WAVELET FRAMES: SYMMETRY, PERIODICITY, AND APPLICATIONS

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Summary

Symmetric or antisymmetric compactly supported wavelets are very much desirable in various applications, since they preserve linear phase properties and also allow symmetric boundary conditions in wavelet algorithms which normally perform better. However, there does not exist any real-valued symmetric or antisymmetric compactly supported orthonormal wavelet with dyadic dilation except for the Haar wavelet. We resolve the problem here by relaxing the orthogonality and non-redundancy condition. At the other end of the spectrum lies the question of whether redundancy could be exploited fully so that localized information at distinct scales or frequencies could be fully captured by the wavelet system. This question is partially answered here in the setting of periodic wavelets using time-localized wavelet frames. In addition, a completely affirmative solution is obtained here in the setting of periodic wavelets using bandlimited wavelet frames that resemble Shannon and Meyer wavelets (see [38]) and possess the frequency segmentation features of wavelet packets (see [46]). Here, we have managed to combine translation and modulation operations into a multiresolution analysis structure, thereby allowing for fast wavelet algorithms to be utilized in applications.

In the first section of Chapter 1, we introduce the concept of frames and briefly review the general properties of frames and the frame operator. In the second section, we introduce the affine system $X(\Psi)$, the shift-invariant quasi-affine system $X^q_K(\Psi)$ at level K and the concept of multiresolution analysis and their respective periodic equivalents. In the third section we present an overview of the results found in this thesis.

The approach in Chapter 2 (published in [23]) is developed under the most general setting of $L^2(\mathbb{R}^s)$. We begin in Section 2.1 by showing that both the frame property and frame bounds of affine systems are preserved under the symmetrization process. In Section 2.2, we consider the case when the original wavelets are obtainable from a multiresolution analysis (MRA), i.e. the setting of framelets. We prove that given an MRA-based tight frame system, a symmetric and antisymmetric tight frame system can be obtained from a, but possibly different, MRA generated by symmetric or antisymmetric refinable functions.

When the original MRA is generated by a symmetric refinable function, the symmetric and antisymmetric tight frame system is obtained from the same MRA. This enables us to convert the systematic construction of spline tight framelets of [16] to a systematic construction of symmetric and antisymmetric spline tight framelets with given orders of smoothness and vanishing moments. Further, framelets constructed via the oblique or unitary extension principle are also considered in Section 2.2. Finally, in Section 2.3, we illustrate with examples the constructions given by our method. We also discuss practical issues related to minimizing the supports of the resulting refinable functions and wavelets as well as improving their spreads in the time domain.

In the first section of Chapter 3, we briefly review the coset representation of lattices and we show that the affine system $X(\Psi)$ is a frame for $L^2(\mathbb{R}^s)$ if and only if the quasi-affine system $X^q_K(\Psi)$ is also a frame for $L^2(\mathbb{R}^s)$ with the same frame bounds. Next, we prove certain elementary results concerning the frame multiresolution analysis (FMRA), which is an MRA with uniform frame bounds.

In the second section which is on $L^2(\mathbb{T}^s)$, we formulate the polyphase space of harmonics. We show that if the periodic affine system $X_{2\pi}$ is a frame for $L^2(\mathbb{T}^s)$, then the periodic quasi-affine system $X^q_{2\pi,K}$ at level K is a frame for $L^2(\mathbb{T}^s)$. Further, this implies that $X_{2\pi}$ is a frame for all the polyphase space of harmonics. We also show analogous results for the restricted periodic affine system $X^R_{2\pi}$ and the restricted periodic quasi-affine system $X^{q,R}_{2\pi,K}$. In addition, we review certain fundamental results from [24] concerning periodic MRAs. Then in Section 3.3, we review periodic extension principles from [25] for tight wavelet frames and generalize these principles under unitary transformations.

In the last section of Chapter 3, we establish the connection between Euclidean space wavelets and periodic wavelets through the Poisson Summation formula. Here we focus on obtaining results that relate shift-invariant spaces of $L^2(\mathbb{R}^s)$ with periodized shift-invariant spaces of $L^2(\mathbb{T}^s)$ constructed from uniform frequency samples of functions from the former. We show that frame properties of shift-invariant spaces of the former are carried over to the periodized systems of the latter. We also show the correspondence of multiresolution properties, in particular that of FMRAs for the two systems. We review the construction of periodic wavelets from periodic FMRAs and show that such constructions could be used for the Euclidean setting. In particular, we could characterize the existence of semi-orthogonal tight wavelet frames for the Euclidean space setting, generalizing the characterization result in [39] to FMRAs constructed from multiple refinable functions. We end the chapter with the connection of the affine system in $L^2(\mathbb{R}^s)$ and the periodic affine system in $L^2(\mathbb{T}^s)$ using extension principles.

In Chapter 4, we construct periodic bandlimited wavelet systems and periodic time-localized wavelet systems with the aim of achieving a flexible time-frequency representation that could also emulate the short-time Fourier transform, i.e. inclusion of modulation information into an MRA structure. The main approach used here is to add additional number of wavelet functions that captures the desired modulation information to the wavelet system. The bandlimited wavelet systems constructed in Section 4.1 are generic and allows for a flexible partitioning of the time-frequency plane while the time-localized wavelet systems of Section 4.2 are constructed from modifying and enlarging existing time-localized orthonormal wavelet bases or tight wavelet frames while retaining most of their original properties such as approximation orders and compact support.

The bandlimited wavelet systems are constructed from either Shannon or Meyer kinds of refinable functions except that we allow freedom of choice on their bandwidths. The only requirement in the design of the wavelet masks is that they must satisfy the minimum energy tight frame condition of the periodic unitary extension principle (UEP), i.e. the perfect reconstruction equation and the anti-aliasing equation.

We begin with a general construction of complex wavelets where we incrementally increase the number of wavelet masks until the entire spectrum of the multiresolution analysis is covered. The wavelet masks share the decay properties of Shannon or Meyer wavelet masks. Some degrees of overlaps in the masks are unavoidable if we are to allow for their smooth decay in the frequency domain. To achieve real and symmetric (antisymmetric) properties, the masks are designed to be symmetric (antisymmetric) in the frequency domain and some mild restrictions on the bandwidths of some of the masks are imposed so that the anti-aliasing condition could be satisfied. We cancel out aliasing chiefly by using corresponding pairs of symmetric and antisymmetric wavelet masks at frequencies where the anti-aliasing condition could not be satisfied by default and this usually occurs at the middle bands.

The methods used in the construction of time-localized wavelet systems generally involves manipulation of the masks of existing orthonormal wavelet bases or tight wavelet frames so that the enlarged and modified wavelet system still satisfies the minimum energy tight frame condition. A direct and naive construction by the diagonal extension of the original wavelet masks with modulated masks allows for only a fixed and limited modulation range and it requires the addition of more refinable functions to the MRA.

We remedy this by considering that the equations of the periodic UEP are modulation invariant and adding the modulated versions of these equations to the original equations, thereby expanding the wavelet system without changing the MRA. In the event that symmetry (antisymmetry) is absent from the original masks, symmetric (antisymmetric) properties could also be added by means of reflection in the frequency domain and applying unitary transformations to the masks. The latter comes at the cost of using twice the number of masks and using a vector MRA. The modulation range of these constructions is required to be bounded in order for the wavelet system to be a tight frame.

We remedy the problem of having a bounded modulation range by splitting some of the wavelet subbands into "packets" using a different set of masks. This idea generalizes orthogonal wavelet representation by requiring the "packetized" masks to satisfy the perfect reconstruction equation, i.e. the energy of the packetized masks must satisfy a sum of constant norm. The frame approximation order is preserved as the MRA is unchanged and we could choose the packetized masks to be modulated versions of some existing wavelet masks such as that of the Haar system. The representation is therefore computationally efficient since the desired representation of the signal could be obtained adaptively and almost directly.

In Chapter 5, we study the uniqueness of representation by wavelet frames for $L^2(\mathbb{T}^s)$ and derive decomposition and reconstruction algorithms for the coefficients of the representation. We also study the stationary wavelet transform and its relation to the periodic quasi-affine system and we analyze the time-frequency properties of some Gabor atoms and chirp signals using our generic bandlimited wavelet systems.

In Section 5.1, we establish the uniqueness of representation by wavelet frames using the wavelet expansion in the frequency domain by polyphase harmonics of wavelets. Essentially, we diagonalize the Gramians of these polyphase harmonics by applying unitary transformations to the wavelet coefficients and the polyphase harmonics. Using these uniqueness results we derive the reconstruction algorithm.

In Section 5.2, we assume that the multiresolution subspaces and wavelet subspaces are orthogonal, i.e. we consider the semi-orthogonal setting of FMRA wavelets. We show that we could represent polyphase harmonics of a finer multiresolution subspace by polyphase harmonics of a coarser multiresolution subspace and its corresponding wavelet subspace using decomposition masks. Next, we derive decomposition algorithms using these masks and establish sufficient conditions for perfect reconstruction.

In Section 5.3, we consider the nonorthogonal setting of MRA wavelets, i.e. we do not assume the sum of multiresolution subspaces and wavelet subspaces as a direct sum. We derive the decomposition algorithms using the minimum energy tight frame condition of the periodic UEP. Here we find that the conjugate transpose of the reconstruction masks play the role of decomposition masks.

In Section 5.4, we study the derivation of the stationary wavelet transform by considering the time domain version of our algorithms. We verify the translation invariant nature of the transform by showing that the transform includes all the coefficients of various versions of the decimated wavelet transform. We also derive the quasi-affine representation of the wavelet expansion based on the stationary wavelet transform.

In Section 5.5, we show that the collection of generic bandlimited refinable functions constructed in Section 4.1 possesses spectral frame approximation order if the multiresolution subspaces grow sufficiently fast, and that the bandlimited wavelet systems derived from them based on the periodic UEP have global vanishing moments of arbitrarily high order. We also review an example of compactly supported pseudo-splines, which when periodized, also provide spectral frame approximation order and global vanishing moments of arbitrarily high order. We conclude the thesis by explaining the process of plotting the time-frequency representations of some Gabor atoms and chirp signals using the transforms based on our bandlimited wavelet systems. These time-frequency representations demonstrate that the transforms designed successfully incorporate strengths of both the wavelet transforms and the short-time Fourier transform.

Chapter 1

Introduction

In modern signal processing, digital samples of signals are often used to represent or reconstruct the signals. Therefore, it is practical to expect that if the samples are "close" to each other, the signals should also be "close" to each other and vice versa. This is important so that when some terms in the representation of the signal in terms of its samples are neglected, we can be sure that the reconstructed signal will not differ much from the original signal. Such requirements are best understood in the context of frames, where the coefficients of a frame expansion replace the role of the samples of the signals.

1.1 Frames of Hilbert Spaces

A countable system X in a separable Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants A, $B > 0$ such that for every $f \in \mathcal{H}$,

$$A \|f\|_{\mathcal{H}}^2 \leq \sum_{g \in X} |\langle f, g \rangle_{\mathcal{H}}|^2 \leq B \|f\|_{\mathcal{H}}^2 . \tag{1.1}$$

A frame is a special case of a *Bessel system*, in which only the right inequality of (1.1) is required to hold for every $f \in \mathcal{H}$. The constants A and B are *lower* and *upper bounds* of the frame. The supremum of A and the infimum of B for (1.1) to hold are called *frame bounds*. The elements of a frame must satisfy $\|g\|_{\mathcal{H}} \leq \sqrt{B}$. A frame X is said to be *tight* if we may take $A = B$. A tight frame with bound 1 is sometimes referred to as a *normalized tight frame* in the literature, see for instance [32]. A frame is a *Riesz basis* if every $f \in \mathcal{H}$ could be represented uniquely by elements of the frame. A tight frame X for $\mathcal{H}$ becomes an *orthonormal basis* when all the vectors in X have their norms equal to 1.

Let $l^2(\mathbb{Z}^s)$ be the space of all complex-valued square-summable sequences on $\mathbb{Z}^s$ endowed with the standard inner product $\langle a,b\rangle_{l^2(\mathbb{Z}^s)} := \sum\limits_{n\in\mathbb{Z}^s} a(n)\overline{b(n)}$ and norm $\|\cdot\|_{l^2(\mathbb{Z}^s)} := \langle\cdot,\cdot\rangle^{\frac{1}{2}}_{l^2(\mathbb{Z}^s)}$. For our purposes in the construction of multiresolution analyses and wavelets, we shall review the following standard properties of frames which could be found in the books [6], [14] and [26].

Adding the zero element to a frame does not change the frame condition (1.1). A sequence $\{f_n\}$ of vectors in a Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$ if and only if there exists a positive constant C such that, for every $h\in\mathcal{H}$, $\sum\limits_{n\in\mathbb{Z}} |\langle h,f_n\rangle_{\mathcal{H}}|^2$ is finite and there exists a sequence $a=\{a_n\}\in l^2(\mathbb{Z})$ such that $h=\sum\limits_{n\in\mathbb{Z}} a_n f_n$ in $\mathcal{H}$ and $\|a\|_{l^2(\mathbb{Z})} \le C\|h\|_{\mathcal{H}}$, i.e. the closure of the span of $\{f_n\}$ must be $\mathcal{H}$. Let $\{f_n\}$ be a frame for $\mathcal{H}$ with frame bounds A and B. For any sequence $a=\{a_n\}\in l^2(\mathbb{Z})$, $h:=\sum\limits_{n\in\mathbb{Z}} a_n f_n$ converges in $\mathcal{H}$ and $\|h\|^2_{\mathcal{H}} \le B\|a\|^2_{l^2(\mathbb{Z})}$.

In a Hilbert space $\mathcal{H}$, the *frame operator* $S:\mathcal{H}\to\mathcal{H}$ of a frame $\{f_n\}$ for $\mathcal{H}$ is defined for each $f\in\mathcal{H}$ by

$$Sf = \sum_{n\in\mathbb{Z}} \langle f,f_n\rangle_{\mathcal{H}} f_n \text{ in } \mathcal{H}. \tag{1.2}$$

The frame operator S is a positive operator satisfying $AI\le S\le BI$, where A and B are frame bounds, i.e. $\|S^{-1}\|^{-1}_{\mathcal{H}} = A$ and $\|S\|_{\mathcal{H}} = B$, and I is the identity mapping on $\mathcal{H}$. Therefore, the frame operator S is bounded and continuous and is an invertible operator satisfying $B^{-1}I\le S^{-1}\le A^{-1}I$. For each $f\in\mathcal{H}$, the element f can be decomposed into

$$f = \sum_{n\in\mathbb{Z}} \langle f,S^{-1}f_n\rangle_{\mathcal{H}} f_n = \sum_{n\in\mathbb{Z}} \langle f,f_n\rangle_{\mathcal{H}} S^{-1}f_n \text{ in } \mathcal{H}. \tag{1.3}$$

The sequence $\{S^{-1}f_n\}$ is also a frame with frame operator S^{-1} and frame bounds B^{-1} and A^{-1} and is known as the *canonical dual frame*. In particular the canonical dual of a tight frame is $\{A^{-1}f_n\}$. The formula (1.3) suggests that we can reconstruct f from the sequences $\{\langle f,f_n\rangle\}$ and $\{\langle f,S^{-1}f_n\rangle\}$. The frame operator S commutes with all unitary operators T that are permutations on $\{f_m\}_{m\in\mathbb{Z}}$.

If $\{f_n\}$ is a frame but not a basis, then there exist nonzero sequences $\{a_n\}\in l^2(\mathbb{Z})$ such that $\sum\limits_{n\in\mathbb{Z}} a_n f_n = 0$. Therefore $f=\sum\limits_{n\in\mathbb{Z}} [\langle f,S^{-1}f_n\rangle_{\mathcal{H}} + a_n] f_n$ can be represented in many different ways by the frame elements. During a signal transmission process, suppose that the frame coefficients $\{\langle f,S^{-1}f_n\rangle_{\mathcal{H}}\}$ of the signal are transmitted and are perturbated into $\{\langle f,S^{-1}f_n\rangle_{\mathcal{H}} + b_n\}$ by noise. There exists this possibility that parts of the noise perturbation might sum to zero and cancel out. This never occurs if $\{f_n\}$ is an orthonormal basis since $\left\|\sum\limits_{n\in\mathbb{Z}} b_n f_n\right\|_{\mathcal{H}} = \|b\|_{l^2(\mathbb{Z})}$, where $b = \{b_n\}_{n\in\mathbb{Z}}$, i.e. additional perturbations make the reconstruction worse. As in information theory, there is a tradeoff between signal size and error reduction using the redundancy of a frame.

The preference for using the canonical dual frame in reconstruction could be seen in the following way. Suppose that $\{f_n\}$ is a frame for a subspace V of the Hilbert space $\mathcal{H}$. Then the orthogonal projection of $\mathcal{H}$ onto V is given by

$$Pf = \sum_{n\in\mathbb{Z}} \langle f, S^{-1} f_n\rangle_{\mathcal{H}} f_n \text{ in } \mathcal{H},$$

i.e. the coefficients $\{\langle f, S^{-1} f_n\rangle_{\mathcal{H}}\}$ have minimal l^2-norm among all sequences $\{a_n\} \in l^2(\mathbb{Z})$ such that $f = \sum\limits_{n\in\mathbb{Z}} a_n f_n$.

Frames possess better stability properties under the application of operators when compared to bases. If $\{e_n\}$ is a basis, then only bounded bijective operators U could be applied to preserve the basis property, i.e. ensure that $\{Ue_n\}$ remains a basis. In contrast, the application of bounded surjective operators will preserve the frame property. The surjective property could be extended to any operator with closed range property if we only require the transformed collection to be a frame for a smaller subspace of the original space. For example, if $\{f_n\}$ is a frame for $\mathcal{H}$ and $\{g_n\}$ is a sequence in $\mathcal{H}$ such that $g_n = f_n$ except for a finite set of $n \in \mathbb{Z}$, then $\{g_n\}$ is a frame for its closed linear span.

We briefly describe an approach to determine all frames for $\mathcal{H}$ as given in [1]. Given any two frames $\{f_n\}$ and $\{g_m\}$ for $\mathcal{H}$, the bi-infinite matrix U with (m,n)-th entry given by $u_{mn} = \langle g_m, S^{-1} f_n\rangle_{\mathcal{H}}$ defines a bounded operator on $l^2(\mathbb{Z})$. Given a frame $\{f_n\}$ and a bi-infinite matrix $U = \{u_{mn}\}$ that defines a bounded operator on $l^2(\mathbb{Z})$, the sequence $\{h_m\}$ defined by $h_m = \sum\limits_{n\in\mathbb{Z}} u_{mn} f_n$ is well defined, in particular $\{h_m\}$ is a frame for $\mathcal{H}$ if and only if there exists a constant $C > 0$ such that for every $f \in \mathcal{H}$, $\sum\limits_{m\in\mathbb{Z}} |\langle f, h_m\rangle_{\mathcal{H}}|^2 \geq C \sum\limits_{n\in\mathbb{Z}} |\langle f, f_n\rangle_{\mathcal{H}}|^2$. This illustrates the possibility of using appropriate transformations to obtain new frames from existing frames, which is one of the themes of our thesis.

Riesz bases for $\mathcal{H}$ are characterized as collections $\{Ue_n\}$ where $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \mapsto \mathcal{H}$ is a bounded and invertible operator. In a similar way, frames for $\mathcal{H}$ are exactly the collections $\{Ue_n\}$ where $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \mapsto \mathcal{H}$ is a bounded and surjective operator.

The development of frames arises naturally from applications in time-frequency analysis. Continuous time-frequency representations of signals based on the short-time Fourier transform and the continuous wavelet transform are helpful from the theoretical perspectives of time-frequency analysis though not always useful for practical applications. The discretization of these representations by sampling operations lead to non-orthogonal series expansions in general. The collection of time-frequency atoms used to represent the signal may not form a Riesz basis and in the event they do, they may have comparably much poorer time-frequency localization as in the case of the Gabor system, rendering them not utilizable in time-frequency analysis. We shall be studying the construction of wavelet frames with useful properties such as symmetry, periodicity and good time-frequency localization in this thesis.

1.2 Affine Systems and Multiresolution Analysis

Let $L^2(\mathbb{R}^s)$ be the space of all complex-valued square-integrable functions on the s-dimensional Euclidean space $\mathbb{R}^s$ endowed with the normalized inner product $\langle f, g\rangle := (2\pi)^{-s}\int_{\mathbb{R}^s} f(t)\overline{g(t)}\mathrm{d}t$ and norm $\|\cdot\| := \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. The Fourier transform $\widehat{f}$ of a function f in $L^1(\mathbb{R}^s)$, the space of all complex-valued integrable functions on $\mathbb{R}^s$, is defined as $\widehat{f}(\omega) := (2\pi)^{-s}\int_{\mathbb{R}^s} f(t)\mathrm{e}^{-\mathrm{i}\omega\cdot t}\mathrm{d}t$, and is extended in the standard manner to a unitary operator $\mathcal{F}$ on $L^2(\mathbb{R}^s)$.

Let Ψ be a finite ordered subset of $L^2(\mathbb{R}^s)$. We use Ψ to denote both a set and a column vector. Following [44], we define the *affine system* $X(\Psi)$ generated by Ψ to be

$$X(\Psi) := \{d^{\frac{k}{2}} E_k^l \psi(M^k\cdot) : \psi \in \Psi, l \in \mathbb{Z}^s, k \in \mathbb{Z}\}, \tag{1.4}$$

where $E_k^l : L^2(\mathbb{R}^s) \to L^2(\mathbb{R}^s)$ is the *shift operator* given by

$$E_k^l : f \mapsto f(\cdot - M^{-k}l),$$

with M being a $s \times s$ invertible matrix with integer entries such that M is expansive, i.e. all the eigenvalues of M are greater than 1, and $d := |\det M|$. An affine system that forms a frame for $L^2(\mathbb{R}^s)$ is known as a *wavelet frame*. For a wavelet frame, the functions $\psi \in \Psi$ in (1.4) are known as *mother wavelets* or simply *wavelets*. As the affine system $X(\Psi)$ comprises shifts of dilates of mother wavelets $\psi \in \Psi$, it is sometimes called a stationary wavelet frame.

For a fixed $K \geq 0$, the $\mathbb{Z}^s$ shift-invariant *truncated-affine system* $X_K(\Psi)$ of an affine system $X(\Psi)$ is defined to be

$$X_K(\Psi) := E(\{d^{\frac{k}{2}} E_k^l \psi(M^k\cdot) : \psi \in \Psi, l \in \mathcal{L}_k, k \geq K\}), \tag{1.5}$$

where $E(\Lambda) := E_0(\Lambda)$ is the collection of all integer $\mathbb{Z}^s$ shift operations applied to Λ with $\mathcal{L}_k$ denoting a full collection of coset representatives of $\mathbb{Z}^s/M^k\mathbb{Z}^s$.

The $M^{-K}\mathbb{Z}^s$ shift-invariant *quasi-affine system* $X^q_K(\Psi)$ of an affine system $X(\Psi)$ at level K is defined to be

$$X^q_K(\Psi) := E_K(\Lambda_K) \tag{1.6}$$

which consists of all the $M^{-K}\mathbb{Z}^s$ shifts of

$$\Lambda_K := \{d^{k-\frac{K}{2}}\psi(M^k\cdot) : \psi \in \Psi, k < K\} \cup \{d^{\frac{k}{2}} E^l_k \psi(M^k\cdot) : \psi \in \Psi, l \in \mathcal{L}_{k-K}, k \geq K\}. \tag{1.7}$$

Unlike the quasi-affine system $X^q(\Psi) := X^q_0(\Psi)$ introduced in [44], the affine system $X(\Psi)$ is not invariant under any lattice shifts since only the $M^{-k}\mathbb{Z}^s$ shifts of $\psi(M^k\cdot)$ are included in $X(\Psi)$ and these shifts become sparser as k becomes smaller. The smallest closed linear subspace $V^K(\Lambda_K)$ of $L^2(\mathbb{R}^s)$ that contains $E_K(\Lambda_K)$ is the $M^{-K}\mathbb{Z}^s$ shift-invariant space generated by Λ_K, i.e.

$$V^K(\Lambda_K) := \overline{\operatorname{span}}\, E_K(\Lambda_K).$$

Let $\Phi \subset L^2(\mathbb{R}^s)$ be a finite set and let $V(\Phi)$ be the closed shift-invariant linear subspace generated by Φ, i.e. $V(\Phi) = \overline{\operatorname{span}}\,\{E^l\phi : \phi \in \Phi, l \in \mathbb{Z}^s\}$ (where $E^l := E^l_0$). Following [4], the cardinality of a minimal generating set Φ for $V(\Phi)$ is called the *length* of V which is denoted by $\operatorname{len} V$. The space $V(\Phi)$ is said to be *finitely generated shift-invariant* (FSI) if $\operatorname{len} V$ is finite and is said to be *principal shift-invariant* (PSI) space if $\operatorname{len} V = 1$.

Next, we recall some fundamental results on stationary and nonstationary wavelet frames derived from a *multiresolution analysis* (MRA) of $L^2(\mathbb{R}^s)$, i.e. *framelets*. Generalizing [3], an MRA of $L^2(\mathbb{R}^s)$ is a sequence of closed subspaces $\{V^k(\Phi_k)\}$ generated by finite ordered subsets Φ_k of $L^2(\mathbb{R}^s)$ with $|\Phi_k| = \rho$ for all k such that (i) $V^k(\Phi_k) \subset V^{k+1}(\Phi_{k+1})$, (ii) $\bigcup_{k\in\mathbb{Z}} V^k(\Phi_k)$ is dense in $L^2(\mathbb{R}^s)$.

In the event that there exist $A, B > 0$ such that $E_k(\Phi_k)$ is a frame for $V^k(\Phi_k)$ with uniform bounds A and B for every $k \in \mathbb{Z}$, then the MRA is known as a *frame multiresolution analysis* (FMRA) with bounds A and B. If for every $k \in \mathbb{Z}$, $V^k(\Phi_k) := \{d^{\frac{k}{2}} f(M^k\cdot) : f \in V^0(\Phi_0)\}$, the MRA or FMRA is known as a *stationary* MRA or FMRA respectively and is denoted by $\{V^k(\Phi)\}$, where $\Phi := \Phi_0$. For this stationary case, the above notions of MRA and FMRA are introduced in [3] and [2] respectively. In such a case, we also have (iii) $\bigcap_{k\in\mathbb{Z}} V^k(\Phi) = \{0\}$ since Φ is a finite subset of $L^2(\mathbb{R}^s)$ (see Corollary 4.14 of [3] and Theorem 2.2 and Remark 2.6 of [36]).

Condition (i) requires the vector Φ_k to be *refinable* for every $k \in \mathbb{Z}$, i.e.

$$\widehat{\Phi}_k = \widehat{H}_{k+1}\widehat{\Phi}_{k+1}, \tag{1.8}$$

where $\widehat{H}_{k+1}$ is a $2\pi(M^T)^{k+1}\mathbb{Z}^s$-periodic matrix-valued measurable function known as the *refinement mask.* The vector Φ_k is known as a *refinable vector* and (1.8) is the *refinement equation.* For a stationary MRA, the refinement equation simplifies to

$$\widehat{\Phi}(M^T\cdot) = \widehat{H}\widehat{\Phi}, \tag{1.9}$$

where $\widehat{H}$ is a $2\pi\mathbb{Z}^s$-periodic matrix-valued measurable function.

When Φ_k satisfies (i) for every $k \in \mathbb{Z}$, Condition (ii) requires $\bigcap_{k\in\mathbb{Z}}\bigcap_{\phi_k\in\Phi_k}\{\omega \in \mathbb{R}^s : \widehat{\phi}_k(\omega) = 0\}$ to be a set of measure zero (see Theorem 4.3 of [3] and Theorem 2.1 and Remark 2.6 of [36]), which always holds in the stationary case if there exists $\phi \in \Phi$ such that ϕ is compactly supported (see [35]). This means that the entire frequency domain is fully "covered" by the MRA.

Suppose that $\{V^k(\Phi_k)\}$ is an MRA of $L^2(\mathbb{R}^s)$. Let Ψ_k be a finite ordered subset of $V^{k+1}(\Phi_{k+1})$. Then there exists a $2\pi(M^T)^{k+1}\mathbb{Z}^s$-periodic matrix-valued measurable function $\widehat{G}_{k+1}$ known as the *wavelet mask* such that

$$\widehat{\Psi}_k = \widehat{G}_{k+1}\widehat{\Phi}_{k+1}. \tag{1.10}$$

Equation (1.10) defines a vector of *pre-wavelets* Ψ_k and is called the *wavelet equation.* For a stationary MRA $\{V^k(\Phi)\}$ of $L^2(\mathbb{R}^s)$ with Ψ being a finite ordered subset of $V^1(\Phi)$, the wavelet equation (see [16]) simplifies to

$$\widehat{\Psi}(M^T\cdot) = \widehat{G}\widehat{\Phi}, \tag{1.11}$$

where the wavelet mask $\widehat{G}$ is a $2\pi\mathbb{Z}^s$-periodic matrix-valued measurable function.

We define the *combined MRA mask* to be the $|\Phi_k \cup \Psi_k| \times |\Phi_k|$ matrix

$$\widehat{L}_k := \begin{bmatrix} \widehat{H}_k \\ \widehat{G}_k \end{bmatrix}, \tag{1.12}$$

and in the event of Φ_k being a singleton set, i.e. $\Phi_k := \{\phi_k\}$, we denote $\widehat{h}_k := \widehat{H}_k$.

Under the assumption that the entries of $\widehat{L}_k$ lie in $L^\infty(\mathbb{T}^s)$, the space of all essentially bounded complex-valued functions on the s-dimensional circle group $\mathbb{T}^s := \mathbb{R}^s/2\pi\mathbb{Z}^s$, we define the Fourier coefficients of the masks $\widehat{H}_k$ and $\widehat{G}_k$, which we shall term simply as *lowpass filter* H_k and *highpass filter* G_k, by

$$\widehat{H}_k(\omega) = \sum_{n\in\mathbb{Z}^s} H_k(n)\mathrm{e}^{-\mathrm{i}n\cdot\omega}, \quad \widehat{G}_k(\omega) = \sum_{n\in\mathbb{Z}^s} G_k(n)\mathrm{e}^{-\mathrm{i}n\cdot\omega}.$$

We shall generally use the notations $\widehat{h}_k$, $\widehat{g}_k$, h_k and g_k in place of $\widehat{H}_k$, $\widehat{G}_k$, H_k and G_k respectively when $H_k(n)$ and $G_k(n)$, $n \in \mathbb{Z}^s$, are scalars. The refinement and wavelet equations (1.8) and (1.10) are equivalent to

$$\Phi_k = \sum_{n\in\mathbb{Z}^s} H_{k+1}(n)\Phi_{k+1}(\cdot - n), \quad \Psi_k = \sum_{n\in\mathbb{Z}^s} G_{k+1}(n)\Phi_{k+1}(\cdot - n), \tag{1.13}$$

while the refinement and wavelet equations (1.9) and (1.11) for the stationary case are equivalent to

$$\Phi = |\det M| \sum_{n\in\mathbb{Z}^s} H(n)\Phi(M \cdot -n), \quad \Psi = |\det M| \sum_{n\in\mathbb{Z}^s} G(n)\Phi(M \cdot -n). \tag{1.14}$$

Defining Ψ_k by (1.10), if the system

$$X_{\mathbb{R}} := \{E_k^l \psi_k : \psi_k \in \Psi_k, l \in \mathbb{Z}^s, k \in \mathbb{Z}\}$$

forms a frame for $L^2(\mathbb{R}^s)$, then $X_{\mathbb{R}}$ is known as a *wavelet frame*. Comparing with (1.4), this is a more general formulation as the functions $\psi_k \in \Psi_k$ need not be dilates of functions in some basic set Ψ, i.e. it includes both stationary and nonstationary cases.

The notions of MRAs and wavelets also have counterparts for 2π-periodic functions (see for instance [24] and [25]). Let $L^2(\mathbb{T}^s)$ be the space of all complex-valued square-integrable functions on $\mathbb{T}^s$ endowed with the normalized inner product $\langle f, g\rangle_{L^2(\mathbb{T}^s)} := (2\pi)^{-s}\int_{\mathbb{T}^s} f(t)\overline{g(t)}\mathrm{d}t$ and norm $\|\cdot\|_{L^2(\mathbb{T}^s)} := \langle\cdot,\cdot\rangle_{L^2(\mathbb{T}^s)}^{\frac{1}{2}}$. Reusing notations, we define the Fourier coefficients $\{\widehat{f}(n)\}_{n\in\mathbb{Z}^s}$ of a function $f \in L^2(\mathbb{T}^s)$ as $\widehat{f}(n) := \langle f, \mathrm{e}^{\mathrm{i}n\cdot}\rangle_{L^2(\mathbb{T}^s)}$. We define the *periodic affine system* $X_{2\pi}$ to be

$$X_{2\pi} := \{\phi_0 : \phi_0 \in \Phi_0\} \cup \{T_k^l \psi_k : \psi_k \in \Psi_k, l \in \mathcal{L}_k, k \geq 0\}, \tag{1.15}$$

where $T_k^l : L^2(\mathbb{T}^s) \to L^2(\mathbb{T}^s)$ is the *shift operator* given by

$$T_k^l : f \mapsto f(\cdot - 2\pi M^{-k} l).$$

A periodic affine system that forms a frame for $L^2(\mathbb{T}^s)$ is known as a *periodic wavelet frame*. For a periodic wavelet frame, the functions $\psi_k \in \Psi_k$ in (1.15) are known as *wavelets*. Due to the periodic nature of functions in $L^2(\mathbb{T}^s)$, affine systems in $L^2(\mathbb{T}^s)$ are generally nonstationary, i.e. different wavelets for different levels k, which will be the context that we are dealing with here.

For a fixed $K \geq 0$, we introduce the notion of $2\pi M^{-K}\mathbb{Z}^s$ shift-invariant *periodic quasi-affine system* $X_{2\pi,K}^q$ of an affine system $X_{2\pi}$ at level K as

$$X_{2\pi,K}^q := T_K(\Omega_K) \tag{1.16}$$

which consists of all the $2\pi M^{-K}\mathbb{Z}^s$ shifts of

$$\begin{aligned}\Omega_K \;&:=\; \{d^{-\frac{K}{2}}\phi_0 : \phi_0 \in \Phi_0\} \cup \{d^{\frac{k}{2}-\frac{K}{2}}\psi_k : \psi_k \in \Psi_k : 0 \leq k < K\} \cup \\ &\quad\; \{T_k^l \psi_k : \psi_k \in \Psi_k, l \in \mathcal{L}_{k-K}, k \geq K\}. \end{aligned} \tag{1.17}$$

The smallest closed linear subspace $V_{2\pi}^K(\Omega_K)$ of $L^2(\mathbb{T}^s)$ that contains $T_K(\Omega_K)$ is the $2\pi M^{-K}\mathbb{Z}^s$ shift-invariant space generated by Ω_K, i.e.

$$V_{2\pi}^K(\Omega_K) := \overline{\operatorname{span}}\, T_K(\Omega_K).$$

The cardinality of a smallest generating set for $V_{2\pi}^K$ is called the *length* of $V_{2\pi}^K$ which is denoted by $\operatorname{len} V_{2\pi}^K$. The space $V_{2\pi}^K$ is said to be *finitely generated shift-invariant* (FSI) if $\operatorname{len} V_{2\pi}^K$ is finite and is said to be a *principal shift-invariant* (PSI) space if $\operatorname{len} V_{2\pi}^K = 1$.

For $R \geq K \geq 0$, we shall also define the *restricted periodic affine system* $X_{2\pi}^R$ to be

$$X_{2\pi}^R := \{\phi_0 : \phi_0 \in \Phi_0\} \cup \{T_k^l \psi_k : \psi_k \in \Psi_k, l \in \mathcal{L}_k, 0 \leq k \leq R\} \tag{1.18}$$

and the $2\pi M^{-K}\mathbb{Z}^s$ shift-invariant *restricted periodic quasi-affine system* $X_{2\pi,K}^{q,R}$ of an affine system $X_{2\pi}$ at level $K \leq R$ to be

$$X_{2\pi,K}^{q,R} := T_K(\Omega_K^R) \tag{1.19}$$

which consists of all the $2\pi M^{-K}\mathbb{Z}^s$ shifts of

$$\begin{aligned}\Omega_K^R \;&:=\; \{d^{-\frac{K}{2}}\phi_0 : \phi_0 \in \Phi_0\} \cup \{d^{\frac{k}{2}-\frac{K}{2}}\psi_k : \psi_k \in \Psi_k : 0 \leq k < K\} \cup \\ &\quad\; \{T_k^l \psi_k : \psi_k \in \Psi_k, l \in \mathcal{L}_{k-K}, K \leq k \leq R\}. \end{aligned} \tag{1.20}$$

The notions of the restricted periodic affine and quasi-affine systems are useful in the context of applications where signals are usually periodic and are of finite dimensions.

Let $\mathcal{S}(M^k)^{r\times\rho}$ denote the class of M^k-periodic sequences of $r \times \rho$ complex-valued matrices, i.e. $H_k(l + M^k p) = H_k(l)$ for all $H_k \in \mathcal{S}(M^k)^{r\times\rho}$ with $l, p \in \mathbb{Z}^s$. We shall also denote $\mathcal{S}(M^k) := \mathcal{S}(M^k)^{1\times 1}$.

A *periodic* MRA $\{V_{2\pi}^k(\Phi_k)\}$ of $L^2(\mathbb{T}^s)$ is a sequence of closed subspaces generated by finite ordered subsets Φ_k of $L^2(\mathbb{T}^s)$ with $|\Phi_k| = \rho$ such that (i) $V_{2\pi}^k(\Phi_k) \subseteq V_{2\pi}^{k+1}(\Phi_{k+1})$ and (ii) $\bigcup_{k\geq 0} V_{2\pi}^k(\Phi_k)$ is dense in $L^2(\mathbb{T}^s)$. In the event that there exist $A, B > 0$ such that for every $k \geq 0$, $T_k(\Phi_k)$ forms a frame for $V_{2\pi}^k(\Phi_k)$ with uniform bounds A and B, the periodic MRA is known as a *periodic* FMRA with bounds A and B.

Condition (i) requires the vector Φ_k to be *refinable* for every $k \geq 0$, i.e. there exists $\widehat{H}_{k+1} \in \mathcal{S}((M^T)^{k+1})^{\rho\times\rho}$ known as the *periodic refinement mask* such that

$$\widehat{\Phi}_k(n) = \widehat{H}_{k+1}(n)\widehat{\Phi}_{k+1}(n), \quad n \in \mathbb{Z}^s. \tag{1.21}$$

Equation (1.21) is the *periodic refinement equation* and is equivalent to

$$\Phi_k = \sum_{l \in \mathcal{L}_{k+1}} H_{k+1}(l) T_{k+1}^l \Phi_{k+1}. \tag{1.22}$$

in the time domain with $H_{k+1} \in \mathcal{S}(M^{k+1})^{\rho \times \rho}$. When Condition (i) is satisfied, Condition (ii) is equivalent to the requirement that $\bigcap_{k \geq 0} \bigcap_{\phi_k \in \Phi_k} \{n \in \mathbb{Z}^s : \widehat{\phi}_k(n) = 0\}$ is empty (see Theorem 3.1 of [24]). This means that the entire frequency domain is fully "covered" by the MRA.

Suppose that $\{V_{2\pi}^k(\Phi_k)\}$ is an MRA of $L^2(\mathbb{T}^s)$. Let Ψ_k be a finite ordered subset of $V_{2\pi}^{k+1}(\Phi_{k+1})$. Then there exists $\widehat{G}_{k+1} \in \mathcal{S}((M^T)^{k+1})^{\varrho_k \times \rho}$ known as the *periodic wavelet mask* such that

$$\widehat{\Psi}_k(n) = \widehat{G}_{k+1}(n) \widehat{\Phi}_{k+1}(n), \quad n \in \mathbb{Z}^s. \tag{1.23}$$

Equation (1.23) defines a vector of pre-wavelets and is called the *periodic wavelet equation* and is equivalent to

$$\Psi_k = \sum_{l \in \mathcal{L}_{k+1}} G_{k+1}(l) T_{k+1}^l \Phi_{k+1} \tag{1.24}$$

in the time domain with $G_{k+1} \in \mathcal{S}(M^{k+1})^{\varrho_k \times \rho}$. Likewise to the real line case, we also define the *combined MRA mask* to be the $|\Phi_k \cup \Psi_k| \times |\Phi_k|$ matrix

$$\widehat{L}_k := \begin{bmatrix} \widehat{H}_k \\ \widehat{G}_k \end{bmatrix}, \tag{1.25}$$

and in the event of Φ_k being a singleton set, i.e. $\Phi_k := \{\phi_k\}$, we denote $\widehat{h}_k := \widehat{H}_k$.

1.3 Overview of Thesis

Most of the results in this thesis are developed for the general multidimensional multiwavelet setting with arbitrary integer dilation matrices. However, in order to provide an easily accessible overview of the main results, we shall present them in this section by only considering the one-dimensional scenario, i.e. $s = 1$ with the dilation matrix $M = 2$. References to the full versions of these results in subsequent chapters are indicated. Most of the time we shall also assume that the generic MRA used here is generated by a single refinable function, i.e. $\Phi := \phi$ for the stationary case and $\Phi_k := \phi_k$ for the nonstationary case with $k \geq 0$. In the following, we shall set the notations $\mathcal{L}_k = \mathcal{R}_k = \{0, \ldots, 2^k - 1\}$.

Chapter 2 is on the construction of symmetric or antisymmetric compactly supported wavelets on the real line. The main idea behind our method involves utilizing unitary transformations of existing wavelet frames with compact support. Given that $\Psi := \left[\psi^m\right]_{m=1}^{\varrho}$ is a vector-valued function satisfying the wavelet equation (1.11) of the MRA $\{V^k(\phi)\}$ of $L^2(\mathbb{R})$, we define

$$\Xi := \frac{1}{\sqrt{2}}\begin{bmatrix}\phi\\ \phi(\eta-\cdot)\end{bmatrix}, \quad \Upsilon := \frac{1}{\sqrt{2}}\begin{bmatrix}\psi^m\\ \psi^m(\kappa_m-\cdot)\end{bmatrix}_{m=1}^{\varrho}, \quad \Phi' := U_0\Xi, \quad \Psi' := U_{2\varrho}\Upsilon, \quad (1.26)$$

where η, $\kappa_m \in \mathbb{Z}$ and $U_{2\varrho}$ being a $2\varrho \times 2\varrho$ block diagonal matrix with the matrix $U_0 := \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ as their blocks. Therefore, for a given set of wavelets Ψ, we provide a general, and yet simple, method to derive a new set of wavelets Ψ' such that each wavelet in Ψ' is either symmetric or antisymmetric. The affine system generated by Ψ' is a tight frame for $L^2(\mathbb{R})$ whenever the affine system generated by Ψ is so.

Theorem 1.1. (Theorem 2.7) *Let Ψ be a finite set of tight framelets obtained from the MRA $\{V^k(\phi)\}$ of $L^2(\mathbb{R})$. Define Φ' and Ψ' as in* (1.26). *Then Ψ' is a finite set of symmetric or antisymmetric tight framelets obtained from the MRA generated by Φ'.*

In particular, we show that when Ψ is constructed via an MRA, Ψ' can also be derived from a, but possibly different, MRA. If moreover the MRA for constructing Ψ is generated by a symmetric refinable function, then we prove that Ψ' is obtained from the same MRA. The proof involves applying unitary transformations to the perfect reconstruction condition and anti-aliasing condition of the oblique extension principle (OEP) (Theorem 2.8) and the unitary extension principle (UEP) (Theorem 2.10).

Theorem 1.2. (Theorem 2.9) *If $X(\Psi)$ is a tight frame for $L^2(\mathbb{R})$ derived from an MRA generated by a symmetric refinable function using the OEP, then $X(\Psi')$ is also a tight frame for $L^2(\mathbb{R})$ derived from the same MRA using the OEP.*

Theorem 1.3. (Theorem 2.11) *If $X(\Psi)$ is a tight frame for $L^2(\mathbb{R})$ derived from an MRA generated by a real-valued function ϕ using the UEP, then $X(\Psi')$ is also a tight frame for $L^2(\mathbb{R})$ derived from the MRA $\{V^k(\Phi')\}$ using the UEP.*

In Chapter 3, we study the connection of wavelet frames of the real line with that of their periodizations. This involves establishing results concerning affine systems, quasi-affine systems and MRAs for both the real line and the periodic formulation. We extend the result of Ron and Shen in [44] concerning quasi-affine systems and affine systems for

$K \geq 0$ as follows. The proof involves ensuring that the frame condition (1.1) is satisfied for both systems.

Proposition 1.4. (Corollary 3.8) *The affine system $X(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R})$ if and only if its quasi-affine counterpart $X_K^q(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R})$. Further, the two systems have identical (Bessel) frame bounds. In particular, the affine system $X(\Psi)$ is a tight frame if and only if the quasi-affine system $X_K^q(\Psi)$ is a tight frame.*

We also show that for a finite set Φ in $L^2(\mathbb{R})$, $E(\Phi)$ being a frame for $V(\Phi)$ is sufficient for the MRA $\{V^k(\Phi)\}$ to be an FMRA with uniform bounds. Establishing this result involves the use of the dilation factor to ensure that the frame condition (1.1) holds across the different scales.

Proposition 1.5. (Proposition 3.9) *Let $\Phi \subset L^2(\mathbb{R})$ be finite. If $E(\Phi)$ is a (Bessel system) frame for $V(\Phi)$, then $E(\{2^{\frac{k}{2}} E_k^l \phi(2^k \cdot) : \phi \in \Phi, l \in \mathcal{L}_k\})$ is a (Bessel system) frame for $V^k(\Phi)$ with the same (Bessel) frame bounds as $E(\Phi)$.*

Proposition 1.6. (Proposition 3.12) *Let $\Phi \subset L^2(\mathbb{R})$ be finite. Let $\{V^k(\Phi)\}$ be an FMRA of $L^2(\mathbb{R})$ and W^k be the orthogonal complement of $V^k(\Phi)$ in $V^{k+1}(\Phi)$. Let $\Psi \subset W^0$ be finite. Then $X(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R})$ if and only if $E(\Psi)$ is a (Bessel system) frame for W^0 with the same (Bessel) frame bounds.*

The above result states that a sufficient and necessary condition for a semi-orthogonal affine system derived from an FMRA to be a frame is the existence of a shift-invariant system to be a frame for W^0. The proof involves the orthogonal decomposition of $L^2(\mathbb{R})$ by the wavelet subspaces and the use of the dilation factor across the scales.

Next, we move on to results similar to Proposition 1.4 for the periodic setting.

Proposition 1.7. (Proposition 3.15) *Fix $K \geq 0$. If the periodic affine system $X_{2\pi}$ is a (Bessel system) frame for $L^2(\mathbb{T})$, then the periodic quasi-affine system $X_{2\pi,K}^q$ is a (Bessel system) frame for $L^2(\mathbb{T})$ with the same (Bessel) frame bounds.*

Proposition 1.8. (Proposition 3.17) *Fix $R \geq K \geq 0$. If the restricted periodic affine system $X_{2\pi}^R$ is a (Bessel system) frame for its closed linear span $V_{2\pi}^R$, then the restricted periodic quasi-affine system $X_{2\pi,K}^{q,R}$ is a (Bessel system) frame for $V_{2\pi}^R$ with the same (Bessel) frame bounds.*

For the construction of wavelet frames in $L^2(\mathbb{T})$, the periodic analogues of the UEP and the OEP are derived in [25]. Here, we extend them to the generalized oblique extension principle (GOEP) for $L^2(\mathbb{T})$. This is in the theme of using appropriate transformation matrices to obtain new wavelet frames from existing ones. Like the periodic UEP, the GOEP based on the MRA $\{V^k(\Phi_k)\}$ with $\Phi_k := \{\phi_k\}$ requires the assumption of

$$\lim_{k\to\infty} 2^k \left|\widehat{\phi}_k(n)\right|^2 = A > 0, \quad n \in \mathbb{Z}^s, \tag{1.27}$$

which ensures that $\{\phi_k\}_{k\in\mathbb{N}}$ eventually covers the frequency domain uniformly as $k \to \infty$.

Theorem 1.9. (Theorem 3.28) *For each $k \geq 0$, let $\Phi_k, \Psi_k \subset V_{2\pi}^{k+1}(\Phi_{k+1})$ with $\Phi_k := \{\phi_k\}$ and $|\Psi_k| = \varrho_k$ satisfying the periodic refinement equation* (1.21) *and periodic wavelet equation* (1.23) *for some $\widehat{H}_{k+1} \in \mathcal{S}(2^{k+1})$ and $\widehat{G}_{k+1} \in \mathcal{S}(2^{k+1})^{\varrho_k \times 1}$ respectively and* (1.27) *holds. Define $\widehat{\Phi}'_k := \widehat{\Theta}_k \widehat{\Phi}_k$ and $\widehat{\Psi}'_k := \widehat{\Omega}_k \widehat{\Psi}_k$, where $\widehat{\Theta}_k \in \mathcal{S}(2^k)$ and $\widehat{\Omega}_k \in \mathcal{S}(2^k)^{\varrho'_k \times \varrho_k}$ with $\widehat{\Theta}_k(n) \neq 0$ and $\lim_{k\to\infty} \left|\widehat{\Theta}_k(n)\right|^2 = 1$ for every $n \in \mathbb{Z}$. Suppose that for every $k \geq 0$, the $(\varrho'_k + 1) \times 2$ matrix*

$$\widehat{\mathbb{L}}'_k := \operatorname{diag}(\widehat{\Theta}_k, \widehat{\Omega}_k)\widehat{\mathbb{L}}_k \operatorname{diag}(\widehat{\Theta}_{k+1}^{-1}, \widehat{\Theta}_{k+1}^{-1}) \tag{1.28}$$

with $\widehat{\mathbb{L}}_k(j) := \begin{bmatrix} \widehat{L}_k(j) & \widehat{L}_k(j+2^k) \end{bmatrix}$ satisfies $\widehat{\mathbb{L}}_k'^{*}\widehat{\mathbb{L}}'_k = 2I_2$. Then the periodic affine system $X'_{2\pi} := \Phi'_0 \cup \{T_k^l \psi'_k : \psi'_k \in \Psi'_k, l \in \mathcal{L}_k, k \geq 0\}$ forms a tight wavelet frame with frame bound A for $L^2(\mathbb{T})$ derived from the MRA $\{V_{2\pi}^k(\Phi'_k)\}_{k\geq 0}$.

Suitable choices for $\widehat{\Omega}_k$ are matrices with unitary columns. The choice of $\widehat{\Theta}_0(0) = 1$ and $\widehat{\Omega}_k = I_{\varrho_k}$ for every $k \geq 0$ gives the periodic OEP, while the choice of $\widehat{\Theta}_k = 1$ and $\widehat{\Omega}_k = I_{\varrho_k}$ for every $k \geq 0$ gives the periodic UEP.

We define the *polyphase harmonics* of a function $\varphi \in L^2(\mathbb{T})$ at level K for $j \in \mathcal{R}_K := \{0, \ldots, 2^K - 1\}$ to be

$$\varphi_{K,j}(t) := \sum_{n\in\mathbb{Z}} \widehat{\varphi}(j + 2^K n) \mathrm{e}^{\mathrm{i}(j+2^K n)t}.$$

We also introduce subspaces of polyphase harmonics, i.e. $\Theta_{2\pi}^{K,j} := \{f_{K,j} : f \in L^2(\mathbb{T})\}$, $V_{2\pi}^{K,j} := \Theta_{2\pi}^{K,j} \cap V_{2\pi}^K$ and also $W_{2\pi}^{K,j} := \Theta_{2\pi}^{K,j} \cap W_{2\pi}^K$. For periodization purposes, we let

$$L^{2,\alpha}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f(t) = O((1 + |t|)^{-(1+\alpha)}), \alpha > 0\}$$

Next, for $k \geq 0$, we define the periodization of functions φ, $\varphi_k \in L^{2,\alpha}(\mathbb{R})$ at $\omega \in \mathbb{T}^s$ by

$$\varphi_{\omega,k}(t) = \sum_{n\in\mathbb{Z}} \widehat{\varphi_{\omega,k}}(n) \mathrm{e}^{\mathrm{i}nt}$$

with $\widehat{\varphi_{\omega,k}}(n) := 2^{-\frac{k}{2}}\widehat{\varphi}(2^{-k}(\omega + 2\pi n))$ for the stationary case and $\widehat{\varphi_{\omega,k}}(n) := \widehat{\varphi}_k(\omega + 2\pi n)$ for the nonstationary case. We employ similar notations for the periodization of a finite set of functions. We also denote

$$V^K_{2\pi,\omega}(\Lambda_{\omega,k}) := V^K_{2\pi}(\Lambda_{\omega,k}) := \overline{\operatorname{span}}\, T_K(\Lambda_{\omega,k}) \quad \text{and} \quad V^{K,j}_{2\pi,\omega}(\Lambda_{\omega,k}) := V^{K,j}_{2\pi}(\Lambda_{\omega,k})$$

and we leave out writing the generating set $\Lambda_{\omega,k}$ when the space $V^K_{2\pi,\omega}$ could be inferred from its context. Here, we make use of the Poisson Summation Formula (see [42]) which shows that periodization in the time domain is equivalent to sampling in the frequency domain. We state the connection results of MRAs on the real line with that of the periodic case. The proof of the following result involves verifying both the MRA conditions of $L^2(\mathbb{R})$ and that of $L^2(\mathbb{T})$.

Theorem 1.10. (Theorem 3.43) *For $k \geq 0$, let $\Phi_k \subset L^{2,\alpha}(\mathbb{R})$ be finite. The collection $\{V^k(\Phi_k)\}$ is an MRA of $L^2(\mathbb{R})$ if and only if $\{V^k_{2\pi}(\Phi_{\omega,k})\}$ is an MRA of $L^2(\mathbb{T})$ for almost every $\omega \in \mathbb{T}$. In particular, $\{V^k(\Phi_k)\}$ is an FMRA of $L^2(\mathbb{R})$ if and only if $\{V^k_{2\pi}(\Phi_{\omega,k})\}$ is a periodic FMRA of $L^2(\mathbb{T})$ with the same bounds for almost every $\omega \in \mathbb{T}$.*

Let $\eta_k := \operatorname{ess\,sup}\{\eta_{\omega,k} : \omega \in \mathbb{T}\}$ with $\eta_{\omega,k} := \max\{\dim W^{k,j}_{2\pi,\omega} : j \in \mathcal{R}_k\}$, where $W^{k,j}_{2\pi,\omega} = W^k_{2\pi,\omega} \cap \Theta^{k,j}_{2\pi}$ and here $W^k_{2\pi,\omega}$ is the orthogonal complement of $V^k_{2\pi,\omega}$ in $V^{k+1}_{2\pi,\omega}$. The collection $\{\eta_k\}_{k\geq 0}$ is known as the *index* of an FMRA $\{V^k(\Phi_k)\}$ for the nonstationary case. For a stationary FMRA $\{V^k(\Phi)\}$ it suffices to consider its index as η_0,

Theorem 1.11. (Corollary 3.47) *For $k \geq 0$, let $\Phi_k \subset L^{2,\alpha}(\mathbb{R})$ be finite. Suppose that $\{V^k(\Phi_k)\}$ is an FMRA of $L^2(\mathbb{R})$. Let W^k be the orthogonal complement of $V^k(\Phi_k)$ in $V^{k+1}(\Phi_{k+1})$. There exists $\Psi_k = \{\psi^m_k\}^{\eta_k}_{m=1} \subset W^k$ such that $E_k(\Psi_k)$ is a tight frame for W^k with $\langle E^l_k\psi^m_k, E^r_k\psi^n_k\rangle = 0$ for all $m, n = 1, \ldots, \eta_k$, $m \neq n$ and $l, r \in \mathbb{Z}^s$.*

The proof of the above result involves obtaining the existence of $\Psi_{\omega,k}$ such that $T_k(\Psi_{\omega,k})$ is a tight frame for $W^k_{2\pi,\omega}$ for almost every $\omega \in \mathbb{T}$. The minimum number of wavelets required is determined by computing $\dim W^k_{2\pi,\omega}$ through the use of $\eta_{\omega,k}$ for almost every $\omega \in \mathbb{T}$.

Theorem 1.12. (Theorem 3.48) *For $k \geq 0$, let $\Phi_k \subset L^{2,\alpha}(\mathbb{R})$ be finite. Suppose that $\{V^k(\Phi_k)\}$ is an FMRA of $L^2(\mathbb{R})$ with index $\{\eta_k\}_{k\geq 0}$. Let W^k be the orthogonal complement of $V^k(\Phi_k)$ in $V^{k+1}(\Phi_{k+1})$ and ρ_k be the number of pre-wavelets in W^k. Then the following are equivalent for each $k \geq 0$.*

(i) *The set $\Sigma_{\varrho_k} := \bigcup_{j\in\mathcal{R}_k}\{\omega \in \mathbb{T} : \dim W^{k,j}_{2\pi,\omega} > \varrho_k\}$ is of measure zero.*

(ii) *There holds* $\eta_k \leq \varrho_k$.

(iii) *There exists* $\Psi_k = \{\psi_k^m\}_{m=1}^{\varrho_k} \subset W^k$ *with* $\langle E_k^l \psi_k^m, E_k^r \psi_k^n \rangle = 0$ *for all* $m, n = 1, \ldots, \varrho_k$, $m \neq n$ *and* $l, r \in \mathbb{Z}^s$ *such that* $E_k(\Psi_k)$ *is a tight frame for* W^k.

(iv) *There exists* $\Psi_k = \{\psi_k^m\}_{m=1}^{\varrho_k} \subset W^k$ *such that* $E_k(\Psi_k)$ *is a frame for* W^k.

For a stationary FMRA, it suffices to compute $\dim W^0_{2\pi,\omega}$ for almost every $\omega \in \mathbb{T}$ to determine the minimum number of wavelets.

Corollary 1.13. (Corollary 3.49) *Let* $\Phi \subset L^{2,\alpha}(\mathbb{R})$ *be finite. Suppose that* $\{V^k(\Phi)\}$ *is an FMRA of* $L^2(\mathbb{R})$ *with index* η_0. *Let* W^k *be the orthogonal complement of* $V^k(\Phi)$ *in* $V^{k+1}(\Phi)$. *Then the following are equivalent.*

(i) *The set* $\Sigma_{\varrho_0} := \{\omega \in \mathbb{T} : \dim W^{0,0}_{2\pi,\omega} > \varrho_0\}$ *is of measure zero.*

(ii) *There holds* $\eta_0 \leq \varrho_0$.

(iii) *There exists* $\Psi = \{\psi^m\}_{m=1}^{\varrho_0} \subset W^0$ *with* $\langle E^l \psi^m, E^r \psi^n \rangle = 0$ *for all* $m, n = 1, \ldots, \varrho_0$, $m \neq n$ *and* $l, r \in \mathbb{Z}^s$ *such that* $E(\Psi)$ *is a tight frame for* W^0.

(iv) *There exists* $\Psi = \{\psi^m\}_{m=1}^{\varrho_0} \subset W^0$ *such that* $E(\Psi)$ *is a frame for* W^0.

For the general case of MRA wavelets (i.e. the underlying MRA is not an FMRA), we could also examine the connection between the constructions based on the UEP of $L^2(\mathbb{R})$ with those based on the UEP of $L^2(\mathbb{T})$. To this end, it suffices to ensure that the periodic UEP holds for the periodized affine system X_ω.

Theorem 1.14. (Theorem 3.52) *Let* $\Phi \subset L^{2,\alpha}(\mathbb{R})$ *be finite. The affine system* $X(\Psi)$ *is a tight frame for* $L^2(\mathbb{R})$ *obtained from the MRA* $\{V^k(\Phi)\}$ *by the UEP if and only if the corresponding periodized affine system* X_ω *is a tight frame for* $L^2(\mathbb{T})$ *obtained from the MRA* $V^k_{2\pi}(\Phi_{\omega,k})$ *by the periodic UEP for almost every* $\omega \in \mathbb{T}$.

With the above results concerning the connection of real line wavelets with periodic wavelets, we shall look at periodic constructions of wavelets in Chapter 4. We begin first with bandlimited constructions of wavelet masks. First, let

$$\tilde{\beta}_k^n(j) := \beta\left(\frac{N_{k,n} j}{L_{k,n} - N_{k,n}}\right),$$

where β is the cumulative distribution function of a Beta distribution and $0 \leq N_{k,n} < L_{k,n} < N_{k,n+1}$ for $n \in \{1, \ldots, \varrho_k + 1\}$, are used to indicate the bandwidths of our refinement and wavelet masks. We have $\tilde{\beta}_k^n\left(\frac{N_{k,n}}{N_{k,n}}-1\right)=\beta(0)=0$ and $\tilde{\beta}_k^n\left(\frac{L_{k,n}}{N_{k,n}}-1\right)=\beta(1)=1$ for $N_{k,n}<L_{k,n}$. For purposes of convenience, we shall also refer to the refinable mask as $\widehat{g}_{k+1}^0$ for $k\geq 0$.

Construction 1.15. (Construction 4.1) *For* $k\geq 0$, *let* $\phi_k=\sum_{n=-L_{k,1}}^{L_{k,1}}\widehat{\phi}_k(n)\mathrm{e}^{-\mathrm{i}n\cdot}$, *where*

$$\widehat{\phi}_k(j)=\begin{cases} 2^{-\frac{k}{2}} & \text{if } j\in\{-N_{k,1},\dots,N_{k,1}\},\\ 2^{-\frac{k}{2}}\cos\left[\frac{\pi}{2}\tilde{\beta}_k^1\left(\frac{|j|}{N_{k,1}}-1\right)\right] & \text{if } \begin{array}{l} j\in\{-L_{k,1},\dots,-N_{k,1}-1\}\\ \cup\{N_{k,1}+1,\dots,L_{k,1}\},\end{array}\\ 0 & \text{otherwise},\end{cases}$$

and $L_{k,1}<N_{k+1,1}$ *and* $L_{k,1}\leq 2^k$. *For* $k\geq 0$, *let*

$$\widehat{h}_{k+1}(j)=\begin{cases} \sqrt{2} & \text{if } j\in\{-N_{k,1},\dots,N_{k,1}\},\\ \sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}_k^1\left(\frac{|j|}{N_{k,1}}-1\right)\right] & \text{if } \begin{array}{l} j\in\{-L_{k,1},\dots,-N_{k,1}-1\}\\ \cup\{N_{k,1}+1,\dots,L_{k,1}\},\end{array}\\ 0 & \text{if } j\in\mathcal{R}_{k+1}\backslash\{-L_{k,1},\dots,L_{k,1}\}.\end{cases}$$

Using the above refinement mask and refinable function, we first construct bandlimited complex wavelets since they require less conditions to be fulfilled.

Construction 1.16. (Construction 4.4) *For* $n\in\{1,\dots,\varrho_k\}$, *let*

$$\widehat{g}_{k+1}^n(j)=\begin{cases} \sqrt{2}\sin\left[\frac{\pi}{2}\tilde{\beta}_k^n\left(\frac{|j|}{N_{k,n}}-1\right)\right] & \text{if } j\in\{N_{k,n}+1,\dots,L_{k,n}\},\\ \sqrt{2} & \text{if } j\in\{L_{k,n},\dots,N_{k,n+1}\},\\ \sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}_k^{n+1}\left(\frac{|j|}{N_{k,n+1}}-1\right)\right] & \text{if } j\in\{N_{k,n+1}+1,\dots,L_{k,n+1}\},\\ 0 & \text{if } j\in\mathcal{R}_{k+1}\backslash\{N_{k,n}+1,\dots,L_{k,n+1}\},\end{cases}$$

with the conditions $0\leq N_{k,n}\leq L_{k,n}<N_{k,n+1}$, $L_{k,n+1}-N_{k,n}<2^k$, $N_{k,\varrho_k+1}=2^{k+1}-L_{k,1}$ *and* $L_{k,\varrho_k+1}=2^{k+1}-N_{k,1}$ *and the additional condition* $L_{k,n+1}\leq L_{k+1,1}$ *or* $N_{k,n}\geq 2^{k+1}-L_{k+1,1}$ *if* $L_{k+1,1}<2^k$.

Proposition 1.17. (Proposition 4.5) *The periodic affine system* $X_{2\pi}$ *constructed from the refinement and wavelet masks* $\widehat{h}_{k+1}$ *and* $\widehat{g}_{k+1}^n$, $n\in\{1,\dots,\varrho_k\}$, *in Constructions* 1.15 *and* 1.16 *satisfy the periodic UEP* (*Theorem* 1.9) *and forms a tight frame for* $L^2(\mathbb{T})$. *The masks generally have smooth decay with overlapping supports that can be controlled.*

All our subsequent constructions of symmetric and antisymmetric real bandlimited wavelets are based on variations of the above constructions with additional conditions imposed to ensure the periodic UEP holds. We refer the reader to Section 4.1 for the details. Next, we look at time-localized constructions which are modifications of existing wavelet systems so that they contain modulation information in an MRA structure.

Construction 1.18. (Construction 4.16) *For* $0 \le k < K$, *define* $\widehat{\widetilde{\Phi}}_k := L^{\frac{k}{2}}\widehat{\Phi}_k$ *and* $\widehat{\widetilde{\Psi}}_k := \widehat{\widetilde{G}}_{k+1}\widehat{\widetilde{\Phi}}_{k+1}$, *where the combined MRA mask* $\widehat{\widetilde{L}}_{k+1}(j) := \begin{bmatrix} \widehat{\widetilde{g}}^{0,0,0}_{k+1}(j) \\ \widehat{\widetilde{G}}_{k+1}(j) \end{bmatrix}$ *is a* $2(\varrho_k+1)L\times 1$ *vector with* $\widehat{\widetilde{G}}_{k+1}(j) := \begin{bmatrix} \widehat{\widetilde{g}}^{0,0,1}_{k+1}(j) \\ \left[\widehat{\widetilde{g}}^{0,\mu}_{k+1}(j)\right]_{\mu=1}^{L-1} \\ \left[\widehat{\widetilde{g}}^{m}_{k+1}(j)\right]_{m=1}^{\varrho_k} \end{bmatrix}$, $\widehat{\widetilde{g}}^{m}_{k+1}(j) := \left[\widehat{\widetilde{g}}^{m,\mu}_{k+1}(j)\right]_{\mu=0}^{L-1}$ *and*

$$\widehat{\widetilde{g}}^{m,\mu}_{k+1}(j) = \begin{bmatrix} \widehat{\widetilde{g}}^{m,\mu,0}_{k+1}(j) \\ \widehat{\widetilde{g}}^{m,\mu,1}_{k+1}(j) \end{bmatrix} = (2L)^{-\frac{1}{2}} \begin{bmatrix} \widehat{g}^{m}_{k+1}(j - C_k\mu) \\ \widehat{g}^{m}_{k+1}(j + C_k\mu) \end{bmatrix},$$

for $m \in \{0,\dots,\varrho_k\}$, $\mu \in \{0,\dots,L-1\}$ *and* $j \in \mathcal{R}_{k+1}$ *and we let* $C_kL = 2^k$ *with* $\log_2 L$ *being a nonnegative integer. For* $k \ge K$, *define* $\widehat{\widetilde{\Phi}}_k := L^{\frac{K}{2}}\widehat{\Phi}_k$ *and* $\widehat{\widetilde{\Psi}}_k := L^{\frac{K}{2}}\widehat{\Psi}_k$ *with* $\widehat{\widetilde{L}}_{k+1}(j) := \widehat{L}_{k+1}(j)$ *as the original combined MRA mask.*

Theorem 1.19. (Theorem 4.17) *Let the affine system* $X_{2\pi}$ *be a tight frame for* $L^2(\mathbb{T})$ *derived from the periodic UEP with* $\{V^k_{2\pi}(\Phi_k)\}_{k\ge 0}$ *as the underlying MRA of* $L^2(\mathbb{T})$. *Suppose that* $\widetilde{\Phi}_k$ *and* $\widetilde{\Psi}_k$ *with the combined MRA mask* $\widehat{\widetilde{L}}_{k+1} := \left[\widehat{\widetilde{g}}^{m}_{k+1}\right]_{m=0}^{\varrho_k}$ *are constructed as in Construction* 1.18. *Then* $\widetilde{X}_{2\pi} := \{\phi_0\} \cup \{T^l_k\widetilde{\psi}_k : \widetilde{\psi}_k \in \widetilde{\Psi}_k, l \in \mathcal{L}_k, k \ge 0\}$ *is a tight frame for* $L^2(\mathbb{T})$ *derived from the same MRA* $\{V^k_{2\pi}(\Phi_k)\}_{k\ge 0}$ *using the periodic UEP.*

If the original masks lack symmetry, we are able to introduce symmetry and antisymmetry by means of unitary transformation and by making this change of definition, i.e. $\widehat{\widetilde{g}}^{m,\mu,0}_{k+1}(j) = (2L)^{-\frac{1}{2}}\widehat{g}^{m}_{k+1}(-j - C_k\mu)$. We refer the reader to Section 4.2 for the details.

It is required in the above construction that the modulation range be bounded in order for the wavelet system to be a tight frame. This is remedied by using the idea of splitting the wavelet subbands into "packets" using a different set of masks.

Construction 1.20. (Construction 4.22) *For* $0 \le k < K$, *define* $\widehat{\widetilde{\Phi}}_k := \widehat{\Phi}_k := \widehat{\phi}_k$ *and* $\widehat{\widetilde{\Psi}}_k := \widehat{\Psi}_k$ *with* $\widehat{\widetilde{L}}_{k+1} := \widehat{L}_{k+1}$ *being the original combined MRA mask. For* $k \ge K$, *define* $\widehat{\widetilde{\Phi}}_k := \widehat{\Phi}_k := \widehat{\phi}_k$ *and* $\widehat{\widetilde{\Psi}}_k := \widehat{\widetilde{G}}_{k+1}\widehat{\widetilde{\Phi}}_{k+1}$, *where the combined MRA mask*

$$\widehat{\widetilde{L}}_{k+1}(j) := \begin{bmatrix} \widehat{g}^{0}_{k+1}(j) \\ \widehat{\widetilde{G}}_{k+1}(j) \end{bmatrix} \text{ with } \widehat{\widetilde{G}}_{k+1}(j) := \left[\widehat{\widetilde{g}}^{m}_{k+1}(j)\right]_{m=1}^{\varrho_k},\ \widehat{\widetilde{g}}^{m}_{k+1}(j) := \left[\widehat{\widetilde{g}}^{m,\mu}_{k+1}(j)\right]_{\mu=0}^{r_k-1},$$

$\widehat{\widehat{g}}^{m,\mu}_{k+1}(j) := \widehat{\alpha}^{m,\mu}_k(j)\widehat{g}^m_{k+1}(j)$, *for* $m \in \{1,\dots,\varrho_k\}$ *and* $\mu \in \{0,\dots,r_k-1\}$ *and* $j \in \mathcal{R}_{k+1}$ *with* $\widehat{\alpha}^{m,\mu}_k \in \mathcal{S}(2^k)$ *and* $\sum_{\mu=0}^{r_k-1} |\widehat{\alpha}^{m,\mu}_k(\nu)|^2 = 1$ *for all* $\nu \in \mathcal{R}_k$.

Theorem 1.21. (Theorem 4.23) *Let the affine system* $X_{2\pi}$ *as defined in* (1.15) *be a tight frame for* $L^2(\mathbb{T})$ *derived from the periodic UEP with* $\{V^k_{2\pi}(\Phi_k)\}_{k\ge 0}$ *as the underlying MRA of* $L^2(\mathbb{T})$ *and* $\widehat{L}_{k+1} := \left[\widehat{g}^m_{k+1}\right]^{\varrho_k}_{m=0}$ *as the combined MRA mask. Suppose that* $\widetilde{\Phi}_k$ *and* $\widetilde{\Psi}_k$ *with the combined MRA mask* $\widehat{\widehat{L}}_{k+1} := \left[\widehat{\widehat{g}}^m_{k+1}\right]^{\varrho_k}_{m=0}$ *are constructed as in Construction* 1.20. *Then* $\widetilde{X}_{2\pi} := \{\phi_0\} \cup \{T^l_k\widetilde{\psi}_k : \widetilde{\psi}_k \in \widetilde{\Psi}_k, l \in \mathcal{L}_k, k \ge 0\}$ *is a tight frame for* $L^2(\mathbb{T})$ *derived from the same MRA* $\{V^k_{2\pi}(\Phi_k)\}_{k\ge 0}$ *using the periodic UEP.*

The observant reader will notice that it is actually possible to leverage on portions of the different constructions to derive other constructions. We have indeed shown this in an example at the end of Chapter 4.

In order to apply the wavelet frames in $L^2(\mathbb{T})$ to practical problems, in Chapter 5, we first obtain results concerning the periodic decomposition and reconstruction algorithms using polyphase harmonics of ϕ_k and Ψ_k. Let us define $v_{k,j} := (\phi_k)_{k,j}$ and $u^m_{k,j} := (\psi^m_k)_{k,j}$ for $m \in \{1,\dots,\varrho_k\}$. Let $f_{k+1} = f_k + g_k \in V^{k+1}_{2\pi}$, where $f_k = \sum_{j\in\mathcal{R}_k} \widehat{s}_k(j)^* v_{k,j} \in V^k_{2\pi}$ and $g_k = \sum_{j\in\mathcal{R}_k} \widehat{t}_k(j)^* u_{k,j} \in W^k_{2\pi}$ for some $\widehat{s}_k \in \mathcal{S}(2^k)$ and $\widehat{t}_k \in \mathcal{S}(2^k)^{\varrho_k\times 1}$, which are the discrete Fourier transforms of $s_k \in \mathcal{S}(2^k)$ and $t_k \in \mathcal{S}(2^k)^{\varrho_k\times 1}$ and $V^k_{2\pi} = \bigoplus^{\perp}_{j\in\mathcal{R}_k} \operatorname{span}\{v_{k,j}\}$ and $W^k_{2\pi} = \bigoplus^{\perp}_{j\in\mathcal{R}_k} \operatorname{span}\{u^m_{k,j} : m = 1,\dots,\varrho_k\}$. Our results are obtained chiefly by making use of the perfect reconstruction condition and anti-aliasing condition of the UEP, i.e. $\widehat{\mathbb{L}}_k(j)^*\widehat{\mathbb{L}}_k(j) = 2I_2$ with $j \in \mathcal{R}_k$.

Proposition 1.22. (Proposition 5.10) *If*

$$\begin{aligned} f_{k+1} &= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}(j+2^k r)^* v_{k+1,j+2^k r} \\ &= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \left[\widehat{s}_k(j)^*\widehat{H}_{k+1}(j+2^k r) + \widehat{t}_k(j)^*\widehat{G}_{k+1}(j+2^k r)\right] v_{k+1,j+2^k r}, \end{aligned} \tag{1.29}$$

then there exists $\left[\widehat{\widehat{s}}_k(j)^* \quad \widehat{\widehat{t}}_k(j)^*\right]^* \in \operatorname{Ker}\left[\widehat{\mathbb{L}}_k(j)\operatorname{diag}\left[\mathbf{1}_{\operatorname{supp}\widehat{\phi}_{k+1}}(j+2^k r)\right]^1_{r=0}\right]^*$ *for each* $j \in \mathcal{R}_k$ *such that*

$$2\left[\widehat{s}_k - \widehat{\widehat{s}}_k\right]^* \mathbf{1}_{\operatorname{supp}\widehat{\phi}_k}(j) = \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}^* \mathbf{1}_{\operatorname{supp}\widehat{\phi}_{k+1}} \widehat{H}^*_{k+1}(j+2^k r)\mathbf{1}_{\operatorname{supp}\widehat{\phi}_k}(j),$$

$$\begin{aligned} &2\left[\widehat{t}_k - \widehat{\widehat{t}}_k\right]^* \operatorname{diag}\left[\mathbf{1}_{\operatorname{supp}\widehat{\psi}^m_k}\right]^{\varrho_k}_{m=1}(j) \\ &= \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}^* \mathbf{1}_{\operatorname{supp}\widehat{\phi}_{k+1}} \widehat{G}^*_{k+1}(j+2^k r)\operatorname{diag}\left[\mathbf{1}_{\operatorname{supp}\widehat{\psi}^m_k}\right]^{\varrho_k}_{m=1}(j), \end{aligned}$$

and for $r \in \mathcal{R}_1$,

$$\widehat{s_{k+1}}^* \mathbf{1}_{\operatorname{supp}\widehat{\phi}_{k+1}}(j{+}2^k r) = \Big[\widehat{s_k}^* \mathbf{1}_{\operatorname{supp}\widehat{\phi}_k}(j)\widehat{H}_{k+1}(j{+}2^k r) \\ +\widehat{t_k}^* \operatorname{diag}\left[\mathbf{1}_{\operatorname{supp}\widehat{\psi}_k^m}\right]_{m=1}^{\varrho_k}(j)\widehat{G}_{k+1}(j{+}2^k r)\Big]\mathbf{1}_{\operatorname{supp}\widehat{\phi}_{k+1}}(j{+}2^k r).$$

For convenience, we define

$$\widehat{H}'_{k+1}(j+2^k r) = \mathbf{1}_{\operatorname{supp}\widehat{\phi}_k}(j)\widehat{H}_{k+1}(j+2^k r)\mathbf{1}_{\operatorname{supp}\widehat{\phi}_{k+1}}(j+2^k r),$$
$$\widehat{G}'_{k+1}(j+2^k r) = \operatorname{diag}\left[\mathbf{1}_{\operatorname{supp}\widehat{\psi}_k^m}\right]_{m=1}^{\varrho_k}(j)\widehat{G}_{k+1}(j+2^k r)\mathbf{1}_{\operatorname{supp}\widehat{\phi}_{k+1}}(j+2^k r),$$
$$\widehat{s_k}'(j) = \mathbf{1}_{\operatorname{supp}\widehat{\phi}_k}(j)\widehat{s_k}(j), \quad \widehat{t_k}'(j) = \operatorname{diag}\left[\mathbf{1}_{\operatorname{supp}\widehat{\psi}_k^m}\right]_{m=1}^{\varrho_k}(j)\widehat{t_k}(j),$$
$$\widehat{s_{k+1}}'(j+2^k r) = \mathbf{1}_{\operatorname{supp}\widehat{\phi}_{k+1}}(j+2^k r)\widehat{s_{k+1}}(j+2^k r).$$

We shall also let

$$\widehat{\mathbb{L}}'_k(j) = \operatorname{diag}(\mathbf{1}_{\operatorname{supp}\widehat{\phi}_k}(j), \operatorname{diag}\left[\mathbf{1}_{\operatorname{supp}\widehat{\psi}_k^m}\right]_{m=1}^{\varrho_k}(j))\widehat{\mathbb{L}}_k(j)\operatorname{diag}\left[\mathbf{1}_{\operatorname{supp}\widehat{\phi}_{k+1}}(j+2^k r)\right]_{r=0}^{1}.$$

The *upsampling operator* $\uparrow_k\colon \mathcal{S}(2^k) \to \mathcal{S}(2^{k+1})$ is given by

$$\uparrow_k\colon \{s_k(l)\}_{l\in\mathcal{L}_k} \mapsto \{\uparrow_k s_k(r)\}_{r\in\mathcal{L}_{k+1}} := \{s_k(l)\mathbf{1}_{\{r=2l\}}\}_{r\in\mathcal{L}_{k+1}}.$$

We shall also write the composition $\uparrow_K^{K+k}\colon \mathcal{S}(2^K) \mapsto \mathcal{S}(2^{K+k})$ as

$$\uparrow_K^{K+k} := \uparrow_{K+k-1}\uparrow_{K+k-2}\cdots\uparrow_K .$$

We define the periodic convolution $\otimes : \mathcal{S}(2^k)\times\mathcal{S}(2^k) \to \mathcal{S}(2^k)$ of $a_k \in \mathcal{L}_k$ and $b_k \in \mathcal{L}_k$ as

$$\{a_k \otimes b_k(l)\}_{l\in\mathcal{L}_k} = \left\{\sum_{r\in\mathcal{L}_k} a_k(l-r)b_k(r)\right\}_{l\in\mathcal{L}_k}.$$

In the time domain, Proposition 1.22 is given as

Proposition 1.23. (Proposition 5.11) *If* f_{k+1} *is given by* (1.29), *then*

$$f_{k+1} = \sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l)^* T_{k+1}^l \phi_{k+1} = \sum_{l\in\mathcal{L}_k}\left[s_k(l)^* T_k^l\phi_k + t_k(l)^* T_k^l \Psi_k\right]$$
$$= \sum_{l\in\mathcal{L}_{k+1}}\left[(\uparrow_k s_k^*)\otimes H_{k+1}(l) + (\uparrow_k t_k^*)\otimes G_{k+1}(l)\right] T_{k+1}^l\phi_{k+1}.$$

Further, there exist $\left[\widehat{\widetilde{s}_k}'(j)^* \ \ \widehat{\widetilde{t}_k}'(j)^*\right]^* \in \operatorname{Ker}\widehat{\mathbb{L}}'_k(j)^*$, $j \in \mathcal{R}_k$, *such that for every* $l \in \mathcal{L}_k$ *and* $n \in \mathcal{L}_{k+1}$,

$$\left[s'_k(l) - \widetilde{s}_k{}'(l)\right]^* = [s'^*_{k+1}\otimes H'^*_{k+1}](2l),$$
$$\left[t'_k(l) - \widetilde{t}_k{}'(l)\right]^* = [s'^*_{k+1}\otimes G'^*_{k+1}](2l),$$
$$s'_{k+1}(n)^* = [(\uparrow_k s'^*_k)\otimes H'_{k+1}](n) + [(\uparrow_k t'^*_k)\otimes G'_{k+1}](n).$$

Next, we state the quasi-affine representation for the stationary wavelet transform. Notice that the representation is translation invariant due to the absence of downsampling.

Proposition 1.24. (Proposition 5.17) *Fix* $0 \leq K \leq K+L$. *If*

$$f_{K+L} = \sum_{j \in \mathcal{R}_{K+L-1}} \sum_{r \in \mathcal{R}_1} \widehat{s_{K+L}}(j + 2^{K+L-1}r)^* v_{K+L, j+2^{K+L-1}r},$$

then for a given $\delta \in \mathcal{L}_L$,

$$\begin{aligned}
f_{K+L} &= \sum_{l \in \mathcal{L}_{K+L}} s_{K+L}(l+\delta)^* T_{K+L}^{l+\delta} \phi_{K+L} \\
&= \sum_{l \in \mathcal{L}_K} s_K^\delta(l)^* T_{K+L}^{2^L l+\delta} \phi_K + \sum_{k=K}^{K+L} \sum_{l \in \mathcal{L}_k} t_k^\delta(l)^* T_{K+L}^{2^{K+L-k}l+\delta} \Psi_k \\
&= \sum_{\delta_L \in \mathcal{L}_L} \sum_{l \in \mathcal{L}_K} 2^{-L} a_K(2^L l + \delta_L)^* T_{K+L}^{2^L l+\delta_L} \phi_K + \sum_{k=K}^{K+L} \sum_{\delta_k \in \mathcal{L}_k} \sum_{l \in \mathcal{L}_{K+L-k}} 2^{-k} b_{K+L-k}(2^k l + \delta_k)^* T_{K+L}^{2^k l+\delta_k} \Psi_{K+L-k} \\
&= \sum_{l \in \mathcal{L}_{K+L}} 2^{-L} a_K(l)^* T_{K+L}^l \phi_K + \sum_{k=K}^{K+L} \sum_{l \in \mathcal{L}_{K+L}} 2^{-k} b_{K+L-k}(l)^* T_{K+L}^l \Psi_{K+L-k},
\end{aligned}$$

with $a_{k-1} = (\uparrow_k^{K+L} H_k) \otimes a_k$, $b_{k-1} = (\uparrow_k^{K+L} G_k) \otimes a_k$ *for* $k \in \{K, \ldots, K+L\}$ *and* $a_{K+L} = s_{K+L}$. *Further, for every* $k \in \{K, \ldots, K+L-1\}$, *there exist* $\left[\widehat{\widetilde{s_k^\delta}}'(j)^* \ \ \widehat{\widetilde{t_k^\delta}}'(j)^*\right]^* \in \operatorname{Ker} \widehat{\mathbb{L}}_k'(j)^*$, $j \in \mathcal{R}_k$, *such that for every* $l \in \mathcal{L}_k$ *and* $n \in \mathcal{L}_{k+1}$,

$$\begin{aligned}
\left[s_k^{\delta\prime}(l) - \widetilde{s_k^\delta}'(l)\right]^* &= a_k'(2^{K+L-k} l + \delta)^*, \\
\left[t_k^{\delta\prime}(l) - \widetilde{t_k^\delta}'(l)\right]^* &= b_k'(2^{K+L-k} l + \delta)^*, \\
a_{k+1}'(2^{K+L-k-1} n + \delta)^* &= [(\uparrow_k s_k^{\delta\prime *}) \otimes H_{k+1}' + (\uparrow_k t_k^{\delta\prime *}) \otimes G_{k+1}'](n)
\end{aligned}$$

with $\widehat{s_k^\delta}' = \mathbf{1}_{\operatorname{supp} \widehat{\phi}_k} \widehat{s_k^\delta}$, $\widehat{t_k^\delta}' = \operatorname{diag}\left[\mathbf{1}_{\operatorname{supp} \widehat{\psi}_k^m}\right]_{m=1}^{\varrho_k} \widehat{t_k^\delta}$, $\widehat{a_k}'(j + 2^k\nu) = \mathbf{1}_{\operatorname{supp} \widehat{\phi}_k}(j) \widehat{a_k}(j + 2^k \nu)$ *and* $\widehat{b_k}'(j + 2^k \nu) = \operatorname{diag}\left[\mathbf{1}_{\operatorname{supp} \widehat{\psi}_k^m}\right]_{m=1}^{\varrho_k}(j) \widehat{b_k}(j + 2^k\nu)$, *where* $\nu \in \mathcal{R}_{K+L-k}$.

We conclude the thesis with results concerning the approximation properties of our periodic constructions as well as the time-frequency representations they provide. The next result is used to justify the sparsity of representations of bandlimited signals by our bandlimited tight wavelet frame.

Proposition 1.25. (Proposition 5.22) *The tight wavelet frame* $X_{2\pi} := \{\phi_0\} \cup \{T_k^l \psi_k : \psi_k \in \Psi_k, l \in \mathcal{L}_k, k \geq 0\}$ *constructed from the MRA* $\{V_{2\pi}^k(\phi_k)\}$ *via the periodic UEP with* $\{\phi_k\}_{k \geq 0}$ *given in Construction* 4.1 *satisfying* $\liminf_{k \to \infty} 2^{-k} N_{k,1} > 0$, *has spectral frame approximation order. Hence* $X_{2\pi}$ *also has global vanishing moments of arbitrarily high order.*

Towards the end of Chapter 5, we also review the approximation properties of some of our time-localized constructions. Using our decomposition and reconstruction algorithms, examples of time-frequency representations of signals based on our bandlimited constructions are provided. These time-frequency representations highlight the strengths of these nonstationary periodic transforms which capture the features of both the traditional wavelet transform and the short-time Fourier transform. For more details, the reader is referred to Section 5.5 of this thesis.

Chapter 2

Symmetric and Antisymmetric Tight Wavelet Frames

Linear phase filtering is important in that it preserves the relative positions of signals without distortion after convolution, i.e. the filtering process, up to a phase shift. The design of linear phase filters involves the inclusion of symmetry or antisymmetry in the filters. With the exception of the Haar wavelet, real-valued orthogonal conjugate quadrature mirror filters do not preserve linear phase as they are not symmetric. Many constructions sought to remedy this problem by relaxing some restrictions. The resolution of this problem in this thesis, which is published in [23], involves relaxing the orthogonality and non-redundancy condition so that symmetrization of the filters could be performed.

2.1 Symmetric and Antisymmetric Construction

Symmetry is obtained in [11] by using two compactly supported dual refinable functions only one of which could be a spline function. In [10], similar dual symmetric spline wavelet bases are used with only one of them being compactly supported. Symmetry, orthonormality and compact support are achieved in [21] and [20] by using a vector MRA and in [34] by using non-dyadic dilations. In [44], symmetry and compact support are obtained by relaxing the non-redundancy condition with one of the wavelets having a vanishing moment of order one. In [16] and [8], examples of symmetric compactly supported tight wavelet frames with high orders of vanishing moments are obtained but those from systematic constructions are not symmetric. This is remedied in [8], [15], [16] and [27] at the cost of using two dual frame systems. In [29], three compactly supported symmetric or antisymmetric tight frame wavelets are constructed from B-splines with the order of vanishing moments being the same as that of the B-spline. This construction is extended in [30] using compactly supported symmetric functions with stable shifts. In [28] and [41], the authors focus on finding conditions that the refinement and wavelet masks should satisfy for the construction of compactly supported symmetric tight wavelet frames and this reveals the difficulties of obtaining such a systematic construction.

The approach in this thesis is entirely different and overcomes the above difficulties. Our objective is to obtain symmetric and antisymmetric wavelets through appropriate modifications and transformations of known wavelets. The main idea here originates from the following simple, but highly useful, observation for the case on the real line. Consider a wavelet $\psi \in L^2(\mathbb{R})$ that is not symmetric. Assume that the affine system $X(\psi) := \{2^{k/2}\psi(2^k \cdot -l) : k, l \in \mathbb{Z}\}$ generated by ψ forms a tight frame for $L^2(\mathbb{R})$. Let $\Psi' := \{\psi^{1'}, \psi^{2'}\}$, where

$$\psi^{1'} := \frac{1}{2}(\psi + \psi(-\cdot)), \quad \psi^{2'} := \frac{1}{2}(\psi - \psi(-\cdot)).$$

Then $\psi^{1'}$ is symmetric and $\psi^{2'}$ is antisymmetric about the origin. Further, the orders of the smoothness and vanishing moments of ψ are not reduced. It turns out that $X(\Psi') := X(\psi^{1'}) \cup X(\psi^{2'})$ also forms a tight frame for $L^2(\mathbb{R})$. Therefore this method converts any nonsymmetric wavelet that generates an affine tight frame to a pair of symmetric and antisymmetric wavelets that generate an affine tight frame. The idea here can be refined to ensure that the supports of the new wavelets $\psi^{1'}$ and $\psi^{2'}$ are almost the same, if not identical, as that of ψ. In particular, if we begin with an orthonormal basis generated by one wavelet ψ, then the method gives a tight frame generated by two wavelets $\psi^{1'}$ and $\psi^{2'}$ with symmetry and of similar support as ψ. It can also be adjusted easily to suit the case when the original affine tight frame is generated by more than one wavelet. The number of new wavelets is at most twice the number of the original wavelets. The general setup is as follows.

Construction 2.1. *Let* $\Psi := \left[\psi^m\right]_{m=1}^{\varrho} \subset L^2(\mathbb{R}^s)$ *be a finite set of functions. Consider*

$$\Upsilon := \begin{bmatrix} \frac{1}{\sqrt{2}}\psi^m \\ \frac{1}{\sqrt{2}}\psi^m(\kappa_m - \cdot) \end{bmatrix}_{m=1}^{\varrho},$$

where $\kappa_m \in \mathbb{Z}^s$, *as a* $2\varrho \times 1$ *vector arranged in the order of* $\frac{1}{\sqrt{2}}\psi^m$ *followed by* $\frac{1}{\sqrt{2}}\psi^m(\kappa_m - \cdot)$

for $m = 1, \ldots, \varrho$. Define $\Psi' := U_{2\varrho}\Upsilon$, where $U_{2\varrho}$ is the $2\varrho \times 2\varrho$ unitary matrix given by

$$U_{2\varrho} := \begin{bmatrix} U_0 & & \\ & \ddots & \\ & & U_0 \end{bmatrix}, \quad U_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.1}$$

Then Ψ' consists of symmetric and antisymmetric functions, where a typical symmetric function $\frac{1}{2}(\psi^m + \psi^m(\kappa_m - \cdot))$ is symmetric about $\frac{\kappa_m}{2}$ and a typical antisymmetric function $\frac{1}{2}(\psi^m - \psi^m(\kappa_m - \cdot))$ is antisymmetric about $\frac{\kappa_m}{2}$.

The above is a very natural way of obtaining symmetric and antisymmetric functions from a given collection of functions. The main issue here is to show that whenever $X(\Psi)$ is a frame for $L^2(\mathbb{R}^s)$, Construction 2.1 gives a frame $X(\Psi')$ for $L^2(\mathbb{R}^s)$ with the same frame bounds. Our proof will utilize the following elementary lemma obtained from the frame condition (1.1) and a change of variables.

Lemma 2.2. *Let the ordered set $\Psi := \left[\psi^m\right]_{m=1}^{\varrho}$ be a subset of $L^2(\mathbb{R}^s)$. If the affine system $X(\Psi)$ as in (1.4) is a frame for $L^2(\mathbb{R}^s)$, then the affine system $X([\psi^m(\kappa_m - \cdot)]_{m=1}^{\varrho})$, where $\kappa_m \in \mathbb{Z}^s$, is also a frame for $L^2(\mathbb{R}^s)$ with the same frame bounds.*

The next lemma will also be used. Although it is a special case of Theorem 4 in [1], we include its simple proof for completeness.

Lemma 2.3. *Let $\{g_n\}_{n\in K}$ be a frame for $L^2(\mathbb{R}^s)$. Then $\{h_n\}_{n\in K} := \mathcal{U}\{g_n\}_{n\in K}$, where $\mathcal{U}$ is a unitary matrix with finitely many nonzero entries in each row and column, is also a frame for $L^2(\mathbb{R}^s)$ with the same frame bounds as $\{g_n\}_{n\in K}$.*

Proof. The matrix $\mathcal{U}$ defines a unitary operator from $l_2(K)$, the space of all complex square-summable sequences indexed by K, onto $l_2(K)$ by

$$\mathcal{U} : \{c_k\}_{k\in K} \to \{\sum_{k\in K} u_{jk}c_k\}_{j\in K}.$$

Indeed, $\|\mathcal{U}\{c_k\}_{k\in K}\|^2_{l^2(K)} = \|\{c_k\}_{k\in K}\|^2_{l^2(K)}$ for all finite sequences $\{c_k\}_{k\in K}$, which also holds for all sequences in $l_2(K)$ since $\mathcal{U}$ is a bounded linear operator on the densely defined subspace of finite sequences in $l_2(K)$. For $f \in L^2(\mathbb{R}^s)$, since

$$\{\langle h_j, f\rangle\}_{j\in K} = \{\langle \sum_{k\in K} u_{jk}g_k, f\rangle\}_{j\in K} = \{\sum_{k\in K} u_{jk}\langle g_k, f\rangle\}_{j\in K} = \mathcal{U}\{\langle g_k, f\rangle\}_{k\in K},$$

the result follows from the fact that $\mathcal{U}$ is a unitary operator on $l_2(K)$ and the frame condition (1.1). $\square$

Theorem 2.4. *Let* $\Psi := \left[\psi^m\right]_{m=1}^{\varrho}$ *such that the affine system* $X(\Psi)$ *as in* (1.4) *is a frame for* $L^2(\mathbb{R}^s)$*. Let* Ψ' *be constructed from* Ψ *as in Construction* 2.1*. Then the affine system* $X(\Psi')$ *is also a frame for* $L^2(\mathbb{R}^s)$ *with the same frame bounds as* $X(\Psi)$*. In particular, if* $X(\Psi)$ *is a tight frame for* $L^2(\mathbb{R}^s)$*, then* $X(\Psi')$ *is also a tight frame for* $L^2(\mathbb{R}^s)$*.*

Proof. Let $\widetilde{\Psi} := \left[\psi^m(\kappa_m - \cdot)\right]_{m=1}^{\varrho}$, $\kappa_m \in \mathbb{Z}^s$, and Υ be as in Construction 2.1. Lemma 2.2 shows that $X(\widetilde{\Psi})$ is a frame for $L^2(\mathbb{R}^s)$ with the same frame bounds as $X(\Psi)$. When we combine $X(\Psi)$ with $X(\widetilde{\Psi})$ under the appropriate normalization as $X(\Upsilon)$, $X(\Upsilon)$ remains a frame for $L^2(\mathbb{R}^s)$ with the same frame bounds. This is because the frame condition (1.1) implies that

$$A\|f\|^2 \le \sum_{g\in X(\Psi)} \left|\langle f, \frac{1}{\sqrt{2}}g\rangle\right|^2 + \sum_{g\in X(\widetilde{\Psi})} \left|\langle f, \frac{1}{\sqrt{2}}g\rangle\right|^2 \le B\|f\|^2, \quad f \in L^2(\mathbb{R}^s),$$

where A and B are the frame bounds of $X(\Psi)$.

We order the functions in $X(\Upsilon)$ such that the 2ϱ wavelets $\psi^1, \psi^1(\kappa_1 - \cdot), \ldots, \psi^\varrho, \psi^\varrho(\kappa_\varrho - \cdot)$ are always grouped together under the various applications of the dilation matrix M and the shift operator E_k^l. By selecting the same ordering for the functions in $X(\Psi')$, it follows that $X(\Psi') = \mathcal{U}X(\Upsilon)$, where $\mathcal{U}$ is the block diagonal matrix of bi-infinite order with the matrix $U_{2\varrho}$ as the diagonal blocks. Then we apply Lemma 2.3 to $X(\Upsilon)$ to conclude that $X(\Psi')$ is a frame with the same frame bounds as $X(\Upsilon)$. □

2.2 Construction of Framelets

A straightforward calculation gives explicit expressions of the lowpass and highpass filters for refinable functions and wavelets under certain affine transformations. We record them in the following proposition, which will be used in our subsequent construction of symmetric and antisymmetric framelets.

Proposition 2.5. *Let* $\Phi := \left[\phi^m\right]_{m=1}^{\rho}$ *and* $\Psi := \left[\psi^m\right]_{m=1}^{\varrho}$ *satisfy the refinement and wavelet equations in* (1.14) *respectively with matrix filters* $H := \left[H^{m,r}\right]_{m,r=1}^{\rho}$ *and* $G := \left[G^{m,r}\right]_{m=1,r=1}^{\varrho,\rho}$*. Let* $\widetilde{\Phi} := \left[\phi^m(\lambda \cdot + \eta_m)\right]_{m=1}^{\rho}$ *and* $\widetilde{\Psi} := \left[\psi^m(\lambda \cdot + \kappa_m)\right]_{m=1}^{\varrho}$*, where* $\lambda \in \{\pm 1\}$ *and* $\eta_m, \kappa_m \in \mathbb{Z}^s$*. Then*

$$\widetilde{\Phi} = |\det M| \sum_{n\in\mathbb{Z}^s} \widetilde{H}(n)\widetilde{\Phi}(M\cdot - n), \quad \widetilde{\Psi} = |\det M| \sum_{n\in\mathbb{Z}^s} \widetilde{G}(n)\widetilde{\Phi}(M\cdot - n),$$

where $\widetilde{H}(n) := \Big[H^{m,r}(M\eta_m - \eta_r + \lambda n)\Big]_{m,r=1}^{\rho}$ *and* $\widetilde{G}(n) := \Big[G^{m,r}(M\kappa_m - \eta_r + \lambda n)\Big]_{m=1,r=1}^{\varrho,\rho}$ *for* $n \in \mathbb{Z}^s$. *Further,*

$$\begin{aligned}\widehat{\widetilde{H}}(\omega) &= \operatorname{diag}\Big[\mathrm{e}^{\mathrm{i}M\eta_m\cdot\lambda\omega}\Big]_{m=1}^{\rho}\widehat{H}(\lambda\omega)\operatorname{diag}\Big[\mathrm{e}^{-\mathrm{i}\eta_r\cdot\lambda\omega}\Big]_{r=1}^{\rho},\\ \widehat{\widetilde{G}}(\omega) &= \operatorname{diag}\Big[\mathrm{e}^{\mathrm{i}M\kappa_m\cdot\lambda\omega}\Big]_{m=1}^{\varrho}\widehat{G}(\lambda\omega)\operatorname{diag}\Big[\mathrm{e}^{-\mathrm{i}\eta_r\cdot\lambda\omega}\Big]_{r=1}^{\rho}.\end{aligned}$$

Now, consider the affine system $X(\Psi)$ in (1.4) generated by Ψ. Theorem 2.4 shows that if $X(\Psi)$ is a tight frame for $L^2(\mathbb{R}^s)$, then $X(\Psi')$ is also a tight frame for $L^2(\mathbb{R}^s)$, where Ψ' is constructed from Ψ as in Construction 2.1. Given, in addition, that $X(\Psi)$ is derived from an MRA, we are interested to know whether $X(\Psi')$ comes from an MRA, and further, the same MRA or a different MRA. In this connection, we need the following lemma.

Lemma 2.6. *Suppose that* $\{V^k(\Phi)\}$ *is an MRA of* $L^2(\mathbb{R}^s)$, *where* $\Phi := \Big[\phi^m\Big]_{m=1}^{\rho}$. *Let* $\widetilde{\Phi} := \Big[\phi^m(\eta_m - \cdot)\Big]_{m=1}^{\rho}$, *where* $\eta_m \in \mathbb{Z}^s$. *Then* $\{V^k(\Phi \cup \widetilde{\Phi})\}$ *is an MRA of* $L^2(\mathbb{R}^s)$.

Proof. Proposition 2.5 shows that $\widetilde{\Phi}$ is a refinable vector-valued function. By (1.9) for both Φ and $\widetilde{\Phi}$, $\Phi \cup \widetilde{\Phi}$ is also refinable. The density of $\bigcup_{k\in\mathbb{Z}} V^k(\Phi)$ in $L^2(\mathbb{R}^s)$ implies the density of $\bigcup_{k\in\mathbb{Z}} V^k(\Phi \cup \widetilde{\Phi})$. Therefore $\{V^k(\Phi \cup \widetilde{\Phi})\}$ is an MRA of $L^2(\mathbb{R}^s)$. ☐

We shall build upon Construction 2.1 in the following way. Given that $\Psi := \Big[\psi^m\Big]_{m=1}^{\varrho}$ is a vector-valued function satisfying the wavelet equation (1.11) of the MRA $\{V^k(\Phi)\}$ of $L^2(\mathbb{R}^s)$, let $\widetilde{\Phi} := \Big[\phi^m(\eta_m - \cdot)\Big]_{m=1}^{\rho}$ and $\widetilde{\Psi} := \Big[\psi^m(\kappa_m - \cdot)\Big]_{m=1}^{\varrho}$, for some $\eta_m, \kappa_m \in \mathbb{Z}^s$. Then we define

$$\Xi := \frac{1}{\sqrt{2}}\begin{bmatrix}\phi^m\\ \phi^m(\eta_m - \cdot)\end{bmatrix}_{m=1}^{\rho},\ \Upsilon := \frac{1}{\sqrt{2}}\begin{bmatrix}\psi^m\\ \psi^m(\kappa_m - \cdot)\end{bmatrix}_{m=1}^{\varrho},\ \Phi' := U_{2\rho}\Xi,\ \Psi' := U_{2\varrho}\Upsilon, \tag{2.2}$$

where $U_{2\rho}$ and $U_{2\varrho}$ are $2\rho \times 2\rho$ and $2\varrho \times 2\varrho$ block diagonal matrices respectively with the matrix U_0 in (2.1) as their blocks.

Theorem 2.7. *Let* $\Psi := \Big[\psi^m\Big]_{m=1}^{\varrho}$ *be a finite set of tight framelets obtained from the MRA* $\{V^k(\Phi)\}$ *of* $L^2(\mathbb{R}^s)$ *generated by* $\Phi := \Big[\phi^m\Big]_{m=1}^{\rho}$. *Define* Φ' *and* Ψ' *as in (2.2). Then* Ψ' *is a finite set of symmetric or antisymmetric tight framelets obtained from the MRA generated by* Φ'.

Proof. Let $\widetilde{\Phi} := \Big[\phi^m(\eta_m - \cdot)\Big]_{m=1}^{\rho}$ and $\widetilde{\Psi} := \Big[\psi^m(\kappa_m - \cdot)\Big]_{m=1}^{\varrho}$. From Lemma 2.6, we know that $\{V^k(\Xi)\}$ is an MRA of $L^2(\mathbb{R}^s)$. By Proposition 2.5,

$$\widehat{\widetilde{\Phi}}(M^T\cdot) = \widehat{\widetilde{H}}\widehat{\widetilde{\Phi}}, \quad \widehat{\widetilde{\Psi}}(M^T\cdot) = \widehat{\widetilde{G}}\widehat{\widetilde{\Phi}}.$$

Combining with (1.9) and (1.11), we obtain

$$\begin{bmatrix} \widehat{\Phi}(M^T\cdot) \\ \widehat{\widetilde{\Phi}}(M^T\cdot) \end{bmatrix} = \begin{bmatrix} \widehat{H} & 0 \\ 0 & \widehat{\widetilde{H}} \end{bmatrix} \begin{bmatrix} \widehat{\Phi} \\ \widehat{\widetilde{\Phi}} \end{bmatrix}, \quad \begin{bmatrix} \widehat{\Psi}(M^T\cdot) \\ \widehat{\widetilde{\Psi}}(M^T\cdot) \end{bmatrix} = \begin{bmatrix} \widehat{G} & 0 \\ 0 & \widehat{\widetilde{G}} \end{bmatrix} \begin{bmatrix} \widehat{\Phi} \\ \widehat{\widetilde{\Phi}} \end{bmatrix}. \tag{2.3}$$

Rearranging the rows of the vectors in (2.3) based on the ordering in Ξ and Υ gives

$$\widehat{\Xi}(M^T\cdot) = \widehat{P}\widehat{\Xi}, \quad \widehat{\Upsilon}(M^T\cdot) = \widehat{Q}\widehat{\Xi}, \tag{2.4}$$

where $\widehat{P}$ and $\widehat{Q}$ are the refinement and wavelet masks of Ξ and Υ respectively. By Theorem 2.4, $X(\Psi')$ is a tight frame for $L^2(\mathbb{R}^s)$. Note that Φ' generates the same MRA as Ξ with refinement mask $\widehat{H}' := U_{2\rho}\widehat{P}U_{2\rho}^*$ because Φ' is obtained from a unitary transformation of Ξ. Similarly, the wavelet mask of Ψ' is $\widehat{G}' := U_{2\varrho}\widehat{Q}U_{2\rho}^*$ with the tight frame $X(\Psi')$ arising from the MRA $\{V^k(\Phi')\}$. □

In practice, fast wavelet decomposition and reconstruction algorithms are needed. These algorithms exist for tight framelets derived from the *oblique extension principle* (OEP) (see [44], [8] and [16]). In [16], tight framelets are constructed from an MRA generated by a refinable B-spline with the desired approximation order using the OEP. However, the framelets are not symmetric even though B-splines are symmetric. Next, we shall prove that when the refinable function in the OEP is symmetric, Construction 2.1 gives symmetric and antisymmetric tight framelets arising from the same MRA, and the corresponding new fundamental function in the OEP can also be found. Knowing the fundamental function is important in applying the fast decomposition and reconstruction algorithms (see [16]) for tight framelets derived from the OEP.

Before we state the OEP, recall that the *spectrum* of a shift-invariant space $V(\Phi)$ is defined (up to measure zero sets) as

$$\sigma(V(\Phi)) := \{\omega \in \mathbb{T}^s : \sum_{j\in 2\pi\mathbb{Z}^s} |\widehat{\phi}(\omega + j)|^2 > 0 \text{ for some } \phi \in \Phi\},$$

where $\sum_{j\in 2\pi\mathbb{Z}^s} |\widehat{\phi}(\omega + j)|^2$ is well defined for almost every $\omega \in \mathbb{T}^s$ since $\phi \in L^2(\mathbb{R}^s)$. The spectrum of $V(\Phi)$ only depends on the space and is independent of the choice of generators of the space (see [4] and [43]). In all our discussion that follows, we shall assume that every $\phi \in \Phi$ satisfies

$$\sigma(V(\phi)) = \sigma(V(\phi(\eta - \cdot))) \tag{2.5}$$

for some $\eta \in \mathbb{Z}^s$. Equation (2.5) holds when all the functions $\phi \in \Phi$ are compactly supported (since $\sigma(V(\phi)) = \mathbb{T}^s$) or satisfy $\left|\widehat{\phi}(\omega)\right|^2 = \left|\widehat{\phi}(-\omega)\right|^2$ a.e. on $\mathbb{R}^s$, which is valid for real-valued or symmetric ϕ.

The following theorem is known as the *oblique extension principle* (OEP). It is stated in the setting of Φ being a singleton set $\{\phi\}$.

Theorem 2.8. [16] (Oblique Extension Principle) *Let $\{V^k(\phi)\}$ be an MRA of $L^2(\mathbb{R}^s)$ with combined mask $\widehat{L}$ defined as in* (1.12) *having entries in $L^\infty(\mathbb{T}^s)$ and such that $E(\phi)$ is a Bessel system. Suppose that $\lim_{\omega\to 0}\widehat{\phi}(\omega)=1$ and there exists a $2\pi\mathbb{Z}^s$-periodic nonnegative essentially bounded function Θ, which is continuous at the origin, with $\Theta(0)=1$ and satisfies*

$$\overline{\widehat{h}(\omega)}\Theta(M^T\omega)\widehat{h}(\omega+\nu)+\widehat{G}(\omega)^*\widehat{G}(\omega+\nu)=\delta_\nu\Theta(\omega), \tag{2.6}$$

whenever $\omega\in\sigma(V(\phi))$ and $\nu\in 2\pi(M^{-T}\mathbb{Z}^s/\mathbb{Z}^s)$ is such that $\omega+\nu\in\sigma(V(\phi))$. Then the affine system $X(\Psi)$ as in (1.4) *defined by $\widehat{L}$ is a tight frame for $L^2(\mathbb{R}^s)$.*

The function Θ in Theorem 2.8 is known as the *fundamental function.* The OEP is also proved independently in [8]. We shall now show that if $X(\Psi)$ is a tight frame for $L^2(\mathbb{R}^s)$ derived from an MRA generated by a symmetric refinable function using the OEP, then for Ψ' constructed from Ψ as in Construction 2.1, $X(\Psi')$ is also a tight frame for $L^2(\mathbb{R}^s)$ derived from the same MRA using the OEP. In view of various available examples in the literature (see also Section 2.3), instead of the more general case as discussed in Theorem 2.7, here we only deal with the situation in which the MRA is generated by a single symmetric refinable function.

Theorem 2.9. *Let $\Psi:=\Big[\psi^m\Big]_{m=1}^{\varrho}$ such that $X(\Psi)$ as in* (1.4) *is a tight frame for $L^2(\mathbb{R}^s)$ derived from the OEP with $\{V^k(\phi)\}$ as the underlying MRA of $L^2(\mathbb{R}^s)$, ϕ being symmetric about $\frac{\eta}{2}$, where $\eta\in\mathbb{Z}^s$, Θ as the fundamental function, and $\widehat{L}:=\begin{bmatrix}\widehat{h}\\ \widehat{G}\end{bmatrix}$ as the combined MRA mask. Let the set of symmetric and antisymmetric wavelets Ψ' be constructed from Ψ as in Construction* 2.1. *Then $X(\Psi')$ is a tight frame for $L^2(\mathbb{R}^s)$ derived from the same MRA $\{V^k(\phi)\}$ using the OEP with the fundamental function $\Theta':=\frac{1}{2}[\Theta+\Theta(-\cdot)]$ and the combined MRA mask $\widehat{L}':=\begin{bmatrix}\widehat{h}\\ \widehat{G}'\end{bmatrix}$, where $\widehat{G}'$ is the $2\varrho\times 1$ vector given by*

$$\widehat{G}'(\omega):=\frac{1}{2}\begin{bmatrix}\widehat{g}^m(\omega)+\mathrm{e}^{-\mathrm{i}(M\kappa_m-\eta)\cdot\omega}\widehat{g}^m(-\omega)\\ \widehat{g}^m(\omega)-\mathrm{e}^{-\mathrm{i}(M\kappa_m-\eta)\cdot\omega}\widehat{g}^m(-\omega)\end{bmatrix}_{m=1}^{\varrho}, \tag{2.7}$$

$\kappa_m\in\mathbb{Z}^s$.

Proof. We first apply Proposition 2.5 to see that for $\widetilde{\phi}:=\phi(\eta-\cdot)$ and $\widetilde{\Psi}:=\Big[\psi^m(\kappa_m-\cdot)\Big]_{m=1}^{\varrho}$,

$$\overline{\widehat{\widetilde{h}}(\omega)}\widehat{\widetilde{h}}(\omega+\nu)=\mathrm{e}^{\mathrm{i}\eta\cdot\nu}\overline{\widehat{h}(-\omega)}\widehat{h}(-\omega-\nu),\ \ \widehat{\widetilde{G}}(\omega)^*\widehat{\widetilde{G}}(\omega+\nu)=\mathrm{e}^{\mathrm{i}\eta\cdot\nu}\widehat{G}(-\omega)^*\widehat{G}(-\omega-\nu), \tag{2.8}$$

where $\nu \in 2\pi(M^{-T}\mathbb{Z}^s/\mathbb{Z}^s)$, since $\mathrm{e}^{-\mathrm{i}M\eta\cdot\nu} = \mathrm{e}^{-\mathrm{i}M\kappa_m\cdot\nu} = 1$. By the symmetry of ϕ, $\omega \in \sigma(V(\phi))$ if and only if $-\omega \in \sigma(V(\phi))$ for almost every $\omega \in \mathbb{T}^s$. Thus

$$\overline{\widehat{\widetilde{h}}(\omega)}\Theta(-M^T\omega)\widehat{\widetilde{h}}(\omega+\nu) + \widehat{\widetilde{G}}(\omega)^*\widehat{\widetilde{G}}(\omega+\nu) = \delta_\nu\Theta(-\omega) \tag{2.9}$$

holds for $\omega \in \sigma(V(\phi))$ and $\nu \in 2\pi((M^{-T}\mathbb{Z}^s/\mathbb{Z}^s)$ such that $\omega+\nu \in \sigma(V(\phi))$ as we may replace ω by $-\omega$ and ν by $-\nu$ in (2.6). Let Υ be as in Construction 2.1. Since $\widetilde{\phi} = \phi$, adding (2.6) and (2.9) leads to

$$\overline{\widehat{h}(\omega)}\Theta'(M^T\omega)\widehat{h}(\omega+\nu) + \widehat{Q}(\omega)^*\widehat{Q}(\omega+\nu) = \delta_\nu\Theta'(\omega), \tag{2.10}$$

with $\widehat{Q}$ given as in (2.4) whenever $\omega \in \sigma(V(\phi))$ and $\nu \in 2\pi(M^{-T}\mathbb{Z}^s/\mathbb{Z}^s)$ is such that $\omega+\nu \in \sigma(V(\phi))$.

Next, as $\Psi' := U_{2\varrho}\Upsilon$, where $U_{2\varrho}$ is the constant unitary matrix in (2.1), it follows that the final wavelet mask is given by $\widehat{G'} := U_{2\varrho}\widehat{Q}$. Let $\omega \in \sigma(V(\phi))$ and $\nu \in 2\pi(M^{-T}\mathbb{Z}^s/\mathbb{Z}^s)$ such that $\omega+\nu \in \sigma(V(\phi))$. Then $\widehat{G'}(\omega)^*\widehat{G'}(\omega+\nu) = \widehat{Q}(\omega)^*\widehat{Q}(\omega+\nu)$ and so (2.10) yields

$$\overline{\widehat{h}(\omega)}\Theta'(M^T\omega)\widehat{h}(\omega+\nu) + \widehat{G'}(\omega)^*\widehat{G'}(\omega+\nu) = \delta_\nu\Theta'(\omega).$$

Hence by Theorem 2.8, $X(\Psi')$ is a tight frame for $L^2(\mathbb{R}^s)$ derived from the MRA $\{V^k(\phi)\}$ using the OEP with the fundamental function Θ'. $\square$

Let us highlight an application of Theorem 2.9 which gives a systematic approach to constructing symmetric and antisymmetric framelets, with given approximation order, for the univariate case with dilation factor 2. In Section 3.2 of [16], starting from a B-spline ϕ of order m (which is symmetric), tight frame systems are constructed by choosing appropriate trigonometric polynomials Θ to be the fundamental function in the OEP, according to m and the approximation order of the system required. The approximation order is closely related to the order of vanishing moments of the framelets, which in turn depends on ϕ and Θ (see Theorems 2.8 and 2.11 of [16]). One choice of the fundamental function Θ gives a total of three mother wavelets, while another choice produces two. None of the wavelets is symmetric, though both fundamental functions are symmetric. Applying Theorem 2.9 to these two sets of wavelets, we see that Construction 2.1 gives three symmetric and three antisymmetric wavelets for the first set, and two symmetric and two antisymmetric wavelets for the second. In both instances, since ϕ and Θ are unchanged, the approximation order of the resulting tight frame system remains the same.

In [29], three symmetric and antisymmetric framelets are constructed directly from the B-spline of order m. This method is extended to constructions based on a compactly supported symmetric refinable function with stable shifts in [30]. Our construction does not require the stability assumption of the refinable function and reduces the construction of symmetric tight framelets to the construction of tight framelets, which is easier. It combines the procedure in [16] with Construction 2.1 to give a systematic procedure for obtaining symmetric and antisymmetric framelets with at least the same vanishing moments, smoothness and approximation orders as the original wavelets. While the construction in [29] results in framelets with the highest possible order of vanishing moments, the flexibility of our construction allows us to tailor the approximation order of our framelet system and the order of vanishing moments of the framelets according to the needs of our application.

Let us now return to the general setting of $L^2(\mathbb{R}^s)$ and arbitrary dilation matrix M. We have shown that when $\phi \in L^2(\mathbb{R}^s)$ is symmetric, the new set of symmetric and antisymmetric framelets is obtained from the same MRA generated by ϕ. However, in many cases, the scaling function ϕ such as one of the Daubechies scaling functions or a pseudo-spline (see [16]) is not symmetric, and the corresponding wavelets are obtainable from the *unitary extension principle* (UEP), i.e. the OEP with fundamental function $\Theta = 1$. We shall see that in these instances, notwithstanding that the scaling function ϕ is not symmetric, it is still possible to construct a symmetric and antisymmetric tight frame system from the UEP. However, the set of framelets comes from an MRA generated by two functions, which is different from the original MRA $\{V^k(\phi)\}$, and the proof requires the following vector version of the UEP (see [44]).

Theorem 2.10. [44] (Unitary Extension Principle). *Let $\{V^k(\Phi)\}$ be an MRA of $L^2(\mathbb{R}^s)$ with combined mask $\widehat{L}$ defined as in* (1.12) *having entries in $L^\infty(\mathbb{T}^s)$ and such that $E(\Phi)$ is a Bessel system. Suppose that $\lim_{\omega\to 0}(\widehat{\Phi}^*\widehat{\Phi})(\omega) = 1$ and*

$$\widehat{L}(\omega)^*\widehat{L}(\omega+\nu) = \delta_\nu I, \tag{2.11}$$

whenever $\omega \in \sigma(V(\Phi))$ and $\nu \in 2\pi(M^{-T}\mathbb{Z}^s/\mathbb{Z}^s)$ is such that $\omega+\nu \in \sigma(V(\Phi))$. Then the affine system $X(\Psi)$ defined by $\widehat{L}$ is a tight frame for $L^2(\mathbb{R}^s)$.

Our next result is analogous to Theorem 2.9 for the UEP setting, except that the refinable function ϕ may not be symmetric but satisfies (2.5). Again, based on examples of interest (see Section 2.3), we focus on the case when the original MRA is generated by a single refinable function.

Theorem 2.11. *Let* $\Psi := \left[\psi^m\right]_{m=1}^{\varrho}$ *such that* $X(\Psi)$ *as in* (1.4) *is a tight frame for* $L^2(\mathbb{R}^s)$ *derived from the UEP with* $\{V^k(\phi)\}$ *as the underlying MRA of* $L^2(\mathbb{R}^s)$ *under the condition that* ϕ *satisfies* (2.5) *and* $\widehat{L} := \begin{bmatrix}\widehat{h}\\ \widehat{G}\end{bmatrix}$ *as the combined MRA mask. Let* $\Xi := \frac{1}{\sqrt{2}}\begin{bmatrix}\phi\\ \phi(\eta-\cdot)\end{bmatrix}$, *where* $\eta \in \mathbb{Z}^s$. *Suppose that* $\Phi' := U_0\Xi$, *where* U_0 *is the unitary matrix in* (2.1), *and the set of symmetric and antisymmetric wavelets* Ψ' *is constructed from* Ψ *as in Construction* 2.1. *Then* $X(\Psi')$ *is a tight frame for* $L^2(\mathbb{R}^s)$ *derived from the MRA* $\{V^k(\Phi')\}$ *using the UEP with the combined MRA mask* $\widehat{L}' := \begin{bmatrix}\widehat{H}'\\ \widehat{G}'\end{bmatrix}$, *where* $\widehat{H}'$ *and* $\widehat{G}'$ *are the* 2×2 *and* $2\varrho\times 2$ *matrices given by*

$$\widehat{H}'(\omega) := \frac{1}{2}\begin{bmatrix}\widehat{h}(\omega)+\mathrm{e}^{-\mathrm{i}(M\eta-\eta)\cdot\omega}\widehat{h}(-\omega) & \widehat{h}(\omega)-\mathrm{e}^{-\mathrm{i}(M\eta-\eta)\cdot\omega}\widehat{h}(-\omega)\\ \widehat{h}(\omega)-\mathrm{e}^{-\mathrm{i}(M\eta-\eta)\cdot\omega}\widehat{h}(-\omega) & \widehat{h}(\omega)+\mathrm{e}^{-\mathrm{i}(M\eta-\eta)\cdot\omega}\widehat{h}(-\omega)\end{bmatrix}, \tag{2.12}$$

$$\widehat{G}'(\omega) := \frac{1}{2}\begin{bmatrix}\widehat{g}^m(\omega)+\mathrm{e}^{-\mathrm{i}(M\kappa_m-\eta)\cdot\omega}\widehat{g}^m(-\omega) & \widehat{g}^m(\omega)-\mathrm{e}^{-\mathrm{i}(M\kappa_m-\eta)\cdot\omega}\widehat{g}^m(-\omega)\\ \widehat{g}^m(\omega)-\mathrm{e}^{-\mathrm{i}(M\kappa_m-\eta)\cdot\omega}\widehat{g}^m(-\omega) & \widehat{g}^m(\omega)+\mathrm{e}^{-\mathrm{i}(M\kappa_m-\eta)\cdot\omega}\widehat{g}^m(-\omega)\end{bmatrix}_{m=1}^{\varrho}, \tag{2.13}$$

$\kappa_m \in \mathbb{Z}^s$, *respectively.*

Proof. By Lemma 2.6, $\{V^k(\Xi)\}$ is an MRA of $L^2(\mathbb{R}^s)$. Further, $E(\Xi)$ is also a Bessel system. Let Υ be as in Construction 2.1. The combined MRA mask $\begin{bmatrix}\widehat{P}\\ \widehat{Q}\end{bmatrix}$ has entries in $L^\infty(\mathbb{T}^s)$ and $\widehat{P}$ is a 2×2 diagonal matrix and $\widehat{Q}$ is a vector of ϱ 2×2 diagonal matrices given as in (2.4). In addition, $\lim_{\omega\to 0}(\hat{\Xi}^*\hat{\Xi})(\omega) = 1$. We shall show that

$$\widehat{P}(\omega)^*\widehat{P}(\omega+\nu)+\widehat{Q}(\omega)^*\widehat{Q}(\omega+\nu) = \delta_\nu I, \tag{2.14}$$

whenever $\omega \in \sigma(V(\Xi))$ and $\nu \in 2\pi(M^{-T}\mathbb{Z}^s/\mathbb{Z}^s)$ is such that $\omega+\nu \in \sigma(V(\Xi))$. We note from (2.5) that $\sigma(V(\phi)) = \sigma(V(\widetilde{\phi}))$, where $\widetilde{\phi} := \phi(\eta-\cdot)$, and hence $\sigma(V(\Xi)) = \sigma(V(\phi))\cup\sigma(V(\widetilde{\phi})) = \sigma(V(\phi))$.

The $(1,1)$-entry of (2.14) is exactly (2.11). By the structure of the 2×2 diagonal matrices in $\widehat{P}$ and $\widehat{Q}$, we see that the $(1,2)$- and $(2,1)$-entries of (2.14) are both zero. It remains to prove the equality of the $(2,2)$-entry on both sides of (2.14), i.e.

$$\overline{\widehat{\widetilde{h}}(\omega)}\widehat{\widetilde{h}}(\omega+\nu)+\widehat{\widetilde{G}}(\omega)^*\widehat{\widetilde{G}}(\omega+\nu) = \delta_\nu, \tag{2.15}$$

where $\widetilde{\Psi} := \left[\psi^m(\kappa_m-\cdot)\right]_{m=1}^{\varrho}$. As in the proof of Theorem 2.9, we use Proposition 2.5 to obtain (2.8). Since $\omega \in \sigma(V(\phi))$ if and only if $-\omega \in \sigma(V(\widetilde{\phi}))$ for almost every $\omega \in \mathbb{T}^s$, it follows from (2.5) that $\omega \in \sigma(V(\phi))$ if and only if $-\omega \in \sigma(V(\phi))$ for almost every $\omega \in \mathbb{T}^s$.

Thus in view of (2.8), (2.15) holds for $\omega \in \sigma(V(\phi))$ and $\nu \in 2\pi(M^{-T}\mathbb{Z}^s/\mathbb{Z}^s)$ such that $\omega+\nu \in \sigma(V(\phi))$, because we can replace ω by $-\omega$ and ν by $-\nu$ in (2.11).

Now, let $\Psi' := U_{2\varrho}\Upsilon$, where $U_{2\varrho}$ is the constant unitary matrix in (2.1). We first observe from the refinement equation (1.9) that the vector Φ' is refinable with refinement mask $\widehat{H}' := U_0\widehat{P}U_0^*$, generating the same MRA as Ξ. Using the wavelet equation (1.11), the final wavelet mask is given by $\widehat{G}' := U_{2\varrho}\widehat{Q}U_0^*$. Clearly, the entries of the combined MRA mask $\widehat{L}' := \begin{bmatrix} \widehat{H}' \\ \widehat{G}' \end{bmatrix}$ lie in $L^\infty(\mathbb{T}^s)$. Also, we have $\lim_{\omega\to 0}(\widehat{\Phi}'^*\widehat{\Phi}')(\omega) = 1$. Let $\omega \in \sigma(V(\Xi))$ and $\nu \in 2\pi(M^{-T}\mathbb{Z}^s/\mathbb{Z}^s)$ such that $\omega+\nu \in \sigma(V(\Xi))$. Then $\widehat{H}'(\omega)^*\widehat{H}'(\omega+\nu) = U_0\widehat{P}(\omega)^*\widehat{P}(\omega+\nu)U_0^*$ and $\widehat{G}'(\omega)^*\widehat{G}'(\omega+\nu) = U_0\widehat{Q}(\omega)^*\widehat{Q}(\omega+\nu)U_0^*$. This enables us to conclude from (2.14) that (2.11) holds for $\widehat{L}'$, i.e.

$$\widehat{H}'(\omega)^*\widehat{H}'(\omega+\nu) + \widehat{G}'(\omega)^*\widehat{G}'(\omega+\nu) = \delta_\nu I.$$

Applying Theorem 2.10 to $\widehat{L}'$ gives the result. $\square$

2.3 Examples

We shall now illustrate the results in Section 2.2 with concrete examples for the univariate case with dilation factor 2. We begin with a discussion on practical issues related to the flexibility we have in the construction of symmetric and antisymmetric wavelets. When we utilize Construction 2.1 to construct our wavelets, we need to consider the positions of reflection of the original wavelets. Since we have the freedom of reflecting the wavelets about any half-integer point, we may choose to reflect them about half-integer points around the midpoints of their individual supports. This minimizes the supports of the resulting wavelets, in the sense that they are almost the same as the supports of the original wavelets. However, this may not always be ideal since we may obtain more than one peak or have more oscillations when we essentially take the sum and difference, using the matrix U_0 in (2.1), of the original wavelets and their reflections. Therefore it could be more desirable to reflect about the positions where their peaks occurred so that the resulting wavelets will have better spreads in the time domain. It should also be mentioned that in some cases, other positions may be even more appropriate, depending on the graphs of the original wavelets. For situations when the original refinable functions are not symmetric, similar considerations in choosing the positions of reflection apply.

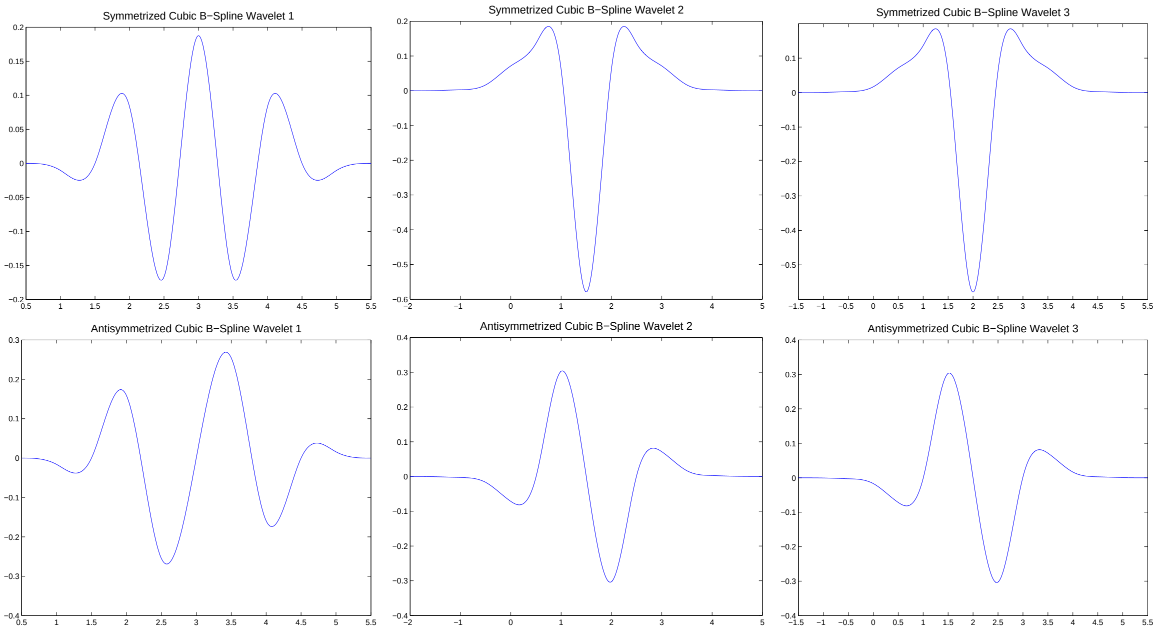

Figure 2.1: Symmetric and antisymmetric wavelets obtained in Example 2.3.1 from a systematic construction based on the cubic B-spline.

Example 2.3.1. This example, illustrated in Figure 2.1, is based on the systematic construction in Example 3.7 of [16]. The original wavelets are obtained from an MRA generated by a symmetric refinable function using the OEP, and we apply Theorem 2.9. Here the lowpass filter h is that of the cubic B-spline ϕ supported on $[0,4]$, and there are three wavelets ψ^1, ψ^2 and ψ^3 in the construction with filters g^1, g^2 and g^3 respectively. The approximation order of the framelet system generated by ψ^1, ψ^2 and ψ^3 is 4. We define $\Psi := \frac{1}{\sqrt{2}}\left[\psi^1, \psi^1(6-\cdot), \psi^2, \psi^2(3-\cdot), \psi^3, \psi^3(4-\cdot)\right]^T$ and $\Psi' := U_6\Psi$, where U_6 is the 6×6 block diagonal matrix with the matrix U_0 defined in (2.1) as its blocks. For ψ^1, we reflect at the midpoint of its support as this happens to reduce the oscillations in the resulting antisymmetric wavelet. As for ψ^2 and ψ^3, we choose to reflect at the nearest half-integers where their peaks occur. It follows from (2.7) that the matrix filter of Ψ' is given by $G' := \left[G^{m\prime}\right]_{m=1}^{3}$, where $G^{m\prime}(n) := \frac{1}{2}\begin{bmatrix} g^m(n) + g^m(\mu_m - n) \\ g^m(n) - g^m(\mu_m - n)\end{bmatrix}$ for $m = 1,2,3$ with $\mu_1 = 8$, $\mu_2 = 2$ and $\mu_3 = 4$.

Example 2.3.2. Consider the Daubechies-4 refinable function ϕ with filter h supported on $\{1,\dots,4\}$ and the corresponding wavelet ψ with filter g given by $g(n) := (-1)^{3-n}h(3-n)$ (see [13] and [14]). As ϕ is not symmetric, we apply Theorem 2.11. Let $\Phi := \frac{1}{\sqrt{2}}\left[\phi, \phi(4-\cdot)\right]^T$, $\Psi := \frac{1}{\sqrt{2}}\left[\psi, \psi(3-\cdot)\right]^T$, $\Phi' := U_0\Phi$ and $\Psi' := U_0\Psi$, where U_0 is as

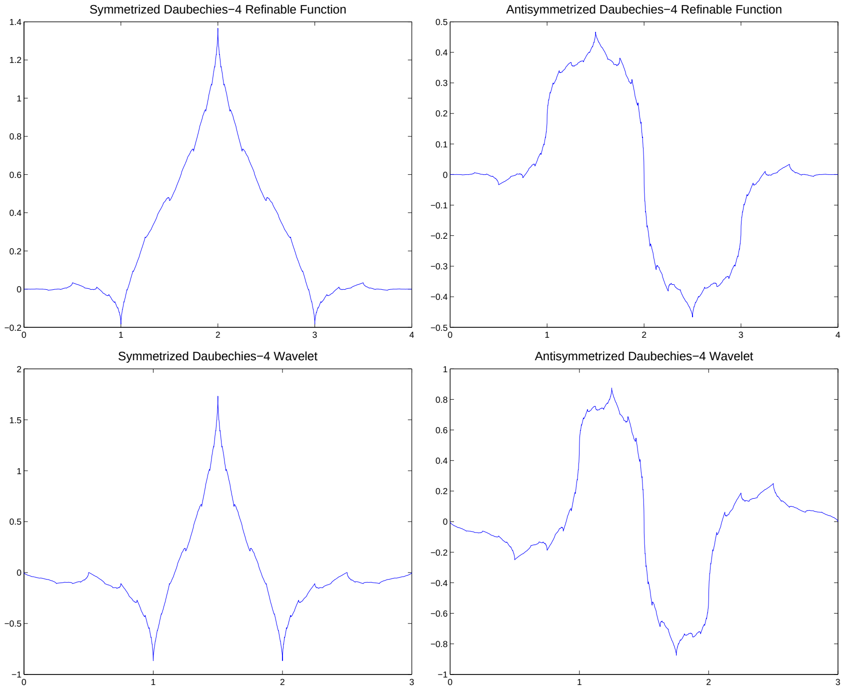

Figure 2.2: Symmetric and antisymmetric refinable functions and wavelets obtained in Example 2.3.2 from the Daubechies-4 refinable function and wavelet.

defined in (2.1). Using (2.13), the matrix filter G' of Ψ' can be expressed as $G'(n) = (-1)^{3-n}H'(3-n)$, where the matrix filter H' of Φ' is given by

$$H'(n) := \frac{1}{2}\begin{bmatrix} h(n)+h(4-n) & h(n)-h(4-n) \\ h(n)-h(4-n) & h(n)+h(4-n) \end{bmatrix}$$

from (2.12). The graphs of the resulting refinable functions and wavelets are shown in Figure 2.2. Both the original refinable function and wavelet are reflected around their peaks. The supports of the resulting wavelets are the same as that of the original, since the reflection point occurs at the midpoint.

Chapter 3

Connection Between Wavelet Frames of $L^2(\mathbb{R}^s)$ and $L^2(\mathbb{T}^s)$

The Poisson summation formula is the bridge connecting the theory of wavelet frames of the real line to that of periodic ones. This makes it necessary to study the harmonics or uniform samples of functions in the frequency domain. The conditions for obtaining periodic wavelet frames from periodic MRAs could be expressed in terms of the harmonics of periodic functions. The wavelet frames could either be semi-orthogonal or nonorthogonal to the MRA subspaces. These wavelet frames could then be extended back to the real line by ensuring that these conditions hold for arbitrary collection of harmonics, i.e. the construction of real line wavelets subtly involves the construction of periodic wavelets.

3.1 Euclidean Space Formulation

Let M be a $s \times s$ invertible matrix with integer entries such that M is expansive, i.e. all the eigenvalues of M are greater than 1. We set

$$D := M^T, \quad d := |\det(M)| = |\det(D)|\,.$$

For $k \geq 0$, let $\mathcal{L}_k$ denote a full collection of coset representatives of $\mathbb{Z}^s/M^k\mathbb{Z}^s$ and $\mathcal{R}_k$ denote a full collection of coset representatives of $\mathbb{Z}^s/D^k\mathbb{Z}^s$. Then $d^k = |\mathcal{L}_k| = |\mathcal{R}_k|$,

$$\mathbb{Z}^s = \bigcup_{l\in\mathcal{L}_k}(l + M^k\mathbb{Z}^s) = \bigcup_{j\in\mathcal{R}_k}(j + D^k\mathbb{Z}^s), \tag{3.1}$$

and for any distinct $l_1,\, l_2 \in \mathcal{L}_k,\, j_1,\, j_2 \in \mathcal{R}_k$,

$$(l_1 + M^k\mathbb{Z}^s) \cap (l_2 + M^k\mathbb{Z}^s) = \emptyset = (j_1 + D^k\mathbb{Z}^s) \cap (j_2 + D^k\mathbb{Z}^s).$$

As preparation of our study in this multidimensional setup, let us first derive several lemmas on the properties of the collections $\mathcal{L}_k$ and $\mathcal{R}_k$.

Lemma 3.1. *Let $s \in \mathbb{N}$ and $\{\lambda_i, \mu_i\}_{i=1}^s \subset \mathbb{N}$. Suppose that for every $i \in \{2, \ldots, s\}$, $\lambda_{i-1}|\lambda_i$, $\mu_{i-1}|\mu_i$ and $\prod_{i=1}^s \lambda_i | \prod_{i=1}^s \mu_i$. Then $\lambda_i|\mu_i$ for every $i \in \{1, \ldots, s\}$.*

Proof. The lemma is clearly true for $s = 1$. Suppose that the result is true for all $i < s$. Given the hypothesis for the case of $i = s$, we have $\lambda_1|(\prod_{i=2}^s \lambda_i)$ and $\mu_1|(\prod_{i=2}^s \mu_i)$ and $(\lambda_1 \prod_{i=2}^s \lambda_i)|(\mu_1 \prod_{i=2}^s \mu_i)$. By induction, we would have $\lambda_1|\mu_1$ and $(\prod_{i=2}^s \lambda_i)|(\prod_{i=2}^s \mu_i)$ and consequently $\lambda_i|\mu_i$ for $\{2, \ldots, s\}$. □

Lemma 3.2. *For $k \geq K \geq 0$, there exists a choice of coset representatives of $\mathbb{Z}^s/M^k\mathbb{Z}^s$ and $\mathbb{Z}^s/D^k\mathbb{Z}^s$ such that $\mathcal{L}_k = \mathcal{L}_{k-K} + M^{k-K}\mathcal{L}_K$ and $\mathcal{R}_k = \mathcal{R}_{k-K} + D^{k-K}\mathcal{R}_K$ respectively.*

Proof. We construct a canonical choice of $\mathcal{L}_k$ and $\mathcal{R}_k$ as follows. Let $\{e_i\}_{i=1}^s$ be the standard basis of the free abelian group $\mathbb{Z}^s$ and $f_{k,i}$ and $g_{k,i}$ be the i^{th} columns of the matrices M^k and D^k respectively. Therefore, $\{f_{k,i}\}_{i=1}^s$ and $\{g_{k,i}\}_{i=1}^s$ are generators of $M^k\mathbb{Z}^s$ and $D^k\mathbb{Z}^s$ respectively. There exist invertible matrices $P_k, Q_k \in M_s(\mathbb{Z})$, the ring of all $s \times s$ matrices with integer entries, such that

$$Q_k D^k P_k^{-1} = N_k = (P_k^{-1})^T M^k Q_k^T = P_k^{-T} M^k Q_k^T, \tag{3.2}$$

with $N_k := \operatorname{diag}\left[\lambda_i^{(k)}\right]_{i=1}^s \in M_s(\mathbb{Z})$ having positive diagonal entries such that $\lambda_{i-1}^{(k)}|\lambda_i^{(k)}$ for $i \in \{2, \ldots, s\}$ and $\det N_k = d^k$ is the invariant factor form of M^k and D^k (see [33]). Let $e'_{k,i}$ and $e''_{k,i}$ be the i^{th} columns of the matrices P_k^T and Q_k^{-1} respectively, i.e. $\{e'_{k,i}\}_{i=1}^s$ and $\{e''_{k,i}\}_{i=1}^s$ are bases of $\mathbb{Z}^s$. We define the set of s-tuples $\{f'_{k,i}\}_{i=1}^s$ and $\{g'_{k,i}\}_{i=1}^s$ by

$$\left[f'_{k,1} \cdots f'_{k,s}\right] = M^k Q_k^T, \quad \left[g'_{k,1} \cdots g'_{k,s}\right] = D^k P_k^{-1},$$

i.e. $\{f'_{k,i}\}_{i=1}^s$ and $\{g'_{k,i}\}_{i=1}^s$ generate $M^k\mathbb{Z}^s$ and $D^k\mathbb{Z}^s$ respectively. Since $M^k Q_k^T = P_k^T N_k$ and $D^k P_k^{-1} = Q_k^{-1} N_k$, we conclude that $f'_{k,i} = \lambda_i^{(k)} e'_{k,i}$ and $g'_{k,i} = \lambda_i^{(k)} e''_{k,i}$ for $i \in \{1, \ldots, s\}$, respectively. We choose $\mathcal{L}_k$ and $\mathcal{R}_k$ such that

$$\mathcal{L}_k = \{\sum_{i=1}^s m_i e'_{k,i} : 0 \leq m_i < \lambda_i^{(k)}, i = 1, \ldots, s\},$$
$$\mathcal{R}_k = \{\sum_{i=1}^s r_i e''_{k,i} : 0 \leq r_i < \lambda_i^{(k)}, i = 1, \ldots, s\}, \tag{3.3}$$

and order them in such a way that

$$\sum_{i=1}^s m_i e'_{k,i} < \sum_{i=1}^s n_i e'_{k,i}, \quad \sum_{i=1}^s r_i e''_{k,i} < \sum_{i=1}^s t_i e''_{k,i}$$

if and only if there exists a least integer $i \in \{1, \dots, s\}$ for which $m_i < n_i$ and $r_i < t_i$ respectively. Therefore, we could list $\mathcal{L}_k = \{l_{k,i}\}_{i=0}^{d^k-1}$ and $\mathcal{R}_k = \{r_{k,i}\}_{i=0}^{d^k-1}$ in ordered sets.

We claim that for all $k \in \mathbb{N}$, $N_k = N^k$, where $N := N_1$. This is clearly true for $k = 1$. For ease of writing, we let $\lambda_i := \lambda_i^{(1)}$ for every $i \in \{1, \dots, s\}$. Suppose that the result is also true for all $i < k$. We have $\det N_k = d^{k-1}d$, i.e. $\prod_{i=1}^{s} \lambda_i^{(k)} = \prod_{i=1}^{s} \lambda_i^{(k-1)}\lambda_i = \prod_{i=1}^{s} \lambda_i^{k-1}\lambda_i = \prod_{i=1}^{s} \lambda_i^{k}$ by the induction hypothesis. By Lemma 3.1, we have $\lambda_i^{(k)} = \lambda_i^k$ for every $i \in \{1, \dots, s\}$.

Using our canonical choice of $\mathcal{L}_k$ and $\mathcal{R}_k$ for $k \geq K$, we choose l and j such that $l \in P_{k-K}^T \mathcal{L}_{k-K}$ and $j \in Q_{k-K}^T \mathcal{L}_K$. Consider the coset representatives $P_{k-K}^{-T} l = \sum_{i=1}^{s} m_i e'_{k-K,i}$ and $Q_{k-K}^{-T} j = \sum_{i=1}^{s} r_i e'_{K,i}$, where $0 \leq m_i \leq \lambda_i^{k-K} - 1$ and $0 \leq r_i \leq \lambda_i^K - 1$ for $i \in \{1, \dots, s\}$. This shows that $0 \leq m_i + \lambda_i^{k-K} r_i \leq \lambda_i^k - 1$ and $P_{k-K}^{-T} l + N^{k-K} Q_{k-K}^{-T} j$ lies in $\mathcal{L}_k$. Therefore $l + M^{k-K} j$ lies in $P_{k-K}^T \mathcal{L}_k$. In the event that $l + M^{k-K} j = 0$, i.e. $P_{k-K}^{-T} l + N^{k-K} Q_{k-K}^{-T} j = 0$, since for $i \in \{1, \dots, s\}$, each m_i is a multiple of λ_i^{k-K}, $P_{k-K}^{-T} l$ must be the zero element and this shows that any possible representation is unique. Finally, since $|\mathcal{L}_k| = d^k = d^{k-K} d^K = |\mathcal{L}_{k-K}|\,|\mathcal{L}_K|$, the representation existence is verified. $\square$

Lemma 3.3. *For $0 \leq k \leq K$, the kernel of the surjective mapping $\iota : \mathcal{L}_K \times \mathcal{L}_k \to \mathcal{L}_K$ given by $\iota : (l, j) \mapsto l + M^{K-k} j$ has d^k elements.*

Proof. Using the canonical construction as described in Lemma 3.2, let the set of coset representatives $\mathcal{L}_k := \{l_{k,i}\}_{i=0}^{d^k-1}$ be chosen as in (3.3). We shall also use the invariant factor form N^k of M^k as given in (3.2), where $N^k = N_1^k$. Let $P_{K-k}^{-T} l = \sum_{i=1}^{s} m_i e'_{K,i} \in \mathcal{L}_K$ and $Q_{K-k}^{-T} j = \sum_{i=1}^{s} r_i e'_{k,i} \in \mathcal{L}_k$, where $0 \leq m_i \leq \lambda_i^K - 1$ and $0 \leq r_i \leq \lambda_i^k - 1$ for $i \in \{1, \dots, s\}$ and the matrices P_{K-k} and Q_{K-k} are given in (3.2). We shall consider the corresponding epimorphism $\tau : P_{K-k}^T \mathcal{L}_K \times Q_{K-k}^T \mathcal{L}_k \to P_{K-k}^T \mathcal{L}_K$ given by $\tau : (l, j) \mapsto l + M^{K-k} j$. Suppose that $P_{K-k}^{-T} \tau(l, j) = 0$. Since $P_{K-k}^{-T} l = N^{K-k}(-Q_{K-k}^{-T} j)$, for each j, there are exactly d^k choices for l and hence the kernel of τ has d^k elements. $\square$

Let the s-dimensional circle group at level K be $\mathbb{T}_K^s := \mathbb{R}^s / D^K(2\pi\mathbb{Z}^s)$ and for each $\omega \in \mathbb{T}_K^s$, define the *pre-Gramian* of the $M^{-K}\mathbb{Z}^s$ shift-invariant system $E_K(\Lambda_K)$ at level K, given by (1.6) and (1.7), to be the matrix-valued function

$$J_{K,\Lambda_K}(\omega) := \left[\widehat{\varphi}_{K,\omega,0}\right]_{\varphi \in \Lambda_K},$$

where the sequence

$$\widehat{\varphi}^o_{K,\omega,k} := d^{-\frac{k}{2}} \{\widehat{\varphi}(D^{-k}(\omega + 2\pi D^K n))\}_{n \in \mathbb{Z}^s}, \tag{3.4}$$

reduces to the standard *fibre* of φ at ω (denoted by $\widehat{\varphi}_{||\omega}$) when $K = k = 0$ (see [4]) and satisfies

$$\begin{aligned}
\widehat{\varphi}_{0,\omega,k} &= \sum_{j\in\mathcal{R}_K} \widehat{\varphi}_{K,\omega+2\pi j,k}, \\
\widehat{\varphi}_{K,\omega+2\pi j,k} &= \{\widehat{\varphi}^o_{0,\omega,k}(n)\mathbf{1}_{j+D^K\mathbb{Z}^s}(n)\}_{n\in\mathbb{Z}^s}, \\
\widehat{\varphi}^o_{K,\omega,k} &= d^{-\frac{K}{2}} d^{\frac{K}{2}-\frac{k}{2}}\{\widehat{\varphi}(D^{-(k-K)}(D^{-K}\omega+2\pi n))\}_{n\in\mathbb{Z}^s} = d^{-\frac{K}{2}}\widehat{\varphi}_{0,D^{-K}\omega,k-K}. \quad (3.5)
\end{aligned}$$

The pre-Gramian $J_{K,\Lambda_K}(\omega)$ is well defined for almost every ω since $\Lambda_K \subset L^2(\mathbb{R}^s)$ implies that $\|\widehat{\varphi}_{K,\omega,0}\|_{l^2(\mathbb{Z}^s)}$ is well defined for almost every $\omega \in \mathbb{T}^s_K$ and every $\varphi \in \Lambda_K$ as shown later in Lemma 3.5. The K-*fibre* of a closed $M^{-k}\mathbb{Z}^s$ shift-invariant subspace $V^k(\Lambda_k)$ of $L^2(\mathbb{R}^s)$ for $k \geq K$ generated by some countable set $\Lambda_k \subset V^k \equiv V^k(\Lambda_k)$ ($V^k(\Lambda_k)$ is written as V^k when the generating set is inferred from the context) at $\omega \in \mathbb{T}^s_K$ is defined to be

$$\widehat{V}^k_{K||\omega}(\Lambda_k) := \overline{\operatorname{span}}\,\{\widehat{\varphi}_{K,\omega,0} : \varphi \in \Lambda_k\}, \tag{3.6}$$

and the definition is independent of the generating set and is well-defined for almost every $\omega \in \mathbb{T}^s_K$ (see [4]). In the event that $\Lambda_k = \{d^{\frac{k}{2}-\frac{K}{2}}E^l_{k-K}\varphi(M^{k-K}\cdot) : \varphi \in \Lambda_K, l \in \mathcal{L}_{k-K}\}$, then

$$\widehat{V}^k_{K||\omega}(\Lambda_k) = \overline{\operatorname{span}}\,\{\mathrm{e}^{-\mathrm{i}\omega\cdot M^{K-k}l}\mathcal{M}^l_{K,k-K}\widehat{\varphi}_{K,\omega,k-K} : \varphi \in \Lambda_K, l \in \mathcal{L}_{k-K}\}, \tag{3.7}$$

where the *modulation operator* $\mathcal{M}^l_{K,k} : l^2(\mathbb{Z}^s) \to l^2(\mathbb{Z}^s)$ at level K is given by

$$\mathcal{M}^l_{K,k} : a \mapsto \{\mathrm{e}^{-\mathrm{i}2\pi D^K n\cdot M^{-k}l}a(n)\}_{n\in\mathbb{Z}^s}.$$

The *Gramian* of the set $E_K(\Lambda_K)$ at level K for each $\omega \in \mathbb{T}^s_K$ is defined to be

$$\mathrm{M}_{K,\Lambda_K}(\omega) := J_{K,\Lambda_K}(\omega)J^*_{K,\Lambda_K}(\omega) = \left[\langle\widehat{\varphi}_{K,\omega,0}, \widehat{\phi}_{K,\omega,0}\rangle_{l^2(\mathbb{Z}^s)}\right]_{\varphi,\phi\in\Lambda_K}.$$

Like the pre-Gramian, the Gramian $\mathrm{M}_{K,\Lambda_K}(\omega)$ is well defined for almost every $\omega \in \mathbb{T}^s_K$. The *spectrum* of the $M^{-K}\mathbb{Z}^s$ shift-invariant space $V^K(\Lambda_K)$ is defined (up to modulo measure zero sets) as

$$\sigma_K(V^K(\Lambda_K)) := \{\omega \in \mathbb{T}^s_K : \|\widehat{\varphi}_{K,\omega,0}\|_{l^2(\mathbb{Z}^s)} > 0 \text{ for some } \varphi \in \Lambda_K\} \tag{3.8}$$

and only depends on the space and is independent of the choice of generators of the space (see [4] and [43]).

We view the bi-infinite matrices $\mathrm{M}_{K,\Lambda_K}(\omega)$, $\omega \in \mathbb{T}^s_K$, as linear operators and in the event of they being boundedly invertible, we denote their bounded inverses by $\mathrm{M}_{K,\Lambda_K}(\omega)^{-1}$. (For those ω such that the underlying operator is not well defined or is unbounded, we take the norm of the underlying operator to be ∞.) For the functions

$$\mathsf{M}_{K,\Lambda_K} : \omega \mapsto \|\mathrm{M}_{K,\Lambda_K}(\omega)\|, \qquad \mathsf{M}^{-}_{K,\Lambda_K} : \omega \mapsto \left\|\mathrm{M}_{K,\Lambda_K}(\omega)^{-1}\right\|$$

defined on measurable subsets $F \subseteq \mathbb{T}^s_K$, we consider their $L_\infty(F)$-norm, where $L_\infty(F)$ denotes the space of all essentially bounded complex-valued functions on F. These $L_\infty(F)$-norms are used to characterize the Bessel and frame properties of $E_K(\Lambda_K)$. With the exception of the modulation operator $\mathcal{M}^l_{K,k}$ and the sequence $\widehat{\varphi}_{K,\omega,0}$, we shall leave out writing the level K when K is 0. We shall let $K \geq 0$ for the rest of this chapter.

Theorem 3.4. [43, 45] *Let $\Lambda_K \subset L^2(\mathbb{R}^s)$ be countable and consider the $M^{-K}\mathbb{Z}^s$ shift-invariant system $E_K(\Lambda_K)$.*

(i) *The system $E_K(\Lambda_K)$ is a Bessel system if and only if $\|\mathsf{M}_{K,\Lambda_K}\|_{L^\infty(\mathbb{T}^s_K)} < \infty$. Further, the Bessel bound is equal to $d^K \|\mathsf{M}_{K,\Lambda_K}\|_{L^\infty(\mathbb{T}^s_K)}$.*

(ii) *Assume that $E_K(\Lambda_K)$ is a Bessel system. The system $E_K(\Lambda_K)$ is a frame for $V^K(\Lambda_K)$ if and only if $1/\left\|\mathsf{M}^{-}_{K,\Lambda_K}\right\|_{L^\infty(\sigma_K(V^K(\Lambda_K)))} < \infty$. Further, the lower frame bound is given by $d^K/\left\|\mathsf{M}^{-}_{K,\Lambda_K}\right\|_{L^\infty(\sigma_K(V^K(\Lambda_K)))}$. In particular, $E_K(\Lambda_K)$ is a tight frame for $V^K(\Lambda_K)$ if and only if its Gramian $\mathrm{M}_{K,\Lambda_K}(\omega)$ is an orthogonal projector for almost every $\omega \in \mathbb{T}^s$.*

Theorem 3.4 outlines the main approach in our attempt to establish the connection of real line signals to their periodization, i.e. we shall look at the properties of frequency samples of functions, which in this case is the Gramian of fibres.

Lemma 3.5. *Let $K \geq 0$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}^s$, $f, g \in L^2(\mathbb{R}^s)$ and $f_k := d^{\frac{k}{2}} f(M^k \cdot)$. We have*

(i) *for almost every $\omega \in \mathbb{R}^s$, $\widehat{(f_k)}^o_{K,\omega,0}(n) = \widehat{f}^o_{K,\omega,k}(n)$,*

(ii) *for almost every $\omega \in \mathbb{R}^s$, $(E^l_k f)^\wedge(\omega) = \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l}\widehat{f}(\omega)$,*

(iii) *for almost every $\omega \in \mathbb{T}^s_K$, $(E^l_k f)^{\wedge o}_{K,\omega,0} = \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l}\mathcal{M}^l_{K,k}\widehat{f}^o_{K,\omega,0}$,*

(iv) *$\langle E^l_k f, g\rangle = \int_{\mathbb{T}^s_K} \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l}\langle \mathcal{M}^l_{K,k}\widehat{f}^o_{K,\omega,0}, \widehat{g}^o_{K,\omega,0}\rangle_{l^2(\mathbb{Z}^s)}\mathrm{d}\omega$,*

(v) *$\|f\|^2 = \int_{\mathbb{T}^s_K} \left\|\widehat{f}^o_{K,\omega,0}\right\|^2_{l^2(\mathbb{Z}^s)} \mathrm{d}\omega$.*

Proof. We show (i) by evaluating for $n \in \mathbb{Z}^s$,

$$\widehat{(f_k)}^o_{K,\omega,0}(n) = \widehat{f}_k(\omega + 2\pi D^K n) = d^{-\frac{k}{2}}\widehat{f}(D^{-k}(\omega + 2\pi D^K n)) = \widehat{f}^o_{K,\omega,k}(n). \tag{3.9}$$

Part (ii) is established from

$$\begin{aligned}(E^l_k f)^\wedge(\omega) &= \frac{1}{(2\pi)^s}\int_{\mathbb{R}^s} f(t - M^{-k}l)\mathrm{e}^{-\mathrm{i}\omega\cdot t}\mathrm{d}t = \frac{1}{(2\pi)^s}\int_{\mathbb{R}^s} f(t)\mathrm{e}^{-\mathrm{i}\omega\cdot(t+M^{-k}l)}\mathrm{d}t \\ &= \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l}\widehat{f}(\omega).\end{aligned}$$

Letting $n \in \mathbb{Z}^s$, we show (iii) using (ii), i.e.

$$\begin{aligned}&(E^l_k f)^{\wedge\, o}_{K,\omega,0}(n) = (E^l_k f)^\wedge(\omega + 2\pi D^K n) = \mathrm{e}^{-\mathrm{i}(\omega+2\pi D^K n)\cdot M^{-k}l}\widehat{f}(\omega + 2\pi D^K n) \\ &= \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l}\mathcal{M}^l_{K,k}\widehat{f}^o_{K,\omega,0}(n).\end{aligned}$$

By utilizing parts (ii) and (iii) and Plancherel's theorem (see [42]), we show

$$\begin{aligned}&\langle E^l_k f, g\rangle = (2\pi)^s\langle \mathrm{e}^{-\mathrm{i}\gamma\cdot M^{-k}l}\widehat{f}(\gamma), \widehat{g}(\gamma)\rangle = \sum_{n\in\mathbb{Z}^s}\int_{\mathbb{T}^s_K+2\pi D^K n}\mathrm{e}^{-\mathrm{i}\gamma\cdot M^{-k}l}\widehat{f}(\gamma)\overline{\widehat{g}(\gamma)}\mathrm{d}\gamma \\ &= \sum_{n\in\mathbb{Z}^s}\int_{\mathbb{T}^s_K}\mathrm{e}^{-\mathrm{i}(\omega+2\pi D^K n)\cdot M^{-k}l}\widehat{f}(\omega + 2\pi D^K n)\overline{\widehat{g}(\omega + 2\pi D^K n)}\mathrm{d}\omega \\ &= \int_{\mathbb{T}^s_K}\sum_{n\in\mathbb{Z}^s}\mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l}\mathcal{M}^l_{K,k}\widehat{f}^o_{K,\omega,0}(n)\overline{\widehat{g}^o_{K,\omega,0}(n)}\mathrm{d}\omega,\end{aligned}$$

and hence (iv) holds and is justified by the verification of (v). The proof of (v) using Parseval's identity (see [42]) is as follows:

$$\begin{aligned}&\int_{\mathbb{T}^s_K}\left\|\widehat{f}^o_{K,\omega,0}\right\|^2_{l^2(\mathbb{Z}^s)}\mathrm{d}\omega = \int_{\mathbb{T}^s_K}\sum_{n\in\mathbb{Z}^s}\left|\widehat{f}(\omega + 2\pi D^K n)\right|^2\mathrm{d}\omega = \sum_{n\in\mathbb{Z}^s}\int_{\mathbb{T}^s_K}\left|\widehat{f}(\omega + 2\pi D^K n)\right|^2\mathrm{d}\omega \\ &= \sum_{n\in\mathbb{Z}^s}\int_{\mathbb{T}^s_K+2\pi D^K n}\left|\widehat{f}(\omega)\right|^2\mathrm{d}\omega = \int_{\mathbb{R}^s}\left|\widehat{f}(\omega)\right|^2\mathrm{d}\omega = \|f\|^2.\end{aligned}$$

□

Now, recall the quasi-affine system $X^q_K(\Psi)$ at level K as defined in (1.6) and (1.7). We observe that it can be expressed as follows:

$$\begin{aligned}X^q_K(\Psi) &= \{d^{k-\frac{K}{2}}\psi(M^k(\cdot - M^{-K}r)) : \psi \in \Psi, r \in \mathbb{Z}^s, k < K\} \cup \\ &\quad \{d^{\frac{k}{2}}\psi(M^k \cdot - (M^{k-K}r + l)) : \psi \in \Psi, l \in \mathcal{L}_{k-K}, r \in \mathbb{Z}^s, k \geq K\} \\ &= \{d^{k-\frac{K}{2}}\psi(M^k \cdot - M^{k-K}r) : \psi \in \Psi, r \in \mathbb{Z}^s, k < K\} \cup \\ &\quad \{d^{\frac{k}{2}}\psi(M^k \cdot - r) : \psi \in \Psi, r \in \mathbb{Z}^s, k \geq K\}.\end{aligned}$$

In particular, $X^q(\Psi) := X_0^q(\Psi)$ is given as

$$X^q(\Psi)=\{d^k\psi(M^k(\cdot - r)):\psi\in\Psi, r\in\mathbb{Z}^s, k<0\}\cup\{d^{\frac{k}{2}}\psi(M^k\cdot - r):\psi\in\Psi, r\in\mathbb{Z}^s, k\geq 0\}.$$

The following result shows that the quasi-affine system $X^q(\Psi)$ has identitical frame properties as the affine system $X(\Psi)$.

Theorem 3.6. [44] *The affine system $X(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R}^s)$ if and only if its quasi-affine counterpart $X^q(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R}^s)$. Further, the two systems have identical (Bessel) frame bounds. In particular, the affine system $X(\Psi)$ is a tight frame if and only if the quasi-affine system $X^q(\Psi)$ is a tight frame.*

We shall illustrate below the similarity in the structure of the quasi-affine systems $X^q(\Psi)$ and $X_K^q(\Psi)$.

Proposition 3.7. *The quasi-affine system $X^q(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R}^s)$ if and only if the quasi-affine system $X_K^q(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R}^s)$ with the same (Bessel) frame bounds. In particular, $X^q(\Psi)$ is a tight frame for $L^2(\mathbb{R}^s)$ if and only if $X_K^q(\Psi)$ is a tight frame for $L^2(\mathbb{R}^s)$.*

Proof. Suppose that $X^q(\Psi)$ is a Bessel system with Bessel bound B. Using the right inequality of (1.1) on the function $g := d^{-\frac{K}{2}} f(M^{-K}\cdot) \in L^2(\mathbb{R}^s)$, where $f \in L^2(\mathbb{R}^s)$, we have

$$\begin{aligned}
B\,\|g\|^2 &\geq \sum_{k<0}\sum_{r\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left|\langle g, d^k E^r\psi(M^k\cdot)\rangle\right|^2 + \sum_{k=0}^{\infty}\sum_{r\in\mathbb{Z}^s}\sum_{l\in\mathcal{L}_k}\sum_{\psi\in\Psi}\left|\langle g, d^{\frac{k}{2}} E^r E_k^l\psi(M^k\cdot)\rangle\right|^2 \\
&= \sum_{r\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left[\sum_{k<0}\left|\langle g, d^k\psi(M^k(\cdot - r))\rangle\right|^2 + \sum_{k=0}^{\infty}\sum_{l\in\mathcal{L}_k}\left|\langle g, d^{\frac{k}{2}}\psi(M^k(\cdot - r) - l)\rangle\right|^2\right] \qquad (3.10)\\
&= \sum_{r\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left[\sum_{k<0}\left|\langle f, d^{\frac{K}{2}+k}\psi(M^{K+k}\cdot - M^k r)\rangle\right|^2 + \sum_{k=0}^{\infty}\sum_{l\in\mathcal{L}_k}\left|\langle f, d^{\frac{K}{2}+\frac{k}{2}}\psi(M^{K+k}\cdot - M^k r - l)\rangle\right|^2\right] \\
&= \sum_{r\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left[\sum_{k<K}\left|\langle f, d^{k-\frac{K}{2}}\psi(M^k\cdot - M^{k-K} r)\rangle\right|^2 + \sum_{k=K}^{\infty}\sum_{l\in\mathcal{L}_{k-K}}\left|\langle f, d^{\frac{k}{2}}\psi(M^k\cdot - M^{k-K} r - l)\rangle\right|^2\right].
\end{aligned}$$

Hence, we obtain

$$B\|f\|^2 \geq \sum_{r\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left[\sum_{k<K}\left|\langle f, d^{k-\frac{K}{2}} E_K^r\psi(M^k\cdot)\rangle\right|^2 + \sum_{k=K}^{\infty}\sum_{l\in\mathcal{L}_{k-K}}\left|\langle f, d^{\frac{k}{2}} E_K^r E_k^l\psi(M^k\cdot)\rangle\right|^2\right]. \qquad (3.11)$$

As f is arbitrary, $X_K^q(\Psi)$ is a Bessel system with the same Bessel bound as $X^q(\Psi)$. In a similar manner, the lower frame bound condition in (1.1) is shown to hold for $X_K^q(\Psi)$

in the event that $X^q(\Psi)$ is a frame for $L^2(\mathbb{R}^s)$. For the converse, the inequality (3.11) is equivalent to (3.10), if we let $f := d^{\frac{K}{2}} g(M^K \cdot)$. $\square$

Hence, the quasi-affine system $X_K^q(\Psi)$ is a frame for $L^2(\mathbb{R}^s)$ for every $K \geq 0$ if and only if the quasi-affine system $X^q(\Psi)$ is a frame for $L^2(\mathbb{R}^s)$ with the same frame bounds. Therefore, Theorem 3.6 is extended in the following corollary as a consequence of Proposition 3.7.

Corollary 3.8. *The affine system $X(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R}^s)$ if and only if its quasi-affine counterpart $X_K^q(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R}^s)$. Further, the two systems have identical frame bounds. In particular, the affine system $X(\Psi)$ is a tight frame if and only if the quasi-affine system $X_K^q(\Psi)$ is a tight frame.*

In other words, Corollary 3.8 shows that the wavelet representation of a function could be expressed either in terms of the affine system $X(\Psi)$ or in terms of the many choices of the shift-invariant quasi-affine system $X_K^q(\Psi)$.

Next, we consider the semi-orthogonal setup of obtaining wavelets for $L^2(\mathbb{R}^s)$ from FMRAs. The following proposition (found in [2] for the 1-dimensional single-generator case of dilation factor $M = 2$) shows that as long as $E(\Phi)$ is a frame for $V(\Phi)$, then the MRA $\{V^k(\Phi)\}$ is an FMRA with uniform bounds.

Proposition 3.9. *If $E(\Phi)$ is a (Bessel system) frame for $V(\Phi)$, then $E(\{d^{\frac{k}{2}} E_k^l \phi(M^k \cdot) : \phi \in \Phi, l \in \mathcal{L}_k\})$ is a (Bessel system) frame for $V^k(\Phi)$ with the same (Bessel) frame bounds as $E(\Phi)$.*

Proof. Let $g \in V^k(\Phi)$ and $f = d^{-\frac{k}{2}} g(M^{-k} \cdot) \in V(\Phi)$ and $E(\Phi)$ be a Bessel system for $V(\Phi)$ with Bessel bound B. Then

$$\begin{aligned}
\sum_{l \in \mathbb{Z}^s} \sum_{\phi \in \Phi} \left| \langle f, E^l \phi \rangle \right|^2 &= \sum_{l \in \mathbb{Z}^s} \sum_{\phi \in \Phi} \left| \langle d^{-\frac{k}{2}} g(M^{-k} \cdot), E^l \phi \rangle \right|^2 = \sum_{l \in \mathbb{Z}^s} \sum_{\phi \in \Phi} \left| \langle g, d^{\frac{k}{2}} (E^l \phi)(M^k \cdot) \rangle \right|^2 \\
&= \sum_{l \in \mathbb{Z}^s} \sum_{\phi \in \Phi} \left| \langle g, d^{\frac{k}{2}} E_k^l \phi(M^k \cdot) \rangle \right|^2 = \sum_{r \in \mathbb{Z}^s} \sum_{l \in \mathcal{L}_k} \sum_{\phi \in \Phi} \left| \langle g, d^{\frac{k}{2}} E^r E_k^l \phi(M^k \cdot) \rangle \right|^2,
\end{aligned}$$

where the last two sums follows from (3.1). Using the right inequality of (1.1), we have

$$\sum_{r \in \mathbb{Z}^s} \sum_{l \in \mathcal{L}_k} \sum_{\phi \in \Phi} \left| \langle g, d^{\frac{k}{2}} E^r E_k^l \phi(M^k \cdot) \rangle \right|^2 \leq B \|f\|^2 = B \|g\|^2,$$

and so $E(\{d^{\frac{k}{2}} E_k^l \phi(M^k \cdot) : \phi \in \Phi, l \in \mathcal{L}_k\})$ is a Bessel system for $V^k(\Phi)$ with the same Bessel bound as $E(\Phi)$. Similarly, in the event that $E(\Phi)$ is also a frame, we could show that the lower frame bound inequality of (1.1) carries over as well. $\square$

Lemma 3.10. *Let W^k be the orthogonal complement of $V^k(\Phi)$ in $V^{k+1}(\Phi)$. We have*

$$W^k = \{f \in L^2(\mathbb{R}^s) : f(M^{-k}\cdot) \in W^0\}. \tag{3.12}$$

Proof. For any $f \in V^k(\Phi)$ and $g \in W^k \subseteq V^{k+1}(\Phi)$, we have $f(M^{-k}\cdot) \in V(\Phi)$ and $g(M^{-k}\cdot) \in V^1(\Phi)$. Since $f(M^{-k}\cdot)$ is arbitrary in $V(\Phi)$ and $\langle d^{-\frac{k}{2}} f(M^{-k}\cdot), d^{-\frac{k}{2}} g(M^{-k}\cdot)\rangle = \langle f, g\rangle = 0$, we deduce that $g(M^{-k}\cdot)$ lies in W^0. □

Lemma 3.11. *Let $\{V^k(\Phi)\}$ be an MRA of $L^2(\mathbb{R}^s)$. For each $k \in \mathbb{Z}$, let W^k be the orthogonal complement of $V^k(\Phi)$ in $V^{k+1}(\Phi)$. Then the subspaces W^k are pairwise orthogonal and $L^2(\mathbb{R}^s) = \bigoplus^{\perp}_{k\in\mathbb{Z}} W^k$.*

Proof. For $k < n$, and given any $f_k \in W^k \subseteq V^{k+1}(\Phi) \subseteq V^n(\Phi)$ and $f_n \in W^n$, clearly we have $\langle f_k, f_n\rangle = 0$. Let P_k be the orthogonal projector from $L^2(\mathbb{R}^s)$ onto $V^k(\Phi)$. Then $W^k = \{f - P_k f : f \in V^{k+1}(\Phi)\}$. Observing that $\lim_{k\to\infty} P_k f = f$ and $\lim_{k\to-\infty} P_k f = 0$, we deduce that for any $f \in L^2(\mathbb{R}^s)$, we have

$$f = \sum_{k\in\mathbb{Z}} (P_{k+1} f - P_k f).$$

Therefore, the result of the direct sum follows since $P_{k+1} - P_k$ is the orthogonal projector from $L^2(\mathbb{R}^s)$ onto W^k. □

Proposition 3.12. *Let $\{V^k(\Phi)\}$ be an FMRA of $L^2(\mathbb{R}^s)$ and W^k be the orthogonal complement of $V^k(\Phi)$ in $V^{k+1}(\Phi)$. Let $\Psi \subset W^0$ be finite. Then $X(\Psi)$ is a (Bessel system) frame for $L^2(\mathbb{R}^s)$ if and only if $E(\Psi)$ is a (Bessel system) frame for W^0 with the same (Bessel) frame bounds.*

Proof. ($\Rightarrow$) Assume that $X(\Psi)$ is a Bessel system with Bessel bound B. By Lemma 3.10, for every $\psi \in \Psi$, $d^{\frac{k}{2}} E^l_k \psi(M^k\cdot)$ lies in W^k. Let $f \in W^0$ be an arbitrary function. Applying Lemma 3.11 shows that $\sum_{k\in\mathbb{Z}} \sum_{r\in\mathbb{Z}^s} \sum_{l\in\mathcal{L}_k} \left|\langle f, d^{\frac{k}{2}} E^r E^l_k \psi(M^k\cdot)\rangle\right|^2 = \sum_{r\in\mathbb{Z}^s} |\langle f, E^r\psi\rangle|^2$, where $\psi \in \Psi$. Consequently, using the right inequality of (1.1) for $X(\Psi)$ on f, we obtain

$$\sum_{r\in\mathbb{Z}^s} \sum_{\psi\in\Psi} |\langle f, E^r\psi\rangle|^2 \leq B \|f\|^2,$$

and it follows that $E(\Psi)$ is a Bessel system with the same Bessel bound as $X(\Psi)$. In a similar manner, the lower frame bound condition in (1.1) is shown to hold for $E(\Psi)$ in the event that $X(\Psi)$ is also a frame for $L^2(\mathbb{R}^s)$.

($\Leftarrow$) Suppose that $E(\Psi)$ is a Bessel system with Bessel bound B. Lemma 3.10 shows that given a fixed $k \in \mathbb{Z}$, for any $f \in W^k$, we have $f(M^{-k}\cdot) \in W^0$. Using the fact that $\langle f, d^{\frac{k}{2}} E_k^l \psi(M^k\cdot)\rangle = \langle f, d^{\frac{k}{2}} (E^l\psi)(M^k\cdot)\rangle = \langle d^{-\frac{k}{2}} f(M^{-k}\cdot), E^l\psi\rangle$ for every $\psi \in \Psi$ and applying (3.1) and the right inequality of (1.1) for $E(\Psi)$ on $d^{-\frac{k}{2}} f(M^{-k}\cdot)$, we have

$$\begin{aligned}\sum_{r\in\mathbb{Z}^s}\sum_{l\in\mathcal{L}_k}\sum_{\psi\in\Psi}\left|\langle f, d^{\frac{k}{2}} E^r E_k^l \psi(M^k\cdot)\rangle\right|^2 &= \sum_{l\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left|\langle f, d^{\frac{k}{2}} E_k^l \psi(M^k\cdot)\rangle\right|^2 \\ \le B\left\|d^{-\frac{k}{2}} f(M^{-k}\cdot)\right\|^2 &= B\left\|f\right\|^2. \end{aligned} \tag{3.13}$$

Hence, for a given $k \in \mathbb{Z}$, $E(\{d^{\frac{k}{2}} E_k^l \psi(M^k\cdot) : \psi \in \Psi, l \in \mathcal{L}_k\})$ is a Bessel system for W^k with the Bessel bound B.

Next, for an arbitrary $f \in L^2(\mathbb{R}^s)$, by Lemma 3.11, $f = \sum_{k\in\mathbb{Z}} f_k$, where for each $k \in \mathbb{Z}$, $f_k \in W^k$, and if $k \neq n$, $\langle f_n, d^{\frac{k}{2}} E_k^l \psi(M^k\cdot)\rangle = 0$ for all $\psi \in \Psi$ and $l \in \mathcal{L}_k$. Therefore,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left|\langle f, d^{\frac{k}{2}} E_k^l \psi(M^k\cdot)\rangle\right|^2 &= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left|\sum_{n\in\mathbb{Z}}\langle f_n, d^{\frac{k}{2}} E_k^l \psi(M^k\cdot)\rangle\right|^2 \\ = \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left|\langle f_k, d^{\frac{k}{2}} E_k^l \psi(M^k\cdot)\rangle\right|^2. \end{aligned} \tag{3.14}$$

It follows from (3.13) and Lemma 3.11 that

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^s}\sum_{\psi\in\Psi}\left|\langle f_k, d^{\frac{k}{2}} E_k^l \psi(M^k\cdot)\rangle\right|^2 \le B\sum_{k\in\mathbb{Z}}\left\|f_k\right\|^2 = B\left\|f\right\|^2. \tag{3.15}$$

In a similar manner, the lower frame bound condition in (1.1) is shown to hold for $X(\Psi)$ in the event that $E(\Psi)$ is also a frame for W^0.

□

Proposition 3.12 (found in [2] for the 1-dimensional single-generator case of dilation factor $M = 2$) shows that it suffices to ensure that $E(\Psi)$ is a frame for W^0 in order for the affine system $X(\Psi)$ derived from an FMRA to be a frame. We shall describe in Section 3.4 on the construction of such frames using the corresponding periodic analogue of FMRAs.

3.2 Periodic Formulation

For each $j \in \mathcal{R}_K$, define the *pre-Gramian* of the $2\pi M^{-K}\mathbb{Z}^s$ shift-invariant system $T_K(\Omega_K)$ at level K given by (1.16) and (1.17) to be the matrix-valued function

$$J_{K,\Omega_K}(j) := \left[\widehat{\varphi}_{K,j}\right]_{\varphi\in\Omega_K},$$

where

$$\widehat{\varphi}_{K,j} := \{\widehat{\varphi}(n)\mathbf{1}_{j+D^K\mathbb{Z}^s}(n)\}_{n\in\mathbb{Z}^s}.$$

Based on the sequence $\widehat{\varphi}_{K,j}$, we define the j^{th}-*polyphase harmonic* of the function φ at level K as

$$\varphi_{K,j}(t) := \sum_{n\in\mathbb{Z}^s} \widehat{\varphi}_{K,j}(n)\mathrm{e}^{\mathrm{i}n\cdot t}. \tag{3.16}$$

We also define the j^{th}-space of polyphase harmonics at level K, $\Theta_{2\pi}^{K,j} \subset L^2(\mathbb{T}^s)$ consisting of all functions with Fourier coefficients sampled on the lattice $j + D^K\mathbb{Z}^s$ to be

$$\Theta_{2\pi}^{K,j} := \{f_{K,j} : f \in L^2(\mathbb{T}^s)\}. \tag{3.17}$$

We further define the j^{th}-$V_{2\pi}^K$ subspace of polyphase harmonics at level K to be

$$V_{2\pi}^{K,j} := \Theta_{2\pi}^{K,j} \cap V_{2\pi}^K, \tag{3.18}$$

where $V_{2\pi}^K$ is a $2\pi M^{-K}\mathbb{Z}^s$ shift-invariant subspace of $L^2(\mathbb{T}^s)$ generated by some countable subset Ω_K of $L^2(\mathbb{T}^s)$, i.e. $V_{2\pi}^K := V_{2\pi}^K(\Omega_K)$.

The *Gramian* of the set $T_K(\Omega_K)$ at level K for each $j \in \mathcal{R}_K$ is defined to be

$$\mathrm{M}_{K,\Omega_K}(j) := J_{K,\Omega_K}(j)J^*_{K,\Omega_K}(j) = \left[\langle\widehat{\varphi}_{K,j}, \widehat{\phi}_{K,j}\rangle_{l^2(\mathbb{Z}^s)}\right]_{\varphi,\phi\in\Omega_K}.$$

The *spectrum* of the $2\pi M^{-K}\mathbb{Z}^s$ shift-invariant space $V_{2\pi}^K(\Omega_K)$ is defined as

$$\sigma_K(V_{2\pi}^K(\Omega_K)) := \{j \in \mathcal{R}_K : \|\widehat{\varphi}_{K,j}\|_{l^2(\mathbb{Z}^s)} > 0 \text{ for some } \varphi \in \Omega_K\}. \tag{3.19}$$

We view the matrices $\mathrm{M}_{K,\Omega_K}(j)$, $j \in \mathcal{R}_K$, as linear operators and in the event of they being boundedly invertible, we denote their bounded inverses by $\mathrm{M}_{K,\Omega_K}(j)^{-1}$. (For those j such that the underlying operator is not well defined or is unbounded, we take the norm of the underlying operator to be ∞.) For the $\mathcal{R}_K$-periodic sequences

$$\mathsf{M}_{K,\Omega_K} : j \mapsto \|\mathrm{M}_{K,\Omega_K}(j)\|, \qquad \mathsf{M}^-_{K,\Omega_K} : j \mapsto \left\|\mathrm{M}_{K,\Omega_K}(j)^{-1}\right\|$$

on $S \subseteq \mathcal{R}_K$, we consider their $L_\infty(S)$-norm, where $L_\infty(S)$ denotes the space of all bounded complex-valued $\mathcal{R}_K$-periodic sequences on S. These $L_\infty(S)$-norms are used to characterize the Bessel and frame properties of $T_K(\Omega_K)$.

The next theorem is the periodic analogue of Theorem 3.4 and likewise we shall look at properties of the frequency samples of the periodic functions.

Theorem 3.13. [5] *Let $\Omega_K \subset L^2(\mathbb{T}^s)$ be countable and consider the shift-invariant system $T_K(\Omega_K)$.*

(i) *The system $T_K(\Omega_K)$ is a Bessel system if and only if $\|\mathsf{M}_{K,\Omega_K}\|_{L^\infty(\mathcal{R}_K)}$ is finite. Further, the Bessel bound is equal to $d^K \|\mathsf{M}_{K,\Omega_K}\|_{L^\infty(\mathcal{R}_K)}$.*

(ii) *Assume that $T_K(\Omega_K)$ is a Bessel system. The system $T_K(\Omega_K)$ is a frame for $V_{2\pi}^K(\Omega_K)$ if and only if $1/\left\|\mathsf{M}_{K,\Omega_K}^-\right\|_{L^\infty(\sigma_K(V_{2\pi}^K(\Omega_K)))}$ is finite. Further, the lower frame bound is given by $d^K/\left\|\mathsf{M}_{K,\Omega_K}^-\right\|_{L^\infty(\sigma_K(V_{2\pi}^K(\Omega_K)))}$. In particular, $T_K(\Omega_K)$ is a tight frame for $V_{2\pi}^K(\Omega_K)$ if and only if its Gramian $\mathrm{M}_{K,\Omega_K}(j)$ is d^{-K} times an orthogonal projector for every $j \in \mathcal{R}_K$.*

Lemma 3.14. *Let $K, k \geq 0$, and $f, g \in L^2(\mathbb{T}^s)$. We have*

(i) $(T_k^l f)^\wedge = \mathcal{M}_{0,k}^l \widehat{f}$,

(ii) *for every $l \in \mathcal{L}_k$ and $j \in \mathcal{R}_K$,* $(T_k^l f)^{\wedge o}_{K,j} = \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-k}l} \mathcal{M}_{K,k}^l \widehat{f}^o_{K,j}$,

(iii) *for every $l \in \mathcal{L}_k$,* $T_k^l f = \sum_{j\in\mathcal{R}_K} T_k^l f_{K,j} = \sum_{j\in\mathcal{R}_K} \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-k}l} \sum_{n\in\mathbb{Z}^s} (\mathcal{M}_{K,k}^l \widehat{f}_{K,j})(n) \mathrm{e}^{\mathrm{i}(j+D^K n)\cdot}$.

(iv) *for every $l \in \mathcal{L}_k$, where $k \leq K$,* $T_k^l f = \sum_{j\in\mathcal{R}_K} T_k^l f_{K,j} = \sum_{j\in\mathcal{R}_K} \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-k}l} f_{K,j}$.

(v) *for every $j \in \mathcal{R}_K$,* $\langle f_{K,j}, g\rangle_{L^2(\mathbb{T}^s)} = \langle f_{K,j}, g_{K,j}\rangle_{L^2(\mathbb{T}^s)}$.

(vi) $\sum_{l\in\mathcal{L}_K} \left|\langle T_K^l f, g\rangle_{L^2(\mathbb{T}^s)}\right|^2 = d^K \sum_{j\in\mathcal{R}_K} \left|\langle f_{K,j}, g\rangle_{L^2(\mathbb{T}^s)}\right|^2$ *and* $\|f\|^2_{L^2(\mathbb{T}^s)} = \sum_{j\in\mathcal{R}_K} \|f_{K,j}\|^2_{L^2(\mathbb{T}^s)}$.

Proof. Part (i) is shown using (ii) for the case of $K = j = 0$. For (ii), let $n \in \mathbb{Z}^s$. Then we have

$$\begin{aligned}
(T_k^l f)^{\wedge o}_{K,j}(n) &= (T_k^l f)^\wedge(j + D^K n) = \langle T_k^l f, \mathrm{e}^{\mathrm{i}(j+D^K n)\cdot}\rangle_{L^2(\mathbb{T}^s)} \\
&= \mathrm{e}^{-\mathrm{i}2\pi(j+D^K n)\cdot M^{-k}l} \langle f, \mathrm{e}^{\mathrm{i}(j+D^K n)\cdot}\rangle_{L^2(\mathbb{T}^s)} = \mathrm{e}^{-\mathrm{i}2\pi(j+D^K n)\cdot M^{-k}l} \widehat{f}(j + D^K n) \\
&= \mathrm{e}^{-\mathrm{i}2\pi(j+D^K n)\cdot M^{-k}l} \widehat{f}^o_{K,j}(n).
\end{aligned}$$

For (iii), using (3.16) and (3.17), it is clear that $T_k^l f = \sum_{j\in\mathcal{R}_K} T_k^l f_{K,j}$. It remains to check that

$$T_k^l f_{K,j}(t) = \sum_{n\in\mathbb{Z}^s} \widehat{f}_{K,j}(n) \mathrm{e}^{\mathrm{i}(j+D^K n)\cdot(t-2\pi M^{-k}l)} = \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-k}l} \sum_{n\in\mathbb{Z}^s} \mathrm{e}^{-\mathrm{i}2\pi D^K n\cdot M^{-k}l} \widehat{f}_{K,j}(n) \mathrm{e}^{\mathrm{i}(j+D^K n)\cdot t}.$$

We obtain (iv) as a consequence of (iii). Part (v) follows from Plancherel's theorem, i.e.

$$\langle f_{K,j}, g\rangle_{L^2(\mathbb{T}^s)} = \langle \{\widehat{f_{K,j}(n)}\}_{n\in\mathbb{Z}^s}, \{\widehat{g}(n)\}_{n\in\mathbb{Z}^s}\rangle_{l^2(\mathbb{Z}^s)} = \langle \widehat{f}_{K,j}, \widehat{g}_{K,j}\rangle_{l^2(\mathbb{Z}^s)} = \langle f_{K,j}, g_{K,j}\rangle_{L^2(\mathbb{T}^s)}.$$

For (vi), we use (iv) to show that

$$\sum_{l\in\mathcal{L}_K} \left|\langle T_K^l f, g\rangle_{L^2(\mathbb{T}^s)}\right|^2 = \sum_{l\in\mathcal{L}_K}\langle \sum_{j\in\mathcal{R}_K} \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-K}l} f_{K,j}, g\rangle_{L^2(\mathbb{T}^s)}\langle g, \sum_{r\in\mathcal{R}_K} \mathrm{e}^{-\mathrm{i}2\pi r\cdot M^{-K}l} f_{K,r}\rangle_{L^2(\mathbb{T}^s)}$$
$$= \sum_{j\in\mathcal{R}_K}\sum_{r\in\mathcal{R}_K}\sum_{l\in\mathcal{L}_K} \mathrm{e}^{\mathrm{i}2\pi(r-j)\cdot M^{-K}l}\langle f_{K,j}, g\rangle_{L^2(\mathbb{T}^s)}\langle g, f_{K,r}\rangle_{L^2(\mathbb{T}^s)}$$
$$= \sum_{j\in\mathcal{R}_K}\sum_{r\in\mathcal{R}_K} d^K\delta_{j,r}\langle f_{K,j}, g\rangle_{L^2(\mathbb{T}^s)}\langle g, f_{K,r}\rangle_{L^2(\mathbb{T}^s)}.$$

Moreover, $\|f\|^2_{L^2(\mathbb{T}^s)} = \langle \sum_{j\in\mathcal{R}_K} f_{K,j}, f\rangle_{L^2(\mathbb{T}^s)} = \sum_{j\in\mathcal{R}_K}\langle f_{K,j}, f\rangle_{L^2(\mathbb{T}^s)} = \sum_{j\in\mathcal{R}_K}\langle f_{K,j}, f_{K,j}\rangle_{L^2(\mathbb{T}^s)}$. □

Next, we examine the periodic quasi-affine system $X^q_{2\pi,K}$ at level K given in (1.16) and (1.17). This system could be expressed as follows:

$$\begin{aligned}X^q_{2\pi,K} = \{&d^{-\frac{K}{2}}\phi_0(\cdot - 2\pi M^{-K}r) : \phi_0\in\Phi_0, r\in\mathcal{L}_K\}\cup\\ \{&d^{\frac{k}{2}-\frac{K}{2}}\psi_k(\cdot - 2\pi M^{-K}r) : \psi_k\in\Psi_k, r\in\mathcal{L}_K, 0\le k<K\}\cup\\ \{&\psi_k(\cdot - 2\pi M^{-k}r) : \psi_k\in\Psi_k, r\in\mathcal{L}_k, k\ge K\},\end{aligned}$$

since Lemma 3.2 shows that for $k \ge K$, $j\in\mathcal{L}_K$ and $l\in\mathcal{L}_{k-K}$, $\psi_k(\cdot - 2\pi M^{-k}(M^{k-K}j + l)) = \psi_k(\cdot - 2\pi M^{-k}r)$ for some $r\in\mathcal{L}_k$. In contrast with the quasi-affine systems of $L^2(\mathbb{R}^s)$, we already have $X^q_{2\pi,0} = X_{2\pi}$, i.e. the quasi-affine structure of $X^q_{2\pi,K}$ for $K>0$ is different from that of $X^q_{2\pi,0}$. Therefore, we cannot expect to obtain results in the periodic setting fully analogous to that of $L^2(\mathbb{R}^s)$.

The next proposition is a partial periodic analogue of Corollary 3.8.

Proposition 3.15. *Fix $K \ge 0$. The periodic quasi-affine system $X^q_{2\pi,K}$ is a (Bessel system) frame for $L^2(\mathbb{T}^s)$ if the periodic affine system $X_{2\pi}$ is a (Bessel system) frame for $L^2(\mathbb{T}^s)$ with the same (Bessel) frame bounds. In particular, $X^q_{2\pi,K}$ is a tight frame for $L^2(\mathbb{T}^s)$ if $X_{2\pi}$ is a tight frame for $L^2(\mathbb{T}^s)$.*

Proof. Suppose that $X_{2\pi}$ is a Bessel system for $L^2(\mathbb{T}^s)$ with Bessel bound B. Using the right inequality of (1.1) on a function $f\in L^2(\mathbb{T}^s)$, we have

$$B\left\|T_K^{-r}f\right\|^2_{L^2(\mathbb{T}^s)} \ge \sum_{\phi_0\in\Phi_0}\left|\langle f, T_K^r\phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{\infty}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_K^r T_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2.$$

Then we have

$$\begin{aligned}Bd^K\|f\|^2_{L^2(\mathbb{T}^s)} \ \ge\ \sum_{r\in\mathcal{L}_K}\Bigg[&\sum_{\phi_0\in\Phi_0}\left|\langle f, T_K^r\phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{K-1}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_K^r T_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2 +\\ &\sum_{k=K}^{\infty}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_K^r T_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2\Bigg].\end{aligned} \tag{3.20}$$

Since Lemma 3.3 shows that each element in the codomain of $\iota : \mathcal{L}_K \times \mathcal{L}_k \to \mathcal{L}_K$ given by $\iota : (r,l) \mapsto r + M^{K-k} l$ has a preimage of d^k elements, we could express the sum of the second summand on the right hand side of (3.20) as

$$\begin{aligned}
&\sum_{k=0}^{K-1} \sum_{r\in\mathcal{L}_K} \sum_{l\in\mathcal{L}_k} \sum_{\psi_k\in\Psi_k} \left|\langle f, T_K^r T_k^l \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2 \\
&= \sum_{k=0}^{K-1} \sum_{r\in\mathcal{L}_K} \sum_{l\in\mathcal{L}_k} \sum_{\psi_k\in\Psi_k} \left|\langle f, \psi_k(\cdot - 2\pi M^{-K}(r + M^{K-k} l))\rangle_{L^2(\mathbb{T}^s)}\right|^2 \qquad (3.21)\\
&= \sum_{k=0}^{K-1} \sum_{r\in\mathcal{L}_K} \sum_{l\in\mathcal{L}_k} \sum_{\psi_k\in\Psi_k} \left|\langle f, \psi_k(\cdot - 2\pi M^{-K} r)\rangle_{L^2(\mathbb{T}^s)}\right|^2 = \sum_{k=0}^{K-1} \sum_{r\in\mathcal{L}_K} \sum_{\psi_k\in\Psi_k} \left|\langle f, d^{\frac{k}{2}} T_K^r \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2,
\end{aligned}$$

independently of our choice of coset representatives. Next, the sum of the third summand on the right hand side of (3.20) could be expressed as

$$\sum_{k=K}^{\infty} \sum_{r\in\mathcal{L}_K} \sum_{l\in\mathcal{L}_k} \sum_{\psi_k\in\Psi_k} \left|\langle f, T_K^r T_k^l \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2 = d^K \sum_{k=K}^{\infty} \sum_{l\in\mathcal{L}_k} \sum_{\psi_k\in\Psi_k} \left|\langle f, T_k^l \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2.$$

Therefore, (3.20) is equivalent to

$$\begin{aligned}
B\,\|f\|^2_{L^2(\mathbb{T}^s)} \geq &\sum_{r\in\mathcal{L}_K} \left[\sum_{\phi_0\in\Phi_0} \left|\langle f, d^{-\frac{K}{2}} T_K^r \phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{K-1} \sum_{\psi_k\in\Psi_k} \left|\langle f, d^{\frac{k}{2}-\frac{K}{2}} T_K^r \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2 \right] \\
&+ \sum_{k=K}^{\infty} \sum_{l\in\mathcal{L}_k} \sum_{\psi_k\in\Psi_k} \left|\langle f, T_k^l \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2. \qquad (3.22)
\end{aligned}$$

As f is arbitrary, $X^q_{2\pi,K}$ is a Bessel system for $L^2(\mathbb{T}^s)$ with the same Bessel bound as $X_{2\pi}$. In a similar manner, the lower frame bound condition in (1.1) is shown to hold for $X^q_{2\pi,K}$ in the event that $X_{2\pi}$ is a frame for $L^2(\mathbb{T}^s)$. □

Proposition 3.16 shows that the periodic affine system $X_{2\pi}$ must satisfy the frame condition for all the j^{th} spaces of polyphase harmonics $\Theta^{K,j}_{2\pi}$ given in (3.17), i.e.

$$A\|f_{K,j}\|^2_{L^2(\mathbb{T}^s)} \leq \sum_{\phi_0\in\Phi_0} \left|\langle f_{K,j}, \phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{\infty} \sum_{l\in\mathcal{L}_k} \sum_{\psi_k\in\Psi_k} \left|\langle f_{K,j}, T_k^l \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2 \leq B\|f_{K,j}\|^2_{L^2(\mathbb{T}^s)} \qquad (3.23)$$

for all $f_{K,j} \in \Theta^{K,j}_{2\pi}$, in order for the periodic quasi-affine system $X^q_{2\pi,K}$ to be a frame for $L^2(\mathbb{T}^s)$.

Proposition 3.16. *Fix $K \geq 0$. The periodic affine system $X_{2\pi}$ satisfies the (Bessel) frame condition for $\Theta^{K,j}_{2\pi}$ for every $j \in \mathcal{R}_K$ if the periodic quasi-affine system $X^q_{2\pi,K}$ is* *a (Bessel system) frame for $L^2(\mathbb{T}^s)$ with the same (Bessel) frame bounds. In particular, $X_{2\pi}$ satisfies the tight frame condition for $\Theta_{2\pi}^{K,j}$ for every $j \in \mathcal{R}_K$ if $X_{2\pi,K}^q$ is a tight frame for $L^2(\mathbb{T}^s)$.*

Proof. Suppose that $X_{2\pi,K}^q$ is a Bessel system for $L^2(\mathbb{T}^s)$ with Bessel bound B. Next, with the help of Lemma 3.14, the equivalence of (3.20) and (3.22) and using the right inequality of (1.1) on a function $f \in \Theta_{2\pi}^{K,j}$, where $j \in \mathcal{R}_K$, we have

$$\begin{aligned} B\|f\|^2_{L^2(\mathbb{T}^s)} &\geq \sum_{r\in\mathcal{L}_K}\left[\sum_{\phi_0\in\Phi_0}\left|\langle f, d^{-\frac{K}{2}}T_K^r\phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{\infty}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, d^{-\frac{K}{2}}T_K^rT_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2\right] \\ &= \sum_{r\in\mathcal{R}_K}\left[\sum_{\phi_0\in\Phi_0}\left|\langle f_{K,r}, \phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{\infty}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f_{K,r}, T_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2\right], \end{aligned}$$

and so (3.23) holds. As f is arbitrary, $X_{2\pi}$ satisfies the Bessel condition for $\Theta_{2\pi}^{K,j}$ with the same Bessel bound as $X_{2\pi,K}^q$. In a similar manner, the lower frame bound condition for $\Theta_{2\pi}^{K,j}$ in (3.23) is shown to hold for $X_{2\pi}$ in the event that $X_{2\pi,K}^q$ is a frame for $L^2(\mathbb{T}^s)$. $\square$

In practical applications, we could only utilize the restricted periodic affine system $X_{2\pi}^R$ given in (1.18) and also the restricted periodic quasi-affine system $X_{2\pi,K}^{q,R}$ given in (1.19) and (1.20). Henceforth, it is also desirable to establish the analogue of Propositions 3.15 and 3.16 for these systems.

Proposition 3.17. *Fix $R \geq K \geq 0$. The restricted periodic quasi-affine system $X_{2\pi,K}^{q,R}$ is a (Bessel system) frame for its closed linear span $V_{2\pi}^R$ if the restricted periodic affine system $X_{2\pi}^R$ is a (Bessel system) frame for $V_{2\pi}^R$ with the same (Bessel) frame bounds. In particular, $X_{2\pi,K}^{q,R}$ is a tight frame for $V_{2\pi}^R$ if $X_{2\pi}^R$ is a tight frame for $V_{2\pi}^R$.*

Proof. Suppose that $X_{2\pi}^R$ is a Bessel system for $V_{2\pi}^R$ with Bessel bound B. Using the right inequality of (1.1) on a function $f \in V_{2\pi}^R$, we have

$$B\left\|T_K^{-r}f\right\|^2_{L^2(\mathbb{T}^s)} \geq \sum_{\phi_0\in\Phi_0}\left|\langle f, T_K^r\phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{R}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_K^rT_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2.$$

Then we have

$$\begin{aligned} Bd^K\|f\|^2_{L^2(\mathbb{T}^s)} \;\geq\; &\sum_{r\in\mathcal{L}_K}\left[\sum_{\phi_0\in\Phi_0}\left|\langle f, T_K^r\phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{K-1}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_K^rT_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \right. \\ &\left. \sum_{k=K}^{R}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_K^rT_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2\right]. \end{aligned} \tag{3.24}$$

Computing in a manner similar to (3.21), the sum of the second summand on the right hand side of (3.24) could be expressed as

$$\sum_{k=0}^{K-1}\sum_{r\in\mathcal{L}_K}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_K^r T_k^l \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2 = \sum_{k=0}^{K-1}\sum_{r\in\mathcal{L}_K}\sum_{\psi_k\in\Psi_k}\left|\langle f, d^{\frac{k}{2}} T_K^r \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2,$$

independently of our choice of coset representatives. Next, the sum of the third summand on the right hand side of (3.24) could be expressed as

$$\sum_{k=K}^{R}\sum_{r\in\mathcal{L}_K}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_K^r T_k^l \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2 = d^K\sum_{k=K}^{R}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_k^l \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2.$$

Therefore, (3.24) is equivalent to

$$\begin{aligned}B\,\|f\|^2_{L^2(\mathbb{T}^s)} \geq &\sum_{r\in\mathcal{L}_K}\left[\sum_{\phi_0\in\Phi_0}\left|\langle f, d^{-\frac{K}{2}} T_K^r \phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{K-1}\sum_{\psi_k\in\Psi_k}\left|\langle f, d^{\frac{k}{2}-\frac{K}{2}} T_K^r \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2\right]\\ &+\sum_{k=K}^{R}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, T_k^l \psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2.\end{aligned} \tag{3.25}$$

As f is arbitrary, $X^{q,R}_{2\pi,K}$ is a Bessel system $V^R_{2\pi}$ with the same Bessel bound as $X^R_{2\pi}$. In a similar manner, the lower frame bound condition in (1.1) is shown to hold for $X^{q,R}_{2\pi,K}$ in the event that $X^R_{2\pi}$ is a frame for $V^R_{2\pi}$. □

Likewise Proposition 3.18 shows that the restricted periodic affine system $X^R_{2\pi}$ must satisfy the frame condition for all the j^{th} spaces of polyphase harmonics $\Theta^{K,j}_{2\pi}\cap V^R_{2\pi}$ given in (3.17), i.e.

$$A\|f_{K,j}\|^2_{L^2(\mathbb{T}^s)}\leq\sum_{\phi_0\in\Phi_0}\left|\langle f_{K,j},\phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2+\sum_{k=0}^{R}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f_{K,j}, T_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2\leq B\|f_{K,j}\|^2_{L^2(\mathbb{T}^s)} \tag{3.26}$$

for all $f_{K,j}\in\Theta^{K,j}_{2\pi}\cap V^R_{2\pi}$, in order for the restricted periodic quasi-affine system $X^{q.R}_{2\pi,K}$ to be a frame for $V^R_{2\pi}$.

Proposition 3.18. *Fix $R\geq K\geq 0$. The restricted periodic affine system $X^R_{2\pi}$ satisfies the* (*Bessel*) *frame condition for $\Theta^{K,j}_{2\pi}\cap V^R_{2\pi}$ for every $j\in\mathcal{R}_K$ if the restricted periodic quasi-affine system $X^{q,R}_{2\pi,K}$ is a* (*Bessel system*) *frame for its closed linear span $V^R_{2\pi}$ with the same* (*Bessel*) *frame bounds. In particular, $X^R_{2\pi}$ satisfies the tight frame condition for $\Theta^{K,j}_{2\pi}\cap V^R_{2\pi}$ for every $j\in\mathcal{R}_K$ if $X^{q,R}_{2\pi,K}$ is a tight frame for $V^R_{2\pi}$.*

Proof. Suppose that $X_{2\pi,K}^{q,R}$ is a Bessel system $V_{2\pi}^R$ with Bessel bound B. With the help of Lemma 3.14, the equivalence of (3.24) and (3.25) and using the right inequality of (1.1) on a function $f \in \Theta_{2\pi}^{K,j} \cap V_{2\pi}^R$, where $j \in \mathcal{R}_K$, we have

$$\begin{aligned} B\,\|f\|_{L^2(\mathbb{T}^s)}^2 &\geq \sum_{r\in\mathcal{L}_K}\left[\sum_{\phi_0\in\Phi_0}\left|\langle f, d^{-\frac{K}{2}}T_K^r\phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{R}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f, d^{-\frac{K}{2}}T_K^rT_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2\right] \\ &= \sum_{r\in\mathcal{R}_K}\left[\sum_{\phi_0\in\Phi_0}\left|\langle f_{K,r}, \phi_0\rangle_{L^2(\mathbb{T}^s)}\right|^2 + \sum_{k=0}^{R}\sum_{l\in\mathcal{L}_k}\sum_{\psi_k\in\Psi_k}\left|\langle f_{K,r}, T_k^l\psi_k\rangle_{L^2(\mathbb{T}^s)}\right|^2\right], \end{aligned}$$

and (3.26) holds. As f is arbitrary, $X_{2\pi}^R$ satisfies the Bessel condition for $\Theta_{2\pi}^{K,j} \cap V_{2\pi}^R$ with the same Bessel bound as $X_{2\pi,K}^{q,R}$. In a similar manner, the lower frame bound condition for $\Theta_{2\pi}^{K,j} \cap V_{2\pi}^R$ in (3.26) is shown to hold for $X_{2\pi}^R$ in the event that $X_{2\pi,K}^{q,R}$ is a frame for $V_{2\pi}^R$. □

We remark that a finite dimensional spanning set always forms a frame for its linear span. Hence the conditions and results of Propositions 3.17 and 3.18 always hold. The additional information supplied by these two propositions is about the preservation of the (Bessel) frame bounds of the respective systems.

Let us now review several results on periodic MRAs and periodic affine systems constructed from them. The following states the requirements for Condition (i), i.e. nesting property, of a periodic MRA to be satisfied.

Proposition 3.19. [24] *For each $k \geq 0$, let $\Phi_k := \left[\phi_k^m\right]_{m=1}^{\rho}$ be a subset of $L^2(\mathbb{T}^s)$ and $v_{k,j}^m := (\phi_k^m)_{k,j}$ be the corresponding polyphase harmonics given by* (3.16). *Then the following are equivalent for each $k \geq 0$.*

(i) $V_{2\pi}^k(\Phi_k) \subseteq V_{2\pi}^{k+1}(\Phi_{k+1})$.

(ii) *There exists $H_{k+1} \in \mathcal{S}(M^{k+1})^{\rho\times\rho}$ such that*

$$\Phi_k = \sum_{l\in\mathcal{L}_{k+1}} H_{k+1}(l)T_{k+1}^l\Phi_{k+1}. \tag{3.27}$$

(iii) *There exists $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\rho}$ such that*

$$\widehat{\Phi}_k(n) = \widehat{H}_{k+1}(n)\widehat{\Phi}_{k+1}(n), \quad n \in \mathbb{Z}^s. \tag{3.28}$$

(iv) *There exists $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\rho}$ such that*

$$v_{k,j} = \sum_{r\in\mathcal{R}_1} \widehat{H}_{k+1}(j + D^kr)v_{k+1,j+D^kr}, \quad j \in \mathcal{R}_k, \tag{3.29}$$

where $v_{k,j} := \left[v_{k,j}^1, \ldots, v_{k,j}^\rho\right]^T$.

Our next proposition gives conditions which enable the affine system $X_{2\pi}$ to be derived from the MRA $\{V_{2\pi}^k(\Phi_k)\}$.

Proposition 3.20. [24] *For each $k \geq 0$, let $\Phi_k := \left[\phi_k^m\right]_{m=1}^{\rho}$ and $\Psi_k := \left[\psi_k^n\right]_{n=1}^{\varrho_k}$ be subsets of $L^2(\mathbb{T}^s)$ with $W_{2\pi}^k(\Psi_k) := \operatorname{span} T_k(\Psi_k)$ and $v_{k,j}^m := (\phi_k^m)_{k,j}$ and $u_{k,j}^n := (\psi_k^n)_{k,j}$ be the corresponding polyphase harmonics given by* (3.16)*. Then the following are equivalent for each $k \geq 0$.*

(i) $W_{2\pi}^k(\Psi_k) \subseteq V_{2\pi}^{k+1}(\Phi_{k+1})$.

(ii) *There exists $G_{k+1} \in \mathcal{S}(M^{k+1})^{\varrho_k \times \rho}$ such that*

$$\Psi_k = \sum_{l \in \mathcal{L}_{k+1}} G_{k+1}(l) T_{k+1}^l \Phi_{k+1}. \tag{3.30}$$

(iii) *There exists $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\varrho_k \times \rho}$ such that*

$$\widehat{\Psi}_k(n) = \widehat{G}_{k+1}(n) \widehat{\Phi}_{k+1}(n), \quad n \in \mathbb{Z}^s. \tag{3.31}$$

(iv) *There exists $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\varrho_k \times \rho}$ such that*

$$u_{k,j} = \sum_{r \in \mathcal{R}_1} \widehat{G}_{k+1}(j + D^k r) v_{k+1, j+D^k r}, \quad j \in \mathcal{R}_k, \tag{3.32}$$

where $u_{k,j} := \left[u_{k,j}^1, \ldots, u_{k,j}^{\varrho_k}\right]^T$ and $v_{k,j} := \left[v_{k,j}^1, \ldots, v_{k,j}^{\rho}\right]^T$.

It is shown in [24] that

$$\begin{aligned} V_{2\pi}^k(\Phi_k) &= \bigoplus_{j \in \mathcal{R}_k}^{\perp} \operatorname{span}\{v_{k,j}^m : m = 1, \ldots, \rho\}, \\ W_{2\pi}^k(\Psi_k) &= \bigoplus_{j \in \mathcal{R}_k}^{\perp} \operatorname{span}\{u_{k,j}^m : m = 1, \ldots, \varrho_k\}. \end{aligned} \tag{3.33}$$

For $j \in \mathcal{R}_k$, the Gramians of the sets $T_k(\Phi_k)$ and $T_k(\Psi_k)$ are given by

$$\begin{aligned} \mathrm{M}_k(j) &= \left[\langle v_{k,j}^m, v_{k,j}^n \rangle\right]_{m,n=1}^{\rho}, \\ \mathrm{N}_k(j) &= \left[\langle u_{k,j}^m, u_{k,j}^n \rangle\right]_{m,n=1}^{\varrho_k} \end{aligned} \tag{3.34}$$

respectively. As in (3.18), for $j \in \mathcal{R}_k$, let us define the following subspaces of polyphase harmonics

$$\begin{aligned} U_{2\pi}^{k+1,j}(\Phi_{k+1}) &:= \bigoplus_{r \in \mathcal{R}_1}^{\perp} \left[\Theta_{2\pi}^{k+1, j+D^k r} \cap V_{2\pi}^{k+1}\right] = \operatorname{span}\{v_{k+1, j+D^k r}^m : m = 1, \ldots, \rho, r \in \mathcal{R}_1\}, \\ V_{2\pi}^{k,j}(\Phi_k) &:= \Theta_{2\pi}^{k,j} \cap V_{2\pi}^k = \operatorname{span}\{v_{k,j}^m : m = 1, \ldots, \rho\}, \\ W_{2\pi}^{k,j}(\Psi_k) &:= \Theta_{2\pi}^{k,j} \cap W_{2\pi}^k = \operatorname{span}\{u_{k,j}^m : m = 1, \ldots, \varrho_k\}. \end{aligned} \tag{3.35}$$

It is further shown in [24] that $V_{2\pi}^{k+1}(\Phi_{k+1}) = V_{2\pi}^{k}(\Phi_k) \oplus^{\perp} W_{2\pi}^{k}(\Psi_k)$ if and only if

$$U_{2\pi}^{k+1,j}(\Phi_{k+1}) = V_{2\pi}^{k,j}(\Phi_k) \oplus^{\perp} W_{2\pi}^{k,j}(\Psi_k) \tag{3.36}$$

for all $j \in \mathcal{R}_k$ and (3.36) is equivalent to

$$\sum_{r\in\mathcal{R}_1} \widehat{G}_{k+1}(j + D^k r)\mathrm{M}_{k+1}(j + D^k r)\widehat{H}_{k+1}(j + D^k r)^* = 0 \tag{3.37}$$

and

$$\dim U_{2\pi}^{k+1,j}(\Phi_{k+1}) = \dim V_{2\pi}^{k,j}(\Phi_k) + \dim W_{2\pi}^{k,j}(\Psi_k) \tag{3.38}$$

for $j \in \mathcal{R}_k$. It is also inferred from (3.29) and (3.32) that

$$\mathrm{M}_k(j) = \sum_{r\in\mathcal{R}_1} \widehat{H}_{k+1}(j + D^k r)\mathrm{M}_{k+1}(j + D^k r)\widehat{H}_{k+1}(j + D^k r)^*, \tag{3.39}$$

$$\mathrm{N}_k(j) = \sum_{r\in\mathcal{R}_1} \widehat{G}_{k+1}(j + D^k r)\mathrm{M}_{k+1}(j + D^k r)\widehat{G}_{k+1}(j + D^k r)^*. \tag{3.40}$$

We cite below from [24] the characterization of $T_k(\Phi_k)$ being a tight frame in terms of polyphase harmonics and the existence of a canonical choice of generators to satisfy the criterion.

Theorem 3.21. [24] *For each $k \geq 0$, let $\Phi_k := \left[\phi_k^m\right]_{m=1}^{\rho}$ be a subset of $L^2(\mathbb{T}^s)$ and $v_{k,j}^m := \phi_{k\,k,j}^m$ be the corresponding polyphase harmonics given by* (3.16). *Then $T_k(\Phi_k)$ is a tight frame for $V_{2\pi}^k(\Phi_k)$ and $\langle T_k^l \phi_k^m, T_k^r \phi_k^n\rangle_{L^2(\mathbb{T}^s)} = 0$ for all $m, n = 1, \dots, \rho$, $m \neq n$, and $l, r \in \mathcal{L}_k$ if and only if for all $j \in \mathcal{R}_k$, $\langle v_{k,j}^m, v_{k,j}^n\rangle_{L^2(\mathbb{T}^s)} = 0$ if $m \neq n$ and $\left\|v_{k,j}^m\right\|_{L^2(\mathbb{T}^s)}^2 = 0$ or d^{-k}, for all $m, n = 1, \dots, \rho$, that is, $\mathrm{M}_k(j)$ given in* (3.34) *is a diagonal matrix with diagonal entries* 0 *or d^{-k} for $j \in \mathcal{R}_k$.*

Theorem 3.22. [24] *For each $k \geq 0$, let $\Phi_k := \left[\phi_k^m\right]_{m=1}^{\rho}$ be a subset of $L^2(\mathbb{T}^s)$. There exist functions $\theta_k^1, \dots, \theta_k^\rho$ in $V_{2\pi}^k(\Phi_k)$ such that $\{T_k^l \theta_k^m : m = 1, \dots, \rho, l \in \mathcal{L}_k\}$ forms a tight frame for $V_{2\pi}^k(\Phi_k)$, and for all $m, n = 1, \dots, \rho$ and $l, r \in \mathcal{L}_k$,*

$$\langle T_k^l \theta_k^m, T_k^r \theta_k^n\rangle_{L^2(\mathbb{T}^s)} = 0 \quad \text{if } m \neq n. \tag{3.41}$$

Theorem 3.23. [24] *For each $k \geq 0$, let $\Phi_k := \left[\phi_k^m\right]_{m=1}^{\rho}$ be a subset of $L^2(\mathbb{T}^s)$. The length of $V_{2\pi}^k(\Phi_k)$ is given by*

$$\mathrm{len}(V_{2\pi}^k(\Phi_k)) = \max\{\dim V_{2\pi}^{k,j}(\Phi_k) : j \in \mathcal{R}_k\}. \tag{3.42}$$

With $\rho_k = \mathrm{len}(V_{2\pi}^k(\Phi_k))$, there exist functions $\theta_k^1, \dots, \theta_k^{\rho_k}$ in $V_{2\pi}^k(\Phi_k)$ such that $\{T_k^l \theta_k^m : m = 1, \dots, \rho_k, l \in \mathcal{L}_k\}$ forms a tight frame for $V_{2\pi}^k(\Phi_k)$, and for all $m, n = 1, \dots, \rho_k$ and $l, r \in \mathcal{L}_k$, (3.41) *holds.*

Theorem 3.24. [24] *Let $\{V_{2\pi}^k(\Phi_k)\}$ be an MRA of $L^2(\mathbb{T}^s)$ such that $|\Phi_k| = \rho$, and for $m = 1, \ldots, \rho$, with $\phi_k^m \in \Phi_k$ and associated $v_{k,j} := (\phi_k^m)_{k,j}$ given by* (3.16) *and $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\rho}$. Suppose that for each $k \geq 0$, $\mathrm{M}_k(j)$ given in* (3.34) *is a diagonal matrix with diagonal entries* 0 *or d^{-k} for all $j \in \mathcal{R}_k$. Then for every $k \geq 0$, there exists $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho d\times\rho}$ that satisfies the conditions* (3.37) *and* (3.38)*, and that $\mathrm{N}_k(j)$ given in* (3.34) *and* (3.40) *is a diagonal matrix with diagonal entries* 0 *or d^{-k} for all $j \in \mathcal{R}_k$.*

As remarked in [24], Theorem 3.24 holds even when we begin with an arbitrary MRA of $L^2(\mathbb{T}^s)$ as we can always change the spanning set $T_k(\Phi_k)$ of the space $V_{2\pi}^k(\Phi_k)$ to a tight frame satisfying Theorem 3.21. Consequently, by Proposition 3.20, there always exists Ψ_k such that $T_k(\Psi_k)$ is a tight frame for its closed linear span $W_{2\pi}^k$ whenever $W_{2\pi}^k$ is the orthogonal complement of $V_{2\pi}^k(\Phi_k)$ in $V_{2\pi}^{k+1}(\Phi_{k+1})$.

3.3 Extension Principles

The conditions described from Theorem 3.21 to Theorem 3.24 are rather stringent for obtaining tight wavelet frames since they eventually require (3.36) to hold for all possible cases. Here, we shall describe extension principles for constructing tight wavelet frames that allow us to preserve properties of the original MRA by relaxing the condition which requires the finite dimensional spanning sets to be tight frames.

The following theorem which is essential to the proof of the unitary extension principle for $L^2(\mathbb{T}^s)$ has an equivalent formulation in Proposition 3.26 which requires weaker conditions (i.e. (3.47) instead of (3.43) needs to be satisfied). The conditions given in the theorem are known as *minimum energy tight frame conditions.*

Theorem 3.25. [25] *For each $k \geq 0$, let $\Phi_k := \left[\phi_k^m\right]_{m=1}^{\rho}$ and $\Psi_k := \left[\psi_k^n\right]_{n=1}^{\varrho_k}$ be subsets of $V_{2\pi}^{k+1}(\Phi_{k+1})$. Then the following are equivalent.*

(i) *There exist $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\rho}$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\varrho_k\times\rho}$ such that* (3.28) *and* (3.31) *hold respectively, and*

$$\widehat{\mathbb{L}}_k(j)^*\widehat{\mathbb{L}}_k(j) = dI_{\rho d}, \quad j \in \mathcal{R}_k, \tag{3.43}$$

where for $j \in \mathcal{R}_k$, the $(\rho + \varrho_k) \times \rho d$ matrix $\widehat{\mathbb{L}}_k(j)$ is given by

$$\begin{aligned}
\widehat{\mathbb{L}}_k(j) &:= \begin{bmatrix} \widehat{\mathbb{H}}_k(j) \\ \widehat{\mathbb{G}}_k(j) \end{bmatrix} = \begin{bmatrix} \widehat{L}_{k+1}(j + D^k r_1) & \cdots & \widehat{L}_{k+1}(j + D^k r_d) \end{bmatrix}, \\
\widehat{\mathbb{H}}_k(j) &:= \begin{bmatrix} \widehat{H}_{k+1}(j + D^k r_1) & \cdots & \widehat{H}_{k+1}(j + D^k r_d) \end{bmatrix}, \\
\widehat{\mathbb{G}}_k(j) &:= \begin{bmatrix} \widehat{G}_{k+1}(j + D^k r_1) & \cdots & \widehat{G}_{k+1}(j + D^k r_d) \end{bmatrix},
\end{aligned} \tag{3.44}$$

and $r_1, \ldots, r_d$ denote all the elements of $\mathcal{R}_1$.

(ii) *For all $f \in L^2(\mathbb{T}^s)$ with $\varrho_k \geq \rho(d-1)$, we have*

$$\sum_{m=1}^{\rho} \sum_{l \in \mathcal{L}_{k+1}} \left| \langle f, T_{k+1}^l \phi_{k+1}^m \rangle \right|^2 = \sum_{m=1}^{\rho} \sum_{l \in \mathcal{L}_k} \left| \langle f, T_k^l \phi_k^m \rangle \right|^2 + \sum_{m=1}^{\varrho_k} \sum_{l \in \mathcal{L}_k} \left| \langle f, T_k^l \psi_k^m \rangle \right|^2. \tag{3.45}$$

(iii) *For all $f \in L^2(\mathbb{T}^s)$ with $\varrho_k \geq \rho(d-1)$, we have*

$$\sum_{m=1}^{\rho} \sum_{l \in \mathcal{L}_{k+1}} \langle f, T_{k+1}^l \phi_{k+1}^m \rangle T_{k+1}^l \phi_{k+1}^m = \sum_{l \in \mathcal{L}_k} \left[\sum_{m=1}^{\rho} \langle f, T_k^l \phi_k^m \rangle T_k^l \phi_k^m + \sum_{m=1}^{\varrho_k} \langle f, T_k^l \psi_k^m \rangle T_k^l \psi_k^m \right]. \tag{3.46}$$

Proposition 3.26. *Given that there exist $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho \times \rho}$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\varrho_k \times \rho}$ such that* (3.28) *and* (3.31) *hold. For each $j \in \mathcal{R}_k$, suppose that $\operatorname{rank} \mathrm{M}_k(j) = q(j)$, $\operatorname{rank} \mathrm{N}_k(j) = p(j) - q(j)$ and $\operatorname{rank} \mathrm{M}_{k+1}(j + D^k r) = p(j,r)$, where $r \in \mathcal{R}_k$, and there exist $\rho \times \rho$ unitary matrices $U_{k+1}(j + D^k r)$ such that the 1^{st} $p(j,r) \times p(j,r)$ block of $\mathrm{M}_{k+1}(j+D^k r)' = U_{k+1}(j+D^k r)\mathrm{M}_{k+1}(j+D^k r)U_{k+1}(j+D^k r)^*$ consists of nonzero diagonal entries with the remaining blocks being zero matrices. Let $r_1, \ldots, r_d \in \mathcal{R}_1$ and define the $\rho \times \rho$ block diagonal matrix $I'_{q(j)} = \operatorname{diag}(I_{q(j)}, 0_{\rho - q(j)})$, the $\varrho_k \times \varrho_k$ block diagonal matrix $I'_{p(j)-q(j)} = \operatorname{diag}(I_{p(j)-q(j)}, 0_{\varrho_k - (p(j)-q(j))})$ and the $\rho d \times \rho d$ block diagonal matrix $I'_{p(j)} = \operatorname{diag}(I'_{p(j,r_1)}, \ldots, I'_{p(j,r_d)})$, where for $\mu = 1, \ldots, d$, the $\rho \times \rho$ block diagonal matrix $I'_{p(j,r_\mu)} = \operatorname{diag}(I_{p(j,r_\mu)}, 0_{\rho - p(j,r_\mu)})$. Assume that*

$$I'_{p(j)} \widehat{\mathbb{L}}'_k(j)^* \begin{bmatrix} I'_{q(j)} & 0 \\ 0 & I'_{p(j)-q(j)} \end{bmatrix} \begin{bmatrix} I'_{q(j)} & 0 \\ 0 & I'_{p(j)-q(j)} \end{bmatrix} \widehat{\mathbb{L}}'_k(j) I'_{p(j)} = d I'_{p(j)}, \tag{3.47}$$

for all $j \in \mathcal{R}_k$ with $\widehat{\mathbb{L}}_k(j)$ defined as in (3.44) *where*

$$\widehat{\mathbb{L}}'_k(j) = \operatorname{diag}(U_k(j), V_k(j)) \widehat{\mathbb{L}}_k(j) \operatorname{diag}(U_{k+1}(j + D^k r_1), \ldots, U_{k+1}(j + D^k r_d))^*,$$

$U_k(j)$ and $V_k(j)$ are $\rho \times \rho$ and $\varrho_k \times \varrho_k$ unitary matrices such that the 1^{st} $q(j) \times q(j)$ block of $U_k(j)\mathrm{M}_k(j)U_k(j)^*$ and the 1^{st} $p(j) - q(j) \times p(j) - q(j)$ block of $V_k(j)N_k(j)V_k(j)^*$ consist of nonzero diagonal entries with the remaining blocks being zero matrices respectively. Then the equivalent conditions of Theorem 3.25 *are satisfied.*

Proof. It suffices to verify that (3.43) holds for any given $j \in \mathcal{R}_k$. We define the $\rho \times \rho$ matrices $\widehat{\widetilde{G}}^m_{k+1}(j + D^k r)$ for $m = 1, \dots, d$ and $r \in \mathcal{R}_1$, such that $\widehat{\widetilde{G}}^m_{k+1}(j + D^k r_m) = \sqrt{d}(I - I'_{p(j,r_m)}) = \sqrt{d}\,\mathrm{diag}\,(0_{p(j,r_m)}, I_{\rho - p(j,r_m)})$ and $\widehat{\widetilde{G}}^m_{k+1}(j + D^k r) = 0$ for $r \neq r_m$. Next we define

$$\widehat{\mathbb{L}}_k(j) := \begin{bmatrix} \widehat{\widetilde{H}}'_{k+1}(j + D^k r_1) & \cdots & \widehat{\widetilde{H}}'_{k+1}(j + D^k r_d) \\ \widehat{\widetilde{G}}'_{k+1}(j + D^k r_1) & \cdots & \widehat{\widetilde{G}}'_{k+1}(j + D^k r_d) \end{bmatrix},$$

where $\widehat{\widetilde{H}}'_{k+1}(j + D^k r) = I'_{q(j)} \widehat{H}'_{k+1}(j + D^k r) I'_{p(j,r)}$, $\widehat{\widetilde{G}}'_{k+1}(j + D^k r) = I'_{p(j)-q(j)} \widehat{G}'_{k+1}(j + D^k r) I'_{p(j,r)}$, $\widehat{H}'_{k+1}(j + D^k r) = U_k(j) \widehat{H}_{k+1}(j + D^k r) U_{k+1}(j + D^k r)^*$, $\widehat{G}'_{k+1}(j + D^k r) = V_k(j) \widehat{G}_{k+1}(j + D^k r) U_{k+1}(j + D^k r)^*$ and the extended wavelet mask

$$\widehat{\widetilde{G}}'_{k+1}(j + D^k r)^* = \begin{bmatrix} \widehat{\widetilde{G}}'_{k+1}(j + D^k r)^* & \widehat{\widetilde{G}}^1_{k+1}(j + D^k r)^* & \cdots & \widehat{\widetilde{G}}^d_{k+1}(j + D^k r)^* \end{bmatrix}.$$

We could verify that

$$\begin{aligned}
&\widehat{\mathbb{L}}_k(j)^* \widehat{\mathbb{L}}_k(j) = I'_{p(j)} \widehat{\mathbb{L}}'_k(j)^* \begin{bmatrix} I'_{q(j)} & 0 \\ 0 & I'_{p(j)-q(j)} \end{bmatrix} \begin{bmatrix} I'_{q(j)} & 0 \\ 0 & I'_{p(j)-q(j)} \end{bmatrix} \widehat{\mathbb{L}}'_k(j) I'_{p(j)} + \\
&\begin{bmatrix} \widehat{\widetilde{G}}^1_{k+1}(j + D^k r_1)^* & \cdots & \widehat{\widetilde{G}}^d_{k+1}(j + D^k r_1)^* \\ \vdots & & \vdots \\ \widehat{\widetilde{G}}^1_{k+1}(j + D^k r_d)^* & \cdots & \widehat{\widetilde{G}}^d_{k+1}(j + D^k r_d)^* \end{bmatrix} \begin{bmatrix} \widehat{\widetilde{G}}^1_{k+1}(j + D^k r_1) & \cdots & \widehat{\widetilde{G}}^1_{k+1}(j + D^k r_d) \\ \vdots & & \vdots \\ \widehat{\widetilde{G}}^d_{k+1}(j + D^k r_1) & \cdots & \widehat{\widetilde{G}}^d_{k+1}(j + D^k r_d) \end{bmatrix} \\
&= d I'_{p(j)} + d(I - I'_{p(j)}) = d I_{\rho d}.
\end{aligned}$$

□

We remark that the unitary matrices $U_{k+1}(j + D^k r)$, $U_k(j)$ and $V_k(j)$ always exists as the matrices $M_{k+1}(j + D^k r)$, $M_k(j)$ and $N_k(j)$ are Hermitian matrices. We also note from (3.40) that the number of wavelets generating the wavelet subspace remain unchanged since

$$\widetilde{\mathrm{N}}'_k(j) = \sum_{r \in \mathcal{R}_1} \widehat{\widetilde{G}}'_{k+1}(j + D^k r) \mathrm{M}'_{k+1}(j + D^k r) \widehat{\widetilde{G}}'_{k+1}(j + D^k r)^* = \mathrm{diag}(\mathrm{N}_k(j), 0_{\rho d}),$$

i.e. the rank of the matrix $\widetilde{\mathrm{N}}'_k(j)$ is the same as the rank of $\mathrm{N}_k(j)$.

We state the *unitary extension principle* (UEP) for $L^2(\mathbb{T}^s)$ here. The main requirements for the UEP to hold are the refinable functions eventually "covering" the frequency domain "uniformly" and the columns of the extended mask $\widehat{\mathbb{L}}_k(j)$ being orthonormal on the spectrum of $V^{k+1}_{2\pi}(\Phi_{k+1})$ for every $k \geq 0$.

Theorem 3.27. [25] *For each $k \geq 0$, let $\Phi_k := \left[\phi_k^m\right]_{m=1}^{\rho}$ and $\Psi_k := \left[\psi_k^n\right]_{n=1}^{\varrho_k}$ be subsets of $V_{2\pi}^{k+1}(\Phi_{k+1})$ with $\varrho_k \geq \rho(d-1)$ satisfying* (3.28) *and* (3.31) *for some $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\rho}$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\varrho_k\times\rho}$ respectively, and*

$$\lim_{k\to\infty} d^k \sum_{m=1}^{\rho} \left|\widehat{\phi}_k^m(n)\right|^2 = A > 0, \quad n \in \mathbb{Z}^s. \tag{3.48}$$

If for every $k \geq 0$ and for each $j \in \mathcal{R}_k$, the $(\rho + \varrho_k) \times \rho d$ matrix $\widehat{\mathbb{L}}_k(j)$ as defined in (3.44) *satisfies $\widehat{\mathbb{L}}_k(j)^*\widehat{\mathbb{L}}_k(j) = dI_{\rho d}$, then the periodic affine system $X_{2\pi} := \{\phi_0 : \phi_0 \in \Phi_0\} \cup \{T_k^l\psi_k : \psi_k \in \Psi_k, l \in \mathcal{L}_k, k \geq 0\}$ as defined in* (1.15) *forms a tight wavelet frame for $L^2(\mathbb{T}^s)$ with frame bound A derived from the MRA $\{V_{2\pi}^k(\Phi_k)\}_{k\geq 0}$.*

Next, in the theme of using appropriate transformations to obtain new wavelet frames from existing ones, we derive the *generalized oblique extension principle* (GOEP) for $L^2(\mathbb{T}^s)$.

Theorem 3.28. *For each $k \geq 0$, let $\Phi_k := \left[\phi_k^m\right]_{m=1}^{\rho}$ and $\Psi_k := \left[\psi_k^n\right]_{n=1}^{\varrho_k}$ be subsets of $V_{2\pi}^{k+1}(\Phi_{k+1})$ with $\varrho_k \geq \rho(d-1)$ satisfying* (3.28) *and* (3.31) *for some $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\rho}$ and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\varrho_k\times\rho}$ respectively, and suppose that* (3.48) *holds. Define $\widehat{\Phi}'_k := \widehat{\Theta}_k\widehat{\Phi}_k$ and $\widehat{\Psi}'_k := \widehat{\Omega}_k\widehat{\Psi}_k$, where $\widehat{\Theta}_k \in \mathcal{S}(D^k)^{\rho\times\rho}$ and $\widehat{\Omega}_k \in \mathcal{S}(D^k)^{\varrho'_k\times\varrho_k}$ with $\varrho'_k \geq \rho(d-1)$, $\widehat{\Theta}_k(j)$ being invertible for each $j \in \mathcal{R}_k$ and*

$$\lim_{k\to\infty} \widehat{\Theta}_k(j)^*\widehat{\Theta}_k(j) = I_\rho, \quad j \in \mathbb{Z}^s. \tag{3.49}$$

If for every $k \geq 0$ and for each $j \in \mathcal{R}_k$, the $(\rho + \varrho'_k) \times \rho d$ matrix

$$\widehat{\mathbb{L}}'_k(j) := \operatorname{diag}(\widehat{\Theta}_k(j), \widehat{\Omega}_k(j))\widehat{\mathbb{L}}_k(j)\operatorname{diag}\left[\widehat{\Theta}_{k+1}(j)^{-1}\right]_{m=1}^{d} \tag{3.50}$$

with $\widehat{\mathbb{L}}_k(j)$ as defined in (3.44) *satisfies $\widehat{\mathbb{L}}'_k(j)^*\widehat{\mathbb{L}}'_k(j) = dI_{\rho d}$, then the periodic affine system $X'_{2\pi} := \{\phi'_0 : \phi'_0 \in \Phi'_0\} \cup \{T_k^l\psi'_k : \psi'_k \in \Psi'_k, l \in \mathcal{L}_k, k \geq 0\}$ forms a tight wavelet frame with frame bound A for $L^2(\mathbb{T}^s)$ derived from the MRA $\{V_{2\pi}^k(\Phi'_k)\}_{k\geq 0}$.*

Proof. It is clear that $\widehat{\Phi}'_k$ and $\widehat{\Psi}'_k$ satisfy (3.28) and (3.31) with

$$\widehat{\Phi}'_k(j) = \widehat{\Theta}_k(j)\widehat{\Phi}_k(j) = \widehat{\Theta}_k(j)\widehat{H}_{k+1}(j)\widehat{\Theta}_{k+1}(j)^{-1}\widehat{\Theta}_{k+1}(j)\widehat{\Phi}_{k+1}(j) = \widehat{H}'_{k+1}(j)\widehat{\Phi}'_{k+1}(j) \tag{3.51}$$

and

$$\widehat{\Psi}'_k(j) = \widehat{\Omega}_k(j)\widehat{\Psi}_k(j) = \widehat{\Omega}_k(j)\widehat{G}_{k+1}(j)\widehat{\Theta}_{k+1}(j)^{-1}\widehat{\Theta}_{k+1}(j)\widehat{\Phi}_{k+1}(j) = \widehat{G}'_{k+1}(j)\widehat{\Phi}'_{k+1}(j) \tag{3.52}$$

for $j \in \mathbb{Z}^s$.

For a given $j \in \mathbb{Z}^s$ and $\epsilon > 0$, (3.48) implies that there exists $K > 0$ such that for all $k \geq K$,

$$\left| d^k \widehat{\Phi}_k(j)^* \widehat{\Phi}_k(j) - A \right| < \epsilon. \tag{3.53}$$

The condition (3.49) implies that there exists $K' > 0$ such that for every $k \geq K'$,

$$\left\| \widehat{\Theta}_k(j)^* \widehat{\Theta}_k(j) - I_\rho \right\|_{\max} := \max_{m,n \in \{1,\ldots,\rho\}} \left| (\widehat{\Theta}_k(j)^* \widehat{\Theta}_k(j))_{m,n} - \delta_{m,n} \right| < \epsilon \tag{3.54}$$

From (3.53) and (3.54), we are able to make the following estimate

$$\left| d^k \widehat{\Phi}'_k(j)^* \widehat{\Phi}'_k(j) - A \right| \leq \left| d^k \widehat{\Phi}_k(j)^* (\widehat{\Theta}_k(j)^* \widehat{\Theta}_k(j) - I_\rho) \widehat{\Phi}_k(j) \right| + \left| d^k \widehat{\Phi}_k(j)^* \widehat{\Phi}_k(j) - A \right| . \tag{3.55}$$

For the sake of convenience, let us denote $\widehat{\Phi}_k(j) = \left[\widehat{\phi_k^m}(j) \right]_{m=1}^{\rho}$ as a vector in $\mathbb{R}^\rho$. For all $k \geq \max\{K, K'\}$, we could utilize (3.54) and the Cauchy-Schwarz inequality to bound

$$\begin{aligned} \left| d^k \widehat{\Phi}_k(j)^* (\widehat{\Theta}_k(j)^* \widehat{\Theta}_k(j) - I_\rho) \widehat{\Phi}_k(j) \right| &\leq d^k \left\| \widehat{\Phi}_k(j) \right\|_{\mathbb{R}^\rho} \left\| (\widehat{\Theta}_k(j)^* \widehat{\Theta}_k(j) - I_\rho) \widehat{\Phi}_k(j) \right\|_{\mathbb{R}^\rho} \\ \leq \rho d^k \left\| (\widehat{\Theta}_k(j)^* \widehat{\Theta}_k(j) - I_\rho) \right\|_{\max} \left\| \widehat{\Phi}_k(j) \right\|_{\mathbb{R}^\rho}^2 &< \rho \epsilon d^k \widehat{\Phi}_k(j)^* \widehat{\Phi}_k(j). \end{aligned}$$

Since (3.53) and (3.55) imply that $\left| d^k \widehat{\Phi}'_k(j)^* \widehat{\Phi}'_k(j) - A \right| < \epsilon[\rho(A+\epsilon)+1]$ for all $k \geq \max\{K, K'\}$, consequently

$$\lim_{k\to\infty} d^k \sum_{m=1}^{\rho} \left| \widehat{\phi_k'^m}(j) \right|^2 = \lim_{k\to\infty} d^k \left\| \widehat{\Phi}'_k(j) \right\|_{\mathbb{R}^\rho}^2 = A > 0, \quad j \in \mathbb{Z}^s.$$

Hence the MRA $\{V_{2\pi}^k(\Phi'_k)\}_{k\geq 0}$ satisfies the hypothesis of Theorem 3.27 and $X'_{2\pi}$ will be a tight wavelet frame for $L^2(\mathbb{T}^s)$ if $\widehat{\mathbb{L}}'_k(j)^* \widehat{\mathbb{L}}'_k(j) = dI_{\rho d}$ for every $k \geq 0$ and each $j \in \mathcal{R}_k$. $\square$

Suitable choices for $\widehat{\Theta}_k$ and $\widehat{\Omega}_k$ from $\mathcal{S}(D^k)^{\rho\times\rho}$ and $\mathcal{S}(D^k)^{\varrho'_k \times \varrho_k}$ in Theorem 3.28 could be unitary matrices and matrices with unitary columns respectively. This leads us to the following construction and corollary.

Corollary 3.29. *For each $k \geq 0$, let $\Phi_k := \left[\phi_k^m \right]_{m=1}^{\rho}$ and $\Psi_k := \left[\psi_k^n \right]_{n=1}^{\varrho_k}$ be subsets of $V_{2\pi}^{k+1}(\Phi_{k+1})$ with $\varrho_k \geq \rho(d-1)$. Let the affine system $X_{2\pi}$ as defined in* (1.15) *be a tight frame for $L^2(\mathbb{T}^s)$ derived from the UEP with $\{V_{2\pi}^k(\Phi_k)\}_{k\geq 0}$ as the underlying MRA of $L^2(\mathbb{T}^s)$ and $\widehat{L}_{k+1} := \begin{bmatrix} \widehat{H}_{k+1} \\ \widehat{G}_{k+1} \end{bmatrix}$ as the combined MRA mask. Define $\widehat{\Phi}'_k := \widehat{U}_{\Phi_k} \widehat{\Phi}_k$ and $\widehat{\Psi}'_k := \widehat{U}_{\Psi_k} \widehat{\Psi}_k$, where $\widehat{U}_{\Phi_k} \in \mathcal{S}(D^k)^{\rho\times\rho}$ and $\widehat{U}_{\Psi_k} \in \mathcal{S}(D^k)^{\varrho'_k \times \varrho_k}$ are unitary matrices and* *matrices with unitary columns such that* $\varrho'_k \geq \rho(d-1)$ *respectively. Then* $X'_{2\pi} := \{\phi'_0 : \phi'_0 \in \Phi'_0\} \cup \{T^l_k \psi'_k : \psi'_k \in \Psi'_k, l \in \mathcal{L}_k, k \geq 0\}$ *is a tight frame for* $L^2(\mathbb{T}^s)$ *derived from the MRA* $\{V^k_{2\pi}(\Phi'_k)\}_{k\geq 0}$ *using the GOEP with the combined MRA mask* $\widehat{L}'_{k+1} := \begin{bmatrix} \widehat{H}'_{k+1} \\ \widehat{G}'_{k+1} \end{bmatrix}$, *where* $\widehat{H}'_{k+1} = \widehat{U}_{\Phi_k} \widehat{H}_{k+1} \widehat{U}^*_{\Phi_{k+1}}$ *and* $\widehat{G}'_{k+1} = \widehat{U}_{\Psi_k} \widehat{G}_{k+1} \widehat{U}^*_{\Phi_{k+1}}$.

Proof. In order to utilize Theorem 3.28, we let $\widehat{\Theta}_k = \widehat{U}_{\Phi_k}$ and $\widehat{\Omega}_k = \widehat{U}_{\Psi_k}$. Since $\widehat{\Theta}_k(j)$ is unitary for each $j \in \mathcal{R}_k$, (3.49) holds. Next, it is clear from (3.51) and (3.52) that $\widehat{\Phi}'_k$ and $\widehat{\Psi}'_k$ satisfy (3.28) and (3.31) with

$$\widehat{\Phi}'_k(j) = \widehat{H}'_{k+1}(j)\widehat{\Phi}'_{k+1}(j), \quad \widehat{\Psi}'_k(j) = \widehat{G}'_{k+1}(j)\widehat{\Phi}'_{k+1}(j), \quad j \in \mathbb{Z}^s.$$

This shows that by Propositions 3.19 and 3.20, the affine system $X'_{2\pi}$ is obtained from the MRA $\{V^k_{2\pi}(\Phi'_k)\}_{k\geq 0}$. Let $\widehat{\mathbb{L}}_k(j)$ and $\widehat{\mathbb{L}}'_k(j)$ be given as in (3.44) and (3.50). We verify that for all $k \geq 0$,

$$\widehat{\mathbb{L}}'_k(j)^*\widehat{\mathbb{L}}'_k(j) = \widehat{\mathbb{H}}'_k(j)^*\widehat{\mathbb{H}}'_k(j) + \widehat{\mathbb{G}}'_k(j)^*\widehat{\mathbb{G}}'_k(j) = dI_{\rho d}$$

holds for all $j \in \mathcal{R}_k$. This is true since for a given $k \geq 0$ and $j \in \mathcal{R}_{k+1}$,

$$\begin{aligned} &\widehat{H}'_{k+1}(j)^*\widehat{H}'_{k+1}(j) + \widehat{G}'_{k+1}(j)^*\widehat{G}'_{k+1}(j) = \widehat{U}_{\Phi_{k+1}}(j)\left[\widehat{H}_{k+1}(j)^*\widehat{U}_{\Phi_k}(j)^*\widehat{U}_{\Phi_k}(j)\widehat{H}_{k+1}(j)\right. \\ &\left.+\widehat{G}_{k+1}(j)^*\widehat{U}_{\Psi_k}(j)^*\widehat{U}_{\Psi_k}(j)\widehat{G}_{k+1}(j)\right]\widehat{U}_{\Phi_{k+1}}(j)^* = \widehat{U}_{\Phi_{k+1}}(j)dI_\rho\widehat{U}_{\Phi_{k+1}}(j)^* = dI_\rho, \end{aligned}$$

and for a given $k \geq 0$, $j \in \mathcal{R}_k$ and $r, s \in \mathcal{R}_1$ with $r \neq s$,

$$\begin{aligned} &\widehat{H}'_{k+1}(j + D^k r)^*\widehat{H}'_{k+1}(j + D^k s) + \widehat{G}'_{k+1}(j + D^k r)^*\widehat{G}'_{k+1}(j + D^k s) \\ &= \widehat{U}_{\Phi_{k+1}}(j + D^k r)\left[\widehat{H}_{k+1}(j + D^k r)^*\widehat{U}_{\Phi_k}(j)^*\widehat{U}_{\Phi_k}(j)\widehat{H}_{k+1}(j + D^k s)\right. \\ &\left.+\widehat{G}_{k+1}(j + D^k r)^*\widehat{U}_{\Psi_k}(j)^*\widehat{U}_{\Psi_k}(j)\widehat{G}_{k+1}(j + D^k s)\right]\widehat{U}_{\Phi_{k+1}}(j + D^k s)^* \\ &= \widehat{U}_{\Phi_{k+1}}(j + D^k r)0_\rho\widehat{U}_{\Phi_k}(j + D^k s)^* = 0_\rho. \end{aligned}$$

Therefore, by Theorem 3.28 (GOEP), our result is true. □

The choice of the matrices $\widehat{\Theta}_0(0) = I_\rho$ and $\widehat{\Omega}_k(j) = I_{\varrho_k}$ for all $j \in \mathcal{R}_k$ and $k \geq 0$ in Theorem 3.28 leads to the following *oblique extension principle* for $L^2(\mathbb{T}^s)$.

Corollary 3.30. [25] *For each* $k \geq 0$, *let* $\Phi_k := \left[\phi^m_k\right]^{\rho_k}_{m=1}$ *and* $\Psi_k := \left[\psi^n_k\right]^{\varrho_k}_{n=1}$ *be subsets of* $V^{k+1}_{2\pi}(\Phi_{k+1})$ *with* $\varrho_k \geq \rho(d-1)$ *satisfying* (3.28) *and* (3.31) *for some* $\widehat{H}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\rho}$

and $\widehat{G}_{k+1} \in \mathcal{S}(D^{k+1})^{\varrho_k \times \rho}$ *respectively, and suppose that* (3.48) *holds. Define* $\widehat{\Phi}'_k := \widehat{\Theta}_k \widehat{\Phi}_k$ *where* $\widehat{\Theta}_k \in \mathcal{S}(D^k)^{\rho \times \rho}$ *are invertible matrices such that*

$$\lim_{k \to \infty} \widehat{\Theta}_k(j)^* \widehat{\Theta}_k(j) = I_\rho, \quad j \in \mathbb{Z}^s, \tag{3.56}$$

and $\widehat{\Theta}_0(0) = I_\rho$*. If for every* $k \geq 0$ *and for each* $j \in \mathcal{R}_k$*, the* $(\rho + \varrho_k) \times \rho d$ *matrix*

$$\widehat{\mathbb{L}}'_k(j) := \mathrm{diag}\,(\widehat{\Theta}_k(j), I_{\varrho_k}) \widehat{\mathbb{L}}_k(j) \mathrm{diag}\, \left[\widehat{\Theta}_{k+1}(j)^{-1}\right]_{m=1}^{d} \tag{3.57}$$

with $\widehat{\mathbb{L}}_k(j)$ *as defined in* (3.44) *satisfies* $\widehat{\mathbb{L}}'_k(j)^* \widehat{\mathbb{L}}'_k(j) = dI_{\rho d}$*, then the periodic affine system* $X_{2\pi} := \{\phi_0 : \phi_0 \in \Phi_0\} \cup \{T_k^l \psi_k : \psi_k \in \Psi_k, l \in \mathcal{L}_k, k \geq 0\}$ *as defined in* (1.15) *forms a tight wavelet frame for* $L^2(\mathbb{T}^s)$ *with frame bound* A*, derived from the MRA* $\{V_{2\pi}^k(\Phi'_k)\}_{k \geq 0}$*.*

Proof. Since $\widehat{\Theta}_0 \in \mathcal{S}(D^0)^{\rho \times \rho}$, it follows that $\widehat{\Theta}_0(j) = \widehat{\Theta}_0(0) = I_\rho$ for all $j \in \mathbb{Z}^s$. Consequently, $\widehat{\Phi}'_0(j) = \widehat{\Phi}_0(j)$ for all $j \in \mathbb{Z}^s$, i.e. $\Phi'_0 = \Phi_0$ and the result is verified. □

3.4 Periodization Connection

The Poisson summation formula (see [42]) states that periodization in the time domain is the same as sampling in the frequency domain and this will be our chief motivation of this section. Since the formula requires a certain amount of decay in the time domain, we need to impose a decay condition on our function spaces. To this end, let $K, k \geq 0$ and for every $\varphi_k \in \Lambda_k \subset L^{2,\alpha}(\mathbb{R}^s)$, where Λ_k is given as in (1.7) and

$$L^{2,\alpha}(\mathbb{R}^s) := \{f \in L^2(\mathbb{R}^s) : f(t) = O((1 + |t|)^{-(1+\alpha)}), \alpha > 0\}$$

we define the $2\pi M^{-K}\mathbb{Z}^s$-periodic function

$$\varphi_{K,\omega,k}(t) := \sum_{n \in \mathbb{Z}^s} \widehat{\varphi_{K,\omega,k}}(n) \mathrm{e}^{\mathrm{i}n \cdot M^K t}, \tag{3.58}$$

where $\omega \in \mathbb{T}_K^s \backslash \Delta$ is such that $|\Delta| = 0$ and the Fourier coefficients

$$\widehat{\varphi_{K,\omega,k}}(n) := \overset{o}{\widehat{(\varphi_k)}}_{K,\omega,0}(n) \tag{3.59}$$

given in (3.4) lie in $l^2(\mathbb{Z}^s)$. In the event that $\varphi_k = d^{\frac{k}{2}} \varphi(M^k \cdot)$, then according to Lemma 3.5, $\widehat{\varphi_{K,\omega,k}} = \widehat{\varphi}^o_{K,\omega,k}$. Formally, the Poisson summation formula shows that $\varphi_{K,\omega,k} = \mathbb{P}_{2\pi M^{-K}} \left[\varphi_k \left(\frac{\cdot}{2\pi}\right) \mathrm{e}^{-\mathrm{i}\frac{\omega \cdot}{2\pi}}\right]$, where $\mathbb{P}_{2\pi M^{-K}} : L^1(\mathbb{R}^s) \to L^1(\mathbb{T}^s)$ is the $2\pi M^{-K}\mathbb{Z}^s$-periodization operator given by

$$\mathbb{P}_{2\pi M^{-K}} : f \mapsto (2\pi)^{-1} \sum_{n \in \mathbb{Z}^s} f(\cdot - 2\pi M^{-K} n).$$

The verification formally is as follows:

$$\widehat{\varphi_{K,\omega,k}}(n) = \frac{1}{(2\pi)^{s+1}} \int_{\mathbb{T}^s_{-K}} \sum_{m\in\mathbb{Z}^s} \varphi_k\left(\frac{1}{2\pi}(t-2\pi M^{-K}m)\right) \mathrm{e}^{\frac{-\mathrm{i}}{2\pi}\omega\cdot(t-2\pi M^{-K}m)} \mathrm{e}^{-\mathrm{i}D^K n\cdot t}\mathrm{d}t$$

$$= \frac{1}{(2\pi)^{s+1}} \sum_{m\in\mathbb{Z}^s} \int_{\mathbb{T}^s_{-K}} \varphi_k\left(\frac{1}{2\pi}(t-2\pi M^{-K}m)\right) \mathrm{e}^{\frac{-\mathrm{i}}{2\pi}\omega\cdot(t-2\pi M^{-K}m)} \mathrm{e}^{-\mathrm{i}D^K n\cdot t}\mathrm{d}t$$

$$= \frac{1}{(2\pi)^{s+1}} \sum_{m\in\mathbb{Z}^s} \int_{\mathbb{T}^s_{-K}-2\pi M^{-K}m} \varphi_k\left(\frac{x}{2\pi}\right) \mathrm{e}^{\frac{-\mathrm{i}}{2\pi}\omega\cdot x} \mathrm{e}^{-\mathrm{i}D^K n\cdot(x+2\pi M^{-K}m)}\mathrm{d}x$$

$$= \frac{1}{(2\pi)^{s+1}} \int_{\mathbb{R}^s} \varphi_k\left(\frac{x}{2\pi}\right) \mathrm{e}^{\frac{-\mathrm{i}}{2\pi}(\omega+2\pi D^K n)\cdot x}\mathrm{d}x = \frac{1}{(2\pi)^s}\int_{\mathbb{R}^s} \varphi_k(t)\, \mathrm{e}^{-\mathrm{i}(\omega+2\pi D^K n)\cdot t}\mathrm{d}t.$$

In the event that $\varphi_k = d^{\frac{k}{2}}\varphi(M^k\cdot)$, we have

$$(2\pi)^s\widehat{\varphi_{K,\omega,k}}(n) = d^{\frac{k}{2}}\int_{\mathbb{R}^s} \varphi\left(M^k t\right) \mathrm{e}^{-\mathrm{i}(\omega+2\pi D^K n)\cdot t}\mathrm{d}t = d^{-\frac{k}{2}}\int_{\mathbb{R}^s}\varphi(x)\,\mathrm{e}^{-\mathrm{i}(\omega+2\pi D^K n)\cdot M^{-k}x}\mathrm{d}x$$

$$= d^{-\frac{k}{2}}\int_{\mathbb{R}^s}\varphi(t)\,\mathrm{e}^{-\mathrm{i}D^{-k}(\omega+2\pi D^K n)\cdot t}\mathrm{d}t = d^{-\frac{k}{2}}\widehat{\varphi}(D^{-k}(\omega+2\pi D^K n)).$$

(Note that $\widehat{(\varphi_{\omega,k})}^{o}_{K,j}(n) = \widehat{\varphi_{\omega,k}}(j+D^K n) = \widehat{(\varphi_k)}_{0,\omega,0}(j+D^K n)$.) Since $2\pi M^{-K}\mathbb{Z}^s$-periodic functions are $2\pi\mathbb{Z}^s$-periodic functions, it suffices to study the periodization connection for the $2\pi\mathbb{Z}^s$-periodic case. To this end, let us denote $\varphi_{\omega,k} := \varphi_{0,\omega,k}$ and define the set of functions $\Lambda_{\omega,k}$ by

$$\Lambda_{\omega,k} := \{\varphi_{\omega,k} : \varphi_k \in \Lambda_k\}$$

and its closed $2\pi M^{-K}\mathbb{Z}^s$ shift-invariant span $V^K_{2\pi,\omega}(\Lambda_{\omega,k})$ by

$$V^K_{2\pi,\omega}(\Lambda_{\omega,k}) := V^K_{2\pi}(\Lambda_{\omega,k}) := \overline{\operatorname{span}}\, T_K(\Lambda_{\omega,k}).$$

We shall now consider the shift-invariant system $E(\Phi)\cup X_0(\Psi)$ obtained from the MRA $\{V^k(\Phi)\}$, where $X_0(\Psi)$ is given in (1.5). Define the *periodized affine system* X_ω of a shift-invariant system $E(\Phi)\cup X_0(\Psi)$, where $\omega\in\mathbb{T}^s$, to be

$$X_\omega := \{\phi_{\omega,0} : \phi\in\Phi\}\cup\{T^l_k\psi_{\omega,k} : \psi\in\Psi, l\in\mathcal{L}_k, k\ge 0\}. \tag{3.60}$$

The corresponding *periodized quasi-affine system* $X^q_{K,\omega}$ of a shift-invariant system $E(\Phi)\cup X_0(\Psi)$ at level $K\ge 0$ is defined to be

$$X^q_{K,\omega} := T_K(\Omega_{K,\omega})$$

which consists of all the $2\pi M^{-K}\mathbb{Z}^s$ shifts of

$$\begin{aligned}\Omega_{K,\omega} \;:=\;& \{d^{-\frac{K}{2}}\phi_{\omega,0} : \phi\in\Phi\}\cup\{d^{\frac{k}{2}-\frac{K}{2}}\psi_{\omega,k} : \psi\in\Psi : 0\le k<K\}\cup\\ & \{T^l_k\psi_{\omega,k} : \psi\in\Psi, l\in\mathcal{L}_{k-K}, k\ge K\}.\end{aligned}$$

Lemma 3.31. *Let $k, l \in \mathbb{Z}^s$ and $f_k \in L^{2,\alpha}(\mathbb{R}^s)$. Then*

(i) $\mathbb{P}_{2\pi}\left[E_k^l f_k\left(\frac{\cdot}{2\pi}\right)\mathrm{e}^{-\mathrm{i}\frac{\omega\cdot}{2\pi}}\right] = \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l} T_k^l f_{\omega,k}$.

(ii) *for almost every $\omega \in \mathbb{T}^s$,* $(\mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l} T_k^l f_{\omega,k})^\wedge = \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l} \mathcal{M}_{0,k}^l \widehat{f_{\omega,k}}$.

Proof. Part (i) follows from $\mathbb{P}_{2\pi}\left[E_k^l f_k\left(\frac{\cdot}{2\pi}\right)\mathrm{e}^{-\mathrm{i}\frac{\omega\cdot}{2\pi}}\right] = \mathbb{P}_{2\pi}\left[f_k(\frac{\cdot}{2\pi} - M^{-k}l)\mathrm{e}^{-\mathrm{i}\frac{\omega\cdot}{2\pi}}\right]$. Since Lemma 3.14 shows that $\mathcal{M}_{0,k}^l \widehat{f_{\omega,k}}$ is the sequence of Fourier coefficients of $T_k^l f_{\omega,k}$ for almost every $\omega \in \mathbb{T}^s$ and Lemma 3.5 shows that $(E_k^l f_k)_{0,\omega,0}^\wedge = \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l}\mathcal{M}_{0,k}^l \widehat{f_{\omega,k}}$, part (ii) holds. □

A *range function* is a mapping $\mathcal{J} : \mathbb{T}^s \to \{\text{closed subspaces of } l^2(\mathbb{Z}^s)\}$. The mapping $\mathcal{J}$ is *measurable* if $\omega \mapsto \langle \mathcal{P}(\omega)a, b\rangle_{l^2(\mathbb{Z}^s)}$ is a measurable function for each a, $b \in l^2(\mathbb{Z}^s)$, where $\mathcal{P}(\omega)$ is the associated orthogonal projection from $l^2(\mathbb{Z}^s)$ onto $\mathcal{J}(\omega)$. Therefore, this means that measurability of $\mathcal{J}$ depends on the measurability of the projection of uniform samples in the frequency domain.

Theorem 3.32. [4] *The closed subspace S of $L^{2,\alpha}(\mathbb{R}^s)$ is shift-invariant if and only if*

$$S = \{f \in L^{2,\alpha}(\mathbb{R}^s) : \widehat{f}_{0,\omega,0} \in \mathcal{J}(\omega) \text{ for a.e. } \omega \in \mathbb{T}^s\}, \tag{3.61}$$

where $\mathcal{J}$ is a measurable range function. There is a one-to-one correspondence between S and $\mathcal{J}$ by identifying range functions which are equal almost everywhere. Furthermore if $\Lambda \subset S$ is a countable set that generates S, then

$$\mathcal{J}(\omega) = \overline{\operatorname{span}}\,\{\widehat{\varphi}_{0,\omega,0} : \varphi \in \Lambda\} \quad \text{for a.e. } \omega \in \mathbb{T}^s.$$

Theorem 3.32 essentially says that two functions f and g lie in the same closed shift-invariant space if and only if their corresponding uniform frequency samples differ by a set of measure zero.

Theorem 3.33. [4] *Let V be a closed shift-invariant subspace of a closed shift-invariant subspace S of $L^{2,\alpha}(\mathbb{R}^s)$ and let W be the orthogonal complement of V in S. Then W is a closed shift-invariant space, $\widehat{S}_{||\omega}$ and $S_{2\pi,\omega}$ are the orthogonal sums of $\widehat{V}_{||\omega}$ and $\widehat{W}_{||\omega}$, and $V_{2\pi,\omega}$ and $W_{2\pi,\omega}$ respectively for almost every $\omega \in \mathbb{T}^s$.*

Theorem 3.33 says that uniform frequency samples and the corresponding periodization of two signals are orthogonal except for a set of measure zero if the signals are orthogonal to each other.

Corollary 3.34. *Let V be a closed shift-invariant subspace of a closed shift-invariant subspace S of $L^{2,\alpha}(\mathbb{R}^s)$. Then $\widehat{V}_{||\omega}$ and $V_{2\pi,\omega}$ are subspaces of $\widehat{S}_{||\omega}$ and $S_{2\pi,\omega}$ respectively for almost every $\omega \in \mathbb{T}^s$.*

Proposition 3.35. *Let V and S be closed shift-invariant subspaces of $L^{2,\alpha}(\mathbb{R}^s)$. If $\widehat{V}_{||\omega}$ is a subspace of $\widehat{S}_{||\omega}$ or $V_{2\pi,\omega}$ is a subspace of $S_{2\pi,\omega}$ for almost every $\omega \in \mathbb{T}^s$, then V is a subspace of S.*

Proof. Suppose that $\widehat{V}_{||\omega} \subseteq \widehat{S}_{||\omega}$ for almost every $\omega \in \mathbb{T}^s$ and let $f \in V$. By Theorem 3.32, $\widehat{f}_{0,\omega,0} \in \mathcal{J}(\omega)$ for almost every $\omega \in \mathbb{T}^s$, where $\mathcal{J}$ is a measurable range function given as in (3.61). If $f \notin S$, then there exists $\Delta \subseteq \mathbb{T}^s$ such that $|\Delta| > 0$ and $\widehat{f}_{0,\omega,0} \notin \mathcal{J}(\omega)$ for all $\omega \in \Delta$, which is a contradiction. $\square$

Corollary 3.34 and Proposition 3.35 state that periodized subspaces constructed using uniform frequency samples of signals satisfy the nesting property except on a set of measure zero if and only if the subspaces containing the signals also satisfy the nesting property.

Theorem 3.36 and Corollary 3.37 state that a set of functions is a frame for their closed linear span if and only if the periodization of these functions is a frame for the corresponding periodized subspaces constructed using their uniform frequency samples for almost all possible samples.

Theorem 3.36. *Let $V(\Lambda)$ be a closed shift-invariant space generated by some countable set $\Lambda \subset L^{2,\alpha}(\mathbb{R}^s)$. Then $E(\Lambda)$ is a (Bessel system) frame for $V(\Lambda)$ if and only if $\Lambda_{\omega,0}$ is a (Bessel system) frame for $V_{2\pi}(\Lambda_{\omega,0})$ with the same bounds for almost every $\omega \in \mathbb{T}^s$. In particular, the former is a tight frame if and only if the latter is a tight frame for almost every $\omega \in \mathbb{T}^s$.*

Proof. Suppose that $E(\Lambda)$ is a Bessel system with bound B. By Theorem 3.4, the norm of its Gramian $\mathrm{M}_\Lambda(\omega) = \left[\langle \widehat{\varphi_{\omega,0}}, \widehat{\phi_{\omega,0}} \rangle_{l^2(\mathbb{Z}^s)}\right]_{\varphi,\phi \in \Lambda}$ is bounded above by B for almost every $\omega \in \mathbb{T}^s$. Since the Gramian of $\Lambda_{\omega,0}$ satisfy $\mathrm{M}_{\Lambda_{\omega,0}}(0) = \mathrm{M}_\Lambda(\omega)$, we conclude using Theorem 3.13 that $\Lambda_{\omega,0}$ is a Bessel system with bound B for almost every $\omega \in \mathbb{T}^s$. Similarly, if $E(\Lambda)$ is a frame for $V(\Lambda)$ with lower bound A, Theorem 3.4 shows that $\|\mathrm{M}_\Lambda(\omega)^{-1}\|$ is bounded above by A^{-1} for almost every $\omega \in \sigma(V(\Lambda))$ and $\Lambda_{\omega,0}$ is deduced to be a frame for $V_{2\pi}(\Lambda_{\omega,0})$ with lower bound A for almost every $\omega \in \mathbb{T}^s$ after a repeated application of Theorem 3.13. For the converse, from Theorem 3.13, we shall have $\left\|\mathrm{M}_{\Lambda_{\omega,0}}(0)\right\|$ bounded above by B in the former and $\left\|\mathrm{M}_{\Lambda_{\omega,0}}(0)^{-1}\right\|$ bounded above by A^{-1} in the latter for almost every $\omega \in \mathbb{T}^s$ and the argument is reversed using Theorem 3.4. $\square$

For the rest of this chapter on the periodization of wavelets from FMRAs, we shall consider the more general setting of nonstationary wavelets as periodic wavelets are nonstationary.

Corollary 3.37. *For $k \geq 0$, let $\Phi_k \subset L^{2,\alpha}(\mathbb{R}^s)$ be countable. Then $E(\{E_k^l \varphi_k : \varphi_k \in \Phi_k, l \in \mathcal{L}_k\})$ is a frame for $V^k(\Phi_k)$ if and only if $T_k(\Phi_{\omega,k})$ is a frame for $V_{2\pi}^k(\Phi_{\omega,k})$ with the same bounds for almost every $\omega \in \mathbb{T}^s$. In particular, the former is a tight frame if and only if the latter is a tight frame for almost every $\omega \in \mathbb{T}^s$. In addition, $V_{2\pi}^k(\Phi_{\omega,k}) = \{f_{\omega,0} : \widehat{f_{\omega,0}} \in \widehat{V}_{||\omega}^k(\Phi_k)\}$ for almost every $\omega \in \mathbb{T}^s$.*

Proof. For a given $k \geq 0$, let $\Lambda = \{E_k^l \varphi_k : \varphi_k \in \Phi_k, l \in \mathcal{L}_k\}$. Using Lemma 3.5 and according to (3.59), the fibre $\widehat{V}_{||\omega}^k(\Phi_k)$ is the closed linear span of $\{\mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l}\mathcal{M}_{0,k}^l \widehat{\varphi_{\omega,k}} : \varphi_k \in \Phi_k, l \in \mathcal{L}_k\}$, which relates to the Fourier coefficients of $\Lambda_{\omega,0} = \{\mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l} T_k^l \varphi_{\omega,k} : \varphi_k \in \Phi_k, l \in \mathcal{L}_k\}$ for almost every $\omega \in \mathbb{T}^s$, as verified by Lemma 3.31. Therefore, Theorem 3.36 shows that $E(\Lambda)$ is a frame for $V^k(\Phi_k)$ if and only if $T_k(\Phi_{\omega,k})$ is a frame for $V_{2\pi}^k(\Phi_{\omega,k})$ with the same bounds for almost every $\omega \in \mathbb{T}^s$. $\square$

Lemma 3.38. [43] *Let $A(\omega)$ be a measurable Hermitian matrix-valued function for almost every $\omega \in \mathbb{T}^s$. Then there exists a measurable unitary matrix-valued function $U(\omega)$ such that $U(\omega)^* A(\omega) U(\omega)$ is a diagonal matrix for almost every $\omega \in \mathbb{T}^s$.*

Corollary 3.39. *For $k \geq 0$, let $\Phi_k \subset L^{2,\alpha}(\mathbb{R}^s)$ with $|\Phi_k| = \rho$. There exist functions $\{\theta_k^m\}_{m=1}^\rho \subset V^k(\Phi_k)$ such that $E(\{E_k^l \theta_k^m : m = 1, \ldots, \rho, l \in \mathcal{L}_k\})$ is a tight frame for $V^k(\Phi_k)$ and for all $m, n = 1, \ldots, \rho$ and $l, r \in \mathcal{L}_k$,*

$$\langle E_k^l \theta_k^m, E_k^r \theta_k^n \rangle = 0, \quad \textit{if } m \neq n. \tag{3.62}$$

Proof. The proof is essentially that of Theorem 3.22 with the additional requirement that the functions $\{\theta_k^m\}_{m=1}^\rho$ are constructed in the following way to be measurable. With Corollary 3.37 in mind, we shall only consider an arbitrary $\omega \in \mathbb{T}^s \backslash \Delta$, where $\Delta \subset \mathbb{T}^s$ with $|\Delta| = 0$ such that for every $\omega \in \mathbb{T}^s \backslash \Delta$ and $k \geq 0$, $V_{2\pi}^k(\Phi_{\omega,k})$ is a subspace of $L^2(\mathbb{T}^s)$. Fix $j \in \mathcal{R}_k$. By the positive semi-definiteness and Hermitian property of $\mathrm{M}_{\omega,k}(j)$ expressed in (3.34), Lemma 3.38 shows that there exists a $\rho \times \rho$ measurable unitary matrix $U_{\omega,k}(j)$ such that

$$U_{\omega,k}(j)\mathrm{M}_{\omega,k}(j)U_{\omega,k}(j)^* = \mathrm{diag}\left[\lambda_{\omega,k}^m(j)\right]_{m=1}^\rho, \tag{3.63}$$

where $\lambda^1_{\omega,k}(j),\ldots,\lambda^\rho_{\omega,k}(j)$ are the eigenvalues of $\mathrm{M}_{\omega,k}(j)$ which are always nonnegative. For $m=1,\ldots,\rho$, define $\beta^m_{\omega,k}(j)$ by

$$\beta^m_{\omega,k}(j) := \begin{cases} d^{-\frac{k}{2}}[\lambda^m_{\omega,k}(j)]^{-\frac{1}{2}} & \text{if } \lambda^m_{\omega,k}(j)\neq 0, \\ 1 & \text{if } \lambda^m_{\omega,k}(j)=0. \end{cases}$$

Letting $B_{\omega,k}(j) := \operatorname{diag}\left[\beta^m_{\omega,k}(j)\right]^\rho_{m=1}$ and $C_{\omega,k}(j) := B_{\omega,k}(j)U_{\omega,k}(j) = \left[c^{m,n}_{\omega,k}(j)\right]^\rho_{m,n=1}$, (3.63) shows that

$$C_{\omega,k}(j)\mathrm{M}_{\omega,k}(j)C_{\omega,k}(j)^* = \operatorname{diag}\left[\delta^m_{\omega,k}(j)\right]^\rho_{m=1}, \tag{3.64}$$

where $\delta^m_{\omega,k}(j)=0$ or d^{-k} for $m=1,\ldots,\rho$. For $m=1,\ldots,r$, we define

$$w^m_{\omega,k,j} := \sum_{n=1}^{\rho} c^{m,n}_{\omega,k}(j)v^n_{\omega,k,j}, \tag{3.65}$$

which lies in $V^k_{2\pi}(\Phi_{\omega,k})$ using (3.33). With the invertibility of $C_{\omega,k}(j)$, we have $V^k_{2\pi}(\Phi_{\omega,k}) := \operatorname{span}\{w^m_{\omega,k,j} : m=1,\ldots,\rho, j\in\mathcal{R}_k\}$. For $m=1,\ldots,\rho$, define

$$\theta^m_{\omega,k} := \sum_{j\in\mathcal{R}_k} w^m_{\omega,k,j}. \tag{3.66}$$

Therefore $\{\theta^m_{\omega,k}\}^\rho_{m=1}\subset V^k_{2\pi}(\Phi_{\omega,k})$ and $\{w^m_{\omega,k,j} : m=1,\ldots,\rho, j\in\mathcal{R}_k\}$ is the corresponding collection of polyphase harmonics. Since (3.64) and (3.65) show that the matrix

$$\left[\langle w^m_{\omega,k,j}, w^n_{\omega,k,j}\rangle_{L^2(\mathbb{T}^s)}\right]^\rho_{m,n=1} = C_{\omega,k}(j)\mathrm{M}_{\omega,k}(j)C_{\omega,k}(j)^* \tag{3.67}$$

is diagonal with diagonal entries 0 or d^{-k} for all $j\in\mathcal{R}_k$, Theorem 3.21 implies that for almost every $\omega\in\mathbb{T}^s$, $\{T^l_k\theta^m_{\omega,k} : m=1,\ldots,\rho, l\in\mathcal{L}_k\}$ forms a tight frame for $V^k_{2\pi}(\Phi_{\omega,k})$, and for all $m,n=1,\ldots,\rho$ and $l,r\in\mathcal{L}_k$,

$$\langle T^l_k\theta^m_{\omega,k}, T^r_k\theta^n_{\omega,k}\rangle_{L^2(\mathbb{T}^s)} = 0 \quad \text{if } m\neq n.$$

Consequently, Lemmas 3.5 and 3.14 show that for all $m,n=1,\ldots,\rho$ with $m\neq n$ and $l,r\in\mathcal{L}_k$,

$$\begin{aligned}\langle E^l_k\theta^m_k, E^r_k\theta^n_k\rangle &= \int_{\mathbb{T}^s} \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}(l-r)}\langle \mathcal{M}^l_{0,k}\widehat{\theta^m_k}_{0,\omega,0}, \mathcal{M}^r_{0,k}\widehat{\theta^n_k}_{0,\omega,0}\rangle_{l^2(\mathbb{Z}^s)}\mathrm{d}\omega \\ &= \int_{\mathbb{T}^s} \mathrm{e}^{\mathrm{i}\omega\cdot M^{-k}(r-l)}\langle \mathcal{M}^l_{0,k}\widehat{\theta^m_{\omega,k}}, \mathcal{M}^r_{0,k}\widehat{\theta^n_{\omega,k}}\rangle_{l^2(\mathbb{Z}^s)}\mathrm{d}\omega \\ &= \int_{\mathbb{T}^s} \mathrm{e}^{\mathrm{i}\omega\cdot M^{-k}(r-l)}\langle T^l_k\theta^m_{\omega,k}, T^r_k\theta^n_{\omega,k}\rangle_{L^2(\mathbb{T}^s)}\mathrm{d}\omega = 0,\end{aligned}$$

and $E(\{E^l_k\theta^m_k : m=1,\ldots,\rho, l\in\mathcal{L}_k\})$ being a tight frame for $V^k(\Phi_k)$ follows from Corollary 3.37. □

We emphasize that the "almost every" condition in Corollary 3.37 is essential. The result does not hold, for instance, if the lower frame bound of $T_k(\Phi_{\omega,k})$ is arbitrarily close to zero for some ω belonging to a set of positive measure. This could be seen from the following example for $s=1$ and $M=2$.

Example 3.4.1. Let $c, N \in \mathbb{N}$ be such that $c \geq 3$ and $c \mod 2 = 1$. Noting that $\lceil \log_2 c2^N \rceil \geq \lceil \log_2 2^N \rceil = N$ and $k \geq \lceil \log_2 c2^N \rceil$ implies that $2^k > c2^N$, construct the $L^2(\mathbb{R})$ functions $\phi_k = 2^{\frac{k}{2}} \mathbf{1}_{[0,c2^N)}(2^k \cdot) = 2^{\frac{k}{2}} \mathbf{1}_{[0,c2^{N-k})}$, whose corresponding Fourier transforms are given by

$$\widehat{\phi_k}(\omega) = 2^{-\frac{k}{2}} \frac{1}{\mathrm{i}2^{-k}\omega} \left[1 - \mathrm{e}^{-\mathrm{i}2^{-k}\omega c2^N}\right] = \frac{2^{\frac{k}{2}}}{\mathrm{i}\omega} \left[1 - \mathrm{e}^{-\mathrm{i}c2^{N-k}\omega}\right],$$

which leads to

$$\left|\widehat{\phi_k}(\omega)\right|^2 = \frac{2^k}{\omega^2} \left[2 - 2\cos c2^{N-k}\omega\right] = \frac{2^{k+2}}{\omega^2} \left[\sin^2 c2^{N-k}\frac{\omega}{2}\right],$$

whose zeros are located at $\omega = 2\pi \frac{n}{c} 2^{k-N}$, where $n \in \mathbb{Z}\backslash\{0\}$. Observe that $\omega + 2\pi 2^k = 2\pi \frac{n+c2^N}{c} 2^{k-N}$ and this shows that the zeros are $2\pi 2^k$ periodic.

Since

$$\begin{aligned} \phi_k &= 2^{\frac{k}{2}} \left[\mathbf{1}_{[0,c2^{N-(k+1)})} + \mathbf{1}_{[c2^{N-(k+1)},c2^{N-k})}\right] \\ &= 2^{-\frac{1}{2}}\phi_{k+1} + 2^{-\frac{1}{2}}\phi_{k+1}(\cdot - c2^N 2^{-(k+1)}) = 2^{-\frac{1}{2}}\phi_{k+1} + 2^{-\frac{1}{2}} E_{k+1}^{c2^N}\phi_{k+1}, \end{aligned}$$

we have refinability for the $\phi_k's$.

Since $\left|\widehat{\phi_k}(\omega + 2\pi 2^k n)\right|^2 \leq \frac{2^k(4)}{(2\pi 2^k n)^2}$ for $n \in \mathbb{N}$, $\left|\widehat{\phi_k}(\omega + 2\pi 2^k n)\right|^2 \leq \frac{2^k(4)}{[2\pi 2^k(1+n)]^2}$ for $n \leq -2$ and $\left|\widehat{\phi_k}(\omega - 2\pi 2^k)\right|^2 \leq 2^{k+2}\left[\frac{c2^{N-k}}{2}\right]^2$, by Weierstrass M-Test, the Gramian $\mathrm{M}_{k,\phi_k}(\omega)$ of the set $E_k(\phi_k)$ is a continuous function. For $k \geq 0$, as $\widehat{\phi_k}$ has $2\pi 2^k$-periodic zeros $\mathrm{M}_{k,\phi_k}(\omega)$ is not bounded below away from zero and by Theorem 3.4, the Bessel system $E_k(\phi_k)$ is not a frame. Using the periodization method given in (3.58), we obtain

$$(2\pi)\phi_{\omega,k}(t) = \begin{cases} 2^{\frac{k}{2}}(1 - \mathrm{e}^{-\mathrm{i}\omega c2^{N-k}})(1 - \mathrm{e}^{-\mathrm{i}\omega})^{-1}\mathbf{1}_{[0,2\pi)}(t)\mathrm{e}^{-\mathrm{i}\frac{\omega t}{2\pi}} & \text{if } 0 \leq k < \lceil \log_2 c2^N \rceil, \\ 2^{\frac{k}{2}}\mathbf{1}_{[0,2\pi c2^{N-k})}(t)\mathrm{e}^{-\mathrm{i}\frac{\omega t}{2\pi}} & \text{if } k \geq \lceil \log_2 c2^N \rceil. \end{cases}$$

Noting that for $0 \leq k < \lceil \log_2 c2^N \rceil$, we have

$$(2\pi)\mathbb{P}_{2\pi}[2^{\frac{k}{2}}\mathbf{1}_{[0,c2^{N-k})}((2\pi)^{-1}\cdot)\mathrm{e}^{-\frac{\mathrm{i}\omega\cdot}{2\pi}}](t) = 2^{\frac{k}{2}} \sum_{m=0}^{c2^{N-k}-1} \mathbf{1}_{[-2\pi m,-2\pi m+2\pi c2^{N-k})}(t)\mathrm{e}^{-\mathrm{i}\omega m}\mathrm{e}^{-\frac{\mathrm{i}\omega t}{2\pi}}$$

$$= \left[2^{\frac{k}{2}} \sum_{m=0}^{c2^{N-k}-1} \mathrm{e}^{-\mathrm{i}\omega m}\right] \mathrm{e}^{-\frac{\mathrm{i}\omega t}{2\pi}}.$$

For $\omega \in (0, 2\pi)$ or $n \neq 0$, the Fourier coefficients of $\phi_{\omega,k}$ are given as

$$(2\pi)\widehat{\phi_{\omega,k}}(n) = \begin{cases} \frac{2^{\frac{k}{2}}}{\mathrm{i}(\omega+2\pi n)}\left[1-\mathrm{e}^{-\mathrm{i}\omega c2^{N-k}}\right] & \text{if } 0 \leq k < \lceil \log_2 c2^N \rceil, \\ \frac{2^{\frac{k}{2}}}{\mathrm{i}(\omega+2\pi n)}\left[1-\mathrm{e}^{-\mathrm{i}(\omega+2\pi n)c2^{N-k}}\right] & \text{if } k \geq \lceil \log_2 c2^N \rceil, \end{cases}$$

which leads to

$$\begin{aligned} \left|(2\pi)\widehat{\phi_{\omega,k}}(n)\right|^2 &= \begin{cases} \frac{2^k(2)}{(\omega+2\pi n)^2}\left[1-\cos\omega c2^{N-k}\right] & \text{if } 0 \leq k < \lceil \log_2 c2^N \rceil, \\ \frac{2^k(2)}{(\omega+2\pi n)^2}\left[1-\cos c2^{N-k}(\omega+2\pi n)\right] & \text{if } k \geq \lceil \log_2 c2^N \rceil, \end{cases} \\ &= \begin{cases} \frac{2^k(4)}{(\omega+2\pi n)^2}\left[\sin^2\frac{1}{2}\omega c2^{N-k}\right] & \text{if } 0 \leq k < \lceil \log_2 c2^N \rceil, \\ \frac{2^k(4)}{(\omega+2\pi n)^2}\left[\sin^2\frac{1}{2}c2^{N-k}(\omega+2\pi n)\right] & \text{if } k \geq \lceil \log_2 c2^N \rceil. \end{cases} \end{aligned}$$

We also confirm that for $k = \lceil \log_2 c2^N \rceil - 1$, $\phi_{\omega,k}$ is refinable, i.e.

$$\phi_{\omega,k} = \sum_{l \in \mathcal{L}_{k+1}} (1-\mathrm{e}^{-\mathrm{i}\omega c2^{N-k}})(1-\mathrm{e}^{-\mathrm{i}\omega})^{-1} T_{k+1}^l \phi_{\omega,k+1}.$$

Let $j \in \mathcal{R}_k \backslash \{0\}$ and $p \in \mathbb{Z}$. For $0 \leq k < \lceil \log_2 c2^N \rceil$, we have

$$(2\pi)\widehat{\phi_{\omega,k}}(j+2^k p) = \frac{2^{\frac{k}{2}}}{\mathrm{i}\left[\omega+2\pi(j+2^k p)\right]}\left[1-\mathrm{e}^{-\mathrm{i}\omega c2^{N-k}}\right],$$

and $\widehat{\phi_{\omega,k}}(j+2^k p) = 0$ if $\omega = 0$. For $k \geq \lceil \log_2 c2^N \rceil$, we have

$$(2\pi)\widehat{\phi_{\omega,k}}(j+2^k p) = \frac{2^{\frac{k}{2}}}{\mathrm{i}\left[\omega+2\pi(j+2^k p)\right]}\left[1-\mathrm{e}^{-\mathrm{i}(\omega+2\pi j)c2^{N-k}}\right].$$

Therefore $\widehat{\phi_{\omega,k}}(j+2^k p) = 0$ if $\omega + 2\pi j = 2\pi\frac{n}{c}(2^{k-N})$. Since $j \in \mathcal{R}_k \backslash \{0\}$, we must have $\omega = 0$ and $j = \frac{n}{c}2^{k-N}$, where $n \in \{c, 2c, \ldots, (2^N-1)c\}$, i.e. $j \in \{2^{k-N}, \ldots, (2^N-1)2^{k-N}\}$. Hence for such j's, the polyphase harmonics $(\phi_{0,k})_{k,j} = 0$, which also shows that even when $\phi_{0,k}$ is compactly supported within $[0, 2\pi]$, $\sigma_k(V_{2\pi}^k(\phi_{0,k}))$ is only a proper subset of $\mathcal{R}_k$. Using Weierstrass M-Test again, we conclude that for $j \in \{2^{k-N}, \ldots, (2^N-1)2^{k-N}\}$, $\|(\phi_{\omega,k})_{k,j}\|^2_{L^2(\mathbb{T})}$ is a continuous function having zeros in $[0, 2\pi)$ whenever $\omega = 0$ and this shows that $\|(\phi_{\omega,k})_{k,j}\|^2_{L^2(\mathbb{T})}$ is not bounded below on $\omega \in [0, 2\pi)$. Therefore, by Theorem 3.13, $T_k(\phi_{\omega,k})$ is not a frame for almost every $\omega \in [0, 2\pi)$ with uniform bounds since the lower bound of $T_k(\phi_{\omega,k})$ is arbitrarily close to zero for ω belonging to some subset of $[0, 2\pi)$ with positive measure.

In Example 3.4.1, we observe that for $k \geq \lceil \log_2 c2^N \rceil$, even when $\phi_{0,k}$ is compactly supported within $[0, 2\pi]$ and refinable, $\sigma_k(V_{2\pi}^k(\phi_{0,k}))$ is only a proper subset of $\mathcal{R}_k$. The implication is that we could not represent functions belonging to $\mathcal{R}_k \backslash \sigma_k(V_{2\pi}^k(\phi_{0,k}))$ by using $E_k(\phi_{0,k})$ alone. This situation does not occur in the nonperiodic setting of compactly supported functions $\phi_k \in L^2(\mathbb{R})$ with $\widehat{\phi_k}$ having mild decay properties such as $\phi_k' \in L_1(\mathbb{R})$, i.e. we have $\sigma_k(V^k(\phi_k)) = \mathbb{T}_k$ (modulo measure zero sets). In particular, if $E_k(\phi_k)$ is a Riesz basis, we have $\sigma_k(V^k(\phi_k)) = \mathbb{T}_k$ exactly since $\widehat{\phi_k}$ is an entire function and M_{k,ϕ_k} will be continuous on $\mathbb{T}^s$ and bounded below away from zero everywhere. In this case, $\sigma_k(V_{2\pi}^k(\phi_{\omega,k}))$ is always the set $\mathcal{R}_k$.

A consequence of Corollaries 3.37 and 3.39 is an alternative proof for a similar result from [4] for the stationary setting concerning the orthogonal decomposition of an FSI space into PSI spaces. Our formulation takes care of both stationary and nonstationary cases.

Theorem 3.40. *For $k \geq 0$, let $V^k(\Phi_k)$ be a closed $M^{-k}\mathbb{Z}^s$ shift-invariant space of $L^{2,\alpha}(\mathbb{R}^s)$ with $|\Phi_k| = \rho$. Then* $\operatorname{len} V^k(\Phi_k) = \rho_k := \operatorname{ess\,sup}\{\dim V_{2\pi,\omega}^{k,j} : j \in \mathcal{R}_k, \omega \in \mathbb{T}^s\}$ *and there exist functions $\{\theta_k^m\}_{m=1}^{\rho_k} \subset V^k(\Phi_k)$ such that $E_k(\{\theta_k^m\}_{m=1}^{\rho_k})$ is a tight frame for $V^k(\Phi_k)$ and for all $m, n = 1, \ldots, \rho_k$, and $l, r \in \mathcal{L}_k$,* (3.62) *holds. Consequently, $V^k(\Phi_k)$ can be written as the orthogonal sum of ρ_k $M^{-k}\mathbb{Z}^s$ PSI spaces.*

Proof. The proof is essentially that of Theorem 3.23 with the additional requirement that the functions $\{\theta_k^m\}_{m=1}^{\rho_k}$ are constructed in the following way to be measurable. Corollary 3.39 shows that there exist functions $\{\theta_k^m\}_{m=1}^{\rho} \subset V^k(\Phi_k)$ such that $E_k(\{\theta_k^m\}_{m=1}^{\rho})$ is a tight frame for $V^k(\Phi_k)$ and for all $m, n = 1, \ldots, \rho$ and $l, r \in \mathcal{L}_k$, (3.62) holds. Let $\rho_{\omega,k} := \max\{\dim V_{2\pi}^{k,j}(\Phi_{\omega,k}) : j \in \mathcal{R}_k\}$ and hence $\rho_k = \operatorname{ess\,sup}\{\rho_{\omega,k} : \omega \in \mathbb{T}^s\}$. Observe from (3.34) that the number of nonzero eigenvalues of $\mathrm{M}_{\omega,k}(j)$ is bounded by $\rho_{\omega,k}$. For each $j \in \mathcal{R}_k$, after interchanging rows followed by columns on both sides of (3.63) by multiplying on the left and on the right a $\rho \times \rho$ permutation matrix and its transpose respectively, the resulting eigenvalues satisfy $\lambda_{\omega,k}^m(j) = 0$ for all $m = \rho_{\omega,k} + 1, \ldots, \rho$. As a result, (3.64) and (3.67) shows that the corresponding $\left\|w_{\omega,k,j}^m\right\|_{L^2(\mathbb{T}^s)}^2 = 0$, which means, by (3.66), that $\theta_{\omega,k}^m = 0$ for $m = \rho_{\omega,k} + 1, \ldots, \rho$ for almost every $\omega \in \mathbb{T}^s$. Consequently, (3.59) together with Lemma 3.5 shows that $\theta_k^m = 0$ for $m = \rho_k + 1, \ldots, \rho$. Therefore, $V^k(\Phi_k) = \bigoplus_{m=1}^{\rho_k \perp} V^k(\theta_k^m)$ and $L_k := \operatorname{len} V^k(\Phi_k) \leq \rho_k$. There exists $\{\vartheta_k^m\}_{m=1}^{L_k}$ such that $E(\{E_k^l \vartheta_k^m : m = 1, \ldots, L_k, l \in \mathcal{L}_k\})$ spans $V^k(\Phi_k)$. Let $y_{\omega,k,j}^m$, $m = 1, \ldots, L_k$, $j \in \mathcal{R}_k$ be the polyphase harmonics of $\{\vartheta_{\omega,k}^m\}_{m=1}^{L_k}$, which by Corollary 3.37, spans $V_{2\pi}^k(\Phi_{\omega,k})$. Using (3.35), for $j \in \mathcal{R}_k$, $\operatorname{span}\{y_{\omega,k,j}^m\}_{m=1}^{L_k} = V_{2\pi}^{k,j}(\Phi_{\omega,k})$. Hence, $L_k \geq \rho_k$.

□

Our next result shows that two function spaces are the same if their corresponding periodized spaces constructed from uniform frequency samples are the same for almost all possible samples.

Corollary 3.41. *Let V be the FSI space generated by a finite subset Λ of a FSI space S of $L^{2,\alpha}(\mathbb{R}^s)$. Suppose that $\dim \widehat{V}_{||\omega} = \dim \widehat{S}_{||\omega}$ (or $\dim V_{2\pi,\omega} = \dim S_{2\pi,\omega}$) for almost every $\omega \in \mathbb{T}^s$. Then $V = S$.*

Proof. Since $\Lambda \subset S$ and S is a closed shift-invariant space, therefore $V \subseteq S$. By Theorem 3.33, $W := S \ominus V$ is also a FSI space and $\widehat{S}_{||\omega} = \widehat{V}_{||\omega} \oplus^{\perp} \widehat{W}_{||\omega}$ for almost every $\omega \in \mathbb{T}^s$. Using our hypothesis that $\dim \widehat{W}_{||\omega} = 0$ for almost every $\omega \in \mathbb{T}^s$, Theorem 3.40 shows that W is an FSI space with $\operatorname{len} W = 0$. Therefore, $W = \{0\}$ and $V = S$. $\square$

The rest of the results are on wavelets obtained from the semi-orthogonal setting of FMRAs and the nonorthogonal setting of MRAs using the UEP.

Lemma 3.42. *For $k \geq 0$, let $\Phi_k \subset L^{2,\alpha}(\mathbb{R}^s)$ with $|\Phi_k| = \rho$. The union $\bigcup_{k\geq 0} V^k(\Phi_k)$ is dense in $L^2(\mathbb{R}^s)$ if and only if $\bigcup_{k\geq 0} V^k_{2\pi}(\Phi_{\omega,k})$ is dense in $L^2(\mathbb{T}^s)$ for almost every $\omega \in \mathbb{T}^s$.*

Proof. ($\Rightarrow$) Corollary 3.34 shows that there exists a set $\Delta \subset \mathbb{T}^s$ with $|\Delta| = 0$ such that for every $\omega \in \mathbb{T}^s \backslash \Delta$ and $k \geq 0$, $V^k_{2\pi}(\Phi_{\omega,k})$ is a subspace of $L^2(\mathbb{T}^s)$. Let $F := \bigcap_{k\in\mathbb{Z}} \bigcap_{\phi_k \in \Phi_k} \{\omega \in \mathbb{R}^s : \widehat{\phi_k}(\omega) = 0\}$ and for $n \in \mathbb{Z}^s$, define $F_n := F - 2\pi n = \bigcap_{k\in\mathbb{Z}} \bigcap_{\phi_k\in\Phi_k} \{\omega \in \mathbb{R}^s : \widehat{\phi_k}(\omega + 2\pi n) = 0\}$. Since $|F| = 0$ by Condition (ii) of an MRA, therefore $|F_n| = 0$. For $\omega \in \mathbb{T}^s \backslash (\Delta \cup \bigcup_{n\in\mathbb{Z}^s} F_n)$ and $n \in \mathbb{Z}^s$, there exist $k \geq 0$ and $\phi_k \in \Phi_k$ depending on ω and n such that $\widehat{\phi_{\omega,k}}(n) \neq 0$. By Condition (ii) of a periodic MRA, $\bigcup_{k\geq 0} V^k_{2\pi}(\Phi_{\omega,k})$ is dense in $L^2(\mathbb{T}^s)$.

($\Leftarrow$) By Condition (ii) of a periodic MRA, there exists a set $F \subset \mathbb{T}^s$ with $|F| = 0$ such that for $\omega \in \mathbb{T}^s \backslash F$, $\bigcap_{k\geq 0} \bigcap_{\phi_{\omega,k}\in\Phi_{\omega,k}} \{n \in \mathbb{Z}^s : \widehat{\phi_{\omega,k}}(n) = 0\} = \emptyset$. For $n \in \mathbb{Z}^s$, define $F_n := F + 2\pi n$. For a given $l \in \mathbb{Z}^s$ and $\omega \in (\mathbb{T}^s + 2\pi l) \backslash F_l$, equivalently $\omega - 2\pi l \in \mathbb{T}^s \backslash F$, since $\bigcap_{k\geq 0} \bigcap_{\phi_k\in\Phi_k} \{m + l \in \mathbb{Z}^s : \widehat{\phi_k}(\omega + 2\pi m) = 0\}$ is an empty set, it follows that the set $\bigcap_{k\geq 0} \bigcap_{\phi_k\in\Phi_k} \{m \in \mathbb{Z}^s : \widehat{\phi_k}(\omega + 2\pi m) = 0\}$ is also empty. Therefore, for $\omega \in \mathbb{R}^s \backslash \bigcup_{n\in\mathbb{Z}^s} F_n$, there exist $k \geq 0$ and $\phi_k \in \Phi_k$ depending on ω such that $\left|\widehat{\phi_k}(\omega)\right| > 0$ and the result holds by Condition (ii) of an MRA. $\square$

Theorem 3.43. *For $k \geq 0$, let $\Phi_k \subset L^{2,\alpha}(\mathbb{R}^s)$ with $|\Phi_k| = \rho$. The collection $\{V^k(\Phi_k)\}$ is an MRA of $L^2(\mathbb{R}^s)$ if and only if $\{V^k_{2\pi}(\Phi_{\omega,k})\}$ is an MRA of $L^2(\mathbb{T}^s)$ for almost every $\omega \in \mathbb{T}^s$. In particular, $\{V^k(\Phi_k)\}$ is an FMRA of $L^2(\mathbb{R}^s)$ if and only if $\{V^k_{2\pi}(\Phi_{\omega,k})\}$ is a periodic FMRA of $L^2(\mathbb{T}^s)$ with the same bounds for almost every $\omega \in \mathbb{T}^s$.*

Proof. In our proof, by virtue of Corollary 3.34, we shall only consider an arbitrary $\omega \in \mathbb{T}^s \backslash \Delta$, where $|\Delta| = 0$, such that for every $k \geq 0$, $V_{2\pi}^k(\Phi_{\omega,k})$ is a subspace of $L^2(\mathbb{T}^s)$ and $\widehat{V}_{||\omega}^k(\Phi)$ is a subspace of $l^2(\mathbb{Z}^s)$.
($\Rightarrow$) Corollary 3.34 implies that for every $k \geq 0$, $\widehat{V}_{||\omega}^k(\Phi_k) \subseteq \widehat{V}_{||\omega}^{k+1}(\Phi_{k+1})$. Consequently, Corollary 3.37 shows that for every $k \geq 0$, $V_{2\pi}^k(\Phi_{\omega,k}) \subseteq V_{2\pi}^{k+1}(\Phi_{\omega,k+1})$. The density requirement of $\bigcup_{k\geq 0} V_{2\pi}^k(\Phi_{\omega,k})$ in $L^2(\mathbb{T}^s)$ is satisfied using Lemma 3.42. Therefore, $\{V_{2\pi}^k(\Phi_{\omega,k})\}$ is a periodic MRA of $L^2(\mathbb{T}^s)$. If $\{V^k(\Phi_k)\}$ is also an FMRA of $L^2(\mathbb{R}^s)$, then Corollary 3.37 shows that $\{V_{2\pi}^k(\Phi_{\omega,k})\}$ is a periodic FMRA of $L^2(\mathbb{T}^s)$ with the same bounds as $\{V^k(\Phi_k)\}$.
($\Leftarrow$) By Corollary 3.37, since for every $k \geq 0$, $\widehat{V}_{||\omega}^k(\Phi_k) \subseteq \widehat{V}_{||\omega}^{k+1}(\Phi_{k+1})$, Proposition 3.35 shows that $V^k(\Phi_k) \subset V^{k+1}(\Phi_{k+1})$. Next, Lemma 3.42 shows that $\bigcup_{k\geq 0} V^k(\Phi_k)$ is dense in $L^2(\mathbb{R}^s)$. Therefore, $\{V^k(\Phi_k)\}$ is an MRA of $L^2(\mathbb{R}^s)$. If $\{V_{2\pi}^k(\Phi_{\omega,k})\}$ is also a periodic FMRA of $L^2(\mathbb{T}^s)$, then Corollary 3.37 shows that $E_k(\Phi_k)$ is a frame for $V^k(\Phi_k)$ with the same bounds as $\{V_{2\pi}^k(\Phi_{\omega,k})\}$ and hence $\{V^k(\Phi_k)\}$ is an FMRA of $L^2(\mathbb{R}^s)$ with the same bounds as $\{V_{2\pi}^k(\Phi_{\omega,k})\}$. □

Theorem 3.43 states that a collection of subspaces is an MRA if and only if their corresponding periodized subspaces constructed from uniform frequency samples is an MRA for almost all possible samples. Our next result is the analogue of Proposition 3.12 after periodization.

Theorem 3.44. *Let $\Phi \subset L^{2,\alpha}(\mathbb{R}^s)$ be a finite set. Suppose that $\{V^k(\Phi)\}$ is an MRA of $L^2(\mathbb{R}^s)$. Let W^k be the orthogonal complement of $V^k(\Phi)$ in $V^{k+1}(\Phi)$ and $\Psi \subset W^0$ be a finite set. Then $E(\Psi)$ forms a frame for W^0 if and only if for every $k \geq 0$, $T_k(\Psi_{\omega,k})$ is a frame for $W_{2\pi,\omega}^k$ with the same bounds for almost every $\omega \in \mathbb{T}^s$.*

Proof. In our proof, by virtue of Theorem 3.33 and Corollary 3.34, we only consider an arbitrary $\omega \in \mathbb{T}^s \backslash \Delta$, where $|\Delta| = 0$, such that for every $k \geq 0$, $V_{2\pi}^k(\Phi_{\omega,k})$ and $W_{2\pi,\omega}^k$ are subspaces of $L^2(\mathbb{T}^s)$ and $\widehat{V}_{||\omega}^k(\Phi)$ and $\widehat{W}_{||\omega}^k$ are subspaces of $l^2(\mathbb{Z}^s)$.
($\Rightarrow$) The proof of Proposition 3.12 shows that $E(\{d^{\frac{k}{2}} E_k^l \psi(M^k \cdot) : \psi \in \Psi, l \in \mathcal{L}_k\})$ is a frame for W^k with the same bounds as $E(\Psi)$. The fibre $\widehat{W}_{||\omega}^k$ according to (3.59) is the closed linear span of $\{\mathrm{e}^{-\mathrm{i}\omega \cdot M^{-k} l} \mathcal{M}_{0,k}^l \widehat{\psi_{\omega,k}} : \psi \in \Psi, l \in \mathcal{L}_k\}$, which corresponds to the Fourier coefficients of $\{\mathrm{e}^{-\mathrm{i}\omega \cdot M^{-k} l} T_k^l \psi_{\omega,k} : \psi \in \Psi, l \in \mathcal{L}_k\}$ as verified by Lemma 3.31. In addition, Lemma 3.10 shows that the subspace W^k satisfies (3.12). Hence, Corollary 3.37 implies that $T_k(\Psi_{\omega,k})$ is a frame for $W_{2\pi,\omega}^k$ with the same bounds as $E(\Psi)$.
($\Leftarrow$) Corollary 3.37 shows that $E(\Psi)$ is a frame for the subspace W^0 with the same bounds as $\Psi_{\omega,0}$. □

We shall adapt the proof of a result in [37] concerning the extension of columns of a matrix with Laurent polynomial entries. With the same essential steps, this gives Proposition 3.45, which is on extending columns of a matrix with measurable functions as entries.

Proposition 3.45. *Let $A(\omega) := [A_1(\omega)|\cdots|A_q(\omega)]$ be a $p \times q$ measurable matrix-valued function with orthonormal columns for almost every $\omega \in F \subseteq \mathbb{T}^s$, where $p > q$. Then there exists a $p \times p$ measurable unitary matrix $Q(\omega)$ such that for almost every $\omega \in F$,*

$$Q(\omega)A(\omega) = \begin{bmatrix} I_q \\ 0 \end{bmatrix},$$

where $Q = Q_q Q_{q-1} \cdots Q_1$ and for $i = 1, \ldots, q$, $Q_i = \begin{bmatrix} I_{i-1} & 0 \\ 0 & P_i \end{bmatrix}$ is a $p \times p$ measurable unitary matrix and P_i is a $(p-i+1) \times (p-i+1)$ measurable unitary matrix.

Proof. Consider an arbitrary $\omega \in F \backslash \Delta$ where $|\Delta| = 0$ such that the columns of $A(\omega)$ are orthonormal while on the set Δ, for $i \in \{1, \ldots, q\}$, Q_i and P_i are defined to be the identity matrix. Let $Y_1(\omega) := A_1(\omega) - \|A_1(\omega)\|_{\mathbb{C}^p}\, e_p^1$, $f_1(\omega) := \|Y_1(\omega)\|_{\mathbb{C}^p}^{-2}\, \mathbf{1}_{\{\|Y_1(\omega)\|_{\mathbb{C}^p} > 0\}}$ and $Q_1(\omega) := I - 2f_1(\omega)Y_1(\omega)Y_1(\omega)^*$, where $e_p^1 = [1, 0, \ldots, 0]^T$ in $\mathbb{C}^p$ endowed with the usual Euclidean 2-norm and I is the identity matrix. Then $Q_1(\omega)$ is a unitary $p \times p$ Householder reflector matrix and Q_1 is measurable as f_1 is measurable. By our rhombus construction, $Q_1(\omega)A_1(\omega) = e_p^1$. Since $Q_1(\omega)$ is unitary, it follows that for $i, j \in \{1, \ldots, q\}$,

$$\langle Q_1(\omega)A_i(\omega), Q_1(\omega)A_j(\omega)\rangle = \delta_{ij}.$$

Thus, the first entry $(Q_1(\omega)A_k(\omega))_1$ of $Q_1(\omega)A_k(\omega)$ is zero for $k \in \{2, \ldots, q\}$. Consequently, we have

$$Q_1(\omega)A(\omega) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & A_2^{(2)}(\omega) & \cdots & A_q^{(2)}(\omega) \end{bmatrix},$$

where for $k \in \{2, \ldots, q\}$, $A_k^{(2)}(\omega)$ are $(p-1) \times 1$ matrices with $\langle A_i^{(2)}(\omega), A_j^{(2)}(\omega)\rangle_{\mathbb{C}^{p-1}} = \delta_{ij}$.

Suppose that there are measurable unitary matrices $Q_1(\omega), \ldots, Q_{k-1}(\omega)$ such that

$$Q_{k-1}(\omega) \cdots Q_2(\omega)Q_1(\omega)A(\omega) = \begin{bmatrix} I_{k-1} & 0 & \cdots & 0 \\ 0 & A_k^{(k)}(\omega) & \cdots & A_q^{(k)}(\omega) \end{bmatrix},$$

where $A_i^{(k)}(\omega)$ are $(p-k+1) \times 1$ matrices with $\langle A_i^{(k)}(\omega), A_j^{(k)}(\omega)\rangle_{\mathbb{C}^{p-k+1}} = \delta_{ij}$ for $i, j \in \{k, \ldots, q\}$. For induction purposes, let us define $Y_k(\omega) := A_k^{(k)}(\omega) - \left\|A_k^{(k)}(\omega)\right\|_{\mathbb{C}^{p-k+1}} e_{p-k+1}^1$,

$f_k(\omega) := \|Y_k(\omega)\|^{-2}_{\mathbb{C}^{p-k+1}} \mathbf{1}_{\{\|Y_k(\omega)\|_{\mathbb{C}^{p-k+1}}>0\}}$ and $P_k(\omega)$ be a unitary $(p-k+1)\times(p-k+1)$ Householder reflector matrix given by $P_k(\omega) := I - 2f_k(\omega)Y_k(\omega)Y_k(\omega)^*$. Consequently, we obtain $P_k A_k^{(k)}(\omega) = e^1_{p-k+1}$ and $\langle P_k A_i^{(k)}(\omega), P_k A_j^{(k)}(\omega)\rangle_{\mathbb{C}^{p-k+1}} = \delta_{ij}$ for $i,j \in \{k,\dots,q\}$ and $(P_k A_i^{(k)}(\omega))_1 = 0$ for $i \in \{k+1,\dots,q\}$. Let $Q_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & P_k \end{bmatrix}$. Then $Q_k(\omega)$ is measurable and unitary and

$$Q_k(\omega)Q_{k-1}(\omega)\cdots Q_1 A(\omega) = \begin{bmatrix} I_k & 0 & \cdots & 0 \\ 0 & A^{(k+1)}_{k+1}(\omega) & \cdots & A^{(k+1)}_q(\omega) \end{bmatrix},$$

where $A_i^{(k+1)}(\omega)$ are $(p-k)\times 1$ matrices with $\langle A_i^{(k+1)}(\omega), A_j^{(k+1)}(\omega)\rangle_{\mathbb{C}^{p-k}} = \delta_{ij}$ for $i,j \in \{k+1,\dots,q\}$. Letting $Q = Q_q Q_{q-1}\cdots Q_1$, we have $Q(\omega)A(\omega) = \begin{bmatrix} I_q \\ 0 \end{bmatrix}$. $\square$

Our next theorem establishes the existence of tight frames in $L^2(\mathbb{R}^s)$ from periodized tight frames in $L^2(\mathbb{T}^s)$ both of which are derived from FMRAs.

Theorem 3.46. *For $k \ge 0$, let $\Phi_k \subset L^{2,\alpha}(\mathbb{R}^s)$ with $|\Phi_k| = \rho$. Suppose that $\{V^k(\Phi_k)\}$ is an FMRA of $L^2(\mathbb{R}^s)$. Let W^k be the orthogonal complement of $V^k(\Phi_k)$ in $V^{k+1}(\Phi_{k+1})$. Suppose that for almost every $\omega \in \mathbb{T}^s$, there exists $G_{\omega,k+1} \in \mathcal{S}(M)^{\varrho_k\times\rho}$ such that $T_k(\Psi_{\omega,k})$ is a frame for $W^k_{2\pi,\omega}$ with bounds C and D, where $\Psi_{\omega,k} := \sum_{l\in\mathcal{L}_{k+1}} G_{\omega,k+1}(l)T^l_{k+1}\Phi_{\omega,k+1}$. In addition, if the $2\pi D^{k+1}\mathbb{Z}^s$-periodic matrix-valued function $\widehat{G}_{k+1}$ defined by $\widehat{G}_{k+1}(\omega + 2\pi j) := \widehat{G}_{\omega,k+1}(j)$, where $j \in \mathcal{R}_{k+1}$, lies in $L^2(\mathbb{T}^s)$, then $E_k(\Psi_k)$ is a frame for W^k with bounds C and D, where $\widehat{\Psi}_k = \widehat{G}_{k+1}\widehat{\Phi}_{k+1}$.*

Proof. Assuming that the matrix-valued function $\widehat{G}_{k+1}$ lies in $L^2(\mathbb{T}^s)$, we have $\Psi_k \subset V^{k+1}(\Phi_{k+1})$. In our proof, by virtue of Theorem 3.33 and Corollary 3.34, we only consider an arbitrary $\omega \in \mathbb{T}^s\backslash\Delta$, where $|\Delta| = 0$, such that for every $k \ge 0$, $V^k_{2\pi}(\Phi_{\omega,k})$ and $W^k_{2\pi,\omega}$ are a pair of orthogonal subspaces of $L^2(\mathbb{T}^s)$ and $\widehat{V}^k_{||\omega}(\Phi_k)$ and $\widehat{W}^k_{||\omega}$ are a pair of orthogonal subspaces of $l^2(\mathbb{Z}^s)$ and for every $j \in \mathcal{R}_{k+1}$, $\widehat{G}_{\omega,k+1}(j)$ is finite. For almost every $\omega \in \mathbb{T}^s$, according to (3.59), $\{\mathcal{M}^l_{0,k}\widehat{\psi_{k0,\omega,0}} : \psi_k \in \Psi_k, l \in \mathcal{L}_k\}$ forms a frame for $\widehat{W}^k_{||\omega}$ with bounds C and D. By Lemma 3.5 and Theorem 3.33, since for any $f \in V^k(\Phi_k)$ and $\psi_k \in \Psi_k$ and $l \in \mathcal{L}_k$, we have

$$\langle E^l_k\psi_k, f\rangle = \int_{\mathbb{T}^s} \mathrm{e}^{-\mathrm{i}\omega\cdot M^{-k}l}\langle \mathcal{M}^l_{0,k}\widehat{\psi_{k0,\omega,0}}, \widehat{f}_{0,\omega,0}\rangle_{l^2(\mathbb{Z}^s)}\mathrm{d}\omega = \int_{\mathbb{T}^s} 0\mathrm{d}\omega = 0,$$

therefore $E_k(\Psi_k) \subset W^k$. Finally, Corollaries 3.37 and 3.41 show that $E_k(\Psi_k)$ is a frame for W^k with bounds C and D. $\square$

We now describe the construction of $\widehat{G}_{k+1}$ in Theorem 3.46 using the proof of Theorem 3.24 as follows. Without loss of generality, using Theorem 3.22, for every $k \geq 0$, we may assume that $T_k(\Phi_{\omega,k})$ with $|\Phi_{\omega,k}| = \rho$ is a tight frame satisfying Theorem 3.21 for every $\omega \in \mathbb{T}^s \backslash \Delta$ with $|\Delta| = 0$ as in the proof of Theorem 3.46. Note that in steps (1) and (2) of the algorithm that follows, we shall make use of the result in the proof of Lemma 3.38 which shows that the rank of a matrix is a measurable function.

(1) Fix $k \geq 0$ and $j \in \mathcal{R}_k$. For $\omega \in \mathbb{T}^s \backslash \Delta$, let $S_{\omega,k}$ and $\widehat{\mathbb{H}}_{\omega,k}$ be the $\rho d \times \rho d$ and $\rho \times \rho d$ matrices defined by

$$\begin{aligned} S_{\omega,k}(j) &:= \operatorname{diag}\left[\sqrt{d^k \mathrm{M}_{\omega,k+1}(j + D^k r)}\right]_{r=r_1}^{r_d}, \\ \widehat{\mathbb{H}}_{\omega,k}(j) &:= \left[\widehat{H}_{\omega,k+1}(j + D^k r_1)\middle| \ \ldots \ \middle|\widehat{H}_{\omega,k+1}(j + D^k r_d)\right] \end{aligned} \tag{3.68}$$

respectively, where $r_1, \cdots, r_d$ denote all the elements of $\mathcal{R}_1$. Consider $\mathbb{T}^s \backslash \Delta$ as a finite disjoint union of measurable subsets (up to measure zero sets) with a typical subset of the form $\Delta_{S,p} = \{\omega \in \mathbb{T}^s \backslash \Delta : \operatorname{rank}(S_{\omega,k}(j)) = p_k(j)\}$.

(2) For $\omega \in \Delta_{S,p}$, let $A^*_\omega := S_{\omega,k}\widehat{\mathbb{H}}^*_{\omega,k}$ be the $\rho d \times \rho$ matrix whose nonzero columns are all orthonormal such that

$$A_\omega(j) A^*_\omega(j) = d^k \mathrm{M}_{\omega,k}(j). \tag{3.69}$$

Consider $\Delta_{S,p}$ as a finite disjoint union of measurable subsets (up to measure zero sets) such that a typical subset is of the form $\Delta_{A,q} = \{\omega \in \Delta_{S,p} : \operatorname{rank}(A^*_\omega) = q_k(j)$ and A^*_ω has nonzero entries in specific rows and columns$\}$. Perform column interchange operations (justified by the specific positions of nonzero entries) using a $\rho \times \rho$ permutation matrix $F(\Delta_{A,q})$ such that

$$A^*_\omega F = \begin{bmatrix} A^*_{q,\omega} & 0 \end{bmatrix},$$

where $A_{q,\omega}$ is a $q \times \rho d$ matrix such that $A_{q,\omega} A^*_{q,\omega} = I_q$, the $q \times q$ identity matrix.

(3) Perform row interchange operations using a $\rho d \times \rho d$ permutation matrix $E(\Delta_{A,q})$ such that $\sqrt{d}(ES_{\omega,k})$ is in reduced row-echelon form and let $A'^*_\omega = EA^*_\omega F$. Hence

$$A'^*_\omega = E\begin{bmatrix} A^*_{q,\omega} & 0 \end{bmatrix} = \begin{bmatrix} EA^*_{q,\omega} & 0 \end{bmatrix} = \begin{bmatrix} A^*_{qp,\omega} & 0 \\ 0 & 0 \end{bmatrix},$$

where $A_{qp,\omega}$ is a $q \times p$ matrix of measurable functions on $\Delta_{A,q}$ such that $A_{qp,\omega} A^*_{qp,\omega} = I_q$.

(4) Construct a $p \times p$ unitary matrix Q_ω of measurable functions on $\Delta_{A,q}$ using the procedure given in the proof of Proposition 3.45 such that

$$Q_\omega A^*_{qp,\omega} = \begin{bmatrix} I_q \\ 0 \end{bmatrix}.$$

Since Q_ω is an extension of $A_{qp,\omega}$, we can replace its first q rows by $A_{qp,\omega}$ and write

$$\begin{aligned} Q_\omega &= \begin{bmatrix} A_{qp,\omega} \\ B_\omega \end{bmatrix}, \\ Q_\omega Q^*_\omega &= \begin{bmatrix} A_{qp,\omega} A^*_{qp,\omega} & A_{qp,\omega} B^*_\omega \\ B_\omega A^*_{qp,\omega} & B_\omega B^*_\omega \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & I_{p-q} \end{bmatrix}, \end{aligned}$$

where B_ω is a $(p-q) \times p$ matrix with orthonormal rows. In the event that $q = 0$, we let B_ω be the $p \times p$ identity matrix and in the event that $p = q$, we skip the steps involving the construction of B_ω and its utility.

(5) Define a $\rho d \times \rho d$ matrix B'_ω for $\omega \in \Delta_{A,q}$ by

$$B'_\omega := \begin{bmatrix} B_\omega & 0 \\ 0 & 0 \end{bmatrix} E.$$

(6) Construct the matrix $\widehat{\mathbb{G}}_{\omega,k}$ for $\omega \in \Delta_{A,q}$ in this manner. If $(i_1, i_1), \ldots, (i_p, i_p)$ entries of $S_{\omega,k}$ are the nonzero diagonal entries, where $1 \le i_1 < \cdots < i_p \le \rho d$, then for each $\alpha \in \{1, \ldots, p\}$, set the first $p - q$ entries of the i_α^{th} column of $\widehat{\mathbb{G}}_{\omega,k}$ to be the α^{th} column of the matrix $\sqrt{d} B_\omega$ and the remaining $\rho d - (p - q)$ entries to be zero. The entries of each of the remaining $\rho d - p$ columns of $\widehat{\mathbb{G}}_{\omega,k}$ are chosen arbitrarily so that $\widehat{G}_{k+1}$ lies in $L^2(\Delta_{A,q})$. Since $\sqrt{d}(S_{\omega,k} E^T)$ is in reduced column-echelon form, $\widehat{\mathbb{G}}_{\omega,k}$ satisfies

$$\widehat{\mathbb{G}}_{\omega,k} S_{\omega,k} E^T = \begin{bmatrix} B_\omega & 0 \\ 0 & 0 \end{bmatrix},$$

i.e. $B'_\omega = \widehat{\mathbb{G}}_{\omega,k} S_{\omega,k}$.

(7) Define the $\rho d \times \rho$ matrices $\widehat{G}_{\omega,k+1}(j + D^k r)$ for $r \in \mathcal{R}_1$ and for $\omega \in \Delta_{A,q}$ by

$$\widehat{\mathbb{G}}_{\omega,k}(j) = \begin{bmatrix} \widehat{G}_{\omega,k+1}(j + D^k r_1) & \cdots & \widehat{G}_{\omega,k+1}(j + D^k r_d) \end{bmatrix}.$$

Since we have

$$\widehat{\mathbb{G}}_{\omega,k} S_{\omega,k} S^*_{\omega,k} \widehat{\mathbb{H}}^*_{\omega,k} = B'_\omega A^*_\omega = (B'_\omega E^T)(E A^*_\omega F) F^T = \begin{bmatrix} B_\omega A^*_{qp,\omega} & 0 \\ 0 & 0 \end{bmatrix} F^T = 0,$$

$$\widehat{\mathbb{G}}_{\omega,k} S_{\omega,k} S^*_{\omega,k} \widehat{\mathbb{G}}^*_{\omega,k} = B'_\omega B'^*_\omega = (B'_\omega E^T)(B'_\omega E^T)^* = \begin{bmatrix} B_\omega B^*_\omega & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{p-q} & 0 \\ 0 & 0 \end{bmatrix}, \quad (3.70)$$

hence (3.37) is satisfied and from (3.40), we infer that $\mathrm{N}_{\omega,k}(j)$ is a $\rho d\times\rho d$ diagonal matrix with diagonal entries 0 or d^{-k}. Since

$$\begin{aligned}\dim U^{k+1,j}_{2\pi,\omega} &= \sum_{r\in\mathcal{R}_1}\operatorname{rank}(\mathrm{M}_{\omega,k+1}(j+D^k r)) = \operatorname{rank}(S_{\omega,k}(j)) = p_k(j), && (3.71)\\ \dim V^{k,j}_{2\pi,\omega} &= \operatorname{rank}(\mathrm{M}_{\omega,k}(j)) = q_k(j), && (3.72)\\ \dim W^{k,j}_{2\pi,\omega} &= \operatorname{rank}(\mathrm{N}_{\omega,k}(j)) = p_k(j)-q_k(j), && (3.73)\end{aligned}$$

follow from (3.68), (3.69) and (3.70), we conclude that (3.38) is satisfied.

(8) Repeat all the above steps for all measurable subsets $\Delta_{S,p}$ and $\Delta_{A,q}$.

(9) Finally, extend periodically the values of the matrices $\widehat{G}_{\omega,k+1}(j+D^k r)$ for $r\in\mathcal{R}_1$ to obtain $\widehat{G}_{\omega,k+1}\in\mathcal{S}(D^{k+1})^{\rho d\times\rho}$ on $\mathbb{T}^s\backslash\Delta$.

We define the *index* $\{\eta_k\}_{k\geq 0}$ of the FMRA $\{V^k(\Phi_k)\}$ as

$$\eta_k := \operatorname{ess\,sup}\{\eta_{\omega,k}:\omega\in\mathbb{T}^s\}, \tag{3.74}$$

where $\eta_{\omega,k} := \max\{\dim W^{k,j}_{2\pi,\omega} : j\in\mathcal{R}_k\}$. Therefore, the index $\{\eta_k\}_{k\geq0}$ of an FMRA $\{V^k(\Phi_k)\}$ consists of positive integers satisfying $\eta_k\leq|\Phi_k|\,d$.

As a consequence of our construction given above, we are able to establish the existence of a tight wavelet frame in $L^2(\mathbb{R}^s)$ derived from an FMRA.

Corollary 3.47. *For $k\geq 0$, let $\Phi_k\subset L^{2,\alpha}(\mathbb{R}^s)$ with $|\Phi_k|=\rho$. Suppose that $\{V^k(\Phi_k)\}$ is an FMRA of $L^2(\mathbb{R}^s)$ with index $\{\eta_k\}_{k\geq0}$. Let W^k be the orthogonal complement of $V^k(\Phi_k)$ in $V^{k+1}(\Phi_{k+1})$. There exists $\Psi_k=\{\psi_k^m\}_{m=1}^{\eta_k}\subset W^k$ such that $E_k(\Psi_k)$ is a tight frame for W^k with $\langle E_k^l\psi_k^m, E_k^r\psi_k^n\rangle=0$ for all $m,n=1,\ldots,\eta_k$, $m\neq n$ and $l,r\in\mathbb{Z}^s$.*

Proof. It is shown in (3.40), (3.70) and (3.73) that for $m=\eta_k+1,\ldots,\rho d$ and almost every $\omega\in\mathbb{T}^s$, we have $\left\|\psi^m_{\omega,k}\right\|^2_{L^2(\mathbb{T}^s)}=0$ as $\left\|(\psi^m_{\omega,k})_{k,j}\right\|^2_{L^2(\mathbb{T}^s)}=0$ for all $j\in\mathcal{R}_k$. Since $\widehat{(\psi_k^m)}_{0,\omega,0}=\widehat{\psi^m_{\omega,k}}$, by Lemma 3.5, $\|\psi_k^m\|^2=\int_{\mathbb{T}^s}\left\|\widehat{(\psi_k^m)}_{0,\omega,0}\right\|^2_{l^2(\mathbb{Z}^s)}\mathrm{d}\omega=0$ and we have $\psi_k^m=0$. Furthermore, for all $m,n=1,\ldots,\eta_k$, $m\neq n$ and $l,r\in\mathbb{Z}^s$, by Lemma 3.5 and Theorem 3.21 and Lemma 3.31, we have

$$\begin{aligned}\langle E_k^l\psi_k^m, E_k^r\psi_k^n\rangle &= \int_{\mathbb{T}^s}\mathrm{e}^{-\mathrm{i}\omega\cdot(l-r)}\langle\mathcal{M}^{l-r}_{0,k}\widehat{(\psi_k^m)}_{0,\omega,0},\widehat{(\psi_k^n)}_{0,\omega,0}\rangle_{l^2(\mathbb{Z}^s)}\mathrm{d}\omega\\ &= \int_{\mathbb{T}^s}\mathrm{e}^{\mathrm{i}\omega\cdot(r-l)}\langle T_k^{l-r}\psi^m_{\omega,0},\psi^n_{\omega,0}\rangle_{L^2(\mathbb{T}^s)}\mathrm{d}\omega = 0.\end{aligned}$$

Finally, using Theorem 3.13, since (3.40), (3.70) and (3.73) imply that $T_k(\Psi_{\omega,k})$ is a tight frame of $W^k_{2\pi,\omega}$ for almost every $\omega\in\mathbb{T}^s$, Corollary 3.37 implies that $E_k(\Psi_k)$ is a tight frame for W^k. □

Theorem 3.48. *For $k \geq 0$, let $\Phi_k \subset L^{2,\alpha}(\mathbb{R}^s)$ with $|\Phi_k| = \rho$. Suppose that $\{V^k(\Phi_k)\}$ is an FMRA of $L^2(\mathbb{R}^s)$ with index $\{\eta_k\}_{k\geq 0}$. Let W^k be the orthogonal complement of $V^k(\Phi_k)$ in $V^{k+1}(\Phi_{k+1})$. Then the following are equivalent for each $k \geq 0$.*

(i) *The set $\Sigma_{\varrho_k} := \bigcup_{j\in\mathcal{R}_k}\{\omega \in \mathbb{T}^s : \dim U^{k+1,j}_{2\pi,\omega} - \dim V^{k,j}_{2\pi,\omega} > \varrho_k\}$ is of measure zero.*

(ii) *There holds $\eta_k \leq \varrho_k$.*

(iii) *There exists $\Psi_k = \{\psi_k^m\}_{m=1}^{\varrho_k} \subset W^k$ with $\langle E_k^l\psi_k^m, E_k^r\psi_k^n\rangle = 0$ for all $m, n = 1, \ldots, \varrho_k$, $m \neq n$ and $l, r \in \mathbb{Z}^s$ such that $E_k(\Psi_k)$ is a tight frame for W^k.*

(iv) *There exists $\Psi_k = \{\psi_k^m\}_{m=1}^{\varrho_k} \subset W^k$ such that $E_k(\Psi_k)$ is a frame for W^k.*

Proof. Assume that (i) holds. Then for almost every $\omega \in \mathbb{T}^s$ and any $j \in \mathcal{R}_k$, using (3.71) and (3.72), since

$$\dim U^{k+1,j}_{2\pi,\omega} - \dim V^{k,j}_{2\pi,\omega} = \sum_{r\in\mathcal{R}_1} \operatorname{rank}(\mathrm{M}_{\omega,k+1}(j + D^k r)) - \operatorname{rank}(\mathrm{M}_{\omega,k}(j)) \leq \varrho_k,$$

it follows from (3.73) and (3.74) that (ii) holds. Next, we apply the algorithm after Theorem 3.46 and also Corollary 3.47 to construct the required ψ_k^m for $m = 1, \ldots, \eta_k$. For $m = \eta_k + 1, \ldots, \varrho_k$, we set $\psi_k^m := 0$. Hence, (iii) holds and implies (iv). To establish that (iv) implies (i), we again utilize (3.71), (3.72) and (3.73) to show that in the event that there exist $j \in \mathcal{R}_k$ and a set $\sigma_{\varrho_k} \subset \Sigma_{\varrho_k}$ such that $|\sigma_{\varrho_k}| > 0$ and

$$\dim W^{k,j}_{2\pi,\omega} = \dim U^{k+1,j}_{2\pi,\omega} - \dim V^{k,j}_{2\pi,\omega} > \varrho_k$$

for almost every $\omega \in \sigma_{\varrho_k}$, this will lead to a contradiction with (iv) since Corollary 3.37 and Theorem 3.40 show otherwise. □

For the case of a stationary FMRA $\{V^k(\Phi)\}$, Theorem 3.48 leads to the following corollary.

Corollary 3.49. *Let $\Phi \subset L^{2,\alpha}(\mathbb{R}^s)$ be finite. Suppose that $\{V^k(\Phi)\}$ is an FMRA of $L^2(\mathbb{R}^s)$ with index η_0. Let W^k be the orthogonal complement of $V^k(\Phi)$ in $V^{k+1}(\Phi)$. Then the following are equivalent.*

(i) *The set $\Sigma_{\varrho_0} := \{\omega \in \mathbb{T}^s : \dim U^{1,0}_{2\pi,\omega} - \dim V^{0,0}_{2\pi,\omega} > \varrho_0\}$ is of measure zero.*

(ii) *There holds $\eta_0 \leq \varrho_0$.*

(iii) *There exists $\Psi = \{\psi^m\}_{m=1}^{\varrho_0} \subset W^0$ with $\langle E^l\psi^m, E^r\psi^n\rangle = 0$ for all $m, n = 1, \ldots, \varrho_0$, $m \neq n$ and $l, r \in \mathbb{Z}^s$ such that $E(\Psi)$ is a tight frame for W^0.*

(iv) *There exists* $\Psi = \{\psi^m\}_{m=1}^{\varrho_0} \subset W^0$ *such that* $E(\Psi)$ *is a frame for* W^0.

Corollary 3.49 generalizes a similar result in [39]. In [39], the set $\Phi := \{\phi\}$ is a singleton and the dilation matrix $M = 2I$. For $k \geq 0$, we define

$$\Gamma := \{\omega \in \mathbb{T}^s : \operatorname{rank}(\mathrm{M}_{\omega,0}(0)) = 0\} = \{\omega \in \mathbb{T}^s : \mathrm{M}_{\omega,0}(0) = 0\}, \tag{3.75}$$

$$\Delta_p := \{\omega \in \mathbb{T}^s : \sum_{r \in \mathcal{R}_1} \operatorname{rank}(\mathrm{M}_{\omega,1}(r)) = p\}, \tag{3.76}$$

where $\mathrm{M}_{\omega,1}(r)$ is a scalar. If an integer p satisfies both $|\Delta_p| > 0$ and $|\Delta_r| = 0$ for $r > p$, then the *index* η_0 of the stationary FMRA $\{V^k(\Phi)\}$ is given as follows:

$$\eta_0 := \begin{cases} p & \text{if } |\Delta_p \cap \Gamma| > 0, \\ p-1 & \text{if } |\Delta_p \cap \Gamma| = 0, \end{cases}$$

i.e. either for every $r > 0$, $|\Delta_{\eta_0+r}| = 0$ with $|\Delta_{\eta_0}| > 0$ or for every $r > 1$, $|\Delta_{\eta_0+r}| = 0$ with $|\Delta_{\eta_0+1}| > 0$ and $|\Delta_{\eta_0+1} \cap \Gamma| = 0$.

Corollary 3.50. [39] *Suppose that* $\{V^k(\phi)\}$ *is an FMRA of* $L^2(\mathbb{R}^s)$ *with index* η_0 *and dilation matrix* $M = 2I$. *Let* W^k *be the orthogonal complement of* $V^k(\phi)$ *in* $V^{k+1}(\phi)$. *If there exists* $\Psi = \{\psi^m\}_{m=1}^{\varrho_0} \subset W^0$ *such that* $E(\Psi)$ *is a frame for* W^0, *then* $\varrho_0 \geq \eta_0$. *If* $\varrho_0 \geq \eta_0$, *then there exists* $\Psi = \{\psi^m\}_{m=1}^{\varrho_0} \subset W^0$ *such that* $E(\Psi)$ *is a tight frame for* W^0.

We conclude this chapter with the connection of the affine system in $L^2(\mathbb{R}^s)$ and the periodic affine system in $L^2(\mathbb{T}^s)$ using extension principles. Let us state a lemma concerning the minimum energy tight frame condition for wavelets derived from an MRA for $L^2(\mathbb{R}^s)$.

Lemma 3.51. [6] *Let* $\Phi, \Psi \subset L^{2,\alpha}(\mathbb{R}^s)$ *be finite with* $|\Phi| = \rho$, $|\Psi| = \varrho$ *satisfying* (1.8) *and* (1.10) *for some* $\widehat{H}_{k+1} := d^{\frac{1}{2}} \widehat{H}(D^{-(k+1)}\cdot)$ *and* $\widehat{G}_{k+1} := d^{\frac{1}{2}} \widehat{G}(D^{-(k+1)}\cdot)$ *respectively, where* $\widehat{H}, \widehat{G} \in L^2(\mathbb{T}^s)$ *are* $2\pi\mathbb{Z}^s$*-periodic matrix-valued measurable functions. For* $k \in \mathbb{Z}$, *define* $\widehat{H}_{\omega,k+1}(j) := \widehat{H}_{k+1}(\omega + 2\pi j)$ *and* $\widehat{G}_{\omega,k+1}(j) := \widehat{G}_{k+1}(\omega + 2\pi j)$. *Assume that* $\lim_{\omega \to 0} \sum_{\phi \in \Phi} \left|\widehat{\phi}(\omega)\right|^2 = A > 0$ *and the* $(\rho + \varrho) \times \rho d$ *matrix* $\widehat{\mathbb{L}}_{\omega,k}(j)$ *as defined in a manner similar to* (3.44) *satisfies* $\widehat{\mathbb{L}}_{\omega,k}(j)^* \widehat{\mathbb{L}}_{\omega,k}(j) = dI_{\rho d}$ *for each* $j \in \mathcal{R}_k$ *and almost every* $\omega \in \mathbb{R}^s$. *Then for all* $k \in \mathbb{Z}$ *and all* $f \in L^2(\mathbb{R}^s)$ *for which* $\widehat{f}$ *is a compactly supported continuous function, we have the minimum energy tight frame condition, i.e.*

$$\sum_{l \in \mathbb{Z}^s} \sum_{\phi \in \Phi} \left| \langle f, d^{\frac{k+1}{2}} E^l_{k+1} \phi(M^{k+1}\cdot) \rangle \right|^2 = \sum_{l \in \mathbb{Z}^s} \left[\sum_{\phi \in \Phi} \left| \langle f, d^{\frac{k}{2}} E^l_k \phi(M^k\cdot) \rangle \right|^2 + \sum_{\psi \in \Psi} \left| \langle f, d^{\frac{k}{2}} E^l_k \psi(M^k\cdot) \rangle \right|^2 \right]$$

and for all $f \in L^2(\mathbb{R}^s)$, *we have* $\lim_{k \to -\infty} \sum_{l \in \mathbb{Z}^s} \sum_{\phi \in \Phi} \left| \langle f, d^{\frac{k}{2}} E^l_k \phi(M^k\cdot) \rangle \right|^2 = 0$.

Theorem 3.52. *Let $\Phi \subset L^{2,\alpha}(\mathbb{R}^s)$ with $|\Phi| = \rho$. The affine system $X(\Psi)$ as defined in* (1.4) *is a tight frame for $L^2(\mathbb{R}^s)$ obtained from the MRA $\{V^k(\Phi)\}$ by the UEP if and only if the corresponding periodized affine system X_ω as defined in* (3.60) *is a tight frame for $L^2(\mathbb{T}^s)$ obtained from the MRA $V^k_{2\pi}(\Phi_{\omega,k})$ by the periodic UEP for almost every $\omega \in \mathbb{T}^s$.*

Proof. We shall consider only an arbitrary $\omega \in \mathbb{T}^s \backslash \Delta$ where $|\Delta| = 0$ such that for every $k \geq 0$, $V^k_{2\pi}(\Phi_{\omega,k})$ is a subspace of $L^2(\mathbb{T}^s)$ and $\widehat{H}_{\omega,k+1} := \widehat{H}_{k+1}(\omega+2\pi\cdot) := d^{\frac{1}{2}}\widehat{H}(D^{-(k+1)}(\omega+2\pi\cdot))$ and $\widehat{G}_{\omega,k+1} := \widehat{G}_{k+1}(\omega+2\pi\cdot) := d^{\frac{1}{2}}\widehat{G}(D^{-(k+1)}(\omega+2\pi\cdot))$ lie in $l^2(\mathbb{Z}^s)$ for some $2\pi D^{k+1}\mathbb{Z}^s$-periodic matrix-valued measurable functions $\widehat{H}_{k+1}, \widehat{G}_{k+1} \in L^2(\mathbb{T}^s)$ satisfying (1.8) and (1.10) respectively. Therefore, we obtain

$$\widehat{\Phi}_{\omega,k} = \widehat{(\Phi_k)}_{0,\omega,0} = \widehat{\Phi}_k(\omega+2\pi\cdot) = \widehat{H}_{k+1}(\omega+2\pi\cdot)\widehat{\Phi}_{k+1}(\omega+2\pi\cdot) = \widehat{H}_{\omega,k+1}\widehat{\Phi}_{\omega,k+1},$$
$$\widehat{\Psi}_{\omega,k} = \widehat{(\Psi_k)}_{0,\omega,0} = \widehat{\Psi}_k(\omega+2\pi\cdot) = \widehat{G}_{k+1}(\omega+2\pi\cdot)\widehat{\Phi}_{k+1}(\omega+2\pi\cdot) = \widehat{G}_{\omega,k+1}\widehat{\Phi}_{\omega,k+1},$$

where $\widehat{\Phi}_k := d^{-\frac{k}{2}}\widehat{\Phi}(D^{-k}\cdot)$ and $\widehat{\Psi}_k := d^{-\frac{k}{2}}\widehat{\Psi}(D^{-k}\cdot)$, i.e. (3.28) and (3.31) are satisfied. Here, we have utilized Theorem 3.43, Corollary 3.34, Propositions 3.35, 3.19 and 3.20 to confirm the MRA structure and derive the corresponding affine system from the MRA.

In addition, the conditions of the UEP, i.e. $\widehat{\mathbb{L}}_{\omega,k}(j)$ as defined in a manner similar to (3.44) satisfies $\widehat{\mathbb{L}}_{\omega,k}(j)^*\widehat{\mathbb{L}}_{\omega,k}(j) = dI_{\rho d}$ for each $j \in \mathcal{R}_k$ and

$$\lim_{k\to\infty}\sum_{\phi\in\Phi}\left|\widehat{\phi}(D^{-k}\omega)\right|^2 = \lim_{k\to\infty}\sum_{\phi\in\Phi}\left|\widehat{\phi}(D^{-k}(\omega+2\pi n))\right|^2$$
$$= \lim_{k\to\infty} d^k \sum_{\phi\in\Phi}\left|\widehat{\phi_{\omega,k}}(n)\right|^2 = A > 0$$

for all $n \in \mathbb{Z}^s$ are satisfied for both directions. By Theorem 3.36, the shift-invariant system $E(\Phi) \cup X_0(\Psi)$ as defined in (1.5) is a frame for $L^2(\mathbb{R}^s)$ if and only if X_ω is a frame for $L^2(\mathbb{T}^s)$ with the same frame bounds for almost every $\omega \in \mathbb{T}^s$. It remains to see using Lemma 3.51 that the affine system $X(\Psi)$ is also a frame for $L^2(\mathbb{R}^s)$ could be inferred by "telescoping" from the shift-invariant system $E(\Phi) \cup X_0(\Psi)$. With this in mind, we confirm that

$$\sum_{l\in\mathbb{Z}^s}\sum_{\phi\in\Phi}\left|\langle f, d^{\frac{K}{2}}E^l_K\phi(M^K\cdot)\rangle\right|^2 = \sum_{l\in\mathbb{Z}^s}\left[\sum_{\phi\in\Phi}\left|\langle f, d^{\frac{k}{2}}E^l_k\phi(M^k\cdot)\rangle\right|^2 + \sum_{p=k}^{K-1}\sum_{\psi\in\Psi}\left|\langle f, d^{\frac{p}{2}}E^l_p\psi(M^p\cdot)\rangle\right|^2\right]$$

for all $f \in L^2(\mathbb{R}^s)$ such that $\widehat{f}$ is a compactly supported continuous function. By letting $k \to -\infty$, we obtain

$$\sum_{l\in\mathbb{Z}^s}\sum_{\phi\in\Phi}\left|\langle f, d^{\frac{K}{2}}E^l_K\phi(M^K\cdot)\rangle\right|^2 = \sum_{l\in\mathbb{Z}^s}\sum_{k=-\infty}^{K-1}\sum_{\psi\in\Psi}\left|\langle f, d^{\frac{k}{2}}E^l_k\psi(M^k\cdot)\rangle\right|^2,$$

which implies that

$$\sum_{l\in\mathbb{Z}^s}\left[\sum_{\phi\in\Phi}\left|\langle f,E^l\phi\rangle\right|^2+\sum_{k=0}^{\infty}\sum_{\psi\in\Psi}\left|\langle f,d^{\frac{k}{2}}E_k^l\psi(M^k\cdot)\rangle\right|^2\right]=\|f\|^2\,.$$

This is equivalent to

$$\sum_{l\in\mathbb{Z}^s}\sum_{k=-\infty}^{\infty}\sum_{\psi\in\Psi}\left|\langle f,d^{\frac{k}{2}}E_k^l\psi(M^k\cdot)\rangle\right|^2=\|f\|^2\,,$$

and it holds on a dense subset of $L^2(\mathbb{R}^s)$. Thus the relation holds for all $f\in L^2(\mathbb{R}^s)$. $\square$

We remark that Theorem 3.52 could be generalized to nonstationary MRAs that preserve a dilation structure, i.e. for instance $\widehat{\phi}_k(2^k\cdot)=\widehat{h}_{k+1}\widehat{\phi}_{k+1}$ for $k\in\mathbb{N}$, for the case of the dilation matrix $D=2I$. This may be achieved by using a recent result from [31] that generalizes the UEP for $L^2(\mathbb{R})$ to nonstationary settings. This will be helpful in the event that we utilize the periodic constructions in Chapter 4 to obtain constructions on the real line.

Chapter 4

Constructions in $L^2(\mathbb{T})$

The Gabor system is based on the short-time Fourier transform of shifts and modulates of a window function to represent signals with regular time-frequency atoms. To achieve a similar time-frequency representation with the wavelet system, one usually applies the wavelet decomposition process repeatedly on wavelet subbands and obtains "packets" of wavelet atoms. We could achieve a similar and possibly more flexible representation if we introduce modulation to the wavelet system by means of using additional number of wavelet functions, i.e. we combine translation and modulation into an MRA structure. Thus, this incorporates the strengths of both the wavelet transform and the short-time Fourier transform. All the constructions in this chapter are for the one-dimensional periodic case with dilation factor 2.

4.1 Bandlimited Construction

We shall construct a bandlimited multiresolution $\{V_{2\pi}^k(\phi_k)\}$ of $L^2(\mathbb{T})$ where $\{T_k^l \phi_k : l \in \mathcal{L}_k\}$ forms a tight frame for a subspace of $V_{2\pi}^k(\phi_k)$ and $\widehat{\phi}_k(0) = 2^{-\frac{k}{2}}$. For $k \geq 0$, let

$$\tilde{\beta}_k^n(j) := \beta\left(\frac{N_{k,n} j}{L_{k,n} - N_{k,n}}\right), \quad j \in \mathbb{Q},$$

where β is the cumulative distribution function of a Beta distribution and $0 \leq N_{k,n} \leq L_{k,n} \leq N_{k,n+1}$ for $n \in \{1, \ldots, \varrho_k + 1\}$ are used to indicate the bandwidths of our refinement and wavelet masks with ϱ_k being the number of wavelet masks. We will impose additional conditions on the bandwidths of the masks in our constructions later. If $N_{k,n} = L_{k,n}$, we shall let $\tilde{\beta}_k^n = 0$ instead. Since

$$\beta(\omega) + \beta(1 - \omega) = 1,$$

we have $\widetilde{\beta}_k^n\left(\frac{N_{k,n}}{N_{k,n}}-1\right) = \beta(0) = 0$ and $\widetilde{\beta}_k^n\left(\frac{L_{k,n}}{N_{k,n}}-1\right) = \beta(1) = 1$ for $N_{k,n} < L_{k,n}$. We describe our construction of the refinable function and its mask below, which we shall use exclusively for our bandlimited constructions of wavelets.

Construction 4.1. *For $k \geq 0$, let $\phi_k = \sum_{n=-L_{k,1}}^{L_{k,1}} \widehat{\phi}_k(n)\mathrm{e}^{-\mathrm{i}n\cdot}$, where*

$$\widehat{\phi}_k(j) = \begin{cases} 2^{-\frac{k}{2}} & \text{if } j \in \{-N_{k,1}, \dots, N_{k,1}\}, \\ 2^{-\frac{k}{2}} \cos\left[\frac{\pi}{2}\widetilde{\beta}_k^1\left(\frac{|j|}{N_{k,1}}-1\right)\right] & \text{if } \begin{array}{l} j \in \{-L_{k,1}, \dots, -N_{k,1}-1\} \\ \cup \{N_{k,1}+1, \dots, L_{k,1}\}, \end{array} \\ 0 & \text{otherwise,} \end{cases}$$

and $L_{k,1} < N_{k+1,1}$ and $L_{k,1} \leq 2^k$. For $k \geq 0$, let

$$\widehat{h}_{k+1}(j) = \begin{cases} \sqrt{2} & \text{if } j \in \{-N_{k,1}, \dots, N_{k,1}\}, \\ \sqrt{2}\cos\left[\frac{\pi}{2}\widetilde{\beta}_k^1\left(\frac{|j|}{N_{k,1}}-1\right)\right] & \text{if } \begin{array}{l} j \in \{-L_{k,1}, \dots, -N_{k,1}-1\} \\ \cup \{N_{k,1}+1, \dots, L_{k,1}\}, \end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

In the event that $N_{k,1} = L_{k,1}$, we would redefine

$$\widehat{h}_{k+1}(j) = \begin{cases} \sqrt{2} & \text{if } j \in \{-N_{k,1}+1, \dots, N_{k,1}-1\}, \\ 1 & \text{if } j \in \{-N_{k,1}\} \cup \{N_{k,1}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

For purposes of convenience, we shall also refer to $\widehat{h}_{k+1}$ as $\widehat{g}_{k+1}^0$. Note that $\widehat{\phi}_k(n) = \widehat{h}_{k+1}(n)\widehat{\phi}_{k+1}(n)$ for all $n \in \mathbb{Z}$.

Remark. In this chapter, when we set the values of a mask in $\mathcal{S}(2^{k+1})$ such as in (4.1), under the case indicated as "otherwise", we refer to the remaining values of $\mathcal{R}_{k+1}$ from those already defined.

The mask $\widehat{h}_{k+1}$ in Construction 4.1 filters away completely high frequency data belonging to the bands $\{-2^k, \dots, -L_{k,1}\} \cup \{L_{k,1}, \dots, 2^k\}$, dampens data belonging to the transition bands $\{-L_{k,1}, \dots, -N_{k,1}-1\} \cup \{N_{k,1}+1, \dots, L_{k,1}\}$ and allows low frequency data belonging to $\{-N_{k,1}, \dots, N_{k,1}\}$ to pass through unchanged, i.e. behaves like the frequency response of the ideal filter. We verify that for $j \in \{-L_{k,1}, \dots, L_{k,1}\}$ or equivalently $j - 2^k \in \{-2^k, \dots, L_{k,1} - 2^k\}$ and $j + 2^k \in \{2^k - L_{k,1}, \dots, 2^k\}$, we have $\widehat{h}_{k+1}(j \pm 2^k) = 0$, since $L_{k,1} \leq 2^{k+1} - L_{k,1}$.

Lemma 4.2. *Assume that $N_{k,\lambda} < L_{k,\lambda}$ and $N_{k,\mu} < L_{k,\mu}$. Suppose that $N_{k,\lambda} + L_{k,\mu} = 2^{k+1}$ and $N_{k,\mu} + L_{k,\lambda} = 2^{k+1}$. Then*

$$\cos\left[\frac{\pi}{2}\tilde{\beta}_k^{\lambda}\left(\frac{2^{k+1}-|j|}{N_{k,\lambda}}-1\right)\right] = \sin\left[\frac{\pi}{2}\tilde{\beta}_k^{\mu}\left(\frac{|j|}{N_{k,\mu}}-1\right)\right].$$

Proof. The verification is as follows. Indeed,

$$\begin{aligned}
\cos\left[\frac{\pi}{2}\tilde{\beta}_k^{\lambda}\left(\frac{2^{k+1}-|j|}{N_{k,\lambda}}-1\right)\right] &= \cos\left[\frac{\pi}{2}\beta\left(\frac{N_{k,\lambda}}{L_{k,\lambda}-N_{k,\lambda}}\left(\frac{2^{k+1}-|j|}{N_{k,\lambda}}-1\right)\right)\right] \\
= \cos\left[\frac{\pi}{2}\beta\left(\frac{L_{k,\mu}-|j|}{L_{k,\mu}-N_{k,\mu}}\right)\right] &= \sin\left[\frac{\pi}{2}\left[1-\beta\left(\frac{L_{k,\mu}-|j|}{L_{k,\mu}-N_{k,\mu}}\right)\right]\right] \\
= \sin\left[\frac{\pi}{2}\beta\left(1-\frac{L_{k,\mu}-|j|}{L_{k,\mu}-N_{k,\mu}}\right)\right] &= \sin\left[\frac{\pi}{2}\beta\left(\frac{|j|-N_{k,\mu}}{L_{k,\mu}-N_{k,\mu}}\right)\right] \\
= \sin\left[\frac{\pi}{2}\beta\left(\frac{N_{k,\mu}}{L_{k,\mu}-N_{k,\mu}}\left(\frac{|j|}{N_{k,\mu}}-1\right)\right)\right] &= \sin\left[\frac{\pi}{2}\tilde{\beta}_k^{\mu}\left(\frac{|j|}{N_{k,\mu}}-1\right)\right].
\end{aligned}$$

□

Lemma 4.3. *If $N_{k,\lambda} < L_{k,\lambda}$ and $N_{k,\lambda} + L_{k,\lambda} = 2^k$, then we have*

$$\cos\frac{\pi}{2}\tilde{\beta}_k^{\lambda}\left(\frac{2^k-|j|}{N_{k,\lambda}}-1\right) = \sin\frac{\pi}{2}\tilde{\beta}_k^{\lambda}\left(\frac{|j|}{N_{k,\lambda}}-1\right).$$

Proof. Since

$$\begin{aligned}
1-\tilde{\beta}_k^{\lambda}\left(\frac{2^k-|j|}{N_{k,\lambda}}-1\right) &= 1-\beta\left(\frac{N_{k,\lambda}}{L_{k,\lambda}-N_{k,\lambda}}\left(\frac{2^k-|j|}{N_{k,\lambda}}-1\right)\right) \\
= \beta\left(\frac{L_{k,\lambda}-N_{k,\lambda}-2^k+|j|+N_{k,\lambda}}{L_{k,\lambda}-N_{k,\lambda}}\right) &= \beta\left(\frac{|j|-N_{k,\lambda}}{L_{k,\lambda}-N_{k,\lambda}}\right) = \tilde{\beta}_k^{\lambda}\left(\frac{|j|}{N_{k,\lambda}}-1\right),
\end{aligned}$$

this implies our result. □

We shall make use of Theorem 3.25 to construct framelets by ensuring the masks $\widehat{h}_{k+1}$, $\widehat{g}_{k+1}^n$ satisfy (3.47) or equivalently

$$\left|\widehat{h}_{k+1}(j)\right|^2 + \sum_{n=1}^{\varrho_k}\left|\widehat{g}_{k+1}^n(j)\right|^2 = 2, \tag{4.3}$$

$$\overline{\widehat{h}_{k+1}(j)}\widehat{h}_{k+1}(j+2^k) + \sum_{n=1}^{\varrho_k}\overline{\widehat{g}_{k+1}^n(j)}\widehat{g}_{k+1}^n(j+2^k) = 0. \tag{4.4}$$

for all $j \in \sigma_{k+1}(V_{2\pi}^{k+1}(\phi_{k+1}))$. Equation (4.3) says that the masks must cover the frequency domain "uniformly" while (4.4) requires the masks to be orthogonal to their modulates with a shift of 2^k in the frequency domain. In signal processing literature (see [38]), the former is known as the perfect reconstruction condition while the latter is known as the anti-aliasing condition, which is necessary to remove aliasing caused by downsampling.

We shall now describe the construction of complex framelets with smooth decay and controlled overlap in the frequency domain. The masks are essentially like that of the frequency response of the ideal filter except that only data belonging to certain high frequency bands are allowed through unchanged or dampened. Such a construction allows us to introduce modulation to the wavelet system by partitioning the frequency domain into the required subbands.

Construction 4.4. *Let $\widehat{\phi}_{k+1}$ and $\widehat{h}_{k+1}$ be defined as in Construction 4.1. For $n \in \{1, \ldots, \varrho_k\}$, let*

$$\widehat{g}^n_{k+1}(j) = \begin{cases} \sqrt{2}\sin\left[\frac{\pi}{2}\tilde{\beta}^n_k\left(\frac{|j|}{N_{k,n}} - 1\right)\right] & \text{if } j \in \{N_{k,n}+1, \ldots, L_{k,n}\}, \\ \sqrt{2} & \text{if } j \in \{L_{k,n}, \ldots, N_{k,n+1}\}, \\ \sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}^{n+1}_k\left(\frac{|j|}{N_{k,n+1}} - 1\right)\right] & \text{if } j \in \{N_{k,n+1}+1, \ldots, L_{k,n+1}\}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.5}$$

with the conditions $0 \le N_{k,n} \le L_{k,n} < N_{k,n+1}$, $L_{k,n+1} - N_{k,n} < 2^k$, $N_{k,\varrho_k+1} = 2^{k+1} - L_{k,1}$ and $L_{k,\varrho_k+1} = 2^{k+1} - N_{k,1}$ and the additional condition $L_{k,n+1} \le L_{k+1,1}$ or $N_{k,n} \ge 2^{k+1} - L_{k+1,1}$ if $L_{k+1,1} < 2^k$. In the event that $N_{k,n} = L_{k,n}$ for all $n \in \{1, \ldots, \varrho_k\}$, we would redefine

$$\widehat{g}^n_{k+1}(j) = \begin{cases} 1 & \text{if } j \in \{L_{k,n}\}, \\ \sqrt{2} & \text{if } j \in \{L_{k,n}+1, \ldots, N_{k,n+1}-1\}, \\ 1 & \text{if } j \in \{N_{k,n+1}\}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.6}$$

In Construction 4.4, the last wavelet mask $\widehat{g}^{\varrho_k}_{k+1}$ is constructed so that it complements the refinement mask $\widehat{h}_{k+1}$. The additional condition of $L_{k,n+1} \le L_{k+1,1}$ or $N_{k,n} \ge 2^{k+1} - L_{k+1,1}$ is used to ensure that the bandwidths of the wavelet masks lie within $\sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$. The masks are in general complex as we did not impose any additional conditions of conjugate symmetry in the frequency domain.

Proposition 4.5. *The refinement and wavelet masks $\widehat{h}_{k+1}$, $\widehat{g}^n_{k+1}$ for $n \in \{1, \ldots, \varrho_k\}$ defined by* (4.1) *and* (4.5) *or* (4.2) *and* (4.6) *respectively as in Construction* 4.4 *satisfy* (4.3) *and* (4.4)*. They generally have smooth decay with overlapping supports that can be controlled. Hence by the periodic UEP, the affine system $X_{2\pi}$ is a tight frame for $L^2(\mathbb{T})$.*

Proof. Since $L_{k,n+1} - N_{k,n} < 2^k$, for $\widehat{g}^n_{k+1}(j) \neq 0$ such that $N_{k,n} \leq 2^k$ and $L_{k,n+1} \geq 2^k$, we have $\widehat{g}^n_{k+1}(j + 2^k) = 0$ and $\widehat{g}^n_{k+1}(j - 2^k) = 0$ respectively. For other $\widehat{g}^l_{k+1}(j) \neq 0$ such that $L_{k,l+1} \leq 2^k$ or $N_{k,l} \geq 2^k$, we have $\widehat{g}^l_{k+1}(j + 2^k) = 0$ and $\widehat{g}^l_{k+1}(j - 2^k) = 0$ respectively. Let $n \in \{1, \ldots, \varrho_k\}$. If $N_{k,n} < L_{k,n}$, then for $j \in \{N_{k,n}, \ldots, L_{k,n}\}$, we verify that

$$\left|\widehat{g}^n_{k+1}(j)\right|^2 + \left|\widehat{g}^{n-1}_{k+1}(j)\right|^2 = 2\sin^2 \frac{\pi}{2}\tilde{\beta}^n_k \left(\frac{|j|}{N_{k,n}} - 1\right) + 2\cos^2 \frac{\pi}{2}\tilde{\beta}^n_k \left(\frac{|j|}{N_{k,n}} - 1\right) = 2$$

and all $\widehat{g}^l_{k+1}(j) = 0$ for $l \notin \{n-1, n\}$. If $j = N_{k,n} = L_{k,n}$, we check that

$$\left|\widehat{g}^n_{k+1}(j)\right|^2 + \left|\widehat{g}^{n-1}_{k+1}(j)\right|^2 = 1^2 + 1^2 = 2$$

and all $\widehat{g}^l_{k+1}(j) = 0$ for $l \notin \{n-1, n\}$. For $j \in \{L_{k,n} + 1, \ldots, N_{k,n+1} - 1\}$, we have $\left|\widehat{g}^n_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^l_{k+1}(j) = 0$ for all $l \neq n$. If $N_{k,\varrho_k+1} < L_{k,\varrho_k+1}$, then for $j \in \{N_{k,\varrho_k+1}, \ldots, L_{k,\varrho_k+1}\}$, using the condition that $N_{k,\varrho_k+1} = 2^{k+1} - L_{k,1}$ and $L_{k,\varrho_k+1} = 2^{k+1} - N_{k,1}$, since $\left|\widehat{h}_{k+1}(j)\right|^2 = 2\cos^2 \frac{\pi}{2}\tilde{\beta}^1_k \left(\frac{2^{k+1}-|j|}{N_{k,1}} - 1\right)$, we apply Lemma 4.2 to deduce that

$$\left|\widehat{g}^{\varrho_k}_{k+1}(j)\right|^2 + \left|\widehat{h}_{k+1}(j)\right|^2 = 2\left[\cos^2 \frac{\pi}{2}\tilde{\beta}^{\varrho_k+1}_k \left(\frac{|j|}{N_{k,\varrho_k+1}} - 1\right) + \sin^2 \frac{\pi}{2}\tilde{\beta}^{\varrho_k+1}_k \left(\frac{|j|}{N_{k,\varrho_k+1}} - 1\right)\right] = 2$$

with all other $\widehat{g}^n_{k+1}(j) = 0$. If $j = N_{k,\varrho_k+1} = L_{k,\varrho_k+1}$ instead, we could verify that

$$\left|\widehat{g}^{\varrho_k}_{k+1}(j)\right|^2 + \left|\widehat{h}_{k+1}(j)\right|^2 = 1^2 + 1^2 = 2$$

with all other $\widehat{g}^n_{k+1}(j) = 0$. For $j \in \{0, \ldots, N_{k,1} - 1\} \cup \{L_{k,\varrho_k+1} + 1, \ldots, 2^{k+1} - 1\}$, we have $\left|\widehat{h}_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^n_{k+1}(j) = 0$ for $n \in \{1, \ldots, \varrho_k\}$. Since (4.3) and (4.4) hold for all $j \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$, we conclude that they also hold for $j + 2^k \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$. $\square$

The next construction involves real symmetric framelets with controlled overlap and smooth decay in the frequency domain. Here, we impose conditions of symmetry on the masks in the frequency domain. In order for the anti-aliasing condition to be satisfied by the masks at the middle bands around $j = \pm 2^{k-1}$, i.e. $\widehat{g}^{\lambda_0}_{k+1}$ and $\widehat{g}^{\mu_0}_{k+1}$, where $\lambda_0 = \left\lfloor \frac{\varrho_k}{2} \right\rfloor$ and $\mu_0 = \lambda_0 + 1$, we impose a "balancing condition" on their supports.

Construction 4.6. *Let $\widehat{\phi}_{k+1}$ and $\widehat{h}_{k+1}$ be given as in Construction* 4.1. *Let $\lambda_0 = \left\lfloor \frac{\varrho_k}{2} \right\rfloor$, $\mu_0 = \lambda_0 + 1$ and $0 \leq N_{k,n} \leq L_{k,n} < N_{k,n+1}$, $N_{k,\mu_0} < L_{k,\mu_0}$, $N_{k,\mu_0} + L_{k,\mu_0} = 2^k$ and*

$N_{k,\varrho_k+1} = L_{k,\varrho_k+1} \leq L_{k+1,1}$. *For* $n \in \{1,\dots,\varrho_k\}\backslash\{\mu_0\}$, *let*

$$\widehat{g}^n_{k+1}(j) = \begin{cases} \sqrt{2}\sin\left[\frac{\pi}{2}\tilde{\beta}^n_k\left(\frac{|j|}{N_{k,n}}-1\right)\right] & \textit{if } \begin{array}{l} j \in \{-L_{k,n},\dots,-N_{k,n}-1\} \\ \cup\{N_{k,n}+1,\dots,L_{k,n}\}, \end{array} \\ \sqrt{2} & \textit{if } \begin{array}{l} j \in \{-N_{k,n+1},\dots,-L_{k,n}\} \\ \cup\{L_{k,n},\dots,N_{k,n+1}\}, \end{array} \\ \sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}^{n+1}_k\left(\frac{|j|}{N_{k,n+1}}-1\right)\right] & \textit{if } \begin{array}{l} j \in \{-L_{k,n+1},\dots,-N_{k,n+1}-1\} \\ \cup\{N_{k,n+1}+1,\dots,L_{k,n+1}\}, \end{array} \\ 0 & \textit{otherwise.} \end{cases} \tag{4.7}$$

For $n = \mu_0$, *let*

$$\widehat{g}^n_{k+1}(j) = \begin{cases} \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\sqrt{2}\sin\left[\frac{\pi}{2}\tilde{\beta}^n_k\left(\frac{|j|}{N_{k,n}}-1\right)\right] & \textit{if } \begin{array}{l} j \in \{-L_{k,n},\dots,-N_{k,n}-1\} \\ \cup\{N_{k,n}+1,\dots,L_{k,n}\}, \end{array} \\ \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\sqrt{2} & \textit{if } \begin{array}{l} j \in \{-N_{k,n+1},\dots,-L_{k,n}\} \\ \cup\{L_{k,n},\dots,N_{k,n+1}\}, \end{array} \\ \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}^{n+1}_k\left(\frac{|j|}{N_{k,n+1}}-1\right)\right] & \textit{if } \begin{array}{l} j \in \{-L_{k,n+1},\dots,-N_{k,n+1}-1\} \\ \cup\{N_{k,n+1}+1,\dots,L_{k,n+1}\}, \end{array} \\ 0 & \textit{otherwise.} \end{cases} \tag{4.8}$$

If $N_{k,n} = L_{k,n}$ *for* $n \in \{1,\dots,\varrho_k\}\backslash\{\mu_0\}$, *then for* $n \in \{1,\dots,\varrho_k-1\}\backslash\{\lambda_0,\mu_0\}$, *let*

$$\widehat{g}^n_{k+1}(j) = \begin{cases} 1 & \textit{if } j \in \{-L_{k,n}\}\cup\{L_{k,n}\}, \\ \sqrt{2} & \textit{if } j \in \{-N_{k,n+1}+1,\dots,-L_{k,n}-1\}\cup\{L_{k,n}+1,\dots,N_{k,n+1}-1\}, \\ 1 & \textit{if } j \in \{-N_{k,n+1}\}\cup\{N_{k,n+1}\}, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.9}$$

For $n = \lambda_0$, *let*

$$\widehat{g}^n_{k+1}(j) = \begin{cases} 1 & \textit{if } j \in \{-L_{k,n}\}\cup\{L_{k,n}\}, \\ \sqrt{2} & \textit{if } \begin{array}{l} j \in \{-N_{k,n+1},\dots,-L_{k,n}-1\} \\ \cup\{L_{k,n}+1,\dots,N_{k,n+1}\}, \end{array} \\ \sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}^{n+1}_k\left(\frac{|j|}{N_{k,n+1}}-1\right)\right] & \textit{if } \begin{array}{l} j \in \{-L_{k,n+1},\dots,-N_{k,n+1}-1\} \\ \cup\{N_{k,n+1}+1,\dots,L_{k,n+1}\}, \end{array} \\ 0 & \textit{otherwise.} \end{cases} \tag{4.10}$$

For $n=\mu_0$, let

$$\widehat{g}^n_{k+1}(j)=\begin{cases} \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\sqrt{2}\sin\left[\frac{\pi}{2}\tilde{\beta}^n_k\left(\frac{|j|}{N_{k,n}}-1\right)\right] & \text{if } j\in\{-L_{k,n},\dots,-N_{k,n}-1\} \\ & \quad\cup\{N_{k,n}+1,\dots,L_{k,n}\}, \\ \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\sqrt{2} & \text{if } j\in\{-N_{k,n+1}+1,\dots,-L_{k,n}\} \\ & \quad\cup\{L_{k,n},\dots,N_{k,n+1}-1\}, \\ \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}} & \text{if } j\in\{-N_{k,n+1}\}\cup\{N_{k,n+1}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Finally, under the condition that $\mu_0<\varrho_k$, let

$$\widehat{g}^{\varrho_k}_{k+1}(j)=\begin{cases} 1 & \text{if } j\in\{-L_{k,\varrho_k}\}\cup\{L_{k,\varrho_k}\}, \\ \sqrt{2} & \text{if } j\in\{-N_{k,\varrho_k+1},\dots,-L_{k,\varrho_k}-1\}\cup\{L_{k,\varrho_k}+1,\dots,N_{k,\varrho_k+1}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

Remark. In the event that $N_{k,\mu_0}=2^{k-1}-1$ and $L_{k,\mu_0}=2^{k-1}+1$, we have the liberty of setting $\widehat{g}^{\mu_0}_{k+1}(j)=\mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\widehat{g}^{\lambda_0}_{k+1}(j)=\mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}$ for $j=\pm2^{k-1}$.

Proposition 4.7. *The refinement and wavelet masks $\widehat{h}_{k+1}$, $\widehat{g}^n_{k+1}$ for $n\in\{1,\dots,\varrho_k\}$ defined by* (4.1)*,* (4.7) *and* (4.8) *or* (4.2)*,* (4.9)*,* (4.10)*,* (4.11) *and* (4.12) *respectively as in Construction* 4.6 *satisfy* (4.3) *and* (4.4)*. They are real and symmetric and generally have smooth decay with overlapping supports that can be controlled. Hence, by the periodic UEP, the affine system $X_{2\pi}$ is a tight frame for $L^2(\mathbb{T})$.*

Proof. The condition $N_{k,\mu_0}+L_{k,\mu_0}=2^k$ implies that for $m=1,\dots,\lambda_0$ and $n=\mu_0+1,\dots,\varrho_k$, we have

$$2L_{k,m}\le N_{k,\mu_0}+L_{k,\mu_0}=2^k=N_{k,\mu_0}+L_{k,\mu_0}\le N_{k,n}+N_{k,n},$$

i.e. we have the following increasing sequences

$$\begin{gathered}\{N_{k,m}-2^k,L_{k,m}-2^k,-2^{k-1},-L_{k,m},-N_{k,m},N_{k,m},L_{k,m},2^{k-1},2^k-L_{k,m},2^k-N_{k,m}\}\\ \{-L_{k,n},-N_{k,n},-2^{k-1},N_{k,n}-2^k,L_{k,n}-2^k,2^k-L_{k,n},2^k-N_{k,n},2^{k-1},N_{k,n},L_{k,n}\}\\ \{N_{k,\lambda_0}-2^k,-L_{k,\mu_0},-N_{k,\mu_0},-N_{k,\lambda_0},N_{k,\lambda_0},N_{k,\mu_0},L_{k,\mu_0},2^k-N_{k,\lambda_0}\}\\ \{-L_{k,\mu_0+1},-L_{k,\mu_0},-N_{k,\mu_0},L_{k,\mu_0+1}-2^k,2^k-L_{k,\mu_0+1},N_{k,\mu_0},L_{k,\mu_0},L_{k,\mu_0+1}\}\end{gathered}$$

of integers. The result is that for $n=1,\dots,\lambda_0-1$, with $j\in\{-L_{k,n+1},\dots,-N_{k,n}\}\cup\{N_{k,n},\dots,L_{k,n+1}\}$, or equivalently $j-2^k\in\{N_{k,n}-2^k,\dots,L_{k,n+1}-2^k\}$ and $j+2^k\in\{2^k-L_{k,n+1},\dots,2^k-N_{k,n}\}$, we have $\widehat{g}^n_{k+1}(j\pm2^k)=0$. Also, for $n=\mu_0+1,\dots,\varrho_k$, with $j \in \{-L_{k,n+1}, \ldots, -N_{k,n}\} \cup \{N_{k,n}, \ldots, L_{k,n+1}\}$, or equivalently $j - 2^k \in \{N_{k,n} - 2^k, \ldots, L_{k,n+1} - 2^k\}$ and $j + 2^k \in \{2^k - L_{k,n+1}, \ldots, 2^k - N_{k,n}\}$, we have $\widehat{g}^n_{k+1}(j \pm 2^k) = 0$.

Since $N_{k,\mu_0} < L_{k,\mu_0}$, for $j \in \{-N_{k,\mu_0}, \ldots, -N_{k,\lambda_0}\} \cup \{N_{k,\lambda_0}, \ldots, N_{k,\mu_0}\}$, or equivalently $j - 2^k \in \{N_{k,\lambda_0} - 2^k, \ldots, -L_{k,\mu_0}\}$ and $j + 2^k \in \{L_{k,\mu_0}, \ldots, 2^k - N_{k,\lambda_0}\}$, we have $\widehat{g}^{\lambda_0}_{k+1}(j \pm 2^k) = 0 = \widehat{g}^{\mu_0}_{k+1}(j)$. For $j \in \{-L_{k,\mu_0+1}, \ldots, -L_{k,\mu_0}\} \cup \{L_{k,\mu_0}, \ldots, L_{k,\mu_0+1}\}$, or equivalently $j - 2^k \in \{-N_{k,\mu_0}, \ldots, L_{k,\mu_0+1} - 2^k\}$ and $j + 2^k \in \{2^k - L_{k,\mu_0+1}, \ldots, N_{k,\mu_0}\}$, we have $\widehat{g}^{\lambda_0}_{k+1}(j) = 0 = \widehat{g}^{\mu_0}_{k+1}(j \pm 2^k)$.

Since $N_{k,\mu_0} < L_{k,\mu_0}$, for $j \in \{-L_{k,\mu_0}, \ldots, -N_{k,\mu_0}\} \cup \{N_{k,\mu_0}, \ldots, L_{k,\mu_0}\}$, which is essentially equivalent to $j - 2^k \in \{-L_{k,\mu_0}, \ldots, -N_{k,\mu_0}\}$ and $j + 2^k \in \{N_{k,\mu_0}, \ldots, L_{k,\mu_0}\}$, we have

$$\overline{\widehat{g}^{\lambda_0}_{k+1}(j)}\widehat{g}^{\lambda_0}_{k+1}(j \pm 2^k) = 2\cos\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{|j|}{N_{k,\mu_0}} - 1\right)\cos\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{2^k - |j|}{N_{k,\mu_0}} - 1\right),$$
$$\overline{\widehat{g}^{\mu_0}_{k+1}(j)}\widehat{g}^{\mu_0}_{k+1}(j \pm 2^k) = -2\sin\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{|j|}{N_{k,\mu_0}} - 1\right)\sin\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{2^k - |j|}{N_{k,\mu_0}} - 1\right),$$

and hence by Lemma 4.3,

$$\overline{\widehat{g}^{\lambda_0}_{k+1}(j)}\widehat{g}^{\lambda_0}_{k+1}(j \pm 2^k) + \overline{\widehat{g}^{\mu_0}_{k+1}(j)}\widehat{g}^{\mu_0}_{k+1}(j \pm 2^k) = 0. \tag{4.13}$$

We have shown that for $n \in \{1, \ldots, \lambda_0 - 1, \mu_0 + 1, \ldots, \varrho_k\}$ such that $\widehat{g}^n_{k+1}(j) \neq 0$, we have $\widehat{g}^n_{k+1}(j \pm 2^k) = 0$, and for $\widehat{g}^{\lambda_0}_{k+1}(j) \neq 0$ or $\widehat{g}^{\mu_0}_{k+1}(j) \neq 0$, (4.13) must hold. Therefore, (4.4) holds for all $j \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$.

Let $n \in \{1, \ldots, \varrho_k\}$. If $N_{k,n} < L_{k,n}$, then for $j \in \{-L_{k,n}, \ldots, -N_{k,n}\} \cup \{N_{k,n}, \ldots, L_{k,n}\}$, we verify that

$$\left|\widehat{g}^n_{k+1}(j)\right|^2 + \left|\widehat{g}^{n-1}_{k+1}(j)\right|^2 = 2\sin^2\frac{\pi}{2}\tilde{\beta}^n_k\left(\frac{|j|}{N_{k,n}} - 1\right) + 2\cos^2\frac{\pi}{2}\tilde{\beta}^n_k\left(\frac{|j|}{N_{k,n}} - 1\right) = 2$$

and all $\widehat{g}^l_{k+1}(j) = 0$ for $l \notin \{n-1, n\}$. If $j = N_{k,n} = L_{k,n}$ or $j = -N_{k,n} = -L_{k,n}$, then clearly

$$\left|\widehat{g}^n_{k+1}(j)\right|^2 + \left|\widehat{g}^{n-1}_{k+1}(j)\right|^2 = 1 + 1 = 2$$

and all $\widehat{g}^l_{k+1}(j) = 0$ for $l \notin \{n-1, n\}$. For $j \in \{-N_{k,n+1} + 1, \ldots, -L_{k,n} - 1\} \cup \{L_{k,n} + 1, \ldots, N_{k,n+1} - 1\}$, we have $\left|\widehat{g}^n_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^l_{k+1}(j) = 0$ for all $l \neq n$. For $j \in \{-N_{k,\varrho_k+1}\} \cup \{N_{k,\varrho_k+1}\}$, clearly, $\left|\widehat{g}^{\varrho_k}_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^l_{k+1}(j) = 0$ for all $l \neq \varrho_k$. For $j \in \{-N_{k,1} + 1, \ldots, N_{k,1} - 1\}$, we have $\left|\widehat{h}_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^n_{k+1}(j) = 0$ for $n \in \{1, \ldots, \varrho_k\}$. Since (4.3) holds for all $j \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$, we conclude that it also holds for $j + 2^k \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$. □

Remark. We may vary our symmetric construction in several ways. Although the proofs of these variations may be similar, each one of them has its delicate details. For completeness and clarity, we shall include them here.

Construction 4.6 could be modified by permitting one of our framelet masks, i.e. $\widehat{g}_{k+1}^{\mu_0}$ in (4.8) to be antisymmetric instead. To this end let us define

$$\operatorname{sgn}_{k+1}(j) := \operatorname{sign}(j \bmod 2^{k+1} - 2^k).$$

Construction 4.8. *Let $\widehat{\phi}_{k+1}$ and $\widehat{h}_{k+1}$ be given as in Construction* 4.1. *Let $\lambda_0 = \left\lfloor \frac{\varrho_k}{2} \right\rfloor$, $\mu_0 = \lambda_0 + 1$ and $0 \leq N_{k,n} \leq L_{k,n} < N_{k,n+1}$, $N_{k,\mu_0} < L_{k,\mu_0}$, $N_{k,\mu_0} + L_{k,\mu_0} = 2^k$ and $N_{k,\varrho_k+1} = L_{k,\varrho_k+1} \leq L_{k+1,1}$. For $n \in \{1, \dots, \varrho_k\} \backslash \{\mu_0\}$, let $\widehat{g}_{k+1}^n$ be given as in* (4.7). *For $n = \mu_0$, let*

$$\widehat{g}_{k+1}^n(j) = \begin{cases} \mathrm{i}\operatorname{sgn}_{k+1}(j)\sqrt{2}\sin\left[\frac{\pi}{2}\tilde{\beta}_k^n\left(\frac{|j|}{N_{k,n}} - 1\right)\right] & \text{if } \begin{array}{l} j \in \{-L_{k,n}, \dots, -N_{k,n} - 1\} \\ \cup \{N_{k,n} + 1, \dots, L_{k,n}\}, \end{array} \\ \mathrm{i}\operatorname{sgn}_{k+1}(j)\sqrt{2} & \text{if } \begin{array}{l} j \in \{-N_{k,n+1}, \dots, -L_{k,n}\} \\ \cup \{L_{k,n}, \dots, N_{k,n+1}\}, \end{array} \\ \mathrm{i}\operatorname{sgn}_{k+1}(j)\sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}_k^{n+1}\left(\frac{|j|}{N_{k,n+1}} - 1\right)\right] & \text{if } \begin{array}{l} j \in \{-L_{k,n+1}, \dots, -N_{k,n+1} - 1\} \\ \cup \{N_{k,n+1} + 1, \dots, L_{k,n+1}\}, \end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

If $N_{k,n} = L_{k,n}$ for $n \in \{1, \dots, \varrho_k\} \backslash \{\mu_0\}$, then for $n \in \{1, \dots, \varrho_k - 1\} \backslash \{\lambda_0, \mu_0\}$, let $\widehat{g}_{k+1}^n$ be given as in (4.9). *For $n = \lambda_0$, let $\widehat{g}_{k+1}^n$ be given as in* (4.10). *For $n = \mu_0$, let*

$$\widehat{g}_{k+1}^n(j) = \begin{cases} \mathrm{i}\operatorname{sgn}_{k+1}(j)\sqrt{2}\sin\left[\frac{\pi}{2}\tilde{\beta}_k^n\left(\frac{|j|}{N_{k,n}} - 1\right)\right] & \text{if } \begin{array}{l} j \in \{-L_{k,n}, \dots, -N_{k,n} - 1\} \\ \cup \{N_{k,n} + 1, \dots, L_{k,n}\}, \end{array} \\ \mathrm{i}\operatorname{sgn}_{k+1}(j)\sqrt{2} & \text{if } \begin{array}{l} j \in \{-N_{k,n+1} + 1, \dots, -L_{k,n}\} \\ \cup \{L_{k,n}, \dots, N_{k,n+1} - 1\}, \end{array} \\ \mathrm{i}\operatorname{sgn}_{k+1}(j) & \text{if } j \in \{-N_{k,n+1}\} \cup \{N_{k,n+1}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

Finally, let $\widehat{g}_{k+1}^{\varrho_k}$ be given as in (4.12) *only if $\mu_0 < \varrho_k$.*

Remark. In the event that $N_{k,\mu_0} = 2^{k-1} - 1$ and $L_{k,\mu_0} = 2^{k-1} + 1$, we are at the liberty of setting $\widehat{g}_{k+1}^{\mu_0}(-2^{k-1}) = -\mathrm{i}$ with $\widehat{g}_{k+1}^{\lambda_0}(-2^{k-1}) = 1$ and $\widehat{g}_{k+1}^{\mu_0}(2^{k-1}) = \mathrm{i}$ with $\widehat{g}_{k+1}^{\lambda_0}(2^{k-1}) = 1$.

Proposition 4.9. *The refinement and wavelet masks $\widehat{h}_{k+1}$, $\widehat{g}_{k+1}^n$ for $n \in \{1, \dots, \varrho_k\}$ defined by* (4.1), (4.7) *and* (4.14) *or* (4.2), (4.9), (4.10), (4.15) *and* (4.12) *respectively as in Construction* 4.8 *satisfy* (4.3) *and* (4.4) *and are real and symmetric except for* *one antisymmetric framelet mask. They generally have smooth decay with overlapping supports that can be controlled. Hence, by the periodic UEP, the affine system $X_{2\pi}$ is a tight frame for $L^2(\mathbb{T})$.*

Proof. The condition $N_{k,\mu_0} + L_{k,\mu_0} = 2^k$ implies that for $m = 1, \ldots, \lambda_0$ and $n = \mu_0 + 1, \ldots, \varrho_k$, we have

$$2L_{k,m} \le N_{k,\mu_0} + L_{k,\mu_0} = 2^k = N_{k,\mu_0} + L_{k,\mu_0} \le N_{k,n} + N_{k,n},$$

i.e. we have the following increasing sequences

$$\begin{aligned}
&\{N_{k,m} - 2^k, L_{k,m} - 2^k, -2^{k-1}, -L_{k,m}, -N_{k,m}, N_{k,m}, L_{k,m}, 2^{k-1}, 2^k - L_{k,m}, 2^k - N_{k,m}\}\\
&\{-L_{k,n}, -N_{k,n}, -2^{k-1}, N_{k,n} - 2^k, L_{k,n} - 2^k, 2^k - L_{k,n}, 2^k - N_{k,n}, 2^{k-1}, N_{k,n}, L_{k,n}\}\\
&\{N_{k,\lambda_0} - 2^k, -L_{k,\mu_0}, -N_{k,\mu_0}, -N_{k,\lambda_0}, N_{k,\lambda_0}, N_{k,\mu_0}, L_{k,\mu_0}, 2^k - N_{k,\lambda_0}\}\\
&\{-L_{k,\mu_0+1}, -L_{k,\mu_0}, -N_{k,\mu_0}, L_{k,\mu_0+1} - 2^k, 2^k - L_{k,\mu_0+1}, N_{k,\mu_0}, L_{k,\mu_0}, L_{k,\mu_0+1}\}
\end{aligned}$$

of integers. The result is that for $n = 1, \ldots, \lambda_0 - 1$, with $j \in \{-L_{k,n+1}, \ldots, -N_{k,n}\} \cup \{N_{k,n}, \ldots, L_{k,n+1}\}$, or equivalently $j - 2^k \in \{N_{k,n} - 2^k, \ldots, L_{k,n+1} - 2^k\}$ and $j + 2^k \in \{2^k - L_{k,n+1}, \ldots, 2^k - N_{k,n}\}$, we have $\widehat{g}^n_{k+1}(j \pm 2^k) = 0$. Also, for $n = \mu_0 + 1, \ldots, \varrho_k$, with $j \in \{-L_{k,n+1}, \ldots, -N_{k,n}\} \cup \{N_{k,n}, \ldots, L_{k,n+1}\}$, or equivalently $j - 2^k \in \{N_{k,n} - 2^k, \ldots, L_{k,n+1} - 2^k\}$ and $j + 2^k \in \{2^k - L_{k,n+1}, \ldots, 2^k - N_{k,n}\}$, we have $\widehat{g}^n_{k+1}(j \pm 2^k) = 0$.

Since $N_{k,\mu_0} < L_{k,\mu_0}$, for $j \in \{-N_{k,\mu_0}, \ldots, -N_{k,\lambda_0}\} \cup \{N_{k,\lambda_0}, \ldots, N_{k,\mu_0}\}$, or equivalently $j - 2^k \in \{N_{k,\lambda_0} - 2^k, \ldots, -L_{k,\mu_0}\}$ and $j + 2^k \in \{L_{k,\mu_0}, \ldots, 2^k - N_{k,\lambda_0}\}$, we have $\widehat{g}^{\lambda_0}_{k+1}(j \pm 2^k) = 0 = \widehat{g}^{\mu_0}_{k+1}(j)$. For $j \in \{-L_{k,\mu_0+1}, \ldots, -L_{k,\mu_0}\} \cup \{L_{k,\mu_0}, \ldots, L_{k,\mu_0+1}\}$, or equivalently $j - 2^k \in \{-N_{k,\mu_0}, \ldots, L_{k,\mu_0+1} - 2^k\}$ and $j + 2^k \in \{2^k - L_{k,\mu_0+1}, \ldots, N_{k,\mu_0}\}$, we have $\widehat{g}^{\lambda_0}_{k+1}(j) = 0 = \widehat{g}^{\mu_0}_{k+1}(j \pm 2^k)$.

Since $N_{k,\mu_0} < L_{k,\mu_0}$, for $j \in \{-L_{k,\mu_0}, \ldots, -N_{k,\mu_0}\} \cup \{N_{k,\mu_0}, \ldots, L_{k,\mu_0}\}$, which is essentially equivalent to $j - 2^k \in \{-L_{k,\mu_0}, \ldots, -N_{k,\mu_0}\}$ and $j + 2^k \in \{N_{k,\mu_0}, \ldots, L_{k,\mu_0}\}$, we have

$$\begin{aligned}
\overline{\widehat{g}^{\lambda_0}_{k+1}(j)}\widehat{g}^{\lambda_0}_{k+1}(j \pm 2^k) &= 2\cos\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{|j|}{N_{k,\mu_0}} - 1\right)\cos\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{2^k - |j|}{N_{k,\mu_0}} - 1\right),\\
\overline{\widehat{g}^{\mu_0}_{k+1}(j)}\widehat{g}^{\mu_0}_{k+1}(j \pm 2^k) &= -2\sin\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{|j|}{N_{k,\mu_0}} - 1\right)\sin\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{2^k - |j|}{N_{k,\mu_0}} - 1\right),
\end{aligned}$$

and hence by Lemma 4.3,

$$\overline{\widehat{g}^{\lambda_0}_{k+1}(j)}\widehat{g}^{\lambda_0}_{k+1}(j \pm 2^k) + \overline{\widehat{g}^{\mu_0}_{k+1}(j)}\widehat{g}^{\mu_0}_{k+1}(j \pm 2^k) = 0. \tag{4.16}$$

We have shown that for $n \in \{1, \ldots, \lambda_0 - 1, \mu_0 + 1, \ldots, \varrho_k\}$ such that $\widehat{g}^n_{k+1}(j) \neq 0$, we have $\widehat{g}^n_{k+1}(j \pm 2^k) = 0$, and for $\widehat{g}^{\lambda_0}_{k+1}(j) \neq 0$ or $\widehat{g}^{\mu_0}_{k+1}(j) \neq 0$, (4.16) must hold. Therefore, (4.4) holds for all $j \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$.

Let $n \in \{1, \ldots, \varrho_k\}$. If $N_{k,n} < L_{k,n}$, then for $j \in \{-L_{k,n}, \ldots, -N_{k,n}\} \cup \{N_{k,n}, \ldots, L_{k,n}\}$, we verify that

$$\left|\widehat{g}^n_{k+1}(j)\right|^2 + \left|\widehat{g}^{n-1}_{k+1}(j)\right|^2 = 2\sin^2 \frac{\pi}{2}\tilde{\beta}^n_k \left(\frac{|j|}{N_{k,n}} - 1\right) + 2\cos^2 \frac{\pi}{2}\tilde{\beta}^n_k \left(\frac{|j|}{N_{k,n}} - 1\right) = 2$$

and all $\widehat{g}^l_{k+1}(j) = 0$ for $l \notin \{n-1, n\}$. If $j = N_{k,n} = L_{k,n}$ or $j = -N_{k,n} = -L_{k,n}$, then clearly

$$\left|\widehat{g}^n_{k+1}(j)\right|^2 + \left|\widehat{g}^{n-1}_{k+1}(j)\right|^2 = 1 + 1 = 2$$

and all $\widehat{g}^l_{k+1}(j) = 0$ for $l \notin \{n-1, n\}$. For $j \in \{-N_{k,n+1} + 1, \ldots, -L_{k,n} - 1\} \cup \{L_{k,n} + 1, \ldots, N_{k,n+1} - 1\}$, we have $\left|\widehat{g}^n_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^l_{k+1}(j) = 0$ for all $l \neq n$. For $j \in \{-N_{k,\varrho_k+1}\} \cup \{N_{k,\varrho_k+1}\}$, clearly, $\left|\widehat{g}^{\varrho_k}_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^l_{k+1}(j) = 0$ for all $l \neq \varrho_k$. For $j \in \{-N_{k,1} + 1, \ldots, N_{k,1} - 1\}$, we have $\left|\widehat{h}_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^n_{k+1}(j) = 0$ for $n \in \{1, \ldots, \varrho_k\}$. Since (4.3) holds for all $j \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$, we conclude that it also holds for $j + 2^k \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$. □

We may remove the restriction that $N_{k,\mu_0} + L_{k,\mu_0} = 2^k$ by using two pairs of symmetric and antisymmetric framelets instead, i.e. we replace the masks at the middle bands around $j = \pm 2^{k-1}$ by symmetric masks $\widehat{g}^{\lambda_0}_{k+1}$ and $\widehat{g}^{\mu_0}_{k+1}$ and their corresponding antisymmetric ones $\widehat{g}^{\widetilde{\lambda}_0}_{k+1}$ and $\widehat{g}^{\widetilde{\mu}_0}_{k+1}$. The redundancy provided by the antisymmetric masks is used for anti-aliasing purposes.

Construction 4.10. *Let $\widehat{\phi}_{k+1}$ and $\widehat{h}_{k+1}$ be given as in Construction* 4.1*. Let $\mu_0 = \lambda_0 + 1$ such that $0 \leq N_{k,n} \leq L_{k,n} < N_{k,n+1}$, $N_{k,\mu_0} < 2^{k-1} < L_{k,\mu_0}$ and $N_{k,\varrho_k+1} = L_{k,\varrho_k+1} \leq L_{k+1,1}$. For $n \in \{1, \ldots, \varrho_k\} \backslash \{\lambda_0, \mu_0\}$, let $\widehat{g}^n_{k+1}$ be given as in* (4.7)*. For the labels $n \in \{\lambda_0, \mu_0\}$, let*

$$\widehat{g}^n_{k+1}(j) = \begin{cases} \sin\left[\frac{\pi}{2}\tilde{\beta}^n_k \left(\frac{|j|}{N_{k,n}} - 1\right)\right] & \text{if } \begin{aligned} j \in &\{-L_{k,n}, \ldots, -N_{k,n} - 1\} \\ &\cup \{N_{k,n} + 1, \ldots, L_{k,n}\}, \end{aligned} \\ 1 & \text{if } \begin{aligned} j \in &\{-N_{k,n+1}, \ldots, -L_{k,n}\} \\ &\cup \{L_{k,n}, \ldots, N_{k,n+1}\}, \end{aligned} \\ \cos\left[\frac{\pi}{2}\tilde{\beta}^{n+1}_k \left(\frac{|j|}{N_{k,n+1}} - 1\right)\right] & \text{if } \begin{aligned} j \in &\{-L_{k,n+1}, \ldots, -N_{k,n+1} - 1\} \\ &\cup \{N_{k,n+1} + 1, \ldots, L_{k,n+1}\}, \end{aligned} \\ 0 & \text{otherwise,} \end{cases} \tag{4.17}$$

and

$$\widehat{g}_{k+1}^{\widetilde{n}}(j) = \mathrm{i}\,\mathrm{sgn}_{k+1}(j)\widehat{g}_{k+1}^{n}(j). \tag{4.18}$$

If $N_{k,n} = L_{k,n}$ *for* $n \in \{1, \ldots, \varrho_k\} \backslash \{\mu_0\}$, *then for* $n \in \{1, \ldots, \varrho_k - 1\} \backslash \{\lambda_0, \mu_0\}$, *let* $\widehat{g}_{k+1}^{n}$ *be given as in* (4.9). *For* $n = \lambda_0$, *let*

$$\widehat{g}_{k+1}^{n}(j) = \begin{cases} 2^{-\frac{1}{2}} & \text{if } j \in \{-L_{k,n}\} \cup \{L_{k,n}\}, \\ 1 & \text{if } \begin{array}{l} j \in \{-N_{k,n+1}, \ldots, -L_{k,n} - 1\} \\ \quad \cup \{L_{k,n} + 1, \ldots, N_{k,n+1}\}, \end{array} \\ \cos\left[\frac{\pi}{2}\tilde{\beta}_k^{n+1}\left(\frac{|j|}{N_{k,n+1}} - 1\right)\right] & \text{if } \begin{array}{l} j \in \{-L_{k,n+1}, \ldots, -N_{k,n+1} - 1\} \\ \quad \cup \{N_{k,n+1} + 1, \ldots, L_{k,n+1}\}, \end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{4.19}$$

For $n = \mu_0$, *let*

$$\widehat{g}_{k+1}^{n}(j) = \begin{cases} \sin\left[\frac{\pi}{2}\tilde{\beta}_k^{n}\left(\frac{|j|}{N_{k,n}} - 1\right)\right] & \text{if } \begin{array}{l} j \in \{-L_{k,n}, \ldots, -N_{k,n} - 1\} \\ \quad \cup \{N_{k,n} + 1, \ldots, L_{k,n}\}, \end{array} \\ 1 & \text{if } \begin{array}{l} j \in \{-N_{k,n+1} + 1, \ldots, -L_{k,n}\} \\ \quad \cup \{L_{k,n}, \ldots, N_{k,n+1} - 1\}, \end{array} \\ 2^{-\frac{1}{2}} & \text{if } j \in \{-N_{k,n+1}\} \cup \{N_{k,n+1}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.20}$$

For the labels $n \in \{\lambda_0, \mu_0\}$, *let*

$$\widehat{g}_{k+1}^{\widetilde{n}}(j) = \mathrm{i}\,\mathrm{sgn}_{k+1}(j)\widehat{g}_{k+1}^{n}(j). \tag{4.21}$$

Finally, let $\widehat{g}_{k+1}^{\varrho_k}$ *be given as in* (4.12) *only if* $\mu_0 < \varrho_k$.

Proposition 4.11. *The refinement and wavelet masks* $\widehat{h}_{k+1}$, $\widehat{g}_{k+1}^{n}$ *for* $n \in \{1, \ldots, \varrho_k\}$ *defined by* (4.1), (4.7) *and* (4.17), (4.18) *or* (4.2), (4.9), (4.19), (4.20), (4.12) *together with* (4.21) *for* $n \in \{\lambda_0, \mu_0\}$ *respectively as in Construction* 4.10 *satisfy* (4.3) *and* (4.4) *and are real and symmetric except for two antisymmetric framelet masks. They generally have smooth decay with overlapping supports that can be controlled. Hence, by the periodic UEP, the affine system* $X_{2\pi}$ *is a tight frame for* $L^2(\mathbb{T})$.

Proof. For $m = 1, \ldots, \lambda_0$ and $n = \mu_0 + 1, \ldots, \varrho_k$, we have

$$L_{k,m} \leq N_{k,\mu_0} < 2^{k-1} < L_{k,\mu_0} < N_{k,n},$$

i.e. we have the following increasing sequences

$$\{N_{k,m}-2^k, L_{k,m}-2^k, -2^{k-1}, -L_{k,m}, -N_{k,m}, N_{k,m}, L_{k,m}, 2^{k-1}, 2^k-L_{k,m}, 2^k-N_{k,m}\}$$
$$\{-L_{k,n}, -N_{k,n}, -2^{k-1}, N_{k,n}-2^k, L_{k,n}-2^k, 2^k-L_{k,n}, 2^k-N_{k,n}, 2^{k-1}, N_{k,n}, L_{k,n}\}$$

of integers. The result is that for $n=1,\dots,\lambda_0-1$, with $j\in\{-L_{k,n+1},\dots,-N_{k,n}\}\cup\{N_{k,n},\dots,L_{k,n+1}\}$, or equivalently $j-2^k\in\{N_{k,n}-2^k,\dots,L_{k,n+1}-2^k\}$ and $j+2^k\in\{2^k-L_{k,n+1},\dots,2^k-N_{k,n}\}$, we have $\widehat{g}^n_{k+1}(j\pm 2^k)=0$. Also, for $n=\mu_0+1,\dots,\varrho_k$, with $j\in\{-L_{k,n+1},\dots,-N_{k,n}\}\cup\{N_{k,n},\dots,L_{k,n+1}\}$, or equivalently $j-2^k\in\{N_{k,n}-2^k,\dots,L_{k,n+1}-2^k\}$ and $j+2^k\in\{2^k-L_{k,n+1},\dots,2^k-N_{k,n}\}$, we have $\widehat{g}^n_{k+1}(j\pm 2^k)=0$.

Let $n\in\{\lambda_0,\mu_0\}$. For $j\in\{-L_{k,n+1},\dots,-N_{k,n}\}\cup\{N_{k,n},\dots,L_{k,n+1}\}$, we have

$$\overline{\widehat{g}^{\tilde{n}}_{k+1}(j)}\widehat{g}^{\tilde{n}}_{k+1}(j\pm 2^k)=-\overline{\widehat{g}^{n}_{k+1}(j)}\widehat{g}^{n}_{k+1}(j\pm 2^k)$$

and hence

$$\overline{\widehat{g}^{n}_{k+1}(j)}\widehat{g}^{n}_{k+1}(j\pm 2^k)+\overline{\widehat{g}^{\tilde{n}}_{k+1}(j)}\widehat{g}^{\tilde{n}}_{k+1}(j\pm 2^k)=0. \tag{4.22}$$

We have shown that for $n\in\{1,\dots,\lambda_0-1,\mu_0+1,\dots,\varrho_k\}$ such that $\widehat{g}^n_{k+1}(j)\neq 0$, we have $\widehat{g}^n_{k+1}(j\pm 2^k)=0$, and for $\widehat{g}^{\lambda_0}_{k+1}(j)\neq 0$ or $\widehat{g}^{\mu_0}_{k+1}(j)\neq 0$, (4.22) must hold. Therefore, (4.4) holds for all $j\in\mathcal{R}_k$.

Let $n\in\{1,\dots,\lambda_0-1,\mu_0+1,\dots,\varrho_k\}$. If $N_{k,n}<L_{k,n}$, then for $j\in\{-L_{k,n},\dots,-N_{k,n}\}\cup\{N_{k,n},\dots,L_{k,n}\}$, we verify that

$$\left|\widehat{g}^n_{k+1}(j)\right|^2+\left|\widehat{g}^{n-1}_{k+1}(j)\right|^2=2\sin^2\frac{\pi}{2}\widetilde{\beta}^n_k\left(\frac{|j|}{N_{k,n}}-1\right)+2\cos^2\frac{\pi}{2}\widetilde{\beta}^n_k\left(\frac{|j|}{N_{k,n}}-1\right)=2$$

and all $\widehat{g}^l_{k+1}(j)=0$ for $l\notin\{n-1,n\}$. If $j=N_{k,n}=L_{k,n}$ or $j=-N_{k,n}=-L_{k,n}$, then clearly

$$\left|\widehat{g}^n_{k+1}(j)\right|^2+\left|\widehat{g}^{n-1}_{k+1}(j)\right|^2=1+1=2$$

and all $\widehat{g}^l_{k+1}(j)=0$ for $l\notin\{n-1,n\}$. For $j\in\{-N_{k,n+1}+1,\dots,-L_{k,n}-1\}\cup\{L_{k,n}+1,\dots,N_{k,n+1}-1\}$, we have $\left|\widehat{g}^n_{k+1}(j)\right|^2=2$ and $\widehat{g}^l_{k+1}(j)=0$ for all $l\neq n$. For $j\in\{-N_{k,\varrho_k+1}\}\cup\{N_{k,\varrho_k+1}\}$, clearly, $\left|\widehat{g}^{\varrho_k}_{k+1}(j)\right|^2=2$ and $\widehat{g}^l_{k+1}(j)=0$ for all $l\neq\varrho_k$. For $j\in\{-N_{k,1}+1,\dots,N_{k,1}-1\}$, we have $\left|\widehat{h}_{k+1}(j)\right|^2=2$ and $\widehat{g}^n_{k+1}(j)=0$ for $n\in\{1,\dots,\varrho_k\}$.

For $n=\lambda_0$ and $j\in\{-L_{k,n},\dots,-N_{k,n}\}\cup\{N_{k,n},\dots,L_{k,n}\}$, we could show in a likewise manner that

$$\left|\widehat{g}^{\lambda_0-1}_{k+1}(j)\right|^2+\left|\widehat{g}^{\lambda_0}_{k+1}(j)\right|^2+\left|\widehat{g}^{\widetilde{\lambda_0}}_{k+1}(j)\right|^2=2.$$

For $j \in \{-N_{k,n+1}, \ldots, -L_{k,n} - 1\} \cup \{L_{k,n} + 1, \ldots, N_{k,n+1}\}$, we could show that

$$\left|\widehat{g}^{\lambda_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\widetilde{\lambda_0}}_{k+1}(j)\right|^2 = 2.$$

For $n = \mu_0$ and $j \in \{-L_{k,n}, \ldots, -N_{k,n}\} \cup \{N_{k,n}, \ldots, L_{k,n}\}$, we could confirm that

$$\left|\widehat{g}^{\lambda_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\widetilde{\lambda_0}}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\mu_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\widetilde{\mu_0}}_{k+1}(j)\right|^2 = 2.$$

For $j \in \{-N_{k,n+1} + 1, \ldots, -L_{k,n}\} \cup \{L_{k,n}, \ldots, N_{k,n+1} - 1\}$, we could ensure that

$$\left|\widehat{g}^{\mu_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\widetilde{\mu_0}}_{k+1}(j)\right|^2 = 2.$$

For $j \in \{-L_{k,n+1}, \ldots, -N_{k,n+1}\} \cup \{N_{k,n+1}, \ldots, L_{k,n+1}\}$, we are assured that

$$\left|\widehat{g}^{\mu_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\widetilde{\mu_0}}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\mu_0+1}_{k+1}(j)\right|^2 = 2.$$

Since (4.3) holds for all $j \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$, we conclude that it also holds for $j + 2^k \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$. $\square$

We may further remove the redundancy of two pairs of symmetric and antisymmetric framelets by using a pair of symmetric and antisymmetric framelets ψ_{λ_0} and ψ_{μ_0} instead and imposing the condition that the signal is processed unchanged at the middle bands around $j = \pm 2^{k-1}$.

Construction 4.12. *Let $\widehat{\phi}_{k+1}$ and $\widehat{h}_{k+1}$ be given as in Construction* 4.1*. Assume that $0 \le N_{k,n} \le L_{k,n} < N_{k,n+1}$ for $n \ne \mu_0$, $L_{k,\lambda_0} < 2^{k-1} < N_{k,\mu_0} = N_{k,\mu_0+1} \le L_{k,\mu_0} = L_{k,\mu_0+1}$ and $N_{k,\varrho_k+1} = L_{k,\varrho_k+1} \le L_{k+1,1}$ with $\mu_0 = \lambda_0 + 1$. For $n \in \{1, \ldots, \varrho_k\} \backslash \{\lambda_0, \mu_0\}$, let $\widehat{g}^n_{k+1}$ be defined by* (4.7)*. We define (or redefine) $\widehat{g}^n_{k+1}$ for $n \in \{\lambda_0, \mu_0\}$ by*

$$\widehat{g}^n_{k+1}(j) = \begin{cases} \mathrm{i}^{n \bmod 2}(\mathrm{sgn}_{k+1}(j))^n \sin\left[\frac{\pi}{2}\tilde{\beta}^{\lambda_0}_k\left(\frac{|j|}{N_{k,\lambda_0}} - 1\right)\right] & \text{if } \begin{aligned} j \in &\{-L_{k,\lambda_0}, \ldots, -N_{k,\lambda_0} - 1\} \\ &\cup\{N_{k,\lambda_0} + 1, \ldots, L_{k,\lambda_0}\}, \end{aligned} \\ \mathrm{i}^{n \bmod 2}(\mathrm{sgn}_{k+1}(j))^n & \text{if } \begin{aligned} j \in &\{-N_{k,\mu_0}, \ldots, -L_{k,\lambda_0}\} \\ &\cup\{L_{k,\lambda_0}, \ldots, N_{k,\mu_0}\}, \end{aligned} \\ \mathrm{i}^{n \bmod 2}(\mathrm{sgn}_{k+1}(j))^n \cos\left[\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{|j|}{N_{k,\mu_0}} - 1\right)\right] & \text{if } \begin{aligned} j \in &\{-L_{k,\mu_0}, \ldots, -N_{k,\mu_0} - 1\} \\ &\cup\{N_{k,\mu_0} + 1, \ldots, L_{k,\mu_0}\}, \end{aligned} \\ 0 & \text{otherwise.} \end{cases} \tag{4.23}$$

In the event that $N_{k,n} = L_{k,n}$ for all $n \in \{1, \ldots, \varrho_k\}$, we would redefine $\widehat{g}^n_{k+1}$ for $n \in$

$\{1,\dots,\varrho_k-1\}\backslash\{\lambda_0,\mu_0\}$ *by* (4.9) *and* $\widehat{g}^n_{k+1}$ *for* $n\in\{\lambda_0,\mu_0\}$ *by*

$$\widehat{g}^n_{k+1}(j)=\begin{cases} 2^{-\frac{1}{2}}\mathrm{i}^{n \bmod 2}(\mathrm{sgn}_{k+1}(j))^n & \text{if } j\in\{-L_{k,\lambda_0}\}\cup\{L_{k,\lambda_0}\},\\ \mathrm{i}^{n \bmod 2}(\mathrm{sgn}_{k+1}(j))^n & \text{if } \begin{array}{l} j\in\{-N_{k,\mu_0}+1,\dots,-L_{k,\lambda_0}-1\}\\ \cup\{L_{k,\lambda_0}+1,\dots,N_{k,\mu_0}-1\},\end{array}\\ 2^{-\frac{1}{2}}\mathrm{i}^{n \bmod 2}(\mathrm{sgn}_{k+1}(j))^n & \text{if } j\in\{-N_{k,\mu_0}\}\cup\{N_{k,\mu_0}\},\\ 0 & \text{otherwise.}\end{cases}\tag{4.24}$$

We would also redefine $\widehat{g}^{\varrho_k}_{n+1}$ *by* (4.12).

Proposition 4.13. *The masks* $\widehat{h}_{k+1}$, $\widehat{g}^n_{k+1}$ *for* $n\in\{1,\dots,\varrho_k\}\backslash\{\lambda_0,\mu_0\}$ *and* $\widehat{g}^n_{k+1}$ *for* $n\in\{\lambda_0,\mu_0\}$ *defined by* (4.1), (4.7) *and* (4.23) *or* (4.2), (4.9), (4.24) *and* (4.12) *respectively as in Construction* 4.12 *satisfy* (4.3) *and* (4.4) *and are all real with symmetry or antisymmetry. They generally have smooth decay with overlapping supports that can be controlled. Hence, by the periodic UEP, the affine system* $X_{2\pi}$ *is a tight frame for* $L^2(\mathbb{T})$.

Proof. The condition $L_{k,\lambda_0}<2^{k-1}<N_{k,\mu_0}$ implies that for $m=1,\dots,\lambda_0$ and $n=\mu_0,\dots,\varrho_k$, we have $L_{k,m}\le 2^{k-1}\le N_{k,n}$, i.e. we have the following increasing sequences

$$\{N_{k,m}-2^k,L_{k,m}-2^k,-2^{k-1},-L_{k,m},-N_{k,m},N_{k,m},L_{k,m},2^{k-1},2^k-L_{k,m},2^k-N_{k,m}\}$$
$$\{-L_{k,n},-N_{k,n},-2^{k-1},N_{k,n}-2^k,L_{k,n}-2^k,2^k-L_{k,n},2^k-N_{k,n},2^{k-1},N_{k,n},L_{k,n}\}$$

of integers. The consequence is that for $n=1,\dots,\lambda_0-1$, with $j\in\{-L_{k,n+1},\dots,-N_{k,n}\}\cup\{N_{k,n},\dots,L_{k,n+1}\}$, or equivalently $j-2^k\in\{N_{k,n}-2^k,\dots,L_{k,n+1}-2^k\}$ and $j+2^k\in\{2^k-L_{k,n+1},\dots,2^k-N_{k,n}\}$, we have $\widehat{g}^n_{k+1}(j\pm2^k)=0$. Also, for $n=\mu_0+1,\dots,\varrho_k$, with $j\in\{-L_{k,n+1},\dots,-N_{k,n}\}\cup\{N_{k,n},\dots,L_{k,n+1}\}$, or equivalently $j-2^k\in\{N_{k,n}-2^k,\dots,L_{k,n+1}-2^k\}$ and $j+2^k\in\{2^k-L_{k,n+1},\dots,2^k-N_{k,n}\}$, we have $\widehat{g}^n_{k+1}(j\pm2^k)=0$.

For $j\in\{-L_{k,\mu_0},\dots,-N_{k,\lambda_0}\}\cup\{N_{k,\lambda_0},\dots,L_{k,\mu_0}\}$, or equivalently $j-2^k\in\{N_{k,\lambda_0}-2^k,\dots,L_{k,\mu_0}-2^k\}$ and $j+2^k\in\{2^k-L_{k,\mu_0},\dots,2^k-N_{k,\lambda_0}\}$, we have

$$\overline{\widehat{g}^{\lambda_0}_{k+1}(j)}\widehat{g}^{\lambda_0}_{k+1}(j\pm2^k)+\overline{\widehat{g}^{\mu_0}_{k+1}(j)}\widehat{g}^{\mu_0}_{k+1}(j\pm2^k)=0.\tag{4.25}$$

We have shown that for $n\in\{1,\dots,\lambda_0-1,\mu_0+1,\dots,\varrho_k\}$ such that $\widehat{g}^n_{k+1}(j)\ne0$, we have $\widehat{g}^n_{k+1}(j\pm2^k)=0$, and for $\widehat{g}^{\lambda_0}_{k+1}(j)\ne0$ or $\widehat{g}^{\mu_0}_{k+1}(j)\ne0$, (4.25) must hold. Therefore, (4.4) holds for all $j\in\sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$.

Let $n\in\{1,\dots,\lambda_0-1\}$. If $N_{k,n}=L_{k,n}$, then for $j\in\{-L_{k,n}\}\cup\{L_{k,n}\}$, we have

$$\left|\widehat{g}^n_{k+1}(j)\right|^2+\left|\widehat{g}^{n-1}_{k+1}(j)\right|^2=1+1=2$$

and all $\widehat{g}^{l}_{k+1}(j) = 0$ for $l \notin \{n-1, n\}$. If $N_{k,n} < L_{k,n}$, then for $j \in \{-L_{k,n}, \dots, -N_{k,n}\} \cup \{N_{k,n}, \dots, L_{k,n}\}$, we verify that

$$\left|\widehat{g}^{n}_{k+1}(j)\right|^2 + \left|\widehat{g}^{n-1}_{k+1}(j)\right|^2 = 2\sin^2\frac{\pi}{2}\tilde{\beta}^{n}_{k}\left(\frac{|j|}{N_{k,n}} - 1\right) + 2\cos^2\frac{\pi}{2}\tilde{\beta}^{n}_{k}\left(\frac{|j|}{N_{k,n}} - 1\right) = 2$$

and all $\widehat{g}^{l}_{k+1}(j) = 0$ for $l \notin \{n-1, n\}$. For $j \in \{-N_{k,n+1}+1, \dots, -L_{k,n}-1\} \cup \{L_{k,n}+1, \dots, N_{k,n+1}-1\}$, we have $\left|\widehat{g}^{n}_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^{l}_{k+1}(j) = 0$ for all $l \neq n$. Let $n \in \{\mu_0+1, \dots, \varrho_k - 1\}$. For $j \in \{-N_{k,n+1}+1, \dots, -L_{k,n}-1\} \cup \{L_{k,n}+1, \dots, N_{k,n+1}-1\}$, we have $\left|\widehat{g}^{n}_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^{l}_{k+1}(j) = 0$ for all $l \neq n$. If $N_{k,n+1} < L_{k,n+1}$, then for $j \in \{-L_{k,n+1}, \dots, -N_{k,n+1}\} \cup \{N_{k,n+1}, \dots, L_{k,n+1}\}$, we verify that

$$\left|\widehat{g}^{n}_{k+1}(j)\right|^2 + \left|\widehat{g}^{n+1}_{k+1}(j)\right|^2 = 2\left[\cos^2\frac{\pi}{2}\tilde{\beta}^{n+1}_{k}\left(\frac{|j|}{N_{k,n+1}} - 1\right) + \sin^2\frac{\pi}{2}\tilde{\beta}^{n+1}_{k}\left(\frac{|j|}{N_{k,n+1}} - 1\right)\right] = 2$$

and all $\widehat{g}^{l}_{k+1}(j) = 0$ for $l \notin \{n, n+1\}$. If $N_{k,n+1} = L_{k,n+1}$, then for $j \in \{-N_{k,n+1}\} \cup \{N_{k,n+1}\}$, we have

$$\left|\widehat{g}^{n}_{k+1}(j)\right|^2 + \left|\widehat{g}^{n+1}_{k+1}(j)\right|^2 = 1 + 1 = 2$$

and all $\widehat{g}^{l}_{k+1}(j) = 0$ for $l \notin \{n, n+1\}$. If $N_{k,\lambda_0} < L_{k,\lambda_0}$, then for $j \in \{-L_{k,\lambda_0}, \dots, -N_{k,\lambda_0}\} \cup \{N_{k,\lambda_0}, \dots, L_{k,\lambda_0}\}$, we have

$$\left|\widehat{g}^{\mu_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\lambda_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\lambda_0-1}_{k+1}(j)\right|^2 = 2\left[\sin^2\frac{\pi}{2}\tilde{\beta}^{\lambda_0}_{k}\left(\frac{|j|}{N_{k,\lambda_0}} - 1\right) + \cos^2\frac{\pi}{2}\tilde{\beta}^{\lambda_0}_{k}\left(\frac{|j|}{N_{k,\lambda_0}} - 1\right)\right] = 2$$

and all other $\widehat{g}^{l}_{k+1}(j) = 0$ for $l \notin \{\lambda_0 - 1, \lambda_0, \mu_0\}$. If $N_{k,\lambda_0} = L_{k,\lambda_0}$, then for $j \in \{-L_{k,\lambda_0}\} \cup \{L_{k,\lambda_0}\}$, we have

$$\left|\widehat{g}^{\mu_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\lambda_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\lambda_0-1}_{k+1}(j)\right|^2 = \frac{1}{2} + \frac{1}{2} + 1 = 2$$

and all other $\widehat{g}^{l}_{k+1}(j) = 0$ for $l \notin \{\lambda_0 - 1, \lambda_0, \mu_0\}$. If $N_{k,\mu_0} = L_{k,\mu_0+1}$, then for $j \in \{-N_{k,\mu_0}\} \cup \{N_{k,\mu_0}\}$, we have

$$\sum_{n=\lambda_0}^{\mu_0+1}\left|\widehat{g}^{n}_{k+1}(j)\right|^2 = \frac{1}{2} + \frac{1}{2} + 1 = 2$$

and all other $\widehat{g}^{l}_{k+1}(j) = 0$ for $l \notin \{\lambda_0, \mu_0, \mu_0+1\}$. If $N_{k,\mu_0} < L_{k,\mu_0}$, then for $j \in \{-L_{k,\mu_0}, \dots, -N_{k,\mu_0}\} \cup \{N_{k,\mu_0}, \dots, L_{k,\mu_0}\}$, we have

$$\sum_{n=\lambda_0}^{\mu_0+1}\left|\widehat{g}^{n}_{k+1}(j)\right|^2 = 2\cos^2\frac{\pi}{2}\tilde{\beta}^{\mu_0}_{k}\left(\frac{|j|}{N_{k,\mu_0}} - 1\right) + 2\sin^2\frac{\pi}{2}\tilde{\beta}^{\mu_0}_{k}\left(\frac{|j|}{N_{k,\mu_0}} - 1\right) = 2$$

and all other $\widehat{g}^{l}_{k+1}(j) = 0$ for $l \notin \{\lambda_0, \mu_0, \mu_0+1\}$. For $j \in \{-N_{k,\mu_0}+1, \ldots, -L_{k,\lambda_0}-1\} \cup \{L_{k,\lambda_0}+1, \ldots, N_{k,\mu_0}-1\}$, we have

$$\left|\widehat{g}^{\lambda_0}_{k+1}(j)\right|^2 + \left|\widehat{g}^{\mu_0}_{k+1}(j)\right|^2 = 1 + 1 = 2$$

and all $\widehat{g}^{l}_{k+1}(j) = 0$ for $l \notin \{\lambda_0, \mu_0\}$. For $j \in \{-N_{k,\varrho_k+1}, \ldots, -L_{k,\varrho_k}-1\} \cup \{L_{k,\varrho_k}+1, \ldots, N_{k,\varrho_k+1}\}$, we have $\left|\widehat{g}^{\varrho_k}_{k+1}(j)\right|^2 = 2$ and all $\widehat{g}^{n}_{k+1}(j) = 0$ for $n < \varrho_k$. For $j \in \{-N_{k,1}+1, \ldots, N_{k,1}-1\}$, we have $\left|\widehat{h}_{k+1}(j)\right|^2 = 2$ and $\widehat{g}^{n}_{k+1}(j) = 0$ for $n \in \{1, \ldots, \varrho_k\}$. Since (4.3) holds for all $j \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$, we conclude that it also holds for $j + 2^k \in \sigma_{k+1}(V^{k+1}_{2\pi}(\phi_{k+1}))$. □

Remark. As the reader would have already observed, there can be variations in the constructions by assuming that $N_{k,n} = L_{k,n}$ for only some of the $n \in \{1, \ldots, \varrho_k\}$ subject to the usual constraints of the respective constructions.

4.2 Time-Localized Construction

Time-localized wavelets in $L^2(\mathbb{T})$ are analogous to compactly supported wavelets in $L^2(\mathbb{R})$, i.e. they could be obtained by periodizing compactly supported wavelets in $L^2(\mathbb{R})$. The techniques discussed in this section are used to include modulation information into the wavelet system while preserving the time-localized nature of the wavelets. These techniques are also applicable to the bandlimited case but they are not necessary for the inclusion of modulation information.

Suppose that the periodic affine system $X_{2\pi}$ as defined in (1.15) is a tight frame of real functions for $L^2(\mathbb{T})$ derived from the MRA $\{V^k_{2\pi}(\Phi_k)\}$ such that the minimum energy condition (3.45) holds for each $k \in \mathbb{N}$, i.e.

$$\sum_{m=1}^{\rho} \sum_{l \in \mathcal{L}_{k+1}} \left|\langle f, T^l_{k+1}\phi^m_{k+1}\rangle\right|^2 = \sum_{m=1}^{\rho} \sum_{l \in \mathcal{L}_{k}} \left|\langle f, T^l_{k}\phi^m_{k}\rangle\right|^2 + \sum_{m=1}^{\varrho_k} \sum_{l \in \mathcal{L}_{k}} \left|\langle f, T^l_{k}\psi^m_{k}\rangle\right|^2, \quad f \in L^2(\mathbb{T}).$$

Hence for any $K \geq 0$, the collection of real functions $\{T^l_K\phi_K : \phi_K \in \Phi_K, l \in \mathcal{L}_K,\} \cup \{T^l_k\psi_k : \psi_k \in \Psi_k, l \in \mathcal{L}_k, k \geq K\}$ is also a tight frame for $L^2(\mathbb{T})$. For simplicity, we assume that $\rho = 1$, i.e. for each $k \geq 0$, let $\Phi_k := \phi_k$ and $\Psi_k := \left[\psi^m_k\right]^{\varrho_k}_{m=1}$ be subsets of $L^2(\mathbb{T})$ satisfying (3.28) and (3.31) for some $\widehat{H}_{k+1} = \widehat{h}_{k+1} = \widehat{g}^0_{k+1} \in \mathcal{S}(2^{k+1})$ and $\widehat{G}_{k+1} = \left[\widehat{g}^m_{k+1}\right]^{\varrho_k}_{m=1} \in \mathcal{S}(2^{k+1})^{\varrho_k \times 1}$ respectively. We shall also assume in this section that (3.48) holds.

In our following construction, we shall add modulation features to our wavelet system by enlarging the MRA using "diagonal" extension of the masks.

Construction 4.14. *For $k \geq 0$, define* $\widehat{\widetilde{\Phi}}_k^T := \begin{bmatrix} \widehat{\widetilde{\phi}}_k^{-T} & \widehat{\widetilde{\phi}}_k^{+T} \end{bmatrix}$ *with* $\widehat{\widetilde{\phi}}_k^{-} = \Big[\widehat{\widetilde{\phi}}_k^{\lambda}\Big]_{\lambda=-L+1}^{0}$ *and* $\widehat{\widetilde{\phi}}_k^{+} = \Big[\widehat{\widetilde{\phi}}_k^{\mu}\Big]_{\mu=0}^{L-1}$*, where* $\widehat{\widetilde{\phi}}_k^{\lambda}(j) = \widehat{\phi}_k(-j - c\lambda)$ *and* $\widehat{\widetilde{\phi}}_k^{\mu}(j) = \widehat{\phi}_k(j + c\mu)$*. Similarly, we define* $\widehat{\widetilde{\Psi}}_k^T := \begin{bmatrix} \widehat{\widetilde{\psi}}_k^{-T} & \widehat{\widetilde{\psi}}_k^{+T} \end{bmatrix}$ *with* $\widehat{\widetilde{\psi}}_k^{-} = \Big[\widehat{\widetilde{\psi}}_k^{\lambda}\Big]_{\lambda=-L+1}^{0}$ *and* $\widehat{\widetilde{\psi}}_k^{+} = \Big[\widehat{\widetilde{\psi}}_k^{\mu}\Big]_{\mu=0}^{L-1}$*, where* $\widehat{\widetilde{\psi}}_k^{\lambda}(j) = \Big[\widehat{\psi}_k^m(-j - c\lambda)\Big]_{m=1}^{\varrho_k}$ *and* $\widehat{\widetilde{\psi}}_k^{\mu}(j) = \Big[\widehat{\psi}_k^m(j + c\mu)\Big]_{m=1}^{\varrho_k}$*. For $k \geq 0$, the masks* $\widehat{\widetilde{H}}_{k+1} \in \mathcal{S}(2^{k+1})^{\rho \times \rho}$ *and* $\widehat{\widetilde{G}}_{k+1} \in \mathcal{S}(2^{k+1})^{\widetilde{\varrho}_k \times \rho}$ *where $\rho = 2L$ and $\widetilde{\varrho}_k = 2L\varrho_k$ are defined such that*

$$\widehat{\widetilde{H}}_{k+1} := \begin{bmatrix} \operatorname{diag}\Big[\widehat{\widetilde{h}}_{k+1}^{\lambda}\Big]_{\lambda=-L+1}^{0} & 0 \\ 0 & \operatorname{diag}\Big[\widehat{\widetilde{h}}_{k+1}^{\mu}\Big]_{\mu=0}^{L-1} \end{bmatrix}$$

$$\widehat{\widetilde{G}}_{k+1} := \begin{bmatrix} \operatorname{diag}\Big[\widehat{\widetilde{g}}_{k+1}^{\lambda}\Big]_{\lambda=-L+1}^{0} & 0 \\ 0 & \operatorname{diag}\Big[\widehat{\widetilde{g}}_{k+1}^{\mu}\Big]_{\mu=0}^{L-1} \end{bmatrix} \tag{4.26}$$

with $\widehat{\widetilde{h}}_{k+1}^{\lambda}(j) = \widehat{h}_{k+1}(-j - c\lambda)$, $\widehat{\widetilde{h}}_{k+1}^{\mu}(j) = \widehat{h}_{k+1}(j + c\mu)$, $\widehat{\widetilde{g}}_{k+1}^{\lambda}(j) = \Big[\widehat{g}_{k+1}^m(-j - c\lambda)\Big]_{m=1}^{\varrho_k}$ *and* $\widehat{\widetilde{g}}_{k+1}^{\mu}(j) = \Big[\widehat{g}_{k+1}^m(j + c\mu)\Big]_{m=1}^{\varrho_k}$ *for all $j \in \mathcal{R}_{k+1}$.*

Theorem 4.15. *For each $k \geq 0$, let $\widetilde{\Phi}_k$ and $\widetilde{\Psi}_k$ be constructed from Φ_k and Ψ_k as in Construction* 4.14*. Then* $\widetilde{X}_{2\pi} := \{\widetilde{\phi}_0^{\lambda}, \widetilde{\phi}_0^{\mu} : \phi_0 \in \Phi_0, \lambda = -L+1, \ldots, 0, \mu = 0, \ldots, L-1\} \cup \{T_k^l \widetilde{\psi}_k^{\lambda,m}, T_k^l \widetilde{\psi}_k^{\mu,m} : \psi_k \in \Psi_k, l \in \mathcal{L}_k, \lambda = -L+1, \ldots, 0, \mu = 0, \ldots, L-1, m = 1, \ldots, \varrho_k, k \geq 0\}$ *is a tight frame for $L^2(\mathbb{T})$ derived from the MRA $\{V_{2\pi}^k(\widetilde{\Phi}_k)\}_{k \geq 0}$ using the periodic UEP.*

Proof. The density of $\bigcup_{k \in \mathbb{Z}} V_{2\pi}^k(\Phi_k)$ in $L_2(\mathbb{T})$ implies the density of $\bigcup_{k \in \mathbb{Z}} V_{2\pi}^k(\widetilde{\Phi}_k)$ in $L_2(\mathbb{T})$. The refinability condition (3.28) is shown by verifying that for $k \geq 0$, $\lambda \in \{-L+1, \ldots, 0\}$ and $\mu \in \{0, \ldots, L-1\}$, we have

$$\begin{aligned} \widehat{\widetilde{\phi}}_k^{\lambda}(j) &= \widehat{\phi}_k(-j - c\lambda) = \widehat{h}_{k+1}(-j - c\lambda)\widehat{\phi}_{k+1}(-j - c\lambda) = \widehat{\widetilde{h}}_{k+1}^{\lambda}(j)\widehat{\widetilde{\phi}}_{k+1}^{\lambda}(j), \\ \widehat{\widetilde{\phi}}_k^{\mu}(j) &= \widehat{\phi}_k(j + c\mu) = \widehat{h}_{k+1}(j + c\mu)\widehat{\phi}_{k+1}(j + c\mu) = \widehat{\widetilde{h}}_{k+1}^{\mu}(j)\widehat{\widetilde{\phi}}_{k+1}^{\mu}(j). \end{aligned}$$

Thus $\{V_{2\pi}^k(\widetilde{\Phi}_k)\}_{k \geq 0}$ is an MRA of $L_2(\mathbb{T})$. Clearly, (3.48) holds for the MRA $\{V_{2\pi}^k(\widetilde{\Phi}_k)\}_{k \geq 0}$. We also have for $k \geq 0$, $\lambda \in \{-L+1, \ldots, 0\}$, $\mu \in \{0, \ldots, L-1\}$, and $m \in \{1, \ldots, \varrho_k\}$,

$$\begin{aligned} \widehat{\widetilde{\psi}}_k^{\lambda,m}(j) &= \widehat{\psi}_k^m(-j - c\lambda) = \widehat{g}_{k+1}^m(-j - c\lambda)\widehat{\phi}_{k+1}(-j - c\lambda) = \widehat{\widetilde{g}}_{k+1}^{\lambda,m}(j)\widehat{\widetilde{\phi}}_{k+1}^{\lambda}(j), \\ \widehat{\widetilde{\psi}}_k^{\mu,m}(j) &= \widehat{\psi}_k^m(j + c\mu) = \widehat{g}_{k+1}^m(j + c\mu)\widehat{\phi}_{k+1}(j + c\mu) = \widehat{\widetilde{g}}_{k+1}^{\mu,m}(j)\widehat{\widetilde{\phi}}_{k+1}^{\mu}(j). \end{aligned}$$

Therefore (3.31) holds and $\{T_k^l \widetilde{\psi}_k^{\lambda,m}, T_k^l \widetilde{\psi}_k^{\mu,m} : \psi_k \in \Psi_k, l \in \mathcal{L}_k, \lambda = -L+1, \ldots, 0, \mu = 0, \ldots, L-1, m = 1, \ldots, \varrho_k\}$ is derived from the MRA $\{V_{2\pi}^k(\widetilde{\Phi}_k)\}_{k \geq 0}$. The masks $\widehat{\widetilde{H}}_{k+1}$ and

$\widehat{\widehat{G}}_{k+1}$ satisfy

$$\widehat{\widehat{H}}_{k+1}(j+\nu)^*\widehat{\widehat{H}}_{k+1}(j)+\widehat{\widehat{G}}_{k+1}(j+\nu)^*\widehat{\widehat{G}}_{k+1}(j)=2\delta_{0,\nu}I_\rho,$$

where $\nu\in\{0,2^k\}$, $2\delta_{0,\nu}I_L=\operatorname{diag}\left[\widehat{\widehat{h}}{}^{\lambda}_{k+1}(j+\nu)^*\widehat{\widehat{h}}{}^{\lambda}_{k+1}(j)+\widehat{\widehat{g}}{}^{\lambda}_{k+1}(j+\nu)^*\widehat{\widehat{g}}{}^{\lambda}_{k+1}(j)\right]_{\lambda=-L+1}^{0}$ and $2\delta_{0,\nu}I_L=\operatorname{diag}\left[\widehat{\widehat{h}}{}^{\mu}_{k+1}(j+\nu)^*\widehat{\widehat{h}}{}^{\mu}_{k+1}(j)+\widehat{\widehat{g}}{}^{\mu}_{k+1}(j+\nu)^*\widehat{\widehat{g}}{}^{\mu}_{k+1}(j)\right]_{\mu=0}^{L-1}$. Consequently, (3.43) of Theorem 3.25 holds and hence the conditions of the periodic UEP are satisfied. Thus $\widetilde{X}_{2\pi}$ is a tight frame for $L^2(\mathbb{T})$ derived from the MRA $\{V_{2\pi}^k(\widetilde{\Phi}_k)\}_{k\geq 0}$ using the periodic UEP. □

Construction 4.14 only allows for a fixed and limited range of modulation and requires the expansion in the MRA. We shall remedy this by constructing minimum energy time-localized wavelets $\{\widetilde{\psi}_k^{m,\mu}\}_{m,\mu=1}^{\varrho_k,2L}$ which, like our bandlimited construction, contribute modulation information to the wavelet system and satisfy

$$\sum_{l\in\mathcal{L}_{k+1}}\left|\langle f,T_{k+1}^l\phi_{k+1}\rangle\right|^2=\sum_{\mu=1}^{2L}\sum_{l\in\mathcal{L}_k}\left|\langle f,T_k^l\widetilde{\phi}_k^{\mu}\rangle\right|^2+\sum_{m=1}^{\varrho_k}\sum_{\mu=1}^{2L}\sum_{l\in\mathcal{L}_k}\left|\langle f,T_k^l\widetilde{\psi}_k^{m,\mu}\rangle\right|^2,\quad f\in L^2(\mathbb{T}).$$

First we look at complex constructions, where the additional masks constructed are modulates of the original masks.

Construction 4.16. *For $0\leq k<K$, define $\widehat{\widehat{\Phi}}_k:=L^{\frac{k}{2}}\widehat{\Phi}_k$ and $\widehat{\widehat{\Psi}}_k:=\widehat{\widehat{G}}_{k+1}\widehat{\widehat{\Phi}}_{k+1}$, where the combined MRA mask*

$$\widehat{\widehat{L}}_{k+1}(j):=\begin{bmatrix}\widehat{\widehat{g}}{}^{0,0,0}_{k+1}(j)\\ \widehat{\widehat{G}}_{k+1}(j)\end{bmatrix}\tag{4.27}$$

is a $2(\varrho_k+1)L\times 1$ vector with $\widehat{\widehat{G}}_{k+1}(j):=\begin{bmatrix}\widehat{\widehat{g}}{}^{0,0,1}_{k+1}(j)\\ \left[\widehat{\widehat{g}}{}^{0,\mu}_{k+1}(j)\right]_{\mu=1}^{L-1}\\ \left[\widehat{\widehat{g}}{}^{m}_{k+1}(j)\right]_{m=1}^{\varrho_k}\end{bmatrix}$, $\widehat{\widehat{g}}{}^{m}_{k+1}(j):=\left[\widehat{\widehat{g}}{}^{m,\mu}_{k+1}(j)\right]_{\mu=0}^{L-1}$ *and*

$$\widehat{\widehat{g}}{}^{m,\mu}_{k+1}(j)=\begin{bmatrix}\widehat{\widehat{g}}{}^{m,\mu,0}_{k+1}(j)\\ \widehat{\widehat{g}}{}^{m,\mu,1}_{k+1}(j)\end{bmatrix}=(2L)^{-\frac{1}{2}}\begin{bmatrix}\widehat{g}^{m}_{k+1}(j-C_k\mu)\\ \widehat{g}^{m}_{k+1}(j+C_k\mu)\end{bmatrix},$$

for $m\in\{0,\dots,\varrho_k\}$, $\mu\in\{0,\dots,L-1\}$ and $j\in\mathcal{R}_{k+1}$ and we let $C_kL=2^k$. For $k\geq K$, define $\widehat{\widehat{\Phi}}_k:=L^{\frac{K}{2}}\widehat{\Phi}_k$ and $\widehat{\widehat{\Psi}}_k:=L^{\frac{K}{2}}\widehat{\Psi}_k$ with $\widehat{\widehat{L}}_{k+1}(j):=\widehat{L}_{k+1}(j)$ as the original combined MRA mask.

Theorem 4.17. *For each $k \geq 0$, let $\widetilde{\Phi}_k$ and $\widetilde{\Psi}_k$ be constructed from Φ_k and Ψ_k as in Construction 4.16 with $\widehat{\widetilde{L}}_{k+1}$ as in (4.27) and $\widehat{L}_{k+1} := \left[\widehat{g}^m_{k+1}\right]_{m=0}^{\varrho_k}$ as their respective combined MRA masks. Then $\widetilde{X}_{2\pi} := \{\phi_0\} \cup \{T^l_k \widetilde{\psi}_k : \widetilde{\psi}_k \in \widetilde{\Psi}_k, l \in \mathcal{L}_k, k \geq 0\}$ is a tight frame for $L^2(\mathbb{T})$ derived from the same MRA $\{V^k_{2\pi}(\Phi_k)\}_{k\geq 0}$ using the periodic UEP.*

Proof. By verifying that $\widehat{\widetilde{\Phi}}_k = L^{-\frac{1}{2}} \widehat{g}^0_{k+1} \widehat{\widetilde{\Phi}}_{k+1}$ for $0 \leq k < K$ and $\widehat{\widetilde{\Phi}}_k = \widehat{g}^0_{k+1} \widehat{\widetilde{\Phi}}_{k+1}$ for $k \geq K$, we confirm that $\widetilde{\Phi}_k$ and $\widetilde{\Psi}_k$ of Construction 4.16 satisfy (3.28) and (3.31) respectively for all $k \geq 0$. Clearly, (3.48) holds for the MRA $\{V^k_{2\pi}(\widetilde{\Phi}_k)\}_{k\geq 0}$. The UEP condition $\widehat{\mathbb{L}}_k(j)^* \widehat{\mathbb{L}}_k(j) = 2I_2$ with $\widehat{\mathbb{L}}_k(j)$ as defined in (3.44) is equivalent to

$$\sum_{m=0}^{\varrho_k} \overline{\widehat{g}^m_{k+1}(j)} \widehat{g}^m_{k+1}(j+\nu) = 2\delta_{0,\nu}$$

for $j \in \mathcal{R}_{k+1}$ and $\nu \in \{0, 2^k\}$, which leads to the following condition

$$\sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \overline{\widehat{g}^m_{k+1}(j \pm C_k\mu)} \widehat{g}^m_{k+1}(j \pm C_k\mu + \nu) = 2L\delta_{0,\nu} \tag{4.28}$$

that is independent of the choice of $C_k L$. Using (4.28) for $0 \leq k < K$, we deduce that

$$\begin{aligned}
&\sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left[\left| \widehat{\widetilde{g}}^{m,\mu,0}_{k+1}(j) \right|^2 + \left| \widehat{\widetilde{g}}^{m,\mu,1}_{k+1}(j) \right|^2 \right] \\
&= \frac{1}{2L} \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left[\left| \widehat{g}^m_{k+1}(j - C_k\mu) \right|^2 + \left| \widehat{g}^m_{k+1}(j + C_k\mu) \right|^2 \right] = \frac{1}{2L}(2L + 2L) = 2
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left[\overline{\widehat{\widetilde{g}}^{m,\mu,0}_{k+1}(j)} \widehat{\widetilde{g}}^{m,\mu,0}_{k+1}(j+2^k) + \overline{\widehat{\widetilde{g}}^{m,\mu,1}_{k+1}(j)} \widehat{\widetilde{g}}^{m,\mu,1}_{k+1}(j+2^k) \right] \\
&= \frac{1}{2L} \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left[\overline{\widehat{g}^m_{k+1}(j - C_k\mu)} \widehat{g}^m_{k+1}(j - C_k\mu + 2^k) + \overline{\widehat{g}^m_{k+1}(j + C_k\mu)} \widehat{g}^m_{k+1}(j + C_k\mu + 2^k) \right] \\
&= 0 + 0 = 0,
\end{aligned}$$

and it follows that $\widehat{\widetilde{\mathbb{L}}}_k(j)^* \widehat{\widetilde{\mathbb{L}}}_k(j) = 2I_2$, where $\widehat{\widetilde{\mathbb{L}}}_k(j) = \left[\widehat{\widetilde{L}}_{k+1}(j) \quad \widehat{\widetilde{L}}_{k+1}(j+2^k) \right]$ and $\widehat{\widetilde{L}}_{k+1}$ is defined as in (4.27). Therefore by Theorem 3.27 (periodic UEP), $\widetilde{X}_{2\pi}$ is a tight frame for $L^2(\mathbb{T})$ derived from the MRA $\{V^k_{2\pi}(\Phi_k)\}_{k\geq 0}$. □

We say that a function $f \in L^2(\mathbb{T})$ is symmetric (up to linear phase κ) if

$$\widehat{f}(-j) = \mathrm{e}^{-\mathrm{i}j\cdot\kappa} \widehat{f}(j)$$

for some $\kappa \in \mathbb{T}$. Correspondingly, a periodic sequence $g_k \in \mathcal{S}(2^k)$ is symmetric (up to linear phase κ) if

$$\widehat{g}_k(-j) = \mathrm{e}^{\frac{-\mathrm{i}2\pi j\cdot\kappa}{2^k}}\widehat{g}_k(j)$$

for some $\kappa \in \mathbb{Z}$.

The wavelets in Construction 4.16 are not real and symmetric. It is possible to achieve this requirement if we modify Construction 4.16 slightly.

Construction 4.18. *For $0 \le k < K$, define $\widehat{\widetilde{\Phi}}_k := L^{\frac{k}{2}}\begin{bmatrix}\widehat{\phi}_k(-\cdot) & \widehat{\phi}_k\end{bmatrix}^T$ and $\widehat{\widetilde{\Psi}}_k := \widehat{\widetilde{G}}_{k+1}\widehat{\widetilde{\Phi}}_{k+1}$, where the combined MRA mask*

$$\widehat{\widetilde{L}}_{k+1}(j) := \begin{bmatrix}\widehat{\widetilde{g}}^{0,0}_{k+1}(j)\\ \widehat{\widetilde{G}}_{k+1}(j)\end{bmatrix} \tag{4.29}$$

is a $2(\varrho_k+1)L \times 2$ matrix with $\widehat{\widetilde{G}}_{k+1}(j) := \begin{bmatrix}\begin{bmatrix}\widehat{\widetilde{g}}^{0,\mu}_{k+1}(j)\end{bmatrix}_{\mu=1}^{L-1}\\ \begin{bmatrix}\widehat{\widetilde{g}}^{m}_{k+1}(j)\end{bmatrix}_{m=1}^{\varrho_k}\end{bmatrix}$, $\widehat{\widetilde{g}}^{m}_{k+1}(j) := \begin{bmatrix}\widehat{\widetilde{g}}^{m,\mu}_{k+1}(j)\end{bmatrix}_{\mu=0}^{L-1}$ *and*

$$\widehat{\widetilde{g}}^{m,\mu}_{k+1}(j) := \begin{bmatrix}\widehat{\widetilde{g}}^{m,\mu,0}_{k+1}(j) & 0\\ 0 & \widehat{\widetilde{g}}^{m,\mu,1}_{k+1}(j)\end{bmatrix} := L^{-\frac{1}{2}}\begin{bmatrix}\widehat{g}^{m}_{k+1}(-j-C_k\mu) & 0\\ 0 & \widehat{g}^{m}_{k+1}(j+C_k\mu)\end{bmatrix},$$

for $m \in \{0,\dots,\varrho_k\}$, $\mu \in \{0,\dots,L-1\}$ and $j \in \mathcal{R}_{k+1}$ and we let $C_kL = 2^k$. For $k \ge K$, define $\widehat{\widetilde{\Phi}}_k := L^{\frac{K}{2}}\begin{bmatrix}\widehat{\phi}_k(-\cdot) & \widehat{\phi}_k\end{bmatrix}^T$ and $\widehat{\widetilde{\Psi}}_k := \widehat{\widetilde{G}}_{k+1}\widehat{\widetilde{\Phi}}_{k+1}$ with $\widehat{\widetilde{g}}^{0,0}_{k+1} := \begin{bmatrix}\widehat{g}^0_{k+1}(-\cdot) & 0\\ 0 & \widehat{g}^0_{k+1}\end{bmatrix}$ as the refinement mask and $\widehat{\widetilde{G}}_{k+1} := \begin{bmatrix}\widehat{\widetilde{g}}^{m,0}_{k+1}\end{bmatrix}_{m=1}^{\varrho_k} := \begin{bmatrix}\widehat{g}^m_{k+1}(-\cdot) & 0\\ 0 & \widehat{g}^m_{k+1}\end{bmatrix}_{m=1}^{\varrho_k}$ as the wavelet mask.

Theorem 4.19. *For each $k \ge 0$, let $\widetilde{\Phi}_k$ and $\widetilde{\Psi}_k$ be constructed from Φ_k and Ψ_k as in Construction 4.18 with $\widehat{\widetilde{L}}_{k+1} := \begin{bmatrix}\widehat{\widetilde{g}}^{m}_{k+1}\end{bmatrix}_{m=0}^{\varrho_k}$ and $\widehat{L}_{k+1} := \begin{bmatrix}\widehat{g}^{m}_{k+1}\end{bmatrix}_{m=0}^{\varrho_k}$ as their respective combined MRA masks. Then $\widetilde{X}_{2\pi} := \{\phi_0, \phi_0(-\cdot)\} \cup \{T^l_k\widetilde{\psi}_k : \widetilde{\psi}_k \in \widetilde{\Psi}_k, l \in \mathcal{L}_k, k \ge 0\}$ is a tight frame for $L^2(\mathbb{T})$ derived from the MRA $\{V^k_{2\pi}(\widetilde{\Phi}_k)\}_{k\ge 0}$ using the periodic UEP.*

Proof. By verifying that $\widehat{\widetilde{\Phi}}_k = \widehat{\widetilde{g}}^{0,0}_{k+1}\widehat{\widetilde{\Phi}}_{k+1}$ for $0 \le k < K$ and for $k \ge K$, we confirm that $\widetilde{\Phi}_k$ and $\widetilde{\Psi}_k$ of Construction 4.18 satisfy (3.28) and (3.31) respectively for all $k \ge 0$. Clearly, (3.48) holds for the MRA $\{V^k_{2\pi}(\widetilde{\Phi}_k)\}_{k\ge 0}$. The UEP condition $\widehat{\mathbb{L}}_k(j)^*\widehat{\mathbb{L}}_k(j) = 2I_2$ with $\widehat{\mathbb{L}}_k(j)$ as defined in (3.44) is equivalent to

$$\sum_{m=0}^{\varrho_k}\overline{\widehat{g}^m_{k+1}(\pm j)}\widehat{g}^m_{k+1}(\pm j+\nu) = 2\delta_{0,\nu}$$

for $j \in \mathcal{R}_{k+1}$ and $\nu \in \{0, 2^k\}$, which leads to the following condition

$$\sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \overline{\widehat{g}_{k+1}^{m}(\pm(j + C_k\mu))}\widehat{g}_{k+1}^{m}(\pm(j + C_k\mu) + \nu) = 2L\delta_{0,\nu} \tag{4.30}$$

that is independent of the choice of C_kL. Using (4.30) for $0 \leq k < K$, we deduce that

$$\sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left|\widetilde{\widehat{g}}_{k+1}^{m,\mu,0}(j)\right|^2 = L^{-1} \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left|\widehat{g}_{k+1}^{m}(-j - C_k\mu)\right|^2 = L^{-1}(2L) = 2,$$

$$\sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left|\widetilde{\widehat{g}}_{k+1}^{m,\mu,1}(j)\right|^2 = L^{-1} \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left|\widehat{g}_{k+1}^{m}(j + C_k\mu)\right|^2 = L^{-1}(2L) = 2,$$

and

$$\sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \overline{\widetilde{\widehat{g}}_{k+1}^{m,\mu,0}(j)}\widetilde{\widehat{g}}_{k+1}^{m,\mu,0}(j + 2^k) = L^{-1} \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \overline{\widehat{g}_{k+1}^{m}(-j - C_k\mu)}\widehat{g}_{k+1}^{m}(-j - C_k\mu + 2^k) = 0,$$

$$\sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \overline{\widetilde{\widehat{g}}_{k+1}^{m,\mu,1}(j)}\widetilde{\widehat{g}}_{k+1}^{m,\mu,1}(j + 2^k) = L^{-1} \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \overline{\widehat{g}_{k+1}^{m}(j + C_k\mu)}\widehat{g}_{k+1}^{m}(j + C_k\mu + 2^k) = 0.$$

It follows that $\widehat{\widetilde{\mathbb{L}}}_k(j)^*\widehat{\widetilde{\mathbb{L}}}_k(j) = 2I_4$, where $\widehat{\widetilde{\mathbb{L}}}_k(j) = \begin{bmatrix} \widehat{\widetilde{L}}_{k+1}(j) & \widehat{\widetilde{L}}_{k+1}(j + 2^k) \end{bmatrix}$, since

$$\widehat{\widetilde{L}}_{k+1}(j)^*\widehat{\widetilde{L}}_{k+1}(j) = \begin{bmatrix} \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left|\widetilde{\widehat{g}}_{k+1}^{m,\mu,0}(j)\right|^2 & 0 \\ 0 & \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \left|\widetilde{\widehat{g}}_{k+1}^{m,\mu,1}(j)\right|^2 \end{bmatrix} = 2I_2,$$

$$\widehat{\widetilde{L}}_{k+1}(j)^*\widehat{\widetilde{L}}_{k+1}(j + 2^k) = \begin{bmatrix} \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \overline{\widetilde{\widehat{g}}_{k+1}^{m,\mu,0}(j)}\widetilde{\widehat{g}}_{k+1}^{m,\mu,0}(j + 2^k) & 0 \\ 0 & \sum_{m=0}^{\varrho_k} \sum_{\mu=0}^{L-1} \overline{\widetilde{\widehat{g}}_{k+1}^{m,\mu,1}(j)}\widetilde{\widehat{g}}_{k+1}^{m,\mu,1}(j + 2^k) \end{bmatrix} = 0$$

and in a similar manner, $\widehat{\widetilde{L}}_{k+1}(j + 2^k)^*\widehat{\widetilde{L}}_{k+1}(j) = 0$ and $\widehat{\widetilde{L}}_{k+1}(j + 2^k)^*\widehat{\widetilde{L}}_{k+1}(j + 2^k) = 2I_2$ with $\widehat{\widetilde{L}}_k$ defined as in (4.29). Similarly, for $k \geq K$, we have $\widehat{\widetilde{\mathbb{L}}}_k(j)^*\widehat{\widetilde{\mathbb{L}}}_k(j) = 2I_4$ since

$$\widehat{\widetilde{L}}_{k+1}(j)^*\widehat{\widetilde{L}}_{k+1}(j) = \begin{bmatrix} \sum_{m=0}^{\varrho_k} \left|\widehat{g}_{k+1}^{m}(-j)\right|^2 & 0 \\ 0 & \sum_{m=0}^{\varrho_k} \left|\widehat{g}_{k+1}^{m}(j)\right|^2 \end{bmatrix} = 2I_2,$$

$$\widehat{\widetilde{L}}_{k+1}(j)^*\widehat{\widetilde{L}}_{k+1}(j + 2^k) = \begin{bmatrix} \sum_{m=0}^{\varrho_k} \overline{\widehat{g}_{k+1}^{m}(-j)}\widehat{g}_{k+1}^{m}(-j + 2^k) & 0 \\ 0 & \sum_{m=0}^{\varrho_k} \overline{\widehat{g}_{k+1}^{m}(j)}\widehat{g}_{k+1}^{m}(j + 2^k) \end{bmatrix} = 0$$

and in a similar manner, $\widehat{\widetilde{L}}_{k+1}(j+2^k)^*\widehat{\widetilde{L}}_{k+1}(j)=0$ and $\widehat{\widetilde{L}}_{k+1}(j+2^k)^*\widehat{\widetilde{L}}_{k+1}(j+2^k)=2I_2$. Therefore by Theorem 3.27 (periodic UEP), $\widetilde{X}_{2\pi}$ is a tight frame for $L^2(\mathbb{T})$ derived from the MRA $\{V_{2\pi}^k(\widetilde{\Phi}_k)\}_{k\geq 0}$. $\square$

We shall now symmetrize Construction 4.18 by the same procedure found in Chapter 2. Although this general procedure could be fully developed for the periodic setting, we shall only consider its specific application here.

Construction 4.20. *For each $k \geq 0$, let $\widetilde{\Phi}_k$ and $\widetilde{\Psi}_k$ be constructed from $\Phi_k := \phi_k$ and $\Psi_k := \left[\psi_k^m\right]_{m=1}^{\varrho_k}$ as in Construction 4.18. For $0 \leq k < K$ and $k \geq K$, consider the new combined MRA masks $\widehat{L}'_k(j) := U_{2(\varrho_k+1)L}\widehat{\widetilde{L}}_k(j)U_0^*$ and $\widehat{L}'_k(j) := U_{2(\varrho_k+1)}\widehat{\widetilde{L}}_k(j)U_0^*$ respectively, where $\widehat{\widetilde{L}}_k(j)$ is given as in Construction 4.18 and $U_{2(\varrho_k+1)L} := \operatorname{diag}\left[U_0\right]_{m=1}^{2(\varrho_k+1)L}$ and $U_{2(\varrho_k+1)} := \operatorname{diag}\left[U_0\right]_{m=1}^{2(\varrho_k+1)}$ are $2(\varrho_k+1)L \times 2(\varrho_k+1)L$ and $2(\varrho_k+1)\times 2(\varrho_k+1)$ unitary matrices respectively with $U_0 := \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. Define $\widehat{\Phi}'_k := U_0\widehat{\widetilde{\Phi}}_k$ and $\widehat{\Psi}'_k := U_{2\varrho_k L}\widehat{\widetilde{\Psi}}_k$ for $0 \leq k < K$ and $\widehat{\Phi}'_k := U_0\widehat{\widetilde{\Phi}}_k$ and $\widehat{\Psi}'_k := U_{2\varrho_k}\widehat{\widetilde{\Psi}}_k$ for $k \geq K$.*

Theorem 4.21. *For each $k \geq 0$, let Φ'_k and Ψ'_k be constructed from Φ_k and Ψ_k as in Construction 4.20 with $\widehat{L}'_{k+1}$ and $\widehat{L}_{k+1}$ as their respective combined MRA masks. Then $X'_{2\pi} := \{\phi_0\} \cup \{T_k^l\psi'_k : \psi'_k \in \Psi'_k, l \in \mathcal{L}_k, k \geq 0\}$ is a tight frame with real and symmetric or antisymmetric elements (up to linear phase) for $L^2(\mathbb{T})$ derived from the MRA $\{V_{2\pi}^k(\Phi'_k)\}_{k\geq 0}$ using the periodic UEP.*

Proof. Theorem 4.19 shows that $\widetilde{X}_{2\pi}$ is a tight frame for $L^2(\mathbb{T})$ derived from the MRA $\{V_{2\pi}^k(\widetilde{\Phi}_k)\}_{k\geq 0}$, and $\{V_{2\pi}^k(\Phi'_k)\}_{k\geq 0}$ is the same MRA as $\{V_{2\pi}^k(\widetilde{\Phi}_k)\}_{k\geq 0}$ as $\widehat{\Phi}'_k$ is obtained from a unitary transformation of $\widehat{\widetilde{\Phi}}_k$. By Corollary 3.29, $X'_{2\pi}$ is a tight frame for $L^2(\mathbb{T}^s)$ derived from the MRA $\{V_{2\pi}^k(\Phi'_k)\}_{k\geq 0}$ with combined MRA mask $\widehat{L}'_k$ given as in Construction 4.20. The symmetric and antisymmetric properties of the frame elements is clear from the choice of the unitary matrices. $\square$

Although Constructions 4.16, 4.18 and 4.20 give the flexibility of extending the range of modulation by the wavelet system, it is required that the modulation range be bounded in order for the wavelet system to be a tight frame. We shall remedy this by introducing a slight modification to the wavelet masks based on the idea of splitting the wavelet subbands into "packets" using a different set of masks.

Construction 4.22. *For $0 \le k < K$, define $\widehat{\widehat{\Phi}}_k := \widehat{\Phi}_k$ and $\widehat{\widehat{\Psi}}_k := \widehat{\Psi}_k$ with $\widehat{\widehat{L}}_{k+1} := \widehat{L}_{k+1}$ being the original combined MRA mask. For $k \ge K$, define $\widehat{\widehat{\Phi}}_k := \widehat{\Phi}_k$ and $\widehat{\widehat{\Psi}}_k := \widehat{\widehat{G}}_{k+1}\widehat{\widehat{\Phi}}_{k+1}$, where the combined MRA mask*

$$\widehat{\widehat{L}}_{k+1}(j) := \begin{bmatrix} \widehat{g}^0_{k+1}(j) \\ \widehat{\widehat{G}}_{k+1}(j) \end{bmatrix} \text{ with } \widehat{\widehat{G}}_{k+1}(j) := \left[\widehat{\widehat{g}}^m_{k+1}(j)\right]_{m=1}^{\varrho_k}, \ \widehat{\widehat{g}}^m_{k+1}(j) := \left[\widehat{\widehat{g}}^{m,\mu}_{k+1}(j)\right]_{\mu=0}^{r_k-1},$$

$\widehat{\widehat{g}}^{m,\mu}_{k+1}(j) := \widehat{\alpha}^{m,\mu}_k(j)\widehat{g}^m_{k+1}(j)$, for $m \in \{1, \dots, \varrho_k\}$ and $\mu \in \{0, \dots, r_k - 1\}$ and $j \in \mathcal{R}_{k+1}$ with $\widehat{\alpha}^{m,\mu}_k \in \mathcal{S}(2^k)$ and $\sum_{\mu=0}^{r_k-1} |\widehat{\alpha}^{m,\mu}_k(\nu)|^2 = 1$ for all $\nu \in \mathcal{R}_k$.

Theorem 4.23. *For each $k \ge 0$, let $\widetilde{\Phi}_k$ and $\widetilde{\Psi}_k$ be constructed from Φ_k and Ψ_k as in Construction 4.22 with $\widehat{\widehat{L}}_{k+1}$ and $\widehat{L}_{k+1}$ as their respective combined MRA masks. Then $\widetilde{X}_{2\pi} := \{\phi_0\} \cup \{T^l_k \widetilde{\psi}_k : \widetilde{\psi}_k \in \widetilde{\Psi}_k, l \in \mathcal{L}_k, k \ge 0\}$ is a tight frame for $L^2(\mathbb{T})$ derived from the same MRA $\{V^k_{2\pi}(\Phi_k)\}_{k\ge0}$ using the periodic UEP.*

Proof. It is clear that $\widetilde{\Phi}_k$ and $\widetilde{\Psi}_k$ of Construction 4.22 satisfy (3.28) and (3.31) respectively for all $k \ge 0$ and (3.48) holds for the MRA $\{V^k_{2\pi}(\widetilde{\Phi}_k)\}_{k\ge0}$. Next, since for $m \in \{1, \dots, \varrho_k\}$,

$$\sum_{\mu=0}^{r_k-1} \left|\widehat{g}^{m,\mu}_{k+1}(j)\right|^2 = \sum_{\mu=0}^{r_k-1} \left|\widehat{\alpha}^{m,\mu}_k(j)\widehat{g}^m_{k+1}(j)\right|^2 = \left|\widehat{g}^m_{k+1}(j)\right|^2,$$

$$\sum_{\mu=0}^{r_k-1} \overline{\widehat{g}^{m,\mu}_{k+1}(j)}\widehat{g}^{m,\mu}_{k+1}(j+2^k) = \sum_{\mu=0}^{r_k-1} \overline{\widehat{g}^m_{k+1}(j)\widehat{\alpha}^{m,\mu}_k(j)}\widehat{\alpha}^{m,\mu}_k(j+2^k)\widehat{g}^m_{k+1}(j+2^k)$$

$$= \overline{\widehat{g}^m_{k+1}(j)}\widehat{g}^m_{k+1}(j+2^k) \sum_{\mu=0}^{r_k-1} |\widehat{\alpha}^{m,\mu}_k(j)|^2 = \overline{\widehat{g}^m_{k+1}(j)}\widehat{g}^m_{k+1}(j+2^k),$$

therefore $\sum_{m=0}^{\varrho_k} \widehat{\widehat{g}}^m_{k+1}(j)^* \widehat{\widehat{g}}^m_{k+1}(j+\nu) = 2\delta_{0,\nu}$, where $\widehat{\widehat{g}}^0_{k+1} = \widehat{g}^0_{k+1}$ and it follows that (3.43) holds. Therefore, by Theorem 3.27, $\widetilde{X}_{2\pi}$ is a tight frame for $L^2(\mathbb{T})$ derived from the same MRA $\{V^k_{2\pi}(\Phi_k)\}_{k\ge0}$ using the periodic UEP. □

In the event that the wavelets of the packetized system $\widetilde{X}_{2\pi}$ of Theorem 4.23 do not possess properties of symmetry or antisymmetry, we could symmetrize $\widetilde{X}_{2\pi}$ by the procedure found in Chapter 2 using the application of Corollary 3.29 and unitary transformations. We shall now see as follows that orthogonal wavelet packet representation using conjugate mirror filters as described in [38] and [46] is a special case of Construction 4.22.

Theorem 4.24. *For $k \ge 0$, let $\{T^l_k\theta_k : l \in \mathcal{L}_k\}$ be an orthonormal basis of a space $S_k \subset L^2(\mathbb{T})$ and $\widehat{h}_k := \widehat{g}^0_k \in \mathcal{S}(2^k)$ and $\widehat{g}_k := \mathrm{e}^{-\frac{i2\pi\cdot}{2^k}}\overline{\widehat{h}_k(\cdot + 2^{k-1})} := \widehat{g}^1_k \in \mathcal{S}(2^k)$ satisfy*

(3.43). *For* $m \in \{0,1\}$*, define* $\widehat{\theta}^m_{k-1} := \widehat{g}^m_k \widehat{\theta}_k$*. The family* $\{T^l_{k-1}\theta^m_{k-1} : m = 0,1, l \in \mathcal{L}_{k-1}\}$ *is an orthonormal basis of* S_k*.*

Proof. By Theorem 3.13, the Gramian, as given in (3.34), $\mathrm{M}_k(j) := \left\|\widehat{\theta}_{k,j}\right\|^2_{l^2(\mathbb{Z})} = 2^{-k}$ for every $j \in \mathcal{R}_k$. Similarly, the two families $\{T^l_{k-1}\theta^m_{k-1} : l \in \mathcal{L}_{k-1}\}$ for $m \in \{0,1\}$ are orthonormal bases if $\left\|\widehat{\theta}^0_{k-1,j}\right\|^2_{l^2(\mathbb{Z})} = \left\|\widehat{\theta}^1_{k-1,j}\right\|^2_{l^2(\mathbb{Z})} = 2^{-k+1}$ for every $j \in \mathcal{R}_{k-1}$ and they yield orthogonal spaces if $\langle \widehat{\theta}^0_{k-1,j}, \widehat{\theta}^1_{k-1,j}\rangle_{l^2(\mathbb{Z})} = 0$ for every $j \in \mathcal{R}_{k-1}$. The former could be seen easily since for $m \in \{0,1\}$,

$$\begin{aligned}
&\left\|\widehat{\theta}^m_{k-1,j}\right\|^2_{l^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \left|\widehat{\theta}^m_{k-1}(j+2^{k-1}n)\right|^2 = \sum_{n\in\mathbb{Z}} \left|\widehat{g}^m_k(j+2^{k-1}n)\right|^2 \left|\widehat{\theta}^m_k(j+2^{k-1}n)\right|^2 \\
&= \sum_{n\in\mathbb{Z}} \left|\widehat{g}^m_k(j+2^k n)\right|^2 \left|\widehat{\theta}^m_k(j+2^k n)\right|^2 + \sum_{n\in\mathbb{Z}} \left|\widehat{g}^m_k(j+2^{k-1}+2^k n)\right|^2 \left|\widehat{\theta}^m_k(j+2^{k-1}+2^k n)\right|^2 \\
&= \left|\widehat{g}^m_k(j)\right|^2 (2^{-k}) + \left|\widehat{g}^m_k(j+2^{k-1})\right|^2 (2^{-k}) = 2(2^{-k}) = 2^{-k+1}.
\end{aligned}$$

Similarly, the latter could be shown by

$$\begin{aligned}
\langle \widehat{\theta}^0_{k-1,j}, \widehat{\theta}^1_{k-1,j}\rangle_{l^2(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} \widehat{\theta}^0_{k-1}(j+2^{k-1}n)\overline{\widehat{\theta}^1_{k-1}(j+2^{k-1}n)} \\
&= \sum_{n\in\mathbb{Z}} \widehat{g}^0_k(j+2^{k-1}n)\widehat{\theta}_k(j+2^{k-1}n)\overline{\widehat{g}^1_k(j+2^{k-1}n)}\overline{\widehat{\theta}_k(j+2^{k-1}n)} \\
&= \sum_{n\in\mathbb{Z}} \widehat{g}^0_k(j+2^k n)\overline{\widehat{g}^1_k(j+2^k n)} \left|\widehat{\theta}_k(j+2^k n)\right|^2 \\
&+ \sum_{n\in\mathbb{Z}} \widehat{g}^0_k(j+2^{k-1}+2^k n)\overline{\widehat{g}^1_k(j+2^{k-1}+2^k n)} \left|\widehat{\theta}_k(j+2^{k-1}+2^k n)\right|^2 \\
&= \widehat{g}^0_k(j)\overline{\widehat{g}^1_k(j)}(2^{-k}) + \widehat{g}^0_k(j+2^{k-1})\overline{\widehat{g}^1_k(j+2^{k-1})}(2^{-k}) = 0.
\end{aligned}$$

Finally, the two families span S_k since

$$\sum_{l\in\mathcal{L}_k} s_k(l)T^l_k\theta_k = \sum_{l\in\mathcal{L}_{k-1}} s^0_{k-1}(l)T^l_{k-1}\theta^0_{k-1} + \sum_{l\in\mathcal{L}_{k-1}} s^1_{k-1}(l)T^l_{k-1}\theta^1_{k-1},$$

where $2\widehat{s}^m_{k-1}(j) = \left[\widehat{s}_k(j)\widehat{g}^m_k(j) + \widehat{s}_k(j+2^{k-1})\widehat{g}^m_k(j+2^{k-1})\right]$. The coefficients are computed according to the decomposition and reconstruction algorithms given in Chapter 5. □

Corollary 4.25. *For* $k \geq 0$*, let* $\{T^l_k\theta_k : l \in \mathcal{L}_k\}$ *be an orthonormal basis of a space* $S_k \subset L^2(\mathbb{T})$ *and* $\widehat{h}_k := \widehat{g}^0_k \in \mathcal{S}(2^k)$ *and* $\widehat{g}_k := \mathrm{e}^{-\frac{\mathrm{i}2\pi\cdot}{2^k}}\overline{\widehat{h}_k(\cdot+2^{k-1})} := \widehat{g}^1_k \in \mathcal{S}(2^k)$ *satisfy* (3.43)*. For* $i \leq k$ *and* $\boldsymbol{\epsilon} \in \{0,\dots,2^i-1\}$*, define* $\widehat{\theta}^{\boldsymbol{\epsilon}}_{k-i} := \widehat{g}^{\epsilon_{i-1}}_{k-i+1}\cdots\widehat{g}^{\epsilon_0}_k\widehat{\theta}_k$*, where the binary representation of* $\boldsymbol{\epsilon}$ *is* $\epsilon_0\cdots\epsilon_{i-1}$*. Then the family* $\{T^l_{k-i}\theta^{\boldsymbol{\epsilon}}_{k-i} : \boldsymbol{\epsilon} = 0,\dots,2^i-1, l \in \mathcal{L}_{k-i}\}$ *is an orthonormal basis of* S_k *that satisfies* $2^{-i}\sum_{\boldsymbol{\epsilon}=0}^{2^i-1}\left|\widehat{g}^{\epsilon_{i-1}}_{k-i+1}\right|^2\cdots\left|\widehat{g}^{\epsilon_0}_k\right|^2 = 1$*.*

Proof. We are only required to verify that $2^{-i}\sum_{\epsilon=0}^{2^i-1}\left|\widehat{g}_{k-i+1}^{\epsilon_{i-1}}\right|^2\cdots\left|\widehat{g}_k^{\epsilon_0}\right|^2=1$ since the rest of the proof is a consequence of Theorem 4.24 applied iteratively. We shall obtain our proof by induction. Clearly, $\left|\widehat{\theta}_{k-1}^0(j)\right|^2+\left|\widehat{\theta}_{k-1}^1(j)\right|^2=2\left|\widehat{\theta}_k(j)\right|^2$ since $2^{-1}\left[\left|\widehat{g}_k^0(j)\right|^2+\left|\widehat{g}_k^1(j)\right|^2\right]=1$ for every $j\in\mathcal{R}_k$. Without loss of generality, let us assume that $\sum_{\epsilon=0}^{2^{i-1}-1}\left|\widehat{\theta}_{k-i+1}^{\epsilon}(j)\right|^2=2^{i-1}\left|\widehat{\theta}_k(j)\right|^2$. Next, we check that

$$\sum_{\epsilon=0}^{2^i-1}\left|\widehat{\theta}_{k-i}^{\epsilon}(j)\right|^2=\left[\left|\widehat{g}_{k-i+1}^0(j)\right|^2+\left|\widehat{g}_{k-i+1}^1(j)\right|^2\right]\sum_{\epsilon=0}^{2^{i-1}-1}\left|\widehat{\theta}_{k-i+1}^{\epsilon}(j)\right|^2=2(2^{i-1})\left|\widehat{\theta}_k(j)\right|^2$$

and this confirms our result. □

Although Construction 4.22 may not be as flexible as our bandlimited constructions, it is certainly more flexible than orthogonal wavelet packet representations typified by that of Corollary 4.25 since there are no special constraints on the packet filters other than the requirement that the energy of the packet masks must satisfy a sum of unit norm. This means that the packet filters of Construction 4.22 could be chosen to be either time-localized or bandlimited and the representation is computationally efficient as a desired representation of a signal could be obtained almost directly without going through the iterative process of applying orthogonal wavelet packets. Furthermore, since the refinable function and hence the MRA remains unchanged, the frame approximation order is preserved and at the same time, finer partitioning in the frequency domain could be obtained by modifying the number of wavelet masks adaptively.

The linear phase preserving time-limited $L^2(\mathbb{R})$ constructions of [7] and [16] using the UEP typically involves a symmetric refinable function and three wavelets with the first wavelet being an orthogonal flip of the refinable function. We could visualize their time-frequency plot by comparing with their analogous bandlimited counterpart after considering where they are localized in the frequency domain. The refinable function is mainly localized in a subset of $[-\frac{\pi}{2},\frac{\pi}{2}]$ and the first wavelet mainly occupies a subset of $[-\pi,\frac{\pi}{2}]\cup[\frac{\pi}{2},\pi]$. The frequency localization of the remaining wavelets varies with constructions. For illustration purposes, let us just assume that they are basically localized in the middle bands, i.e. $[-\frac{\pi}{2}-\omega_0,-\frac{\pi}{2}+\omega_0]\cup[\frac{\pi}{2}-\omega_0,\frac{\pi}{2}+\omega_0]$ with $|\omega_0|<\frac{\pi}{2}$. For a reasonable construction of the bandlimited analogue of the lowpass filter, we shall utilize the restriction that the refinable mask is localized in $[0,\omega_0]$.

Construction 4.26. *Let $\widehat{h}_{k+1}$ be given as in Construction* 4.1*, i.e.*

$$\widehat{h}_{k+1}(j) = \begin{cases} \sqrt{2} & \text{if } j \in \{-N_k, \ldots, N_k\}, \\ \sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j|}{N_k} - 1\right)\right] & \text{if } j \in \{-L_k, \ldots, -N_k - 1\} \cup \{N_k + 1, \ldots, L_k\}, \\ 0 & \text{otherwise,} \end{cases}$$

with $\tilde{\beta}_k = \beta\left(\frac{N_k \cdot}{L_k - N_k}\right)$, $N_k = \left\lfloor \frac{\omega_0}{\pi} \right\rfloor \times 2^k \le 2^{k-1} \le L_k \le 2^k - N_k$ to reflect the time-limited nature of the original filters. For $j \in \mathcal{R}_{k+1}$, let $\widehat{g}^1_{k+1}(j) = \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}} \overline{\widehat{h}_{k+1}(j + 2^k)}$, i.e.

$$\widehat{g}^1_{k+1}(j) = \begin{cases} \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\sqrt{2} & \text{if } j \in \{-2^k, \ldots, N_k - 2^k\}, \\ \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j+2^k|}{N_k} - 1\right)\right] & \text{if } j \in \{N_k + 1 - 2^k, \ldots, L_k - 2^k\}, \\ \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j-2^k|}{N_k} - 1\right)\right] & \text{if } j \in \{2^k - L_k, \ldots, 2^k - N_k - 1\}, \\ \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}}\sqrt{2} & \text{if } j \in \{2^k - N_k, \ldots, 2^k\}, \\ 0 & \text{otherwise,} \end{cases}$$

and $4\,|A_{k+1}(j)|^2 = 2 - \left|\widehat{h}_{k+1}(j)\right|^2 - \left|\widehat{h}_{k+1}(j + 2^k)\right|^2$, i.e.

$$2\,|A_{k+1}(j)|^2 = \begin{cases} \sin^2\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j+2^k|}{N_k} - 1\right) & \text{if } j \in \{N_k + 1 - 2^k, \ldots, -L_k\}, \\ \sin^2\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j+2^k|}{N_k} - 1\right) - \cos^2\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j|}{N_k} - 1\right) & \text{if } j \in \{1 - L_k, \ldots, L_k - 2^k - 1\}, \\ \sin^2\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j|}{N_k} - 1\right) & \text{if } \begin{array}{l} j \in \{L_k - 2^k, \ldots, -N_k\} \\ \cup \{N_k, \ldots, 2^k - L_k\}, \end{array} \\ \sin^2\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j-2^k|}{N_k} - 1\right) - \cos^2\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j|}{N_k} - 1\right) & \text{if } j \in \{2^k - L_k + 1, \ldots, L_k - 1\}, \\ \sin^2\frac{\pi}{2}\tilde{\beta}_k\left(\frac{|j-2^k|}{N_k} - 1\right) & \text{if } j \in \{L_k, \ldots, 2^k - N_k - 1\}, \\ 0 & \text{otherwise,} \end{cases}$$

with $A_{k+1}(j) = \sum_{r=-2^k}^{2^k-1} a_r \mathrm{e}^{-\frac{\mathrm{i}2\pi jr}{2^k}}$, $A_{k+1}(j + 2^k) = A_{k+1}(j)$ and all the computed a_r being real, and let $\widehat{g}^2_{k+1}(j) = A_{k+1}(j) + \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}} A_{k+1}(-j)$ and $\widehat{g}^3_{k+1}(j) = \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}} A_{k+1}(-j) - A_{k+1}(j)$.

The standard orthogonal flip ensures that $\overline{\widehat{g}^1_{k+1}(j)}\widehat{g}^1_{k+1}(j + 2^k) = -\widehat{h}_{k+1}(j + 2^k)\overline{\widehat{h}_{k+1}(j)}$. We remark that both $\widehat{h}_{k+1}$ and $\widehat{g}^1_{k+1}$ could be modified by the fundamental mask $\widehat{\Theta}_k$ of the periodic OEP (Corollary 3.30) acting as a Fourier multiplier with the corresponding $|A_{k+1}|^2$ modified appropriately without changing their localizations. Observe that if $N_k \ge 2^{k-1}$, we automatically have $|A_{k+1}|^2 \le 0$ and there is no need to construct $\widehat{g}^2_{k+1}$ and

$\widehat{g}^3_{k+1}$ as above. Both $\widehat{g}^2_{k+1}$ and $\widehat{g}^3_{k+1}$ are symmetric and antisymmetric respectively up to linear phase since $\widehat{g}^2_{k+1}(-j) = A_{k+1}(-j) + \mathrm{e}^{\frac{\mathrm{i}2\pi j}{2^{k+1}}} A_{k+1}(j) = \mathrm{e}^{\frac{\mathrm{i}2\pi j}{2^{k+1}}} \widehat{g}^2_{k+1}(j)$ and $\widehat{g}^3_{k+1}(-j) = \mathrm{e}^{\frac{\mathrm{i}2\pi j}{2^{k+1}}} A_{k+1}(j) - A_{k+1}(-j) = -\mathrm{e}^{\frac{\mathrm{i}2\pi j}{2^{k+1}}} \widehat{g}^3_{k+1}(j)$. Furthermore, $\widehat{g}^3_{k+1}(j) = \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}} \overline{\widehat{g}^2_{k+1}(j+2^k)}$ with $A_{k+1}(-j) = \overline{A_{k+1}(j)}$, which implies that $\overline{\widehat{g}^3_{k+1}(j)}\widehat{g}^3_{k+1}(j+2^k) = -\widehat{g}^2_{k+1}(j+2^k)\overline{\widehat{g}^2_{k+1}(j)}$. We observe that for $j \in \mathcal{R}_{k+1}$,

$$\begin{aligned}
&\left|\widehat{g}^2_{k+1}(j)\right|^2 + \left|\widehat{g}^3_{k+1}(j)\right|^2 = \left|A_{k+1}(j) + \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}} A_{k+1}(-j)\right|^2 + \left|A_{k+1}(j) - \mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}} A_{k+1}(-j)\right|^2 \\
&= |A_{k+1}(j)|^2 + |A_{k+1}(-j)|^2 + A_{k+1}(j)\overline{A_{k+1}(-j)}\mathrm{e}^{\frac{\mathrm{i}2\pi j}{2^{k+1}}} + A_{k+1}(-j)\overline{A_{k+1}(j)}\mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}} \\
&+ |A_{k+1}(j)|^2 + |A_{k+1}(-j)|^2 - A_{k+1}(j)\overline{A_{k+1}(-j)}\mathrm{e}^{\frac{\mathrm{i}2\pi j}{2^{k+1}}} - A_{k+1}(-j)\overline{A_{k+1}(j)}\mathrm{e}^{-\frac{\mathrm{i}2\pi j}{2^{k+1}}} \\
&= 4\,|A_{k+1}(j)|^2,
\end{aligned}$$

which leads to

$$\left|\widehat{h}_{k+1}(j)\right|^2 + \left|\widehat{g}^1_{k+1}(j)\right|^2 + \left|\widehat{g}^2_{k+1}(j)\right|^2 + \left|\widehat{g}^3_{k+1}(j)\right|^2 = 2.$$

Therefore, (3.43) is satisfied and Construction 4.26 provides the masks of a tight frame.

We shall utilize Construction 4.26 to typify a construction of time-localized masks and we shall modify the Haar refinable mask and wavelet mask according to Construction 4.16 for use as packet masks to illustrate an actual implementation of packet filters in Construction 4.22.

Example 4.2.1. Let $\widehat{h}_{k+1} := \widehat{g}^0_{k+1}$, $\widehat{g}^m_{k+1}$ for $m \in \{1,2,3\}$ be given as in Construction 4.26 with $N_k = 2^{k-2}$ and $L_k = 3 \cdot 2^{k-2}$ and let $C_k L = 2^{k-1}$. For $\mu = 0, \ldots, L-1$, let the modulated Haar refinable and wavelet masks be given as

$$\widehat{\alpha}^{\mu+}_k(j) = (4L)^{-\frac{1}{2}}\left[1 + \mathrm{e}^{-\mathrm{i}2\pi 2^{-k}(j+C_k\mu)}\right], \quad \widehat{\alpha}^{\mu-}_k(j) = (4L)^{-\frac{1}{2}}\left[1 - \mathrm{e}^{-\mathrm{i}2\pi 2^{-k}(j+C_k\mu)}\right]$$

respectively. We could easily verify that

$$\begin{aligned}
&\sum_{\mu=0}^{L-1}\left[\left|\widehat{\alpha}^{\mu+}_k(j)\right|^2 + \left|\widehat{\alpha}^{\mu-}_k(j)\right|^2\right] = \sum_{\mu=0}^{L-1}(4L)^{-1}\left\{\left[1 + \mathrm{e}^{-\mathrm{i}2\pi 2^{-k}(j+C_k\mu)}\right]\left[1 + \mathrm{e}^{\mathrm{i}2\pi 2^{-k}(j+C_k\mu)}\right]\right. \\
&\left. + \left[1 - \mathrm{e}^{-\mathrm{i}2\pi 2^{-k}(j+C_k\mu)}\right]\left[1 - \mathrm{e}^{\mathrm{i}2\pi 2^{-k}(j+C_k\mu)}\right]\right\} \\
&= \sum_{\mu=0}^{L-1}(4L)^{-1}\left[2 + 2\cos 2\pi 2^{-k}(j+C_k\mu) + 2 - 2\cos 2\pi 2^{-k}(j+C_k\mu)\right] \\
&= \sum_{\mu=0}^{L-1}(4L)^{-1}(4) = 1,
\end{aligned}$$

which shows that $\{\widehat{\alpha}^{\mu+}_k, \widehat{\alpha}^{\mu-}_k : \mu = 0, \ldots, L-1\}$ satisfies the criteria for utilization as packet masks for $\{\widehat{g}^m_{k+1} : m = 1, \ldots, 3\}$ as in Construction 4.22.

Since $\widehat{g}^1_{k+1}$ is "supported" on $\{-2^k, \ldots, L_k - 2^k\} \cup \{2^k - L_k, \ldots, 2^k\}$ and A_{k+1} is "supported" on $\{N_k - 2^k, \ldots, -N_k\} \cup \{N_k, \ldots, 2^k - N_k\}$, we determine the bandwidths of the original masks g^m_{k+1} for $m \in \{1, 2, 3\}$ to be the following:

$$\text{bandwidth of}\, \widehat{g}^1_{k+1} \,\text{is localized on}\, \{-2^k, \ldots, -2^{k-2}\} \cup \{2^{k-2}, \ldots, 2^k\},$$
$$\text{bandwidth of}\, \widehat{g}^2_{k+1} \,\text{and}\, \widehat{g}^3_{k+1} \,\text{is localized on}\, \{-3 \cdot 2^{k-2}, \ldots, -2^{k-2}\} \cup \{2^{k-2}, \ldots, 3 \cdot 2^{k-2}\}.$$

Next, we determine the bandwidths of the packetized masks $\{\widehat{\alpha}^{\mu+}_k \widehat{g}^m_{k+1}, \widehat{\alpha}^{\mu-}_k \widehat{g}^m_{k+1} : \mu = 0, \ldots, L-1, m = 1, 2, 3\}$. Without loss of generality, we assume that $\widehat{\alpha}^{0+}_k$ and $\widehat{\alpha}^{0-}_k$ are essentially localized on $\{-2^{k-2}, \ldots, 2^{k-2}\}$ and $\{-2^{k-1}, \ldots, -2^{k-2}\} \cup \{2^{k-2}, \ldots, 2^{k-1}\}$ respectively since they are conjugate mirror masks.

For $C_k\mu \leq 2^{k-2}$ and $m \in \{2, 3\}$,

$$\text{bandwidth of}\, \widehat{\alpha}^{\mu+}_k \,\text{is localized on} \begin{cases} \{-2^k, \ldots, -3 \cdot 2^{k-2} - C_k\mu\}, \\ \{-2^{k-2} - C_k\mu, \ldots, 2^{k-2} - C_k\mu\}, \\ \{3 \cdot 2^{k-2} - C_k\mu, \ldots, 2^k\}, \end{cases}$$
$$\text{bandwidth of}\, \widehat{\alpha}^{\mu-}_k \,\text{is localized on} \begin{cases} \{-3 \cdot 2^{k-2} - C_k\mu, \ldots, -2^{k-2} - C_k\mu\}, \\ \{2^{k-2} - C_k\mu, \ldots, 3 \cdot 2^{k-2} - C_k\mu\}, \end{cases}$$

$$\text{bandwidth of}\, \widehat{\alpha}^{\mu+}_k \widehat{g}^1_{k+1} \,\text{is localized on} \begin{cases} \{-2^k, \ldots, -3 \cdot 2^{k-2} - C_k\mu\}, \\ \{-2^{k-2} - C_k\mu, \ldots, -2^{k-2}\}, \\ \{3 \cdot 2^{k-2} - C_k\mu, \ldots, 2^k\}, \end{cases}$$
$$\text{bandwidth of}\, \widehat{\alpha}^{\mu-}_k \widehat{g}^1_{k+1} \,\text{is localized on} \begin{cases} \{-3 \cdot 2^{k-2} - C_k\mu, \ldots, -2^{k-2} - C_k\mu\}, \\ \{2^{k-2}, \ldots, 3 \cdot 2^{k-2} - C_k\mu\}, \end{cases}$$

$$\text{bandwidth of}\, \widehat{\alpha}^{\mu+}_k \widehat{g}^m_{k+1} \,\text{is localized on} \begin{cases} \{-2^{k-2} - C_k\mu, \ldots, -2^{k-2}\}, \\ \{3 \cdot 2^{k-2} - C_k\mu, \ldots, 3 \cdot 2^{k-2}\}, \end{cases}$$
$$\text{bandwidth of}\, \widehat{\alpha}^{\mu-}_k \widehat{g}^m_{k+1} \,\text{is localized on} \begin{cases} \{-3 \cdot 2^{k-2}, \ldots, -2^{k-2} - C_k\mu\}, \\ \{2^{k-2}, \ldots, 3 \cdot 2^{k-2} - C_k\mu\}. \end{cases}$$

For $2^{k-2} \leq C_k\mu \leq 2^{k-1}$ and $m \in \{2, 3\}$,

$$\text{bandwidth of}\, \widehat{\alpha}^{\mu+}_k \,\text{is localized on} \begin{cases} \{-2^{k-2} - C_k\mu, \ldots, 2^{k-2} - C_k\mu\}, \\ \{3 \cdot 2^{k-2} - C_k\mu, \ldots, 5 \cdot 2^{k-2} - C_k\mu\}, \end{cases}$$
$$\text{bandwidth of}\, \widehat{\alpha}^{\mu-}_k \,\text{is localized on} \begin{cases} \{-2^k, \ldots, -2^{k-2} - C_k\mu\}, \\ \{2^{k-2} - C_k\mu, \ldots, 3 \cdot 2^{k-2} - C_k\mu\}, \\ \{5 \cdot 2^{k-2} - C_k\mu, \ldots, 2^k\}, \end{cases}$$

$$\text{bandwidth of } \widehat{\alpha}_k^{\mu+}\widehat{g}_{k+1}^1 \text{ is localized on } \begin{cases} \{-2^{k-2} - C_k\mu, \ldots, -2^{k-2}\}, \\ \{3 \cdot 2^{k-2} - C_k\mu, \ldots, 5 \cdot 2^{k-2} - C_k\mu\}, \end{cases}$$

$$\text{bandwidth of } \widehat{\alpha}_k^{\mu-}\widehat{g}_{k+1}^1 \text{ is localized on } \begin{cases} \{-2^{k}, \ldots, -2^{k-2} - C_k\mu\}, \\ \{2^{k-2}, \ldots, 3 \cdot 2^{k-2} - C_k\mu\}, \\ \{5 \cdot 2^{k-2} - C_k\mu, \ldots, 2^k\}, \end{cases}$$

$$\text{bandwidth of } \widehat{\alpha}_k^{\mu+}\widehat{g}_{k+1}^m \text{ is localized on } \begin{cases} \{-2^{k-2} - C_k\mu, \ldots, -2^{k-2}\}, \\ \{3 \cdot 2^{k-2} - C_k\mu, \ldots, 3 \cdot 2^{k-2}\}, \end{cases}$$

$$\text{bandwidth of } \widehat{\alpha}_k^{\mu-}\widehat{g}_{k+1}^m \text{ is localized on } \begin{cases} \{-3 \cdot 2^{k-2}, \ldots, -2^{k-2} - C_k\mu\}, \\ \{2^{k-2}, \ldots, 3 \cdot 2^{k-2} - C_k\mu\}. \end{cases}$$

Remark. Since the combined bandwidths of $\widehat{\alpha}_k^{\mu+}$ and $\widehat{\alpha}_k^{\mu-}$ is from $\{-2^k, \ldots, 2^k\}$, i.e. $\mathcal{R}_{k+1}$, this means that the combined bandwidths of $\widehat{\alpha}_k^{\mu+}\widehat{g}_{k+1}^m$ and $\widehat{\alpha}_k^{\mu-}\widehat{g}_{k+1}^m$ for $m \in \{1, 2, 3\}$ must be the bandwidth of the respective $\widehat{g}_{k+1}^m$. Therefore, although the input signal is now processed differently by $\widehat{\alpha}_k^{\mu+}\widehat{g}_{k+1}^m$ and $\widehat{\alpha}_k^{\mu-}\widehat{g}_{k+1}^m$ for $m \in \{1, 2, 3\}$, no information concerning the signal is lost and new insight into the processed data is available.

Chapter 5

Applications

The representation of a signal as a function of time fails to provide the spectrum of frequencies present while its Fourier analysis hides the point of transmission and the duration of each of the signal's harmonics. The preferred approach should aim to combine the advantages of these two complementary representations, i.e. constructing an instantaneous spectrum as a function of time. The instantaneous spectrum should also be easily discretized by fast algorithms so that it is more compatible with modern digital communication theory. Due to the uncertainty principle, the design of such a spectrum using wavelet representations is only possible provided that the observation of the signal as a function of time and frequency is not arbitrarily precise.

This chapter explains how wavelet frames on $L^2(\mathbb{T}^s)$ could be applied to practical situations. Periodic wavelets are considered as signals occuring in practice are often extended periodically. Sections 5.1 to 5.3 describe the decomposition and reconstruction algorithms for different setups of the general multidimensional multiwavelet setting of $L^2(\mathbb{T}^s)$. For practical purposes, Section 5.4 narrows down to the 1-dimensional setting with arbitrary integer dilation factor M and Section 5.5 further restricts to the setting of a single refinable function with dilation factor $M = 2$ for the time-frequency analysis of some Gabor atoms and chirp signals.

5.1 Uniqueness of Representation

We shall first focus on understanding the representation of a function in $V_{2\pi}^{k+1}$ using its underlying subspaces $V_{2\pi}^k$ and $W_{2\pi}^k$ given in (3.33) . Let $f_{k+1} = f_k + g_k \in V_{2\pi}^{k+1}$, where $f_k = \sum_{j\in\mathcal{R}_k} \widehat{s_k}(j)^* v_{k,j} \in V_{2\pi}^k$ and $g_k = \sum_{j\in\mathcal{R}_k} \widehat{t_k}(j)^* u_{k,j} \in W_{2\pi}^k$ for some $\widehat{s_k} \in \mathcal{S}(D^k)^{\rho\times 1}$

and $\widehat{t_k} \in \mathcal{S}(D^k)^{\varrho_k \times 1}$, which are the discrete Fourier transforms of $s_k \in \mathcal{S}(M^k)^{\rho\times 1}$ and $t_k \in \mathcal{S}(M^k)^{\varrho_k \times 1}$, with the polyphase harmonics $v_{k,j}$ and $u_{k,j}$ given as in Propositions 3.19 and 3.20 respectively. Therefore,

$$\begin{aligned} f_{k+1} &= \sum_{j\in\mathcal{R}_{k+1}} \widehat{s_{k+1}}(j)^* v_{k+1,j} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}(j+D^k r)^* v_{k+1,j+D^k r} \\ &= \sum_{j\in\mathcal{R}_k} \widehat{s_k}(j)^* v_{k,j} + \sum_{j\in\mathcal{R}_k} \widehat{t_k}(j)^* u_{k,j} \\ &= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \left[\widehat{s_k}(j)^* \widehat{H}_{k+1}(j+D^k r) + \widehat{t_k}(j)^* \widehat{G}_{k+1}(j+D^k r)\right] v_{k+1,j+D^k r}, \end{aligned} \tag{5.1}$$

with $\widehat{H}_{k+1}$ and $\widehat{G}_{k+1}$ given as in Propositions 3.19 and 3.20 respectively.

Our next two results show that the representation of $f_k \in V_{2\pi}^k$ and $g_k \in W_{2\pi}^k$ by polyphase harmonics is not unique and there is a minimal representation (up to ordering) by equivalent polyphase harmonics whose Gramian is diagonal and consists of eigenvalues of M_k and N_k respectively, where M_k and N_k are given as in (3.34). First, we shall state the assumptions used in the following common setups.

Setup 5.1. *For a given $k \geq 0$, let $j \in \mathcal{R}_k$ and $r \in \mathcal{R}_1$. Suppose that* $\operatorname{rank}\mathrm{M}_k(j) = q(j)$, $\operatorname{rank}\mathrm{N}_k(j) = p(j) - q(j)$, $\operatorname{rank}\mathrm{M}_{k+1}(j+D^k r) = p(j,r)$. *There exist unitary matrices $U_k(j) \in \mathbb{C}^{\rho\times\rho}$, $V_k(j) \in \mathbb{C}^{\varrho_k\times\varrho_k}$ and $U_{k+1}(j+D^k r) \in \mathbb{C}^{\rho\times\rho}$ such that*

$$\begin{aligned} &U_k(j)\mathrm{M}_k(j)U_k(j)^* = \operatorname{diag}(\mathrm{M}'_k(j), 0_{\rho-q(j)}), \\ &V_k(j)\mathrm{N}_k(j)V_k(j)^* = \operatorname{diag}(\mathrm{N}'_k(j), 0_{\varrho_k-[p(j)-q(j)]}), \\ &U_{k+1}(j+D^k r)\mathrm{M}_{k+1}(j+D^k r)U_{k+1}(j+D^k r)^* = \operatorname{diag}(\mathrm{M}'_{k+1}(j+D^k r), 0_{\rho-p(j,r)}), \end{aligned} \tag{5.2}$$

where $\mathrm{M}'_k(j)$, $\mathrm{N}'_k(j)$ and $\mathrm{M}'_{k+1}(j+D^k r)$ are invertible diagonal $q(j)\times q(j)$, $p(j)-q(j)\times p(j)-q(j)$ and $p(j,r)\times p(j,r)$ matrices respectively. We shall also define the following diagonal matrices, i.e. the $\rho\times\rho$ diagonal matrix $I'_{q(j)} = \operatorname{diag}(I_{q(j)}, 0_{\rho-q(j)})$, the $\varrho_k\times\varrho_k$ diagonal matrix $I'_{p(j)-q(j)} = \operatorname{diag}(I_{p(j)-q(j)}, 0_{\varrho_k-[p(j)-q(j)]})$, the $\rho\times\rho$ diagonal matrix $I'_{p(j,r)} = \operatorname{diag}(I_{p(j,r)}, 0_{\rho-p(j,r)})$ and the $\rho d\times\rho d$ diagonal matrix $I'_{p(j)} = \operatorname{diag}(I'_{p(j,r_1)}, \ldots, I'_{p(j,r_d)})$, where $r_1, \ldots, r_d \in \mathcal{R}_1$ are distinct coset representatives of $\mathbb{Z}^s/D\mathbb{Z}^s$.

Setup 5.2. *Assume Setup 5.1. Let $\widehat{s_k}'(j) = U_k(j)\widehat{s_k}(j)$, $\widehat{t_k}'(j) = V_k(j)\widehat{t_k}(j)$, $\widehat{s_{k+1}}'(j+D^k r) = U_{k+1}(j+D^k r)\widehat{s_{k+1}}(j+D^k r)$, $v'_{k,j} = U_k(j)v_{k,j}$, $u'_{k,j} = V_k(j)u_{k,j}$, $v'_{k+1,j+D^k r} = U_{k+1}(j+D^k r)v_{k+1,j+D^k r}$,*

$$\begin{aligned} &\widehat{H}'_{k+1}(j+D^k r) = U_k(j)\widehat{H}_{k+1}(j+D^k r)U_{k+1}(j+D^k r)^* \quad \textit{and} \\ &\widehat{G}'_{k+1}(j+D^k r) = V_k(j)\widehat{G}_{k+1}(j+D^k r)U_{k+1}(j+D^k r)^*, \end{aligned}$$

where $\widehat{s_k}$, $\widehat{t_k}$, $v_{k,j}$, $u_{k,j}$, $\widehat{H}_{k+1}$ *and* $\widehat{G}_{k+1}$ *are given as in* (5.1).

Setup 5.3. *Assume Setup* 5.1. *Let* $\widehat{s_k}'(j) = I'_{q(j)}U_k(j)\widehat{s_k}(j)$, $\widehat{t_k}'(j) = I'_{p(j)-q(j)}V_k(j)\widehat{t_k}(j)$, $\widehat{s_{k+1}}'(j+D^kr) = I'_{p(j,r)}U_{k+1}(j+D^kr)\widehat{s_{k+1}}(j+D^kr)$, $v'_{k,j} = U_k(j)v_{k,j}$, $u'_{k,j} = V_k(j)u_{k,j}$, $v'_{k+1,j+D^kr} = U_{k+1}(j+D^kr)v_{k+1,j+D^kr}$,

$$\widehat{H}'_{k+1}(j+D^kr) = I'_{q(j)}U_k(j)\widehat{H}_{k+1}(j+D^kr)U_{k+1}(j+D^kr)^*I'_{p(j,r)} \quad \text{and}$$
$$\widehat{G}'_{k+1}(j+D^kr) = I'_{p(j)-q(j)}V_k(j)\widehat{G}_{k+1}(j+D^kr)U_{k+1}(j+D^kr)^*I'_{p(j,r)},$$

where $\widehat{s_k}$, $\widehat{t_k}$, $v_{k,j}$, $u_{k,j}$, $\widehat{H}_{k+1}$ *and* $\widehat{G}_{k+1}$ *are given as in* (5.1). *We shall also let*

$$\widehat{\mathbb{L}}'_k(j) = \mathrm{diag}(I'_{q(j)}U_k(j), I'_{p(j)-q(j)}V_k(j))\widehat{\mathbb{L}}_k(j)\mathrm{diag}(U_{k+1}(j+D^kr_1),\ldots,U_{k+1}(j+D^kr_d))^*I'_{p(j)}$$

with the assumption that $\widehat{\mathbb{L}}_k(j)$ *defined as in* (3.44) *satisfies* (3.43).

The difference between Setups 5.2 and 5.3 is that in the latter, values of the masks and frame coefficients of signals outside the spectrum of the refinable functions and wavelets have been set to zero and the masks also satisfy the minimum energy tight frame condition.

Lemma 5.4. *Assume Setup* 5.2. *If* $\sum_{j\in\mathcal{R}_k}\widehat{s_k}(j)^*v_{k,j} = 0$ *and* $\sum_{j\in\mathcal{R}_k}\widehat{t_k}(j)^*u_{k,j} = 0$, *then for each* $j\in\mathcal{R}_k$, *the first* $q(j)$ *entries and the first* $p(j)-q(j)$ *entries of* $\widehat{s_k}'(j)$ *and* $\widehat{t_k}'(j)$ *are identically zero respectively with the remaining entries being arbitrary. In particular, we have*

$$\sum_{j\in\mathcal{R}_k}\widehat{s_k}'(j)^*I'_{q(j)}v'_{k,j} = \sum_{j\in\mathcal{R}_k}\widehat{s_k}(j)^*v_{k,j}, \text{ and } \sum_{j\in\mathcal{R}_k}\widehat{t_k}'(j)^*I'_{p(j)-q(j)}u'_{k,j} = \sum_{j\in\mathcal{R}_k}\widehat{t_k}(j)^*u_{k,j}. \tag{5.3}$$

Proof. For a given $l\in\mathcal{R}_k$, since $\left[\langle\widehat{s_k}(l)^*v_{k,l}, v^\mu_{k,l}\rangle\right]^\rho_{\mu=1} = \sum_{j\in\mathcal{R}_k}\left[\langle\widehat{s_k}(j)^*v_{k,j}, v^\mu_{k,l}\rangle\right]^\rho_{\mu=1} = 0$, we infer our result from the observation that

$$[U_k(l)\widehat{s_k}(l)]^*\begin{bmatrix}\mathrm{M}'_k(l) & 0\\ 0 & 0\end{bmatrix} = 0.$$

and that $\mathrm{M}_k(l)'$ is a $q(l)\times q(l)$ diagonal matrix. The proof for $\widehat{t_k}'(j)$ is similar. $\square$

The reconstruction algorithm is derived using Lemma 5.4 without the use of Proposition 3.26 and Theorem 3.27 (periodic UEP).

Proposition 5.5. *Assume Setup* 5.2. *If*

$$\sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}(j+D^kr)^*v_{k+1,j+D^kr} =$$
$$\sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\left[\widehat{s_k}(j)^*\widehat{H}_{k+1}(j+D^kr) + \widehat{t_k}(j)^*\widehat{G}_{k+1}(j+D^kr)\right]v_{k+1,j+D^kr}, \tag{5.4}$$

then for $j \in \mathcal{R}_k$ and $r \in \mathcal{R}_1$,

$$\widehat{s_{k+1}}'(j + D^k r)^* I'_{p(j,r)} = \left[\widehat{s_k}'(j)^* \widehat{H}'_{k+1}(j + D^k r) + \widehat{t_k}'(j)^* \widehat{G}'_{k+1}(j + D^k r)\right] I'_{p(j,r)} \tag{5.5}$$

$$\sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j + D^k r)^* I'_{p(j,r)} v'_{k+1,j+D^k r} =$$

$$\sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \left[\widehat{s_k}'(j)^* \widehat{H}'_{k+1}(j + D^k r) + \widehat{t_k}'(j)^* \widehat{G}'_{k+1}(j + D^k r)\right] I'_{p(j,r)} v'_{k+1,j+D^k r}. \tag{5.6}$$

The $p(j,r)+1,\ldots,\rho$ entries of $\widehat{s_{k+1}}'(j + D^k r)$ could be arbitrary and (5.4) *is equivalent to* (5.6)*. In particular, for $j \in \mathcal{R}_k$ and $r \in \mathcal{R}_1$, we have*

$$\widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} = \left[\widehat{s_k}'(j)^* I'_{q(j)} \widehat{H}'_{k+1}(j+D^k r) + \widehat{t_k}'(j)^* I'_{p(j)-q(j)} \widehat{G}'_{k+1}(j+D^k r)\right] I'_{p(j,r)} \tag{5.7}$$

$$\sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j + D^k r)^* I'_{p(j,r)} v'_{k+1,j+D^k r} =$$

$$\sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\left[\widehat{s_k}'(j)^* I'_{q(j)} \widehat{H}'_{k+1}(j + D^k r) + \widehat{t_k}'(j)^* I'_{p(j)-q(j)} \widehat{G}'_{k+1}(j + D^k r)\right] I'_{p(j,r)} v'_{k+1,j+D^k r} \tag{5.8}$$

and (5.4) *is equivalent to* (5.8)*.*

Proof. By the proof of Lemma 5.4, the $\rho \times \rho$ unitary matrix $U_{k+1}(j + D^k r)$ diagonalizes $M_{k+1}(j + D^k r)$ as in (5.2) and results in

$$\left[\widehat{s_{k+1}}'(j+D^k r)^* - \widehat{s_k}'(j)^* \widehat{H}'_{k+1}(j+D^k r) - \widehat{t_k}'(j)^* \widehat{G}'_{k+1}(j+D^k r)\right] \begin{bmatrix} M'_{k+1}(j+D^k r) & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Hence (5.5) and (5.6) hold and (5.6) is equivalent to (5.4). This is true even if $U_k(j)$ and $V_k(j)$ are arbitrary $\rho \times \rho$ and $\varrho_k \times \varrho_k$ unitary matrices respectively.

Next, assuming that $U_k(j)$ and $V_k(j)$ are chosen as in Setup 5.1, we make use of Lemma 5.4, (5.1), and (5.6) to show that

$$f_{k+1} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} v'_{k+1,j+D^k r} = \sum_{j\in\mathcal{R}_k} \widehat{s_k}'(j)^* I'_{q(j)} v'_{k,j} + \sum_{j\in\mathcal{R}_k} \widehat{t_k}'(j)^* I'_{p(j)-q(j)} u'_{k,j}$$

$$= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\left[\widehat{s_k}'(j)^* I'_{q(j)} \widehat{H}'_{k+1}(j + D^k r) + \widehat{t_k}'(j)^* I'_{p(j)-q(j)} \widehat{G}'_{k+1}(j + D^k r)\right] v'_{k+1,j+D^k r}, \tag{5.9}$$

where $v'_{k,j} = \sum_{r\in\mathcal{R}_1} \widehat{H}'_{k+1}(j+D^k r) I'_{p(j,r)} v'_{k+1,j+D^k r}$ and $u'_{k,j} = \sum_{r\in\mathcal{R}_1} \widehat{G}'_{k+1}(j+D^k r) I'_{p(j,r)} v'_{k+1,j+D^k r}$. Apply Lemma 5.4 to (5.9) again gives us our result. $\square$

The masks $\widehat{H}'_{k+1}$ and $\widehat{G}'_{k+1}$ are therefore known as *reconstruction masks* since they are used to reconstruct $\widehat{s_{k+1}}'$ from $\widehat{s_k}'$ and $\widehat{t_k}'$ in (5.5).

5.2 Semi-Orthogonal Representation

For the results in this section, we shall assume that $V_{2\pi}^{k+1} = V_{2\pi}^{k} \oplus^{\perp} W_{2\pi}^{k}$ and they are applicable to wavelet frame constructions derived from FMRAs. Our next proposition represents the spanning members of $U_{2\pi}^{k+1,j}$ given in (3.35) by spanning members of $V_{2\pi}^{k,j}$ and $W_{2\pi}^{k,j}$.

Proposition 5.6. *Assume Setup* 5.2. *There exist* $\widehat{P}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\rho}$ *and* $\widehat{Q}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\varrho_k}$ *such that for* $j \in \mathcal{R}_k$ *and* $r \in \mathcal{R}_1$,

$$v_{k+1,j+D^k r} = \widehat{P}_{k+1}(j+D^k r)v_{k,j} + \widehat{Q}_{k+1}(j+D^k r)u_{k,j}. \tag{5.10}$$

In particular, we have

$$\widehat{P}'_{k+1}(j+D^k r)\begin{bmatrix}\mathrm{M}'_k(j) & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\mathrm{M}'_{k+1}(j+D^k r) & 0\\ 0 & 0\end{bmatrix}\widehat{H}'_{k+1}(j+D^k r)^*, \tag{5.11}$$

$$\widehat{Q}'_{k+1}(j+D^k r)\begin{bmatrix}\mathrm{N}'_k(j) & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\mathrm{M}'_{k+1}(j+D^k r) & 0\\ 0 & 0\end{bmatrix}\widehat{G}'_{k+1}(j+D^k r)^*, \tag{5.12}$$

where $\widehat{P}'_{k+1}(j+D^k r) = U_{k+1}(j+D^k r)\widehat{P}_{k+1}(j+D^k r)U_k(j)^*$ *and* $\widehat{Q}'_{k+1}(j+D^k r) = U_{k+1}(j+D^k r)\widehat{Q}_{k+1}(j+D^k r)V_k(j)^*$. *Therefore, the first* $q(j)$ *and* $p(j)-q(j)$ *columns of* $\widehat{P}'_{k+1}(j+D^k r)$ *and* $\widehat{Q}'_{k+1}(j+D^k r)$ *respectively are uniquely determined.*

Proof. Since $V_{2\pi}^{k+1} = V_{2\pi}^{k} \oplus^{\perp} W_{2\pi}^{k}$, for a given $j \in \mathcal{R}_k$ and $r \in \mathcal{R}_1$, there exist $A(l) \in \mathbb{C}^{\rho\times\rho}$ and $B(l) \in \mathbb{C}^{\rho\times\varrho_k}$, where $l \in \mathcal{R}_k$, such that

$$\begin{aligned} v_{k+1,j+D^k r} &= \sum_{l\in\mathcal{R}_k}\left[A(l)v_{k,l} + B(l)u_{k,l}\right]\\ &= \sum_{l\in\mathcal{R}_k}\sum_{n\in\mathcal{R}_1}\left[A(l)\widehat{H}_{k+1}(l+D^k n) + B(l)\widehat{G}_{k+1}(l+D^k n)\right]v_{k+1,l+D^k n}. \end{aligned}$$

Let $j_0 \in \mathcal{R}_k$ and $r_0 \in \mathcal{R}_1$. We have

$$\begin{aligned} &\delta_{j+D^k r, j_0+D^k r_0}M_{k+1}(j_0+D^k r_0) = \left[\langle v^m_{k+1,j+D^k r}, v^\mu_{k+1,j_0+D^k r_0}\rangle\right]_{m,\mu=1}^{\rho} =\\ &\sum_{l\in\mathcal{R}_k}\sum_{n\in\mathcal{R}_1}\left[A(l)\widehat{H}_{k+1}(l+D^k n) + B(l)\widehat{G}_{k+1}(l+D^k n)\right]\left[\langle v^m_{k+1,l+D^k n}, v^\mu_{k+1,j_0+D^k r_0}\rangle\right]_{m,\mu=1}^{\rho}\\ &= \left[A(j_0)\widehat{H}_{k+1}(j_0+D^k r_0) + B(j_0)\widehat{G}_{k+1}(j_0+D^k r_0)\right]\mathrm{M}_{k+1}(j_0+D^k r_0). \end{aligned} \tag{5.13}$$

Since $\operatorname{rank}\mathrm{M}_{k+1}(j_0+D^k r_0) = p(j_0,r_0)$ and the unitary matrix $U_{k+1}(j_0+D^k r_0) \in \mathbb{C}^{\rho\times\rho}$ diagonalizes $\mathrm{M}_{k+1}(j_0+D^k r_0)$ such that

$$U_{k+1}(j_0+D^k r_0)\mathrm{M}_{k+1}(j_0+D^k r_0)U_{k+1}(j_0+D^k r_0)^* = \begin{bmatrix}\mathrm{M}'_{k+1}(j_0+D^k r_0) & 0\\ 0 & 0\end{bmatrix},$$

with $\mathrm{M}'_{k+1}(j_0 + D^k r_0)$ being an invertible $p(j_0, r_0) \times p(j_0, r_0)$ diagonal matrix, we let $A'(j_0) = U_{k+1}(j_0 + D^k r_0)A(j_0)U_k(j_0)^*$, $B'(j_0) = U_{k+1}(j_0 + D^k r_0)B(j_0)V_k(j_0)^*$, $\widehat{H}'_{k+1}(j_0 + D^k r_0) = U_k(j_0)\widehat{H}_{k+1}(j_0 + D^k r_0)U_{k+1}(j_0 + D^k r_0)^*$ and $\widehat{G}'_{k+1}(j_0 + D^k r_0) = V_k(j_0)\widehat{G}_{k+1}(j_0 + D^k r_0)U_{k+1}(j_0 + D^k r_0)^*$ to obtain

$$\left[A'(j_0)\widehat{H}'_{k+1}(j_0 + D^k r_0) + B'(j_0)\widehat{G}'_{k+1}(j_0 + D^k r_0)\right] \begin{bmatrix} \mathrm{M}'_{k+1}(j_0 + D^k r_0) & 0 \\ 0 & 0 \end{bmatrix} = 0$$

for $j_0 \neq j$ or $r_0 \neq r$. We conclude that the first $p(j_0, r_0)$ columns of $A'(j_0)\widehat{H}'_{k+1}(j_0 + D^k r_0) + B'(j_0)\widehat{G}'_{k+1}(j_0 + D^k r_0)$ are equal to zero for $j_0 \neq j$ or $r_0 \neq r$. For convenience, we choose the $A'(j_0)$ and $B'(j_0)$ to be zero matrices whenever $j_0 \neq j$ or $r_0 \neq r$ hold. This leads to

$$v_{k+1,j+D^k r} = A(j)v_{k,j} + B(j)u_{k,j}, \tag{5.14}$$

Let $m, \mu \in \{1, \ldots, \rho\}$. Since the refinement equation (3.29) shows that

$$\begin{aligned}
\langle v^m_{k+1,j+D^k r}, v^\mu_{k,j}\rangle &= \langle v^m_{k+1,j+D^k r}, \sum_{l\in\mathcal{R}_1}\sum_{i=1}^{\rho} \widehat{H}^{\mu,i}_{k+1}(j + D^k l) v^i_{k+1,j+D^k l}\rangle \\
&= \sum_{l\in\mathcal{R}_1}\sum_{i=1}^{\rho}\langle v^m_{k+1,j+D^k r}, v^i_{k+1,j+D^k l}\rangle \widehat{H}^{\mu,i}_{k+1}(j + D^k l)^* \\
&= \sum_{i=1}^{\rho}\langle v^m_{k+1,j+D^k r}, v^i_{k+1,j+D^k r}\rangle \widehat{H}^{\mu,i}_{k+1}(j + D^k r)^* \\
&= \left(\left[\langle v^m_{k+1,j+D^k r}, v^i_{k+1,j+D^k r}\rangle\right]_{i=1}^{\rho}\right)^T \left[\widehat{H}^{\mu,i}_{k+1}(j + D^k r)^*\right]_{i=1}^{\rho},
\end{aligned}$$

which leads to

$$\begin{aligned}
&\left[\langle v^m_{k+1,j+D^k r}, v^\mu_{k,j}\rangle\right]_{m,\mu=1}^{\rho} = \left[\langle v^m_{k+1,j+D^k r}, v^i_{k+1,j+D^k r}\rangle\right]_{m,i=1}^{\rho} \left[\widehat{H}^{\mu,i}_{k+1}(j + D^k r)^*\right]_{i,\mu=1}^{\rho}, \\
&= \mathrm{M}_{k+1}(j + D^k r)\widehat{H}_{k+1}(j + D^k r)^*,
\end{aligned}$$

we deduce using (5.14) that

$$\left[\langle v^m_{k+1,j+D^k r}, v^\mu_{k,j}\rangle\right]_{m,\mu=1}^{\rho} = \widehat{P}_{k+1}(j + D^k r)\left[\langle v^m_{k,j}, v^\mu_{k,j}\rangle\right]_{m,\mu=1}^{\rho} = \widehat{P}_{k+1}(j + D^k r)\mathrm{M}_k(j),$$

where $\widehat{P}_{k+1}(j + D^k r) = A(j)$. Consequently, (5.11) follows from diagonalizing $\mathrm{M}_k(j)$ with $U_k(j)$.

In a similar manner, let $m \in \{1, \dots, \rho\}$ and $\mu \in \{1, \dots, \varrho_k\}$. Since (3.32) shows that

$$\begin{aligned}
\langle v^m_{k+1,j+D^k r}, u^\mu_{k,j}\rangle &= \langle v^m_{k+1,j+D^k r}, \sum_{l\in\mathcal{R}_1}\sum_{i=1}^{\rho} \widehat{G}^{\mu,i}_{k+1}(j+D^k l) v^i_{k+1,j+D^k l}\rangle \\
&= \sum_{l\in\mathcal{R}_1}\sum_{i=1}^{\rho} \langle v^m_{k+1,j+D^k r}, v^i_{k+1,j+D^k l}\rangle \widehat{G}^{\mu,i}_{k+1}(j+D^k l)^* \\
&= \sum_{i=1}^{\rho} \langle v^m_{k+1,j+D^k r}, v^i_{k+1,j+D^k r}\rangle \widehat{G}^{\mu,i}_{k+1}(j+D^k r)^* \\
&= \left(\left[\langle v^m_{k+1,j+D^k r}, v^i_{k+1,j+D^k r}\rangle\right]_{i=1}^{\rho}\right)^T \left[\widehat{G}^{\mu,i}_{k+1}(j+D^k r)^*\right]_{i=1}^{\rho},
\end{aligned}$$

which leads to

$$\begin{aligned}
&\left[\langle v^m_{k+1,j+D^k r}, u^\mu_{k,j}\rangle\right]_{m,\mu=1}^{\rho,\varrho_k} = \left[\langle v^m_{k+1,j+D^k r}, v^i_{k+1,j+D^k r}\rangle\right]_{m,i=1}^{\rho} \left[\widehat{G}^{\mu,i}_{k+1}(j+D^k r)^*\right]_{i,\mu=1}^{\rho,\varrho_k} \\
&= \mathrm{M}_{k+1}(j+D^k r)\widehat{G}_{k+1}(j+D^k r)^*,
\end{aligned}$$

we deduce using (5.14) that

$$\left[\langle v^m_{k+1,j+D^k r}, u^\mu_{k,j}\rangle\right]_{m,\mu=1}^{\rho,\varrho_k} = \widehat{Q}_{k+1}(j+D^k r)\left[\langle u^m_{k,j}, u^\mu_{k,j}\rangle\right]_{m,\mu=1}^{\varrho_k} = \widehat{Q}_{k+1}(j+D^k r)\mathrm{N}_k(j),$$

where $\widehat{Q}_{k+1}(j+D^k r) = B(j)$. Consequently, (5.12) follows from diagonalizing $\mathrm{N}_k(j)$ with $V_k(j)$. □

As we shall observe why from the following results, the masks $\widehat{P}'_{k+1}$ and $\widehat{Q}'_{k+1}$ are known as *decomposition masks*. The next proposition essentially corresponds to Theorem 3.25 and Proposition 3.26 for the semi-orthogonal case.

Proposition 5.7. *Assume Setup* 5.2. *The masks* $\widehat{P}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\rho}$ *and* $\widehat{Q}_{k+1} \in \mathcal{S}(D^{k+1})^{\rho\times\varrho_k}$ *in* (5.10) *satisfy*

$$\left[\widehat{P}'_{k+1}(j+D^k n)I'_{q(j)}\widehat{H}'_{k+1}(j+D^k r)+\widehat{Q}'_{k+1}(j+D^k n)I'_{p(j)-q(j)}\widehat{G}'_{k+1}(j+D^k r)\right]I'_{p(j,r)}=\delta_{n,r}I'_{p(j,r)} \quad (5.15)$$

for $j \in \mathcal{R}_k$, $n, r \in \mathcal{R}_1$, *where* $\widehat{P}'_{k+1}(j+D^k n) = U_{k+1}(j+D^k n)\widehat{P}_{k+1}(j+D^k n)U_k(j)^*$ *and* $\widehat{Q}'_{k+1}(j+D^k n) = U_{k+1}(j+D^k n)\widehat{Q}_{k+1}(j+D^k n)V_k(j)^*$.

Proof. For a given $j \in \mathcal{R}_k$ and $r_1 \in \mathcal{R}_1$, by (5.10) of Proposition 5.6, (3.29) and (3.32), we have

$$\begin{aligned}
v_{k+1,j+D^k r_1} &= \widehat{P}_{k+1}(j+D^k r_1)v_{k,j} + \widehat{Q}_{k+1}(j+D^k r_1)u_{k,j} \\
&= \sum_{n\in\mathcal{R}_1}\left[\widehat{P}_{k+1}(j+D^k r_1)\widehat{H}_{k+1}(j+D^k n) + \widehat{Q}_{k+1}(j+D^k r_1)\widehat{G}_{k+1}(j+D^k n)\right] v_{k+1,j+D^k n}.
\end{aligned}$$

Let $r_2 \in \mathcal{R}_1$. Using a similar reasoning as in (5.13), we have

$$\delta_{r_1,r_2}\mathrm{M}_{k+1}(j+D^k r_2) = \left[\langle v^m_{k+1,j+D^k r_1}, v^\mu_{k+1,j+D^k r_2}\rangle\right]^\rho_{m,\mu=1} =$$
$$\sum_{n\in\mathcal{R}_1}\left[\widehat{P}_{k+1}(j+D^k r_1)\widehat{H}_{k+1}(j+D^k n) + \widehat{Q}_{k+1}(j+D^k r_1)\widehat{G}_{k+1}(j+D^k n)\right]$$
$$\left[\langle v^m_{k+1,j+D^k n}, v^\mu_{k+1,j+D^k r_2}\rangle\right]^\rho_{m,\mu=1}$$
$$=\left[\widehat{P}_{k+1}(j+D^k r_1)\widehat{H}_{k+1}(j+D^k r_2)+\widehat{Q}_{k+1}(j+D^k r_1)\widehat{G}_{k+1}(j+D^k r_2)\right]\mathrm{M}_{k+1}(j+D^k r_2). \tag{5.16}$$

The unitary matrix $U_{k+1}(j+D^k r_2) \in \mathbb{C}^{\rho\times\rho}$ diagonalizes $\mathrm{M}_{k+1}(j+D^k r_2)$ such that

$$U_{k+1}(j+D^k r_2)\mathrm{M}_{k+1}(j+D^k r_2)U_{k+1}(j+D^k r_2)^* = \begin{bmatrix}\mathrm{M}'_{k+1}(j+D^k r_2) & 0\\ 0 & 0\end{bmatrix},$$

where $\mathrm{M}'_{k+1}(j+D^k r_2)$ is an invertible $p(j,r_2)\times p(j,r_2)$ diagonal matrix. Premultiplying and postmultiplying (5.16) by $U_{k+1}(j+D^k r_2)$ and $U_{k+1}(j+D^k r_2)^*$ respectively with the appropriate normalization leads to

$$\left[\widehat{P}'_{k+1}(j+D^k n)\widehat{H}'_{k+1}(j+D^k r) + \widehat{Q}'_{k+1}(j+D^k n)\widehat{G}'_{k+1}(j+D^k r)\right] I'_{p(j,r)} = \delta_{n,r} I'_{p(j,r)}.$$

Using the observation from Proposition 5.6 that only the first $q(j)$ and $p(j)-q(j)$ columns of $\widehat{P}'_{k+1}(j+D^k r_1)$ and $\widehat{Q}'_{k+1}(j+D^k r_1)$ are respectively unique shows that (5.15) is valid. □

Assuming Setup 5.2 and referring to (5.1), if (5.10) holds as well, then we also have

$$f_{k+1} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}(j+D^k r)^*\left[\widehat{P}_{k+1}(j+D^k r)v_{k,j} + \widehat{Q}_{k+1}(j+D^k r)u_{k,j}\right]$$
$$= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)}\left[\widehat{P}'_{k+1}(j+D^k r)v'_{k,j} + \widehat{Q}'_{k+1}(j+D^k r)u'_{k,j}\right], \tag{5.17}$$

with $\widehat{P}'_{k+1}(j+D^k r)$ and $\widehat{Q}'_{k+1}(j+D^k r)$ given as in Proposition 5.7.

We derive below the decomposition algorithms of the low pass and high pass coefficients for the semi-orthogonal case in the frequency domain.

Proposition 5.8. *Assume Setup* 5.2 *and* (5.1) *and* (5.17) *to hold. If*

$$\sum_{j\in\mathcal{R}_k}\widehat{s_k}(j)^* v_{k,j} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}(j+D^k r)^*\widehat{P}_{k+1}(j+D^k r)v_{k,j} \text{ and} \tag{5.18}$$
$$\sum_{j\in\mathcal{R}_k}\widehat{t_k}(j)^* u_{k,j} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}(j+D^k r)^*\widehat{Q}_{k+1}(j+D^k r)u_{k,j}, \tag{5.19}$$

then

$$\widehat{s_k}'(j)^* I'_{q(j)} = \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* \widehat{P}'_{k+1}(j+D^k r) I'_{q(j)}, \quad j\in\mathcal{R}_k,$$

$$\sum_{j\in\mathcal{R}_k} \widehat{s_k}'(j)^* I'_{q(j)} v'_{k,j} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* \widehat{P}'_{k+1}(j+D^k r) I'_{q(j)} v'_{k,j} \text{ and} \tag{5.20}$$

$$\widehat{t_k}'(j)^* I'_{p(j)-q(j)} = \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* \widehat{Q}'_{k+1}(j+D^k r) I'_{p(j)-q(j)}, \quad j\in\mathcal{R}_k,$$

$$\sum_{j\in\mathcal{R}_k} \widehat{t_k}'(j)^* I'_{p(j)-q(j)} u'_{k,j} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* \widehat{Q}'_{k+1}(j+D^k r) I'_{p(j)-q(j)} u'_{k,j}, \tag{5.21}$$

where $\widehat{P}'_{k+1}(j+D^k r) = U_{k+1}(j+D^k r)\widehat{P}_{k+1}(j+D^k r)U_k(j)^*$ *and* $\widehat{Q}'_{k+1}(j+D^k r) = U_{k+1}(j+D^k r)\widehat{Q}_{k+1}(j+D^k r)V_k(j)^*$. *The* $q(j)+1,\dots,\rho$ *entries of* $\widehat{s_k}'(j)$ *and the* $p(j)-q(j)+1,\dots,\varrho_k$ *entries of* $\widehat{t_k}'(j)$ *could be arbitrary and* (5.18) *is equivalent to* (5.20) *and* (5.19) *is equivalent to* (5.21). *In particular, we have*

$$\widehat{s_k}'(j)^* I'_{q(j)} = \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} \widehat{P}'_{k+1}(j+D^k r) I'_{q(j)}, \quad j\in\mathcal{R}_k,$$

$$\sum_{j\in\mathcal{R}_k} \widehat{s_k}'(j)^* I'_{q(j)} v'_{k,j} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} \widehat{P}'_{k+1}(j+D^k r) I'_{q(j)} v'_{k,j} \text{ and}$$

$$\widehat{t_k}'(j)^* I'_{p(j)-q(j)} = \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} \widehat{Q}'_{k+1}(j+D^k r) I'_{p(j)-q(j)}, \quad j\in\mathcal{R}_k,$$

$$\sum_{j\in\mathcal{R}_k} \widehat{t_k}'(j)^* I'_{p(j)-q(j)} u'_{k,j} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} \widehat{Q}'_{k+1}(j+D^k r) I'_{p(j)-q(j)} u'_{k,j}.$$

Proof. We use Lemma 5.4 to obtain (5.20) and (5.21). Using Lemma 5.4, Proposition 5.5, (5.1) and (5.17), since (5.4) and (5.6) are equivalent, we show that

$$f_{k+1} = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} v'_{k+1,j+D^k r} = \sum_{j\in\mathcal{R}_k} \widehat{s_k}'(j)^* I'_{q(j)} v'_{k,j} + \sum_{j\in\mathcal{R}_k} \widehat{t_k}'(j)^* I'_{p(j)-q(j)} u'_{k,j}$$
$$= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} \left[\widehat{P}'_{k+1}(j+D^k r) v'_{k,j} + \widehat{Q}'_{k+1}(j+D^k r) u'_{k,j}\right]$$

and the remaining result follows from the fact that $V^{k+1}_{2\pi} = V^k_{2\pi} \oplus^\perp W^k_{2\pi}$. □

We show below that we have perfect reconstruction under certain scenarios.

Proposition 5.9. *Let* $j\in\mathcal{R}_k$, $r\in\mathcal{R}_1$ *and assume Setup* 5.2. *If the following information* $\left[\widehat{s_k}'(j)\right]^* e_m$, $\left[\widehat{t_k}'(j)\right]^* e_m$ *and* $\left[\widehat{s_{k+1}}'(j+D^k r)\right]^* e_m$ *are known a priori for* $m \in \{q(j)+$

$1,\dots,\rho\}$, $m \in \{p(j)-q(j)+1,\dots,\varrho_k\}$ *and* $m \in \{p(j,r)+1,\dots,\rho\}$ *respectively, where* e_m *is the* m^{th} *unit vector, then we have perfect reconstruction and*

$$\widehat{s_{k+1}}'(j+D^kr)^*I'_{p(j,r)} = \widehat{s_{k+1}}'(j+D^kr)^*\widehat{P}'_{k+1}(j+D^kr)I'_{q(j)}\widehat{H}'_{k+1}(j+D^kr)I'_{p(j,r)} + \widehat{s_{k+1}}'(j+D^kr)^*\widehat{Q}'_{k+1}(j+D^kr)I'_{p(j)-q(j)}\widehat{G}'_{k+1}(j+D^kr)I'_{p(j,r)}. \tag{5.22}$$

Proof. Using (5.20), (5.21) and (5.7), we obtain

$$\widehat{s_{k+1}}'(j+D^kr)^*I'_{p(j,r)} = \sum_{l\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^kl)^*\widehat{P}'_{k+1}(j+D^kl)I'_{q(j)}\widehat{H}'_{k+1}(j+D^kr)I'_{p(j,r)} + \sum_{l\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^kl)^*\widehat{Q}'_{k+1}(j+D^kl)I'_{p(j)-q(j)}\widehat{G}'_{k+1}(j+D^kr)I'_{p(j,r)},$$

which leads to (5.22) with the application of Proposition 5.7. □

5.3 Nonorthogonal Representation

For the following results, we shall only assume that $V_{2\pi}^{k+1} = V_{2\pi}^k + W_{2\pi}^k$. The decomposition algorithm is derived using Proposition 3.26 and the decomposed coefficients are not unique, i.e. they are dependent on the choice of masks. However, their variations do not affect the reconstruction process.

Proposition 5.10. *Assume Setup* 5.2. *For* $j\in\mathcal{R}_k$, *let*

$$\widehat{\mathbb{L}}'_k(j) = \operatorname{diag}(U_k(j),V_k(j))\widehat{\mathbb{L}}_k(j)\operatorname{diag}(U_{k+1}(j+D^kr_1),\dots,U_{k+1}(j+D^kr_d))^*$$

with the assumption that $\widehat{\mathbb{L}}_k(j)$ *defined as in* (3.44) *satisfies* (3.43). *If*

$$\begin{aligned} f_{k+1} &= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}(j+D^kr)^*v_{k+1,j+D^kr} \\ &= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\left[\widehat{s_k}(j)^*\widehat{H}_{k+1}(j+D^kr)+\widehat{t_k}(j)^*\widehat{G}_{k+1}(j+D^kr)\right]v_{k+1,j+D^kr}, \end{aligned} \tag{5.23}$$

then for each $j\in\mathcal{R}_k$, *there exists* $\begin{bmatrix}\widehat{\widetilde{s}_k}'(j)^* & \widehat{\widetilde{t}_k}'(j)^*\end{bmatrix}^* \in \operatorname{Ker}\left[\widehat{\mathbb{L}}'_k(j)I'_{p(j)}\right]^*$ *such that*

$$d\left[\widehat{s_k}'(j)-\widehat{\widetilde{s}_k}'(j)\right]^*I'_{q(j)} = \sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^kr)^*I'_{p(j,r)}\widehat{H}'_{k+1}(j+D^kr)^*I'_{q(j)}, \tag{5.24}$$

$$d\left[\widehat{t_k}'(j)-\widehat{\widetilde{t}_k}'(j)\right]^*I'_{p(j)-q(j)} = \sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^kr)^*I'_{p(j,r)}\widehat{G}'_{k+1}(j+D^kr)^*I'_{p(j)-q(j)}, \tag{5.25}$$

and for $r\in\mathcal{R}_1$,

$$\widehat{s_{k+1}}'(j+D^kr)^*I'_{p(j,r)} = \left[\widehat{s_k}'(j)^*I'_{q(j)}\widehat{H}'_{k+1}(j+D^kr)+\widehat{t_k}'(j)^*I'_{p(j)-q(j)}\widehat{G}'_{k+1}(j+D^kr)\right]I'_{p(j,r)}. \tag{5.26}$$

The $p(j,r)+1,\dots,\rho$ *entries of* $\widehat{s_{k+1}}'(j+D^kr)$ *could be arbitrary.*

Proof. Let $r_1, \ldots, r_d \in \mathcal{R}_1$ be distinct coset representatives of $\mathbb{Z}^s/D\mathbb{Z}^s$. Using (5.7), we have

$$d\left[\widehat{s_k}'(j)^* I'_{q(j)} \quad \widehat{t_k}'(j)^* I'_{p(j)-q(j)}\right] \widehat{\mathbb{L}}'_k(j) I'_{p(j)} = d\left[\widehat{s_{k+1}}'(j+D^k r_\mu)\right]_{\mu=1}^{d*} I'_{p(j)}$$
$$= \left[\widehat{s_{k+1}}'(j+D^k r_\mu)\right]_{\mu=1}^{d*} I'_{p(j)} \widehat{\mathbb{L}}'_k(j)^* \widehat{\mathbb{L}}'_k(j) I'_{p(j)}. \tag{5.27}$$

As $\rho d = \operatorname{rank}(\widehat{\mathbb{L}}'_k(j)^*\widehat{\mathbb{L}}'_k(j)) \le \operatorname{rank}\widehat{\mathbb{L}}'_k(j)^*$, $\widehat{\mathbb{L}}'_k(j)^*$ has full column rank and $p(j) = \operatorname{rank}\left[\widehat{\mathbb{L}}'_k(j) I'_{p(j)}\right]^*$. Thus

$$d\left[\widehat{s_k}'(j)^* I'_{q(j)} \quad \widehat{t_k}'(j)^* I'_{p(j)-q(j)}\right] = \left[\widehat{s_{k+1}}'(j+D^k r_\mu)\right]_{\mu=1}^{d*} I'_{p(j)} \widehat{\mathbb{L}}'_k(j)^* + d\left[\widehat{\widehat{s_k}}'(j)^* \quad \widehat{\widehat{t_k}}'(j)^*\right],$$

where $\left[\widehat{\widehat{s_k}}'(j)^* \quad \widehat{\widehat{t_k}}'(j)^*\right]^* \in \operatorname{Ker}\left[\widehat{\mathbb{L}}'_k(j) I'_{p(j)}\right]^*$. This leads to

$$d\widehat{s_k}'(j)^* I'_{q(j)} = \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} \widehat{H}'_{k+1}(j+D^k r)^* I'_{q(j)} + d\widehat{\widehat{s_k}}'(j)^* I'_{q(j)},$$

$$d\widehat{t_k}'(j)^* I'_{p(j)-q(j)} = \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)} \widehat{G}'_{k+1}(j+D^k r)^* I'_{p(j)-q(j)} + d\widehat{\widehat{t_k}}'(j)^* I'_{p(j)-q(j)},$$

which we rewrite as

$$d\left[(\widehat{s_k}'(j) - \widehat{\widehat{s_k}}'(j))^* I'_{q(j)} \; (\widehat{t_k}'(j) - \widehat{\widehat{t_k}}'(j))^* I'_{p(j)-q(j)}\right] = \left[\widehat{s_{k+1}}'(j+D^k r_\mu)\right]_{\mu=1}^{d*} I'_{p(j)} \widehat{\mathbb{L}}'_k(j)^* \begin{bmatrix} I'_{q(j)} & 0 \\ 0 & I'_{p(j)-q(j)} \end{bmatrix} \tag{5.28}$$

Since $\left[\widehat{\widehat{s_k}}'(j)^* \; \widehat{\widehat{t_k}}'(j)^*\right]^* \in \operatorname{Ker}\left[\begin{bmatrix} I'_{q(j)} & \\ & I'_{p(j)-q(j)} \end{bmatrix} \widehat{\mathbb{L}}'_k(j) I'_{p(j)}\right]^*$ by observing from (5.5) and (5.7) that $\left[\widehat{\widehat{s_k}}'(j)^* \quad \widehat{\widehat{t_k}}'(j)^*\right] \widehat{\mathbb{L}}'_k(j) I'_{p(j)} = \left[\widehat{\widehat{s_k}}'(j)^* I'_{q(j)} \quad \widehat{\widehat{t_k}}'(j)^* I'_{p(j)-q(j)}\right] \widehat{\mathbb{L}}'_k(j) I'_{p(j)}$, we make use of either Proposition 3.26 or (5.27) to show that

$$d\left[(\widehat{s_k}'(j) - \widehat{\widehat{s_k}}'(j))^* \;\; (\widehat{t_k}'(j) - \widehat{\widehat{t_k}}'(j))^*\right] \begin{bmatrix} I'_{q(j)} & 0 \\ 0 & I'_{p(j)-q(j)} \end{bmatrix} \widehat{\mathbb{L}}'_k(j) I'_{p(j)} = d\left[\widehat{s_{k+1}}'(j+D^k r_\mu)\right]_{\mu=1}^{d*} I'_{p(j)}. \tag{5.29}$$

Without loss of generality, we shall assume that $\left[\widehat{\widehat{s_k}}'(j)^* \quad \widehat{\widehat{t_k}}'(j)^*\right]^*$ is the zero vector. Now,

$$d\left[\widehat{s_k}'(j)^* I'_{q(j)} \quad \widehat{t_k}'(j)^* I'_{p(j)-q(j)}\right] \begin{bmatrix} \widehat{H}'_{k+1}(j+D^k r_1) I'_{p(j,r_1)} & \cdots & \widehat{H}'_{k+1}(j+D^k r_d) I'_{p(j,r_d)} \\ \widehat{G}'_{k+1}(j+D^k r_1) I'_{p(j,r_1)} & \cdots & \widehat{G}'_{k+1}(j+D^k r_d) I'_{p(j,r_d)} \end{bmatrix}$$
$$= \begin{bmatrix} \widehat{s_{k+1}}'(j+D^k r_1) \\ \vdots \\ \widehat{s_{k+1}}'(j+D^k r_d) \end{bmatrix}^* \begin{bmatrix} I'_{p(j,r_1)} \widehat{H}'_{k+1}(j+D^k r_1)^* I'_{q(j)} & I'_{p(j,r_1)} \widehat{G}'_{k+1}(j+D^k r_1)^* I'_{p(j)-q(j)} \\ \vdots & \vdots \\ I'_{p(j,r_d)} \widehat{H}'_{k+1}(j+D^k r_d)^* I'_{q(j)} & I'_{p(j,r_d)} \widehat{G}'_{k+1}(j+D^k r_d)^* I'_{p(j)-q(j)} \end{bmatrix}$$
$$\begin{bmatrix} I'_{q(j)} \widehat{H}'_{k+1}(j+D^k r_1) I'_{p(j,r_1)} & \cdots & I'_{q(j)} \widehat{H}'_{k+1}(j+D^k r_d) I'_{p(j,r_d)} \\ I'_{p(j)-q(j)} \widehat{G}'_{k+1}(j+D^k r_1) I'_{p(j,r_1)} & \cdots & I'_{p(j)-q(j)} \widehat{G}'_{k+1}(j+D^k r_d) I'_{p(j,r_d)} \end{bmatrix}.$$

This implies that

$$d\left[\widehat{s_k}'(j)^* I'_{q(j)} \quad \widehat{t_k}'(j)^* I'_{p(j)-q(j)}\right]\begin{bmatrix}\widehat{H}'_{k+1}(j+D^k r_1)I'_{p(j,r_1)} & \cdots & \widehat{H}'_{k+1}(j+D^k r_d)I'_{p(j,r_d)} \\ \widehat{G}'_{k+1}(j+D^k r_1)I'_{p(j,r_1)} & \cdots & \widehat{G}'_{k+1}(j+D^k r_d)I'_{p(j,r_d)}\end{bmatrix}$$

$$=\left[\sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)}\widehat{H}'_{k+1}(j+D^k r)^* I'_{q(j)} \ \sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)}\widehat{G}'_{k+1}(j+D^k r)^* I'_{p(j)-q(j)}\right]$$

$$\begin{bmatrix}I'_{q(j)}\widehat{H}'_{k+1}(j+D^k r_1)I'_{p(j,r_1)} & \cdots & I'_{q(j)}\widehat{H}'_{k+1}(j+D^k r_d)I'_{p(j,r_d)} \\ I'_{p(j)-q(j)}\widehat{G}'_{k+1}(j+D^k r_1)I'_{p(j,r_1)} & \cdots & I'_{p(j)-q(j)}\widehat{G}'_{k+1}(j+D^k r_d)I'_{p(j,r_d)}\end{bmatrix}.$$

Hence, for a given $r_\mu \in \mathcal{R}_1$,

$$\begin{aligned}
& d\widehat{s_k}'(j)^* I'_{q(j)}\widehat{H}'_{k+1}(j+D^k r_\mu)I'_{p(j,r_\mu)} + d\widehat{t_k}'(j)^* I'_{p(j)-q(j)}\widehat{G}'_{k+1}(j+D^k r_\mu)I'_{p(j,r_\mu)} \\
&= \sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)}\widehat{H}'_{k+1}(j+D^k r)^* I'_{q(j)}\widehat{H}'_{k+1}(j+D^k r_\mu)I'_{p(j,r_\mu)} \\
&+ \sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)}\widehat{G}'_{k+1}(j+D^k r)^* I'_{p(j)-q(j)}\widehat{G}'_{k+1}(j+D^k r_\mu)I'_{p(j,r_\mu)}.
\end{aligned}$$

Therefore

$$\left[d\widehat{s_k}'(j)^* I'_{q(j)} - \sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)}\widehat{H}'_{k+1}(j+D^k r)I'_{q(j)}\right]\widehat{H}'_{k+1}(j+D^k r_\mu)I'_{p(j,r_\mu)}$$

$$=\left[\Big(\sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^k r)^* I'_{p(j,r)}\widehat{G}'_{k+1}(j+D^k r) - d\widehat{t_k}'(j)^*\Big)I'_{p(j)-q(j)}\right]\widehat{G}'_{k+1}(j+D^k r_\mu)I'_{p(j,r_\mu)}$$

and the expressions in brackets must be equal to zero for a fixed $j \in \mathcal{R}_k$ and a given $r_\mu \in \mathcal{R}_1$ that could vary freely. So we infer from (5.28) and (5.29) that (5.24), (5.25) and (5.26) hold. □

Comparing the nonorthogonal representation with the semi-orthogonal representation, we find that this time $\widehat{H}'^*_{k+1}$ and $\widehat{G}'^*_{k+1}$ play the role of decomposition masks while the role of $\widehat{H}'_{k+1}$ and $\widehat{G}'_{k+1}$ as reconstruction masks remain unchanged.

5.4 Stationary Wavelet Transform

In practice, we apply the wavelet transform to 1-dimensional data and for 2-dimensional data, we construct tensor product equivalents of the 1-dimensional wavelets and apply the corresponding wavelet transform. To simplify matters, we shall now restrict our discussion to the 1-dimensional setting with arbitrary integer dilation factor M.

The analysis of the data using the decomposition algorithm given in Proposition 5.10 is not translation invariant in time, i.e. not modulation invariant in frequency, due to downsampling of the filtered coefficients in the time domain. If we apply a single level transform of such kind in regression, then this will lead to only the even numbered coefficients being processed for the case of the dilation factor $M = 2$. Due to the misalignments between features in the signal and in the transform representation, artifacts near neighbourhoods of discontinuities will be introduced with the thresholding of the wavelet coefficients for the elimination of high frequency noise. The artifacts introduced in the processing could be eliminated with the use of the *stationary wavelet transform* (SWT) [40] by an averaging process described in [12].

First, let us define the *shift operator* $S_k^r : \mathcal{S}(M^k) \to \mathcal{S}(M^k)$ given by

$$\{(S_k^r s_k)(l)\}_{l\in\mathcal{L}_k} = \{s_k(l-r)\}_{l\in\mathcal{L}_k},$$

where $r \in \mathcal{L}_k$. We shall also identify S_K^r with $S_{K+k}^r|_{\mathcal{S}(M^K)} : \mathcal{S}(M^K) \to \mathcal{S}(M^K)$ for every $r \in \mathcal{L}_K$. The *upsampling operator* $\uparrow_k\colon \mathcal{S}(M^k) \to \mathcal{S}(M^{k+1})$ is given by

$$\uparrow_k\colon \{s_k(l)\}_{l\in\mathcal{L}_k} \mapsto \{\uparrow_k s_k(r)\}_{r\in\mathcal{L}_{k+1}} := \{s_k(l)\mathbf{1}_{\{r=Ml\}}\}_{r\in\mathcal{L}_{k+1}}.$$

We shall also write the composition $\uparrow_K^{K+k}\colon \mathcal{S}(M^K) \mapsto \mathcal{S}(M^{K+k})$ as

$$\uparrow_K^{K+k}:=\uparrow_{K+k-1}\uparrow_{K+k-2}\cdots\uparrow_K .$$

The *general downsampling operator* $\downarrow_{k,r}\colon \mathcal{S}(M^k) \to \mathcal{S}(M^{k-1})$ for a given $r \in \mathcal{L}_k$ is given by

$$\downarrow_{k,r}\colon \{s_k(l)\}_{l\in\mathcal{L}_k} \mapsto \{\downarrow_{k,r} s_k(l)\}_{l\in\mathcal{L}_{k-1}} := \{\downarrow_{k,0} S_k^{-r} s_k(l)\}_{l\in\mathcal{L}_{k-1}} = \{s_k(Ml+r)\}_{l\in\mathcal{L}_k}.$$

In the event that $r = 0$, then we simply write $\downarrow_{k,0}$ as $\downarrow_k$. For a given $r \in \mathcal{L}_K$, we shall also write the composition $\downarrow_{K+k,r}^K\colon \mathcal{S}(M^{K+k}) \to \mathcal{S}(M^K)$ as

$$\downarrow_{K+k,r}^K:=\downarrow_{K+1,r}\downarrow_{K+2,r}\cdots\downarrow_{K+k,r} .$$

We define the periodic convolution $\otimes : \mathcal{S}(M^k) \times \mathcal{S}(M^k) \to \mathcal{S}(M^k)$ of $a_k \in \mathcal{L}_k$ and $b_k \in \mathcal{L}_k$ as

$$\{a_k \otimes b_k(l)\}_{l\in\mathcal{L}_k} = \left\{\sum_{r\in\mathcal{L}_k} a_k(l-r)b_k(r)\right\}_{l\in\mathcal{L}_k}.$$

Recall also that the discrete Fourier transform of $s_k \in \mathcal{S}(M^k)^{\rho\times 1}$ is defined as

$$\widehat{s_k}(j) := \sum_{l\in\mathcal{L}_k} s_k(l)\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-k}l}.$$

We shall now rewrite the decomposition and reconstruction process of (5.24), (5.25) and (5.26) given in Proposition 5.10 in the time domain using the discrete Fourier transform.

Proposition 5.11. *Assume Setup 5.3. If f_{k+1} is given by (5.1), then*

$$
\begin{aligned}
f_{k+1} &= \sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l)^* T^l_{k+1}\Phi_{k+1} = \sum_{l\in\mathcal{L}_k}\left[s_k(l)^* T^l_k \Phi_k + t_k(l)^* T^l_k \Psi_k\right] \\
&= \sum_{l\in\mathcal{L}_{k+1}} \left[(\uparrow_k s_k^*)\otimes H_{k+1}(l) + (\uparrow_k t_k^*)\otimes G_{k+1}(l)\right] T^l_{k+1}\Phi_{k+1}. \qquad (5.30)
\end{aligned}
$$

Further, there exist $\left[\widehat{\widetilde{s}}_k{}'(j)^* \;\; \widehat{\widetilde{t}}_k{}'(j)^*\right]^* \in \operatorname{Ker}\widehat{\mathbb{L}}'_k(j)^*$, $j\in\mathcal{R}_k$, *such that for every* $l\in\mathcal{L}_k$ *and* $n\in\mathcal{L}_{k+1}$,

$$
\begin{aligned}
\left[s'_k(l) - \widetilde{s}_k{}'(l)\right]^* &= [s'^*_{k+1}\otimes H'^*_{k+1}](Ml), \qquad (5.31)\\
\left[t'_k(l) - \widetilde{t}_k{}'(l)\right]^* &= [s'^*_{k+1}\otimes G'^*_{k+1}](Ml), \qquad (5.32)\\
s'_{k+1}(n)^* &= [(\uparrow_k s'^*_k)\otimes H'_{k+1}](n) + [(\uparrow_k t'^*_k)\otimes G'_{k+1}](n). \qquad (5.33)
\end{aligned}
$$

Proof. First, let us apply Lemma 3.2 to the definition of polyphase harmonics given in (3.16) to express (5.1) as

$$
\begin{aligned}
f_{k+1}(t) &= \sum_{j\in\mathcal{R}_{k+1}} \widehat{s_{k+1}}(j)^* \left[\sum_{n\in\mathbb{Z}^s} \widehat{\Phi}_{k+1}(j+D^{k+1}n)\mathrm{e}^{\mathrm{i}(j+D^{k+1}n)\cdot t}\right] \\
&= \sum_{n\in\mathbb{Z}^s}\sum_{j\in\mathcal{R}_{k+1}} \widehat{s_{k+1}}(j+D^{k+1}n)^*\widehat{\Phi}_{k+1}(j+D^{k+1}n)\mathrm{e}^{\mathrm{i}(j+D^{k+1}n)\cdot t} = \sum_{n\in\mathbb{Z}^s}\widehat{s_{k+1}}(n)^*\widehat{\Phi}_{k+1}(n)\mathrm{e}^{\mathrm{i}n\cdot t} \\
&= \sum_{n\in\mathbb{Z}^s}\sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l)^*\mathrm{e}^{-\mathrm{i}2\pi n\cdot M^{-(k+1)}l}(2\pi)^{-s}\int_{\mathbb{T}^s}\Phi_{k+1}(x)\mathrm{e}^{-\mathrm{i}n\cdot x}\mathrm{d}x\mathrm{e}^{\mathrm{i}n\cdot t} \qquad (5.34)\\
&= \sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l)^*\sum_{n\in\mathbb{Z}^s}(2\pi)^{-s}\int_{\mathbb{T}^s}\Phi_{k+1}(x)\mathrm{e}^{-\mathrm{i}n\cdot\left[x+2\pi M^{-(k+1)}l\right]}\mathrm{d}x\mathrm{e}^{\mathrm{i}n\cdot t} \\
&= \sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l)^*\sum_{n\in\mathbb{Z}^s}(2\pi)^{-s}\int_{\mathbb{T}^s}\Phi_{k+1}(x-2\pi M^{-(k+1)}l)\mathrm{e}^{-\mathrm{i}n\cdot x}\mathrm{d}x\mathrm{e}^{\mathrm{i}n\cdot t} \\
&= \sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l)^* T^l_{k+1}\Phi_{k+1}(t).
\end{aligned}
$$

A similar computation of (5.34) with s_k and t_k and ϕ_k and ψ_k respectively leads to the second equality of (5.30). To show the third equality of (5.30), we express (5.1) as

$$
\begin{aligned}
f_{k+1}(t) &= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\sum_{n\in\mathbb{Z}^s}\Big[\widehat{s_k}(j+D^kr+D^{k+1}n)^*\widehat{H}_{k+1}(j+D^kr+D^{k+1}n)\\
&+\widehat{t_k}(j{+}D^kr{+}D^{k+1}n)^*\widehat{G}_{k+1}(j{+}D^kr{+}D^{k+1}n)\Big]\Big[\widehat{\Phi}_{k+1}(j{+}D^kr{+}D^{k+1}n)\mathrm{e}^{\mathrm{i}(j+D^kr+D^{k+1}n)\cdot t}\Big]\\
&= \sum_{n\in\mathbb{Z}^s}\Big[\widehat{s_k}(n)^*\widehat{H}_{k+1}(n)+\widehat{t_k}(n)^*\widehat{G}_{k+1}(n)\Big]\widehat{\Phi}_{k+1}(n)\mathrm{e}^{\mathrm{i}n\cdot t}\\
&= \sum_{n\in\mathbb{Z}^s}\sum_{r\in\mathcal{L}_k}\sum_{l\in\mathcal{L}_{k+1}}\left[s_k(r)^*H_{k+1}(l)+t_k(r)^*G_{k+1}(l)\right]\mathrm{e}^{-\mathrm{i}2\pi n\cdot M^{-k}r}\mathrm{e}^{-\mathrm{i}2\pi n\cdot M^{-(k+1)}l}\widehat{\Phi}_{k+1}(n)\mathrm{e}^{\mathrm{i}n\cdot t}\\
&= \sum_{r\in\mathcal{L}_k}\sum_{l\in\mathcal{L}_{k+1}}\left[s_k(r)^*H_{k+1}(l)+t_k(r)^*G_{k+1}(l)\right]\\
&\quad\sum_{n\in\mathbb{Z}^s}(2\pi)^{-s}\int_{\mathbb{T}^s}\Phi_{k+1}(x)\mathrm{e}^{-\mathrm{i}n\cdot x}\mathrm{d}x\mathrm{e}^{\mathrm{i}n\cdot t}\mathrm{e}^{-\mathrm{i}2\pi n\cdot M^{-(k+1)}(l+Mr)}\\
&= \sum_{l\in\mathcal{L}_{k+1}}\sum_{r\in\mathcal{L}_k}\left[s_k(r)^*H_{k+1}(l-Mr)+t_k(r)^*G_{k+1}(l-Mr)\right]\\
&\quad\sum_{n\in\mathbb{Z}^s}(2\pi)^{-s}\int_{\mathbb{T}^s}\Phi_{k+1}(x)\mathrm{e}^{-\mathrm{i}n\cdot\left[x+2\pi M^{-(k+1)}l\right]}\mathrm{d}x\mathrm{e}^{\mathrm{i}n\cdot t}\\
&= \sum_{l\in\mathcal{L}_{k+1}}\left[(\uparrow_k s_k^*)\otimes H_{k+1}(l)+(\uparrow_k t_k^*)\otimes G_{k+1}(l)\right]T_{k+1}^l\Phi_{k+1}(t).
\end{aligned}
$$

To confirm (5.31), we apply Lemma 3.2 and the inverse DFT to (5.24) in Proposition 5.10 to get

$$
\begin{aligned}
d^{k+1}\left[s_k'(l)-\widetilde{s_k}'(l)\right]^* &= \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\widehat{s_{k+1}}'(j+D^kr)^*\widehat{H}_{k+1}'(j+D^kr)^*\mathrm{e}^{\mathrm{i}2\pi(j+D^kr)\cdot M^{-k}l}\\
&= \sum_{j\in\mathcal{R}_{k+1}}\widehat{s_{k+1}}'(j)^*\widehat{H}_{k+1}'(j)^*\mathrm{e}^{\mathrm{i}2\pi j\cdot M^{-k}l}\\
&= \sum_{j\in\mathcal{R}_{k+1}}\sum_{r\in\mathcal{L}_{k+1}}s_{k+1}'(r)^*\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}r}\sum_{n\in\mathcal{L}_{k+1}}H_{k+1}'(n)^*\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}n}\mathrm{e}^{\mathrm{i}2\pi j\cdot M^{-k}l}\\
&= \sum_{j\in\mathcal{R}_{k+1}}\sum_{n\in\mathcal{L}_{k+1}}\sum_{r\in\mathcal{L}_{k+1}}s_{k+1}'(r)^*H_{k+1}'(n-r)^*\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}n}\mathrm{e}^{\mathrm{i}2\pi j\cdot M^{-k}l}\\
&= \sum_{j\in\mathcal{R}_{k+1}}\sum_{n\in\mathcal{L}_{k+1}}[s_{k+1}'^*\otimes H_{k+1}'^*](n)\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}n}\mathrm{e}^{\mathrm{i}2\pi j\cdot M^{-(k+1)}Ml}\\
&= d^{k+1}[s_{k+1}'^*\otimes H_{k+1}'^*](Ml).
\end{aligned}
$$

Similarly, we could show that (5.25) is equivalent to (5.32). Finally, (5.33) could be shown to be equivalent to (5.26) since

$$
\begin{aligned}
& d^{k+1} s'_{k+1}(l)^* = \sum_{j\in\mathcal{R}_k}\sum_{r\in\mathcal{R}_1}\Big[\widehat{s_k}'(j+D^k r)^*\widehat{H}'_{k+1}(j+D^k r) \\
& +\widehat{t_k}'(j+D^k r)^*\widehat{G}'_{k+1}(j+D^k r)\Big]\,\mathrm{e}^{\mathrm{i}2\pi(j+D^k r)\cdot M^{-(k+1)}l} \\
& = \sum_{j\in\mathcal{R}_{k+1}}\Big[\widehat{s_k}'(j)^*\widehat{H}'_{k+1}(j)+\widehat{t_k}'(j)^*\widehat{G}'_{k+1}(j)\Big]\,\mathrm{e}^{\mathrm{i}2\pi j\cdot M^{-(k+1)}l} \\
& = \sum_{j\in\mathcal{R}_{k+1}}\sum_{r\in\mathcal{L}_k}\sum_{n\in\mathcal{L}_{k+1}}\Big[s'_k(r)^*\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}Mr}H'_{k+1}(n)\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}n} \\
& +t'_k(r)^*\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}Mr}G'_{k+1}(n)\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}n}\Big]\,\mathrm{e}^{\mathrm{i}2\pi j\cdot M^{-(k+1)}l} \\
& = \sum_{j\in\mathcal{R}_{k+1}}\sum_{n\in\mathcal{L}_{k+1}}\sum_{r\in\mathcal{L}_k}\left[s'_k(r)^*H'_{k+1}(n)+t'_k(r)^*G'_{k+1}(n)\right]\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}(n+Mr-l)} \\
& = \sum_{j\in\mathcal{R}_{k+1}}\sum_{n\in\mathcal{L}_{k+1}}\sum_{r\in\mathcal{L}_k}\big[s'_k(r)^*H'_{k+1}(n-Mr) \\
& +t'_k(r)^*G'_{k+1}(n-Mr)\big]\,\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}n}\mathrm{e}^{\mathrm{i}2\pi j\cdot M^{-(k+1)}l} \\
& = \sum_{j\in\mathcal{R}_{k+1}}\sum_{n\in\mathcal{L}_{k+1}}\big[[(\uparrow_k s'^*_k)\otimes H'_{k+1}](n)+[(\uparrow_k t'^*_k)\otimes G'_{k+1}](n)\big]\,\mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}(n-l)} \\
& = d^{k+1}\left[[(\uparrow_k s'^*_k)\otimes H'_{k+1}](l)+[(\uparrow_k t'^*_k)\otimes G'_{k+1}](l)\right].
\end{aligned}
$$

□

We shall have uniqueness in the reconstruction process in Proposition 5.11 if for every $j\in\mathcal{R}_k$ and $r\in\mathcal{R}_1$ the matrices $\mathrm{M}_k(j)$, $\mathrm{N}_k(j)$ and $\mathrm{M}_{k+1}(j+D^k r)$ as defined in Setup 5.1 have full rank.

Corollary 5.12. *Assume Setup* 5.3 *with* $\operatorname{rank}\mathrm{M}_{k+1}(j+D^k r)=\rho$, $\operatorname{rank}\mathrm{M}_k(j)=\rho$ *and* $\operatorname{rank}\mathrm{N}_k(j)=\varrho_k$ *for every* $j\in\mathcal{R}_k$ *and* $r\in\mathcal{R}_1$*. If* f_{k+1} *is given by* (5.1)*, then*

$$
\begin{aligned}
f_{k+1} &= \sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l)^*T^l_{k+1}\Phi_{k+1} = \sum_{l\in\mathcal{L}_k}\left[s_k(l)^*T^l_k\Phi_k+t_k(l)^*T^l_k\Psi_k\right] \\
&= \sum_{l\in\mathcal{L}_{k+1}}\left[(\uparrow_k s^*_k)\otimes H_{k+1}(l)+(\uparrow_k t^*_k)\otimes G_{k+1}(l)\right]T^l_{k+1}\Phi_{k+1}. \qquad (5.35)
\end{aligned}
$$

Further, for every $l\in\mathcal{L}_k$ *and* $n\in\mathcal{L}_{k+1}$*, we have*

$$s_k(l)^*=[s^*_{k+1}\otimes H^*_{k+1}](Ml), \qquad (5.36)$$

$$t_k(l)^*=[s^*_{k+1}\otimes G^*_{k+1}](Ml), \qquad (5.37)$$

$$s_{k+1}(n)^*=[(\uparrow_k s^*_k)\otimes H_{k+1}](n)+[(\uparrow_k t^*_k)\otimes G_{k+1}](n). \qquad (5.38)$$

The next proposition, which concerns the upsampling, downsampling, periodic convolution and shift operations, will be used to relate the stationary wavelet transform with the ordinary wavelet transform.

Proposition 5.13. *The upsampling, downsampling, periodic convolution and shift operations satisfy*

$$S_{k-1}^{r} \downarrow_k s_k = \downarrow_k S_k^{Mr} s_k \tag{5.39}$$

$$\downarrow_{K+k}^{K} [(\uparrow_K^{K+k} H_K) \otimes s_{K+k}] = H_K \otimes (\downarrow_{K+k}^{K} s_{K+k}) \tag{5.40}$$

$$S_k^r (H_k \otimes s_k) = H_k \otimes (S_k^r s_k). \tag{5.41}$$

Proof. The verification of (5.39), (5.40) and (5.41) is as follows:

$$\begin{aligned}
&S_{k-1}^{r} \downarrow_k \{s_k(l)\}_{l \in \mathcal{L}_k} = S_{k-1}^{r} \{s_k(Ml)\}_{l \in \mathcal{L}_k} = \{s_k(M(l-r))\}_{l \in \mathcal{L}_k} = \downarrow_k \{S_k^{Mr} s_k(l)\}_{l \in \mathcal{L}_k},\\
&\downarrow_{K+k}^{K} [(\uparrow_K^{K+k} H_K) \otimes s_{K+k}](l) = \downarrow_{K+k}^{K} \{[\sum_{r \in \mathcal{L}_{K+k}} (\uparrow_K^{K+k} H_K)(r) s_{K+k}(l-r)]\}_{l \in \mathcal{L}_{K+k}}(l)\\
&= \downarrow_{K+k}^{K} \{[\sum_{r \in \mathcal{L}_K} (\uparrow_K^{K+k} H_K)(M^k r) s_{K+k}(l - M^k r)]\}_{l \in \mathcal{L}_{K+k}}(l)\\
&= \sum_{r \in \mathcal{L}_K} (\uparrow_K^{K+k} H_K)(M^k r) s_{K+k}(M^k l - M^k r) = \sum_{r \in \mathcal{L}_K} H_K(r) s_{K+k}(M^k(l-r))\\
&= H_K \otimes (\downarrow_{K+k}^{K} s_{K+k})(l),\\
&S_k^r (H_k \otimes s_k)(l) = H_k \otimes s_k(l-r) = \sum_{\nu \in \mathcal{L}_k} H_k(\nu) s_k(l-r-\nu) = \sum_{\nu \in \mathcal{L}_k} H_k(\nu)(S_k^r s_k)(l-\nu)\\
&= H_k \otimes (S_k^r s_k)(l).
\end{aligned}$$

□

Therefore, we observe that (i) a shift of Mr units followed by downsampling is the same as downsampling followed by a shift of r units, (ii) performing convolution with an upsampled filter followed by downsampling is the same as performing downsampling first followed by convolution with the filter, and (iii) shifts applied before convolution is equivalent to shifts applied after convolution.

Following the convention used by [40], let us also define the *$\boldsymbol{\epsilon}$-decimated discrete wavelet transform* ($\boldsymbol{\epsilon}$-DWT), where the M-nary representation of $\boldsymbol{\epsilon}$ is $\epsilon_0 \cdots \epsilon_{K+L}$. Let the M-nary representations of $\epsilon_0\epsilon_1 \cdots \epsilon_K$ and $\epsilon_{K+1} \cdots \epsilon_{K+L}$ be denoted by $\mathbf{l}_1$ and $\mathbf{l}_2$ respectively. For the standard discrete wavelet transform (DWT), we deal with the sequence $t_K = \downarrow_{K+1} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_k H_k \otimes s_{K+L}$. In the $\boldsymbol{\epsilon}$-DWT, using Proposition 5.13, we handle the sequence $t_K^{\boldsymbol{\epsilon}}$ given by

$$
\begin{aligned}
t_K^{\boldsymbol{\epsilon}} := & \downarrow_{K+1,\epsilon_{K+1}} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_{k,\epsilon_k} H_k \otimes s_{K+L} = \downarrow_{K+1} S_{K+1}^{-\epsilon_{K+1}} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_k S_k^{-\epsilon_k} H_k \otimes s_{K+L} \\
&= \downarrow_{K+1} S_{k+1}^{-\epsilon_{k+1}} G_{K+1} \bigotimes_{k=K+2}^{k=K+L} S_{k-1}^{-\epsilon_{k-1}} \downarrow_k H_k \otimes (S_{K+L}^{-\epsilon_{K+L}} s_{K+L}) \\
&= \downarrow_{K+1} S_{k+1}^{-\epsilon_{k+1}} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_k S_{k-1}^{-M\epsilon_{k-1}} H_k \otimes (S_{K+L}^{-\epsilon_{K+L}} s_{K+L}) \\
&= \downarrow_{K+1} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_k H_k \otimes (S_{K+1}^{-M^{L-1}\epsilon_{K+1}} \cdots S_{K+L-1}^{-M\epsilon_{K+L-1}} S_{K+L}^{-\epsilon_{K+L}} s_{K+L}) \\
&= \downarrow_{K+1} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_k H_k \otimes (S_{K+L}^{-\mathbf{l}_2} s_{K+L}).
\end{aligned}
$$

If we apply the operator $S_{K+L}^{-\mathbf{l}_1}$ to $t_K^{\boldsymbol{\epsilon}}$, then we have

$$
\begin{aligned}
S_{K+L}^{-\mathbf{l}_1} t_K^{\boldsymbol{\epsilon}} &= S_{K+L}^{-\mathbf{l}_1} \downarrow_{K+1} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_k H_k \otimes (S_{K+L}^{-\mathbf{l}_2} s_{K+L}) \\
&= \downarrow_{K+1} S_{K+L}^{-M\mathbf{l}_1} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_k H_k \otimes (S_{K+L}^{-\mathbf{l}_2} s_{K+L}) \\
&= \downarrow_{K+1} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_k H_k \otimes S_{K+L}^{-M^L\mathbf{l}_1} (S_{K+L}^{-\mathbf{l}_2} s_{K+L}) \\
&= \downarrow_{K+1} G_{K+1} \bigotimes_{k=K+2}^{K+L} \downarrow_k H_k \otimes (S_{K+L}^{-\boldsymbol{\epsilon}} s_{K+L}).
\end{aligned}
$$

Therefore $S_{K+L}^{-\mathbf{l}_1} t_K^{\boldsymbol{\epsilon}}$ is the $\mathbf{l}_1$-shifted K^{th} detail sequence of the standard DWT applied to $S_{K+L}^{-\boldsymbol{\epsilon}} s_{K+L}$. Similarly, we would obtain the $\mathbf{l}_1$-shifted K^{th} *smooth part* $S_{K+L}^{-\mathbf{l}_1} s_K^{\boldsymbol{\epsilon}}$ as

$$
S_{K+L}^{-\mathbf{l}_1} s_K^{\boldsymbol{\epsilon}} = \bigotimes_{k=K+1}^{K+L} \downarrow_k H_k \otimes (S_{K+L}^{-\boldsymbol{\epsilon}} s_{K+L}).
$$

We define the *stationary wavelet transform* (SWT) of $a_{K+L} := s_{K+L}$ at level $k \in \{K+L, \ldots, 1\}$ recursively by

$$
a_{k-1} = (\uparrow_k^{K+L} H_k) \otimes a_k \quad \text{and} \quad b_{k-1} = (\uparrow_k^{K+L} G_k) \otimes a_k. \tag{5.42}
$$

Due to the absence of downsampling, a_{k-1} and b_{k-1} for $k \in \{1, \ldots, K+L\}$ are still in $\mathcal{S}(M^{K+L})$. Using (5.42), we could show that the SWT of a_{K+L} contains the coefficients of the $\boldsymbol{\epsilon}$-DWT for every choice of $\boldsymbol{\epsilon}$. Indeed, for each $k \in \{K+L-1, \dots, 1\}$,

$$
\begin{aligned}
& S_{K+L}^{-M^{-(K-k+1)}\boldsymbol{l}_1} \downarrow_{K+L}^{k} S_{K+L}^{\boldsymbol{\epsilon}} b_k = \downarrow_{K+L}^{k} S_{K+L}^{-M^L \boldsymbol{l}_1+\boldsymbol{\epsilon}} b_k = \downarrow_{K+L}^{k} S_{K+L}^{\boldsymbol{l}_2} b_k \\
= & \downarrow_{K+L}^{k} S_{K+L}^{\boldsymbol{l}_2} (\uparrow_{k+1}^{K+L} G_{k+1}) \otimes (\uparrow_{k+2}^{K+L} H_{k+2}) \otimes \cdots \otimes H_{K+L} \otimes s_{K+L} \\
= & \downarrow_{K+L}^{k} [(\uparrow_{k+1}^{K+L} G_{k+1}) \otimes (\uparrow_{k+2}^{K+L} H_{k+2}) \otimes \cdots \otimes H_{K+L} \otimes S_{K+L}^{\boldsymbol{l}_2} s_{K+L}] \\
= & \downarrow_{k+1}^{k} \downarrow_{K+L}^{k+1} [(\uparrow_{k+1}^{K+L} G_{k+1}) \otimes (\uparrow_{k+2}^{K+L} H_{k+2}) \otimes \cdots \otimes H_{K+L} \otimes S_{K+L}^{\boldsymbol{l}_2} s_{K+L}] \\
= & \downarrow_{k+1}^{k} [G_{k+1} \otimes \downarrow_{K+L}^{k+1} \left[(\uparrow_{k+2}^{K+L} H_{k+2}) \otimes \cdots \otimes H_{K+L} \otimes S_{K+L}^{\boldsymbol{l}_2} s_{K+L}]\right] \\
= & \downarrow_{k+1}^{k} \left[G_{k+1} \otimes \downarrow_{k+2}^{k+1} \downarrow_{K+L}^{k+2} [(\uparrow_{k+2}^{K+L} H_{k+2}) \otimes \cdots \otimes H_{K+L} \otimes S_{K+L}^{\boldsymbol{l}_2} s_{K+L}]\right] \\
= & \downarrow_{k+1}^{k} \left[G_{k+1} \otimes \downarrow_{k+2}^{k+1} \left[H_{k+2} \otimes \cdots \otimes \downarrow_{K+L}^{K+L-1} [H_{K+L} \otimes S_{K+L}^{\boldsymbol{l}_2} s_{K+L}]\right] \cdots\right] \\
= & \downarrow_{k+1} \left[G_{k+1} \otimes \downarrow_{k+2} \left[H_{k+2} \otimes \cdots \otimes \downarrow_{K+L} [H_{K+L} \otimes S_{K+L}^{\boldsymbol{l}_2} s_{K+L}]\right] \cdots\right] \\
= & \downarrow_{k+1} G_{k+1} \bigotimes_{n=k+2}^{K+L} \downarrow_n H_n \otimes S_{K+L}^{\boldsymbol{l}_2} s_{K+L} = S_{K+L}^{M^{-(K+L-k+1)}\boldsymbol{l}_2} t_k^{\boldsymbol{\epsilon}}.
\end{aligned}
$$

The SWT contains all the $2\pi M^{-K-L}$ shifts of the refinable function and the corresponding wavelet system. We shall see that this essentially leads to the quasi-affine representation of f_{K+L}.

Proposition 5.14. *Assume Setup* 5.3. *If* f_{k+1} *is given by* (5.1), *then for* $\epsilon \in \mathcal{L}_1$,

$$
\begin{aligned}
f_{k+1} &= \sum_{l \in \mathcal{L}_{k+1}} s_{k+1}(l+\epsilon)^* T_{k+1}^{l+\epsilon} \Phi_{k+1} = \sum_{l \in \mathcal{L}_k} \left[s_k^{\epsilon}(l)^* T_{k+1}^{Ml+\epsilon} \Phi_k + t_k^{\epsilon}(l)^* T_{k+1}^{Ml+\epsilon} \Psi_k\right] \\
&= d^{-1} \sum_{\varepsilon \in \mathcal{L}_1} \sum_{l \in \mathcal{L}_k} \left[s_k^{\varepsilon}(l)^* T_{k+1}^{Ml+\varepsilon} \Phi_k + t_k^{\varepsilon}(l)^* T_{k+1}^{Ml+\varepsilon} \Psi_k\right] \\
&= d^{-1} \sum_{\varepsilon \in \mathcal{L}_1} \sum_{l \in \mathcal{L}_k} \left[a_k(Ml+\varepsilon)^* T_{k+1}^{Ml+\varepsilon} \Phi_k + b_k(Ml+\varepsilon)^* T_{k+1}^{Ml+\varepsilon} \Psi_k\right], \\
&= \sum_{l \in \mathcal{L}_{k+1}} [(\uparrow_k s_k^{\epsilon *}) \otimes H_{k+1}(l) + (\uparrow_k t_k^{\epsilon *}) \otimes G_{k+1}(l)] T_{k+1}^{l+\epsilon} \Phi_{k+1}. \qquad (5.43)
\end{aligned}
$$

Further, there exist $\left[\widehat{\widetilde{s}_k^{\epsilon}}'(j)^* \ \widehat{\widetilde{t}_k^{\epsilon}}'(j)^*\right]^* \in \operatorname{Ker} \widehat{\mathbb{L}}_k'(j)^*$, $j \in \mathcal{R}_k$ *such that*

$$
\left[s_k^{\epsilon\prime}(l) - \widetilde{s}_k^{\epsilon\prime}(l)\right]^* = [s_{k+1}'^* \otimes H_{k+1}'^*](Ml+\epsilon) = a_k'(Ml+\epsilon)^*, \qquad (5.44)
$$

$$
\left[t_k^{\epsilon\prime}(l) - \widetilde{t}_k^{\epsilon\prime}(l)\right]^* = [s_{k+1}'^* \otimes G_{k+1}'^*](Ml+\epsilon) = b_k'(Ml+\epsilon)^*, \qquad (5.45)
$$

$$
s_{k+1}'(n+\epsilon)^* = [(\uparrow_k s_k^{\epsilon\prime *}) \otimes H_{k+1}' + (\uparrow_k t_k^{\epsilon\prime *}) \otimes G_{k+1}'](n), \qquad (5.46)
$$

for every $l \in \mathcal{L}_k$ *and* $n \in \mathcal{L}_{k+1}$, *where* $\widehat{s_k^{\epsilon}}'(j) = I_{q(j)}' U_k(j) \widehat{s_k^{\epsilon}}(j)$, $\widehat{t_k^{\epsilon}}'(j) = I_{p(j)-q(j)}' V_k(j) \widehat{t_k^{\epsilon}}(j)$, $\widehat{a_k}'(j+D^k r) = I_{q(j)}' U_k(j) \widehat{a_k}(j+D^k r)$, $\widehat{b_k}'(j+D^k r) = I_{p(j)-q(j)}' V_k(j) \widehat{b_k}(j+D^k r)$ *and* a_k *and* b_k *are given as in* (5.42).

Proof. By Proposition 5.11, we could utilize (5.30) and (5.33) to obtain

$$\begin{aligned} T_{k+1}^{-\epsilon} f_{k+1} &= \sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l)^* T_{k+1}^{l-\epsilon}\Phi_{k+1} = \sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l+\epsilon)^* T_{k+1}^{l}\Phi_{k+1} \\ &= \sum_{l\in\mathcal{L}_k} \left[s_k^\epsilon(l)^* T_k^l \Phi_k + t_k^\epsilon(l)^* T_k^l \Psi_k \right] \\ &= \sum_{l\in\mathcal{L}_{k+1}} \left[(\uparrow_k s_k^{\epsilon *}) \otimes H_{k+1}(l) + (\uparrow_k t_k^{\epsilon *}) \otimes G_{k+1}(l) \right] T_{k+1}^l \Phi_{k+1} \end{aligned}$$

and so

$$f_{k+1} = \sum_{l\in\mathcal{L}_{k+1}} s_{k+1}(l+\epsilon)^* T_{k+1}^{l+\epsilon}\Phi_{k+1} = \sum_{l\in\mathcal{L}_k} s_k^\epsilon(l)^* T_{k+1}^{Ml+\epsilon}\Phi_k + \sum_{l\in\mathcal{L}_k} t_k^\epsilon(l)^* T_{k+1}^{Ml+\epsilon}\Psi_k.$$

which justifies (5.43) and (5.46). Consequently, we infer from Lemma 5.4, (5.31) and (5.32) of Proposition 5.11 that

$$\begin{aligned} \left[s_k^{\epsilon\prime}(l) - \widetilde{s_k^\epsilon}'(l) \right]^* &= [s_{k+1}'(\cdot+\epsilon)^* \otimes H_{k+1}'^*](Ml) = \sum_{n\in\mathcal{L}_{k+1}} s_{k+1}'(n+\epsilon)^* H_{k+1}'(Ml-n)^* \\ &= \sum_{n\in\mathcal{L}_{k+1}} s_{k+1}(n)^* H_{k+1}'(Ml+\epsilon-n)^* = [s_{k+1}'^* \otimes H_{k+1}'^*](Ml+\epsilon) = a_{k+1}'(Ml+\epsilon)^* \end{aligned}$$

and

$$\begin{aligned} \left[t_k^{\epsilon\prime}(l) - \widetilde{t_k^\epsilon}'(l) \right]^* &= [s_{k+1}'(\cdot+\epsilon)^* \otimes G_{k+1}'^*](Ml) = \sum_{n\in\mathcal{L}_{k+1}} s_{k+1}'(n+\epsilon)^* G_{k+1}'(Ml-n)^* \\ &= \sum_{n\in\mathcal{L}_{k+1}} s_{k+1}(n)^* G_{k+1}'(Ml+\epsilon-n)^* = [s_{k+1}'^* \otimes G_{k+1}'^*](Ml+\epsilon) = b_{k+1}'(Ml+\epsilon)^* \end{aligned}$$

and the above computation proves (5.44) and (5.45). $\square$

Before we proceed further, it is necessary to consider the frequency domain formulation of Proposition 5.14 in order to establish the uniqueness aspect of the transform.

Proposition 5.15. *Assume Setup 5.3. If f_{k+1} is given by (5.1), then for $\epsilon \in \mathcal{L}_1$,*

$$\begin{aligned} f_{k+1} &= \sum_{j\in\mathcal{R}_{k+1}} \left[\widehat{s_k^\epsilon}(j)^* \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}\epsilon} (\Phi_k)_{k+1,j} + \widehat{t_k^\epsilon}(j)^* \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}\epsilon} (\Psi_k)_{k+1,j} \right], \\ &= d^{-1} \sum_{\varepsilon\in\mathcal{L}_1} \sum_{j\in\mathcal{R}_{k+1}} \left[\widehat{s_k^\varepsilon}(j)^* \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}\varepsilon} (\Phi_k)_{k+1,j} + \widehat{t_k^\varepsilon}(j)^* \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-(k+1)}\varepsilon} (\Psi_k)_{k+1,j} \right]. \quad (5.47) \end{aligned}$$

Further, for each $j \in \mathcal{R}_k$, there exists $\left[\widehat{\widetilde{s_k^\epsilon}}'(j)^* \;\; \widehat{\widetilde{t_k^\epsilon}}'(j)^* \right]^* \in \operatorname{Ker} \widehat{\mathbb{L}}_k'(j)^*$ such that

$$d \left[\widehat{s_k^\epsilon}'(j) - \widehat{\widetilde{s_k^\epsilon}}'(j) \right]^* = \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* \mathrm{e}^{\mathrm{i}2\pi(j+D^k r)\cdot M^{-(k+1)}\epsilon} \widehat{H}_{k+1}'(j+D^k r)^*, \quad (5.48)$$

$$d \left[\widehat{t_k^\epsilon}'(j) - \widehat{\widetilde{t_k^\epsilon}}'(j) \right]^* = \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+D^k r)^* \mathrm{e}^{\mathrm{i}2\pi(j+D^k r)\cdot M^{-(k+1)}\epsilon} \widehat{G}_{k+1}'(j+D^k r)^*, \quad (5.49)$$

and for $r \in \mathcal{R}_1$,

$$\widehat{s_{k+1}}'(j+D^k r)^* \mathrm{e}^{\mathrm{i}2\pi(j+D^k r)\cdot M^{-(k+1)}\epsilon} = \widehat{s_k^\epsilon}'(j)^* \widehat{H}'_{k+1}(j+D^k r) + \widehat{t_k^\epsilon}'(j)^* \widehat{G}'_{k+1}(j+D^k r), \qquad (5.50)$$

where $\widehat{s_k^\epsilon}'(j) = I'_{q(j)} U_k(j) \widehat{s_k^\epsilon}(j)$, $\widehat{t_k^\epsilon}'(j) = I'_{p(j)-q(j)} V_k(j) \widehat{t_k^\epsilon}(j)$.

Proof. We show the two equalities of (5.47) by applying Lemma 3.2 to (5.43) of Proposition 5.14, i.e. $f_{k+1} = f_k^\epsilon + g_k^\epsilon$, where $f_k^\epsilon = \sum_{l \in \mathcal{L}_k} s_k^\epsilon(l)^* T_{k+1}^{Ml+\epsilon} \Phi_k$, $g_k^\epsilon = \sum_{l \in \mathcal{L}_k} t_k^\epsilon(l)^* T_{k+1}^{Ml+\epsilon} \Psi_k$, so that

$$\begin{aligned}
f_k^\epsilon &= \sum_{l \in \mathcal{L}_k} s_k^\epsilon(l)^* \sum_{j \in \mathcal{R}_{k+1}} \sum_{n \in \mathbb{Z}^s} \widehat{(\Phi_k)}_{k+1,j}(n) \mathrm{e}^{\mathrm{i}(j+D^{k+1}n)\cdot[t - 2\pi M^{-(k+1)}(Ml+\epsilon)]} \\
&= \sum_{j \in \mathcal{R}_{k+1}} \sum_{n \in \mathbb{Z}^s} \sum_{l \in \mathcal{L}_k} s_k^\epsilon(l)^* \mathrm{e}^{-\mathrm{i}j\cdot 2\pi M^{-k} l} \widehat{(\Phi_k)}_{k+1,j}(n) \mathrm{e}^{\mathrm{i}(j+D^{k+1}n)\cdot t} \mathrm{e}^{-\mathrm{i}j\cdot 2\pi M^{-(k+1)}\epsilon} \\
&= \sum_{j \in \mathcal{R}_{k+1}} \sum_{n \in \mathbb{Z}^s} \widehat{s_k^\epsilon}(j)^* \widehat{(\Phi_k)}_{k+1,j}(n) \mathrm{e}^{\mathrm{i}(j+D^{k+1}n)\cdot t} \mathrm{e}^{-\mathrm{i}j\cdot 2\pi M^{-(k+1)}\epsilon} \\
&= \sum_{j \in \mathcal{R}_{k+1}} \widehat{s_k^\epsilon}(j)^* \mathrm{e}^{-\mathrm{i}j\cdot 2\pi M^{-(k+1)}\epsilon} (\Phi_k)_{k+1,j}
\end{aligned}$$

and similarly $g_k^\epsilon = \sum_{j \in \mathcal{R}_{k+1}} \widehat{t_k^\epsilon}(j)^* \mathrm{e}^{-\mathrm{i}j\cdot 2\pi M^{-(k+1)}\epsilon} (\Psi_k)_{k+1,j}$. Using Lemma 3.2, we show that (5.44) of Proposition 5.14 is equivalent to

$$\begin{aligned}
& d\left[\widehat{s_k^\epsilon}'(j) - \widehat{\widehat{s_k^\epsilon}}'(j)\right]^* = d \sum_{l \in \mathcal{L}_k} \sum_{n \in \mathcal{L}_{k+1}} s'_{k+1}(n+\epsilon)^* H'_{k+1}(Ml-n)^* \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-k} l} \\
&= \sum_{l \in \mathcal{L}_k} \sum_{n \in \mathcal{L}_{k+1}} d^{-k} \sum_{\nu \in \mathcal{R}_{k+1}} \widehat{s_{k+1}}'(\nu)^* \mathrm{e}^{\mathrm{i}2\pi\nu\cdot M^{-(k+1)}(n+\epsilon)} H'_{k+1}(Ml-n)^* \mathrm{e}^{-\mathrm{i}2\pi j\cdot M^{-k} l} \\
&= \sum_{l \in \mathcal{L}_k} \sum_{\nu \in \mathcal{R}_{k+1}} \sum_{n \in \mathcal{L}_{k+1}} d^{-k} \widehat{s_{k+1}}'(\nu)^* H'_{k+1}(Ml-n)^* \mathrm{e}^{-\mathrm{i}2\pi\nu\cdot M^{-(k+1)}(Ml-n)} \mathrm{e}^{\mathrm{i}2\pi\nu\cdot M^{-(k+1)}\epsilon} \mathrm{e}^{-\mathrm{i}2\pi(j-\nu)\cdot M^{-k} l} \\
&= \sum_{l \in \mathcal{L}_k} \sum_{\nu \in \mathcal{R}_{k+1}} d^{-k} \widehat{s_{k+1}}'(\nu)^* \widehat{H}'_{k+1}(\nu)^* \mathrm{e}^{\mathrm{i}2\pi\nu\cdot M^{-(k+1)}\epsilon} \mathrm{e}^{-\mathrm{i}2\pi(j-\nu)\cdot M^{-k} l} \\
&= \sum_{l \in \mathcal{L}_k} \sum_{\nu \in \mathcal{R}_k} \sum_{r \in \mathcal{R}_1} d^{-k} \widehat{s_{k+1}}'(\nu + D^k r)^* \widehat{H}'_{k+1}(\nu + D^k r)^* \mathrm{e}^{\mathrm{i}2\pi(\nu+D^k r)\cdot M^{-(k+1)}\epsilon} \mathrm{e}^{-\mathrm{i}2\pi(j-\nu-D^k r)\cdot M^{-k} l} \\
&= \sum_{\nu \in \mathcal{R}_k} \sum_{r \in \mathcal{R}_1} \widehat{s_{k+1}}'(\nu + D^k r)^* \widehat{H}'_{k+1}(\nu + D^k r)^* \mathrm{e}^{\mathrm{i}2\pi(\nu+D^k r)\cdot M^{-(k+1)}\epsilon} \delta_{j,\nu}
\end{aligned}$$

and this confirms that (5.48) is valid. Similarly, we could show that (5.45) is equivalent to (5.49). Finally, (5.50) is easily shown by following the same proof found in Proposition 5.10. □

As in Corollary 5.12, we shall have uniqueness in the reconstruction process in Proposition 5.14 if for every $j \in \mathcal{R}_k$ and $r \in \mathcal{R}_1$ the matrices $\mathrm{M}_k(j)$, $\mathrm{N}_k(j)$ and $\mathrm{M}_{k+1}(j + D^k r)$ as defined in Setup 5.1 have full rank.

Corollary 5.16. *Assume Setup* 5.3 *with* $\operatorname{rank} \mathrm{M}_{k+1}(j + D^k r) = \rho$, $\operatorname{rank} \mathrm{M}_k(j) = \rho$ *and* $\operatorname{rank} \mathrm{N}_k(j) = \varrho_k$ *for every* $j \in \mathcal{R}_k$ *and* $r \in \mathcal{R}_1$. *If* f_{k+1} *is given by* (5.1), *then for* $\epsilon \in \mathcal{L}_1$,

$$
\begin{aligned}
f_{k+1} &= \sum_{l \in \mathcal{L}_{k+1}} s_{k+1}(l+\epsilon)^* T_{k+1}^{l+\epsilon} \Phi_{k+1} = \sum_{l \in \mathcal{L}_k} \left[s_k^\epsilon(l)^* T_{k+1}^{Ml+\epsilon} \Phi_k + t_k^\epsilon(l)^* T_{k+1}^{Ml+\epsilon} \Psi_k \right] \\
&= d^{-1} \sum_{\varepsilon \in \mathcal{L}_1} \sum_{l \in \mathcal{L}_k} \left[s_k^\varepsilon(l)^* T_{k+1}^{Ml+\varepsilon} \Phi_k + t_k^\varepsilon(l)^* T_{k+1}^{Ml+\varepsilon} \Psi_k \right] \\
&= d^{-1} \sum_{\varepsilon \in \mathcal{L}_1} \sum_{l \in \mathcal{L}_k} \left[a_k(Ml+\varepsilon)^* T_{k+1}^{Ml+\varepsilon} \Phi_k + b_k(Ml+\varepsilon)^* T_{k+1}^{Ml+\varepsilon} \Psi_k \right], \\
&= \sum_{l \in \mathcal{L}_{k+1}} \left[(\uparrow_k s_k^{\epsilon *}) \otimes H_{k+1}(l) + (\uparrow_k t_k^{\epsilon *}) \otimes G_{k+1}(l) \right] T_{k+1}^{l+\epsilon} \Phi_{k+1}.
\end{aligned} \tag{5.51}
$$

Further, for every $l \in \mathcal{L}_k$ *and* $n \in \mathcal{L}_{k+1}$, *we have*

$$ s_k^\epsilon(l)^* = [s_{k+1}^* \otimes H_{k+1}^*](Ml+\epsilon) = a_k(Ml+\epsilon)^*, \tag{5.52} $$

$$ t_k^\epsilon(l)^* = [s_{k+1}^* \otimes G_{k+1}^*](Ml+\epsilon) = b_k(Ml+\epsilon)^*, \tag{5.53} $$

$$ s_{k+1}(n+\epsilon)^* = [(\uparrow_k s_k^{\epsilon *}) \otimes H_{k+1}](n) + [(\uparrow_k t_k^{\epsilon *}) \otimes G_{k+1}](n). \tag{5.54} $$

Using Proposition 5.14 and Corollary 5.16, we could derive the quasi-affine representation for the stationary wavelet transform in the following proposition.

Proposition 5.17. *Fix* $0 \leq K \leq L$. *Assume Setup* 5.3 *with* $\operatorname{rank} \mathrm{M}_k(j) = q_k(j)$, $\operatorname{rank} \mathrm{N}_k(j) = p_k(j) - q_k(j)$ *and* $\operatorname{rank} \mathrm{M}_{k+1}(j + D^k r) = p_k(j, r)$ *for every* $k \in \{K, \ldots, K + L - 1\}$, $j \in \mathcal{R}_k$ *and* $r \in \mathcal{R}_1$. *If*

$$ f_{K+L} = \sum_{j \in \mathcal{R}_{K+L-1}} \sum_{r \in \mathcal{R}_1} \widehat{s_{K+L}}(j + D^{K+L-1} r)^* v_{K+L, j+D^{K+L-1} r}, $$

then for $\delta \in \mathcal{L}_L$,

$$
\begin{aligned}
f_{K+L} =& \sum_{l \in \mathcal{L}_{K+L}} s_{K+L}(l+\delta)^* T_{K+L}^{l+\delta} \Phi_{K+L} \\
=& \sum_{l \in \mathcal{L}_K} s_K^\delta(l)^* T_{K+L}^{M^L l+\delta} \Phi_K + \sum_{k=K}^{K+L} \sum_{l \in \mathcal{L}_k} t_k^\delta(l)^* T_{K+L}^{M^{K+L-k} l+\delta} \Psi_k \qquad (5.55) \\
=& \sum_{\delta_L \in \mathcal{L}_L} \sum_{l \in \mathcal{L}_K} d^{-L} a_K(M^L l + \delta_L)^* T_{K+L}^{M^L l+\delta_L} \Phi_K + \sum_{k=K}^{K+L} \sum_{\delta_k \in \mathcal{L}_k} \sum_{l \in \mathcal{L}_{K+L-k}} d^{-k} b_{K+L-k}(M^k l + \delta_k)^* T_{K+L}^{M^k l+\delta_k} \Psi_{K+L-k} \\
=& \sum_{l \in \mathcal{L}_{K+L}} d^{-L} a_K(l)^* T_{K+L}^{l} \Phi_K + \sum_{k=K}^{K+L} \sum_{l \in \mathcal{L}_{K+L}} d^{-k} b_{K+L-k}(l)^* T_{K+L}^{l} \Psi_{K+L-k},
\end{aligned}
$$

for a given $\delta \in \mathcal{L}_L$. *Further, for any* $k \in \{K, \dots, K+L-1\}$, *there exist* $\left[\widehat{\widetilde{s_k^\delta}}'(j)^* \ \widehat{\widetilde{t_k^\delta}}'(j)^*\right]^* \in$ $\operatorname{Ker}\widehat{\mathbb{L}}'_k(j)^*$, $j \in \mathcal{R}_k$, *such that for every* $l \in \mathcal{L}_k$ *and* $n \in \mathcal{L}_{k+1}$,

$$\left[s_k^{\delta\prime}(l) - \widetilde{s_k^\delta}'(l)\right]^* = a'_k(M^{K+L-k}l + \delta)^*, \tag{5.56}$$

$$\left[t_k^{\delta\prime}(l) - \widetilde{t_k^\delta}'(l)\right]^* = b'_k(M^{K+L-k}l + \delta)^*, \tag{5.57}$$

$$a'_{k+1}(M^{K+L-k-1}n + \delta)^* = [(\uparrow_k s_k^{\delta\prime *}) \otimes H'_{k+1} + (\uparrow_k t_k^{\delta\prime *}) \otimes G'_{k+1}](n) \tag{5.58}$$

with $\widehat{s_k^\delta}'(j) = I'_{q_k(j)} U_k(j)\widehat{s_k^\delta}(j)$, $\widehat{t_k^\delta}'(j) = I'_{p_k(j)-q_k(j)} V_k(j)\widehat{t_k^\delta}(j)$, $\widehat{a_k}'(j+D^k\nu) = I'_{q_k(j)} U_k(j)\widehat{a_k}(j+D^k\nu)$ *and* $\widehat{b_k}'(j + D^k\nu) = I'_{p_k(j)-q_k(j)} V_k(j)\widehat{b_k}(j + D^k\nu)$, *where* $\nu \in \mathcal{R}_{K+L-k}$ *and* a_k *and* b_k *are given as in* (5.42).

Proof. Utilizing Proposition 5.11, we are led to

$$\begin{aligned} T_{K+L}^{-\delta} f_{K+L} &= \sum_{l \in \mathcal{L}_{K+L}} s_{K+L}(l)^* T_{K+L}^{l-\delta} \Phi_{K+L} = \sum_{l \in \mathcal{L}_{K+L}} s_{K+L}(l+\delta)^* T_{K+L}^{l} \Phi_{K+L} \\ &= \sum_{l \in \mathcal{L}_K} s_K^\delta(l)^* T_K^l \Phi_K + \sum_{k=K}^{K+L} \sum_{l \in \mathcal{L}_k} t_k^\delta(l)^* T_k^l \Psi_k. \end{aligned}$$

Therefore,

$$\begin{aligned} f_{K+L} &= \sum_{l \in \mathcal{L}_{K+L}} s_{K+L}(l+\delta)^* T_{K+L}^{l+\delta} \Phi_{K+L} \\ &= \sum_{l \in \mathcal{L}_K} s_K^\delta(l)^* T_{K+L}^{M^L l+\delta} \Phi_K + \sum_{k=K}^{K+L} \sum_{l \in \mathcal{L}_k} t_k^\delta(l)^* T_{K+L}^{M^{K+L-k}l+\delta} \Psi_k, \end{aligned}$$

which justifies the first two equalities of (5.55). Applying Lemma 3.2 to the above computation gives us

$$f_{K+L} = f_{K+L-1}^{\delta_1} + g_{K+L-1}^{\delta_1},$$

where $\delta_1 = \epsilon_1 \in \mathcal{L}_1$ and

$$\begin{aligned} f_{K+L-1}^{\delta_1} &= \sum_{l \in \mathcal{L}_{K+L-1}} a_{K+L-1}(Ml + \delta_1)^* T_{K+L}^{Ml+\delta_1} \Phi_{K+L-1}, \\ g_{K+L-1}^{\delta_1} &= \sum_{l \in \mathcal{L}_{K+L-1}} b_{K+L-1}(Ml + \delta_1)^* T_{K+L}^{Ml+\delta_1} \Psi_{K+L-1}. \end{aligned}$$

We show in a similar manner that

$$f_{K+L-1}^{\delta_1} = f_{K+L-2}^{\delta_2} + g_{K+L-2}^{\delta_2},$$

where $\delta_2 = M\epsilon_1 + \epsilon_2 \in \mathcal{L}_2$ for some $\epsilon_2 \in \mathcal{L}_1$ and

$$f_{K+L-2}^{\delta_2} = \sum_{l \in \mathcal{L}_{K+L-2}} a_{K+L-2}(M^2 l + \delta_2)^* T_{K+L}^{M^2 l + \delta_2} \Phi_{K+L-2},$$
$$g_{K+L-2}^{\delta_2} = \sum_{l \in \mathcal{L}_{K+L-2}} b_{K+L-2}(M^2 l + \delta_2)^* T_{K+L}^{M^2 l + \delta_2} \Psi_{K+L-2}.$$

If we sum over the ϵ, i.e. δ, for each level separately, then we would obtain

$$\begin{aligned}
f_{K+L} &= \sum_{\epsilon_1 \in \mathcal{L}_1} d^{-1} f_{K+L-1}^{\delta_1} + \sum_{\delta_1 \in \mathcal{L}_1} d^{-1} g_{K+L-1}^{\delta_1} \\
&= \sum_{\epsilon_1 \in \mathcal{L}_1} d^{-1} [f_{K+L-2}^{\delta_2} + g_{K+L-2}^{\delta_2}] + \sum_{\delta_1 \in \mathcal{L}_1} d^{-1} g_{K+L-1}^{\delta_1} \\
&= \sum_{\epsilon_1 \in \mathcal{L}_1} \sum_{\epsilon_2 \in \mathcal{L}_1} d^{-2} [f_{K+L-2}^{\delta_2} + g_{K+L-2}^{\delta_2}] + \sum_{\delta_1 \in \mathcal{L}_1} d^{-1} g_{K+L-1}^{\delta_1} \\
&= \sum_{\delta_2 \in \mathcal{L}_2} d^{-2} f_{K+L-2}^{\delta_2} + \sum_{\delta_2 \in \mathcal{L}_2} d^{-2} g_{K+L-2}^{\delta_2} + \sum_{\delta_1 \in \mathcal{L}_1} d^{-1} g_{K+L-1}^{\delta_1} \\
&= \sum_{\delta_L \in \mathcal{L}_L} d^{-L} f_K^{\delta_L} + \sum_{k=K}^{K+L} \sum_{\delta_k \in \mathcal{L}_k} d^{-k} g_{K+L-k}^{\delta_k},
\end{aligned}$$

where $\delta_k = M^{k-1}\epsilon_1 + \ldots + M^{k-k}\epsilon_k$ for some $\epsilon_1, \ldots, \epsilon_k \in \mathcal{L}_1$ and

$$f_K^{\delta_L} = \sum_{l \in \mathcal{L}_K} a_K(M^L l + \delta_L)^* T_{K+L}^{M^L l + \delta_L} \Phi_K,$$
$$g_{K+L-k}^{\delta_k} = \sum_{l \in \mathcal{L}_{K+L-k}} b_{K+L-k}(M^k l + \delta_k)^* T_{K+L}^{M^k l + \delta_k} \Psi_{K+L-k}.$$

This shows the penultimate equality of (5.55). Next, Lemma 3.2 shows the last equality of (5.55). Consequently, we infer from Proposition 5.13, (5.44), (5.45) and (5.46) of Proposition 5.14 that

$$\begin{aligned}
{s_K^{\delta}}'(l) - \widetilde{s_K^{\delta}}'(l) =&\downarrow_{K+1} H'_{K+1} \bigotimes_{n=K+2}^{K+L} \downarrow_n H'_n \otimes a'_{K+L}(l+\delta) \\
=&\downarrow_{K+1} H'_{K+1} \bigotimes_{n=K+2}^{K+L-1} \downarrow_n H'_n \otimes a'_{K+L-1}(Ml+\delta) \\
=&\downarrow_{K+1} H'_{K+1} \otimes \downarrow_{K+2} H'_{K+2} \otimes a'_{K+2}(M^{L-2} l + \delta) = a'_K(M^L l + \delta),
\end{aligned}$$

$$
\begin{aligned}
t_k^{\delta'}(l) - \widetilde{t_k^{\delta}}'(l) =& \downarrow_{k+1} G'_{k+1} \bigotimes_{n=k+2}^{K+L} \downarrow_n H'_n \otimes a'_{K+L}(l+\delta) \\
=& \downarrow_{k+1} G'_{k+1} \bigotimes_{n=k+2}^{K+L-1} \downarrow_n H'_n \otimes a'_{K+L-1}(Ml+\delta) \\
=& \downarrow_{k+1} G'_{k+1} \otimes \downarrow_{k+2} H'_{k+2} \otimes a'_{k+2}(M^{K+L-k-2}l+\delta) \\
=& \downarrow_{k+1} G'_{k+1} \otimes a'_{k+1}(M^{K+L-k-1}l+\delta) = b'_k(M^{K+L-k}l+\delta)
\end{aligned}
$$

and

$$
a'_{k+1}(M^{K+L-k-1}n+\delta)^* = s'_{k+1}(n+\delta)^* = \left[(\uparrow_k s_k^{\delta'*}) \otimes H'_{k+1} + (\uparrow_k t_k^{\delta'*}) \otimes G'_{k+1}\right](n),
$$

and the above computation proves (5.56), (5.57) and (5.58). $\square$

5.5 Time-Frequency Analysis

We shall now look at the simplified 1-dimensional setting of dilation factor $M = 2$. For $\nu \in \mathbb{R}$, let $H^\nu(\mathbb{T})$ be the Sobolev space of all 2π-periodic tempered distributions f such that $\|f\|^2_{H^\nu(\mathbb{T})} := \sum_{n\in\mathbb{Z}} (1+n^2)^\nu \left|\widehat{f}(n)\right|^2$ is finite. For $\nu \geq 0$, the Sobolev seminorm is defined by $|f|^2_{H^\nu(\mathbb{T})} := \sum_{n\in\mathbb{Z}} n^{2\nu} \left|\widehat{f}(n)\right|^2$, where $f \in H^\nu(\mathbb{T})$. For $\nu \geq 0$, the Sobolev norm and seminorm satisfy $2^{\min\{1-\nu,0\}} \|f\|^2_{H^\nu(\mathbb{T})} \leq \|f\|^2_{L^2(\mathbb{T})} + |f|^2_{H^\nu(\mathbb{T})} \leq 2^{\max\{1-\nu,0\}} \|f\|^2_{H^\nu(\mathbb{T})}$.

Following [22], for $R \geq 1$, the *frame approximation operator* Q_R associated with the restricted periodic affine system $X^R_{2\pi}$ mentioned in (1.18) derived from a single refinable function ϕ_0 is defined to be

$$
Q_R(f) = \sum_{l\in\mathcal{L}_k} \langle f, \phi_0\rangle_{L^2(\mathbb{T})}\phi_0 + \sum_{k=0}^{R-1}\sum_{m=1}^{\varrho_k}\sum_{l\in\mathcal{L}_k} \langle f, T^l_k\psi^m_k\rangle_{L^2(\mathbb{T})} T^l_k\psi^m_k, \quad f \in L^2(\mathbb{T}).
$$

The periodic affine system $X_{2\pi}$ as defined in (1.15) is said to provide *frame approximation order p* if there exist a positive constant C and a positive integer K such that for all $k \geq K$,

$$
\|f - Q_k(f)\|_{L^2(\mathbb{T})} \leq 2^{-kp} C\, |f|_{H^p(\mathbb{T})}, \quad f \in H^p(\mathbb{T}). \tag{5.59}
$$

Furthermore, the periodic affine system provides *spectral frame approximation order* if it provides frame approximation order p for every $p > 0$.

The concept of vanishing moments for functions on the real line is extended to functions in $L^2(\mathbb{T})$ in [22]. The authors define that $\Psi_k := \left[\psi^n_k\right]_{n=1}^{\varrho_k} \subset L^2(\mathbb{T})$ for $k \geq 0$ has *p vanishing* *moments* for some $p \geq 0$ if there exist positive constants C and K, independent of k and j such that

$$\sum_{m=1}^{\varrho_k} 2^k \left|\widehat{\psi}_k^m(j)\right|^2 \leq C \left|2^{-k} j\right|^{2p}, \quad j \in \mathcal{R}_k,\ k \geq K. \tag{5.60}$$

Following a more general extension from [22], $\Psi_k \subset L^2(\mathbb{T})$ is said to have *global vanishing moments of order* p for some $p \geq 0$ if there exist positive constants C and K, independent of k and n such that

$$\sum_{m=1}^{\varrho_k} 2^k \left|\widehat{\psi}_k^m(0)\right|^2 \leq C 2^{-2kp} \text{ and } \sum_{m=1}^{\varrho_k} |n|^{-2p} 2^k \left|\widehat{\psi}_k^m(n)\right|^2 \leq C 2^{-2kp},\ n \in \mathbb{Z}\backslash\{0\},\ k \geq K. \tag{5.61}$$

We cite the following lemma, which says that in the limiting case, the refinable function and the wavelets derived from the corresponding MRA must cover the frequency domain "uniformly".

Lemma 5.18. [22] *For each $k \geq 0$, let $\Phi_k := \phi_k$ and $\Psi_k := \left[\psi_k^n\right]_{n=1}^{\varrho_k}$ be subsets of $L^2(\mathbb{T})$ satisfying Theorem 3.27 with frame bound 1. Then*

$$2^k \left|\widehat{\phi}_k(n)\right|^2 + \sum_{r=k}^{\infty} \sum_{m=1}^{\varrho_r} 2^r \left|\widehat{\psi}_r^m(n)\right|^2 = 1$$

for each $n \in \mathbb{Z}$.

The next theorem supplies a sufficient condition on the frame approximation order for smooth functions provided by the tight wavelet frames derived from the periodic UEP.

Theorem 5.19. [22] *For each $k \geq 0$, let $\Phi_k := \phi_k$ and $\Psi_k := \left[\psi_k^n\right]_{n=1}^{\varrho_k}$ be subsets of $L^2(\mathbb{T})$ satisfying Theorem 3.27 with frame bound 1. The tight wavelet frame $X_{2\pi}$ as defined in (1.15) provides frame approximation order p as in (5.59) if there exist positive constants ϵ, C, K with $\epsilon \in (0, 2^{-1}]$ such that for all $k \geq K$,*

$$2^{2kp} \max\left\{ |j|^{-2p} \left(1 - 2^k \left|\widehat{\phi}_k(j)\right|^2\right) : j \in (\mathcal{R}_k \cap (-2^k\epsilon, 2^k\epsilon])\backslash\{0\} \right\} \leq C. \tag{5.62}$$

We present the following result that relates frame approximation order and vanishing moments.

Theorem 5.20. [22] *For each $k \geq 0$, let $\Phi_k := \phi_k$ and $\Psi_k := \left[\psi_k^n\right]_{n=1}^{\varrho_k}$ be subsets of $L^2(\mathbb{T})$ satisfying Theorem 3.27 with frame bound 1. The tight wavelet frame $X_{2\pi}$ as defined in (1.15) provides frame approximation order at least $p > 0$ as in (5.59) if Ψ_k has p vanishing moments. Conversely, if the tight wavelet frame $X_{2\pi}$ provides frame approximation order p, then Ψ_k has at least $p/2$ vanishing moments.*

Next, we point out how sparsity of frame expansion coefficients is influenced by global vanishing moments.

Theorem 5.21. [22] *Let* $\Psi_k := \left[\psi_k^n\right]_{n=1}^{\varrho_k} \subset L^2(\mathbb{T})$ *possess global vanishing moments of order* $p > 0$ *as in* (5.61), *where* C *and* K *are positive constants. Then for any* $q > p+2^{-1}$, *there exists a positive constant* $\widetilde{C} := \max\{4C, 4C', 2C''\}$ *such that*

$$\sum_{m=1}^{\varrho_k} \left|\langle f, T_k^l \psi_k^m \rangle_{L^2(\mathbb{T})}\right|^2 \leq \widetilde{C} 2^{-(2p+1)k} \left(\left|\widehat{f}(0)\right|^2 + |f|^2_{H^q(\mathbb{T})} \right), \quad f \in H^q(\mathbb{T}),\ k \geq K, \quad (5.63)$$

where $C' := C \sum_{n\in\mathbb{Z}\backslash\{0\}} |n|^{-2(q-p)}$, $C'' := C \sup \left\{ \sum_{n\in\mathbb{Z}\backslash\{0\}} |\omega+n|^{-2(q-p)} : \omega \in [-2^{-1}, 2^{-1}] \right\}$ *and* $l \in \mathcal{L}_k$.

We apply the above results to show that the bandlimited constructions in Chapter 4 have spectral frame approximation order, global vanishing moments of arbitrarily high order, and sparse representation.

Proposition 5.22. *Any bandlimited tight wavelet frame* $X_{2\pi}$ *constructed from the MRA* $\{V_{2\pi}^k(\phi_k)\}$ *with* $\{\phi_k\}_{k\geq 0}$ *given in Construction* 4.1 *such that* $\liminf_{k\to\infty} 2^{-k} N_{k,1} > 0$ *holds and satisfies Theorem* 3.27 *with frame bound* 1 *has spectral frame approximation order. Hence* $X_{2\pi}$ *also has global vanishing moments of arbitrarily high order and* (5.63) *holds.*

Proof. The additional condition of $\liminf_{k\to\infty} 2^{-k} N_{k,1} > 0$ implies that there exist $\epsilon \in (0, 2^{-1})$ and $K > 0$ such that $2^{-k} N_{k.1} \geq \epsilon$ for every $k \geq K$, i.e. $N_{k,1} \geq 2^k \epsilon$. Since for all $j \in \{-N_{k,1}, \ldots, N_{k,1}\}$, we have $2^k \left|\widehat{\phi}_k(j)\right|^2 = 1$, it means that $2^k \left|\widehat{\phi}_k(j)\right|^2 = 1$ for all $j \in \mathcal{R}_k \cap (-2^k\epsilon, 2^k\epsilon]$. Therefore, for any $p > 0$, (5.62) holds and by Theorem 5.19, the tight wavelet frame $X_{2\pi}$ has frame approximation p, i.e. it possesses spectral frame approximation order. By Theorem 5.20, it also has p vanishing moments for any $p > 0$. By Lemma 5.18, for every $k \geq K$, we have

$$2^k \sum_{m=1}^{\varrho_k} \left|\widehat{\psi}_k^m(n)\right|^2 \leq 1 - 2^k \left|\widehat{\phi}_k(n)\right|^2 \leq 1$$

for all $n \in \mathbb{Z}$. In particular, $2^k \sum_{m=1}^{\varrho_k} \left|\widehat{\psi}_k^m(0)\right|^2 = 0$. Furthermore for $n = j + 2^k r \in \mathbb{Z}\backslash\mathcal{R}_k$, where $j \in \mathcal{R}_k$ and $r \in \mathbb{Z}\backslash\{0\}$, since $\left|2^{-k}j + r\right| \geq 2^{-1}$, it also follows that

$$2^k \sum_{m=1}^{\varrho_k} \left|j + 2^k r\right|^{-2p} \left|\widehat{\psi}_k^m(j + 2^k r)\right|^2 \leq 2^k 2^{2p} 2^{-2kp} \sum_{m=1}^{\varrho_k} \left|\widehat{\psi}_k^m(j + 2^k r)\right|^2 \leq 2^{2p} 2^{-2kp}.$$

Consequently, the tight wavelet frame has global vanishing moments of order p as defined in (5.61) for any $p > 0$. Therefore by Theorem 5.21, (5.63) holds. □

Next, we look at the frame approximation order of some time-localized constructions. For positive integers s, l, we denote

$$P_{s,l}(t) := \sum_{\kappa=0}^{l-1} \binom{s+\kappa-1}{\kappa} t^{\kappa} = \sum_{\kappa=0}^{l-1} \frac{(s+\kappa-1)!}{\kappa!(s-1)!} t^{\kappa}, \quad t \in \mathbb{R}.$$

The masks of the compactly supported filters for pseudo-splines of type II with order (s,l) in [19] are given by

$$A_{s,l}(\omega) := \cos^{2s}(\omega/2) P_{s,l}(\sin^2(\omega/2)), \quad \omega \in \mathbb{R}, s \geq 1, l \in \{1,\dots,s\}.$$

For $k \geq 0$, we define $\widehat{h}_{k+1} \in \mathcal{S}(2^{k+1})$ by setting

$$\widehat{h}_{k+1}(j) := A_{s_{k+1},l_{k+1}}(2\pi 2^{-(k+1)} j), \quad j \in \mathcal{R}_k, \tag{5.64}$$

where $l_{k+1} \in \{1,\dots,s_{k+1}\}$, $\lim_{k\to\infty} s_{k+1} = \infty$, $\sum\limits_{k=1}^{\infty} 2^{-k} s_k < \infty$. Then $\widehat{h}_{k+1}(0) = 1$ and $\left|\widehat{h}_{k+1}(j)\right|^2 + \left|\widehat{h}_{k+1}(j+2^k)\right|^2 \leq 1$.

It is shown in [22] that the infinite products

$$\widehat{\varphi}_k(n) := 2^{\frac{-k}{2}} \prod_{r=k+1}^{\infty} \widehat{h}_r(n), \quad n \in \mathbb{Z}, k \geq 0, \tag{5.65}$$

are well defined and $\left|1 - 2^k \left|\widehat{\varphi}_k(n)\right|^2\right| \leq \sum\limits_{r=k+1}^{\infty} \left|1 - \left|\widehat{h}_r(n)\right|^2\right|$ for every $n \in \mathbb{Z}$. As noted in [22], this formulation arising from pseudo-splines includes many of the time-localized refinable functions in $L^2(\mathbb{T})$ that are of interest.

Proposition 5.23. *The time-localized tight wavelet frame $X_{2\pi}$ constructed as in Construction* 4.22 *from the MRA $\{V_{2\pi}^k(\varphi_k)\}$ with $\{\varphi_k\}_{k\geq 0}$ given in* (5.65) *such that it satisfies Theorem* 3.27 *with frame bound* 1 *has spectral frame approximation order. Hence $X_{2\pi}$ also has global vanishing moments of arbitrarily high order and* (5.63) *holds.*

Proof. It is shown in Lemma 3.3 of [31] that for any $p > 0$, there exist $C, K \geq 0$ such that for all $k \geq K$,

$$0 \leq 1 - \left|A_{s_k,l_k}(\omega)\right|^2 \leq C \left|\omega\right|^{2p}, \omega \in [-\pi,\pi].$$

It follows from (5.64) that for $k \geq K$ and $j \in \mathcal{R}_k \backslash \{0\}$,

$$\sum_{r=k+1}^{\infty} \left|1 - \left|\widehat{h}_r(j)\right|^2\right| \leq C \sum_{r=k+1}^{\infty} \left|2\pi 2^{-r} j\right|^{2p} = 2^{-2kp} \left|j\right|^{2p} (2\pi)^{2p} C/(2^{2p}-1),$$

i.e. $2^{2kp}|j|^{-2p}\left[\sum_{r=k+1}^{\infty}\left|1-\left|\widehat{h}_r(j)\right|^2\right|\right] \leq (2\pi)^{2p}C/(2^{2p}-1)$. Hence (5.62) is satisfied for any $p > 0$ with $\epsilon := \frac{1}{2}$ and by Theorem 5.19, the tight wavelet frame $X_{2\pi}$ has frame approximation order p, i.e. it possesses spectral frame approximation order. Using the reasoning similar to Proposition 5.22, we conclude that $X_{2\pi}$ has global vanishing moments of arbitrarily high order and hence (5.63) holds. □

We shall follow the convention as described in [38] to visualize the time-frequency (TF) representation of a signal. In order to visualize the time-frequency plots of signals using the decimated wavelet transform and the stationary wavelet transform of the bandlimited wavelet frames of Section 4.1, we sample the signals to be plotted at the rate of N samples per unit time on a prescribed time interval, where the sampling rate $N = 2^K$. Next, we collect the sampled data into a finite sequence. We consider only using the bandlimited constructions as we intend to utilize the fast Fourier transform in our implementations.

We construct the refinement mask $\widehat{h}_{k+1}$ as in (4.1) from Construction 4.1, i.e.

$$\widehat{h}_{k+1}(j) = \begin{cases} \sqrt{2} & \text{if } j \in\{-N_{k,1},\dots,N_{k,1}\}, \\ \sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}_k^1\left(\frac{|j|}{N_{k,1}}-1\right)\right] & \text{if } \begin{array}{l} j \in\{-L_{k,1},\dots,-N_{k,1}-1\} \\ \quad\cup\{N_{k,1}+1,\dots,L_{k,1}\}, \end{array} \\ 0 & \text{if } j \in\mathcal{R}_{k+1}\backslash\{-L_{k,1},\dots,L_{k,1}\}. \end{cases}$$

Here, we choose the regularized β-function to be

$$\beta[4,4](t) := \sum_{j=4}^{4+4-1} \frac{\Gamma(4+4)}{\Gamma(j+1)\Gamma(4+4-j)} t^j(1-t)^{4+4-1-j},$$

where the Γ-function is given as $\Gamma(t) = \int_0^\infty x^{t-1}\mathrm{e}^{-x}\mathrm{d}x$.

For the wavelet masks, we shall utilize Constructions 4.10 and 4.12 which will be chosen appropriately depending on our partitioning of the frequency domain, i.e. for $n \in \{1,\dots,\varrho_k\}\backslash\{\lambda_0,\mu_0\}$ with $\mu_0 = \lambda_0 + 1$, let

$$\widehat{g}_{k+1}^n(j) = \begin{cases} \sqrt{2}\sin\left[\frac{\pi}{2}\tilde{\beta}_k^n\left(\frac{|j|}{N_{k,n}}-1\right)\right] & \text{if } \begin{array}{l} j \in\{-L_{k,n},\dots,-N_{k,n}-1\} \\ \quad\cup\{N_{k,n}+1,\dots,L_{k,n}\}, \end{array} \\ \sqrt{2} & \text{if } \begin{array}{l} j \in\{-N_{k,n+1},\dots,-L_{k,n}\} \\ \quad\cup\{L_{k,n},\dots,N_{k,n+1}\}, \end{array} \\ \sqrt{2}\cos\left[\frac{\pi}{2}\tilde{\beta}_k^{n+1}\left(\frac{|j|}{N_{k,n+1}}-1\right)\right] & \text{if } \begin{array}{l} j \in\{-L_{k,n+1},\dots,-N_{k,n+1}-1\} \\ \quad\cup\{N_{k,n+1}+1,\dots,L_{k,n+1}\}, \end{array} \\ 0 & \text{if } \begin{array}{l} j \in\mathcal{R}_{k+1}\backslash\{-L_{k,n+1},\dots,-N_{k,n}-1\} \\ \quad\cap\mathcal{R}_{k+1}\backslash\{N_{k,n}+1,\dots,L_{k,n+1}\}. \end{array} \end{cases}$$

If $N_{k,\mu_0} < 2^{k-1} < L_{k,\mu_0}$, for $n \in \{\lambda_0, \mu_0\}$, choose

$$\widehat{g}^n_{k+1}(j) = \begin{cases} \sin\left[\frac{\pi}{2}\tilde{\beta}^n_k\left(\frac{|j|}{N_{k,n}} - 1\right)\right] & \text{if } \begin{array}{l} j \in\{-L_{k,n},\dots,-N_{k,n}-1\} \\ \quad\cup\{N_{k,n}+1,\dots,L_{k,n}\}, \end{array} \\ 1 & \text{if } \begin{array}{l} j \in\{-N_{k,n+1},\dots,-L_{k,n}\} \\ \quad\cup\{L_{k,n},\dots,N_{k,n+1}\}, \end{array} \\ \cos\left[\frac{\pi}{2}\tilde{\beta}^{n+1}_k\left(\frac{|j|}{N_{k,n+1}} - 1\right)\right] & \text{if } \begin{array}{l} j \in\{-L_{k,n+1},\dots,-N_{k,n+1}-1\} \\ \quad\cup\{N_{k,n+1}+1,\dots,L_{k,n+1}\}, \end{array} \\ 0 & \text{if } \begin{array}{l} j \in\mathcal{R}_{k+1}\backslash\{-L_{k,n+1},\dots,-N_{k,n}-1\} \\ \quad\cap\mathcal{R}_{k+1}\backslash\{N_{k,n}+1,\dots,L_{k,n+1}\}, \end{array} \end{cases}$$

and

$$\widehat{g}^{\widetilde{n}}_{k+1}(j) = \mathrm{i}\,\mathrm{sgn}_{k+1}(j)\widehat{g}^n_{k+1}(j), \quad \widetilde{n} \in \{\widetilde{\lambda_0}, \widetilde{\mu_0}\}.$$

If $L_{k,\lambda_0} < 2^{k-1} < N_{k,\mu_0} = N_{k,\mu_0+1} \leq L_{k,\mu_0} = L_{k,\mu_0+1}$, for $n \in \{\lambda_0, \mu_0\}$, choose

$$\widehat{g}^n_{k+1}(j)=\begin{cases} \mathrm{i}^{n \bmod 2}(\mathrm{sgn}_{k+1}(j))^n \sin\left[\frac{\pi}{2}\tilde{\beta}^{\lambda_0}_k\left(\frac{|j|}{N_{k,\lambda_0}} - 1\right)\right] & \text{if } \begin{array}{l} j \in\{-L_{k,\lambda_0},\dots,-N_{k,\lambda_0}-1\} \\ \quad\cup\{N_{k,\lambda_0}+1,\dots,L_{k,\lambda_0}\}, \end{array} \\ \mathrm{i}^{n \bmod 2}(\mathrm{sgn}_{k+1}(j))^n & \text{if } \begin{array}{l} j \in\{-N_{k,\mu_0},\dots,-L_{k,\lambda_0}\} \\ \quad\cup\{L_{k,\lambda_0},\dots,N_{k,\mu_0}\}, \end{array} \\ \mathrm{i}^{n \bmod 2}(\mathrm{sgn}_{k+1}(j))^n \cos\left[\frac{\pi}{2}\tilde{\beta}^{\mu_0}_k\left(\frac{|j|}{N_{k,\mu_0}} - 1\right)\right] & \text{if } \begin{array}{l} j \in\{-L_{k,\mu_0},\dots,-N_{k,\mu_0}-1\} \\ \quad\cup\{N_{k,\mu_0}+1,\dots,L_{k,\mu_0}\}, \end{array} \\ 0 & \text{if } \begin{array}{l} j \in\mathcal{R}_{k+1}\backslash\{-L_{k,\mu_0},\dots,-N_{k,\lambda_0}-1\} \\ \quad\cap\mathcal{R}_{k+1}\backslash\{N_{k,\lambda_0}+1,\dots,L_{k,\mu_0}\}. \end{array} \end{cases}$$

The stationary wavelet transform is used as in (5.42) in the frequency domain and it is related to the decimated wavelet transform as in Proposition 5.17. For the decimated wavelet transform, we use the algorithm as given in Proposition 5.10, which is the frequency domain version of Proposition 5.11. The quasi-affine representation of the stationary wavelet transform is given as

$$f_{K+L} = \sum_{l\in\mathcal{L}_{K+L}} 2^{-L}a_K(l)^* T^l_{K+L}\phi_K + \sum_{k=K}^{K+L}\sum_{l\in\mathcal{L}_{K+L}} 2^{-k}b_{K+L-k}(l)^* T^l_{K+L}\Psi_{K+L-k}$$

with $a'_{k-1} = (\uparrow_k^{K+L} H'_k) \otimes a'_k$ and $b'_{k-1} = (\uparrow_k^{K+L} G'_k) \otimes a'_k$, which is equivalent to

$$\begin{aligned}
\widehat{a}'_{k-1}(j)^* &= \sum_{l\in\mathcal{L}_{K+L}} \sum_{r\in\mathcal{L}_{K+L}} a'_k(l-r)^*(\uparrow_k^{K+L} H'_k)(r)^* \mathrm{e}^{-\mathrm{i}2\pi 2^{-(K+L)}lj} \\
&= \sum_{l\in\mathcal{L}_{K+L}} \sum_{r\in\mathcal{L}_{K}} a'_k(l-2^{K+L-k}r)^* H'_k(r)^* \mathrm{e}^{-\mathrm{i}2\pi 2^{-(K+L)}lj} \mathrm{e}^{-\mathrm{i}2\pi 2^{-k}rj} \mathrm{e}^{\mathrm{i}2\pi 2^{-k}rj} \\
&= \sum_{l\in\mathcal{L}_{K+L}} a'_k(l)^* \mathrm{e}^{-\mathrm{i}2\pi 2^{-(K+L)}(l+2^{K+L-k}r)j} \mathrm{e}^{\mathrm{i}2\pi 2^{-k}rj} \sum_{r\in\mathcal{L}_K} H'_k(r)^* \mathrm{e}^{-\mathrm{i}2\pi 2^{-k}rj} \\
&= \widehat{a}'_k(j)^* \widehat{H}'_k(j)^*, \\
\widehat{b}'_{k-1}(j)^* &= \widehat{a}'_k(j)^* \widehat{G}'_k(j)^*,
\end{aligned}$$

with $\widehat{H}'_{k+1}(j+2^k r) = \mathbf{1}_{\mathrm{supp}\,\widehat{\phi}_k}(j)\widehat{H}_{k+1}(j+2^k r)\mathbf{1}_{\mathrm{supp}\,\widehat{\phi}_{k+1}}(j+2^k r)$ and $\widehat{G}'_{k+1}(j+2^k r) = \mathrm{diag}\left[\mathbf{1}_{\mathrm{supp}\,\widehat{\psi}_k^m}(j)\right]_{m=1}^{\varrho_k} \widehat{G}_k(j+2^k r)\mathbf{1}_{\mathrm{supp}\,\widehat{\phi}_{k+1}}(j+2^k r)$ and $H_k \in \mathcal{S}(2^k)$, $G_k \in \mathcal{S}(2^k)^{\varrho_k\times 1}$, $a_k \in \mathcal{S}(2^{K+L})$ and $b_k \in \mathcal{S}(2^{K+L})^{\varrho_k\times 1}$. The decimated wavelet transform used is given as

$$f_{k+1} = \sum_{l\in\mathcal{L}_{k+1}} \left[(\uparrow_k s_k^*) \otimes H_{k+1}(l) + (\uparrow_k t_k^*) \otimes G_{k+1}(l)\right] T_{k+1}^l \phi_{k+1},$$

with $s'_k(l)^* = s'^*_{k+1} \otimes H'^*_{k+1}(2l)$ and $t'_k(l)^* = s'^*_{k+1} \otimes G'^*_{k+1}(2l)$, which is equivalent to

$$\begin{aligned}
2\widehat{s_k}'(j)^* &= \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+2^k r)^* \mathbf{1}_{\mathrm{supp}\,\widehat{\phi}_{k+1}}(j+2^k r)\widehat{H}'_{k+1}(j+2^k r)^*, \\
2\widehat{t_k}'(j)^* &= \sum_{r\in\mathcal{R}_1} \widehat{s_{k+1}}'(j+2^k r)^* \mathbf{1}_{\mathrm{supp}\,\widehat{\phi}_{k+1}}(j+2^k r)\widehat{G}'_{k+1}(j+2^k r)^*.
\end{aligned}$$

In the above computations, we note that $\widehat{s_k}' = \mathbf{1}_{\mathrm{supp}\,\widehat{\phi}_k}\widehat{s_k}$, $\widehat{t_k}' = \mathrm{diag}\left[\mathbf{1}_{\mathrm{supp}\,\widehat{\psi}_k^m}\right]_{m=1}^{\varrho_k}\widehat{t_k}$, $\widehat{a_k}'(j+2^k\nu) = \mathbf{1}_{\mathrm{supp}\,\widehat{\phi}_k}(j)\widehat{a_k}(j+2^k\nu)$ and $\widehat{b_k}'(j+2^k\nu) = \mathrm{diag}\left[\mathbf{1}_{\mathrm{supp}\,\widehat{\psi}_k^m}\right]_{m=1}^{\varrho_k}(j)\widehat{b_k}(j+2^k\nu)$, where $\nu \in \mathcal{R}_{K+L-k}$.

Since our filters are real and they preserve linear phase, it suffices to consider only positive frequencies and since the magnitude of the antisymmetric band coefficients is the same as the corresponding symmetric band coefficients in the Fourier domain for $[0, \frac{\pi}{2}]$, we shall utilize only the symmetric band data for our time-frequency plots and normalize their values by multiplying by two. For the stationary wavelet transform, the time axis is divided into 2^K intervals of constant step length $2\pi 2^{-K}$. For the decimated wavelet transform, the time axis is divided into 2^{K-1} intervals of step length $2\pi 2^{-(K-1)}$ and at the k^{th} level ($k < K$), the time intervals are collated into partitions with step lengths of $2\pi 2^{-k}$ and they become larger as k decreases, i.e. the time step is multiplied by 2 each time. For both transforms, the frequency axis is divided into 2^{K-1} bands with the angular

Nyquist frequency π identified with 2^{K-1}. The collation of the frequency bands depends on the frequency localization of the respective filters.

We consider $\widehat{h}_{k+1}$ and $\widehat{g}_{k+1}^m$ to be *localized* on $\{0,\ldots,N_{k,1}\}$ and $\{L_{k,m},\ldots,N_{k,m+1}\}$ respectively for $m \in \{1,\ldots,\varrho_k\}$ since "most" of the energy of the mask is located in this band. Let $TF^s(f_K)(l,j)$ be the *time-frequency content* of f_K at time $l \in \mathcal{L}_K$ and frequency $j \in \mathcal{R}_K$ using the stationary wavelet transform and let $TF^d(f_K)(l,j)$ be the *time-frequency content* of f_K at time $l \in \mathcal{L}_k$ and frequency $j \in \mathcal{R}_K$ and $0 \le k < K$ using the decimated wavelet transform. For the former, we assign $TF^s(f_K)(l,j) := a_k^m(l)$ for $(l,j) \in [2\pi 2^{-K} l, 2\pi 2^{-K}(l+1)] \times [0, \pi 2^{-K} N_{k,1}]$ and $TF^s(f_K)(l,j) := b_k^m(l)$ for $(l,j) \in [2\pi 2^{-K} l, 2\pi 2^{-K}(l+1)] \times [\pi 2^{-K} L_{k,m}, \pi 2^{-K} N_{k,m+1}]$, where $0 \le k \le K$ denotes the decomposition level. In a similar way, for the latter, we assign $TF^d(f_K)(l,j) := s_k^m(l)$ for $(l,j) \in [2\pi 2^{-k} l, 2\pi 2^{-k}(l+1)] \times [0, \pi 2^{-K} N_{k,1}]$ and $TF^d(f_K)(l,j) := t_k^m(l)$ for $(l,j) \in [2\pi 2^{-k} l, 2\pi 2^{-k}(l+1)] \times [\pi 2^{-K} L_{k,m}, \pi 2^{-K} N_{k,m+1}]$.

In Figures 5.1 to 5.4, time-frequency representations of the decimated and stationary wavelet transforms using our bandlimited wavelet frames are compared with those using decimated and stationary wavelet bases, wavelet packets, short-time Fourier transform, analytic wavelet transform, Wigner-Ville distribution and Choi-William distribution. Test signals are two Gabor atoms, two linear chirps, a combination of one linear chirp with one quadratic chirp and two Gabor atoms, and two hyperbolic chirps. The two Gabor atoms in Figure 5.1 are given by

$$f_1(t) = 3\mathrm{e}^{-100N^{-2}(t-2^{-1}N)^2} \cos 16^{-1}\pi t, \quad f_2(t) = 3\mathrm{e}^{-100N^{-2}(t-4^{-1}3N)^2} \cos 4^{-1}\pi t.$$

The two linear chirps considered in Figure 5.2 are

$$\begin{aligned} f_1(t) &= \left[(t-N^{-1})(1-t)\right]^{-\frac{1}{2}} \cos 250\pi 1024^{-1} N^{-1} t^2, \\ f_2(t) &= \left[(t-N^{-1})(1-t)\right]^{-\frac{1}{2}} \left[\cos 100\pi 1024^{-1} t + \cos 250\pi 1024^{-1} N^{-1} t^2\right]. \end{aligned}$$

In Figure 5.3, the signal analyzed comprises of one linear chirp, one quadratic chirp and two Gabor atoms given by

$$\begin{aligned} f_1(t) &= F(t) \cos 100\pi^2 1024^{-1} N^{-1} t^2, \quad f_2(t) = F(t) \cos 30\pi^3 1024^{-1} N^{-2} (N-t)^3, \\ f_3(t) &= F(t) \mathrm{e}^{-1600N^{-1}1024^{-1}(t-2^{-1}N)^2} \cos 50\pi 1024^{-1} t, \\ f_4(t) &= F(t) \mathrm{e}^{-1600N^{-1}1024^{-1}(t-8^{-1}7N)^2} \cos 350\pi 1024^{-1} t, \end{aligned}$$

where the envelope

$$F(t) = \begin{cases} 1 + \sin\left[\pi\left(0.125 - N^{-1}\right)^{-1}\left(t - N^{-1}\right) - 2^{-1}\pi\right] & \text{if } t \in [N^{-1}, 0.125], \\ 2 & \text{if } t \in [0.125, 0.875 + N^{-1}], \\ 1 + \sin\left[\pi\left(0.125 - N^{-1}\right)^{-1}(1 - t) - 2^{-1}\pi\right] & \text{if } t \in [0.875 + N^{-1}, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the two hyperbolic chirps in Figure 5.4 are

$$f_1(t) = E(t)\mathbf{1}_{[0.1, 0.75 - N^{-1}]}\left[\sin 15N\pi 1024^{-1}(0.8 - t)^{-1}\right]\mathbf{1}_{(0.1, 0.68)},$$
$$f_2(t) = E(t)\mathbf{1}_{[0.1, 0.75 - N^{-1}]}\left[\sin 5N\pi 1024^{-1}(0.8 - t)^{-1}\right]\mathbf{1}_{(0.1, 0.75)},$$

where the envelope

$$E(t) = \begin{cases} 1 + \sin\left[\pi\left(0.1625 - N^{-1}\right)^{-1}\left(t - N^{-1}\right) - 2^{-1}\pi\right] & \text{if } t \in [N^{-1}, 0.1625], \\ 2 & \text{if } t \in [0.1625, 0.4875 + N^{-1}], \\ 1 + \sin\left[\pi\left(0.1625 - N^{-1}\right)^{-1}(0.65 - t) - 2^{-1}\pi\right] & \text{if } t \in [0.4875 + N^{-1}, 0.65], \\ 0 & \text{otherwise.} \end{cases}$$

The time-frequency representations of the Gabor atoms, linear chirps, multichirp signals computed using the WAVELAB toolbox (http://www-stat.stanford.edu/˜wavelab/) are shown in Figures 5.1, 5.2, 5.3 (left to right order). The transforms using wavelet bases are unable to resolve the three sets of signals properly due to poor frequency resolutions. This is in particular more severe at high frequencies and this also occurs with the analytic wavelet transform. Our bandlimited wavelet frame transforms, in particular the stationary version, resolve the chirps and Gabor atoms as well as the continuous short-time Fourier transform and do not create complex interference patterns present in the representations using the Wigner-Ville and Choi-William distributions. Our transforms also preserve most of the features of the signals unlike that of the wavelet packet transform. This is due to the choice of partitioning the frequency domain into subbands of the same bandwidth by setting the number of bands noBands = 32, the bandwidth $\Delta\omega = \text{samplesize}/(2 \times \text{noBands})$, $L_{k,m} = m\Delta\omega$ and $N_{k,m} = L_{k,m} - 15$ for $m = 1, \ldots, \varrho_{k+1}$ where $\varrho_k = \text{noBands}$.

Our bandlimited wavelet frame transforms perform fairly well for the hyperbolic chirps as shown in Figure 5.4. The analytic wavelet transform performs much better for the hyperbolic chirps due to its continuous nature even though the choice of the partitioning of the frequency domain in our transforms behave like that of the analytic wavelet transform. However, unlike the continuous transforms, the inverse of our transforms are easily computed by our wavelet algorithms and are not computationally intensive.

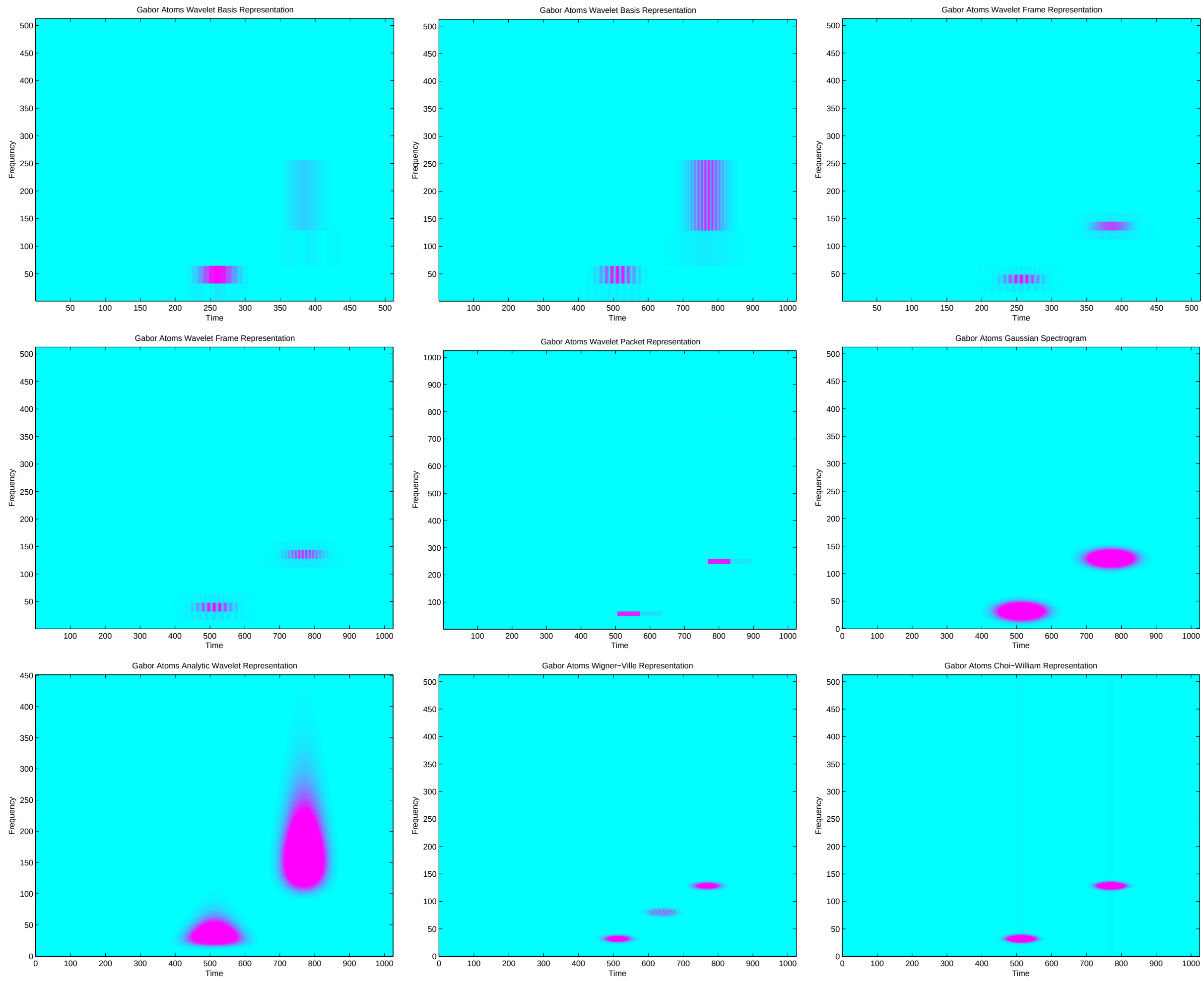

Figure 5.1: Gabor atoms signal representations using (a) Decimated Wavelet Basis Transform, (b) Stationary Wavelet Basis Transform, (c) Decimated Wavelet Frame Transform, (d) Stationary Wavelet Frame Transform, (e) Wavelet Packet Transform, (f) Short-Time Fourier Transform, (g) Analytic Wavelet Transform, (h) Wigner-Ville Distribution, (i) Choi-William Distribution.

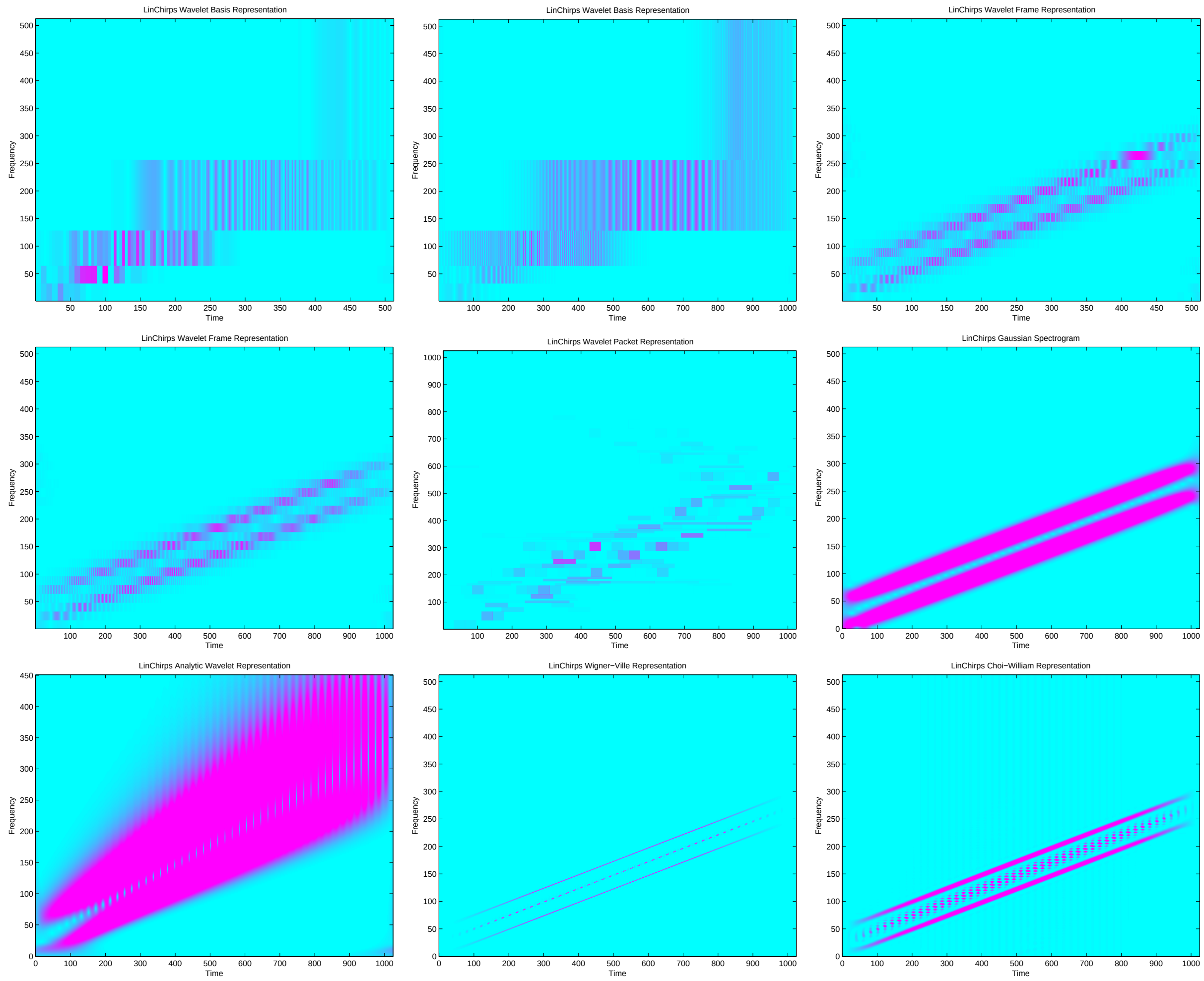

Figure 5.2: Linear chirps signal representations using (a) Decimated Wavelet Basis Transform, (b) Stationary Wavelet Basis Transform, (c) Decimated Wavelet Frame Transform, (d) Stationary Wavelet Frame Transform, (e) Wavelet Packet Transform, (f) Short-Time Fourier Transform, (g) Analytic Wavelet Transform, (h) Wigner-Ville Distribution, (i) Choi-William Distribution.

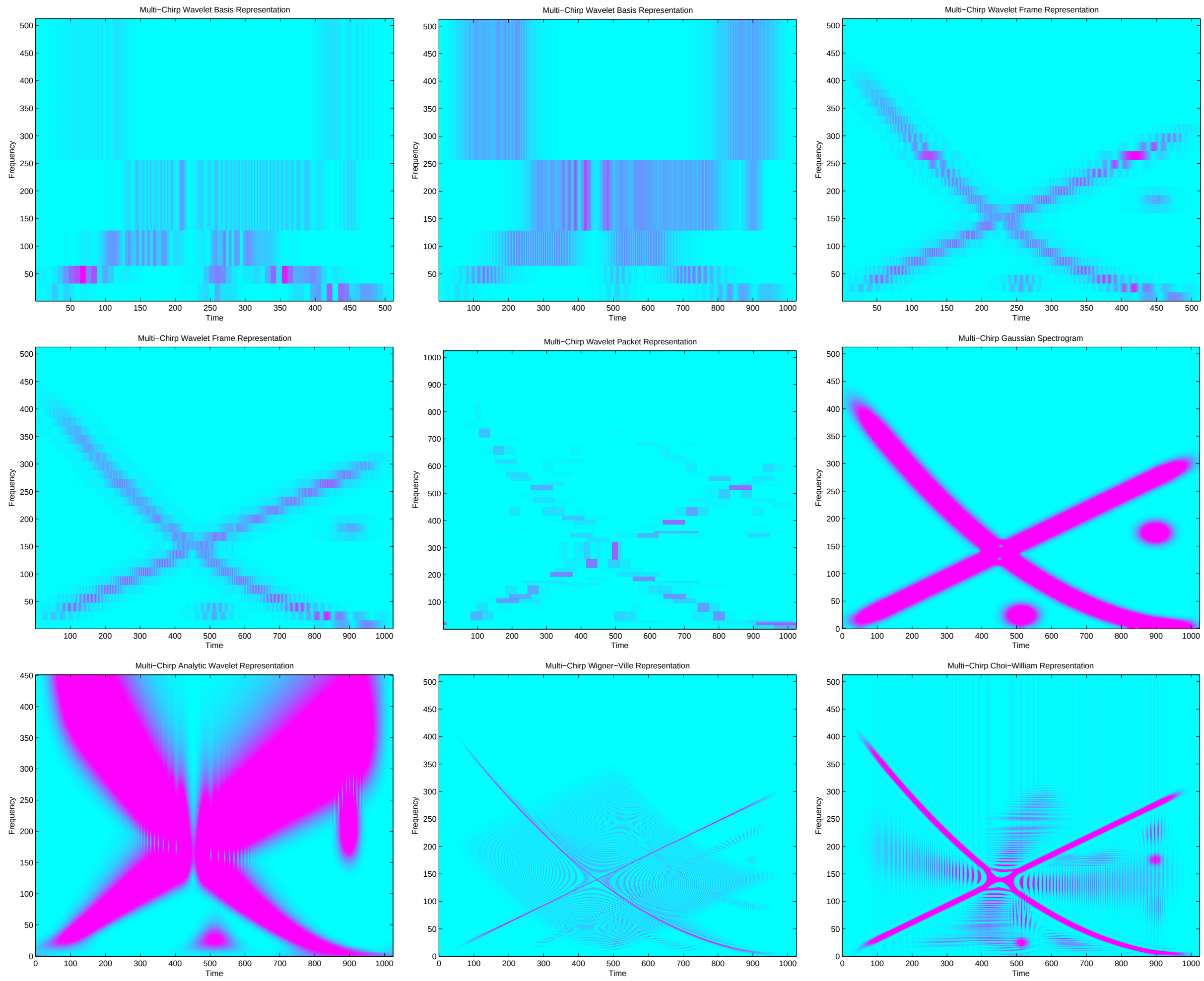

Figure 5.3: Multichirp signal representations using (a) Decimated Wavelet Basis Transform, (b) Stationary Wavelet Basis Transform, (c) Decimated Wavelet Frame Transform, (d) Stationary Wavelet Frame Transform, (e) Wavelet Packet Transform, (f) Short-Time Fourier Transform, (g) Analytic Wavelet Transform, (h) Wigner-Ville Distribution, (i) Choi-William Distribution.

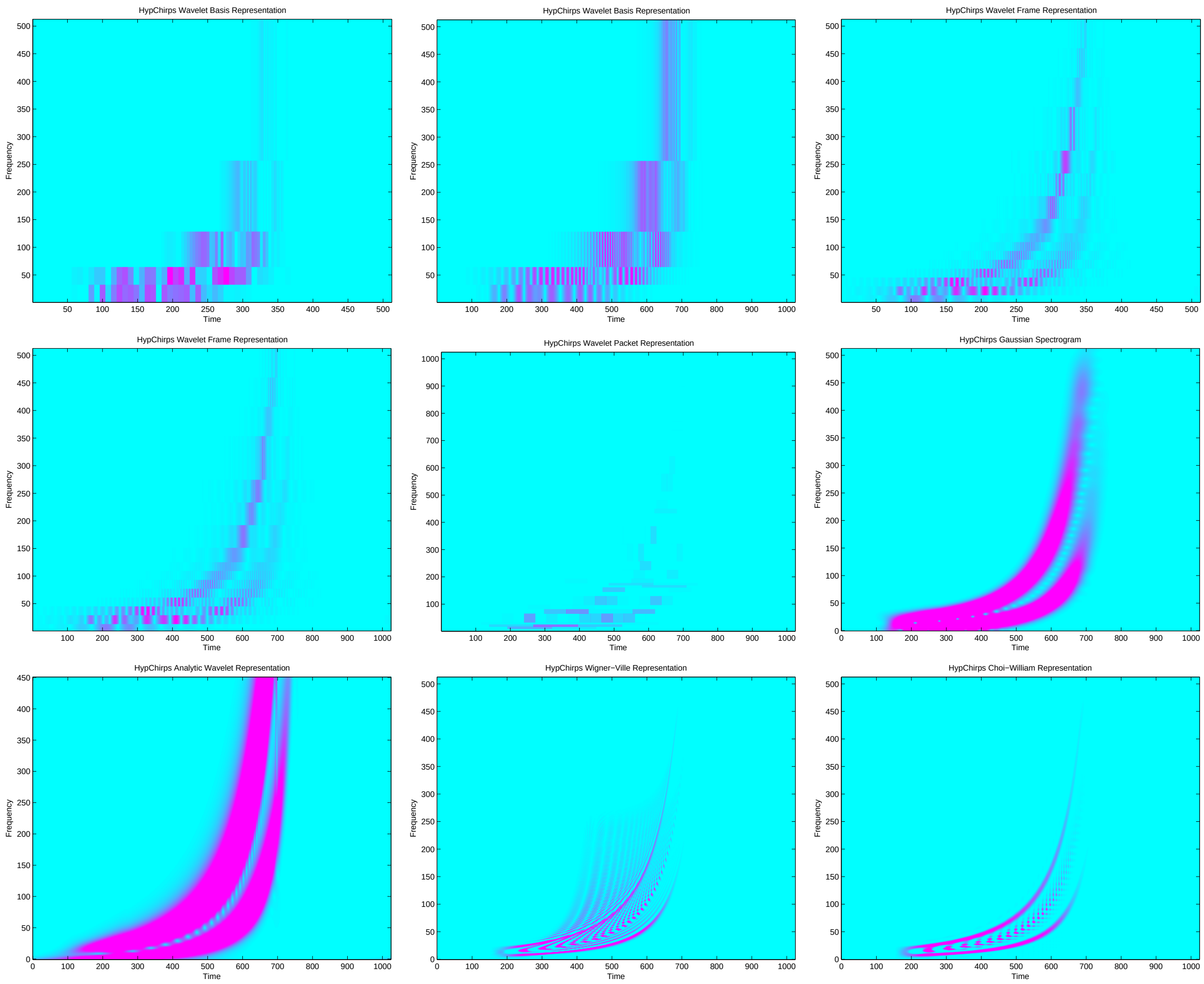

Figure 5.4: Hyperbolic chirps signal representations using (a) Decimated Wavelet Basis Transform, (b) Stationary Wavelet Basis Transform, (c) Decimated Wavelet Frame Transform, (d) Stationary Wavelet Frame Transform, (e) Wavelet Packet Transform, (f) Short-Time Fourier Transform, (g) Analytic Wavelet Transform, (h) Wigner-Ville Distribution, (i) Choi-William Distribution.

We remark that in all the representations, the stationary version of our transforms performs better than the decimated version by improving the time resolution with translation invariant sampling. The good time-frequency representations of the different signals in Figures 5.1 to 5.4 also demonstrate that our transforms incorporate the strengths of both the wavelet transform and the short-time Fourier transform.

We conclude the thesis by describing the partitioning of the frequency domain for our bandlimited wavelet frame transforms as an algorithm below. Ideally, we would keep $\Delta\omega/\omega$ as an invariant so that our transform approximates the analytic wavelet transform. Due to the discretized nature of our transforms, we use the recurrence formula $\Delta\omega_{4m}/\omega_{4m} = \Delta\omega_{4m-4}/(\omega_{4m-4} - \Delta\omega_{4m-4})$ with $\omega_{4m-4} = \omega_{4m} - \Delta\omega_{4m}/2$ and we use a fixed $\Delta\omega = 16$ when $\omega \leq 32$.

(1) Set $\Delta\omega = 64$, $\text{noBands} = 4$, $\omega = \text{samplesize}/2 + \lfloor \Delta\omega/2 \rfloor$, $m = \varrho_{k+1}$.

(2) While $\omega > 32$ and $\Delta\omega > 32$, repeat the following steps:
Set $\gamma = \omega - \lfloor \Delta\omega/2 \rfloor$, $\Delta\omega = \lfloor \gamma\Delta\omega/(\omega + \Delta\omega) \rfloor$, $\omega = \gamma$.
For $i = 1$ to noBands, set $L_{k,m} = \omega$, $N_{k,m} = L_{k,m} - \lfloor \Delta\omega/4 \rfloor$, $\omega = \omega - \Delta\omega$, $m = m - 1$.

(3) Set $\Delta\omega = 16$. While $N_{k,m} \geq 4$, repeat the following steps:
Set $L_{k,m} = L_{k,m+1} - \Delta\omega$, $N_{k,m} = N_{k,m+1} - \Delta\omega$, $m = m - 1$.

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