho(r,y)B'P(A+BF)\tilde{x}(k). \tag{3.97}$$

Note that the additional term $\rho(r, y)$ does not affect the stability of the estimators. It is now clear that eigenvalues of the closed-loop system in (3.97) can be changed by the function $\rho(r, y)$. In fact, for such a situation, it follows from Case 1 in the proof of Theorem 3.1 that the nonlinear gain matrix ρ is not necessary to be in a diagonal form. It is only required to satisfy the following condition

$$-2(B'PB)^{-1} \le \rho \le 0. \tag{3.98}$$

Assuming that $h(0) \ne r$ (for the trivial case when $h = r$, there is no need to add any nonlinear gain to the control), let us propose the following nonlinear gain

$$\rho(r,h) = (B'PB)^{-\frac{1}{2}}\mathrm{diag}\Big\{\tilde{\rho}_1(r,h),\cdots,\tilde{\rho}_m(r,h)\Big\}(B'PB)^{-\frac{1}{2}}, \tag{3.99}$$

with

$$\tilde{\rho}_i(r,y) = \tilde{\rho}_i(r,h) = -\beta_i\frac{2}{\pi}\mathrm{arctan}\Big(\alpha_i\Big|||h(k)-r||-||h(0)-r||\Big|\Big) \tag{3.100}$$

where $0 \le \beta_i \le 2, \quad i = 1,\ldots,m$. Obviously the value of ρ_i starts from 0 and gradually decreases to a constant

$$-2\beta_i(B'PB)^{-1}\mathrm{arctan}(\alpha_i\,|h(0)-r|)/\pi > -\beta_i(B'PB)^{-1}$$

as h approaches to the target reference r. The parameter α_i is used to determine the speed of change in ρ_i.

To examine the behavior of the closed-loop system (3.97) more explicitly, let us define an auxiliary system $G_{\mathrm{aux}}(z)$ as

$$G_{\mathrm{aux}}(z) := C_{\mathrm{aux}}(zI - A_{\mathrm{aux}})^{-1}B_{\mathrm{aux}} := B'P(zI - A - BF)^{-1}B. \tag{3.101}$$

Obviously, $G_{\mathrm{aux}}(z)$ is stable. Note that

$$C_{\mathrm{aux}}B_{\mathrm{aux}} = B'PB > 0, \tag{3.102}$$

which implies $G_{\mathrm{aux}}(z)$ is a square, invertible and uniform rank system with m infinite zeros of order 1 and with $n-m$ invariant zeros. I will show that this auxiliary system is in fact of minimum phase, i.e., all its invariant zeros are stable. Note that for such a system, it follows from the result reported in Chapter 5 of Chen *et al.* [20] that there exist nonsingular transformations $\Gamma_s \in \mathbb{R}^{n\times n}$, $\Gamma_i \in \mathbb{R}^m$ and $\Gamma_o \in \mathbb{R}^p$ such that the transformed system has the following special form,

$$\Big(\Gamma_s^{-1}A_{\mathrm{aux}}\Gamma_s,\ \Gamma_s^{-1}B_{\mathrm{aux}}\Gamma_i,\ \Gamma_o^{-1}C_{\mathrm{aux}}\Gamma_s\Big) = \left(\begin{bmatrix} A_{aa} & L_{ad} \\ E_{da} & A_{dd}\end{bmatrix}, \begin{bmatrix} 0 \\ I_m \end{bmatrix}, [0 \quad I_m]\right), \tag{3.103}$$

where the eigenvalues of A_{aa} are the invariant zeros of the auxiliary system $G_{\mathrm{aux}}(z)$, L_{ad}, E_{da} and A_{dd} are some constant matrices. Next, I will proceed to show that all the eigenvalues of A_{aa} are inside the unit circle and thus $G_{\mathrm{aux}}(z)$ is of minimum phase. Note that at the steady state when $h = r$, the nonlinear function matrix ρ of (6.48) with an appropriately chosen β can be set to $\rho = -(B'PB)^{-1}$ and the closed-loop system of (3.97) can be expressed as

$$\begin{aligned}
\tilde{x}(k+1) &= (A+BF)\tilde{x}(k) - B(B'PB)^{-1}B'P(A+BF)\tilde{x}(k) \\
&= [I - B(B'PB)^{-1}B'P(A+BF)]\tilde{x}(k) \\
&= [I - B_{\text{aux}}(C_{\text{aux}}B_{\text{aux}})^{-1}C_{\text{aux}}]A_{\text{aux}}\tilde{x}(k) \\
&= \left[I - \gamma_s \begin{bmatrix} 0 \\ I \end{bmatrix} \Gamma_i^{-1} \left(\Gamma_o [0 \quad I] \Gamma_s^{-1} \Gamma_s \begin{bmatrix} 0 \\ I \end{bmatrix} \Gamma_i^{-1} \right)^{-1} \Gamma_o [0 \quad I] \Gamma_s^{-1} \right] \\
&\quad \times \Gamma_s \begin{bmatrix} A_{aa} & L_{ad} \\ E_{da} & A_{dd} \end{bmatrix} \Gamma_s^{-1} \tilde{x}(k) \\
&= \left(\Gamma_s \begin{bmatrix} A_{aa} & L_{ad} \\ 0 & 0 \end{bmatrix} \Gamma_s^{-1} \right) \tilde{x}(k). \qquad (3.104)
\end{aligned}$$

Clearly, the closed-loop system has $n-m$ eigenvalues at $\lambda(A_{aa})$ and the rest at 0. Thus, the stability of the closed-loop system with $\rho = -(B'Pb)^{-1}$ implies the eigenvalues of A_{aa} are all inside the unit circle. This shows that $G_{\text{aux}}(z)$ is indeed of minimum phase.

It should be noted that there is freedom in pre-selecting the locations of these invariant zeros by selecting an appropriate W in (3.8). In general, one should select the invariant zeros of $G_{\text{aux}}(z)$, which are corresponding to the closed-loop poles of (3.97) for the steady state nonlinear gain matrix, with dominating ones having a large damping ratio, which in turn generally yield a smaller overshoot. The following procedure might be used for such a purpose.

1. Given a set of $n-m$ self-conjugated complex scalars, which should include all the uncontrollable modes, if any, of (A, B), one must determine an appropriate $W > 0$ such that the resulting auxiliary system $G_{\text{aux}}(z)$ has its invariant zeros placed exactly at the locations given in the set.

   Firstly, use the singular value decomposition technique to find a unitary matrix $U \in \mathbb{R}^{n\times n}$ and a non-singular matrix $T_i \in \mathbb{R}^{m\times m}$ such that

$$\tilde{B}_{\text{aux}} = U'B_{\text{aux}}T_i = U'BT_i = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \qquad (3.105)$$

and partition accordingly

$$\tilde{A}_{\text{aux}} = U' A_{\text{aux}} U = U'(A + BF)U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{3.106}$$

It is straightforward to verify that the stabilizability of (A, B) implies the stabilizability of (A_{11}, A_{12}). In fact, their uncontrollable modes, if any, are identical.

Next, for determining an appropriate matrix $P = P' > 0$, let us partition it accordingly as follows

$$\tilde{P} = U' P U = \begin{bmatrix} P_{11} & P_{11}' \\ P_{21} & P_{22} \end{bmatrix}. \tag{3.107}$$

Then, C_{aux} can be expressed as

$$\begin{aligned} C_{\text{aux}} &= B'P = (T_i^{-1})'[0 \quad I_m]U'U \begin{bmatrix} P_{11} & P_{11}' \\ P_{21} & P_{22} \end{bmatrix} U' = T_i^{-1})'[P_{21} \quad P_{22}]U' \\ &= [T_i^{-1})'P_{22}][P_{22}^{-1}P_{21} \quad I_m]U' := T_o[P_{22}^{-1}P_{21} \quad I_m]U'. \end{aligned} \tag{3.108}$$

Using the results of Chen *et al.* [20] (see *e.g.*, Chapters 8 and 9), one can show that the invariant zeros of the auxiliary system $G_{\text{aux}}(z)$ are given by the eigenvalues of $A_{11} - A_{12}P_{22}^{-1}P_{21}$. Since (A_{11}, A_{12}) is stabilizable and the given set of conjugated complex scalars include all uncontrollable modes, there exists a constant matrix, say F_* such that $A_{11} - A_{12}F_*$ has its eigenvalues placed exactly at the locations given in the set. Obviously, one can select P_{22} and P_{21} such that

$$P_{22}^{-1}P_{21} = F_*. \tag{3.109}$$

2. Select an appropriate $P_{22} = P_{22}' > 0$, $P_{21} = P_{22} \times F_*$, and an appropriate $P_{11} = P_{11}' > P_{21}'P_{22}^{-1}P_{21}$ to ensure that

$$P = U \begin{bmatrix} P_{11} & P_{11}' \\ P_{21} & P_{22} \end{bmatrix} U' > 0. \tag{3.110}$$

3. Compute

$$W = P - (A + BF)'P(A + BF). \tag{3.111}$$

If W is not positive definite, one has to go back to Step 2 to choose another solution of P or go to the first step to re-select another set of desired invariant zeros.

Another method for selecting W is based on a trial and error approach by limiting the choice of W to a diagonal matrix and adjusting its diagonal weights through simulation. Generally, such an approach would yield a satisfactory result as well. I will illustrate such a design approach in the following section.

It is noted that there are different types of nonlinear gains that have been suggested in the literature (see *e.g.*, [19,53,74]). One can also propose the following nonlinear gains,

$$\rho_i(r_i, h_i) = -\beta_i \Big| ||h_i(k) - r_i||^{\alpha_i} - ||h_i(0) - r_i||^{\alpha_i} \Big|, \quad i = 1, \cdots, m, \tag{3.112}$$

or

$$\rho_i(r, h) = -\beta_i \Big| ||h(k) - r||^{\alpha_i} - ||h(0) - r||^{\alpha_i} \Big|, \quad i = 1, \cdots, m. \tag{3.113}$$

If $h(0) \neq r$, it is also possible to choose

$$\rho_i(r, h) = \frac{-\kappa_{1i}(B'PB)^{-1}}{||h(0) - r||^{\kappa_2}} \Big| ||h(k) - r||^{\kappa_{2i}} - ||h(0) - r||^{\kappa_{2i}} \Big|, \quad 0 \leq \kappa_{1i} \leq 1, \quad i = 1, \cdots, m. \tag{3.114}$$

However, in order to make sure the closed-loop system (3.97) to remain stable, all the poles should be inside the unit circle during the whole transient as well as steady state periods.

3.5 A Design Example

To illustrate the concept of the CNF control, let us apply the technique to design a Magnetic-Tape-Drive servo system. The dynamics of the system are given in Franklin *et al.* [24]. The goal of the control system is to enable commanding the tape to specific positions over the read/write head while maintaining a specified tension in the tape at all times. The time-scaled dynamics of the drive is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 10 \\ 3.315 & -3.315 & -0.5882 & -0.5882 \\ 3.315 & -3.315 & -0.5882 & -0.5882 \end{bmatrix} x(t)$$

$$+\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 8.533 & 0 \\ 0 & 8.533 \end{bmatrix} \text{sat}(u) \tag{3.115}$$

where $x = (\, x_1 \quad x_2 \quad \omega_1 \quad \omega_2 \,)'$ with x_1 and x_2 being the positions of the tape at capstans (in mm), and ω_1 and ω_2 being angular rates of motors/capstan assemblies (in rad/sec); and $u = (\, i_1 \quad i_2 \,)'$ with i_1 and i_2 being electric currents supplied to drive motors (in A). The saturation levels of the actuators are $\bar{i}_1 = \bar{i}_2 = 1$ A. The measurement of the system is the positions of the tape, *i.e.*,

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t), \tag{3.116}$$

and the controlled output of the system is given by

$$h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} \bar{x}(t) \\ T_e(t) \end{pmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ -2.113 & 2.113 & 0.375 & 0.375 \end{bmatrix} x(t) \tag{3.117}$$

where $\bar{x} = (x_1 + x_2)/2$ is the position of the tape over read/write head (in mm), and T_e is the tension in the tape (in N).

The design specifications are as follows: (i) the 1% settling time due to a 1 mm step change in position of the tape head, $\bar{x}$, should be less than 2.5 seconds for the time-scaled system of (3.115), which is equivalent to 250 ms for the actual system; (ii) overshoot should be less than 20%; (iii) the tape tension, T_e, should be controlled to 2 N with the constraint that $0 < T_e < 4$ N; and (iv) the input current should not exceed 1 A at each drive motor.

As suggested in [24], let us follow and select a sampling $T = 0.05$ sec to carry out the controller design. The discretized dynamical equation is then given by

$$x(k+1) = \begin{bmatrix} 0.95992 & 0.04008 & -0.48614 & 0.01386 \\ 0.04008 & 0.95992 & -0.01386 & 0.48614 \\ 0.15656 & -0.15656 & 0.93214 & -0.06786 \\ 0.15656 & -0.15656 & -0.06786 & 0.93214 \end{bmatrix} x(k)$$

$$+\begin{bmatrix} -0.10492 & 0.00175 \\ -0.00175 & 0.10492 \\ 0.41482 & -0.01183 \\ -0.01183 & 0.41482 \end{bmatrix} \text{sat}(u(k)). \tag{3.118}$$

The measurement output and controlled output are respectively given by

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(k), \tag{3.119}$$

and

$$h(k) = \begin{pmatrix} h_1(k) \\ h_2(k) \end{pmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ -2.113 & 2.113 & 0.375 & 0.375 \end{bmatrix} x(k). \tag{3.120}$$

The aim is to design appropriate CNF controllers with full state feedback, full order measurement feedback and reduced order measurement feedback, which would control the controlled output of the system to track the command reference as fast as possible and as smooth as possible. For easy comparison, the linear state feedback gain, F, is selected precisely the same as that given by [24]. The following are detailed parameters for the CNF controllers:

1. CNF controller with full state feedback:

$$u(k) = Fx(k) + Gr + \rho(r, y)F_{\rm n}[x(k) - x_{\rm e}], \tag{3.121}$$

where

$$F = \begin{bmatrix} 0.210 & -0.018 & -0.744 & -0.074 \\ 0.018 & -0.210 & -0.074 & -0.744 \end{bmatrix}, \quad G = \begin{bmatrix} -0.192 & 0.2378 \\ 0.192 & 0.2378 \end{bmatrix},$$

$$F_{\rm n} = \begin{bmatrix} -1.387214 & -1.045337 & 2.442275 & -1.673712 \\ 0.762558 & 0.998011 & -1.582986 & 1.881035 \end{bmatrix},$$

$$x_{\rm e} = (\,0.526739 \quad 1.473261 \quad 0 \quad 0\,)'$$

and

$$\rho(r, y) = \rho(r, h) = (B'PB)^{-\frac{1}{2}} \text{diag}\Big\{\tilde{\rho}_1(r, h), \tilde{\rho}_2(r, h)\Big\}(B'PB)^{-\frac{1}{2}}, \tag{3.122}$$

with

$$\tilde{\rho}_i(r, h) = -\beta_i \frac{2}{\pi} \arctan\Big(\alpha_i \Big| |h(k) - r| - |h(0) - r| \Big|\Big), \quad i = 1, 2,$$

where $\alpha_1 = \alpha_2 = 8$, $\beta_1 = 0.4$ and $\beta_2 = 0.15$.

2. CNF controller with full order measurement feedback:

$$\begin{cases} x_{\rm v}(k+1) = (A+KC_1)x_{\rm v}(k) - Ky(k) + B\ {\rm sat}(u(k)) \\ u(k) \quad = F(x_{\rm v}(k)-x_{\rm e}) + Hr + \rho(r,y)F_{\rm n}(x_{\rm v}(k)-x_{\rm e}), \end{cases} \tag{3.123}$$

where F, $F_{\rm n}$, $x_{\rm e}$, $\rho(r,y)$ are as given in the state feedback case, and

$$K = \begin{bmatrix} -1.754890 & -0.197135 \\ -0.201561 & -1.721682 \\ 1.384781 & 0.517239 \\ -0.528328 & -1.322550 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0.183858 \\ 0 & 0.183858 \end{bmatrix}.$$

3. CNF controller with reduced order measurement feedback:

$$x_{\rm v}(k+1) = A_{\rm cmp}x_{\rm v}(k) + K_{\rm cmp}y + B_{\rm cmp}\ {\rm sat}(u(k)) \tag{3.124}$$

and

$$u(k) = F\left[\begin{pmatrix} y(k) \\ x_{\rm v}(k)-K_{\rm R}y(k) \end{pmatrix} - x_{\rm e}\right] + Hr + \rho(r,y)F_{\rm n}\left[\begin{pmatrix} y(k) \\ x_{\rm v}(k)-K_{\rm R}y(k) \end{pmatrix} - x_{\rm e}\right], \tag{3.125}$$

where

$$A_{\rm cmp} = \begin{bmatrix} 0.023518 & 0 \\ 0 & 0.011109 \end{bmatrix}, \quad K_{\rm cmp} = \begin{bmatrix} 1.907907 & -0.000872 \\ -0.000533 & -1.955276 \end{bmatrix},$$

$$B_{\rm cmp} = \begin{bmatrix} 0.218834 & 0.000492 \\ 0.000460 & 0.216155 \end{bmatrix}, \quad K_{\rm R} = \begin{bmatrix} 1.866598 & 0.086366 \\ -0.085638 & -1.892142 \end{bmatrix},$$

and F, H, $x_{\rm e}$, $\rho(r,y)$ and $F_{\rm n}$ are the same as those given in the previous two cases.

Using Simulink in Matlab, one obtains a set of simulation results in Figures 3.1–3.3, which are done under the following initial condition,

$$x_0 = (\,-0.1 \quad 0.1 \quad 0 \quad 0\,)', \tag{3.126}$$

together with initial conditions for both full and reduced order controllers being set to zero. The reference, r, is chosen as

$$r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{3.127}$$

The results clearly show that the control laws with the nonlinear components, *i.e.*, the CNF controllers, outperform their linear counterparts a great deal. The first channel, the step response of the position of the tape head, has almost no overshoot with faster settling time and the second one, the response of the tape tension, has smaller overshoot and is kept within the neighborhood of 2 N. Finally, note that for all three cases, the step responses of the position of the tape head have a 1% settling time of 0.65 seconds under the CNF control. The settling time under the linear control laws is 1.55 seconds. The resulting overall improvement of the step responses in the first channel is more than 50%.

Although the tension of the tape is not critical for this magnetic-tape-drive system so long as it is kept within 0 and 4N, I will present in Figures 3.4–3.6 the results of the linear and CNF control with $\alpha_1 = \alpha_2 = 6$, $\beta_1 = \beta_2 = 1$, to demonstrate the powerfulness of the CNF control technique. For this case, both the position and the tension of the tape under the CNF control have quite impressively fast settling times (0.95 and 0.4 seconds, respectively) and have no overshoot at all.

The results of the linear and CNF control with $\alpha_1 = \alpha_2 = 4$, $\beta_1 = \beta_2 = 0.8$ are also shown in Figures 3.7–3.9.

Lastly, one sees that control inputs in the previous situations are actually unsaturated, therefore it is reasonable for one to choose the ρ as in (3.99). Here, however I would like to choose ρ as a diagonal matrix following the proof of Theorem 3.1, which is

$$\rho(r, y) = \text{diag}\Big\{\rho_1(r_1, y),\ 0\Big\},$$

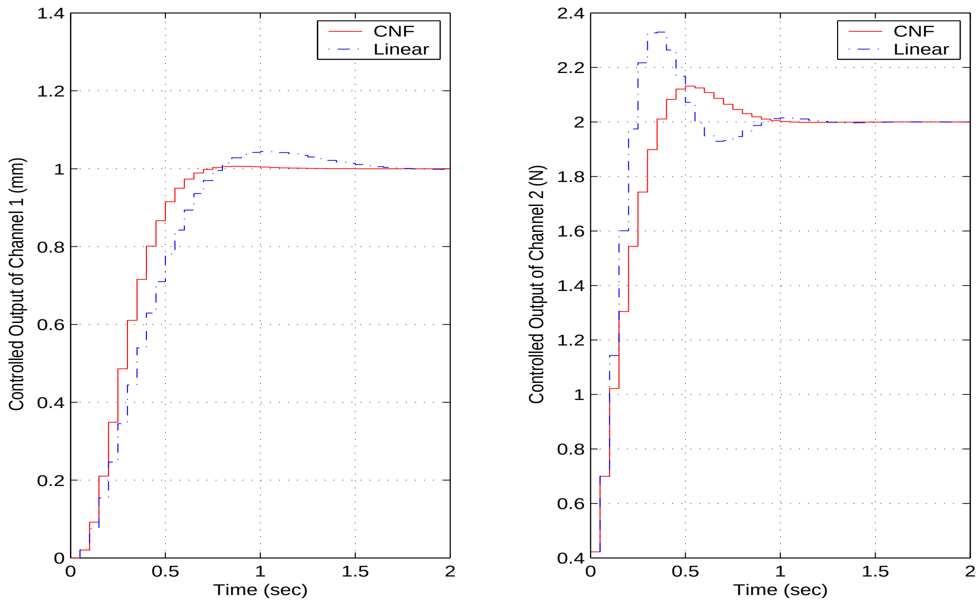
and where

$$\rho_1(r_1, y) = -0.35 \left\| \left|\frac{1}{2}[y_1(k) + y_2(k)] - r_1\right|^3 - \left|\frac{1}{2}[y_1(0) + y_2(0)] - r_1\right|^3 \right\|.$$

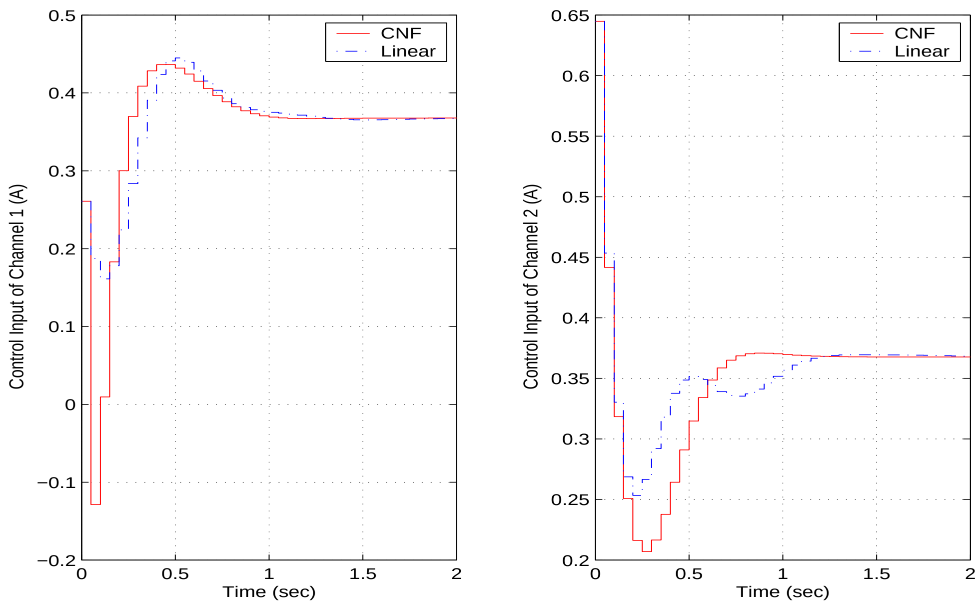
The results are shown in Figures 3.10–3.12.

3.6 Conclusion

I have presented a nonlinear tracking control technique, *i.e.*, the CNF control design for discrete-time multivariable systems. The CNF control law consists of two parts, a linear component and a nonlinear component. The former is chosen to give fast rising

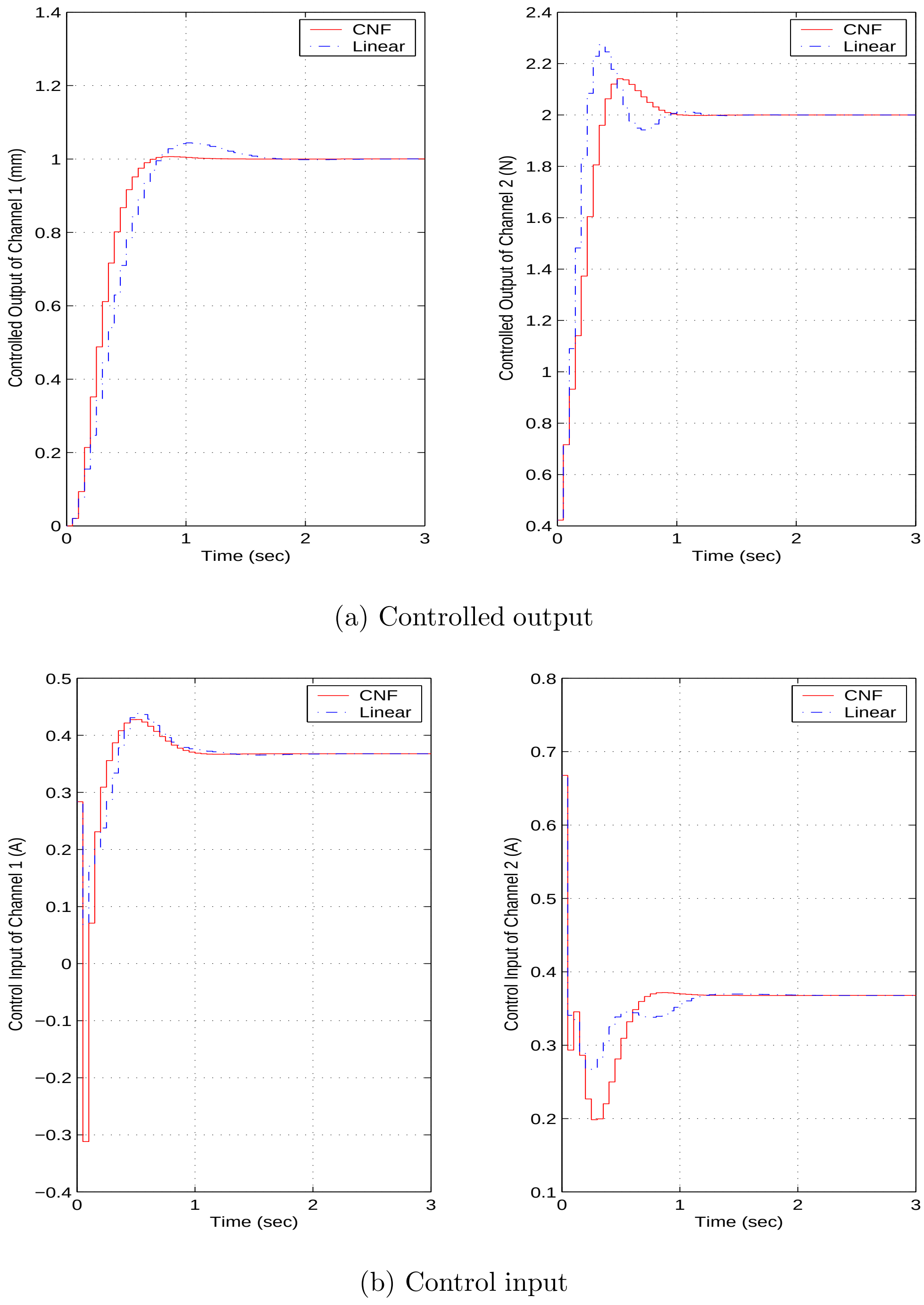


(a) Controlled output



(b) Control input

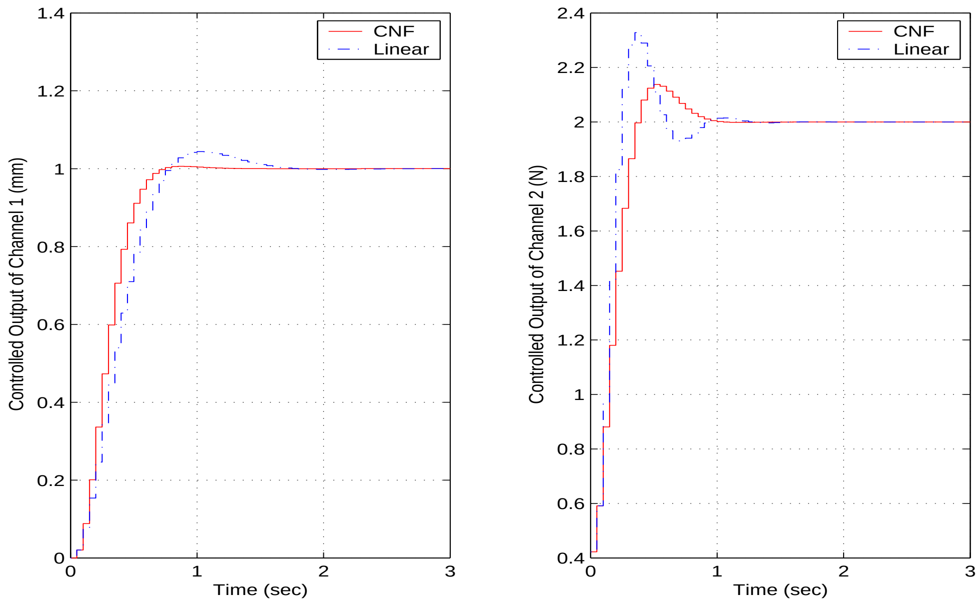
Figure 3.1: Input and output responses under state feedback.



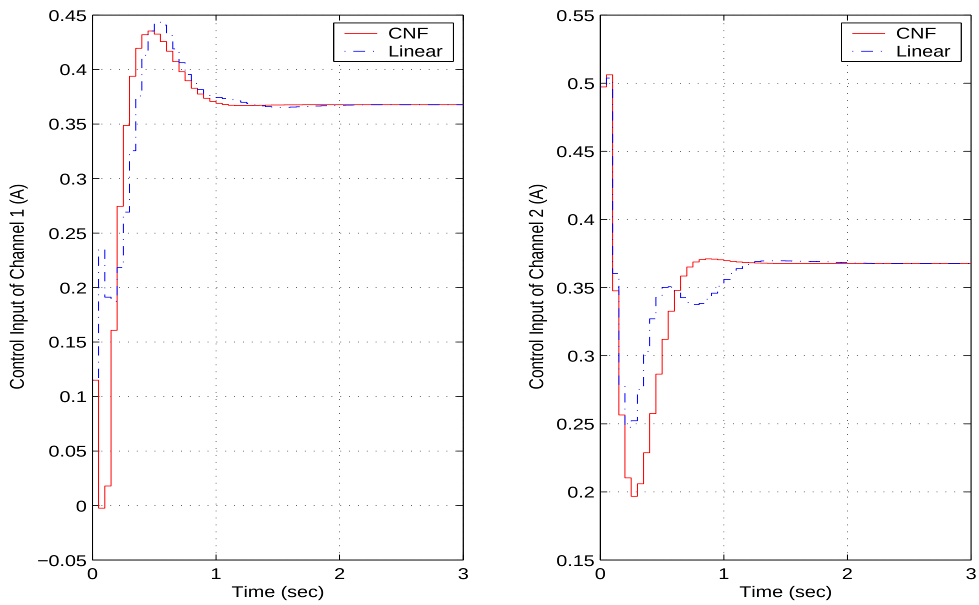
(a) Controlled output

(b) Control input

Figure 3.2: Input and output responses under full order measurement feedback.

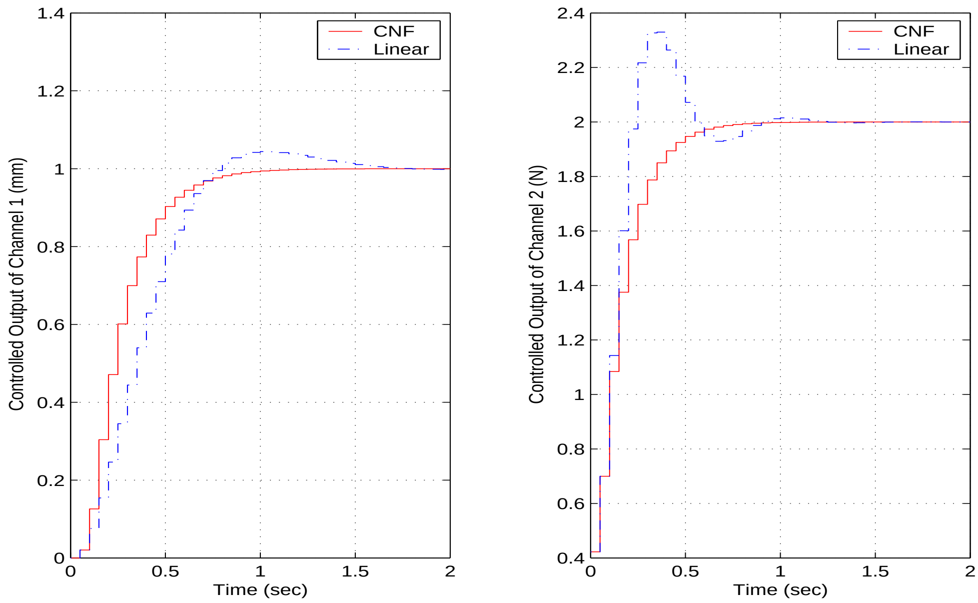


(a) Controlled output

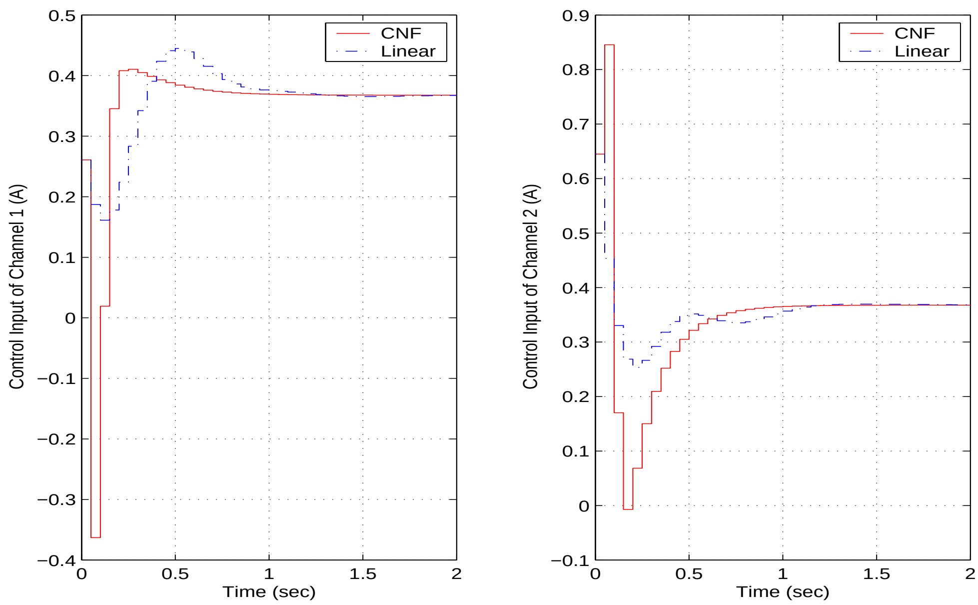


(b) Control input

Figure 3.3: Input and output responses under reduced order measurement feedback.

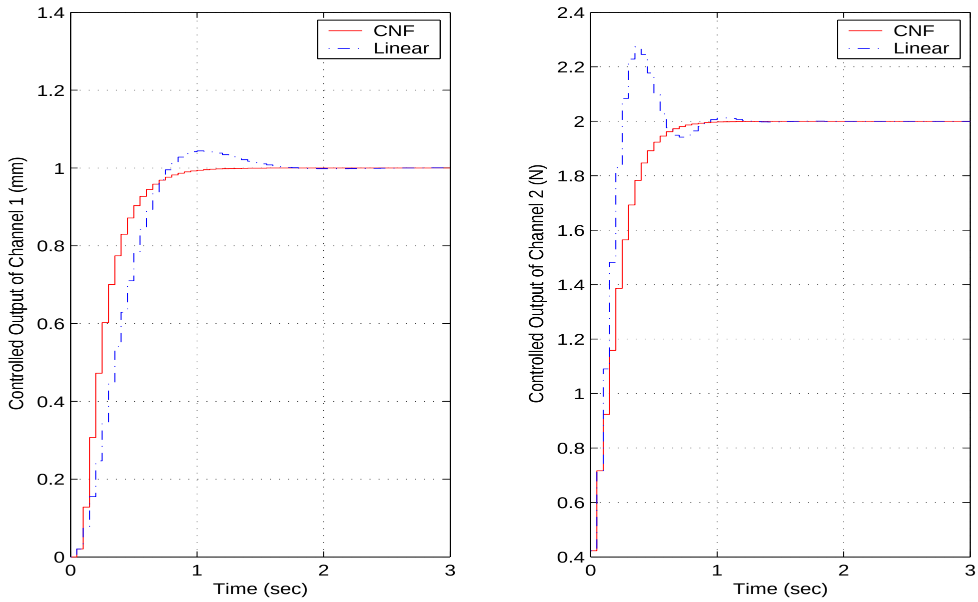


(a) Controlled output

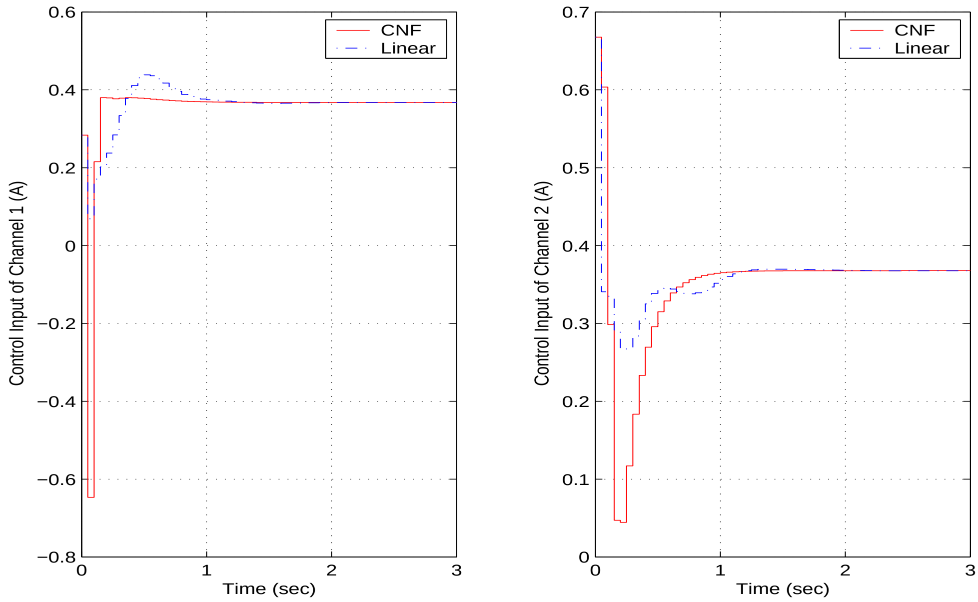


(b) Control input

Figure 3.4: Input and output responses under state feedback: $\alpha_1 = \alpha_2 = 6$, $\beta_1 = \beta_2 = 1$

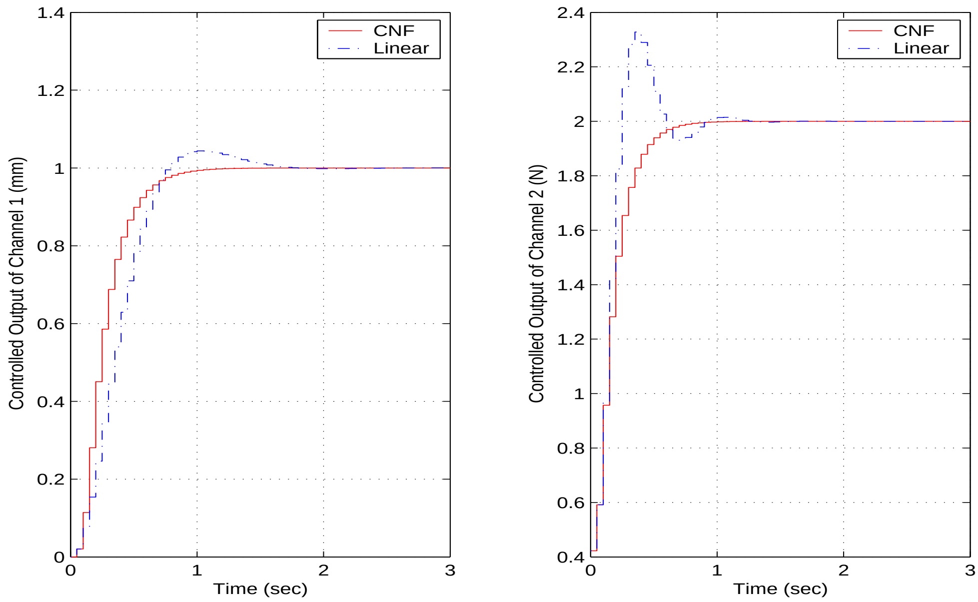


(a) Controlled output

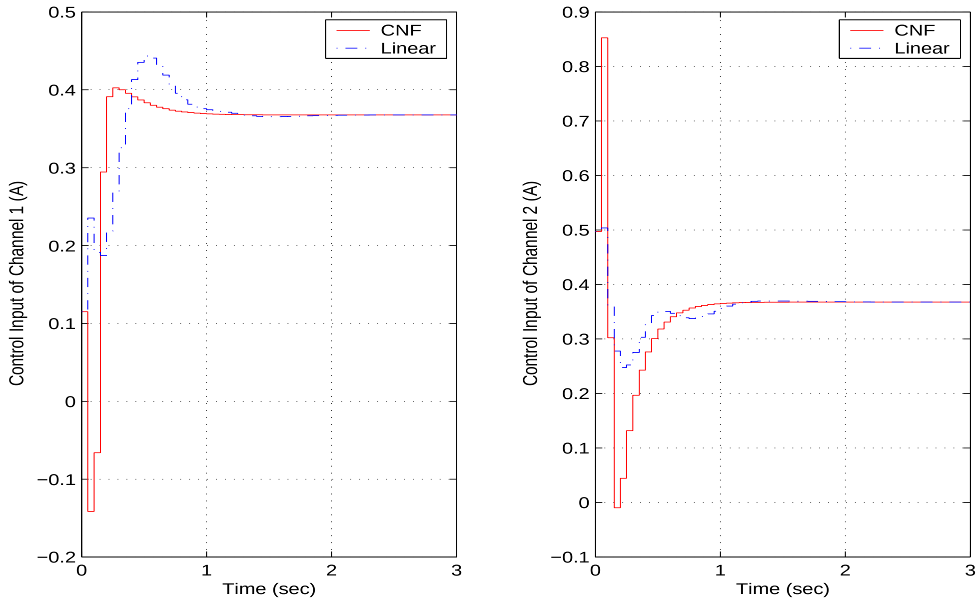


(b) Control input

Figure 3.5: Input and output responses under full order measurement feedback: $\alpha_1 = \alpha_2 = 6$, $\beta_1 = \beta_2 = 1$

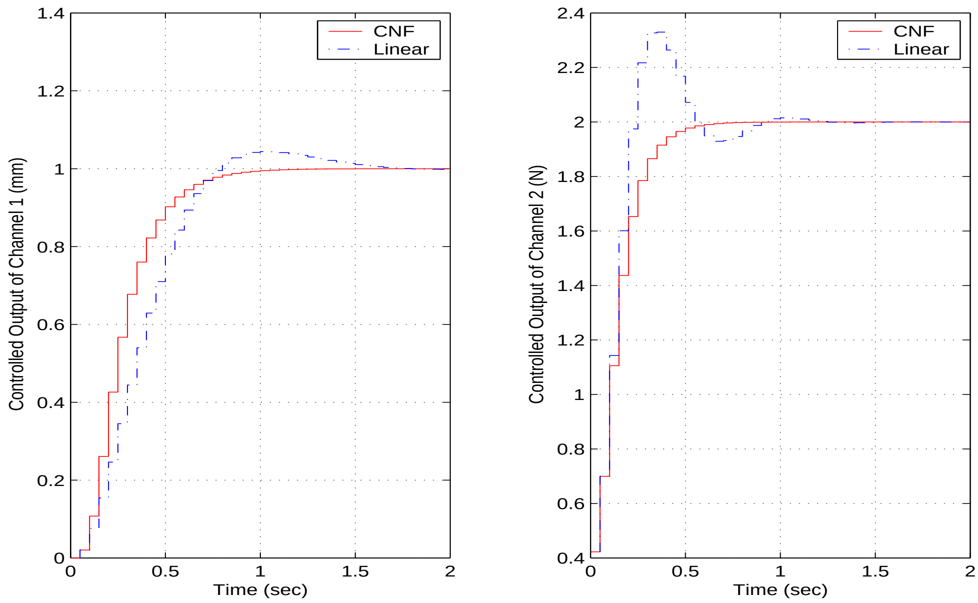


(a) Controlled output

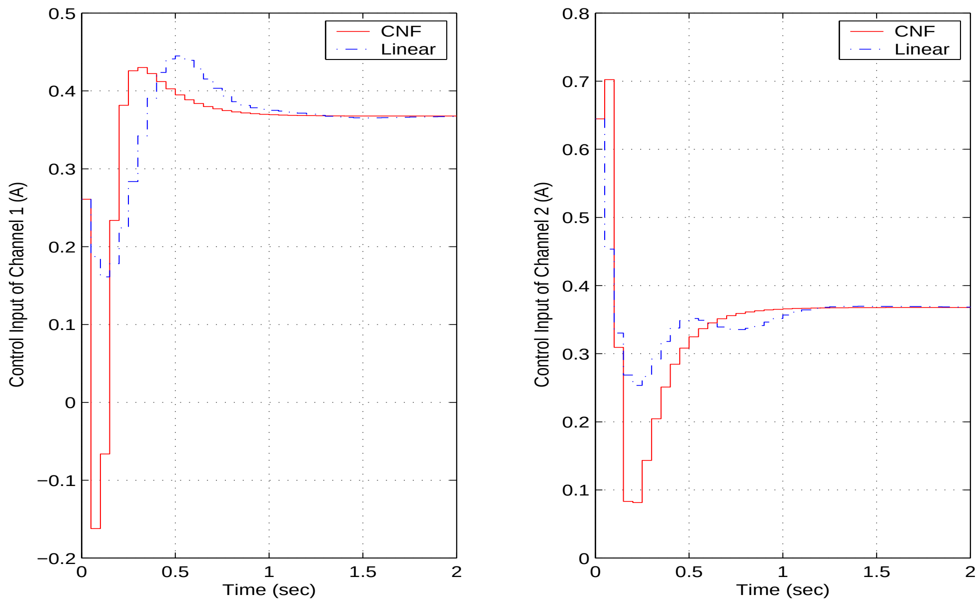


(b) Control input

Figure 3.6: Input and output responses under reduced order measurement feedback: $\alpha_1 = \alpha_2 = 6$, $\beta_1 = \beta_2 = 1$

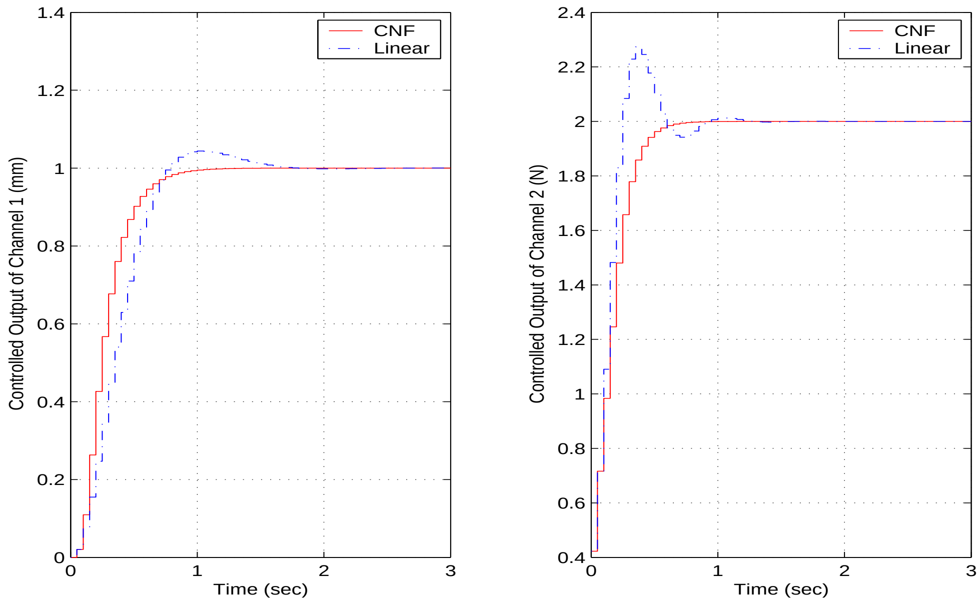


(a) Controlled output

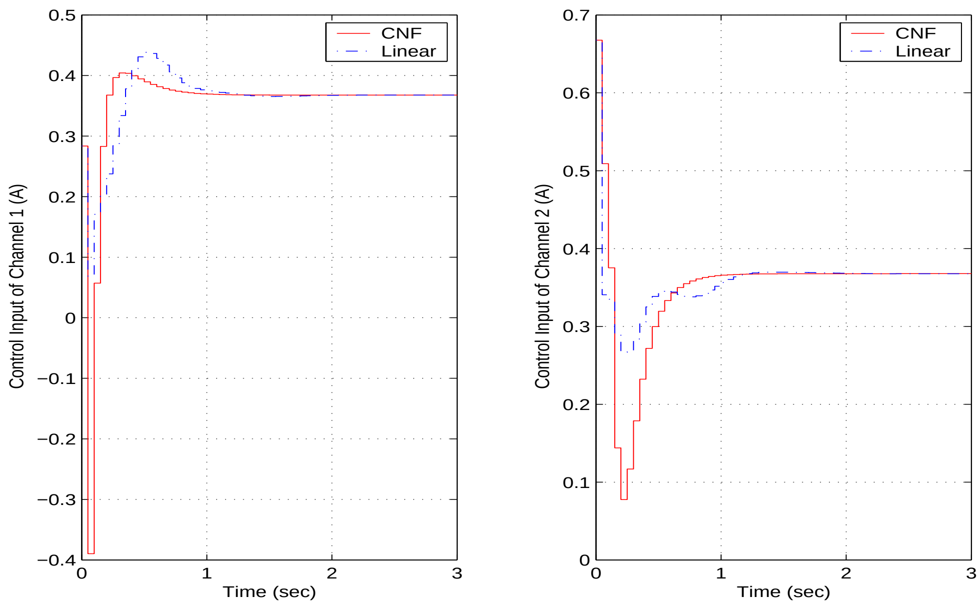


(b) Control input

Figure 3.7: Input and output responses under state feedback: $\alpha_1 = \alpha_2 = 4$, $\beta_1 = \beta_2 = 0.8$

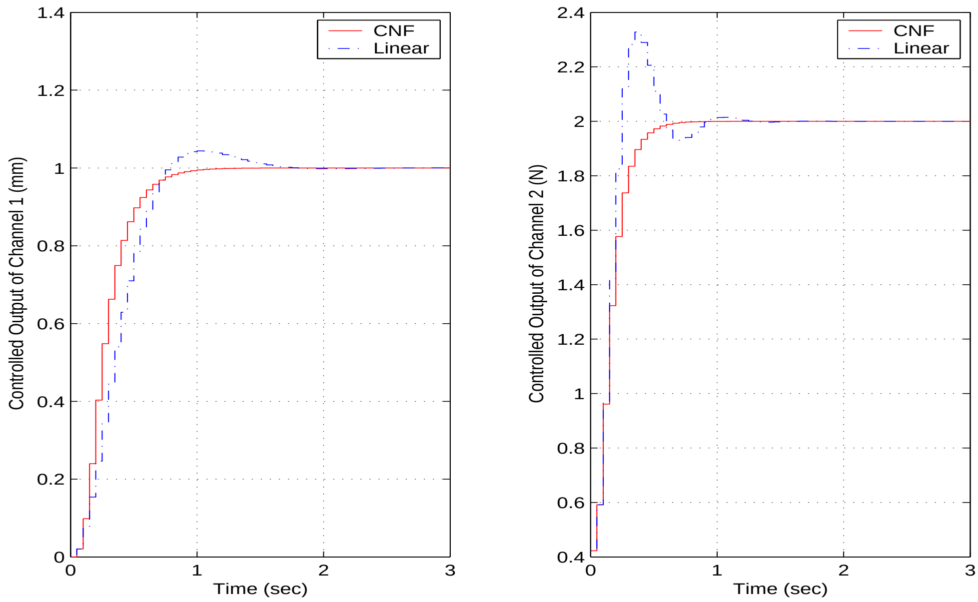


(a) Controlled output

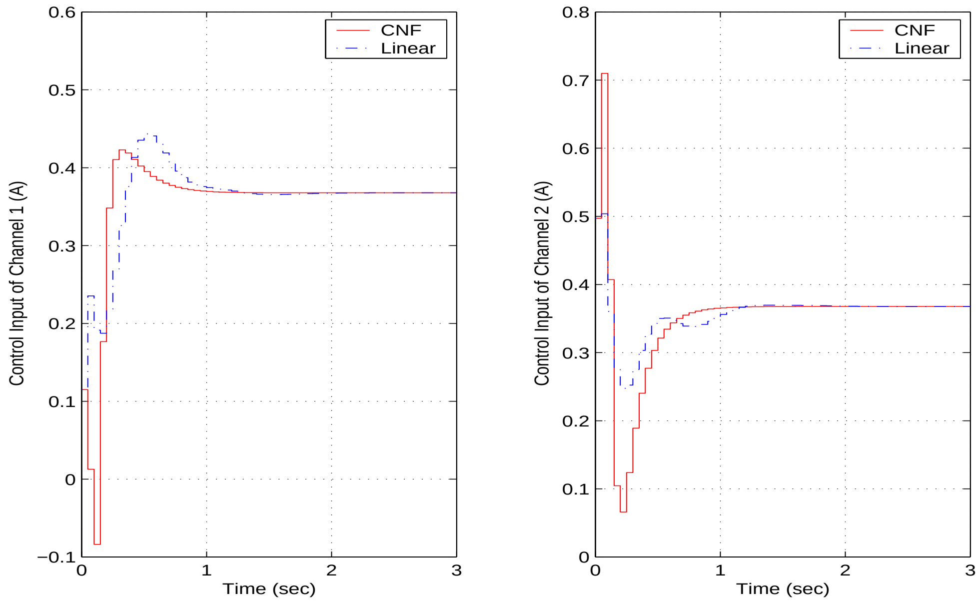


(b) Control input

Figure 3.8: Input and output responses under full order measurement feedback: $\alpha_1 = \alpha_2 = 4$, $\beta_1 = \beta_2 = 0.8$

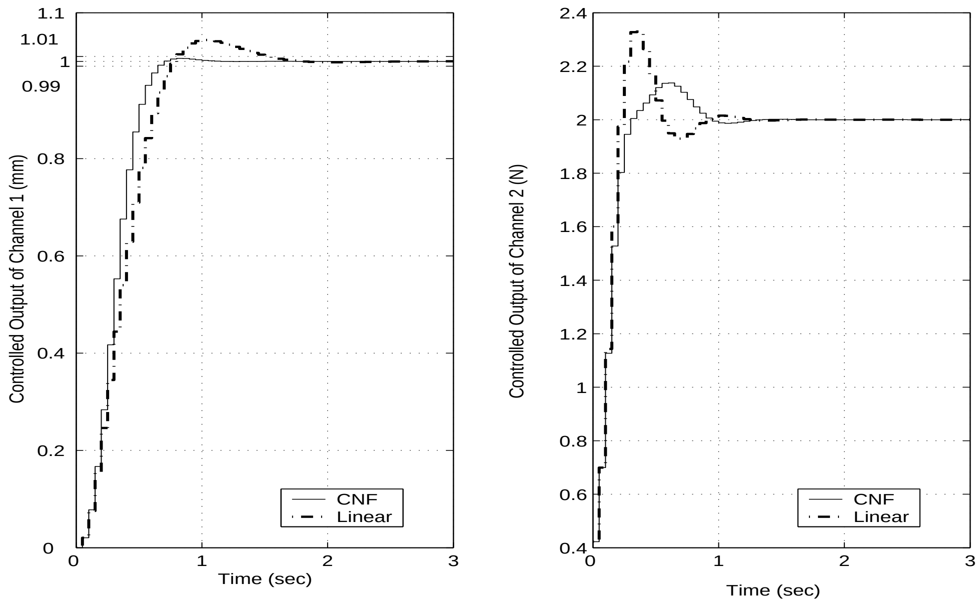


(a) Controlled output

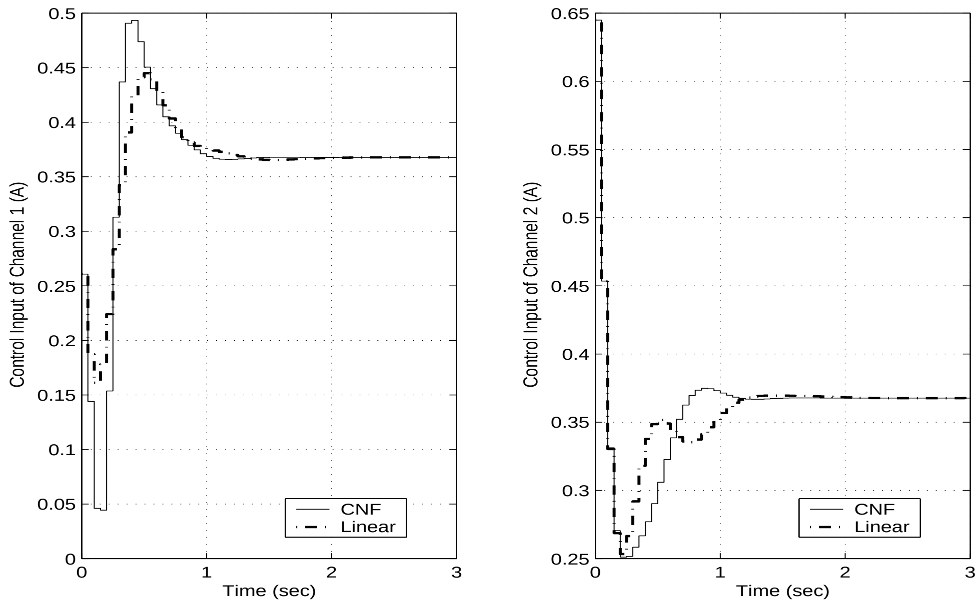


(b) Control input

Figure 3.9: Input and output responses under reduced order measurement feedback: $\alpha_1 = \alpha_2 = 4$, $\beta_1 = \beta_2 = 0.8$

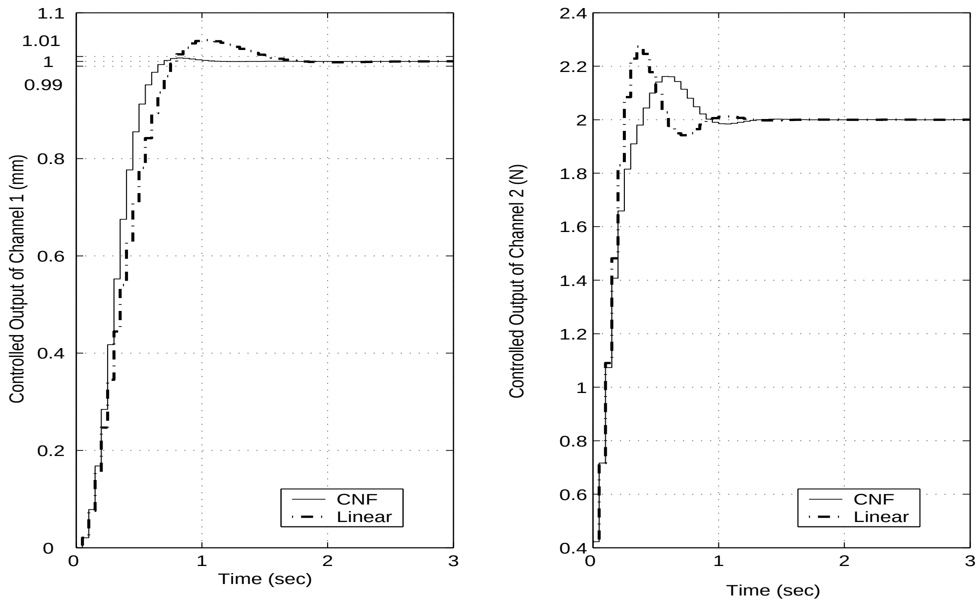


(a) Controlled output

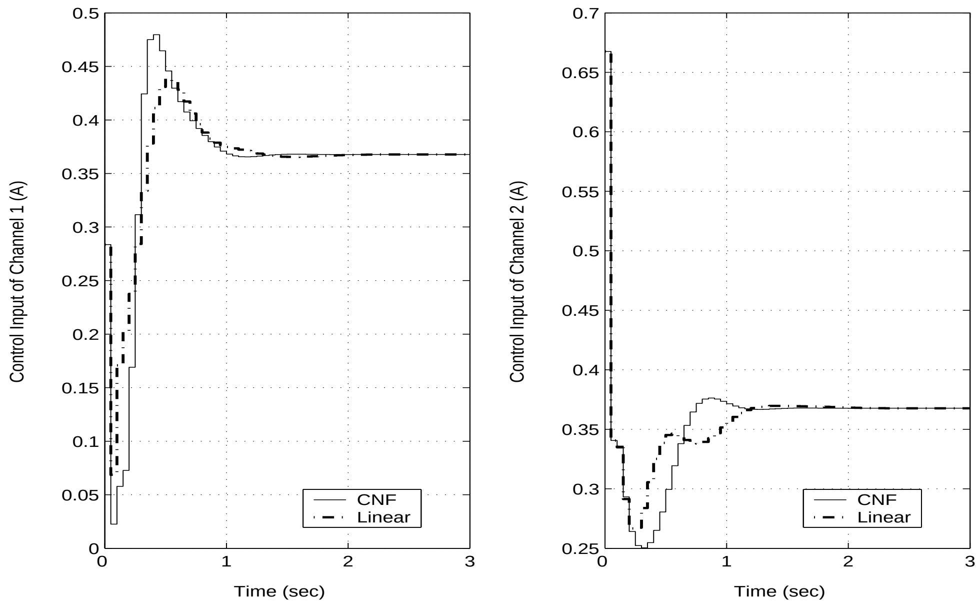


(b) Control input

Figure 3.10: Input and output responses under state feedback.

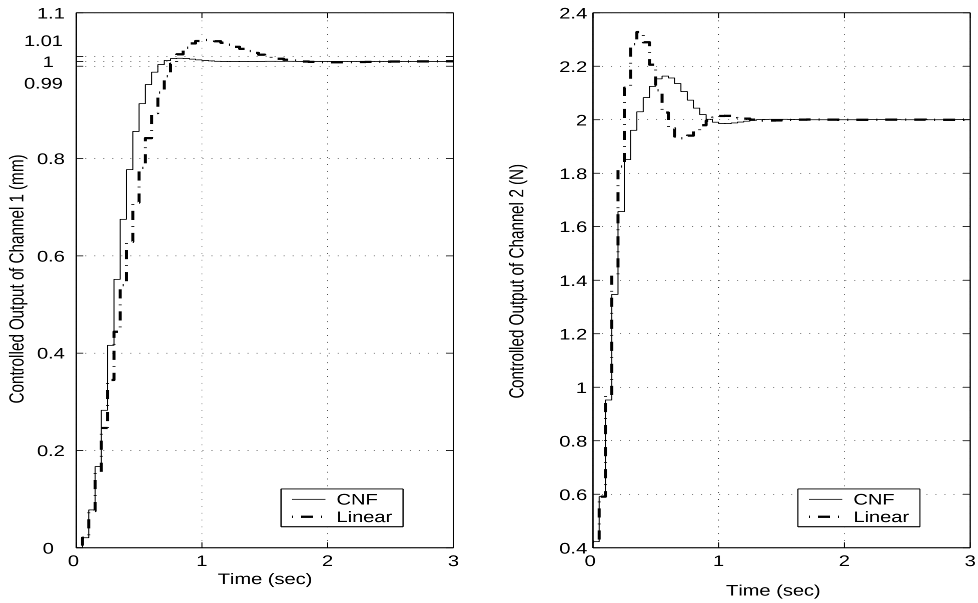


(a) Controlled output

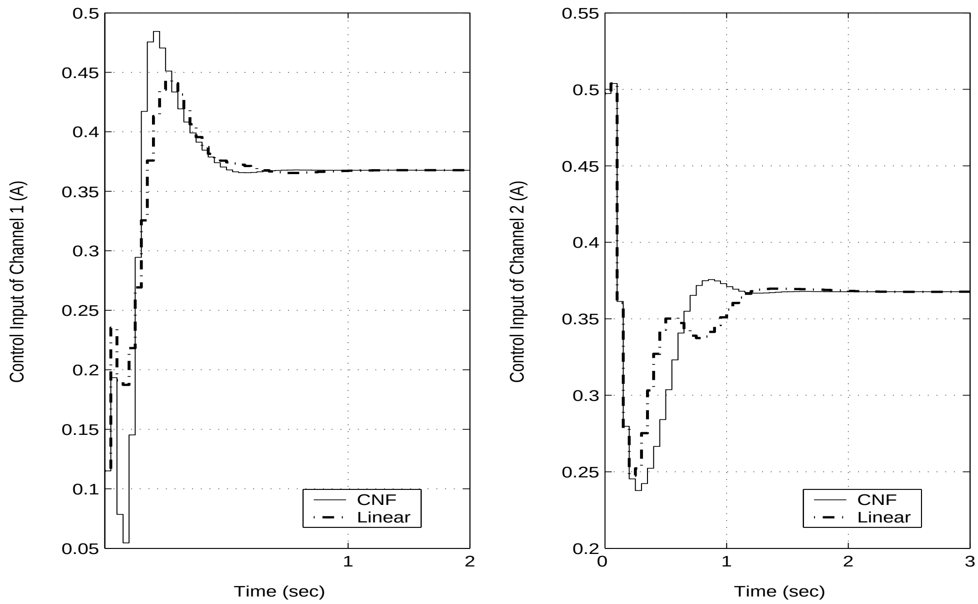


(b) Control input

Figure 3.11: Input and output responses under full order measurement feedback.



(a) Controlled output



(b) Control input

Figure 3.12: Input and output responses under reduced order measurement feedback.

time while the latter is added to smooth out the transient peaks or overshoots when the controlled output is approaching the target reference. The technique is applicable to linear general multivariable system with some standard assumptions, and has been successfully demonstrated to yield a nice tracking performance in a real application.

Chapter 4

CNF Control for Linearizable Systems with Input Saturation

In previous chapters, I have addressed the complete theory for the CNF control methodology. This completeness is limited to linear systems. Yet one knows there are quite a lot of nonlinear systems which are similar to linear systems in nature after a diffeomorphism and/or state transformations, called nonlinear linearizable systems. It seems obvious that one may apply CNF scheme to this kind of systems also. Therefore, in this chapter, as an application of CNF control technique, I will explore the possibility of its use in nonlinear linearizable systems with actuator saturation. It turns out that a certain condition relating the actual control input for original systems and the one after transformation has to be established. So long as this condition is met, the CNF controller design can be carried out easily. For simplicity, I will consider only the state feedback case. It can be easily extended to measurement feedback cases. Also, the extension to multivariable systems is possible although the results may be rather restrictive. The application of the technique to a pendulum system will be addressed in order to show the effectiveness of the extension, which shows that the proposed design method indeed yields a very satisfactory performance.

4.1 Introduction

So far one knows that saturation may cause the performance of linear saturated systems degrade and sometimes even the stability may be lost. Similarly, for nonlinear systems with actuator saturation, the closed-loop system performance may deteriorate as well if not more severely. Unfortunately little has be done on this topic although nonlinear control theory has been explored for a long time and is a very hot research area today. Nevertheless, since tracking theory for general linear systems with input saturation has been established during the past few years, it is possible for one to extend it to feedback linearizable nonlinear systems. Typically researchers in nonlinear control consider stabilization and/or regulation problems, or, based on inverse dynamics of the original system, they consider tracking certain signals which can be produced by a linear neutrally stable reference model while very few consider step signals tracking. The reason may lie in that by inverse dynamics one need to know some orders of derivatives of the reference signal which should be bounded yet this is not the case for a step function.

In this chapter, I will present a design procedure of composite nonlinear feedback (CNF) control for SISO nonlinear feedback linearizable systems with actuator saturation. I will only consider the state feedback case. After a feedback linearizable transformation, the original nonlinear system becomes a linear system. As in the earlier works [19,53,74], the CNF control consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time not exceeding the actuator limits for the desired command input levels. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part.

This chapter is organized as follows. In Section **??**, the theory of the composite nonlinear feedback control is developed. The application of the CNF technique to an air-to-air missile autopilot system will be presented in Section 4.3, which shows that the proposed design method yields a very satisfactory performance. Finally, I will draw some concluding remarks in Section 4.4.

4.2 Problem Formulation and Controller Design

Let us consider the following single variable nonlinear system Σ with an amplitude-constrained actuator characterized by

$$\begin{cases} \dot{x} &= f(x) + g(x)\mathrm{sat}(v), \quad x(0) = x_0 \\ h &= h(x) \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^n$, $v \in \mathbb{R}$, $y \in \mathbb{R}^p$ and $h \in \mathbb{R}$ are respectively the state, control input, measurement output and controlled output of the given system Σ. The saturation function is defined by

$$\mathrm{sat}(v) = \mathrm{sign}(v)\,\min(|v|,\ \bar{v}), \tag{4.2}$$

where $\bar{v}$ is the maximum amplitude of the control channel.

The aim to design certain controller, with all the state information known, which renders the whole (closed-loop) system will track a step function with amplitude of r under the input constraint. Due to the difficulty in solving this problem, instead of dealing with general case let us consider a more specific case where

$$f(x) = Ax - g(x)\alpha(x) \tag{4.3}$$

and also

$$g(x) = B(x). \tag{4.4}$$

If one considers further by confining $g(x)$ to

$$g(x) = B\gamma(x), \tag{4.5}$$

one actually gets into a standard form of nonlinear linearizable system described by

$$\begin{cases} \dot{x} = A\ x + B\gamma(x)[\mathrm{sat}(v) - \alpha(x)], \quad x(0) = x_0 \\ y = \quad x \\ h = C_2\ x \end{cases} \tag{4.6}$$

where $x \in \mathbb{R}^n$, $v \in \mathbb{R}$, $y \in \mathbb{R}^p$ and $h \in \mathbb{R}$ are respectively the state, control input, measurement output and controlled output of the given system Σ, and the functions $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $\gamma : \mathbb{R}^n \to \mathbb{R}$ are defined in a domain $D \subset \mathbb{R}^n$ of interest, and $\gamma(x) \neq 0$

for every $x \in D$. In addition, A, B, C_1 and C_2 are appropriate dimensional constant matrices and the saturation function, again, is defined by

$$\text{sat}(v) = \text{sign}(v)\ \min(|v|,\ \bar{v}), \tag{4.7}$$

where $\bar{v}$ is the maximum amplitude of the control channel.

For this simplified system, I will present the so-called CNF controller based on the results regarding linear systems with input saturation.

It is easily seen that one can linearize this system via the state feedback

$$\text{sat}(v) = \alpha(x) + \beta(x)u \tag{4.8}$$

where $\beta(x) = \gamma^{-1}(x)$, to obtain the linear state equation

$$\dot{x} = Ax + Bu \tag{4.9}$$

Further, assuming that for $x \in D_a$ where $D_a \subset D$ denotes the largest domain of attraction of the system containing the origin, one has

$$|\alpha(x)| \leq \alpha \tag{4.10}$$

where $\alpha \geq 0$, and

$$|\gamma(x)| \geq \gamma \tag{4.11}$$

where again $\gamma \geq 0$.

If u is subjected to

$$|u| \leq \gamma(\bar{v} - \alpha), \tag{4.12}$$

where $\bar{v} - \alpha \geq 0$, then one has

$$|\alpha(x) + \beta(x)u| \leq \bar{v}. \tag{4.13}$$

Under the above condition, one has

$$v = \ \text{sat}(v) = \alpha(x) + \beta(x)u = \alpha(x) + \beta(x)\ \text{sat}(u), \tag{4.14}$$

with

$$\text{sat}(u) = \text{sign}(u)\ \min(|u|,\ \gamma(\bar{v} - \alpha)), \tag{4.15}$$

and the following linear saturated system

$$\dot{x} = Ax + B\text{sat}(u) \tag{4.16}$$

is equivalent to the original plant (4.6). Furthermore, (4.8) becomes

$$v = \alpha(x) + \beta(x)\text{sat}(u). \tag{4.17}$$

For tracking purpose, the following assumptions on the given system are required: i) (A, B) is stabilizable; and ii) (A, B, C_2) is right invertible and has no invariant zeros at $s = 0$. The objective here is to design control laws that are capable of achieving fast tracking of target references under input saturation. As such, it is well understood in the literature that these assumptions are standard and necessary.

Let us proceed to develop a composite nonlinear feedback control technique for the case when all the state variables of the plant Σ are measurable, *i.e.*, $y = x$. The design will be done in three steps, which is a natural extension of the results of Chen *et al.* [19]. One has the following step-by-step design procedure.

STEP S.1: Design a linear feedback law,

$$u_{\text{L}} = Fx + Gr, \tag{4.18}$$

where $r \in \mathbb{R}$ is the step reference. The state feedback gain matrix $F \in \mathbb{R}^{1\times n}$ is chosen such that the closed-loop system matrix $A + BF$ is asymptotically stable and the resulting closed-loop system transfer matrix, *i.e.*, $C_2(sI - A - BF)^{-1}B$, has certain desired properties, *e.g.*, having a small dominating damping ratio in each channel. Note that such an F can be worked out using some well-studied methods such as the LQR, H_∞ and H_2 optimization approaches (see, *e.g.*, Anderson and Moore [1], Chen [17] and Saberi *et al.* [62]). Furthermore, G is a scalar constant and is given by

$$G := G_0' \left(G_0 G_0'\right)^{-1}, \tag{4.19}$$

with $G_0 := -C_2(A + BF)^{-1}B$. Here note that both G_0 and G are well defined because $A + BF$ is stable, and (A, B, C_2) is right invertible and has no invariant zeros at $s = 0$, which implies $(A+BF, B, C)$ is right invertible and has no invariant zeros at $s = 0$ (see *e.g.*, Lemma 2.5.1 of Chen [17]).

Step s.2: Next, compute

$$H := \left[I - F(A + BF)^{-1}B\right] G \tag{4.20}$$

and

$$x_{\rm e} := G_{\rm e}\, r := -(A + BF)^{-1}BG\, r. \tag{4.21}$$

Note that the definitions of H, $G_{\rm e}$ and $x_{\rm e}$ would become transparent later in the derivation. Given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, solve the following Lyapunov equation:

$$(A + BF)'P + P(A + BF) = -W, \tag{4.22}$$

for $P > 0$. Such a P exists since $A + BF$ is asymptotically stable. Then, the nonlinear feedback control law $u_{\rm N}$ is given by

$$u_{\rm N} = \rho(r, y)B'P(x - x_{\rm e}), \tag{4.23}$$

where $\rho(r, y)$ is some nonpositive function, locally Lipschitz in y, which is used to change the closed-loop system damping ratio as the output approaches the target. The choice of this nonlinear function will be discussed at the end of this section.

Step s.3: The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller:

$$u = u_{\rm L} + u_{\rm N} = Fx + Gr + \rho(r, y)B'P(x - x_{\rm e}). \tag{4.24}$$

Finally, one obtains

$$v = \alpha(x) + \beta(x)\ \mathrm{sat}(u) = \alpha(x) + \beta(x)\ \mathrm{sat}[Fx + Gr + \rho(r, y)B'P(x - x_{\rm e})]. \tag{4.25}$$

This completes the design of the CNF controller for the state feedback case.

The following theorem shows that the closed-loop system comprising the given plant in (4.6) and the CNF control law of (4.25) is asymptotically stable. It also determines the magnitudes of the step functions in r that can be tracked by such a control law without exceeding the control limit.

Theorem 4.1. *Consider the given system in (4.6) with $y = x$, which satisfies the assumptions i) and ii), and also satisfies (4.10) and (4.11), and the composite nonlinear feedback control law of (4.25). For any $\delta \in (0, 1)$, let $c_\delta > 0$ be the largest positive scalar such that for all $x \in \boldsymbol{X}_\delta$, where*

$$\boldsymbol{X}_\delta := \left\{ x : x'Px \le c_\delta \right\} \subset D_a, \tag{4.26}$$

the following property holds,

$$| \, F \, x \, | \le (1 - \delta)\bar{u}, \tag{4.27}$$

where

$$\bar{u} = \gamma(\bar{v} - \alpha) \ge 0. \tag{4.28}$$

Then, for any nonpositive function $\rho(r, y)$, locally Lipschitz in y, the composite nonlinear feedback law in (4.25) is capable of driving the system controlled output $h(t)$ to track asymptotically the step command input of amplitude r, provided that the initial state x_0 and r satisfy:

$$\tilde{x}_0 := (x_0 - x_\mathrm{e}) \in \boldsymbol{X}_\delta, \quad |H \, r| \le \delta \bar{u}. \tag{4.29}$$

Proof. It is straightforward to show that the system represented by (4.6) is equivalent to that by (4.16) and the third (output) equation of (4.6). It remains to show that the latter meets the tracking goal.

Let us first define a new state variable $\tilde{x} = x - x_\mathrm{e}$. It is simple to verify that the linear feedback control law of (4.18) can be rewritten as

$$u_\mathrm{L}(t) = F\tilde{x}(t) + [I - F(A + BF)^{-1}B]Gr \tag{4.30}$$

$$= F\tilde{x}(t) + Hr, \tag{4.31}$$

and hence for all $\tilde{x} \in \boldsymbol{X}_\delta$ and, provided that $|H\,r| \le \delta \bar{u}$, the closed-loop system is linear and is given by

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + Ax_{\rm e} + BHr. \tag{4.32}$$

Noting that

$$\begin{aligned} Ax_{\rm e} + BHr &= \Big\{B[I - F(A+BF)^{-1}B]G - A(A+BF)^{-1}BG\Big\}r \\ &= \Big\{[I - BF(A+BF)^{-1}]BG - A(A+BF)^{-1}BG\Big\}r \\ &= \Big\{I - BF(A+BF)^{-1} - A(A+BF)^{-1}\Big\}BGr \\ &= 0, \end{aligned} \tag{4.33}$$

the closed-loop system in (4.32) can then be simplified as

$$\dot{\tilde{x}} = (A + BF)\tilde{x}. \tag{4.34}$$

Similarly, the closed-loop system comprising the given plant in (4.6) and the CNF control law of (4.24) can be expressed as

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + Bw, \tag{4.35}$$

where

$$w = \operatorname{sat}(F\tilde{x} + Hr + u_{\rm N}) - F\tilde{x} - Hr. \tag{4.36}$$

Clearly, for the given x_0 satisfying (4.29), one has $\tilde{x}_0 = (x_0 - x_{\rm e}) \in \boldsymbol{X}_\delta$. Note that (4.35) is reduced to (4.34) if $\rho(r, y) = 0$.

Next, define a Lyapunov function $V = \tilde{x}'P\tilde{x}$ and evaluate the derivative of V along the trajectories of the closed-loop system in (4.35), *i.e.*,

$$\begin{aligned} \dot{V} &= \dot{\tilde{x}}'P\tilde{x} + \tilde{x}'P\dot{\tilde{x}} \\ &= \tilde{x}'(A+BF)'P\tilde{x} + \tilde{x}'P(A+BF)\tilde{x} + 2\tilde{x}'PBw \\ &= -\tilde{x}'W\tilde{x} + 2\tilde{x}'PBw. \end{aligned} \tag{4.37}$$

Note that for all

$$\tilde{x} \in \boldsymbol{X}_\delta = \{\tilde{x} : \tilde{x}'P\tilde{x} \le c_\delta\} \quad \Rightarrow \quad |F\,\tilde{x}| \le (1-\delta)\bar{u}. \tag{4.38}$$

In the remainder of this proof, I will adopt similar lines of reasoning as those of Turner *et al.* [74] by considering the following different scenarios. For simplicity, let us drop the dependent variables of the nonlinear function ρ in the rest of this proof.

Case 1. The input is unsaturated. It is obvious that one has

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\tilde{x}'PB\rho B'P\tilde{x} \leq -\tilde{x}'W\tilde{x}. \tag{4.39}$$

Case 2. The input is exceeding its upper limit. In this case, one has

$$F\tilde{x} + Hr + \rho B'P\tilde{x} \geq \bar{u}. \tag{4.40}$$

For all $\tilde{x} \in \boldsymbol{X}_\delta$, which implies (4.38) holds, and r satisfying (4.29), one has

$$F\tilde{x} + Hr \leq \bar{u}, \tag{4.41}$$

and thus

$$w = \text{sat}(F\tilde{x} + Hr + \rho B'P\tilde{x}) - F\tilde{x} - Hr = \bar{u} - F\tilde{x} - Hr \geq 0 \tag{4.42}$$

and

$$\rho B'P\tilde{x} \geq \bar{u} - (F\tilde{x} + Hr) \geq 0 \quad \Rightarrow \quad B'P\tilde{x} = \tilde{x}'PB \leq 0. \tag{4.43}$$

Hence,

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\tilde{x}'PB\bar{w} \leq -\tilde{x}'W\tilde{x}. \tag{4.44}$$

Case 3. The input is exceeding its lower limit. For this case, one has

$$F\tilde{x} + Hr + \rho B'P\tilde{x} \leq -\bar{u}. \tag{4.45}$$

For all $\tilde{x} \in \boldsymbol{X}_\delta$, which implies (4.38) holds, and r satisfying (4.29), one has

$$F\tilde{x} + Hr \geq -\bar{u}, \tag{4.46}$$

and thus

$$w = \text{sat}(F\tilde{x} + Hr + \rho B'P\tilde{x}) - F\tilde{x} - Hr = -u - F\tilde{x} - Hr \leq 0 \tag{4.47}$$

and

$$\rho B'P\tilde{x} \leq -\bar{u} - (F\tilde{x} + Hr) \leq 0 \quad \Rightarrow \quad B'P\tilde{x} = \tilde{x}'PB \geq 0. \tag{4.48}$$

Hence,

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\tilde{x}'PBw \le -\tilde{x}'W\tilde{x}. \tag{4.49}$$

In conclusion, I have shown that

$$\dot{V} \le -\tilde{x}'W\tilde{x}, \quad \tilde{x} \in \boldsymbol{X}_\delta, \tag{4.50}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (4.35). Noting that $W > 0$, all trajectories of (4.35) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial state x_0 and the step command input r that satisfy (4.29), one has

$$\lim_{t\to\infty} x(t) = x_e, \tag{4.51}$$

which implies

$$\lim_{t\to\infty} u(t) = F \lim_{t\to\infty} x(t) + Gr + \lim_{t\to\infty} \rho B'P[x(t) - x_{\mathrm{e}}] = Fx_{\mathrm{e}} + Gr. \tag{4.52}$$

Hence,

$$\begin{aligned}\lim_{t\to\infty} h(t) &= C_2 \lim_{t\to\infty} x(t) \\ &= C_2 x_{\mathrm{e}} \\ &= -C_2(A+BF)^{-1}BGr \\ &= [C_2(A+BF)^{-1}B]Gr \\ &= G_0G_0'(G_0G_0')^{-1}r = r.\end{aligned} \tag{4.53}$$

This completes the proof of Theorem 4.1. □

Remark 4.1. When $h(x)$ is not in the form just considered, things become more difficult. However, if one can solve the equation

$$h(x) = r \tag{4.54}$$

explicitly so that one knows the target state x_e, one can still solve this problem. For simplicity, let us assume that this x_e is unique otherwise one has to confine the considering state region to certain neighborhood of each x_e in question. As a matter of fact, one can transform this problem into a standard one solved in Theorem (4.1). It is possible that one can simply redefine a tracking target as $h_n(x) = C_{2n}x$ and force it to track the target $r_n = C_{2n}x_e$. Thus eventually the state x will evolve to approach and finally stay at x_e so that one recovers $h(x_e) = r$.

4.3 An Example

Consider the pendulum equation (taken from [47] p. 542):

$$\dot{x}_1 = x_2 \tag{4.55}$$

$$\dot{x}_2 = -a\text{sin}x_1 - bx_2 + cv \tag{4.56}$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$ and $u = T$ is a torque input. The goal is to stabilize the pendulum at the angle $\theta = \delta_r$. A linearizing-stabilizing feedback control is given by

$$v = \left(\frac{a}{c}\right)\text{sin}x_1 + \left(\frac{1}{c}\right)u, \tag{4.57}$$

which transforms the original nonlinear system into a linear one

$$\dot{x}_1 = x_2 \tag{4.58}$$

$$\dot{x}_2 = -bx_2 + u \tag{4.59}$$

or

$$\dot{x} = Ax + Bu \tag{4.60}$$

$$h = C_2x \tag{4.61}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = [1 \quad 0] \tag{4.62}$$

By using pole-placement approach one designs the controller

$$u = Fx + G\theta_r = (f_1x_1 + f_2x_2) + G\theta_r \tag{4.63}$$

where f_1 and f_2 are chosen such that

$$A + BF = \begin{bmatrix} 0 & 1 \\ f_1 & f_2 - b \end{bmatrix} \tag{4.64}$$

is Hurwitz, which guarantees the system to be stable.

To demonstrate the result, let us choose a command reference:

$$\theta_r = 4(\approx \frac{5}{4}\pi). \tag{4.65}$$

and choose $a = 10$, $b = 1$ and $c = 10$. The saturation level is set to $\bar{v} = 1.25$, thus one may choose $\bar{u} = 2$.

The aim is to design appropriate CNF controller with full state feedback, which would control the controlled output of the system to track the command reference as fast as possible and as smooth as possible. Following the procedures given in the previous section and with appropriate selections of design parameters, I have obtained the following CNF control law. Note that the linear part of the control law is carried out using the standard pole-placement design.

CNF controller design:

$$v = \left(\frac{a}{c}\right)\sin x_1 + \left(\frac{1}{c}\right)u = \sin x_1 + \left(\frac{1}{10}\right)u, \tag{4.66}$$

with

$$u = Fx + G\theta_r + \rho(\theta_r, y)F_{\rm n}(x - x_{\rm e}), \tag{4.67}$$

where

$$F = [\,-15 \quad -2.5\,],$$

$$G = 15,$$

$$F_{\rm n} = [\,-4530 \quad -760\,],$$

$$x_{\rm e} = [\,4 \quad 0\,]'$$

and

$$\rho(\theta_r, y) = -\left| e^{-0.0025|h(t)-\theta_r|} - e^{-0.0025|h(0)-\theta_r|} \right|.$$

Using SIMULINK in MATLAB, I obtain the simulation result in Figure (4.1), which is done under the following initial condition

$$x_0 = [\,0 \quad 0\,]'. \tag{4.68}$$

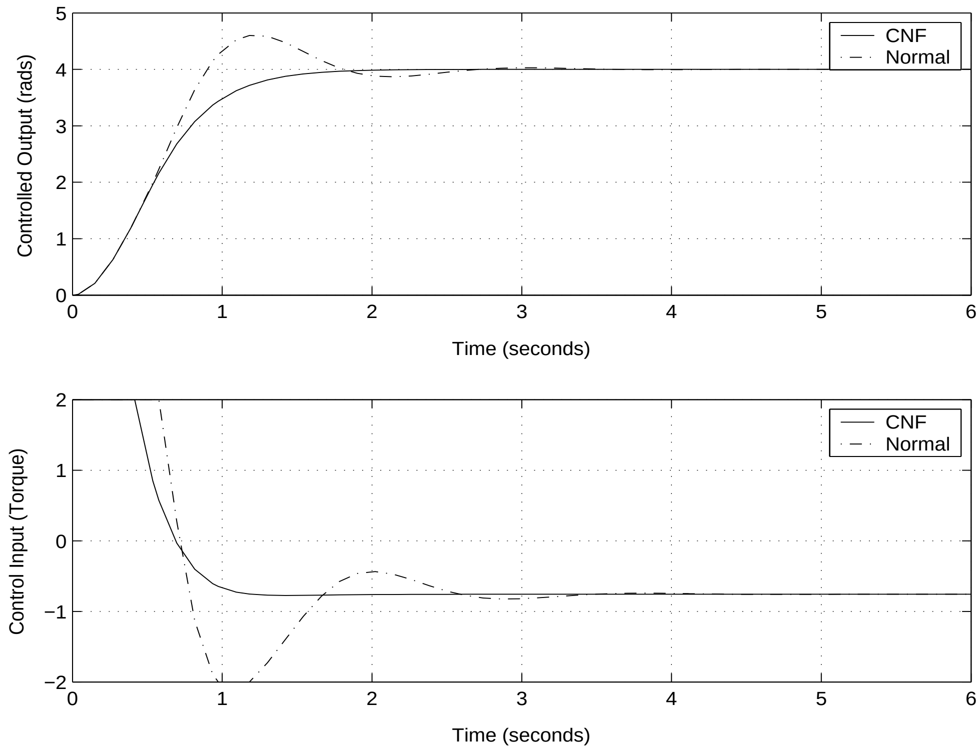


Figure 4.1: Input and output signals under state feedback: $h(x) = x_1$.

The result clearly shows that the control laws with the nonlinear components, *i.e.*, the CNF controller, outperform its conventional counterpart a great deal.

Now let us consider tracking a nonlinear control output $h(x) = x_1^3$. Let $r = 8$ so that $x_{e1} = 2$ and one can still use the above system but let it track a reference $\theta_r = 2$ instead. Nothing else changes but of course ρ should read as

$$\rho(\theta_r, y) = -\left| e^{-0.0005|h(t)-r|} - e^{-0.0005|h(0)-r|} \right|$$

and x_e as

$$x_{\rm e} = [2 \quad 0]' .$$

as the reference signal for h is no longer θ_r. Below is the resulting figure, Figure (4.2).

If one let $r = 4$ so that $x_{e1} = 4^{1/3}$ and use the same system as before but let it track another reference $\theta_r = 4^{1/3}$. Again, nothing else changes but ρ should read as

$$\rho(\theta_r, y) = -\left| e^{-0.0025|h(t)-r|} - e^{-0.0025|h(0)-r|} \right|$$

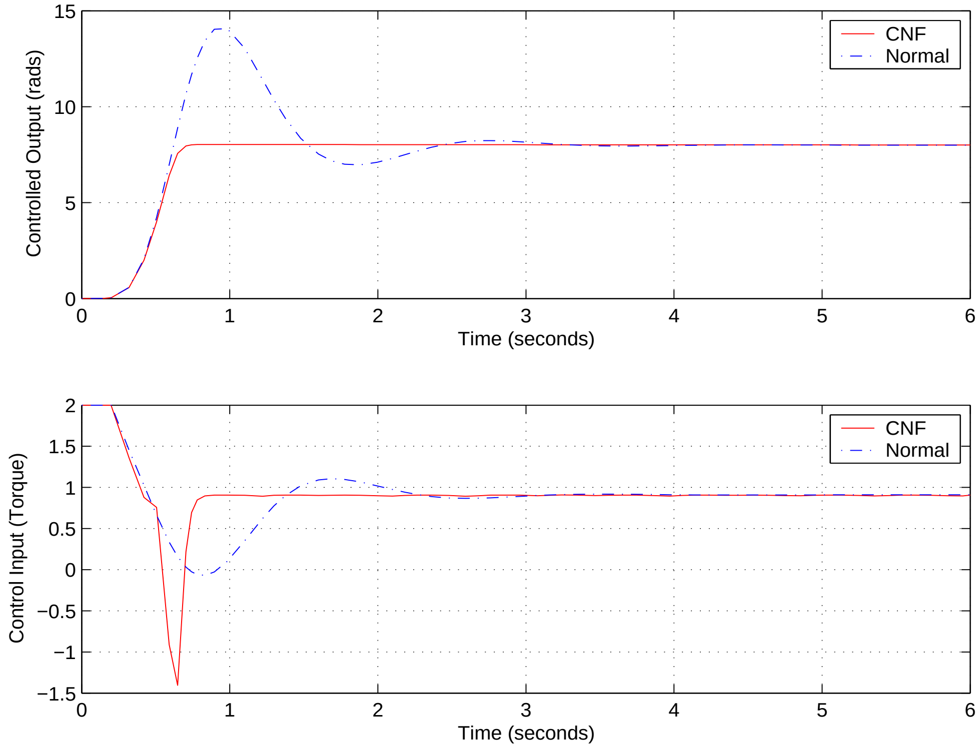


Figure 4.2: Input and output signals under state feedback: $h(x) = x_1^3 \to 8$.

and x_e as

$$x_e = [\,1.587401 \quad 0\,]' .$$

See Figure (4.3) for the result.

4.4 Conclusion

I have extended the so-called CNF control techniques for linear input-saturated systems to SISO nonlinear feedback linearizable systems with actuator saturation. The closed-loop is able to track step function signal which is rarely considered in the literature for nonlinear systems. It has been shown that the performance is better compared to normal linear approaches. Further extension to MIMO case and more general nonlinear systems is possible and is still under investigation. Besides, output feedback can also be obtained for feedback linearizable nonlinear systems either for SISO or MIMO linear systems by imposing some observability conditions on the transformed linear systems. Obviously the theory for discrete-time nonlinear feedback linearizable systems can also be established

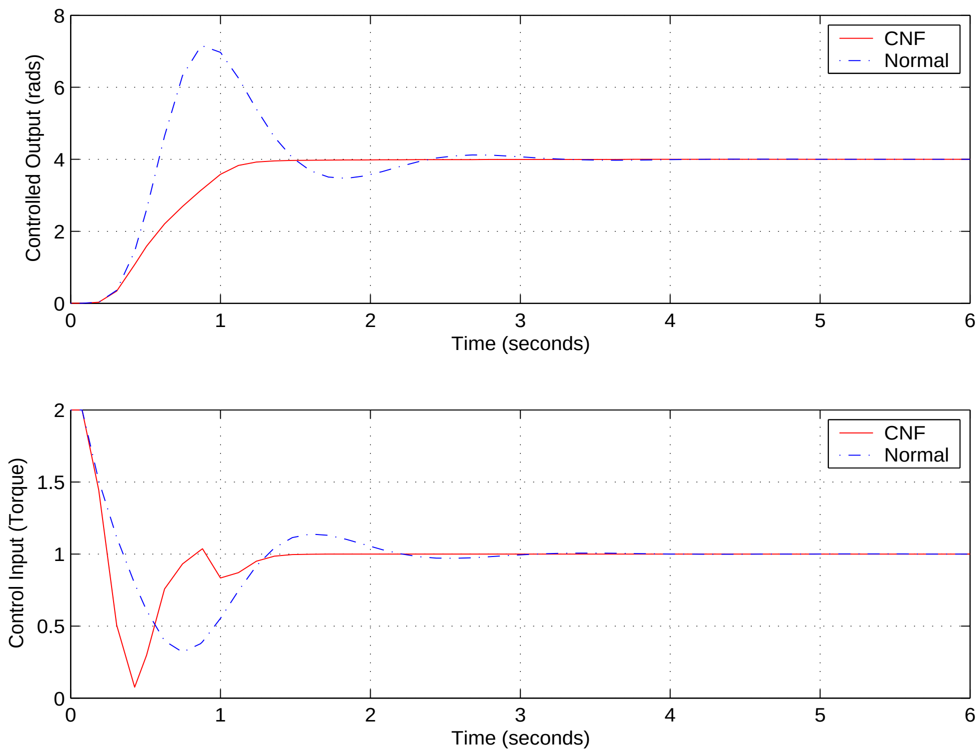


Figure 4.3: Input and output signals under state feedback: $h(x) = x_1^3 \to 4$.

similarly.

Chapter 5

CNF Control for Continuous-Time Partial Linear Composite Systems with Input Saturation

This chapter studies the technique of the composite nonlinear feedback (CNF) control for a class of cascade nonlinear systems with input saturation. In particular, the class of systems under consideration consists of two parts, a linear portion and a nonlinear portion with the output of the linear part connecting to the input of the nonlinear part and with the input of the given system being saturated. The objective of this chapter is to design a composite nonlinear feedback control law based on the linear portion such that the output of the system tracks a step input rapidly with small overshoot and at the same time maintains the stability of the whole cascade system. The specific attention should be paid as the peaking phenomenon in the linear part may cause nonlinear zero dynamics go to infinity within finite time period. Typically when one drives the linear dynamics too fast it may destabilize the whole system. However, as indicated in [70], if the output of the linear part can be made small enough, the nonlinear part will stay in the domain of attraction.

The result has been successfully demonstrated by numerical and application examples including a flight control system for a fighter aircraft.

5.1 Introduction

When people talk about tracking control for nonlinear systems, it is natural that one thinks about the semi-global and global stabilization problems, and output regulation problem. Due to the vast diversity of nonlinear phenomenon, it is not possible for a specific method to be applied to general nonlinear systems. Rather, many different methods and schemes have been proposed to deal with different kinds of systems for different purposes including tracking certain desired signals. In extending the CNF methodology to more general systems, even nonlinear systems, without exception I have to limit my scope to a very specific type of systems for a very specific control purpose. Note that for nonlinear systems with input saturation, to my good knowledge so far, very little work has been done if not any. Therefore, it is simpler and natural as well for one to start with a class of partially linear composite systems with input saturation. In line with CNF control for linear systems, let us consider only constant signal tracking and the focus is, again, improvement of transient performance.

The class of systems under consideration consists of two parts, a linear portion and a nonlinear portion with the output of the linear part connecting to the input of the nonlinear part and with the input of the given system being saturated. Many nonlinear systems can be transformed into partially linear composite systems via a state-space diffeomorphism and/or a preliminary feedback transformations (see, for example, [42]). In recent two decades, the semi-global and global stabilization problems for partially linear composite systems have been extensively studied by many researchers such as [12], [44], [45], [54], [55], [70] and [72], to name just a few. In particular, it was shown in [70] that a nonlinear system which is zero input globally asymptotically stable (GAS) will preserve its GAS property if its input decreases to zero with a very fast exponential rate. It is not difficult to make the output of the linear part, which is the input of the nonlinear part, to converge to zero with some exponential rate. However, the peaking phenomenon in linear systems may destroy the stability of the nonlinear systems before the output rapidly decays to zero.

When constructing CNF controller for the linear part, particular attention is paid to improve the transient performance of the closed-loop system. The research on nonlinear output regulation problems has made great progress since the 1990s. Related results have been extensively reported in the literature (see, for example, [14], [38], [40] and [41]). However, the transient performance is not considered in most of these works. It is a tough task to study the transient performance of the nonlinear output regulation problem, especially when the reference inputs are time-varying signals. I will consider in this chapter a tracking control problem with a constant (or step) reference. To improve the tracking performance, Lin *et al.* proposed the CNF control technique in their pioneer work [53] for a class of second order linear systems. Turner *et al.* [74] later extended the results of [53] to higher order and multiple input systems under a restrictive assumption on the system. However, both [53] and [74] considered only the state feedback case. Recently, Chen *et al.* [19] have developed a CNF control to a more general class of systems with measurement feedback, and successfully applied the technique to solve a hard disk drive servo problem. The CNF control consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time not exceeding the actuator limits for desired command input levels. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part. This chapter aims to design a CNF control law for partially linear composite systems with input saturation based on the linear part of the composite system such that the closed-loop system has desired performances, *e.g.*, quick response and small overshoot, and the tracking error decays to zero with sufficiently large exponential rate to guarantee the stability of the whole system. The result will be illustrated by two examples, one is an output regulation problem and the other is a step tracking problem for a fighter aircraft.

The remaining part of this chapter is organized as follows. Section 5.2 describes the control problem and presents some relevant preliminary results. The CNF control law design for the partially linear composite systems is given in Section 5.3. Section 5.4 illustrates the proposed design technique with numerical and application examples where the performances of the closed-loop system are compared between the CNF control and the corresponding linear control. Finally, Section 5.5 draws some concluding remarks.

5.2 Problem Description and Preliminaries

Consider a partially linear composite system with input saturation characterized by

$$\dot{\xi} = f(\xi, y), \quad \xi(0) = \xi_0 \tag{5.1}$$

$$\dot{x} = Ax + B\,\mathrm{sat}(u), \quad x(0) = x_0 \tag{5.2}$$

$$y = Cx \tag{5.3}$$

where $(\xi, x) \in \mathbb{R}^m \times \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ the control input, and $y \in \mathbb{R}$ the output of the system, f is a smooth (i.e., C^∞) function, A, B and C are appropriate dimensional constant matrices, and sat : $\mathbb{R} \to \mathbb{R}$ represents the actuator saturation defined as

$$\mathrm{sat}(u) = \mathrm{sgn}(u)\min\{u_{\max}, |u|\} \tag{5.4}$$

with $u_{\max}$ being the saturation level of the input. I will aim to design a control law for (5.1)–(5.3) such that the resulting closed-loop system is stable and the output of the closed-loop system will track a step reference input r rapidly without experiencing large overshoot. The CNF control law consists of a linear feedback control and a nonlinear feedback control. The linear feedback law is designed to stabilize the system with a small closed-loop damping ratio for quick tracking. The nonlinear feedback law is to increase the closed-loop damping ratio as the system output approaches the reference input to reduce the overshoot while it keeps the closed-loop stability. This problem is an extension of the recent work of [19] and [53] on composite nonlinear feedback control for linear systems by connecting a nonlinear zero dynamics (5.1) to the linear system (5.2). To design a CNF control law, assume that

A1: (A, B) is controllable;

A2: (A, B, C) is invertible and has no invariant zeros at $s = 0$; and

A3: There exists a C^1 positive definite function $V_\xi(\xi)$ and class K_∞ functions α_1 and α_2 such that

$$\alpha_1(\|\xi\|) \le V_\xi(\xi) \le \alpha_2(\|\xi\|), \tag{5.5}$$

$$\frac{\partial V_\xi(\xi)}{\partial \xi} f(\xi, r) < 0, \tag{5.6}$$

for all $\xi \in \mathbb{R}^m$.

Remark 5.1. Assumption A3 is to ensure that the nonlinear system (5.1) is stable when the system output y tracks exactly the step command input r.

Lemma 5.1. *Consider the nonlinear control system of the form*

$$\dot{\xi} = f(\xi, r + \eta(t)), \tag{5.7}$$

which satisfies Assumption A3. Given any $\gamma > 0$ and $\beta > 0$, there exists a scalar $a > 0$ such that for any

$$|\eta(t)| \le \beta e^{-at}, \quad t \ge 0, \tag{5.8}$$

the solution $\xi(t)$ of (5.7) exists and is bounded for all $t \ge 0$ provided that $\xi(0) \in \Omega_\gamma := \{\xi : \|\xi\| \le \gamma\}$.

Proof. The proof of this Lemma follows the lines of reasoning as in Theorem 4.1 of [70]. Noting that $V_\xi(\xi)$ is a C^1 positive definite function, let

$$c = \max\{V_\xi(\xi) \ : \ \xi \in \Omega_\gamma\}$$

for any given $\gamma > 0$. Since $V_\xi(\xi)$ is C^1 and $f(\xi, y)$ smooth, there exists a constant $h > 0$ such that, for all $\xi \in \Omega_\gamma$ and $|v| \le \beta$,

$$\left| \frac{\partial V_\xi(\xi)}{\partial \xi} f(\xi, r + v) \right| \le h.$$

Let $\tau = \frac{1}{h}$. Then for every solution $\xi(t)$ of (5.7) under any admissible input such that $|\eta(t)| \le \beta$ and $\xi(0) \in \Omega_\gamma$,

$$V_\xi(\xi(t)) \le c + 1, \ \ 0 \le t \le \tau.$$

By the continuity of $\frac{\partial V_\xi(\xi)}{\partial \xi} f(\xi, r)$ and (5.6), there exists an $\alpha > 0$ such that

$$\frac{\partial V_\xi(\xi)}{\partial \xi} f(\xi, r + v) < 0 \tag{5.9}$$

when $c \leq V_\xi(\xi) \leq c + 1$ and $|v| \leq \alpha$.

Next, choose a such that

$$\beta e^{-a\tau} \leq \alpha. \tag{5.10}$$

If η is an input satisfying (5.8), and $\xi(t)$ is the solution of (5.7) with $\xi(0) \in \Omega_\gamma$, one can claim that

$$V_\xi(\xi(t)) \leq c + 1, \;\; t \geq 0. \tag{5.11}$$

In fact, I have proved that $V_\xi(\xi(t)) \leq c + 1$ for $0 \leq t \leq \tau$. For $t > \tau$, (5.8) and (5.10) implies $|\eta(t)| < \alpha$, and then by (5.9), one has

$$\frac{\partial V_\xi(\xi)}{\partial \xi} f(\xi, r + \eta) < 0.$$

Thus,

$$V_\xi(\xi(t)) \leq V_\xi(\xi(\tau)) \leq c + 1, \;\; t > \tau.$$

Moreover, $\xi(t)$ is bounded by

$$\|\xi(t)\| \leq \alpha_1^{-1}(V_\xi(\xi)) \leq \alpha_1^{-1}(c + 1).$$

This completes the proof of Lemma 5.1. ∎

Remark 5.2. a is said to be good for (γ, β) if a satisfies Lemma 5.1, which was introduced in [70].

Remark 5.3. Assumption A3 can be relaxed to be satisfied locally, *e.g.*, in $\Omega_{\bar{\gamma}}$. In this case, it is clear that, from the proof of Lemma 5.1, by selecting $0 < \gamma < \bar{\gamma}$, and $\beta > 0$ such that

$$\{\xi \; : \; V_\xi(\xi) \leq c + 1\} \subset \Omega_{\bar{\gamma}},$$

then there exists an $a > 0$ which is good for (γ, β).

5.3 Design of the Composite Nonlinear Feedback Control Law

In this section, I will proceed to design a CNF control law for the system (5.1)–(5.3). Let us assume that the given system (5.1)–(5.3) satisfies Assumptions A1 to A3, and all the states of the linear system (5.2) are available for feedback. The CNF control law can be constructed by the following step-by-step procedure.

STEP S.1. Select appropriate scalars $\gamma > 0$, $\beta > 0$ and $a > 0$ such that a is good for (γ, β). γ and β can be selected arbitrarily if Assumption A3 is satisfied globally. Moreover, by Remark 5.3, γ, β and a can also be appropriately selected even Assumption A3 is only satisfied locally.

STEP S.2. Design a linear feedback law

$$u_{\mathrm{L}} = Fx + Gr \tag{5.12}$$

where r is a step command input and F is chosen such that

1. $A + BF$ is Hurwitz and the output of the following system,

$$\dot{x} = (A + BF)x, \quad y = Cx, \tag{5.13}$$

has $\|y(t)\| \le ke^{-at}$ for some $k > 0$; and

2. The closed-loop system $C(sI - A - BF)^{-1}B$ has certain desired properties, *e.g.*, having a small damping ratio.

The existence of such an F is guaranteed by Assumption A1, i.e., (A, B) is controllable. In fact, it can be designed using methods such as the H_2 and H_∞ optimization approaches, as well as the robust and perfect tracking technique. G is a scalar given by

$$G = -[C(A + BF)^{-1}B]^{-1}. \tag{5.14}$$

Note that G is well defined since $A + BF$ is Hurwitz and the triple (A, B, C) is invertible and has no invariant zeros at $s = 0$. Also, let

$$H := [1 - F(A + BF)^{-1}B]G \tag{5.15}$$

and

$$x_{\mathrm{e}} := G_{\mathrm{e}} r := -(A+BF)^{-1} BGr. \tag{5.16}$$

STEP S.3. Given a positive-define matrix $W \in \mathbb{R}^{n \times n}$, solve the Lyapunov equation

$$(A+BF)'P + P(A+BF) = -W \tag{5.17}$$

for $P > 0$. Note that such a P exists since $A+BF$ is asymptotically stable. Then, the nonlinear feedback control law $u_{\mathrm{N}}(t)$ is given by

$$u_{\mathrm{N}} = \rho(r,y) B'P(x - x_{\mathrm{e}}) \tag{5.18}$$

where $\rho(r,y)$ is any non-positive function locally Lipschitz in y. This nonlinear control law is used to change the system closed-loop damping ratio as the output approaches the step command input.

STEP S.4. The CNF control law is given by combining the linear and nonlinear feedback law derived in the previous steps,

$$u = u_{\mathrm{L}} + u_{\mathrm{N}} = Fx + Gr + \rho(r,y) B'P(x - x_{\mathrm{e}}). \tag{5.19}$$

Theorem 5.1. *Consider the given system (5.1)–(5.3) satisfies Assumptions A1 to A3. Let scalars $\gamma > 0$, $\beta > 0$ and $a > 0$ be selected such that a is good for (γ, β), and let*

$$\mathcal{N} := \left\{ x \in \mathbb{R}^n : \ \|x\| \le \frac{\beta}{||C||} \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} \right\}. \tag{5.20}$$

For any $\delta \in (0,1)$, let $c_\delta > 0$ be the largest positive scalar satisfying the following condition:

$$|Fx| \le u_{\max}(1-\delta) \tag{5.21}$$

for all $x \in \mathbf{X}_\delta$, where

$$\mathbf{X}_\delta := \left\{ x \ : \ x'Px \le c_\delta, \ x \in \mathcal{N} \right\}.$$

Then for any non-positive function $\rho(r,y)$, locally Lipschitz in y, the state trajectory of the closed-loop system comprising the given system (5.1)–(5.3) and the CNF control law

(5.19) is bounded for all $t \geq 0$, provided that the initial states ξ_0 and x_0, and amplitude of step input r satisfy

$$\xi_0 \in \Omega_\gamma, \quad \tilde{x}_0 := (x_0 - x_e) \in \mathbf{X}_\delta, \quad |Hr| \leq u_{\max}. \tag{5.22}$$

Moreover, the system output y tracks asymptotically the step command input of amplitude r.

Proof. The closed-loop system comprising the given plant (5.1)–(5.3) and the CNF control law (5.19) is given by

$$\dot{\xi} = f(\xi, y) \tag{5.23}$$

$$\dot{x} = Ax + B\,\mathrm{sat}(Fx + Gr + \rho(r,y)B'P(x - x_{\mathrm{e}})) \tag{5.24}$$

$$y = Cx \tag{5.25}$$

Let $\tilde{x} = x - x_{\mathrm{e}}$. The closed-loop system (5.23)–(5.25) can be expressed as

$$\dot{\xi} = f(\xi, r + C\tilde{x}) \tag{5.26}$$

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + Bw \tag{5.27}$$

where

$$w = \mathrm{sat}(F\tilde{x} + Hr + \rho(r,y)B'P\tilde{x}) - F\tilde{x} - Hr. \tag{5.28}$$

Define a Lyapunov function $V_{\tilde{x}}(\tilde{x}) = \tilde{x}'P\tilde{x}$, then

$$\lambda_{\min}(P)\|\tilde{x}\|^2 \leq V_{\tilde{x}}(\tilde{x}) \leq \lambda_{\max}(P)\|\tilde{x}\| \tag{5.29}$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of P, respectively. Then,

$$\begin{aligned} \dot{V}_{\tilde{x}}(\tilde{x}) &= \frac{\partial V_{\tilde{x}}(\tilde{x})}{\partial \tilde{x}}\left((A + BF)\tilde{x} + Bw\right) \\ &= -\tilde{x}'W\tilde{x} + \frac{\partial V_{\tilde{x}}(\tilde{x})}{\partial \tilde{x}}Bw. \end{aligned}$$

It have been shown in [19] that,

$$\frac{\partial V_{\tilde{x}}(\tilde{x})}{\partial \tilde{x}}Bw = 2\tilde{x}'PBw \leq 0$$

for all $\tilde{x} \in \mathbf{X}_\delta$ and $|Hr| \le \delta u_{\max}$. Thus

$$\dot{V}_{\tilde{x}}(\tilde{x}) \le -\tilde{x}' W \tilde{x}, \quad \tilde{x} \in \mathbf{X}_\delta \tag{5.30}$$

i.e., $\mathbf{X}_\delta$ is an invariant set of the system (5.27). Thus the solution of (5.27) exists and is bounded for all $t \ge 0$ and $\tilde{x}_0 \in \mathbf{X}_\delta$. Noting that $x = x_{\rm e} + \tilde{x}$, x exists and is bounded for all $t \ge 0$ and x_0 satisfies (5.22).

To show the existence and boundedness of the solution ξ of (5.26), it is sufficient to show that $\|\tilde{y}\| := \|C\tilde{x}\| \le \beta e^{-at}$. Noting that (5.30) gives

$$\dot{V}_{\tilde{x}}(\tilde{x}) \le -\tilde{x}' W \tilde{x} \le -\lambda_{\min}(W)\|\tilde{x}\|^2 \tag{5.31}$$

for all $\tilde{x} \in \mathbf{X}_\delta$. According to the proof of Theorem 4.10 of [47], (5.29) and (5.31) yields that

$$\begin{aligned} \|\tilde{x}(t)\| &\le \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} \|\tilde{x}(0)\| e^{-\frac{\lambda_{\min}(W)}{2\lambda_{\max}(P)}t} \\ &\le \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} \|\tilde{x}(0)\| e^{-at} \end{aligned}$$

since a is selected such that $0 < a \le \lambda_{\min}(W)/(2\lambda_{\max}(P))$. Then

$$\begin{aligned} \|\tilde{y}(t)\| = \|C\tilde{x}(t)\| &\le \|C\|\|\tilde{x}(t)\| \\ &\le \|C\| \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} \|\tilde{x}(0)\| e^{-at} \\ &\le \beta e^{-at} \end{aligned}$$

for all $\tilde{x}(0) \in \mathbf{X}_\delta$. Thus, by Remark 5.2, the solution of (5.26) exists and is bounded for all $t \ge 0$.

Moreover, noting that $W > 0$, all trajectories of (5.27) starting from $\mathbf{X}_\delta$ will converge to the origin. Thus,

$$\lim_{t\to\infty} x(t) = x_{\rm e} \tag{5.32}$$

for all initial state x_0 and the step command input of amplitude r that satisfy (5.22). Therefore,

$$\lim_{t\to\infty} y(t) = Cx_{\rm e} = -C(A + BF)^{-1}BGr = r. \tag{5.33}$$

This completes the proof of Theorem 5.1. ∎

Remark 5.4. The CNF control law (5.19) is reduced to the linear feedback control law (5.12) when the function $\rho(r, y) = 0$. Thus, Theorem 5.1 shows that the additional nonlinear feedback control law (5.18) does not affect the ability of the closed-loop system to track the command input. Any command input that can be asymptotically tracked by the linear control law (5.12) can also be asymptotically tracked by the CNF control law (5.19). However, this additional term u_{N} in the CNF control law can be used to improve the performance of the overall closed-loop system. This is the key property of the control technique studied in this manuscript.

Remark 5.5. The main purpose of adding the nonlinear part to the CNF control law is to speed up the settling time, or equivalently to contribute a significant value to the control input when the tracking error, $r - y$, is small. The nonlinear part, in general, will be in action when the control signal is far away from its saturation level and, thus, it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising (5.2) and (5.19) can be expressed as

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + \rho(r, y)BB'P\tilde{x}. \tag{5.34}$$

It is clear that eigenvalues of the closed-loop system (5.34) can be changed by the function $\rho(r, y)$. In fact, define the auxiliary system $G_{\text{aux}}(s)$ as

$$G_{\text{aux}}(s) := C_{\text{aux}}(sI - A_{\text{aux}})^{-1}B_{\text{aux}} := B'P(sI - A - BF)^{-1}B. \tag{5.35}$$

Then, the system (5.34) can be expressed as Figure 5.1. Using the well-known classical root-locus theory. The poles of the closed-loop system (5.34) approach the location of the invariant zeros of $G_{\text{aux}}(s)$ as $|\rho|$ becomes larger and larger.

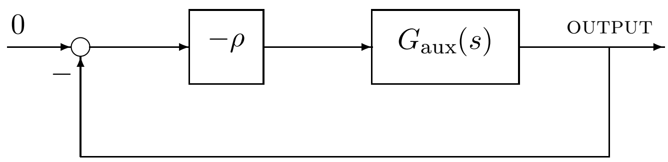


Figure 5.1: Interpretation of the nonlinear function $\rho(r, y)$.

Remark 5.6. It is shown in [19] that the auxiliary system G_{aux} is stable and invertible with a relative degree equal to 1, and is of minimum phase with $n-1$ stable invariant zeros. It should be noted that there is freedom in pre-selecting the locations of these invariant zeros by selecting an appropriate W in (5.17). In general, one should select the invariant zeros of G_{aux}, which are corresponding to the closed-loop poles of (5.34) for large $|\rho|$, such that the dominated ones have a large damping ratio, which in turn will yield a smaller overshoot. Interested readers are referred to [19] for the detailed procedure for the selecting of such a W. Another important step in designing the CNF control law is the selection of the non-positive nonlinear function $\rho(r, y)$. It is common that one chooses $\rho(r, y)$ as a function of the tracking error $r - y$, which in most practical situations is known and available for feedback, such that $\rho(r, y)$ has the following two properties, 1) when the output y is far away from the final set point, $|\rho(r, y)|$ is small and thus the effect of the nonlinear part on the overall system is very limited; and 2) when the output approaches the set point, $|\rho(r, y)|$ becomes larger and larger, and the nonlinear control law will become effective. Of course, the choice of $\rho(r, y)$ is non-unique. The following choice is one of the suitable candidates,

$$\rho(r, y) = -\beta_{\text{n}} \left| e^{-\alpha_{\text{n}}|y(t)-r|} - e^{-\alpha_{\text{n}}|y(0)-r|} \right|, \tag{5.36}$$

where $\beta_{\text{n}} > 0$ and $\alpha_{\text{n}} > 0$ are tuning parameters.

5.4 Illustrative Examples

In this section, I will illustrate the CNF design method with two examples. To compare the performance of the CNF control law and the linear control law, let us first take the example from [55] where the semi-global stabilization problem is solved by a linear state feedback. Based on the linear control law given by [55], I will design a CNF control law to improve the performance of the closed-loop system. The second example is the design of a flight control system for a simplified model of a fighter aircraft reported in [81].

Example 5.1. Consider a partially linear composite system (see [55]) characterized by

$$\dot{\xi} = -\xi + \xi^2 y, \tag{5.37}$$

$$\dot{x} = Ax + B\,\mathrm{sat}(u) \tag{5.38}$$

$$y = Cx \tag{5.39}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5.40}$$

and $u_{\max} = 0.2$. For the stabilization problem of (5.37)–(5.39), let $r = 0$. It is simple to verify that the triple (A, B, C) is controllable and has a relative degree of 1 and four invariant zeros at $\{j, -j, j, -j\}$. Thus, Assumptions A1 and A2 are satisfied. Let $\gamma = 1$ and $\beta = 1$, then it can be shown that any $a > 0$ is weakly good for (γ, β). To design the CNF control law, let us use the linear feedback control law,

$$u_{\mathrm{L}} = Fx = \begin{bmatrix} 0.403 & -0.0001 & -0.204 & -4.06 & -10.4 \end{bmatrix} x, \tag{5.41}$$

reported in [55]. Next, select $W = I_5$ and solve the following Lyapunov equation

$$(A + BF)'P + P(A + BF) = -W,$$

which yields a solution

$$P = \begin{bmatrix} 12.7439 & -0.5000 & -8.2902 & -25.8924 & -2.4813 \\ -0.5000 & 12.8221 & 26.6781 & 4.4923 & 0.1934 \\ -8.2902 & 26.6781 & 75.5045 & 26.7835 & 1.9341 \\ -25.8924 & 4.4923 & 26.7835 & 70.7732 & 6.7201 \\ -2.4813 & 0.1934 & 1.9341 & 6.7201 & 0.6942 \end{bmatrix} > 0.$$

The nonlinear function $\rho(r, y)$ is chosen as

$$\rho(r, y) = -15(e^{-5|y-r|} - e^{-5|y(0)-r|}).$$

Finally, the CNF control law is given by

$$u = Fx + \rho(r, y)B'Px. \tag{5.42}$$

The responses of the state variables of the closed-loop systems of the given systems with the linear control law and with the CNF control law, respectively, are given in Figure 5.2 under the same initial conditions $\xi(0) = -0.2$ and $x(0) = [-0.1,\ 0.1,\ -0.05,\ -0.08,\ 0.05]'$. Clearly, the CNF control has outperformed the linear counterpart significantly.

Example 5.2. Consider a simplified model of a fighter aircraft reported in [81], which is characterized by

$$\dot{v} = 1.8254\cos(0.0175(\alpha + 11.3404)) - 1.9821 \times 10^{-3}(0.0886 + 0.0175\alpha)v^2 \quad (5.43)$$

$$\dot{\alpha} = -0.5923\alpha + 50.7296q - 0.1145\text{sat}(u) \quad (5.44)$$

$$\dot{q} = -0.0178\alpha - 0.3636q - 0.0676\text{sat}(u) \quad (5.45)$$

where the airspeed v (m/s), angle of attack α (deg), and pitch angular rate q (rad/s) are state variables, deflection of elevator u (deg) is control input with a saturation level $u_{\max} = 10°$. The model is extracted from the nonlinear model of six degree of freedoms based on a steady flight condition with Mach = 0.3, height = 1000 meters, and with a straight and horizontal flight. The control objective is to set the angle of attack to a reference attitude 5° quickly without experiencing large overshoot.

Let $\xi = v$ and $x = (\alpha,\ q)'$, and let $y = \alpha$. Then, the dynamics of the aircraft can be rewritten as the form of (5.1)–(5.3), i.e.,

$$\dot{\xi} = 1.8254\cos(0.0175(y + 11.3404)) - 1.9821 \times 10^{-3}(0.0886 + 0.0175y)\xi^2 \quad (5.46)$$

$$\dot{x} = Ax + B\text{sat}(u) \quad (5.47)$$

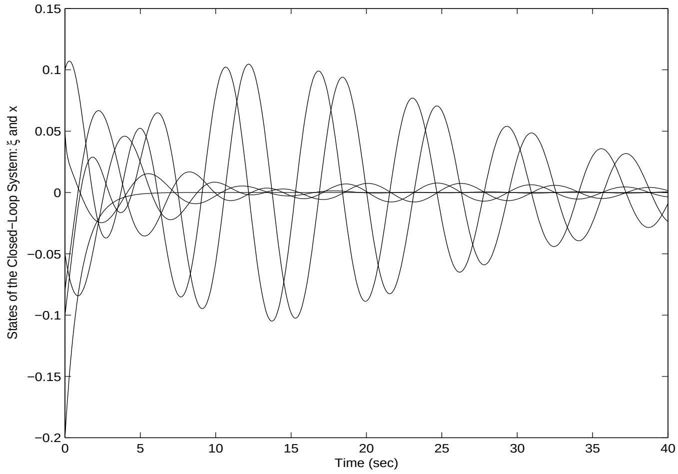
$$y = Cx \quad (5.48)$$

where

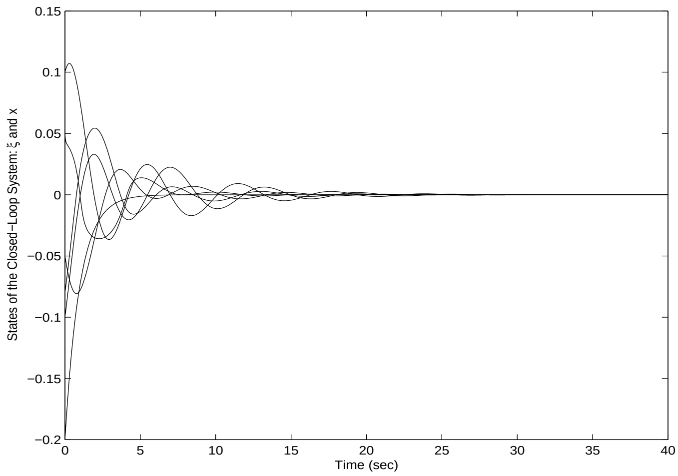
$$A = \begin{bmatrix} -0.5923 & 50.7296 \\ -0.0178 & -0.3636 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1145 \\ -0.0676 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The triple (A, B, C) is controllable, and has a relative degree of 1 and an invariant zero at -30.3140. Thus, Assumptions A1 and A2 are satisfied. Let $r = 5$, then the nonlinear system (5.46) with $y = r$ has an equilibrium point $\xi = v_0 = 70.8328$. Let $\tilde{\xi} = \xi - v_0$. One has

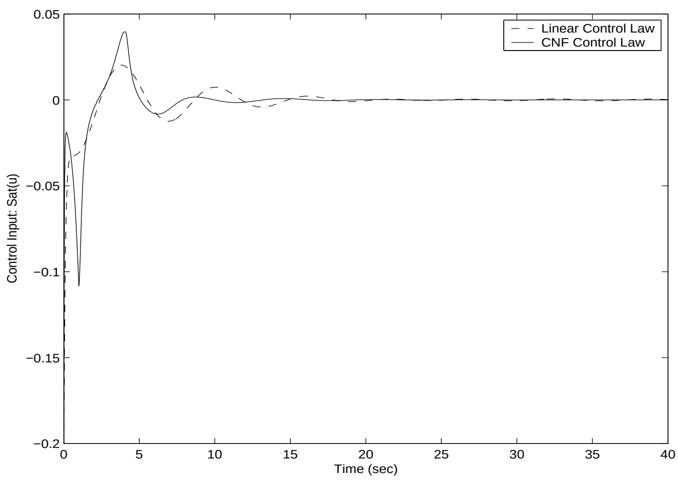
$$\dot{\tilde{\xi}} = -0.0495\tilde{\xi} - 3.4912 \times 10^{-4}\tilde{\xi}^2. \quad (5.49)$$



(a) State responses with the linear control law.



(b) State responses with the CNF control law.



(c) Control signals.

Figure 5.2: State responses and control signals of the closed-loop systems.

It is simple to verify that (5.49) is regionally asymptotically stable, *e.g.*, Assumption A3 is satisfied locally in $\Omega_{\bar{\gamma}} = \{\tilde{\xi} \ : \ \|\tilde{\xi}\| \le 60\}$. Thus, a CNF control law can be constructed, which is given as follows

$$u = Fx + Gr + \rho(r, y)B'P(x - G_{\mathrm{e}}r) \tag{5.50}$$

with $F = [\,0.9253 \quad 35.5945\,]$ placing the eigenvalues of $A + BF$ at $-1.7677 \pm j1.7677$, $G = -1.5966$, $G_{\mathrm{e}} = [\,1 \quad 0.0097\,]'$,

$$\rho(r, y) = -(e^{-|y-r|} - e^{-|y(0)-r|}), \tag{5.51}$$

and P is the positive define solution of the following Lyapunov equation

$$(A + BF)'P + P(A + BF) = -W$$

where

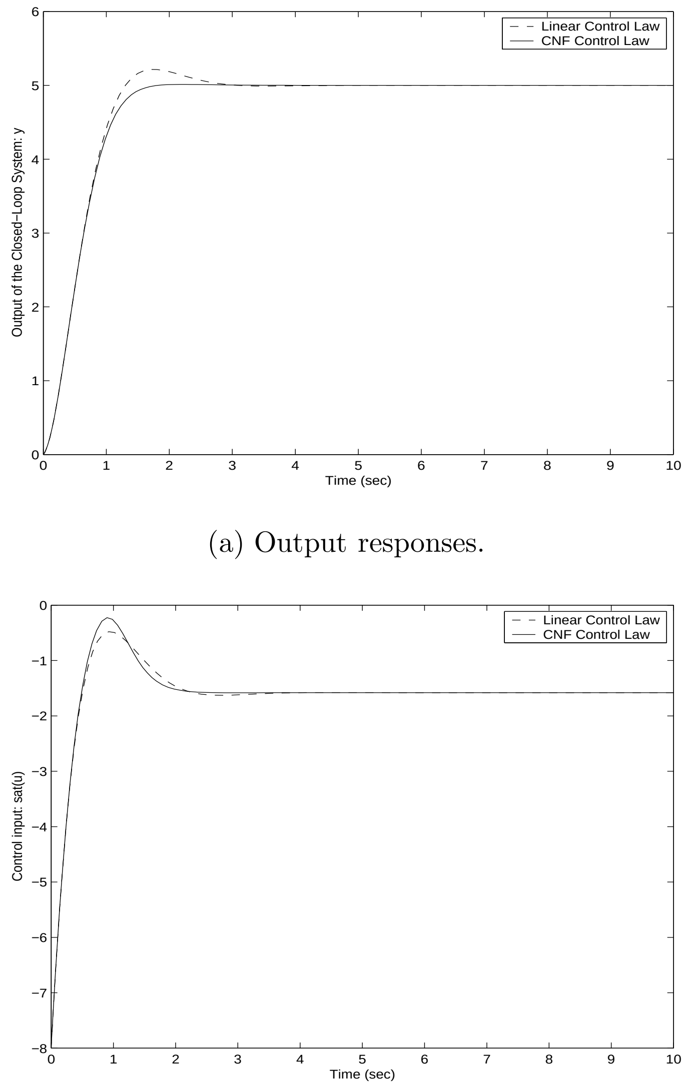
$$W = \begin{bmatrix} 0.4 & 9.4 \\ 9.4 & 2568.7 \end{bmatrix} > 0 \tag{5.52}$$

is selected, according to [19] and [21], such that the invariant zeros of $G_{\mathrm{aux}}(s) = B'P(sI - A - BF)^{-1}B$ is -0.5.

The simulation results shown in Figure 5.3(a) shows the system output (angle of attack) under the CNF control law (5.50) and the linear control law which switches off the nonlinear part of the CNF control law (5.50) under the initial conditions $\xi(0) = 100$ and $x(0) = 0$. Thanks to the nonlinear part of the CNF control law, the output can track the reference command input rapidly, and the overshoot is reduced evidently, 4.31% for the linear control law, 0.26% for the CNF control law. Figure 5.3(b) shows the control input applied to the system under these two control laws.

5.5 Conclusion

The composite nonlinear feedback control technique is extended to the partially linear composite system with input saturation. Simulation result shows that the nonlinear control law greatly improved the performance of the closed-loop system. It should be noted that, in this chapter, I have assumed that the linear part of the composite system



(a) Output responses.

(b) Control signals.

Figure 5.3: Output responses and control signals of the flight control system.

is SISO, and all the states of the linear part are available to feedback. It should not be too difficult to extend the result of this chapter to MIMO systems with measurement feedback using the result reported in [32].

Chapter 6

CNF Control for Discrete-Time Partial Linear Composite Systems with Input Saturation

In this chapter, the design procedure of composite nonlinear feedback control for SISO discrete-time partially linear composite systems with actuator saturation will be addressed. Only the state feedback case will considered. The composite nonlinear feedback control serves to improve the transient performance of the system output without exciting the peaking phenomenon in linear part so that nonlinear zero dynamics will not become unstable and hence guarantees the internal stability of the whole system. Although no works in the literature discuss the so-called peaking phenomenon and its possible destabilizing effect in discrete-time systems, it exists naturally as for many sampled-data systems, during a single sampling period, the system behavior is itself a continuous-time process. Thus, it is necessary for one to consider this peaking phenomenon and its possible destabilizing effect in discrete-time systems. As such, the CNF scheme for discrete-time partially linear composite systems will be addressed. Conditions on how to use CNF control law are derived and an application of this technique to two examples is presented, which shows that the proposed design method yields a very satisfactory performance.

6.1 Introduction

From previous chapter, one knows that a nonlinear system which is zero input globally asymptotically stable (GAS) will preserve its GAS property if its input decreases to zero with a very fast exponential rate [70]. However a bad transient performance may destroy the stability of the nonlinear part before the output rapidly decays to zero. This is also true for discrete-time systems since the inter-sampling behavior is equivalent to the response of a continuous-time system with unchanging input. One also knows that, for set-point tracking, settling time and overshoot are two important transient performance indices, and quick response and small overshoot are desirable in the most of the target tracking control problems. However, it is well known that, in general, quick response results in a large overshoot. Thus, most of the design schemes have to make a trade-off between these two transient performance indices.

In this chapter, I aim to design a CNF control law for discrete-time partially linear composite systems with input saturation. Based on the linear part of the composite system, the CNF control is designed such that the closed-loop system has desired performances, *e.g.*, quick response and small overshoot. Moreover, I will show that the closed-loop system with improved transient performance preserves the stability of the nonlinear part of the partially linear composite system. The result will be illustrated by a numerical example and a fighter example.

This chapter is organized as follows. In Section 6.2, the theory of the composite nonlinear feedback control for discrete-time partially linear SISO systems is developed. The application of this technique to a numerical example will be presented in Section 6.4, which shows that the proposed design method yields a very satisfactory performance. Finally, some concluding remarks will be drawn in Section 6.5.

6.2 Problem Formulation and Preliminaries

Consider a partially linear composite discrete-time systems with input saturation, Σ, characterized by

$$\xi(k+1) = f(\xi(k), y(k)), \quad \xi(0) = \xi_0 \tag{6.1}$$

$$x(k+1) = Ax(k) + B\text{sat}(u(k)), \quad x(0) = x_0 \tag{6.2}$$

$$y(k) = Cx(k) \tag{6.3}$$

where $(\xi, x) \in \mathbb{R}^m \times \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are respectively the state, control input and control output of the given system Σ, f is a smooth (i.e., C^∞) function, A, B, C are appropriate dimensional constant matrices, and the saturation function is defined by

$$\text{sat}(u) = \text{sign}(u)\ \min(|u|,\ u_{\max}), \tag{6.4}$$

where $u_{\max}$ is the maximum amplitude of the control channel.

The aim is to design a certain controller, with all the state information known, which renders the whole (closed-loop) system track a step function with amplitude of r under the input constraint. Without loss of generality, let us assume $f(0, r) = 0$. In fact, if $f(\xi^*, r) = 0$ with $\xi^* \neq 0$, the state transformation $\tilde{\xi} = \xi - \xi^*$ gives

$$\dot{\tilde{\xi}} = f(\tilde{\xi} + \xi^*, r) := \tilde{f}(\tilde{\xi}, r)$$

then, one has $\tilde{f}(0, r) = 0$. For tracking purpose, the following assumptions on the given system are required:

A1: (A, B) is controllable;

A2: (A, B, C) is invertible and has no invariant zeros at $z = 1$; and

A3: There exists a C^1 positive definite function $V_\xi(\cdot)$ and class K_∞ functions α_1, α_2 and α_3 such that

$$\alpha_1(||\xi(k)||) \leq V_\xi(\xi(k)) \leq \alpha_2(||\xi(k)||), \tag{6.5}$$

$$V_\xi(\xi(k+1)) - V_\xi(\xi(k)) \leq \alpha_3(||\xi(k)||), \tag{6.6}$$

where $\xi(k) \in \mathbb{R}^m$ is the solution of

$$\xi(k+1) = f(\xi(k), r), \quad \xi(0) = \xi_0. \tag{6.7}$$

Remark 6.1. The objective here is to design control laws that are capable of achieving fast tracking of target references under input saturation. As such, it is well understood in the literature that assumptions **A1-A2** are standard and necessary. Assumption **A3** is to ensure that the nonlinear system (6.1) is asymptotically stable when the system output y tracks exactly the step command input r.

Lemma 6.1. *Consider the nonlinear control system of the form*

$$\xi(k+1) = f(\xi(k), r + \eta(k)), \tag{6.8}$$

which satisfies Assumption **A3**. *Given any* $\gamma > 0$ *and* $0 < a < 1$, *there exists a scalar* $\beta > 0$ *such that for any*

$$|\eta(k)| \leq \beta \cdot a^k, \quad k \geq 0, \tag{6.9}$$

the solution $\xi(k)$ *of (6.8) exists and is bounded for all* $k \geq 0$ *provided that* $\xi(0) \in \Omega_\gamma := \{\xi \ : \ ||\xi|| \leq \gamma\}$.

Proof. The proof of this Lemma follows the similar lines of reasoning as in Theorem 4.1 of [70]. Noting that $V_\xi(\xi)$ is a C^1 positive definite function, let

$$c = \max\{V_\xi(\xi) \ : \ \xi \in \Omega_\gamma\} \tag{6.10}$$

for any given $\gamma > 0$. Since $V_\xi(\xi)$ is C^1 and $f(\xi, y)$ is smooth, for all time instants $k \geq 0$, there exists a constant $h > 0$ such that, for all $\xi \in \Omega_\gamma$ and $|v| \leq \beta$, where $\beta > 0$ is any positive real number,

$$|V_\xi(\xi(k+1)) - V_\xi(\xi(k))| = |V_\xi(f(\xi(k), r+v)) - V_\xi(\xi(k))| \leq h. \tag{6.11}$$

Let $\tau > 0$ be a specific time instant of interest. Then for every solution $\xi(k)$ of (6.8) under any input such that $|\eta(k)| \leq \beta$ and $\xi(0) \in \Omega_\gamma$,

$$V_\xi(\xi(k)) \leq c + \tau h, 0 \leq k \leq \tau. \tag{6.12}$$

On the other hand, by the continuity of $V_\xi(\xi)$ and (6.6), there exists an $\alpha > 0$ such that

$$V_\xi(\xi(k+1)) - V_\xi(\xi(k)) = V_\xi(f(\xi(k), r+v)) - V_\xi(\xi(k)) \le \alpha_3(||\xi(k)||), \quad k \ge 0, \tag{6.13}$$

for all $|v| \le \alpha$. Then, one can specify β such that

$$\beta \cdot a^k \le \alpha, \quad k \ge \tau. \tag{6.14}$$

If η is an input satisfying (6.9), and $\xi(k)$ is the solution of (6.8) with $\xi(0) \in \Omega_\gamma$, one can claim that

$$V_\xi(\xi(k)) \le c + \tau h, \quad k \ge 0. \tag{6.15}$$

In fact, I have proved that $V_\xi(\xi(k)) \le c + \tau h$ for $0 \le k \le \tau$. For $k \ge \tau$, (6.9) and (6.14) imply $|\eta(k)| < \alpha$, and then by (6.13), one has

$$V_\xi(\xi(k)) \le V_\xi(\xi(\tau)) \le c + \tau h, \quad k > \tau. \tag{6.16}$$

Moreover, $\xi(k)$ is bounded by

$$||\xi(k)|| \le \alpha_1^{-1}(V_\xi(\xi(k))) \le \alpha_1^{-1}(c + \tau h). \tag{6.17}$$

This completes the proof of Lemma 6.1. □

Remark 6.2. In Sussmann and Kokotović [70], they say a is good for (γ, β) if a satisfies Lemma 6.1. Here, however, I propose a similar discrete-time version. In fact, this lemma considers when γ and a are given, there indeed exists a β such that a is good for (γ, β).

Remark 6.3. Assumption **A3** can be relaxed to be satisfied locally, *e.g.*, in $\Omega_{\bar{\gamma}} := \{\xi \ : \ ||\xi|| \le \bar{\gamma}\}$. In this case, it is clear that, from the proof of Lemma 6.1, by selecting $0 < \gamma < \bar{\gamma}$, and $0 < a < 1$ such that

$$\{\xi \ : \ V_\xi(\xi) \le c + \tau h\} \subset \Omega_{\bar{\gamma}}, \tag{6.18}$$

for some integer $\tau > 0$, then there exists a $\beta > 0$ such that a is good for (γ, β).

6.3 Design of The Composite Nonlinear Feedback Control Law

In this section, let us proceed to develop a composite nonlinear feedback control technique for the case when all the state variables of the linear part of the plant Σ are measurable.

The design will be done in four steps described in the following step-by-step design procedure which is a natural extension of the results of [19].

STEP S.1: Design a linear feedback law,

$$u_{\mathrm{L}}(k) = Fx(k) + Gr, \tag{6.19}$$

where $r \in \mathbb{R}$ is the step reference. The state feedback gain matrix $F \in \mathbb{R}^{1\times n}$ is chosen such that

1. the output of the closed-loop system (6.2) and (6.3) under the state feedback $u = Fx$ is such that $A + BF$ is Schur,
2. the resulting closed-loop system transfer matrix, *i.e.*, $C_2(zI - A - BF)^{-1}B$, has certain desired properties, *e.g.*, having a small dominating damping ratio.

Let G be a scalar constant and is given by

$$G := [C(I - A - BF)^{-1}B]^{-1}. \tag{6.20}$$

Here note that G is well defined because $A + BF$ is stable, and (A, B, C) is right invertible and has no invariant zeros at $z = 1$, which implies $(A+BF, B, C)$ is right invertible and has no invariant zeros at $z = 1$ (see *e.g.*, Theorem 3.8.1 of Chen *et al.* [20]).

STEP S.2: Next, one computes

$$H := \left[I + F(I - A - BF)^{-1}B\right] G \tag{6.21}$$

and

$$x_{\mathrm{e}} := G_{\mathrm{e}}\, r := (I - A - BF)^{-1}BG\, r. \tag{6.22}$$

Note that the definitions of H, G_{e} and x_{e} would become transparent later in the following derivation. Given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, solve the following Lyapunov equation:

$$P = (A + BF)'P(A + BF) + W, \tag{6.23}$$

for $P > 0$. Such a P exists since $A + BF$ is asymptotically stable. Then, the nonlinear feedback control law $u_{\rm N}$ is given by

$$u_{\rm N}(k) = \rho(r, y)B'P(A + BF)(x(k) - x_{\rm e}), \tag{6.24}$$

where $\rho(r, y)$ is some nonpositive function, locally Lipschitz in y, which is used to change the closed-loop system damping ratio as the output approaches the target. The choice of this nonlinear function will be discussed at the end of this section.

Step s.3: Given $\gamma > 0$ and $a = \sqrt{1 - \frac{\lambda_{\min}(W)}{\lambda_{\max}(P)}}$, select β such that a is good for (γ, β).

Step s.4: The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller:

$$u(k) = u_{\rm L}(k) + u_{\rm N}(k) = Fx(k) + Gr + \rho(r, y)B'P(A + BF)(x(k) - x_{\rm e}). \tag{6.25}$$

This completes the design of the CNF controller.

Theorem 6.1. *Consider the given system (6.1) to (6.3) satisfying assumptions A1 to A3. Define*

$$\mathcal{N} := \left\{ x \in \mathbb{R}^n : ||x|| \leq \frac{\beta}{\|C\|} \left(\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)} \right)^{1/2} \right\} \tag{6.26}$$

For any $\delta \in (0, 1)$, let $c_\delta > 0$ be the largest positive scalar satisfying the following property:

$$| \, F \, x \, | \leq (1 - \delta) u_{\max}, \tag{6.27}$$

for all $x \in \boldsymbol{X}_\delta$, where

$$\boldsymbol{X}_\delta := \left\{ x : x'Px \leq c_\delta, x \in \mathcal{N} \right\}. \tag{6.28}$$

Then, for any nonpositive function $\rho(r, y)$, locally Lipschitz in y and $|\rho(r, y)| \leq \rho^* := 2(B'PB)^{-1}$, the solution of the closed-loop system under the CNF control law (6.25) exists and is bounded for all $k \geq 0$, provided that the initial state $x_0 = x(0)$ and r satisfy:

$$\tilde{x}_0 = \tilde{x}(0) := (x_0 - x_{\rm e}) \in \boldsymbol{X}_\delta, \quad |H \, r| \leq \delta u_{\max}. \tag{6.29}$$

Moreover, the system output y tracks asymptotically the step command input of amplitude r.

Proof. The closed-loop system compromising the given plant (6.1)–(6.3) and the CNF control law (6.25) is given by

$$\xi(k+1) = f(\xi(k), y(k)) \tag{6.30}$$

$$x(k+1) = Ax(k) + B\text{sat}(Fx(k) + Gr + \rho(r,y)B'P(A+BF)(x(k)-x_{\text{e}})) \tag{6.31}$$

$$y(k) = Cx(k). \tag{6.32}$$

Let $\tilde{x}(k) = x(k) - x_{\text{e}}$. The closed-loop system (6.1)-(6.3) can be expressed as

$$\xi(k+1) = f(\xi(k), r + C\tilde{x}(k)) \tag{6.33}$$

$$\tilde{x}(k+1) = (A+BF)\tilde{x}(k) + Bw, \tag{6.34}$$

where

$$w = \text{sat}(F\tilde{x}(k) + Gr + \rho(r,y)B'P(A+BF)(x(k)-x_{\text{e}})) - F\tilde{x}(k) - Hr. \tag{6.35}$$

Define a Lyapunov function $V(\tilde{x}) = \tilde{x}'P\tilde{x}$, then one has

$$\lambda_{\min}(P)||\tilde{x}||^2 \le V(\tilde{x}) \le \lambda_{\max}(P)||\tilde{x}||^2 \tag{6.36}$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of P, respectively. Then,

$$\triangle V(\tilde{x}(k)) = V(\tilde{x}(k+1)) - V(\tilde{x}(k)) = -\tilde{x}'(k)W\tilde{x}(k) + 2\tilde{x}'(k)(A+BF)'PBw(k) + w'(k)B'PBw(k). \tag{6.37}$$

It has been shown in [75] that,

$$2\tilde{x}'(k)(A+BF)'PBw(k) + w'(k)B'PBw(k) \le 0, \tag{6.38}$$

for all $\tilde{x} \in \boldsymbol{X}_\delta$, $|Hr| \le \delta u_{\max}$ and $-\rho^* \le \rho(r,y) \le 0$. Thus

$$\triangle V(\tilde{x}(k)) = V(\tilde{x}(k+1)) - V(\tilde{x}(k)) \le -\tilde{x}'(k)W\tilde{x}(k) \le 0, \tag{6.39}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (6.34). Thus the solution of (6.34) exists and is bounded for all $k \ge 0$ and $\tilde{x}_0 \in \boldsymbol{X}_\delta$. Nothing that $x(k) = x_{\text{e}} + \tilde{x}(k)$, $x(k)$ exists and is bounded for all $k \ge 0$ and x_0 satisfies (6.29).

To show the existence and boundedness of the solution ξ of (6.30), it suffices to show that $||\tilde{y}(k)|| = ||y(k) - Cx_e|| = ||C\tilde{x}(k)|| \leq \beta \cdot a^k$. To this end, by recalling a lemma from [67], page 447, Lemma 13.2, one has $0 < \lambda_{\min}(W) \leq \lambda_{\max}(P)$ and $V(\tilde{x}(k+1)) \leq \varrho \cdot V(\tilde{x}(k))$ where $\varrho = 1 - \frac{\lambda_{\min}(W)}{\lambda_{\max}(P)}$.

Therefore, one gets

$$V(\tilde{x}(k+1)) \leq \varrho^k \cdot V(\tilde{x}(0)) \tag{6.40}$$

and then

$$\lambda_{\min}(P)||\tilde{x}(k+1)||^2 \leq \varrho^k \cdot V(\tilde{x}(0)) \tag{6.41}$$

so that

$$||\tilde{x}(k+1)|| \leq \left(\frac{V(0)}{\lambda_{\min}(P)}\right)^{1/2} \cdot (\sqrt{\varrho})^k \leq \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} \cdot ||\tilde{x}(0)|| \cdot (\sqrt{\varrho})^k, \quad \forall k \geq 0. \tag{6.42}$$

Finally, note that $a = \sqrt{\varrho}$,

$$||\tilde{y}(k+1)|| = ||C\tilde{x}(k+1)|| \leq \|C\| \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} ||\tilde{x}(0)||a^k \leq \beta \cdot a^k, \tag{6.43}$$

for all $\tilde{x}(0) \in \boldsymbol{X}_\delta$. By Lemma 6.1, the solution of (6.30) exists and is bounded for all $k \geq 0$ and $x_0 \in \Omega_\gamma$.

Moreover, noting that $W > 0$, all trajectories of (6.34) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial state x_0 and the step command input r that satisfy (6.29), one has

$$\lim_{k\to\infty} x(k) = x_{\mathrm{e}}, \tag{6.44}$$

which implies

$$\lim_{k\to\infty} u(k) = F \lim_{k\to\infty} x(k) + Gr + \lim_{k\to\infty} \rho B' P[x(k) - x_{\mathrm{e}}] = F x_{\mathrm{e}} + Gr. \tag{6.45}$$

Hence,

$$\lim_{k\to\infty} y(k) = C \lim_{k\to\infty} x(k) = Cx_{\mathrm{e}} = C(I - A - BF)^{-1} BGr = G^{-1}Gr = r. \tag{6.46}$$

This completes the proof of Theorem 6.1. □

Next, note that the key component in designing the CNF controllers is the selection of ρ and W. The freedom in choosing the nonlinear function ρ is used to tune the control laws so as to improve the performance of the closed-loop system as the controlled output y approaches the target reference. Since the main purpose of adding the nonlinear part to the CNF controller is to speed up the settling time and to reduce the overshoot, or equivalently to contribute a significant value to the control input when the tracking error, $r - y$, is small, it is appropriate for one to select a nonlinear gain matrix such that the nonlinear part will be in action when the control signal is far away from its saturation level, and thus it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising the linear part of the plant, i.e., (6.2), and the CNF control law (6.25) can be expressed as

$$\tilde{x}(k+1) = (A + BF)\tilde{x}(k) + \rho BB'P(A + BF)\tilde{x}(k). \tag{6.47}$$

It is clear that eigenvalues of the closed-loop system in (6.47) can be changed by the nonlinear function ρ. Assuming that $y(0) \neq r$ (for the trivial case when $y = r$, there is no need to add any nonlinear gain to the control), let us propose the following nonlinear gain

$$\rho(r, y) = -\beta(B'PB)^{-1}\frac{2}{\pi}\arctan\Big(\alpha\Big||y(k) - r| - |y(0) - r|\Big|\Big) \tag{6.48}$$

with $0 \le \beta \le 2$.

Obviously ρ starts from 0 and gradually decreases to a constant

$$-2\beta(B'PB)^{-1}\arctan(\alpha\,|y(0) - r|)/\pi > -\beta(B'PB)^{-1}$$

as y approaches to the target reference r. The parameter α is used to determine the speed of change in ρ.

It can be shown that the closed-loop poles of (6.47) are related to the invariant zeros of an auxiliary system characterized by

$$G_{\text{aux}}(z) := C_{\text{aux}}(zI - A_{\text{aux}})^{-1}B_{\text{aux}} := B'P(zI - A - BF)^{-1}B, \tag{6.49}$$

which is obviously stable, and which was shown in [34] to be a square, invertible and uniform rank system with one infinite zero of order 1 and with $n - 1$ stable invariant zeros. In fact, if one selects $\beta = 1$, the closed-loop poles of (5.34) in the steady state when $y = r$ are precisely given by the invariant zeros of $G_{\text{aux}}(z)$ together with additional one at $z = 0$. Generally, the invariant zeros of $G_{\text{aux}}(z)$ can be pre-assigned by the appropriate choice of W, which can also be selected using a trial and error approach by limiting it to be in a diagonal matrix and adjusting its diagonal weights through simulation. I would like to refer interested readers to [34] for detail.

6.4 Design Examples

Example 6.1. To illustrate the effectiveness of the developed design methodology, consider a simple discrete-time system characterized by

$$\xi(k+1) = 0.99\xi(k) + \xi(k)\tilde{y}(k) \tag{6.50}$$

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.5 \\ 0.125 \end{bmatrix} \text{sat}\ (u(k)) \tag{6.51}$$

$$y(k) = [0 \quad 1]x(k) \quad \text{or} \tag{6.52}$$

$$\tilde{y}(k) = r - y(k). \tag{6.53}$$

with $u_{\max} = 1$.

The aim is to design appropriate CNF controller with full state feedback, which would control the controlled output of the system to track the command reference as fast as possible and as smooth as possible while at the same time the zero dynamics keeps bounded and stable. It is easy to see that when $y(k) = r$ or equivalently $\tilde{y}(k) = 0$, (6.50) becomes $\xi(k+1) = 0.99\xi(k)$ which is asymptotically stable. Also it can be easily verified that the linear part is controllable and right invertible and has no zero at $z = 1$. Therefore, conditions **A1** to **A3** are met. Similarly, one can easily choose $\Omega_{\bar{\gamma}}$ for (6.50), say $\Omega_{\bar{\gamma}} = \{\xi : |\xi| \leq 100\}$ for the following design. By following the procedures given in the previous section and with appropriate selections of design parameters, I have obtained the following CNF control law. Please note that the linear part of the control law is carried out using the standard LQR design.

CNF controller design for $r = 1$:

$$u = Fx + Gr + \rho(r, y)F_{\rm n}(x - x_{\rm e}), \tag{6.54}$$

where

$$F = [\,-1.18614066163451 \quad -0.70346483459137\,],$$

$$G = 0.70346483459137,$$

$$F_{\rm n} = [\,0.81213814213564 \quad 0.46331007325138\,],$$

$$x_{\rm e} = [\,0 \quad 1\,]'.$$

The nonlinear function $\rho(r, y)$ is chosen as in (6.48) with $\alpha = 1$ and $\beta = 1$.

Using SIMULINK in MATLAB, one obtains the simulation result in Figure (6.1), which is done under the following initial condition

$$x_0 = x(0) = [\,0 \quad 0\,]' \quad \text{and} \quad \xi_0 = \xi(0) = 1. \tag{6.55}$$

The result clearly shows that the control laws with the nonlinear components, *i.e.*, the CNF controller, outperform its conventional counterpart a great deal.

Also, the zero dynamics is indeed bounded, see Figure (6.2).

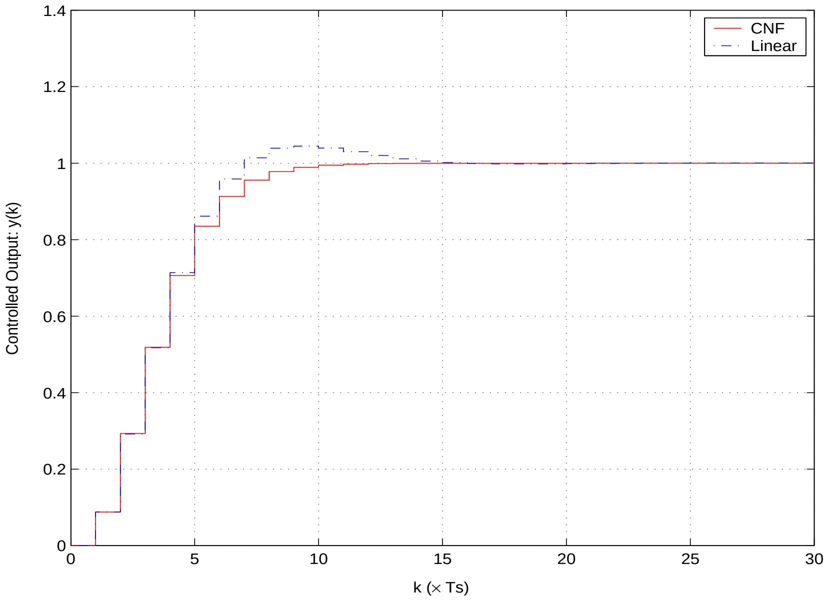
Example 6.2. Next, let us consider a system characterized by

$$\xi(k+1) = 0.9\xi(k) + 0.1\xi^2(k)y(k) \tag{6.56}$$

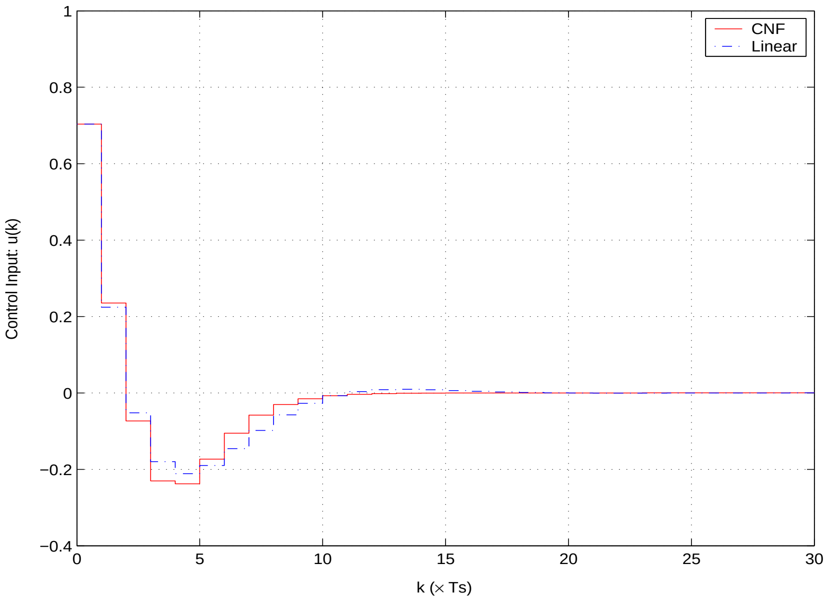
$$x(k+1) = \begin{bmatrix} 1 & 0.1 & 0 & 0 & 0 \\ -0.1 & 1 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0.1 & 0 \\ 0 & 0 & -0.1 & 1 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix} \text{sat}(u) \tag{6.57}$$

$$y(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(k) \tag{6.58}$$

with $u_{\max} = 0.2$. This model is obtained by discretizing the continuous-time model of the example in [55] via Euler's method with sampling period $T = 0.1$. I will consider a tracking problem of the system (6.56)-(6.58) with constant reference $r = 0.16$. The aim

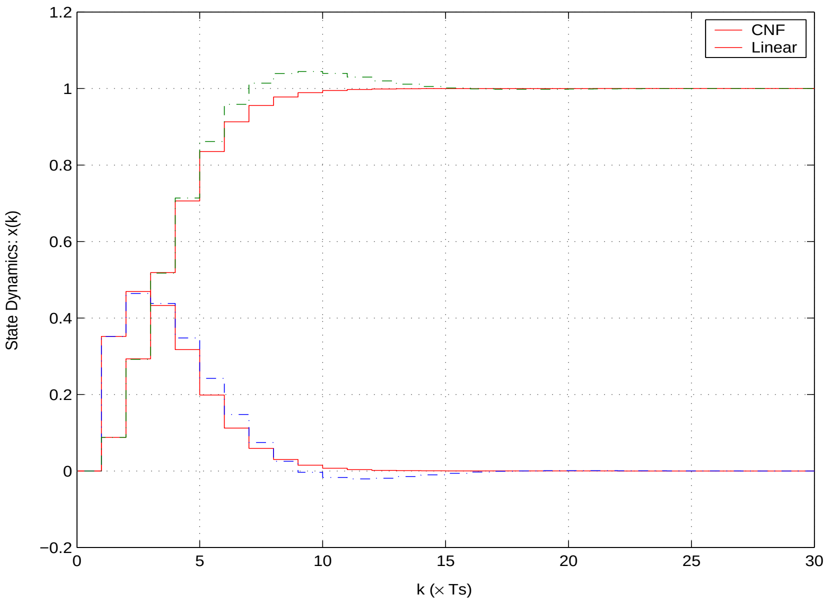


(a) Controlled output: $y(k)$

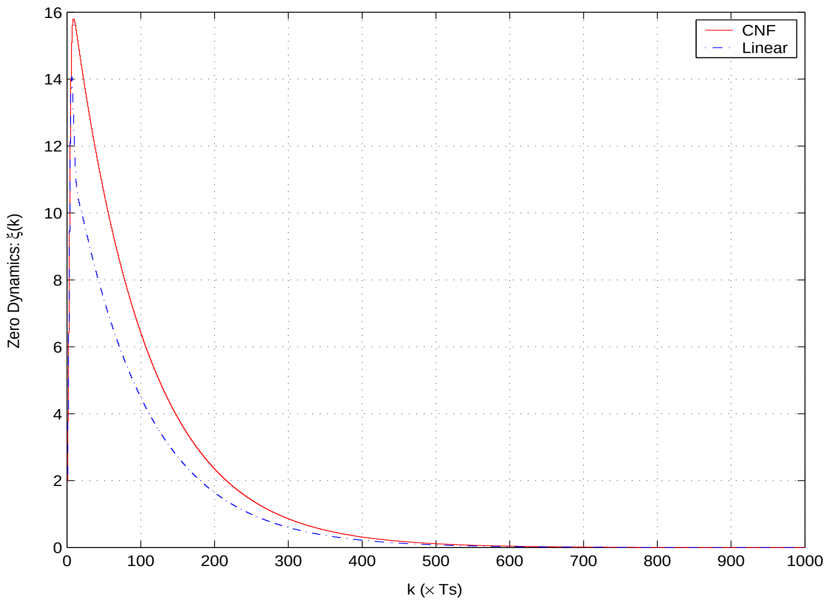


(b) Control input: $u(k)$

Figure 6.1: Output and input signals: $r = 1$.



(a) State and Zero dynamics: $x(k)$



(b) Zero dynamics: $\xi(k)$

Figure 6.2: State and Zero dynamics: $\xi(k)$.

is to design an appropriate CNF controller with state feedback to improve the transient performance of the closed-loop system. It is not difficult to verify that Assumptions A1-A3 are satisfied for the system (6.56)-(6.58). A linear feedback control law is firstly designed using the low gain feedback technique [52]. Thus one obtains a linear control law $u_{\rm L}(k) = Fx(k) + Gr$ with

$$F = [\,0.7851 \quad 0.1370 \quad -0.0432 \quad -4.1191 \quad -5.7906\,], \quad G = 5.0487.$$

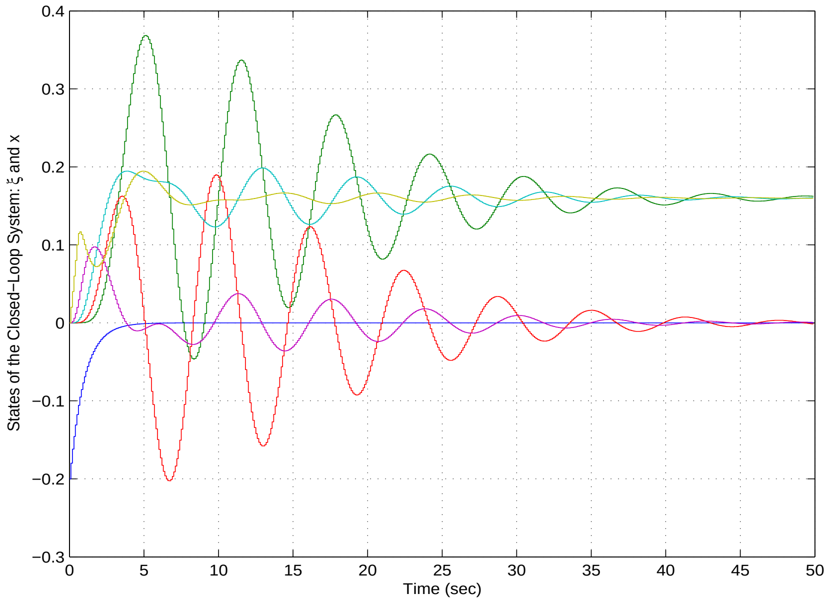
Next, let us select $W = I_5$ and solve the following discrete-time Lyapunov equation (6.23), which gives a solution

$$P = \begin{bmatrix} 124.60 & -10.99 & -105.66 & -167.56 & -28.33 \\ -10.99 & 120.46 & 163.87 & 33.12 & 0.98 \\ -105.66 & 163.87 & 349.33 & 185.78 & 23.82 \\ -167.56 & 33.11 & 185.78 & 300.34 & 50.24 \\ -28.33 & 0.98 & 23.82 & 50.24 & 10.01 \end{bmatrix} > 0.$$

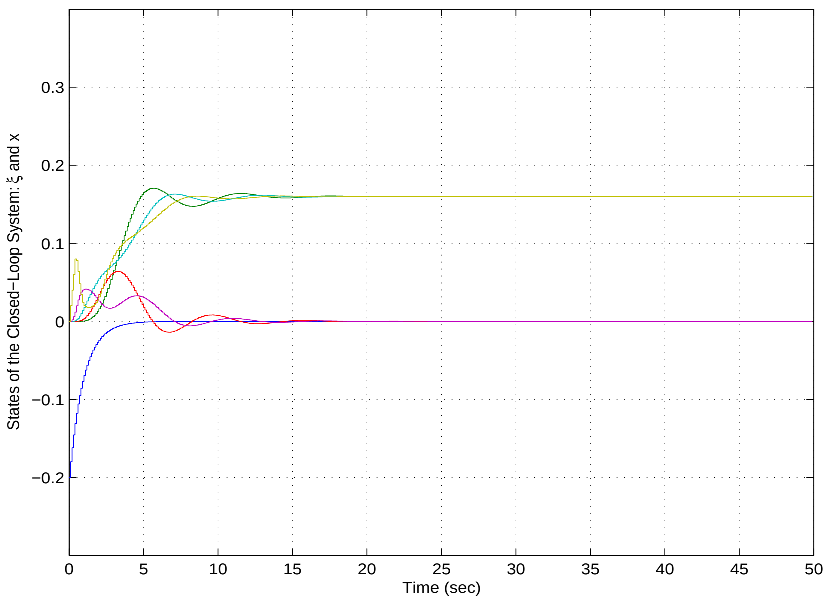
The nonlinear function $\rho(r, y)$ is chosen as in (6.48) with $\alpha = 6$ and $\beta = 1$. Finally, the CNF control law is given by

$$u(k) = Fx(k) + Gr + \rho(r, y)B'P(A + BF)(x(k) - x_{\rm e}). \tag{6.59}$$

where $x_{\rm e} = (I - A - BF)^{-1}BG\,r$. Using SIMULINK in MATLAB, one obtains the simulation result in Figure 6.3 and 6.4, which is done under the following initial condition $x(0) = 0$ and $\xi(0) = -0.2$. The simulation result shows that the control law with the nonlinear components, *i.e.*, the CNF controller, improved the transient performance significantly. Specifically, Figure 6.3.(a) and 6.3.(b) show the trajectories of the closed-loop systems under the linear control law and the CNF control law respectively. All the states of the closed-loop system under the CNF control law convergence to the steady state quickly in 15 seconds with much smaller amplitude. However, under the linear control law, more than 45 seconds are required for all the trajectories convergence to the steady state. Figure 6.4.(c) and 6.4.(d) compare the system outputs of the closed-loop systems and the control inputs under the linear control and the CNF control respectively. The overshoot under the linear control is 21.58%, but for the CNF control, it is only 0.45%.

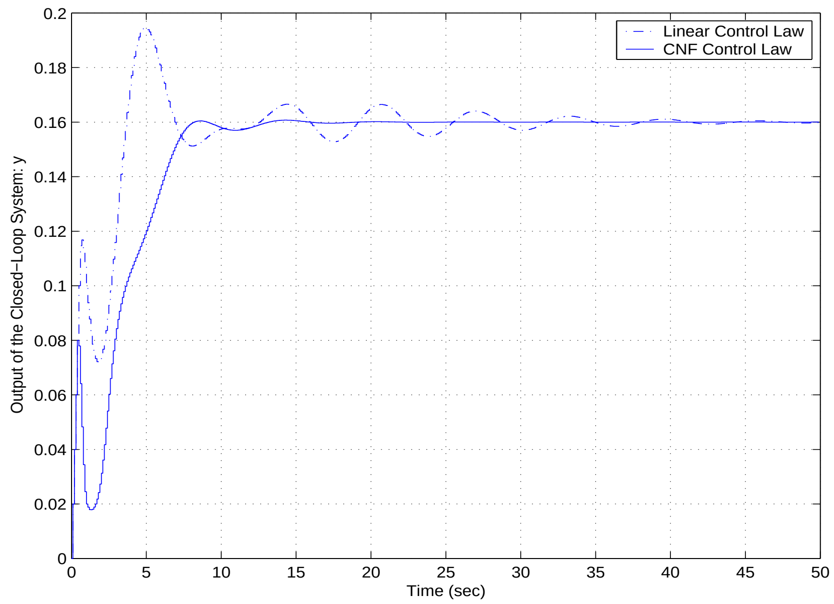


(a) State responses with the linear control law.

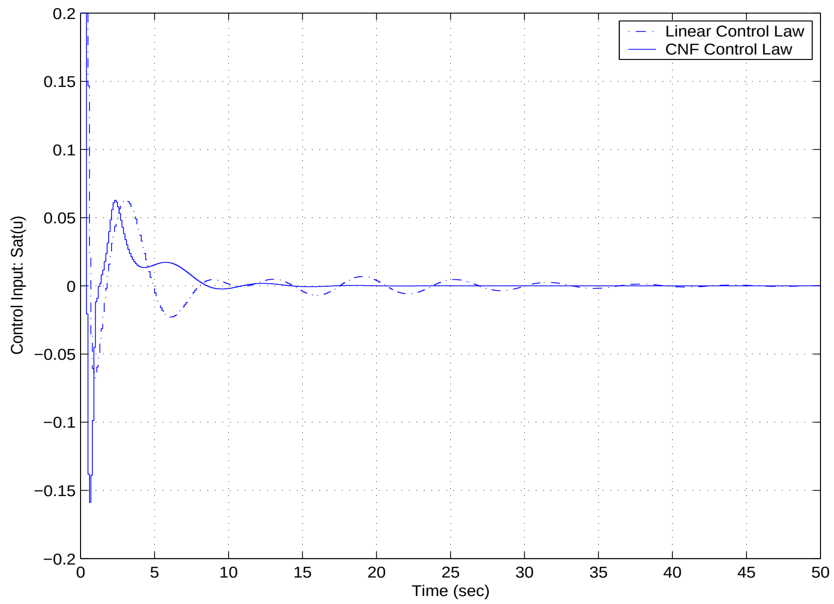


(b) State responses with the CNF control law.

Figure 6.3: (State responses of the closed-loop system.



(c) System output of the closed-loop system.



(d) Control input of the closed-loop system.

Figure 6.4: Output and input of the closed-loop system.

6.5 Conclusion

I have extended the so-called CNF control techniques for linear input-saturated discrete-time systems to a class of SISO partially linear composite discrete-time systems with actuator saturation. The closed-loop system is able to track step function signals yet the whole system is stable. It has been shown that the transient performance is improved comparing to normal linear approaches. Both CNF and linear controllers avoid adverse effect of peaking-phenomenon. Further extension to MIMO case can be established similarly by provoking the results of CNF control for linear MIMO discrete-time systems (see [34]).

Chapter 7

Asymptotic Time Optimal Tracking of a Class of Linear Systems with Input Saturation

This chapter proposes the so-called asymptotic time-optimal tracking (ATOT) problem. Typically one deals with "point-to-point" tracking, while in practice one usually needs asymptotic tracking, or "point-to-region" tracking. As a matter of fact, the ATOT problem was posed as an open problem in the book [18] about Hard Disk Drive servo control. A simplified model for typical hard disk drives can be a double integrator and the authors of [18] found that when using the CNF control the model shows faster tracking performance than time-optimal control. I will rigorously define this ATOT problem and propose a formula giving the optimal-settling time for this problem. Ideal controller design as well as practical controller design will be explored. The interesting part lies in that the CNF control technique can be used to approximate the optimal settling time and it will be demonstrated by an illustrative example.

7.1 Introduction and Problem Statement

It is well known that the actuator saturation in a hard disk drive has seriously limited the performance of its overall servo system, see Chen *et al.* [18, 19]. Traditionally, the most popular nonlinear control technique used in the design of servo systems, especially the hard disk drive servo systems, is the so-called proximate time-optimal servomechanism (PTOS) proposed by Workman [78], which achieves near time-optimal performance for a large class of motion control systems characterized by a double integrator, *e.g.*, hard disk drives and spring-mass mechanical systems. The PTOS was actually modified from the well-known time-optimal control or bang-bang control. However, it is made to yield a minimum variance with smooth switching from the track seeking to track following modes via a mode switching controller. It was shown in Workman [78] that by properly adjusting the controller parameters, the settling time for tracking a step reference in the resulting servo system with the PTOS controller can be made as close as possible to the optimal time achieved by the bang-bang control.

Note that the time-optimal control or bang-bang control indeed yields the best performance in point-to-point tracking, although such a technique cannot be used in practical situations. It is well known that the resulting system is very sensitive to the uncertainties and noises. Moreover, it is generally not necessary to have a precise point to point tracking in practical situations. Instead, it would be more preferable to consider asymptotic tracking in which the tracking target is defined as a small neighborhood of a given set point. I believe that such a consideration is very practical. For example, in a hard disk drive servo system (see *e.g.*, [18, 19]), it is a common practice to activate its read/write head to read or write data once it enters $\pm 5\%$ of the data track-width of the target set point.

Interestingly, it has been recently demonstrated by an example in [18, 19] that the time-optimal control or bang-bang control, and consequently the PTOS, do not necessarily yield the best performance in asymptotic tracking situations. There are control laws that would yield a better performance than that of the time-optimal control. This is actually the motivation for the work of this paper. Our goals or contributions are two-fold: 1) to derive the optimal settling for asymptotic tracking; and 2) to find a control law that achieves this optimal performance.

To be more specific, let us consider a class of second order linear systems Σ characterized by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ a \end{bmatrix} \text{sat}(u), \quad y = [\,1 \quad 0\,]\, x, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x(0) = x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}, \tag{7.1}$$

where x is the state, y is the measurement output, a is a constant and $\text{sat}(u)$ is control input to the system with

$$\text{sat}(u) = \text{sign}(u) \times \min\{u_{\max}, |u|\}. \tag{7.2}$$

As pointed out earlier, there are a large class of real life problems, such as hard disk drives and spring-mass mechanical systems, can be approximately modeled as a double-integrator system characterized by (7.1). The problem to be considered and solved in this chapter is the following:

Definition 7.1. Consider the system of (7.1) with actuator nonlinearities. Let r be a reference target and δ be a positive scalar and $\delta \in [0, 1]$. Let

$$u = \phi(y, r, \delta) \tag{7.3}$$

be an internally stabilizing controller for the system, i.e., the closed-loop system comprising of the given system Σ of (7.1) and the control law of (7.3) is asymptotically stable. Let $t_s(x_0, r, \delta, \phi)$ be the corresponding settling time for the resulting system output $y(t, \phi)$ to enter the δ-neighborhood of the target reference, i.e, $t_s(x_0, r, \delta, \phi)$ is the smallest scalar such that for all $t \geq t_s(x_0, r, \delta, \phi)$,

$$|y(t, \phi) - r| \leq \delta \cdot |r| \quad \text{and} \quad \lim_{t\to\infty} y(t, \phi) = r. \tag{7.4}$$

Finally, let $t_s^*(x_0, r, \delta)$ be the optimal settling time over all the internally stabilizing controllers, i.e.,

$$t_s^*(x_0, r, \delta) := \inf\left\{ t_s(x_0, r, \delta, \phi) \;\middle|\; \phi(y, r, \delta) \text{ internally stabilizes } \Sigma \right\}. \tag{7.5}$$

The asymptotic time-optimal tracking (ATOT) control problem is to find a stabilizing measurement feedback control law $\phi^*(y, r, \delta)$ such that $t_s(x_0, r, \delta, \phi^*) = t_s^*(x_0, r, \delta)$.

The detailed derivations for the optimal asymptotic tracking performance t_s^* and the optimal controller ϕ^* are given respectively in Sections 7.2 and 7.3.

7.2 Optimal Settling Time

I will derive in this section the optimal settling time $t_s^*(x_0, r, \delta)$ for the asymptotic time-optimal tracking problem defined in Definition 7.1. The focus will be on the case when the target reference r is a step function, *i.e.*, r is a constant. First, note that x_1 in (7.1) usually represents the displacement of its corresponding physical system, while x_2 represents its velocity. For simplicity of presentation, assume that the initial velocity of the system is zero, i.e., $x_{20} = 0$. Without loss of generality, one can also assume that the initial displacement is zero $x_{10} = 0$. If $x_{10} \neq 0$, thus one can re-define a new target reference $r_{\text{new}} = r - x_{10}$. Nevertheless, the problem of tracking r with nonzero initial condition is not equivalent to that of tracking r_{new} with zero initial condition. I will deal with this case and other more general cases in the remarks following Theorem 7.1. Similarly, for simplicity, let us assume $a = 1$ and $u_{\max} = 1$ in (7.1). This can be done by a proper scaling on u and r. The first main result follows.

Theorem 7.1. *Consider the given system Σ of (7.1) with $a = 1$, $u_{\max} = 1$ and $x_0 = 0$. Given a step target reference r (for simplicity, assume $r \geq 0$) and a positive scalar $\delta \in [0, 1]$, the optimal settling time for Σ under all possible stabilizing control laws (see, e.g., Definition 7.1) is given by:*

$$t_s^*(r, \delta) = \begin{cases} 2(\sqrt{r(1+\delta)} - \sqrt{r\delta}), & 0 \leq \delta < \frac{1}{3}, \\ \sqrt{2r(1-\delta)}, & \frac{1}{3} \leq \delta \leq 1. \end{cases} \tag{7.6}$$

Note that x_0 is dropped from the above expression as x_0 is assumed to be zero.

Proof. Since the system is a double integrator system, if one figures out x_2 versus time t (see figure (7.1)), then the ouput $y = x_1 = \int_0^{T_t} x_2(\tau) d\tau$, where $T_t \geq 0$ is the desired time instant, is simply the net area (with $\pm$ signs) enclosed by $t = 0$, $t = T_t$, $x_2(t)$ and the time axis $x_2 = 0$.

Let us construct $\triangle OAB$ as shown in the figure (7.2) where $OA = AB$ and the slope of OA is equal to $\max(u) = u_{\max} = +1$ while the slope of AB is equal to $\min(u) = u_{\min} = -u_{\max} = -1$.

For the case of $\frac{1}{3} \leq \delta \leq 1$, one first applies $u(t) = u_{\max} = +1$ from $t = 0$ to $t = t_A = 2\sqrt{\frac{1}{3}r}$ and then apply $u(t) = u_{\min} = -u_{\max} = -1$ till $t = t_B = 4\sqrt{\frac{1}{3}r}$, as shown in

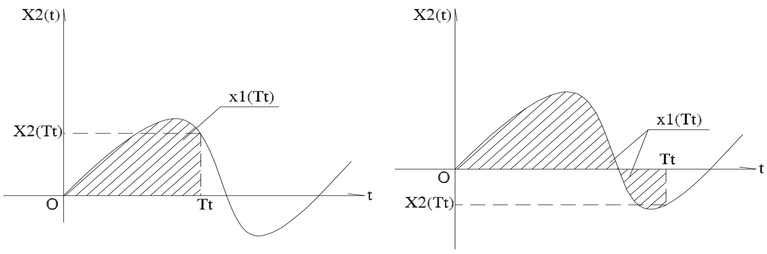


Figure 7.1: Plot of $x_2(t)$ versus t

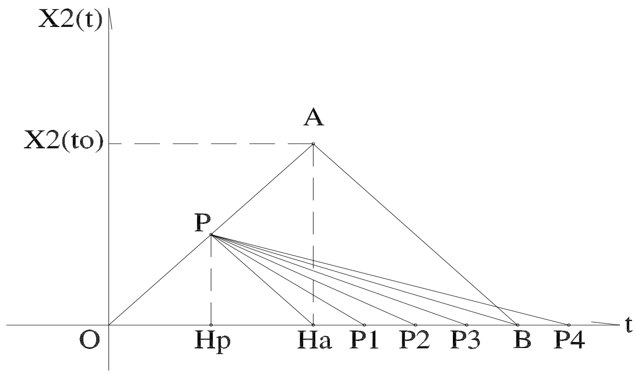


Figure 7.2: Case 1: $1/3 \leq \delta \leq 1$

figure (7.2). Since $u_{\max} = 1$, $x_1 = \int_0^t au(\tau)d\tau \leq \int_0^t au_{\max}(\tau)d\tau = \int_0^t ad\tau = \frac{1}{2}at^2 = \frac{1}{2}t^2$, or $t \geq \sqrt{2x_1}$ for $x_1 \geq 0$, t_s^* is the time at which x_1 arrives at $(1-\delta)r$ along OA, which is $\sqrt{2r(1-\delta)}$. At $t = t_B = 4\sqrt{\frac{1}{3}r}$, the output $x_1 = \frac{4}{3}r \leq (1+\delta)r$ as $\frac{1}{3} \leq \delta \leq 1$, so the ouput is within the region of $[(1-\delta)r,\ (1+\delta)r]$. After that, if one removes any control, $x_2 = 0$ and x_1 keeps unchanged, *i.e.*, the ouput is always within the region of $[(1-\delta)r,\ (1+\delta)r]$. This justifies the calculation of t_s^* for the case of $\frac{1}{3} \leq \delta \leq 1$.

For the case of $0 \leq \delta < \frac{1}{3}$, one first applies $u(t) = +1$ from $t = 0$ to $t = t_A = \sqrt{(1+\delta)r}$ where the time coordinate t_A correponds to A, and then apply $u(t) = -1$ till $t = t_B = 2\sqrt{(1+\delta)r}$ where, again, the time coordinate t_B correponds to B, as shown in figure (7.3). In this case, t_s^* is the time at which x_1 arrives at $(1-\delta)r$ along OAB, which is, after some simple calculations, exactly $2(\sqrt{(1+\delta)r} - \sqrt{\delta r})$. One must prove that there exists no shorter settling time. First I claim that $t_s^* > t_A$. $x_1 = \int_0^{t_A} au(\tau)d\tau \leq \int_0^{t_A} au_{\max}(\tau)d\tau = \int_0^{t_A} ad\tau = \frac{1}{2}at_A^2 = \frac{1}{2}t_A^2 = \frac{1}{2}(1+\delta)r < (1-\delta)r$ for $0 \leq \delta < \frac{1}{3}$. Therefore, at t_A, x_1 will not arrive at $(1-\delta)r$ and hence $t_s^* > t_A$.

Suppose there is another settling time t_s' which satisfies $t_s' < t_s^*$, then, if let us indicate the point corresponding to t_s as P, there are only three possible cases for the location of the point corresponding to t_s', namely P_a, P_o or P_b, see figure (7.3), where H_a, H_p' and H_p are projection points corresponding to A, P_o (or P_a and P_b) and P respectively. Now one must prove that all these cases are impossible. To this sequel, I will first introduce a proposition. This proposition shows that the trajectories leaving or enetring some point $x_2(t_0)$ can only take the slope between $-a$ and $+a$, which complies with $\frac{d}{dt}x_2(t) = au(t)$.

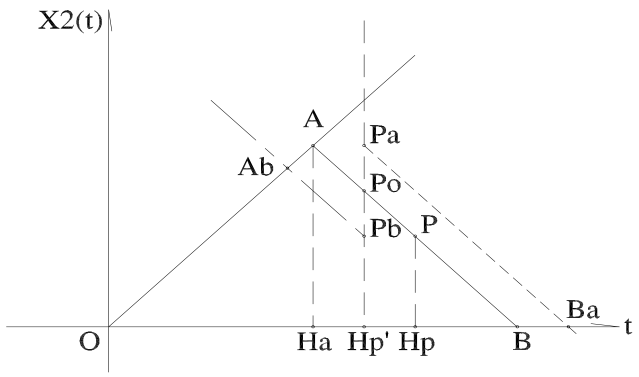


Figure 7.3: Case 2: $0 \leq \delta < 1/3$

Proposition Suppose $x_2(t_0)$ is located at some point A, then the trajectories leaving $(t > t_0)$ or entering $(t > t_0)$ A will be confined to the slanted shade area shown in the figure (7.4).

Proof of the Proposition First assume $t > t_0$. $x_2(t) = \int_{t_0}^t au(\tau)d\tau$, but $-1 \leq u(\tau) \leq +1$, which means $\int_{t_0}^t -ad\tau \leq x_2(t) = \int_{t_0}^t au(\tau)d\tau \leq \int_{t_0}^t ad\tau$ or, $x_2(t_0) - a(t-t_0) \leq x_2(t) \leq x_2(t_0) + a(t-t_0)$. Hence the result for the trajectories leaving A. For the case of $t < t_0$, one has $\int_{t_0}^t ad\tau \leq x_2(t) = \int_{t_0}^t au(\tau)d\tau \leq \int_{t_0}^t -ad\tau$ or, $x_2(t_0) + a(t-t_0) \leq x_2(t) \leq x_2(t_0) - a(t-t_0)$. Hence the result for the trajectories entering A.

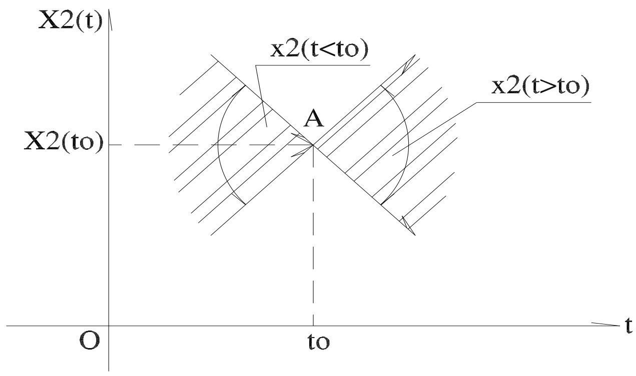


Figure 7.4: The Trajectories leaving or entering $x_2(t_0)$

Now let us go on with the proof of the theorem. Suppose that $x_2(t_s')$ stays at P_a, let us draw a line P_aB_a parallel to PB. According to the above proposition, trajectories leaving P_a will be on or above the line P_aB_a, which implies that the area of $\triangle H_p'P_aB_a$ is the infimum for all possible $x_2(t),\ t \geq t_s'$. Since at t_s', the area is already $1-\delta$, the area or the output x_1 will definitely exceed $1+\delta$ as the area of $\triangle H_p'P_aB_a$ is larger than that of $\triangle H_pPB$, which contradicts the definition of settling time, see Definition 7.1.

Suppose now that $x_2(t_s')$ stays at P_b, let us draw a line P_bA_b parallel to BP. Again, according to the above proposition, trajectories entering P_b will be on or below the line P_bA_b, which implies that the area of the polygon $OA_bP_bH_p'O$ is the supremum for all possible $x_2(t),\ 0 \leq t \leq t_s'$. Since at t_s, the area is already $(1-\delta)r$, one sees that the area of $OA_bP_bH_p'O$ or the output $x_1(t_s')$ will be smaller than $(1-\delta)r$, which, again, contradicts the definition of settling time in Definition 7.1.

For the last case that $x_2(t_s')$ stays at P_o, using the same argument as the case of $x_2(t_s')$ staying at P_b shown above, one can claim too, that there doesn't exist such a t_s' which satisfies $t_s' < t_s$. In summary, t_s^* is indeed the desired optimal settling time for the system.

Therefore, one has

$$t_s^* = \begin{cases} 2(\sqrt{r(1+\delta)} - \sqrt{r\delta}), & 0 \leq \delta < \frac{1}{3}, \\ \sqrt{2r(1-\delta)}, & \frac{1}{3} \leq \delta \leq 1. \end{cases} \tag{7.7}$$

This completes the proof of the theorem. □

In order to see clearly the relationships between t_s and δ, one can plot the figure. For example, for the case of $r = 1$, the relationship between t_s and δ is plotted in Figure (7.5):

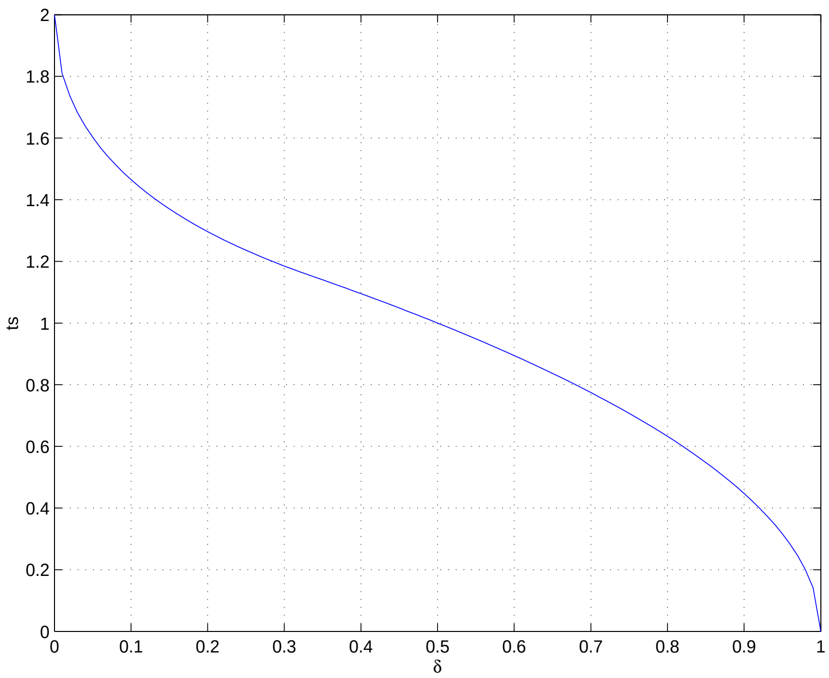


Figure 7.5: The Relationship between δ and t_s, r=1

Assuming $\delta = 0.01$, the corresponding optimal settling time is $t_s^* = 1.8100$, which will be used in the illustrative example in Section 7.4. Furthermore, assuming $\delta = \frac{1}{3}$, the corresponding optimal settling time is $t_s = 1.1547$, which is the joint point for the two different cases of δ.

Remarks

1. As shown in the proof for the case of $\frac{1}{3} \leq \delta \leq 1$, let the output stay at $x_1 = \frac{4}{3}r \in [(1-\delta)r,\ (1+\delta)r]$ while $x_2 = 0$. As a matter of fact, one can set it to be any $x_1 \in [2(1-\delta)r,\ (1+\delta)r]$, which can be realized by let $x_2(t)$ go along the lines PH_a, PP_1, PP_2, PP_3, PB, $PP_4 \cdots$ as shown in figure (7.2), corresponding to the decreasing amplitude of control input gradually. Obviously, one has infinitely many choices.

2. For the general case when $a > 0$, $r > 0$, $x_{20} = 0$, $x_{10} < (1-\delta)r$ where $\delta \geq 0$ is desired tracking bound, and $\max(u) = u_+ > 0$, $\min(u) = -u_- < 0$ where u_+ doesn't

necessarily equal u_-, by introducing new tracking area of $[(1-\delta)r - x_{10},\ (1+\delta)r - x_{10}]$ and hence artificially set a new zero initial condition for x_1, one has the following formula:

$$t_s^*(r,\delta,x_{10},u_+,u_-) = \begin{cases} \sqrt{\frac{2[(1+\delta)r-x_{10}]u_-}{au_+(u_++u_-)}} + \sqrt{\frac{2[(1+\delta)r-x_{10}]u_+}{au_-(u_++u_-)}} - 2\sqrt{\frac{r\delta}{au_-}}, \\ \qquad 0 \le \delta < \frac{u_+(r-x_{10})}{(u_++2u_-)r}; \\ \sqrt{\frac{2[(1-\delta)r-x_{10}]}{au_+}}, \quad \frac{u_+(r-x_{10})}{(u_++2u_-)r} \le \delta \le 1. \end{cases} \tag{7.8}$$

By applying max(u) first and then min(u), one obtains the desired control input.

3. For the case when $a > 0$, $r > 0$, $x_{20} = 0$, $(1-\delta)r \le x_{10} \le (1+\delta)r$ where $\delta \ge 0$ is desired tracking bound, obviously $t_s^* = 0$.

4. For the case when $a > 0$, $r > 0$, $x_{20} = 0$, $x_{10} > (1+\delta)r$ where $\delta \ge 0$ is desired tracking bound, the settling time shall be the infimum of the time instant at which the system output reaches $(1+\delta)r$. The formula for t_s^* can be revised as follows.

$$t_s^*(r,\delta,x_{10},u_-,u_+) = \begin{cases} \sqrt{\frac{-2[(1-\delta)r-x_{10}]u_+}{au_-(u_-+u_+)}} + \sqrt{\frac{-2[(1-\delta)r-x_{10}]u_-}{au_+(u_-+u_+)}} - 2\sqrt{\frac{r\delta}{au_+}}, \\ \qquad 0 \le \delta < \frac{-u_-(r-x_{10})}{(u_-+2u_+)r}; \\ \sqrt{\frac{-2[(1+\delta)r-x_{10}]}{au_-}}, \quad \frac{-u_-(r-x_{10})}{(u_-+2u_+)r} \le \delta \le 1. \end{cases} \tag{7.9}$$

By applying min(u) first and then max(u), one obtains the desired control input.

5. For the case when $a > 0$, $r < 0$, $x_{20} = 0$, $x_{10} > (1-\delta)r$ where $\delta \ge 0$, by introducing new tracking area of $[(1+\delta)r - x_{10},\ (1-\delta)r - x_{10}]$ and hence artificially set a new zero initial condition for x_1, apply the following formula (7.10) to get the optimal settling time.

$$t_s^*(r,\delta,x_{10},u_-,u_+) = \begin{cases} \sqrt{\frac{-2[(1+\delta)r-x_{10}]u_+}{au_-(u_-+u_+)}} + \sqrt{\frac{-2[(1+\delta)r-x_{10}]u_-}{au_+(u_-+u_+)}} - 2\sqrt{\frac{-r\delta}{au_+}}, \\ \qquad 0 \le \delta < \frac{u_-(r-x_{10})}{(u_-+2u_+)r}; \\ \sqrt{\frac{-2[(1-\delta)r-x_{10}]}{au_-}}, \quad \frac{u_-(r-x_{10})}{(u_-+2u_+)r} \le \delta \le 1. \end{cases} \tag{7.10}$$

By applying min(u) first and then max(u), one obtains the desired control input.

6. For the case when $a > 0$, $r < 0$, $x_{20} = 0$, $(1+\delta)r \le x_{10} \le (1-\delta)r$ where $\delta \ge 0$ is desired tracking bound, obviously $t_s^* = 0$.

7. For the case when $a > 0$, $r < 0$, $x_{20} = 0$, $x_{10} < (1+\delta)r$ where $\delta \ge 0$, apply the following formula (7.11) to get the optimal settling time. Again, the settling time shall

be the infimum of the time instant at which the system output reaches $(1+\delta)r$.

$$t_s^*(r,\delta,x_{10},u_-,u_+) = \begin{cases} \sqrt{\frac{2[(1-\delta)r-x_{10}]u_-}{au_+(u_++u_-)}} + \sqrt{\frac{2[(1-\delta)r-x_{10}]u_+}{au_-(u_++u_-)}} - 2\sqrt{\frac{-r\delta}{au_-}}, \\ \qquad 0 \le \delta < \frac{-u_+(r-x_{10})}{(u_++2u_-)r}; \\ \sqrt{\frac{2[(1+\delta)r-x_{10}]}{au_+}}, \qquad \frac{-u_+(r-x_{10})}{(u_++2u_-)r} \le \delta \le 1. \end{cases} \tag{7.11}$$

By applying $\max(u)$ first and then $\min(u)$, one obtains the desired control input.

8. For the case when $a < 0$, $r < 0$, $x_{20} = 0$, $x_{10} > (1-\delta)r$ where $\delta \ge 0$, one has the following formula (7.8). By applying $\max(u)$ first and then $\min(u)$, one obtains the desired control input.

9. For the case when $a < 0$, $r < 0$, $x_{20} = 0$, $(1+\delta)r \le x_{10} \le (1-\delta)r$ where $\delta \ge 0$ is desired tracking bound, obviously $t_s^* = 0$.

10. For the case when $a < 0$, $r < 0$, $x_{20} = 0$, $x_{10} < (1+\delta)r$ where $\delta \ge 0$, the settling time shall be the infimum of the time instant at which the system output reaches $(1+\delta)r$. The formula for t_s^* is exactly the same as formula (7.9). By applying $\min(u)$ first and then $\max(u)$, one obtains the desired control input.

11. For the case when $a < 0$, $r > 0$, $x_{20} = 0$, $x_{10} < (1-\delta)r$ where $\delta \ge 0$, apply formula (7.10) to get the optimal settling time. By applying $\min(u)$ first and then $\max(u)$, one obtains the desired control input.

12. For the case when $a < 0$, $r > 0$, $x_{20} = 0$, $(1-\delta)r \le x_{10} \le (1+\delta)r$ where $\delta \ge 0$ is desired tracking bound, obviously $t_s^* = 0$.

13. For the case when $a > 0$, $r > 0$, $x_{20} = 0$, $x_{10} > (1+\delta)r$ where $\delta \ge 0$, one applies formula (7.11) to get the optimal settling time. Again, the settling time shall be the infimum of the time instant at which the system output reaches $(1+\delta)r$. By applying $\max(u)$ first and then $\min(u)$, one obtains the desired control input.

14. So far one has given the formulae for all the possible cases when $x_{20} = 0$. When $x_{20} \ne 0$, things become more complicated as there are too many different combinations of conditions regarding a, x_{10}, r, and $\max(u) = u_+ > 0$, $\min(u) = -u_- < 0$. However, for each specified case, using almost the same reasoning as the proof of Theorem 7.1, one can obtain corresponding results accordingly.

7.3 Asymptotic Time-Optimal Tracking Controller Design

Now let us proceed to design a controller that would achieve the optimal settling time as given in Theorem 7.1.

I have already shown in the proof of Theorem 7.1 that by applying $u = +1$ from $t = 0$ to $t = t_A = \sqrt{(1+\delta)r}$ and then apply $u = -1$ till $t = t_B = 2\sqrt{(1+\delta)r}$ for the case of $0 \leq \delta \leq \frac{1}{3}$, one ends up with $x_1(t_B) = (1+\delta)r$ and $x_2(t_B) = 0$. For the case of $\frac{1}{3} < \delta \leq 1$, apply $u = +1$ from $t = 0$ to $t = t_A = 2\sqrt{\frac{1}{2}r}$ and then apply $u = -1$ till $t = t_B = 4\sqrt{\frac{1}{2}r}$ and end up with $x_1(t_B) = (1-\delta)r$ and $x_2(t_B) = 0$.

The next step to drive the system output to the target r is a trivial design problem. There are many available methods which can reach this goal, which further drives x_1 to r and x_2 to 0 asymptotically without making x_1 exceeding the tracking region of $[(1-\delta)r,\ (1+\delta)r]$. A simple choice is to use time-optimal control. It drives the system output to the target monotonically and hence will never exceed the tracking bound while at the same time x_2 reaches 0. One can use $u_{\max}$ and $u_{\min}$ for the time-optimal control design or even one can use smaller control signals, say $\alpha u_{\max}$ and $\alpha u_{\min}$ where $0 < \alpha < 1$, as saturation levels, which only makes the time to the target longer.

However, the above designed controller can not be used in practical situations as it is a non-robust controller, almost the same as time-optimal controller. One may appeal to other design methods although one may only obtain sub-optimal ATOT controllers. I will try the CNF control scheme as indeed in [18] the authors give an example with an (SISO) CNF controller. Along the same line, the following design procedure is adopted from Chen *et al.* [18, 19] which was developed based on Lin *et al.* [53] .

Rewrite (7.1) in the following form:

$$\begin{cases} \dot{x} &= Ax + B\text{sat}(u) \\ y &= Cx \end{cases} \tag{7.12}$$

where A, B, C are the corresponding matrices in (7.1).

The CNF control consists of linear part control and nonlinear part control. I will present the control algorithm step by step as following:

Step 1: Linear part control

$$u_L = Fx + Gr \tag{7.13}$$

where F and G are chosen such that (1) $(A + BF)$ is an asymptotically stable matrix, (2) The closed system $C(sI - A - BF)^{-1}B$ has certain properties, such as having a small damping ratio, (3) G is a scalar given by $G = -[C(A + BF)^{-1}B]^{-1}$ and r is the command input.

Step 2: Nonlinear part control

$$u_N = \rho B^T P(x - x_e) \tag{7.14}$$

where ρ is a nonpositive, Lipschitz continuous function and P is the solution of the following Lyapunov equation,

$$(A + BF)^T P + P(A + BF) = -W \tag{7.15}$$

W is a positive definite matrix, $x_e = -(A + BF)^{-1}BGr$ and $H := [1 - F(A + BF)^{-1}B]G$. For any $\delta \in (0, 1)$, let c_δ be the largest positive scalar satisfying the following conditions:

$$|Fx| \le (1 - \delta)\bar{u}, \forall x \in X_\delta := \{x'Px \le c_\delta\} \tag{7.16}$$

The following two conditions should be guaranteed in the CNF controller design.

$$\hat{x}_0 = x_0 - x_e \in X_\delta \tag{7.17}$$

$$|Hr| \le \delta\bar{u} \tag{7.18}$$

Step 3: Composite control

$$\begin{aligned} u &= \phi_{cnf}(y, r, \delta, \varepsilon) = u_L + u_N \\ &= Fx + Gr + \rho B^T P(x - x_e) \end{aligned} \tag{7.19}$$

The following theorem is adopted from Chen *et al.* [18].

Theorem 7.2. *The control law (7.19) is capable of driving the controlled output y, to track asymptotically a step command input r, provided that conditions (7.17) and (7.18) are satisfied.*

There are many choices for ρ, only if ρ is a non-positive function, locally Lipschitz. In Lin *et al.* ([53]), it gave some ideas on how to choose the nonlinear part for a second order SISO system, such that the damping ratio goes to infinity asymptotically. For this purpose, let us choose ρ in (7.19) as follows, which is a non-positive function, locally Lipschitz in y,

$$\rho = \varepsilon(e^{-r} - e^{-|r-y|}), \quad \varepsilon > 0 \tag{7.20}$$

The transient performance of this system can be improved dramatically: a faster rise time, a shorter settling time, with less overshoot, which is inherently the advantage for CNF control over the linear feedback control. Note that the above CNF controller (7.19) is parameterized by another additional tuning parameter ε, which is to be adjusted to achieve the optimal settling time. In Section (7.4), the simulation will show how this parameter affects the settling time. Figure (7.9) shows that there seems to be one point, where $\varepsilon = \varepsilon^*$, and $t_s = t_s^*$, although no rigorous proof can be given at the moment. Nevertheless, it is easy to tune only one parameter in order to approximate the optimal settling time by simulation.

In addition, I provide some guidelines to choose the parameters to achieve faster tracking,

1. Choose F such that the closed-loop system has small damping ratio and the conditions (7.17), (7.18) are satisfied.

2. First randomly choose an ε, if the overshoot is beyond the scope you expect, then choose a smaller one ε accordingly. If the output reaches the destination increasingly at infinity, choose a bigger one. However for the ε you have chosen, there should have overshoot in order to get a faster settling time. When the overshoot enters the tracking bounds, tune this parameter ε gradually and slightly around this value.

Since one dynamic term has been added in the control signal, the system will move the eigenvalues away from the imaginary axis, thanks to the nonlinear part, which will enhance the robustness of the system. And the only part one need to change is the coefficient term ε in ρ after one chooses the feedback gain F.

7.4 Simulations

I now illustrate the results of previous section in the following example. I will use the model in (7.1) with $a = 1$, $\delta = 0.01$ and $r = 1$. I will also compare the results with those of time optimal control.

The parameters chosen are:

$$W = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, F = [-50 \quad -10], \quad \varepsilon = 133.5 \tag{7.21}$$

Figure (7.6) gives the controlled output y under the TOC (dot-dash line) and ATOT (solid line) approaches. The settling time under ATOT is $t_s = 1.8110$, which is very close

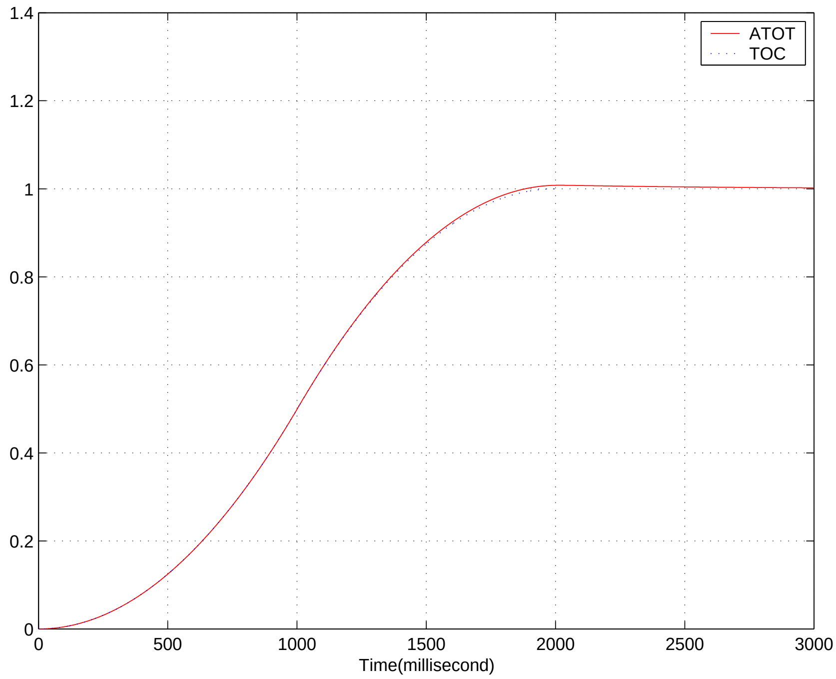


Figure 7.6: Controlled output for the whole process

to the optimal value $t_s = 1.8100$. While the settling time with TOC is 1.8586. One can see there exists much difference.

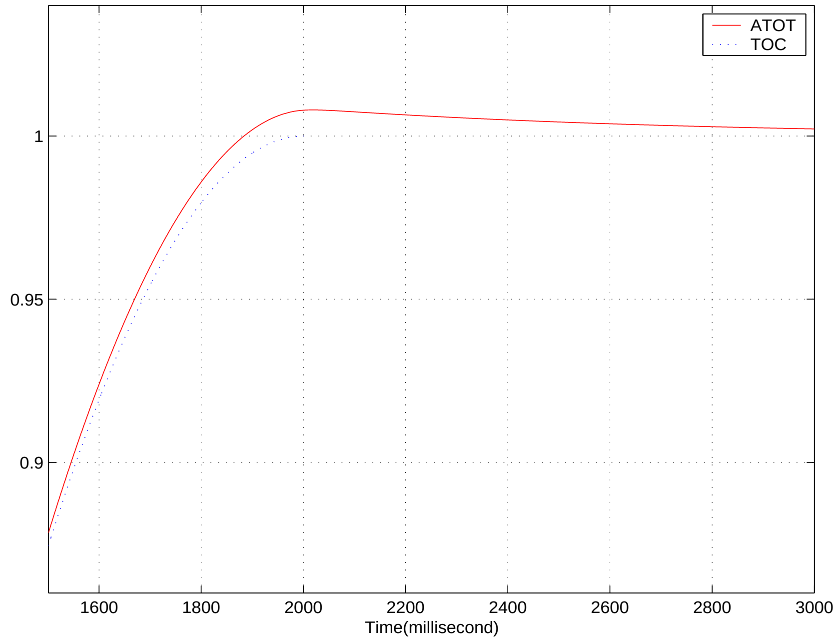


Figure 7.7: Controlled output for a selected period

In Figure (7.7), one sees that the ATOT is faster that the TOC under the same definition of settling time. Although the time one can spare is very short, this little improvement will be very useful in some actual physical systems, such as the hard disk drive servo systems. Furthermore, the controller of ATOT is robust and is able to reject noise as well. It shows the advantage over the TOC.

Figure (7.8) gives the controlled signal, which is continuous and will decay when the output converges to the desired position. Both linear part and nonlinear part contribute different weight to the CNF control law at different stages of the control.

Moreover, let us present a figure in (7.9) about the relationship between different values of ε and settling time. It gives one some clues on how to choose appropriate ε for practical use.

7.5 Conclusion

In this chapter, I proposed and defined the ATOT problem, and presented the formula of the optimal settling time under ATOT control. The composite nonlinear feedback control

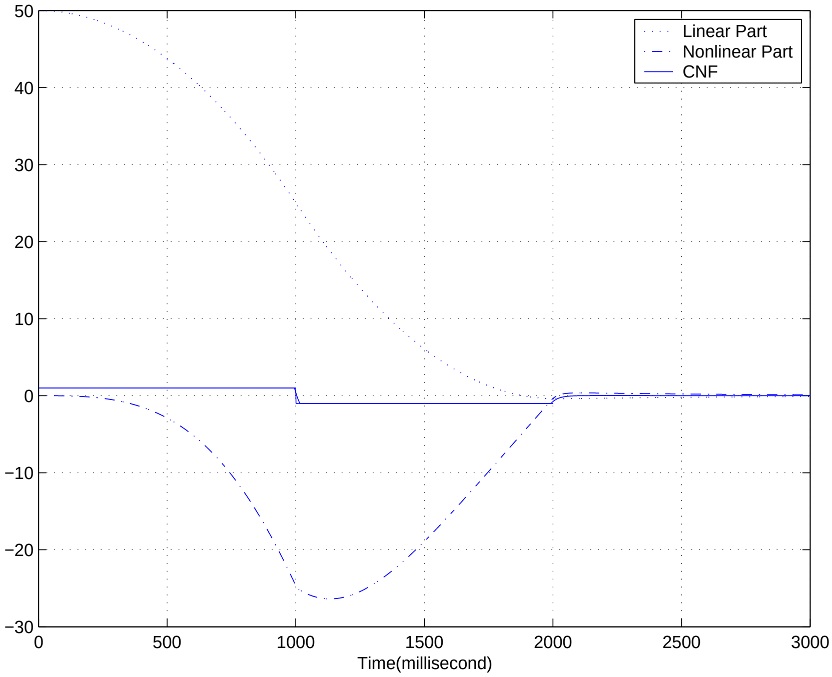


Figure 7.8: The control signal

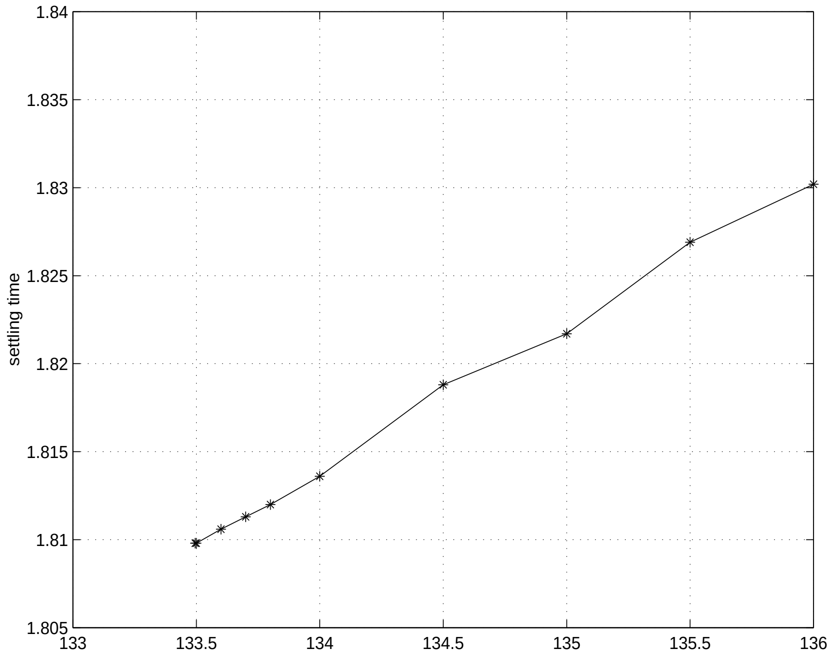


Figure 7.9: Relationship between ε and settling time

serves as a solution to approximate the optimal settling time. Further research will be focused on finding the possible rigorous relationship between the optimal settling time and the adjusting parameter in CNF controllers. It will also be of interest to investigate the ATOT problem for higher order systems and more general systems if applicable.

Chapter 8

Conclusion

In this study, the author developed a new method with a simple structure in order to track set-point signals under actuator saturation for general continuous-time and discrete-time linear systems under state feedback and measurement feedback. I proposed a combination of a linear state feedback controller, $u_{\rm L}$, and a nonlinear controller, $u_{\rm N}$ with a tuning parameter ρ so that by tuning the parameter ρ, I were able to get better performance than that obtained by using only a linear controller $u_{\rm L}$. Simulation examples clearly showed the improvement of system performance and in some cases there was very significant improvement. In this chapter, I will give a broader view of the CNF scheme, refer to its possible applications and propose some possible future research directions.

8.1 Tuning Mechanism of ρ

The expectation for improved performance by using CNF control is reasonable as when the parameter ρ vanishes, the combined controller, $u_{\rm L} + u_{\rm N}$, reverts to its linear counterpart $u_{\rm L}$ as if no additional nonlinear part $u_{\rm N}$ has been added. Thus, by carefully tuning the parameter ρ in $u_{\rm N}$, it is quite probable that one can get better performance. The added nonlinear part, $u_{\rm N}$, changes the root loci of the closed-loop system by the tuning parameter ρ and $u_{\rm N}$'s effect on the system performance has clear physical meaning in SISO case as shown in Chen *et al.* (2003) [19] but in general, this clear physical meaning cannot be carried over to MIMO case.

The root loci of MIMO systems are changed but their effects on the system performance are not clear due to the coupling of different channels. In the literature, no clear physical meaning has been found for the relationship between the poles of the closed-loop systems and the system performance in each channel (Skogestad and Postlethwaite, 1996) [65]. Thus, in general, the popular concepts like gain margin and phase margin in SISO case are not similarly defined in MIMO systems.

In fact, Skogestad and Postlethwaite (1996) [65] also show that no generally good methods have been developed to take care of channel coupling in MIMO systems. Nevertheless, one can still seek assistance from decoupling control (Wang, 2002) [76] or some conventional design methods like Rank Dominance Compensation Design (Stephanopoulos, 1986 [68]; Skogestad and Postlethwaite, 1996 [65]). However one has to add a certain pre-compensator to the plant and then design the controller based on this pre-compensated plant. Consequently the controller including the pre-compensator becomes more complex and will cause more difficulties when tuning ρ in u_{N}. It is thus necessary for one to make a reasonable trade-off between the merits of decoupling control and the difficulties of parameter tuning.

On the other hand, the conditions regarding ρ are more mathematical although I did give some detailed procedures on how to tune it. More practical guidelines are needed which can only be obtained through further research. The reason is that for multivariable control systems, no generally good loop gain tuning methods have been developed. All currently available methods are typically rather problem-specific (Stephanopoulos, 1986) [68]. However, one may still follow some of these methods such as sequential loop closing method which takes care of each loop one by one according to certain loop index and hopefully, one can tune ρ satisfactorily for his problem at hand.

Because of possible difficulties in tuning ρ, intensive simulations become very important in practice as simulation usually gives very useful information about system behavior so that one can avoid certain adverse responses due to improper parameter selection. Note that the MATLAB toolkit has been developed for this purpose, see Cheng *et al.* 2004 [22]. The academic trial version can be downloaded from http://bmchen.net.

8.2 Choice of Linear Controller

The CNF controllers are based on linear state feedback controllers which do not violate control constraints. Therefore, for designing this linear controller, several methods developed in the literature (Gutman and Hagander, 1986 [30]; Lin, 1998 [52] and Blanchini and Miani, 2000 [11]) can be used. The methods in Gutman and Hagander (1986) [30] and Blanchini and Miani (2000) [11] are of more theoretical significance although they are applicable to more general cases as the authors did not propose highly efficient algorithms to find the controller. In other words, one can be sure that a controller exists so long as certain conditions are met but one may not find a proper one.

The so-called low gain feedback control design methodology proposed in Lin (1998) [52], however, is of much interest to the author. This systematic method with clear easy-to-follow algorithms can be used to find a family of feedback gain matrices and thus gives one more freedom to choose an appropriate one for his use. Therefore, low-gain feedback design may well serve to expand the domain of attraction in the CNF design. In some cases, the domain of the CNF controller may be too small, especially under measurement feedback cases. One possible solution is to change the linear state feedback gain F so that the domain of attraction will be expanded. Low-gain feedback thus offers a very good choice of different linear controllers for use.

On the other hand, in the CNF designs I propose not only CNF controller $u_{\mathrm{L}} + u_{\mathrm{N}}$, but also the linear controller u_{L} which does not violate the control constraints. The conditions imposing on the system for controllers design are some which connect initial conditions of the plant, initial conditions of the observer (for measurement feedback cases only), the reference levels and the saturation levels. They must all be checked in order to get a proper CNF controller. When ρ is set to zero(s), one gets a linear controller u_{L}. By closely investigating the proposed conditions for CNF controller design, one may find some effective algorithm. It is basically a problem of the determination of domain of attraction. For this aim, several methods dealing with ellipsoidal, polyhedral or smoothed domains suggested in Blanchini and Miani (2000) [11] may be used.

Obviously, the above-mentioned low-gain feedback may also be used to expand domain of attraction so that one can get a controller with a larger domain of attraction which is rather important in real applications.

8.3 Dealing with Asymmetric Saturation

Although the CNF schemes serve to offer improved performance for general linear systems with actuator saturation, the actuator saturation under investigation is only symmetric saturation while asymmetric saturation is not considered. This asymmetry may cause certain problems which add difficulties to controller design (Gutman and Hagander, 1986) [30]. For example, the domain of attraction will be distorted due to this asymmetry.

In a recent paper, Hu *et al.* (2002) [37] reported that a totally different method than that used for their previous results on linear SISO ANCBC systems with symmetric saturation in Hu and Lin (2001) [36] had to be adopted when they considered asymmetric saturation. The reason is that the symmetry property often simplifies theoretical development but asymmetric saturation may cause great difficulties in this development.

One possible method to deal with asymmetric saturation is to cut down the saturation levels if both saturation limits are of opposite signs so that one gets a restricted symmetric saturation. One then carries out the CNF controller design but it can be easily seen that the overall performance will not be good enough as one does not make use of the full potential of the actual saturation levels. For example, since the magnitudes of both limits are significantly different, say the upper limit is 1 while the lower limit is either 0.1 or -0.1, if one forces both limits to the same limit levels (in this example, 0 or 0.1) one will end up with very bad performance as the control input has been confined to a very small level or even zero.

Another possible modification to deal with this asymmetry for the CNF controller design could be a shift of saturation center, which is the average of both saturation limits. In that case, the effects of different control levels should be carefully examined and simulation is important in order to understand how control limits affect the system behavior. Trial and error seems unavoidable and further investigations are needed before a systematic method may be obtained. In the case when one has to seek for a possible specific solution for the specific control problem at hand, simulations also play a key role as one need to know the possible effects of change of certain parameters on the system performance, which may give one some guidelines for controller design.

8.4 Potential Applications

Since I have proposed CNF controllers for both state feedback case and measurement feedback cases, and for both continuous-time and discrete-time linear systems, I believe that this method can be widely used in practice.

Measurement feedback is quite commonly used in practice, as it is rarely seen or almost impossible that all states can be obtained. Also, digital computers and special purpose digital control chips have been used extensively so far, and it seems that almost no modern controllers use only continuous-time processing elements (Aström and Wittenmark, 1997 [3]). Just as easy setup and convenient parameter tuning of PID control leads to its usage in almost 85% loops in modern chemical plants and other large-scale plants (Aström and Wittenmark, 1997 [3]), I believe that the CNF controller can offer field control engineers a new choice of simple controller with improved performance.

Also, actuator saturation is almost unavoidable in practical situations. Thus the CNF schemes may also be used to take good care of actuator saturation in many practical control loops. In fact, even when no actuator saturation exists or the control signal can never exceed the saturation limits, the CNF schemes may still offer improved performance compared to those using only linear controller.

As shown previously, in order for the CNF schemes to play a more important role in practice, more research should be focused on the tuning method of ρ in $u_{\rm N}$ so that convenient methods may be proposed. At least good tuning methods should be proposed for certain specific commonly used control processes.

On the other hand, possible modifications for the CNF schemes to deal with disturbance reduction or elimination should be pursued. PID has excellent property of elimination of constant bias widely occurring in practical control processes by error integral control. It is possible also to introduce this error control in order to reduce or eliminate constant bias. The method is to augment the plant to include the error signals of reference signals and controller outputs as augmented states and design an enhanced CNF controller for this augmented plant in order to force the whole state vector to stay within a compact neighborhood of the origin and thus recover the almost accurate tracking of the reference signals. If under some conditions, the augmented state vector does stop at some point so that the error signal is forced to be zero(s), the accurate tracking of the reference signals is achieved.

For disturbances other than constant bias, so long as they are slowly changing, this enhanced CNF schemes can be still used but the performance may not be the same as that for the case of constant bias.

For fast changing disturbances, further investigation must be done in order to see whether the CNF schemes can be tailored to tackle them. Methods used in output regulation may be tried as they are good at tackling fast changing disturbances so long as they are produced by some linear exo-systems. For other type of fast changing disturbances, other methods like PID control with input and output constraints (Glattfelder and Schaufelberger, 2003 [28]), model predictive control with constraints (Maciejowski, 2002 [59]) seem quite promising. If these disturbances are of stochastic nature, methods developed for stochastic control may be attempted (Aström, 1970 [2]).

8.5 Nonlinear Extension

Finally, I have extended the CNF schemes to nonlinear linearizable SISO systems under state feedback. Extension to nonlinear linearizable MIMO systems under state feedback is possible but the theoretical results may be rather restrictive and further research is certainly needed to get a less restricted result. I also extended the state feedback CNF scheme to partial linear systems which have nonlinear zero dynamics.

In practice it is quite possible that even though one cannot find rigorous stability analysis for some controller design they work very well. This phenomenon occurs even more often in simulations (Walkman, 1986 [78]). Therefore, in order to get theoretical results which apply to more general cases, one must pay close attention to these practically workable designs in order to generalize current results as they are typically not based on rigorous theoretical analysis. These practically workable designs should be investigated closely in order to generalize current results as they are typically not based on rigorous theoretical analysis.

Till now, although there are some promising results on stabilization and output regulation of nonlinear systems (Byrnes *et al.*, 1996 [13]; Kokotović and Arcak, 2001 [48]), there are almost no discussions on the improvement of system performance. Further research should be conducted on applying CNF control to other possible classes of nonlinear systems in order to provide some insights into providing improved set-point tracking performance for even more general nonlinear systems.

The basic ideas for CNF control may be modified for this improvement. A basic controller should be found to solve the set-point tracking first as done in the literature (Isidori, 1995 [42]; Khalil, 2002 [47]; and so on). The next step should be to include additional controller action properly to get improved performance. Due to the complicated system behavior of nonlinear systems, there is still a very long way to go before a possible solution can be found.

8.6 Future: Towards Transient Performance Improvement for More General Systems

Addtition to the possible refinement mentioned above, it is instructive also for one to see the CNF scheme from a broader viewpoint. Specifically, from the point view of feedback, this CNF simply explores the possibility of time-changing feedback laws in improving system performance. By setting the saturation levels to be infinity, it can be used in general linear systems without actuator saturation. This idea is not unusual in time-varying systems where due to the time-varing nature of systems dynamics the feedback gain may change accordingly, and in some finite-time discrete-time optimal control systems, where a time-series of feedback gain must be sought to reach certain optimal performance index. For linear time-invariant systems, fixed feedback gain is usually adopted and most methods like pole placement, LQR, and H_2, H_∞ methods consider fixed feedback gain control only.

With today's software and hardware capabilities such as high-potential calculating capability and low prices of advanced control components, it is time for one to consider using time-varying feedback gains in order to get better system performance especially in very stringent situations like NANO dimension manufacturing. Similar to loop shaping, it is possible for one to shape system performance stage by stage. Obviously, this should be based on exact prediction of closed-loop system behavior. Nevertheless, it can be loosen to be effective to certain range of performance so that performance robustness and hence structural and controller robustness may be considered also. All these considerations are based on the idea of changing feedback gains under different conditions, which is common in gain scheduling in adaptive control. However, for each specific operating condition of gain schedule control, the gain is still a fixed one. I hope that gradually, with further research, the mechanism of how to tune the feedback gains will be more and more evident so that it can be used easily and broadly in practice.

Published/Submitted Papers

1. Y. He, B. M. Chen and W. Lan, "Improving transient performance in tracking control for a class of nonlinear discrete-time systems with input saturation," *the 44th IEEE Conference on Decision and Control*, Seville, Spain, pp. 8094-8099, December 2005.

2. Y. He, B. M. Chen and C. Wu, "Composite nonlinear feedback control for general discrete-time multivariable systems with actuator nonlinearities," Submitted to journal.

3. Y. He, B. M. Chen and C. Wu, "Composite nonlinear control with state and measurement feedback for general multivariable systems with input saturation," *Systems & Control Letters*, Vol. 54, No. 5, pp. 455-469, May 2005.

4. Y. He, B. M. Chen and C. Wu, "Composite nonlinear feedback control for general discrete-time multivariable systems with actuator nonlinearities," *Proceedings of the 5th Asian Control Conference*, Melbourne, Australia, pp. 539-544, July 2004.

5. Y. He, B. M. Chen and C. Wu, "Composite nonlinear control with state and measurement feedback for general multivariable systems with input saturation," *Proceedings of the 42nd IEEE Conference on Control and Decision*, Maui, Hawaii, USA, pp. 4469-4474, December 2003.

6. W. Lan, B. M. Chen and Y. He, "On improvement of transient performance in

tracking control for a class of nonlinear systems with input saturation," *Systems & Control Letters*, Vol. 55, No. 2, pp. 132-138, February 2006.

7. W. Lan, B. M. Chen and Y. He, "Composite nonlinear feedback control for a class of nonlinear systems with input saturation," Presented at *the 16th IFAC World Congress*, Prague, Czech, July 2005.

8. C. Wu, B. M. Chen and Y. He, "Asymptotic time optimal tracking of a class of linear systems with actuator nonlinearities," *Proceedings of the 4th International Conference on Control and Automation*, Montreal, Quebec, Canada, pp. 58-62, June 2003.

Bibliography

[1] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice Hall, Englewood Cliffs, 1989.

[2] K. J. Aström, Introduction to stochastic control theory, Academic Press, New York, 1970.

[3] K. J. Aström and B. Wittenmark, Computer-controlled systems: theory and design, 3/e, Prentice Hall, Upper Saddle River, NJ, 1997.

[4] M. Athans and P. L. Falb, *Optimal control: an introduction to the theory and its applications*, McGraw-Hill, New York, 1966.

[5] A. Bemporad, A. Casavola and E. Mosca, "Nonlinear control of constrained linear systems via predictive reference management," *IEEE Transactions on Automatic Control*, Vol. 42, pp. 340-349, 1997.

[6] A. Bemporad and E. Mosca, "Fulfilling hard constraints in uncertain linear systems by reference managing," *Automatica*, Vol. 34, pp. 451-461, 1998.

[7] S. Bennett, *A history of control engineering: 1930-1955*, P. Peregrinus on behalf of the Institution of Electrical Engineers, London, 1993.

[8] D. S. Berstein and A. N. Michel, "A chonological bibliography on saturation actuators," *International Journal of Robust and Nonlinear Control*, Vol. 5, pp. 375-380, 1995.

[9] G. Bitsoris, "On the positive invariance of polyhedral sets for discrete time systems," *Systems and Control Letters*, Vol. 11, pp. 243-348, 1988.

[10] G. Bitsoris, "Positively invariant polyhedral sets for discrete-time linear systems," *International Journal of Control*, Vol. 47, pp. 1713-1726, 1988.

[11] F. Blanchini and S. Miani, "Any domain of attraction for a linear constrained system is a tracking domain of attraction," *SIAM Journal of Control and Optimization*, Vol. 38, pp. 971-994, 2000.

[12] C. I. Byrnes and A. Isidori, "Asymptotic stabilization of minimum phase nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 36, pp. 1122-1137, 1991.

[13] C. I. Byrnes, F. D. Priscoli and A. Isidori, *Output regulation of uncertain nonlinear systems*, Birkhäuser, Boston, 1997.

[14] C. I. Byrnes, F. D. Priscoli, A. Isidori and W. Kang, "Structurally stable output regulation of nonlinear systems," *Automatica*, vol. 33, pp. 369-385, 1997.

[15] P. J. Campo, M. Morari and C. N. Nett, "Multivariable anti-windup and bumpless transfer: a general theory," *Proceedings of American Control Conference*, pp. 1706-1711, 1989.

[16] P. J. Campo and M. Morari, "Robust control of processes subject to saturation nonlinearities," *Computers and Chemical Engineering*, Vol. 14, pp. 343-358, 1990.

[17] B. M. Chen, *Robust and H_∞ Control*, Springer, London, 2000.

[18] B. M. Chen, T. H. Lee and V. Venkataramanan, *Hard Disk Drive Servo Systems*, Springer, London, 2002.

[19] B. M. Chen, T. H. Lee, K. Peng and V. Venkataramanan, "Composite nonlinear feedback control for linear systems with input saturation: Theory and an application," *IEEE Transactions on Automatic Control*, Vol. 48, No. 3, pp. 427-439, March 2003.

[20] B. M. Chen, Z. Lin and Y. Shamash, *Linear Systems Theory: A Structural Decomposition Approach*, Birkhäuser, Boston, 2004.

[21] B. M. Chen and D. Z. Zheng, "Simultaneous finite- and infinite-zero assignments of linear systems," *Automatica*, Vol. 31, No. 4, pp. 643-648, 1995.

[22] G. Cheng, B. M. Chen, K. Peng and T. H. Lee, "A MATLAB toolkit for composite nonlinear feedback control," *Proceedings of the 8th International Conference on Control, Automation, Robotics and Vision*, Kunming, China, pp. 878-883, December 2004.

[23] R. C. Dorf and R. H. Bishop, *Modern Control Systems*, 7th Edition, Addison Wesley, Reading, Massachusetts, 1995.

[24] G. F. Franklin, J. D. Powell and M. L. Workman, *Digital Control of Dynamic Systems*, 3/e, Addison-Wesley, Reading, Massachusetts, 1998.

[25] G. Garcia, S. Tarbouriech, R. Suarez and J. Alvarez-Ramirez, "Nonlinear bounded control for norm-bounded uncertain systems," *IEEE Transactions on Automatic Control*, Vol. 44, pp. 1254-1258, 1999.

[26] E. G. Gilbert and K. T. Tin, "Linear systems with state and control constraints: The theory and application of maximal output admissible sets," *IEEE Transactions on Automatic Control*, Vol. 36, pp. 1008-1020, 1991.

[27] E. G. Gilbert, I. Kolmanovsky and K. T. Tin, "Discrete-time reference governors and the nonlinear control of systems with state and control constraints," *International Journal of Robust and Nonlinear Control*, Vol. 5, pp. 487-504, 1995.

[28] A.H. Glattfelder and W. Schaufelberger, Control systems with input and output constraints, Springer, New York, 2003.

[29] T. J. Graettinger and B. H. Krogh, "On the computation of reference signal constraints for guaranteed tracking performance," *Automatica*, Vol. 28, pp. 1125-1141, 1992.

[30] P. Gutman and P. Hagander, "A new design of constrained controllers for linear systems," *IEEE Transactions on Automatic Control*, Vol. 30, pp. 22-33, 1985.

[31] R. Hanus, M. Kinnaert and J. L. Henrotte, "Conditioning technique, a general anti-windup and bumpless transfer method," *Automatica*, Vol. 23, pp. 729-739, 1987.

[32] Y. He, B. M. Chen and C. Wu, "Composite nonlinear control with state and measurement feedback for general multivariable systems with input saturation," *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, USA, pp. 4469-4474, December 2003.

[33] Y. He, B. M. Chen and C. Wu, "Composite nonlinear feedback control for general discrete-time multivariable systems with actuator nonlinearities," *Systems & Control Letters*, Vol. 54, No. 5, pp. 455-469, May 2005.

[34] Y. He, B. M. Chen and C. Wu, "Composite nonlinear feedback control for general discrete-time multivariable systems with actuator nonlinearities," *Proceedings of the 5th Asian Control Conference*, Melbourne, Australia, pp. 539-544, 2004.

[35] D. Henrion, G. Garcia and S. Tarbouriech, "Piecewise linear robust control of systems with input saturation," *European Journal of Control*, Vol. 5, pp. 157-166, 1999.

[36] T. Hu and Z. Lin, *Control Systems with Actuator Saturation: Analysis and Design*, Birkhäuser, Boston, 2001.

[37] T. Hu, A. N. Pitsillides and Z. Lin, "Null Controllability and Stabilization of Linear Systems Subject to Asymmetric Actuator Saturation," in *Actuator Saturation Control*, edited by V. Kapila and K. M. Grigoriadis, Marcel Dekker, New York, 2002.

[38] J. Huang, "Remarks on the robust output regulation problem for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 46, no. 12, pp. 2028-2031, 2001.

[39] J. Huang, *Nonlinear Output Regulation: Theory and Applications*, SIAM, Philadelphia, 2004.

[40] J. Huang and Z. Chen, "A general framework for tackling output regulation problem," *Proceedings of the 2002 American Control Conference*, pp. 102-109, May, 2002.

[41] J. Huang and C. F. Lin, "On a robust nonlinear servomechanism problem," *IEEE Transactions on Automatic Control*, vol. 39, pp. 1510-1513, 1994.

[42] A. Isidori, Nonlinear Control Systems, 3/e, Springer, New York, 1995.

Nonlinear Control Systems II, Springer, London, 1999.

[43] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 35, pp. 131-140, 1990

[44] Z. P. Jiang and I. Mareels, "Robust nonlinear integral control," *IEEE Transactions on Automatic Control*, Vol. 46, No. 8, pp. 1336-1342, 2001.

[45] Z. P. Jiang, I. Mareels, D. J. Hill and J. Huang, "A unifying framework for global regulation via nonlinear output feedback," *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 1047-1052, Maui, Hawaii, 2003.

[46] P. Kapasouris, M. Athans, and G. Stein, "Design of feedback control systems for stable plants with saturating actuators," *Proceedings of the 27th IEEE Conference on Decision and Control*, pp. 469-479, 1988.

[47] H. K. Khalil, *Nonlinear Systems*, 3rd edition, Prentice Hall, New Jersey, 2000.

[48] P. Kokotović and M. Arcak, Constructive nonlinear control: a historical perspective, Automatica, Vol. 37:5, pp. 637-662, May 2001.

[49] M. V. Kothare, P. J. Campo, M. Morari and C. N. Nett, "A unified framework for the study of anti-windup designs," *Automatica*, Vol. 30, pp. 1869-1883, 1994.

[50] N. J. Krikelis and S. K. Barkas, "Design of tracking systems subject to actuator saturation and integrator windup," *International Journal of Control*, Vol. 39, pp. 667-682, 1984.

[51] W. Lan, B. M. Chen and Y. He, "Composite nonlinear feedback control for a class of nonlinear systems with input saturation," Presented at the *IFAC World Congress*, Prague, Czech, July 2005.

[52] Z. Lin, *Low Gain Feedback*, Lecture Notes in Control and Information Sciences, Vol. 240, Springer, London, 1998.

[53] Z. Lin, M. Pachter and S. Banda, "Toward improvement of tracking performance — Nonlinear feedback for linear system," *International Journal of Control*, Vol. 70, pp.1-11, 1998.

[54] Z. Lin and A. Saberi, "Semi-global stabilization of partially linear composite systems via feedback of the state of the linear part," *Proceedings of the 31st Conference on Decision and Control*, pp. 3431-3433, Arizona, USA, December, 1992.

[55] Z. Lin and A. Saberi, "Semi-global stabilization of partially linear composite system via linear dynamic feedback," *Proceedings of the 31st Conference on Decision and Control*, pp. 2538-2543, Texas, USA, December, 1993.

[56] Z. Lin and A. Saberi, "A semi-global low and high gain design technique for linear systems with input saturation — Stabilization and Disturbance rejection," *International Journal of Robust and Nonlinear Control*, Vol. 5, pp.381-398, 1995.

[57] Z. Lin, A. A. Stoorvogel and A. Saberi, "Output regulation for linear systems subject to input saturation," *Automatica*, Vol. 32, pp. 29-47, 1996.

[58] R. Mantri, A. Saberi, Z. Lin and A. A. Stoorvogel, "Output regulation for linear discrete-time systems subject to input saturation," *International Journal of Robust and Nonlinear Control*, Vol. 7, pp. 1003-1021, 1997.

[59] J.M. Maciejowski, Predictive control: with constraints, Prentice Hall, New York, 2002.

[60] J. Macki and M. Strauss, *Introduction to Optimal Control*, Springer, Berlin, 1982.

[61] B. Romanchuck, *Analysis of saturated feedback loops: an input-output approach*, PhD thesis, Department of Engineering, University of Cambridge, 1995.

[62] A. Saberi, P. Sannuti and B. M. Chen, *H_2 Optimal Control*, Prentice Hall, London, 1995.

[63] A. Saberi, A. A. Stroovogel and P. Sannuti, *Control of linear systems with regulation and input constraints*, Springer, London, 2000.

[64] W. E. Schmitendorf and B. R. Barmish, "Null controllability of linear systems with constrained controls," *SIAM Journal on Control and Optimization*, Vol. 18, pp. 327-345, 1980.

[65] S. Skogestad and I. Postlethwaite, Multivariable feedback control: analysis and design, Wiley, New York, 1996.

[66] E. D. Sontag, "An algebraic approach to bounded controllability of linear systems," *International Journal of Control*, Vol. 39, pp. 181-188, 1984.

[67] J. T. Spooner, M. Maggiore, R. Ordóñez and K. M. Passino, *Stable and Adaptive Control and Estimation for Nonlinear Systems: Neural and Fuzzy Approximator Techniques*, John Wiley & Sons, New York, 2002.

[68] G. Stephanopoulos, *Chemical Process Control: An Introduction to Theory and Practice*, Prentice Hall, Englewood Cliffs, 1984.

[69] R. Suarez, J. Alvarez-Ramirez, M. Sznaier and C. Ibarra-Valdez, "L2 disturbance attenuation for linear systems with bounded controls: an ARE-based approach," in *Control of Uncertain Systems with Bounded Inputs*, Eds, S. Tarbouriech and G. Garcia, pp. 25-38 Lecture Notes in Control and Information Sciences, Vol. 227, Springer, London, 1998.

[70] H. J. Sussmann and P. V. Kokotović, "The peaking phenomenon and the global stabilization of nonlinear systems," *IEEE Transactions on Automatic Control*, Vol. 36, No. 4, pp. 424-440, 1991.

[71] A. R. Teel, “Global stabilization and restricted tracking for multiple integrators with bounded controls,” *Systems & Control Letters*, Vol. 17, pp. 165-171, 1992.

[72] A. R. Teel, “Semi-global stabilization of minimum phase nonlinear systems in special normal forms,” *System & Control Letters*, vol. 19, pp. 187-192, 1992.

[73] M. C. Turner and I. Postlethwaite, “Guaranteed stability region of linear systems with actuator saturation using the low-and-high gain technique,” *International Journal of Control*, Vol 74, pp. 1425-1434, 2001.

[74] M. C. Turner, I. Postlethwaite and D. J. Walker, “Nonlinear tracking control for multivariable constrained input linear systems,” *International Journal of Control*, Vol 73, pp. 1160-1172, 2000.

[75] V. Venkataramanan, K. Peng, B. M. Chen and T. H Lee, “Discrete-time composite nonlinear feedback control with an application in design of a hard disk drive servo system,” *IEEE Transactions on Control Systems Technology*, Vol. 11, pp. 16-23, 2003.

[76] Qing-Guo Wang, *Decoupling control*, Springer, York, 2003.

[77] R. F. Wilson, J. R. Cloutier and R. K. Yedavalli, “Lyapunov-constrained eigenstructure assignment for the design of robust mode-coupled roll-yaw missile autopilots,” *Proceedings of the IEEE Conference on Control Applications*, pp. 994-999, 1992.

[78] M. L. Workman, *Adaptive Proximate Time Optimal Servomechanisms*, Ph.D. Thesis, Stanford University, 1987.

[79] G. F. Wredenhagen and B. Belanger, “Piecewise linear LQ control for constrained input systems,” *Automatica*, Vol. 30, pp. 403-416, 1994.

[80] G. F. Wredenhagen and B. Belanger, “Piecewise linear LQ control for constrained input systems,” *Automatica*, Vol. 30, pp. 403-416, 1994.

[81] X. Yang, X.-H. Wang, G.-Z. Shen and C.-Y. Wen, “Modeling and simulation of a six-degree-of-freedom aircraft,” *Journal of Systems Simulation*, vol. 12, pp. 210-213, 2000 (in Chinese).