RATIONALIZABILITY IN GENERAL GAMES

CHEN QU

(B.Econs., Fudan U)

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Summary

In this paper, we present a unified framework to analyze rationalizable strategic behavior in any arbitrary games by using Harsanyi's notion of type. (i) We investigate properties of rationalizability in general games. Specifically, we show that the set of all the rationalizable strategy profiles is the largest rationalizable set in product form. Moreover, we show that the largest rationalizable set can be derived by the (possibly transfinite) iterated elimination of never-best responses (IENBR). In particular, IENBR is a well-defined order-independent procedure. (ii) We investigate the relationship between rationalizability and Nash equilibrium in general games. We provide a sufficient and necessary condition to guarantee no spurious Nash equilibria in the reduced game after the IENBR procedure. (iii) We formulate and prove that rationalizability is the strategic implication of common knowledge of rationality in general games.

Keywords: Type space; Monotonicity; Rationalizability; Iterated elimination; Nash equilibrium; Common knowledge; Rationality

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1. Introduction

The notion of rationalizability was introduced independently by Bernheim (1984) and Pearce (1984) and thus far has become one of the most important solution concepts in non-cooperative games. The basic idea behind this notion is that rational behavior must be justified by "rational beliefs" and conversely, "rational beliefs" must be based on rational behavior. The notion of rationalizability captures the strategic implications of the assumption of "common knowledge of rationality" (see Tan and Werlang (1988)), which is very different from the assumption of "commonality of beliefs" or "correct conjectures" behind an equilibrium (see Aumann and Brandenburger (1995)).

In the literature, the study of rationalizable strategic behavior is restricted to finite games with continuous payoff functions. Since many important models arising in economic applications are games with infinite strategy spaces and discontinuous payoff functions, e.g., models of price and spatial competition, auctions, and mechanism design, it is clearly important and practically relevant to extend the notion of rationalizability to very general situations with various modes of behavior. Epstein (1997) provided a unified "model of preference" to allow for different categories of preferences such as subjective expected utility, probabilistically sophisticated preference, Choquet expected utility and the multi-priors model, and presented the notion of $\mathcal{P}^*$-rationalizability. However, from a technical point of view, Epstein's (1997) analysis relies on topological assumptions on the game structure and, in particular, most of his discussion on rationalizability is restricted to finite games. Apt (2007) relaxed the finite set-up of games and studied rationalizability by an iterative procedure, but

Apt's (2007) analysis requires players' preferences to have a form of expected utility. In this paper we study rationalizable strategic behavior in arbitrary games with general preferences.

We offer a definition of rationalizability in general situations (Definition 1). Roughly speaking, a set of strategy profiles is regarded as "rationalizable" if every strategy in this set is justified by a type from the set. We show that rationalizable strategies can be derived from an iterated elimination of never-best responses (IENBR) (Theorem 1).

To define the notion of rationalizability, we need to consider a system of preferences/beliefs in every subgame. Following Epstein's (1997) notion of "model of preference," by using Harsanyi's (1967-68) notion of type, we introduce the "model of type," which specifies a set of admissible and feasible types for each of players in every contingencies. For each type of a player, the player is able to make a decision over his own strategies. Our approach is topology-free and is applicable to any arbitrary games.

In this paper, we investigate the relationship between rationalizability and Nash equilibrium and present a necessary and sufficient condition for no spurious Nash equilibria (Theorem 2). This paper is thereby closely related to Chen et al.'s (2007) work on strict dominance in general games. In this paper, we also study the epistemic foundation of rationalizability in general games; in particular, we formulate and prove that rationalizability is the strategic implication of common knowledge of rationality in general settings (Theorem 3).

The rest of this paper is organized into five sections. Section 2 is the set-up. Sections 3 and 4 present the main results concerning rationalizability with IENBR and Nash equilibrium respectively. Section 5 provides the epistemic foundation for rationalizability. Section 6 offers some concluding remarks.

2. Set-up

Consider a normal-form game

$$G \equiv (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}),$$

where N is an (in)finite set of players, S_i is an (in)finite set of player i's strategies, and $u_i\colon S \to \mathbb{R}$ is player i's arbitrary payoff function where $S \equiv \times_{i\in N} S_i$. For $s \in S$ let $s \equiv (s_i, s_{-i})$. A strategy profile s^* is a *(pure) Nash equilibrium* in G if for every player i,

$$u_i\left(s^*\right) \geq u_i\left(s_i, s^*_{-i}\right) \ \forall s_i \in S_i.$$

The notion of "type" by Harsanyi (1967-68) is a simple and parsimonious description of the exhaustive uncertainty facing a player, including the player's knowledge, preferences/beliefs, etc. Given one player i's type, he has one corresponding preference over his own strategies, according to which he can make his decision. We consider a *model of type (on G)*:[1]

$$T(\cdot) \equiv \{T_i(\cdot)\}_{i\in N},$$

[1] This is in the same spirit of Epstein's (1997) "model of preference"; see also Chen and Luo (2010).

where $T_i(\cdot)$ is defined for every (nonempty) subset $S' \subseteq S$ and every player $i \in N$. The set $T_i(S')$ is interpreted as player i's *type space* in the reduced game $G|_{S'} \equiv (N, \{S'_i\}_{i\in N}, \{u_i|_{S'}\}_{i\in N})$, where $u_i|_{S'}$ is the payoff function u_i restricted on S'. Each type $t_i \in T_i(S')$ has a corresponding *preference relation* $\succsim_{t_i}$ over player i's strategies in S_i. The indifference relation, $\sim_{t_i}$, is defined as usual, i.e., $s_i \sim_{t_i} s'_i$ iff $s_i \succsim_{t_i} s'_i$ and $s'_i \succsim_{t_i} s_i$. For instance, we may consider $T_i(S')$ as a probability space or a regular preference space defined on S'. The following example demonstrates that the analytical framework can be applied to finite games where the players have the standard subjective expected utility (SEU) preferences.

Example 1. Consider a finite game G. Player i's belief about the strategies his opponents play in the reduced game $G|_{S'}$ is defined as a probability distribution μ_i over S'_{-i}, i.e., $\mu_i \in \Delta\left(S'_{-i}\right)$ where $\Delta\left(S'_{-i}\right)$ is the set of probability distributions over S'_{-i}. For any μ_i, the expected payoff of s_i can be calculated by $U_i\left(s_i, \mu_i\right) = \sum_{s_{-i}\in S'_{-i}} u_i\left(s_i, s_{-i}\right) \cdot \mu_i\left(s_{-i}\right)$ where $\mu_i\left(s_{-i}\right)$ is the probability assigned by μ_i to s_{-i}. That is, μ_i generates an SEU preference over S_i. For our purpose we may define a model of type (on G) as follows:

$$T(\cdot) \equiv \{T_i(\cdot)\}_{i\in N},$$

where, for every player $i \in N$, $T_i(S') = \Delta\left(S'_{-i}\right)$ for every (nonempty) subset $S' \subseteq S$.

Throughout this paper, we impose the following two conditions, C1 and C2, for the model of type.

C1 (Monotonicity) $\forall i$, $T_i(S') \subseteq T_i(S'')$ if $S' \subseteq S''$.

The monotonicity condition states that when one player faces greater strategic uncertainty, the player possesses more types to be used for resolving uncertainty. Under C1, $T_i \equiv T_i(S)$ can be viewed as the "universal" type space of player i.

For $s \in S$, player i's *Dirac type* $\delta_i(s)$ is a type with the property:

$$\forall s_i', s_i'' \in S_i, u_i(s_i', s_{-i}) \geq u_i(s_i'', s_{-i}) \text{ iff } s_i' \succsim_{\delta_i(s)} s_i''.$$

A Dirac type $\delta_i(s)$ is a degenerated type with which player i behaves as if he faces a certain play s_{-i} of his opponents. The following condition states that the type space on a singleton contains only a Dirac type. This condition seems to be a rather natural requirement when strategic uncertainty is reduced to the case of certainty.

C2 (Diracability) $\forall i$, $T_i(\{s\}) = \{\delta_i(s)\}$ if $s \in S$.

In finite games, it is easy to see that C1 and C2 are satisfied for the standard SEU preference model defined in Example 1. Note that C1 and C2 imply that $\forall i$, $\delta_i(s) \in T_i(S')$ if $s \in S'$, i.e., the type space on S' contains all the possible Dirac types on S'.

A strategy $s_i \in S_i$ is a *best response* to $t_i \in T_i(S')$ if $s_i \succsim_{t_i} s_i'$ for any $s_i' \in S_i$. Notice that even if a reduced game $G|_{S'}$ is concerned, any strategy of player i in the original game G can be a candidate for the best response. Let $BR(t_i)$ denote the set of best responses to t_i. The following lemma states that Nash equilibrium can be defined by the Dirac type.

Lemma 1. *s^* is a Nash equilibrium iff, for every player i, s_i^* is a best response to $\delta_i(s^*)$.*

Proof. s^* is a Nash equilibrium iff, for every player i, $u_i(s^*) \geq u_i(s_i, s_{-i}^*)$ $\forall s_i \in S_i$ iff, for every player i, $s_i^* \succsim_{\delta_i(s^*)} s_i$ $\forall s_i \in S_i$ iff, for every player i, s_i^* is a best response to $\delta_i(s^*)$. □

Next we provide the formal definition of rationalizability in general games. The spirit of this definition is that for every strategy in a rationalizable set, the player can always find some type in the type space defined over this set to support his choice of strategy.

Definition 1. A subset $R \subseteq S$ is *rationalizable* if $\forall i$ and $\forall s \in R$, there exists some $t_i \in T_i(R)$ such that $s_i \in BR(t_i)$.

The following lemma asserts that there is the largest rationalizable set.

Lemma 2. *Let $R^* \equiv \cup_{R \text{ is rationalizable}} R$. Then R^* is the largest rationalizable set.*

Proof: It suffices to show that R^* is a rationalizable set. Let $s \in R^*$. Then, there exists a rationalizable set R such that $s \in R$. Thus, for every player i, there exists some $t_i \in T_i(R)$ such that $s_i \in BR(t_i)$. Since $R \subseteq R^*$, by C1, $t_i \in T_i(R^*)$. □

Although Cartesian-product form is not imposed on rationalizable sets, the following lemma shows that the largest rationalizable set must be in this form.

Lemma 3. *If R is rationalizable, then $\times_{i\in N}R_i$ is rationalizable, where $R_i \equiv \{s_i |\ s \in R\}$. Hence, $R^* = \times_{i\in N}\left(\cup_{R \text{ is rationalizable}} R_i\right)$.*

Proof: Let $s \in \times_{i\in N}R_i$. Then, for every player i, there exists $t_i \in T_i(R)$ such that $s_i \in BR(t_i)$. Since $R \subseteq \times_{i\in N}R_i$, by C1, $t_i \in T_i(\times_{i\in N}R_i)$.

Since $R_i^* = \cup_{R \text{ is rationalizable}} R_i$ for all i, by Lemma 2, $\times_{i\in N}\left(\cup_{R \text{ is rationalizable}} R_i\right)$ is rationalizable and, hence, $R^* = \times_{i\in N}\left(\cup_{R \text{ is rationalizable}} R_i\right)$. □

3. IENBR and rationalizability

In the literature, rationalizability is also defined as the outcome of an iterated elimination of never-best responses. We define a transfinite elimination process that can be used for any arbitrary game.[2] Let λ^0 denote the first element in an ordinal Λ, and let $\lambda + 1$ denote the successor to λ in Λ. For any S' and S'' with $S'' \subseteq S' \subseteq S$, S' is said to be *reduced to* S'' (notation: $S' \to S''$) if, $\forall s \in S' \backslash S''$, there exists some player i such that $s_i \notin BR(t_i)$ for any $t_i \in T_i(S')$.

Definition 2. An *iterated elimination of never-best responses (IENBR)* is a finite, countably infinite, or uncountably infinite family $\{R^\lambda\}_{\lambda\in\Lambda}$ such that $R^{\lambda^0} = S$, $R^\lambda \to R^{\lambda+1}$, (and $R^\lambda = \cap_{\lambda'<\lambda} R^{\lambda'}$ for a limit ordinal λ), and $R^\infty \equiv \cap_{\lambda\in\Lambda} R^\lambda \to R'$ only for $R' = R^\infty$.

A central result of this paper is provided below, which tells that Definitions 1 and 2 are equivalent.

[2] See Chen et al.'s (2007) Example 1 for the reason why we need a transfinite process in general games.

Theorem 1. $R^{\infty} = R^{*}$.

Proof. (i) By Definition 2, $\forall s \in R^{\infty}$, every player i has some $t_i \in T_i(R^{\infty})$ such that $s_i \in BR(t_i)$. So R^{∞} is a rationalizable set and, hence, $R^{\infty} \subseteq R^{*}$.

(ii) By Lemma 2, R^{*} is a rationalizable set and, by C1, survives every round of elimination in Definition 2. So $R^{*} \subseteq R^{\infty}$. $\square$

The definition of IENBR procedure does not require the elimination of *all* never-best response strategies in each round of elimination. This flexibility raises a question whether any IENBR procedure results a unique outcome. Theorem 1 implies that IENBR is a well-defined order-independent procedure.

Corollary 1. *R^{∞} exists and is unique. Moreover, R^{∞} is nonempty if G has a Nash equilibrium.*

Proof. R^{*} exists and is unique and, by Theorem 1, R^{∞} exists and is unique for any game.

Let s^{*} be a Nash equilibrium in G. Since s_i^{*} is a best response to $\delta_i(s^{*})$ for every player i, by C2, $\{s^{*}\}$ is a rationalizable set. By Theorem 1, $s^{*} \in R^{\infty}$. $\square$

4. Nash equilibrium and rationalizability

Corollary 1 shows that every Nash equilibrium survives IENBR and hence every Nash equilibrium is a rationalizable strategy profile. However, the following example taken from Chen et al. (2007) demonstrates that a Nash equilibrium in the reduced game after an IENBR procedure may be a spurious Nash equilibrium, i.e., it is not a Nash equilibrium in the original game.

Example 2. Consider a two-person symmetric game: $G \equiv \left(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, where $N = \{1, 2\}$, $S_1 = S_2 = [0, 1]$, and for all $x_i, x_j \in [0, 1]$, $i, j = 1, 2$, and $i \neq j$ (cf. Fig. 1)

$$u_i(x_i, x_j) = \begin{cases} 1, & \text{if } x_i \in [1/2, 1] \text{ and } x_j \in [1/2, 1], \\ 1 + x_i, & \text{if } x_i \in [0, 1/2) \text{ and } x_j \in (2/3, 5/6), \\ x_i, & \text{otherwise.} \end{cases}$$

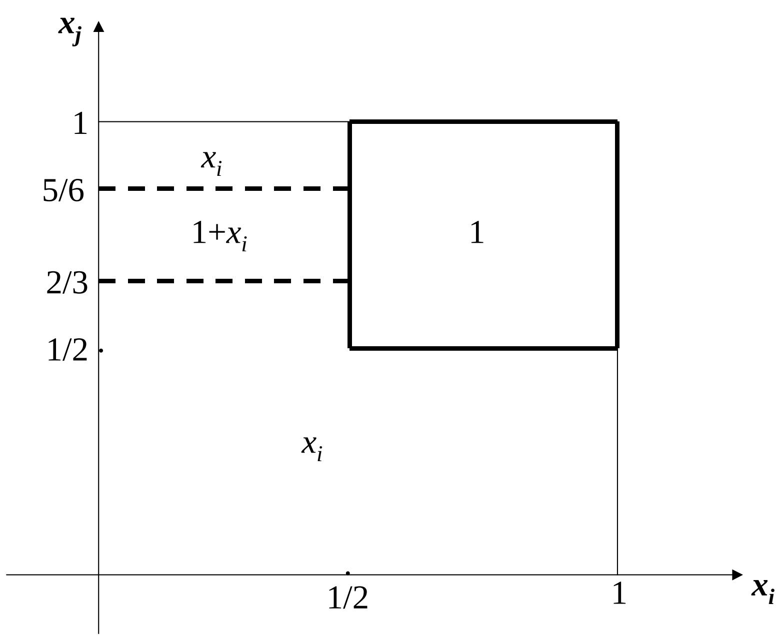

Figure 1. Payoff function $u_i\left(x_i, x_j\right)$.

It is easily verified that $R^{\infty} = [1/2, 1] \times [1/2, 1]$ since every strategy $s_i \in [0, 1/2)$ is strictly dominated and hence never a best response. That is, IENBR leaves the reduced game $G|_{R^{\infty}} \equiv \left(N, \{R_i^{\infty}\}_{i\in N}, \{u_i|_{R^{\infty}}\}_{i\in N}\right)$ that cannot be further reduced. Clearly, R^∞ is the set of Nash equilibria in the reduced game $G|_{R^\infty}$. However, it is easy to see that the set of Nash equilibria in game G is $\{s \in R^\infty |\ s_1, s_2 \notin (2/3, 5/6)\}$. Thus, IENBR generates spurious Nash equilibria $s \in R^\infty$ where some $s_i \in (2/3, 5/6)$.

For any subset $S' \subseteq S$, we say that $G \equiv (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ has *well-defined best responses on* S' if $\forall i$ and $\forall s \in S'$, $BR(\delta_i(s)) \neq \varnothing$. Let NE denote the set of Nash equilibria in G, and $NE|_{R^\infty}$ the set of Nash equilibria in the reduced game $G|_{R^\infty} \equiv (N, \{R_i^\infty\}_{i\in N}, \{u_i|_{R^\infty}\}_{i\in N})$. A sufficient and necessary condition under which rationalizability generates no spurious Nash equilibria is provided below.

Theorem 2. *$NE = NE|_{R^\infty}$ iff G has well-defined best responses on $NE|_{R^\infty}$.*

Proof. ("Only if" part.) Let $s^* \in NE|_{R^\infty}$. Since $NE|_{R^\infty} = NE$, $s_i^* \in BR(\delta_i(s^*))$ $\forall i$. Thus, $BR(\delta_i(s^*)) \neq \varnothing$ for all i.

("If" part.) (i) Let $s^* \in NE$. By Corollary 1, $s^* \in R^\infty$ and, hence, $s^* \in NE|_{R^\infty}$. So $NE \subseteq NE|_{R^\infty}$. (ii) Let $s^* \in NE|_{R^\infty}$. Since G has well-defined best responses on $NE|_{R^\infty}$, for every player i there exists some $\hat{s}_i \in S_i$ such that $\hat{s}_i \in BR(\delta_i(s^*))$, which implies that $\hat{s}_i \succsim_{\delta_i(s^*)} s_i^*$ and $(\hat{s}_i, s_{-i}^*) \in R^\infty$. Since $s^* \in NE|_{R^\infty}$, $s_i^* \succsim_{\delta_i(s^*)} \hat{s}_i$. Therefore, $s_i^* \sim_{\delta_i(s^*)} \hat{s}_i$ and, hence, $s_i^* \in BR(\delta_i(s^*))$. That is, $s^* \in NE$. So $NE|_{R^\infty} \subseteq NE$. $\square$

In Example 2, it is easy to verify that (i) G has no well-defined best response on the set of spurious Nash equilibria – i.e. $\{s \in R^\infty |\ s_1 \in (2/3, 5/6) \text{ or } s_2 \in (2/3, 5/6)\}$ and (ii) G has well-defined best responses on the set of non spurious Nash equilibria – i.e. $\{s \in R^\infty |\ s_1, s_2 \notin (2/3, 5/6)\}$. This sufficient and necessary condition in Theorem 2 does not involve any topological assumption on the original or the reduced games. In Chen et al.'s (2007) Corollary 4, some classes of games with special topological structures were proved to "preserve Nash equilibria" for the iterated elimination of strictly dominated strategies. These results are also applicable to the IENBR procedure.

The following corollary asserts that if one game is solvable by the IENBR procedure, the unique rationalizable strategy profile is the only Nash equilibrium.

Corollary 2. *$NE = R^{\infty}$ if $|R^{\infty}| = 1$.*

Proof. Let $R^{\infty} = \{s^*\}$. By C2, s_i^* is a best response to $\delta_i(s^*)$ for every player i. So $s^* \in NE$ and hence $R^{\infty} \subseteq NE$. By Corollary 1, $NE \subseteq R^{\infty}$. $\square$

5. Epistemic foundation

In this section we provide epistemic conditions for rationalizability in general games. A model of knowledge for a game G is given by

$$\mathcal{M}(G) \equiv (\Omega, \{\mathbf{s}_i\}_{i \in N}, \{\mathbf{t}_i\}_{i \in N}),$$

where Ω is the space of states with typical element $\omega \in \Omega$, $\mathbf{s}_i(\omega) \in S_i$ is player i's strategy at state ω, and $\mathbf{t}_i(\omega) \in T_i$ is player i's type at state ω.

A subset $E \subseteq \Omega$ is referred to as an *event*. Denote by $\mathbf{s}(\omega)$ the strategy profile at ω and let

$$S^E \equiv \{\mathbf{s}(\omega) \mid \omega \in E\}.$$

We extend the model of type in Section 2 to the space of states as follows. Consider a *model of type on* Ω:

$$\widetilde{T}(\cdot) \equiv \{\widetilde{T}_i(\cdot)\}_{i \in N},$$

where $\widetilde{T}_i(\cdot)$ is defined over every (nonempty) subset $E \subseteq \Omega$. The set $\widetilde{T}_i(E)$ is referred to as player i's type space for event E, and each type $t_i \in \widetilde{T}_i(E)$ has a preference relation $\succsim_{t_i}$ on player i's strategies in S_i. For our purpose, we need the following conditions for the model of type on Ω.

C3 (Continuity) For any sequence of events $\left\{E^l\right\}_{l=1}^{\infty}$, $\cap_{l=1}^{\infty}\widetilde{T}_i\left(E^l\right) \subseteq \widetilde{T}_i\left(\cap_{l=1}^{\infty}E^l\right)$ $\forall i$.

C4 (Consistency) For any event $E \subseteq \Omega$, $\widetilde{T}_i(E) = T_i(S^E)$ $\forall i$.

The continuity condition C3 requires that the intersection of type spaces on a sequence of events is included in the type space on the intersection of the sequence of events. This kind of condition is related to the property of knowledge structure termed "limit closure" in Fagin et al. (1999), which is satisfied by most of type models discussed in the literature, e.g., (countably additive) probability measure spaces and regular preference models. The consistency condition C4 requires that the type space on an event is consistent with the type space on the strategies projected from the event. This condition is much in the same spirit of "marginal consistency" imposed on preference models (see Epstein (1997)).

Say "*player i knows an event E at ω*" if $\mathbf{t}_i(\omega) \in \widetilde{T}_i(E)$. Let

$$K_i E \equiv \{\omega \in \Omega |\ i \text{ knows } E \text{ at } \omega\}.$$

For simplicity, we assume the knowledge operator satisfies the axiom of knowledge, i.e., $K_i E \subseteq E$. An event $\boxed{E}$ is called a *self-evident event* in E, if $K\boxed{E} = \boxed{E} \subseteq E$. Define the event "*$E$ is mutual knowledge*" as:

$$KE \equiv \cap_{i \in N} K_i E,$$

and the event "*E is common knowledge*" as:

$$CKE \equiv \cap_{l=1}^{\infty} K^l E$$

where $K^1 E = KE$ and $K^l E = K\left(K^{l-1}E\right)$ for $l \geq 2$. The following lemma shows that some useful properties about the knowledge operator K and the common knowledge operator CK. It is easy to see that these properties are satisfied by the standard semantic model of knowledge with partitional information structures (see, e.g., Osborne and Rubinstein (1994, Chapter 5)).

Lemma 4. *The operators K and CK satisfy the following properties:*

1. $E \subseteq F \Rightarrow KE \subseteq KF$.

2. $\cap_{l=1}^{\infty} KE^l = K\left(\cap_{l=1}^{\infty} E^l\right)$.

3. $\omega \in CKE$ *iff* $\omega \in \boxed{E}$ *for some self-evident* $\boxed{E} \subseteq E$.

Proof. (1) Let $\omega \in KE$. Then $\mathbf{t}_i(\omega) \in \widetilde{T}_i(E)\ \forall i$. If $E \subseteq F$, by C1 and C4, $\widetilde{T}_i(E) \subseteq \widetilde{T}_i(F)$. Hence $\mathbf{t}_i(\omega) \in \widetilde{T}_i(F)\ \forall i$, i.e., $\omega \in KF$.

(2) By Lemma 4.1, it suffices to show $\cap_{l=1}^{\infty} KE^l \subseteq K\left(\cap_{l=1}^{\infty} E^l\right)$. Let $\omega \in \cap_{l=1}^{\infty} KE^l$. Then $\omega \in KE^l$ for all $l \geq 1$ iff, for all i, $\mathbf{t}_i(\omega) \in \widetilde{T}_i\left(E^l\right)$ for all $l \geq 1$ iff, for all i, $\mathbf{t}_i(\omega) \in \cap_{l=1}^{\infty} \widetilde{T}_i\left(E^l\right)$. By C3, $\mathbf{t}_i(\omega) \in \widetilde{T}_i\left(\cap_{l=1}^{\infty} E^l\right)$ for all i, i.e., $\omega \in K\left(\cap_{l=1}^{\infty} E^l\right)$.

(3) ("Only if" part.) Let $\omega \in CKE$. By $KE \subseteq E$ and Lemma 4.1, $K^{l+1}E \subseteq K^l E$ for all $l \geq 1$. By Lemma 4.2, $K(CKE) = K\left(\cap_{l=1}^{\infty} K^l E\right) = \cap_{l=2}^{\infty} K^l E = CKE$. Let $\boxed{E} = CKE$. Then $\boxed{E}$ is self-evident and $\omega \in \boxed{E}$.

("If" part.) Let $\omega \in \boxed{E} = K\boxed{E} \subseteq E$. By Lemma 4.1, $K^{l+1}\boxed{E} = K^l\boxed{E} \subseteq K^l E$ for all $l \geq 1$. So $\omega \in \boxed{E} \subseteq \cap_{l=1}^{\infty} K^l E = CKE$. $\square$

Say player i is *"rational at ω"* if $\mathbf{s}_i(\omega)$ is a best response to $\mathbf{t}_i(\omega)$. Let

$$\mathfrak{R}_i \equiv \{\omega \in \Omega |\ i \text{ is rational at } \omega\}$$

and

$$\mathfrak{R} \equiv \cap_{i \in N} \mathfrak{R}_i.$$

That is, $\mathfrak{R}$ is the event "everyone is rational." The following Theorem 3 provides epistemic conditions for the notion of rationalizability. This result shows that rationalizability is the strategic implication of common knowledge of rationality.

Theorem 3. *For any model of knowledge, $S^{CK\mathfrak{R}} \subseteq R^*$. Moreover, there is a model of knowledge such that $S^{CK\mathfrak{R}} = R^*$.*

Proof. Let $s \in S^{CK\mathfrak{R}}$. Then there exists $\omega \in CK\mathfrak{R}$ such that $\mathbf{s}(\omega) = s$. By Lemma 4.3, $\omega \in \boxed{\mathfrak{R}}$ for some self-evident event $\boxed{\mathfrak{R}} \subseteq \mathfrak{R}$. Therefore, for any $\omega' \in \boxed{\mathfrak{R}}$, $\mathbf{s}_i(\omega') \in BR(\mathbf{t}_i(\omega'))$ and $\mathbf{t}_i(\omega') \in \widetilde{T}_i\left(\boxed{\mathfrak{R}}\right)$ for all i. By C4, $\mathbf{t}_i(\omega') \in T_i\left(S^{\boxed{\mathfrak{R}}}\right)$. Thus, $S^{\boxed{\mathfrak{R}}}$ is rationalizable and hence $s \in S^{\boxed{\mathfrak{R}}} \subseteq R^*$.

Define $\Omega \equiv R^*$. For any $\omega = \{s_i\}_{i \in N} \in \Omega$, for every player i define $\mathbf{s}_i(\omega) = s_i$ and $\mathbf{t}_i(\omega) = t_i \in T_i(R^*)$ such that $s_i \in BR(t_i)$. Clearly, every player i is rational across states in Ω. By C4, $\Omega \subseteq K\Omega$. Therefore, $\Omega = CK\mathfrak{R}$ and, hence, $S^{CK\mathfrak{R}} = R^*$. □

6. Concluding remarks

In this paper we have presented a unified framework to analyze rationalizable strategic behavior in any arbitrary game. In particular, we introduce the "model of type" to define the notion of rationalizability in games with (in)finite players, arbitrary strategy spaces, and arbitrary payoff functions. One important feature of this paper is that the framework allows the players to have various preferences which include subjective expected utility as a special case.

We have investigated properties about rationalizability in general situations. More specifically, we have shown that the union of all the rationalizable sets is the largest rationalizable set (Lemma 2) and is in the Cartesian-product form (Lemma 3). Moreover, we have shown that the largest rationalizable set can be derived by the (possibly transfinite) iterated elimination process – i.e., IENBR (Theorem 1). As a by-product, we have obtained that IENBR is a well-defined order-independent procedure in general situations (Corollary 1).

In this paper we have investigated the relationship between rationalizability and Nash equilibrium in general games. While every Nash equilibrium survives the IENBR procedure, a Nash equilibrium in the final reduced game after IENBR may fail to be a Nash equilibrium in the original game. That is, the IENBR procedure may generate spurious Nash equilibria in infinite games. We have thus provided a sufficient and necessary condition to guarantee no spurious Nash equilibria (Theorem 2). In this paper we have also formulated and proved that rationalizability is the strategic implication of common knowledge of rationality in general settings (Theorem 3).

To close this paper, we would like to point out some extensions of this paper for future research. The exploration of the notion of extensive-form rationalizability in dynamic games remains an interesting subject for further study. The extension of this paper to games with incomplete information is clearly an important subject for further research. The extension of this paper to permit social and coalitional interactions in the notion of rationalizability is also an intriguing topic worth further investigation; cf. Ambrus (2006) and Luo and Yang (2009) for the related research on coalitional rationalizability.

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