BANDLIMITED WAVELETS

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Summary

Bandlimited wavelets are members from a finite set $\Psi := \{\psi_i \in L^2(\mathbb{R}) : i = 1, \ldots, n\}$ of bandlimited functions for which the collection $X(\Psi) := \{\psi_i(2^j \cdot -k) : j, k \in \mathbb{Z}, i = 1, \ldots, n\}$ forms a frame for $L^2(\mathbb{R})$. Classical examples are the Shannon and Meyer's wavelets. However for the past decade, the emphasis is on the construction of compactly supported wavelets and not bandlimited ones, so little is known about the systematic construction of bandlimited wavelets. Thus the main objective of this thesis is to provide relatively simple ways in constructing large families of bandlimited wavelets so that the resulting collections of $X(\Psi)$ form orthonormal bases, Riesz bases, tight frames or dual frames of $L^2(\mathbb{R})$.

In the first chapter, some preliminary results regarding the fundamentals of wavelet theory are given. They provide foundation materials for the thesis.

Subsequently in Chapter 2, we first characterize the generation of bandlimited scaling functions via a special class of even real-valued 2π-periodic functions $\mathcal{A}_{\delta,\Omega}$, where on the interval $[-\pi, \pi]$, the functions are supported on $[-\Omega, \Omega] \subseteq [-2\pi/3, 2\pi/3]$ and take the value 1 on $[-\delta, \delta]$, $\delta > 0$. Next, we provide characterizations of a function $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ such that the integer shifts of the resulting scaling function ϕ form (a) an orthonormal basis for V_0, (b) a Riesz basis for V_0, (c) a frame for V_0, where $V_0 := \overline{\text{span}\{\phi(\cdot - k)\,;\, k \in \mathbb{Z}\}}$. Several examples are given to illustrate the theory in this chapter.

Regularity and decay of functions in $L^2(\mathbb{R})$ as well as interpolatory properties of scaling functions and wavelet functions are discussed in Chapter 3. This is to facilitate subsequent construction of bandlimited wavelets with good time and frequency localization.

A study of bell functions and orthonormal bandlimited wavelets is made in Chapter 4. In particular, we illustrate the construction of Meyer's wavelets through the use of bell functions and the class $\mathcal{A}_{\delta,\Omega}$, $0 < \delta \leq \Omega \leq 2\pi/3$.

In Chapter 5, we utilize two special setups based on the Mixed Unitary Extension Principle (Mixed UEP) as mentioned in [5], [7] and [9] to explicitly create bandlimited dual frames and bandlimited tight frames. Moreover, these wavelets can be constructed using bell functions such that they belong to the Schwartz class.

Finally, in the last chapter, we adapt a method used in [15] to construct bandlimited biorthogonal wavelets. Technical proofs are also adapted carefully from [6] and [8]. The chapter ends with examples of bandlimited biorthogonal interpolatory wavelets.

Notations

$L^p(E)$	The space of all complex-valued p-integrable functions on a measurable set E.
supp	The closure of the set for which the associated function takes nonzero values.
δ_{jk}	The Kronecker delta function.
$\mathbf{1}_E(\cdot)$	The characteristic function over the set E.
$C^k(\mathbb{R})$	The set of all complex-valued functions whose kth derivative is continuous over the real line.
$\ell^p(\mathbb{Z})$	The set of all p-summable infinite complex sequences.
$\equiv$	Equality of two functions pointwise up to a set of Lebesgue measure zero. sense.
$\lceil x \rceil$	The smallest integer greater than or equal to x.
$\mathbb{Z}$	The set of all integers.
$\mathbb{N}$	The set of all natural numbers.
$\mathbb{R}$	The set of all real numbers.
$\mathbb{C}$	The set of all complex numbers.
D^k	The kth derivative of a function.
$\text{sgn}(x)$	The sign of the variable x.
$m(E)$	The Lebesgue measure of the measurable set E.

Contents

Chapter 1

Introduction to Wavelet Theory

1.1 Introduction to Fourier analysis

The reader is assumed to be familiar with basic concepts of Lebesgue measure, integration theory, normed spaces and Hilbert spaces. In particular, the Lebesgue dominated convergence theorem will be used several times throughout this thesis. Since the thesis also requires background knowledge of Fourier analysis and wavelet theory, let us present some basic concepts of these areas here. The space $L^2(\mathbb{R})$ is a Hilbert space with inner product defined as

$$\langle f, g \rangle := \int_{x \in \mathbb{R}} f(x)\overline{g(x)}\, dx, \quad f, g \in L^2(\mathbb{R}).$$

Its norm is given by

$$\| f \|_2 := \langle f, f \rangle^{1/2}, \quad f \in L^2(\mathbb{R}).$$

We consider the Fourier transform in the following.

Theorem 1.1.1 *Let $f \in L^1(\mathbb{R})$. The Fourier transform $\tilde{\mathcal{F}}$ of f, $f \mapsto \widehat{f}$, defined as*

$$(\tilde{\mathcal{F}}(f))(\xi) := \widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx,$$

has the following properties.

(1) *The function $\hat{f}$ is bounded and continuous.*

(2) *Extending the Fourier transform* $\tilde{\mathcal{F}}$ *on* $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *by taking closure, the extended operator* $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $f \mapsto \hat{f}$ *is a continuous linear bijection.*

(3) $(\mathcal{F}^{-1}(\widehat{f}))(x) := f(x) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} e^{ix\xi}\widehat{f}(\xi)\, d\xi$ *a.e. if* $\widehat{f} \in L^1(\mathbb{R})$.

The extension of the operator $\tilde{\mathcal{F}}$ is based on the fact that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. By applying appropriate Cauchy limits, we get the extended operator $\mathcal{F}$ which will be referred to as the Fourier transform. Details about the proof of this theorem can be found in [6]. There are some minor variations of the definition of the Fourier and inverse Fourier transforms. For consistency, the version above will be used throughout this thesis. Next, we have two important results, namely, the Plancherel's theorem and Parseval's identity given below respectively.

Theorem 1.1.2 *For all* f, $g \in L^2(\mathbb{R})$, *the following relation holds:*

$$\langle f, g\rangle = \frac{1}{2\pi}\langle \widehat{f}, \widehat{g}\rangle.$$

In particular, $\| f\|_2 = \dfrac{1}{\sqrt{2\pi}}\| \widehat{f}\|_2$.

We will now state some basic properties of the Fourier transform.

Proposition 1.1.1 *The Fourier transform* $\mathcal{F}$ *has the following properties:*

(1) $\mathcal{F}(f(\alpha\cdot))(\xi) = \frac{1}{|\alpha|}\widehat{f}(\frac{\xi}{\alpha})$, *where* $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

(2) $\mathcal{F}(f(\cdot - x_0))(\xi) = \widehat{f}(\xi)e^{-i\xi x_0}$, *where* $x_0 \in \mathbb{R}$.

(3) $\mathcal{F}(fe^{i\xi_0\cdot})(\xi) = \widehat{f}(\xi - \xi_0)$, *where* $\xi_0 \in \mathbb{R}$.

(4) $\mathcal{F}(\overline{f})(\xi) = \overline{\widehat{f}(-\xi)}$.

1.2 Fundamental wavelet theory

Now we introduce some fundamental concepts in wavelet theory. We say that a sequence of functions $\{v_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ is a *frame* for $L^2(\mathbb{R})$ if there exist constants A,

$B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, v_n\rangle|^2 \leq B\|f\|_2^2, \tag{1.2.1}$$

for all $f \in L^2(\mathbb{R})$. A frame is a special case of a *Bessel* system, i.e. the right inequality of (1.2.1) holds for every $f \in L^2(\mathbb{R})$. The supremum of A and the infimum of B for (1.2.1) to hold are called *frame bounds*. A frame $\{v_n\}_{n\in\mathbb{Z}}$ is said to be *tight* if we may take $A = B = 1$. Such a frame is sometimes referred to as a *normalized tight frame* in the literature. A tight frame for $L^2(\mathbb{R})$ becomes an orthonormal basis when $\|v_n\|_2 = 1$ for every $n \in \mathbb{Z}$. More information about the theory of frames can be found in the book [5].

We say that $\{v_n\}_{n\in\mathbb{Z}}$ is a *Riesz sequence* in $L^2(\mathbb{R})$ if there exist constants $A, B > 0$ such that

$$A\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \|\sum_{n\in\mathbb{Z}} c_n v_n\|_2^2 \leq B\sum_{n\in\mathbb{Z}} |c_n|^2$$

for all $\{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. If in addition, the linear span of $\{v_n\}_{n\in\mathbb{Z}}$ is dense in $L^2(\mathbb{R})$, then $\{v_n\}_{n\in\mathbb{Z}}$ is said to be a *Riesz basis* for $L^2(\mathbb{R})$. In particular, if $A = B = 1$, we say that $\{v_n\}_{n\in\mathbb{Z}}$ forms an *orthonormal basis* for $L^2(\mathbb{R})$.

Define the *affine system* $X(\Psi) := \{2^{j/2}\psi_i(2^j \cdot -k) : j, k \in \mathbb{Z}, i = 1, \ldots, n\}$, where $\Psi = \{\psi_i \in L^2(\mathbb{R}) : i = 1, \ldots, n\}$. If $X(\Psi)$ forms a frame or Riesz basis for $L^2(\mathbb{R})$, then Ψ is commonly referred to as a set of *wavelets* or *mother wavelets* for $L^2(\mathbb{R})$.

Next, it is well known that wavelets are usually constructed by means of a multiresolution analysis (MRA).

Definition 1.2.1 *A multiresolution analysis (MRA) of $L^2(\mathbb{R})$ with dilation factor* 2 *is a doubly infinite nested sequence of closed subspaces of $L^2(\mathbb{R})$,*

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots,$$

with the following properties:

(M1) $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in $L^2(\mathbb{R})$.*

(M2) $f \in V_j$ *if and only if* $f(2\cdot) \in V_{j+1}$ *, for every* $j \in \mathbb{Z}$.

(M3) *$f \in V_j$ if and only if $f(\cdot - 2^{-j}k) \in V_j$, for every j, $k \in \mathbb{Z}$.*

(M4) *There exists a function $\phi \in L^2(\mathbb{R})$ such that*

$$\overline{\text{span}\{\phi(\cdot - k) \,:\, k \in \mathbb{Z}\}} = V_0.$$

The function ϕ is called a scaling function and V_0 is called an integer shift-invariant subspace of $L^2(\mathbb{R})$.

We say that a function $\phi \in L^2(\mathbb{R})$ is *refinable* if it satisfies the following *two-scale relation*: there exists a coefficient sequence $\{h_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\phi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \phi(2x - k) \quad a.e. \tag{1.2.2}$$

It is well known that if $\phi \in L^2(\mathbb{R})$ and is refinable, then the subspaces $\{V_j\}_{j\in\mathbb{Z}}$ defined by

$$V_j = \overline{\text{span}\{\phi(2^j \cdot - k) : k \in \mathbb{Z}\}} \tag{1.2.3}$$

satisfies properties (M2), (M3), (M4) in Definition 1.2.1 automatically. The subspaces $\{V_j\}_{j\in\mathbb{Z}}$ are usually termed as *shift-invariant subspaces* of $L^2(\mathbb{R})$. The interested reader can refer to the book [23] for details. Lastly, we require three results regarding the characterizations of integer shift-invariant subspaces, refinability and the density of union of shift-invariant subspaces in $L^2(\mathbb{R})$.

Theorem 1.2.1 *Given that f, $\phi \in L^2(\mathbb{R})$ and $\overline{\text{span}\{\phi(\cdot - k) \,:\, k \in \mathbb{Z}\}} = V$, $f \in V$ if and only if there exists a 2π-periodic measurable function $\hat{m}$ such that $\hat{f} = \hat{m}\hat{\phi}$ a.e.*

This characterization is proved by deBoor, DeVore and Ron in [4]. Consequently, we have a characterization of the refinability of a function $\phi \in L^2(\mathbb{R})$.

Theorem 1.2.2 *Given that $\phi \in L^2(\mathbb{R})$, ϕ is refinable if and only if there exists a 2π-periodic measurable function $\hat{a}$ such that*

$$\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi) \quad a.e., \tag{1.2.4}$$

where $\hat{a}(\xi) = \sum_{k\in\mathbb{Z}} h_k e^{-ik\xi}$ and $\{h_k\}_{k\in\mathbb{Z}}$ is the coefficient sequence in (1.2.2).

Note that $\hat{a}$ is usually referred to as the *mask* of the scaling function ϕ. We now state a characterization of the density of the union of shift-invariant subspaces in $L^2(\mathbb{R})$.

Theorem 1.2.3 *Let $\phi \in L^2(\mathbb{R})$ be refinable and $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspaces of $L^2(\mathbb{R})$ defined by ϕ as in* (1.2.3)*. We have*

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$$

if and only if $\bigcap_{j\in\mathbb{Z}} Z_j(\hat{\phi})$ has Lebesgue measure zero where

$$Z_j(\hat{\phi}) := \{\xi \in \mathbb{R} \,:\, \hat{\phi}(2^{-j}\xi) = 0\}.$$

In particular, if $\hat{\phi}(0) \neq 0$ and $\hat{\phi}$ is continuous at the origin, then

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

For details of the proof, the interested reader can refer to [4]. We will now proceed to the next chapter where we define and investigate bandlimited scaling functions and their masks.

Chapter 2

Analysis on Bandlimited Scaling Functions

2.1 Masks of bandlimited scaling functions

In this section, we first characterize refinability of a large class of bandlimited functions. A function ϕ is *bandlimited* if $\phi \in L^2(\mathbb{R})$ and its Fourier transform $\hat{\phi}$ has compact support in some interval $[-\Omega, \Omega]$, where $\Omega > 0$. Subsequently, we will investigate the generation of bandlimited scaling functions through a product involving bandlimited 2π-periodic functions. With some abuse of notation, we say a 2π-periodic function $\hat{a}$ is 2π-*bandlimited* if $\operatorname{supp} \hat{a}\mathbf{1}_{[-\pi,\,\pi]} \subseteq [-\Omega, \Omega]$ for some positive $\Omega < \pi$. First, we introduce special classes of 2π-bandlimited 2π-periodic functions and bandlimited functions to facilitate the discussion.

Definition 2.1.1 *For $0 < \delta \le \Omega < \pi$, let $\mathcal{A}_{\delta,\Omega}$ be the set of all 2π-periodic, bounded, even, nonnegative functions $\hat{a}$ with the following properties:*

(a) *$\hat{a}$ is continuous everywhere except possibly at the set of points $\{\Omega+2\pi l,\ -\Omega+2\pi l : l \in \mathbb{Z}\}$.*

(b) *$\hat{a}(\xi) = 0$ for $\xi \in [-\pi, \pi]\backslash[-\Omega, \Omega]$.*

(c) *$\hat{a}$ is totally positive in the following sense: $\hat{a}(\xi) > 0$ for $\xi \in (-\Omega, \Omega)$.*

(d) *$\hat{a}$ has an interval of constancy* (IOC), *i.e.* $\hat{a}(\xi) = 1$ *for* $\xi \in [-\delta, \delta]$.

Here, we mention two possibilities of the function $\hat{a}$ which will be useful in subsequent sections. In fact, due to the special structure of the function $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, we have the following characterizations.

(1) $\lim_{\xi \to \Omega^-} \hat{a}(\xi) > 0$ if and only if $\hat{a}$ is discontinuous at the point Ω.

(2) $\lim_{\xi \to \Omega^-} \hat{a}(\xi) = 0$ if and only if $\hat{a}$ is continuous everywhere.

These two properties can be easily shown by standard calculus techniques, so we will omit the proof here. Analogously, we define another special class of functions in $L^2(\mathbb{R})$.

Definition 2.1.2 *For $0 < \delta \leq \Omega < \pi$, let $\mathcal{B}_{\delta,\Omega}$ consist of all functions $\phi \in L^2(\mathbb{R})$ such that the following hold.*

(a) *Its Fourier transform $\hat{\phi}$ is bounded, even, nonnegative and continuous everywhere except possibly at the points $\pm 2\Omega$.*

(b) $\hat{\phi}(\xi) = 0$ *for* $\xi \in \mathbb{R} \backslash [-2\Omega, 2\Omega]$.

(c) *$\hat{\phi}$ is totally positive in the following sense:* $\hat{\phi}(\xi) > 0$ *for* $\xi \in (-2\Omega, 2\Omega)$.

(d) *$\hat{\phi}$ has an interval of constancy* (IOC), *i.e.* $\hat{\phi}(\xi) = 1$ *for* $\xi \in [-2\delta, 2\delta]$.

Similarly, we have the following.

(1) $\lim_{\xi \to 2\Omega^-} \hat{\phi}(\xi) > 0$ if and only if $\hat{\phi}$ is discontinuous at the point 2Ω.

(2) $\lim_{\xi \to 2\Omega^-} \hat{\phi}(\xi) = 0$ if and only if $\hat{\phi}$ is continuous everywhere.

While the properties of $\mathcal{A}_{\delta,\Omega}$ and $\mathcal{B}_{\delta,\Omega}$ look cumbersome, it shall be shown that they are in fact mild and natural assumptions by providing a large class of examples at the end of this chapter. We shall see later that many well-known bandlimited scaling functions like the Shannon's scaling function and the Meyer's scaling function arise from appropriate functions in $\mathcal{A}_{\delta,\Omega}$ for some δ and Ω.

In the literature, properties about scaling functions are often discussed in terms of bracket product and spectrum which we define below. For $f,\, g \in L^2(\mathbb{R})$, define

$$[f,\, g](\xi) := \sum_{l\in\mathbb{Z}} f(\xi + 2\pi l)\overline{g(\xi + 2\pi l)},$$

which is known to be a well-defined $L^1[-\pi,\, \pi]$ function. Then for a scaling function ϕ, its *bracket product* is defined as

$$[\hat{\phi},\, \hat{\phi}](\xi) := \sum_{l\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi l)|^2$$

and its *spectrum* is

$$\sigma(\phi) := \{\xi \in [-\pi, \pi] \,:\, [\hat{\phi},\, \hat{\phi}](\xi) > 0\}.$$

It is interesting and easy to note that for $0 < \delta \leq \Omega < \pi$ and $\phi \in \mathcal{B}_{\delta,\Omega}$, $\sigma(\phi) = [-\pi,\, \pi]$ if and only if $\Omega \geq \pi/2$, where equality of the sets is up to a set of measure zero. Indeed, if $\Omega \geq \pi/2$, it follows easily from the definition of $\mathcal{B}_{\delta,\Omega}$ that $\sigma(\phi) = [-\pi,\, \pi]$. Conversely, if $\sigma(\phi) = [-\pi, \pi]$, suppose on the contrary that $\Omega < \pi/2$. Then $[\hat{\phi},\, \hat{\phi}](\xi) = |\hat{\phi}(\xi)|^2 = 0$ for $\xi \in [-\pi, -\Omega] \cup [\Omega, \pi]$, which is a contradiction. In fact,

$$\sigma(\phi) = \begin{cases} [-2\Omega, 2\Omega], & \text{if } \Omega < \pi/2, \\ [-\pi, \pi], & \text{if } \pi/2 \leq \Omega < \pi. \end{cases} \tag{2.1.1}$$

This chapter is organized as follows:

(1) Characterize $\mathcal{A}_{\delta,\Omega}$ in terms of $\mathcal{B}_{\delta,\Omega}$ when $0 < \delta \leq \Omega \leq 2\pi/3$.

(2) Show that for any $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ with $0 < \delta \leq \Omega \leq 2\pi/3$, the resulting scaling function ϕ generates an MRA of $L^2(\mathbb{R})$.

(3) Find characterizations of functions $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ such that they give scaling functions with orthonormal shifts, shifts that form a frame for V_0 and a Riesz sequence in $L^2(\mathbb{R})$, where V_0 is defined by ϕ as in (1.2.3).

(4) Provide examples of functions $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ which generate classical examples of scaling functions like the Shannon and Meyer's scaling functions.

Before we start to characterize $\mathcal{A}_{\delta,\Omega}$ in terms of $\mathcal{B}_{\delta,\Omega}$ when $0 < \delta \leq \Omega \leq 2\pi/3$, we require a theorem from [22] for one of the uniqueness results in Proposition 2.1.1.

Theorem 2.1.1 *Let $\phi \in L^2(\mathbb{R})$ satisfy (1.2.4) and $\lim_{\xi\to 0} \hat{\phi}(\xi) = \hat{\phi}(0) \neq 0$. Then any solution $\varphi \in L^2(\mathbb{R})$ with $\lim_{\xi\to 0} \hat{\varphi}(\xi) = \hat{\varphi}(0)$ satisfying (1.2.4) can be written as*

$$\varphi(x) = \frac{\hat{\varphi}(0)}{\hat{\phi}(0)}\phi(x).$$

Proposition 2.1.1 *Let $0 < \delta \leq \Omega \leq 2\pi/3$. Then for every $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, there exists a unique function $\phi \in \mathcal{B}_{\delta,\Omega}$ such that (1.2.4) holds. Conversely, for every $\phi \in \mathcal{B}_{\delta,\Omega}$, there exists a function $\hat{a}$ which is unique up to a set of measure zero in the class $\mathcal{A}_{\delta,\Omega}$ such that (1.2.4) holds. Furthermore $\hat{a}$ is continuous if and only if $\hat{\phi}$ is continuous.*

Proof: Let $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ and we define ϕ by its Fourier transform $\hat{\phi}$ given by

$$\hat{\phi}(\xi) := \Big[\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)\Big]\mathbf{1}_{[-2\Omega,\,2\Omega]}(\xi), \tag{2.1.2}$$

where

$$N := \begin{cases} \lceil \log_2(\Omega/\delta)\rceil, & \text{if } \delta < \Omega, \\ 1, & \text{if } \delta = \Omega. \end{cases} \tag{2.1.3}$$

First we consider the function

$$q(\xi) := \frac{\hat{\phi}(2\xi)}{\hat{\phi}(\xi)}\mathbf{1}_{[-\Omega,\,\Omega]}(\xi) = \frac{\hat{a}(\xi)}{\hat{a}(2^{-N}\xi)}\mathbf{1}_{[-\Omega,\,\Omega]}(\xi) = \hat{a}(\xi)\mathbf{1}_{[-\Omega,\,\Omega]}(\xi) = \hat{a}(\xi)\mathbf{1}_{[-\pi,\,\pi]}(\xi)$$

because $\hat{a}(2^{-N}\xi) = 1$ for $|\xi| \leq \Omega$. This arises from the following: $2^{-N}|\xi| \leq \delta$ for $|\xi| \leq \Omega$, and $\hat{a}(\xi) = 1$ for $|\xi| \leq \delta$. Note that the function q is well defined since $\hat{\phi}(\xi) > 0$ for $|\xi| \leq \Omega$. Observe that $\hat{a}$ is the 2π-periodic extension of the function q in the sense that,

$$\hat{a}(\xi) = \sum_{l\in\mathbb{Z}} q(\xi + 2\pi l).$$

We claim that ϕ, defined in (2.1.2) by $\hat{a}$, is a solution to the refinement equation (1.2.4) and furthermore, $\phi \in \mathcal{B}_{\delta,\Omega}$. For almost all $\xi \in \mathbb{R}$,

$$\begin{aligned} \hat{a}(\xi)\hat{\phi}(\xi) &= \sum_{l\in\mathbb{Z}} q(\xi + 2\pi l)\hat{\phi}(\xi) \\ &= [q(\xi)\hat{\phi}(\xi)]\mathbf{1}_{[-\Omega,\,\Omega]}(\xi) = \hat{\phi}(2\xi)\mathbf{1}_{[-\Omega,\,\Omega]}(\xi) = \hat{\phi}(2\xi), \end{aligned}$$

because the functions $\hat{a}$ and $\hat{\phi}$ has supports $\bigcup_{l\in\mathbb{Z}}[-\Omega+2\pi l,\,\Omega+2\pi l]$ and $[-2\Omega,\,2\Omega]$ respectively and

$$\Big(\bigcup_{l\in\mathbb{Z}}[-\Omega+2\pi l,\,\Omega+2\pi l]\Big)\bigcap[-2\Omega,\,2\Omega]=[-\Omega,\,\Omega]$$

up to a set of measure zero since $\Omega\le 2\pi/3$. Thus $\hat{\phi}$ is a solution to the refinement equation (1.2.4).

To show that $\phi\in\mathcal{B}_{\delta,\Omega}$, first notice that the product function $\hat{\phi}$ in (2.1.2) can be written as $\hat{\phi}(\xi)=\prod_{j=1}^{N}w_j(\xi)$ where $w_j(\xi)=\hat{a}(2^{-j}\xi)\mathbf{1}_{[-\Omega,\,\Omega]}(2^{-j}\xi)$, $j=1,\,\ldots,N$. Now, $w_j(\xi)$ is continuous everywhere except possibly at the points $\pm2^j\Omega$. Since the product function is identically zero outside the interval $[-2\Omega,\,2\Omega]$, it follows that the product function $\hat{\phi}$ is continuous everywhere except possibly at $\pm2\Omega$. Next, $\hat{\phi}(\xi)=1$ for $|\xi|\le2\delta$ because $\hat{a}(2^{-j}\xi)=1$ for $|\xi|\le2\delta$ and every $j\in\mathbb{N}$.

We also have $\hat{\phi}(\xi)>0$ for $|\xi|<2\Omega$ because $\hat{a}(2^{-j}\xi)>0$ for $|\xi|<2\Omega$, and all $j\in\mathbb{N}$. It is clear from the definition of $\hat{\phi}$ in (2.1.2) that $\hat{\phi}$ is bounded everywhere whenever $\hat{a}$ is bounded everywhere. Lastly, note that if $\hat{a}$ is an even function, then $\hat{\phi}$ is also even because

$$\hat{\phi}(-\xi)=\Big[\prod_{j=1}^{N}\hat{a}(-2^{-j}\xi)\Big]\mathbf{1}_{[-2\Omega,\,2\Omega]}(-\xi)=[\prod_{j=1}^{N}\hat{a}(2^{-j}\xi)]\mathbf{1}_{[-2\Omega,\,2\Omega]}(\xi)=\hat{\phi}(\xi)$$

for almost everywhere $\xi\in\mathbb{R}$. We conclude that $\phi\in\mathcal{B}_{\delta,\Omega}$.

Furthermore if $\hat{a}$ is continuous everywhere, then $\hat{a}(\Omega)=0=\hat{a}(-\Omega)$ and so $\hat{a}\mathbf{1}_{[-\Omega,\,\Omega]}$ is also continuous everywhere. Now note that

$$\hat{\phi}(\xi)=\Big[\prod_{j=1}^{N}\hat{a}(2^{-j}\xi)\Big]\mathbf{1}_{[-2\Omega,\,2\Omega]}(\xi)=\prod_{j=1}^{N}\Big[\hat{a}(2^{-j}\xi)\mathbf{1}_{[-\Omega,\,\Omega]}(2^{-j}\xi)\Big],$$

and thus $\hat{\phi}$ must be continuous everywhere.

To show uniqueness, let φ be another function in $\mathcal{B}_{\delta,\Omega}$ such that (1.2.4) holds. Then by Theorem 2.1.1,

$$\varphi(x)=\frac{\hat{\varphi}(0)}{\hat{\phi}(0)}\phi(x)=\phi(x)$$

since ϕ, $\varphi\in\mathcal{B}_{\delta,\Omega}$ and $\hat{\varphi}(0)=1=\hat{\phi}(0)$.

Conversely, if $\phi \in \mathcal{B}_{\delta,\Omega}$, consider the function

$$p(\xi) := \begin{cases} \frac{\hat{\phi}(2\xi)}{\hat{\phi}(\xi)}, & \text{if } \xi \in [-\Omega,\, \Omega], \\ 0, & \text{otherwise}, \end{cases} \tag{2.1.4}$$

which is well defined because $\hat{\phi}(\xi) > 0$ for $|\xi| \leq \Omega$. Define $\hat{a}(\xi) := \sum_{l\in\mathbb{Z}} p(\xi + 2\pi l)$ which is also the 2π-periodic extension of the function p because the support of p is $[-\Omega,\, \Omega] \subset [-\pi,\, \pi]$. We shall show that $\hat{a}$ is a solution to the refinement equation (1.2.4) and indeed $\hat{a} \in \mathcal{A}_{\delta,\Omega}$.

Now, for almost everywhere $\xi \in \mathbb{R}$,

$$\begin{aligned} \hat{a}(\xi)\hat{\phi}(\xi) &= \sum_{l\in\mathbb{Z}} p(\xi + 2\pi l)\hat{\phi}(\xi) = [\sum_{l\in\mathbb{Z}} p(\xi + 2\pi l)\hat{\phi}(\xi)]\mathbf{1}_{[-\Omega,\Omega]}(\xi) \\ &= [p(\xi)\hat{\phi}(\xi)]\mathbf{1}_{[-\Omega,\Omega]}(\xi) = \hat{\phi}(2\xi)\mathbf{1}_{[-\Omega,\Omega]}(\xi) = \hat{\phi}(2\xi), \end{aligned}$$

because the functions $\hat{a}$ and $\hat{\phi}$ has supports $\bigcup_{l\in\mathbb{Z}}[-\Omega + 2\pi l,\, \Omega + 2\pi l]$ and $[-2\Omega,\, 2\Omega]$ respectively and

$$\Big(\bigcup_{l\in\mathbb{Z}}[-\Omega + 2\pi l,\, \Omega + 2\pi l]\Big) \bigcap [-2\Omega,\, 2\Omega] = [-\Omega,\, \Omega]$$

up to a set of measure zero since $\Omega \leq 2\pi/3$. Thus $\hat{a}$ is a solution to the refinement equation (1.2.4).

Since $\hat{\phi}$ is an even function with $\hat{\phi}(\xi) > 0$ and continuous for $|\xi| \leq \Omega$, then $1/\hat{\phi}(\xi)$ is also continuous, bounded, even and strictly positive for $|\xi| \leq \Omega$. Thus by the definition of the function p in (2.1.4), p is also an even, bounded function which is strictly positive and continuous whenever $\xi \in (-\Omega,\, \Omega)$. Lastly, since both $\hat{\phi}(\xi)$ and $\hat{\phi}(2\xi)$ equal 1 whenever $\xi \in [-\delta,\, \delta]$, again by (2.1.4), $p(\xi) = 1$ for $\xi \in [-\delta,\, \delta]$. As $\hat{a}$ is the 2π-periodic extension of the function p, it follows that $\hat{a} \in \mathcal{A}_{\delta,\Omega}$.

If furthermore, $\hat{\phi}$ is continuous everywhere, $\hat{\phi}(2\Omega) = 0 = \hat{\phi}(-2\Omega)$ and by the definition of p in (2.1.4), this implies that $p(\Omega) = 0 = p(-\Omega)$. Since p is continuous in $(-\Omega,\, \Omega)$ and $p(\xi) = 0$ for $|\xi| > \Omega$, p must be continuous everywhere. This makes its 2π-periodic extension function $\hat{a}$ continuous everywhere as well.

To show uniqueness, let $\widetilde{a} \in \mathcal{A}_{\delta,\Omega}$ be another solution to (1.2.4). We claim that $\widetilde{a}(\xi) = \hat{\phi}(2\xi)/\hat{\phi}(\xi)$ for $|\xi| \leq \Omega$, for otherwise it violates the refinement equation (1.2.4) on the interval $[-\Omega, \Omega]$. Furthermore, by the definition of $\mathcal{A}_{\delta,\Omega}$, $\widetilde{a}(\xi) = 0$ for $\Omega < |\xi| \leq \pi$. Thus $\widetilde{a}\mathbf{1}_{[-\pi,\pi]} = p$. By the periodicity of $\widetilde{a}$, we have $\widetilde{a} \equiv \hat{a}$. This finishes the proof of the proposition. ■

It is known in the wavelet literature (see [2]) that for a scaling function ϕ, if $\sigma(\phi) \neq [-\pi, \pi]$, then ϕ has infinitely many masks which satisfy the refinement equation. For example, define ϕ_Ω by its Fourier transform $\hat{\phi}_\Omega := \mathbf{1}_{[-2\Omega,2\Omega]}(\xi)$ where $\Omega < \pi/2$. Then the family of masks $\{\hat{a}_c\}_c$ defined by the 2π-periodic extension of the functions $\hat{q}_c(\xi) := \mathbf{1}_{[-\Omega,\Omega]}(\xi) + \mathbf{1}_{[-2\Omega,-c]}(\xi) + \mathbf{1}_{[c,2\Omega]}(\xi)$, $\pi/2 < c \leq 2\Omega$, satisfy the refinement equation. However these masks do not lie in $\mathcal{A}_{\delta,\Omega}$.

Masks with good low-pass properties are desired in applications, take for instance, the Shannon's ideal low-pass filter and the masks corresponding to Meyer's scaling functions (see [19]). Incidentally, such masks belong to $\mathcal{A}_{\delta,\Omega}$ for some $0 < \delta \leq \Omega \leq 2\pi/3$, which motivates us to only consider masks belonging to $\mathcal{A}_{\delta,\Omega}$ for a bandlimited scaling function ϕ in $\mathcal{B}_{\delta,\Omega}$. Furthermore, the uniqueness result proved in Proposition 2.1.1 leaves no ambiguities when we discuss the associated mask $\hat{a}$ in $\mathcal{A}_{\delta,\Omega}$ for a given $\phi \in \mathcal{B}_{\delta,\Omega}$.

For $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, we have constructed a scaling function $\phi \in \mathcal{B}_{\delta,\Omega}$ in the proof of Proposition 2.1.1. We shall next show that this scaling function ϕ actually coincides with the conventional scaling function φ defined by the infinite product

$$\hat{\varphi}(\xi) := \prod_{j=1}^{\infty} \hat{a}(2^{-j}\xi). \tag{2.1.5}$$

In doing so, we also illustrate the importance of the interval of constancy $[-\delta, \delta]$ of the mask $\hat{a}$ and the number N as defined in (2.1.3).

Proposition 2.1.2 *For $0 < \delta \leq \Omega \leq 2\pi/3$, if $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, then φ defined by $\hat{a}$ as in the conventional infinite product in* (2.1.5) *coincides almost everywhere with ϕ defined by $\hat{a}$ as in the finite product* (2.1.2).

Proof: For $\nu \geq 1$, define $g_\nu(\xi) := \prod_{j=1}^{\nu} \hat{a}(2^{-j}\xi)$. We have $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} \hat{a}(2^{-j}\xi) = \lim_{\nu\to\infty} g_\nu(\xi)$. We shall analyze the behavior of $\hat{\varphi}(\xi) = \lim_{\nu\to\infty} g_\nu(\xi)$ on the following subintervals of $\mathbb{R}$: $[-2\Omega, 2\Omega]$ and $[-2\Omega, 2\Omega]^c$ and thus show the pointwise equality almost everywhere between $\hat{\varphi}$ and $\hat{\phi}$ on each subinterval.

For $|\xi| \le 2\Omega$ and $j \ge N+1$, $\hat{a}(2^{-j}\xi) = 1$, where N is as defined in (2.1.3). Then for $|\xi| \le 2\Omega$ and $\nu \ge N+1$, $g_\nu(\xi) = \prod_{j=1}^{\nu} \hat{a}(2^{-j}\xi) = \prod_{j=1}^{N} \hat{a}(2^{-j}\xi)$. Thus, for $|\xi| \le 2\Omega$, $\lim_{\nu\to\infty} g_\nu(\xi) = \prod_{j=1}^{N} \hat{a}(2^{-j}\xi)$ and this means that $\hat{\varphi}(\xi) = \hat{\phi}(\xi)$.

Next, we show that for almost everywhere $|\xi| > 2\Omega$, $\lim_{\nu\to\infty} g_\nu(\xi) = 0$. Since $\Omega \le 2\pi/3$ and $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, $\hat{a}(2^{-j}\xi) = 0$ for $|\xi| \in (2^j\Omega, 2^j(-\Omega+2\pi)) \supseteq (2^j\Omega, 2^{j+1}\Omega)$. So for $\nu \in \mathbb{N}$, $g_\nu(\xi) = \prod_{j=1}^{\nu} \hat{a}(2^{-j}\xi) = 0$, $|\xi| \in \bigcup_{j=1}^{\nu} (2^j\Omega, 2^{j+1}\Omega)$. Then

$$\hat{\varphi}(\xi) = \lim_{\nu\to\infty} g_\nu(\xi) = 0, \quad |\xi| \in (2\Omega, \infty)\, a.e.,$$

which means that for $|\xi| > 2\Omega$, $\hat{\varphi}(\xi) = \hat{\phi}(\xi)$ almost everywhere. This concludes the proof. ■

We emphasize that from the proof of Proposition 2.1.2, global pointwise equality between the functions $\hat{\phi}$ and $\hat{\varphi}$ can be achieved if either of the following conditions holds:

(1) $\Omega < 2\pi/3$,

(2) $\hat{a}$ is continuous everywhere.

Indeed, Condition (1) implies that $\hat{a}(2^{-j}\xi) = 0$ for $\xi \in (2^j\Omega, 2^j(-\Omega+2\pi)) \supset [2^j 2\pi/3, 2^{j+1}2\pi/3]$. Thus, from the proof of Proposition 2.1.2, $\hat{\varphi}(\xi) = \lim_{\nu\to\infty} g_\nu(\xi) = 0$ for $|\xi| \in (2\Omega, \infty)$ everywhere, which gives the desired result. If Condition (2) holds, then $\hat{a}(\pm\Omega) = 0$. Given this and the proof of Proposition 2.1.2, we see that for $\nu \in \mathbb{N}$, $g_\nu(\xi) = \prod_{j=1}^{\nu} \hat{a}(2^{-j}\xi) = 0$, $|\xi| \in \bigcup_{j=1}^{\nu} [2^j\Omega, 2^{j+1}\Omega]$. Then

$$\hat{\varphi}(\xi) = \lim_{\nu\to\infty} g_\nu(\xi) = 0, \quad |\xi| \in (2\Omega, \infty),$$

which means that for $|\xi| > 2\Omega$, $\hat{\varphi}(\xi) = \hat{\phi}(\xi)$ everywhere.

Proposition 2.1.1 means that if $\Omega \le 2\pi/3$, a function $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ uniquely determines a scaling function $\phi \in \mathcal{B}_{\delta,\Omega}$ satisfying (1.2.4) and vice-versa. A natural question arising from Proposition 2.1.1 is that whether the assumption $\Omega \le 2\pi/3$ could be relaxed. We shall show in the following that it cannot be the case.

Proposition 2.1.3 *For $0 < \delta \le \Omega < \pi$, let $\phi \in \mathcal{B}_{\delta,\Omega}$. Then ϕ is refinable if and only if $\Omega \le 2\pi/3$.*

Proof: The sufficiency statement is clear from Proposition 2.1.1. It suffices to prove the necessity. Suppose that $\Omega > 2\pi/3$ and assume on the contrary that ϕ is refinable. Then there exists a 2π-periodic function $\hat{a}$ such that $\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ a.e. Furthermore, from the proof of Proposition 2.1.1, $\hat{a}$ must necessarily take the following form

$$\hat{a}(\xi) = \frac{\hat{\phi}(2\xi)}{\hat{\phi}(\xi)} \neq 0, \quad |\xi| < \Omega.$$

Thus $[-\Omega + 2\pi l, \Omega + 2\pi l] \subseteq \operatorname{supp} \hat{a}$ for all $l \in \mathbb{Z}$.

On the other hand, since $\Omega > 2\pi/3$,

$$\operatorname{supp} \hat{a} \cap \operatorname{supp} \hat{\phi} \supseteq [-2\Omega,\, \Omega - 2\pi] \cup [-\Omega,\, \Omega] \cup [2\pi - \Omega,\, 2\Omega],$$

which contradicts the fact that $\operatorname{supp} \hat{\phi}(2\cdot) = [-\Omega,\, \Omega]$. ■

In view of Proposition 2.1.3, we may only consider the generation of bandlimited refinable functions ϕ by functions $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ when $0 < \delta \le \Omega \le 2\pi/3$. Due to Propositions 2.1.1 and 2.1.3, throughout this thesis, for $0 < \delta \le \Omega \le 2\pi/3$, given $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, we can always define a scaling function ϕ by

$$\hat{\phi}(\xi) = [\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)]\mathbf{1}_{[-2\Omega,\, 2\Omega]}(\xi) = \prod_{j=1}^{N} \left[\hat{a}(2^{-j}\xi)\mathbf{1}_{[-\Omega,\, \Omega]}(2^{-j}\xi)\right], \tag{2.1.6}$$

where $0 < \delta \le \Omega \le 2\pi/3$, $\operatorname{supp} \hat{a}\mathbf{1}_{[-\pi,\, \pi]} = [-\Omega,\, \Omega]$, $\hat{a}(\xi) = 1$ for $|\xi| \le \delta$ and

$$N = \begin{cases} \lceil \log_2(\Omega/\delta) \rceil, & \text{if } \delta < \Omega, \\ 1, & \text{if } \delta = \Omega, \end{cases} \tag{2.1.7}$$

and we can refer $\hat{a}$ as the mask of ϕ. Finally we have the following theorem which essentially says that whenever $0 < \delta \le \Omega \le 2\pi/3$, functions in $\mathcal{A}_{\delta,\Omega}$ generate an MRA of $L^2(\mathbb{R})$.

Theorem 2.1.2 *Let $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, and let ϕ be defined by $\hat{a}$ as in (2.1.6) and (2.1.7). Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is the sequence of subspaces generated by ϕ as in (1.2.3). Then $\{V_j\}_{j\in\mathbb{Z}}$ forms an MRA of $L^2(\mathbb{R})$.*

Proof: By Proposition 2.1.1, $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ generates a scaling function $\phi \in \mathcal{B}_{\delta,\Omega}$ which is a refinable function. Furthermore, $\lim_{\xi\to 0} \hat{\phi}(\xi) = \hat{\phi}(0) = 1 \neq 0$. Thus by Theorem 1.2.3, property (M1) of Definition 1.2.1 is satisfied. Since ϕ is refinable, the other properties (M2), (M3), (M4) are automatically satisfied by the definition of $\{V_j\}_{j\in\mathbb{Z}}$, thus giving us the desired conclusion. ■

2.2 Characterizations of frames, orthonormal and Riesz bases

We shall characterize the functions $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ where $0 < \delta \leq \Omega \leq 2\pi/3$, so that the integer shifts of the resulting scaling function ϕ form (a) an orthonormal basis for V_0 , (b) a Riesz basis for V_0, or (c) a frame for V_0, where

$$V_0 = \overline{\text{span}\{\phi(\cdot - k) : k \in \mathbb{Z}\}}. \tag{2.2.1}$$

Before we do so, we require general characterizations of the integer shifts of a scaling function ϕ satisfying (a), (b), (c) respectively in terms of its bracket product $[\hat{\phi}, \hat{\phi}]$.

It is not hard to observe that if $\hat{\phi}$ is compactly supported and bounded everywhere, then $[\hat{\phi}, \hat{\phi}]$ must be bounded everywhere. As $[\hat{\phi}, \hat{\phi}]$ is a 2π-periodic function, it suffices to consider $[\hat{\phi}, \hat{\phi}]$ on the interval $[-\pi, \pi]$. Then, on $[-\pi, \pi]$, the summation $\sum_{l\in\mathbb{Z}} |\hat{\phi}(\cdot + 2\pi l)|^2$ involves only a finite number of bounded terms due to the compact support of $\hat{\phi}$. Thus $[\hat{\phi}, \hat{\phi}]$ is bounded everywhere on $[-\pi, \pi]$ and thus $[\hat{\phi}, \hat{\phi}]$ is bounded everywhere on $\mathbb{R}$.

Proposition 2.2.1 *Let ϕ, $\tilde{\phi} \in L^2(\mathbb{R})$. Then we have the following.*

(a) $\langle \phi, \tilde{\phi}(\cdot - k)\rangle = \delta_{0k}$ *for all* $k \in \mathbb{Z}$ *if and only if* $[\hat{\phi}, \hat{\tilde{\phi}}] \equiv 1$. *In particular, setting* $\phi = \tilde{\phi}$, $\langle \phi, \phi(\cdot - k)\rangle = \delta_{0k}$ *for all* $k \in \mathbb{Z}$ *if and only if* $[\hat{\phi}, \hat{\phi}] \equiv 1$.

(b) *The integer shifts of ϕ form a Riesz sequence i.e. there exist constants A, $B > 0$ such that*

$$A\|c\|_{\ell^2}^2 \le \|\sum_{k\in\mathbb{Z}} c_k\phi(\cdot - k)\|_{L^2}^2 \le B\|c\|_{\ell^2}^2$$

for $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ if and only if $A \le [\hat{\phi}, \hat{\phi}] \le B$ a.e.

(c) *The integer shifts of ϕ form a frame for V_0 i.e. there exist constants A, $B > 0$ such that*

$$A\|f\|_{L^2}^2 \le \sum_{k\in\mathbb{Z}} |\langle f, \phi(\cdot - k)\rangle|^2 \le B\|f\|_{L^2}^2$$

for all $f \in V_0$, if and only if $A \le [\hat{\phi}, \hat{\phi}] \le B$ a.e. in $\sigma(\phi)$, where V_0 is defined by ϕ in (1.2.3).

This is a well-known result in wavelet theory and the interested reader can refer to [5] for detailed proofs.

Lemma 2.2.1 *For $0 < \delta \le \Omega \le 2\pi/3$, $0 < \tilde{\delta} \le \tilde{\Omega} \le 2\pi/3$, let $\hat{a}$, $\hat{\tilde{a}}$ belong to $\mathcal{A}_{\delta,\Omega}$ and $\mathcal{A}_{\tilde{\delta},\tilde{\Omega}}$ respectively. If $\hat{a}\hat{\tilde{a}}(\cdot) + \hat{a}\hat{\tilde{a}}(\cdot + \pi) \equiv 1$, then Ω, $\tilde{\Omega} \ge \pi/2$ and δ, $\tilde{\delta} \ge \pi/3$.*

Proof: Let $k(\xi) := \hat{a}\hat{\tilde{a}}(\xi)$. Then $k \in \mathcal{A}_{\delta_0,\Omega_0}$ where $\delta_0 := \min\{\delta, \tilde{\delta}\}$, $\Omega_0 := \min\{\Omega, \tilde{\Omega}\}$. Suppose on the contrary that $\Omega_0 < \pi/2$. Then on $[-\pi/2, \pi/2]$, $\hat{a}\hat{\tilde{a}}(\xi) + \hat{a}\hat{\tilde{a}}(\xi + \pi) = 0$ for $|\xi| \in [\Omega_0, \pi/2]$ which is a contradiction. Thus $\Omega_0 \ge \pi/2$ and so Ω, $\tilde{\Omega} \ge \pi/2$.

Recall that the 2π-periodic functions $\hat{a}$ and $\hat{\tilde{a}}$ have compact supports $[-\Omega, \Omega]$ and $[-\tilde{\Omega}, \tilde{\Omega}]$ respectively when they are restricted to the fundamental interval $[-\pi, \pi]$. Thus the function k has compact support $[-\Omega_0, \Omega_0]$ when it is restricted to $[-\pi, \pi]$, where $\Omega_0 = \min\{\Omega, \tilde{\Omega}\} \le 2\pi/3$. Let $g(\xi) := k\mathbf{1}_{[-\pi,\pi]}(\xi) = k\mathbf{1}_{[-\Omega_0,\Omega_0]}(\xi)$. As $\Omega_0 \le 2\pi/3$, it follows that $k(\xi) + k(\xi + \pi) = g(\xi)$ on $(-\pi/3, \pi/3)$. Since $k(\cdot) + k(\cdot + \pi) \equiv 1$, $g(\xi) = 1$ for $\xi \in (-\pi/3, \pi/3)$, which means that $k = g\mathbf{1}_{[-\pi,\pi]}$ has an interval of constancy $[-\delta_0, \delta_0] \supseteq (-\pi/3, \pi/3)$. Thus $\delta_0 \ge \pi/3$ and so δ, $\tilde{\delta} \ge \pi/3$. ∎

Corollary 2.2.1 *For $0 < \delta \le \Omega \le 2\pi/3$, $0 < \tilde{\delta} \le \tilde{\Omega} \le 2\pi/3$, let $\hat{a}$, $\hat{\tilde{a}}$ belong to $\mathcal{A}_{\delta,\Omega}$ and $\mathcal{A}_{\tilde{\delta},\tilde{\Omega}}$ respectively. Define ϕ, $\tilde{\phi}$ by $\hat{a}$ and $\hat{\tilde{a}}$ respectively as in* (2.1.6) *and* (2.1.7). *If $[\hat{\phi}, \hat{\tilde{\phi}}] \equiv 1$, then Ω, $\tilde{\Omega} \ge \pi/2$, δ, $\tilde{\delta} \ge \pi/3$, and N, $\tilde{N}$ defined by* (2.1.7) *are both* 1.

Proof: By standard arguments provided in [6] regarding the biorthogonality of dual scaling functions and duality of their masks, $[\hat{\phi}, \hat{\tilde{\phi}}] \equiv 1$ implies that

$$\hat{a}(\cdot)\hat{\tilde{a}}(\cdot) + \hat{a}(\cdot + \pi)\hat{\tilde{a}}(\cdot + \pi) \equiv 1.$$

Henceforth we apply Lemma 2.2.1 to get Ω, $\tilde{\Omega} \geq \pi/2$, δ, $\tilde{\delta} \geq \pi/3$. Lastly, as Ω, $\tilde{\Omega} \leq 2\pi/3$ and δ, $\tilde{\delta} \geq \pi/3$, it follows that from (2.1.7) that $N = 1 = \tilde{N}$. ■

Next, we derive an interesting consequence of orthonormal scaling functions and interpolatory functions in $\mathcal{B}_{\delta,\Omega}$. Recall that a scaling function $\phi \in L^2(\mathbb{R})$ is said to be *interpolatory* if

$$\phi(j) = \delta_{0j}, \qquad j \in \mathbb{Z}. \tag{2.2.2}$$

It is widely known that this definition is equivalent to the condition

$$\sum_{l \in \mathbb{Z}} \hat{\phi}(\xi + 2\pi l) \equiv 1. \tag{2.2.3}$$

The consequence is the following.

Corollary 2.2.2 *For $0 < \delta \leq \Omega \leq 2\pi/3$, if $\phi \in \mathcal{B}_{\delta,\Omega}$ has orthonormal shifts or is interpolatory, then $\Omega \geq \pi/2$, $\delta \geq \pi/3$ and N defined by* (2.1.7) *is* 1.

Proof: For the case of the orthonormal shifts, we simply apply Proposition 2.2.1 and Corollary 2.2.1. As for the other case, consider $\phi \in \mathcal{B}_{\delta,\Omega}$ which is interpolatory. Define φ by its Fourier transform $\hat{\varphi}(\xi) := \hat{\phi}^{1/2}(\xi)$. One can easily check that $\varphi \in \mathcal{B}_{\delta,\Omega}$ and $[\hat{\varphi}, \hat{\varphi}] \equiv 1$. Applying Corollary 2.2.1 then gives the result. ■

With the information given by Corollary 2.2.1, for $\Omega \geq \pi/2$ and $\delta \geq \pi/3$, we provide the following useful characterization between masks in $\mathcal{A}_{\delta,\Omega}$ and corresponding scaling functions in $\mathcal{B}_{\delta,\Omega}$. It should be emphasized that characterization of this type need not hold for masks in general.

Lemma 2.2.2 *For $\pi/3 \leq \delta \leq \Omega \leq 2\pi/3$, $\pi/3 \leq \tilde{\delta} \leq \tilde{\Omega} \leq 2\pi/3$ with Ω, $\tilde{\Omega} \geq \pi/2$, let $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, $\hat{\tilde{a}} \in \mathcal{A}_{\tilde{\delta},\tilde{\Omega}}$ and ϕ, $\tilde{\phi}$ be defined by $\hat{a}$ and $\hat{\tilde{a}}$ respectively as in* (2.1.6) *and* (2.1.7)*, where N and $\tilde{N}$ defined by* (2.1.7) *are both* 1*. Then $[\hat{\phi}, \hat{\tilde{\phi}}] \equiv 1$ if and only if*

$$\hat{a}(\cdot)\hat{\tilde{a}}(\cdot) + \hat{a}(\cdot + \pi)\hat{\tilde{a}}(\cdot + \pi) \equiv 1.$$

Furthermore, for each $m > 0$, $\sum_{l\in\mathbb{Z}} \hat{\phi}^m(\cdot + 2\pi l) \equiv 1$ *if and only if* $\hat{a}^m(\cdot) + \hat{a}^m(\cdot + \pi) \equiv 1$.

Proof: As noted in the proof of Corollary 2.2.1, standard arguments show that $[\hat{\phi}, \hat{\phi}] \equiv 1$ gives $\hat{a}\hat{\tilde{a}}(\cdot) + \hat{a}\hat{\tilde{a}}(\cdot + \pi) \equiv 1$.

Conversely, we first denote $k(\xi) := \hat{a}(\xi)\hat{\tilde{a}}(\xi)$, $g(\xi) := k(\xi)\mathbf{1}_{[-\Omega_0,\Omega_0]}(\xi)$, where $\Omega_0 = \min\{\Omega, \tilde{\Omega}\}$. As $N = \tilde{N} = 1$, the function φ defined by $\hat{\varphi}(\xi) := \hat{\phi}(\xi)\hat{\tilde{\phi}}(\xi)$ simplifies to $\hat{\varphi}(\xi) = k(\xi/2)\mathbf{1}_{[-2\Omega_0,2\Omega_0]}(\xi) = g(\xi/2)$. Due to the bandlimited structures of the masks $\hat{a}$, $\hat{\tilde{a}}$, one can write $g(\xi) = \hat{a}\hat{\tilde{a}}(\xi)\mathbf{1}_{[-\Omega_0,\Omega_0]}(\xi) = \hat{a}\hat{\tilde{a}}(\xi)\mathbf{1}_{[-\pi,\pi]}(\xi)$ and $\hat{a}\hat{\tilde{a}}(\xi) = \sum_{l\in\mathbb{Z}} g(\xi+2\pi l)$.

Note that $[\hat{\phi}, \hat{\tilde{\phi}}](\xi) = \sum_{l\in\mathbb{Z}} \hat{\varphi}(\xi + 2\pi l)$ and thus

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \hat{\varphi}(2\xi + 2\pi l) &= \sum_{l\in\mathbb{Z}} g(\xi + \pi l)\\ &= \sum_{l\in\mathbb{Z}} g(\xi + 2\pi l) + \sum_{l\in\mathbb{Z}} g(\xi + \pi + 2\pi l)\\ &= \hat{a}\hat{\tilde{a}}(\xi) + \hat{a}\hat{\tilde{a}}(\xi + \pi) \equiv 1,\end{aligned}$$

which gives the desired result.

To show the second part of this lemma, we simply replace both $\hat{a}$ and $\hat{\tilde{a}}$ by $\hat{a}^{m/2}$ in the first part. Note that by the definition of $\mathcal{A}_{\delta,\Omega}$, for $m > 0$, $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ if and only if $\hat{a}^{m/2} \in \mathcal{A}_{\delta,\Omega}$. In view of (2.1.6) and (2.1.7), the corresponding scaling function ϕ_m of each $\hat{a}^{m/2}$ is given exactly by

$$\hat{\phi}_m(\xi) = \hat{a}^{m/2}(2^{-j}\xi)\mathbf{1}_{[-2\Omega,\, 2\Omega]}(\xi) = \hat{\phi}^{m/2}(\xi).$$

This concludes the proof of the lemma. ■

There are three important consequences of Lemma 2.2.2. It gives characterizations of the relation between dual masks and dual scaling functions, the relation between interpolatory masks and interpolatory scaling functions and lastly, the relation between conjugate quadrature filters and scaling functions with orthonormal shifts. These will be explained in subsequent chapters. Next, we establish another useful lemma.

Lemma 2.2.3 *For* $0 < \delta \le \Omega \le 2\pi/3$, *let* $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ *and* ϕ *be defined by* $\hat{a}$ *as in* (2.1.6) *and* (2.1.7). *Then the following are equivalent.*

(a) $\lim_{\xi \to 2\Omega^-} \hat{\phi}(\xi) > 0.$

(b) $\lim_{\xi \to \Omega^-} \hat{a}(\xi) > 0.$

(c) *There exists a constant $M > 0$ such that $M \leq \hat{a}(\xi)$ for all $\xi \in (-\Omega, \Omega)$.*

(d) *There exists a constant $A > 0$ such that $A \leq \hat{\phi}(\xi)$ for all $\xi \in (-2\Omega, 2\Omega)$.*

Proof: Recall from (2.1.6) that $\hat{\phi}(\xi) = [\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)]\mathbf{1}_{[-2\Omega,\,2\Omega]}(\xi)$, where $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ and N is as defined in (2.1.7). We shall prove this lemma in the following order: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

Firstly, we show that statement (a) implies statement (b). Assume that $\lim_{\xi \to 2\Omega^-} \hat{\phi}(\xi) > 0$. Since $\hat{a}(\xi) > 0$ and continuous for $\xi \in (-2\Omega,\, 2\Omega)$, this implies that $\lim_{\xi \to 2\Omega^-} \hat{a}(2^{-j}\xi) = \hat{a}(2^{-j}2\Omega) > 0$ for $j \geq 2$. Then by (2.1.6), this means that

$$\lim_{\xi \to \Omega^-} \hat{a}(\xi)\mathbf{1}_{[-\Omega,\,\Omega]}(\xi) = \Big[\lim_{\xi \to 2\Omega^-} \hat{\phi}(\xi) / \prod_{j=2}^{N} \hat{a}(2^{-j}2\Omega)\Big]\mathbf{1}_{[-\Omega,\,\Omega]}(\xi) > 0.$$

Now we prove statement (b) implies statement (c). Assume $\lim_{\xi \to \Omega^-} \hat{a}(\xi) > 0$. Then by the definition of $\mathcal{A}_{\delta,\Omega}$, we know that $\hat{a}$ is continuous and positive on $[-\Omega,\, \Omega]$ and $\hat{a} > 0$ on $[-\Omega,\, \Omega]$. For the last part, suppose on the contrary that for every $A > 0$, there exists $\xi_A \in (-\Omega,\, \Omega)$ such that $A > \hat{a}(\xi_A)$. Choose $A = \frac{1}{n}$, $n \geq 1$. Then there exists $\xi_n \in (-\Omega,\, \Omega)$ such that

$$0 < \hat{a}(\xi_n) < \frac{1}{n}. \tag{2.2.4}$$

Note that $\{\xi_n\}_{n=1}^{\infty}$ is a sequence in $(-\Omega,\, \Omega)$. Letting $n \to \infty$ in (2.2.4), we have

$$0 \leq \lim_{n \to \infty} \hat{a}(\xi_n) \leq \lim_{n \to \infty} \frac{1}{n},$$

so $\lim_{n \to \infty} \hat{a}(\xi_n) = 0$. Since $\{\xi_n\}_{n=1}^{\infty} \subseteq (-\Omega, \Omega) \subset [-\Omega,\, \Omega]$, there exists a subsequence $\{\xi_{n_k}\}_{k=1}^{\infty}$ that converges to ξ^* in $[-\Omega,\, \Omega]$. Since $\{\hat{a}(\xi_n)\}_{n=1}^{\infty}$ converges, its subsequence $\{\hat{a}(\xi_{n_k})\}_{k=1}^{\infty}$ also converges and indeed $\lim_{k \to \infty} \hat{a}(\xi_{n_k}) = 0$. If $\xi^* \in (-\Omega,\, \Omega)$, then $\hat{a}(\lim_{k \to \infty} \xi_{n_k}) = \hat{a}(\xi^*) = 0$ which is impossible. So $\xi^* = \pm\Omega$.

Since $\hat{a}$ is an even function, $\lim_{\xi\to\Omega^-}\hat{a}(\xi) > 0$ if and only if $\lim_{\xi\to-\Omega^+}\hat{a}(\xi) > 0$. So it suffices to consider the case where $\xi^* = \Omega$. Since $\lim_{k\to\infty}\xi_{n_k} = \xi^* = \Omega$, it follows that

$$0 = \lim_{k\to\infty}\hat{a}(\xi_{n_k}) = \lim_{\xi\to\Omega^-}\hat{a}(\xi) > 0,$$

which is a contradiction.

To show that statement (c) implies statement (d), we simply deduce from the structure of $\hat{\phi}$ in (2.1.6) that whenever statement (c) holds, we have $0 < M^N \leq \hat{\phi}(\xi)$ for all $\xi \in (-2\Omega,\, 2\Omega)$.

We see that statement (d) implies statement (a) by simply taking considering the left-hand limit $\lim_{\xi\to 2\Omega^-}\hat{\phi}(\xi)$. This is justifiable as $\hat{\phi}$ is continuous on $(-2\Omega,\, 2\Omega)$ and this concludes the proof of the lemma. ■

Finally, we are ready to begin the characterization and we shall find the two previous lemmas handy.

Theorem 2.2.1 *For* $0 < \delta \leq \Omega \leq 2\pi/3$, *let* $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ *and* ϕ *be defined by* $\hat{a}$ *as in* (2.1.6) *and* (2.1.7). *Define* V_0 *as in* (2.2.1). *We have the following.*

(a) *For* $\Omega \geq \pi/2$,

- (i) *the integer shifts of* ϕ *form a Riesz basis for* V_0 *if and only if either* $\Omega > \pi/2$ *or* $\Omega = \pi/2$ *and* $\lim_{\xi\to\pi/2^-}\hat{a}(\xi) > 0$,
- (ii) *under the assumption that* $\delta \geq \pi/3$, *the integer shifts of* ϕ *form an orthonormal basis for* V_0 *if and only if the Conjugate Quadrature Filter (CQF) condition is satisfied, i.e.*

$$\hat{a}^2(\cdot) + \hat{a}^2(\cdot + \pi) \equiv 1.$$

(b) *For* $\Omega < \pi/2$,

- (i) *the integer shifts of* ϕ *form a frame for* V_0 *if and only if* $\lim_{\xi\to\Omega^-}\hat{a}(\xi) > 0$,
- (ii) *the integer shifts of* ϕ *cannot form a Riesz basis for* V_0.

Proof: To prove statement (a)(i), we first show that, if the integer shifts of ϕ form a Riesz basis for V_0, then $\Omega \geq \pi/2$. If the integer shifts of ϕ form a Riesz basis for V_0, by statement (b) in Proposition 2.2.1, there exist constants A, $B > 0$ such that

$$A \leq [\hat{\phi}, \hat{\phi}](\xi) \leq B$$

for ξ a.e in $\mathbb{R}$. It then follows from standard arguments which can be found in [6] that there exist constants C, $D > 0$ such that

$$C \leq |\hat{a}(\xi)|^2 + |\hat{a}(\xi + \pi)|^2 \leq D \tag{2.2.5}$$

for ξ a.e in $\mathbb{R}$. Suppose on the contrary that $\Omega < \pi/2$. Then on the interval $[-\pi/2, \pi/2]$, we have

$$|\hat{a}(\xi)|^2 + |\hat{a}(\xi + \pi)|^2 = \begin{cases} 0, & \xi \in [-\pi/2, -\Omega), \\ \hat{a}^2(\xi), & \xi \in [-\Omega, \Omega], \\ 0, & \xi \in (\Omega, \pi/2], \end{cases} \tag{2.2.6}$$

which contradicts (2.2.5). Therefore $\Omega \geq \pi/2$. Furthermore, if $\Omega = \pi/2$, then (2.2.6) gives

$$(|\hat{a}(\xi)|^2 + |\hat{a}(\xi + \pi)|^2)^{1/2} \mathbf{1}_{[-\pi/2, \pi/2]}(\xi) = \hat{a}(\xi) \mathbf{1}_{[-\pi/2, \pi/2]}(\xi). \tag{2.2.7}$$

On $(-\pi/2, \pi/2)$, by (2.2.5), $C \leq \hat{a}(\xi) \leq D$ a.e. Now since $\hat{a}$ is continuous on $(-\pi/2, \pi/2)$, $C \leq \hat{a}(\xi) \leq D$ everywhere on $(-\pi/2, \pi/2)$. Taking limit as $\xi \to \Omega = \pi/2$, $\lim_{\xi \to \Omega^-} \hat{a}(\xi) \geq C > 0$.

For the converse, first consider the case when $\Omega > \pi/2$. Since $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, it follows from Proposition 2.1.1, that $\hat{\phi} > 0$ and is continuous on the interval $(-2\Omega, 2\Omega)$. Since $\Omega > \pi/2$, $\hat{\phi}$ is continuous on $[-\pi, \pi]$. This means that there exist constants A , $B > 0$ for which

$$A \leq \hat{\phi}(\xi) \leq B$$

for all $\xi \in [-\pi, \pi]$. Next we show the bracket product $[\hat{\phi}, \hat{\phi}]$ must be bounded above and below by positive constants everywhere. Since $[\hat{\phi}, \hat{\phi}]$ is a 2π-periodic function, it suffices to analyze its behavior on the interval $[-\pi, \pi]$. Now $\Omega > \pi/2$ implies that $[-\pi, \pi] \subset [-2\Omega, 2\Omega]$, and on the interval $[-\pi, \pi]$, we have

$$[\hat{\phi}, \hat{\phi}](\xi) \geq \hat{\phi}^2(\xi) \geq A$$

Since $[\hat{\phi},\, \hat{\phi}](\xi)$ involves a sum of at most two positive and bounded functions in each on subintervals of $[-\pi,\, \pi]$ above, then $[\hat{\phi},\, \hat{\phi}]$ must be bounded on $[-\pi,\, \pi]$. Since $[\hat{\phi}, \hat{\phi}]$ is 2π-periodic, we get the result.

When $\Omega = \pi/2$ and $\lim_{\xi\to\Omega^-} \hat{a}(\xi) > 0$, the function

$$\Big(\sum_{l\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi l)|^2\Big)^{1/2} \cdot \mathbf{1}_{[-\pi,\,\pi]}(\xi) = \hat{\phi}(\xi)\mathbf{1}_{[-2\Omega,\,2\Omega]}(\xi) = \hat{\phi}(\xi)\mathbf{1}_{[-\pi,\,\pi]}(\xi)$$

is bounded below everywhere on $(-\pi,\, \pi)$ by some constant $A > 0$ by invoking Lemma 2.2.3 with $\Omega = \pi/2$. The upper bound of $[\hat{\phi}, \hat{\phi}]$ follows from the boundedness of $\hat{\phi}$. Thus by periodicity again, we get the result.

The proof for statement (a)(ii) follows from Lemma 2.2.2.

Next we will settle statement (b)(i). Since $\Omega < \pi/2$, on the interval $[-\pi,\, \pi]$, we have

$$[\hat{\phi}, \hat{\phi}]^{1/2}(\xi) = \hat{\phi}(\xi)$$

and the spectrum of ϕ, $\sigma(\phi) = [-2\Omega,\, 2\Omega] \subsetneq [-\pi,\, \pi]$. Recall that the integer shifts of ϕ form a frame if and only if there exist constants A, $B > 0$ such that

$$A \leq \hat{\phi}(\xi) \leq B$$

a.e. on $\sigma(\phi)$. Note that $\hat{\phi}$ is bounded on $[-2\Omega,\, 2\Omega]$. We will only consider the part for the lower bound. As by Lemma 2.2.3, we see that this is equivalent to $\lim_{\xi\to\Omega^-} \hat{a}(\xi) > 0$, establishing statement (b)(i).

Statement (b)(ii) is shown earlier by the proof of (a)(i) and this finishes the proof of the proposition. ■

We will show some immediate consequences arising from the proposition. In fact, the characterization in Proposition 2.2.1 becomes simpler if we further allow continuity of the function $\hat{a} \in \mathcal{A}_{\delta,\Omega}$.

Corollary 2.2.3 *For $0 < \delta \leq \Omega \leq 2\pi/3$, let $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C(\mathbb{R})$ and ϕ be defined by $\hat{a}$ as in* (2.1.6) *and* (2.1.7)*. Define V_0 as in* (2.2.1)*. Then the following are equivalent.*

(a) $\Omega > \pi/2$.

(b) *The integer shifts of ϕ form a Riesz basis for V_0.*

(c) *For any positive m, there exist constants $A, B > 0$ such that*

$$A \leq \sum_{l\in\mathbb{Z}} \hat{\phi}^m(\xi + 2\pi l) \leq B$$

for all $\xi \in \mathbb{R}$.

(d) *For any positive m, there exists constants A, $B > 0$ such that*

$$A \leq \hat{a}^m(\xi) + \hat{a}^m(\xi + \pi) \leq B$$

for all $\xi \in \mathbb{R}$.

(e) *There exist constants A, $B > 0$ such that*

$$A \leq \hat{a}(\xi) + \hat{a}(\xi + \pi) \leq B$$

for all $\xi \in \mathbb{R}$.

Proof: Since $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ is continuous everywhere, it is necessary that $\lim_{\xi\to\Omega^-} \hat{a}(\xi) = 0$. For any $m > 0$, we define ϕ_m by its Fourier transform $\hat{\phi}_m = \hat{\phi}^{m/2}$. Since $\phi \in \mathcal{B}_{\delta,\Omega}$ for some positive δ and Ω and $\hat{\phi}$ is even and continuous, by the definition of $\mathcal{B}_{\delta,\Omega}$ and ϕ_m, it follows that $\phi_m \in \mathcal{B}_{\delta,\Omega}$ with $\hat{\phi}_m$ continuous and even. Note that by the proof in Lemma 2.2.2, ϕ_m has a corresponding mask $\hat{a}_m = \hat{a}^{m/2} \in \mathcal{A}_{\delta,\Omega}$.

We shall prove this corollary in the following order: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a). From Theorem 2.2.1, since $\lim_{\xi\to\Omega^-} \hat{a}(\xi) = 0$, statement (a) is equivalent to statement (b).

Suppose that statement (b) holds, then by the equivalence of statements (a) and (b) shown earlier, $\Omega > \pi/2$. Next, recall from Proposition 2.1.1 that $\phi_m \in \mathcal{B}_{\delta,\Omega}$ if and only if $\hat{a}_m \in \mathcal{A}_{\delta,\Omega}$. Furthermore $\hat{a}_m = \hat{a}^{m/2}$ is also even, bounded and continuous because $\hat{a}$ is. So $\hat{a}_m$ satisfies the assumptions of this corollary with $\Omega > \pi/2$, thus we invoke statement (b) of this corollary by replacing ϕ by ϕ_m in statement (b) which gives us statement (c). This is because if ϕ_m satisfies statement (b), it follows from Proposition 2.2.1 that there exist constants A, $B > 0$ for which

$$A \leq \sum_{l \in \mathbb{Z}} |\hat{\phi}_m(\xi + 2\pi l)|^2 \leq B \tag{2.2.8}$$

for ξ a.e in $\mathbb{R}$. As illustrated in the proof of Proposition 2.2.1,

$$\sum_{l \in \mathbb{Z}} |\hat{\phi}_m(\xi + 2\pi l)|^2 = \sum_{l \in \mathbb{Z}} \hat{\phi}^m(\xi + 2\pi l)$$

is a finite sum of continuous functions on the fundamental interval $[-\pi, \pi]$. Hence, (2.2.8) must hold for all $\xi \in [-\pi, \pi]$. Since the function $\sum_{l \in \mathbb{Z}} \hat{\phi}^m(\xi + 2\pi l)$ is 2π-periodic, then (2.2.8) must hold for all $\xi \in \mathbb{R}$.

As mentioned above, the refinable function ϕ_m has a corresponding refinement mask $\hat{a}_m = \hat{a}^{m/2} \in \mathcal{A}_{\delta,\Omega}$, for any positive m. Note that

$$\sum_{l \in \mathbb{Z}} \hat{\phi}^m(\xi + 2\pi l) = \sum_{l \in \mathbb{Z}} |\hat{\phi}^{m/2}(\xi + 2\pi l)|^2 = [\hat{\phi}^{m/2}, \hat{\phi}^{m/2}](\xi) = [\hat{\phi}_m, \hat{\phi}_m](\xi)$$

and $|\hat{a}_m|^2 = \hat{a}^m$. So, if for any positive m, statement (c) holds, then Proposition 2.2.1 implies that there exist constants A, $B > 0$ such that

$$A \leq [\hat{\phi}_m, \hat{\phi}_m] \leq B$$

for all $\xi \in \mathbb{R}$. Since $\hat{a}_m$ is the refinement mask of $\hat{\phi}_m$, it is well-known that there exist constants C, $D > 0$ such that

$$C \leq |\hat{a}_m(\xi)|^2 + |\hat{a}_m(\xi + \pi)|^2 \leq D$$

for all $\xi \in \mathbb{R}$. One can check [23] for the details. Thus statement (c) implies statement (d).

By setting $m = 1$ in statement (d), we get statement (e).

Now let us show that statement (e) implies statement (a). Assume statement (e) and let $\xi = \pi/2$. Then taking $m = 1$, we have

$$\hat{a}(\pi/2) + \hat{a}(3\pi/2) = \hat{a}(\pi/2) + \hat{a}(-\pi/2) = 2\hat{a}(\pi/2),$$

due to the 2π-periodicity and symmetry about the origin of the function $\hat{a}$. From statement (d), we see that $\hat{a}(\pi/2) > 0$. Suppose on the contrary that $\Omega \leq \pi/2$. Since $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C(\mathbb{R})$, $\hat{a}(\xi) = 0$ for $\xi \in [-\pi, \pi]\backslash(-\Omega, \Omega) \supseteq [-\pi, -\pi/2] \cup [\pi/2, \pi]$. This contradicts the fact that $\hat{a}(\pi/2) > 0$, and ends the proof. ∎

In the next section, we shall show some examples to illustrate our results.

2.3 Some examples

We introduce two important categories of functions $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ where $0 < \delta \leq \Omega \leq 2\pi/3$. Under appropriate conditions, these functions $\hat{a}$ generate scaling functions ϕ whose integer shifts form either a frame or Riesz basis for V_0, where V_0 is as defined in (2.2.1).

The first category of functions is the set of functions $\{\hat{a}_\Omega\}$, where $\hat{a}_\Omega$ is the 2π-periodic extension of the function

$$\hat{s}_\Omega(\xi) := \mathbf{1}_{[-\Omega,\,\Omega]}(\xi)$$

with $0 < \Omega \leq 2\pi/3$. Define ϕ_Ω by $\hat{a}_\Omega$ as in (2.1.6) and (2.1.7). Then explicitly, we get

$$\hat{\phi}_\Omega(\xi) = \mathbf{1}_{[-2\Omega,2\Omega]}(\xi)$$

and

$$\phi_\Omega(x) = \begin{cases} \frac{\sin 2\Omega x}{2\Omega x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Clearly $\hat{a} \in \mathcal{A}_{\Omega,\Omega}$. Using the characterization provided in Theorem 2.2.1, we have the following.

(1) If $\Omega < \pi/2$, the integer shifts of ϕ_Ω form a frame but not a Riesz basis for V_0. In fact, $[\hat{\phi}_\Omega, \hat{\phi}_\Omega] = 1$ in $\sigma(\phi_\Omega)$ which implies that it is a tight frame.

(2) If $\Omega = \pi/2$, the integer shifts of $\phi_{\pi/2}$ form an orthonormal basis for V_0 since the CQF condition is satisfied. In fact, $\phi_{\pi/2}$ is the well-known sinc function in the Shannon's sampling theorem.

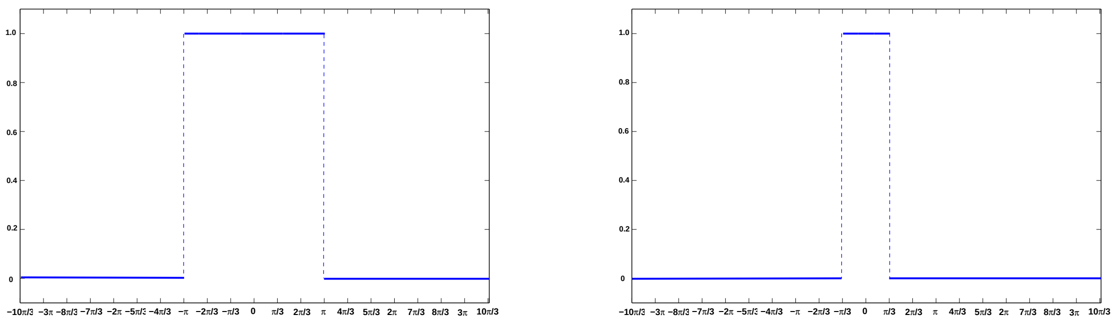

Figure 2.1: Graphs of $\hat{\phi}_{\frac{\pi}{2}}$ and $\hat{\phi}_{\frac{\pi}{6}}$.

(3) If $\Omega > \pi/2$, the integer shifts of ϕ_Ω form a Riesz basis for V_0. In fact, the lower and upper Riesz bounds are 1 and 2 respectively. This is because on the interval $[-\pi,\, \pi]$,

$$\sum_{l\in\mathbb{Z}} \hat{\phi}^2(\xi + 2\pi l) = \begin{cases} 2, & \text{if } \xi \in [-\pi,\, \Omega - 2\pi] \cup [2\pi - \Omega,\, \pi], \\ 1, & \text{if } \xi \in (-\Omega + 2\pi,\, 2\pi - \Omega). \end{cases}$$

Figure 2.1 illustrates $\hat{\phi}_\Omega$ when $\Omega = \pi/2$ and $\Omega = \pi/6$ respectively. However, $\hat{\phi}_\Omega$ is discontinuous and we shall see later that this discontinuity gives ϕ_Ω poor decay in the time domain, i.e. $\phi_\Omega(x)$ decays in the order of $\frac{1}{x}$. Thus it is natural to seek other bandlimited functions ϕ which are well-localized in both time and frequency domains. The second category of functions shows us that this is certainly possible.

Let $\{\hat{a}_{\delta,\Omega,m}\}$ be the 2π-periodic extension of the bell functions

$$b_{\delta,\Omega,m}(\xi) := \cos^m(\frac{\pi}{2} g(\frac{1}{\Omega - \delta}(|\xi| - \delta))),$$

where $0 < \delta < \Omega \leq 2\pi/3$, $m \in \mathbb{N}$ and g is a continuous function satisfying the following properties:

(a) $g(x) = 0$ for all $x \leq 0$ and $g(x) = 1$ for all $x \geq 1$.

(b) $g(x) + g(1 - x) = 1$ for all $x \in \mathbb{R}$.

(c) $g(x)$ is strictly increasing on the interval $(0,\, 1)$.

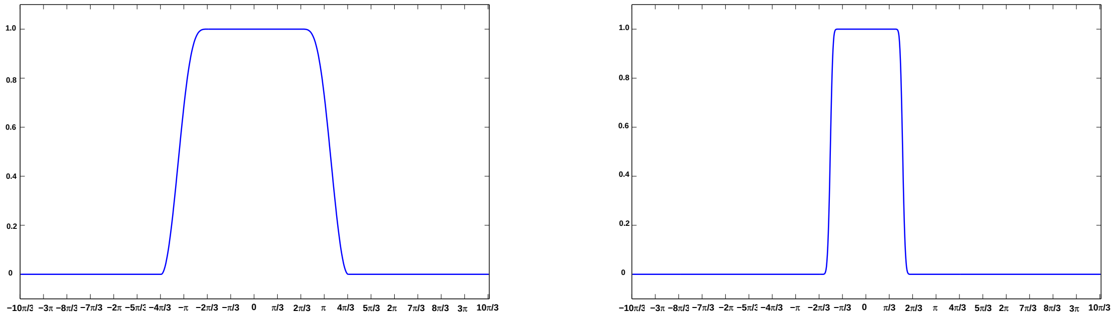

Figure 2.2: Graphs of $\hat{\phi}_{\frac{\pi}{3},\frac{2\pi}{3},1}$ and $\hat{\phi}_{\frac{\pi}{6},\frac{\pi}{3},1}$.

For simplicity, let $2\delta \geq \Omega$. We shall see in Chapters 3 and 4 that $\hat{a}_{\delta,\Omega,m} \in \mathcal{A}_{\delta,\Omega} \cap C(\mathbb{R})$ for each $m \in \mathbb{N}$. Define $\phi_{\delta,\Omega,m}$ by $\hat{a}_{\delta,\Omega,m}(\xi)$ as in (2.1.6) and (2.1.7). Since $2\delta \geq \Omega$, N as defined in (2.1.7) and $\hat{\phi}_{\delta,\Omega,m}(\xi) = b_{\delta,\Omega,m}(\xi/2)$. Using the characterization in Theorem 2.2.1, we have the following.

(1) If $\Omega \leq \pi/2$, the integer shifts of $\phi_{\delta,\Omega,m}$ neither form a frame nor a Riesz basis for V_0.

(2) If $\Omega > \pi/2$, the integer shifts of $\phi_{\delta,\Omega,m}$ form a Riesz basis for V_0.

(3) If $\delta = \pi/2 - \epsilon/2$, $\Omega = \pi/2 + \epsilon/2$ and $m = 1$ with $0 < \epsilon \leq \pi/3$, then $\hat{a}_{\delta,\Omega,m}$ satisfies the CQF condition and the integer shifts of $\phi_{\delta,\Omega,m}$ form an orthonormal basis for V_0. In fact, when $\epsilon \leq \pi/3$, we have $2\delta \geq \Omega$ and thus $\hat{\phi}_{\delta,\Omega,m}(\xi) = b_{\delta,\Omega,m}(\xi/2)$.

For $g = p_1 \in C^1(\mathbb{R})$ where p_1 is defined in Theorem 4.1.2, Figure 2.1 illustrates the two cases of $\hat{\phi}_{\delta,\Omega,1}$ where the first case has $\delta = \frac{\pi}{3}$, $\Omega = \frac{2\pi}{3}$ and the second case has $\delta = \frac{\pi}{6}$ and $\Omega = \frac{\pi}{3}$. The wavelets constructed from the scaling functions in statement (3) are called the Meyer's wavelets. We will see in Chapters 3 and 4 that the function g can be constructed so that it belongs to $C^\infty(\mathbb{R})$. Consequently, the function $\phi_{\delta,\Omega,m}$ has excellent time and frequency localization, i.e. $\phi_{\delta,\Omega,m}$ decays faster than any inverse polynomial and $\hat{\phi}_{\delta,\Omega,m}$ is compactly supported. Although statement (1) looks rather useless at first glance, they are in fact crucial in constructing bandlimited tight frames and biframes of $L^2(\mathbb{R})$! Details will be provided in Chapter 5. In fact, it is later discovered in [11] that it is possible to construct scaling functions with subexponential decay that is far better than any previous decay rate! The interested reader can refer to [11] for details.

Chapter 3

Regularity and Interpolatory Properties

In this chapter, we will discuss some technical properties about regularity and decay of functions in $L^2(\mathbb{R})$, as well as interpolatory properties of scaling functions and wavelet functions. Before we review some well known properties regarding regularity and decay of functions in $L^2(\mathbb{R})$, we start with a definition of a special class of differentiable functions.

3.1 Regularity properties

The Schwartz class S is defined as follows.

Definition 3.1.1 *A function $f : \mathbb{R} \to \mathbb{C}$ lies in the Schwartz class S if it is a $C^\infty(\mathbb{R})$-function and for any nonnegative integers k and l, there exists a constant $C = C(k, l) > 0$ such that*

$$\left|(D^k f)(x)\right| \leq C(1 + |x|)^{-l}$$

for all $x \in \mathbb{R}$.

We proceed to give sufficient conditions for a function to lie in S. Firstly, we have the following theorem.

Theorem 3.1.1 *If $f \in L^2(\mathbb{R})$ such that $\hat{f}$ is compactly supported and bounded, then $f \in C^\infty(\mathbb{R})$ and $D^k f \in L^2(\mathbb{R})$ for all $k \in \mathbb{N} \cup \{0\}$. Furthermore, for $k \in \mathbb{N}$, if $\hat{f} \in C^k(\mathbb{R})$, then $x^l f(\cdot) \in L^2(\mathbb{R})$ for all $l \leq k$ and there exists a constant $M = M(k)$ such that*

$$|f(x)| \leq M(1 + |x|)^{-k}$$

for all $x \in \mathbb{R}$.

Proof: Firstly, from standard techniques in Fourier analysis, if $f \in L^2(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then f is bounded everywhere and continuous. Define $\hat{f}_k(\xi) := (i\xi)^k \hat{f}(\xi)$, $k \in \mathbb{N} \cup \{0\}$. Then we have

$$\int_{\mathbb{R}} |\hat{f}_k(\xi)|^2 \, d\xi = \int_{-\Omega}^{\Omega} |\xi|^{2k} |\hat{f}(\xi)|^2 \, d\xi \leq P \int_{-\Omega}^{\Omega} \xi^{2k} \, d\xi < \infty$$

for all $k \in \mathbb{N} \cup \{0\}$, where Ω and P are some positive constants. By the theory of the Fourier transform on Sobolev spaces as given on page 45 of [13], we conclude that $f \in C^\infty(\mathbb{R})$ and $D^k f \in L^2(\mathbb{R})$ for all $k \in \mathbb{N} \cup \{0\}$. Due to the compact support and boundedness of $\hat{f}$, $\hat{f} \in L^1(\mathbb{R})$ and it follows that f is continuous and bounded everywhere.

Furthermore, if $\hat{f} \in C^k(\mathbb{R})$, for $k \in \mathbb{N}$, we define for $l \leq k$, $f_l(x) := (-ix)^l f(x)$. Note that $D^l \hat{f}$ is continuous and compactly supported for all $l \leq k$, thus $D^l \hat{f}$ must be bounded on its compact support which means that $D^l \hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Once again, by an analogous argument given in page 45 of [13], the inverse Fourier transform of the function $D^l \hat{f}$ is given by f_l where $f_l \in L^2(\mathbb{R})$. Thus $f_l = x^l f(\cdot) \in L^2(\mathbb{R})$ and is bounded and continuous everywhere for all $l \leq k$.

Finally, setting $f_0 = f$ and with some simple manipulation,

$$(1 + |x|)^k |f(x)| = \sum_{l=0}^{k} \binom{k}{l} |f_l(x)| \leq M(k)$$

for all $x \in \mathbb{R}$. This completes the proof. ■

Corollary 3.1.1 *If $f \in L^2(\mathbb{R})$ such that $\hat{f}$ is compactly supported and lies in $C^\infty(\mathbb{R})$, then $f \in S$.*

Proof: From Theorem 3.1.1, it suffices to show that $(-i\xi)^k \hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ for all $k \in \mathbb{N} \cup \{0\}$. This is certainly true since $\hat{f}$ is compactly supported and lies in $C^\infty(\mathbb{R})$. Hence we conclude that $f \in S$. ■

Next, we present a simple and yet useful lemma on $C^k(\mathbb{R})$-functions, where $1 \leq k \leq \infty$.

Lemma 3.1.1 *If h, $t \in C^k(\mathbb{R})$, $1 \leq k \leq \infty$, $k \in \mathbb{N}$, then*

(a) $h \cdot t \in C^k(\mathbb{R})$,

(b) $h + t \in C^k(\mathbb{R})$,

(c) $h \circ t \in C^k(\mathbb{R})$ *if $h \circ t$ is well defined,*

(d) $\frac{1}{h} \in C^k(\mathbb{R})$ *if h never vanishes.*

We shall now pause to comment on the regularity of scaling functions $\phi \in \mathcal{B}_{\delta,\Omega}$, where $0 < \delta \leq \Omega < \pi/2$. In this case, the integer shifts of ϕ form a frame for V_0, where V_0 is as defined in (2.2.1). By Lemma 2.2.3, one can see that $\lim_{\xi \to 2\Omega^-} \hat{\phi}(\xi) \neq 0$ which means that $\hat{\phi}$ is necessarily discontinuous. Then ϕ cannot have decay rate better than this rate in the following sense: $x^l f(\cdot) \notin L^2(\mathbb{R})$ for $l \geq 1$ because otherwise, $D^L \hat{f} \in L^2(\mathbb{R})$ for some $L \geq 1$ which contradicts the discontinuity of $\hat{f}$. The details can be checked in [13] as well. This reinforces the comment in Chapter 2 that most bandlimited frames constructed from frame MRAs do not have good decay in the time domain. Next, we prove a proposition which would be useful in subsequent chapters.

Proposition 3.1.1 *For $0 < \delta \leq \Omega \leq 2\pi/3$, let $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ and ϕ be defined by $\hat{a}$ as in* (2.1.6) *and* (2.1.7). *Then $\hat{a} \in C^k(\mathbb{R})$ if and only if $\hat{\phi} \in C^k(\mathbb{R})$, where $1 \leq k \leq \infty$, $k \in \mathbb{N}$.*

Proof: Firstly, we observe that

$$\hat{\phi}(\xi) = \Big[\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)\Big] \mathbf{1}_{[-2\Omega,\, 2\Omega]}(\xi) = \prod_{j=1}^{N} \Big[\hat{a}(2^{-j}\xi) \mathbf{1}_{[-\pi,\, \pi]}(2^{-j}\xi)\Big] = \prod_{j=1}^{N} q(2^{-j}\xi),$$

where N is as defined in (2.1.7) and $q(\xi) = \hat{a}(\xi)\mathbf{1}_{[-\pi,\pi]}(\xi)$. With this setup, we are ready to prove the result.

For $k \in \mathbb{N}$, assuming $\hat{a} \in C^k(\mathbb{R})$, to show that $\hat{\phi} \in C^k(\mathbb{R})$, it suffices to establish that $q \in C^k(\mathbb{R})$ due to the finite product structure of $\hat{\phi}$. Since $\hat{a} \in C^k(\mathbb{R})$, it follows that $q(\xi) = \hat{a}(\xi)\mathbf{1}_{[-\pi,\pi]}(\xi) \in C^k[-\pi,\pi]$. However, since $\operatorname{supp} \hat{a}\mathbf{1}_{[-\pi,\pi]} = [-\Omega, \Omega]$ with $\Omega < \pi$, this means that $q \in C^k(\mathbb{R})$.

Conversely, if $\hat{\phi} \in C^k(\mathbb{R})$, then from the proof of Proposition 2.1.1,

$$\hat{a}(\xi) = \begin{cases} \frac{\hat{\phi}(2\xi)}{\hat{\phi}(\xi)}, & \text{if } \xi \in [-\Omega, \Omega], \\ 0, & \text{if } \xi \in [-\pi, \pi]\backslash[-\Omega, \Omega], \end{cases}$$

Since $\hat{\phi} \in C^k(\mathbb{R})$, $\hat{\phi}(\xi) > 0$ for $\xi \in (-2\Omega, 2\Omega)$ and $\operatorname{supp} \hat{\phi} = [-2\Omega, 2\Omega]$, by Lemma 3.1.1, we deduce that $\hat{a}\mathbf{1}_{[-\pi,\pi]} \in C^k[-\pi, \pi]$. By periodicity and the fact that $\hat{a}(\xi) = 0$ for $\xi \in (\Omega, \pi] \cup (-\pi, -\Omega)$, it can be concluded that $\hat{a} \in C^k(\mathbb{R})$. ■

In the next section, we will present some results on the interpolatory properties of scaling and wavelet functions. We will also show that interpolatory properties are related to sampling formulae.

3.2 Interpolatory properties and sampling formulae

In (2.2.2), we introduce the notion of an interpolatory scaling function. Analogously, for a wavelet ψ in $L^2(\mathbb{R})$, we say that ψ is an *interpolatory wavelet* if

$$\psi(s + 1/2) = \delta_{0s}, \qquad j \in \mathbb{Z}.$$

We say that a 2π-periodic function $\hat{a}$ is an *interpolatory mask* if

$$\hat{a}(\xi) + \hat{a}(\xi + \pi) \equiv 1. \tag{3.2.1}$$

If ϕ is refinable and interpolatory, then it is necessary that its corresponding mask $\hat{a}$ satisfies (3.2.1).

We have the following characterization on our class of bandlimited scaling functions.

Theorem 3.2.1 *Let $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ with $\pi/3 \leq \delta \leq \Omega \leq 2\pi/3$, $\Omega \geq \pi/2$ and ϕ be defined by $\hat{a}$ as in* (2.1.6) *and* (2.1.7)*. Then $\hat{a}$ is an interpolatory mask if and only if ϕ is an interpolatory function.*

Proof: We note that from Theorem 3.1.1 that since $\hat{\phi}$ is compactly supported and bounded everywhere, $\phi \in C^\infty(\mathbb{R})$. This ensures pointwise values of ϕ are well-defined everywhere and the result follows from Lemma 2.2.2 as mentioned in Chapter 2. ■

Proposition 3.2.1 *Let $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C(\mathbb{R})$ with $\pi/3 \leq \delta \leq \Omega \leq 2\pi/3$, $\Omega \geq \pi/2$ and ϕ be defined by $\hat{a}$ as in* (2.1.6) *and* (2.1.7)*. If $\hat{a}$ is an interpolatory mask, then the integer shifts of ϕ form a Riesz sequence and so $\Omega > \pi/2$.*

Proof: By Theorem 3.2.1, if $\hat{a}$ is interpolatory, then ϕ is interpolatory which means equivalently that (2.2.3) holds. Then by Proposition 2.2.1 and Corollary 2.2.3, there exist constants A, $B > 0$ such that

$$A \leq [\hat{\phi}, \hat{\phi}](\xi) \leq B$$

a.e. This is equivalent to the integer shifts of ϕ forming a Riesz sequence. As a result, $\Omega > \pi/2$. ■

Now interpolatory properties of scaling functions are connected to sampling reconstruction forumlae. Take for instance, it is well known that the scaling function $\phi(x) = \operatorname{sinc} x$ is an interpolatory scaling function with orthonormal integer shifts, where

$$\operatorname{sinc} x := \begin{cases} \dfrac{\sin \pi x}{\pi x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

One can easily verify that $\phi(j) = \delta_{0j}$ for $j \in \mathbb{Z}$. Define

$$PW[\Omega_1, \Omega_2] := \{f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subseteq [\Omega_1, \Omega_2]\}.$$

We quote the famous Shannon-Whittaker's sampling theorem in the following.

Theorem 3.2.2 *For any* $f \in PW[-\pi, \pi]$,

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\phi(x-n) \quad a.e., \tag{3.2.2}$$

where $\phi(x) = \operatorname{sinc} x$.

We wonder whether there are other functions $\phi \in L^2(\mathbb{R})$ such that sampling formulae similar to (3.2.2) hold? Furthermore, can ϕ be an interpolatory scaling function? We answer both questions in the affirmative. The following theorem not only answers both questions, but also illustrates the usefulness of having a scaling function ϕ whose Fourier transform $\hat{\phi}$ possesses an interval of constancy. Indeed, the following theorem is a generalization of the Shannon-Whittaker's sampling theorem.

Theorem 3.2.3 *Let* $\hat{\phi} \in \mathcal{B}_{\delta,\Omega}$ *where* $0 < \delta \le \Omega \le 2\pi/3$ *and* $\Omega + \delta \le \pi$. *Then the following reconstruction formula holds.*

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\phi(x-n) \quad a.e. \tag{3.2.3}$$

for all $f \in PW[-2\delta, 2\delta]$. *In particular, if* $\delta = \Omega = \pi/2$, *then* $\phi(x) = \operatorname{sinc} x$ *and we get the Shannon-Whittaker's sampling theorem. Furthermore,*

(a) *There exist constants* A, $B > 0$ *such that*

$$A\sum_{n\in\mathbb{Z}} |f(n)|^2 \le \|f\|_2^2 \le B\sum_{n\in\mathbb{Z}} |f(n)|^2, \tag{3.2.4}$$

for all $f \in PW[-2\delta, 2\delta]$ *if and only if either* $\Omega > \pi/2$ *or* $\Omega = \pi/2$ *and* $\lim_{\xi\to\pi^-} \hat{\phi}(\xi) > 0$.

(b) *The following equality*

$$\sum_{n\in\mathbb{Z}} |f(n)|^2 = \|f\|_2^2$$

holds for all $f \in PW[-2\delta, 2\delta]$ *if and only if* $[\hat{\phi}, \hat{\phi}] \equiv 1$.

(c) *For* $\delta \ge \pi/3$, $\Omega \ge \pi/2$, *if* $\sum_{l\in\mathbb{Z}} \hat{\phi}(\xi + 2\pi l) \equiv 1$, *then* (3.2.4) *holds and* ϕ *is an interpolatory scaling function.*

Proof: Firstly, we note that $2\delta \leq \delta + \Omega \leq \pi$ implies that $\delta \leq \pi/2$. With $\phi \in \mathcal{B}_{\delta,\Omega}$ satisfying $\Omega + \delta \leq \pi$, we claim that we can always find a 2π-periodic function $\hat{m}$ such that

$$\hat{f}(\xi) = \hat{m}(\xi)\hat{\phi}(\xi) \quad a.e., \tag{3.2.5}$$

for $f \in PW[-2\delta, 2\delta]$ and in fact, $\hat{m}(\xi) = \sum_{l\in\mathbb{Z}} \hat{f}(\xi + 2\pi l)$.

Let

$$g(\xi) := \begin{cases} \hat{f}(\xi)/\hat{\phi}(\xi), & \text{if } \xi \in [-2\delta, 2\delta], \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} \hat{f}(\xi), & \text{if } \xi \in [-2\delta, 2\delta], \\ 0, & \text{otherwise,} \end{cases}$$

because $\hat{\phi}(\xi) = 1$ for $|\xi| \leq 2\delta$. We set $\hat{m}(\xi) := \sum_{l\in\mathbb{Z}} g(\xi + 2\pi l) = \sum_{l\in\mathbb{Z}} \hat{f}(\xi + 2\pi l)$. Now since $\Omega + \delta \leq \pi$, this implies that $2\Omega \leq -2\delta + 2\pi l$ for all $l \geq 1$. Consequently,

$$\operatorname{supp} \hat{m} \cap \operatorname{supp} \hat{\phi} = \bigcup_{l\in\mathbb{Z}} [-2\delta + 2\pi l,\, 2\delta + 2\pi l] \cap [-2\Omega,\, 2\Omega] = [-2\delta,\, 2\delta] \tag{3.2.6}$$

up to a null set whenever $2\Omega \leq -2\delta + 2\pi l$ for all $l \geq 1$.

Thus,

$$\hat{m}(\xi)\hat{\phi}(\xi) = \hat{g}(\xi)\hat{\phi}(\xi) = \frac{\hat{f}(\xi)}{\hat{\phi}(\xi)}\hat{\phi}(\xi) = \hat{f}(\xi) \tag{3.2.7}$$

and (3.2.5) holds a.e. Note that since $\operatorname{supp} \hat{f} = [-2\delta, 2\delta] \subseteq [-\pi, \pi]$ implies that

$$[-2\delta + 2\pi l,\, 2\delta + 2\pi l] \cap [-2\delta + 2\pi k,\, 2\delta + 2\pi k] = \emptyset$$

if $l \neq k$, where $l, k \in \mathbb{Z}$. This explains why (3.2.6) and (3.2.7) hold whenever $\Omega + \delta \leq \pi$.

Next we show that $\hat{m} \in L^2[-\pi, \pi]$. Since $f \in PW[-2\delta, 2\delta] \subset PW[-\pi, \pi]$, by periodization and Parseval's identity, we get

$$\int_{-\pi}^{\pi} |\hat{m}(\xi)|^2\, d\xi = \int_{-\pi}^{\pi} |\sum_{l\in\mathbb{Z}} \hat{f}(\xi + 2\pi l)|^2\, d\xi = \int_{-\pi}^{\pi} |\hat{f}(\xi)|^2\, d\xi = \|\hat{f}\|_2^2 < \infty.$$

Let $\{m(n)\}_{n\in\mathbb{Z}}$ be the Fourier coefficients of $\hat{m}$. Then $\{m(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and furthermore,

$$\begin{aligned} m(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{m}(\xi)e^{in\xi}\, d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{l\in\mathbb{Z}} \hat{f}(\xi + 2\pi l)e^{in\xi}\, d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{in\xi}\, d\xi = f(n). \end{aligned}$$

Taking the inverse Fourier transform on (3.2.5), we get $f(x) = \sum_{n\in\mathbb{Z}} f(n)\phi(x-n)$ a.e. Since $\phi \in \mathcal{B}_{\delta,\Omega}$, by previous arguments, there exists a constant $B > 0$ such that $[\hat{\phi}, \hat{\phi}](\xi) \leq B$ a.e. As $\{f(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$,

$$\|f\|_2^2 = \|\sum_{n\in\mathbb{Z}} f(n)\phi(\cdot - n)\|_2^2 \leq B \sum_{n\in\mathbb{Z}} |f(n)|^2 < \infty,$$

which makes the reconstruction formula well-defined with respect to the $L^2(\mathbb{R})$-norm.

If $\delta = \pi/2$, then $\Omega + \delta \leq \pi$ and $\delta \leq \Omega$ imply that $\Omega = \pi/2 = \delta$. Then since $\phi \in \mathcal{B}_{\delta,\Omega}$, this forces $\hat{\phi}(\xi) = \mathbf{1}_{[-\pi,\pi]}(\xi)$. Therefore by taking the inverse Fourier transform, we get $\phi(x) = \operatorname{sinc} x$ and (3.2.3) reduces to the Shannon-Whittaker's sampling theorem.

Taking $L^2(\mathbb{R})$ norm on (3.2.3) and using the characterization of Riesz sequences in Theorem 2.2.1, we get the result for statement (a).

Similarly, we impose $L^2(\mathbb{R})$ norm on (3.2.3) and using the characterization of orthonormal sequences in Theorem 2.2.1, and by Lemma 2.2.2, the CQF condition implies that $[\hat{\phi}, \hat{\phi}] \equiv 1$. This proves statement (b).

It is clear that ϕ is an interpolatory scaling function in view of the equivalent conditions stated at the start of this section. To show (3.2.4) holds, we invoke Proposition 3.2.1. Thus the theorem is proved. ■

We show that if $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$, $1 \leq k \leq \infty$, then it is quite easy to get an interpolatory mask with the same regularity. This is shown in the following.

Lemma 3.2.1 *Suppose that* $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$, *where* $0 < \delta \leq \Omega \leq 2\pi/3$, $1 \leq k \leq \infty$. *Let* $\Omega > \pi/2$ *or* $\Omega = \pi/2$ *and* $\lim_{\xi\to\pi/2^-} \hat{a}(\xi) > 0$. *Define*

$$\hat{\tilde{a}}(\xi) := \frac{\hat{a}(\xi)}{\hat{a}(\xi) + \hat{a}(\xi+\pi)}.$$

Then $\hat{\tilde{a}}$ *is an interpolatory mask and lies in* $\mathcal{A}_{\tilde{\delta},\Omega} \cap C^k(\mathbb{R})$, *for some* $\tilde{\delta} \geq \pi/3$.

Proof: Now since $\Omega > \pi/2$ or $\Omega = \pi/2$ with $\lim_{\xi\to\pi/2^-} \hat{a}(\xi) > 0$, by Theorem 2.2.1 and Corollary 2.2.3, there exist constants A, $B > 0$ such that

$$A \leq \hat{a}(\xi) + \hat{a}(\xi+\pi) \leq B \quad a.e. \tag{3.2.8}$$

Then we verify that $\hat{\tilde{a}}$ is an interpolatory mask:

$$\hat{\tilde{a}}(\xi) + \hat{\tilde{a}}(\xi + \pi) = \frac{\hat{a}(\xi) + \hat{a}(\xi + \pi)}{\hat{a}(\xi) + \hat{a}(\xi + \pi)} = 1$$

for almost everywhere $\xi \in \mathbb{R}$. Since (3.2.8) holds and $\hat{a}(\cdot) + \hat{a}(\cdot + \pi) \in C^k(\mathbb{R})$ whenever $\hat{a} \in C^k(\mathbb{R})$, by Lemma 3.1.1, we see that $\hat{\tilde{a}}$ must be in $C^k(\mathbb{R})$.

Moreover, if (3.2.8) holds, then $\operatorname{supp} \hat{\tilde{a}} \mathbf{1}_{[-\pi,\,\pi]} = [-\Omega,\,\Omega]$. Thus, to see that $\hat{\tilde{a}} \in \mathcal{A}_{\tilde{\delta},\Omega}$ where $\tilde{\delta} \geq \pi/3$, we first show that indeed there exists some $\tilde{\delta} > 0$ such that $\hat{\tilde{a}}(\xi) = 1$ on $[-\tilde{\delta}, \tilde{\delta}]$. Suppose on the contrary that such a $\tilde{\delta}$ does not exist. We can write $\hat{\tilde{a}}(\xi) = \sum_{l \in \mathbb{Z}} \hat{q}(\xi + 2\pi l)$ where $\operatorname{supp} \hat{q} = [-\Omega,\,\Omega]$. Then on the interval $[\Omega - \pi,\, \pi - \Omega]$,

$$\hat{\tilde{a}}(\xi) + \hat{\tilde{a}}(\xi + \pi) = \hat{q}(\xi)$$

which leads to $\hat{\tilde{a}}(\xi) = 1$ on the interval $[\Omega - \pi,\, \pi - \Omega]$. That is a contradiction. Lastly, we invoke Corollary 2.2.2 to conclude that $\tilde{\delta} \geq \pi/3$. ■

Lastly, we will provide a characterization of MRA interpolatory wavelets and provide some easy formulae to always get an interpolatory wavelet from an interpolatory scaling function.

Theorem 3.2.4 *Suppose that ϕ is an refinable interpolatory continuous function in $L^2(\mathbb{R})$. Define ψ by its Fourier transform $\hat{\psi}(\xi) = \hat{b}(\xi/2)\hat{\phi}(\xi/2)$ where $\hat{b}$ is a 2π-periodic function whose Fourier coefficients $\{b_n\}_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$. Then $\psi(s + j/2) = \delta_{0s}$ if and only if $\hat{b}(\xi) - \hat{b}(\xi + \pi) = e^{-ij\xi}$ for $j = 0,\, 1$.*

Proof: Since ψ is defined by the wavelet mask $\hat{b}$, we have

$$\psi(x) = \sum_{n \in \mathbb{Z}} b_n \phi(2x - n),$$

where $\{b_n\}_{n \in \mathbb{Z}}$ are the Fourier coefficient sequence of $\hat{b}$. It follows from the Weiestrass M-test that $\psi \in C(\mathbb{R})$ as $\phi(2 \cdot -n) \in C(\mathbb{R})$ for all $n \in \mathbb{Z}$ and $\{b_n\}_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$. If $\psi(s + j/2) = \delta_{0s}$ for $j = 0, 1$ and $\phi(s) = \delta_{0s}$, then substituting $x = s$, we get

$$\psi(s + j/2) = \sum_{n \in \mathbb{Z}} b_n \phi(2s + j - n) = \sum_{n \in \mathbb{Z}} b_n \delta_{2s+j,n} = b_{2s+j}.$$

Thus $b_{2s+j} = \delta_{0s}$ for $j = 0, 1$. Next we consider

$$\hat{b}(\xi) - \hat{b}(\xi + \pi) = \frac{1}{2}\Big[\sum_{n\in\mathbb{Z}} b_n e^{-in\xi} - \sum_{n\in\mathbb{Z}} b_n(-1)^n e^{-in\xi}\Big] = \sum_{n\in\mathbb{Z}} b_{2n+1} e^{-i(2n+1)\xi} = e^{-ij\xi}.$$

for $j = 0,\ 1$. Conversely, if $\hat{b}(\xi) - \hat{b}(\xi + \pi) = e^{-ij\xi}$ for $j = 0,\ 1$, then this implies that $\sum_{n\in\mathbb{Z}} b_{2n+1} e^{-i(2n+1)\xi} = e^{-ij\xi}$ for $j = 0,\ 1$. By the uniqueness of the Fourier coefficients, $b_{2s+j} = \delta_{0s}$ for $j = 0,\ 1$. Thus $\psi(s + j/2) = b_{2s+j} = \delta_{0s}$ for $j = 0,\ 1$. ■

We note that there is no bandlimited assumption regarding the functions ϕ and ψ and the conditions are rather generic. We will illustrate the usefulness of this theorem in the next section by generating several examples of interpolatory wavelets.

3.3 Examples

First, we provide an example to illustrate Theorem 3.2.3.

Example 3.3.1 Let $\phi_{\epsilon,m}$ be defined by its Fourier transform

$$\hat{\phi}_{\epsilon,m}(\xi) = \cos^m\Big(\frac{\pi}{2} g(\frac{1}{2\epsilon}(|\xi| - \pi + \epsilon))\Big)$$

where $0 < \epsilon \le \pi/3$, $m \in \mathbb{N}$, and g is a $C^k(\mathbb{R})$-function $(0 \le k \le \infty)$ satisfying the following properties.

(a)

$$g(\xi) = 0,\ \xi \le 0 \text{ and } g(\xi) = 1,\ \xi \ge 1. \tag{3.3.1}$$

(b)

$$g \text{ is strictly increasing on the interval } (0,\ 1). \tag{3.3.2}$$

(c)

$$g(\xi) + g(1 - \xi) \equiv 1. \tag{3.3.3}$$

It will be shown later in Chapter 4 that $\operatorname{supp} \hat{\phi}_{\epsilon,m} = [-2\Omega_\epsilon,\ 2\Omega_\epsilon] = [-\pi - \epsilon,\ \pi + \epsilon] \subseteq [-2\pi/3,\ 2\pi/3]$ and $\hat{\phi}_{\epsilon,m}$ has an IOC $[-2\delta_\epsilon,\ 2\delta_\epsilon] = [-\pi + \epsilon,\ \pi - \epsilon]$. Then $\Omega_\epsilon = \pi/2 + \epsilon/2$,

$\delta_\epsilon = \pi/2 - \epsilon$. Thus, $\phi_{\epsilon,m} \in \mathcal{B}_{\delta_\epsilon,\Omega_\epsilon} \cap C^k(\mathbb{R})$. Furthermore, $\Omega_\epsilon + \delta_\epsilon = \pi$ for every $0 < \epsilon \leq \pi/3$. So we invoke Theorem 3.2.3 to get the reconstruction formula

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\phi_{\epsilon,m}(x-n) \quad a.e.$$

for all $m \in \mathbb{N}$, if $f \in PW[-\pi+\epsilon, \pi-\epsilon]$, $0 < \epsilon \leq \pi/3$. Note that since $\Omega_\epsilon = \pi/2+\epsilon/2 > \pi/2$, there exist positive constants $A = A(\epsilon, m)$, $B = B(\epsilon, m)$ such that

$$A\sum_{n\in\mathbb{Z}} |f(n)|^2 \leq \|f\|_2^2 \leq B\sum_{n\in\mathbb{Z}} |f(n)|^2$$

for any $f \in PW[-\pi+\epsilon, \pi-\epsilon]$.

Furthermore, if $m = 1$, it will be shown in Chapter 4 that $[\hat{\phi}_{\epsilon,1}, \hat{\phi}_{\epsilon,1}] \equiv 1$ and thus $\{\phi_{\epsilon,1}\}$ forms a family of scaling functions with orthonormal integer shifts with

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |f(n)|^2$$

for any $f \in PW[-\pi+\epsilon, \pi-\epsilon]$.

If $m = 2$, it will be similarly shown in Chapter 4 that $\sum_{l\in\mathbb{Z}} \hat{\phi}_{\epsilon,2}(\xi+2\pi l) \equiv 1$ and thus $\{\phi_{\epsilon,2}\}$ forms a family of interpolatory scaling functions.

Next, we provide explicit constructions of bandlimited interpolatory wavelets.

Example 3.3.2 Let $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^2(\mathbb{R})$, $\pi/3 \leq \delta \leq \Omega \leq 2\pi/3$, $\Omega \geq \pi/2$ and suppose that $\hat{a}$ is an interpolatory mask. Then by Theorem 3.2.1, ϕ defined by $\hat{a}$ as in (2.1.6) and (2.1.7) is a bandlimited interpolatory scaling function. For $0 < \delta_0 \leq \Omega_0 \leq 2\pi/3$, let $\hat{a}_0 \in \mathcal{A}_{\delta_0,\Omega_0} \cap C^2(\mathbb{R})$ be an interpolatory mask. Then it can be shown that the Fourier coefficients $\{a_n\}_{n\in\mathbb{Z}}$, $\{y_n\}_{n\in\mathbb{Z}}$ of $\hat{a}$ and $\hat{a}_0$ have the following decay rate: $|a_n|$, $|y_n| \leq C(1+|n|)^{-2}$. This ensures $\{a_n\}_{n\in\mathbb{Z}}, \{y_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. Define ψ by its Fourier transform

$$\hat{\psi}(\xi) = \hat{b}(\xi/2)\hat{\phi}(\xi/2),$$

where $\hat{b}(\xi) = e^{-i\xi}\overline{\hat{a}_0(\xi+\pi)}$. Then we can see that

$$\hat{b}(\xi)-\hat{b}(\xi+\pi) = e^{-i\xi}\overline{\hat{a}_0(\xi+\pi)}-(-e^{-i\xi}\overline{\hat{a}_0(\xi)}) = e^{-i\xi}\overline{(\hat{a}_0(\xi)+\hat{a}_0(\xi+\pi))} = e^{-i\xi}. \quad (3.3.4)$$

It will be shown in Chapter 6 that $X(\Psi)$ always forms a Riesz basis for $L^2(\mathbb{R})$. Thus ψ is an interpolatory wavelet by Theorem 3.2.4.

In particular, if we use the alternating flip formula, i.e.

$$\hat{b}(\xi) = e^{i\xi}\overline{\hat{a}(\xi+\pi)},$$

this automatically ensures that $\hat{a}_0 = \hat{a} \in \mathcal{A}_{\delta,\Omega}$, $0 < \delta \leq \Omega \leq 2\pi/3$ and $\hat{a}_0 = \hat{a}$ is an interpolatory mask. Thus the alternating flip formula will *always* get ourselves an interpolatory wavelet for $L^2(\mathbb{R})$.

More explicitly, let $\hat{a}_\epsilon$ be the 2π-periodic extension of the function

$$b_\epsilon(\xi) = \cos^2\left(\frac{\pi}{2}g(\frac{1}{\epsilon}(|\xi| - \frac{\pi}{2} + \frac{\epsilon}{2}))\right)$$

where $0 < \epsilon \leq \pi/3$ and g is a $C^k(\mathbb{R})$-function ($2 \leq k \leq \infty$) satisfying (3.3.1), (3.3.2), (3.3.3).

It can be checked that $\hat{a}_\epsilon \in \mathcal{A}_{\delta_\epsilon,\Omega_\epsilon}$ where $\delta_\epsilon = \pi/2 - \epsilon/2$ and $\Omega_\epsilon = \pi/2 + \epsilon/2$, $0 < \epsilon \leq \pi/3$. In fact, $\hat{a}_\epsilon$ is an interpolatory mask for every $0 < \epsilon \leq \pi/3$. Since $\Omega_\epsilon \leq 2\pi/3$ and $2\delta_\epsilon \leq \Omega_\epsilon$ for $0 < \epsilon \leq \pi/3$, $\hat{a}_\epsilon$ has a corresponding scaling function ϕ_ϵ defined by its Fourier transform as

$$\hat{\phi}_\epsilon(\xi) = b_\epsilon(\xi/2).$$

Define $\hat{b}_\epsilon(\xi) = e^{-i\xi}\overline{\hat{a}_\epsilon(\xi+\pi)}$ and thus ψ_ϵ defined by its Fourier transform

$$\hat{\psi}_\epsilon(\xi) = \hat{b}_\epsilon(\xi/2)\hat{\phi}_\epsilon(\xi/2) = e^{-i\xi/2}[b_\epsilon(\xi/2+\pi) + b_\epsilon(\xi/2-\pi)]b_\epsilon(\xi/4) \tag{3.3.5}$$

is an interpolatory wavelet for $L^2(\mathbb{R})$. In particular, if $g \in C^\infty(\mathbb{R})$, then $\psi_\epsilon \in S$ for every $0 < \epsilon \leq \pi/3$.

Theorem 3.2.4 is also useful in the construction of compactly supported dual interpolatory wavelets. We recall that Ji and Shen have constructed compactly supported dual scaling functions which are both interpolatory and symmetric in [18]. Define the dual wavelet masks as $\hat{b}(\xi) = e^{-i\xi}\overline{\hat{\tilde{a}}(\xi+\pi)}$ and $\hat{\tilde{b}}(\xi) = e^{-i\xi}\overline{\hat{a}(\xi+\pi)}$. By the result we have proved in (3.3.4), we can easily see that ψ and $\tilde{\psi}$ are both interpolatory wavelets.

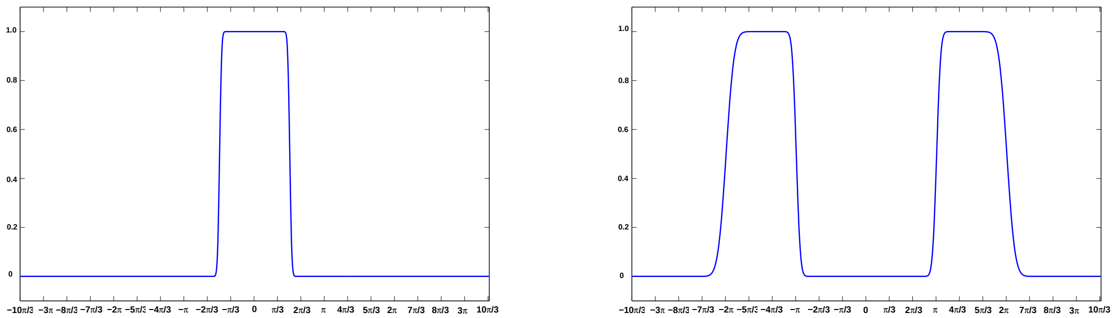

Figure 3.1: Graphs of $\hat{\phi}_{\frac{\pi}{3}}$ and $|\hat{\psi}_{\frac{\pi}{3}}|$.

The bandlimited case has certain advantage and disadvantage in constructing interpolatory wavelets. We will show in Chapter 6 that unlike the case of constructing compactly supported wavelets , the alternating flip formula *always* works in the case of creating a bandlimited interpolatory wavelet so that it forms an affine Riesz basis for $L^2(\mathbb{R})$. In fact, we have shown above that there are explicit constructions of interpolatory scaling functions and wavelets with the Schwartz class. The downside is that it is quite difficult to create dual interpolatory wavelets both with good decay from two dual interpolatory masks $\hat{a}$ and $\hat{\tilde{a}}$ in $\mathcal{A}_{\delta,\Omega}$. This will be illustrated in Chapter 6. This chapter ends off by depicting $\hat{\phi}_{\frac{\epsilon}{3}}$ and $|\hat{\psi}_{\frac{\pi}{3}}|$ from (3.3.5) in Figure 3.1.

Chapter 4

Bell Functions and Orthonormal Wavelets

In this chapter, we will be focusing on the explicit constructions of bell functions and bandlimited orthonormal wavelets. Roughly speaking, a bell function is a compactly supported hump like function. It will be shown in later chapters that we can also adapt these bell functions to construct bandlimited framelets and biorthogonal wavelets with explicit expressions. We note that a family of bell functions was used by Meyer in [19] to prove the existence of bandlimited orthonormal wavelets lying in the Schwartz class. However, we need to discuss the properties of bell functions in detail before the adaptations can take place.

4.1 Construction of bell functions

In this section, we show that it is possible to construct bell functions with arbitrary support, interval of constancy and order of continuous derivatives explicitly. We begin by proving the following theorem on general constructions of bell functions.

Theorem 4.1.1 *Let* $I := (a_1, a_2)$, $J := (b_1, b_2)$. *Define*

$$b_{I,J}(\xi) := \sin\Big(\frac{\pi}{2} g(\frac{1}{a_2 - a_1}(\xi - a_1))\Big) \cdot \cos\Big(\frac{\pi}{2} g(\frac{1}{b_2 - b_1}(\xi - b_1))\Big),$$

where $a_1 < a_2 \leq b_1 < b_2$ *and* g *is a* $C^k(\mathbb{R})$*-function* $(0 \leq k \leq \infty)$ *satisfying the properties:*

(i)

$$g(\xi) = 0,\ \xi \leq 0 \text{ and } g(\xi) = 1,\ \xi \geq 1. \tag{4.1.1}$$

(ii)

$$g \text{ is strictly increasing on the interval } (0,\, 1). \tag{4.1.2}$$

(iii)

$$g(\xi) + g(1-\xi) \equiv 1. \tag{4.1.3}$$

Then we have the following.

(1) $b_{I,J}$ *has support on* $[a_1,\, b_2]$ *and belongs to* $C^k(\mathbb{R})$.

(2) *When* $a_2 < b_1$, $b_{I,J}(\xi)$ *has an* IOC $[a_2,\, b_1]$, *i.e.*

$$b_{I,J}(\xi) = 1, \quad \xi \in [a_2,\, b_1].$$

(3) $b_{I,J}(\xi)$ *is strictly increasing on* $(a_1,\, a_2)$ *and strictly decreasing on* $(b_1,\, b_2)$.

(4) *If* $b_2 - b_1 = a_2 - a_1$, *then* $b_{I,J}(\xi)$ *is symmetric about the point* $\xi^* = \frac{a_2+b_1}{2}$ *and consequently, we can write*

$$b_{I,J}(\xi) = \cos\Big(\frac{\pi}{2} g(\frac{1}{b_2-b_1}|\xi - (\frac{a_2+b_1}{2})| - (\frac{b_1-a_2}{2}))\Big).$$

Proof: Properties (1) and (2) follow from [1].

To see that $b_{I,J}(\xi)$ is strictly increasing on $(a_1,\, a_2)$ and strictly decreasing on $(b_1,\, b_2)$, we first observe that since g is a strictly increasing function in the interval $(0,\, 1)$, $g(\frac{1}{a_2-a_1}(\xi - a_1))$ is strictly increasing in $(a_1,\, a_2)$ and $g(\frac{1}{b_2-b_1}(\xi - b_1))$ is strictly increasing on $(b_1,\, b_2)$. Secondly, note that $\sin(\frac{\pi}{2}x)$ and $\cos(\frac{\pi}{2}x)$ is strictly increasing and decreasing respectively on the interval $(0,\, 1)$. Since $g(\frac{1}{a_2-a_1}(\xi - a_1)) = 0$ when $\xi = a_1$ and

$g(\frac{1}{a_2-a_1}(\xi-a_1))=1$ when $\xi=a_2$ and g is strictly increasing on $(a_1,\, a_2)$, it follows that the range of the function $g(\frac{1}{a_2-a_1}(\cdot-a_1))$ on $(a_1,\, a_2)$ is $(0,\, 1)$. Moreover,

$$\sin\left(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\xi-a_1))\right)=1, \qquad \xi\geq a_2, \tag{4.1.4}$$

and

$$\cos\left(\frac{\pi}{2}g(\frac{1}{b_2-b_1}(\xi-b_1))\right)=1, \qquad \xi\leq b_1. \tag{4.1.5}$$

We can then conclude that $\sin\left(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\xi-a_1))\right)$ is strictly increasing on $(a_1,\, a_2)$ and similarly $\cos\left(\frac{\pi}{2}g(\frac{1}{b_2-b_1}(\xi-b_1))\right)$ is strictly decreasing on $(b_1,\, b_2)$. Since $a_2\leq b_1$, we obtain the result in statement (3).

For $b_2-b_1=a_2-a_1$, we want to show that $b_{I,J}(\xi)$ is symmetric about the point $\xi^*=\frac{a_2+b_1}{2}$, i.e.

$$b_{I,J}(\frac{a_2+b_1}{2}+\xi)=b_{I,J}(\frac{a_2+b_1}{2}-\xi) \quad \text{for all } \xi\in\mathbb{R}.$$

Consider

$$\begin{aligned} & b_{I,J}(\frac{a_2+b_1}{2}+\xi) \\ = \quad & \sin\left(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\frac{a_2+b_1}{2}+\xi-a_1))\right)\cdot\cos\left(\frac{\pi}{2}g(\frac{1}{b_2-b_1}(\frac{a_2+b_1}{2}+\xi-b_1))\right) \\ = \quad & \sin\left(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\frac{a_2+b_1}{2}+\xi-a_1))\right)\cdot\cos\left(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\xi+\frac{a_2-b_1}{2}))\right). \end{aligned}$$

Similarly,

$$\begin{aligned} & b_{I,J}(\frac{a_2+b_1}{2}-\xi) \\ = \quad & \sin\left(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\frac{a_2+b_1}{2}-\xi-a_1))\right)\cdot\cos\left(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(-\xi+\frac{a_2+b_1}{2}-b_1))\right). \end{aligned}$$

Note that since $g(\xi)+g(1-\xi)\equiv 1$, then

$$\cos(\frac{\pi}{2}g(\xi))=\cos(\frac{\pi}{2}(1-g(1-\xi)))=\sin(\frac{\pi}{2}g(1-\xi)),$$

and replacing ξ by $1-\xi$, we obtain

$$\sin(\frac{\pi}{2}g(\xi))=\cos(\frac{\pi}{2}g(1-\xi)).$$

Applying these two identities on $b_{I,J}(\frac{a_2+b_1}{2}-\xi)$, we get

$$\begin{aligned} & b(\frac{a_2+b_1}{2}-\xi) \\ = \quad & \cos\Big(\frac{\pi}{2}g(1-\frac{1}{a_2-a_1}(\frac{a_2+b_1}{2}-\xi-a_1))\Big)\cdot\sin\Big(\frac{\pi}{2}g(1-\frac{1}{a_2-a_1}(-\xi+\frac{a_2+b_1}{2}-b_1))\Big) \\ = \quad & \cos\Big(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\frac{a_2}{2}-\frac{b_1}{2}+\xi))\Big)\cdot\sin\Big(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\xi+\frac{a_2}{2}+\frac{b_1}{2}-a_1))\Big) \\ = \quad & b_{I,J}(\frac{a_2+b_1}{2}+\xi), \end{aligned}$$

for all $\xi\in\mathbb{R}$.

Lastly, in view of property (2) in this proposition and (4.1.4), (4.1.5), we can write

$$\begin{aligned} b_{I,J}(\xi) &= \begin{cases} \sin\Big(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\xi-a_1))\Big), & \text{if } \xi<a_2, \\ 1, & \text{if } a_2\le\xi\le b_1, \\ \cos\Big(\frac{\pi}{2}g(\frac{1}{b_2-b_1}(\xi-b_1))\Big), & \text{if } \xi\ge b_1, \end{cases} \\ &= \begin{cases} \sin\Big(\frac{\pi}{2}g(\frac{1}{a_2-a_1}(\xi-a_1))\Big), & \text{if } \xi<\frac{a_2+b_1}{2}, \\ \cos\Big(\frac{\pi}{2}g(\frac{1}{b_2-b_1}(\xi-b_1))\Big), & \text{if } \xi>\frac{a_2+b_1}{2}, \end{cases} \\ &= \begin{cases} \cos\Big(\frac{\pi}{2}g(1-\frac{1}{a_2-a_1}(\xi-a_1))\Big), & \text{if } \xi\le\frac{a_2+b_1}{2}, \\ \cos\Big(\frac{\pi}{2}g(\frac{1}{b_2-b_1}(\xi-\frac{a_2+b_1}{2}-\frac{b_1-a_2}{2}))\Big), & \text{if } \xi>\frac{a_2+b_1}{2}, \end{cases} \\ &= \begin{cases} \cos\Big(\frac{\pi}{2}g(\frac{1}{b_2-1}(-\xi+\frac{a_2+b_1}{2}-\frac{b_1-a_2}{2}))\Big), & \text{if } \xi\le\frac{a_2+b_1}{2}, \\ \cos\Big(\frac{\pi}{2}g(\frac{1}{b_2-b_1}(\xi-\frac{a_2+b_1}{2}-\frac{b_1-a_2}{2}))\Big), & \text{if } \xi>\frac{a_2+b_1}{2}, \end{cases} \\ &= \cos\Big(\frac{\pi}{2}g(\frac{1}{b_2-b_1}|\xi-(\frac{a_2+b_1}{2})|-(\frac{b_1-a_2}{2}))\Big). \end{aligned}$$

This ends the proof. ■

In particular, when $\frac{a_2+b_1}{2}=0$, $b_{I,J}$ may be written as $\cos(\frac{\pi}{2}g(|\xi|-b_1))$. This is because when $a_2=-b_1$, $\frac{b_1-a_2}{2}=\frac{2b_1}{2}=b_1$ and thus

$$\cos\Big(\frac{\pi}{2}g(|\xi-(\frac{a_2+b_1}{2})|-(\frac{b_1-a_2}{2}))\Big)=\cos\Big(\frac{\pi}{2}g(|\xi|-b_1)\Big)$$

For convenience, we will call $b_{I,J}$ bell functions due to their bell-like structure.

Two key features of this construction not explicitly mentioned in the statement of the above theorem but equally important are

$$\sin^2(\frac{\pi}{2}g(\xi)) + \cos^2(\frac{\pi}{2}g(\xi)) \equiv 1, \tag{4.1.6}$$

$$\cos(\frac{\pi}{2}g(\xi)) = \cos(\frac{\pi}{2}(1-g(1-\xi))) = \sin(\frac{\pi}{2}g(1-\xi)). \tag{4.1.7}$$

The property in (4.1.6) is essential in enabling us to construct the bandlimited orthonormal wavelets, tight frames, biframes and biorthogonal wavelets later on. The property in (4.1.7) allows us to determine simple conditions for symmetry of the bell functions. Note that we have assumed the existence of a function g with certain properties in Theorem 4.1.1. Thus the natural questions are that whether such functions exist in the first place, and furthermore could such functions be explicitly written down? We shall answer in the affirmative and see that such explicit functions not only exist, but also available as a family of with arbitrary order of continuous derivatives.

To this end, we prove the following theorem.

Theorem 4.1.2 *Define a sequence of splines $\{p_k\}_{k\in\mathbb{N}}$ by*

$$p_k(\xi) := \begin{cases} 0, & \text{if } \xi < 0, \\ v_k(\xi), & \text{if } 0 \le \xi \le 1, \\ 1, & \text{if } \xi > 1, \end{cases}$$

where $A_k := \Big(\sum_{j=0}^{k} \binom{k}{j}(-1)^j \frac{1}{2k-j+1}\Big)^{-1}$ *and* $v_k(\xi) := A_k\Big(\sum_{j=0}^{k} \binom{k}{j}(-1)^j \frac{1}{2k-j+1}\xi^{2k-j+1}\Big)$, $k \in \mathbb{N}$*, then the following hold.*

(a) $p_k \in C^k(\mathbb{R})$.

(b) $p_k(\xi) + p_k(1-\xi) \equiv 1$.

(c) $p_k^{(1)}(\xi) > 0$, $\xi \in (0,1)$ *and* $\operatorname{sgn}(A_k) = (-1)^k$.

Proof: We first show that A_k is well defined for all $k \in \mathbb{N}$. It suffices to show that $\Big(\sum_{j=0}^{k} \binom{k}{j}(-1)^j \frac{1}{2k-j+1}\Big) \neq 0$ for all $k \in \mathbb{N}$. Suppose there exists some $K \in \mathbb{N}$ such that $\left(\sum_{j=0}^{K}\binom{K}{j}(-1)^j\frac{1}{2K-j+1}\right) = 0$. Define $w_K(\xi) := \left(\sum_{j=0}^{K}\binom{K}{j}(-1)^j\frac{1}{2K-j+1}\xi^{2K-j+1}\right)$. Then we have $w_K(0) = 0 = w_K(1)$. Clearly w_K is differentiable everywhere and thus by Rolle's Theorem, there exists some $\xi' \in (0, 1)$ such that $w_K^{(1)}(\xi') = 0$. On the other hand, differentiating the polynomial w_K with some simplification, we get $w_K^{(1)}(\xi) = \xi^K(\xi-1)^K$ which never vanishes on the interval $(0, 1)$ giving a contradiction. Thus $\left(\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{1}{2k-j+1}\right) \neq 0$ and A_k is well defined for all $k \in \mathbb{N}$.

To prove statement (a), we show that $p_k^{(i)} \in C(\mathbb{R})$, $1 \leq i \leq k$. Since v_k is a polynomial, it is infinitely differentiable and thus it suffices to show $p_k^{(i)}$ is continuous at the two points $\xi = 0$ and $\xi = 1$. This amounts to proving that

$$\lim_{\xi\to 0^-} p_k^{(i)}(\xi) = \lim_{\xi\to 0^+} p_k^{(i)}(\xi) < \infty$$

and

$$\lim_{\xi\to 1^-} p_k^{(i)}(\xi) = \lim_{\xi\to 1^+} p_k^{(i)}(\xi) < \infty$$

for each $1 \leq i \leq k$.

We consider

$$\lim_{\xi\to 0^+} v_k^{(1)}(\xi) = \lim_{\xi\to 0^+} A_k\xi^k(\xi-1)^k = 0 = \lim_{\xi\to 0^-} v_k^{(1)}(\xi),$$

and similarly

$$\lim_{\xi\to 1^-} v_k^{(1)}(\xi) = \lim_{\xi\to 1^-} A_k\xi^k(\xi-1)^k = 0 = \lim_{\xi\to 1^+} v_k^{(1)}(\xi).$$

Then we get

$$p_k^{(1)}(\xi) = \begin{cases} 0, & \text{if } \xi < 0 \text{ or } \xi > 1, \\ A_k\xi^k(\xi-1)^k, & \text{if } 0 \leq \xi \leq 1, \end{cases}$$

By the basic differentiation properties of chain rule and product rule, it is not hard to see that

$$p_k^{(i)}(\xi) = \begin{cases} 0, & \text{if } \xi < 0 \text{ or } \xi > 1, \\ v_k^{(i)}(\xi), & \text{if } 0 \leq \xi \leq 1, \end{cases}$$

where $v_k^{(i)}$, $1 \le i \le k$, can be written as a finite sum of polynomials with each polynomial having roots 0 and 1. Thus,

$$\lim_{\xi \to 0^+} v_k^{(i)}(\xi) = 0 = \lim_{\xi \to 0^-} v_k^{(i)}(\xi),$$

and similarly

$$\lim_{\xi \to 1^-} v_k^{(i)}(\xi) = 0 = \lim_{\xi \to 1^+} v_k^{(i)}(\xi).$$

for each $1 \le i \le k$. This completes the proof for statement (a).

By the structure of p_k, to show that $p_k(\xi) + p_k(1-\xi) \equiv 1$ is equivalent to showing that $v_k(\xi) + v_k(1-\xi) \equiv 1$, $\xi \in [0, 1]$. Recall that

$$v_k^{(1)}(\xi) = A_k \xi^k (\xi - 1)^k, \quad \text{for all } \xi \in \mathbb{R}.$$

Then

$$\frac{d}{d\xi}(v_k(\xi) + v_k(1-\xi)) = v_k^{(1)}(\xi) - v_k^{(1)}(1-\xi) = A_k[\xi^k(\xi-1)^k - (1-\xi)^k(-\xi)^k] = 0$$

for all $\xi \in \mathbb{R}$ and all $k \in \mathbb{N}$. Thus $v_k(\xi) + v_k(1-\xi) \equiv C$ for some constant C. Letting $\xi = 0$, $C = v_k(0) + v_k(1) = 1$. Thus $v_k(\xi) + v_k(1-\xi) = 1$ for all $\xi \in [0, 1]$.

Note that $v_k^{(1)}(\xi) = A_k \xi^k (\xi - 1)^k$ is either strictly positive or negative on $(0, 1)$. We thus conclude that $v_k^{(1)}(\xi) > 0$ on $(0, 1)$, for otherwise the continuity of p_k will be violated as $v_k(0) = 0$ and $v_k(1) = 1$. Lastly, note that sgn $(\xi^k(\xi-1)^k) = (-1)^k$ whenever $\xi \in (0, 1)$. Since $v_k^{(1)}(\xi) > 0$ in $(0, 1)$, it follows that sgn $(A_k) = (-1)^k$ as well. This concludes the proof of Theorem 4.1.2. ■

It is relatively simple to implement computation of bell functions when we choose $g = p_k$ due to their simplistic expressions. However, Theorem 4.1.2 has a slight drawback in the sense that we can only obtain a sequence of splines $\{p_k\}_{k \in \mathbb{N}}$ where each $p_k \in C^k(\mathbb{R})$ and k is arbitrary but finite. It is desirable that the function g involved in Theorem 4.1.1 belongs to $C^\infty(\mathbb{R})$. As a matter of fact, we will show in Theorem 4.1.3 that there indeed exists a family of functions $q_{a,s}$ in $C^\infty(\mathbb{R})$ that satisfy properties of the function g in Theorem 4.1.1.

Theorem 4.1.3 *Define*

$$f_{a,s}(\xi) := \begin{cases} e^{-\frac{a}{\xi^{2s}}}, & \text{if } \xi > 0 \\ 0, & \text{if } \xi \leq 0 \end{cases}$$

where $a > 0$ *and* $s \in \mathbb{Z}^+$. *Let* $h_{a,s}(\xi) := f_{a,s}(\xi) \cdot f_{a,s}(1-\xi)$ *and* $q_{a,s}(\xi) := \dfrac{\int_{-\infty}^{\xi} h_{a,s}(t)\,dt}{\int_{-\infty}^{\infty} h_{a,s}(t)\,dt}$. *Then the following hold.*

(a) $q_{a,s} \in C^{\infty}(\mathbb{R})$ *for all* $a > 0$, $s \in \mathbb{Z}^+$.

(b) $q_{a,s}(\xi) + q_{a,s}(1-\xi) \equiv 1$ *for all* $a > 0$, $s \in \mathbb{Z}^+$.

(c) $q_{a,s}^{(1)}(\xi) > 0$, $\xi \in (0, 1)$.

(d) $q_{a,s}(\xi) = 0$ *for* $\xi \leq 0$ *and* $q_{a,s}(\xi) = 1$ *for* $\xi \geq 1$.

Theorem 4.1.3 is proved in [23] so we shall omit the proof here.

Example 4.1.1 Let $k = 3$ in Theorem 4.1.2, Then we get $A_3 = -140 < 0$ and

$$v_3(\xi) = -20\xi^7 + 70\xi^6 - 84\xi^5 + 35\xi^4.$$

The spline

$$p_3(\xi) = \begin{cases} 0, & \text{if } \xi < 0, \\ v_3(\xi), & \text{if } 0 \leq \xi \leq 1, \\ 1, & \text{if } \xi > 1, \end{cases}$$

satisfies (4.1.1), (4.1.2), (4.1.3) in Theorem 4.1.1.

Example 4.1.2 Let $a = 1$, $s = 1$ in Theorem 4.1.3. Then we get the function

$$f_{1,1}(\xi) = \begin{cases} e^{-\frac{1}{\xi^2}}, & \text{if } \xi > 0, \\ 0, & \text{if } \xi \leq 0, \end{cases}$$

and the function $q_{1,1}$ satisfies (4.1.1), (4.1.2), (4.1.3) in Theorem 4.1.3.

We note that such constructions are not exhaustive. For all we know, there could exist other families which provide better properties like subexponential decay which is faster than the decay of the Meyer's wavelets. The author notes that construction of bandlimited wavelets with subexponential decay has been discussed in [11] by Dziubánski and Hernández. However, due to time constraint and the unavailability of explicit construction, we will not discuss it in this thesis, and the interested reader can refer to [11] for details.

In the next section, we will use the bell functions to construct bandlimited orthonormal wavelets commonly referred to as Meyer's wavelets.

4.2 Bandlimited orthornormal wavelets

A reader familiar with the Meyer's wavelets would hardly be surprised with the constructions involved in this section. Nevertheless, there are several key techniques useful for later chapters in the underlying constructions. We shall construct Meyer's wavelets in a different and yet equivalent way. We begin by defining a function $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ with $\pi/3 \leq \delta \leq \Omega \leq \frac{2\pi}{3}$, $\Omega \geq \pi/2$, satisfying the CQF condition, i.e.

$$|\hat{a}(\cdot)|^2 + |\hat{a}(\cdot + \pi)|^2 \equiv 1.$$

Then by (2.1.6), (2.1.7) in Chapter 2, its corresponding scaling function ϕ defined by its Fourier transform

$$\hat{\phi}(\xi) = \big[\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)\big]\mathbf{1}_{[-2\Omega,\, 2\Omega]}(\xi)$$

satisfies

$$\sum_{l\in\mathbb{Z}} |\hat{\phi}(\cdot + 2\pi l)|^2 \equiv 1.$$

Consequently, defining

$$\hat{\psi}(\xi) = e^{\frac{i\xi}{2}}\overline{\hat{a}(\frac{\xi}{2} + \pi)}\hat{\phi}(\frac{\xi}{2}), \tag{4.2.1}$$

the theory of multiresolution analysis ensures us that $X(\Psi)$ forms an orthonormal wavelet basis for $L^2(\mathbb{R})$. The following proposition lists formal properties of the wavelet ψ defined by (4.2.1) which most of the proof will be quoted from [23] without proof.

Proposition 4.2.1 For any function $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ with $\pi/3 \leq \delta \leq \Omega \leq \frac{2\pi}{3}$, $\Omega \geq \pi/2$, satisfying the CQF condition, one can associate an orthonormal wavelet given by (4.2.1) which has the following properties:

(a) $\operatorname{supp} \hat{\psi} \subseteq [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}]$.

(b) ψ is a real-valued $C^\infty(\mathbb{R})$-function.

(c) $\psi(-\frac{1}{2} - x) = \psi(-\frac{1}{2} + x)$ for all $x \in \mathbb{R}$.

Proof: By the characterization of the functions $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ with $\pi/3 \leq \delta \leq \Omega \leq \frac{2\pi}{3}$, $\Omega \geq \pi/2$, satisfying the CQF condition in Proposition 2.2.1, $[\hat{\phi}, \hat{\phi}] \equiv 1$. We note that $\hat{a} \equiv 1$ in the set of intervals $\bigcup_{l\in\mathbb{Z}} [\Omega - \pi + 2\pi l, \pi - \Omega + 2\pi l]$. Furthermore, $2(\pi - \Omega) \geq \Omega$ because $\Omega \leq \frac{2\pi}{3}$. Thus

$$\hat{\phi}(\xi) = \Big[\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)\Big] \mathbf{1}_{[-2\Omega,\, 2\Omega]}(\xi) = \hat{a}(\frac{\xi}{2})\, \mathbf{1}_{[-2\Omega,\, 2\Omega]}(\xi)$$

and obviously, $\operatorname{supp} \hat{\phi} = [-2\Omega, 2\Omega] \subseteq [-\frac{4\pi}{3}, \frac{4\pi}{3}]$. Thus ϕ satisfies the conditions of Proposition 3.2 in [23] (up to a difference of a scalar constant $\frac{1}{\sqrt{2\pi}}$) and this proves our result. ■

The most important special case of the above construction is when the function $\hat{a}$, in addition to being in class $\mathcal{A}_{\delta,\Omega}$ with $\pi/3 \leq \delta \leq \Omega \leq \frac{2\pi}{3}$, $\Omega \geq \pi/2$ and the CQF condition, also lies in $C^\infty(\mathbb{R})$. In fact, in the following theorem, we shall give explicit constructions of such a function $\hat{a}$.

Theorem 4.2.2 *There exist real-valued wavelets ψ_ϵ, $0 < \epsilon \leq \pi/3$, such that the following hold.*

(a) *Each ψ_ϵ is in the Schwartz class.*

(b) *Each $\psi_\epsilon(-\frac{1}{2} + x) = \psi_\epsilon(-\frac{1}{2} - x)$ for all $x \in \mathbb{R}$.*

(c) $\operatorname{supp} \hat{\psi}_\epsilon \subseteq [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \frac{8\pi}{3}]$.

Proof: For $0 < \epsilon \leq \pi/3$, define $I_\epsilon := [-\frac{\pi}{2} - \frac{\epsilon}{2}, -\frac{\pi}{2} + \frac{\epsilon}{2}]$ and $J_\epsilon := [\frac{\pi}{2} - \frac{\epsilon}{2}, \frac{\pi}{2} + \frac{\epsilon}{2}]$. Let $\hat{a}_\epsilon$ be the 2π-periodic extension of the bell function b_ϵ, where

$$b_\epsilon(\xi) = b_{I_\epsilon, J_\epsilon}(\xi) = \cos\Big(\frac{\pi}{2} g(\frac{1}{\epsilon}(|\xi| - \frac{\pi}{2} + \frac{\epsilon}{2}))\Big), \tag{4.2.2}$$

and $g = q_{a,s} \in C^\infty(\mathbb{R})$ as in Theorem 4.1.3. Since $b_2 - b_1 = \epsilon = a_2 - a_1$, b_ϵ is symmetric about the point $\xi^* = [(-\frac{\pi}{2} + \frac{\epsilon}{2}) + (\frac{\pi}{2} - \frac{\epsilon}{2})] = 0$. Thus b_ϵ is an even function and so is $\hat{a}_\epsilon$. Note that $\hat{a}_\epsilon \in \mathcal{A}_{\delta_\epsilon, \Omega_\epsilon}$, where $0 < \delta_\epsilon = \frac{\pi}{2} - \frac{\epsilon}{2} < \frac{\pi}{2} + \frac{\epsilon}{2} = \Omega_\epsilon \leq \frac{2\pi}{3}$. Thus we can define the corresponding scaling function $\phi_\epsilon \in \mathcal{B}_{\pi-\epsilon, \pi+\epsilon}$ by its Fourier transform

$$\hat{\phi}_\epsilon(\xi) = \Big[\prod_{j=1}^{N} \hat{a}_\epsilon(2^{-j}\xi)\Big] \mathbf{1}_{[-\pi-\epsilon,\, \pi+\epsilon]}(\xi) = \hat{a}_\epsilon(\frac{\xi}{2}) \mathbf{1}_{[-\pi-\epsilon,\, \pi+\epsilon]}(\xi) = b_\epsilon(\xi/2) \tag{4.2.3}$$

because N as defined in (2.1.7) is equal to 1 since $2\delta_\epsilon \geq \Omega_\epsilon$ whenever $0 < \epsilon \leq \pi/3$.

Next, the support of $\hat{a}_\epsilon = \bigcup_{l \in \mathbb{Z}} [-\frac{\pi}{2} - \frac{\epsilon}{2} + 2\pi l,\, \frac{\pi}{2} + \frac{\epsilon}{2} + 2\pi l] \subseteq \bigcup_{l \in \mathbb{Z}} [-\frac{2\pi}{3} + 2\pi l,\, \frac{2\pi}{3} + 2\pi l]$, since $0 < \epsilon \leq \frac{\pi}{3}$. Note that it suffices to verify that $\hat{a}_\epsilon$ satisfies the CQF condition in order to apply the results in Proposition 4.2.1 to get properties (b) and (c) in this theorem. Property (a) is easily verified as $g = q_{a,s} \in C^\infty(\mathbb{R})$, $\hat{a}_\epsilon$, $\hat{\phi}_\epsilon \in C^\infty(\mathbb{R})$ and consequently $\hat{\psi}_\epsilon = e^{-i\cdot/2} \hat{a}_\epsilon(\cdot/2 + \pi) \hat{\phi}_\epsilon(\cdot/2) \in C^\infty(\mathbb{R})$ by Lemma 3.1.1. It is shown in [1] and [23] that indeed $\hat{a}_\epsilon$ satisfies the CQF condition.

Now $\hat{a}_\epsilon(\xi+\pi) = \sum_{l \in \mathbb{Z}} b_\epsilon(\xi+\pi+2\pi l)$ where $\operatorname{supp} b_\epsilon(\cdot+\pi+2\pi l) \subseteq [-5\pi/3 + 2\pi l,\, -\pi/3 + 2\pi l]$. Note that $\operatorname{supp} \hat{\phi}_\epsilon \subseteq [-4\pi/3,\, 4\pi/3]$ and $\operatorname{supp} b_\epsilon(\cdot + \pi + 2\pi l) \cap \operatorname{supp} \hat{\phi}_\epsilon$ is a set of measure zero if and only if $l \leq -2$ or $l \geq 1$. Consider $b_\epsilon(-\xi/2 + \pi) = b_\epsilon(\xi/2 - \pi)$ because b_ϵ is even. Therefore $b_\epsilon(\xi/2 + \pi) + b_\epsilon(\xi/2 - \pi) = b_\epsilon(-\xi/2 - \pi) + b_\epsilon(\xi/2 - \pi)$. Since $\operatorname{supp} b_\epsilon(\cdot - \pi) \subseteq [\pi/3,\, 5\pi/3]$ and $[\pi/3,\, 5\pi/3] \cap \mathbb{R}^- = \emptyset$, then we can write $b_\epsilon(|\xi|/2 - \pi) = b_\epsilon(-\xi/2 - \pi) + b_\epsilon(\xi/2 - \pi)$. In view of all these, we have

$$\hat{\psi}_\epsilon(\xi) = e^{i\xi/2} \hat{a}_\epsilon(\xi/2 + \pi) \hat{\phi}_\epsilon(\xi/2) = e^{i\xi/2} [b_\epsilon(\frac{\xi}{2} - \pi) + b_\epsilon(\frac{\xi}{2} + \pi)] \hat{\phi}_\epsilon(\frac{\xi}{2}), \tag{4.2.4}$$

where its support is $[\pi - \epsilon,\, 2\pi + 2\epsilon] \cup [-2\pi - 2\epsilon,\, -\pi + \epsilon] \subseteq [\frac{2\pi}{3},\, \frac{8\pi}{3}] \cup [-\frac{8\pi}{3},\, -\frac{2\pi}{3}]$. Note that the argument used in showing (4.2.4) will be used repeatedly in other chapters. This concludes the proof of the theorem. ■

In fact, if one replaces the function g to be any one of the functions from the family of splines $\{p_k\}_{k \in \mathbb{N}}$ in Theorem 4.1.2, then statement (a) in Theorem 4.2.2 will

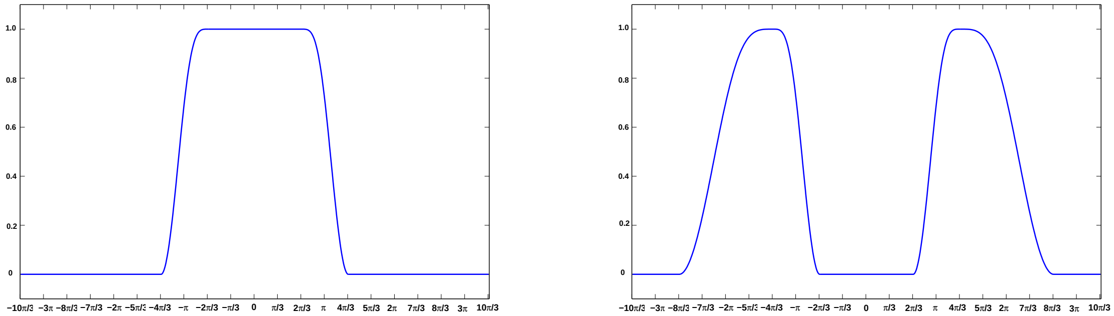

Figure 4.1: Graphs of $\hat{\phi}_{\frac{\pi}{3}}$ and $|\hat{\psi}_{\frac{\pi}{3}}|$.

be replaced by the following statement: There exists a sequence of real-valued wavelets ψ_ϵ such that

(a') each $\hat{\psi}_\epsilon \in C^k(\mathbb{R})$ and $|\psi_\epsilon(x)| \leq C(1+|x|)^{-l}$ for all $x \in \mathbb{R}$, where $l < k$, $0 \leq k < \infty$ and C is a constant depending on k, ϵ and l only.

This is because when $g = p_k \in C^k(\mathbb{R})$, $b_\epsilon \in C^k(\mathbb{R})$ and so is $\hat{\psi}_\epsilon(\xi) = e^{\frac{i\xi}{2}}[b_\epsilon(\frac{\xi}{2} - \pi) + b_\epsilon(\frac{\xi}{2}+\pi)]b_\epsilon(\frac{\xi}{4})$. Then by the properties of the Fourier transform in Theorem 3.1.1, $|\psi_\epsilon(x)| \leq C(1+|x|)^{-l}$ for all $x \in \mathbb{R}$ whenever $l < k$. Properties (b) and (c) remain unchanged due to the proof of Theorem 4.2.2. Figure 4.1 illustrates both $\hat{\phi}_{\frac{\pi}{3}}$ and $\hat{\psi}_{\frac{\pi}{3}}$ with $g = p_{10} \in C^{10}(\mathbb{R})$.

Finally, we are better equipped to tackle the explicit constructions of bandlimited tight framelets in the next chapter. While the existence of Meyer's wavelets is rather well understood over the past decade, the construction of bandlimited tight framelets is not as well understood and many constructions provided seem ad-hoc. We will use a rather systematic way in the next chapter to construct these framelets, which is via the Mixed Unitary Extension Principle (Mixed UEP).

Chapter 5

Bandlimited Biframelets

5.1 Construction by the mixed UEP

The Unitary Extension Principle was first established by Ron and Shen in [20] for the construction of tight wavelet frames. It was further generalized into the Mixed Extension Principle and the Oblique Extension Principle in [21] and [9] respectively. In this chapter, we will construct bandlimited *biframes* of $L^2(\mathbb{R})$ from one or two bandlimited refinable scaling functions using the Mixed Unitary Extension Principle (Mixed UEP). Define $\Psi := \{\psi_i \in L^2(\mathbb{R}) : i = 1, \ldots, n\}$ and $\tilde{\Psi} := \{\tilde{\psi}_i \in L^2(\mathbb{R}) : i = 1, \ldots, n\}$. We say that $X(\Psi)$ and $X(\tilde{\Psi})$ are *biframes* for $L^2(\mathbb{R})$ or form a *biframelet system* of $L^2(\mathbb{R})$ if for every $f \in L^2(\mathbb{R})$,

$$f = \sum_{i=1}^{n} \sum_{j,k\in\mathbb{Z}} \langle f, \tilde{\psi}_{i,j,k} \rangle \psi_{i,j,k} = \sum_{i=1}^{n} \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{i,j,k} \rangle \tilde{\psi}_{i,j,k},$$

and $X(\Psi)$, $X(\tilde{\Psi})$ each forms a frame for $L^2(\mathbb{R})$. As noted in [9], the Mixed UEP does not even require the integer shifts of the refinable functions ϕ and $\tilde{\phi}$ to form a frame for V_0 and $\tilde{V}_0$ respectively, where V_0 and $\tilde{V}_0$ are defined by ϕ and $\tilde{\phi}$ respectively as in (2.1.6) and (2.1.7). This is fortunate because we know from Theorem 2.2.1 that for $0 < \delta \le \Omega \le \pi/2$, a function $\phi \in \mathcal{B}_{\delta,\Omega}$ must necessarily have its Fourier transform $\hat{\phi}$ discontinuous if the integer shifts of ϕ form a frame for V_0. Consequently, the decay of ϕ is poor. If the integer shifts of ϕ are not required to form a frame for V_0, in view of

Theorem 2.2.1, we could now allow the scaling function ϕ to be a function such that its Fourier transform $\hat{\phi}$ is a bell function belonging to $C^\infty(\mathbb{R})$ which implies that ϕ is in the Schwartz class. This leads to construction of framelets which are in the Schwartz class as well. Thus the mixed UEP provides added flexibility to construct bandlimited framelets with excellent time and frequency localization.

There are a few general things to take note of. Firstly, similar to the case of employing the Mixed UEP to construct compactly supported biframelets of $L^2(\mathbb{R})$, constructions of bandlimited biframelets of $L^2(\mathbb{R})$ via the Mixed UEP are more flexible than those of bandlimited tight frames of $L^2(\mathbb{R})$. Secondly, although there are some existing constructions of bandlimited framelets in the wavelet literature in [3] and [14], they are not created by the Mixed UEP, so it would be interesting to investigate a systematic approach of constructing bandlimited framelets by the Mixed UEP.

This chapter will be outlined as follows. We will quote two special formulae related to the Mixed UEP and construct bandlimited biframes of $L^2(\mathbb{R})$ from a special pair of refinable bandlimited functions ϕ, $\tilde{\phi}$ based on these formulae. We will also provide a characterization of $X(\Psi)$ and $X(\tilde{\Psi})$ forming a biframelet system of $L^2(\mathbb{R})$ with Ψ and $\tilde{\Psi}$ each being a singleton set in $L^2(\mathbb{R})$ based on the second formula. Lastly, we construct explicit examples of bandlimited biframes of $L^2(\mathbb{R})$ with excellent time and frequency localization using bell functions in Chapter 4.

We quote a result in [5] which is based on [7] from Chui, He, Stöckler and [9] by Daubechies, Han, Ron and Shen. This result is a consequence of the mixed UEP.

Theorem 5.1.1 *Let ϕ, $\tilde{\phi}$ be a pair of refinable scaling functions in $L^2(\mathbb{R})$ and $\hat{a}$, $\hat{\tilde{a}}$ be their respective refinement masks. Define the sets Ψ and $\tilde{\Psi}$ in either one of the following two setups.*

(A) *Let $\Psi := \{\psi_l\}_{l=1}^3$, $\tilde{\Psi} := \{\tilde{\psi}_l\}_{l=1}^3$ where*

$$\begin{aligned}
&\hat{\psi}_l(\xi) := \hat{a}_l(\xi/2)\hat{\phi}(\xi/2), \qquad \hat{\tilde{\psi}}_l(\xi) := \hat{\tilde{a}}_l(\xi/2)\hat{\tilde{\phi}}(\xi/2), \qquad l = 1,\, 2,\, 3,\\
&\hat{a}_1(\xi) := e^{i\xi}\overline{\hat{a}(\xi+\pi)}, \qquad \hat{\tilde{a}}_1(\xi) := e^{i\xi}\overline{\hat{\tilde{a}}(\xi+\pi)},\\
&\hat{a}_l(\xi) := e^{i(l-2)\xi}\hat{m}(\xi), \qquad \hat{\tilde{a}}_l(\xi) := e^{i(l-2)\xi}\hat{\tilde{m}}(\xi), \qquad l = 2,\, 3,\\
&P(\xi) := 1 - \left(\hat{a}(\xi)\overline{\hat{\tilde{a}}(\xi)} + \hat{a}(\xi+\pi)\overline{\hat{\tilde{a}}(\xi+\pi)}\right),
\end{aligned} \tag{5.1.1}$$

and $(\hat{m}, \hat{\tilde{m}})$ *is a factorization pair of* P *taking the form*

$$P = 2\hat{m}\hat{\tilde{m}}, \qquad \hat{m}(0) = 0 = \hat{\tilde{m}}(0). \tag{5.1.2}$$

(B) *Let* $\Psi := \{\psi_l\}_{l=1}^2$, $\tilde{\Psi} := \{\tilde{\psi}_l\}_{l=1}^2$ *where*

$$\begin{aligned} &\hat{\psi}_l(\xi) := \hat{a}_l(\xi/2)\hat{\phi}(\xi/2), \qquad \hat{\tilde{\psi}}_l(\xi) := \hat{\tilde{a}}_l(\xi/2)\hat{\tilde{\phi}}(\xi/2), \qquad l = 1, 2, \\ &\hat{a}_1(\xi) := e^{i\xi}\overline{\hat{\tilde{a}}(\xi+\pi)}, \qquad \hat{\tilde{a}}_1(\xi) := e^{i\xi}\overline{\hat{a}(\xi+\pi)}, \\ &\hat{a}_2(\xi) := \hat{a}(\xi)\hat{m}(2\xi), \qquad \hat{\tilde{a}}_2(\xi) := \hat{\tilde{a}}(\xi)\hat{\tilde{m}}(2\xi) \end{aligned}$$

and $(\hat{m}, \hat{\tilde{m}})$ *is a real factorization pair of* P *as defined in* (5.1.1) *and* (5.1.2).

If there exist $\rho > \frac{1}{2}$ *and a constant* $B > 0$ *such that*

$$|\hat{\psi}_l(\xi)|, |\hat{\tilde{\psi}}_l(\xi)| \leq B(1 + |\xi|)^{-\rho}$$

for almost everywhere $\xi \in \mathbb{R}$ *and for all* $l = 1, \ldots, n$, *where* $n = 3$ *in setup (A) and* $n = 2$ *in setup (B), then* $X(\Psi)$ *and* $X(\tilde{\Psi})$ *form a biframelet system of* $L^2(\mathbb{R})$. *In particular, if* $\phi = \tilde{\phi}$, $\hat{a} = \hat{\tilde{a}}$, $\hat{m} = \hat{\tilde{m}}$, *then* $\Psi = \tilde{\Psi}$ *and* $X(\Psi)$ *forms a tight frame for* $L^2(\mathbb{R})$.

Now we state our main focus of this chapter.

Theorem 5.1.2 *For* $0 < \delta < \Omega \leq 2\pi/3$, $0 < \tilde{\delta} < \tilde{\Omega} \leq 2\pi/3$, *let* $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, $\hat{\tilde{a}} \in \mathcal{A}_{\tilde{\delta},\tilde{\Omega}}$ *such that* $\hat{a}(\xi)\hat{\tilde{a}}(\xi) + \hat{a}(\xi+\pi)\hat{\tilde{a}}(\xi+\pi) \leq 1$ *for all* $\xi \in \mathbb{R}$. *Let* ϕ *and* $\tilde{\phi}$ *be the corresponding scaling functions of the masks* $\hat{a}$ *and* $\hat{\tilde{a}}$ *respectively as defined in* (2.1.6) *and* (2.1.7). *Define the sets* Ψ, $\tilde{\Psi}$ *by the functions* $\hat{a}$, $\hat{\tilde{a}}$, ϕ, $\tilde{\phi}$, $\hat{m}$, $\hat{\tilde{m}}$ *by either setup* (A) *or* (B) *of* Theorem 5.1.1, *where* $\hat{m} = \frac{1}{\sqrt{2}}P^{\nu/n}$ *and* $\hat{\tilde{m}} = \frac{1}{\sqrt{2}}P^{1-\nu/n}$, $1 \leq \nu < n$, $\nu, n \in \mathbb{N}$. *Then* $X(\Psi)$ *and* $X(\tilde{\Psi})$ *form a biframelet system of* $L^2(\mathbb{R})$.

Proof: Firstly, observe that $\phi \in \mathcal{B}_{\delta,\Omega}$ and $\tilde{\phi} \in \mathcal{B}_{\tilde{\delta},\tilde{\Omega}}$ with the respective refinement masks $\hat{a}$ and $\hat{\tilde{a}}$. Next $P \geq 0$, which ensures nonnegativity of the pair $(\hat{m}, \hat{\tilde{m}})$ in both setups (A) and (B). Now since $\hat{\phi}$, $\hat{\tilde{\phi}}$ are compactly supported and bounded, so are $\hat{\psi}_i$, $\hat{\tilde{\psi}}_i$ for all i in both setups (A) and (B). The compact support of $\hat{\psi}_i$, $\hat{\tilde{\psi}}_i$, $i = 1, \ldots, n$ in both setups ensures that the decay conditions in Theorem 5.1.1 are satisfied. Then we apply Theorem 5.1.1 to obtain the result. ■

For applications, we desire Ψ and $\tilde{\Psi}$ to have as few generators as possible. We shall illustrate that it is possible to obtain only one generator for each set. When $\Omega > \pi/2$ and $\tilde{\Omega} > \pi/2$, we can employ a different setup in Chapter 6 to create Ψ and $\tilde{\Psi}$ that consist of only one generator each and the resulting $X(\Psi)$ and $X(\tilde{\Psi})$ form a pair of biorthogonal Riesz bases for $L^2(\mathbb{R})$. In view of this, we consider a special case of Theorem 5.1.2.

Corollary 5.1.1 *For $0 < \delta < \Omega < \pi/2$, $0 < \tilde{\delta} < \tilde{\Omega} < \pi/2$, let $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, $\hat{\tilde{a}} \in \mathcal{A}_{\tilde{\delta},\tilde{\Omega}}$ such that $\hat{a}(\xi)$, $\hat{\tilde{a}}(\xi) \leq 1$ for all $\xi \in \mathbb{R}$. Define the sets Ψ and $\tilde{\Psi}$ by $\hat{a}$, $\hat{\tilde{a}}$, ϕ, $\tilde{\phi}$, $\hat{m}$, $\hat{\tilde{m}}$ as in* Theorem 5.1.2. *Then $X(\Psi)$ and $X(\tilde{\Psi})$ form a biframelet system of $L^2(\mathbb{R})$. Furthermore we have the following:*

(A) *For setup* (A),

(1) $\psi_1 \equiv 0$ *if and only if* $2\Omega \leq -\tilde{\Omega} + \pi$.
$\tilde{\psi}_1 \equiv 0$ *if and only if* $2\tilde{\Omega} \leq -\Omega + \pi$.

(2) *If* $2\Omega > -\tilde{\Omega} + \pi$, $\operatorname{supp} \hat{\psi}_1 = [-2\tilde{\Omega} + 2\pi,\ 4\Omega] \cup [-4\Omega,\ 2\tilde{\Omega} - 2\pi]$.
If $2\tilde{\Omega} > -\Omega + \pi$, $\operatorname{supp} \hat{\tilde{\psi}}_1 = [-2\Omega + 2\pi,\ 4\tilde{\Omega}] \cup [-4\tilde{\Omega},\ 2\Omega - 2\pi]$.

(3) $\operatorname{supp} \hat{\psi}_2 = \operatorname{supp} \hat{\psi}_3 = [2\delta',\ \min\{4\Omega,\ 2\pi - 2\delta'\}] \cup [-\min\{4\Omega,\ 2\pi - 2\delta'\},\ -2\delta']$,
$\operatorname{supp} \hat{\tilde{\psi}}_2 = \operatorname{supp} \hat{\tilde{\psi}}_3 = [2\delta',\ \min\{4\tilde{\Omega},\ 2\pi - 2\delta'\}] \cup [-\min\{4\tilde{\Omega},\ 2\pi - 2\delta\},\ -2\delta']$,
where $\delta' = \min\{\delta,\ \tilde{\delta}\}$.

(B) *For setup* (B), *statements* (1) *and* (2) *in* (A) *hold with statement* (3) *replaced by* (3').

(3') $\operatorname{supp} \hat{\psi}_2 = [\delta',\ \min\{-\delta' + \pi,\ 2\Omega\}] \cup [-\min\{-\delta' + \pi,\ 2\Omega\},\ -\delta']$,
$\operatorname{supp} \hat{\tilde{\psi}}_2 = [\delta',\ \min\{-\delta' + \pi,\ 2\tilde{\Omega}\}] \cup [-\min\{-\delta' + \pi,\ 2\tilde{\Omega}\},\ -\delta']$,
where $\delta' = \min\{\delta,\ \tilde{\delta}\}$.

Proof: Note that $\hat{\phi} \in \mathcal{B}_{\delta,\Omega}$, $\hat{\tilde{\phi}} \in \mathcal{B}_{\tilde{\delta},\tilde{\Omega}}$ where $0 < \delta < \Omega < \pi/2$, $0 < \tilde{\delta} < \tilde{\Omega} < \pi/2$. Without loss of generality, let us assume that $\tilde{\Omega} \leq \Omega$. Since $\Omega < \frac{\pi}{2}$, on the interval

$[-\pi, \pi]$, $\hat{a}\hat{\tilde{a}}$ has support $[-\tilde{\Omega}, \tilde{\Omega}]$ and has the interval of constancy $[-\delta', \delta']$ where $\delta' = \min\{\delta, \tilde{\delta}\}$. Then we have

$$\text{supp } \hat{a}\hat{\tilde{a}} \cap \text{supp } \hat{a}\hat{\tilde{a}}(\cdot + \pi) = \emptyset.$$

Since $0 \le \hat{a}(\xi)\hat{\tilde{a}}(\xi) \le 1$ and supp $\hat{a}\hat{\tilde{a}} \cap$ supp $\hat{a}\hat{\tilde{a}}(\cdot + \pi) = \emptyset$, P takes value in $[0, 1]$.

For statement (1), supp $\hat{\psi}_1 =$ supp $\hat{a}_1(\frac{\cdot}{2})\hat{\phi}(\frac{\cdot}{2})$ and supp $\hat{a}_1 = [-\tilde{\Omega} + \pi, \tilde{\Omega} + \pi] \cup [-\tilde{\Omega} - \pi, \tilde{\Omega} - \pi]$ on the interval $[-2\pi, 2\pi]$. With supp $\hat{\phi} = [-2\Omega, 2\Omega] \subsetneq [-\pi, \pi]$, we get $\psi_1 \equiv 0$ if and only if $\hat{a}_1\hat{\phi} \equiv 0$ which is equivalent to supp $\hat{a}_1 \cap$ supp $\hat{\phi}$ is a set of measure zero. This is tantamount to $2\Omega \le -\tilde{\Omega} + \pi$. Similarly, $\tilde{\psi}_1 \equiv 0$ if and only if $2\tilde{\Omega} \le -\Omega + \pi$. Otherwise, we get result (2).

Next, on the interval $[-\pi, \pi]$, supp $P = [\delta', -\delta' + \pi] \cup [\delta' - \pi, -\delta']$ because $\hat{a}\hat{\tilde{a}}(\xi) + \hat{a}\hat{\tilde{a}}(\xi + \pi) = 1$, $\xi \in [-\pi, \pi] \cap \Big([\delta', -\delta' + \pi] \cup [\delta' - \pi, -\delta']\Big)^c$, and $P(\xi) = 1$ for $\xi \in [\tilde{\Omega}, -\tilde{\Omega} + \pi] \cup [\tilde{\Omega} - \pi, -\tilde{\Omega}]$ because $\hat{a}\hat{\tilde{a}}(\xi) + \hat{a}\hat{\tilde{a}}(\xi + \pi) = 0$, $|\xi| \in [\tilde{\Omega}, -\tilde{\Omega} + \pi]$.

Since $\hat{m} = \frac{1}{\sqrt{2}}P^{\frac{\nu}{n}}$, $1 \le \nu < n$, $\nu, n \in \mathbb{N}$, it follows that supp $\hat{m} = [\delta', -\delta' + \pi] \cup [\delta' - \pi, -\delta']$ on the interval $[-\pi, \pi]$.

Note that $\delta' = \min\{\delta, \tilde{\delta}\} \le \delta < \Omega < 2\Omega$, and supp $\hat{\phi} = [-2\Omega, 2\Omega]$. Then

$$\text{supp } \hat{a}_2\hat{\phi} = [\delta', \min\{2\Omega, -\delta' + \pi\}] \cup [-\min\{2\Omega, -\delta' + \pi\}, -\delta'],$$

thus

$$\begin{aligned}\text{supp } \hat{\psi}_3 = \text{supp } \hat{\psi}_2 &= \text{supp } \hat{m}(\frac{\cdot}{2})\hat{\phi}(\frac{\cdot}{2}) \\ &= [2\delta', \min\{4\Omega, -2\delta' + 2\pi\}] \cup [-\min\{4\Omega, -2\delta' + 2\pi\}, -2\delta'].\end{aligned}$$

Since $\hat{\tilde{m}} = \frac{1}{\sqrt{2}}P^{\frac{n-\nu}{n}}$ we have supp $\hat{\tilde{m}} = [\delta', -\delta' + \pi] \cup [\delta' - \pi, -\delta']$ on the interval $[-\pi, \pi]$. With supp $\hat{\phi} = [-2\tilde{\Omega}, 2\tilde{\Omega}]$, we can compute similarly that

$$\text{supp } \hat{\tilde{\psi}}_3 = \text{supp } \hat{\tilde{\psi}}_2 = [2\delta', \min\{4\tilde{\Omega}, -2\delta' + 2\pi\}] \cup [-\min\{4\tilde{\Omega}, -2\delta' + 2\pi\}, -2\delta'].$$

Recall that in setup (B),

$$\hat{\psi}_2(\xi) = \hat{a}_2(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2}), \qquad \hat{\tilde{\psi}}_2(\xi) = \hat{\tilde{a}}_2(\frac{\xi}{2})\hat{\tilde{\phi}}(\frac{\xi}{2}),$$

where $\hat{a}_2(\xi) = \hat{a}(\xi)\hat{m}(2\xi)$, $\hat{\tilde{a}}_2(\xi) = \hat{\tilde{a}}(\xi)\hat{\tilde{m}}(2\xi)$. On the interval $[-2\pi, 2\pi]$,

$$\text{supp } \hat{m} = \text{supp } \hat{\tilde{m}} = [\delta', -\delta' + \pi] \cup [\delta' + \pi, -\delta' + 2\pi].$$

On the interval $[-2\pi,\, 2\pi]$,

$$\text{supp } \hat{a}(\frac{\cdot}{2}) = [-2\Omega,\, 2\Omega] \subsetneq [-\pi,\, \pi], \qquad \text{supp } \hat{\tilde{a}}(\frac{\cdot}{2}) = [-2\tilde{\Omega},\, 2\tilde{\Omega}] \subsetneq [-\pi,\, \pi].$$

Thus on the interval $[-2\pi,\, 2\pi]$,

$$\text{supp } \hat{a}_2(\frac{\cdot}{2}) = \text{supp } \hat{a}(\frac{\cdot}{2})\hat{m}(\cdot) = [\delta',\, \min\{-\delta' + \pi,\, -2\Omega\}] \cup [-\min\{-\delta' + \pi,\, -2\Omega\},\, -\delta'],$$
$$\text{supp } \hat{\tilde{a}}_2(\frac{\cdot}{2}) = \text{supp } \hat{a}(\frac{\cdot}{2})\hat{m}(\cdot) = [\delta',\, \min\{-\delta' + \pi,\, -2\tilde{\Omega}\}] \cup [-\min\{-\delta' + \pi,\, -2\tilde{\Omega}\},\, -\delta'].$$

Now, $\text{supp}\,\hat{\phi}(\frac{\cdot}{2}) = [-4\Omega,\, 4\Omega] \subsetneq [-2\pi,\, 2\pi]$ and $\text{supp}\,\hat{\tilde{\phi}}(\frac{\cdot}{2}) = [-4\tilde{\Omega},\, 4\tilde{\Omega}] \subsetneq [-2\pi,\, 2\pi]$. Noting that $\text{supp}\,\hat{\psi}_2 = \text{supp}\,\hat{a}_2(\frac{\cdot}{2}) \cap \text{supp}\,\hat{\phi}(\frac{\cdot}{2})$ and $\text{supp}\,\hat{\tilde{\psi}}_2 = \text{supp}\,\hat{\tilde{a}}_2(\frac{\cdot}{2}) \cap \text{supp}\,\hat{\tilde{\phi}}(\frac{\cdot}{2})$, we get the desired result in (3'). ■

Corollary 5.1.2 *With the assumptions of* Corollary 5.1.1, *if it is further assumed that* $\hat{a} = \hat{\tilde{a}}$ *and* $\hat{m} = \hat{\tilde{m}}$, *then* $\Psi = \tilde{\Psi}$ *and* $X(\Psi)$ *forms a tight frame for* $L^2(\mathbb{R})$. *We also have the following.*

(A) *For setup* (A),

(1) $\psi_1 \equiv 0$ *if and only if* $\Omega \le \pi/3$.

(2) *If* $\Omega > \pi/3$, $\text{supp}\,\hat{\psi}_1 = [-2\Omega + 2\pi,\, 4\Omega] \cup [-4\Omega,\, 2\Omega - 2\pi]$.

(3) $\text{supp}\,\hat{\psi}_2 = \text{supp}\,\hat{\psi}_3 = [2\delta,\, \min\{4\Omega,\, 2\pi - 2\delta\}] \cup [-\min\{4\Omega,\, 2\pi - 2\delta\},\, -2\delta]$.

(B) *For setup* (B), *statements* (1) *and* (2) *in* (A) *hold with statement* (3) *replaced by* (3').

(3') $\text{supp}\,\hat{\psi}_2 = [\delta,\, \min\{-\delta + \pi,\, 2\Omega\}] \cup [-\min\{-\delta + \pi,\, 2\Omega\},\, -\delta]$.

Proof: The proof follows directly by setting $\Omega = \tilde{\Omega}$, $\delta = \tilde{\delta}$ in Corollary 5.1.1. ■

We note that there is a slight tradeoff of having just one generator for $X(\Psi)$ to form a tight frame of $L^2(\mathbb{R})$ in Corollary 5.1.2: the closed linear span of the integer shifts of ϕ consist of bandlimited functions f whose Fourier transform $\hat{f}$ has compact support up to $[-\pi/3,\, \pi/3]$ and does not contain other bandlimited functions. This can be easily seen by Theorem 3.2.2. On the other hand, less generators means less computational cost. We shall see in the following that though we have a single generator, this generator cannot form a basis for $L^2(\mathbb{R})$.

Proposition 5.1.1 *With the assumptions of setup* (B) *in* Corollary 5.1.1 *with* Ω, $\tilde{\Omega} < \pi/2$ *and* $2\Omega \leq -\tilde{\Omega} + \pi$, $2\tilde{\Omega} \leq -\Omega + \pi$, *then* $X(\Psi)$ *and* $X(\tilde{\Psi})$ *cannot form a pair of biorthogonal Riesz wavelet bases for* $L^2(\mathbb{R})$. *In particular, if* $\hat{a} = \hat{\tilde{a}}$, $\hat{m} = \hat{\tilde{m}}$, $\Omega \leq \pi/3$, *then* $X(\Psi)$ *cannot be an orthonormal basis for* $L^2(\mathbb{R})$.

Proof: Now $\underline{\operatorname{supp} \hat{\tilde{\psi}}_2, \operatorname{supp}}\, \hat{\psi}_2 \subsetneq [-\pi, \pi]$, so $m(\operatorname{supp} \hat{\tilde{\psi}})$, $m(\operatorname{supp} \hat{\psi}) < 2\pi$ which means $\sum_{l\in\mathbb{Z}} \hat{\psi}_2(\xi + 2\pi l)\overline{\hat{\tilde{\psi}}_2}(\xi + 2\pi l)$ equal zero on a set of positive measure. By Theorem 2.2.1, we obtain the result. ■

5.2 Explicit constructions

We shall see below that the construction of explicit bandlimited biframes generally is more flexible than the construction of explicit bandlimited tight frames. This is due to the need to impose a square root factorization of the function P in Theorem 5.1.2 for constructing a tight frame. We note that in the bandlimited context, the Féjer-Riesz lemma is not required to perform the factorization on P, and yet we can still obtain tight framelets. However a potential problem is that the factorization of the function P may affect the differentiability of the resulting pair $(\hat{m}, \hat{\tilde{m}})$ which could lead to poor time localization of the resulting wavelets. So we take caution in the following to eliminate this problem. The following construction is based on working backwards so as to obtain explicit expressions of the mother wavelets in terms of their Fourier transforms and to preserve the differentiability properties of the Fourier transforms of the wavelets.

Let $I = [-\Omega, -\delta]$, $J = [\delta, \Omega]$ and $\hat{a}$ be the 2π-periodic extension of the function

$$\eta(\xi) = b^m_{I,J}(\xi) = \cos^m\Big(\frac{\pi}{2} g(\frac{1}{\Omega} - \delta(|\xi| - \delta))\Big),$$

as in Theorem 4.1.1, where $0 < \delta < \Omega < \pi/2$, $2\delta \geq \Omega$, $m \in \mathbb{N}$, $g \in C^k(\mathbb{R})$.

Let $I' = [\tilde{\delta}, \tilde{\Omega}]$, $J' = [\pi - \tilde{\Omega}, \pi - \tilde{\delta}]$, $\hat{\tilde{a}}$ be the 2π-periodic extension of the function

$$\tilde{\eta}(\xi) = [1 - \gamma_1(\xi)]\mathbf{1}_{[-\tilde{\Omega}, \tilde{\Omega}]}(\xi),$$

as in Theorem 4.1.1, where $\gamma_1(\xi) = \gamma_2(\xi) + \gamma_2(\xi + \pi)$, $\gamma_2(\xi) = b^n_{I',J'}(\xi)$ where $0 < \tilde{\delta} < \tilde{\Omega} < \frac{\pi}{2}$ and $2\tilde{\delta} \geq \tilde{\Omega}$, $2 \leq n < \infty$, $n \in \mathbb{N}$, $g \in C^k(\mathbb{R})$. One can verify that indeed $\hat{\tilde{a}} \in \mathcal{A}_{\delta,\Omega}$ and that $\hat{a}$, $\hat{\tilde{a}}$ satisfy the assumptions on $\hat{a}$ and $\hat{\tilde{a}}$ in Corollary 5.1.1.

Let us further assume that $\tilde{\Omega} \leq \delta$. Then we claim that in the interval $[0, \pi]$, $P(\xi) = \gamma_2(\xi)$ which we will prove in due course. The upshot is that we can easily perform factorization on the function P by simply letting $\hat{m}$ and $\hat{\tilde{m}}$ to be the π-periodic extension of the following functions

$$z_\nu(\xi) := \frac{1}{\sqrt{2}} b^\nu_{I',J'}(\xi), \qquad z_{n-\nu}(\xi) := \frac{1}{\sqrt{2}} b^{n-\nu}_{I',J'}(\xi),$$

where $1 \leq \nu < n$, $\nu \in \mathbb{N}$. Due to this factorization, we shall observe that the properties of differentiability of z_ν and $z_{n-\nu}$ will be passed onto the Fourier transforms of the resulting framelets.

Now let us establish the claim that $P(\xi) = \gamma_2(\xi)$ on $[0, \pi]$. Since $\operatorname{supp} \hat{\tilde{a}} = \bigcup_{l\in\mathbb{Z}} [-\tilde{\Omega} + 2\pi l, \tilde{\Omega} + 2\pi l]$, $\tilde{\Omega} \leq \delta$, and $\hat{a}(\xi) = 1$, $\xi \in \bigcup_{l\in\mathbb{Z}} [-\delta + 2\pi l, \delta + 2\pi l]$, it follows that $P(\xi) = 1 - [\hat{a}(\xi)\hat{\tilde{a}}(\xi) + \hat{a}(\xi + \pi)\hat{\tilde{a}}(\xi + \pi)] = 1 - [\hat{\tilde{a}}(\xi) + \hat{\tilde{a}}(\xi + \pi)]$ for all $\xi \in \mathbb{R}$.

Next when $\tilde{\Omega} < \frac{\pi}{2}$, $\operatorname{supp} \hat{\tilde{a}}\mathbf{1}_{[-\pi, \pi]} = [-\tilde{\Omega}, \tilde{\Omega}] \subsetneq [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus $\operatorname{supp} \hat{\tilde{a}}(\cdot + \pi)\mathbf{1}_{[-\pi, \pi]} = [-\tilde{\Omega} + \pi, \pi] \cup [-\pi, \tilde{\Omega} - \pi] \subsetneq [\frac{\pi}{2}, \pi] \cup [-\pi, -\frac{\pi}{2}]$ which implies that $\operatorname{supp} \hat{\tilde{a}} \cap \operatorname{supp} \hat{\tilde{a}}(\cdot + \pi) = \emptyset$. Since P is π-periodic, it suffices to just consider P on $[0, \pi]$. Now on the interval $[0, \pi]$,

$$\hat{\tilde{a}}(\xi) + \hat{\tilde{a}}(\xi + \pi) = \begin{cases} \tilde{\eta}(\xi), & \text{if } 0 \leq \xi < \tilde{\Omega}, \\ 0, & \text{if } \tilde{\Omega} \leq \xi \leq \pi - \tilde{\Omega}, \\ \tilde{\eta}(\xi - \pi), & \text{if } \pi - \tilde{\Omega} < \xi \leq \pi, \end{cases}$$

where we note that $\tilde{\Omega} < \pi - \tilde{\Omega}$ because $\tilde{\Omega} < \frac{\pi}{2} < \pi - \tilde{\Omega}$. Thus,

$$
\begin{aligned}
P(\xi) = 1 - \Big[\hat{\tilde{a}}(\xi) + \hat{\tilde{a}}(\xi+\pi)\Big] &= \begin{cases} \gamma_1(\xi), & \text{if } 0 \le \xi < \tilde{\Omega}, \\ 1, & \text{if } \tilde{\Omega} \le \xi \le \pi - \tilde{\Omega}, \\ \gamma_1(\xi - \pi), & \text{if } \pi - \tilde{\Omega} < \xi \le \pi, \end{cases} \\
&= \begin{cases} \gamma_2(\xi), & \text{if } 0 \le \xi < \tilde{\Omega}, \\ 1, & \text{if } \tilde{\Omega} \le \xi \le \pi - \tilde{\Omega}, \\ \gamma_2((\xi+\pi) - \pi), & \text{if } \pi - \tilde{\Omega} < \xi \le \pi, \end{cases} \\
&= \gamma_2(\xi)
\end{aligned}
$$

because $\gamma_2(\xi) = \gamma_1(\xi) + \gamma_1(\xi+\pi)$ where $\operatorname{supp}\gamma_1 = [\tilde{\delta}, \pi - \tilde{\delta}]$.

We now proceed to compute explicit expressions of the wavelet functions in terms of their Fourier transforms using setups (A) and (B) in Theorem 5.1.2. Since $2\tilde{\delta} \ge \tilde{\Omega}$ and $2\delta \ge \Omega$,

$$
\begin{aligned}
\hat{\phi}(\xi) &= \Big[\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)\Big]\mathbf{1}_{[-2\Omega,\,2\Omega]} = \hat{a}(\xi/2)\mathbf{1}_{[-2\Omega,\,2\Omega]}(\xi) = b^m_{I,J}(\xi/2), \\
\hat{\tilde{\phi}}(\xi) &= \Big[\prod_{j=1}^{N} \hat{\tilde{a}}(2^{-j}\xi)\Big]\mathbf{1}_{[-2\tilde{\Omega},\,2\tilde{\Omega}]}(\xi) = \hat{\tilde{a}}(\xi/2)\mathbf{1}_{[-2\tilde{\Omega},\,2\tilde{\Omega}]}(\xi) = \tilde{\eta}(\frac{\xi}{2}).
\end{aligned}
$$

We first consider setup (A). With some calculation similar to the justification of (4.2.4), it is not hard to see that

$$
\begin{aligned}
\hat{\psi}_1(\xi) &= e^{\frac{i\xi}{2}}\hat{\tilde{a}}(\frac{\xi}{2}+\pi)\hat{\phi}(\frac{\xi}{2}) \\
&= e^{\frac{i\xi}{2}}\Big(\tilde{\eta}(\frac{\xi}{2}-\pi) + \tilde{\eta}(\frac{\xi}{2}+\pi)\Big)\hat{\phi}(\frac{\xi}{2}) \\
&= e^{\frac{i\xi}{2}}\Big(\tilde{\eta}(\frac{\xi}{2}-\pi) + \tilde{\eta}(\frac{\xi}{2}+\pi)\Big)b^m_{I,J}(\xi/4),
\end{aligned} \tag{5.2.1}
$$

$$
\hat{\tilde{\psi}}_1(\xi) = e^{\frac{i\xi}{2}}\hat{a}(\frac{\xi}{2}+\pi)\hat{\tilde{\phi}}(\frac{\xi}{2}) = e^{\frac{i\xi}{2}}\Big(\tilde{\eta}(\frac{\xi}{2}-\pi) + \tilde{\eta}(\frac{\xi}{2}+\pi)\Big)\tilde{\eta}(\frac{\xi}{4}), \tag{5.2.2}
$$

$$
\hat{\psi}_2(\xi) = \hat{m}(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2}) = \frac{1}{\sqrt{2}}[z_\nu(\frac{\xi}{2}) + z_\nu(\frac{\xi}{2}+\pi)]b^m_{I,J}(\xi/4), \qquad \hat{\psi}_3(\xi) = e^{\frac{i\xi}{2}}\hat{\psi}_2(\xi),
$$

$$
\hat{\tilde{\psi}}_2(\xi) = \hat{\tilde{m}}(\frac{\xi}{2})\hat{\tilde{\phi}}(\frac{\xi}{2}) = \frac{1}{\sqrt{2}}[z_{n-\nu}(\frac{\xi}{2}) + z_{n-\nu}(\frac{\xi}{2}+\pi)]\tilde{\eta}(\xi/4), \qquad \hat{\tilde{\psi}}_3(\xi) = e^{\frac{i\xi}{2}}\hat{\tilde{\psi}}_2(\xi),
$$

where the supports of $\hat{\psi}_l$, $\hat{\tilde{\psi}}_l$, $l = 1, 2, 3$, are given in Corollary 5.1.1.

For setup (B), $\hat{\psi}_1$, $\hat{\tilde{\psi}}_1$ remain the same as in (5.2.1) and (5.2.2), whereas

$$\begin{aligned}\hat{\psi}_2(\xi) = \hat{a}_2(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2}) &= \frac{1}{\sqrt{2}} b_{I,J}(\frac{\xi}{2}) b_{I,J}(\frac{\xi}{4})\Big[z_\nu(\xi) + z_\nu(\xi+\pi)\Big] \\ &= \frac{1}{\sqrt{2}} b_{I,J}(\frac{\xi}{2})\Big[z_\nu(\xi) + z_\nu(\xi+\pi)\Big], \\ \hat{\tilde{\psi}}_2(\xi) = \hat{\tilde{a}}_2(\frac{\xi}{2})\hat{\tilde{\phi}}(\frac{\xi}{2}) &= \frac{1}{\sqrt{2}} \tilde{\eta}(\xi/2)\tilde{\eta}(\xi/4)[z_{n-\nu}(\xi) + z_{n-\nu}(\xi+\pi)] \\ &= \frac{1}{\sqrt{2}} \tilde{\eta}(\xi/2)[z_{n-\nu}(\xi) + z_{n-\nu}(\xi+\pi)],\end{aligned} \tag{5.2.3}$$

where the supports of $\hat{\psi}_l$, $\hat{\tilde{\psi}}_l$, $l = 1, 2$ are given in Corollary 5.1.1.

Although the computations may seem tedious, the upshot is that these explicit expressions could prove useful in applications.

Next, we show that if $g \in C^k(\mathbb{R})$ where $0 \le k \le \infty$, the Fourier transforms of the mother wavelets defined in Theorem 5.1.2 will also belong to $C^k(\mathbb{R})$. In view of Lemma 3.1.1 and the formulae of the mother wavelets in terms of their Fourier transforms in Theorem 5.1.2, it suffices to verify that $\tilde{\eta} \in C^k(\mathbb{R})$ whenever $g \in C^k(\mathbb{R})$.

Note that $\gamma_2 = \gamma_1 + \gamma_1(\cdot + \pi) \in C^k(\mathbb{R})$ because $\gamma_1 \in C^k(\mathbb{R})$ whenever $g \in C^k(\mathbb{R})$. Recall that $\tilde{\eta}(\xi) = \Big[1 - \gamma_2(\xi)\Big]\mathbf{1}_{[-\tilde{\Omega},\tilde{\Omega}]}(\xi)$. Clearly, $1-\gamma_2 \in C^k(\mathbb{R})$, so to show $\tilde{\eta} \in C^k(\mathbb{R})$, it suffices to show that $1 - \gamma_2(\xi) = 0$ on $[\tilde{\Omega}, \tilde{\Omega} + \epsilon]$ for some $\epsilon > 0$. Indeed one can verify that $1 - \gamma_2(\xi) = 0$ for $|\xi| \in [\tilde{\Omega}, \pi - \tilde{\Omega}]$ where $\tilde{\Omega} < \pi/2 < \pi - \tilde{\Omega}$.

Example 5.2.1 *We use setup* (B) *here. Let* $\Omega = \frac{2\pi}{9}$, $\tilde{\Omega} = \frac{\pi}{9}$, $\tilde{\delta} = \frac{\pi}{18}$, $\delta = \frac{\pi}{9}$, $m = 4$, $n = 3$, $\nu = 1$. *Then* $2\Omega < -\tilde{\Omega} + \pi$, $2\tilde{\Omega} < -\Omega + \pi$. *We either take* $g = q_{1,1} \in C^\infty(\mathbb{R})$ *as defined in* Theorem 4.1.3 *or* $p_{10} \in C^{10}(\mathbb{R})$ *as defined in* Theorem 4.1.2. *According to* Corollary 5.1.1, $\hat{\psi}_1 \equiv 0 \equiv \hat{\tilde{\psi}}_1$. *Using* (5.2.3),

$$\hat{\psi}_2(\xi) = b^m_{I,J}(\xi/2) b^m_{I,J}(\xi/4)\Big[z_\nu(\xi) + z_\nu(\xi+\pi)\Big],$$

$$\hat{\tilde{\psi}}_2(\xi) = \tilde{\eta}(\xi/2)\tilde{\eta}(\xi/4)\Big[z_{n-\nu}(\xi) + z_{n-\nu}(\xi+\pi)\Big],$$

where $\operatorname{supp}\hat{\psi}_2 = [\pi/18, 4\pi/9] \cup [-4\pi/9, -\pi/18]$, $\operatorname{supp}\hat{\tilde{\psi}}_2 = [\pi/18, 2\pi/9] \cup [-2\pi/9, -\pi/18]$. Figure 5.1 *depicts* $\hat{\psi}_2$ *and* $\hat{\tilde{\psi}}_2$ *when* $g = p_{10}$.

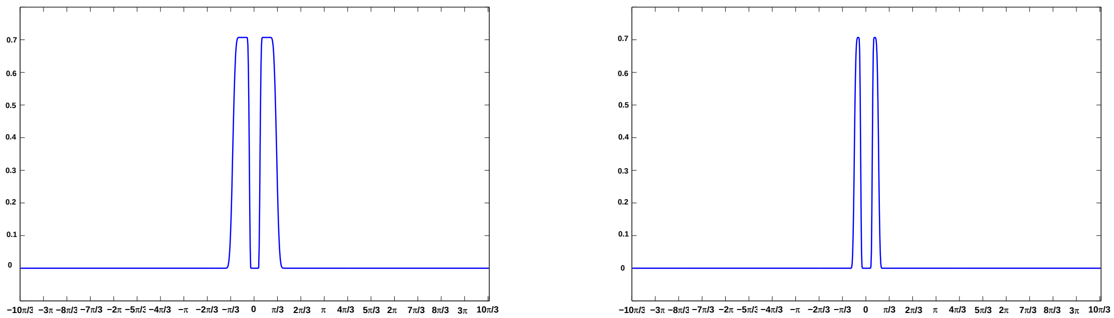

Figure 5.1: Graphs of $|\hat{\psi}_2|$ and $|\hat{\tilde{\psi}}_2|$.

In the second part of this section, we offer an easier construction of bandlimited tight frames for $L^2(\mathbb{R})$. Let $I = [-\Omega, -\delta]$, $J = [\delta, \Omega]$, $\hat{a}$ be the 2π-periodization of the bell function $b_{I,J}$ where $0 < \delta < \Omega < \frac{\pi}{2}$ and $2\delta \geq \Omega$. Define ψ_1, ψ_2, ψ_3 as in part (A) of Theorem 5.1.2 and set $\Psi = \{\psi_1, \psi_2, \psi_3\}$. Not only will we derive explicit expressions for $\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3$, we can also show that if the function g in the bell function belongs to $C^k(\mathbb{R})$, $1 \leq k \leq \infty$, $k \in \mathbb{N}$, so do $\hat{\psi}_i$ for $i = 1, 2, 3$.

Firstly, on the fundamental interval $[-\pi, \pi]$,

$$\begin{aligned}
\hat{a}(\xi) &= \cos\left(\frac{\pi}{2} g(\frac{1}{\Omega-\delta}(|\xi|-\delta))\right) \\
&= \begin{cases} \sin\left(\frac{\pi}{2} g(\frac{1}{\Omega-\delta}(\xi+\Omega))\right), & \text{if } -\Omega \leq \xi < -\delta, \\ 1, & \text{if } -\delta \leq \xi \leq \delta, \\ \cos\left(\frac{\pi}{2} g(\frac{1}{\Omega-\delta}(\xi-\delta))\right), & \text{if } \delta < \xi \leq \Omega, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Now $\hat{a}(\xi)\mathbf{1}_{[-\pi,\pi]}(\xi) = \cos\left(\frac{\pi}{2} g(\frac{1}{\Omega-\delta}(|\xi|-\delta))\right) < 1$, $|\xi| \in (\delta, \Omega]$ and clearly $0 \leq \hat{a} \leq 1$ due to the construction of bell functions in Chapter 4. So $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ with $\Omega < \frac{\pi}{2}$ and all other assumptions of Corollary 5.1.2 are satisfied.

Now we will overcome the main difficulty in deriving useful explicit expressions for $\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3$ which is to find an explicit expression for

$$\hat{m}(\xi) = \sqrt{1 - (\hat{a}^2(\xi) + \hat{a}^2(\xi+\pi))}.$$

Thanks to the property of $\sin^2\left(\frac{\pi}{2} g(\frac{1}{\Omega-\delta}(\xi-\delta))\right) + \cos^2\left(\frac{\pi}{2} g(\frac{1}{\Omega-\delta}(\xi-\delta))\right) \equiv 1$, we can achieve this. Indeed on the interval $[0,\, \pi]$, we obtain

$$\hat{m}(\xi) = \begin{cases} \sin\left(\frac{\pi}{2} g(\frac{1}{\Omega-\delta}(\xi-\delta))\right), & \text{if } \delta \le \xi < \Omega, \\ 1, & \text{if } \Omega \le \xi \le \pi - \Omega, \\ \cos\left(\frac{\pi}{2} g(\frac{1}{\Omega-\delta}(\xi+\Omega-\pi))\right), & \text{if } \pi-\Omega < \xi \le \pi-\delta, \\ 0, & \text{otherwise.} \end{cases} = b_{I_1,J_1}(\xi)$$

where $I_1 := [\delta,\, \Omega]$ and $I_2 := [\pi-\Omega,\, \pi-\delta]$ and b_{I_1,J_1} is as defined in Theorem 4.1.1. On the interval $[-\pi,\, \pi]$,

$$\hat{a}_1(\xi) = e^{i\xi}\overline{\hat{a}(\xi+\pi)} = e^{i\xi}\Big[b_{I,J}(\xi-\pi) + b_{I,J}(\xi+\pi)\Big].$$

Next, $\hat{\phi}(\xi) = \Big[\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)\Big]\mathbf{1}_{[-2\Omega,\, 2\Omega]}(\xi) = \hat{a}(\frac{\xi}{2})\mathbf{1}_{[-2\Omega,\, 2\Omega]} = b_{I,J}(\xi/2)$, since N as defined in (2.1.7) is equal to 1 as $2\delta \ge \Omega$. If $\Omega \le \frac{\pi}{3}$, we already know from Corollary 5.1.2 that $\hat{\psi}_1 \equiv 0$. For setup (A), if we assume $\frac{\pi}{3} < \Omega < \frac{\pi}{2}$, we get

$$\begin{aligned} \hat{\psi}_1(\xi) &= \hat{a}_1(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2}) \\ &= e^{\frac{i\xi}{2}}\hat{a}(\frac{\xi}{2}+\pi)\hat{\phi}(\frac{\xi}{2}) \\ &= e^{i\xi/2}\Big[b_{I,J}(\xi/2-\pi) + b_{I,J}(\xi/2+\pi)\Big] \cdot b_{I,J}(\xi/4), \end{aligned}$$

where $\operatorname{supp}\hat{\psi}_1 = [-2\Omega+2\pi,\, 4\Omega] \cup [-4\Omega,\, 2\Omega-2\pi]$ by Corollary 5.1.2.

Note that $\hat{m}$ is π-periodic, thus on the interval $[-\pi,\, \pi]$,

$$\hat{m}(\xi) = b_{I_1,J_1}(\xi) + b_{I_1,J_1}(\xi+\pi).$$

Since $\operatorname{supp}\hat{\phi} = [-2\Omega,\, 2\Omega] \subsetneq [-\pi,\, \pi]$, it follows that

$$\hat{\psi}_2(\xi) = \frac{\hat{m}(\frac{\xi}{2})}{\sqrt{2}}\hat{\phi}(\frac{\xi}{2}) = \frac{1}{\sqrt{2}}\Big[b_{I_1,J_1}(\frac{\xi}{2}) + b_{I_1,J_1}(\frac{\xi}{2}+\pi)\Big] \cdot b_{I,J}(\frac{\xi}{4}),$$

where $\operatorname{supp}\hat{\psi}_2 = [2\delta,\, \min\{4\Omega,\, 2\pi-2\delta\}] \cup [-\min\{4\Omega,\, 2\pi-2\delta\},\, -2\delta]$, and $\hat{\psi}_3 = e^{\frac{i\xi}{2}}\hat{\psi}_2(\xi)$.

For setup (B): Considering the interval $[-\pi,\, \pi]$ again, since $\operatorname{supp}\hat{a}\mathbf{1}_{[-\pi,\,\pi]} = [-\Omega,\, \Omega] \subseteq [-\pi/2,\, \pi/2]$, then

$$\hat{a}_2(\xi) = \frac{1}{\sqrt{2}}\hat{a}(\xi)\hat{m}(2\xi) = \frac{1}{\sqrt{2}} b_{I,J}(\xi) \cdot \sum_{l=-1}^{2} b_{I_1,J_1}(2\xi + \pi l)$$

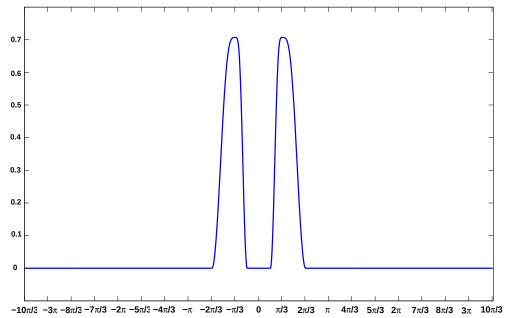

Figure 5.2: Graph of $\hat{\psi}_2$.

Since $\operatorname{supp} \hat{\psi}_2 \subsetneq [-\pi, \pi]$ as given in part (B) of Corollary 5.1.2

$$\begin{aligned} \hat{\psi}_2(\xi) = \hat{a}_2(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2}) &= \frac{1}{\sqrt{2}} b_{I,J}(\frac{\xi}{2}) \sum_{l=-1}^{2} b_{I_1,J_2}(\xi + \pi l) \cdot b_{I,J}(\frac{\xi}{4}) \\ &= \frac{1}{\sqrt{2}} b_{I,J}(\frac{\xi}{2}) \sum_{l=0}^{1} b_{I_1,J_2}(\xi + \pi l), \end{aligned} \tag{5.2.4}$$

where $\operatorname{supp} \hat{\psi}_2$ is as evaluated in part (B) of Corollary 5.1.2. Now $\hat{\psi}_1$, $\hat{\psi}_2$, $\hat{\psi}_3$ in setup (A) and $\hat{\psi}_1$, $\hat{\psi}_2$ obtained in setup (B) are formed by sums and products of functions in $C^k(\mathbb{R})$ since $g \in C^k(\mathbb{R})$. Thus by product rule and chain rule, all of the above functions belong to $C^k(\mathbb{R})$.

Example 5.2.2 *Let* $\Omega = \frac{\pi}{3}$, $\delta = \frac{\pi}{6}$, $g = q_{1,1} \in C^\infty(\mathbb{R})$ *as defined in* Theorem 4.1.3 *or* $g = p_{10}$ *as defined in* Theorem 4.1.2. *Then by* Corollary 5.1.2, $\hat{\psi}_1 \equiv 0$. *Furthermore, by* (5.2.4), $I = [-\pi/3, -\pi/6]$, $J = [\pi/6, \pi/3]$, $I_1 = [\pi/6, \pi/3]$, $J_1 = [2\pi/3, 5\pi/6]$ *and*

$$\hat{\psi}_2(\xi) = \frac{1}{\sqrt{2}} b_{I,J}(\frac{\xi}{2})[b_{I_1,J_1}(\xi) + b_{I_1,J_1}(\xi + \pi)].$$

Letting $\Psi = \{\psi_2\}$, *by* Corollary 5.1.2, $X(\Psi)$ *forms a tight frame for* $L^2(\mathbb{R})$. Figure 5.2 *illustrates* $\hat{\psi}_2$ *when* $g = p_{10}$.

Chapter 6

Bandlimited Biorthogonal Wavelets

In this chapter, we illustrate that it is rather simple to construct a pair of dual bandlimited biorthogonal Riesz wavelets. Our construction is via a good choice of the refinement mask and the wavelet mask. Note that conventional constructions involve carefully picking two refinement masks $\hat{a}$ and $\hat{\tilde{a}}$ so that they are dual to each other, and then followed by defining the wavelet masks as $\hat{b} := e^{-i\cdot}\overline{\hat{\tilde{a}}(\cdot+\pi)}$ and $\hat{\tilde{b}} := e^{-i\cdot}\overline{\hat{a}(\cdot+\pi)}$. Instead of doing that, we choose the masks $\hat{a}$ and $\hat{b}$ first in a particular way, and then define $\hat{\tilde{a}}$ and $\hat{\tilde{b}}$. We will see that the former method is a special case of the latter, and furthermore, there is much more freedom in constructing the desired wavelets. It is well known in the literature that Riesz wavelet bases are constructed from scaling functions with integer shifts that form a Riesz sequence. Thus, by the characterization of integer shifts of a bandlimited function forming a Riesz sequence in the second chapter, we will only consider functions from the set $\mathcal{A}_{\delta,\Omega}$, $0 < \delta < \Omega \leq 2\pi/3$ and $\Omega > \pi/2$, throughout this chapter.

It is well known that if one starts from a scaling function ϕ with orthonormal integer shifts, using the alternating flip formula

$$\hat{b}(\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)}, \tag{6.0.1}$$

and defining $\psi \in L^2(\mathbb{R})$ by

$$\hat{\psi}(\xi) := \hat{b}(\xi/2)\hat{\phi}(\xi/2), \tag{6.0.2}$$

and letting $\Psi := \{\psi\}$, then $X(\Psi)$ forms an orthonormal basis of $L^2(\mathbb{R})$. Moreover, the alternating flip formula in (6.0.1) gives us an explicit formulation of the wavelet functions in terms of their Fourier transforms. This motivates us to investigate whether the alternating flip formulae still works if one starts from a bandlimited scaling function lying in the Schwartz class with integer shifts forming a Riesz sequence instead. Indeed we will see in the following that this is true. We are ready to state our main result in this chapter.

Theorem 6.0.1 *For $\pi/3 \leq \delta < \Omega \leq 2\pi/3$, $\Omega > \pi/2$, $\pi/3 \leq \delta_0 < \Omega_0 \leq 2\pi/3$, $\Omega_0 > \pi/2$, let $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$ and $\hat{a}_0 \in \mathcal{A}_{\delta_0,\Omega_0} \cap C^k(\mathbb{R})$ where $k \geq 2$. Define*

$$\hat{b}(\xi) := e^{-i\xi}\hat{a}_0(\xi+\pi), \qquad \hat{\tilde{a}}(\xi) := \frac{\overline{\hat{b}(\xi+\pi)}}{\overline{\hat{d}(\xi)}}, \qquad \hat{\tilde{b}}(\xi) := -\frac{\overline{\hat{a}(\xi+\pi)}}{\overline{\hat{d}(\xi)}}, \tag{6.0.3}$$

where

$$\hat{d}(\xi) := \hat{a}(\xi)\hat{b}(\xi+\pi) - \hat{b}(\xi)\hat{a}(\xi+\pi), \tag{6.0.4}$$

and

$$\hat{\phi}(\xi) := \Big[\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)\Big]\mathbf{1}_{[-2\Omega,\,2\Omega]}(\xi), \quad \hat{\psi}(\xi) := \hat{b}(\xi/2)\hat{\phi}(\xi/2),$$

$$\hat{\tilde{\phi}}(\xi) := \Big[\prod_{j=1}^{\tilde{N}} \hat{\tilde{a}}(2^{-j}\xi)\Big]\mathbf{1}_{[-2\Omega_0,\,2\Omega_0]}(\xi), \quad \hat{\tilde{\psi}}(\xi) := \hat{\tilde{b}}(\xi/2)\hat{\tilde{\phi}}(\xi/2),$$

where N and $\tilde{N}$ are as defined in (2.1.7). *Then $\hat{\phi}$, $\hat{\tilde{\phi}}$, $\hat{\psi}$, $\hat{\tilde{\phi}} \in C^k(\mathbb{R})$, $\operatorname{supp}\hat{\psi} = [2(-\Omega_0+\pi),\, 4\Omega] \cup [-4\Omega,\, 2(\Omega_0-\pi)]$ and $\operatorname{supp}\hat{\tilde{\psi}} = [2(-\Omega+\pi),\, 4\Omega_0] \cup [-4\Omega_0,\, 2(\Omega-\pi)]$. Furthermore, for $\Psi = \{\psi\}$ and $\tilde{\Psi} = \{\tilde{\psi}\}$, the systems $X(\Psi)$ and $X(\tilde{\Psi})$ form a pair of bandlimited biorthogonal Riesz wavelet bases for $L^2(\mathbb{R})$.*

Corollary 6.0.1 *For $0 < \delta < \Omega \leq 2\pi/3$, $\Omega > \pi/2$, let $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$. Define ψ by the alternating flip formula* (6.0.1) *and* (6.0.2). *Let $\Psi := \{\psi\}$, then $X(\Psi)$ forms a Riesz basis for $L^2(\mathbb{R})$.*

Proof: We choose $\hat{a}_0 = \hat{a}$ and apply Theorem 6.0.1 to obtain the result. ■

Note that when $\hat{a}_0 = \hat{a}$ in Theorem 6.0.1, then we have the simplifications $\hat{b}(\xi) = e^{-i\xi}\hat{a}(\xi+\pi)$, $\hat{\tilde{a}}(\xi) = \hat{a}(\xi)\Big[\hat{a}^2(\xi)+\hat{a}^2(\xi+\pi)\Big]^{-1}$, $\hat{\tilde{b}}(\xi) = e^{i\xi}\hat{a}(\xi+\pi)\Big[\hat{a}^2(\xi)+\hat{a}^2(\xi+\pi)\Big]^{-1}$.

Define ψ, $\tilde{\psi}$, Ψ, $\tilde{\Psi}$ as in Theorem 6.0.1, then by Theorem 6.0.1, $X(\Psi)$ and $X(\tilde{\Psi})$ form a pair of biorthogonal Riesz bases for $L^2(\mathbb{R})$. We remark that Corollary 6.0.1 adds on another family of Riesz wavelets created by the alternating flip formula (6.0.1) to those families discussed in [15].

We mention that the masks defined in (6.0.1) and (6.0.2) are similarly used in [15] for constructing compactly supported Riesz wavelet bases. As such, the result above is very much inspired by [15]. Although biorthogonal wavelet theory is relatively well understood in the past decade, the focus has been on constructing compactly supported biorthogonal wavelets and not bandlimited ones. Therefore, we will need a considerable amount of preparatory work in the next two sections before we can prove Theorem 6.0.1.

6.1 Direct sum decompositions of $L^2(\mathbb{R})$

The general theory of biorthogonal wavelets makes some natural assumptions on the scaling function ϕ.

(A1) $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

(A2) $\hat{\phi}(0) = 1$ and $\hat{\phi}(2\pi k) = 0$, $k \in \mathbb{Z} \setminus \{0\}$.

(A3) There exist positive constants A and B such that

$$A \leq [\hat{\phi}, \hat{\phi}] \leq B \qquad \text{a.e.}$$

(A4) There exists a coefficient sequence $\{a_k\}_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ such that the two-scale relation holds, i.e.

$$\phi(x) = \sum_{k\in\mathbb{Z}} a_k \phi(2x - k) \qquad \text{a.e.}$$

We shall show later that the assumptions made in Theorem 6.0.1 will always ensure ϕ and $\tilde{\phi}$ to satisfy properties (A1) to (A4). Define $V_0 := \overline{\text{span}\ \{\phi(\cdot - k) : k \in \mathbb{Z}\}}$, then certainly the integer shifts of ϕ form a Riesz basis for V_0. Thus by a simple dilation argument, $\{\phi_{1,k}\}_{k\in\mathbb{Z}}$ forms a Riesz basis for $V_1 := \overline{\text{span}\{\phi(2 \cdot -k) : k \in \mathbb{Z}\}}$. We note that property (A4) assures us that the mask $\hat{a}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{ik\xi}$ of ϕ is a continuous function due to the Weierstrass M-test. The following is a theory of scaling functions with two-scale sequences to be in $\ell^1(\mathbb{Z})$. The reader should note that most of the results for the rest of this section are quoted from [6] without proof and that [6] uses assumptions (A1) to (A4) in the development of biorthogonal wavelet theory.

Definition 6.1.1 *A Laurent series is said to belong to the* Wiener class $\mathcal{W}$ *if its coefficient sequence is in* $\ell^1(\mathbb{Z})$.

Since the discrete convolution of two $\ell^1(\mathbb{Z})$-sequences is again a sequence in $\ell^1(\mathbb{Z})$, this makes $\mathcal{W}$ into an algebra. The truth is that $\mathcal{W}$ is even more than an algebra, as seen in the following well-known theorem due to N. Wiener.

Theorem 6.1.1 *Let* $f \in \mathcal{W}$ *and suppose that* $f(z) \neq 0$ *for all* z *on the unit circle* $|z| = 1$. *Then* $\frac{1}{f} \in \mathcal{W}$ *as well.*

In this context, the Laurent series involved take the form

$$p(z) = \sum_{k\in\mathbb{Z}} p_k z^k = \sum_{k\in\mathbb{Z}} p_k e^{-ik\xi} = \hat{p}(\xi),$$

where $z = e^{-i\xi}$, and

$$p(-z) = \sum_{k\in\mathbb{Z}} p_k (-z)^k = \sum_{k\in\mathbb{Z}} p_k e^{ik(\xi+\pi)} = \hat{p}(\xi + \pi).$$

Although in [6], the z-symbol is used for representing the Fourier series, we shall use notations like $\hat{p}(\xi) = \sum_{k\in\mathbb{Z}} p_k e^{-ik\xi}$ throughout the chapter. Let ϕ be a scaling function whose mask

$$\hat{a}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}$$

is in $\mathcal{W}$. Recall that $\hat{a}$ governs the relation of $V_0 \subset V_1$ in the sense that

$$\phi(x) = \sum_{k\in\mathbb{Z}} a_k \phi(2x - k),$$

and the integer shifts of ϕ generate V_0. Let us now consider any other $\ell^1(\mathbb{Z})$-sequence $\{b_k\}_{k\in\mathbb{Z}}$ and its Fourier series

$$\hat{b}(\xi) = \sum_{k\in\mathbb{Z}} b_k e^{-ik\xi}.$$

Then $\hat{b}$ is also in $\mathcal{W}$ and defines a function

$$\psi(x) := \sum_{k\in\mathbb{Z}} b_k \phi(2x-k) \tag{6.1.1}$$

in V_1. This function ψ also generates a closed subspace W_0 in the same manner as ϕ generates V_0, namely:

$$W_0 := \overline{\text{span }\{\psi(\cdot - k) \,:\, k \in \mathbb{Z}\}}. \tag{6.1.2}$$

Hence, analogous to the function $\hat{a}$, the function $\hat{b}$ governs the relation $W_0 \subset V_1$ in the sense that (6.1.1) and (6.1.2) are satisfied.

Our main concern in the construction of wavelets is at least to ensure that V_0 and W_0 are complementary subspaces of V_1, in the sense that

$$V_0 \cap W_0 = \{0\} \qquad \text{and} \qquad V_1 = V_0 + W_0, \tag{6.1.3}$$

which means

$$V_1 = V_0 \dot{+} W_0, \tag{6.1.4}$$

and this notation will be used in place of (6.1.3). In the following we will see that the matrix

$$M_{\hat{a},\hat{b}}(\xi) := \begin{bmatrix} \hat{a}(\xi) & \hat{b}(\xi) \\ \hat{a}(\xi+\pi) & \hat{b}(\xi+\pi) \end{bmatrix} \tag{6.1.5}$$

plays an essential role in characterizing (6.1.4). Hence, we must consider the determinant $\Delta_{\hat{a},\hat{b}}(\xi) := \det M_{\hat{a},\hat{b}}(\xi)$ of the matrix in (6.1.5). Since $\hat{a}$, $\hat{b}$ are in $\mathcal{W}$ and $\mathcal{W}$ is an algebra, we have $\Delta_{\hat{a},\hat{b}} \in \mathcal{W}$ as well. In addition, if $\Delta_{\hat{a},\hat{b}}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, then by Theorem 6.1.1, we also have $\frac{1}{\Delta_{\hat{a},\hat{b}}} \in \mathcal{W}$. So under the condition $\Delta_{\hat{a},\hat{b}}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, the two functions

$$\hat{\tilde{a}}(\xi) = \overline{\frac{\hat{b}(\xi+\pi)}{\Delta_{\hat{a},\hat{b}}(\xi)}}, \qquad \hat{\tilde{b}}(\xi) = \overline{\frac{\hat{a}(\xi+\pi)}{\Delta_{\hat{a},\hat{b}}(\xi)}} \tag{6.1.6}$$

are both in the Wiener class $\mathcal{W}$. The reason for considering the functions $\hat{\tilde{a}}$ and $\hat{\tilde{b}}$ is that the resulting matrix $M_{\hat{\tilde{a}},\hat{\tilde{b}}}$ is the inverse of $M_{\hat{a},\hat{b}}$, namely, for any $\xi \in \mathbb{R}$,

$$M_{\hat{a},\hat{b}}(\xi) M^*_{\hat{\tilde{a}},\hat{\tilde{b}}}(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = M^*_{\hat{\tilde{a}},\hat{\tilde{b}}}(\xi) M_{\hat{a},\hat{b}}(\xi). \tag{6.1.7}$$

The first identity in (6.1.7) is equivalent to the pair of identities

$$\hat{a}(\xi)\overline{\hat{\tilde{a}}(\xi)} + \hat{b}(\xi)\overline{\hat{\tilde{b}}(\xi)} \equiv 1, \qquad \hat{a}(\xi)\overline{\hat{\tilde{a}}(\xi+\pi)} + \hat{b}(\xi)\overline{\hat{\tilde{b}}(\xi+\pi)} \equiv 0, \tag{6.1.8}$$

while the second identity in (6.1.7) is equivalent to the following set of four identities:

$$\begin{cases} \hat{a}(\xi)\overline{\hat{\tilde{a}}(\xi)} + \hat{a}(\xi+\pi)\overline{\hat{\tilde{a}}(\xi+\pi)} \equiv 1, & \hat{a}(\xi)\overline{\hat{\tilde{b}}(\xi)} + \hat{a}(\xi+\pi)\overline{\hat{\tilde{b}}(\xi+\pi)} \equiv 0, \\ \overline{\hat{\tilde{a}}(\xi)}\hat{b}(\xi) + \overline{\hat{\tilde{a}}(\xi+\pi)}\hat{b}(\xi+\pi) \equiv 0, & \hat{b}(\xi)\overline{\hat{\tilde{b}}(\xi)} + \hat{b}(\xi+\pi)\hat{\tilde{b}}(\xi+\pi) \equiv 1. \end{cases} \tag{6.1.9}$$

For $L^2(\mathbb{R})$ decomposition, we do not need the identities in (6.1.9). However, this set of identities will be crucial to our discussion of 'duality' in the next section.

Since $\hat{\tilde{a}}$, $\hat{\tilde{b}} \in \mathcal{W}$, we may write

$$\hat{\tilde{a}}(\xi) = \sum_{k\in\mathbb{Z}} \tilde{a}_k e^{-ik\xi}, \qquad \hat{\tilde{b}}(\xi) = \sum_{k\in\mathbb{Z}} \tilde{b}_k e^{-ik\xi}, \tag{6.1.10}$$

where $\{\tilde{a}_k\}_{k\in\mathbb{Z}}$, $\{\tilde{b}_k\}_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, whenever $\Delta_{\hat{a},\hat{b}}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. We are now ready to formulate the following decomposition result.

Theorem 6.1.2 *A necessary and sufficient condition for the direct-sum decomposition* (6.1.4) *to hold is that the function $\Delta_{\hat{a},\hat{b}}$ never vanishes on the real line $\mathbb{R}$. Furthermore, if $\Delta_{\hat{a},\hat{b}}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, then the family $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$, governed by $\hat{b}$ as in* (6.1.1)*, is a Riesz basis of W_0, and the decomposition relation*

$$\phi(2x-l) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \{\bar{\tilde{a}}_{l-2k}\phi(x-k) + \bar{\tilde{b}}_{l-2k}\psi(x-k)\}, \qquad l \in \mathbb{Z},$$

holds for all $x \in \mathbb{R}$ where $\{\tilde{a}_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are defined as in (6.1.10)*.*

Let us pause for a moment and comment on the decomposition of $L^2(\mathbb{R})$ via Theorem 6.1.2. Let $\Delta_{\hat{a},\hat{b}}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$ and define

$$W_j := \overline{\text{span}\ \{\psi(2^j \cdot -k) : k \in \mathbb{Z}\}}, \qquad j \in \mathbb{Z}.$$

Then in view of the definition of V_j, $j \in \mathbb{Z}$, and the assertion $V_1 = V_0 \dot{+} W_0$ in Theorem 6.1.2, we have

$$V_{j+1} = V_j \dot{+} W_j, \qquad j \in \mathbb{Z}.$$

Hence, since $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA of $L^2(\mathbb{R})$, it follows that the family $\{W_j\}_{j\in\mathbb{Z}}$ constitutes a direct sum decomposition of $L^2(\mathbb{R})$, namely,

$$L^2(\mathbb{R}) = \cdots \dot{+} W_{-1} \dot{+} W_0 \dot{+} \cdots$$

However, $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ may not constitute a Riesz basis of $L^2(\mathbb{R})$ if we just assume $\Delta_{\hat{a},\hat{b}}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Furthermore we recall that in any series representation

$$f(x) = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}(x), \qquad f \in L^2(\mathbb{R}),$$

we need a dual $\tilde{\psi}$ of ψ to extract any time-frequency information of f from the coefficients $c_{j,k}$.

6.2 Wavelets and their duals

We continue our discussion of the decomposition of $L^2(\mathbb{R})$ and extend our effort to ensure that the decompositions are 'wavelet decompositions'. We assume that $\hat{a}$, $\hat{b} \in \mathcal{W}$,

$$\hat{a}(0) = 1, \qquad \hat{a}(\pi) = 0, \qquad \hat{b}(0) = 0. \tag{6.2.1}$$

Let $\hat{\tilde{a}}$ and $\hat{\tilde{b}}$ be defined by (6.1.6). Then we have $\hat{\tilde{a}}$, $\hat{\tilde{b}} \in \mathcal{W}$ and the four trigonometric functions satisfy (6.1.8). Therefore it follows from this set of identities and (6.2.1) that

$$\hat{\tilde{a}}(0) = 1, \qquad \hat{\tilde{a}}(\pi) = 0. \tag{6.2.2}$$

The similarity between $\hat{a}$ and $\hat{\tilde{a}}$, as described by (6.2.1) and (6.2.2), suggests that

$$\hat{\tilde{a}}(\xi) = \sum_{k\in\mathbb{Z}} \tilde{a}_k e^{-ik\xi}$$

should also be chosen as the mask of some scaling function that generates a possibly different MRA of $L^2(\mathbb{R})$.

This motivates the following strategy for constructing wavelets and their duals. In our context, we will start from two 2π-periodic functions $\hat{a}$, $\hat{\tilde{a}}$ in $\mathcal{A}_{\delta,\Omega}$ and $\mathcal{A}_{\tilde{\delta},\tilde{\Omega}}$ respectively, where $0 < \delta < \Omega \leq 2\pi/3$, $0 < \tilde{\delta} < \tilde{\Omega} \leq 2\pi/3$ and Ω, $\tilde{\Omega} > \pi/2$. Define

ϕ and $\tilde{\phi}$ by $\hat{a}$ and $\hat{\tilde{a}}$ respectively as in (2.1.6). Then ϕ generates an MRA $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and $\tilde{\phi}$ generates another MRA $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$. According to Theorem 6.1.2, selecting any two 2π-periodic functions $\hat{b}$, $\hat{\tilde{b}}$ that satisfy

$$\Delta_{\hat{a},\hat{b}}(\xi) \neq 0 \text{ and } \Delta_{\hat{\tilde{a}},\hat{\tilde{b}}}(\xi) \neq 0 \qquad \text{for all } \xi \in \mathbb{R},$$

will result in two totally unrelated direct-sum decompositions of $L^2(\mathbb{R})$. In view of the discussion in the previous section, we will make use of the first identity in (6.1.9) to make a connection between these two decompositions.

Definition 6.2.1 *The two-scale masks $\hat{a}$ and $\hat{\tilde{a}}$ are said to be 'duals' of each other if they satisfy*

$$\hat{a}(\xi)\overline{\hat{\tilde{a}}(\xi)} + \hat{a}(\xi+\pi)\overline{\hat{\tilde{a}}(\xi+\pi)} \equiv 1. \tag{6.2.3}$$

Hence, if the two trigonometric functions $\hat{b}$, $\hat{\tilde{b}}$ are chosen so that the two nonsingular matrices $M_{\hat{a},\hat{b}}(\xi)$ and $M^*_{\hat{\tilde{a}},\hat{\tilde{b}}}(\xi)$ are inverses of each other for all $\xi \in \mathbb{R}$, (6.2.3) holds for the pair of functions $\hat{a}$ and $\hat{\tilde{a}}$. We remark that by (6.2.2) and the second identity in (6.1.9), the pair $(\hat{\tilde{a}}, \hat{\tilde{b}})$ satisfies the condition

$$\hat{\tilde{a}}(0) = 1, \qquad \hat{\tilde{a}}(\pi) = 0, \qquad \hat{\tilde{b}}(0) = 0$$

which is the same set of conditions as in (6.2.1).

Recall that the two masks $\hat{a}$ and $\hat{\tilde{a}}$ give rise to two scaling functions ϕ and $\tilde{\phi}$. Although ϕ and $\tilde{\phi}$ might generate two different MRAs of $L^2(\mathbb{R})$, they could still be related in the following sense.

Definition 6.2.2 *Two scaling functions ϕ and $\tilde{\phi}$, generating possibly different MRAs $\{V_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$ respectively of $L^2(\mathbb{R})$, are said to be* 'dual scaling functions'*, if they satisfy the condition*

$$\langle \phi(\cdot - j), \tilde{\phi}(\cdot - k)\rangle = \int_{-\infty}^{\infty} \phi(x-j)\overline{\tilde{\phi}(x-k)}\,dx = \delta_{jk}, \qquad j,\, k \in \mathbb{Z}.$$

We next obtain some important intermediate results about the functions given in Theorem 6.0.1, and $k \geq 2$.

Proposition 6.2.1 *Let* $\hat{a}$, $\hat{\tilde{a}}$, $\hat{b}$, $\hat{\tilde{b}}$, ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$, k *be as defined in* Theorem 6.0.1. *Then we have the following.*

(a) $\hat{a}$, $\hat{\tilde{a}}$, $\hat{b}$, $\hat{\tilde{b}}$ *are in* $C^k(\mathbb{R})$. *Consequently,* $\hat{a}$, $\hat{\tilde{a}} \in \mathcal{W}$.

(b) $\hat{a}$ *and* $\hat{\tilde{a}}$ *are dual to each other, and* $M_{\hat{a},\hat{b}}(\xi)M^*_{\hat{\tilde{a}},\hat{\tilde{b}}}(\xi) = I_{2\times 2} = M_{\hat{\tilde{a}},\hat{\tilde{b}}}(\xi)M^*_{\hat{a},\hat{b}}(\xi)$ *for all* $\xi \in \mathbb{R}$.

(c) ϕ, $\tilde{\phi}$ *belong to* $\mathcal{B}_{\delta,\Omega}$ *and* $\mathcal{B}_{\tilde{\delta},\Omega_0}$ *respectively with both* $\hat{\phi}$, $\hat{\tilde{\phi}} \in C^k(\mathbb{R})$, *where* $\tilde{\delta} = \min\{\delta_0, \delta, \pi - \min\{\Omega, \Omega_0\}\}$.

(d) ϕ *and* $\tilde{\phi}$ *are dual scaling functions and both* ϕ *and* $\tilde{\phi}$ *satisfy criteria* (A1)-(A4).

(e) ψ *and* $\tilde{\psi}$ *are bandlimited with* $\hat{\psi}$, $\hat{\tilde{\psi}} \in C^k(\mathbb{R})$.

Proof: For (a), clearly, $\hat{a}$, $\hat{b} \in C^k(\mathbb{R})$, $k \geq 2$. Denote the Fourier coefficients of the $L^2_{2\pi}$ functions $\hat{a}$, $\hat{\tilde{a}}$, $\hat{b}$, $\hat{\tilde{b}}$ to be $\{a_n\}_{n\in\mathbb{Z}}$, $\{\tilde{a}_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}}$, $\{\tilde{b}_n\}_{n\in\mathbb{Z}}$. Then there exists a constant $M > 0$ such that

$$|a_n|,\ |b_n| \leq M(1 + |n|)^{-2}$$

for all $n \in \mathbb{Z}$. Thus, $\hat{a}$, $\hat{b} \in \mathcal{W}$. To show that $\hat{\tilde{a}}$, $\hat{\tilde{b}} \in \mathcal{W}$, we first show that $\hat{\tilde{a}}$, $\hat{\tilde{b}} \in C^k(\mathbb{R})$, $k \geq 2$.

In view of Lemma 3.1.1, it suffices to show that

$$\hat{d}(\xi) = \hat{a}(\xi)\hat{b}(\xi + \pi) - \hat{b}(\xi)\hat{a}(\xi + \pi) = e^{-i\xi}\Big[\hat{a}(\xi)\hat{a}_0(\xi) + \hat{a}(\xi + \pi)\hat{a}_0(\xi + \pi)\Big]$$

never vanishes. Let $r(\xi) = \hat{a}(\xi)\hat{a}_0(\xi)$. Then $r \in \mathcal{A}_{\delta',\Omega'}$, where $0 < \delta' = \min\{\delta, \delta_0\}$, $\pi/2 < \Omega' = \min\{\Omega, \Omega_0\} \leq 2\pi/3$. Thus, by Theorem 2.2.1 and Corollary 2.2.3, since $\Omega' > \pi/2$, there exist constants A, $B > 0$ such that for all $\xi \in \mathbb{R}$, $A \leq r(\xi) + r(\xi + \pi) \leq B$. Thus $\hat{d}$ never vanishes and $\hat{\tilde{a}}(\xi) = \dfrac{\overline{\hat{b}(\xi + \pi)}}{\overline{\hat{d}(\xi)}} \in C^k(\mathbb{R})$. In addition, $\hat{\tilde{b}}(\xi) = -\dfrac{\overline{\hat{a}(\xi + \pi)}}{\overline{\hat{d}(\xi)}} \in C^k(\mathbb{R})$. Therefore $\hat{\tilde{a}}$, $\hat{\tilde{b}} \in \mathcal{W}$, completing the proof of (a).

By the π-periodicity of $\hat{d}$ and the definitions of $\hat{a}$, $\hat{b}$, $\hat{\tilde{a}}$, $\hat{\tilde{b}}$, it is easy to verify statement (b).

To show statement (c), it suffices to show that $\hat{\tilde{a}} \in \mathcal{A}_{\tilde{\delta},\Omega_0} \cap C^k(\mathbb{R})$ since we already know that $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$. Note that $\hat{\tilde{a}}(\xi) = \dfrac{\hat{a}_0(\xi)}{r(\xi) + r(\xi+\pi)}$. Since $r(\xi) + r(\xi + \pi)$ never vanishes, it is clear that $\hat{\tilde{a}}(\xi) = 0$, $\xi \in [-\pi, \pi]\backslash(-\Omega_0, \Omega_0)$. To establish $\hat{\tilde{a}}(\xi) = 1$, $\xi \in [-\tilde{\delta}, \tilde{\delta}]$, it suffices to show that $r(\xi) + r(\xi + \pi) = 1$, $\xi \in [-\tilde{\delta}, \tilde{\delta}]$. Let $q(\xi) = r(\xi)\mathbf{1}_{[-\pi,\pi]}(\xi)$. On the interval $[-\pi/2, \pi/2]$, we have

$$r(\xi) + r(\xi+\pi) = \begin{cases} q(\xi) + q(\xi+\pi), & \text{if } \xi \in [-\pi/2, \Omega' - \pi), \\ q(\xi), & \text{if } \xi \in [\Omega' - \pi, \pi - \Omega'], \\ q(\xi) + q(\xi - \pi), & \text{if } \xi \in (\pi - \Omega', \pi/2], \end{cases}$$

where $\pi/2 < \Omega' = \min\{\Omega, \Omega_0\} \leq 2\pi/3$. Since $r(\xi) = 1$ for $\xi \in [-\delta_0, \delta_0]$, we conclude that $r(\xi) = 1$ for $\xi \in [-\tilde{\delta}, \tilde{\delta}]$ where $\tilde{\delta} = \min\{\delta, \delta_0, \pi - \Omega'\}$. Thus $\hat{\tilde{a}}(\xi) = 1$ for $\xi \in [-\tilde{\delta}, \tilde{\delta}]$. Therefore $\hat{\tilde{a}} \in \mathcal{B}_{\tilde{\delta},\Omega_0} \cap C^k(\mathbb{R})$ giving us the desired result.

As for (d), it follows from Lemma 2.2.2 and part (b) that ϕ and $\tilde{\phi}$ are dual scaling functions. We shall see that criterion (A1)-(A4) are satisfied by both the scaling functions ϕ, $\tilde{\phi}$ defined in Theorem 6.0.1. Indeed from statement (c), we have $\hat{\phi}, \hat{\tilde{\phi}} \in C^k(\mathbb{R})$, $k \geq 2$. Then by Chapter 3, we see that there exists a constant $A > 0$ such that $|\phi(x)|, |\tilde{\phi}(x)| \leq A(1+|x|)^{-2}$ for every $x \in \mathbb{R}$, which ensures that $\phi, \tilde{\phi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, i.e. (A1) holds. Since $\phi \in \mathcal{B}_{\delta,\Omega}$ and $\tilde{\phi} \in \mathcal{B}_{\tilde{\delta},\Omega_0}$, where $0 < \delta < \Omega < \pi/3$, $0 < \tilde{\delta} < \Omega_0 \leq 2\pi/3$, it is clear that $\hat{\phi}(0) = 1 = \hat{\tilde{\phi}}(0)$ and $\hat{\phi}(2\pi k) = 0 = \hat{\tilde{\phi}}(2\pi k)$, $k \in \mathbb{Z} \setminus \{0\}$, giving (A2).

Statement (A3) follows from the characterization in Theorem 2.2.1 since $\Omega, \Omega_0 > \pi/2$. Statement (A4) is a consequence of the earlier result that $\phi, \tilde{\phi}$ are refinable with their respective masks $\hat{a}$ and $\hat{\tilde{a}} \in \mathcal{W}$.

Finally, for (e), since $\hat{a}, \hat{\tilde{a}}, \hat{b}, \hat{\tilde{b}}, \hat{\phi}, \hat{\tilde{\phi}} \in C^k(\mathbb{R})$, it follows from the definition of $\hat{\psi}$ and $\hat{\tilde{\psi}}$ in Theorem 6.0.1 that $\hat{\psi}, \hat{\tilde{\psi}} \in C^k(\mathbb{R})$. Since $\hat{\phi}, \hat{\tilde{\phi}}$ have compact support, it is not hard to see that $\hat{\psi}, \hat{\tilde{\psi}}$ must be compactly supported too, which means that $\psi, \tilde{\psi}$ are bandlimited. This completes the proof. ■

By considering the functions

$$\psi(x) = \sum_{k\in\mathbb{Z}} \tilde{b}_k \phi(2x-k), \qquad \tilde{\psi}(x) = \sum_{k\in\mathbb{Z}} b_k \tilde{\phi}(2x-k),$$

where

$$\hat{\tilde{b}}(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{b}_k e^{-ik\xi}, \qquad \hat{b}(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} b_k e^{-ik\xi},$$

and setting

$$\psi_{j,k} := 2^{\frac{j}{2}}\psi(2^j\cdot -k), \qquad \tilde{\psi}_{j,k} := 2^{\frac{j}{2}}\tilde{\psi}(2^j\cdot -k),$$

as well as

$$W_j := \overline{\operatorname{span}\{\psi_{j,k} \,:\, k\in\mathbb{Z}\}}, \qquad \tilde{W}_j := \overline{\operatorname{span}\{\tilde{\psi}_{j,k} \,:\, k\in\mathbb{Z}\}},$$

we have

$$V_{j+1} = V_j \dot{+} W_j, \qquad \tilde{V}_{j+1} = \tilde{V}_j \dot{+} \tilde{W}_j, \qquad j\in\mathbb{Z}.$$

Here, as usual, we set

$$V_j := \overline{\operatorname{span}\{\phi_{j,k} \,:\, k\in\mathbb{Z}\}}, \qquad \tilde{V}_j := \overline{\operatorname{span}\{\tilde{\phi}_{j,k} \,:\, k\in\mathbb{Z}\}},$$

where

$$\phi_{j,k} := 2^{\frac{j}{2}}\phi(2^j\cdot -k), \qquad \tilde{\phi}_{j,k} := 2^{\frac{j}{2}}\tilde{\phi}(2^j\cdot -k),$$

with ϕ and $\tilde{\phi}$ being the scaling functions whose masks $\hat{a}$ and $\hat{\tilde{a}}$ are defined in Theorem 6.0.1.

We shall next show that if these masks $\hat{a}$ and $\hat{\tilde{a}}$ are dual to each other, then not only are $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ and $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ dual to each other, but additional orthogonality properties are achieved as well.

Theorem 6.2.1 *Let $\hat{a}$, $\hat{\tilde{a}}$, $\hat{b}$, $\hat{\tilde{b}}$, ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ be defined as in* Theorem 6.0.1. *Then*

$$\langle \psi_{j,k}, \tilde{\psi}_{l,m}\rangle = \delta_{jl}\delta_{km}, \qquad j,\, k,\, l,\, m\in\mathbb{Z}, \tag{6.2.4}$$

and

$$\langle \phi_{j,k}, \tilde{\psi}_{j,l}\rangle = 0, \qquad \langle \tilde{\phi}_{j,k}, \psi_{j,l}\rangle = 0, \qquad j,\, k,\, l\in\mathbb{Z},$$

that is, $V_j \perp \tilde{W}_j$ and $\tilde{V}_j \perp W_j$ for all $j\in\mathbb{Z}$.

The proof follows word by word from [6] since under the hypothesis in Theorem 6.0.1, we have shown that the scaling functions ϕ and $\tilde{\phi}$ are dual to each other in

Proposition 6.2.1. As a consequence of the biorthogonality property in (6.2.4), both families are $\ell^2(\mathbb{Z}^2)$-linearly independent which we will show later. Therefore since

$$L^2(\mathbb{R}) = \cdots \dot{+} W_{-1} \dot{+} W_0 \dot{+} W_1 \dot{+} \cdots = \cdots \dot{+} \tilde{W}_{-1} \dot{+} \tilde{W}_0 \dot{+} \tilde{W}_1 \dot{+} \cdots,$$

both $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ and $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ are bases of $L^2(\mathbb{R})$. In fact, under the hypothesis of Theorem 6.0.1, it follows that both $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ and $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ are frames of $L^2(\mathbb{R})$ too. In the following section, we shall show that $X(\Psi)$ and $X(\tilde{\Psi})$ form frames for $L^2(\mathbb{R})$ and that coupled with $\ell^2(\mathbb{Z}^2)$-linear independence, we may conclude that $X(\Psi)$ and $X(\tilde{\Psi})$ each forms a Riesz basis of $L^2(\mathbb{R})$.

6.3 Frames, Riesz bases and linear independence

In this section, we shall present the details to show that indeed $X(\Psi)$ and $X(\tilde{\Psi})$ form a pair of dual frames and further, they form a pair of biorthogonal Riesz bases. We need to introduce a series of technical lemmas to achieve this aim. The reader should note that most of them are adapted from ideas in [8].

Proposition 6.3.1 *For $c > 0$, let $F_c(\xi) := \frac{1}{2\pi c}\hat{f}(\xi)\overline{\hat{\phi}(\frac{\xi}{c})}$ where $\xi \in \mathbb{R}$, and define the $2\pi c$-periodic function*

$$G_c(\xi) := \sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2\pi ck)\overline{\hat{\phi}(\frac{\xi + 2\pi ck}{c})},$$

where $\hat{f} \in L^2(\mathbb{R})$ and $\hat{\phi}$ is a compactly supported and bounded function on $\mathbb{R}$. Then the Fourier coefficients of G_c are given by $\hat{G}_c(n) = \langle f, \phi(c\cdot -n)\rangle_{L^2(\mathbb{R})} = \hat{F}_c(\frac{n}{c})$, $n \in \mathbb{Z}$. Furthermore, $G_c \in L^2[0, 2\pi c]$.

Proof: Consider

$$\hat{G}_c(n) = \frac{1}{2\pi c}\int_{-\pi c}^{\pi c} G_c(\xi)e^{-in\frac{\xi}{c}}\,d\xi = \frac{1}{2\pi c}\int_{-\pi c}^{\pi c}\sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2\pi ck)\overline{\hat{\phi}(\frac{\xi + 2\pi ck}{c})}e^{-in\frac{\xi}{j}}\,d\xi.$$

Now since $\hat{\phi}$ is compactly supported, so is the function $F_c(\cdot) = \hat{f}_c(\cdot)\hat{\phi}(\frac{\cdot}{c})$. As such, only a finite number of terms in the summation of the series is being considered in the interval $[-\pi c,\, \pi c]$. Thus we can interchange the sum and integral to give

$$\begin{aligned}
\hat{G}_c(n) &= \frac{1}{2\pi c}\sum_{k\in\mathbb{Z}}\int_{-\pi c}^{\pi c}\hat{f}(\xi+2\pi ck)\overline{\hat{\phi}(\frac{\xi+2\pi ck}{c})}e^{-in\frac{\xi}{c}}\,d\xi \\
&= \frac{1}{2\pi c}\sum_{k\in\mathbb{Z}}\int_{-\pi c+2\pi ck}^{\pi c+2\pi ck}\hat{f}(\xi')\overline{\hat{\phi}\,(\frac{\xi'}{c})}e^{-in\frac{\xi'-2\pi ck}{c}}\,d\xi' \\
&= \frac{1}{2\pi c}\sum_{k\in\mathbb{Z}}\int_{-\pi c+2\pi ck}^{\pi c+2\pi ck}\hat{f}(\xi)\overline{\hat{\phi}(\frac{\xi}{c})}e^{-in\frac{\xi}{c}}\,d\xi \\
&= \frac{1}{2\pi c}\int_{\mathbb{R}}\hat{f}(\xi)\overline{\hat{\phi}(\frac{\xi}{c})}e^{-in\frac{\xi}{c}}\,d\xi = \langle f,\, \phi(c\cdot -n)\rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

Therefore

$$\sum_{n\in\mathbb{Z}}|\hat{G}_c(n)|^2 = \sum_{n\in\mathbb{Z}}|\langle f,\, \phi(c\cdot -n)\rangle|^2 \le \sup_{\xi\in[0,\,2c\pi]}[\hat{\phi},\, \hat{\phi}](\frac{\xi}{c})\|f\|_{L^2}^2 \le M\|f\|_{L^2}^2 < \infty,$$

for $f \in L^2(\mathbb{R})$ since $\hat{\phi}$ is compactly supported and bounded. Since the Fourier coefficients of $G_c(\cdot)$ are square-summable for every $f \in L^2(\mathbb{R})$, we conclude that $G_c(\cdot) \in L^2[0,\, 2\pi c]$ for every $f \in L^2(\mathbb{R})$.

Note that $F \in L^1(\mathbb{R})$. Indeed,

$$\int_{\mathbb{R}}|F_c(\xi)|\,d\xi = \int_{\mathbb{R}}|\hat{f}(\xi)\hat{\phi}(\frac{\xi}{c})|\,d\xi \le \|\hat{f}\|_{L^2}\Big[\int_{\mathbb{R}}|\hat{\phi}(\frac{\xi}{c})|^2\,d\xi\Big]^{\frac{1}{2}} = c\|\hat{f}\|_{L^2}\|\hat{\phi}\|_{L^2} < \infty$$

since $\hat{f},\, \hat{\phi} \in L^2(\mathbb{R})$. Thus

$$\hat{F}_c(\frac{n}{c}) = \frac{1}{2\pi c}\int_{\mathbb{R}}\hat{f}(\xi)\overline{\hat{\phi}(\frac{\xi}{c})}e^{-in\frac{\xi}{c}}\,d\xi = \langle f,\, \phi(c\cdot -n)\rangle = \hat{G}_c(n).$$

This completes the proof. ■

Corollary 6.3.1 *Let $\phi,\, \tilde{\phi},\, f_1,\, f_2 \in L^2(\mathbb{R})$ such that $\hat{\phi},\, \hat{\tilde{\phi}}$ are compactly supported and bounded. Then for $j \in \mathbb{N}$, we have*

$$\sum_{k\in\mathbb{Z}}\langle f_1,\, \phi_{j,k}\rangle\,\langle\tilde{\phi}_{j,k},\, f_2\rangle = 2\pi\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\hat{f}_1(\xi)\overline{\hat{\phi}(\frac{\xi}{2^j})}\hat{\tilde{\phi}}(\frac{\xi}{2^j}+2\pi k)\overline{\hat{f}_2(\xi+2\pi 2^j k)}\,d\xi,$$

for all $f_1,\, f_2 \in L^2(\mathbb{R})$.

Proof: Set $c = 2^j$ in Proposition 6.3.1 and let $F_j^1(\xi) := \frac{1}{2\pi 2^j}\hat{f}_1(\xi)\overline{\hat{\phi}(\frac{\xi}{2^j})}$ and $F_j^2(\xi) := \frac{1}{2\pi 2^j}\hat{f}_2(\xi)\overline{\hat{\tilde{\phi}}(\frac{\xi}{2^j})}$ and $G_j^1(\xi) := 2\pi 2^j \sum_{k\in\mathbb{Z}} F_j^1(\xi + 2\pi 2^j k)$, $G_j^2(\xi) := 2\pi 2^j \sum_{k\in\mathbb{Z}} F_j^2(\xi + 2\pi 2^j k)$, where $\hat{f}_1,\ \hat{f}_2 \in L^2(\mathbb{R})$ and $\hat{\phi},\ \hat{\tilde{\phi}}$ are compactly supported and bounded functions. Note that by Proposition 6.3.1, $G_j^1,\ G_j^2 \in L^2[0, 2\pi 2^j]$ functions. Then by Proposition 6.3.1, and Parseval's identity for $L^2[0, 2\pi 2^j]$ functions,

$$\sum_{n\in\mathbb{Z}} \langle f_1,\ \phi_{j,n}\rangle\langle\tilde{\phi}_{j,n},\ f_2\rangle = \sum_{n\in\mathbb{Z}} \hat{G}_j^1(n)\overline{\hat{G}_j^2(n)} = 2\pi\langle G_j^1,\ G_j^2\rangle_{L^2[0,\,2\pi 2^j]}. \tag{6.3.1}$$

On the other hand,

$$\begin{aligned}&\int_0^{2\pi 2^j} \Big(\sum_{k\in\mathbb{Z}} \hat{f}_1(\xi + 2\pi 2^j k)\overline{\hat{\phi}(\frac{\xi + 2\pi 2^j k}{2^j})}\Big)\Big(\sum_{l\in\mathbb{Z}} \overline{\hat{f}_2(\xi + 2\pi 2^j l)}\hat{\tilde{\phi}}(\frac{\xi + 2\pi 2^j l}{2^j})\Big)\, d\xi \\ &= \langle G_j^1,\ G_j^2\rangle_{L^2[0,\,2\pi 2^j]} = \int_0^{2\pi j} G_j^1(\xi)\overline{G_j^2(\xi)}\, d\xi\end{aligned} \tag{6.3.2}$$

Since $\hat{\phi},\ \hat{\tilde{\phi}}$ are compactly supported, both series in the integral consist only of finite number of terms on the interval $[0,\ 2\pi 2^j]$ and thus the integral and summations can be interchanged.

Hence

$$\begin{aligned}&\langle G_j^1,\ G_j^2\rangle_{L^2[0,\,2\pi 2^j]} \\ =\ &\sum_{k\in\mathbb{Z}} \int_0^{2\pi 2^j} \sum_{l\in\mathbb{Z}} \hat{f}_1(\xi + 2\pi 2^j k)\overline{\hat{\phi}(\frac{\xi + 2\pi 2^j k}{2^j})}\,\overline{\hat{f}_2(\xi + 2\pi 2^j l)}\hat{\tilde{\phi}}(\frac{\xi + 2\pi 2^j l}{2^j})\, d\xi \\ =\ &\sum_{k\in\mathbb{Z}} \int_{2\pi 2^j k}^{2\pi 2^j + 2\pi 2^j k} \Big(\sum_{l\in\mathbb{Z}} \overline{\hat{f}_2(\xi - 2\pi 2^j k + 2\pi 2^j l)}\hat{\tilde{\phi}}(\frac{\xi - 2\pi 2^j k + 2\pi 2^j l}{2^j})\Big)\overline{\hat{\phi}(\frac{\xi}{2^j})}\hat{f}_1(\xi)\, d\xi \\ =\ &\sum_{k\in\mathbb{Z}} \int_{2\pi 2^j k}^{2\pi 2^j + 2\pi 2^j k} \Big(\sum_{l\in\mathbb{Z}} \overline{\hat{f}_2(\xi + 2\pi 2^j l)}\hat{\tilde{\phi}}(\frac{\xi + 2\pi 2^j l}{2^j})\Big)\overline{\hat{\phi}(\frac{\xi}{2^j})}\hat{f}_1(\xi)\, d\xi \\ =\ &\sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \overline{\hat{f}_2(\xi + 2\pi 2^j l)}\hat{\tilde{\phi}}(\frac{\xi + 2\pi 2^j l}{2^j})\overline{\hat{\phi}(\frac{\xi}{2^j})}\hat{f}_1(\xi)\, d\xi.\end{aligned}$$

This completes the proof of the corollary. ■

Finally we show that

Lemma 6.3.1 *Let $\hat{a}$ and $\hat{a}_0$, ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ be defined as in* Theorem 6.0.1. *Then for all $f_1,\ f_2 \in L^2(\mathbb{R})$,*

$$\sum_{k\in\mathbb{Z}} \langle f_1,\, \phi_{1,k}\rangle\langle\tilde{\phi}_{1,k},\, f_2\rangle = \sum_{l\in\mathbb{Z}} \Big(\langle f_1,\, \phi_{0,l}\rangle\langle\tilde{\phi}_{0,l},\, f_2\rangle + \langle f_1,\, \psi_{0,l}\rangle\langle\tilde{\psi}_{0,l},\, f_2\rangle\Big).$$

Proof: Firstly, ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$ are well-defined in $L^2(\mathbb{R})$ because $\hat{\phi}$, $\hat{\tilde{\phi}}$, $\hat{\psi}$, $\hat{\tilde{\psi}}$ have compact support and are bounded. Next, we show that the sum defined above do make sense as well. The reader can refer to [5] that given $f,\, \varphi \in L^2(\mathbb{R})$, the infinite series $\sum_{k\in\mathbb{Z}} |\langle f,\, \varphi_{0,k}\rangle|^2$ is bounded if and only if the 2π-periodic function $\sum_{l\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\pi l)|^2$ is bounded above a.e. As established earlier, $[\hat{\phi}, \hat{\phi}]$, $[\hat{\tilde{\phi}}, \hat{\tilde{\phi}}]$, $[\hat{\tilde{\psi}}, \hat{\tilde{\psi}}]$, $[\hat{\psi}, \hat{\psi}]$ must be bounded above a.e. Then by a simple application of the Cauchy-Schwartz inequality, all the terms stated above in the lemma make sense.

Set $j = 1$ in Corollary 6.3.1, we get

$$\sum_{k\in\mathbb{Z}} \langle f_1,\, \phi_{1,k}\rangle\langle\tilde{\phi}_{1,k},\, f_2\rangle = 2\pi \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_1(\xi)\overline{\hat{\phi}(\frac{\xi}{2})}\hat{\tilde{\phi}}(\frac{\xi}{2} + 2\pi k)\overline{\hat{f}_2(\xi + 4\pi k)}\, d\xi.$$

Setting $j = 0$ in Corollary 6.3.1 for ϕ, $\tilde{\phi}$ and ψ, $\tilde{\psi}$ gives

$$\sum_{l\in\mathbb{Z}} \langle f_1,\, \phi_{0,l}\rangle\langle\tilde{\phi}_{0,l},\, f_2\rangle = 2\pi \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_1(\xi)\overline{\hat{\phi}(\frac{\xi}{2})}\hat{\tilde{\phi}}(\xi + 2\pi l)\overline{\hat{f}_2(\xi + 2\pi l)}\, d\xi,$$

$$\sum_{l\in\mathbb{Z}} \langle f_1,\, \psi_{0,l}\rangle\langle\tilde{\psi}_{0,l},\, f_2\rangle = 2\pi \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_1(\xi)\overline{\hat{\psi}(\frac{\xi}{2})}\hat{\tilde{\psi}}(\xi + 2\pi l)\overline{\hat{f}_2(\xi + 2\pi l)}\, d\xi.$$

Thus

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}} \Big(\langle f_1,\, \phi_{0,l}\rangle\langle\tilde{\phi}_{0,l},\, f_2\rangle + \langle f_1,\, \psi_{0,l}\rangle\langle\tilde{\psi}_{0,l},\, f_2\rangle\Big) \\
&= 2\pi \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_1(\xi)\overline{\hat{f}_2(\xi + 2\pi l)}\Big[\overline{\hat{\phi}(\xi)}\hat{\tilde{\phi}}(\xi + 2\pi l) + \overline{\hat{\psi}(\xi)}\hat{\tilde{\psi}}(\xi + 2\pi l)\Big]\, d\xi \\
&= 2\pi \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_1(\xi)\overline{\hat{f}_2(\xi + 4\pi k)}\Big[\overline{\hat{\phi}(\xi)}\hat{\tilde{\phi}}(\xi + 4\pi k) + \overline{\hat{\psi}(\xi)}\hat{\tilde{\psi}}(\xi + 4\pi k)\Big]\, d\xi \\
&\quad + 2\pi \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_1(\xi)\overline{\hat{f}_2(\xi + 2\pi + 4\pi k)}\Big[\overline{\hat{\phi}(\xi)}\hat{\tilde{\phi}}(\xi + 2\pi + 4\pi k) + \overline{\hat{\psi}(\xi)}\hat{\tilde{\psi}}(\xi + 2\pi + 4\pi k)\Big]\, d\xi.
\end{aligned}$$

Using the two-scale relation of ϕ, $\tilde{\phi}$, ψ, $\tilde{\psi}$, we have

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}}\Big(\langle f_1,\,\phi_{0,l}\rangle\langle\tilde{\phi}_{0,l},\,f_2\rangle+\langle f_1,\,\psi_{0,l}\rangle\langle\tilde{\psi}_{0,l},\,f_2\rangle\Big)\\
&=2\pi\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\hat{f}_1(\xi)\overline{\hat{f}_2(\xi+4\pi k)}\Big[\overline{\hat{a}(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2})}\hat{\tilde{a}}(\frac{\xi}{2})\hat{\tilde{\phi}}(\frac{\xi}{2}+2\pi k)+\overline{\hat{b}(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2})}\hat{\tilde{b}}(\frac{\xi}{2})\hat{\tilde{\phi}}(\frac{\xi}{2}+2\pi k)\Big]\\
&\quad+2\pi\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\hat{f}_1(\xi)\overline{\hat{f}_2(\xi+2\pi+4\pi k)}\Big[\overline{\hat{a}(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2})}\hat{\tilde{a}}(\frac{\xi}{2}+\pi)\hat{\tilde{\phi}}(\frac{\xi}{2}+\pi+2\pi k)\\
&\qquad\qquad+\overline{\hat{b}(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2})}\hat{\tilde{b}}(\frac{\xi}{2}+\pi)\hat{\tilde{\phi}}(\frac{\xi}{2}+\pi+2\pi k)\Big]\\
&=2\pi\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\hat{f}_1(\xi)\overline{\hat{f}_2(\xi+4\pi k)}\overline{\hat{\phi}(\frac{\xi}{2})}\hat{\tilde{\phi}}(\frac{\xi}{2}+2\pi k)\Big[\overline{\hat{a}(\frac{\xi}{2})}\hat{\tilde{a}}(\frac{\xi}{2})+\overline{\hat{b}(\frac{\xi}{2})}\hat{\tilde{b}}(\frac{\xi}{2})\Big]\,d\xi\\
&\quad+2\pi\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\hat{f}_1(\xi)\overline{\hat{f}_2(\xi+2\pi+4\pi k)}\overline{\hat{\phi}(\frac{\xi}{2})}\hat{\tilde{\phi}}(\frac{\xi}{2}+\pi+2\pi k)\Big[\overline{\hat{a}(\frac{\xi}{2})}\hat{\tilde{a}}(\frac{\xi}{2}+\pi)+\overline{\hat{b}(\frac{\xi}{2})}\hat{\tilde{b}}(\frac{\xi}{2}+\pi)\Big]\,d\xi\\
&=2\pi\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\hat{f}_1(\xi)\overline{\hat{f}_2(\xi+4\pi k)}\overline{\hat{\phi}(\frac{\xi}{2})}\hat{\tilde{\phi}}(\frac{\xi}{2}+2\pi k)\,d\xi,
\end{aligned}$$

since $\overline{\hat{a}(\frac{\xi}{2})}\hat{\tilde{a}}(\frac{\xi}{2})+\overline{\hat{b}(\frac{\xi}{2})}\hat{\tilde{b}}(\frac{\xi}{2})\equiv 1$ and $\overline{\hat{a}(\frac{\xi}{2})}\hat{\tilde{a}}(\frac{\xi}{2}+\pi)+\overline{\hat{b}(\frac{\xi}{2})}\hat{\tilde{b}}(\frac{\xi}{2}+\pi)\equiv 0$. This completes the proof. ■

Next, we prove a lemma to show that $X(\Psi)$ and $X(\widetilde{\Psi})$ are complete and Bessel in $L^2(\mathbb{R})$.

Lemma 6.3.2 *Under the assumptions in* Theorem 6.0.1, *we have for all* $f_1,\ f_2\in L^2(\mathbb{R})$,

$$\sum_{j,k\in\mathbb{Z}}\langle f_1,\psi_{j,k}\rangle\langle\widetilde{\psi}_{j,k},f_2\rangle=\langle f_1,f_2\rangle. \tag{6.3.3}$$

Proof: We note that the proof is almost entirely the same as that provided by Cohen, Daubechies and Feauveau in [8] for compactly supported biorthogonal wavelets. However, we will highlight parts of the proof where the arguments could be simplified due to the bandlimited assumptions of Theorem 6.0.1.

We shall show that the left-hand side of (6.3.3) makes sense by proving $\sum_{j,k\in\mathbb{Z}}|\langle f_1,\psi_{j,k}\rangle|^2$ and $\sum_{j,k\in\mathbb{Z}}|\langle\widetilde{\psi}_{j,k},\ f_2\rangle|^2$ are bounded for all $f_1,\ f_2\in L^2(\mathbb{R})$. Indeed, by Parseval's identity and the Poisson's summation formula, we have the following. By letting $\phi = \psi = \tilde{\phi}$ in (6.3.1) and (6.3.2),

$$\sum_{k\in\mathbb{Z}} |\langle f_1, \psi_{j,k}\rangle|^2 = \int_0^{2\pi 2^{-j}} |\sum_{l\in\mathbb{Z}} \hat{f}_1(\xi + 2\pi 2^{-j} l)\overline{\hat{\psi}(2^j\xi + 2\pi l)}|^2 \, d\xi.$$

Applying the Cauchy-Schwartz inequality,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} |\langle f_1, \psi_{j,k}\rangle|^2 \\
&= 2\pi \int_0^{2\pi 2^{-j}} |\sum_{l\in\mathbb{Z}} (\hat{f}_1(\xi + 2\pi l 2^{-j})(\overline{\hat{\psi}(2^j\xi + 2\pi l)})^{\delta}(\overline{\hat{\psi}(2^j\xi + 2\pi l)})^{1-\delta}|^2 \, d\xi \\
&\le 2\pi \int_0^{2\pi 2^{-j}} [\sum_{l\in\mathbb{Z}} |\hat{\psi}(2^j\xi + 2\pi l)|^{2(1-\delta)}][\sum_{l\in\mathbb{Z}} |\hat{f}_1(\xi + 2\pi l 2^{-j})|^2 |\hat{\psi}(2^j\xi + 2\pi l)|^{2\delta}] \, d\xi \\
&= 2\pi \int_{\mathbb{R}} |\hat{f}_1(\xi)|^2 |\hat{\psi}(2^j\xi)|^{2\delta} \sum_{m\in\mathbb{Z}} |\hat{\psi}(2^j\xi + 2\pi m)|^{2(1-\delta)} \, d\xi
\end{aligned} \tag{6.3.4}$$

for any $\delta \in (0, 1)$.

Since $\hat{\psi}$ is compactly supported in $[-8\pi/3,\, -2\pi/3] \cup [2\pi/3,\, 8\pi/3]$ and bounded, we have

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} |\hat{\psi}(2^j\xi + 2\pi m)|^{2(1-\delta)} &\le \sum_{m\in\mathbb{Z}} M^{2(1-\delta)} \mathbf{1}_{[-8\pi/3,\, -2\pi/3]\cup[2\pi/3,\, 8\pi/3]}(\xi + 2\pi m) \\
&\le \sum_{m\in\mathbb{Z}} M^2 \mathbf{1}_{[-8\pi/3,\, -2\pi/3]\cup[2\pi/3,\, 8\pi/3]}(\xi + 2\pi m) \le 4M^2
\end{aligned} \tag{6.3.5}$$

for all ξ and for all $\delta \in (0, 1)$.

Then

$$\begin{aligned}
\sum_{j,k\in\mathbb{Z}} |\langle f_1, \psi_{j,k}\rangle|^2 &\le 4\pi M^2 \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\hat{f}_1(\xi)|^2 |\hat{\psi}(2^j\xi)|^{2\delta} \, d\xi = 4\pi M^2 \int_{\mathbb{R}} |\hat{f}_1(\xi)|^2 \sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^{2\delta} \, d\xi \\
&\le 4\pi M^2 \int_{\mathbb{R}} |\hat{f}_1(\xi)|^2 \sum_{j\in\mathbb{Z}} M^{2\delta} \mathbf{1}_{[-8\pi/3,\, -2\pi/3]\cup[2\pi/3,\, 8\pi/3]}(2^j\xi) \, d\xi \\
&\le 4\pi M^2 \int_{\mathbb{R}} |\hat{f}_1(\xi)|^2 M^2 \sum_{j\in\mathbb{Z}} \mathbf{1}_{[-8\pi/3,\, -2\pi/3]}(2^j\xi) + \mathbf{1}_{[2\pi/3,\, 8\pi/3]}(2^j\xi) \, d\xi \\
&\le 8\pi M^4 \int_{\mathbb{R}} |\hat{f}_1(\xi)|^2 \, d\xi = 4M^4 \|f\|_2^2
\end{aligned} \tag{6.3.6}$$

Similarly, there exists a constant $B > 0$ such that

$$\sum_{j,k\in\mathbb{Z}} |\langle f_2, \tilde{\psi}_{j,k}\rangle|^2 \leq B\|f_2\|_2^2$$

for all $f_2 \in L^2(\mathbb{R})$. ■

Next, we have the strong L^2 convergence result by Cohen, Daubechies and Feauveau in [8]. As the proof holds perfectly true under our assumptions, we shall not prove it here. The interested reader can refer to [8] for the proof.

Lemma 6.3.3 *Under the assumptions of* Theorem 6.0.1, *we have, for all* $f \in L^2(\mathbb{R})$,

$$\lim_{J,K\to\infty} \sum_{|j|\leq J, |k|\leq K} \langle f, \psi_{j,k}\rangle \widetilde{\psi}_{j,k} = \lim_{J,K\to\infty} \sum_{|j|\leq J, |k|\leq K} \langle f, \widetilde{\psi}_{j,k}\rangle \psi_{j,k} = f$$

where the limits are in the strong L^2 *topology.*

Under our assumptions, we note that both $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ and $\{\widetilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ constitute a frame in $L^2(\mathbb{R})$. The upper bound is simply the Bessel bound proved earlier and the lower bound follows from the following argument:

$$\begin{aligned}
\|f\| = \sup_{\|g\|=1} |\langle f,\, g\rangle| &\leq \sup_{\|g\|=1} \sum_{j,k\in\mathbb{Z}} |\langle f,\, \psi_{j,k}\rangle||\langle \widetilde{\psi}_{j,k},\, g\rangle| \\
&\leq (\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2)^{1/2} \sup_{\|g\|=1} (\sum_{j,k\in\mathbb{Z}} |\langle \widetilde{\psi}_{j,k},\, g\rangle|^2)^{1/2} \\
&\leq C(\sum_{j,k\in\mathbb{Z}} |\langle f,\, \psi_{j,k}\rangle|^2)^{1/2}.
\end{aligned}$$

We need the following theorem from [6] to establish that $X(\Psi)$ and $X(\tilde{\Psi})$ both form Riesz bases of $L^2(\mathbb{R})$.

Theorem 6.3.2 *Let* $\psi \in L^2(\mathbb{R})$. *Then the following two statements are equivalent.*

(a) $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ *is a Riesz basis of* $L^2(\mathbb{R})$.

(b) $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ *is a frame of* $L^2(\mathbb{R})$, *and is also an* $\ell^2(\mathbb{Z}^2)$*-linearly independent family, in the sense that if* $\sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k} = 0$ *and* $\{c_{j,k}\}_{j,k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$, *then* $c_{j,k} = 0$ *for all* $j,\, k \in \mathbb{Z}$. *Furthermore, the Riesz bounds and frame bounds agree.*

So, in order to show that the $\{\psi_{j,k}\}_{j,k}$ and $\{\widetilde{\psi}_{j,k}\}_{j,k}$ constitute dual Riesz bases, we therefore only need to establish $\ell^2(\mathbb{Z}^2)$-linear independence.

Lemma 6.3.4 *Let ϕ, $\widetilde{\phi}$, ψ, $\widetilde{\psi}$ be as defined in* Theorem 6.0.1. *Then each of $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$, respectively $\{\widetilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$, are $\ell^2(\mathbb{Z}^2)$-linearly independent if and only if*

$$\langle \psi_{j,k}, \widetilde{\psi}_{j',k'} \rangle = \delta_{jj'}\delta_{kk'}. \tag{6.3.7}$$

Lemma 6.3.5 *Let ϕ, $\widetilde{\phi}$, ψ, $\widetilde{\psi}$ be as defined in* Theorem 6.0.1. *A sufficient condition for* (6.3.7) *to hold is*

$$\langle \phi_{0,k}, \widetilde{\phi}_{0,l} \rangle = \delta_{jl}. \tag{6.3.8}$$

It is noted that these two results above hold even without our bandlimited assumption in Theorem 6.0.1. The reader can refer to [8] for details. Finally, we will prove Theorem 6.0.1.

Proof of Theorem 6.0.1 In view of Lemmas 6.3.1, 6.3.2, 6.3.3, 6.3.4 and Theorem 6.3.2, to show that $X(\Psi)$ and $X(\tilde{\Psi})$ each form a Riesz basis of $L^2(\mathbb{R})$, it suffices to establish the condition (6.3.8) in Lemma 6.3.5. By Proposition 6.2.1, ϕ and $\tilde{\phi}$ are dual scaling functions and thus (6.3.8) is satisfied.

Since $\operatorname{supp}\hat{a}(\cdot/2+\pi) = \bigcup_{l\in\mathbb{Z}}[2(-\Omega+\pi+2\pi l),\, 2(\Omega+\pi+2\pi l)]$, $\operatorname{supp}\hat{a}_0(\cdot/2+\pi) = \bigcup_{l\in\mathbb{Z}}[2(-\Omega_0+\pi+2\pi l),\, 2(\Omega_0+\pi+2\pi l)]$, $\operatorname{supp}\hat{\phi}(\cdot/2) = [-4\Omega,\, 4\Omega]$, $\operatorname{supp}\hat{\tilde{\phi}}(\cdot/2) = [-4\Omega_0,\, 4\Omega_0]$ and $\pi/2 < \Omega,\, \Omega_0 \le 2\pi/3$, we conclude that $\operatorname{supp}\hat{\psi} = [2(-\Omega_0+\pi),\, 4\Omega] \cup [-4\Omega,\, 2(\Omega_0-\pi)]$ and $\operatorname{supp}\hat{\tilde{\psi}} = [2(-\Omega+\pi),\, 4\Omega_0] \cup [-4\Omega_0,\, 2(\Omega-\pi)]$.

Combining all the above results gives Theorem 6.0.1. It is noted that the alternating flip formulae belong to a special case of Theorem 6.0.1 where we select $\hat{a}_0 = \hat{a}$. ■

It was brought to the author's attention that an application of results in a very recent paper [16], can yield us Theorem 6.0.1. However the approach taken in [16] is very different and its proofs rather technical. Therefore it still seems natural that we adapt proofs from standard biorthogonal wavelet theory to suit our purposes.

Alternatively, we could use Corollary 4.18 in [21] to show that under the setting posed in Theorem 6.0.1, $X(\Psi)$ and $X(\tilde{\Psi})$ defined by $\Psi = \{\psi\}$ and $\tilde{\Psi} = \{\tilde{\psi}\}$ form a pair of biorthogonal Riesz bases in $L^2(\mathbb{R})$. To this end, we verify the conditions of Corollary 4.18 in [21]. Firstly, we check that the integer shifts of ϕ and $\tilde{\phi}$ form a Riesz system. This is verified by using the characterization proved in Theorem 2.2.1 and the assumptions that $\Omega,\ \Omega_0 > \pi/2$ and $\hat{a},\ \hat{\tilde{a}} \in C^2(\mathbb{R})$. Secondly, we show that $\langle \phi(\cdot - j),\ \tilde{\phi}(\cdot - j') \rangle = \delta_{j,j'}$ which is equivalent to $[\hat{\phi},\ \hat{\tilde{\phi}}] \equiv 1$ by Proposition 2.2.1. Indeed, by construction of the masks in Theorem 6.0.1, we have $\hat{a}\hat{\tilde{a}}(\cdot) + \hat{a}\hat{\tilde{a}}(\cdot + \pi) \equiv 1$. Then we apply Lemma 2.2.2 to get $[\hat{\phi},\ \hat{\tilde{\phi}}] \equiv 1$. Thirdly, it follows easily from the definition of ψ and $\tilde{\psi}$ in Theorem 6.0.1 that $X(\Psi)$ and $X(\tilde{\Psi})$ are two affine systems constructed by the square version of the Mixed Extension Principle. Lastly, it is required to show that $X(\Psi)$ and $X(\tilde{\Psi})$ are Bessel sets. We apply Proposition 2.6 in Bin Han's paper [14] to see that there exist constants $M_1,\ M_2,\ \tilde{M}_1,\ \tilde{M}_2 > 0$ such that $\sum_{l \in \mathbb{Z}} |\hat{\psi}(\xi + 2\pi l)| \le M_1$, $\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)| \le M_2$ and $\sum_{l \in \mathbb{Z}} |\hat{\tilde{\psi}}(\xi + 2\pi l)| \le \tilde{M}_1$, $\sum_{j \in \mathbb{Z}} |\hat{\tilde{\psi}}(2^j \xi)| \le \tilde{M}_2$. Using arguments in (6.3.4), (6.3.5) and (6.3.6), these inequalities imply that $X(\Psi)$ and $X(\tilde{\Psi})$ are Bessel sets. Hence Corollary 4.18 in [21] gives the result.

Let us now focus our discussion on consequences of Theorem 6.0.1. We shall see that the Meyer's wavelets is indeed a very special case resulting from Theorem 6.0.1. However, this theorem does not include the classical Shannon's wavelets due to the $C^k(\mathbb{R})$, $k \ge 2$, restriction required.

Lastly we have a result which roughly says that a pair of dual scaling functions $\hat{\phi} \in \mathcal{B}_{\delta,\Omega}$ and $\hat{\tilde{\phi}} \in \mathcal{B}_{\delta_0,\Omega_0}$ where $\hat{\tilde{a}}$ is defined by $\hat{a}$ and $\hat{a}_0$ in Theorem 6.0.1, cannot have both interpolatory properties and good regularity simultaneously. Furthermore, it says that if a bandlimited scaling function ϕ resulting from $\hat{a} \in \mathcal{A}_{\delta,\Omega}$, is both interpolatory and has orthonormal shifts, then it cannot have good regularity. Precisely, we have the following.

Proposition 6.3.1 *Suppose that $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ with $\pi/3 \le \delta \le \Omega \le 2\pi/3$, $\Omega \ge \pi/2$, is an interpolatory mask. Define $\hat{\tilde{a}} := \dfrac{\hat{a}_0}{\hat{d}_0}$ where $\hat{a}_0 \in \mathcal{A}_{\delta_0,\Omega_0}$, $0 < \delta_0 \le \Omega_0 \le 2\pi/3$, with*

$$\hat{d}_0(\cdot) = \hat{a}(\cdot)\hat{a}_0(\cdot) + \hat{a}(\cdot + \pi)\hat{a}_0(\cdot + \pi).$$

If $\hat{\tilde{a}}$ is also an interpolatory mask, then $\hat{a}$ is the 2π-periodic extension of the characteristic function $\mathbf{1}_{[-\pi/2,\,\pi/2]}$ and supp $\hat{\tilde{a}} = [-\pi/2,\ \pi/2]$. *In particular, suppose $|\hat{a}(\cdot)|^2 +$*

$|\hat{a}(\cdot+\pi)|^2 \equiv 1$ *and set* $\hat{a}_0 = \hat{a}$*, then* $\hat{\tilde{a}}$ *is an interpolatory mask and* $\hat{a}$ *is the* 2π*-periodic extension of the function* $\mathbf{1}_{[-\pi/2,\,\pi/2]}$*.*

Proof: Note that $\operatorname{supp}\hat{a}\mathbf{1}_{[-\pi,\,\pi]} = [-\Omega,\,\Omega]$ and $\operatorname{supp}\hat{\tilde{a}}\mathbf{1}_{[-\pi,\,\pi]} = [-\Omega_0,\,\Omega_0]$. Then for $\hat{a} \in \mathcal{A}_{\delta,\Omega}$ to be an interpolatory mask, it is necessary that

$$A \le \hat{a}^2(\xi) + \hat{a}^2(\xi+\pi) \le B$$

a.e by Proposition 3.2.1. Then by the characterization provided in Theorem 2.2.1, this means that either $\Omega > \pi/2$ or $\Omega = \pi/2$ and $\lim\limits_{\xi\to\pi/2^-} \hat{a}(\xi) > 0$.

If $\hat{\tilde{a}}$ is an interpolatory mask, then

$$1 = \hat{\tilde{a}}(\cdot) + \hat{\tilde{a}}(\cdot+\pi) = \frac{\hat{a}_0(\cdot) + \hat{a}_0(\cdot+\pi)}{\hat{a}\hat{a}_0(\cdot) + \hat{a}\hat{a}_0(\cdot+\pi)}$$

which gives $\hat{a}(\cdot)\hat{a}_0(\cdot)+\hat{a}(\cdot+\pi)\hat{a}_0(\cdot+\pi) = \hat{a}_0(\cdot)+\hat{a}_0(\cdot+\pi)$ a.e. Since $\hat{a}(\cdot)+\hat{a}(\cdot+\pi) \equiv 1$, multiplying to the right-hand side of the above equation, we get

$$\begin{aligned} \hat{a}(\cdot)\hat{a}_0(\cdot) + \hat{a}(\cdot+\pi)\hat{a}_0(\cdot+\pi) &= [\hat{a}_0(\cdot) + \hat{a}_0(\cdot+\pi)][\hat{a}(\cdot) + \hat{a}(\cdot+\pi)] \\ = \hat{a}(\cdot)\hat{a}_0(\cdot) + \hat{a}_0(\cdot)\hat{a}(\cdot+\pi) &+ \hat{a}(\cdot)\hat{a}_0(\cdot+\pi) + \hat{a}(\cdot+\pi)\hat{a}_0(\cdot+\pi). \end{aligned}$$

Therefore

$$\hat{a}_0(\cdot)\hat{a}(\cdot+\pi) + \hat{a}(\cdot)\hat{a}_0(\cdot+\pi) \equiv 0. \tag{6.3.9}$$

Next, since $\hat{\tilde{a}}$ is also an interpolatory mask, similarly, we have either $\Omega_0 > \pi/2$ or $\Omega_0 = \pi/2$ and $\lim\limits_{\xi\to\pi/2^-} \hat{\tilde{a}}(\xi) > 0$. This would mean that $\hat{d}_0$ never vanishes and thus $\hat{\tilde{a}}$ is well defined and is continuous everywhere except possibly at the points $\pm\Omega_0$.

On the other hand, since $\hat{a}$, $\hat{a}_0$, $\hat{a}(\cdot+\pi)$, $\hat{a}(\cdot+\pi) \ge 0$, for (6.3.9) to hold, $\hat{a}_0(\xi)\hat{a}(\xi+\pi) = 0 = \hat{a}(\xi)\hat{a}_0(\xi+\pi)$ a.e which implies that $\operatorname{supp}\hat{a}_0 \cap \operatorname{supp}\hat{a}_0(\cdot+\pi)$ is of measure zero and $\operatorname{supp}\hat{a} \cap \operatorname{supp}\hat{a}(\cdot+\pi)$ is also of measure zero. Since all these functions are 2π-periodic, we need only to consider the behaviour of these functions in the fundamental interval $[-\pi,\,\pi]$. Thus we have

$$\operatorname{supp}\hat{a}\mathbf{1}_{[-\pi,\,\pi]} = [-\Omega,\,\Omega], \qquad \operatorname{supp}\hat{a}_0\mathbf{1}_{[-\pi,\,\pi]} = [-\Omega_0,\,\Omega_0],$$

$$\operatorname{supp}\hat{a}(\cdot+\pi)\mathbf{1}_{[-\pi,\,\pi]} = [-\pi,\,\Omega-\pi] \cup [\pi-\Omega,\,\pi],$$

$$\operatorname{supp} \hat{a}_0(\cdot + \pi)\mathbf{1}_{[-\pi,\,\pi]} = [-\pi,\, \Omega_0 - \pi] \cup [\pi - \Omega_0,\, \pi].$$

Then $\operatorname{supp} \hat{a}_0 \mathbf{1}_{[-\pi,\,\pi]} \cap \operatorname{supp} \hat{a}(\cdot + \pi)\mathbf{1}_{[-\pi,\,\pi]}$ is a null set if and only if $\Omega_0 \leq \pi - \Omega$. Likewise, $\operatorname{supp} \hat{a}\mathbf{1}_{[-\pi,\,\pi]} \cap \operatorname{supp} \hat{a}_0(\cdot + \pi)\mathbf{1}_{[-\pi,\,\pi]}$ is a null set if and only if $\Omega \leq \pi - \Omega_0$. Thus both coincide to give the condition $\Omega + \Omega_0 \leq \pi$. Since $\Omega,\, \Omega_0 \geq \pi/2$, we must have $\Omega = \pi/2 = \Omega_0$.

Thus $\operatorname{supp} \hat{a} \cap \operatorname{supp} \hat{a}(\cdot + \pi)$ is a null set. On the interval $[-\pi/2,\, \pi/2]$,

$$1 = \hat{a}(\xi) + \hat{a}(\xi + \pi) = \hat{a}(\xi).$$

Since $\operatorname{supp} \hat{a}\mathbf{1}_{[-\pi,\,\pi]} = [-\Omega,\, \Omega] = [-\pi/2,\, \pi/2]$, $\hat{a}$ must be the 2π-periodic extension of the characteristic function $\mathbf{1}_{[-\pi/2,\,\pi/2]}$.

In particular, if $\hat{a}$ is further assumed to satisfy the CQF condition, setting $\hat{a}_0 = \hat{a}$ gives

$$\hat{\tilde{a}}(\xi) + \hat{\tilde{a}}(\xi + \pi) = \Big[\hat{a}^2(\xi) + \hat{a}^2(\xi + \pi)\Big]^{-1}\Big[\hat{a}(\xi) + \hat{a}(\xi + \pi)\Big] \equiv 1.$$

Thus, $\hat{\tilde{a}}$ is an interpolatory mask as well. Then by the first part of this proposition, we have $\hat{a}$ is the 2π-periodic extension of the characteristic function $\mathbf{1}_{[-\pi/2,\,\pi/2]}$. ∎

6.4 Explicit constructions

In this section, we consider the construction of two families of bandlimited biorthogonal wavelets. The first family of wavelets is defined as follows. Define $I := [-\Omega,\, -\delta]$, $J := [\delta,\, \Omega]$, $I' := [-\pi/2 - \epsilon/2,\, -\pi/2 + \epsilon/2]$, $J' := [\pi/2 - \epsilon/2, \pi/2 + \epsilon/2]$. For $m \in \mathbb{N}$, let $\hat{a}_m$ be the 2π-periodic extension of the bell function $b_{\delta,\Omega,m} = b^m_{I,J}$, $0 < \delta < \Omega \leq 2\pi/3$, $2\delta \geq \Omega$, $\Omega > \pi/2$, where $b_{I,J}$ is as defined in Theorem 4.1.1. Let $\hat{a}_0$ be the 2π-periodic extension of the bell function $b_\epsilon = b_{I',J'}$, $0 < \epsilon \leq \pi/3$. We can either set the function $g = q_{1,1}$ as defined in Theorem 4.1.3 to get $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^\infty(\mathbb{R})$ or for $k \geq 1$, set $g = p_k$ where p_k is as defined in Theorem 4.1.2 to get $\hat{a} \in \mathcal{A}_{\delta,\Omega} \cap C^k(\mathbb{R})$ and $\hat{\tilde{a}} \in \mathcal{A}_{\pi/2-\epsilon/2,\pi/2+\epsilon/2} \cap C^\infty(\mathbb{R})$. Lastly, we impose $\pi/2 + \epsilon/2 \leq \delta$ to obtain:

$$\hat{a}(\xi)\overline{\hat{\tilde{a}}(\xi)} + \hat{a}(\xi + \pi)\overline{\hat{\tilde{a}}(\xi + \pi)} = \hat{\tilde{a}}(\xi) + \hat{\tilde{a}}(\xi + \pi) = 1$$

for all $\xi \in \mathbb{R}$. This is because $\hat{a}(\xi) = 1$ for $\xi \in \bigcup_{l\in\mathbb{Z}}[-\delta + 2\pi l,\, \delta + 2\pi l]$, and supp $\hat{\tilde{a}} \subseteq \bigcup_{l\in\mathbb{Z}}[-\delta + 2\pi l,\, \delta + 2\pi l]$ since $\pi/2 + \epsilon/2 \le \delta$.

In view of the definitions of $\hat{a}$, $\hat{\tilde{a}}$, $\hat{b}$, $\hat{\tilde{b}}$, $\hat{d}$ in Theorem 6.0.1, we have the following simplifications:

$$\hat{b}(\xi) = e^{-i\xi}\overline{\hat{a}_0(\xi + \pi)},$$

$$\hat{d}(\xi) = -e^{-i\xi}\Big[\hat{a}(\xi)\hat{a}_0(\xi) + \hat{a}(\xi + \pi)\hat{a}_0(\xi + \pi)\Big] = -e^{-i\xi} \neq 0,$$

$$\hat{\tilde{a}}(\xi) = \hat{a}_0(\xi), \qquad \hat{\tilde{b}}(\xi) = e^{-i\xi}\hat{a}(\xi + \pi).$$

Thus, not only are the functions $\hat{a}$, $\hat{\tilde{a}}$ dual to each other, $\hat{\tilde{a}}$ is also an interpolatory mask. Since $2\delta \ge \Omega$ and $2(\pi/2 - \epsilon/2) \ge \pi/2 + \epsilon/2$ whenever $\epsilon \le \pi/3$, we have $\lceil \log_2(\frac{\Omega}{\delta}) \rceil = 1$ and

$$\hat{\phi}(\xi) = \Big[\prod_{j=1}^{N} \hat{a}(2^{-j}\xi)\Big]\mathbf{1}_{[-2\Omega,\, 2\Omega]}(\xi) = b_{\delta,\Omega,m}(\xi/2),$$

and similarly, $\lceil \log_2(\frac{\tilde{\Omega}}{\tilde{\delta}}) \rceil = 1$,

$$\hat{\tilde{\phi}}(\xi) = b_\epsilon(\xi/2).$$

By applying Theorem 6.0.1, we conclude that $X(\Psi)$ and $X(\tilde{\Psi})$ form a pair of biorthogonal Riesz wavelet bases of $L^2(\mathbb{R})$. Similar to the justification of (4.2.4) ψ and $\tilde{\psi}$ are given in terms of their Fourier transforms as

$$\begin{aligned}\hat{\psi}(\xi) = \hat{b}(\xi/2)\hat{\phi}(\xi/2) &= e^{-i\xi/2}\hat{\tilde{a}}(\xi/2 + \pi)\hat{\phi}(\xi/2) \\ &= e^{-i\xi/2}\Big[b_\epsilon(\xi/2) + b_\epsilon(\xi/2 + \pi)\Big]b_{\delta,\Omega,m}(\xi/4),\end{aligned}$$

$$\begin{aligned}\hat{\tilde{\psi}}(\xi) = \hat{\tilde{b}}(\xi/2)\hat{\tilde{\phi}}(\xi/2) &= e^{-i\xi/2}\hat{a}(\xi/2 + \pi)\hat{\tilde{\phi}}(\xi/2) \\ &= e^{-i\xi/2}\Big[b_{\delta,\Omega,m}(\xi/2) + b_{\delta,\Omega,m}(\xi/2 + \pi)\Big]b_\epsilon(\xi/4).\end{aligned}$$

The second family of wavelets illustrates the usefulness of the alternating flip formula. For $m \in \mathbb{N}$, let $\hat{a}_m$ be the 2π-periodic extension of the bell function b_ϵ^m, $0 < \epsilon \le \pi/3$ and $g = q_{1,1} \in C^\infty(\mathbb{R})$ where $b_{I,J}$ is as defined in Theorem 4.1.1. Let $\hat{a}_0 = \hat{a}_m$ in Theorem 6.0.1. Then $\hat{b}_m(\xi) = e^{-i\xi}\overline{\hat{a}_m(\xi + \pi)}$. Since $\hat{a}_m \in \mathcal{A}_{\pi/2-\epsilon/2,\pi/2+\epsilon/2} \cap C^\infty(\mathbb{R})$ and $\pi/2 + \epsilon/2 > \pi/2$ for all $0 < \epsilon \le \pi/3$, according to Theorem 6.0.1, $X(\Psi)$ and $X(\tilde{\Psi})$

form a pair of biorthogonal dual Riesz wavelet bases of $L^2(\mathbb{R})$, where ψ, $\tilde{\psi}$ are defined by $\hat{a}_m$ as in Theorem 6.0.1.

In particular, when $m = 1$, $\hat{d}(\xi) = -e^{-i\xi}\Big[\hat{a}_1^2(\xi) + \hat{a}_1^2(\xi+\pi)\Big] = e^{-i\xi}$ and $\hat{a}_1$ satisfies the CQF condition. We can check that $\psi = \tilde{\psi}$ and $X(\Psi)$ alone already forms an orthonormal wavelet basis for $L^2(\mathbb{R})$. In fact, ψ is the Meyer's wavelets described in Chapter 4 Theorem 4.2.2 where the CQF condition has already been verified. So we will not discuss the justification here.

When $m = 2$, we have $\hat{a}_2(\xi) + \hat{a}_2(\xi+\pi) \equiv 1$. In view of the previous results established and the alternating flip formula, ψ_2 is an interpolatory wavelet and similar to the justification of (4.2.4), ψ_2 is given by

$$\hat{\psi}(\xi) = \hat{b}(\xi/2)\hat{\phi}(\xi/2) = e^{-i\xi/2}\Big[b_\epsilon^2(\xi/2-\pi) + b_\epsilon^2(\xi/2+\pi)\Big]b_\epsilon^2(\xi/4).$$

Furthermore, we have the following.

$$\hat{a}^2(\xi) + \hat{a}^2(\xi+\pi) = \sum_{l\in\mathbb{Z}} b_\epsilon^4(\xi+\pi l),$$

$$\hat{\tilde{\phi}}(\xi) = \Big[\prod_{j=1}^{N} \hat{\tilde{a}}(2^{-j}\xi)\Big]\mathbf{1}_{[-\pi-\epsilon,\,\pi+\epsilon]}(\xi) = \Big[\frac{\hat{a}(\xi/2)}{\sum_{l\in\mathbb{Z}} b_\epsilon^4(\xi/2+\pi l)}\Big]\mathbf{1}_{[-\pi-\epsilon,\,\pi+\epsilon]}(\xi) = \frac{b_\epsilon^2(\xi/2)}{\sum_{l\in\mathbb{Z}} b_\epsilon^4(\xi/2+\pi l)}$$

and

$$\begin{aligned}\hat{\tilde{\psi}}(\xi) &= e^{-i\xi/2}\frac{\hat{a}(\xi/2+\pi)}{\sum_{l\in\mathbb{Z}} b_\epsilon^4(\xi/2+\pi l)}\hat{\tilde{\phi}}(\xi/2)\\ &= e^{-i\xi/2}\frac{b_\epsilon^2(\xi/2-\pi)+b_\epsilon^2(\xi/2+\pi)}{\sum_{l\in\mathbb{Z}} b_\epsilon^4(\xi/2+\pi l)}\,\frac{b_\epsilon^2(\xi/4)}{\sum_{l\in\mathbb{Z}} b_\epsilon^4(\xi/4+\pi l)}.\end{aligned}$$

Although the dual wavelets look a little complicated, they could still be described explicitly. Again, if we choose the function g associated with the bell functions to be in $C^\infty(\mathbb{R})$, then both ψ and $\tilde{\psi} \in S$.

Example 6.4.1 *Based on the above construction, we choose* $\epsilon = \pi/3$, $m = 2$, $g = p_{10} \in C^{10}(\mathbb{R})$ *where* p_{10} *is as defined in* Theorem 4.1.2, *then* $\hat{\psi}(\xi) = e^{-i\xi/2}\Big[b_{\frac{\pi}{3}}^2(\xi/2-\pi) + b_{\frac{\pi}{3}}^2(\xi/2+\pi)\Big]b_{\frac{\pi}{3}}^2(\xi/4)$, $\hat{\tilde{\psi}}(\xi) = e^{-i\xi/2}\frac{b_{\frac{\pi}{3}}^2(\xi/2-\pi)+b_{\frac{\pi}{3}}^2(\xi/2+\pi)}{\sum_{l\in\mathbb{Z}} b_{\frac{\pi}{3}}^4(\xi/2+\pi l)}\frac{b_{\frac{\pi}{3}}^2(\xi/4)}{\sum_{l\in\mathbb{Z}} b_{\frac{\pi}{3}}^4(\xi/4+\pi l)}$. *Furthermore,* $\hat{\psi}$, $\hat{\tilde{\psi}} \in C^{10}(\mathbb{R})$ *and* $\operatorname{supp}\hat{\psi} = [2\pi/3,\, 8\pi/3] \cup [-8\pi/3,\, -2\pi/3] = \operatorname{supp}\hat{\tilde{\psi}}$. Figure 6.1 *displays the plots of* $|\hat{\psi}|$ *and* $|\hat{\tilde{\psi}}|$.

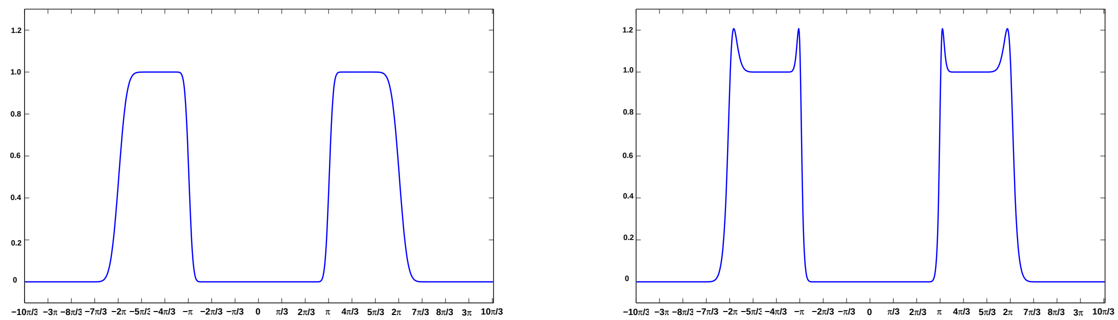

Figure 6.1: Graphs of $|\hat{\psi}|$ and $|\hat{\tilde{\psi}}|$.

6.5 Concluding remarks

This thesis only investigates the construction of univariate bandlimited wavelets with dilation factor two. There is also some research done on bandlimited multidimensional multiwavelets in [12] and [17] but explicit constructions of such wavelets are not given readily. The author feels that much more work could be done in generalizing this work into a multidimensional and multiwavelet setting.

It has come to the author's attention that the existence of bandlimited wavelets with subexponential decay has been proved in [11]. However it is not easy to provide explicit examples of such wavelets for the time being.

When periodized, bandlimited wavelets give rise to trigonometric polynomial wavelets. The trigonometric polynomial structure may prove to be beneficial to signal processing on periodic images and signals. We like to add that there is some renewed interest in the theory of bandlimited wavelets, as Donoho and Raimondo had used the periodized two-dimensional Meyer's wavelets in deconvolution and image deblurring in [10]. It is also the wish of the author that the generalizations provided in this thesis will improve existing methods in applications.

Lastly, frames of local sine and cosine decompositions of $L^2(\mathbb{R})$ have not been investigated yet in the literature. These are areas which the author may like to carry out future research in.

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