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TIGHT WAVELET FRAME PACKET

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Abstract

In this thesis, we study the construction of stationary and nonstationary tight wavelet frame packets and the characterization of Sobolev spaces by them. We also extend our study to the construction of their 2^{-J}-shift invariant counterparts and using them to characterize Sobolev spaces.

Keywords:

tight wavelet frame, stationary tight wavelet frame packet, nonstationary tight wavelet frame packet, Sobolev space, 2^{-J}-shift invariant

Contents

Summary

In this thesis, we study the construction of stationary and nonstationary tight wavelet frame packets and the characterization of Sobolev spaces by them. We also extend our study to the construction of their 2^{-J}-shift invariant counterparts and using them to characterize Sobolev spaces.

After a brief introduction, we provide in Chapter 2 some preliminaries related to the development of this thesis. In Chapter 3, we introduce the construction of stationary tight wavelet frame packet and its characterization of Sobolev spaces. In Chapter 4, we introduce the construction of nonstationary tight wavelet frame packet and its characterization of Sobolev spaces. At last, in Chapter 5, we introduce the construction of 2^{-J}-shift invariant nonstationary tight wavelet frame packet and its characterization of Sobolev spaces.

Basic Notations

- $\ell_p(\mathbb{Z})(1 \le p \le \infty)$ spaces. $\ell_p(\mathbb{Z})$ consists of complex-valued sequences on $\mathbb{Z}$ satisfying

$$\|c\|_{\ell_p(\mathbb{Z})} := \begin{cases} \left(\sum_{k\in\mathbb{Z}} |c(k)|^p\right)^{\frac{1}{p}} < \infty, & 1 \le p < \infty; \\ \sup_{k\in\mathbb{Z}} |c(k)| < \infty, & p = \infty. \end{cases}$$

- $L_p(\mathbb{R})(1 \le p \le \infty)$ spaces. $L_p(\mathbb{R})$ consists of Lebesgue measurable functions satisfying

$$\|f\|_{L_p(\mathbb{R})} := \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p dt\right)^{\frac{1}{p}} < \infty, & 1 \le p < \infty; \\ \operatorname{ess\,sup}\left\{f(x) : x \in \mathbb{R}\right\} < \infty, & p = \infty. \end{cases}$$

- $\mathcal{S}'$, the class of **tempered distributions** which is the **dual space** of the **Schwartz space** $\mathcal{S}$, where

$$\mathcal{S} := \left\{ f \in C^\infty(\mathbb{R}) : \sup_{n\le N} \sup_{x\in\mathbb{R}} (1 + |x|^2)^N |f^{(n)}(x)| < \infty, \text{ for all } n, N \in \mathbb{N} \right\}.$$

- The **inner product** $\langle \cdot, \cdot \rangle$ of the Hilbert space $L_2(\mathbb{R})$ is given by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}dt,$$

which also induced the **norm** $\| \cdot \|_{L_2(\mathbb{R})}$ of $L_2(\mathbb{R})$ by $\|f\|_{L_2(\mathbb{R})} = |\langle f, f \rangle|^{1/2}$.

- For $f, g \in L_1(\mathbb{R})$, the **convolution** of f and g is defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(t)g(x-t)dt.$$

For $a, b \in \ell_1(\mathbb{Z})$, the convolution of a and b is defined by

$$(a * b)(n) := \sum_{m \in \mathbb{Z}} a(m)b(n-m).$$

- The **Fourier transform** of a function $f \in L_1(\mathbb{R})$ is defined by

$$(\mathcal{F}f)(\omega) = \widehat{f}(\omega) := \int_{\mathbb{R}} f(t)e^{-it\omega}dt. \tag{0.1}$$

$\mathcal{F}$ maps the Schwartz space $\mathcal{S}$ onto itself, and extends to all tempered distributions $\mathcal{S}'$ by duality.

- The **Fourier series** of a sequence $c \in \ell_2(\mathbb{Z})$ will be denoted by $\widehat{c}$ and is defined by

$$\widehat{c}(\omega) := \sum_{n \in \mathbb{Z}} c(n)e^{-in\omega}. \tag{0.2}$$

Note that $\widehat{c}(\omega)$ is a complex-valued 2π-periodic continuous function on $\mathbb{R}$ and thus is defined on the torus $\mathbb{T}$.

- For a real number s, we denote by $\mathbb{H}^s(\mathbb{R})$ the Sobolev space consisting of all tempered distributions f such that

$$\|f\|^2_{\mathbb{H}^s(\mathbb{R})} := \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\omega)|^2 (1 + |\omega|^2)^s \, d\omega < \infty.$$

Note that $\mathbb{H}^0(\mathbb{R}) = L_2(\mathbb{R})$ and $\| \cdot \|_{\mathbb{H}^0(\mathbb{R})} = \| \cdot \|_{L_2(\mathbb{R})}$ by the Plancherel's theorem.

- For $f, g \in L_2(\mathbb{R})$, we define the **bracket product** function $[\cdot,\cdot]$ as

$$[f,g] = \sum_{k\in\mathbb{Z}} f(\cdot + 2\pi k)\overline{g(\cdot + 2\pi k)}. \tag{0.3}$$

  And $[f,g] \in L_1(\mathbb{T})$ whenever $f, g \in L_2(\mathbb{R})$.

- For $f, g \in L_2(\mathbb{R})$, $[\cdot,\cdot]_s$ is defined as

$$[f,g]_s = \sum_{k\in\mathbb{Z}} f(\cdot + 2\pi k)\overline{g(\cdot + 2\pi k)}(1 + |\cdot + 2\pi k|^2)^s. \tag{0.4}$$

  Note that $[f,g]_0 = [f,g]$.

- $\mathbf{E}$ is the *translation operator*, i.e., for any $t \in \mathbb{R}$,

$$\mathbf{E}^t f := f(\cdot - t), \tag{0.5}$$

  and $\mathbf{D}$ is the *dyadic dilation operator*, i.e., for any $j \in \mathbb{Z}$,

$$\mathbf{D}^j f := 2^{j/2} f(2^j \cdot). \tag{0.6}$$

Chapter 1

Introduction

Since the formulation of Multiresolution Analysis (MRA) by Mallat and Meyer [60, 59, 61] and the construction of Daubechies' celebrated compactly supported wavelets [21, 22], wavelets theory and its applications have gained enormous popularity in both theory and applications. The success of wavelets leads to the discovery of tight wavelet frames (or tight affine frames) [65, 67, 66, 69, 68, 39, 24, 12, 11] which are more flexible and much easier to construct than wavelets.

Historically, frames were introduced by Duffin and Schaeffer in 1952 to study nonharmonic Fourier series [36]. Univariate wavelet frames (or affine frames) were studied by Daubechies, Grossmann and Meyer in [23] in 1986. A breakthrough on the understanding and systematic construction of orthonormal wavelet frames (or orthonormal wavelet bases) was achieved after the formulation of multiresolution analysis (MRA) formulated in the fall of 1986 by Mallat and Meyer [60, 59, 61] which culminated in the construction of the celebrated Daubechies' compactly supported orthonormal wavelet frames [21, 22] in 1988. However, MRA does not suggest the characterization of orthonormal wavelet frames. Univariate tight wavelet frame characterization implicitly appeared in [48, 37] in the work of Weiss et al in 1996. An explicit multivariate tight wavelet frame characterization was obtained by Han in [39] in 1997. Independently, a general characterization of wavelet frames was obtained by Ron and Shen in [66] in 1997, and by specializing their general theory the characterization of tight wavelet frames was obtained. Furthermore, a characterization of all tight wavelet frames that can be constructed in an MRA was also obtained in [66] (Note that one of its basic theorems [66, Theorem 5.5] was proved under a mild decay condition which was subsequently removed by Chui et al [13]). And MRA-based tight wavelet frames could be constructed via unitary extension principles (UEP) or oblique extension principles (OEP) which makes the construction of tight wavelet frames painless [66, 68, 24].

Compared with the construction of wavelets, which requires a refinable function with orthonormal shifts, tight wavelet frames can be derived from a much larger class of refinable functions which will be detailed in Chapter 2. We do not even need to assume that the shifts of the refinable function form a Riesz basis, or a frame. This flexibility allows us to construct tight wavelet frame that adapts to practical problems. It also gives a wide choice of tight wavelet frames that provide better approximation for a given underlying function.

To further extend the flexibility of tight wavelet frames, we build up the theory and construction of stationary and nonstationary tight wavelet frame packets. Given a tight wavelet frame, associated with it we can either construct a stationary tight wavelet frame packet or construct a nonstationary tight wavelet frame packet, depending on whether we want to change the underlying MRA or not. Compared with other constructions ([58, 8]), our constructions are based on the unitary extension principle (UEP) ([66, 24]). These constructions give rise to a library of tight wavelet frames. Then, by using tight wavelet fame packets we can do the " best basis selection " for a practical problem. And this is appealing for applications. Therefore, tight wavelet frame packets further extend the flexibility of tight wavelet frames.

In frequency domain, tight wavelet frame packets provides more flexibility of partitioning the frequency axis which is desirable in applications, since usually in practice the class of signals to be considered has certain frequency pattern. By using tight wavelet frame packets, we can build a wavelet system that is adapted to the intrinsic frequency pattern of the class of signals to be considered. In this way, we can manage to obtain a sparse representation of the class of signals in time domain.

In fact, stationary or nonstationary tight wavelet frame packets have been applied in the application of high-resolution image reconstruction [6, 7] and in the restoration of chopped and nodded images [3] in the denoising procedure to improve the performance.

In Chapter 2, we will give some preliminaries of tight wavelet frames (or tight affine frames). In Chapter 3, we introduce the construction of stationary tight wavelet frame packet and its characterization of Sobolev spaces. In Chapter 4, we introduce the construction of nonstationary tight wavelet frame packet and its characterization of Sobolev spaces. At last, in Chapter 5, we introduce the construction of 2^{-J}-shift invariant nonstationary tight wavelet frame packet and its characterization of Sobolev spaces.

Chapter 2

Preliminary

In this chapter, we introduce some preliminaries closely related to our study. In section 1, we introduce the **principal shift invariant (PSI)** spaces which serve as the building blocks for the study of wavelet systems (or affine systems). In section 2, we introduce the framework of **multiresolution analysis (MRA)** which is crucial for the understanding and construction of **tight wavelet frames (TWF)**. In section 3, we introduce the theory of wavelet frames (or affine frames) and the construction of TWF via the **unitary extension principal (UEP)** which makes the construction of such systems painless. And also, the class of **pseudo splines**, which is a larger set of **refinable functions** taking B-splines as its special subset, are introduced for the construction of TWF with any prescribed approximation order. In section 4, we introduce the **nonstationary tight wavelet frames (NTWF)** and the construction of NTWF that can achieve spectral approximation order. Finally, in section 5, we introduce the characterization of Sobolev spaces $\mathbb{H}^s(\mathbb{R})$ by NTWF.

2.1 Principal Shift Invariant (PSI) Spaces

In this section, we introduce the **principal shift invariant (PSI)** spaces. Each PSI space is a closed subspace of $L_2(\mathbb{R})$ that can be easily constructed with a single function $\phi \in L_2(\mathbb{R})$. PSI spaces serve as the building blocks for the study of wavelet systems (or affine systems) to be introduced in section 3.

Definition 2.1. *We say that a space S of complex-valued functions on $\mathbb{R}$ is* **shift invariant** *if, for each $f \in S$, S also contains its shifts $\mathbf{E}^k f = f(\cdot - k), k \in \mathbb{Z}$, where $\mathbf{E}$ is the translation operator as defined in* (0.5).

Given $\phi \in L_2(\mathbb{R})$, the set of all shifts of ϕ is denoted by

$$E(\phi) := \{\mathbf{E}^k \phi : k \in \mathbb{Z}\}. \tag{2.1}$$

The shift invariant space generated by ϕ, denoted by $S(\phi)$, is the smallest closed linear subspace in $L_2(\mathbb{R})$ containing $E(\phi)$, i.e.,

$$S(\phi) := \overline{\operatorname{span}} \left\{\mathbf{E}^k \phi : k \in \mathbb{Z}\right\}. \tag{2.2}$$

And $S(\phi)$ is called the **principal shift invariant (PSI)** space generated by ϕ.

The characterization of $S(\phi)$ was obtained by de Boor, Devore and Ron in the Fourier domain.

Theorem 2.1. ([27]) *Let $\phi \in L_2(\mathbb{R})$, then the PSI space $S(\phi)$ as defined in (2.2) is characterized by*

$$\widehat{S(\phi)} = \left\{\tau \widehat{\phi} \in L_2(\mathbb{R}) : \tau \textit{ is } 2\pi \textit{ periodic}\right\}. \tag{2.3}$$

Define $S_j(\phi) := \mathbf{D}^j S(\phi), j \in \mathbb{Z}$, where $\mathbf{D}$ is the dilation operator as defined in (0.6)*, then $S_j(\phi)$ is characterized by*

$$\widehat{S_j(\phi)} = \left\{\left(\tau \widehat{\phi}\right)\left(\frac{\cdot}{2^j}\right) \in L_2(\mathbb{R}) : \tau \textit{ is } 2\pi \textit{ periodic}\right\}. \tag{2.4}$$

Definition 2.2. *Given a PSI space $S(\phi)$, define the* **synthesis operator**

$$\mathbf{T}_{E(\phi)} : \ell_2(\mathbb{Z}) \to S(\phi) : c \mapsto \sum_{k\in\mathbb{Z}} c(k)\mathbf{E}^k\phi,$$

and the **analysis operator**

$$\mathbf{T}^*_{E(\phi)} : S(\phi) \to \ell_2(\mathbb{Z}) : f \mapsto \big(\langle f, \mathbf{E}^k\phi\rangle\big)_{k\in\mathbb{Z}},$$

which is the adjoint of the synthesis operator $\mathbf{T}_{E(\phi)}$.

- *If $\mathbf{T}_{E(\phi)}$ (or $\mathbf{T}^*_{E(\phi)}$) is bounded, then $E(\phi)$ is called a* **Bessel set** *of $S(\phi)$;*

- *If $\mathbf{T}_{E(\phi)}$ is bounded and bounded below, i.e., there exist two positive constants C_1, C_2 such that the inequalities*

$$C_1\|c\|^2_{\ell_2(\mathbb{Z})} \le \Big\|\sum_{k\in\mathbb{Z}} c(k)\mathbf{E}^k\phi\Big\|^2_{L_2(\mathbb{R})} \le C_2\|c\|^2_{\ell_2(\mathbb{Z})}, \tag{2.5}$$

  hold for all $c \in \ell_2(\mathbb{Z})$, then $E(\phi)$ is called a **Riesz basis** *of $S(\phi)$, where C_1 and C_2 are called the* **lower Riesz bound** *and* **upper Riesz bound**, *respectively.*

  In particular, if $C_1 = C_2 = 1$, then $E(\phi)$ is an **orthonormal basis** *of $S(\phi)$;*

- *If $\mathbf{T}^*_{E(\phi)}$ is bounded and bounded below, i.e., there exist two positive constants C_1, C_2 such that the inequalities*

$$C_1\|f\|^2_{L_2(\mathbb{R})} \le \sum_{k\in\mathbb{Z}} |\langle f, \mathbf{E}^k\phi\rangle|^2 \le C_2\|f\|^2_{L_2(\mathbb{R})}, \tag{2.6}$$

  hold for all $f \in S(\phi)$, then $E(\phi)$ is called a **frame** *of $S(\phi)$, where C_1 and C_2 are called the* **lower frame bound** *and* **upper frame bound**, *respectively.*

  In particular, if $C_1 = C_2$, then $E(\phi)$ is called a **tight frame** *of $S(\phi)$.*

Note that when $E(\phi)$ is a Riesz basis of $S(\phi)$, the 2π-periodic function τ in (2.3) is in $L_2(\mathbb{T})$.

Definition 2.3. *Let $\phi \in L_2(\mathbb{R})$. The set*

$$\sigma(S(\phi)) := \left\{\omega \in [-\pi, \pi] : [\widehat{\phi}, \widehat{\phi}](\omega) \neq 0\right\} \tag{2.7}$$

is called the **spectrum** *of the shift-invariant space $S(\phi)$.*

The following bracket product function

$$[\widehat{\phi}, \widehat{\phi}] = \sum_{k \in \mathbb{Z}} |\widehat{\phi}(\cdot + 2\pi k)|^2,$$

is also called **symbol** of ϕ, which plays important roles in the study of PSI spaces.

Theorem 2.2. *Let $\phi \in L_2(\mathbb{R})$, then*

- *$E(\phi)$ is a Bessel set of $S(\phi)$ if and only if there exists some positive constant C such that*

$$[\widehat{\phi}, \widehat{\phi}](\omega) \leq C, \quad a.e. \quad \omega \in \mathbb{R};$$

- *$E(\phi)$ is a Riesz basis of $S(\phi)$ if there exist two positive constants C_1, C_2 such that*

$$C_1 \leq [\widehat{\phi}, \widehat{\phi}](\omega) \leq C_2, \quad a.e. \quad \omega \in \mathbb{R};$$

  In particular, $E(\phi)$ is an orthonormal basis of $S(\phi)$ if and only if

$$[\widehat{\phi}, \widehat{\phi}] = 1, \quad a.e. \quad \omega \in \mathbb{R};$$

- *$E(\phi)$ is a frame of $S(\phi)$ if there exist two positive constants C_1, C_2 such that*

$$C_1 \leq [\widehat{\phi}, \widehat{\phi}](\omega) \leq C_2, \quad a.e. \quad \omega \in \sigma(S(\phi)).$$

  In particular, $E(\phi)$ is a tight frame of $S(\phi)$ if and only if

$$[\widehat{\phi}, \widehat{\phi}](\omega) = C, \quad a.e. \quad \omega \in \sigma(S(\phi)).$$

2.2 Multiresolution Analysis (MRA)

In this section, we first introduce the **multiresolution analysis (MRA)** framework, then introduce its explicit construction from by dilating PSI spaces.

MRA was formulated in the fall of 1986 by Mallat and Meyer [60], it provides a natural framework for the understanding of orthonormal wavelet frames and for the systematic construction of new examples [60, 59, 21, 22]. More precisely, an MRA consists of a nested sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L_2(\mathbb{R})$ satisfying

$$\text{(i).}\quad \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots; \tag{2.8}$$

$$\text{(ii).}\quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R}); \tag{2.9}$$

$$\text{(iii).}\quad \bigcap_{j\in\mathbb{Z}} V_j = \{\emptyset\}, \tag{2.10}$$

as depicted in Figure 2.1. Note that this is not the original MRA introduced by Mallat in [60], it is a generalized version push forward by de Boor, DeVore and Ron [26].

2.2.1 MRA Construction

To construct an MRA $(V_j)_{j\in\mathbb{Z}}$, we start from a PSI spaces $S(\phi), \phi \in L_2(\mathbb{R})$, and define a sequence of closed subspaces by

$$(V_j)_{j\in\mathbb{Z}} := (\mathbf{D}^j S(\phi))_{j\in\mathbb{Z}}. \tag{2.11}$$

With the following result, we can see the MRA condition (2.10) is trivially satisfied.

Theorem 2.3. [26][56] *Let $\phi \in L_2(\mathbb{R})$. Then, $(V_j)_{j\in\mathbb{Z}}$ as defined in (2.11) satisfies the MRA condition (2.10).*

To make $(V_j)_{j\in\mathbb{Z}}$ in (2.11) also satisfies the MRA condition (2.8), we introduce the class of **refinable functions**.

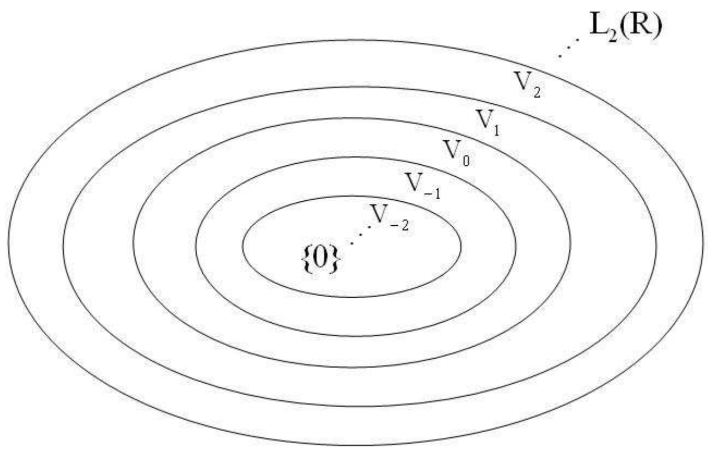


Figure 2.1: The MRA Framework

Definition 2.4. *A function $\phi \in L_2(\mathbb{R})$ is said to be* **refinable** *if ϕ satisfies a* **refinement equation**

$$\phi = \sum_{n \in \mathbb{Z}} 2\, c(n) \phi(2 \cdot -n), \tag{2.12}$$

where the discrete sequence $c \in \ell_2(\mathbb{Z})$ is called the **refinement mask** *of ϕ.*

Note that the refinement equation (2.12) *can be recast in Fourier domain as*

$$\widehat{\phi}(\omega) = \widehat{c}(\omega/2)\widehat{\phi}(\omega/2), \tag{2.13}$$

and the 2π-periodic function $\widehat{c}(\omega) = \sum_{n \in \mathbb{Z}} c(n) e^{-in\omega}$ is also referred to as the refinement mask for notational convenience.

Example 2.1. *The characteristic function $\phi = \chi_{[0,1)}$ is refinable with the refinement mask $c = (\cdots, 0, \mathbf{\frac{1}{2}}, \frac{1}{2}, 0, \cdots)$. The corresponding refinement equation is given by*

$$\phi = \phi(2\cdot) + \phi(2 \cdot -1).$$

Example 2.2. *The hat function $\phi(x) = \begin{cases} 1 - |x| & x \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases}$ is refinable with the*

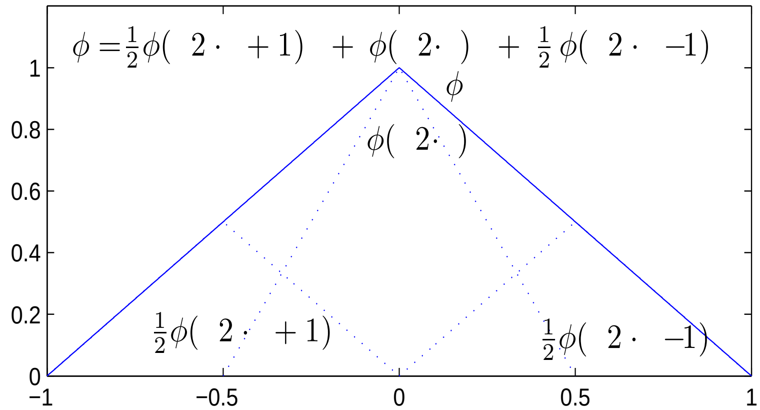


Figure 2.2: The hat function

refinement mask $c = (\cdots, 0, \frac{1}{4}, \mathbf{\frac{1}{2}}, \frac{1}{4}, 0, \cdots)$. *The corresponding refinement equation is given by*

$$\phi = \frac{1}{2}\,\phi(2\cdot+1) + \phi(2\cdot) + \frac{1}{2}\,\phi(2\cdot-1),$$

as depicted in Figure 2.2.

If ϕ is refinable, then by (2.13) and the characterization result (2.4) we can obtain $V_j \subseteq V_{j+1}$. In other words, if $\phi \in L_2(\mathbb{R})$ is refinable, then $(V_j)_{j\in\mathbb{Z}}$ in (2.11) trivially satisfies the MRA condition (2.8) and (2.10). Thus the construction of an MRA in this way is reduced to the problem when $(V_j)_{j\in\mathbb{Z}}$ in (2.11) satisfies the MRA condition (2.9). It is answered by the following result.

Theorem 2.4. [26][56] *Let $\phi \in L_2(\mathbb{R})$ be a refinable function. Then, $(V_j)_{j\in\mathbb{Z}}$ as defined in (2.11) satisfies the MRA condition (2.9) if and only if*

$$\bigcap_{j\in\mathbb{Z}} (2^j Z(\widehat{\phi})) \textit{ is a set of measure zero,} \tag{2.14}$$

where $Z(\widehat{\phi})$ is the zero set of $\widehat{\phi}$.

Theorem 2.4 implies that an MRA $(V_j)_{j\in\mathbb{Z}}$ can be generated by any refinable function $\phi \in L_2(\mathbb{R})$ satisfying (2.14), and such ϕ is also called an **MRA generator**. An interesting special case is worthy to be mentioned:

Corollary 2.1. [26][56] *If a refinable function $\phi \in L_2(\mathbb{R})$ is compactly supported, then $(V_j)_{j\in\mathbb{Z}}$ as defined in (2.11) forms an MRA, i.e., any compactly supported refinable function $\phi \in L_2(\mathbb{R})$ is an MRA generator.*

Proof. Since ϕ has compact support, then $\widehat{\phi}$ is analytic and its zero set is of measure zero (unless $\phi = 0$). The result is immediately followed from Theorem 2.4. $\square$

Corollary 2.1 draws our attention to the class of compactly supported refinable functions. Coming up next, we will review the basic results of refinable functions, especially the subclass of compactly supported refinable functions.

2.2.2 Refinable Functions

Now that we have introduced the PSI spaces and the MRA framework together with its construction from refinable functions satisfying (2.14). In this subsection, we review the basic results of refinable functions. We will see that the properties of a refinable function ϕ are completely determined by its refinement mask c, and also, not surprisingly, the Bessel set, frame and Riesz properties of $E(\phi)$ can be recast in terms of the refinement mask c.

Theorem 2.5. ([22]) *If the refinement mask c is a finitely supported sequence satisfying*

$$\sum_{n=N_1}^{N_2} c(n) = 1,$$

then there exists a compactly supported refinable tempered distribution ϕ supported in $[N_1, N_2]$ unique up to a constant multiple, such that its Fourier transform admits *the infinite product representation*

$$\widehat{\phi}(\omega) = \widehat{\phi}(0) \prod_{j=1}^{\infty} \widehat{c}(2^{-j}\omega), \tag{2.15}$$

where the infinite product converges uniformly on every compact set of $\mathbb{R}$.

Theorem 2.6. ([5, 46]) *Suppose* $\widehat{\phi}(\omega) := \prod_{n=1}^{\infty} \widehat{c}(2^{-n}\omega)$ *is well-defined for a.e.* $\omega \in \mathbb{R}$, *and* $\widehat{c}$ *satisfies the inequality*

$$|\widehat{c}(\omega)|^2 + |\widehat{c}(\omega + \pi)|^2 \leq 1. \tag{2.16}$$

Then

$$[\widehat{\phi}, \widehat{\phi}](\omega) \leq 1, \quad a.e. \quad \omega \in \mathbb{R}, \tag{2.17}$$

i.e., $E(\phi)$ *is a Bessel set of* $S(\phi)$. *Consequently,* $\phi \in L_2(\mathbb{R})$ *with* $\|\phi\|_{L_2(\mathbb{R})} \leq 1$.

By Theorem 2.5 and Theorem 2.6, if the refinement mask c is finitely supported with $\widehat{c}(0) = 1$ and satisfying the inequality (2.16), then we immediately have $E(\phi)$ is a Bessel set of $S(\phi)$ and $\|\phi\|_{L_2(\mathbb{R})} \leq 1$.

Example 2.3. *B-splines* $B_m, m \in \mathbb{N}$, *are compactly supported refinable functions with the corresponding refinement mask*

$$\widehat{c_m}(\omega) = e^{i(m-\mathcal{K})\omega/2} \Big(\frac{1 + e^{-i\omega}}{2}\Big)^m = e^{-i\mathcal{K}\omega/2} \cos^m \omega, \quad m \in \mathbb{N},$$

where $\mathcal{K} = 0$ *if* m *is even,* $\mathcal{K} = 1$ *if* m *is odd.*

Obviously, $|\widehat{c_m}(\omega)|^2 + |\widehat{c_m}(\omega + \pi)|^2 = \cos^{2m} \omega + \sin^{2m} \omega \leq (\cos^2 \omega + \sin^2 \omega)^m = 1$. *By Theorem 2.6, we have* $[\widehat{B_m}, \widehat{B_m}] \leq 1$, *i.e., for every* $m \in \mathbb{N}$, $E(B_m)$ *is a Bessel set of* $S(B_m)$, *and* $\|B_m\|_{L_2(\mathbb{R})} \leq 1$.

Theorem 2.7. ([54]) *If* $\phi \in L_1(\mathbb{R})$ *is refinable, then*

$$\widehat{\phi}(2\pi k) = 0 \ for \ k \in \mathbb{Z}\backslash\{0\}. \tag{2.18}$$

Proof. It follows from (2.13) that

$$\widehat{\phi}(\omega) = \widehat{\phi}(2^{-k}\omega) \prod_{j=1}^{k} \widehat{c}(2^{-j}\omega). \tag{2.19}$$

If $|\,\widehat{c}(0)| = |\sum_{n\in\mathbb{Z}} c(n)| < 1$, then choosing $\omega = 0$ in (2.19), we obtain $\widehat{\phi}(0) = 0$. Moreover, $|\widehat{c}(0)| < 1$ implies that for any fixed $\omega \in \mathbb{R}$ and sufficiently large j,

$$|\widehat{c}(2^{-j}\omega)| < 1.$$

Thus, letting $k \to \infty$ in (2.19), we obtain $\widehat{\phi} = 0$. This is true for any $\omega \in \mathbb{R}$, hence $\phi = 0$. Now suppose $|\,\widehat{c}(0)| \geq 1$. Choosing $\omega = 2^{k+1}k\pi$ in (2.19), where $k \in \mathbb{Z}\backslash\{0\}$, we obtain

$$\widehat{\phi}(2^{k+1}k\pi) = (\widehat{c}(0))^k \widehat{\phi}(2k\pi).$$

It follows that

$$|\widehat{\phi}(2k\pi)| \leq |\phi(2^{k+1}k\pi)|.$$

Letting $k \to \infty$ in the above inequality and applying the Riemann-Lebesgue lemma, we obtain

$$\widehat{\phi}(2k\pi) = 0, \text{ for } k \in \mathbb{Z}\backslash\{0\}.$$

$\square$

Notice that a compactly supported function $\phi \in L_2(\mathbb{R})$ is also in $L_1(\mathbb{R})$, by Theorem 2.7, we quickly have

Corollary 2.2. *If a refinable function $\phi \in L_2(\mathbb{R})$ is compactly supported, then*

$$\widehat{\phi}(2\pi k) = 0 \textit{ for } k \in \mathbb{Z}\backslash\{0\}.$$

Theorem 2.8. ([60, 59, 61]) *Suppose $\phi \in L_2(\mathbb{R})$ is refinable and also assume that $E(\phi)$ is an orthonormal basis of $S(\phi)$, then its refinement mask c satisfies*

$$\widehat{c}(0) = 1, \tag{2.20}$$

and

$$|\widehat{c}(\omega)|^2 + |\widehat{c}(\omega+\pi)|^2 = 1. \tag{2.21}$$

Note that a sequence c which satisfies (2.20) and (2.21) is called a **conjugate quadrature filter** (CQF). And the two conditions, (2.20) and (2.21), are also referred as the **CQF condition** in the wavelets literature.

Note that the CQF condition is not sufficient to define a refinable function ϕ with orthonormal shifts. A counterexample is given by

Example 2.4. *Let ϕ be a refinable function with its refinement mask*

$$\widehat{c}(\omega) = \frac{1+e^{-3i\omega}}{2}, \tag{2.22}$$

then

$$\widehat{\phi}(\omega) = \prod_{j=1}^{\infty} \widehat{c}(2^{-j}\omega) = e^{-3i\omega/2}\frac{\sin(3\omega/2)}{3\omega/2} \tag{2.23}$$

or

$$\phi(x) = \begin{cases} \frac{1}{3} & 0 \le x \le 3, \\ 0 & \textit{otherwise}. \end{cases} \tag{2.24}$$

Obviously, the integer shifts of ϕ are not orthonormal.

Furthermore, we can obtain that $[\widehat{\phi},\widehat{\phi}](\omega) = \frac{1}{3}+\frac{4}{9}\cos(\omega)+\frac{2}{9}\cos(2\omega)$, $[\widehat{\phi},\widehat{\phi}](\frac{2\pi}{3}) = 0$, i.e., $E(\phi)$ is also not a frame of $S(\phi)$ by Theorem 2.2. However, $E(\phi)$ is a Bessel set of $S(\phi)$ by Theorem 2.6. Also, we can observe that $\|\phi\|_{L_2(\mathbb{R})} = \frac{\sqrt{3}}{3} < 1$.

Definition 2.5. *Given $f \in L_2(\mathbb{R})$, we call the function f_{au} defined by*

$$f_{au}(x) := (f * \overline{f}(-\cdot))(x) = \int_{\mathbb{R}} f(t)\overline{f(t-x)}dt$$

the **autocorrelation** *of f. And for a given sequence $c \in \ell_2(\mathbb{Z})$, we also call the sequence c_{au} defined by*

$$c_{au}(n) := (c * \overline{c}(-\cdot))(n) = \sum_{m\in\mathbb{Z}} c(m)\overline{c(m-n)}$$

the autocorrelation of c. Note that $\widehat{f_{au}} = |\widehat{f}|^2, \widehat{c_{au}} = |\widehat{c}|^2$.

Definition 2.6. *A continuous function* ϕ *is called* **interpolatory** *if*

$$\phi(0) = 1, \text{ and } \phi(k) = 0 \text{ for } k \in \mathbb{Z}\backslash\{0\}. \tag{2.25}$$

A refinable function ϕ *which is also interpolatory is called a* **interpolatory refinable** *function.*

Lemma 2.1. (Riesz Lemma)([63]) *If* c *is a finitely supported sequence satisfying*

$$\widehat{c}(\omega) = \widehat{c}(-\omega),$$

then there exist a finitely supported sequence h *such that*

$$\widehat{c} = |\widehat{h}|^2, \tag{2.26}$$

i.e., $c = h_{au}$*,* c *is the autocorrelation of* h*.*

Note that Riesz Lemma can be applied for the construction of a refinable function ϕ by starting from its autocorrelation ϕ_{au}. For example, a refinable function with orthonormal shifts can be constructed from a interpolatory refinable function as shown in the construction of Daubechies orthonormal wavelets [21, 22]. Later we will see that pseudo spline of type I can be constructed from pseudo spline of type II by applying the Riesz Lemma.

By Corollary 2.1, Corollary 2.2, Theorem 2.5 and Theorem 2.6, we can obtain the following result which suffices for the construction of tight wavelet frames and our later construction of tight wavelet frame packets.

Theorem 2.9. *Suppose* c *is a finitely supported sequence supported on* $[N_1, N_2]$ *satisfying*

$$\widehat{c}(0) = 1,$$

and the inequality (2.16), i.e.,

$$|\widehat{c}|^2 + |\widehat{c}(\cdot + \pi)|^2 \le 1.$$

Define $\widehat{\phi}(\omega) := \prod_{n=1}^{\infty} \widehat{c}(2^{-n}\omega)$*. Then,*

- $\phi \in L_2(\mathbb{R})$ *with* $\|\phi\|_{L_2(\mathbb{R})} \le 1$*;*
- $[\widehat{\phi}, \widehat{\phi}] \le 1$, *i.e.,* $E(\phi) = \{\mathbf{E}^k \phi : k \in \mathbb{Z}\}$ *is a Bessel set of* $S(\phi)$*;*
- ϕ *is supported on* $[N_1, N_2]$*;*
- ϕ *is an MRA generator;*
- $\widehat{\phi}(0) = 1$ *and* $\widehat{\phi}(2\pi k) = 0$ *for* $k \in \mathbb{Z}\backslash\{0\}$.

2.3 Wavelet Frames (Affine Frames)

We have introduced PSI spaces and the framework of MRA in the previous two sections. In this section, we first introduce the characterization of **wavelet frames** (also referred to as **affine frames**), then we concentrate on the construction of tight MRA-based wavelet frames via the unitary extension principle (UEP).

Definition 2.7. *A* **wavelet system** *or an* **affine system** $X := X(\Psi) \subset L_2(\mathbb{R})$ *is a collection of functions of the form*

$$X = \cup_{j\in\mathbb{Z}} \mathbf{D}^j E(\Psi),$$

where $\Psi \subset L_2(\mathbb{R})$ *is finite,* $E(\Psi) = \cup_{\psi\in\Psi} E(\psi)$ *is a finite union of the PSI spaces* $E(\psi), \psi \in \Psi$*. The functions in* Ψ *are the generators of* X*, usually referred to as* **mother wavelets**.

Definition 2.8. *Given an affine system (or wavelet system)* $X \subset L_2(\mathbb{R})$*, the analysis operator* $\mathbf{T}_X^*$ *is defined by*

$$\mathbf{T}_X^* : L_2(\mathbb{R}) \to \ell_2(X) : f \to \big(\langle f, g\rangle\big)_{g\in X}.$$

If $\mathbf{T}_X^*$ *is well-defined, bounded and bounded below, i.e., there exist two positive constants* C_1, C_2 *such that the inequalities*

$$C_1\|f\|^2_{L_2(\mathbb{R})} \le \sum_{g\in X} |\langle f, g\rangle|^2 \le C_2\|f\|^2_{L_2(\mathbb{R})}, \tag{2.27}$$

hold for all $f \in L_2(\mathbb{R})$, then X is called an **affine frame** *or a* **wavelet frame** *of $L_2(\mathbb{R})$, where C_1 and C_2 are called the* **lower frame bound** *and the* **upper frame bound***, respectively.*

In particular, if $C_1 = C_2$, then X is called a **tight affine frame** *or a* **tight wavelet frame** *of $L_2(\mathbb{R})$.*

Historically, univariate tight wavelet frame characterization implicitly appeared in [48, 37] in the work of Weiss et al in 1996. An explicit multivariate tight wavelet frame characterization was obtained by Han in [39] in 1997. Independently, a general characterization of wavelet frames was obtained by Ron and Shen in [66] in 1997, they gave a general characterization of all affine frames (wavelet frames), and specialized their results to the case of tight affine frames (tight wavelet frames). Their success is largely due to the **"dual Gramian" analysis** [65] and the **"quasi-affine system"** $X^q(\Psi)$ [66] they invented.

Definition 2.9. [66] *Given an affine system (or wavelet system) $X(\Psi)$, the* **quasi-affine system** *$X^q(\Psi)$ is obtained by replacing, for each $\psi \in \Psi, j < 0$, and $k \in \mathbb{Z}$, the function $\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k)$ in $X(\Psi)$, by the 2^{-j} functions*

$$2^j\psi(2^j(\cdot + \alpha) - k), \;\; \alpha = 0, 1, \cdots, 2^{-j} - 1.$$

Note that, while the wavelet system $X(\Psi)$ is dilation-invariant but not shift-invariant, the situation with the quasi-affine system $X^q(\Psi)$ is complementary.

Theorem 2.10. [66, 13] *$X(\Psi)$ is a wavelet frame if and only if $X^q(\Psi)$ is a wavelet frame. Furthermore, the two frames have identical frame bounds. In particular, $X(\Psi)$ is tight if and only if $X^q(\Psi)$ is tight.*

To do the "dual Gramian" analysis of $X^q(\Psi)$, they first introduce the affine product:

Definition 2.10. [66] *Given wavelet system $X(\Psi)$, the* **affine product** *is the function $\Psi[\cdot,\cdot] : \mathbb{R}\times\mathbb{R}\to\mathbb{C}$,*

$$\Psi[\omega,\omega'] = \sum_{\psi\in\Psi}\sum_{k=\kappa(\omega-\omega')}^{\infty}\widehat{\psi}(2^k\omega)\overline{\widehat{\psi}}(2^k\omega'), \tag{2.28}$$

where κ is the dyadic evaluation

$$\begin{array}{rccl} \kappa: & \mathbb{R} & \to & \mathbb{Z} \\ & \omega & \mapsto & \inf\{k\in\mathbb{Z} : 2^k\omega\in 2\pi\mathbb{Z}\} \end{array}$$

($\kappa(0) := -\infty$, and $\kappa(\omega) := \infty$ unless ω is $2\pi-$periodic).

Then they analyze $X^q(\Psi)$ via the "dual Gramian" fibers $\widetilde{G}(\omega),\quad \omega\in\mathbb{R}$, which may be only almost everywhere defined. Each fiber $\widetilde{G}(\omega)$ is a non-negative definite self-adjoint matrix whose rows and columns are indexed by $2\pi\mathbb{Z}$, and whose (α,β)-entry is

$$\widetilde{G}(\omega)(\alpha,\beta) = \Psi[\omega+\alpha,\omega+\beta].$$

Each fiber $\widetilde{G}(\omega)$ is considered as an endomorphism of $\ell_2(2\pi\mathbb{Z})$ with its norm and inverse norm denoted by $\mathcal{G}^*(\omega)$ and $\mathcal{G}^{*-}(\omega)$ respectively, where

$$\begin{array}{rccl} \mathcal{G}^*: & \mathbb{R} & \to & \mathbb{R}^+ \\ & \omega & \mapsto & \|\widetilde{G}(\omega)\| \\ \mathcal{G}^{*-}: & \mathbb{R} & \to & \mathbb{R}^+ \\ & \omega & \mapsto & \|\widetilde{G}(\omega)^{-1}\| \end{array}$$

are the two norm functions.

Theorem 2.11. [66] *Let $X(\Psi)$ be a wavelet system and $\mathcal{G}^*$ and $\mathcal{G}^{*-}$ be the dual Gramian norm functions defined as above. Then $X(\Psi)$ is a wavelet frame if and only if*

$$\mathcal{G}^*, \mathcal{G}^{*-}\in L_\infty(\mathbb{R}).$$

Furthermore, the upper frame bound of $X(\Psi)$ is $\|\mathcal{G}^*\|_{L_\infty(\mathbb{R})}$ and the lower frame bound of $X(\Psi)$ is $1/\|\mathcal{G}^{*-}\|_{L_\infty(\mathbb{R})}$.

Theorem 2.12. [66] *$X(\Psi)$ is a tight wavelet frame with frame bounds C if and only if*

$$\Psi[\omega,\omega] = C, \tag{2.29}$$

and

$$\Psi[\omega,\omega+2\pi(2m+1)] = 0, \tag{2.30}$$

for a.e. $\omega \in \mathbb{R}$ and $m \in \mathbb{Z}$.

Theorem 2.13. [66, 39, 48] *A wavelet system $X(\Psi)$ generated by a singleton $\Psi = \{\psi\}$ constitutes an orthonormal bases if and only if (2.30) holds, (2.29) holds with $C = 1$, and $\|\psi\|_{L_2(\mathbb{R})} = 1$,i.e.,*

- $\sum_{k=0}^{\infty} \widehat{\psi}(2^k\omega)\overline{\widehat{\psi}(2^k(\omega+2\pi(2m+1)))} = 0$ *for $m \in \mathbb{Z}$ and a.e. $\omega \in \mathbb{R}$;*
- $\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^k\omega)|^2 = 1$ *for a.e. $\omega \in \mathbb{R}$;*
- $\|\psi\|_{L_2(\mathbb{R})} = 1.$

However, the characterization results introduced in this section do not suggest any construction of wavelet frames. To construct wavelet frames that are useful in applications, we introduce the multiresolution analysis (MRA) in the next section.

2.3.1 MRA-based Wavelet Frames

Definition 2.11. [66, 24]. *A wavelet system $X(\Psi)$ is said to be MRA-based if there exists an MRA $(V_j)_{j\in\mathbb{Z}}$ such that $\Psi \subset V_1$. If in addition $X(\Psi)$ is a wavelet frame, we call it an* **MRA-based Wavelet Frame**.

Suppose that $(V_j)_{j\in\mathbb{Z}}$ is an MRA generated by a refinable function $\phi \in L_2(\mathbb{R})$ with its refinement mask $h_0 \in \ell_2(\mathbb{Z})$. Let $\Psi = \{\psi_1, \cdots, \psi_r\}$ and suppose there

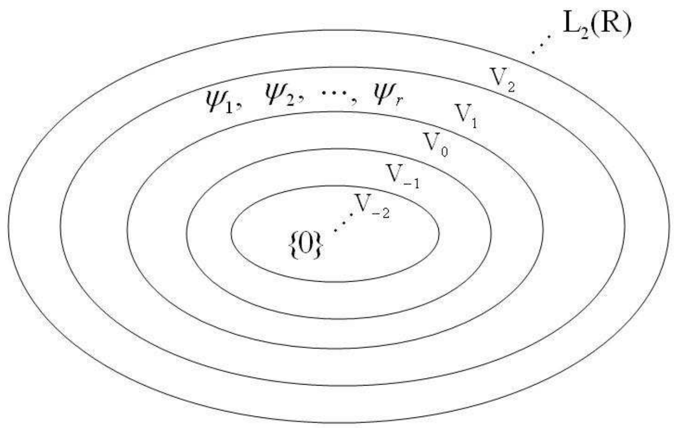


Figure 2.3: MRA-based Wavelet Frames

are r sequences $h_1, \cdots, h_r \in \ell_2(\mathbb{Z})$ also referred as **wavelet masks** such that ψ_i satisfies the **wavelet equation**

$$\psi_i(x) = \sum_{n\in\mathbb{Z}} 2h_i(n)\phi(2x-n), \tag{2.31}$$

for $i = 1, \cdots, r$. Then $\Psi \subset V_1$ by (2.4). Also, we call the vector

$$\mathbf{h} := [h_0, h_1, \cdots, h_r] \tag{2.32}$$

a **combined MRA mask**, and denote its Fourier domain counterpart by

$$\widehat{\mathbf{h}} := [\widehat{h_0}, \widehat{h_1}, \cdots, \widehat{h_r}]. \tag{2.33}$$

Definition 2.12. *Let* $\mathbf{h} = [h_0, h_1, \cdots, h_r]$ *be a combined MRA mask, define*

$$\Theta(w) := \sum_{j=0}^{\infty} \left(\sum_{i=1}^{r} |\widehat{h_i}(2^j\omega)|^2 \right) \prod_{m=0}^{j-1} |\widehat{h_0}(2^m\omega)|^2. \tag{2.34}$$

Note that the definition of Θ *implies* Θ *is a* 2π*-periodic function satisfying the following identity*

$$\Theta(\omega) = \Theta(2\omega)|\widehat{h_0}(\omega)|^2 + \sum_{i=1}^{r} |\widehat{h_i}(\omega)|^2, \quad a.e. \tag{2.35}$$

For the statement of the characterization result of MRA-based wavelet frames, we also impose the following mild conditions.

Assumptions 2.1. [24] *All MRA-based constructions to be considered are assumed to satisfy the following*

- *Each wavelet mask* h_i *satisfies* $\widehat{h_i} \in L_\infty(\mathbb{T}), 1 \le i \le r$*;*
- *The MRA generator* ϕ *satisfies* $\lim_{\omega \to 0} \widehat{\phi}(\omega) = 1$*, with*

$$[\widehat{\phi}, \widehat{\phi}] \in L_\infty(\mathbb{T}),$$

i.e., $E(\phi)$ *is a Bessel set of* $S(\phi)$*.*

$\sigma(V_0) = \sigma(S(\phi))$, i.e., the spectrum of the shift invariant space V_0 defined by (2.7) , plays an important role in the theory of shift-invariant spaces [27, 28, 65]. The values assumed by the combined MRA mask $\widehat{\mathbf{h}}$ outside the set $\sigma(V_0)$ affect neither the MRA nor the resulting wavelet system $X(\Psi)$. In particular, whenever ϕ is compactly supported, we automatically have $\sigma(V_0) = [-\pi, \pi]$ up to a null set.

Theorem 2.14. [66] [24] *Let* $X(\Psi)$ *be an MRA-based wavelet system (or affine system) associated an MRA* $(V_j)_{j \in \mathbb{Z}}$ *generated by a refinable function* ϕ*. Suppose that* ϕ *and the combined MRA mask* $\mathbf{h}$ *as defined in (2.32) satisfies Assumption 2.1. Then* $X(\Psi)$ *is a tight wavelet frame (or tight affine frame) if and only if for almost all* $\omega \in \sigma(V_0)$*, the function* Θ *satisfies*

$$\lim_{j \to -\infty} \Theta(2^j \omega) = 1, \tag{2.36}$$

and

$$\Theta(2\omega)\widehat{h}_0(\omega)\overline{\widehat{h}_0(\omega + \pi)} + \sum_{i=1}^{r} \widehat{h}_i(\omega)\overline{\widehat{h}_i(\omega + \pi)} = 0. \tag{2.37}$$

By restricting $\Theta = 1$ on $\sigma(V_0)$, Theorem 2.14 can be simplified as the well-known **unitary extension principle (UEP)** in the wavelet frame literature which makes the construction of tight wavelet frames painless.

Theorem 2.15. [66] [24] **(UEP)** *Let $X(\Psi)$ be an MRA-based wavelet system (or affine system) associated an MRA $(V_j)_{j\in\mathbb{Z}}$ generated by a refinable function ϕ. Suppose that ϕ and the combined MRA mask* $\mathbf{h}$ *as defined in (2.32) satisfy the Assumption 2.1. If for almost all $\omega \in \sigma(V_0)$,* $\mathbf{h}$ *satisfies*

$$HH^* = I_{2\times 2}, \tag{2.38}$$

where

$$H = \begin{bmatrix} \widehat{h_0}(\omega) & \widehat{h_1}(\omega) & \cdots & \widehat{h_r}(\omega) \\ \widehat{h_0}(\omega+\pi) & \widehat{h_1}(\omega+\pi) & \cdots & \widehat{h_r}(\omega+\pi) \end{bmatrix},$$

then $X(\Psi)$ is a tight wavelet frame (or tight affine frame). And the condition (2.38) is referred as the **UEP condition***.*

Note that the UEP condition (2.38) is sometimes written as the following two conditions

$$\sum_{i=0}^{r} |\widehat{h}_i(\omega)|^2 = 1, \tag{2.39}$$

and

$$\sum_{i=0}^{r} \widehat{h}_i(\omega)\overline{\widehat{h}_i(\omega+\pi)} = 0. \tag{2.40}$$

Definition 2.13. *If a combined MRA mask $\mathbf{h} = [h_0, h_1, \cdots, h_r]$ satisfies the UEP condition (2.38), or equivalently, (2.39) and (2.40), then we call it a* **combined UEP mask***.*

2.3.2 Construction of Tight Wavelet Frames (TWF) via UEP

The UEP condition (2.38) implies a necessary condition on h_0, i.e.,

$$|\widehat{h_0}|^2 + |\widehat{h_0}(\cdot+\pi)|^2 \le 1. \tag{2.41}$$

where h_0 is the refinement mask of the MRA generator ϕ.

And also, Assumption 2.1 implies

$$\widehat{h_0}(0) = 1. \tag{2.42}$$

These two conditions (2.41) and (2.42) turn out to be sufficient for the construction of a tight wavelet frame if we further assume that

$$h_i \text{ is finitely supported for } i = 0, 1, \cdots, r. \tag{2.43}$$

In fact, with (2.41), (2.42) and(2.43), Assumption 2.1 can be removed by Theorem 2.9.

By UEP, the construction of compactly supported tight wavelet frames (or tight affine frames) is reduced to finding a finitely supported sequence h_0 satisfying (2.41) and (2.42). As it can be shown later, such sequences can be easily obtained by taking advantage of the equality

$$\big(\cos(\omega/2)^2 + \sin(\omega/2)^2\big)^n \equiv 1, \text{ for all } n \in \mathbb{N},$$

and the Riesz Lemma.

As a direct application of UEP, the following construction illustrates how UEP makes the construction of MRA-based tight wavelet frames painless.

Construction 2.1. [66] *Let m be a positive integer, and let $\widehat{h_0}(\omega) = e^{-i\mathcal{K}\omega/2}\cos^m(\omega/2)$, where $\mathcal{K} = 0$ if m is even, $\mathcal{K} = 1$ if m is odd. It is the refinement mask of the B-spline ϕ of order m.*

Define

$$\widehat{h_n}(\omega) := \sqrt{\binom{m}{n}} e^{-i\mathcal{K}\omega/2} i^n \sin^n(\omega/2)\cos^{m-n}(\omega/2), 1 \le n \le m,$$

and also define the combined MRA mask $\mathbf{h} := [h_0, h_1, \cdots, h_m]$. We can observe that

$$\sum_{n=0}^{m} |\widehat{h_n}(\omega)|^2 = \big(\cos^2(\omega/2) + \sin^2(\omega/2)\big)^m = 1,$$

$$\sum_{n=0}^{m} \widehat{h_n}(\omega)\overline{\widehat{h_n}(\omega+\pi)} = \big(\sin(\omega/2)\cos(\omega/2)\big)^m (1-1)^m = 0,$$

i.e., **h** *satisfies the UEP condition (2.38) and* **h** *is a combined UEP mask.*

Define $\psi_n, n = 1, \cdots, m$ *by (2.31) and let* $\Psi = \{\psi_1, \psi_2, \cdots, \psi_m\}$. *It follows from UEP that the wavelet system* $X(\Psi)$ *is a compactly supported tight wavelet frame (tight affine frame).*

When $m = 1$, we get the well-known Haar wavelet, which is an orthonormal wavelet frame which was originally discovered by Haar in 1910 [38].

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le t < 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.44}$$

which is piecewise constant. Its refinement mask and wavelet mask are

$$h_0 = (\cdots, 0, \mathbf{\frac{1}{2}}, \frac{1}{2}, 0, \cdots),$$

and

$$h_1 = (\cdots, 0, \mathbf{\frac{1}{2}}, -\frac{1}{2}, 0, \cdots),$$

respectively.

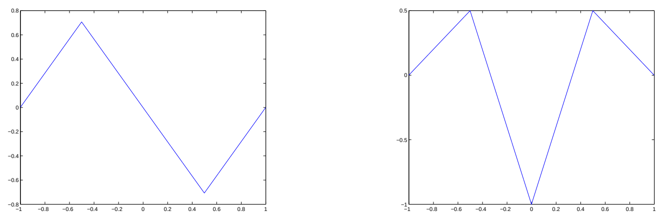
Example 2.5. (Piecewise linear tight wavelet frame) [69] *When* $m = 2$, ϕ *is the B-spline of order* 2. *The refinement mask is*

$$h_0 = (\cdots, 0, \frac{1}{4}, \mathbf{\frac{1}{2}}, \frac{1}{4}, 0, \cdots).$$

The two wavelet masks are

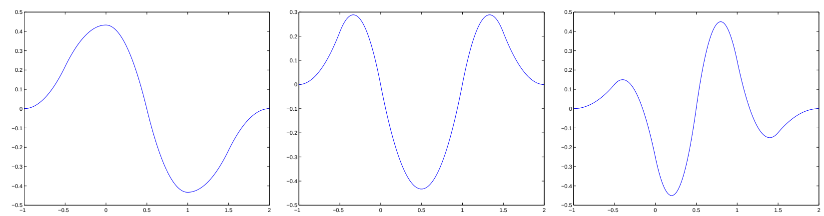
$$h_1 = (\cdots, 0, \frac{\sqrt{2}}{4}, \mathbf{0}, -\frac{\sqrt{2}}{4}, 0, \cdots)$$
$$h_2 = (\cdots, 0, \frac{1}{4}, -\mathbf{\frac{1}{2}}, \frac{1}{4}, 0, \cdots).$$

The plots of the two wavelets ψ_1, ψ_2 *are depicted in Figure 2.4.*



(a) ψ_1 (b) ψ_2

Figure 2.4: Piecewise Linear Tight Wavelet Frame



(a) ψ_1 (b) ψ_2 (c) ψ_3

Figure 2.5: Piecewise Quadratic Tight Wavelet Frame

Example 2.6. (Piecewise quadratic tight wavelet frame) *When $m = 3$, ϕ is the B-spline of order 3. The refinement mask is*

$$h_0 = (\cdots, 0, \frac{1}{8}, \frac{\mathbf{3}}{\mathbf{8}}, \frac{3}{8}, \frac{1}{8}, 0, \cdots).$$

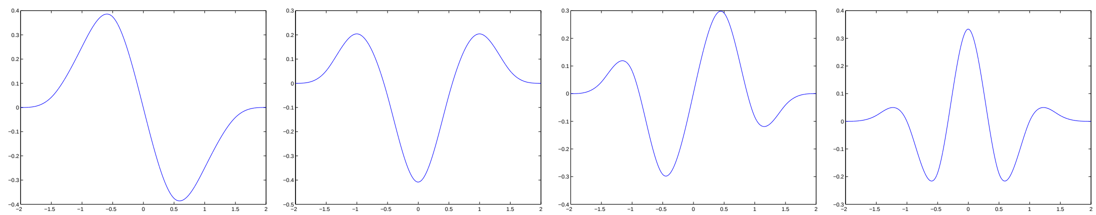
The three wavelet masks are

$$h_1 = (\cdots, 0, \frac{\sqrt{3}}{8}, \frac{\sqrt{\mathbf{3}}}{\mathbf{8}}, -\frac{\sqrt{3}}{8}, -\frac{\sqrt{3}}{8}, 0, \cdots),$$
$$h_2 = (\cdots, 0, \frac{\sqrt{3}}{8}, -\frac{\sqrt{\mathbf{3}}}{\mathbf{8}}, -\frac{\sqrt{3}}{8}, \frac{\sqrt{3}}{8}, 0, \cdots),$$
$$h_3 = (\cdots, 0, \frac{1}{8}, -\frac{\mathbf{3}}{\mathbf{8}}, \frac{3}{8}, -\frac{1}{8}, 0, \cdots).$$

The plots of the 3 wavelets ψ_1, ψ_2, ψ_3 are depicted in Figure 2.5.

Example 2.7. (Piecewise cubic tight wavelet frame) [69] *When $m = 4$, ϕ is the B-spline of order 4. The refinement mask is*

$$h_0 = (\cdots, 0, \frac{1}{16}, \frac{1}{4}, \frac{\mathbf{3}}{\mathbf{8}}, \frac{1}{4}, \frac{1}{16}, 0, \cdots).$$



(a) ψ_1 (b) ψ_2 (c) ψ_3 (d) ψ_4

Figure 2.6: Piecewise Cubic Tight Wavelet Frame

The four wavelet masks are

$$h_1 = (\cdots, 0, \frac{1}{8}, \frac{1}{4}, \mathbf{0}, -\frac{1}{4}, -\frac{1}{8}, 0, \cdots),$$
$$h_2 = (\cdots, 0, \frac{\sqrt{6}}{16}, 0, -\frac{\boldsymbol{\sqrt{6}}}{\mathbf{8}}, 0, \frac{\sqrt{6}}{16}, 0, \cdots),$$
$$h_3 = (\cdots, 0, \frac{1}{8}, -\frac{1}{4}, \mathbf{0}, \frac{1}{4}, -\frac{1}{8}, 0, \cdots),$$
$$h_4 = (\cdots, 0, \frac{1}{16}, -\frac{1}{4}, \frac{\mathbf{3}}{\mathbf{8}}, -\frac{1}{4}, \frac{1}{16}, 0, \cdots).$$

The plots of the 4 wavelets $\psi_1, \psi_2, \psi_3, \psi_4$ are depicted in Figure 2.6.

As can be observed in the plots from Figure 2.4 to Figure 2.6, all the wavelets in Construction 2.1 are symmetric or anti-symmetric.

By applying UEP, it is significantly simpler to construct tight wavelet frames (or tight affine frames) as compared to the construction of orthonormal wavelets. This is largely due to the fact that the construction of orthonormal wavelet frames requires a refinable function ϕ with orthonormal shifts, i.e., $E(\phi)$ is required to be an orthonormal basis of $S(\phi)$, which forces the refinement mask h_0 to satisfy the stringent CQF condition (2.21). In contrast, compactly supported tight wavelet frames can be derived from any compactly supported refinable function ϕ with its refinement mask satisfying the inequality (2.41), i.e., we only require that $E(\phi)$ is a Bessel set of $S(\phi)$. Various constructions of compactly supported tight wavelet frames can be found in [66, 68, 24, 11, 32].

To study the approximation property of the MRA-based tight wavelet frames constructed via UEP, we introduce the notion of **frame approximation order**.

Definition 2.14. *Given a tight wavelet frame $X(\Psi)$, we define the* **truncation operator** $\mathbf{Q}_j$*, $j \in \mathbb{Z}$, by*

$$\mathbf{Q}_j : f \mapsto \sum_{\psi\in\Psi}\sum_{j'<j}\sum_{k\in\mathbb{Z}}\langle f, \psi_{j',k}\rangle\psi_{j',k}, \qquad f \in L_2(\mathbb{R}).$$

We say that $X(\Psi)$ provides **frame approximation order** *s if, for every f in the Sobolev spaces $\mathbb{H}^s(\mathbb{R})$, where $\mathbb{H}^s(\mathbb{R})$ is defined by*

$$\|f\|^2_{\mathbb{H}^s(\mathbb{R})} := \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\omega)|^2(1+|\omega|^2)^s\, d\omega < \infty,$$

there exist a positive constant C, which is independent of f and j, and a positive integer J such that

$$\|f - \mathbf{Q}_j f\|_{L_2(\mathbb{R})} \le C2^{-js}\|f\|_{\mathbb{H}^s(\mathbb{R})}, \quad j \ge J.$$

We say that $X(\Psi)$ provides the **spectral frame approximation order** *if it provides frame approximation order s for any positive integer s.*

The frame approximation order of an MRA-based tight wavelet frame has been extensively studied in [24]. It was shown in [24, Lemma 2.4] that when $X(\Psi)$ is an MRA-based tight wavelet frame constructed via UEP in an MRA generated by a refinable function $\phi \in L_2(\mathbb{R})$, then

$$\mathbf{Q}_j = \mathbf{P}_j, \quad j \in \mathbb{Z},$$

where $\mathbf{P}_j$ is a linear operator for each $j \in \mathbb{Z}$ defined by

$$\mathbf{P}_j : f \mapsto \sum_{k\in\mathbb{Z}}\langle f, \phi_{j,k}\rangle\phi_{j,k}, \qquad f \in L_2(\mathbb{R}). \tag{2.45}$$

The operator $\mathbf{P}_j$ was well studied by Jetter and Zhou [49, 50] in the framework of quasi-interpolation which is the art of assigning suitable dual functionals to a given set of 'approximating' functions. By applying Jetter and Zhou [49] and [50, Theorem 2.1], for an MRA-based tight wavelet frame $X(\Psi)$, which is constructed via UEP in an MRA generated by a refinable function $\phi \in L_2(\mathbb{R})$ with $\widehat{\phi}(0) \neq 0$, $X(\Psi)$ provides frame approximation order s if and only if the following two conditions hold

$$(a). \quad [\widehat{\phi}, \widehat{\phi}] - |\widehat{\phi}|^2 = O(|\cdot|^{2s}); \tag{2.46}$$

$$(b). \quad 1 - |\widehat{\phi}|^2 = O(|\cdot|^{s}). \tag{2.47}$$

For B-splines of order m, since

$$[\widehat{B_m}, \widehat{B_m}] - |\widehat{B_m}|^2 = O(|\cdot|^{2m}),$$

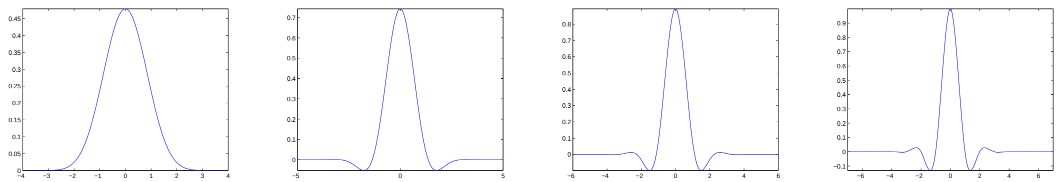
and

$$1 - |\widehat{B_m}(\omega)|^2 = 1 - \frac{\sin^{2m}(\omega/2)}{(\omega/2)^{2m}} = O(|\omega|^2),$$

the MRA-based based frame constructed via UEP in an MRA generated by a B-splines B_m can not exceed 2. As a consequence, the spline tight wavelet frames in Construction 2.1 provide frame approximation order at most 2. To overcome this drawback, we introduce a larger class of refinable functions called **pseudo splines** for the generation of MRA spaces.

2.3.3 Construction of TWF from Pseudo Splines

Pseudo splines offer a rich set of compactly supported refinable functions containing B-splines as a particular interesting subset. Pseudo splines of type I were introduced in [24] to obtained tight wavelet frames with desired approximation order, and pseudo splines of type II were introduced later in [32] and were used to construct symmetric tight wavelet frames, and also the regularity of both types of pseudo splines was analyzed in [32]. Later on, it was shown in [31] that the shifts of both types of pseudo splines are linearly independent[54, 57, 55, 64],



(a) $\phi_2^{4,0}$ (b) $\phi_2^{4,1}$ (c) $\phi_2^{4,2}$ (d) $\phi_2^{4,3}$

Figure 2.7: Pseudo Splines of Type II

which is a necessary condition for the construction of bi-orthogonal wavelet bases [16, 14, 22, 52, 51, 44, 40, 9], and these two types of pseudo splines were consequently used to construct bi-orthogonal wavelet bases in [30].

Let $m \in \mathbb{N}$ and $0 \le l \le m-1$, the refinement mask $a_2^{m,l}$ of a pseudo spline $\phi_2^{m,l}$ of type II is defined as the first $l+1$ terms of the binomial expansion

$$\big(\cos^2(\omega/2) + \sin^2(\omega/2)\big)^{m+l}, \tag{2.48}$$

i.e.,

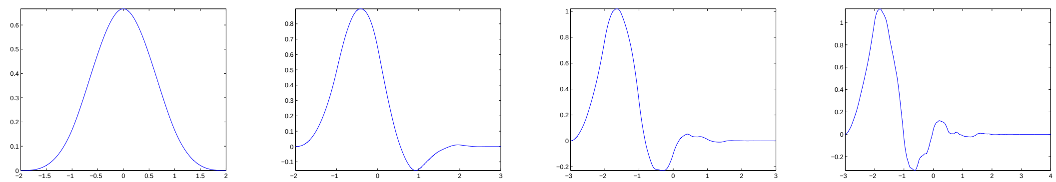
$$\begin{aligned} \widehat{a_2^{m,l}}(\omega) &:= \cos^{2m}(\omega/2) \sum_{n=0}^{l} \binom{m+l}{n} \cos^{2(l-n)}(\omega/2) \sin^{2n}(\omega/2) \\ &= \cos^{2m}(\omega/2) \sum_{n=0}^{l} \binom{m-1+n}{n} \sin^{2n}(\omega/2), \end{aligned} \tag{2.49}$$

and the refinement mask mask $a_1^{m,l}$ of a pseudo spline $\phi_1^{m,l}$ of type I is defined as

$$\widehat{a_1^{m,l}}(\omega) := \sqrt{\widehat{a_2^{m,l}}(\omega)}, \tag{2.50}$$

which is obtained by taking the square root of the mask $a_2^{m,l}$ using the Lemma 2.1 (Riesz Lemma [63]) . It follows from (2.50) that pseudo splines $\phi_2^{m,l}$ of type II are the autocorrelation of their type I counterpart $\phi_1^{m,l}$, i.e., $\widehat{\phi_2^{m,l}} = |\widehat{\phi_1^{m,l}}|^2$.

It can be easily seen that B-spline [25] of order m is the pseudo spline $\phi_1^{m,0}$ of type I, and the scaling function in the construction of Daubechies' orthonormal



(a) $\phi_1^{4,0}$ (b) $\phi_1^{4,1}$ (c) $\phi_1^{4,2}$ (d) $\phi_1^{4,3}$

Figure 2.8: Pseudo Splines of Type I

wavelets with m vanishing moments is the pseudo spline $\phi_1^{m,m-1}$ of type I. In other words, pseudo splines of type I fill the gap between B-splines and orthonormal refinable functions. We can also observe that the pseudo spline $\phi_2^{m,0}$ of type II is the B-spline of order $2m$, and the pseudo spline $\phi_2^{m,m-1}$ of type II is the autocorrelation of $\phi_1^{m,m-1}$. As it is well-known that the translates of $\phi_1^{m,m-1}$ are orthonormal [21, 22], $\phi_2^{m,m-1}$ is thus interpolatory. Note that the refinement masks $a_2^{m,m-1}$ had been used in the stationary subdivision schemes [4] and were called Lagrange interpolation schemes studied by Deslauriers and Dubuc in [29]. Pseudo splines of type II can be similarly understood as filling the gap between B-splines and interpolatory refinable functions. Moreover, pseudo splines of type II are symmetric, and the symmetric property is desirable in applications.

Let c be a refinement mask of a pseudo spline ϕ (type I or type II), it can be easily verified that c satisfies (2.41), (2.42) and(2.43), which are necessary and sufficient conditions for the construction of compactly supported tight wavelet frames by UEP. Consequently, both types of the pseudo splines can be used to construct tight wavelet frames. Dong and Shen used the pseudo splines $\phi_2^{m,m-1}$ of type II to construct compactly supported symmetric tight wavelet frames by applying the Construction 2.2 [32].

Construction 2.2. [10, 32] *Suppose h_0 is a finitely supported sequence satisfying*

(2.41) and (2.42), define

$$\begin{aligned}\widehat{h_1}(\omega) &:= e^{-i\omega}\overline{\widehat{h_0}(\omega+\pi)},\\ \widehat{h_2}(\omega) &:= 2^{-1}[A(\omega)+e^{-i\omega}\overline{A(\omega)}],\\ \widehat{h_3}(\omega) &:= 2^{-1}[A(\omega)-e^{-i\omega}\overline{A(\omega)}],\end{aligned} \tag{2.51}$$

where A is a $\pi-$periodic trigonometric polynomial with real coefficients such that

$$|A(\omega)|^2 = 1 - |\widehat{h_0}(\omega)|^2 - |\widehat{h_0}(\omega+\pi)|^2. \tag{2.52}$$

Then the combined MRA mask $\mathbf{h} = [h_0, h_1, h_2, h_3]$ satisfies the UEP condition (2.38), i.e., $\mathbf{h}$ is a combined UEP mask. Define ψ_1, ψ_2, ψ_3 by (2.31) and let $\Psi = \{\psi_1, \psi_2, \psi_3\}$, then $X(\Psi)$ is a compactly supported tight wavelet frame (or tight affine frame) by UEP. Furthermore, the corresponding wavelets ψ_1, ψ_2 and ψ_3 are symmetric or anti-symmetric whenever ϕ is symmetric.

As it was shown in [32], we can construct symmetric or anti-symmetric tight wavelet frames from pseudo splines of type II by Construction 2.2.

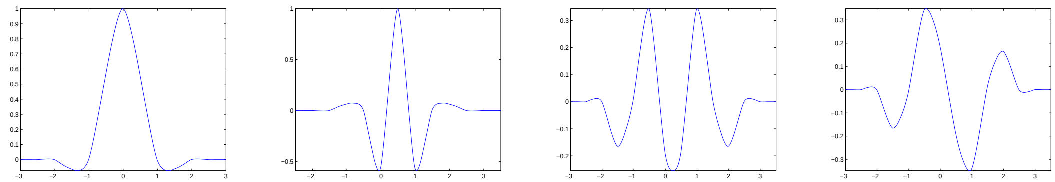
Example 2.8. *Let $\phi_2^{2,1}$ be the pseudo spline of type II, with its refinement mask*

$$h_0 = (\cdots, -1/32, 0, 9/32, \mathbf{1/2}, 9/32, 0, -1/32, \cdots).$$

By Construction 2.2, we can obtain the following 3 wavelet masks

$$\begin{aligned}h_1 &= (\cdots, 1/32, 0, \mathbf{-9/32}, 1/2, -9/32, 0, 1/32, \cdots),\\ h_2 &= (\cdots, -0.08246745105515, 0.00592089659317, 0.17085579870346, \mathbf{-0.09430924424149},\\ &\quad -0.09430924424149, 0.17085579870346, 0.00592089659317, -0.08246745105515, \cdots),\\ h_3 &= (\cdots, -0.08246745105515, -0.00592089659317, 0.17085579870346, \mathbf{0.09430924424149},\\ &\quad -0.09430924424149, -0.17085579870346, 0.00592089659317, 0.08246745105515, \cdots).\end{aligned}$$

The plots of $\phi_2^{2,1}$ and the 3 wavelets ψ_1, ψ_2, ψ_3 are depicted in Figure 2.9.



(a) $\phi_2^{2,1}$ (b) ψ_1 (c) ψ_2 (d) ψ_3

Figure 2.9: Tight Wavelet Frame Constructed by Pseudo Spline $\phi_2^{2,1}$ of type II

Example 2.9. *Let $\phi_2^{3,1}$ be the pseudo spline of type II, with its refinement mask*

$$h_0 = (\cdots, -3/256, -1/32, 3/64, 9/32, \mathbf{55/128}, 9/32, 3/64, -1/32, -3/256, \cdots).$$

By Construction 2.2, we can obtain the following 3 wavelet masks

$$h_1 = (\cdots, -3/256, 1/32, 3/64, \mathbf{-9/32}, 55/128, -9/32, 3/64, 1/32, -3/256, \cdots),$$

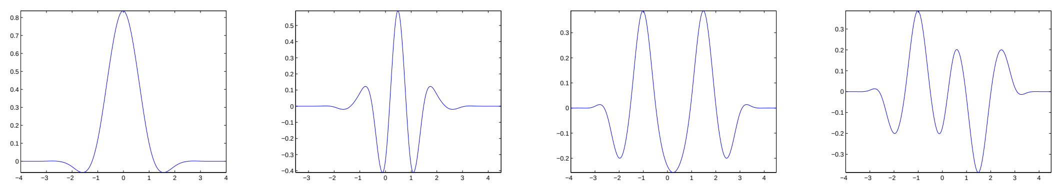
$$\begin{aligned} h_2 = (\cdots, &-0.11081147447130, 0.00061965199100, 0.22356159594985, 0.00069934302526,\\ &\mathbf{-0.11406911649481}, -0.11406911649481, 0.00069934302526, 0.22356159594985,\\ &0.00061965199100, -0.11081147447130, \cdots),\end{aligned}$$

$$\begin{aligned} h_3 = (\cdots, &-0.11081147447130, -0.00061965199100, 0.22356159594985, -0.00069934302526,\\ &\mathbf{-0.11406911649481}, 0.11406911649481, 0.00069934302526, -0.22356159594985,\\ &0.00061965199100, 0.11081147447130, \cdots).\end{aligned}$$

The plots of $\phi_2^{3,1}$ and the 3 wavelets ψ_1, ψ_2, ψ_3 are depicted in Figure 2.10.

2.4 Nonstationary Tight Wavelet Frames (NTWF)

As shown in the previous section, MRA-based tight wavelet frames of higher approximation order can be constructed via UEP by using pseudo splines. However, we still can not achieve spectral frame approximation order. To attain spectral



(a) $\phi_2^{3,1}$ (b) ψ_1 (c) ψ_2 (d) ψ_3

Figure 2.10: Tight Wavelet Frame Constructed by Pseudo Spline $\phi_2^{3,1}$ of type II

frame approximation order we have to switch to the construction of **nonstationary tight wavelet frame (NTWF)** ([45]) to be introduced in this section.

Nonstationary wavelet systems are generally obtained from a sequence of nonstationary refinable functions. Let $\{\phi_{j-1}\}_{j\in\mathbb{N}}$ be a sequence of functions in $L_2(\mathbb{R})$. We say that $\{\phi_{j-1}\}_{j\in\mathbb{N}}$ is a sequence of **nonstationary refinable functions** if

$$\widehat{\phi_{j-1}}(\omega) = \widehat{a_j}(\omega/2)\widehat{\phi_j}(\omega/2), \qquad a.e.\ \omega\in\mathbb{R},\ j\in\mathbb{N}, \tag{2.53}$$

where $\{\widehat{a_j}\}_{j\in\mathbb{N}}$ is a sequence of 2π-periodic trigonometric polynomials. Wavelet functions ψ_{j-1}^{ℓ}, $j\in\mathbb{N}$ and $\ell=1,\dots,\mathcal{J}_j$, are generally obtained from nonstationary refinable functions ϕ_j, $j\in\mathbb{N}$, via

$$\widehat{\psi_{j-1}^{\ell}}(\omega) := \widehat{b_j^{\ell}}(\omega/2)\widehat{\phi_j}(\omega/2), \qquad j\in\mathbb{N},\ \ell=1,\dots,\mathcal{J}_j, \tag{2.54}$$

where $\mathcal{J}_j$ are positive integers depending on j and the 2π-periodic trigonometric polynomials $\widehat{b_j^{\ell}}$ are the corresponding wavelet masks. We start with a nonstationary tight wavelet frame in $L_2(\mathbb{R})$ generated by $\{\phi_0\}\cup\{\psi_j^{\ell}\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathcal{J}_{j+1}\}}$, where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. We call the following wavelet system

$$\begin{aligned} X\big(\phi_0;\{\psi_j^{\ell}\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathcal{J}_{j+1}\}}\big) :=&\\ \{\phi_0(\cdot-k)\ :\ k\in\mathbb{Z}\}\cup\{\psi_{j;j,k}^{\ell} := \mathbf{D}^j\mathbf{E}^k\psi_j^{\ell}\ :\ j\in\mathbb{N}_0, \ell=1,\dots,\mathcal{J}_{j+1}, k\in\mathbb{Z}\}& \end{aligned} \tag{2.55}$$

a **nonstationary tight wavelet frame** for $L_2(\mathbb{R})$ if

$$\|f\|^2_{L_2(\mathbb{R})} = \sum_{k\in\mathbb{Z}} |\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{\mathcal{J}_{j+1}}\sum_{k\in\mathbb{Z}} |\langle f, \psi^{\ell}_{j;j,k}\rangle|^2 \qquad \text{for all} \quad f \in L_2(\mathbb{R}). \tag{2.56}$$

This is equivalent to

$$f = \sum_{k\in\mathbb{Z}}\langle f, \phi_0(\cdot - k)\rangle\phi_0(\cdot - k) + \sum_{j=0}^{\infty}\sum_{\ell=1}^{\mathcal{J}_{j+1}}\sum_{k\in\mathbb{Z}}\langle f, \psi^{\ell}_{j;j,k}\rangle\psi^{\ell}_{j;j,k}, \qquad f \in L_2(\mathbb{R}). \tag{2.57}$$

For a 2π-periodic trigonometric polynomial $\widehat{a}$, we denote $\deg(\widehat{a})$ the smallest nonnegative integer such that the Fourier coefficients of $\widehat{a}$ vanish outside

$$[-\deg(\widehat{a}), \deg(\widehat{a})].$$

The following result which can be regarded as the **generalized unitary extension principle (GUEP)** gave the explicit construction of NTWF.

Theorem 2.16. ([45, Theorem 1.1]) *Let $\widehat{a_j}, j \in \mathbb{N}$, be 2π-periodic trigonometric polynomials satisfying $\widehat{a_j}(0) = 1$ for all $j \in \mathbb{N}$ and*

$$\sum_{j=1}^{\infty} 2^{-j}\deg(\widehat{a_j}) < \infty.$$

Define a sequence of nonstationary refinable functions $\{\phi_{j-1}\}_{j\in\mathbb{N}}$ by

$$\widehat{\phi_{j-1}}(\omega) := \widehat{a_j}(\omega/2)\widehat{\phi_j}(\omega/2) = \prod_{n=1}^{\infty}\widehat{a_{n+j-1}}(2^{-n}\omega), \qquad \omega \in \mathbb{R},\ j \in \mathbb{N}. \tag{2.58}$$

Suppose that there exist 2π-periodic trigonometric polynomials $\widehat{b^{\ell}_j}$, $j \in \mathbb{N}$ and $\ell = 1, \ldots, \mathcal{J}_j$ with each $\mathcal{J}_j$ being a positive integer depending on j, and each $\mathbf{b}_j = [a_j, b^1_j, \cdots, b^{\mathcal{J}_j}_j]$ is a combined UEP mask satisfying the UEP condition (2.38).

Define wavelet functions ψ^{ℓ}_{j-1}, $j \in \mathbb{N}$ and $\ell = 1, \ldots, \mathcal{J}_j$, as in (2.54). Then all functions ϕ_{j-1} and ψ^{ℓ}_{j-1}, $j \in \mathbb{N}$ and $\ell = 1, \ldots, \mathcal{J}_j$, are well-defined compactly supported functions in $L_2(\mathbb{R})$, and the wavelet system $X\big(\phi_0; \{\psi^{\ell}_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\ldots,\mathcal{J}_{j+1}\}}\big)$ defined in (2.55) is an NTWF of $L_2(\mathbb{R})$.

As an application of Theorem 2.16, it was shown in [46, Theorem 1.2]) that the NTWF derived from masks of pseudo-splines can have spectral frame approximation order.

Construction 2.3. ([46, Theorem 1.2]) *Let* $\widehat{a_j} := \widehat{a_2^{m_j,l_j}}$ *(or* $\widehat{a_j} := \widehat{a_1^{m_j,l_j}}$*) be defined in* (2.49) *(or in* (2.50)*), where* $0 \le l_j \le m_j - 1$ *and* m_j $(j \in \mathbb{N})$ *are positive integers satisfying*

$$\lim_{j\to\infty} m_j = \infty, \quad \liminf_{j\to\infty} (l_j+1)/m_j > 0, \quad \textit{and} \quad \sum_{j=1}^{\infty} 2^{-j} m_j < \infty. \tag{2.59}$$

For $j \in \mathbb{N}$, *let* ϕ_{j-1} *be defined in* (2.58) *and* ψ^1_{j-1}, ψ^2_{j-1}, *and* ψ^3_{j-1} *in* (2.54) *with* $\mathcal{J}_j = 3$ *and masks* $\widehat{b^1_j}$, $\widehat{b^2_j}$ *and* $\widehat{b^3_j}$ *being defined in Construction 2.2 from* $\widehat{a_j} := \widehat{a_2^{m_j,l_j}}$ *(or* $\widehat{a_j} := \widehat{a_1^{m_j,l_j}}$*). Then , the wavelet system* $X\big(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0, \ell\in\{1,\dots,\mathcal{J}_{j+1}\}}\big)$ *defined in* (2.55) *is an NTWF of* $L_2(\mathbb{R})$ *with spectral frame approximation order.*

2.5 Characterization of Sobolev Spaces by NTWF

Orthonormal tight wavelet frames had been used to characterize Sobolev spaces $\mathbb{H}^s(\mathbb{R})$ by Meyer [62]. Characterization of $\mathbb{H}^s(\mathbb{R})$ using general tight wavelet frames in terms of the weighted ℓ_2-norm of the analysis wavelet coefficient sequences of the functions was given in [2, 1]. Interestingly, it was shown in [45] that any Sobolev space $\mathbb{H}^s(\mathbb{R})$ with fixed smoothness order s can be characterized in terms of the weighted ℓ_2-norm of the analysis wavelet coefficient sequences of a fixed nonstationary tight wavelet frame constructed in [46] (Construction 2.3).

Before stating the characterization results, we introduce the notation of $[\cdot,\cdot]_s$, i.e., the bracket product with smoothness order s. Precisely, for $f, g \in L_2(\mathbb{R})$, $[\cdot,\cdot]_s$ is defined as

$$[f,g]_s = \sum_{k\in\mathbb{Z}} f(\cdot+2\pi k)\overline{g(\cdot+2\pi k)}(1+|\cdot+2\pi k|^2)^s. \tag{2.60}$$

Note that $[f,g]_0 = [f,g]$.

Theorem 2.17. ([45, Theorem 1.2]) *Let $X\big(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathcal{J}_{j+1}\}}\big)$ be a nonstationary tight wavelet frame in $L_2(\mathbb{R})$ obtained in Theorem 2.16. Assume that for $\alpha>0$ there exist a positive number C and a positive integer J such that*

$$1-|\widehat{a_j}(\omega)|^2 \le C|\omega|^{2\alpha}, \qquad \omega\in\mathbb{R}, j\ge J, \tag{2.61}$$

and

$$[\widehat{\phi}_j,\widehat{\phi}_j]_\alpha(\omega)\le C, \qquad \omega\in\mathbb{R},\ j\in\mathbb{N}_0, \tag{2.62}$$

then for every $-\alpha<s<\alpha$, there exist two positive constants C_1 and C_2 such that

$$C_1\|f\|^2_{\mathbb{H}^s(\mathbb{R})}\le \sum_{k\in\mathbb{Z}}|\langle f,\mathbf{E}^k\phi_0\rangle|^2+\sum_{j=0}^{\infty}\sum_{\ell=1}^{\mathcal{J}_{j+1}}\sum_{k\in\mathbb{Z}}2^{2js}|\langle f,\psi^\ell_{j;j,k}\rangle|^2\le C_2\|f\|^2_{\mathbb{H}^s(\mathbb{R})}, \tag{2.63}$$

for all $f\in\mathbb{H}^s(\mathbb{R})$.

Theorem 2.17 basically says that the weighted ℓ_2-norm of the analysis wavelet coefficient sequence

$$\big\{\langle f,\mathbf{E}^k\phi_0\rangle\big\}_{k\in\mathbb{Z}}\cup\big\{\langle f,\psi^\ell_{j;j,k}\rangle\big\}_{k\in\mathbb{Z},j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathcal{J}_{j+1}\}}$$

of a given function $f\in\mathbb{H}^s(\mathbb{R})$ decomposed under the tight wavelet frame system

$$X\big(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathcal{J}_{j+1}\}}\big)$$

is equivalent to its Sobolev norm in $\mathbb{H}^s(\mathbb{R})$. The right hand side inequality is called the **upper bound** of the characterization in (2.63) and the left hand side inequality is called the **lower bound**.

For a tempered distribution f defined on $\mathbb{R}$, we denote

$$\nu_2(f):=\sup\{s\in\mathbb{R}: f\in\mathbb{H}^s(\mathbb{R})\}. \tag{2.64}$$

If $f\notin\mathbb{H}^s(\mathbb{R})$ for any $s\in\mathbb{R}$, then we simply set $\nu_2(f)=-\infty$.

It was shown in [41, Theorem 2.3] that a compactly supported refinable function $\phi\in L_2(\mathbb{R})$ whose refinement mask is a trigonometric polynomial is in $\mathbb{H}^s(\mathbb{R})$ for some $s > 0$. Generally, for a refinable function ϕ with a trigonometric polynomial refinement mask a, $\nu_2(\phi)$ can be computed via its mask [22, 42, 43, 70, 71, 53].

For a tempered distribution f defined on $\mathbb{R}$, we also define

$$\mu_2(f) := \sup\{s \in \mathbb{R} : [\widehat{f}, \widehat{f}]_s \in L_\infty(\mathbb{R})\}. \tag{2.65}$$

It was shown in [47, Proposition 2.6] that $\nu_2(f) = \mu_2(f)$ for any compactly supported function f.

In particular, when $\phi \in L_2(\mathbb{R})$ is refinable with its refinement mask c is finitely supported, we can write $\widehat{c}(\omega) = (1 + e^{-i\omega})^m \widehat{b}(\omega)$ for some nonnegative integer m and some 2π-periodic trigonometric polynomial $\widehat{b}(\omega)$ with $\widehat{b}(\pi) \neq 0$. Let b_{au} be the autocorrelation of b, i.e., $\widehat{b_{au}}(\omega) = |\widehat{b}(\omega)|^2 = \sum_{n=-N}^{N} b_{au}(n)e^{-in\omega}$, where N is some nonnegative integer. Denote by $\rho(c)$ the spectral radius of the square matrix $\big(b_{au}(2j-k)\big)_{-N\le j,k\le N}$. Define

$$\nu_2(c) := -1/2 - \log_2 \sqrt{\rho(c)}. \tag{2.66}$$

It is known that $\nu_2(c) \le \nu_2(\phi)$ whenever ϕ is the compactly supported refinable function associated with the finitely supported refinement mask c [42, 43, 47]. And the equality holds when $E(\phi)$ forms a Riesz basis of $S(\phi)$ [42, 43, 47, 70, 71, 53].

By applying Theorem 2.17 we can obtain the result for the stationary case:

Theorem 2.18. *Let $X(\Psi)$ be a stationary tight wavelet frame constructed via UEP in an MRA generated by a refinable function $\phi \in L_2(\mathbb{R})$ whose refinement mask h_0 is a finitely supported sequence. Assume that for $\alpha > 0$ there exists a positive constant C such that*

$$1 - |\widehat{h_0}(\omega)|^2 \le C|\omega|^{2\alpha}, \quad \omega \in \mathbb{R},$$

and

$$[\widehat{\phi}, \widehat{\phi}]_\alpha(\omega) \le C, \quad \omega \in \mathbb{R}.$$

If $-\alpha < s < \alpha$, *then* $X(\Psi)$ *can be normalized to a wavelet frame in* $\mathbb{H}^s(\mathbb{R})$, *i.e., there exist positive constants* C_1, C_2 *such that for any* $f \in \mathbb{H}^s(\mathbb{R})$ *we have*

$$C_1\|f\|^2_{\mathbb{H}^s(\mathbb{R})} \leq \sum_{k\in\mathbb{Z}} |\langle f, \mathbf{E}^k\phi\rangle|^2 + \sum_{\psi\in\Psi} \sum_{j\geq 0, k\in\mathbb{Z}} 2^{2js}|\langle f, \psi_{j,k}\rangle|^2 \leq C_2\|f\|^2_{\mathbb{H}^s(\mathbb{R})}. \quad (2.67)$$

Chapter 3

Stationary Tight Wavelet Frame Packet (STWFP)

One dimensional orthonormal wavelet packets were introduced in [18, 17, 20, 19] and their multivariate counterpart can be found in [72]. Similar to the construction of wavelet packets, we can build up a tight wavelet frame packet from a given tight wavelet frame constructed by UEP [66, 24].

3.1 Construction of STWFP

Suppose we have a tight wavelet frame

$$X(\Psi) := \{\psi_{j,k} : \psi \in \Psi, j, k \in \mathbb{Z}\}, \tag{3.1}$$

where $\Psi = \{\psi_1, \cdots, \psi_r\} \subset L_2(\mathbb{R})$. And we fix this tight wavelet frame $X(\Psi)$ thought out this thesis for the discussion of both stationary and nonstationary tight wavelet fame packet.

Let

$$V_j := \overline{\operatorname{span}}\{\phi_{j,k} : k \in \mathbb{Z}\},$$

$$W_{j,i} := \overline{\operatorname{span}}\{\psi_{i;j,k} : k \in \mathbb{Z}\}, \;\; i = 0, 1, \cdots, r.$$

In a tight wavelet frame decomposition, each MRA space V_j is decomposed into a lower resolution space V_{j-1} plus r detail spaces $W_{j-1,i}, i = 1, 2, \cdots, r$, i.e.,

$$V_j = V_{j-1} + \sum_{i=1}^{r} W_{j-1,i}.$$

Note that, these $r + 1$ spaces V_{j-1} and $W_{j-1,i}, i = 1, \cdots, r$, are in general not orthogonal. In other words, tight wavelet frame adds "redundancy".

By recursively splitting the MRA spaces, we obtain the space decomposition

$$\begin{aligned} V_j &= V_{j-1} + \sum_{i=1}^{r} W_{j-1,i} \\ &= V_{j-2} + \sum_{k=j-2}^{j-1} \sum_{i=1}^{r} W_{k,i} \\ &= \cdots \\ &= V_{j_0} + \sum_{k=j_0}^{j-1} \sum_{i=1}^{r} W_{k,i} \\ &= \sum_{k=-\infty}^{j-1} \sum_{i=1}^{r} W_{k,i}, \end{aligned}$$

as shown in Figure 3.1.

With the following result (in the orthonormal case, it is also referred as the "splitting trick"), we can split a given tight wavelet frame to obtain a tight wavelet frame packet.

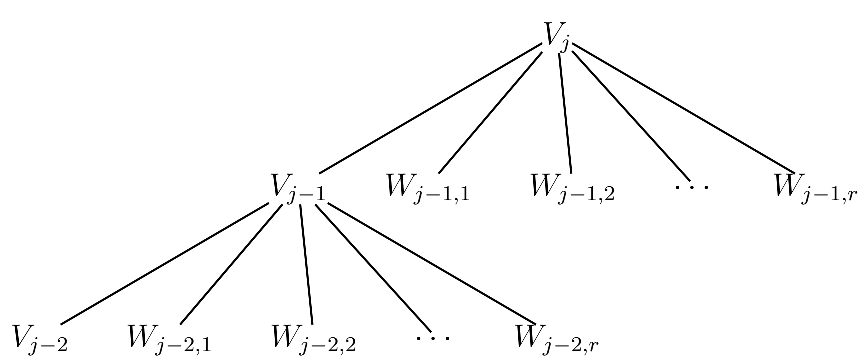


Figure 3.1: Tight wavelet frame space decomposition

Lemma 3.1. *Let $\theta \in L_2(\mathbb{R})$ and $\{\theta_{j,k} : k \in \mathbb{Z}\}$ be a Bessel set of $L_2(\mathbb{R})$ with Bessel bound C_θ for any fixed $j \in \mathbb{Z}$, i.e.,*

$$[\widehat{\theta}, \widehat{\theta}](\omega) \leq C_\theta, \quad \omega \in \mathbb{R}, \tag{3.2}$$

where C_θ is a positive constant. Let $\mathbf{h} = [h_0, h_1, \cdots, h_r]$ be a combined UEP mask satisfying the UEP condition (2.38). Define

$$\theta_i(x) := \sum_n 2\, h_i(n)\theta(2x - n), \tag{3.3}$$

$$U_i := \overline{\operatorname{span}}\{\theta_{i;j-1,k} : k \in \mathbb{Z}\}, \tag{3.4}$$

and $U := \overline{\operatorname{span}}\{\theta_{j,k} : k \in \mathbb{Z}\}$ for $i = 0, \cdots, r$. Then

(i). $\|\theta\|^2_{L_2(\mathbb{R})} \leq C_\theta$. *For $i = 0, \cdots, r$, $\|\theta_i\|^2_{L_2(\mathbb{R})} \leq C_\theta$ and*

$$[\widehat{\theta}_i, \widehat{\theta}_i](\omega) \leq C_\theta, \quad \omega \in \mathbb{R},$$

i.e., each set $\{\theta_{i;j-1,k} : k \in \mathbb{Z}\}$ is a Bessel set.

(ii). *For any sequence $c \in \ell_2(\mathbb{Z})$, there are $r+1$ sequences $c_i, i = 0, \cdots, r$, defined by*

$$c_i(k) = \sqrt{2} \sum_{n\in\mathbb{Z}} \overline{h_i}(n - 2k)\, c(n), \quad k \in \mathbb{Z}, \tag{3.5}$$

such that

$$\|c\|^2_{\ell_2(\mathbb{Z})} = \sum_{i=0}^{r} \|c_i\|^2_{\ell_2(\mathbb{Z})}, \tag{3.6}$$

and

$$\sum_{k\in\mathbb{Z}} c(k)\, \theta_{j,k} = \sum_{i=0}^{r} \sum_{k\in\mathbb{Z}} c_i(k)\, \theta_{i;j-1,k}; \tag{3.7}$$

(iii). *In particular, for any $f \in L_2(\mathbb{R})$, let $c(k) = \langle f, \theta_{j,k}\rangle$ for $k \in \mathbb{Z}$, then $c \in \ell_2(\mathbb{Z})$ and (3.5),(3.6) and (3.7) yield*

$$c_i(k) = \langle f, \theta_{i;j-1,k}\rangle, k \in \mathbb{Z}, i = 0, \cdots, r, \tag{3.8}$$

$$\sum_{k\in\mathbb{Z}} |\langle f, \theta_{j,k}\rangle|^2 = \sum_{i=0}^{r}\sum_{k\in\mathbb{Z}} |\langle f, \theta_{i;j-1,k}\rangle|^2, \tag{3.9}$$

and

$$\sum_{k\in\mathbb{Z}}\langle f, \theta_{j,k}\rangle\, \theta_{j,k} = \sum_{i=0}^{r}\sum_{k\in\mathbb{Z}}\langle f, \theta_{i;j-1,k}\rangle\, \theta_{i;j-1,k}, \tag{3.10}$$

respectively;

(iv). *U has the decomposition*

$$U = U_0 + U_1 + \cdots + U_r. \tag{3.11}$$

Proof. **(i).** It can be easily verified that

$$\|\theta\|^2_{L_2(\mathbb{R})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{\theta}(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\widehat{\theta}, \widehat{\theta}](\omega) d\omega \le C_\theta. \tag{3.12}$$

Notice that (3.3) can be recast in Fourier domain as

$$\widehat{\theta_i} = (\widehat{h_i}\widehat{\theta})(\cdot/2).$$

On the other hand, the UEP condition (2.38) naturally implies

$$\sum_{\nu\in\{0,\pi\}} |\widehat{h_i}(\cdot+\nu)|^2 \le 1, \qquad i = 0, \cdots, r. \tag{3.13}$$

Combined with (3.2) and (3.13), we can deduce that

$$\begin{aligned}
[\widehat{\theta}_i, \widehat{\theta}_i](\omega) &= \left[(\widehat{h_i}\widehat{\theta})(\cdot/2), (\widehat{h_i}\widehat{\theta})(\cdot/2)\right](\omega) \\
&= \sum_{k\in\mathbb{Z}} |(\widehat{h_i}\widehat{\theta})((\omega+2\pi k)/2)|^2 \\
&= \sum_{\nu\in\{0,\pi\}} |\widehat{h_i}(\omega/2+\nu)|^2 \sum_{k\in\mathbb{Z}} |\widehat{\theta}(\omega/2+\nu+2\pi k)|^2 \\
&= \sum_{\nu\in\{0,\pi\}} |\widehat{h_i}(\omega/2+\nu)|^2 [\widehat{\theta}, \widehat{\theta}](\omega/2+\nu) \\
&\le C_\theta \sum_{\nu\in\{0,\pi\}} |\widehat{h_i}(\omega/2+\nu)|^2 \quad a.e. \\
&\le C_\theta, \quad i = 0, \cdots, r.
\end{aligned}$$

It follows that $\|\theta_i\|^2_{L_2(\mathbb{R})} \le C_\theta$ for $i = 0, 1, \cdots, r$. Thus, **(i)** is proved.

(ii). (3.5) can be recast in Fourier domain as

$$\widehat{c_i}(\omega) = 2^{-1/2} \sum_{\nu\in\{0,\pi\}} (\widehat{c}\,\overline{\widehat{h_i}})(\omega/2+\nu), \quad i = 0, 1\cdots, r. \tag{3.14}$$

By (3.14), we have

$$\begin{aligned}
\sum_{i=0}^{r} |\widehat{c_i}(\omega)|^2 &= \frac{1}{2}\sum_{i=0}^{r}\sum_{\nu_1,\nu_2\in\{0,\pi\}} (\widehat{c}\,\overline{\widehat{h_i}})(\omega/2+\nu_1)\overline{(\widehat{c}\,\overline{\widehat{h_i}})(\omega/2+\nu_2)}\\
&= \frac{1}{2}\sum_{i=0}^{r}\sum_{\nu_1,\nu_2\in\{0,\pi\}} \widehat{c}(\omega/2+\nu_1)\overline{\widehat{c}}(\omega/2+\nu_2)\widehat{h_i}(\omega/2+\nu_2)\overline{\widehat{h_i}}(\omega/2+\nu_1)\\
&= \frac{1}{2}\sum_{\nu_1,\nu_2\in\{0,\pi\}} \widehat{c}(\omega/2+\nu_1)\overline{\widehat{c}}(\omega/2+\nu_2)\sum_{i=0}^{r}\widehat{h_i}(\omega/2+\nu_2)\overline{\widehat{h_i}}(\omega/2+\nu_1)\\
&= \frac{1}{2}\sum_{\nu_1,\nu_2\in\{0,\pi\}} \widehat{c}(\omega/2+\nu_1)\overline{\widehat{c}}(\omega/2+\nu_2)\,\delta_{\nu_1,\nu_2}\\
&= \frac{1}{2}\sum_{\nu\in\{0,\pi\}} |\widehat{c}(\omega/2+\nu)|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{i=0}^{r}\|c_i\|^2_{\ell_2(\mathbb{Z})} &= \sum_{i=0}^{r}\sum_{k\in\mathbb{Z}} |c_i(k)|^2\\
&= \sum_{i=0}^{r}\frac{1}{2\pi}\int_0^{2\pi} |\widehat{c_i}(\omega)|^2 d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i=0}^{r} |\widehat{c_i}(\omega)|^2 d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\sum_{\nu\in\{0,\pi\}} |\widehat{c}(\omega/2+\nu)|^2 d\omega\\
&= \frac{1}{2\pi}\int_0^{\pi}\sum_{\nu\in\{0,\pi\}} |\widehat{c}(\omega+\nu)|^2 d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi} |\widehat{c}(\omega)|^2 d\omega\\
&= \sum_{k} |c(k)|^2 = \|c\|^2_{\ell_2(\mathbb{Z})},
\end{aligned}$$

and (3.6) is proved.

Next, we prove (3.7) in a similar way. In Fourier domain, (3.7) becomes

$$2^{-j/2}(\widehat{c}\,\widehat{\theta})(2^{-j}\omega) = 2^{(-j+1)/2}\sum_{i=0}^{r}(\widehat{c_i}\widehat{\theta_i})(2^{-j+1}\omega).$$

It can be shown that the RHS=LHS, explicitly,

$$\begin{aligned}
RHS &= 2^{(-j+1)/2}\sum_{i=0}^{r}(\widehat{c_i}\widehat{\theta_i})(2^{-j+1}\omega)\\
&= 2^{(-j+1)/2}\sum_{i=0}^{r}\widehat{c_i}(2^{-j+1}\omega)\widehat{h_i}(2^{-j}\omega)\widehat{\theta}(2^{-j}\omega)\\
&= 2^{-j/2}\widehat{\theta}(2^{-j}\omega)\sum_{i=0}^{r}\left(\sum_{\nu\in\{0,\pi\}}(\widehat{c}\,\overline{\widehat{h_i}})(2^{-j}\omega+\nu)\right)\widehat{h_i}(2^{-j}\omega)\\
&= 2^{-j/2}\widehat{\theta}(2^{-j}\omega)\sum_{\nu\in\{0,\pi\}}\widehat{c}(2^{-j}\omega+\nu)\left(\sum_{i=0}^{r}\overline{\widehat{h_i}}(2^{-j}\omega+\nu)\widehat{h_i}(2^{-j}\omega)\right)\\
&= 2^{-j/2}\widehat{\theta}(2^{-j}\omega)\sum_{\nu\in\{0,\pi\}}\widehat{c}(2^{-j}\omega+\nu)\,\delta_{0,\nu}\\
&= 2^{-j/2}\widehat{\theta}(2^{-j}\omega)\,\widehat{c}(2^{-j}\omega)\\
&= 2^{-j/2}(\widehat{c\theta})(2^{-j}\omega) = LHS,
\end{aligned}$$

and we obtain (3.7). And **(ii)** is thus proved.

(iii). $c \in \ell_2(\mathbb{Z})$ is easily followed from the Bessel set property of $\{\theta_{j,k} : k \in \mathbb{Z}\}$. In addition, if we can obtain (3.8), then (3.9) and (3.10) are the direct consequence of (3.6) and (3.7), respectively. It boils down to show that (3.8) is true. It can be shown from (3.5) that

$$\begin{aligned}
c_i(k) &= \sum_n \sqrt{2}\,\overline{h_i}(n-2k)\,c(n)\\
&= \sum_n \sqrt{2}\,\overline{h_i}(n-2k)\langle f,\theta_{j,n}\rangle\\
&= \Big\langle f, \sum_n \sqrt{2}\,h_i(n-2k)\theta_{j,n}\Big\rangle\\
&= \Big\langle f, \sum_m \sqrt{2}\,h_i(m)\theta_{j,2k+m}\Big\rangle\\
&= \langle f,\theta_{i;j-1,k}\rangle, \quad i=0,1,\cdots,r.
\end{aligned}$$

And **(iii)** is thus proved.

(iv). It follows from (3.3) that

$$U_0+U_1+\cdots+U_r \subseteq U.$$

Invoking (3.5) by taking $c\in\ell_0(\mathbb{Z})$, where $\ell_0(\mathbb{Z})$ is the spaces of finitely support sequences, we can obtain

$$U\subseteq U_0+U_1+\cdots+U_r.$$

Hence, (3.11) is proved. And we finish the proof of Lemma 3.1. □

Remark 3.1. *Since* $\{\theta_{j,k}:k\in\mathbb{Z}\}$ *is a Bessel set, the space* U^* *defined by*

$$U^* := \Big\{\sum_{k\in\mathbb{Z}} c(k)\theta_{j,k} : c\in\ell_2(\mathbb{Z})\Big\}$$

is generally not closed as $[\widehat{\theta},\widehat{\theta}]$ *may not have a positive lower bound. In other words,* U^* *is just a dense subset of* U *in general. Lemma 3.1 basically says we can always split* U*, the completion of* U^**, into* $r+1$ *closed subspaces* $U_0,U_1,\cdots,U_r$ *by using a combined UEP mask* $\mathbf{h}=[h_0,h_1,\cdots,h_r]$*. Furthermore, if* $f\in U^*$*, i.e.,* $f=\sum_{k\in\mathbb{Z}}c(k)\theta_{j,k}$ *for some* $c\in\ell_2(\mathbb{Z})$*, then we can define* $f_i=\sum_{k\in\mathbb{Z}}c_i(k)\theta_{i;j-1,k}, i=0,\cdots,r$*, where* c_i *is computed from* c *by (3.5), and obtain the decomposition*

$$f_i\in U_i, \quad i=0,\cdots,r,$$

and the reconstruction

$$f = \sum_{i=0}^{r} f_i,$$

together with the desirable norm preserving property

$$\|c\|_{\ell_2(\mathbb{Z})}^2 = \sum_{i=0}^{r} \|c_i\|_{\ell_2(\mathbb{Z})}^2.$$

Lemma 3.1 is the base to build up tight wavelet frame packet. Note that, in the assumption of this lemma, the combined UEP mask $\mathbf{h}$ may have no relation with θ. We only need a arbitrarily given combined UEP mask. By this result, for a given tight wavelet frame $X(\Psi)$, we can further decompose the wavelet spaces $W_{j,i}$ by any combined UEP masks. And then by selectively and recursively decomposing the wavelet spaces $W_{j,i}$, we can obtain various tight wavelet frames which are altogether called **a stationary tight wavelet frame packet**.

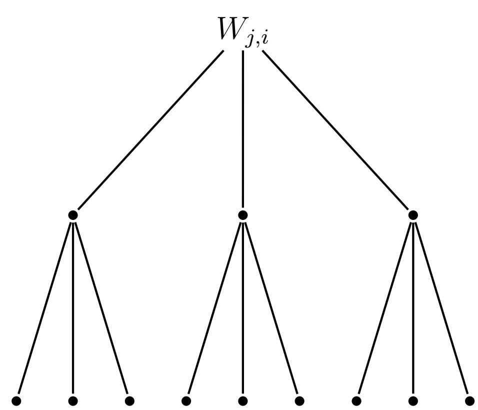


Figure 3.2: Stationary tight wavelet frame packet decomposition

Furthermore, we can also recursively decompose V_j as well as $W_{j,i}$ in the same way, however, in this case we may change the underlying MRA spaces $(V_j)_{j\in\mathbb{Z}}$ associated with $X(\Psi)$ if one of the lowpass filters in the set of combined UEP masks decomposing V_j does not coincide with the refinement mask of ϕ which generates $(V_j)_{j\in\mathbb{Z}}$, and all the tight wavelet frames obtained in this way with the

MRA spaces $(V_j)_{j\in\mathbb{Z}}$ are modified will be called **a nonstationary tight wavelet frame packet**.

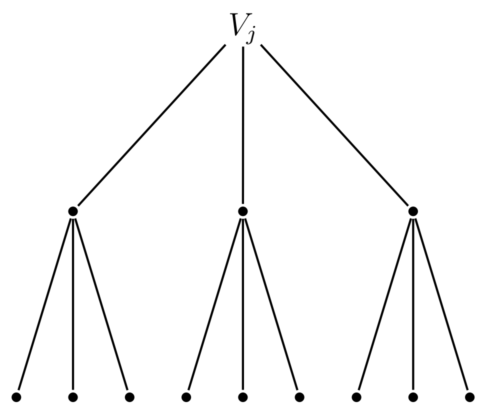


Figure 3.3: Nonstationary tight wavelet frame packet decomposition

As previously mentioned, given a tight wavelet frame $X(\Psi)$, by Lemma 3.1 we can further recursively split the wavelet spaces $W_{j,i}$ by any combined UEP mask. By selectively splitting of $W_{j,i}$, we can obtain tight wavelet frame packets. We can obtain stationary tight wavelet frame packets by only selectively and recursively splitting the wavelet spaces $W_{j,i}$.

For simplicity, we construct stationary tight wavelet frame packets by recursively splitting $W_{j,i}$ with the combined UEP mask $\mathbf{h} = [h_0, h_1, \cdots, h_r]$ associated with the given tight wavelet frame $X(\Psi)$, and set the coarsest scale $j_0 = 0$.

We try two ways to construct a stationary tight wavelet frame packet, one is a recursive way, the other one is the decomposition way. We first describe these two methods, and later show that they are essentially the same.

A Recursive Construction of STWFP

Define $p_0 := \phi$, and for $\kappa \in \mathbb{N}$, κ has the unique representation

$$\kappa = (r+1)l + i$$

for some $l \in \mathbb{N}_0$ and $0 \le i \le r$, and we define p_κ by

$$p_\kappa = p_{(r+1)l+i} := \sum_{n\in\mathbb{Z}} 2h_i(n)p_l(2\cdot -n). \tag{3.15}$$

Note that definition (3.15) implies that $p_0 = \phi$ and $p_i = \psi_i$ for $i = 1, \cdots, r$.

Define

$$P_\kappa := \overline{\operatorname{span}}\{p_{\kappa;0,k} : k \in \mathbb{Z}\}, \kappa \in \mathbb{N}_0. \tag{3.16}$$

(Note that (3.16) implies $P_0 = V_0, P_i = W_{0,i}$ for $i = 1, \cdots, r$.)

Since $X(\Psi)$ is a tight wavelet frame constructed via UEP in an MRA generated by ϕ. We have

$$[\widehat{\phi}, \widehat{\phi}](\omega) \le 1, \quad \omega \in \mathbb{R}.$$

By invoking Lemma 3.1, for $\kappa \in \mathbb{N}$ we have

$$[p_\kappa, p_\kappa](\omega) \le 1, \quad \omega \in \mathbb{R},$$

$$\mathbf{D}\, P_\kappa = \sum_{n=(r+1)\kappa}^{(r+1)(\kappa+1)-1} P_n,$$

and

$$\sum_{k\in\mathbb{Z}} |\langle f, p_{\kappa;1,k}\rangle|^2 = \sum_{n=(r+1)\kappa}^{(r+1)(\kappa+1)-1} \sum_{k\in\mathbb{Z}} |\langle f, p_{n;0,k}\rangle|^2,$$

for any $f \in L_2(\mathbb{R})$.

Generally, for $j \in \mathbb{N}$, we can recursively apply Lemma 3.1 to obtain

$$\mathbf{D}^j\, P_\kappa = \sum_{n=(r+1)^j\kappa}^{(r+1)^j(\kappa+1)-1} P_n; \tag{3.17}$$

and

$$\sum_{k\in\mathbb{Z}} |\langle f, p_{\kappa;j,k}\rangle|^2 = \sum_{n=(r+1)^j\kappa}^{(r+1)^j(\kappa+1)-1} \sum_{k\in\mathbb{Z}} |\langle f, p_{n;0,k}\rangle|^2, \tag{3.18}$$

for any $f \in L_2(\mathbb{R})$.

For $i = 1, \cdots, r$, substituting $\kappa = i$ in (3.17) and (3.18), we can obtain

$$W_{j,i} = \mathbf{D}^j W_{0,i} = \mathbf{D}^j\, P_i = \sum_{n=(r+1)^j i}^{(r+1)^j(i+1)-1} P_n, \tag{3.19}$$

and

$$\sum_{k\in\mathbb{Z}} |\langle f, \psi_{i;j,k}\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle f, p_{i;j,k}\rangle|^2 = \sum_{n=(r+1)^j i}^{(r+1)^j(i+1)-1} \sum_{k\in\mathbb{Z}} |\langle f, p_{n;0,k}\rangle|^2,$$

for any $f \in L_2(\mathbb{R})$, respectively.

Note that (3.19) means that each wavelet space $W_{j,i}(j \geq 1, i = 1, 2, \cdots, r)$ can be further decomposed into $(r+1)^j$ subspaces $P_n, (r+1)^j i \leq n \leq (r+1)^j(i+1)-1$.

Interestingly, if we take $\kappa = 0$ in (3.17) and (3.18), we can obtain,

$$V_j = \sum_{n=0}^{(r+1)^j-1} P_n,$$

and

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{j,k}\rangle|^2 = \sum_{n=0}^{(r+1)^j-1} \sum_{k\in\mathbb{Z}} |\langle f, p_{n;0,k}\rangle|^2, \tag{3.20}$$

for any $f \in L_2(\mathbb{R})$, respectively.

By choosing j to be a fixed level $J > 0$, the MRA property of $(V_j)_{j\in\mathbb{Z}}$ leads to

$$L_2(\mathbb{R}) = \sum_{n=0}^{(r+1)^J-1} P_n + \sum_{i=1}^{r} \sum_{j\geq J} W_{j,i}. \tag{3.21}$$

Theorem 3.1. *For a given tight wavelet frame $X(\Psi)$ constructed via UEP in an MRA generated by ϕ, with the combined UEP mask $\mathbf{h} = [h_0, h_1, \cdots, h_r]$ satisfying the UEP condition (2.38), define $p_0 := \phi$ and define p_n for $n \in \mathbb{N}$ as in (3.15), then for any fixed $J > 0$,*

$$\mathcal{P} := \{p_{n;0,k} : 0 \leq n \leq (r+1)^J - 1, k \in \mathbb{Z}\} \cup \{\psi_{j,k} : \psi \in \Psi, k \in \mathbb{Z}, j \geq J\}$$

is a tight wavelet frame.

Proof. Since $X(\Psi)$ is a tight wavelet frame of $L_2(\mathbb{R})$, by [24, Lemma 2.4], for any $f \in L_2(\mathbb{R})$, we have

$$\begin{aligned}\|f\|^2 &= \sum_{k\in\mathbb{Z}} |\langle f, \phi_{0,k}\rangle|^2 + \sum_{\psi\in\Psi}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \\ &= \sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 + \sum_{\psi\in\Psi}\sum_{j\geq J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2\end{aligned}$$

Combined with (3.20), we can quickly deduce that

$$\begin{aligned}\|f\|^2 &= \sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 + \sum_{\psi\in\Psi}\sum_{j\geq J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 \\ &= \sum_{n=0}^{(r+1)^j-1}\sum_{k\in\mathbb{Z}} |\langle f, \mathbf{E}^k p_n\rangle|^2 + \sum_{\psi\in\Psi}\sum_{j\geq J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2,\end{aligned}$$

for any $f \in L_2(\mathbb{R})$. And Theorem 3.1 is thus proved. ☐

We can observe that Theorem 3.1 is corresponding to the special $L_2(\mathbb{R})$ space decomposition of (3.21). Consequently, based on the tight wavelet frame $\mathcal{P}$ constructed above, we try to obtain other tight wavelet frames by choosing other $L_2(\mathbb{R})$ space decompositions. To do this, we introduce the notation of the disjoint partition Λ_J of a finite set of nonnegative integers

$$\Xi_J := \{n \in \mathbb{N}_0 : 0 \leq n \leq (r+1)^J - 1\} \tag{3.22}$$

into disjoint subsets of the form

$$I_{j,\kappa} := \{(r+1)^j\kappa, \cdots, (r+1)^j(\kappa+1) - 1\}, \quad j, \kappa \in \mathbb{N}_0,$$

i.e.,

$$\Lambda_J := \Big\{I_{j,\kappa} : \bigcup I_{j,\kappa} = \Xi_J\Big\}. \tag{3.23}$$

Then, it follows from (3.21) and (3.17) that

$$
\begin{aligned}
L_2(\mathbb{R}) &= \sum_{n=0}^{(r+1)^J-1} P_n + \sum_{i=1}^{r}\sum_{j\geq J} W_{j,i} \\
&= \sum_{I_{j,\kappa}\in\Lambda_J} \sum_{n=(r+1)^j\kappa}^{(r+1)^j(\kappa+1)-1} P_n + \sum_{i=1}^{r}\sum_{j\geq J} W_{j,i} \\
&= \sum_{I_{j,\kappa}\in\Lambda_J} \mathbf{D}^j P_\kappa + \sum_{i=1}^{r}\sum_{j\geq J} W_{j,i}.
\end{aligned}
$$

Theorem 3.2. *For a given tight wavelet frame $X(\Psi)$ constructed via UEP in an MRA generated by ϕ, with the combined UEP mask $\mathbf{h} = [h_0, h_1, \cdots, h_r]$ satisfying the UEP condition (2.38), define $p_0 := \phi$ and define p_n for $n \in \mathbb{N}$ as in (3.15). For any fixed $J > 0$, Λ_J is a disjoint partition of Ξ_J, where Λ_J and Ξ_J are defined in (3.23) and (3.22), respectively. Then*

$$
\mathcal{P}_{\Lambda_J} := \{p_{n;j,k} : I_{j,n} \in \Lambda_J, k \in \mathbb{Z}\} \cup \{\psi_{j,k} : \psi \in \Psi, j \geq J, k \in \mathbb{Z}\}
$$

is a tight wavelet frame.

Proof. Notice that for any $f \in L_2(\mathbb{R})$, we have

$$
\begin{aligned}
\sum_{I_{j,n}\in\Lambda_J}\sum_{k\in\mathbb{Z}} |\langle f, p_{n;j,k}\rangle|^2 &= \sum_{I_{j,n}\in\Lambda_J}\sum_{k\in\mathbb{Z}} \sum_{n=(r+1)^j n}^{(r+1)^j(n+1)-1} |\langle f, p_{n;0,k}\rangle|^2 \\
&= \sum_{I_{j,n}\in\Lambda_J} \sum_{n=(r+1)^j n}^{(r+1)^j(n+1)-1} \sum_{k\in\mathbb{Z}} |\langle f, p_{n;0,k}\rangle|^2 \\
&= \sum_{n=0}^{(r+1)^J-1} |\langle f, p_{n;0,k}\rangle|^2.
\end{aligned}
$$

By invoking Theorem 3.1, we can obtain $\mathcal{P}_{\Lambda_J}$ is a tight wavelet frame of $L_2(\mathbb{R})$. □

For any fixed $J > 0$ and a disjoint partition Λ_J of Ξ_J, we can obtain a tight wavelet frame $\mathcal{P}_{\Lambda_J}$ by Theorem 3.2. All tight wavelet frames constructed in this way are called are called **a stationary tight wavelet frame packet**.

Example 3.1. *Let $X(\Psi)$ be the piecewise linear tight wavelet frame as introduced in Example 2.5, and let $J = 2$. By Theorem 3.1, we can obtain a stationary tight wavelet frame packet, the plots of $p_0, p_1, \cdots, p_8$ are depicted in Figure 3.1.*

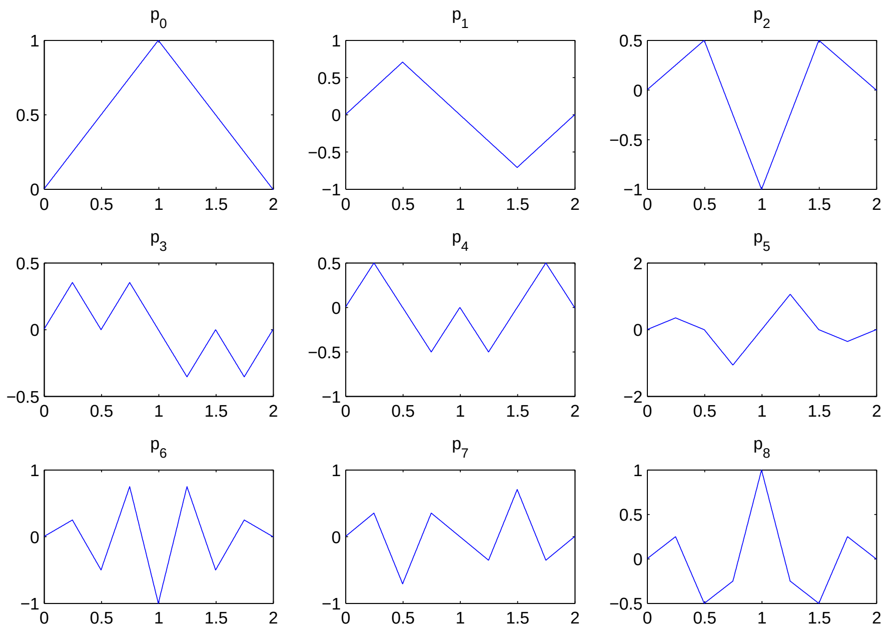


Figure 3.4: Piecewise linear tight wavelet frame packet

Example 3.2. *Let $X(\Psi)$ be the piecewise cubic tight wavelet frame as introduced in Example 2.7, and let $J = 2$. By Theorem 3.1, we can obtain a stationary tight wavelet frame packet, the plots of $p_0, p_1, \cdots, p_{15}$ are depicted in Figure 3.2.*

A Decomposition Construction of STWFP

Besides the recursive derivation of stationary tight wavelet frame packets introduced in the previous section, stationary tight wavelet frame packets can also be constructed by directly decomposing of the MRA space V_J for a fixed level $J > 0$ to the level 0;

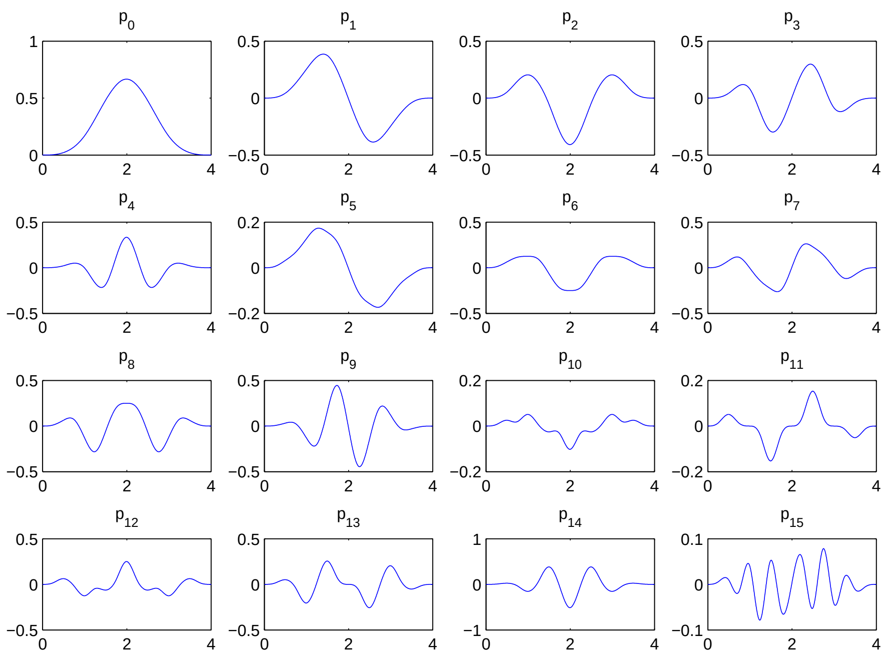


Figure 3.5: Piecewise cubic tight wavelet frame packet

For a given tight wavelet frame $X(\Psi)$ constructed via UEP in an MRA $(V_j)_{j\in\mathbb{Z}}$ generated by ϕ, and the associated combined UEP mask is $\mathbf{h} = [h_0, h_1, \cdots, h_r]$. We now recursively decompose V_J for a fixed level $J > 0$ to level 0 by the fixed combined UEP mask $\mathbf{h}$.

At the first level of decomposition, By Lemma 3.1, V_J is decomposed with $\mathbf{h}$ into the $r+1$ spaces $W_{J-1,\mathbb{i}}, \mathbb{i} \in \Omega_1$, where Ω_1 is a J-tuple index set defined by

$$\Omega_1 := \Big\{\mathbb{i} = (i_J, i_{J-1}, \cdots, i_1) : 0 \le i_J \le r, i_{J-1} = \cdots = i_1 = 0\Big\},$$

and for $\mathbb{i} = (i_J, i_{J-1}, \cdots, i_1)$, we define

$$\mathbb{i}(n) := i_n, \quad n = 1, 2, \cdots, J,$$

and with this notation we have

$$p_{\mathbb{i}} := \sum_{n\in\mathbb{Z}} 2h_{\mathbb{i}(1)}(n)\phi(2\cdot -n),$$

and $W_{J-1,\mathbb{i}}$ is defined by

$$W_{J-1,\mathbb{i}} := \overline{\text{span}}\left\{p_{\mathbb{i};J-1,k} : k \in \mathbb{Z}\right\}.$$

For any $f \in L_2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 = \sum_{\mathbb{i}\in\Omega_1}\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};J-1,k}\rangle|^2.$$

At the second level of decomposition, by Lemma 3.1, each space $W_{J-1,\mathbb{i}}, \mathbb{i} \in \Omega_1$ is decomposed with $\mathbf{h}$ into spaces $W_{J-2,\mathbb{i}'}, \mathbb{i}' \in \Omega_2^{\mathbb{i}}$, where $\Omega_2^{\mathbb{i}}$ is a subset of Ω_2 defined by

$$\Omega_2^{\mathbb{i}} := \{\mathbb{i}' \in \Omega_2 : \mathbb{i}'(1) = \mathbb{i}(1)\}$$

and Ω_2 is a J-tuple index set defined by

$$\Omega_2 := \Big\{(i_J, i_{J-1}, \cdots, i_1) : 0 \le i_J \le r, 0 \le i_{J-1} \le r, i_{J-2} = \cdots = i_1 = 0\Big\},$$

$$p_{\mathbb{i}'} := \sum_{n\in\mathbb{Z}} 2h_{\mathbb{i}'(2)}(n) p_{\mathbb{i}}(2\cdot - n),$$

$$W_{J-2,\mathbb{i}'} := \overline{\text{span}}\left\{p_{\mathbb{i}';J-2,k} : k \in \mathbb{Z}\right\}.$$

And for any $f \in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};J-1,k}\rangle|^2 = \sum_{\mathbb{i}'\in\Omega_2^{\mathbb{i}}}\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i}';J-2,k}\rangle|^2.$$

Generally, at the ℓ-th level $(2 \le \ell \le J)$ of decomposition, by Lemma 3.1, each space $W_{J-\ell+1,\mathbb{i}}, \mathbb{i} \in \Omega_{\ell-1}$ is decomposed with $\mathbf{h}$ into spaces $W_{J-\ell,\mathbb{i}'}$, $\mathbb{i}' \in \Omega_\ell^{\mathbb{i}}$, where $\Omega_\ell^{\mathbb{i}}$ is a subset of Ω_ℓ defined by

$$\Omega_\ell^{\mathbb{i}} := \{\mathbb{i}' \in \Omega_\ell : \mathbb{i}'(n) = \mathbb{i}(n) \text{ for } 1 \le n \le \ell - 1\} \tag{3.24}$$

and Ω_ℓ is a J-tuple index set defined by

$$\Omega_\ell := \Big\{(i_J, i_{J-1}, \cdots, i_1) : 0 \le i_{J-l} \le r, 0 \le l \le \ell, i_{J-\ell} = \cdots = i_1 = 0\Big\},$$

$$p_{\mathbb{i}'} := \sum_{n\in\mathbb{Z}} 2h_{\mathbb{i}'(\ell)}(n)p_{\mathbb{i}}(2\cdot -n), \tag{3.25}$$

$$W_{J-\ell,\mathbb{i}'} := \overline{\operatorname{span}}\left\{p_{\mathbb{i}';J-\ell,k} : k\in\mathbb{Z}\right\}.$$

And for any $f\in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};J-\ell+1,k}\rangle|^2 = \sum_{\mathbb{i}'\in\Omega_\ell^{\mathbb{i}}}\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i}';J-\ell,k}\rangle|^2.$$

In particular, at the J-th level of decomposition, by Lemma 3.1, each space $W_{1,\mathbb{i}}, \mathbb{i}\in\Omega_{J-1}$ is decomposed with $\mathbf{h}$ into spaces $W_{0,\mathbb{i}'}$, $\mathbb{i}'\in\Omega_J^{\mathbb{i}}$, where $\Omega_J^{\mathbb{i}}$ is a subset of Ω_J defined by

$$\Omega_J^{\mathbb{i}} := \{\mathbb{i}'\in\Omega_J : \mathbb{i}'(n) = \mathbb{i}(n) \text{ for } 1\le n\le J-1\}$$

and Ω_J is a J-tuple index set defined by

$$\Omega_J := \Big\{(i_J, i_{J-1}, \cdots, i_1) : 0\le i_l\le r, 1\le l\le J\Big\}, \tag{3.26}$$

$$p_{\mathbb{i}'} := \sum_{n\in\mathbb{Z}} 2h_{\mathbb{i}'(J)}(n)p_{\mathbb{i}}(2\cdot -n),$$

$$W_{0,\mathbb{i}'} := \overline{\operatorname{span}}\left\{p_{\mathbb{i}';0,k} : k\in\mathbb{Z}\right\}.$$

And for any $f\in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};1,k}\rangle|^2 = \sum_{\mathbb{i}'\in\Omega_J^{\mathbb{i}}}\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i}';0,k}\rangle|^2.$$

By combining all the inner product equations in the construction, we can obtain

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 = \sum_{\mathbb{i}\in\Omega_J}\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};0,k}\rangle|^2. \tag{3.27}$$

for any $f\in L_2(\mathbb{R})$, In other words, we obtain another representation of V_J, i.e.,

$$V_J = \overline{\operatorname{span}}\left\{p_{\mathbb{i};0,k} : k\in\mathbb{Z}, \mathbb{i}\in\Omega_J\right\}.$$

Theorem 3.3. *For a given tight wavelet frame $X(\Psi)$, the system*

$$\Pi^{Sta} := \left\{p_{\mathbb{i};0,k}, \mathbb{i} \in \Omega_J\right\} \cup \left\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\right\},$$

is also a tight wavelet frame, where Ω_J is a index set defined in (3.26).

Proof. Since $X(\Psi)$ is a tight wavelet frame of $L_2(\mathbb{R})$, by [24, Lemma 2.4], for any $f \in L_2(\mathbb{R})$ we have

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 + \sum_{\psi\in\Psi}\sum_{j\geq J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2.$$

On the other hand, from (3.27) we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 = \sum_{\mathbb{i}\in\Omega_J}\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};0,k}\rangle|^2.$$

It follows that

$$\|f\|^2 = \sum_{\mathbb{i}\in\Omega_J}\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};0,k}\rangle|^2 + \sum_{\psi\in\Psi}\sum_{j\geq J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2.$$

Hence, Theorem 3.3 is proved. □

Similar to the recursive construction of stationary tight wavelet frame packets, based on the tight wavelet frame Π^{Sta} constructed above, we can obtain a stationary tight wavelet frame packet by performing various disjoint partitions Λ_J of Ω_J with each partition separating Ω_J into disjoint subsets of the form

$$I_{j,\mathbb{i}} := \left\{(i_J, \cdots, i_{j+1}, i'_j, \cdots, i'_1) \in \Omega_J : \mathbb{i} = (i_J, \cdots, i_{j+1}, 0, \cdots, 0) \in \Omega_{J-j}\right\},$$

i.e.,

$$\Lambda_J := \left\{I_{j,\mathbb{i}} : \bigcup I_{j,\mathbb{i}} = \Omega_J\right\}. \tag{3.28}$$

Theorem 3.4. *For a given tight wavelet frame $X(\Psi)$, let Λ_J be a disjoint partition of Ω_J, where Ω_J and Λ_J are defined in (3.26) and (3.28), respectively. Then the system*

$$\Pi^{Sta}_{\Lambda_J} := \left\{p_{\mathbb{i};j,k} : k \in \mathbb{Z}, I_{j,\mathbb{i}} \in \Lambda_J\right\} \cup \left\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\right\}, \tag{3.29}$$

is also a tight wavelet frame.

Proof. Since Λ_J is a disjoint partition of Ω_J, for any $f \in L_2(\mathbb{R})$, we have

$$\sum_{I_{j,\mathbb{i}}\in\Lambda_J}\sum_{k\in\mathbb{Z}}|\langle f, p_{\mathbb{i};j,k}\rangle|^2 = \sum_{I_{j,\mathbb{i}}\in\Lambda_J}\sum_{\mathbb{i}'\in I_{j,\mathbb{i}}}\sum_{k\in\mathbb{Z}}|\langle f, p_{\mathbb{i}';0,k}\rangle|^2$$
$$= \sum_{\mathbb{i}\in\Omega_J}\sum_{k\in\mathbb{Z}}|\langle f, p_{\mathbb{i};0,k}\rangle|^2.$$

By applying Theorem 3.3, Theorem 3.4 is proved. □

Equivalence of the Two Constructions

We may find that these two constructions are quite similar. In fact, we can show that they are equivalent. For convenience, we consider the multi-index $\mathbb{i} = (i_J, \cdots, i_1)$ which is extensively used in our decomposition derivation of a stationary tight wavelet frame packet as a base $r+1$ number $(i_J \cdots i_1)_{r+1}$ and replace it with its decimal representation

$$i = \sum_{n=1}^{J} i_n (r+1)^{n-1},$$

i.e., with the bijection

$$\Upsilon : \mathbb{i} \to \mathbb{N}_0 : (i_J, \cdots, i_1) \mapsto \sum_{n=1}^{J} i_n (r+1)^{n-1},$$

$$\Pi^{Sta} := \left\{p_{\mathbb{i};0,k}, \mathbb{i} \in \Omega_J\right\} \cup \left\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\right\}$$

has a decimal index version

$$\Pi^{Sta} := \left\{p_{n;0,k}, 0 \leq n \leq (r+1)^J - 1\right\} \cup \left\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\right\},$$

which is the same as the one derived in the recursive construction, and vice versa.

By taking into account that ϕ is a refinable function with its refinement mast h_0, we can quickly conclude from (3.25) that

$$p_{\mathbb{i}^*} = \phi,$$

where $\mathbb{i}^* = (\underbrace{0, 0, \cdots, 0}_{J})$.

Furthermore, by the From (3.24) and (3.25), we can observe that in the decomposition way derivation, $p_{\mathbb{i}_2}$, where $\mathbb{i}_2 = (i_J, \cdots, i_{\ell+1}, i_\ell, 0, \cdots, 0)$, is constructed from $p_{\mathbb{i}_1}$, where $\mathbb{i}_1 = (i_J, \cdots, i_{\ell+1}, 0, \cdots, 0)$ through the equation

$$p_{\mathbb{i}_2} := \sum_{n\in\mathbb{Z}} 2\, h_{i_\ell}(n) p_{\mathbb{i}_1}(2\cdot -n).$$

By observing that

$$\Upsilon(\mathbb{i}_2) = \sum_{n=\ell}^{J} i_n(r+1)^{n-1} = (r+1)\sum_{n=\ell+1}^{J} i_n(r+1)^{n-1} + i_\ell = (r+1)\Upsilon(\mathbb{i}_1) + i_\ell,$$

i.e., $\Upsilon(\mathbb{i}_2)$ has the unique representation $\Upsilon(\mathbb{i}_2) = (r+1)\Upsilon(\mathbb{i}_2) + i_\ell$, which is the same as the recursive construction (3.15), we can conclude that these two derivations are totally equivalent for the derivation of stationary tight wavelet frame packets.

For the nonstationary case, we generally do not have a recursive relation as in the stationary case since the original MRA $(V_j)_{j\in\mathbb{Z}}$ associated with $X(\Psi)$ will not be preserved. Hence, we adopt the decomposition method in our later construction of nonstationary tight wavelet frame packets.

3.2 Characterization of Sobolev Spaces by STWFP

Once we build up a stationary tight wavelet frame packet (STWFP)

$$\mathcal{P}^{Sta}_{\Lambda_J} := \{p_{n;j,k} : k \in \mathbb{Z}, I_{j,n} \in \Lambda_J\} \cup \{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\},$$

we can use the weighted ℓ_2-norm of the analysis STWFP coefficient sequence

$$\{\langle f, p_{n;j,k}\rangle\}_{k\in\mathbb{Z}, I_{j,n}\in\Lambda_J} \cup \{\langle f, \psi_{j,k}\rangle\}_{k\in\mathbb{Z}, j\geq J, k\in\mathbb{Z}, \psi\in\Psi}$$

of a given function $f \in \mathbb{H}^s(\mathbb{R})$ to characterize its Sobolev norm in $\mathbb{H}^s(\mathbb{R})$.

Theorem 3.5. *For a given tight wavelet frame $X(\Psi)$ constructed via UEP in an MRA generated by ϕ, with the combined UEP mask $\mathbf{h} = [h_0, h_1, \cdots, h_r]$, define $p_0 := \phi$ and define p_n for $n \in \mathbb{N}$ as in (3.15). Assume that for $\alpha > 0$ there exists a positive constant C such that*

$$1 - |\widehat{h_0}(\omega)|^2 \leq C|\omega|^{2\alpha}, \quad \omega \in \mathbb{R}, \tag{3.30}$$

and

$$[\widehat{\phi}, \widehat{\phi}]_\alpha(\omega) \leq C, \quad \omega \in \mathbb{R}. \tag{3.31}$$

For any fixed $J > 0$, Λ_J is a disjoint partition of Ξ_J, where Λ_J and Ξ_J are defined in (3.23) and (3.22), respectively. If $-\alpha < s < \alpha$, then

$$\mathcal{P}^s_{\Lambda_J} := \{2^{js} p_{n;j,k} : I_{j,n} \in \Lambda_J, k \in \mathbb{Z}\} \cup \{2^{js}\psi_{j,k} : \psi \in \Psi, k \in \mathbb{Z}, j \geq J\}$$

is a wavelet frame of $\mathbb{H}^s(\mathbb{R})$, i.e., there exist two positive constants C_1, C_2 such that

$$\begin{aligned} C_1\|f\|^2_{\mathbb{H}^s(\mathbb{R})} \;\leq\; & \sum_{I_{j,n}\in\Lambda_J} \sum_{k\in\mathbb{Z}} 2^{2js} |\langle f, p_{n;j,k}\rangle|^2 \\ + \; & \sum_{\psi\in\Psi} \sum_{j\geq J} \sum_{k\in\mathbb{Z}} 2^{2js} |\langle f, \psi_{j,k}\rangle|^2 \leq C_2 \|f\|^2_{\mathbb{H}^s(\mathbb{R})}, \end{aligned} \tag{3.32}$$

hold for all $f \in \mathbb{H}^s(\mathbb{R})$.

Proof. For $-\alpha < s < \alpha$, we can show that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 &= \frac{1}{2\pi}\int_{\mathbb{T}} 2^J |[\widehat{f}(2^J\cdot), \widehat{\phi}](\omega)|^2 d\omega \\
&\le \frac{2^{J-1}}{\pi}\int_{\mathbb{T}} [\widehat{f}(2^J\cdot), \widehat{f}(2^J\cdot)]_{-s}(\omega)[\widehat{\phi}, \widehat{\phi}]_s(\omega) d\omega \\
&\le \|[\widehat{\phi}, \widehat{\phi}]_s\|_{L_\infty(\mathbb{R})} \frac{2^{J-1}}{\pi}\int_{\mathbb{T}} [\widehat{f}(2^J\cdot), \widehat{f}(2^J\cdot)]_s(\omega) d\omega \\
&\le \|[\widehat{\phi}, \widehat{\phi}]_\alpha\|_{L_\infty(\mathbb{R})} \frac{2^{J-1}}{\pi}\int_{\mathbb{R}} |\widehat{f}(2^J\omega)|^2 (1+|\omega|^2)^{-s} d\omega \\
&\le \frac{C}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\omega)|^2 (1+|2^{-J}\omega|^2)^{-s} d\omega \\
&= \frac{C}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\omega)|^2 (1+|\omega|^2)^{-s} \left(\frac{1+|\omega|^2}{1+|2^{-J}\omega|^2}\right)^s d\omega \\
&\le C\max\{1, 2^{2Js}\} \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\omega)|^2 (1+|\omega|^2)^{-s} d\omega \\
&= C\max\{1, 2^{2Js}\}\|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})},
\end{aligned}$$

and in the last inequality we used the fact that

$$1 \le \frac{1+|\omega|^2}{1+|2^{-J}\omega|^2} \le 2^{2J}, \quad \omega\in\mathbb{R}, J\in\mathbb{N}.$$

On the other hand,

$$\begin{aligned}
\sum_{I_{j,n}\in\Lambda_J}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle f, p_{n;j,k}\rangle|^2 &\le 2^{2(J-1)|s|} \sum_{I_{j,n}\in\Lambda_J}\sum_{k\in\mathbb{Z}} |\langle f, p_{n;j,k}\rangle|^2 \\
&= 2^{2(J-1)|s|} \sum_{I_{j,n}\in\Lambda_J}\ \sum_{\ell=(r+1)^j n}^{(r+1)^j(n+1)-1}\ \sum_{k\in\mathbb{Z}} |\langle f, p_{\ell;0,k}\rangle|^2 \\
&= 2^{2(J-1)|s|} \sum_{n=0}^{(r+1)^J-1}\ \sum_{k\in\mathbb{Z}} |\langle f, p_{n;0,k}\rangle|^2 \\
&= 2^{2(J-1)|s|} \sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 \\
&\le C 2^{2(J-1)|s|}\max\{1, 2^{2Js}\}\|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})}.
\end{aligned}$$

In addition, it was shown in the proof of [45, Proposition 2.1] that (3.30) and (3.31) yield

$$\sum_{j=J}^{\infty}\sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle f,\psi_{j,k}\rangle|^2 \leq C\|B_{s,t,J}\|_{L_\infty(\mathbb{R})}\|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})},$$

where

$$B_{s,t,J}(\omega) := \sum_{j=J}^{\infty}\frac{2^{-2js}(1+|\omega|^2)^s}{(1+|2^{-J}\omega|^2)^\alpha}\sum_{i=1}^{r}|\widehat{h}_i(2^{-j}\omega)|^2 \in L_\infty(\mathbb{R}).$$

Combining the two inequalities above, we can obtain

$$\sum_{I_{j,n}\in\Lambda_J}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle f,p_{n;j,k}\rangle|^2 + \sum_{j=J}^{\infty}\sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle f,\psi_{j,k}\rangle|^2 \leq C'\|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})},$$

where $C' := C\big(\|B_{s,t,J}\|_{L_\infty(\mathbb{R})} + 2^{2(J-1)|s|}\max\{1, 2^{2Js}\}\big)$.

By a duality argument as in the proof of [45, Theorem 1.2] (Theorem 2.17), we can obtain

$$\frac{1}{C'}\|f\|^2_{\mathbb{H}^{s}(\mathbb{R})} \leq \sum_{I_{j,n}\in\Lambda_J}\sum_{k\in\mathbb{Z}} 2^{2js}|\langle f,p_{n;j,k}\rangle|^2 + \sum_{j=J}^{\infty}\sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} 2^{2js}|\langle f,\psi_{j,k}\rangle|^2 \leq C'\|f\|^2_{\mathbb{H}^{s}(\mathbb{R})},$$

for all $f\in\mathbb{H}^s(\mathbb{R})$, $(-\alpha<s<\alpha)$. Hence, Theorem 3.5 is proved.

$\square$

Chapter 4

Nonstationary Tight Wavelet Frame Packet (NTWFP)

In the stationary tight wavelet frame packet derivation, we decompose V_J by the same combined UEP mask which generates the given tight wavelet frame $X(\Psi)$, so we keep the MRA $(V_j)_{j\in\mathbb{Z}}$ associated with the given tight wavelet frame $X(\Psi)$. However, stationary MRA has its own limitations. As it is widely understood, we can not obtain a compactly supported refinable function ϕ with a finitely supported refinement mask such that $\phi \in C^\infty(\mathbb{R})$ in a stationary MRA [22]. In [15], C^∞ nonstationary orthonormal wavelet bases of $L_2(\mathbb{R})$ are obtained. It was later shown in [46] that compactly supported symmetric C^∞ nonstationary tight wavelet frames of $L_2(\mathbb{R})$ can be similarly obtained. Furthermore, it was pointed out in [46] that such nonstationary tight wavelet frames can achieve spectral frame approximation order. In a recent work [45], such nonstationary tight wavelet frames are used to characterize Sobolev spaces of arbitrary smoothness. In this section, we present the construction of nonstationary tight wavelet frame packet based on a given tight wavelet frame $X(\Psi)$ and then apply it to characterize Sobolve spaces.

4.1 Construction of NTWFP

Given a tight wavelet frame $X(\Psi)$ constructed via UEP in an MRA $(V_j)_{j\in\mathbb{Z}}$ generated by ϕ. We construct the nonstationary tight wavelet frame packets by recursively decomposing V_J with arbitrarily chosen combined UEP masks to the coarsest scale 0. Note that each lowpass filter in the selected combined UEP mask does not coincide with h_0 which is the refinement mask of ϕ.

In the first step, we decompose $V_J = \overline{\text{span}}\{\phi_{J,k} : k \in \mathbb{Z}\}$ with the combined UEP mask $\mathbf{b}_J := \big[b_{\mathbb{i}} : \mathbb{i} \in \Omega_1\big]$ satisfying the UEP condition (2.38), where $\Omega_1 \subset \mathbb{N}_0^J$ is a $J-$tuple index set defined by

$$\Omega_1 := \Big\{(i_J, i_{J-1}, \cdots, i_1) : 0 \le i_J \le \mathcal{J}, i_{J-1} = \cdots = i_1 = 0\Big\},$$

in which $\mathcal{J}$ is a positive constant. By Lemma 3.1, we can decompose V_J into spaces $W_{J-1,\mathbb{i}}$, $\mathbb{i} \in \Omega_1$, where

$$p_{\mathbb{i}} := \sum_{n\in\mathbb{Z}} 2b_{\mathbb{i}}(n)\phi(2\cdot -n),$$
$$W_{J-1,\mathbb{i}} := \overline{\text{span}}\,\big\{p_{\mathbb{i};J-1,k} : k \in \mathbb{Z}\big\}.$$

And for any $f \in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 = \sum_{\mathbb{i}\in\Omega_1}\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};J-1,k}\rangle|^2.$$

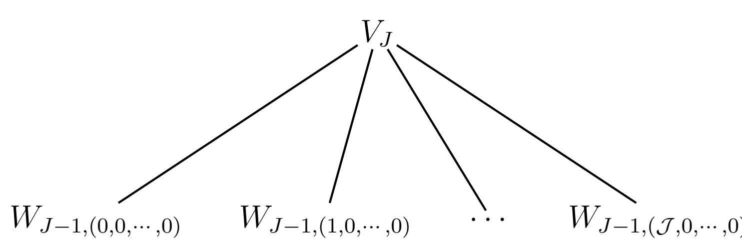


Figure 4.1: First level decomposition of V_J

At the second level of decomposition, by Lemma 3.1, each space $W_{J-1,\mathbb{i}}, \mathbb{i} \in \Omega_1$ is decomposed with a combined UEP mask $\mathbf{b}_{J-1,\mathbb{i}} := \big[b_{\mathbb{i}'} : \mathbb{i}' \in \Omega_2^{\mathbb{i}}\big]$ satisfying the

UEP condition (2.38), where $\Omega_2^{\mathbb{i}}$ is a subset of Ω_2 defined by

$$\Omega_2^{\mathbb{i}} := \{\mathbb{i}' \in \Omega_2 : \mathbb{i}'(1) = \mathbb{i}(1)\}$$

and $\Omega_2 \subset \mathbb{N}_0^J$ is a $J-$tuple index set defined by

$$\Omega_2 := \Big\{(i_J, i_{J-1}, \cdots, i_1) : 0 \le i_J \le \mathcal{J}, 0 \le i_{J-1} \le \mathcal{J}^{(i_J)}, i_{J-2} = \cdots = i_1 = 0\Big\}$$

in which $\mathcal{J}^{(i_J)}$ is a positive constant for each i_J, into spaces $W_{J-2,\mathbb{i}'}$, $\mathbb{i}' \in \Omega_2^{\mathbb{i}}$, where

$$p_{\mathbb{i}'} := \sum_{n\in\mathbb{Z}} 2b_{\mathbb{i}'}(n)p_{\mathbb{i}}(2\cdot -n),$$

$$W_{J-2,\mathbb{i}'} := \overline{\operatorname{span}}\,\big\{p_{\mathbb{i}';J-2,k} : k \in \mathbb{Z}\big\}.$$

And for any $f \in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};J-1,k}\rangle|^2 = \sum_{\mathbb{i}'\in\Omega_2^{\mathbb{i}}} \sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i}';J-2,k}\rangle|^2.$$

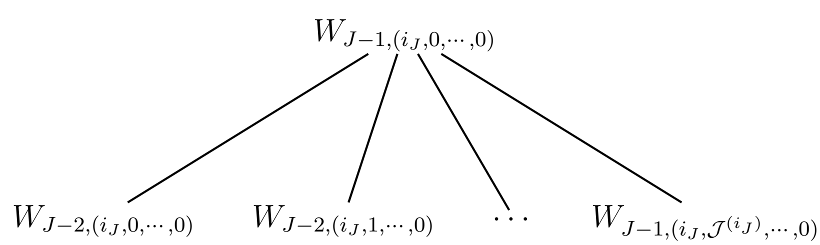


Figure 4.2: Second level decomposition of V_J

Generally, at the ℓ-th level ($2 \le \ell \le J$) of decomposition, by Lemma 3.1, each space $W_{J-\ell+1,\mathbb{i}}, \mathbb{i} \in \Omega_{\ell-1}$ is decomposed with a combined UEP mask $\mathbf{b}_{J-\ell+1,\mathbb{i}} := \big[b_{\mathbb{i}'} : \mathbb{i}' \in \Omega_\ell^{\mathbb{i}}\big]$ satisfying the UEP condition (2.38), where $\Omega_\ell^{\mathbb{i}}$ is a subset of Ω_ℓ defined by

$$\Omega_\ell^{\mathbb{i}} := \{\mathbb{i}' \in \Omega_\ell : \mathbb{i}'(n) = \mathbb{i}(n) \text{ for } 1 \le n \le \ell - 1\}$$

and $\Omega_\ell \subset \mathbb{N}_0^J$ is a $J-$tuple index set defined by

$$\begin{aligned}\Omega_\ell := \Big\{(i_J, i_{J-1}, \cdots, i_1) : 0 \le i_J \le \mathcal{J}, \quad & 0 \le i_{J-l} \le \mathcal{J}^{(i_J, i_{J-1}\cdots, i_{J-l+1})}, \\ & 1 \le l \le \ell, i_{J-\ell} = \cdots = i_1 = 0\Big\}\end{aligned}$$

in which $\mathcal{J}^{(i_J, i_{J-1}\cdots, i_{J-l+1})}$ is a positive constant for each $(i_J, i_{J-1}\cdots, i_{J-l+1})$, into spaces $W_{J-\ell,\mathbb{i}'}$, $\mathbb{i}' \in \Omega^{\mathbb{i}}_{\ell}$, where

$$p_{\mathbb{i}'} := \sum_{n\in\mathbb{Z}} 2b_{\mathbb{i}'}(n)p_{\mathbb{i}}(2\cdot -n),$$

$$W_{J-\ell,\mathbb{i}'} := \overline{\operatorname{span}}\left\{p_{\mathbb{i}';J-\ell,k} : k \in \mathbb{Z}\right\}.$$

And for any $f \in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};J-\ell+1,k}\rangle|^2 = \sum_{\mathbb{i}'\in\Omega^{\mathbb{i}}_{\ell}} \sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i}';J-\ell,k}\rangle|^2.$$

In particular, at the J-th level of decomposition, by Lemma 3.1, each space $W_{1,\mathbb{i}}, \mathbb{i} \in \Omega_{J-1}$ is decomposed with a combined UEP mask $\mathbf{b}_{1,\mathbb{i}} := \left[b_{\mathbb{i}'} : \mathbb{i}' \in \Omega^{\mathbb{i}}_J\right]$ satisfying the UEP condition (2.38), where $\Omega^{\mathbb{i}}_J$ is a subset of Ω_J defined by

$$\Omega^{\mathbb{i}}_J := \{\mathbb{i}' \in \Omega_J : \mathbb{i}'(n) = \mathbb{i}(n) \text{ for } 1 \le n \le J-1\}$$

and $\Omega_J \subset \mathbb{N}_0^J$ is a $J-$tuple index set defined by

$$\Omega_J := \left\{(i_J, i_{J-1}, \cdots, i_1) : 0 \le i_J \le \mathcal{J}, 0 \le i_{J-l} \le \mathcal{J}^{(i_J, i_{J-1}\cdots, i_{J-l+1})}, 1 \le l \le J\right\} \tag{4.1}$$

in which $\mathcal{J}^{(i_J, i_{J-1}\cdots, i_{J-l+1})}$ is a positive constant for each $(i_J, i_{J-1}\cdots, i_{J-l+1})$, into spaces $W_{0,\mathbb{i}'}$, $\mathbb{i}' \in \Omega^{\mathbb{i}}_J$, where

$$p_{\mathbb{i}'} := \sum_{n\in\mathbb{Z}} 2b_{\mathbb{i}'}(n)p_{\mathbb{i}}(2\cdot -n),$$

$$W_{0,\mathbb{i}'} := \overline{\operatorname{span}}\left\{p_{\mathbb{i}';0,k} : k \in \mathbb{Z}\right\}.$$

And for any $f \in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};1,k}\rangle|^2 = \sum_{\mathbb{i}'\in\Omega^{\mathbb{i}}_J} \sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i}';0,k}\rangle|^2.$$

By combining all the inner product equations in the construction, we can obtain

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 = \sum_{\mathbb{i}\in\Omega_J} \sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};0,k}\rangle|^2. \tag{4.2}$$

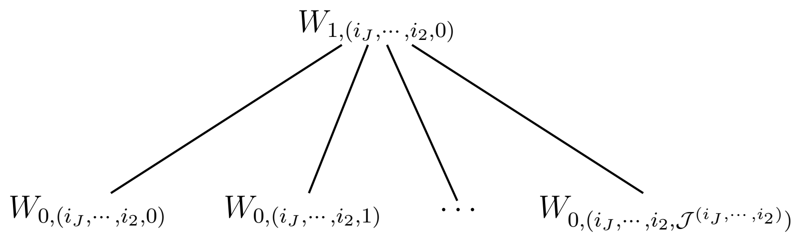


Figure 4.3: J-th level decomposition of V_J

for any $f \in L_2(\mathbb{R})$, In other words, we obtain another representation of V_J, i.e.,

$$V_J = \overline{\text{span}}\left\{p_{\mathbb{i};0,k} : k \in \mathbb{Z}, \mathbb{i} \in \Omega_J\right\}.$$

Theorem 4.1. *For a given tight wavelet frame $X(\Psi)$, the system*

$$\Pi := \left\{p_{\mathbb{i};0,k}, \mathbb{i} \in \Omega_J\right\} \cup \left\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\right\},$$

is also a tight wavelet frame, where Ω_J is a index set defined in (4.1).

Proof. Since $X(\Psi)$ is a tight wavelet frame of $L_2(\mathbb{R})$, by [24, Lemma 2.4], for any $f \in L_2(\mathbb{R})$ we have

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 + \sum_{\psi\in\Psi} \sum_{j\geq J, k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2.$$

On the other hand, from (4.2) we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 = \sum_{\mathbb{i}\in\Omega_J} \sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};0,k}\rangle|^2.$$

It follows that

$$\|f\|^2 = \sum_{\mathbb{i}\in\Omega_J} \sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};0,k}\rangle|^2 + \sum_{\psi\in\Psi} \sum_{j\geq J} \sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2.$$

Hence, Theorem 4.1 is proved. $\square$

As in the stationary case, based on the tight frame

$$\Pi = \left\{p_{\mathbb{i};0,k}, \mathbb{i} \in \Omega_J\right\} \cup \left\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\right\}$$

constructed above, we can obtain a library of tight wavelet frames of $L_2(\mathbb{R})$ by partitioning Ω_J into disjoint subsets of the form

$$I_{j,\mathbb{i}} := \Big\{(i_J, \cdots, i_{j+1}, i'_j, \cdots, i'_1) \in \Omega_J : \mathbb{i} = (i_J, \cdots, i_{j+1}, 0, \cdots, 0) \in \Omega_{J-j}\Big\}, \quad (4.3)$$

i.e.,

$$\Lambda_J = \Big\{I_{j,\mathbb{i}} : \bigcup I_{j,\mathbb{i}} = \Omega_J\Big\}. \quad (4.4)$$

Theorem 4.2. *For a given tight wavelet frame $X(\Psi)$, let Λ_J be a disjoint partition of Ω_J, where Ω_J and Λ_J are defined in (4.1) and (4.4), respectively. Then the system*

$$\Pi_{\Lambda_J} := \big\{p_{\mathbb{i};j,k} : k \in \mathbb{Z}, I_{j,\mathbb{i}} \in \Lambda_J\big\} \cup \big\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\big\},$$

is also a tight wavelet frame.

Proof. Since Λ_J is a disjoint partition of Ω_J, for any $f \in L_2(\mathbb{R})$, we have

$$\begin{aligned}\sum_{I_{j,\mathbb{i}} \in \Lambda_J} \sum_{k \in \mathbb{Z}} |\langle f, p_{\mathbb{i};j,k}\rangle|^2 &= \sum_{I_{j,\mathbb{i}} \in \Lambda_J} \sum_{\mathbb{i}' \in I_{j,\mathbb{i}}} \sum_{k \in \mathbb{Z}} |\langle f, p_{\mathbb{i}';0,k}\rangle|^2 \\ &= \sum_{\mathbb{i} \in \Omega_J} \sum_{k \in \mathbb{Z}} |\langle f, p_{\mathbb{i};0,k}\rangle|^2.\end{aligned}$$

By applying Theorem 4.1, Theorem 4.2 is proved. □

By Theorem 4.2, we can obtain various nonstationary tight wavelet frames Π_{Λ_J} based on various disjoint partitions of Ω_J. All such obtained nonstationary tight wavelet frames Π_{Λ_J} are called a *nonstationary tight wavelet frame packet.*

Example 4.1. *Let $X(\Psi)$ be the piecewise linear tight wavelet frame as introduced in Example 2.5, and let $J = 2$. By Theorem 4.1, we can obtain a stationary tight wavelet frame packet, the plots of $p_0, p_1, \cdots, p_8$ are depicted in Figure 3.1.*

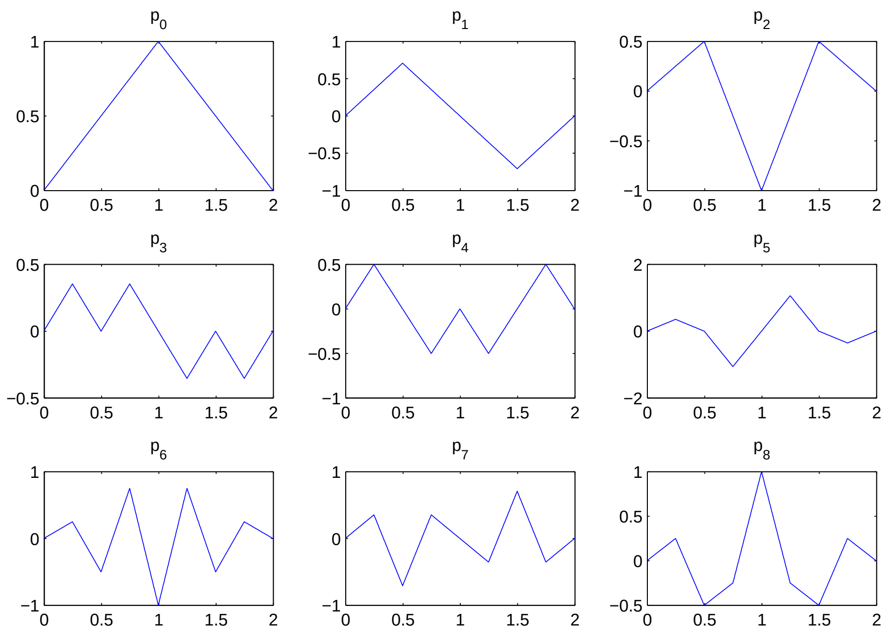


Figure 4.4: Piecewise linear tight wavelet frame packet

4.2 Characterization of Sobolev Spaces by NTWFP

Once we build up a nonstationary tight wavelet frame packet (NTWFP)

$$\Pi_{\Lambda_J} = \{p_{\mathbb{i};j,k} : k \in \mathbb{Z}, I_{j,\mathbb{i}} \in \Lambda_J\} \cup \{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\}$$

we can use the weighted ℓ_2-norm of the analysis NTWFP coefficient sequence

$$\{\langle f, p_{\mathbb{i};j,k}\rangle\}_{k\in\mathbb{Z}, I_{j,\mathbb{i}}\in\Lambda_J} \cup \{\langle f, \psi_{j,k}\rangle\}_{k\in\mathbb{Z}, j\geq J, k\in\mathbb{Z}, \psi\in\Psi}$$

of a given function $f \in \mathbb{H}^s(\mathbb{R})$ to characterize its Sobolev norm in $\mathbb{H}^s(\mathbb{R})$.

Theorem 4.3. *Suppose we have a nonstationary tight wavelet frame packet*

$$\Pi_{\Lambda_J} = \{p_{\mathbb{i};j,k} : k \in \mathbb{Z}, I_{j,\mathbb{i}} \in \Lambda_J\} \cup \{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\},$$

derived from a tight wavelet frame $X(\Psi)$ constructed in an MRA generated by a refinable function ϕ via UEP, with the associated combined UEP mask is $\mathbf{h} =$

$[h_0, h_1, \cdots, h_r]$, where $I_{j,\mathbb{i}}$ and Λ_J are defined in (4.3) and (4.4), respectively. Suppose

$$1 - |\widehat{h_0}(\omega)|^2 \le C|\omega|^{2\alpha}, \quad \omega \in \mathbb{R}, \tag{4.5}$$

$$[\widehat{\phi}, \widehat{\phi}]_\alpha(\omega) \le C, \quad \omega \in \mathbb{R}, \tag{4.6}$$

$$[\widehat{p_{\mathbb{i}}}, \widehat{p_{\mathbb{i}}}]_\alpha(\omega) \le C, \quad \omega \in \mathbb{R}, I_{j,\mathbb{i}} \in \Lambda_J, \tag{4.7}$$

If $-\alpha < s < \alpha$, then

$$\Pi_\Lambda^s := \left\{2^{js} p_{\mathbb{i};j,k} : k \in \mathbb{Z}, I_{j,\mathbb{i}} \in \Lambda_J\right\} \cup \left\{2^{js}\psi_{j,k} : j \ge J, k \in \mathbb{Z}, \psi \in \Psi\right\}$$

is a wavelet frame of $\mathbb{H}^s(\mathbb{R})$, i.e., there exist two positive constants C_1, C_2 such that

$$\begin{aligned} C_1\|f\|^2_{\mathbb{H}^s(\mathbb{R})} &\le \sum_{I_{j,\mathbb{i}} \in \Lambda_J} \sum_{k \in \mathbb{Z}} 2^{2js} |\langle f, p_{\mathbb{i};j,k}\rangle|^2 \\ &+ \sum_{\psi \in \Psi} \sum_{j \ge J} \sum_{k \in \mathbb{Z}} 2^{2js} |\langle f, \psi_{j,k}\rangle|^2 \le C_2 \|f\|^2_{\mathbb{H}^s(\mathbb{R})}, \end{aligned}$$

for all $f \in \mathbb{H}^s(\mathbb{R})$.

Proof. For $-\alpha < s < \alpha$, we can obtain

$$\sum_{k \in \mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 \le C \max\{1, 2^{2Js}\} \|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})},$$

as in the proof of Theorem 3.5.

On the other hand,

$$\begin{aligned} \sum_{I_{j,\mathbb{i}} \in \Lambda_J} \sum_{k \in \mathbb{Z}} 2^{-2js} |\langle f, p_{\mathbb{i};j,k}\rangle|^2 &\le 2^{2(J-1)|s|} \sum_{I_{j,\mathbb{i}} \in \Lambda_J} \sum_{k \in \mathbb{Z}} |\langle f, p_{\mathbb{i};j,k}\rangle|^2 \\ &= 2^{2(J-1)|s|} \sum_{k \in \mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 \\ &\le C 2^{2(J-1)|s|} \max\{1, 2^{2Js}\} \|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})}. \end{aligned}$$

In addition, it was shown in the proof of [45, Proposition 2.1] that (4.5) and (4.6) yield

$$\sum_{j=J}^{\infty} \sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}} 2^{-2js} |\langle f, \psi_{j,k}\rangle|^2 \le C\|B_{s,t,J}\|_{L_\infty(\mathbb{R})} \|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})},$$

where

$$B_{s,t,J}(\omega) := \sum_{j=J}^{\infty} \frac{2^{-2js}(1+|\omega|^2)^s}{(1+|2^{-J}\omega|^2)^{\alpha}} \sum_{i=1}^{r} |\widehat{h}_i(2^{-j}\omega)|^2 \in L_\infty(\mathbb{R}).$$

Combining the two inequalities above, we can obtain

$$\sum_{I_{j,\mathbb{i}}\in\Lambda_J} \sum_{k\in\mathbb{Z}} 2^{-2js} |\langle f, p_{\mathbb{i};j,k}\rangle|^2 + \sum_{j=J}^{\infty} \sum_{\psi\in\Psi} \sum_{k\in\mathbb{Z}} 2^{-2js} |\langle f, \psi_{j,k}\rangle|^2 \le C' \|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})},$$

where $C' := C\big(\|B_{s,t,J}\|_{L_\infty(\mathbb{R})} + 2^{2(J-1)|s|}\max\{1, 2^{2Js}\}\big)$.

By a duality argument as in the proof of [45, Theorem 1.2] (Theorem 2.17), we can obtain

$$\frac{1}{C'}\|f\|^2_{\mathbb{H}^s(\mathbb{R})} \le \sum_{I_{j,\mathbb{i}}\in\Lambda_J} \sum_{k\in\mathbb{Z}} 2^{2js} |\langle f, p_{\mathbb{i};j,k}\rangle|^2 + \sum_{j=J}^{\infty} \sum_{\psi\in\Psi} \sum_{k\in\mathbb{Z}} 2^{2js} |\langle f, \psi_{j,k}\rangle|^2 \le C' \|f\|^2_{\mathbb{H}^s(\mathbb{R})},$$

for all $f \in \mathbb{H}^s(\mathbb{R})$, $(-\alpha < s < \alpha)$. Hence, Theorem 4.3 is proved. $\square$

Chapter 5

2^{-J}-shift Invariant (SI) Tight Wavelet Frame Packet

As pointed out in [66], the wavelets theory is intrinsically centered around the "synthesis operator", while the frame theory is centered around the "analysis operator". Theoretically, this is because the former one is irredundant while the latter one is redundant. In applications, there is also a trend of shifting concentration on these two operators.

In the early applications of wavelet frames, such as denoising [33, 35], image compression, etc., the irredundant systems play a very crucial role. This is largely due to that the application objects such as images have a sparse representation in the wavelets domain in which a thresholding operation can be performed. In other words, we are using a "good" analysis operator in the applications. The synthesis problem is not focused since the system used is irredundant.

However, in the application of denoising by wavelets, Donoho discovered the better performance offer by frames obtained by making the wavelets system to be a 2^{-J}-shift invariant one, and he referred this new technique as "Translation-invariant De-noising" [34]. It suggested that not only "sparsity" but also "redundancy" is crucial in certain applications. It also called for new techniques dealing with "redundancy". Put it differently, we have to pay attention to the synthesis operator since the good performance is due to a redundant system. From then on, there was a widespread exploration on the power of "redundancy" which led to the theory of "compressed sensing" which is the current state-of-the-art in the exploration of the power of redundancy.

5.1 Introduction to Quasi-affine systems

We first introduce the notion of a **quasi-affine system from level** J. A quasi-affine system from level J is defined as

Definition 5.1. *Let $\Psi = \{\psi_1, \ldots, \psi_r\}$ be a finite set of functions in $L_2(\mathbb{R})$. A quasi-affine system from level J is defined as*

$$X^{q,J}(\Psi) := \big\{\psi^{q,J}_{i;j,k} :\ 1 \le i \le r; j,k \in \mathbb{Z}\big\},$$

where $\psi^{q,J}_{i;j,k}$ is defined by

$$\psi^{q,J}_{i;j,k} := \begin{cases} \mathbf{D}^j \mathbf{E}^k \psi_i = 2^{j/2}\psi_i(2^j \cdot - k), & j \ge J; \\ 2^{\frac{j-J}{2}} \mathbf{E}^{2^{-J}k} \mathbf{D}^j \psi_i = 2^{j-J/2}\psi_i(2^j(\cdot - 2^{-J}k)), & j < J. \end{cases} \tag{5.1}$$

The quasi-affine system is obtained by oversampling the affine system. More precisely, we oversample the affine system starting from level $J-1$ and downward to a 2^{-J}-shift invariant system. Hence, the whole quasi-affine system is a 2^{-J}-shift invariant system. The quasi-affine system $X^q_0(\Psi)$ from level 0 was first introduced in [66] to convert a non-shift invariant affine system to a shift invariant system. Further, it was shown in [66, Theorem 5.5] that the affine system $X(\Psi)$ is a tight wavelet frame (or tight affine frame) if and only if $X^{q,0}(\Psi)$ is a tight frame (or tight quasi-affine frame). Note that [66, Theorem 5.5] was proved under a mild decay condition which was subsequently removed by Chui et al [13]). It can be easily observed that

$$X^{q,J}(\Psi) = \mathbf{D}^J X^{q,0}(\Psi),$$

then by [66, Theorem 5.5] we can obtain the following result.

Theorem 5.1. [5]$X(\Psi)$ *is a tight wavelet frame (or tight affine frame) if and only if* $X^{q,J}(\Psi)$ *is a tight frame (or tight* 2^{-J}*-quasi-affine frame).*

5.2 Construction of 2^{-J}-SI STWFP

For a given sequence h, we use $h^{[j]}$ to denote the 2^j upsampling of h, i.e.,

$$h^{[j]}(n) = \begin{cases} h(2^{-j}n), & n \in 2^j\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

With the following result, we can split a given stationary (nonstationary) tight wavelet frame into 2^{-J}-shift invariant stationary (nonstationary) tight wavelet frame packets.

Theorem 5.2. *Let* $\theta \in L_2(\mathbb{R}), j \le J$ *and* $\left\{\theta^{q,J}_{j,k} : k \in \mathbb{Z}\right\}$ *be a Bessel set with the Bessel bound* C^q_θ *which is a positive constant, i.e.,*

$$\sum_{k\in\mathbb{Z}} |\langle f, \theta^{q,J}_{j,k}\rangle|^2 \le C^q_\theta \|f\|^2, \tag{5.3}$$

holds for any $f \in L_2(\mathbb{R})$*. Let* $\mathbf{h} = [h_0, \cdots, h_r]$ *be a combined MRA mask satisfying*

$$\sum_{i=0}^{r} |\widehat{h_i}(\omega)|^2 = 1. \tag{5.4}$$

For $i = 0, \cdots, r$*, define*

$$\theta^{q,J}_{i;j-1,k} := \sum_{n\in\mathbb{Z}} h_i(n)\theta^{q,J}_{j,k}(\cdot - 2^{-j}n), \quad k \in \mathbb{Z}, \tag{5.5}$$

$$U_i^{q,J} := \overline{\operatorname{span}}\left\{\theta_{i;j-1,k}^{q,J} : k \in \mathbb{Z}\right\}. \tag{5.6}$$

And also, define $U^{q,J} := \overline{\operatorname{span}}\left\{\theta_{j,k}^{q,J} : k \in \mathbb{Z}\right\}$. *Then*

(i). *For* $k \in \mathbb{Z}$ *and* $i = 0, \cdots, r$, $\theta_{i;j-1,k}^{q,J} \in L_2(\mathbb{R})$, *with* $\|\theta_{i;j-1,k}^{q,J}\|_{L_2(\mathbb{R})} \leq \|\theta_{j,k}^{q,J}\|_{L_2(\mathbb{R})}$ *and*

$$\sum_{i=0}^{r} \|\theta_{i;j-1,k}^{q,J}\|_{L_2(\mathbb{R})}^2 = \|\theta_{j,k}^{q,J}\|_{L_2(\mathbb{R})}^2;$$

(ii). *For any sequence* $c \in \ell_2(\mathbb{Z})$, *there are* $r+1$ *sequences* $c_i, i = 0, \cdots, r$, *defined by*

$$c_i(k) := \sum_{n \in \mathbb{Z}} \overline{h_i^{[J-j]}}(n)\, c(k+n), \quad k \in \mathbb{Z}, \tag{5.7}$$

such that

$$\sum_{k \in \mathbb{Z}} |c(k)|^2 = \sum_{i=0}^{r} \sum_{k \in \mathbb{Z}} |c_i(k)|^2, \tag{5.8}$$

and

$$\sum_{k \in \mathbb{Z}} c(k)\, \theta_{j,k}^{q,J} = \sum_{i=0}^{r} \sum_{k \in \mathbb{Z}} c_i(k)\, \theta_{i;j-1,k}^{q,J}; \tag{5.9}$$

(iii). *In particular, for any* $f \in L_2(\mathbb{R})$, *let* $c(k) = \langle f, \theta_{j,k}^{q,J} \rangle$ *for* $k \in \mathbb{Z}$, *then* $c \in \ell_2(\mathbb{Z})$ *and (5.7),(5.8) and (5.9) yield*

$$c_i(k) = \langle f, \theta_{i;j-1,k}^{q,J} \rangle, k \in \mathbb{Z}, i = 0, \cdots, r, \tag{5.10}$$

$$\sum_{k \in \mathbb{Z}} |\langle f, \theta_{j,k}^{q,J} \rangle|^2 = \sum_{i=0}^{r} \sum_{k \in \mathbb{Z}} |\langle f, \theta_{i;j-1,k}^{q,J} \rangle|^2, \tag{5.11}$$

and

$$\sum_{k \in \mathbb{Z}} \langle f, \theta_{j,k}^{q,J} \rangle\, \theta_{j,k}^{q,J} = \sum_{i=0}^{r} \sum_{k \in \mathbb{Z}} \langle f, \theta_{i;j-1,k}^{q,J} \rangle\, \theta_{i;j-1,k}^{q,J}, \tag{5.12}$$

respectively;

(iv). $\left\{\theta_{i;j-1,k}^{q,J} : k \in \mathbb{Z}\right\}$ *is a Bessel set for* $i = 0, \cdots, r$, *with the space decomposition*

$$U^{q,J} = U_0^{q,J} + \cdots + U_r^{q,J}. \tag{5.13}$$

Proof. **(i).** (5.5) can be recast in Fourier domain as

$$\widehat{\theta^{q,J}_{i;j-1,k}}(\omega) = \widehat{h}_i(2^{-j}\omega)\widehat{\theta^{q,J}_{j,k}}(\omega), \quad i = 0, \cdots, r. \tag{5.14}$$

Also, by invoking the condition (5.4), we can deduce that

$$\begin{aligned}\sum_{i=0}^{r} \|\theta^{q,J}_{i;j-1,k}\|^2_{L_2(\mathbb{R})} &= \sum_{i=0}^{r} \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{h}_i(2^{-j}\omega)\widehat{\theta^{q,J}_{j,k}}(\omega)|^2 d\omega \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{\theta^{q,J}_{j,k}}(\omega)|^2 \sum_{i=0}^{r} |\widehat{h}_i(2^{-j}\omega)|^2 d\omega \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{\theta^{q,J}_{j,k}}(\omega)|^2 d\omega \\ &= \|\theta^{q,J}_{j,k}\|^2_{L_2(\mathbb{R})}.\end{aligned}$$

It follows that $\|\theta^{q,J}_{i;j-1,k}\|^2_{L_2(\mathbb{R})} \le \|\theta^{q,J}_{j,k}\|^2_{L_2(\mathbb{R})}$. And **(i)** is thus proved.

(ii). (5.7) can be recast in Fourier domain as

$$\widehat{c_i}(\omega) = \widehat{c}(\omega)\overline{\widehat{h}_i}(2^{J-j}\omega), \quad i = 0, \cdots, r. \tag{5.15}$$

By (5.15), we have

$$\sum_{i=0}^{r} |\widehat{c_i}(\omega)|^2 = \sum_{i=0}^{r} |\widehat{c}(\omega)|^2 |\widehat{h}_i(2^{J-j}\omega)|^2 = |\widehat{c}(\omega)|^2 \sum_{i=0}^{r} |\widehat{h}_i(2^{J-j}\omega)|^2 = |\widehat{c}(\omega)|^2.$$

It follows that

$$\sum_{i=0}^{r}\sum_{k\in\mathbb{Z}} |c_i(k)|^2 = \sum_{k} |c(k)|^2.$$

and (5.8) is proved.

In Fourier domain, (5.9) becomes

$$\widehat{c}(2^{-J}\omega)\widehat{\theta^{q,J}_{j,0}}(\omega) = \sum_{i=0}^{r} \widehat{c_i}(2^{-J}\omega)\widehat{\theta^{q,J}_{i;j-1,0}}(\omega).$$

By (5.14) and (5.15), it can be shown that the RHS=LHS, explicitly,

$$\begin{aligned} RHS &= \sum_{i=0}^{r} \widehat{c}(2^{-J}\omega)\overline{\widehat{h_i}}(2^{-j}\omega)\widehat{h_i}(2^{-j}\omega)\widehat{\theta_{j,0}^{q,J}}(\omega) \\ &= \widehat{c}(2^{-J}\omega)\widehat{\theta_{j,0}^{q,J}}(\omega)\sum_{i=0}^{r}|\widehat{h_i}(2^{-j}\omega)|^2 \\ &= LHS, \end{aligned}$$

thus (5.9) is proved.

(iii). By (5.3), we immediately have $c \in \ell_2(\mathbb{Z})$. Also, we can deduce from (5.7) to obtain

$$\begin{aligned} c_i(k) &= \sum_n \overline{h_i^{[J-j]}}(n)\, c(k+n) \\ &= \sum_n \overline{h_i^{[J-j]}}(n)\langle f, \theta_{j,k+n}^{q,J}\rangle \\ &= \sum_n \overline{h_i}(n)\langle f, \theta_{j,k+2^{J-j}n}^{q,J}\rangle \\ &= \Big\langle f, \sum_n h_i(n)\theta_{j,k+2^{J-j}n}^{q,J}\Big\rangle \\ &= \Big\langle f, \sum_n h_i(n)\theta_{j,k}^{q,J}(\cdot - 2^{-j}n)\Big\rangle \\ &= \langle f, \theta_{i;j-1,k}^{q,J}\rangle, \quad i = 0, \cdots, r. \end{aligned}$$

Consequently, (5.11) and (5.12) followed from (5.8) and (5.9), respectively.

(iv). By (5.11) and (5.3), for any $f \in L_2(\mathbb{R})$, we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} |\langle f, \theta_{i;j-1,k}^{q,J}\rangle|^2 &\le \sum_{i=0}^{r}\sum_{k\in\mathbb{Z}} |\langle f, \theta_{i;j-1,k}^{q,J}\rangle|^2 \\ &= \sum_{k\in\mathbb{Z}} |\langle f, \theta_{j,k}^{q,J}\rangle|^2 \\ &\le C_\theta^q \|f\|^2, \quad i = 0, \cdots, r. \end{aligned}$$

And we obtain that for $i = 0, \cdots, r$, each set $\left\{\theta_{i;j-1,k}^{q,J} : k \in \mathbb{Z}\right\}$ is a Bessel set.

From (5.5), we can get

$$\theta^{q,J}_{i;j-1,k} = \sum_{n\in\mathbb{Z}} h_i(n)\theta^{q,J}_{j,k+2^{J-j}n}\,.$$

Consequently, $U^{q,J}_0+\cdots+U^{q,J}_r \subseteq U^{q,J}$. On the other hand, by taking c to be finitely support sequences in (5.9), we can obtain $U^{q,J} \subseteq U^{q,J}_0 + \cdots + U^{q,J}_r$. Hence (5.13) is proved. $\square$

As it is shown in the previous chapter, Lemma 3.1 is the base to construct stationary/nonstationary tight wavelet frame packets. In this chapter we will show that Lemma 5.2 is the base for the construction of 2^{-J}-shift invariant stationary/nonstationary tight wavelet frame packet.

Similar to the construction of the nonstationary tight wavelet frame packet, from a given tight wavelet frame $X(\Psi)$ constructed via UEP in an MRA generated by the refinable function ϕ, we construct 2^{-J}-shift invariant nonstationary tight wavelet frame packet by recursively decomposing the MRA space V_J for a fixed scale J to level 0 with any combined MRA mask $\mathbf{h} = [h_0, \cdots, h_r]$ satisfying the condition (5.4) which is much weaker than the UEP requirement.

In the first step, we decompose $V_J = \overline{\text{span}}\{\phi_{J,k} : k \in \mathbb{Z}\}$ with the combined MRA mask $\mathbf{b}_J := \left[b_{\mathbb{i}} : \mathbb{i} \in \Omega_1\right]$ satisfying the condition (5.4), where $\Omega_1 \subset \mathbb{N}^J_0$ is a $J-$tuple index set defined by

$$\Omega_1 := \Big\{(i_J, i_{J-1}, \cdots, i_1) : 0 \le i_J \le \mathcal{J}, i_{J-1} = \cdots = i_1 = 0\Big\},$$

in which $\mathcal{J}$ is a positive constant. By Lemma 5.2, we can decompose V_J into spaces $W^{q,J}_{J-1,\mathbb{i}}$, $\mathbb{i} \in \Omega_1$, where

$$p^{q,J}_{\mathbb{i};J-1,k} := \sum_{n\in\mathbb{Z}} b_{\mathbb{i}}(n)\phi_{J,k}(\cdot - 2^{-J}n), \quad k \in \mathbb{Z},$$

$$W^{q,J}_{J-1,\mathbb{i}} := \overline{\text{span}}\left\{p^{q,J}_{\mathbb{i};J-1,k} : k \in \mathbb{Z}\right\}.$$

And for any $f \in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 = \sum_{\mathbb{i}\in\Omega_1}\sum_{k\in\mathbb{Z}} |\langle f, p^{q,J}_{\mathbb{i};J-1,k}\rangle|^2.$$

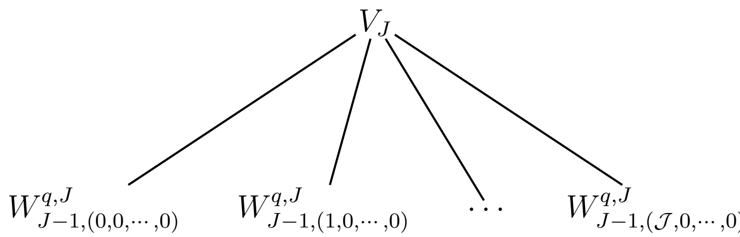


Figure 5.1: First level 2^{-J}-shift invariant decomposition of V_J

At the second level of decomposition, by Lemma 5.2, each space $W^{q,J}_{J-1,\mathbb{i}}$, $\mathbb{i} \in \Omega_1$ is decomposed with a combined MRA mask $\mathbf{b}_{J-1,\mathbb{i}} := \left[b_{\mathbb{i}'} : \mathbb{i}' \in \Omega_2^{\mathbb{i}}\right]$ satisfying the condition (5.4), where $\Omega_2^{\mathbb{i}}$ is a subset of Ω_2 defined by

$$\Omega_2^{\mathbb{i}} := \{\mathbb{i}' \in \Omega_2 : \mathbb{i}'(1) = \mathbb{i}(1)\}$$

and $\Omega_2 \subset \mathbb{N}_0^J$ is a $J-$tuple index set defined by

$$\Omega_2 := \left\{(i_J, i_{J-1}, \cdots, i_1) : 0 \le i_J \le \mathcal{J}, 0 \le i_{J-1} \le \mathcal{J}^{(i_J)}, i_{J-2} = \cdots = i_1 = 0\right\}$$

in which $\mathcal{J}^{(i_J)}$ is a positive constant for each i_J, into spaces $W^{q,J}_{J-2,\mathbb{i}'}$, $\mathbb{i}' \in \Omega_2^{\mathbb{i}}$, where

$$p^{q,J}_{\mathbb{i}';J-2,k} := \sum_{n\in\mathbb{Z}} b_{\mathbb{i}'}(n) p^{q,J}_{\mathbb{i};J-1,k}(\cdot - 2^{-J+1}n), \quad k \in \mathbb{Z},$$

$$W^{q,J}_{J-2,\mathbb{i}'} := \overline{\mathrm{span}}\left\{p^{q,J}_{\mathbb{i}';J-2,k} : k \in \mathbb{Z}\right\}.$$

And for any $f \in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, p^{q,J}_{\mathbb{i};J-1,k}\rangle|^2 = \sum_{\mathbb{i}'\in\Omega_2^{\mathbb{i}}}\sum_{k\in\mathbb{Z}} |\langle f, p^{q,J}_{\mathbb{i}';J-2,k}\rangle|^2.$$

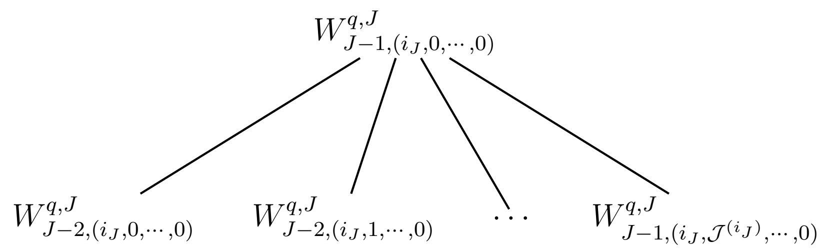


Figure 5.2: Second level 2^{-J}-shift invariant decomposition of V_J

Generally, at the ℓ-th level ($2 \le \ell \le J$) of decomposition, by Lemma 5.2, each space $W^{q,J}_{J-\ell+1,\mathbb{i}}, \mathbb{i} \in \Omega_{\ell-1}$ is decomposed with a combined MRA mask $\mathbf{b}_{J-\ell+1,\mathbb{i}} := \left[b_{\mathbb{i}'} : \mathbb{i}' \in \Omega^{\mathbb{i}}_\ell\right]$ satisfying the condition (5.4), where $\Omega^{\mathbb{i}}_\ell$ is a subset of Ω_ℓ defined by

$$\Omega^{\mathbb{i}}_\ell := \{\mathbb{i}' \in \Omega_\ell : \mathbb{i}'(n) = \mathbb{i}(n) \text{ for } 1 \le n \le \ell - 1\}$$

and $\Omega_\ell \subset \mathbb{N}_0^J$ is a $J-$tuple index set defined by

$$\Omega_\ell := \Big\{(i_J, i_{J-1}, \cdots, i_1) : 0 \le i_J \le \mathcal{J}, \quad 0 \le i_{J-l} \le \mathcal{J}^{(i_J, i_{J-1}\cdots, i_{J-l+1})}, \\ 1 \le l \le \ell, i_{J-\ell} = \cdots = i_1 = 0\Big\}$$

in which $\mathcal{J}^{(i_J, i_{J-1}\cdots, i_{J-l+1})}$ is a positive constant for each $(i_J, i_{J-1}\cdots, i_{J-l+1})$, into spaces $W^{q,J}_{J-\ell,\mathbb{i}'}$, $\mathbb{i}' \in \Omega^{\mathbb{i}}_\ell$, where

$$p^{q,J}_{\mathbb{i}';J-\ell,k} := \sum_{n\in\mathbb{Z}} b_{\mathbb{i}'}(n) p^{q,J}_{\mathbb{i};J-\ell+1,k}(\cdot - 2^{-J+\ell-1}n), \quad k \in \mathbb{Z},$$

$$W^{q,J}_{J-\ell,\mathbb{i}'} := \overline{\text{span}}\left\{p^{q,J}_{\mathbb{i}';J-\ell,k} : k \in \mathbb{Z}\right\}.$$

And for any $f \in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, p^{q,J}_{\mathbb{i};J-\ell+1,k}\rangle|^2 = \sum_{\mathbb{i}'\in\Omega^{\mathbb{i}}_\ell} \sum_{k\in\mathbb{Z}} |\langle f, p^{q,J}_{\mathbb{i}';J-\ell,k}\rangle|^2.$$

In particular, at the J-th level of decomposition, by Lemma 5.2, each space $W^{q,J}_{1,\mathbb{i}}, \mathbb{i} \in \Omega_{J-1}$ is decomposed with a combined MRA mask $\mathbf{b}_{1,\mathbb{i}} := \left[b_{\mathbb{i}'} : \mathbb{i}' \in \Omega^{\mathbb{i}}_J\right]$ satisfying the condition (5.4), where $\Omega^{\mathbb{i}}_J$ is a subset of Ω_J defined by

$$\Omega^{\mathbb{i}}_J := \{\mathbb{i}' \in \Omega_J : \mathbb{i}'(n) = \mathbb{i}(n) \text{ for } 1 \le n \le J - 1\}$$

and $\Omega_J \subset \mathbb{N}_0^J$ is a $J-$tuple index set defined by

$$\Omega_J := \left\{(i_J, i_{J-1}, \cdots, i_1) : 0 \leq i_J \leq \mathcal{J}, 0 \leq i_{J-l} \leq \mathcal{J}^{(i_J, i_{J-1} \cdots, i_{J-l+1})}, 1 \leq l \leq J\right\} \tag{5.16}$$

in which $\mathcal{J}^{(i_J, i_{J-1} \cdots, i_{J-l+1})}$ is a positive constant for each $(i_J, i_{J-1} \cdots, i_{J-l+1})$, into spaces $W_{0,\mathbb{i}'}^{q,J}$, $\mathbb{i}' \in \Omega_J^{\mathbb{i}}$, where

$$p_{\mathbb{i}';0,k}^{q,J} := \sum_{n\in\mathbb{Z}} b_{\mathbb{i}'}(n) p_{\mathbb{i};1,k}^{q,J}(\cdot - 2^{-1}n), \quad k \in \mathbb{Z},$$

$$W_{0,\mathbb{i}'}^{q,J} := \overline{\operatorname{span}} \left\{p_{\mathbb{i}';0,k}^{q,J} : k \in \mathbb{Z}\right\}.$$

And for any $f \in L_2(\mathbb{R})$ we have

$$\sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};1,k}^{q,J}\rangle|^2 = \sum_{\mathbb{i}'\in\Omega_J^{\mathbb{i}}} \sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i}';0,k}^{q,J}\rangle|^2.$$

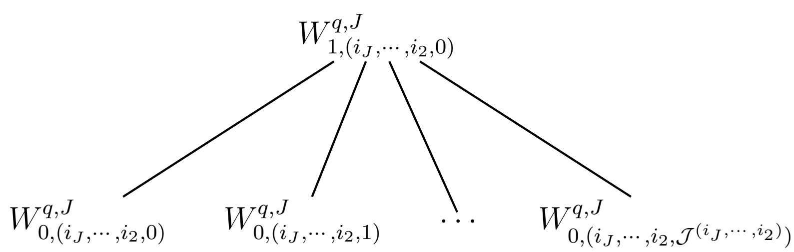


Figure 5.3: J-th level 2^{-J}-shift invariant decomposition of V_J

By combining all the inner product equations in the construction, we can obtain

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 = \sum_{\mathbb{i}\in\Omega_J} \sum_{k\in\mathbb{Z}} |\langle f, p_{\mathbb{i};0,k}^{q,J}\rangle|^2. \tag{5.17}$$

for any $f \in L_2(\mathbb{R})$, In other words, we obtain another representation of V_J, i.e.,

$$V_J = \overline{\operatorname{span}} \left\{p_{\mathbb{i};0,k}^{q,J} : k \in \mathbb{Z}, \mathbb{i} \in \Omega_J\right\}.$$

Theorem 5.3. *For a given tight wavelet frame $X(\Psi)$, the system*

$$\Pi^{q,J} := \left\{p_{\mathbb{i};0,k}^{q,J} : k \in \mathbb{Z}, \mathbb{i} \in \Omega_J\right\} \cup \left\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\right\}$$

is a 2^{-J}-shift invariant tight wavelet frame.

Proof. Since $X(\Psi)$ is a tight wavelet frame of $L_2(\mathbb{R})$, by [24, Lemma 2.4], for any $f \in L_2(\mathbb{R})$ we have

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 + \sum_{\psi\in\Psi}\sum_{j\ge J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2.$$

On the other hand, from (5.17) we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 = \sum_{\mathbb{i}\in\Omega_J}\sum_{k\in\mathbb{Z}} |\langle f, p^{q,J}_{\mathbb{i};0,k}\rangle|^2.$$

It follows that

$$\|f\|^2 = \sum_{\mathbb{i}\in\Omega_J}\sum_{k\in\mathbb{Z}} |\langle f, p^{q,J}_{\mathbb{i};0,k}\rangle|^2 + \sum_{\psi\in\Psi}\sum_{j\ge J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2.$$

Hence, Theorem 5.3 is proved. □

As in the stationary case, based on the 2^{-J}-shift invariant tight wavelet frame

$$\Pi^{q,J} = \left\{p^{q,J}_{\mathbb{i};0,k} : k \in \mathbb{Z}, \mathbb{i} \in \Omega_J\right\} \cup \left\{\psi_{j,k} : j \ge J, k \in \mathbb{Z}, \psi \in \Psi\right\}$$

constructed above, we can obtain a library of 2^{-J}-shift invariant tight wavelet frames by partitioning Ω_J into disjoint subsets of the form

$$I_{j,\mathbb{i}} := \left\{(i_J, \cdots, i_{j+1}, i'_j, \cdots, i'_1) \in \Omega_J : \mathbb{i} = (i_J, \cdots, i_{j+1}, 0, \cdots, 0) \in \Omega_{J-j}\right\}, \tag{5.18}$$

i.e.,

$$\Lambda_J = \left\{I_{j,\mathbb{i}} : \bigcup I_{j,\mathbb{i}} = \Omega_J\right\}. \tag{5.19}$$

Theorem 5.4. *Let Λ_J be a disjoint partition of Ω_J, where Ω_J and Λ_J are defined in (5.16) and (5.19), respectively. Then the system*

$$\Pi^{q,J}_{\Lambda_J} := \left\{p^{q,J}_{\mathbb{i};j,k} : k \in \mathbb{Z}, I_{j,\mathbb{i}} \in \Lambda_J\right\} \cup \left\{\psi_{j,k} : j \ge J, k \in \mathbb{Z}, \psi \in \Psi\right\}$$

is a 2^{-J}-shift invariant tight wavelet frame.

Proof. Taking into account that Λ_J is a disjoint partition of Ω_J, for any $f \in L_2(\mathbb{R})$, we have

$$\sum_{I_{j,\mathbb{i}}\in\Lambda_J}\sum_{k\in\mathbb{Z}}|\langle f, p^{q,J}_{\mathbb{i};j,k}\rangle|^2 = \sum_{I_{j,\mathbb{i}}\in\Lambda_J}\sum_{\mathbb{i}'\in I_{j,\mathbb{i}}}\sum_{k\in\mathbb{Z}}|\langle f, p^{q,J}_{\mathbb{i}';0,k}\rangle|^2$$
$$= \sum_{\mathbb{i}'\in\Omega_J}\sum_{k\in\mathbb{Z}}|\langle f, p^{q,J}_{\mathbb{i}';0,k}\rangle|^2.$$

By applying Theorem 5.3, Theorem 5.4 is proved. □

By Theorem 5.4, we can obtain various 2^{-J}-shift invariant tight wavelet frames $\Pi^{q,J}_{\Lambda_J}$ from various disjoint partitions of Ω_J. All such obtained tight wavelet frames $\Pi^{q,J}_{\Lambda_J}$ are called 2^{-J}*-shift invariant nonstationary tight wavelet frame packets.*

5.3 Characterization of Sobolev Spaces by 2^{-J}-SI NTWFP

Once we build up a 2^{-J}-shift invariant nonstationary tight wavelet frame packet (2^{-J}-SI NTWFP)

$$\Pi^{q,J}_{\Lambda_J} = \left\{p^{q,J}_{\mathbb{i};j,k} : k \in \mathbb{Z}, I_{j,\mathbb{i}} \in \Lambda_J\right\} \cup \left\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\right\}$$

we can use the weighted ℓ_2-norm of the analysis 2^{-J}-SI NTWFP coefficient sequence

$$\left\{\langle f, p^{q,J}_{\mathbb{i};j,k}\rangle\right\}_{k\in\mathbb{Z}, I_{j,\mathbb{i}}\in\Lambda_J} \cup \left\{\langle f, \psi_{j,k}\rangle\right\}_{k\in\mathbb{Z}, j\geq J, k\in\mathbb{Z}, \psi\in\Psi}$$

of a given function $f \in \mathbb{H}^s(\mathbb{R})$ to characterize its Sobolev norm in $\mathbb{H}^s(\mathbb{R})$.

Theorem 5.5. *Suppose we have a* 2^{-J}*-shift invariant stationary tight wavelet frame packet*

$$\Pi^{q,J}_{\Lambda_J} := \left\{p^{q,J}_{\mathbb{i};j,k} : k \in \mathbb{Z}, I_{j,\mathbb{i}} \in \Lambda_J\right\} \cup \left\{\psi_{j,k} : j \geq J, k \in \mathbb{Z}, \psi \in \Psi\right\}$$

derived from a tight wavelet frame $X(\Psi)$ constructed in an MRA generated by a refinable function ϕ via UEP, with the associated combined UEP mask $\mathbf{h} = [h_0, h_1, \cdots, h_r]$, where $I_{j,\mathbb{i}}$ and Λ_J are defined in (5.18) and (5.19), respectively. Suppose

$$1 - |\widehat{h_0}(\omega)|^2 \le C|\omega|^{2\alpha}, \quad \omega \in \mathbb{R}, \tag{5.20}$$

$$[\widehat{\phi}, \widehat{\phi}]_\alpha(\omega) \le C, \quad \omega \in \mathbb{R}, \tag{5.21}$$

$$[\widehat{p^{q,J}_{\mathbb{i};j,0}}, \widehat{p^{q,J}_{\mathbb{i};j,0}}]_\alpha(\omega) \le C, \quad \omega \in \mathbb{R}, I_{j,\mathbb{i}} \in \Lambda_J. \tag{5.22}$$

If $-\alpha < s < \alpha$, then

$$\Pi^{q,J;s}_{\Lambda_J} := \left\{2^{js} p^{q,J}_{\mathbb{i};j,k} : k \in \mathbb{Z}, I_{j,\mathbb{i}} \in \Lambda_J\right\} \cup \left\{2^{js}\psi_{j,k} : j \ge J, k \in \mathbb{Z}, \psi \in \Psi\right\}$$

is a wavelet frame of $\mathbb{H}^s(\mathbb{R})$, i.e., there exist two positive constants C_1, C_2 such that

$$\begin{aligned} C_1\|f\|^2_{\mathbb{H}^s(\mathbb{R})} \quad &\le \quad \sum_{I_{j,\mathbb{i}} \in \Lambda_J} \sum_{k \in \mathbb{Z}} 2^{2js} |\langle f, p^{q,J}_{\mathbb{i};j,k}\rangle|^2 \\ &+ \quad \sum_{\psi \in \Psi} \sum_{j \ge J} \sum_{k \in \mathbb{Z}} 2^{2js} |\langle f, \psi_{j,k}\rangle|^2 \le C_2 \|f\|^2_{\mathbb{H}^s(\mathbb{R})}, \end{aligned}$$

for all $f \in \mathbb{H}^s(\mathbb{R})$.

Proof. For $-\alpha < s < \alpha$, we can obtain

$$\sum_{k \in \mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 \le C \max\{1, 2^{2Js}\} \|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})},$$

as in the proof of Theorem 3.5.

On the other hand,

$$\begin{aligned} \sum_{I_{j,\mathbb{i}} \in \Lambda_J} \sum_{k \in \mathbb{Z}} 2^{-2js} |\langle f, p^{q,J}_{\mathbb{i};j,k}\rangle|^2 &\le 2^{2(J-1)|s|} \sum_{I_{j,\mathbb{i}} \in \Lambda_J} \sum_{k \in \mathbb{Z}} |\langle f, p^{q,J}_{\mathbb{i};j,k}\rangle|^2 \\ &= 2^{2(J-1)|s|} \sum_{k \in \mathbb{Z}} |\langle f, \phi_{J,k}\rangle|^2 \\ &\le C 2^{2(J-1)|s|} \max\{1, 2^{2Js}\} \|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})}. \end{aligned}$$

In addition, it was shown in the proof of [45, Proposition 2.1] that (5.20) and (5.21) yield

$$\sum_{j=J}^{\infty}\sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle f,\psi_{j,k}\rangle|^2 \le C\|B_{s,t,J}\|_{L_\infty(\mathbb{R})}\|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})},$$

where

$$B_{s,t,J}(\omega) := \sum_{j=J}^{\infty} \frac{2^{-2js}(1+|\omega|^2)^s}{(1+|2^{-J}\omega|^2)^\alpha} \sum_{i=1}^{r} |\widehat{h_i}(2^{-j}\omega)|^2 \in L_\infty(\mathbb{R}).$$

Combining the two inequalities above, we can obtain

$$\sum_{I_{j,\mathbb{i}}\in\Lambda_J}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle f, p^{q,J}_{\mathbb{i};j,k}\rangle|^2 + \sum_{j=J}^{\infty}\sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle f,\psi_{j,k}\rangle|^2 \le C'\|f\|^2_{\mathbb{H}^{-s}(\mathbb{R})},$$

where $C' := C\big(\|B_{s,t,J}\|_{L_\infty(\mathbb{R})} + 2^{2(J-1)|s|}\max\{1, 2^{2Js}\}\big)$.

By a duality argument as in the proof of [45, Theorem 1.2] (Theorem 2.17), we can obtain

$$\frac{1}{C'}\|f\|^2_{\mathbb{H}^{s}(\mathbb{R})} \le \sum_{I_{j,\mathbb{i}}\in\Lambda_J}\sum_{k\in\mathbb{Z}} 2^{2js}|\langle f, p^{q,J}_{\mathbb{i};j,k}\rangle|^2 + \sum_{j=J}^{\infty}\sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}} 2^{2js}|\langle f,\psi_{j,k}\rangle|^2 \le C'\|f\|^2_{\mathbb{H}^{s}(\mathbb{R})},$$

for all $f \in \mathbb{H}^s(\mathbb{R})$, $(-\alpha < s < \alpha)$. Hence, Theorem 5.5 is proved. $\square$

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