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# ADAPTIVE CONTROL OF UNCERTAIN CONSTRAINED NONLINEAR SYSTEMS

**By**

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# Summary

Constraints are ubiquitous in physical systems, and manifest themselves as physical stoppages, saturation, as well as performance and safety specifications. Violation of the constraints during operation may result in performance degradation, hazards or system damage. Driven by practical needs and theoretical challenges, the rigorous handling of constraints in control design has become an important research topic in recent decades.

Motivated by this problem, this thesis investigates the use of Barrier Lyapunov Functions (BLFs) for the control of single-input single-output (SISO) nonlinear systems in strict feedback form with constraints in the output and states. Unlike conventional Lyapunov functions, which are well-defined over the entire domain, and radially unbounded for global stability, BLFs possess the special property of finite escape whenever its arguments approach certain limiting values. By ensuring boundedness of the BLFs along the system trajectories, we show that transgression of constraints is prevented, and this embodies the key basis of our control design methodology.

Starting with the simplest case where only the output is constrained, and with known control gain functions, we employ backstepping design with BLF in the first step, and quadratic functions in the remaining steps. It is shown that asymptotic output tracking is achieved without violation of constraint, and all closed-loop signals remain bounded, under a mild restriction on the initial output. Furthermore, we explore the use of asymmetric BLFs as a generalized approach that relaxes the restriction on the initial output. To tackle parametric uncertainties, adaptive versions of the controllers are presented. We provide a comparison study which shows that BLFs require less conservative initial conditions than Quadratic Lyapunov Functions (QLFs) in preventing violation of constraints.

The foregoing method is then extended to the case of full state constraints by employing BLFs in every step of backstepping design. Besides the nominal case where full knowledge of the plant is available, we also tackle scenarios wherein parametric uncertainties are present. It is shown that state constraints cannot be arbitrarily specified, but are subject to feasibility conditions on the initial states and control parameters, which, if satisfied, guarantee asymptotic output tracking without violation of state constraints. In the case of partial state constraints, the design procedure is modified such that BLFs are used in only some of the steps of backstepping, and the feasibility conditions can be relaxed.

In the presence of uncertainty in the control gain functions, we employ domination design instead of the foregoing cancellation based approaches. Within this framework, sufficient conditions that prevent violation of constraints are established to accommodate stability analysis in the practical sense. When dealing with full state constraints, we show that practical output tracking is achieved subject to feasibility conditions on the initial states and control parameters. Additionally, it is shown that, for the special case of output constraint with linearly parameterized nonlinearities, practical output tracking is achieved free from the feasibility conditions.

Finally, we consider, as an application study, single degree-of-freedom uncertain electrostatic microactuators with bi-directional drive, wherein the control objective is to track a reference trajectory within the air gap without any physical contact between the electrodes. Besides the state feedback case, for which the foregoing method for dealing with output constraint can be applied, we also tackle the output feedback problem, and employ adaptive observer backstepping based on asymmetric BLF to ensure asymptotic output tracking without violation of output constraint.

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# Notation

| | |
|---|---|
| $\mathbb{R}$ | Field of real numbers |
| $\mathbb{R}_+$ | Set of non-negative real numbers |
| $\mathbb{R}^n$ | Linear space of $n$-dimensional vectors with elements in $\mathbb{R}$ |
| $\mathbb{R}^{n\times m}$ | Set of $(n \times m)$-dimensional matrices with elements in $\mathbb{R}$ |
| $\lvert a \rvert$ | Absolute value of the scalar $a$; |
| $\lVert x \rVert$ | Euclidean norm of the vector $x$ |
| $\mathbf{A}^T$ | Transpose of the matrix $\mathbf{A}$ |
| $\mathbf{A}^{-1}$ | Inverse of the matrix $\mathbf{A}$ |
| $I_n$ | Identity matrix of dimension $n \times n$ |
| $\lambda_{\min}(\mathbf{A})$ | Minimum eigenvalue of the matrix $\mathbf{A}$ where all eigenvalues are real |
| $\lambda_{\max}(\mathbf{A})$ | Maximum eigenvalue of the matrix $\mathbf{A}$ where all eigenvalues are real |
| $x_i$ | $i$-th element of the vector $x$ |
| $a_{ij}$ | $ij$-th element of the matrix $\mathbf{A}$ |
| $(\hat{\cdot})$ | Estimate of $(\cdot)$ |
| $(\tilde{\cdot})$ | $(\hat{\cdot}) - (\cdot)$ |
| $\sup \alpha(t)$ | Smallest number that is larger than or equal to the maximum value of $\alpha(t)$ |
| $\inf \alpha(t)$ | Largest number that is smaller than or equal to the minimum value of $\alpha(t)$ |
| $\mathrm{diag}(\cdots)$ | Diagonal matrix with the given diagonal elements |
| $\mathrm{blockdiag}(\cdots)$ | Block matrix in which the diagonal blocks are the given square matrices, and the blocks off the diagonal are the zero matrices |
| $f(x;k)$ | Value of the function $f$ at $x$ with parameter $k$ |
| $f : A \to B$ | $f$ maps the domain $A$ into the codomain $B$ |
| $\dot{f}, \dot{\mathbf{A}}$ | Time derivative of the scalar/vector function $f$ or the matrix function $\mathbf{A}$, both defined on $\mathbb{R}$ |
| $a \in A$ | $a$ is an element of the set $A$ |
| $A \subset B$ | Set $A$ is contained in the set $B$ |
| $(a, b)$ | Open subset of the real line |
| $[a, b]$ | Closed subset of the real line |
| $[a, b)$ | Subset of the real line closed at $a$ and open at $b$ |
| $a \to b$ | $a$ tends to $b$ |

# Chapter 1

# Introduction

Adaptive control has progressed through a colorful history to become an established field in modern control that is well-recognized and intensely researched today. Originally motivated by autopilot design for high performance aircraft, which need to deal with large system parameter variations during changing flight conditions, research in adaptive control witnessed a surge in the early 1950s, only to be undermined, albeit momentarily, by an incident with a test flight. With rapid advances in stability theory and the progress of control theory in the 1960s, in part driven by the due discovery of A.M. Lyapunov's pioneering works on stability of motion, understanding of adaptive control grew at a tremendous rate and contributed to the revived interest in the field. After almost three decades of research, a significant breakthrough was made in the form of backstepping design methodology, which overcame many technical restrictions suffered by adaptive controllers and greatly widened their applicability to new classes of systems, including nonlinear ones. Today, although adaptive control and backstepping are considered mature, they are still being actively researched to solve new problems in theory and applications. One such problem involves the consideration of system constraints in adaptive control of uncertain nonlinear systems, which is not only theoretically challenging, particularly in finding ways to contain the effects of the transient adaptation dynamics, but also practically meaningful in face of the ubiquity of constraints in physical and engineering systems.

In the remainder of this chapter, we provide a detailed exposition of the background and motivation, as well as the objectives, scope, and structure of the research presented in this thesis. For clarity of presentation, the background and motivation are separated into four parts, namely Lyapunov Based Control Design, Adaptive Control and Backstepping, Control of Constrained Systems, as well as Control of Microelectromechanical Systems (MEMs). In each part, the related works and background knowledge that motivate the research in this thesis are discussed in detail.

## 1.1 Background and Motivation

### 1.1.1 Lyapunov Based Control Design

Lyapunov's direct method, first introduced in 1892 by A.M. Lyapunov in his seminal work "*The General Problem of Motion Stability*" [109], has, in modern times, become the most important tool in the analysis and control design for nonlinear systems. Based on an analogy with the notion of energy in physical systems, the direct method provides a means of determining stability without the need for explicit knowledge of system solutions, by constructing a scalar "energy-like" function, also known as a Lyapunov function, and then analyzing the properties of its derivative with respect to time. Specifically, for a system represented as follows:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{1.1}$$

where we consider the origin $x = 0$ as an equilibrium, if there exists a positive definite, continuously differentiable function, $V(x)$, such that its derivative along the system trajectories is negative semidefinite, i.e. $\dot{V}(x) \leq 0$, then the origin is (locally) stable, and $V(x)$ is a Lyapunov function. If $V(x)$ is radially unbounded, then global stability can be concluded [156]. The technique is not restricted to the analysis of system stability *per se*, but can also be extended to design controllers that attribute, to the closed loop systems, desirable stability properties, via the concept of Control Lyapunov Functions (CLFs), introduced in [5]. The task of selecting a Lyapunov function candidate, followed by the design of the control law that renders the derivative of the candidate function negative semidefinite along the system trajectories, is, in general, non-trivial, for even if a stabilizing control law exists, we may fail to find it due to an ill-chosen Lyapunov function candidate. On the other hand, once a

CLF is known, many methods can be employed to construct stabilizing control laws [39, 94, 152, 157].

For simplicity, quadratic functions are often proposed as Lyapunov function candidates, as described by the following form

$$V(x) = \frac{1}{2}x^T P x \tag{1.2}$$

where $P$ is a positive definite matrix. In fact, a significant portion of the literature on Lyapunov based control synthesis employs quadratic Lyapunov functions (QLFs). Although QLFs are convenient and often sufficient to solve a large variety of control problems, certain more difficult problems call for more sophisticated forms of Lyapunov functions. One of the most classical examples can be found in early works on control design for robotic manipulators, where energy-like functions were proposed, through physical insight and intuition, as Lyapunov functions described, for example, by the following form:

$$V(x) = \frac{1}{2}(\dot{x}^T M(x)\dot{x} + x^T P x) \tag{1.3}$$

where $M(\cdot)$ and $P$ are symmetric positive definite matrices, with $M(\cdot)$, in particular, being the inertia matrix for the manipulator. This insight paved the way for the proof of closed loop stability with traditional Proportional-Derivative (PD) controllers in a series of independent works [76, 86, 164, 172]. Since then, such physics-motivated approach of constructing Lyapunov functions, has been extended and demonstrated for stable control design in numerous works on mechanical systems [13, 14, 127, 128], spacecraft [108, 155], ocean vessels [37, 167], helicopters [50], and robotics systems [48, 101, 156].

Apart from physically motivated Lyapunov functions, other special forms of Lyapunov functions have also been introduced to handle unknown control gain functions, which are notoriously difficult to handle in adaptive control design. In particular, for the nonlinear system $\dot{x} = f(x) + g(x)u$, where $x \in \mathbb{R}$, $u \in \mathbb{R}$, $f(0) = 0$, and $g(x) \neq 0$ for all $x \in \mathbb{R}$, one can use certainty equivalent feedback linearization control $u = \frac{1}{\hat{g}(x)}(-\hat{f}(x) + v)$, where $\hat{f}(x)$ and $\hat{g}(x)$ are estimates of $f(x)$ and $g(x)$, and measures have to be taken to avoid controller singularity when $\hat{g}(x) = 0$. To avoid this problem, Integral Lyapunov Functions (ILFs), which can be described by the following form

$$V(x) = \int_0^x r \frac{\bar{g}(r)}{g(r)} dr, \tag{1.4}$$

where $\bar{g}(\cdot)$ is a known function satisfying $\bar{g}(\cdot) \geq g(\cdot)$, have been developed in [45, 42], based on the idea that when the derivative of the ILF is taken, the gain function preceding the (virtual) control is canceled reciprocally by an identical term in the ILF. Using this approach, semi-globally stable adaptive controllers have been constructed which elegantly avoids the controller singularity problem. An alternative choice of Lyapunov function is a quadratic-like function with reciprocal of the control gain function, specifically $V = x^2/g(x)$, which operates in a similar manner as ILFs via reciprocal cancelling of the control gain function, but require additional assumptions on the rate of growth of the control gain function [44]. Besides unknown control gain functions, it was shown that nonlinearly parameterized functions can also be handled by using ILFs [43].

Special functionals, known as Lyapunov-Krasovskii functionals, also play a pivotal role in Lyapunov based stability analysis for time-delay systems, based on the well-known Lyapunov-Krasovskii theorem. A particular class of Lyapunov-Krasovskii functionals can be described by the following:

$$V_U = \int_{t-d}^{t} U(x(\tau)) d\tau \tag{1.5}$$

where $d$ is the time delay and $U(\cdot)$ is a positive function. Interested readers can refer to [60] for more in-depth discussion on other classes of functional candidates. These have been applied to time-delay systems that are linear [85, 88, 58, 162], as well those that are nonlinear [32, 72, 177]. With suitably constructed Lyapunov-Krasovskii functionals, terms containing the delayed states can be matched and canceled when the derivative of the Lyapunov function/functional is taken. Following its success in stability analysis, the utility of Lyapunov-Krasovskii functionals in control design for time-delay systems was subsequently explored. Linear systems with nonlinear functions of delayed states were considered (e.g. [176]), along with SISO nonlinear time-delay systems [122], wherein Lyapunov-Krasovskii functionals were used with backstepping to obtain a robust controller. The need for exact knowledge of non-linearities is removed with the use of adaptive NN control in [46], with subsequent extensions tackling the case of completely unknown virtual control coefficients using Nussbaum-type functions [47], as well as multi-input multi-output (MIMO) systems with a more general mixture of delayed states in the unknown nonlinearities [51].

With the celebrated success and rapid development of Lyapunov based design tools in solving challenging academic and practical problems such as time delay systems, nonlinearly parameterized systems, as well as systems with unknown control gain functions, there is a need to carry out investigations within this framework and develop new tools to deal with nonlinear systems with constraints, without the need for explicit solutions for the dynamic equations of the system, which can incur huge computational costs. Furthermore, Lyapunov control synthesis lends itself to the design of stable adaptation laws, and thus provides a promising avenue for fundamental considerations and investigations of the adaptive control problem for high order nonlinear systems with constraints.

### 1.1.2 Adaptive Control and Backstepping

Adaptive control has witnessed more than half a century of intense theoretical research and engineering applications. Originally proposed for aircraft autopilots to deal with parameter variations during changing flight conditions, it has since evolved into an advanced and successful field, culminating from decades of research activities that involve rigorous problem formulation, stability proof, robustness design, performance analysis and applications.

Early research in adaptive control focused on stability issues and on achieving asymptotic tracking properties [33, 56, 97, 117, 120], which laid the cornerstones for a rigorous theory for adaptive systems that emerged later [7, 57, 66, 147]. Accompanying the early results were observations that adaptive controllers had limited robustness properties. Minute disturbances and the presence of unmodelled dynamics can catastrophically destabilize the closed loop systems, as demonstrated by the Rohrs example on a first order plant [142]. Subsequently, robustification techniques have been integrated with adaptive control to improve robustness to unmodelled disturbances and bounded disturbances, and these encompass normalization techniques [67, 91], projection methods [55, 147], dead zone modifications [33, 131], the $\epsilon$-modification

[119], and the $\sigma$-modification [65].

While early works on adaptive control dealt mainly with linear systems and have been highly successful, interest in extensions to nonlinear systems soon grew rapidly, motivated by seminal developments of nonlinear feedback control theory based on differential geometry [69]. Among the important early results for adaptive control of nonlinear systems are works involving feedback linearization techniques [22, 137, 148, 166, 170] and robustification methods [2, 75, 77, 166].

However, global stability cannot be established without some restrictions on the plants, which include the matching condition [166], extended matching condition [78], and growth conditions on system nonlinearities [148]. To this end, the technique of backstepping, rooted in the independent works of [20, 87, 159, 171], and further developed in [21, 79, 126, 144], heralded an important breakthrough for adaptive control that overcame the structural and growth restrictions. Specifically, the marriage of adaptive control and backstepping, i.e. adaptive backstepping, yields a means of applying adaptive control to parametric-uncertain systems with non-matching conditions [94, 114]. As a result, adaptive backstepping can be applied to a large class of nonlinear systems in parametric strict feedback form or pure feedback form. The advantage of adaptive backstepping design is that not only global stability and asymptotic stability can be achieved, but also the transient performance can be explicitly analyzed and guaranteed [94].

Through the collective efforts of many researchers, the adaptive backstepping technique has undergone steady improvements. Although early designs, such as the one in [81], were based on overparameterized schemes that require multiple estimates of the same parameters, this requirement was subsequently obviated with the introduction of tuning functions [93]. For systems that can be represented by the parametric output feedback form, the output feedback adaptive control problem has been solved in [80, 82, 112]. This class of systems is later enlarged to include nonlinearly parameterized output nonlinearities [113], input-to-state stable (ISS) internal dynamics [138], as well zero dynamics that are not necessarily stable [83]. Extended studies of adaptive backstepping control have been performed for nonlinear systems with triangular structures [153], large-scale decentralized systems in strict-feedback form [71], as well as nonholonomic systems [73]. Several robust adaptive backstepping schemes were also proposed in [74] for the systems' uncertainties satisfying an ISS property, and uncertain systems in strict-feedback form with disturbances [34, 95, 104, 129].

Traditional adaptive control techniques rely on the key assumption of linear parametrization, where nonlinearities of the studied plants can be represented in the linear-in-the-parameters form, for which the regressor is exactly known and the uncertainty is parametric and time-invariant. However, many practical systems exhibit nonlinear parametrization in their model representations, including fermentation processes [16], bio-reactor processes [19, 18] and friction dynamics [49]. Departing from the assumption of linear parametrization, several results were presented for different kinds of nonlinearly parameterized systems [4, 16, 17, 18, 19, 38, 43, 107]. Of particular interest are the works in [16, 17], wherein an innovative design approach is provided that appropriately parameterizes the nonlinearly parameterized plant and constructs a suitable Lyapunov function, as well as in [43], where nonlinearly parameterized functions are handled by Integral Lyapunov Functions. Additionally, approximation-based control techniques with guaranteed stability have been proposed [26, 35, 36, 42, 70, 101, 102, 136, 145, 146] to compensate for nonlinearly parameterized functions and general unknown nonlinear functions, based on the Stone-Weierstrass theorem, which states that a universal approximator can approximate, to an arbitrary degree of accuracy, any real continuous function on a compact set [145].

Despite the maturity of backstepping in dealing with such systems, the explicit consideration of constraints within this framework has received little attention, with a few exceptions. In the recent work [92], backstepping control was designed to achieve nonovershooting tracking response for strict feedback systems, by appropriately choosing the control gains such that the initial values for all the error variables are negative. Another work [103] presented modified backstepping based on positively invariant feasibility regions for a class of nonlinear systems with control singularities, such that state trajectories are repelled from regions containing the singularities. The design induces singularities in the Lyapunov functions that coincide with those of the control laws, and this property proved to be instrumental in preventing state trajectories from transgressing the feasibility boundaries. However, there are still fundamental problems about stability, robustness, and other issues for adaptive control of uncertain high-order nonlinear systems with constraints to be further investigated.

### 1.1.3 Control of Constrained Systems

Dealing with constraints in control design has become an important research topic in recent decades, driven by practical needs and theoretical challenges. Many practical systems have constraints on the outputs, inputs, or states, which may appear in the form of physical stoppages, saturation, or performance and safety specifications. Violation of the constraints during operation may result in performance degradation, hazards or system damage. In some cases, it is possible to neglect constraints in control design, but circumvent the problem through mechanical design, modification of operating conditions, or ad-hoc engineering fixes, although such solutions are highly context specific, require substantial human intervention, and do not provide any guarantee of success. A more generic and fundamental approach is to consider the constraints up front in the problem formulation, and then design a controller which ensure that the constraints are met, along with desired stability and performance properties.

Linear systems theory, with its rich set of analytical tools, have laid important foundations for feedback control theory. It is particularly advantageous if plants can be represented by linear systems, for these rich tools can be readily exploited for control design. However, the presence of constraints automatically renders the closed loop system nonlinear, even if the unconstrained system is linear. To handle both state and input constraints in linear systems, many techniques have been developed (see e.g. [27, 54, 59, 63, 64, 106, 143, 175]), most of which are based on notions of set invariance using Lyapunov analysis [11]. When dealing with the simplified problem of only input constraints, many results have also been achieved [6, 24, 30, 89, 105, 163, 168, 169]. The benefit of dealing with linear systems is that positive invariant sets can be obtained constructively.

Another approach is concerned with casting the problem under an optimization framework, which is naturally suited for consideration of constraints. Model predictive control (MPC), also known as receding horizon control, is concerned with solving on-line a finite horizon open-loop optimal control problem, subject to the system dynamics and constraints (see [116] for an excellent overview), and can handle both linear and nonlinear systems. Over the past few decades, MPC has enjoyed widespread popularity and success in industrial applications of process control, with thousands of applications to date that range from chemical to aerospace industries [1]. While linear MPC (i.e. based on linear models of system dynamics) is well established, extension to the nonlinear setting comes with theoretical and computational challenges. Even though many elegant theoretical treatments have been developed, one of the key concerns involve making the optimization algorithms efficient enough to be implemented online, which can be a formidable task considering the possibility of encountering complex or high order nonlinear dynamics [1, 140]. When there is a need to incorporate robustness to uncertainties, the computational complexity increases even more significantly. Notwithstanding these technical difficulties, successful applications have been demonstrated [29, 115, 139].

To extend MPC schemes for tracking of arbitrary reference signals, reference governors have been proposed [9, 10]. The main idea behind reference governors is to have a controller that provide desirable closed loop properties when constraints are neglected, and then modulate the reference signal, which feeds the controller, in such a way as to avoid any violation of system constraints (see e.g. [52, 53]). An early version for linear constrained systems was presented in [53], while a recent generalized version for nonlinear constrained systems was proposed in [52]. For implementation, online optimization algorithms for computing the reference signals are needed. Related to the idea of reference signal modification, an extremum seeking control design has been proposed in [28], with online generation of set points that minimize an uncertain cost function subject to state constraints.

Different from the above-mentioned methods, one can use Barrier Lyapunov Functions (BLFs) to tackle the issue of constraint, which avoids the need for explicit solutions of the system by virtue of being a Lyapunov based control design methodology. For the great majority of works in the literature, the constructed Lyapunov functions are radially unbounded, for global stability, or at least well-defined over the entire domain. In contrast to this convention, the BLF-based method exploits the property that the value of the barrier function approaches infinity whenever its arguments approach certain limits. The design of barrier functions in Lyapunov synthesis has been proposed for constraint handling in Brunovsky-type systems [121]. In their backstepping procedure, the cancelation of cross coupling terms in the Lyapunov function derivative is avoided. Instead, the control gains are carefully chosen to dominate the cross coupling terms. The advantage of this approach is that the control effort is potentially reduced, since the control law does not contain the cross coupling terms that may exhibit large growth rate.

Inspired by the use of barrier functions, it is of interest to investigate and generalize their use for more complex classes of constrained nonlinear systems, which include strict feedback systems, pure feedback systems, mechanical systems, among others. There is also a need to obtain results that remove the need for prior assumptions on the states satisfying some constraints, as an improvement over [121]. Additionally, no attempts have been made for constrained systems with uncertainty using BLF based control design.

### 1.1.4 Control of MEMs

The advent of microelectromechanical systems (MEMs) technology, which allows for micro-scale devices to be batch-produced and processed at low costs, has ignited an interest in how to control these devices effectively to achieve greater precision and speed of response. Electrostatic microactuators have gained widespread acceptance in MEMs applications, due to the simplicity of their structure, ease of fabrication, and the favorable scaling of electrostatic forces into the micro domain.

One of the main problems associated with uni-directional electrostatic actuation with open loop voltage control is the pull-in instability, a saddle node bifurcation phenomenon wherein the movable electrode snaps through to the fixed electrode once its displacement exceeds a certain fraction (typically 1/3) of the full gap. This places a severe limit on the operating range of electrostatic actuators. To overcome this problem, closed loop voltage control with position feedback was proposed to stabilize any point in the gap [25]. An alternative approach, which involves the passive addition of series capacitor, has been found to extend the range of travel without any active feedback control circuitry [23, 150]. Another method is based on charge feedback to stabilize the dynamics of the electrical subsystem, which leads to the stabilization of the minimum phase mechanical subsystem [118, 149]. More advanced nonlinear control techniques have been investigated in [179], including flatness-based control,

Control Lyapunov Function (CLF) synthesis, and backstepping control. In [110], different static and dynamic output feedback control laws have been investigated and compared, including input-output linearization, linear state feedback, feedback passivation, and charge feedback schemes. Under a geometric framework, control for a general class of electrostatic MEMs has been proposed in [111].

Electrostatic micro-actuators with bi-directional drive are less prone to pull-in instability due to the fact that they can be actively controlled in both directions, unlike uni-directional drive actuators where only passive restoring force is provided by mechanical stiffness in one direction. Although less challenging as a theoretical control design problem, the study of micro-actuators with bi-directional drive is nevertheless important since its controllability is an advantage in high performance applications. Open loop control schemes, based on oscillatory switching input, have been proposed in [124, 161] to overcome pull-in instability and extend operation range for bi-directional parallel plate actuators. Recently, the comparative advantages and disadvantages between simple open loop and closed loop control strategies for electrostatic comb actuators with bi-directional drive have been studied [15].

In most of the works on MEMs control, knowledge of model parameters is required and typically estimated through offline system identification methods. However, inconsistencies in bulk micromachining result in variation of parameters across pieces, and may require extensive efforts in parameter identification, with higher costs. Furthermore, some of the parameters, such as the damping constant, are usually difficult to identify accurately, so a viable alternative is to rely on adaptive feedback control for online compensation of parametric uncertainties.

There has been relatively few works in the literature on application of adaptive techniques in MEMs. Adaptive control has been applied in MEMs gyroscopes to compensate for non-ideal coupling effects between the vibratory modes [99, 130, 154]. Another work dealt with electrostatic microactuators by utilizing position, velocity, and acceleration information, to estimate, adaptively, parameters in the inverse model of the system nonlinearities [132].

However, in the above works on adaptive techniques of MEMs, explicit consideration of constraints has been neglected in control design, but instead, control parameters have been chosen to ensure constraint satisfaction via simulations and experiments. With the need to avoid electrode contact for certain continuous tracking operations of electrostatic microactuators, together with the presence of model uncertainties, it is important to design adaptive controllers for electrostatic microactuators with consideration of position constraints. This is a theoretically challenging task, in view of the need to contain the effects of the transient adaptation dynamics and rely on position feedback only.

## 1.2 Objectives, Scope, and Structure of the Thesis

The general objectives of the thesis are to develop constructive and systematic methods of designing adaptive controllers for constrained nonlinear systems, to show system stability, and to obtain performance bounds of the states in the closed-loop systems. In particular, we focus on the tracking problem for nonlinear systems in strict feedback form with output and state constraints, motivated by the fact that many practical systems are subjected to constraints in the form of physical stoppages, saturation, or performance and safety specifications, which must not be violated.

Additionally, uncertainties in the plant are to be accommodated in the control design via adaptive techniques. Not only is the class of linearly parameterized uncertain nonlinearities considered, but general uncertain nonlinearities with known bounded estimates within a compact region of interest are also dealt with. Control gain functions preceding the control input and the virtual controls are not restricted to the unity case, but may also contain uncertainties that need to be compensated for.

Furthermore, the practical relevance of the proposed control design method is to be illustrated. We investigate the effectiveness of the proposed control for single degree-of-freedom uncertain electrostatic microactuators with bi-directional drive. For this application study, the control objective is to track a reference trajectory within the air gap without any physical contact between the electrodes, i.e. position constraint.

Besides problem-oriented objectives as outlined above, we also endeavor to formalize the notion of Barrier Lyapunov Functions in a technically rigorous framework and motivate their use in constructive, systematic control design that ensures non-transgression of constraints in nonlinear systems. Although the use of barrier functions to prevent excursions of variables from a region of interest is not a particularly new idea, as noted by their applications in constrained optimization problems and multi-agent collision avoidance algorithms, a formal treatment of barrier functions in Lyapunov synthesis is currently lacking, and it is the aim of this thesis to reduce this gap.

The thesis is organized as follows. After the introduction, Chapter 2 gives the mathematical preliminaries and design tools for tracking control of uncertain constrained nonlinear systems. We define notions of continuity, differentiability, and smoothness, as well as the classes of systems considered in this thesis, namely the strict feedback form, parametric strict feedback form, and parametric output feedback form. For completeness, concepts of Lyapunov stability and analysis are discussed. Key technicalities underlying the use of Barrier Lyapunov Functions for constraint satisfaction are exposed. Following that, we explore three motivating examples on low order systems to elucidate the benefits and procedure of design.

In Chapter 3, we start with the simplest case where only the output is constrained, and with known control gain functions, we employ backstepping design with BLF in the first step, and quadratic functions in the remaining steps. It is shown that asymptotic output tracking is achieved without violation of constraint, and all closed loop signals remain bounded, under a mild restriction on the initial output. Besides the nominal case where full knowledge of the plant is available, we also tackle scenarios wherein parametric uncertainties are present. Furthermore, we explore the use of asymmetric Barrier Lyapunov Functions as a generalized approach that relaxes the restriction on the initial output.

Chapter 4 extends investigations to the case of full state constraints by employing BLFs in every step of backstepping design. It is shown that state constraints cannot be arbitrarily specified, but are subject to feasibility conditions on the initial states and control parameters, which, if satisfied, guarantee asymptotic output tracking without violation of state constraints. These conditions can be relaxed when handling only partial state constraints. We provide a comparison study which shows that BLFs require less conservative initial conditions than quadratic Lyapunov functions in preventing violation of constraints.

Chapter 5 considers the presence of uncertainty in the control gain functions, and employs domination design instead of the foregoing cancelation based approaches. Within this framework, sufficient conditions that prevent violation of constraints, are established to accommodate stability analysis in the practical sense. When dealing with full state constraints, we show that practical output tracking is achieved subject to feasibility conditions on the initial states and control parameters. Additionally, we show that, for the special case of output constraint with linearly parameterized nonlinearities, practical output tracking is achieved without any feasibility conditions.

In Chapter 6, we consider, as an application study, single degree-of-freedom uncertain electrostatic microactuators with bi-directional drive, wherein the control objective is to track a reference trajectory within the air gap without any physical contact between the electrodes. Besides the state feedback case, for which the foregoing method for dealing with output constraint can be applied, we also tackle the output feedback problem, and employ adaptive observer backstepping based on asymmetric BLF to ensure asymptotic output tracking without violation of output constraint.

Finally, Chapter 7 concludes the contributions of the thesis and makes recommendation on future research work.

# Chapter 2

# Design Tools and Preliminaries

## 2.1 Introduction

In this chapter, we describe in detail the mathematical preliminaries, useful technical lemmata, and design tools for tracking control of uncertain constrained nonlinear systems, which will be used throughout this thesis. We formally define notions of continuity, differentiability, and smoothness, as well as the classes of systems considered in this thesis, namely the strict feedback form, parametric strict feedback form, and parametric output feedback form. For completeness, concepts of Lyapunov stability and analysis are discussed. Most importantly, we introduce formally the notion of Barrier Lyapunov Functions and motivate, through examples for low order systems, their use in control design that ensures non-transgression of output and state constraints.

## 2.2 Mathematical Preliminaries

For the convenience of the reader, this section provides a brief review of the notions of continuity, differentiability, and smoothness, as well as presents a formal description of the classes of systems considered in this thesis, namely the strict feedback form, parametric strict feedback form, and parametric output feedback form. The material covered in this section are largely borrowed from the references [94, 42].

**Definition 1** *[42] A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be continuous at a point $x$ if $f(x + \delta x) \to f(x)$ whenever $\|\delta x\| \to 0$. Equivalently, $f$ is continuous at $x$ if, given $\epsilon > 0$, there is $\delta > 0$ such that*

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon \tag{2.1}$$

*A function $f$ is continuous in a set $S$ if it is continuous at every point of $S$, and it is uniformly continuous in $S$ if given $\epsilon > 0$, there is $\delta(\epsilon) > 0$ (dependent only on $\epsilon$), such that (2.1) holds for all $x, y \in S$.*

**Definition 2** *[42] A function $f : \mathbb{R} \to \mathbb{R}$ is said to be differentiable at a point $x$ if the limit*

$$\frac{df}{dx} := \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \tag{2.2}$$

*exists. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable at a point $x$ (in a set $S$) if the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and are continuous at $x$ (at every point of $S$) for $i = 1, ..., m$, $j = 1, ..., n$.*

**Definition 3** *[90] A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be continuously differentiable of order k, or $C^k$, if*

$$D^a f := \frac{\partial^{a_1}}{\partial x_1^{a_1}} \frac{\partial^{a_2}}{\partial x_2^{a_2}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}} f \tag{2.3}$$

*exists and is continuous for all points $(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$, and all non-negative integers $a_1, a_2, ..., a_n$ satisfying $\sum_{i=1}^{n} a_i \leq k$.*

**Definition 4** *[90] A smooth, or $C^\infty$, function $f : \mathbb{R}^n \to \mathbb{R}^m$ is one that is $C^k$ for every positive $k$.*

**Property 2.2.1** *For any continuous function $f(x) : \mathbb{R}^n \to \mathbb{R}$, if $x$ belongs to a compact set $\Omega_x \subset \mathbb{R}^n$, there exists a positive constant $F$ such that $|f(x)| \leq F$.*

**Definition 5** *[42] A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is said to be*

- *positive definite (denoted by* $\mathbf{A} > 0$*) if* $x^T \mathbf{A} x > 0, \forall x \in \mathbb{R}^n$*,* $x \neq 0$*, or if for some* $\beta > 0$*,* $x^T \mathbf{A} x \geq \beta x^T x = \beta \|x\|^2$ *for all* $x$*;*
- *positive semi-definite (denoted by* $\mathbf{A} \geq 0$*) if* $x^T \mathbf{A} x \geq 0, \forall x \in \mathbb{R}^n$*;*
- *negative semi-definite if* $-\mathbf{A}$ *is positive semi-definite;*
- *negative definite if* $-\mathbf{A}$ *is positive definite;*
- *symmetric if* $\mathbf{A}^T = \mathbf{A}$*;*
- *skew-symmetric if* $\mathbf{A}^T = -\mathbf{A}$*; and*
- *symmetric positive definite (semi-definite) if* $\mathbf{A} > 0 (\geq 0)$ *and* $\mathbf{A} = \mathbf{A}^T$*.*

The classes of systems considered in this thesis include the strict feedback form, parametric strict feedback form, and parametric output feedback form, which are defined in the following. For completeness and relevance of discussion, the class of output feedback systems is also described herewith.

**Definition 6** *[94] A system is said to be in strict feedback form if it can be described by differential equations of the following form:*

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i) x_{i+1}, \quad i = 1, ..., n-1 \\ \dot{x}_n &= f_n(x) + g(x) u \end{aligned} \tag{2.4}$$

*where* $f_i(\cdot)$*,* $g_i(\cdot)$ *are smooth functions,* $x_i \in \mathbb{R}$*,* $i = 1, ..., n$*, are the states,* $\bar{x}_i = [x_1, x_2, ..., x_i]^T$*,* $x = [x_1, x_2, ..., x_n]^T$*, and* $u \in \mathbb{R}$ *is the input.*

**Definition 7** *[94] A system is said to be in parametric strict feedback form if it can be described by differential equations of the following form:*

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \theta^T \varphi_i(\bar{x}_i), \quad i = 1, ..., n-1 \\ \dot{x}_n &= g(x) u + \theta^T \varphi_n(x) \end{aligned} \tag{2.5}$$

*where* $\theta \in \mathbb{R}^l$ *is a vector of unknown constant parameters,* $\varphi_i(\cdot)$*,* $g(\cdot)$ *are smooth functions,* $x_i \in \mathbb{R}$*,* $i = 1, ..., n$*, are the states,* $\bar{x}_i = [x_1, x_2, ..., x_i]^T$*,* $x = [x_1, x_2, ..., x_n]^T$*, and* $u \in \mathbb{R}$ *is the input.*

**Definition 8** *[94] A system is said to be in output feedback form if it can be described by differential equations of the form:*

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_i(y), \quad i = 1, ..., \rho - 1 \\ \dot{x}_j &= x_{j+1} + \varphi_j(y) + b_{j-\rho}\beta(y)u, \quad j = \rho, ..., n-1 \\ \dot{x}_n &= \varphi_n(y) + b_{n-\rho}\beta(y)u \\ y &= x_1 \end{aligned} \tag{2.6}$$

*where $b_0, ..., b_{n-\rho}$ are constant parameters, $\varphi_i(\cdot)$, $\beta(\cdot)$ are smooth functions, $x_i \in \mathbb{R}$, $i = 1, ..., n$, are the states, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the input and output, respectively.*

**Definition 9** *[94] A system is said to be in parametric output feedback form if it can be described by differential equations of the form:*

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \varphi_{0,i}(y) + \sum_{k=1}^{p} \theta_k \varphi_{k,i}(y), \quad i = 1, ..., \rho - 1 \\ \dot{x}_j &= x_{j+1} + \varphi_{0,j}(y) + \sum_{k=1}^{p} \theta_k \varphi_{k,j}(y) + b_{j-\rho}\beta(y)u, \quad j = \rho, ..., n-1 \\ \dot{x}_n &= \varphi_{0,n}(y) + \sum_{k=1}^{p} \theta_k \varphi_{k,n}(y) + b_{n-\rho}\beta(y)u, \\ y &= x_1 \end{aligned} \tag{2.7}$$

*where $\theta_1, ..., \theta_\rho$ and $b_0, ..., b_{n-\rho}$ are unknown constant parameters, $\varphi_{i,j}(\cdot)$, $\beta(\cdot)$ are smooth functions, $x_i \in \mathbb{R}$, $i = 1, ..., n$, are the states, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the input and output respectively.*

Interested readers are referred to [68, 94, 114] for differential geometric conditions under which there exists diffeomorphisms that transform general nonlinear systems into one or more of the above canonical representations.

## 2.3 Lyapunov Stability Analysis

Lyapunov's direct method is an important tool in the analysis (and control design) for nonlinear systems. It provides a means of determining stability of an equilibrium without the need for explicit knowledge of system solutions, by constructing a Lyapunov function, and then analyzing the properties of its time derivative. We briefly review below some well-known notions and tools in Lyapunov stability analysis, borrowed from the references [5, 84, 94, 42, 156], which are important to the results presented in this thesis.

**Definition 10** *[94] A continuous function $\gamma : [0, a) \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\gamma(r) \to \infty$ as $r \to \infty$.*

**Definition 11** *[42] A continuous function $V(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is*

- *locally positive definite if there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that*

$$V(x, t) \geq \alpha(\|x\|) \tag{2.8}$$

  *for all $t \geq 0$ and all $x$ in a neighborhood $\mathcal{N}$ of the origin of $\mathbb{R}^n$;*

- *positive definite if $\mathcal{N} = \mathbb{R}^n$;*

- *(locally) negative definite if $-V$ is (locally) positive definite; and*

- *(locally) decrescent if $V$ is (locally) positive definite and there exists a class $\mathcal{K}$ function $\beta(\cdot)$ such that*

$$V(x, t) \leq \beta(\|x\|) \tag{2.9}$$

  *for all $t \geq 0$ and all $x$ in $\mathbb{R}^n$ (in a neighborhood $\mathcal{N}$ of the origin of $\mathbb{R}^n$).*

**Definition 12** *[42] Given a continuously differential function $V(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, together with a system of differential equations*

$$\dot{x} = f(x, t) \tag{2.10}$$

*the derivative of $V$ along the trajectories of the system is*

$$\dot{V} = \frac{dV(x, t)}{dt} = \frac{\partial V(x, t)}{\partial t} + \left[\frac{\partial V(x, t)}{\partial x}\right] f(x, t) \tag{2.11}$$

**Definition 13** *[84] With respect to the system*

$$\dot{x} = f(x,t), \quad x(0) = x_0 \tag{2.12}$$

*where* $x \in \mathbb{R}^n$ *and* $t \in [0, \infty)$*, a set* $\mathcal{M} \subseteq \mathbb{R}^n$ *is positively invariant if, for every* $x(0) \in \mathcal{M}$*, we have* $x(t) \in \mathcal{M}\ \forall t \geq 0$.

The following theorem provides conditions for the origin to be a stable equilibrium, and presents a clear exposition of the notion of Lyapunov function. Since the conditions are only sufficient, no conclusion on the stability or instability can be drawn if a particular choice of Lyapunov candidate does not meet the conditions on $\dot{V}$.

**Theorem 2.3.1** *[42] (Lyapunov Theorem) Given the non-linear dynamic system*

$$\dot{x} = f(x,t), \quad x(0) = x_0 \tag{2.13}$$

*with an equilibrium point at the origin, and let* $\mathcal{N}$ *be a neighborhood of the origin, e.g..* $\mathcal{N} = \{x : \|x\| \leq \epsilon\}$*, with* $\epsilon > 0$*, then, the origin is*

- *stable in the sense of Lyapunov if, for all* $x \in \mathcal{N}$*, there exists a positive definite scalar function* $V(x,t)$ *such that* $\dot{V}(x,t) \leq 0$*;*
- *uniformly stable if, for all* $x \in \mathcal{N}$*, there exists a positive definite and decrescent scalar function* $V(x,t)$ *such that* $\dot{V}(x,t) \leq 0$*;*
- *asymptotically stable if there exists a positive definite scalar function* $V(x,t)$ *such that* $\dot{V}(x,t) < 0$ *for all* $x \in \mathcal{N}$*,* $x \neq 0$*;*
- *globally asymptotically stable if there exists a positive definite and radially unbounded scalar function* $V(x,t)$ *such that* $\dot{V}(x,t) < 0$ *for all* $x \in \mathbb{R}^n$*,* $x \neq 0$*;*
- *uniformly asymptotically stable if there exists a positive definite and decrescent scalar function* $V(x,t)$ *such that* $\dot{V}(x,t) < 0$ *for all* $x \in \mathcal{N}$*,* $x \neq 0$*;*
- *globally uniformly asymptotically stable if there exists a positive definite, decrescent and radially unbounded scalar function* $V(x,t)$ *such that* $\dot{V}(x,t) < 0$ *for all* $x \in \mathbb{R}^n$*,* $x \neq 0$*;*

- *exponentially stable if there exist positive constants* $\alpha$, $\beta$, *and* $\gamma$ *such that, for all* $x \in \mathcal{N}$, $\alpha\|x\|^2 \leq V(x,t) \leq \beta\|x\|^2$ *and* $\dot{V}(x,t) \leq -\gamma\|x\|^2$;
- *globally exponentially stable if there exist positive constants* $\alpha$, $\beta$, *and* $\gamma$ *such that, for all* $x \in \mathbb{R}^n$, $\alpha\|x\|^2 \leq V(x,t) \leq \beta\|x\|^2$ *and* $\dot{V}(x,t) \leq -\gamma\|x\|^2$.

Lyapunov analysis is a powerful tool that is not restricted to the analysis of system stability only, but can also be extended to design controllers that attribute, to the closed loop systems, desirable stability properties, via the concept of Control Lyapunov Functions (CLFs), which is formalized in the following definition.

**Definition 14** *[5, 94] A positive definite* $C^1$ *function* $V : \mathcal{D} \to \mathbb{R}_+$, *defined on a neighborhood* $\mathcal{D}$ *of the origin, is called a Control Lyapunov Function for the system*

$$\dot{x} = f(x,u), \quad x \in \mathcal{D} \subseteq \mathbb{R}^n, \quad u \in \mathcal{U} \subseteq \mathbb{R}, \quad f(0,0) = 0 \tag{2.14}$$

*if the following inequality holds*

$$\inf_{u \in \mathcal{U}} \left\{ \frac{\partial V(x)}{\partial x} f(x,u) \right\} < 0, \quad \forall x \neq 0 \tag{2.15}$$

For global stabilization, a useful property of $V(x)$ is radial unboundedness, with $\mathcal{D}$ chosen as $\mathbb{R}^n$ and $\mathcal{U}$ as $\mathbb{R}$. Note that there exist many Lyapunov functions for the same system. Depending on the system of interest, specific choices of Lyapunov functions may yield more precise results than others. The task of selecting a Lyapunov function candidate, followed by the design of the control law that renders the derivative of the candidate function negative semidefinite along the system trajectories, is, in general, non-trivial. Different choices of Lyapunov functions may result in different forms of controller, with correspondingly different performance. Further, even if a stabilizing control law exists, we may fail to find it due to an ill-chosen Lyapunov function candidate.

**Lemma 2.3.1** *[156] (Barbalat's Lemma)*
*Consider a differentiable function* $h(t)$. *If* $\lim_{t\to\infty} h(t)$ *is finite and* $\dot{h}$ *is uniformly continuous, then* $\lim_{t\to\infty} \dot{h}(t) = 0$.

Throughout the thesis, the above lemma is useful for establishing asymptotic convergence of signals to zero via analysis of continuity properties of the derivative of the Lyapunov function candidate in the closed loop. In particular, the result $\lim_{t\to\infty} \dot{h}(x(t)) = 0$ will allow us to draw important conclusions on the asymptotic properties of the signal $x(t)$.

We present the existence and uniqueness theorem for ordinary differential equations below. This will be used to prove the subsequent lemma for Barrier Lyapunov Functions.

**Lemma 2.3.2** *Existence and Uniqueness of Solution [158, p.476 Theorem 54]*
*Consider the initial value problem*

$$\dot{\xi} = h(t, \xi(t)), \quad \xi(\sigma^0) = z^0 \tag{2.16}$$

*where $\xi(t) \in \mathcal{Z} \subseteq \mathbb{R}^n$. Assume that $h : \mathcal{I} \times \mathcal{Z} \to \mathbb{R}^n$, where $\mathcal{Z} \subseteq \mathbb{R}^n$ is open and $\mathcal{I} \subseteq \mathbb{R}$ is an interval, satisfies the assumptions:*

$$h(\cdot, z) \;\; : \;\; \mathcal{I} \to \mathbb{R}^n \text{ is measurable for each fixed } z \tag{2.17}$$

$$h(t, \cdot) \;\; : \;\; \mathcal{Z} \to \mathbb{R}^n \text{ is continuous for each fixed } t \tag{2.18}$$

*and the following two conditions also hold:*

1. *$h$ is locally Lipschitz on $z$: that is, there are for each $z^0 \in \mathcal{Z}$ a real number $\rho > 0$ and a locally integrable function $c : \mathcal{I} \to \mathbb{R}_+$ such that the ball $B_\rho(z^0)$ of radius $\rho$ centered at $z^0$ is contained in $\mathcal{Z}$ and*

$$\|h(t, z) - h(t, z^*)\| \leq c(t)\|z - z^*\| \tag{2.19}$$

   *for each $t \in \mathcal{I}$ and $z, z^* \in B_\rho(z^0)$.*

2. *$h$ is locally integrable on $t$; that is, for each fixed $z^0$ there is a locally integrable function $b : \mathcal{I} \to \mathbb{R}_+$ such that*

$$\|h(t, z^0)\| \leq b(t) \tag{2.20}$$

   *for almost all $t$.*

*Then, for each pair* $(\sigma^0, z^0) \in \mathcal{I} \times \mathcal{Z}$ *there is some nonempty subinterval* $J \subseteq \mathcal{I}$ *open relative to* $\mathcal{I}$ *and there exists a solution* $\xi$ *of (2.16) on* $J$*, with the following property: If* $\zeta : J \to \mathcal{Z}$ *is any other solution of (2.16), where* $J' \subseteq J$ *and* $\xi = \zeta$ *on* $J'$*. The solution* $\xi$ *is called the maximal solution of the initial-value problem in the interval* $\mathcal{I}$*.*

With the additional condition that the solution is bounded, the following lemma establishes that the solutions is defined for all time.

**Lemma 2.3.3** *[158, p.481 Proposition C.3.6] Assume that the hypothesis of Lemma 2.3.2 hold and that in addition it is known that there is a compact subset* $K \subseteq \mathcal{Z}$ *such that the maximal solution* $\xi$ *of (2.16) satisfies* $\xi(t) \in K$ *for all* $t \in J$*. Then*

$$J = [\sigma^0, +\infty) \bigcap \mathcal{I} \tag{2.21}$$

*that is, the solution is defined for all times* $t > \sigma^0$*,* $t \in \mathcal{I}$*.*

## 2.4 Barrier Lyapunov Functions

The idea of barrier functions as a means of preventing excursions of variables from a region of interest is not new, and has been a useful tool in constrained optimization problems, where they are used in the cost function to penalize proximity with the boundary of the feasible region [8, 123, 133, 134, 135]. In addition, this idea has also been adopted in the field of robotics, particularly for the problem of collision avoidance, in the form of artificial potential field functions which grow to singularities when the inter-object distance is less than a prescribed value [31, 40, 41, 100, 125, 141, 160, 165].

Motivated by these approaches, we explore the use of barrier functions in Lyapunov synthesis that will pave the way for the development of a systematic control design method for nonlinear constrained systems. When used in this context, we aptly name them *Barrier Lyapunov Functions*, and they are characterized by the property of growing to infinity when the function arguments approach certain limiting values.

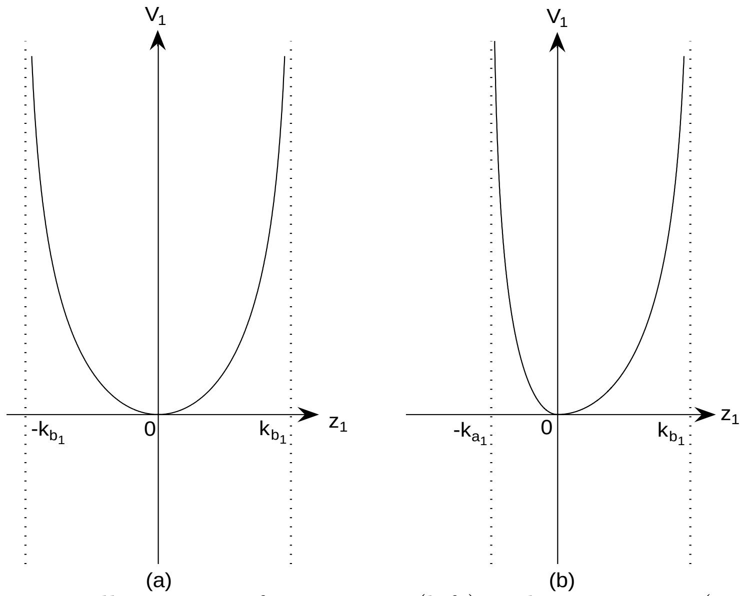


Figure 2.1: Schematic illustration of symmetric (left) and asymmetric (right) barrier functions.

The key principle is that by ensuring boundedness of the BLFs in the closed loop, we also ensure that the constraints are not transgressed.

To this end, we introduce the formal definition of Barrier Lyapunov Functions.

**Definition 15** *A Barrier Lyapunov Function is a scalar function $V(x)$, defined with respect to the system $\dot{x} = f(x)$ on an open region $\mathcal{D}$ containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of $\mathcal{D}$, has the property $V(x) \to \infty$ as $x$ approaches the boundary of $\mathcal{D}$, and satisfies $V(x(t)) \leq b \ \forall t \geq 0$ along the solution of $\dot{x} = f(x)$ for $x(0) \in \mathcal{D}$ and some positive constant $b$.*

General forms of barrier functions $V_1(z_1)$ in Lyapunov synthesis satisfy $V_1(z_1) \to \infty$ as $z_1 \to -k_{a_1}$ or $z_1 \to k_{b_1}$. They may be symmetric ($k_{a_1} = k_{b_1}$) or asymmetric ($k_{a_1} \neq k_{b_1}$), as illustrated in Figure 2.1. Asymmetric barrier functions are more general than their symmetric counterparts, and thus can offer more flexibility for control design to obtain better performance. However, they are considerably more difficult to construct analytically, and to employ for control design.

The existence of a BLF for a system guarantees the stability of the equilibrium at the origin, and that $\mathcal{D}$ is a positively invariant region. The following lemma formalizes this notion for general forms of barrier functions, and is used in the control design and analysis for strict feedback system with output constraint in Chapter 3.

**Lemma 2.4.1** *For any positive constants $k_{a_1}, k_{b_1}$, let $\mathcal{Z}_1 := \{z_1 \in \mathbb{R} : -k_{a_1} < z_1 < k_{b_1}\} \subset \mathbb{R}$ and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z}_1 \subset \mathbb{R}^{l+1}$ be open sets. Consider the system*

$$\dot{\eta} \;=\; h(t, \eta) \tag{2.22}$$

*where $\eta := [w, z_1]^T \in \mathcal{N}$ is the state, and the function $h : \mathbb{R}_+ \times \mathcal{N} \rightarrow \mathbb{R}^{l+1}$ satisfies conditions (2.17)-(2.20). Suppose that there exist functions $U : \mathbb{R}^l \rightarrow \mathbb{R}_+$ and $V_1 : \mathcal{Z}_1 \rightarrow \mathbb{R}_+$, continuously differentiable and positive definite in their respective domains, such that*

$$V_1(z_1) \rightarrow \infty \quad \text{as} \quad z_1 \rightarrow -k_{a_1} \quad \text{or} \quad z_1 \rightarrow k_{b_1} \tag{2.23}$$

$$\gamma_1(\|w\|) \leq U(w) \leq \gamma_2(\|w\|) \tag{2.24}$$

*where $\gamma_1$ and $\gamma_2$ are class $K_\infty$ functions. Let $V(\eta) := V_1(z_1) + U(w)$, and $z_1(0)$ belong to the set $z_1 \in (-k_{a_1}, k_{b_1})$. If the inequality holds:*

$$\dot{V} \;=\; \frac{\partial V}{\partial \eta} h \;\leq\; 0 \tag{2.25}$$

*then $z_1(t)$ remains in the open set $z_1 \in (-k_{a_1}, k_{b_1})$ $\forall t \in [0, \infty)$.*

**Proof:** Since the right hand side of (2.22) satisfies the conditions (2.17)-(2.20), the existence and uniqueness of the solution $\eta(t)$ is ensured on the time interval $[0, \tau_{\max})$ by virtue of Lemma 2.3.2, taking $\sigma^0 = 0$ without loss of generality. This implies that $V(\eta(t))$ exists for $t \in [0, \tau_{\max})$.

Since $V(\eta)$ is positive definite and $\dot{V} \leq 0$, we know that $V(\eta(t)) \leq V(\eta(0))$ for $t \in [0, \tau_{\max})$. From $V(\eta) := V_1(z_1) + U(w)$ and the fact that $V_1(z_1)$ is a positive function, it is clear that $V_1(z_1(t))$ is also bounded for $t \in [0, \tau_{\max})$. Consequently, we know, from (2.23), that $|z_i| \neq k_{b_1}$ and $|z_i| \neq -k_{a_1}$. Given that $-k_{a_1} < z_1(0) < k_{b_1}$, it can be concluded that $z_1(t)$ remains in the set $-k_{a_1} < z_1 < k_{b_1}$ for $t \in [0, \tau_{\max})$.

Therefore, there is a compact subset $K \subseteq \mathcal{N}$ such that the maximal solution of (2.22) satisfies $\eta(t) \in K$ for all $t \in [0, \tau_{\max})$. As a direct consequence of Lemma 2.3.3, we have that $\eta(t)$ is defined for all $t \in [0, \infty)$. It follows that $z_1(t) \in (-k_{a_1}, k_{b_1})$ $\forall t \in [0, \infty)$. ∎

**Remark 2.4.1** *In Lemma 2.4.1, we split the state variable into $z_1$ and $w$, where $z_1$ is the state to be constrained, and $w$ are the free states, along with the adaptive parameters if adaptive control is involved. The constrained state $z_1$ requires the use of a barrier function $V_1$ to prevent it from reaching the limits $-k_{a_1}$ and $k_{b_1}$. The free states require the use of Lyapunov function candidates in the usual sense, i.e. defined over the entire state space, a common choice being quadratic functions.*

Note that Lemma 2.4.1 involves only one BLF, based on the fact that for the output constraint problem, only one BLF is required to contain the output within the region of interest. The following lemma generalizes this result to deal with the problem of state constraints in strict feedback system (Chapter 4), and involve more than one BLF.

**Lemma 2.4.2** *For any positive constant $k_{b_1}$, let $\mathcal{Z} := \{z \in \mathbb{R}^n : |z_i| < k_{b_1}, i = 1, 2, ..., n\} \subset \mathbb{R}^n$, $\mathcal{Z}_i := \{z_i \in \mathbb{R} : |z_i| < k_{b_1}\} \subset \mathbb{R}$, $i = 1, ..., n$, and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z} \subset \mathbb{R}^{n+l}$ be open sets. Consider the system*

$$\dot{\eta} \;=\; h(t, \eta) \tag{2.26}$$

*where $\eta := [w, z]^T \in \mathcal{N}$ is the state, and the function $h : \mathbb{R}_+ \times \mathcal{N} \to \mathbb{R}^{n+l}$ satisfies conditions (2.17)-(2.20). Suppose that there exist functions $U : \mathbb{R}^l \to \mathbb{R}_+$ and $V_i : \mathcal{Z}_i \to \mathbb{R}_+$, $i = 1, ..., n$, continuously differentiable and positive definite in their respective domains, such that*

$$V_i(z_i) \to \infty \;\text{ as }\; z_i \to \pm k_{b_1} \tag{2.27}$$

$$\gamma_1(\|w\|) \leq U(w) \leq \gamma_2(\|w\|) \tag{2.28}$$

*where $\gamma_1$ and $\gamma_2$ are class $K_\infty$ functions. Let $V(\eta) := \sum_{i=1}^{n} V_i(z_i) + U(w)$, and $z_i(0)$ belong to the set $z_i \in (-k_{b_1}, k_{b_1})$, $i = 1, 2, ..., n$. If the inequality holds:*

$$\dot{V} \;=\; \frac{\partial V}{\partial \eta} h \;\leq\; 0 \tag{2.29}$$

*then $z_i(t)$ remains in the open set $z_i \in (-k_{b_1}, k_{b_1})$ $\forall t \in [0, \infty)$.*

**Proof:** First, using Lemma 2.3.2, existence and uniqueness of the solution $\eta(t)$ is ensured for $t \in [0, \tau_{\max})$. This implies that $V(\eta(t))$ exists for $t \in [0, \tau_{\max})$. Then, from the fact that $\dot{V}(\eta) \leq 0$, we know that every $V_i(z_i(t))$, $i = 1, 2, ..., n$, is bounded for $t \in [0, \tau_{\max})$. Thus, $z_i(t)$ remains in the set $|z_i| < k_{b_1}$ for $t \in [0, \tau_{\max})$. We infer that $\eta(t)$ remains in a compact subset $K \subseteq \mathcal{N}$ for all $t \in [0, \tau_{\max})$. Based on Lemma 2.3.3, we conclude that $\eta(t)$ is defined for all $t \in [0, \infty)$, and that $z_i(t) \in (-k_{b_1}, k_{b_1})$ $\forall t \in [0, \infty)$. ■

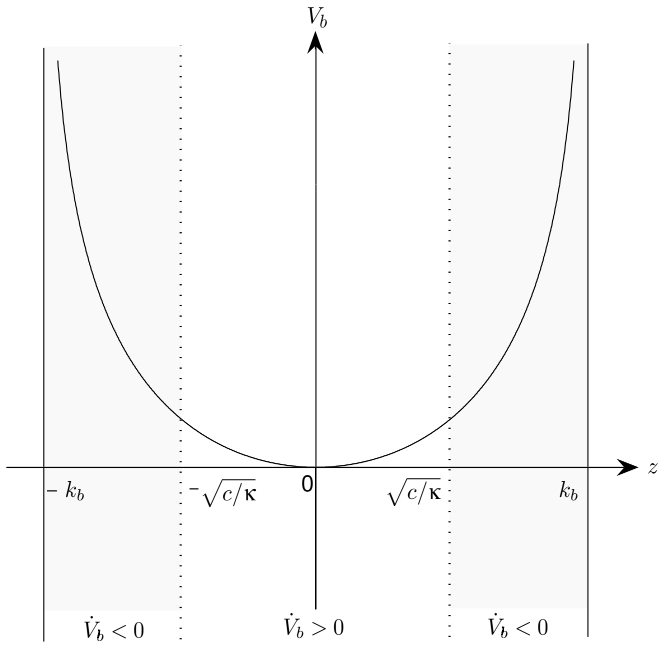


Figure 2.2: Schematic illustration of Barrier Lyapunov Function, $V_b$, and regions in which $\dot{V}_b \leq 0$, based on the inequality $\dot{V}_b \leq -\kappa z^2 + c$ and condition $\kappa > c/k_b^2$.

In Lemmata 2.4.1 and 2.4.2, non-violation of constraint is ensured with the condition that the derivative of the composite Lyapunov function is negative semidefinite, i.e. $\dot{V} \leq 0$. In the following result, we relax this condition to $\dot{V} \leq 0$ for $(z, w) \in \Omega_{zw} := \{z \in \mathbb{R}^n, w \in \mathbb{R}^r | \ \|z\| \leq d_1, \|w\| \leq d_2\}$, such that non-violation of constraint can still be guaranteed under some conditions on $\Omega_{zw}$. This result is useful in establishing conditions for practical stability with guaranteed non-violation of constraints, as detailed in Chapter 5.

**Lemma 2.4.3** *For any positive constant $k_{b_1}$, let $\mathcal{Z} := \{z \in \mathbb{R}^n : |z_i| < k_{b_1}, i = 1, 2, ..., n\} \subset \mathbb{R}^n$, $\mathcal{Z}_i := \{z_i \in \mathbb{R} : |z_i| < k_{b_1}\} \subset \mathbb{R}$, $i = 1, ..., n$, and $\mathcal{N} := \mathbb{R}^l \times \mathcal{Z} \subset \mathbb{R}^{n+l}$ be open sets. Consider the system*

$$\dot{\eta} \;=\; h(t, \eta) \tag{2.30}$$

*where $\eta := [w, z]^T \in \mathcal{N}$ is the state, and the function $h : \mathbb{R}_+ \times \mathcal{N} \to \mathbb{R}^{n+l}$ satisfies conditions (2.17)-(2.20). Suppose that there exist functions $U : \mathbb{R}^l \to \mathbb{R}_+$ and $V_i : \mathcal{Z}_i \to \mathbb{R}_+$, $i = 1, ..., n$, continuously differentiable and positive definite in their respective domains, such that*

$$V_i(z_i) \to \infty \quad \text{as} \quad z_i \to \pm k_{b_1} \tag{2.31}$$

$$\gamma_1(\|w\|) \leq U(w) \leq \gamma_2(\|w\|) \tag{2.32}$$

*where $\gamma_1$ and $\gamma_2$ are class $K_\infty$ functions. Let $V(\eta) := \sum_{i=1}^{n} V_i(z_i) + U(w)$, and $z_i(0)$ belong to the set $z_i \in (-k_{b_1}, k_{b_1})$, $i = 1, 2, ..., n$. If the inequality holds:*

$$\dot{V} \;=\; \frac{\partial V}{\partial \eta} h \;\leq\; -\sum_{i=1}^{n} \kappa_i z_i^2 - \varsigma\|w\|^2 + c \tag{2.33}$$

*where $\kappa_i > c/k_{b_i}^2$ and $c, \varsigma$ are positive constants, then $z_i(t)$ remains in the open set $z_i \in (-k_{b_1}, k_{b_1})$ $\forall t \in [0, \infty)$.*

**Proof:** Existence and uniqueness of the solution $\eta(t)$ of system (2.30), in the interval $t \in [0, \tau_{\max})$, is ensured with the help of Lemma 2.3.2. From (2.33), it is clear that $\dot{V} \leq 0$ whenever $\|w\| \geq \sqrt{c/\varsigma}$ and $|z_i| \geq \sqrt{c/\kappa_i}$, for $i = 1, 2, ..., n$. The condition $\kappa_i > c/k_{b_i}^2$ ensures that there exists a non-empty set

$$\Omega = \{z \in \mathbb{R}^n : \sqrt{c/\kappa_i} \leq |z_i| < k_{b_i}, i = 1, 2, ..., n\} \tag{2.34}$$

in which $\dot{V} \leq 0$. For illustrative purposes, Figure 2.2 shows such a set for a simplified case. Then, due to the fact that $V(\eta)$ is a positive function, we can show that it is upper bounded by the positive constant

$$V_b := V(\eta)\big|_{\{|z_i| = \sqrt{c/\kappa_i},\ \|w\| = \sqrt{c/\varsigma}\}} \tag{2.35}$$

if $V(\eta(0)) \leq V_b$, and bounded by $V(\eta(0))$ if $V(\eta(0)) > V_b$. As $V_i(z_i)$ and $U(w)$ are positive functions, and since $V(\eta)$ is bounded, we infer that each $V_i(z_i(t))$ is also bounded for $t \in [0, \tau\mathrm{max})$. Due to the fact that $V_{b_i}(z_i) \to \infty$ as $|z_i| \to k_{b_1}$, we have that $|z_i(t)| \neq k_{b_1}$. Hence, if $|z_i(0)| < k_{b_1}$, then $|z_i(t)| < k_{b_1}$ for all $t \in [0, \tau_{\max})$.

As a result, $\eta(t)$ belongs to a compact subset $K \subseteq \mathcal{N}$ for all $t \in [0, \tau_{\max})$. Then, based on Lemma 2.3.3, we have $\tau_{\max} = \infty$, such that $\eta(t)$ exists for all $t \in [0, \infty)$, and hence, $|z_i(t)| < k_{b_1}$ for all $t \in [0, \infty)$. ∎

The following lemma will be useful for computing the bounds of stabilizing functions $\alpha_i$ within a compact set to check the sufficient conditions for the case of state constraints.

**Lemma 2.4.4** *For any positive constants $\kappa_i$ and $k_{b_i}$, the following inequality holds for all $z_i$ in the interval $|z_i| \leq k_{b_i}$:*

$$|(k_{b_i}^2 - z_i^2)\kappa_i z_i| \leq \frac{2}{3\sqrt{3}}\kappa_i k_{b_i}^3 \tag{2.36}$$

**Proof:** Denote $p_i(z_i) := (k_{b_i}^2 - z_i^2)\kappa_i z_i$. The maximum value of $p_i(z_i)$ in the interval $|z_i| \leq k_{b_i}$ is obtained at the stationary points or the boundary points.

The stationary points of $p_i(z_i)$ are obtained from the equation:

$$\frac{\mathrm{d}p_i}{\mathrm{d}z_i} = \kappa_i(k_{b_i}^2 - 3z_i^2) = 0 \tag{2.37}$$

which yields $z_i = \frac{k_{b_i}}{\sqrt{3}}$ and $z_i = -\frac{k_{b_i}}{\sqrt{3}}$, both of which lies within the interval $|z_i| \leq k_{b_i}$. The corresponding values of $p_i(z_i)$ at these stationary points are, respectively:

$$p_i = \frac{2}{3\sqrt{3}}\kappa_i k_{b_i}^3 \quad \text{and} \quad p_i = -\frac{2}{3\sqrt{3}}\kappa_i k_{b_i}^3 \tag{2.38}$$

At the boundary points $|z_i| = \pm k_{b_i}$, we have that

$$p_i = 0 \tag{2.39}$$

Taking into account both stationary and boundary points, it is clear that

$$|p_i| \leq \frac{2}{3\sqrt{3}}\kappa_i k_{b_i}^3 \tag{2.40}$$

and thus, inequality (2.36) holds. ■

For clarity of presentation, we outline the method of employing BLF to design a control that does not violate constraints, for first and second order systems as motivating examples. Comparisons with QLFs are provided to show the relative advantage of BLFs in terms of less conservative initial condition requirements for systems with order greater than one. Since the main purpose of these examples is to motivate the use of BLFs, and for the sake of simplicity, we do not consider the presence of uncertainties. In subsequent chapters, the control design based on adaptive techniques for arbitrary $n$-order systems in strict feedback form will be detailed.

### 2.4.1 First Order SISO System

For simplicity, consider the following first order system:

$$\dot{y} = f(y) + g(y)u \tag{2.41}$$

where $f(y)$ and $g(y)$ are known smooth functions, $u \in \mathbb{R}$, and the output $y \in \mathbb{R}$ is required to satisfy $|y| < k_c$, with $k_c$ being a positive constant. Denote, by $z = y - y_d$, the tracking error, with $y_d(t)$ as the desired trajectory satisfying $|y_d| \leq A_0$. To design a control that does not drive $y$ out of the interval $(-k_c, k_c)$, we employ the following BLF candidate, originally proposed in [121]:

$$V = \frac{1}{2}\log\frac{k_b^2}{k_b^2 - z^2} \tag{2.42}$$

where $k_b = k_c - A_0$ denotes the constraint on $z$, that is, we require $|z| < k_b$. As seen from the schematic illustration of $V(z)$ in Figure 2.1a, the BLF escapes to infinity at $|z| = k_b$. It can be shown that $V$ is positive definite and $C^1$ in the open set $|z| < k_b$, and thus a valid Lyapunov function candidate. The derivative of $V$ is given by

$$\dot{V} = \frac{dV}{dz}\dot{z} = \frac{z\dot{z}}{k_b^2 - z^2} = \frac{z(f(y) + g(y)u - \dot{y}_d)}{k_b^2 - z^2} \tag{2.43}$$

for which the design of control

$$u = \frac{1}{g(y)}(-f(y) - (k_b^2 - z^2)\kappa z + \dot{y}_d) \tag{2.44}$$

where $\kappa > 0$ is a constant, yields $\dot{V} = -\kappa z^2$. Since $\dot{V} \leq 0$, it can be shown that $V(t) < V(0)\ \forall t > 0$. According to (2.42), we know that for $V(t)$ to be bounded, it has to be true that $|z(t)| \neq k_b$. Therefore, the tracking error $z$ remains in the region $|z(t)| < k_b$, for all initial conditions $|z(0)| < k_b$. Based on the fact that $y(t) = z(t) + y_d(t)$, and that $|z(t)| < k_b$ and $y_d(t) \leq A_0$, it is clear that the output $y(t)$ remains in the region $|y| < k_c\ \forall t > 0$.

It is interesting to note, for the first order case, that by employing QLFs, we can similarly ensure that the output constraint is satisfied, provided that the initial output satisfies some condition. Specifically, if we consider the Lyapunov function candidate $V = \frac{1}{2}z^2$, and the control $u = \frac{1}{g(y)}(-f(y) - \kappa z + \dot{y}_d)$, we obtain that $\dot{V} \leq 0$. Thus, $|z(t)| \leq |z(0)|\ \forall t > 0$, and, in order to ensure that $|z(t)| < k_b$, it suffices to impose the initial condition $|z(0)| < k_b$. We can see that the condition is the same regardless of whether the BLF or the QLF is used, although the control laws are slightly different. However, for systems of order 2 and above, it will be apparent that employing BLFs results in less conservative initial conditions.

### 2.4.2 Second Order SISO System

Consider the second order system in strict feedback form:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \\ y &= x_1 \end{aligned} \tag{2.45}$$

where $f_1(x_1)$, $f_2(x_1, x_2)$ $g_1(x_1)$ and $g_2(x_1, x_2)$ are smooth functions, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the output, and $x_1, x_2 \in \mathbb{R}$ are the states, with $y(t)$ required to satisfy $|y(t)| < k_{c_1}$ for all $t \geq 0$, with $k_{c_1}$ being a positive constant. We employ backstepping design as follows.

**Step 1** Define the error coordinates $z_1 = y - y_d$ and $z_2 = x_2 - \alpha_1$, where $\alpha_1$ is a stabilizing function to be designed. To design a control that does not drive $y$ out of the interval $|y| < k_{c_1}$, we choose the following BLF candidate in the first step of backstepping:

$$V_1 = \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} \tag{2.46}$$

where $k_{b_1} = k_{c_1} - A_0$. A schematic illustration of $V_1(z_1)$ is shown in Figure 2.1a. The derivative of $V_1$ is given by

$$\dot{V}_1 = \frac{z_1 \dot{z}_1}{k_{b_1}^2 - z_1^2} = \frac{z_1(f_1 + g_1(z_2 + \alpha_1) - \dot{y}_d)}{k_{b_1}^2 - z_1^2} \tag{2.47}$$

Designing the stabilizing function $\alpha_1$ as follows:

$$\alpha_1 = \frac{1}{g_1}(-f_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d) \tag{2.48}$$

where $\kappa_1 > 0$ is a constant, yields the following expression for the $z_1$ dynamics:

$$\dot{z}_1 = -(k_{b_1}^2 - z_1^2)\kappa_1 z_1 + g_1 z_2 \tag{2.49}$$

The derivative of $V_1$ can be rewritten as

$$\dot{V}_1 = -\kappa_1 z_1^2 + \frac{g_1 z_1 z_2}{k_{b_1}^2 - z_1^2} \tag{2.50}$$

with the coupling term $\frac{g_1 z_1 z_2}{k_{b_1}^2 - z_1^2}$ to be canceled in the subsequent step.

As a brief digression, observe that the second term of (2.48), $(k_{b_1}^2 - z_1^2)\kappa_1 z_1$, is designed to cancel the denominator $k_{b_1}^2 - z_1^2$ in the derivative of $V_1$, (2.47), so as to obtain the term $-\kappa_1 z_1^2$ in (2.50), which is crucial since it is negative semidefinite for all $z_1 \in \mathbb{R}$, independent of any condition on $z_1$. If we were to design the $\alpha_1$ as the following function:

$$\alpha_1^* = \frac{1}{g_1}(-f_1 - \kappa_1 z_1 + \dot{y}_d) \tag{2.51}$$

then it would yield

$$\dot{V}_1|_{\alpha_1 = \alpha_1^*} = \frac{-\kappa_1 z_1^2}{k_{b_1}^2 - z_1^2} + \frac{g_1 z_1 z_2}{k_{b_1}^2 - z_1^2} \tag{2.52}$$

where the term $\frac{-\kappa_1 z_1^2}{k_{b_1}^2 - z_1^2}$ is negative only if $|z_1| < k_{b_1}$. This restriction would preclude the use of Lemma 2.4.1 after the final step to assert that $|z_1(t)| < k_{b_1}$ $\forall t > 0$.

**Step 2** Denote $z_3 = x_3 - \alpha_2$, where $\alpha_2$ is a stabilizing function to be designed. Since $x_2$ does not need to be constrained, we choose Lyapunov function candidate by augmenting $V_1$ with a quadratic function as follows

$$V_2 = V_1 + \frac{1}{2}z_2^2 \tag{2.53}$$

The derivative of $V_2$ along the closed loop trajectories is given by

$$\dot{V}_2 = -\kappa_1 z_1^2 + \frac{g_1 z_1 z_2}{k_{b_1}^2 - z_1^2} + z_2(f_2 + g_2 u - \dot{\alpha}_1) \tag{2.54}$$

where $\dot{\alpha}_1$ is given by

$$\dot{\alpha}_1 \quad = \quad \frac{\partial \alpha_1}{\partial x_1}(f_1 + x_2) + \sum_{j=0}^{1} \frac{\partial \alpha_1}{\partial y_d^{(j)}} y_d^{(j+1)} \tag{2.55}$$

The stabilizing function $\alpha_2$ is designed as follows:

$$u = \frac{1}{g_2}\left(-f_2 + \dot{\alpha}_1 - \kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2}\right) \tag{2.56}$$

where $\kappa_2 > 0$ is constant, and the last term on the right hand side is designed to cancel the residual coupling term $\frac{g_1 z_1 z_2}{(k_{b_1}^2 - z_1^2)}$ left over from the first step. Hence, it can be obtained that

$$\dot{z}_2 = -\kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2} \tag{2.57}$$

Then, the derivative of $V_2$ can be rewritten as

$$\dot{V}_2 = -\sum_{i=1}^{2} \kappa_i z_i^2 \tag{2.58}$$

Based on the above expression, the following discussions and insights on the properties of the control and closed loop system are in order.

- **Output Constraint Satisfaction**
  Let the closed loop system (2.49) and (2.57) be written as $\dot{z} = h(t, z)$, where $z := [z_1, z_2]^T$. The right hand side $h(\cdot, \cdot)$ is locally integrable in $t$ and locally Lipschitz in $z \in \mathcal{Z} := \{z \in \mathbb{R}^2 : |z_1| < k_{b_1}\}$. In fact, it satisfies the conditions (2.17)-(2.20) for the existence and uniqueness of the solution $z(t)$. Together

with the fact that $\dot{V}_2 \leq 0$, we infer, from Lemma 2.4.1, that the error signal $z_1$ satisfies $|z_1(t)| < k_{b_1}$ $\forall t > 0$, provided that

$$|z_1(0)| < k_{b_1} \tag{2.59}$$

This can be intuitively understood by noting that since $V_2$ is positive definite and $\dot{V}_2 \leq 0$, it is implied that $V_2$ is bounded $\forall t > 0$. Because $V_2$ is bounded, we know that $|z_1| \neq k_{b_1}$. Given that $|z_1(0)| < k_{b_1}$, and that $z_1(t)$ is continuous, it can be concluded that $|z_1(t)| < k_{b_1}$, $\forall t > 0$. Then, it is straightforward to show, from $y(t) = z_1(t) + y_d(t)$, $|z_1(t)| < k_{b_1}$, and $|y_d(t)| \leq A_0$, that $|y(t)| < k_{b_1} + A_0 = k_{c_1}$. Thus, the output constraint will never be violated.

- **Bounded Control**

  From (2.56), it can be seen that there is a concern of $u$ becoming unbounded whenever $|z_1| = k_{b_1}$. However, we have established that, in the closed loop, the error signal $|z_1(t)|$ will never reach $k_{b_1}$ $\forall t > 0$. As a result, despite the presence of terms comprising $(k_{b_1}^2 - z_1^2)$ in the denominator, the control $u$ remains bounded for all time.

- **Comparison With QLF Based Design**

  By carefully choosing the initial conditions, it is possible to design backstepping control using QLFs to ensure that the output does not violate its constraint. The question that naturally arises is this:

  > *Can the control design based on QLFs meet the output constraint with the same, if not more relaxed, initial condition requirements than those based on BLFs?*

  The answer is negative, as we will demonstrate. Specifically, the initial condition requirement is more stringent when QLFs are employed. If we consider Lyapunov function candidate $V = \frac{1}{2}\sum_{i=1}^{2} z_i^2$, and the following backstepping control

$$\begin{aligned} \alpha_1 &= \frac{1}{g_1}(-f_1 - \kappa_1 z_1 + \dot{y}_d) \\ u &= \frac{1}{g_2}(-f_2 - \kappa_2 z_2 - g_1 z_1 + \dot{\alpha}_1) \end{aligned} \tag{2.60}$$

it can be shown that $\dot{V} \le -\rho V$, where $\rho = 2\min\{\kappa_1, \kappa_2\}$, which implies exponential stability, i.e., $\|z(t)\| \le \|z(0)\|e^{-\rho t}$, $\forall t > 0$. However, if we were to use the initial condition requirement (2.59), then exponential stability is insufficient to ensure that $|z_1(t)| < k_{b_1}$, as illustrated in Figure 2.3. Even though the norm of the vector $z(t)$ diminishes with time, the individual element $z_1(t)$ may still increase and possibly exceed the region $(-k_{b_1}, k_{b_1})$. Noting that $|z_1(t)| \le \|z(t)\| \le \|z(0)\|$, we know that a sufficient condition for $|z_1(t)| < k_{b_1}$ is

$$\begin{aligned} \|z(0)\| &< k_{b_1} \\ \Rightarrow |z_1(0)| &< \sqrt{k_{b_1}^2 - z_2^2(0)} \end{aligned} \tag{2.61}$$

Compared with (2.59), it is apparent that employing BLFs results in less conservative initial conditions. Another disadvantage is that the initial condition requirement (2.61) depends on the stabilizing function $\alpha_1$, due to its dependence on $z_2$, and thus restrict the control parameter selection. Although we have only shown the second order case, systems with order greater than two are also subject to the same limitation when backstepping control, based on QLFs, is employed.

- **Design Principle**

  Although the specific form of Barrier Lyapunov Function (2.46) that we employ in this study is similar to that in [121], there is a difference in the way the control is designed. According to the design methodology of [121], applied to system (2.45) for ease of discussion, cancelation of the residual coupling term $\frac{g_1 z_1 z_2}{k_{b_1}^2 - z_1^2}$, in the derivative of $V_1$, is avoided. Instead, completion of squares is used to separate it into two terms, namely $\frac{1}{2}\frac{g_1 z_1^2}{k_{b_1}^2 - z_1^2}$ and $\frac{1}{2}\frac{g_1 z_2^2}{k_{b_1}^2 - z_1^2}$. Then, based on the conditions that $|z_1(t)| < k_{b_1}$ and $|z_2(t)| < k_{b_2}$, $\forall t > 0$, where $k_{b_1}, k_{b_2}$ are positive constants, the control gains are designed to dominate the two square terms, such that the Lyapunov function derivative is negative semidefinite in the set $\{|z_1| < k_{b_1}, |z_2| < k_{b_2}\}$. In contrast, our method accommodates the cancelation of coupling terms, which involve partial derivatives of the BLF. Thus, the control directly inherits the properties of the BLF.

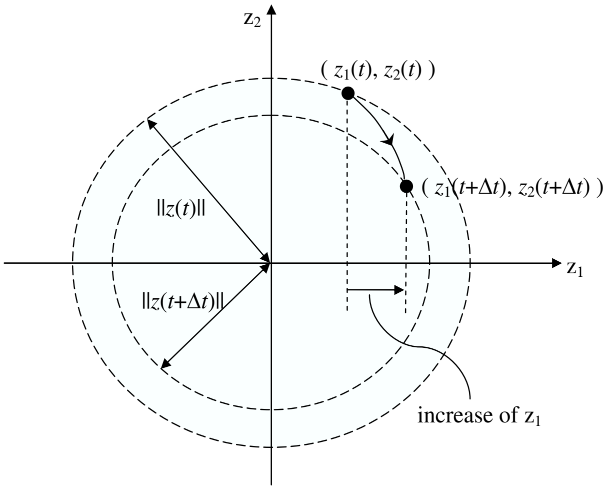


Figure 2.3: Exponential stability does not guarantee non-violation of constraint

### 2.4.3 MIMO Mechanical Systems

In the foregoing discussions, we dealt with the output constraint problem for first and second order SISO nonlinear systems, so as to elucidate the main ideas of using BLFs as a convenient design tool for handling system constraints. This section further provides an exposition on how to deal with the *full state constraint* problem for second order nonlinear systems. As a brief departure from SISO systems, which constitute the systems of interest in this thesis, we provide some insights into how the design tool of BLFs can be applied to a class of multi-input multi-output (MIMO) mechanical systems with constraint on the norm of the position vector.

Consider a class of fully-actuated mechanical system described by:

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) &= \tau \\ y &= q \end{aligned} \tag{2.62}$$

where $M(q) \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $C(q,\dot{q})\dot{q} \in \mathbb{R}^n$ are the Coriolis and centrifugal forces, $G(q) \in \mathbb{R}^n$ are the restoring forces and/or gravity, $q \in \mathbb{R}^n$ is the position vector, $\tau \in \mathbb{R}^n$ is the control input, and $y \in \mathbb{R}^n$ is the output.

For ease of control design, denote $q_1 = q$, $q_2 = \dot{q}$, and rewrite (2.62) into the following form suitable for backstepping:

$$
\begin{aligned}
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= M^{-1}(q_1)(-C(q_1, q_2)q_2 - G(q_1) + \tau) \\
y &= q_1
\end{aligned}
\tag{2.63}
$$

The control objective is to ensure that $q_1$ tracks a desired trajectory $q_d$ while keeping all closed loop signals bounded and preventing the position and velocity constraints from being violated. In other words, the position $q$ is required to remain in the set $\|q\| \le k_{c_1}$, and the velocity $\dot{q}$ in the set $\|\dot{q}\| \le k_{c_2}$, with $k_{c_1}$ and $k_{c_2}$ being positive constants. We make the assumption that the desired trajectory $q_d(t)$ is smooth, i.e. $\|\dot{q}_d\| < Q_1$, $\|\ddot{q}_d\| < Q_2$, where $Q_1$, $Q_2$ are positive constants. In addition, it is bounded by $\|q_d(t)\| \le A_0$, where $A_0$ is a positive constant that satisfies $A_0 < k_{c_1}$.

We follow a similar design procedure as that outlined in the second order SISO example, but in the second step of backstepping, another BLF is employed rather than a quadratic one. Another difference lies in a slight modification of the BLFs that involves the quadratic terms of the error vector, instead of the square term of the scalar error in the SISO case.

<u>**Step 1**</u> Denote $z_1 = q_1 - q_d$ and $z_2 = q_2 - \alpha_1$, where $\alpha_1$ is a stabilizing function to be designed shortly. Choose a Lyapunov function candidate as:

$$V_1 = \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^T z_1} \tag{2.64}$$

where $k_{b_1} = k_c - A_0$. The derivative is given by

$$\dot{V}_1 = \frac{z_1^T \dot{z}_1}{k_{b_1}^2 - z_1^T z_1} = \frac{z_1^T (z_2 + \alpha_1 - \dot{q}_d)}{k_{b_1}^2 - z_1^T z_1} \tag{2.65}$$

Design the stabilizing function $\alpha_1$ as:

$$\alpha_1 = -(k_{b_1}^2 - z_1^T z_1)\kappa_1 z_1 + \dot{q}_d \tag{2.66}$$

which yields

$$\dot{z}_1 = z_2 - (k_{b_1}^2 - z_1^T z_1)\kappa_1 z_1 \tag{2.67}$$

Therefore, the derivative of $V_1$ along the closed loop trajectories can be written as

$$\dot{V}_1 = -\kappa_1\|z_1\|^2 + \frac{z_1^T z_2}{k_{b_1}^2 - z_1^T z_1} \tag{2.68}$$

where the first term on the right hand side is negative definite and the second term is eliminated in the second step.

**Step 2** The control input $\tau$ is designed in this step. Let the Lyapunov function candidate be

$$V_2 = V_1 + \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_2^T M(q_1) z_2} \tag{2.69}$$

The derivative of $V_1$ along the closed loop trajectories is given by

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + \frac{z_2^T}{k_{b_1}^2 - z_2^T M(q_1) z_2}[-C(q_1,q_2)(z_2+\alpha_1) - G(q_1) + \tau - M(q_1)\dot{\alpha}_1 \\
&\quad + \frac{1}{2}\dot{M}(q_1,q_2)z_2] \\
&= \dot{V}_1 + \frac{z_2^T}{k_{b_1}^2 - z_2^T M(q_1) z_2}[-C(q_1,q_2)\alpha_1 - G(q_1) + \tau - M(q_1)\dot{\alpha}_1 \\
&\quad + \frac{1}{2}(\dot{M}(q_1,q_2) - 2C(q_1,q_2))z_2]
\end{aligned} \tag{2.70}$$

Due to skew symmetric property of $\dot{M}(q_1,q_2) - 2C(q_1,q_2)$ [48], the last term is zero, thereby yielding

$$\begin{aligned}
\dot{V}_2 &= -\kappa_1\|z_1\|^2 + \frac{z_2^T}{k_{b_2}^1 - z_2^T M(q) z_2}[-C(q_1,q_2)\alpha_1 - G(q_1) + \tau - M(q_1)\dot{\alpha}_1] \\
&\quad + \frac{z_1^T z_2}{k_{b_1}^2 - z_1^T z_1}
\end{aligned} \tag{2.71}$$

Design the control input $\tau$ as

$$\tau = (k_{b_1}^2 - z_2^T M z_2)\kappa_2 z_2 - \frac{(k_{b_1}^2 - z_2^T M z_2) z_1}{k_{b_1}^2 - z_1^T z_1} + C(q_1,q_2)\alpha_1 + G(q_1) + M(q_1)\dot{\alpha}_1 \tag{2.72}$$

where the derivative of $\alpha_1(q_1, q_d, \dot{q}_d)$ is given by

$$\dot{\alpha}_1 = \frac{\partial\alpha_1}{\partial q_1}q_2 + \sum_{j=0}^{1}\frac{\partial\alpha_1}{\partial q_d^{(j)}}q_d^{(j+1)} \tag{2.73}$$

Thus, it is obtained that

$$\dot{V}_2 \;\; = \;\; -\kappa_1\|z_1\|^2 - \kappa_2\|z_2\|^2 \tag{2.74}$$

which clearly implies that $V_2$ is bounded $\forall t > 0$. Since $V_2$ is bounded, we know that $\|z_1\| \neq k_{b_1}$. From the fact that $\|z_1(0)\| < k_{b_1}$, and that $z_1(t)$ is continuous, it can be concluded that $\|z_1(t)\| < k_{b_1}$, $\forall t > 0$. Then, it is straightforward to show, from $q_1(t) = z_1(t) + q_d(t)$ and $\|q_d(t)\| \leq A_0$, that $\|q_1(t)\| < k_{b_1} + A_0 = k_{c_1}$. Thus, transgression of the position constraint is safely prevented.

We can follow similar arguments to establish that the error vector $z_2(t)$ is also constrained in the region $\|z_2\| < k_{b_1}$ provided that $\|z_2(0)\| < k_{b_1}$. However, before we can conclude that the velocity signal $q_2(t)$ is constrained within $\|q_2\| < k_{c_2}$ via the relationship $q_2(t) = z_2(t)+\alpha_1(t)$, we need to show that there exists a positive constant $A_1$ satisfying the condition

$$k_{c_2} > A_1 + k_{b_1} \geq \sup\|\alpha_1(t)\| + k_{b_1} \tag{2.75}$$

If the above condition is satisfied, then $\|q_2(t)\| \leq k_{b_1} + A_1 < k_{c_2}$.

From the expression of $\alpha_1$ in (2.66), Lemma 2.4.4, and the fact that $\|\dot{q}_d\| \leq Q_1$, we know that an upper bound for $\|\alpha_1\|$ exists, and is given by:

$$\|\alpha_1\| \leq \frac{2}{3\sqrt{3}}\kappa_1 k_{b_1}^3 + Q_1 \tag{2.76}$$

By defining the right hand side of the above inequality as $A_1$, and then verifying, if possible, that there indeed exist some values of $\kappa_1$ such that condition (2.75) is satisfied, we can guarantee that the velocity constraint can be met with the proposed control. Note that the velocity constraint cannot be arbitrarily specified, but is subject to the feasibility condition (2.75) on the control parameter $\kappa_1$.

# Chapter 3

# Control of Output-Constrained Systems

## 3.1 Introduction

In this chapter, we generalize the use of BLFs for SISO nonlinear systems in strict feedback form with output constraint and known control gain functions, motivated by the fact that many practical systems are subject to constraints in the form of physical stoppages, saturation, or performance and safety specifications, wherein violation of the constraints during operation may result in performance degradation, hazards or system damage. Our method is based on constructing BLFs and keeping them bounded in the closed loop, which thereby ensures that the barriers are not transgressed. This is achieved by designing the control to render the derivative of the Lyapunov function negative semidefinite.

Our design of stabilizing functions involve the canceling of cross coupling terms, with post-design analysis revealing that the stabilizing functions remain bounded. This is different from the approach undertaken in [121], where canceling of cross coupling terms was avoided, but instead, control gains were carefully chosen to dominate them. While [121] deals with systems in Brunovsky form, we consider the strict feedback form with nonlinearities appearing in each differential equation. Moreover, we design adaptive controllers to handle the presence of parametric uncertainty in the nonlinearities, while simultaneously preventing constraints from being violated. We also propose novel asymmetrical BLFs for added flexibility in control design as well as performance enhancement, and provide rigorous treatment of the issue of continuously differentiable properties of the stabilizing functions. Furthermore, we provide comparison with QLFs, and show that they result in more conservative initial conditions than those resulting from BLFs.

The remainder of this chapter is organized as follows. Section 3.2 introduces the problem of tracking control for nonlinear strict feedback systems with constraint in the output, with consideration of parametric uncertainties. In Sections 3.3-3.4, we present the control design for the case of output constraints, considering full knowledge of the plant dynamics as well as the presence of parametric uncertainties, based on symmetric and asymmetric BLFs. To put the proposed methods based on BLFs in perspective with conventional methods based on QLFs, a brief comparison study is presented in Section 3.5, where it is shown that QLFs lead to more conservative requirements on initial conditions. Finally, following the simulation study in Section 3.6 to demonstrate the effectiveness of the control, concluding remarks will be made in Section 3.7.

## 3.2 Problem Formulation and Preliminaries

Consider the nonlinear system in strict feedback form:

$$\begin{aligned}
\dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, \quad i = 1, 2, ..., n-1 \\
\dot{x}_n &= f_n(\bar{x}_n) + g_n(\bar{x}_n)u \\
y &= x_1 \qquad (3.1)
\end{aligned}$$

where $x_i \in \mathbb{R}$, $i = 1, 2, ..., n$ are the states, $\bar{x}_i := [x_1, x_2, ..., x_i]^T$, $f_i$ and $g_i$ are smooth functions, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the input and output respectively. We consider the problem of output constraint, where the output is required to remain in the set $|y| \leq k_{c_1}$, with $k_{c_1}$ a positive constant.

The nonlinear functions $f_i(\bar{x}_i)$ may be uncertain, in which case they satisfy the following linear-in-the-parameters (LIP) condition:

$$f_i(\bar{x}_i) = \theta^T \psi_i(\bar{x}_i), \quad i = 1, ..., n \tag{3.2}$$

where $\psi_1, ..., \psi_n$ are smooth functions, and $\theta \in \mathbb{R}^l$ is a vector of uncertain parameters satisfying $\|\theta\| \leq \theta_M$ with known positive constant $\theta_M$.

The control objective is to track a desired trajectory $y_d$ while ensuring that all closed loop signals are bounded and that the *output constraint is not violated.* Note that the output constraint may not necessarily be a physical constraint, but can also be associated with performance requirements.

Throughout this chapter, for notational convenience, we group the derivatives of the desired trajectory in the vector $\bar{y}_{d_i} := [y_d^{(1)}, y_d^{(2)}, ..., y_d^{(i)}]^T$. The following assumptions on the desired trajectory $y_d$, as well as the control gain functions $g_i(\cdot)$, $i = 1, ..., n$, from (3.1), are in order.

**Assumption 3.2.1** *For any $k_{c_1} > 0$, there exist positive constants $\underline{Y}_0$, $\overline{Y}_0$, $A_0$, $Y_1$, $Y_2$,..., $Y_n$ satisfying*

$$\max\{\underline{Y}_0, \overline{Y}_0\} \leq A_0 < k_{c_1} \tag{3.3}$$

*such that the desired trajectory $y_d(t)$ and its time derivatives satisfy*

$$-\underline{Y}_0 \leq y_d(t) \leq \overline{Y}_0, \quad |\dot{y}_d(t)| < Y_1, \quad |\ddot{y}_d(t)| < Y_2, \quad \cdots, \quad |y_d^{(n)}(t)| < Y_n \tag{3.4}$$

*for all $t \geq 0$.*

**Assumption 3.2.2** *The control gain functions $g_i(\cdot)$, $i = 1, 2, ..., n$, are known, and there exists a positive constant $g_0$ such that $0 < g_0 \leq |g_i(\cdot)|$. Without loss of generality, we further assume that the $g_i(\cdot)$ are all positive.*

## 3.3 Control Design

In this section, control design and analysis are presented, based on the fusion of barrier functions, backstepping, and adaptive control techniques. We first consider the case where the system model is known, and employ BLF to ensure that the output remains constrained, along with stability and performance properties. In the subsequent section which deals with the presence of parametric uncertainty, we show that, by incorporating barrier function in adaptive backstepping design, the output constraint is not violated at any time despite the presence of adaptation dynamics.

### 3.3.1 Known Case

First, we consider the case where the functions $f_i(\bar{x}_i)$ and $g_i(\bar{x}_i)$ are known. The control design is based on backstepping, with BLF candidate employed in the first step to impose constraint on the tracking error. Constraint on the output follows from the bounds of the desired trajectory, and the constraint on the tracking error, which is enforced through design. The remaining steps employ QLF candidates.

Since backstepping design has been well studied and mature, we shall omit the detailed procedure for a concise presentation. Denote $z_1 = x_1 - y_d$ and $z_i = x_i - \alpha_{i-1}$, $i = 2, ..., n$. The first two steps of the backstepping design are similar to that presented in Section 2.4.2 for second order strict feedback systems. From Step 3 onwards, the design procedure is identical to the standard backstepping using QLFs.

By designing the stabilizing functions and control law as follows

$$\alpha_1 = \frac{1}{g_1}(-f_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d) \tag{3.5}$$

$$\alpha_2 = \frac{1}{g_2}\left(-f_2 + \dot{\alpha}_1 - \kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2}\right) \tag{3.6}$$

$$\alpha_i = \frac{1}{g_i}(-f_i + \dot{\alpha}_{i-1} - \kappa_i z_i - g_{i-1} z_{i-1}), \quad i = 3, ..., n \tag{3.7}$$

$$u = \alpha_n \tag{3.8}$$

where $k_{b_1} = k_{c_1} - A_0$, $\kappa_i > 0$ is a constant, and $\dot{\alpha}_{i-1}$ is given by

$$\dot{\alpha}_{i-1} = \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{x_j}(f_j(\bar{x}_j) + g_j(\bar{x}_j)x_{j+1}) + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)}, \quad i = 2, ..., n \tag{3.9}$$

the closed loop system can be written as:

$$\begin{aligned}
\dot{z}_1 &= -(k_{b_1}^2 - z_1^2)\kappa_1 z_1 + g_1 z_2 && (3.10)\\
\dot{z}_2 &= -\kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2} + g_2 z_3 && (3.11)\\
\dot{z}_i &= -\kappa_i z_i - g_{i-1} z_{i-1} + z_{i+1}, \quad i = 3, ..., n-1 && (3.12)\\
\dot{z}_n &= -\kappa_n z_n - g_{n-1} z_{n-1} && (3.13)
\end{aligned}$$

Consider the Lyapunov function candidate $V_n$ composed by:

$$\begin{aligned}
V_1 &= \frac{1}{2}\log\frac{k_{b_1}}{k_{b_1}^2 - z_1^2} && (3.14)\\
V_i &= V_{i-1} + \frac{1}{2} z_i^2, \quad i = 2, ..., n && (3.15)
\end{aligned}$$

The derivative of $V_n$ along the closed loop tajectories can be rewritten as

$$\dot{V}_n = -\sum_{j=1}^{n} \kappa_j z_j^2 \leq 0 \tag{3.16}$$

Let the closed loop system (3.10)-(3.13) be written as $\dot{z} = h(t, z)$. The right hand side $h(t, z)$ is locally integrable in $t$ and locally Lipschitz in $z \in \mathcal{Z} := \{z \in \mathbb{R}^n : |z_1| < k_{b_1}\}$. In fact, it satisfies the conditions (2.17)-(2.20) for the existence and uniqueness of the solution $z(t)$. Together with (3.16), we infer, from Lemma 2.4.1, that $|z_1(t)| < k_{b_1}$ $\forall t > 0$, provided that $|z_1(0)| < k_{b_1}$.

**Remark 3.3.1** *It is seen from (3.6) that there is a possibility of $\alpha_2$ becoming unbounded whenever $z_1 = k_{b_1}$. Moreover, the propagation of the derivatives of $\alpha_1$, down to the design of control $u$ in the final step of backstepping, will result in even more terms comprising $(k_{b_1}^2 - z_1^2)$ in the denominator. We address this issue in Theorem 3.3.1, where we formally show that, in the closed loop, under some restrictions on the initial conditions, the error signal $z_1(t)$ never reaches $k_{b_1}$ $\forall t > 0$. As a result, the stabilizing functions $\alpha_2, ..., \alpha_{n-1}$ and the control $u$ does not become unbounded because of the presence of terms comprising $(k_{b_1}^2 - z_1^2)$ in the denominator.*

**Theorem 3.3.1** *Consider the closed loop system (3.1), (3.8) under Assumptions 3.2.1-3.2.2. If the initial conditions are such that $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_1| < k_{b_1}\}$, where $\bar{z}_n := [z_1, ..., z_n]^T$, then the following properties hold.*

*i) The signals $z_i(t)$, $i = 1, 2, ..., n$, remain in the compact set defined by*

$$\Omega_z = \left\{ \bar{z}_n \in \mathbb{R}^n : |z_1| \le D_{z_1},\ \|z_{2:n}\| \le \sqrt{2V_n(0)} \right\} \tag{3.17}$$

$$D_{z_1} = k_{b_1}\sqrt{1 - e^{-2V_n(0)}} < k_{b_1} \tag{3.18}$$

*where $z_{j:k} := [z_j, z_{j+1}, ..., z_{k-1}, z_k]^T$.*

*ii) The output $y(t)$ remains in the set $\Omega_y := \{y \in \mathbb{R} : |y| \le D_{z_1} + A_0 < k_{c_1}\}$ $\forall t > 0$, i.e. output constraint is never violated.*

*iii) All closed loop signals are bounded.*

*iv) The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

**Proof:** The properties $(i) - (iv)$ will be proved in sequence as follows.

i) Note, from $\dot{V}_n \le 0$, that $V_n(t) \le V_n(0)$, which implies that:

$$\frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} \le V_n(0) \tag{3.19}$$

Taking exponentials on both sides of the inequality, it is easy to see that

$$\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} \le e^{2V_n(0)} \tag{3.20}$$

For $|z_1(0)| < k_{b_1}$, we have, from Lemma 2.4.1, that $k_{b_1}^2 - z_1^2(t) > 0\ \forall\ t$. Multiplying both sides by $(k_{b_1}^2 - z_1^2)$ yields

$$k_{b_1}^2 \le e^{2V_n(0)}(k_{b_1}^2 - z_1^2) \tag{3.21}$$

which leads to the inequality

$$|z_1| \le k_{b_1}\sqrt{1 - e^{-2V_n(0)}} \tag{3.22}$$

Similarly, from the fact that $\frac{1}{2}\sum_{j=2}^{n} z_j^2 \le V_n(0)$, it follows that $\|z_{2:n}\| \le \sqrt{2V_n(0)}$. Therefore, $z_i(t)$ remains in the compact set $\Omega_z\ \forall\ t > 0$.

ii) It is straightforward to show, from $y(t) = z_1(t) + y_d(t)$, $|z_1(t)| \le D_{z_1} < k_{b_1}$, and $|y_d(t)| \le A_0$, that $|y(t)| \le D_{z_1} + A_0 < k_{b_1} + A_0 = k_{c_1}$. Hence, we conclude that $y(t) \in \Omega_y \ \forall t > 0$.

iii) From $\dot{V}_n \le 0$ and Lemma 2.4.1, we know that the error signals $z_1(t)$, ..., $z_n(t)$ are bounded. The boundedness of $z_1(t)$ and the reference trajectory $y_d(t)$ imply that the state $x_1(t)$ is bounded. Together with the fact that $\dot{y}_d(t)$ is bounded from Assumption 3.2.1, it is clear that the stabilizing function $\alpha_1(t)$ is also bounded from (3.5). This leads to the boundedness of state $x_2(t) = z_2(t) + \alpha_1(t)$. From (3.17), we have that $|z_1(t)| \le D_{z_1} < k_{b_1} \ \forall t > 0$. Since $\alpha_2$ is a smooth function of the bounded signals $\bar{x}_2(t)$, $\bar{z}_2(t)$, and $\bar{y}_{d_2}(t)$ in the interval $z_1 \in (-k_{b_1}, k_{b_1})$, we know that $\alpha_2(t)$ is bounded. This leads to the boundedness of the state $x_3(t) = z_3(t) + \alpha_2(t)$. Following this line of argument, we can progressively show that each $\alpha_i(t)$, for $i = 3, ..., n-1$, is bounded, since it is a smooth function of the bounded signals $\bar{x}_i(t)$, $\bar{z}_i(t)$, and $\bar{y}_{d_i}(t)$ in the interval $z_1 \in (-k_{b_1}, k_{b_1})$. Thus, the boundedness of the state $x_{i+1} = z_{i+1} + \alpha_i$ can be shown. Since $\bar{x}_n(t)$, $\bar{z}_n(t)$ are bounded, and $|z_1(t)| \le D_{z_1} < k_{b_1}$, we conclude that control $u(t)$ is bounded. Hence, all closed loop signals are bounded.

iv) Based on (3.10), (3.11), (3.12), and (3.13), we write $\ddot{V}_n$ as:

$$\begin{aligned} \ddot{V}_n &= 2(k_{b_1}^2 - z_1^2)\kappa_1^2 z_1^2 + 2\sum_{j=2}^{n} \kappa_j^2 z_j^2 + \frac{2\kappa_2 g_1 z_1 z_2}{k_{b_1}^2 - z_1^2} + 2\sum_{j=3}^{n} \kappa_j g_{j-1} z_{j-1} z_j \\ &\quad -2\sum_{j=1}^{n-1} \kappa_j g_j z_j z_{j+1} \end{aligned}$$

From the fact that $x_i, z_i$, $i = 1, ..., n$ are bounded, and particularly $|z_1(t)| < k_{b_1}$, it is obvious that $\ddot{V}_n(t)$ is bounded, which means that $\dot{V}_n(t)$ is uniformly continuous. Then, by Lemma 2.3.1 (Barbalat's Lemma), $z_i(t) \to 0$ as $t \to \infty$. Since $z_1(t) = y(t) - y_d(t)$, it is clear that $y(t) \to y_d(t)$ as $t \to \infty$. ■

**Remark 3.3.2** *While our investigations focus on (3.14) as the BLF, this choice is by no means unique. Any positive definite, $C^1$ continuously differentiable function, $V_1$, which satisfies the condition $V_1(z_1) \to \infty$ as $z_1 \to \pm k_{b_1}$ is a valid candidate.*

*However, different choices of* $V_1$ *result in different control designs with different performances. An alternative to (3.14) which satisfies these conditions is the barrier function* $V_1 = \frac{k_{b_1}}{\pi}\tan^2(\frac{\pi z_1}{2k_{b_1}})$*, which yields different stabilizing functions in the first 2 steps of backstepping:*

$$\begin{aligned}\alpha_1 &= \frac{1}{g_1}\left(-f_1 + \dot{y}_d - \frac{\kappa_1 z_1^2 \cos^3(\frac{\pi z_1}{2k_{b_1}})}{\sin(\frac{\pi z_1}{2k_{b_1}})}\right)\\ \alpha_2 &= \frac{1}{g_2}\left(-f_2 + \dot{\alpha}_1 - \kappa_2 z_2 - \frac{g_1 \sin(\frac{\pi z_1}{2k_{b_1}})}{\cos^3(\frac{\pi z_1}{2k_{b_1}})}\right)\end{aligned} \tag{3.23}$$

*The remaining stabilizing functions and final control are of identical form to those in the foregoing presentation, and can be shown to finally yield* $\dot{V}_n \leq 0$*, from which closed loop properties similar to those in Theorem 3.3.1 can be achieved. Throughout this thesis, similar arguments may be made for the various cases considered.*

**Remark 3.3.3** *Although we have established, in Theorem 3.3.1, the fact that all signals are bounded, there is a practical concern that the control* $u(t)$ *may grow to a large value when the term* $(k_{b_1}^2 - z_1(t)^2)$ *becomes small. This can be viewed as a drawback of the proposed method. Nevertheless, from (3.17)-(3.18), we know that a computable bound for* $z_1(t)$ *can be obtained, which is dependent on the initial conditions* $z_i(0)$*,* $i = 1, ..., n$*. Thus, by careful selection of the control parameters, it is possible to limit the growth of the control signal within a desirable operating range.*

### 3.3.2 Uncertain Case

In this section, we consider the system (3.1) in which the nonlinear functions $f_i(\bar{x}_i)$ are uncertain, and satisfy the LIP condition (3.2). One advantage of Lyapunov based backstepping designs is that it can be readily modified to accommodate parametric uncertainty via well-established adaptive backstepping techniques. By employing BLF in the first step of backstepping, we can guarantee asymptotic convergence of output tracking error in the presence of parametric uncertainties, and, at the same time, ensure that the output constraint is never violated, especially throughout the transient stages of adaptation. Subsequent steps are still based on quadratic Lyapunov functions.

Since adaptive backstepping design has been well studied and mature, we shall omit giving a detailed step-by-step account of the procedure. Interested readers are referred to [94]. Denote $z_1 = x_1 - y_d$ and $z_i = x_i - \alpha_{i-1}$, $i = 2, ..., n$. Consider the Lyapunov function candidate $V_n$ composed by:

$$V_1 = \frac{1}{2}\log\frac{k_{b_1}}{k_{b_1}^2 - z_1^2} + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \quad (3.24)$$

$$V_i = V_{i-1} + \frac{1}{2}z_i^2, \quad i = 2, ..., n \quad (3.25)$$

where $k_{b_1} = k_{c_1} - A_0$, $\Gamma_1 = \Gamma_1^T > 0$ is constant matrix, and $\tilde{\theta} := \hat{\theta} - \theta$ is the error between $\theta$ and its estimate, $\hat{\theta}$. Note that $V_n$ is positive definite and continuously differentiable in the set $|z_1| < k_{b_1}$. The adaptive backstepping control is designed as follows:

$$\alpha_1 = \frac{1}{g_1}(-\hat{\theta}^T w_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d) \quad (3.26)$$

$$\alpha_2 = \frac{1}{g_2}\left(-\hat{\theta}^T w_2 - \kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2} + \frac{\partial\alpha_1}{\partial x_1}x_2 + \sum_{j=0}^{1}\frac{\partial\alpha_1}{\partial y_d^{(j)}}y_d^{(j+1)} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\tau_2\right) \quad (3.27)$$

$$\alpha_i = \frac{1}{g_i}\left(-\hat{\theta}^T w_i - \kappa_i z_i - g_{i-1}z_{i-1} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{x_j}x_{j+1} + \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i + \sum_{j=2}^{i-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma w_i z_j\right), \quad i = 3, ..., n \quad (3.28)$$

$$w_1 = \psi_1(x_1), \qquad w_i = \psi_i(\bar{x}_i) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\psi_j(\bar{x}_j) \quad (3.29)$$

$$\tau_1 = \frac{w_1 z_1}{k_{b_1}^2 - z_1^2}, \qquad \tau_i = \tau_{i-1} + w_i z_i \quad (3.30)$$

$$u = \alpha_n \quad (3.31)$$

$$\dot{\hat{\theta}} = \Gamma\tau_n \quad (3.32)$$

which yields the closed loop system

$$\dot{z}_1 = -(k_{b_1}^2 - z_1^2)\kappa_1 z_1 + g_1 z_2 - \tilde{\theta}^T\psi_1(x_1) \quad (3.33)$$

$$\dot{z}_2 = -\kappa_2 z_2 - \frac{g_1 z_1}{k_{b_1}^2 - z_1^2} + g_2 z_3 - \tilde{\theta}^T w_2 + \frac{\partial\alpha_1}{\partial\hat{\theta}}(\Gamma\tau_2 - \dot{\hat{\theta}}) \quad (3.34)$$

$$\dot{z}_i = -\kappa_i z_i - g_{i-1}z_{i-1} + g_i z_{i+1} - \tilde{\theta}^T w_i + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}(\Gamma\tau_i - \dot{\hat{\theta}}) + \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_i z_j \tag{3.35}$$

$$\dot{z}_n = -\kappa_n z_n - g_{n-1}z_{n-1} - \tilde{\theta}^T w_n + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}(\Gamma\tau_n - \dot{\hat{\theta}}) + \sum_{j=2}^{n-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_n z_j \tag{3.36}$$

$$\dot{\hat{\theta}} = \Gamma\tau_n \tag{3.37}$$

and the derivative of $V_n$ along (3.33)-(3.37) as

$$\dot{V}_n = -\sum_{j=1}^{n} \kappa_j z_j^2 \tag{3.38}$$

Let the closed loop system (3.33)-(3.37) be written as $\dot{\eta} = h(t, \eta)$, where $\eta = [z^T,\ \tilde{\theta}^T]^T$. The right hand side $h(t, \eta)$ satisfies the conditions (2.17)-(2.20) in the open set $\eta \in \mathcal{Z} := \{z \in \mathbb{R}^n, \tilde{\theta} \in \mathbb{R}^l : |z_1| < k_{b_1}\}$. Thus, the existence and uniqueness of the solution $\eta(t)$ is ensured. Then, we infer from (3.38) and Lemma 2.4.1 that $|z_1(t)| < k_{b_1}\ \forall t > 0$, provided that $|z_1(0)| < k_{b_1}$.

**Theorem 3.3.2** *Consider the closed loop system (3.1), (3.31), (3.32) under Assumptions 3.2.1-3.2.2. If the initial conditions are such that $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_1| < k_{b_1}\}$, where $\bar{z}_n := [z_1, z_2, ..., z_n]^T$, then the following properties hold.*

*i) The signals $z_i(t)$, $i = 1, 2, ..., n$, and $\hat{\theta}(t)$ remain in the compact sets defined by*

$$\Omega_z = \left\{\bar{z}_n \in \mathbb{R}^n : |z_1| \le D_{z_1},\ \|z_{2:n}\| \le \sqrt{2\bar{V}_n}\right\} \tag{3.39}$$

$$\Omega_{\hat{\theta}} = \left\{\hat{\theta} \in \mathbb{R}^l : \|\hat{\theta}\| \le \theta_M + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma^{-1})}}\right\} \tag{3.40}$$

$$\bar{V}_n = \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2(0)} + \frac{1}{2}\sum_{j=2}^{n} z_j^2(0) + \frac{1}{2}\lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}(0)\| + \theta_M)^2 \tag{3.41}$$

$$D_{z_1} = k_{b_1}\sqrt{1 - e^{-2\bar{V}_n}} < k_{b_1} \tag{3.42}$$

*where $z_{j:k} := [z_j, z_{j+1}, ..., z_k]^T$.*

*ii) The output $y(t)$ remains in the set $\Omega_y := \{y \in \mathbb{R} : |y| \le D_{z_1} + A_0 < k_{c_1}\}$ $\forall t > 0$, i.e. the output constraint is never violated.*

*iii) All closed loop signals are bounded.*

*iv) The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

**Proof:** The properties $(i) - (iv)$ will be proved in sequence as follows.

i) From the fact that $\dot{V}_n \leq 0$, it is clear that $V_n(t) \leq V_n(0)$. From (3.24) and the fact that $\|\theta\| \leq \theta_M$, we know that $V_n(0)$ satisfies:

$$\begin{aligned} V_n(0) &\leq \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2(0)} + \frac{1}{2}\sum_{j=2}^{n} z_j^2(0) + \frac{1}{2}\lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}(0)\| + \theta_M)^2 \\ &= \bar{V}_n \end{aligned} \tag{3.43}$$

which implies that

$$\begin{aligned} \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} &\leq \bar{V}_n \\ \Rightarrow \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} &\leq e^{2\bar{V}_n} \end{aligned} \tag{3.44}$$

For $|z_1(0)| < k_{b_1}$, we have, from Lemma 2.4.1, that $k_{b_1}^2 - z_1^2(t) > 0 \; \forall \; t$. Thus, we obtain

$$k_{b_1}^2 \leq e^{2\bar{V}_n}(k_{b_1}^2 - z_1^2) \tag{3.45}$$

which leads to the inequality

$$|z_1| \leq k_{b_1}\sqrt{1 - e^{-2\bar{V}_n}} \tag{3.46}$$

Similarly, from the fact that $\frac{1}{2}\sum_{j=2}^{n} z_j^2 \leq \bar{V}_n(0)$, we easily show that $\|z_{2:n}\| \leq \sqrt{2\bar{V}_n}$. Therefore, we obtain that $z_i(t)$ remains in the compact set $\Omega_z \; \forall \; t > 0$.

Furthermore, from the fact that $V_n(t) \leq V_n \leq \bar{V}_n(0)$, we have that

$$\begin{aligned} \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} &\leq \bar{V}_n \\ \Rightarrow \frac{1}{2}\lambda_{\min}(\Gamma^{-1})\|\hat{\theta} - \theta\|^2 &\leq \bar{V}_n \end{aligned} \tag{3.47}$$

It is straightforward to show that $\|\hat{\theta}\| \leq \theta_M + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma^{-1})}}$ such that $\hat{\theta}$ remains in the compact set $\Omega_{\hat{\theta}} \; \forall \; t$, thus proving $(i)$.

ii) It is straightforward to show, from $y(t) = z_1(t) + y_d(t)$, $|z_1(t)| \leq D_{z_1} < k_{b_1}$, and $|y_d(t)| \leq A_0$, that $|y(t)| < k_{b_1} + A_0 = k_{c_1}$. Hence, we conclude that $y(t) \in \Omega_y$ $\forall t > 0$.

iii) From $\dot{V}_n \leq 0$ and Lemma 2.4.1, we know that the error signals $z_1(t)$, ..., $z_n(t)$, $\tilde{\theta}(t)$ are bounded. Since $\theta$ is constant, we have that $\hat{\theta}(t)$ is bounded. The boundedness of $z_1(t)$ and the reference trajectory $y_d(t)$ imply that the state $x_1(t)$ is bounded. Together with the fact that $\dot{y}_d(t)$ is bounded from Assumption 3.2.1, it is clear that $\alpha_1(t)$ is also bounded from (3.26). This leads to boundedness of the state $x_2(t) = z_2(t) + \alpha_1(t)$. From (3.39), we have that $|z_1(t)| \leq D_{z_1} < k_{b_1}$. Since $\alpha_2$ is a smooth function of the bounded signals $\bar{x}_2(t)$, $\bar{z}_2(t)$, $\bar{y}_{d_2}(t)$, and $\hat{\theta}(t)$ in the set $z_1 \in (-k_{b_1}, k_{b_1})$, we know that $\alpha_2(t)$ is bounded. This leads to boundedness of the state $x_3(t) = z_3(t) + \alpha_2(t)$. Following this line of argument, we can progressively show that each $\alpha_i(t)$, for $i = 1, ..., n-1$, is bounded, since it is a smooth function of the bounded signals $\bar{x}_i(t)$, $\bar{z}_i(t)$, $\bar{y}_{d_i}(t)$, and $\hat{\theta}(t)$ in the set $z_1 \in (-k_{b_1}, k_{b_1})$. Thus, the boundedness of state $x_{i+1} = z_{i+1} + \alpha_i$ can be shown. Since $\bar{x}_n(t)$, $\bar{z}_n(t)$, $\bar{y}_{d_n}(t)$, and $\hat{\theta}(t)$ are bounded, and $|z_1(t)| < k_{b_1}$, we conclude that control $u(t)$ is bounded. Hence, all closed loop signals are bounded.

iv) Lastly, to show that $y(t) \to y_d(t)$ as $t \to \infty$, note, from (3.33), (3.34), (3.35), and (3.36), that $\ddot{V}_n$ can be computed as follows:

$$\begin{aligned}\ddot{V}_n \;=\;& 2(k_{b_1}^2 - z_1^2)\kappa_1^2 z_1^2 + 2\sum_{j=2}^{n}\kappa_j^2 z_j^2 + \frac{2\kappa_2 z_1 z_2}{k_{b_1}^2 - z_1^2} + 2\sum_{j=3}^{n}\kappa_j g_{j-1} z_{j-1} z_j \\ & -2\sum_{j=1}^{n-1}\kappa_j g_j z_j z_{j+1} + 2\sum_{j=1}^{n}\tilde{\theta}^T w_j \kappa_j z_j - \sum_{j=2}^{n}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma(\tau_j - \tau_n)\kappa_j z_j \\ & -2\sum_{k=3}^{n-1}\sum_{j=2}^{k-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma w_k \kappa_k z_k z_j \end{aligned} \tag{3.48}$$

From the fact that $\hat{\theta}, x_i, z_i$, $i = 1, ..., n$ are bounded, and particularly $|z_1(t)| \leq k_{b_1}$, it can be shown that $\omega_j$ and $\tau_j$ are bounded. As a result, $\ddot{V}_n(t)$ is bounded, which means that $\dot{V}_n(t)$ is uniformly continuous. Then, by Barbalat's Lemma, we obtain that $z_i(t) \to 0$ as $t \to \infty$. Since $z_1(t) = x_1(t) - y_d(t)$ and $y(t) = x_1(t)$, it is clear that $y(t) \to y_d(t)$ as $t \to \infty$. ∎

**Remark 3.3.4** *For robustness to unmodeled dynamics, the adaptation law* $\dot{\hat{\theta}} = \Gamma\tau_n$ *may be modified by leakage terms or projection operators, which are commonly employed in robust adaptive control designs [42, 66]. However, a leakage term in the adaptation law destroys the closed loop asymptotic tracking property so that only practical tracking within a neighborhood of the reference trajectory is achievable.*

## 3.4 Asymmetric Barrier Lyapunov Function

Asymmetric barrier functions include symmetric ones as a special class, and thus are, in this sense, more general. To achieve greater flexibility in control design and to relax conditions on starting values of the output, asymmetric barrier functions can be employed. This can be understood by noting that an additional parameter $k_{a_1}$, where $k_{a_1} \neq k_{b_1}$, is now available for consideration in the control design to keep the closed loop tracking error $z_1(t)$ constrained $\forall t > 0$. Consequently, it allows the possible relaxation of $k_{a_1}$, independent of $k_{b_1}$, and vice versa, subject to the upper and lower bounds of the desired trajectory $y_d$.

In the following, we first present the design procedure and results for the case of known systems, and then only state the results for the adaptive case.

<u>**Step 1**</u> Denote $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, where $\alpha_1$ is a stabilizing function to be designed. Choose an asymmetric BLF candidate as:

$$V_1 = \frac{1}{p} q(z_1) \log \frac{k_{b_1}^p}{k_{b_1}^p - z_1^p} + \frac{1}{p}(1 - q(z_1)) \log \frac{k_{a_1}^p}{k_{a_1}^p - z_1^p} \tag{3.49}$$

where $p$ is an even integer satisfying $p \geq n$, the function $q(\cdot) : \mathbb{R} \to \{0, 1\}$ is defined by

$$q(\bullet) = \begin{cases} 1, & \text{if } \bullet > 0 \\ 0, & \text{if } \bullet \leq 0 \end{cases} \tag{3.50}$$

and

$$k_{a_1} = k_{c_1} - \underline{Y}_0, \qquad k_{b_1} = k_{c_1} - \overline{Y}_0 \tag{3.51}$$

are positive constants representing the constraints in the $z_1$ state space, given by $-k_{a_1} < z_1 < k_{b_1}$, induced from the constraints in the $x_1$ state space, given by $|x_1| <$

$k_{c_1}$. For clarity of presentation, a schematic illustration of $V_1(z_1)$ is shown in Figure 2.1b. Throughout this chapter, for ease of notation, we abbreviate $q(z_1)$ by $q$, unless otherwise stated.

**Remark 3.4.1** *For symmetric BLF candidates, $p = 2$ is sufficient. However, for the asymmetric ones, we need an even integer $p \geq n$. The reason will be apparent in Step 2, where the stabilizing function $\alpha_2$ needs to cancel the residual coupling term from the first step. Following the backstepping procedure, to ensure that $\alpha_2$ is $n-1$ times differentiable, we choose $p \geq n$.*

**Lemma 3.4.1** *The Lyapunov function candidate $V_1(z_1)$ in (3.49) is positive definite and $C^1$ in the open interval $z_1 \in (-k_{a_1}, k_{b_1})$.*

**Proof:** For ease of analysis, we rewrite $V_1$ into the following form:

$$V_1(z_1) = \begin{cases} \frac{1}{p} \log \frac{k_{b_1}^p}{k_{b_1}^p - z_1^p}, & 0 < z_1 < k_{b_1} \\ \frac{1}{p} \log \frac{k_{a_1}^p}{k_{a_1}^p - z_1^p}, & -k_{a_1} < z_1 \leq 0 \end{cases} \tag{3.52}$$

It is easy to see that, for $-k_{a_1} < z_1 < k_{b_1}$, we have that $V_1(z_1) \geq 0$ and that $V_1(z_1) = 0$ if and only if $z_1 = 0$, thus implying that $V_1(z_1)$ is positive definite. Additionally, we have that the right and left limits are identical, that is,

$$\lim_{z_1 \to 0^+} \frac{1}{p} \log \frac{k_{b_1}^p}{k_{b_1}^p - z_1^p} = \lim_{z_1 \to 0^-} \frac{1}{p} \log \frac{k_{a_1}^p}{k_{a_1}^p - z_1^p} = 0 \tag{3.53}$$

leading to the fact that $V_1(z_1)$ is continuous in $z_1 \in (-k_{a_1}, k_{b_1})$.

The function $V_1$ is piecewise smooth within each of the two intervals $z_1 \in (-k_{a_1}, 0]$ and $z_1 \in (0, k_{b_1})$. Thus, to show that $V_1$ is a $C^1$ function, we need only to show that $\lim_{z_1 \to 0} \frac{dV_1}{dz_1}$ is identical from both directions. For $0 < z_1 < k_{b_1}$, we have

$$\lim_{z_1 \to 0^+} \frac{dV_1}{dz_1} = \lim_{z_1 \to 0^+} \frac{z_1^{p-1}}{k_{b_1}^p - z_1^p} = 0 \tag{3.54}$$

Similarly, for $-k_{a_1} < z_1 \leq 0$, we obtain that

$$\lim_{z_1 \to 0^-} \frac{dV_1}{dz_1} = \lim_{z_1 \to 0^-} \frac{z_1^{p-1}}{k_{a_1}^p - z_1^p} = 0 \tag{3.55}$$

Hence, we conclude that $V_1(z_1)$ is $C^1$ continuously differentiable. ∎

From (3.49), the derivative of $V_1$ along the closed loop trajectories is given by

$$\dot{V}_1 = \left( \frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p} \right) z_1^{p-1} (f_1 + g_1(z_2 + \alpha_1) - \dot{y}_d) \tag{3.56}$$

from which we can choose the virtual control as

$$\alpha_1 = \frac{1}{g_1} \left[ -f_1 - \left( q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p) \right) \kappa_1 z_1^m + \dot{y}_d \right] \tag{3.57}$$

with $\kappa_1$ being a positive constant, and $m$ any odd integer satisfying

$$m \geq \max\{3, n\} \tag{3.58}$$

The integer $m$ has to be odd in order to yield a negative semidefinite term $-\kappa_1 z_1^{m+p-1}$ in (3.60). Since $n \geq 2$ for system (3.1) considered in this thesis (the case $n = 1$ is trivial), it means that $m$ must be at least 3. Consequently, we obtain the following:

$$\dot{z}_1 = -\left( q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p) \right) \kappa_1 z_1^m + g_1 z_2 \tag{3.59}$$

The derivative of $V_1$ along (3.59) can be rewritten as

$$\dot{V}_1 = -\kappa_1 z_1^{m+p-1} + \left( \frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p} \right) g_1 z_1^{p-1} z_2 \tag{3.60}$$

where the first term is always non-positive and the second term will be canceled in the subsequent step.

**Step 2** Denote $z_3 = x_3 - \alpha_2$, where $\alpha_2$ is a stabilizing function to be designed. Choose the Lyapunov function candidate as $V_2 = V_1 + \frac{1}{2} z_2^2$ and the stabilizing function as:

$$\alpha_2 = \frac{1}{g_2} \left[ -f_2 - \kappa_2 z_2 + \dot{\alpha}_1 - \left( \frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p} \right) g_1 z_1^{p-1} \right] \tag{3.61}$$

which yields

$$\dot{z}_2 = -\kappa_2 z_2 - \left( \frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p} \right) g_1 z_1^{p-1} + g_2 z_3 \tag{3.62}$$

Then, the derivative of $V_2$ along (3.62) is given by

$$\dot{V}_2 = -\kappa_1 z_1^{m+p-1} - \kappa_2 z_2^2 + g_2 z_2 z_3 \tag{3.63}$$

where the coupling term $g_2 z_2 z_3$ will be canceled in the subsequent step.

**Steps 3, ..., n** From Step 3 onwards, the design procedure is identical to standard adaptive backstepping, and the stabilizing functions and control are given by

$$\alpha_i = \frac{1}{g_i}(-f_i + \dot{\alpha}_{i-1} - \kappa_i z_i - g_{i-1} z_{i-1}), \quad i = 3, ..., n \tag{3.64}$$

$$u = \alpha_n \tag{3.65}$$

yielding the closed loop system

$$\dot{z}_i = -\kappa_i z_i - g_{i-1} z_{i-1} + z_{i+1}, \quad i = 3, ..., n-1 \tag{3.66}$$

$$\dot{z}_n = -\kappa_n z_n - g_{n-1} z_{n-1} \tag{3.67}$$

Control design with asymmetric BLFs is more involved as compared with its symmetric counterpart. In general, it is not a trivial task to provide an analytical construct for an asymmetric barrier function that is continuously differentiable and approaches infinity at two different points. A straightforward approach, as we have undertaken, is to assemble piecewise defined functions. The challenge therein lies in not only ensuring that the barrier function be continuously differentiable, but also that the stabilizing functions have the required differentiability properties for the final control law to be well-defined.

According to the backstepping methodology, $\alpha_1$ needs to be differentiated $n-1$ times before appearing in the final control law. In general, $\alpha_i$ needs to be differentiable at least $n-i$ times. A further requirement in our approach is that $\dot{\alpha}_{n-1}$ is continuous, so as to preserve the continuity of the control signal and of the closed loop signals.

As such, $\alpha_1$ must be at least a $C^{n-1}$ function. Due to the presence of the switching function $q(z_1)$, the stabilizing function $\alpha_1$ in (3.57) is designed to contain the $m$th power of $z_1$, where $m \geq \max\{3, n\}$, so as to ensure that its derivative $\alpha_1^{(1)}, ..., \alpha_1^{(n-1)}$, which will be used in the design of the control law in the subsequent steps, are continuous, as will be shown in Lemma 3.4.2. On the other hand, if we let $m = 1$ and design the stabilizing function as

$$\alpha_1 = \frac{1}{g_1}[-f_1 - \left(q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p)\right) \kappa_1 z_1 + \dot{y}_d] \tag{3.68}$$

then it can be checked that

$$\frac{\partial \alpha_1}{\partial z_1} = \begin{cases} -\frac{\kappa_1}{g_1}(k_{b_1}^p - (p+1)z_1^p), & z_1 > 0 \\ -\frac{\kappa_1}{g_1}(k_{a_1}^p - (p+1)z_1^p), & z_1 < 0 \end{cases} \tag{3.69}$$

Since $\lim_{z_1 \to 0^+} \frac{\partial \alpha_1}{\partial z_1} \neq \lim_{z_1 \to 0^-} \frac{\partial \alpha_1}{\partial z_1}$, it is clear that $\alpha_1$ is not even $C^1$, leading to the fact that $\dot{\alpha}_1$ is discontinuous at $z_1 = 0$.

In step 2 of backstepping, we have seen that $\alpha_2$ also contains the switching function $q(z_1)$ in order to cancel the residual coupling term from the first step. As a result, it is essential that the associated $z_1$ term has an order of at least $n-1$ to ensure that $\alpha_2$ is $C^{n-2}$. If $p = 2$ irregardless of $n$, then it is easy to verify that the resulting stabilizing function

$$\alpha_2 = \frac{1}{g_2}[-f_2 - \kappa_2 z_2 + \dot{\alpha}_1 - \left( \frac{q}{k_{b_1}^2 - z_1^2} + \frac{1-q}{k_{a_1}^2 - z_1^2} \right) g_1 z_1] \tag{3.70}$$

is not even $C^1$. However, with $p \geq n$, $\alpha_2$ is at least $C^{n-2}$ in the interval $z_1 \in (-k_{a_1}, k_{b_1})$, as will be shown shortly. The remaining stabilizing functions $\alpha_3, ..., \alpha_{n-1}$ are in standard form as derived from backstepping, and will be $C^{n-i}$ provided that $\alpha_1$ and $\alpha_2$ are, respectively, $C^{n-1}$ and $C^{n-2}$ in $z_1 \in (-k_{a_1}, k_{b_1})$. The following lemma and proof provides a formal treatment of this point.

**Lemma 3.4.2** *Each stabilizing function $\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})$, $i = 1, ..., n-1$, as described in (3.57), (3.61) and (3.7), is at least $C^{n-i}$ in the interval $z_1 \in (-k_{a_1}, k_{b_1})$.*

**Proof:** First, we establish that $\alpha_1$ and $\alpha_2$ are, respectively, at least $C^{n-1}$ and $C^{n-2}$ in $z_1 \in (-k_{a_1}, k_{b_1})$. Then, these two facts imply that $\alpha_i$ is at least $C^{n-i}$. In the following, it is understood that $z_1$ belongs to the interval $z_1 \in (-k_{a_1}, k_{b_1})$, and we shall not repeat it every time.

To prove that $\alpha_1(x_1, z_1, \dot{y}_d)$ is $C^{n-1}$, we need to prove that the $(n-1)$th order partial derivatives exist, and are continuous. Note that (3.57) can be split into three parts as follows

$$\alpha_1(x_1, z_1, \dot{y}_d) = \alpha_{1,a}(x_1) + \alpha_{1,b_1}(x_1)\alpha_{1,b_2}(z_1) + \alpha_{1,c}(\dot{y}_d) \tag{3.71}$$

where $\alpha_{1,a} := -\frac{f_1(x_1)}{g_1(x_1)}$, $\alpha_{1,b_1} := -\frac{\kappa_1}{g_1(x_1)}$, $\alpha_{1,b_2} := [q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p)]z_1^m$, and $\alpha_{1,c} := \frac{\dot{y}_d}{g_1(x_1)}$. Since $\alpha_{1,a}(x_1)$, $\alpha_{1,b_1}(x_1)$ and $\alpha_{1,c}(\dot{y}_d)$ are obviously $C^{n-1}$ functions, our task is reduced to proving that $\alpha_{1,b_2}(z_1)$ is $C^{n-1}$. To this end, note that

$$\alpha_{1,b_2} = \begin{cases} (k_{b_1}^p - z_1^p)z_1^m, & 0 < z_1 < k_{b_1} \\ (k_{a_1}^p - z_1^p)z_1^m, & -k_{a_1} < z_1 \leq 0 \end{cases} \tag{3.72}$$

The function $\alpha_{1,b_2}$ is piecewise $C^{m-1}$ with respect to $z_1$ over the two intervals $z_1 \in (-k_{a_1}, 0)$ and $z_1 \in (0, k_{b_1})$. Thus, to show it is $C^{m-1}$ for $-k_{a_1} < z_1 < k_{b_1}$, we need only to show that

$$\lim_{z_1 \to 0^+} \frac{d^{m-1}\alpha_{1,b_2}}{dz_1^{m-1}} = \lim_{z_1 \to 0^-} \frac{d^{m-1}\alpha_{1,b_2}}{dz_1^{m-1}} \tag{3.73}$$

Taking the piecewise derivative of (3.72), we are able to obtain the following:

$$\frac{d^{m-1}\alpha_{1,b_2}}{dz_1^{m-1}} = \begin{cases} (m!k_{b_1} - \frac{(p+m)!}{(p+1)!}z_1^p)z_1, & 0 < z_1 < k_{b_1} \\ (m!k_{a_1} - \frac{(p+m)!}{(p+1)!}z_1^p)z_1, & -k_{a_1} < z_1 < 0 \end{cases} \tag{3.74}$$

where "!" denotes the factorial operator. From the above, it is clear that (3.73) holds, and thus, we conclude that $\alpha_{1,b_2}(z_1)$ is $C^{m-1}$. Based on the structure of $\alpha_1(x_1, z_1, \dot{y}_d)$ in (3.71), and since $m \geq n$, it follows that $\alpha_1(x_1, z_1, \dot{y}_d)$ is at least $C^{n-1}$.

Following a similar approach as above, by analyzing the limits of the derivative from both sides of 0, we can show that the term $\left(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}\right) z_1^{p-1}$ from (3.61) is $C^{p-2}$ in the interval $z_1 \in (-k_{a_1}, k_{b_1})$. Due to the fact that $\alpha_1$ is $C^{n-1}$, as established above, it follows that $\dot{\alpha}_1(\bar{x}_2, \bar{z}_2, \bar{y}_{d_2})$ is $C^{n-2}$. Furthermore, $p \geq n$ implies that $\alpha_2$ is $C^{n-2}$ in the interval $z_1 \in (-k_{a_1}, k_{b_1})$.

The remaining stabilizing functions are given by

$$\alpha_i = \frac{1}{g_i}(-f_i + \dot{\alpha}_{i-1} - \kappa_i z_i - g_{i-1}z_{i-1})$$

for $i = 3, ..., n-1$, from which we see that $\alpha_i$ is $C^{n-i}$ if $\dot{\alpha}_{i-1}(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})$ is $C^{n-i}$. Following the fact that $\alpha_2$ is $C^{n-2}$, as established above, it can be shown that $\dot{\alpha}_2(\bar{x}_3, \bar{z}_3, \bar{y}_{d_3})$ is $C^{n-3}$, which further implies that $\alpha_3$ is $C^{n-3}$. By iterating this procedure, we can eventually show that every $\alpha_i$ is at least $C^{n-i}$ in $z_1 \in (-k_{a_1}, k_{b_1})$. ∎

The derivative of $V_n$ along (3.59), (3.62), (3.66) and (3.67) is

$$\dot{V}_n = -\kappa_1 z_1^{m+p-1} - \sum_{j=2}^{n} \kappa_j z_j^2 \tag{3.75}$$

Let the closed loop system (3.59), (3.62), (3.66) and (3.67) be written as $\dot{z} = h(t, z)$. The right hand side $h(t, z)$ satisfies the conditions (2.17)-(2.20) for $z \in \mathcal{Z} := \{z \in \mathbb{R}^n : -k_{a_1} < z_1 < k_{b_1}\}$. Then, from (3.75) and Lemma 2.4.1, we conclude that the error signal $z_1$ satisfies $-k_{a_1} < z_1(t) < k_{b_1}$ $\forall t > 0$, provided that $-k_{a_1} < z_1(0) < k_{b_1}$.

We are now ready to summarize the results for the known case in the following theorem.

**Theorem 3.4.1** *Consider the closed loop system (3.1), (3.8), (3.57), (3.61) under Assumptions 3.2.1-3.2.2. If the initial conditions are such that $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : -k_{a_1} < z_1 < k_{b_1}\}$, where $\bar{z}_n := [z_1, z_2, ..., z_n]^T$, then the following properties hold.*

*i) The signals $z_i(t)$, $i = 1, 2, ..., n$, remain in the compact set defined by*

$$\Omega_z = \left\{\bar{z}_n \in \mathbb{R}^n : -\underline{D}_{z_1} \le z_1 \le \overline{D}_{z_1},\ \|z_{2:n}\| \le \sqrt{2V_n(0)}\right\} \tag{3.76}$$

$$\overline{D}_{z_1} = k_{b_1}(1 - e^{-pV_n(0)})^{\frac{1}{p}} < k_{b_1} \tag{3.77}$$

$$\underline{D}_{z_1} = k_{a_1}(1 - e^{-pV_n(0)})^{\frac{1}{p}} < k_{a_1} \tag{3.78}$$

*ii) The output $y(t)$ remains in the set $\Omega_y := \{y \in \mathbb{R} : -k_{c_1} < -\underline{D}_{z_1} - \underline{Y}_0 \le y \le \overline{D}_{z_1} + \overline{Y}_0 < k_{c_1}\}$ $\forall t > 0$, i.e. output constraint is never violated.*

*iii) All closed loop signals are bounded.*

*iv) The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

**Proof:** The properties $(i) - (iv)$ will be proved in sequence as follows.

i) From the result $\dot{V}_n \le 0$, it follows that $V_n(t) \le V_n(0)$, which implies that the following is true:

$$V_n(0) \ge \begin{cases} \frac{1}{p} \log \frac{k_{b_1}^p}{k_{b_1}^p - z_1^p(t)}, & 0 < z_1(t) < k_{b_1} \\ \frac{1}{p} \log \frac{k_{a_1}^p}{k_{a_1}^p - z_1^p(t)}, & -k_{a_1} < z_1(t) \le 0 \end{cases} \tag{3.79}$$

Taking exponentials on both sides of the inequality, and noting, from Lemma 2.4.1, that $k_{b_1}^p - z_1^p(t) > 0$ and $k_{a_1}^p - z_1^p(t) > 0 \ \forall \ t$, we can rearrange the above inequality to yield

$$z_1^p(t) \quad \leq \quad \begin{cases} k_{b_1}^p(1 - e^{-pV_n(0)}), & 0 < z_1(t) < k_{b_1} \\ k_{a_1}^p(1 - e^{-pV_n(0)}), & -k_{a_1} < z_1(t) \leq 0 \end{cases} \tag{3.80}$$

By taking $p$th root on both sides of the inequality, we obtain that $z_1(t) \leq k_{b_1}(1 - e^{-pV_n(0)})^{\frac{1}{p}}$ for positive $z_1(t)$, and that $z_1(t) \geq -k_{a_1}(1 - e^{-pV_n(0)})^{\frac{1}{p}}$ for negative $z_1(t)$. Combining both cases, it is obvious that $-\underline{D}_{z_1} \leq z_1(t) \leq \overline{D}_{z_1}$ $\forall t > 0$.

Similarly, from the fact that $\frac{1}{2}\sum_{j=2}^{n} z_j^2 \leq V_n(0)$, we can easily show that $\|z_{2:n}\| \leq \sqrt{2V_n(0)}$. Therefore, we obtain that $z_i(t)$ remains in the compact set $\Omega_z \ \forall \ t$.

ii) Secondly, it is straightforward to show, from $y(t) = z_1(t) + y_d(t)$, $-\underline{D}_{z_1} \leq z_1(t) \leq \overline{D}_{z_1}$, and $-\underline{Y}_0 \leq y_d(t) \leq \overline{Y}_0$, that

$$-\underline{D}_{z_1} - \underline{Y}_0 \leq y(t) \leq \overline{D}_{z_1} + \overline{Y}_0 \tag{3.81}$$

Since $\underline{D}_{z_1} < k_{a_1}$ and $\overline{D}_{z_1} < k_{b_1}$, we know that

$$\begin{aligned} \overline{D}_{z_1} + \overline{Y}_0 &< k_{b_1} + \overline{Y}_0 = k_{c_1} \\ \underline{D}_{z_1} + \underline{Y}_0 &< k_{a_1} + \underline{Y}_0 = k_{c_1} \end{aligned} \tag{3.82}$$

Hence, we conclude that $y(t) \in \Omega_y \ \forall t > 0$.

iii) To show that all closed loop signals are bounded, we follow the same approach of signal chasing that has been described in detail in Theorem 3.3.1. The only minor difference in the analysis is that the stabilizing functions $\alpha_i$ ($i = 1, ..., n-1$) are now $C^{n-i}$ instead of $C^\infty$.

From the fact that $\dot{V}_n(t) \leq 0 \ \forall t > 0$, we know that the error signals $z_1(t)$, ..., $z_n(t)$ are bounded. Boundedness of $z_1(t)$ and the reference trajectory $y_d(t)$ imply that the state $x_1(t)$ is bounded. Thus, $\alpha_1(t)$ is also bounded from (3.57), which guarantees boundedness of $x_2(t)$. Based on the $C^{n-1}$ property of $\alpha_1(x_1, z_1, \dot{y}_d)$, established in Lemma 3.4.2, it can be shown that $\dot{\alpha}_1(\bar{x}_2, \bar{z}_2, \bar{y}_{d_2})$ is $C^{n-2}$. Then, boundedness of $\bar{x}_2(t), \bar{z}_2(t), \bar{y}_{d2}(t)$ implies that $\dot{\alpha}_1(\bar{x}_2(t), \bar{z}_2(t), \bar{y}_{d2}(t))$ is bounded $\forall t > 0$.

According to Lemma 2.4.1, we have that $-k_{a_1} < z_1(t) < k_{b_1}$ $\forall t > 0$, which, together with the fact that $\dot{\alpha}_1(t)$ is bounded, imply that the stabilizing function $\alpha_2(t)$ from (3.61) is bounded. As a result, we know that the state $x_3(t)$ is also bounded. Then, from the $C^{n-2}$ property of $\alpha_2(\bar{x}_2, \bar{z}_2, \bar{y}_{d_2})$ in the set $-k_{a_1} < z_1 < k_{b_1}$, we know that $\dot{\alpha}_2(\bar{x}_3, \bar{z}_3, \bar{y}_{d_3})$ is $C^{n-3}$ in the set $-k_{a_1} < z_1 < k_{b_1}$. Then, boundedness of $\bar{x}_2(t), \bar{z}_2(t), \bar{y}_{d_2}(t)$, particularly with $-k_{a_1} < z_1(t) < k_{b_1}$, implies that $\dot{\alpha}_2(\bar{x}_3(t), \bar{z}_3(t), \bar{y}_{d_3}(t))$ is bounded $\forall t > 0$.

By induction, from the boundedness of $\dot{\alpha}_{i-1}(t)$, we conclude, from (3.7), that $\alpha_i(t)$ is bounded, which in turn implies boundedness of $x_{i+1}(t)$. From the $C^{n-i}$ property of $\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})$ in the set $-k_{a_1} < z_1 < k_{b_1}$, it can be obtained that $\dot{\alpha}_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})$ is $C^{n-i-1}$ in the set $-k_{a_1} < z_1 < k_{b_1}$. Then, boundedness of $\bar{x}_i(t), \bar{z}_i(t), \bar{y}_{d_i}(t)$, particularly with $-k_{a_1} < z_1(t) < k_{b_1}$, implies that $\dot{\alpha}_i(\bar{x}_i(t), \bar{z}_i(t), \bar{y}_{d_i}(t))$ is bounded. Following this line of argument, it is straightforward to show boundedness of the states $x_1(t)$, ..., $x_n(t)$, the stabilizing functions $\alpha_1(t)$, ..., $\alpha_{n-1}(t)$, and the control $u(t)$. Hence, all closed loop signals are bounded $\forall t > 0$.

iv) Based on (3.59), (3.62), (3.12), and (3.13), we obtain

$$\begin{aligned}
\ddot{V}_n &= 2\left(q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p)\right)\kappa_1^2 z_1^{m+1} + 2\sum_{j=2}^{n}\kappa_j^2 z_j^2 \\
&\quad +2\sum_{j=3}^{n}\kappa_j g_{j-1} z_{j-1} z_j + 2\left(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}\right)\kappa_2 g_1 z_2 z_1^{p-1} \\
&\quad -2\sum_{j=1}^{n-1}\kappa_j g_j z_j z_{j+1}
\end{aligned} \tag{3.83}$$

From the fact that $x_i(t), z_i(t)$, $i = 1, ..., n$ are bounded, and particularly with $z_1(t) \in (-k_{a_1}, k_{b_1})$, it is obvious that $\ddot{V}_n(t)$ is bounded, which means that $\dot{V}_n(t)$ is uniformly continuous. Then, by Barbalat's Lemma, we obtain that $z_i(t) \to 0$ as $t \to \infty$. Since $z_1(t) = y(t) - y_d(t)$, it is clear that $y(t) \to y_d(t)$ as $t \to \infty$. ■

Using a similar design methodology, the results for the uncertain case can be derived.

In particular, the stabilizing functions are designed as:

$$\begin{aligned}
\alpha_1 &= \frac{1}{g_1}\left[-\hat{\theta}^T w_1 - \left(q(k_{b_1}^p - z_1^p) + (1-q)(k_{a_1}^p - z_1^p)\right)\kappa_1 z_1^m + \dot{y}_d\right] \\
\alpha_2 &= \frac{1}{g_2}\left[-\hat{\theta}^T w_2 - \kappa_2 z_2 - \left(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}\right) g_1 z_1^{p-1} + \frac{\partial \alpha_1}{\partial x_1} x_2 \right. \\
&\quad \left. + \sum_{j=0}^{1} \frac{\partial \alpha_1}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2 \right]
\end{aligned}$$

$$\begin{aligned}
\alpha_i &= \frac{1}{g_i}\left[-\hat{\theta}^T w_i - \kappa_i z_i - g_{i-1} z_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{x_j} x_{j+1} \right. \\
&\quad \left. + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_i z_j \right], \quad i = 3, ..., n \quad (3.84)
\end{aligned}$$

with the following intermediate and tuning functions:

$$\begin{aligned}
w_1 &= \psi_1(x_1) \\
w_i &= \psi_i(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{x_j} \psi_j(\bar{x}_j), \qquad i = 2, ..., n \\
\tau_1 &= \left(\frac{q}{k_{b_1}^p - z_1^p} + \frac{1-q}{k_{a_1}^p - z_1^p}\right) w_1 z_1 \\
\tau_i &= \tau_{i-1} + w_i z_i, \qquad i = 2, ..., n \qquad (3.85)
\end{aligned}$$

Then, the actual control and adaptation laws are given by:

$$u = \alpha_n \qquad (3.86)$$

$$\dot{\hat{\theta}} = \Gamma \tau_n \qquad (3.87)$$

Now, we are ready to state the results in a concise manner in the following theorem. The corresponding proofs follow the same lines of argument from the preceding Theorems 3.3.2-3.4.1, and are omitted.

**Theorem 3.4.2** *Consider the closed loop system (3.1),(3.86),(3.87) under Assumptions 3.2.1-3.2.2. If the initial conditions are such that $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : -k_{a_1} < z_1 < k_{b_1}\}$, where $\bar{z}_n = [z_1, z_2, ..., z_n]^T$, then the following properties hold.*

*i) The signals $z_i(t)$, $i = 1, 2, ..., n$, and $\hat{\theta}(t)$ remain in the compact sets defined by*

$$\Omega_z = \left\{ \bar{z}_n \in \mathbb{R}^n : -\underline{D}_{z_1} \leq z_1 \leq \overline{D}_{z_1},\ \|z_{2:n}\| \leq \sqrt{2\bar{V}_n} \right\} \tag{3.88}$$

$$\Omega_{\hat{\theta}} = \left\{ \hat{\theta} \in \mathbb{R}^l : \|\hat{\theta}\| \leq \theta_M + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma^{-1})}} \right\} \tag{3.89}$$

$$\begin{aligned} \bar{V}_n = {} & \frac{q(z_1(0))}{p} \log \frac{k_{b_1}^p}{k_{b_1}^p - z_1^p(0)} + \frac{1 - q(z_1(0))}{p} \log \frac{k_{a_1}^p}{k_{a_1}^p - z_1^p(0)} \\ & + \frac{1}{2} \sum_{j=2}^{n} z_j^2(0) + \frac{1}{2} \lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}(0)\| + \theta_M)^2 \end{aligned} \tag{3.90}$$

$$\overline{D}_{z_1} = k_{b_1}(1 - e^{-p\bar{V}_n})^{\frac{1}{p}} \ < \ k_{b_1} \tag{3.91}$$

$$\underline{D}_{z_1} = k_{a_1}(1 - e^{-p\bar{V}_n})^{\frac{1}{p}} \ < \ k_{a_1} \tag{3.92}$$

*where $z_{j:k} = [z_j, z_{j+1}, ..., z_k]^T$.*

*ii) The output $y(t)$ remains in the set $\Omega_y := \{y \in \mathbb{R} : -k_{c_1} < -\underline{D}_{z_1} - \underline{Y}_0 \leq y \leq \overline{D}_{z_1} + \overline{Y}_0 < k_{c_1}\}$ $\forall t > 0$, i.e. the output constraint is never violated.*

*iii) All closed loop signals are bounded.*

*iv) The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

## 3.5 Comparison With Quadratic Lyapunov Functions

If the initial conditions belong to certain sets, it is possible for backstepping control based on Quadratic Lyapunov Functions to ensure that the output does not violate its constraint. Though this approach is simpler, closer analysis reveals a tradeoff. Specifically, more conservative requirements on the initial conditions may be imposed in order to ensure that output constraint is not violated.

In Section 2.4.2, we have established that, for second order strict feedback systems, the use of BLF in place of a quadratic one leads to relaxation of the initial condition requirement. Although exponential stability can be achieved through simple quadratic Lyapunov functions, they may not ensure that the output constraint is not violated, unless some rather restrictive requirements on the initial conditions are imposed. Here, we extend the investigations and comparisons to strict feedback systems with arbitrary order.

For the known system described by (3.1), consider the quadratic Lyapunov function candidates:

$$V_1 = \frac{1}{2}z_1^2, \quad V_i = V_{i-1} + \frac{1}{2}z_i^2, \quad i = 2, ..., n \tag{3.93}$$

and the standard backstepping control law

$$\begin{aligned} \alpha_1 &= \frac{1}{g_1}(-f_1(x_1) - \kappa_1 z_1 + \dot{y}_d) \\ \alpha_i &= \frac{1}{g_i}(-f_i(\bar{x}_i) - \kappa_i z_i - g_{i-1}z_{i-1} + \dot{\alpha}_{i-1}), \quad i = 2, ..., n-1 \\ u &= \frac{1}{g_n}(-f_n(\bar{x}_n) - \kappa_n z_n - g_{n-1}z_{n-1} + \dot{\alpha}_{n-1}) \end{aligned} \tag{3.94}$$

where $\kappa_1, ..., \kappa_n$ are positive constants. It can be shown that exponential stability is obtained, i.e. $\dot{V}_n \leq -\rho V_n$, where $\rho = 2\min\{\kappa_1, ..., \kappa_n\}$, which leads to the fact that $V_n(t) \leq V_n(0)e^{-\rho t}$. As a result, we have that

$$|z_i(t)| \leq \sqrt{\sum_{j=1}^{n} z_j^2(0)e^{-\rho t} - \sum_{j\neq i} z_j^2(t)} \leq \sqrt{\sum_{j=1}^{n} z_j^2(0)e^{-\rho t}} \leq \|\bar{z}_n(0)\| \tag{3.95}$$

for $t > 0$. For the output constraint case, to ensure $|z_1(t)| \leq k_{b_1}$, we need to ensure that the initial conditions start from the set

$$\Omega_0 = \{\bar{z}_n \in \mathbb{R}^n : \|\bar{z}_n\| \leq k_{b_1}\} \tag{3.96}$$

which is much more restrictive than the condition $|z_1(0)| \leq k_{b_1}$ required when using BLF.

In the presence of parametric uncertainty, consider the following augmented Lyapunov function candidates [94]:

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}_1^T\Gamma_1^{-1}\tilde{\theta}_1, \quad V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2}\tilde{\theta}_i^T\Gamma_i^{-1}\tilde{\theta}_i, \quad i = 2, ..., n \tag{3.97}$$

with the stabilizing functions and control law

$$\begin{aligned}
\alpha_1 &= \frac{1}{g_1}(-\hat{\theta}_1^T\psi_1 - \kappa_1 z_1 + \dot{y}_d) \\
\alpha_i &= \frac{1}{g_i}(-\hat{\theta}_i^T\bar{\psi}_i - \kappa_i z_i - g_{i-1}z_{i-1} + \omega_{i-1}), \quad i = 2, ..., n-1 \\
u &= \frac{1}{g_n}(-\hat{\theta}_n^T\bar{\psi}_n - \kappa_n z_n - g_{n-1}z_{n-1} + \omega_{n-1})
\end{aligned} \tag{3.98}$$

where the modified regressors and intermediate functions are described by

$$\bar{\psi}_i = \psi_i - \sum_{j=1}^{i-1}\psi_j\frac{\partial\alpha_{i-1}}{\partial x_j} \tag{3.99}$$

$$\omega_{i-1} = -\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j - \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(j)}}y_d^{(j+1)}, \quad i = 2, ..., n \tag{3.100}$$

The adaptation laws are given by

$$\dot{\hat{\theta}}_i = \Gamma_i\bar{\psi}_i z_i, \quad i = 1, ..., n \tag{3.101}$$

where $\theta_i = \theta$, $\Gamma_i = \Gamma_i^T > 0$, and $\tilde{\theta}_i := \hat{\theta}_i - \theta$ is the error between $\theta$ and the estimate $\hat{\theta}_i$.

It can be shown that $\dot{V}_n = -\sum_{i=1}^n \kappa_i z_i^2$, from which we know that $V_n(t) \leq V_n(0) \leq \bar{V}_n$, where $\bar{V}_n$ is the upper bound for the initial value of the Lyapunov function, defined by

$$\bar{V}_n := \frac{1}{2}\sum_{i=1}^n[z_i^2(0) + \lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}_i(0)\| + \theta_M)^2] \tag{3.102}$$

This yields $|z_i(t)| \leq \sqrt{2\bar{V}_n}$. For the output constraint case, to ensure that $|z_1(t)| \leq k_{b_1}$, it is necessary to restrict the initial conditions such that $\sqrt{2\bar{V}_n} \leq k_{b_1}$, which implies that

$$\|\bar{z}_n(0)\| \leq \sqrt{k_{b_1}^2 - \lambda_{\max}(\Gamma^{-1})\sum_{i=1}^n(\|\hat{\theta}_i(0)\| + \theta_M)^2} \tag{3.103}$$

Note that the additional condition

$$k_{b_1}^2 > \lambda_{\max}(\Gamma^{-1})\sum_{i=1}^n(\|\hat{\theta}_i(0)\| + \theta_M)^2 \tag{3.104}$$

needs to be satisfied as well. Clearly, these are more restrictive than that required when using BLF, namely $|z_1(0)| < k_{b_1}$.

Though the above equations outline the overparameterized method for simplicity in presentation, it can be easily checked that the tuning functions approach yield similar properties. Although the control design based on quadratic Lyapunov functions is simpler, the drawback is that more restrictive initial conditions are required in comparison with that using BLFs.

## 3.6 Simulation

In this section, we present simulation studies to demonstrate the effectiveness of the proposed control. Consider the second-order nonlinear system

$$\begin{aligned} \dot{x}_1 &= \theta_1 x_1^2 + x_2 \\ \dot{x}_2 &= \theta_2 x_1 x_2 + \theta_3 x_1 + (1 + x_1^2)u \end{aligned} \tag{3.105}$$

where $\theta_1 = 0.1$, $\theta_2 = 0.1$, and $\theta_3 = -0.2$. We consider the output constraint problem, with and without uncertainty in $\theta_1, \theta_2, \theta_3$. The results based on the use of the Symmetric Barrier Lyapunov Function (SBLF), the Asymmetric Barrier Lyapunov Function (ABLF), and the Quadratic Lyapunov Function (QLF) are shown.

The objective is for $x_1$ to track the trajectory $y_d = 0.2 + 0.3 \sin t$, subject to the output constraint $|x_1| < 0.56$. Since $|y_d| \leq A_0 = 0.5$, we have that $k_{b_1} = 0.56 - 0.5 = 0.06$. Further, we have $|\dot{y}_d| \leq Y_1 = 0.3$. Noting that $y_d \geq -0.1$, it is easy to see that $k_{a_1} = 0.56 - 0.1 = 0.46$.

For the known case, the initial conditions are $x_1(0) = 0.25$ and $x_2(0) = 1.5$, and control gains are chosen as $\kappa_1 = \kappa_2 = 2.0$. For the adaptive case, the initial conditions are $x_1(0) = 0.25$, $x_2(0) = 1.5$, and $\hat{\theta}(0) = 0.0$. Control gains are chosen as $\kappa_1 = \kappa_2 = 2.0$ for SBLF and $\kappa_1 = \kappa_2 = 5.0$ for ABLF. The adaptation parameters are selected as $\gamma_1 = \gamma_2 = \gamma_3 = 2.0$ for SBLF and $\gamma_1 = \gamma_2 = \gamma_3 = 5.0$ for ABLF.

Simulation results for the output constraint problem *without uncertainty* are shown in Figures 3.1-3.10. From Figure 3.1, it can be seen that the output $x_1$ stays strictly within the constrained region i.e. $|x_1| < k_{c_1}$ when the SBLF and the ABLF are used. However, when the QLF is used, the constraint is violated. A simple check reveals that $\|\bar{z}_2\| = \sqrt{0.05^2 + 1.306^2} = 1.307 > k_{b_1}$, which violates the condition $\|\bar{z}_n(0)\| \leq k_{b_1}$ for use of QLF in the output constraint problem. On the other hand, we know that $|z_1(0)| = 0.05$ satisfy the less conservative conditions $|z_1(0)| < k_{b_1}$ for SBLF and $-k_{a_1} < z_1(0) < k_{b_1}$ for ABLF.

Another observation is that while good asymptotic tracking performance is achieved, there is larger undershoot in the transient stage for ABLF as compared to SBLF, due to the fact that there is larger allowance for negative tracking error for ABLF, which ensures $-0.46 < z_1(t) < 0.06\ \forall t > 0$.

The output $x_1$ remains in the region $(-k_{c_1}, k_{c_1})$ because the tracking error $z_1$ remains in the regions $(-k_{b_1}, k_{b_1})$ and $(-k_{a_1}, k_{b_1})$, respectively for SBLF and ABLF. Figures 3.2-3.3 show that these constraints for $z_1$ are not violated for various initial values for $x_1$. Note that for the ABLF, the set of allowable starting values of $x_1$ is enlarged.

With various control gains $\kappa_1$ and $\kappa_2$, the constraints are also not violated, as seen in Figures 3.4-3.5. As expected, the tracking error converges faster with larger control gains. Even the tendency for undershoot, in the case of the ABLF, is contained with large control gains, as shown in Figure 3.5.

The phase portraits of $z_1(t)$ and $z_2(t)$ are shown in Figures 3.6-3.7. The error $z_1(t)$ does not transgress its barriers as long as its initial value satisfies $|z_1(0)| < 0.06$ when the SBLF is used, or $-0.46 < z_1(0) < 0.06$ when the ABLF is used. In other words, the region between the barriers is positively invariant. In contrast, with the QLF, the region $|z_1(0)| < 0.06$ is not positively invariant, as witnessed in Figure 3.8. Even though all these cases exhibit convergence of $(z_1(t), z_2(t))$ to 0, the set of admissible initial values of $(z_1, z_2)$ that guarantees output constraint satisfaction is largest for the ABLF, followed by the SBLF, and finally the QLF.

To gain some insights on how the SBLF-based control operates in keeping the output constrained, we observe, from the control law (2.56), that the nonlinear gain term $g_1 z_1/(k_{b_1}^2 - z_1^2)$ is responsible for ensuring that the constraint on the output is satisfied. Whenever $z_1(t)$ approaches the barriers at $z_1 = \pm 0.06$, the gain term grows rapidly and provides a large control action that repels $z_1(t)$ from the barriers. This effect is observed in Figure 3.9, where the control input $u(t)$, based on the SBLF, peaks when the tracking error $z_1(t) \to \pm 0.06$. Similarly, the ABLF-based control pulls $z_1(t)$ away from the barriers with a control input $u(t)$ that grows rapidly when $z_1(t) \to 0.06$ or $z_1(t) \to -0.46$, as seen in Figure 3.10. Interestingly, the negative peaks in $u(t)$, corresponding to $z_1(t) \to 0.06$, are larger than the positive peaks that correspond to $z_1(t) \to -0.46$. This is due to the fact that, with a smaller allowable positive range for $z_1(t)$, the control $u(t)$ needs to grow at a faster rate to ensure that the barrier $z_1 = 0.06$ is not reached. In avoiding the barriers in the $z_1$ dimension, the control action can cause large excursions in the $z_2$ dimension, as seen in Figures 3.6-3.7 for SBLF and ABLF respectively.

Simulation results for output constraint problem *with uncertainty* are shown in Figures 3.11-3.13. As shown in Figure 3.11, good tracking performance is achieved while satisfying the constraint $|x_1| < k_{c_1}$, but with the ABLF, there is a greater tendency to incur negative tracking error due to $k_{a_1} > k_{b_1}$. To diminish undershooting behavior, we can increase the control gains $\kappa_1$ and $\kappa_2$. The tracking error $z_1$ is constrained in the regions $|z_1| < k_{b_1}$ for the SBLF and $-k_{a_1} < z_1 < k_{b_1}$ for the ABLF (Figure 3.12), and the parameter estimate $\hat{\theta}$ remains bounded (Figure 3.13).

## 3.7 Conclusions

In this chapter, we have presented control design for strict feedback systems with constraints on the output, based on Barrier Lyapunov Functions. Besides the nominal case where the plant is fully known, the presence of parametric uncertainties has also been handled. We have shown that asymptotic tracking is achieved without violation of the constraint, and all closed loop signals remain bounded, under a mild condition on the initial output. Further, we have explored the use of asymmetric BLFs as a generalized approach that can provide greater design flexibility and relax the starting conditions. The use of quadratic Lyapunov functions in handling output constraint has been investigated, and it is shown that more conservative restrictions on the initial conditions are required as compared with using BLFs. Finally, the effectiveness of the proposed control has been demonstrated through a simulation example.

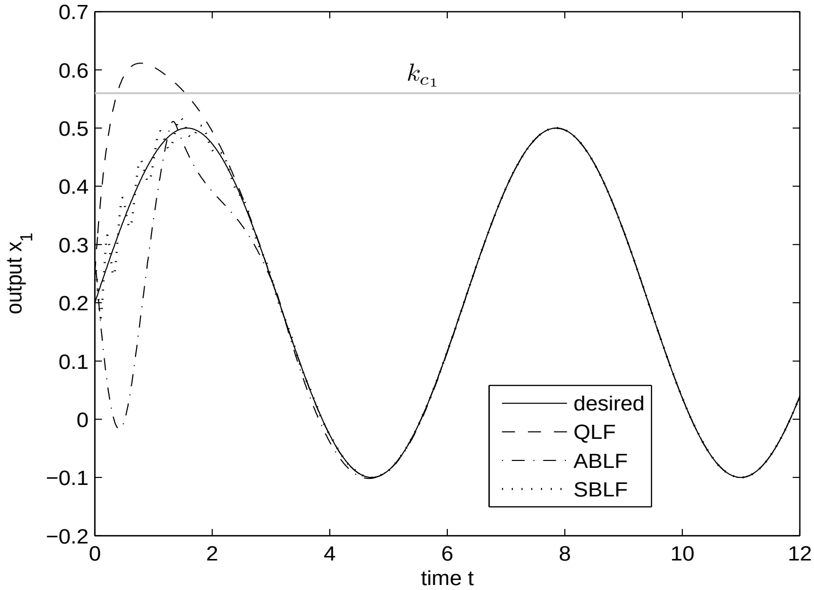


Figure 3.1: Output tracking behavior for output constraint problem based on the use of the QLF, SBLF, and ABLF.

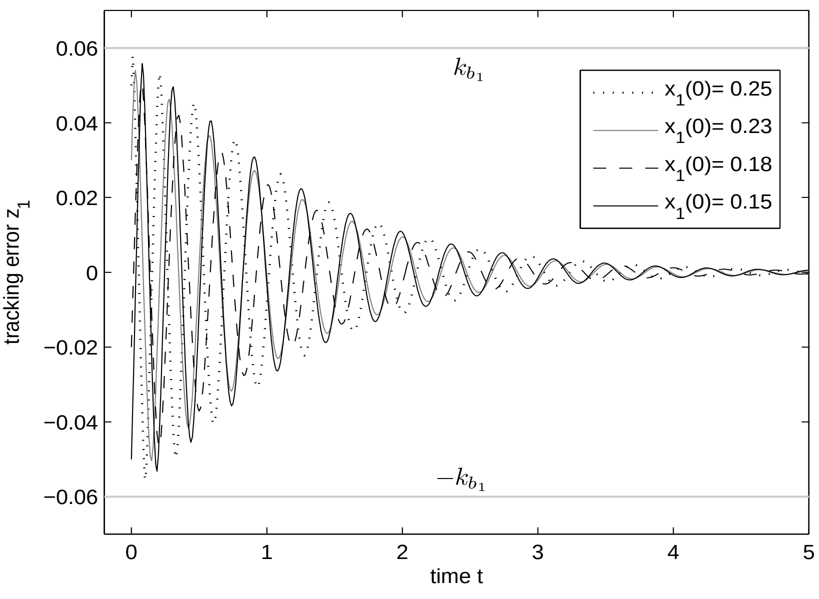


Figure 3.2: Tracking error $z_1$ for various initial conditions satisfying $|z_1(0)| < k_{b_1}$ for the output constraint problem using the SBLF.

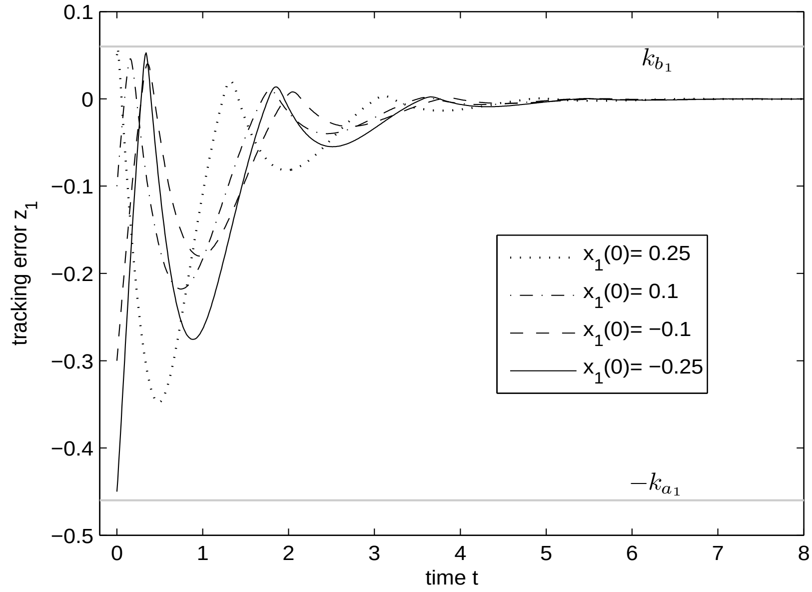


Figure 3.3: Tracking error $z_1$ for various initial conditions satisfying $-k_{a_1} < z_1(0) < k_{b_1}$ for the output constraint problem using the ABLF.

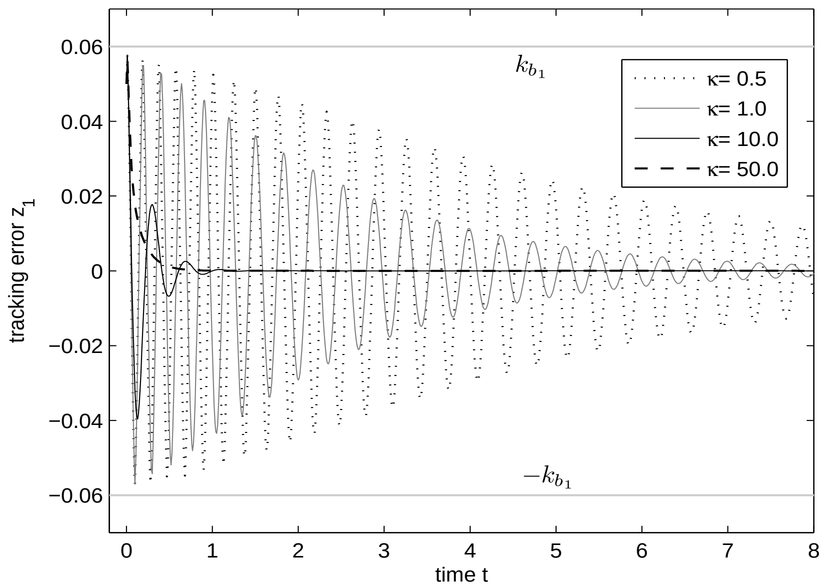


Figure 3.4: Tracking error $z_1$ for various $\kappa = \kappa_1 = \kappa_2$ for the output constraint problem using the SBLF.

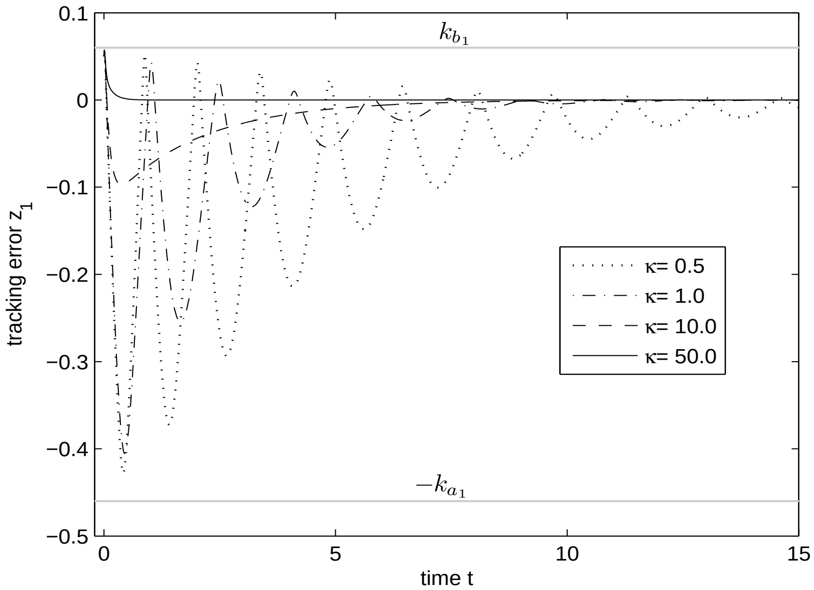


Figure 3.5: Tracking error $z_1$ for various $\kappa = \kappa_1 = \kappa_2$ for the output constraint problem using the ABLF.

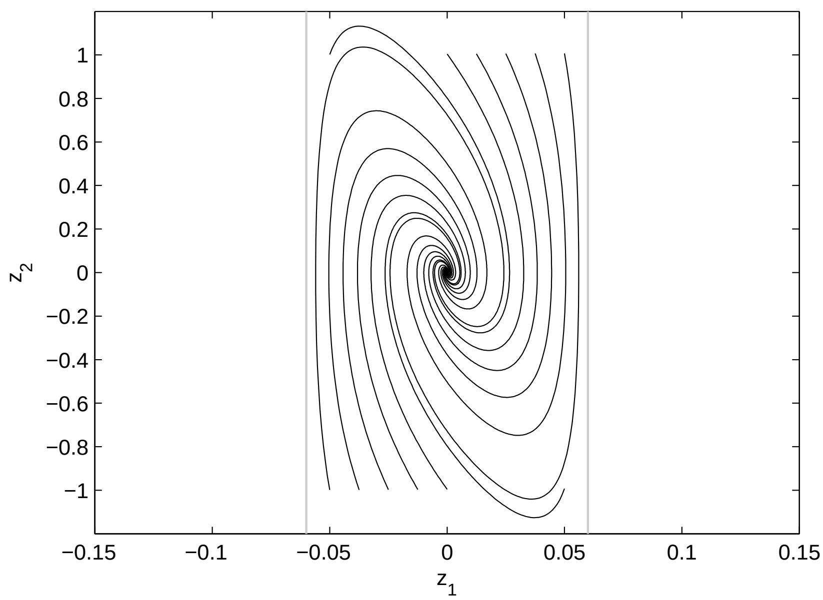


Figure 3.6: Phase portrait of $z_1, z_2$ for the closed loop system when SBLF is used.

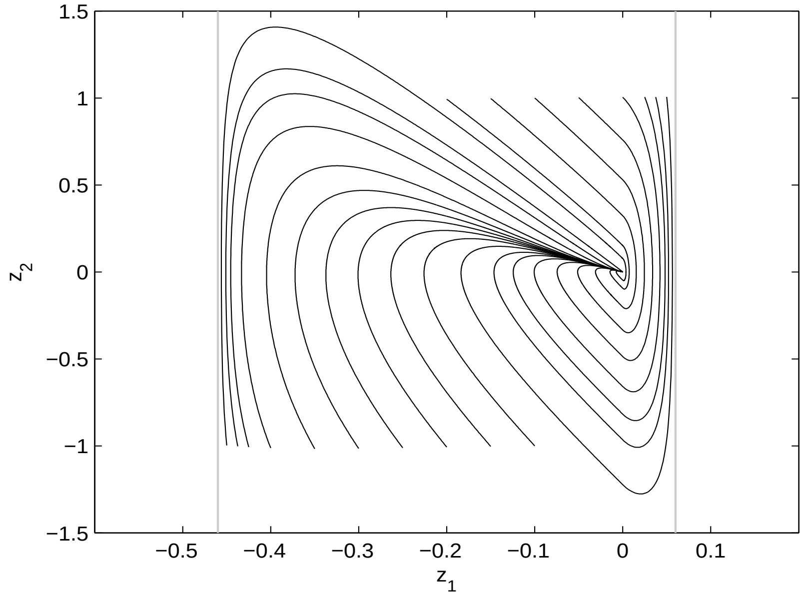


Figure 3.7: Phase portrait of $z_1$ and $z_2$ for the closed loop system when ABLF is used.

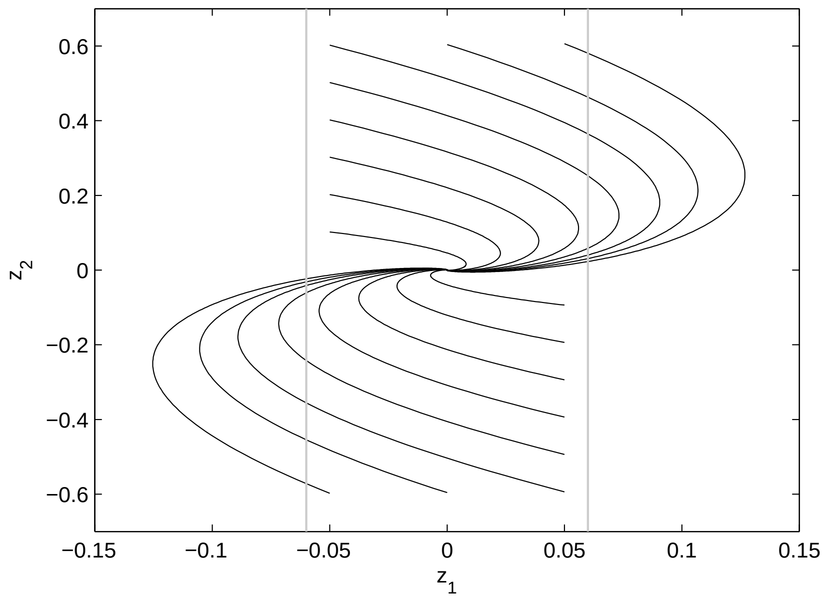


Figure 3.8: Phase portrait of $z_1$ and $z_2$ for the closed loop system when QLF is used.

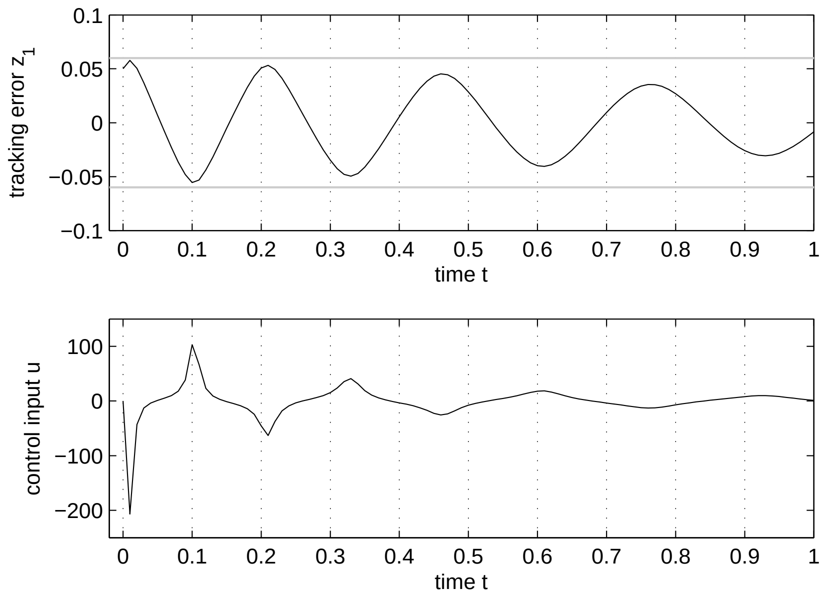


Figure 3.9: Control input $u$ when SBLF is used.

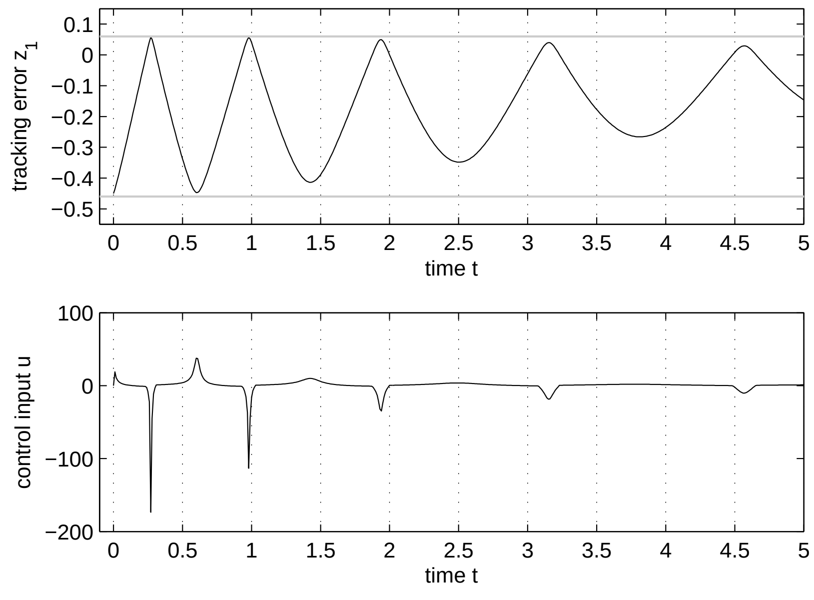


Figure 3.10: Control input $u$ when ABLF is used.

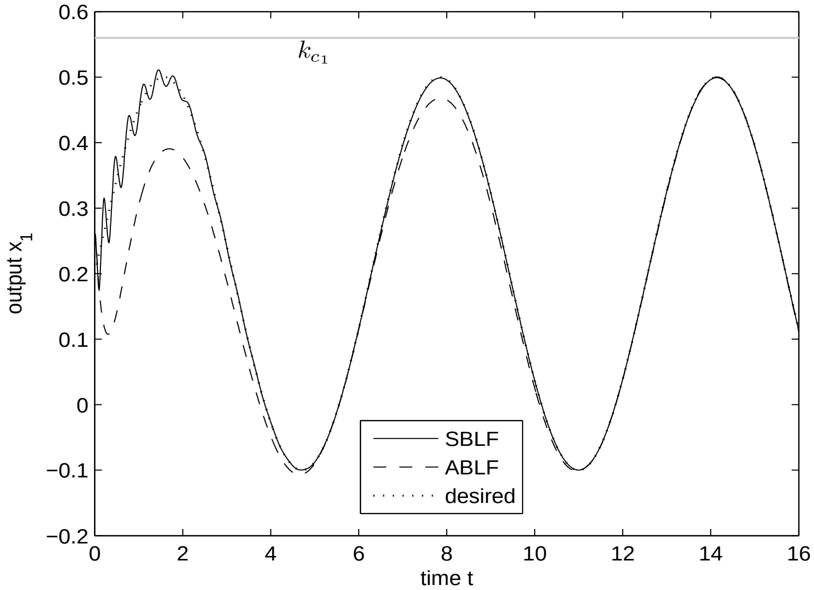


Figure 3.11: Output tracking behavior for the output constraint problem in the presence of uncertainty.

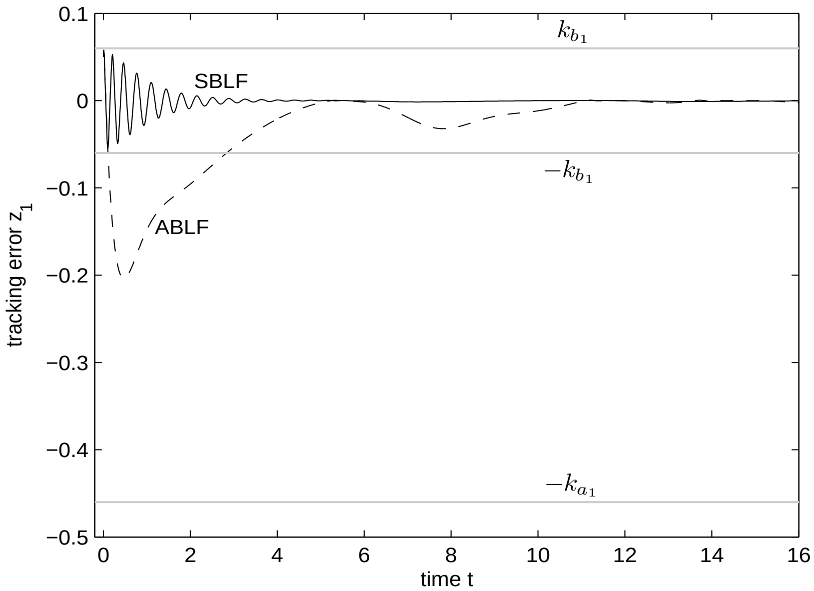


Figure 3.12: Tracking error $z_1$ for the output constraint problem in the presence of uncertainty.

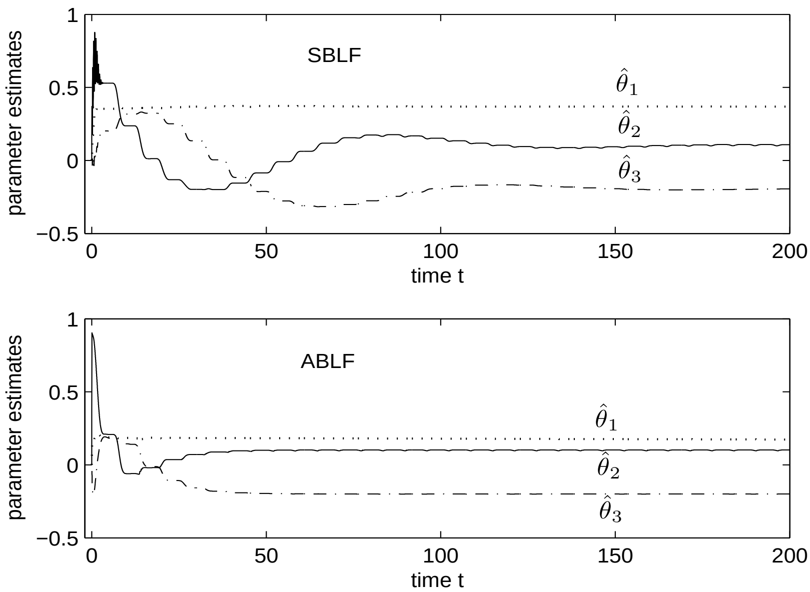


Figure 3.13: Parameter estimates $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ for the output constraint problem in the presence of uncertainty.

# Chapter 4

# Control of State-Constrained Systems

## 4.1 Introduction

In the foregoing exposition on the problem of output constraint with known control gain functions, we employed backstepping design with BLF in the first step, and quadratic functions in the remaining steps. The main principle of the design is based on obtaining the derivative of the Lyapunov function $V(z)$, along the closed loop trajectories, in a negative semidefinite form. With the BLF bounded in closed loop, it is thus guaranteed that the barriers are not transgressed. Cancelation of cross coupling terms is accommodated, with post-design analysis revealing that the stabilizing functions and control remain bounded.

In this chapter, we extend this design approach to SISO nonlinear systems in strict feedback form, with constraints on the states and known control gain functions. For the case of full state constraint, where every state is constrained, we employ BLFs for each step of the backstepping design. In the case where only some of the states have constraints, the design procedure is modified such that BLFs are only used up to the step with the highest order state under constraint, and the feasibility conditions can be relaxed. Feasibility conditions are provided, which can be checked *a priori* to determine if the given problem can be solved under these approaches. Furthermore, we present the design of adaptive controllers to deal with uncertain parameters in the plant model, in face of the simultaneous need of preventing state constraints from being violated.

The remainder of this chapter is organized as follows. In Section 4.2, we formulate the problem of tracking control for nonlinear strict feedback systems with constraints in the states. Following that, in Section 4.3, we present the control design for the case where each state of the plant is to be constrained, and provide conditions for offline checking of the feasibility of the proposed control in achieving its objectives. Section 4.4 extends these results to the case where only some of the states need to be constrained, and shows that the feasibility conditions are relaxed. Finally, simulation results are presented in Section 4.5 to demonstrate the effectiveness of the proposed control, followed by conclusions in Section 4.6.

## 4.2 Problem Formulation and Preliminaries

Consider the nonlinear system in strict feedback form:

$$\begin{aligned} \dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, \quad i = 1, 2, ..., n-1 \\ \dot{x}_n &= f_n(\bar{x}_n) + g_n(\bar{x}_n)u \\ y &= x_1 \end{aligned} \tag{4.1}$$

where $x_1, x_2, ..., x_n$ are the states, $\bar{x}_i = [x_1, x_2, ..., x_i]^T \in \mathbb{R}^i$, $f_i$ and $g_i$ are smooth functions, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the input and output respectively, for $i = 1, 2, ..., n$. For the case of full state constraints, every state $x_i$ is required to remain in the set $|x_i| \leq k_{c_i}$, with $k_{c_i}$ as a positive constant, for $i = 1, ..., n$.

The nonlinear functions $f_i(\bar{x}_i)$ may be uncertain, in which case they satisfy the following linear-in-the-parameters (LIP) condition:

$$f_i(\bar{x}_i) = \theta^T \psi_i(\bar{x}_i), \quad i = 1, ..., n \tag{4.2}$$

where $\psi_1, ..., \psi_n$ are smooth functions, and $\theta \in \mathbb{R}^l$ is a vector of uncertain parameters satisfying $\|\theta\| \leq \theta_M$ with known positive constant $\theta_M$. Due to the continuity property, there exist positive constants $\Psi_i$ such that $|\psi_i(\bar{x}_i)| \leq \Psi_i$ for $|x_i| \leq k_{c_i}$, $i = 1, 2, ..., n$.

The control objective is to track a desired trajectory $y_d$ while ensuring that all closed loop signals are bounded and that state constraints, which may be due to physical constraints as well as performance requirements, are not violated.

For ease of notation, we group the derivatives of the desired trajectory in the vector $\bar{y}_{d_i} := [y_d^{(1)}, y_d^{(2)}, ..., y_d^{(i)}]^T$. In what follows, we present the assumptions on the desired trajectory $y_d$, as well as the control gain functions $g_i(\cdot)$, $i = 1, ..., n$, from (4.1).

**Assumption 4.2.1** *For any $k_{c_1} > 0$, there exist positive constants $A_0$, $Y_1$, $Y_2$,..., $Y_n$ such that the desired trajectory $y_d(t)$ and its time derivatives satisfy*

$$|y_d(t)| \leq A_0 < k_{c_1}, \quad |\dot{y}_d(t)| < Y_1, \quad |\ddot{y}_d(t)| < Y_2, \quad \cdots, \quad |y_d^{(n)}(t)| < Y_n \tag{4.3}$$

*for all $t \geq 0$.*

**Assumption 4.2.2** *The control gain functions $g_i(\cdot)$, $i = 1, 2, ..., n$, are known, and there exists a positive constant $g_0$ such that $0 < g_0 \leq |g_i(\cdot)|$. Without loss of generality, we further assume that the $g_i(\cdot)$ are all positive.*

## 4.3 Full State Constraints

For the case of output constraint in Section 3, only the first step of the backstepping design involves the use of a BLF. By enforcing constraint on the output tracking error $z_1 = y - y_d$, we are able to ensure that the output $y$ itself is constrained within the specified zone, provided that the desired trajectory $y_d$ is also within the same zone. In this chapter, for the case of full state constraints, we extend the use of BLFs to each and every step, in order to keep each error signal $z_i = x_i - \alpha_{i-1}$ $(i = 2, ..., n)$ constrained.

Provided that each stabilizing function $\alpha_{i-1}$ is bounded in the specified constrained region for $x_i$, we can ensure that $x_i$ remains in the constrained region. In view of this, state constraints cannot be arbitrarily specified, but are subject to feasibility conditions, based on the stabilizing functions $\alpha_{i-1}$ designed via backstepping with barrier Lyapunov functions. Nevertheless, the feasibility conditions can be checked *a priori* to determine if the given problem can be solved with this approach.

In the following, we consider the case when the functions $f_i(\bar{x}_i)$ in system (4.1) are known, and also the case when they contain uncertain parameters.

### 4.3.1 Full State Constraints: Known Case

Since $f_i(\bar{x}_i)$ are known, they can be used in the design of the stabilizing functions and final control to cancel the system nonlinearities. In what follows, we outline the design steps, and then provide the sufficient conditions on the design parameters to check for feasibility with respect to the specified state constraints.

**Step 1** Define the error coordinates $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, where $\alpha_1$ is a stabilizing function to be designed. To design a control that does not drive $x_1$ out of the interval $|x_1| < k_{c_1}$, we choose the following symmetric BLF candidate in the first step of backstepping:

$$V_1 = \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} \tag{4.4}$$

where

$$k_{b_1} = k_{c_1} - A_0 \tag{4.5}$$

It can be shown that $V_1$ is positive definite and continuously differentiable in the open set $|z_1| < k_{b_1}$, and thus it is a valid Lyapunov function candidate. The derivative of $V_1$ along the closed loop trajectories is given by

$$\dot{V}_1 = \frac{z_1 \dot{z}_1}{k_{b_1}^2 - z_1^2} = \frac{z_1(f_1 + g_1(z_2 + \alpha_1) - \dot{y}_d)}{k_{b_1}^2 - z_1^2} \tag{4.6}$$

Designing the stabilizing function $\alpha_1$ as:

$$\alpha_1 = \frac{1}{g_1}(-f_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d) \tag{4.7}$$

where $\kappa_1 > 0$ is a constant, yields

$$\dot{z}_1 = -(k_{b_1}^2 - z_1^2)\kappa_1 z_1 + g_1 z_2 \tag{4.8}$$

The derivative of $V_1$ along (4.8) can be written as

$$\dot{V}_1 = -\kappa_1 z_1^2 + \frac{g_1 z_1 z_2}{k_{b_1}^2 - z_1^2} \tag{4.9}$$

where the coupling term $\frac{g_1 z_1 z_2}{k_{b_1}^2 - z_1^2}$ is canceled in the subsequent step.

**Step i** ($i = 2, ..., n-1$)

Denote $z_{i+1} = x_{i+1} - \alpha_i$, where $\alpha_i$ is a stabilizing function to be designed. Choose Lyapunov function candidates as

$$V_i = V_{i-1} + \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2} \tag{4.10}$$

The derivative of $V_i$ along the closed loop trajectories is given by

$$\dot{V}_i = -\sum_{j=1}^{i-1} \kappa_j z_j^2 + \frac{g_{i-1} z_{i-1} z_i}{k_{b_1}^2 - z_{i-1}^2} + \frac{z_i(f_i + g_i(z_{i+1} + \alpha_i) - \dot{\alpha}_{i-1})}{k_{b_1}^2 - z_i^2} \tag{4.11}$$

where $\dot{\alpha}_{i-1}$ is given by

$$\dot{\alpha}_{i-1} \;=\; \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{x_j} (f_j(\bar{x}_j) + g_j(\bar{x}_j) x_{j+1}) + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} \tag{4.12}$$

By designing the stabilizing function as

$$\alpha_i = \frac{1}{g_i} \left( -f_i + \dot{\alpha}_{i-1} - (k_{b_1}^2 - z_i^2) \kappa_i z_i - \frac{k_{b_1}^2 - z_i^2}{k_{b_1}^2 - z_{i-1}^2} g_{i-1} z_{i-1} \right) \tag{4.13}$$

where $\kappa_i > 0$ is constant, it can be obtained that

$$\dot{z}_i = -(k_{b_1}^2 - z_i^2) \kappa_i z_i - \frac{k_{b_1}^2 - z_i^2}{k_{b_1}^2 - z_{i-1}^2} g_{i-1} z_{i-1} + g_i z_{i+1} \tag{4.14}$$

The derivative of $V_i$ along (4.14) can be written as

$$\dot{V}_i = -\sum_{j=1}^{i} \kappa_j z_j^2 + \frac{g_i z_i z_{i+1}}{k_{b_1}^2 - z_i^2} \tag{4.15}$$

where the coupling term $\frac{g_i z_i z_{i+1}}{k_{b_1}^2 - z_i^2}$ is canceled in the subsequent step.

**Remark 4.3.1** *Despite the presence of terms in (4.13) containing $(k_{b_1}^2 - z_{i-1}^2)$ in the denominator, it is shown, in Theorem 4.3.1, that the magnitude of the error signals $z_{i-1}(t)$ is bounded away from $k_{b_1}$ $\forall t > 0$ under some conditions on the initial states and control parameters, resulting in bounded stabilizing function $\alpha_i$.*

**Step n** In the final step, the actual control law is designed. We choose a Lyapunov function candidate as

$$V_n = V_{n-1} + \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_n^2} \tag{4.16}$$

Then, the derivative of $V_n$ along the closed loop trajectories is given by

$$\dot{V}_n = -\sum_{j=1}^{n-1}\kappa_j z_j^2 + \frac{g_{n-1}z_{n-1}z_n}{k_{b_1}^2 - z_{n-1}^2} + \frac{z_n(f_n + g_n u - \dot{\alpha}_{n-1})}{k_{b_1}^2 - z_n^2} \tag{4.17}$$

By designing the actual control as

$$u = \frac{1}{g_n}\left(-f_n + \dot{\alpha}_{n-1} - (k_{b_1}^2 - z_n^2)\kappa_n z_n - \frac{k_{b_1}^2 - z_n^2}{k_{b_1}^2 - z_{n-1}^2}g_{n-1}z_{n-1}\right) \tag{4.18}$$

where $\kappa_n > 0$ is constant, it can be obtained that

$$\dot{z}_n = -(k_{b_1}^2 - z_n^2)\kappa_n z_n - \frac{k_{b_1}^2 - z_n^2}{k_{b_1}^2 - z_{n-1}^2}g_{n-1}z_{n-1} \tag{4.19}$$

The derivative of $V_n$ can be rewritten as

$$\dot{V}_n = -\sum_{j=1}^{n}\kappa_j z_j^2 \tag{4.20}$$

Let the closed loop system (4.8), (4.14) and (4.19) be written as $\dot{z} = h(t, z)$. The right hand side $h(t, z)$ satisfies the conditions (2.17)-(2.20) for $z \in \mathcal{Z} := \{z \in \mathbb{R}^n : |z_i| < k_{b_1}, i = 1, 2, ..., n\}$. Hence, from (4.20) and Lemma 2.4.2, we have that $|z_i(t)| < k_{b_1}$ for all $t > 0$ and $i = 1, ..., n$, provided that $|z_i(0)| < k_{b_1}$.

**Theorem 4.3.1** *Consider the closed loop system (4.1), (4.18) under Assumptions 4.2.1-4.2.2. Denote by $A_i$ an upper bound for $\alpha_i$ in the compact set $\Omega_i$, that is,*

$$A_i \geq \sup_{(\bar{x}_i,\ \bar{z}_i,\ \bar{y}_{d_i})\ \in\ \Omega_i} |\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}; \bar{\kappa}_i)|, \qquad i = 1, ..., n-1 \tag{4.21}$$

*where $\alpha_i$ is parameterized by $\bar{\kappa}_i := [\kappa_1, \kappa_2, ..., \kappa_i]^T$, and $\Omega_i$ is a compact set defined by:*

$$\begin{aligned} \Omega_i &:= \Big\{\bar{x}_i \in \mathbb{R}^i, \bar{z}_i \in \mathbb{R}^i, \bar{y}_{d_i} \in \mathbb{R}^i : \\ &\quad |x_j| \leq D_{z_1} + A_{j-1}, |z_j| \leq D_{z_1}, |y_d^{(j)}| \leq Y_j, j = 1, ..., i\Big\} \qquad (4.22) \\ D_{z_1} &:= k_{b_1}\sqrt{1 - \frac{\Pi_{i=1}^n(k_{b_1}^2 - z_i^2(0))}{k_{b_1}^{2n}}} \qquad (4.23) \end{aligned}$$

*Given the constraints $k_{c_{i+1}} > 0$, $i = 1, ..., n-1$, and that*

*C1) there exist $A_i$, for all $i = 1, ..., n-1$, such that*

$$k_{c_{i+1}} \quad > \quad A_i + k_{b_1} \tag{4.24}$$

*C2) the initial conditions are such that*

$$\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_i| < k_{b_1}, i = 1, 2, ..., n\} \tag{4.25}$$

*then the following properties hold.*

*i) The signals $z_i(t)$, $i = 1, 2, ..., n$, remain in the compact set defined by $\Omega_z = \{\bar{z}_n \in \mathbb{R}^n : |z_i| \le D_{z_1}, i = 1, 2, ..., n\}$.*

*ii) Every state $x_i(t)$ remains in the set $\Omega_x := \{\bar{x}_n \in \mathbb{R}^n : |x_i| \le D_{z_1} + A_{i-1} < k_{c_i},\ i = 1, ..., n\}$ $\forall t > 0$, i.e. the full state constraint is never violated.*

*iii) All closed loop signals are bounded.*

*iv) The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

**Proof:** The properties $(i) - (iv)$ are proved in sequence as follows.

i) From the fact that $\dot{V}_n \le 0$, it is clear that $V_n(t) \le V_n(0)$. Since $z_i^2(0) < k_{b_1}^2$ from Conditions C1 and C2, it is straightforward to conclude that $V_n(0) \le \sum_{i=1}^{n} \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(0)}$ which implies that

$$\frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2} \quad \le \quad \sum_{j=1}^{n} \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_j^2(0)}, \quad i = 1, ..., n \tag{4.26}$$

Using the identity $\log a + \log b = \log ab$, we rewrite the above as

$$\log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2} \quad \le \quad \log \frac{k_{b_1}^{2n}}{\Pi_{j=1}^{n}(k_{b_1}^2 - z_j^2(0))}, \quad i = 1, ..., n \tag{4.27}$$

Furthermore, since $|z_i(0)| < k_{b_1}$, we know, from Lemma 2.4.2, that $k_{b_1}^2 - z_i^2(t) > 0\ \forall\ t$. Then, the above can be rearranged to yield $|z_i(t)| \le D_{z_1}$ hence $z_i(t)$ remains in $\Omega_z\ \forall\ t$.

ii) From $\dot{V}_n \le 0$ and Lemma 2.4.2, we know that $|z_i(t)| \le D_{z_1} < k_{b_1}$, $i = 1, ..., n$, $\forall\ t$, where $k_{b_1} > 0$ due to (4.24). Then, from $|z_1(t)| \le D_{z_1} < k_{c_1} - A_0$, we can show that

$$|x_1(t)| \le D_{z_1} + |y_d(t)| < k_{c_1} - A_0 + |y_d(t)| \tag{4.28}$$

Noting that $|y_d(t)| \le A_0$ from Assumption 4.2.1, we therefore conclude that $|x_1(t)| \le D_{z_1} + A_0 < k_{c_1}$, $\forall\ t$.

To show that $|x_2(t)| \le k_{c_2}$, we need to first verify that there exists a positive constant $A_1$ such that $|\alpha_1(t)| \le A_1$, $\forall\ t$. Since $|x_1(t)| \le D_{z_1} + A_0$, $|z_1(t)| \le D_{z_1}$, and $|\dot{y}_d(t)| \le Y_1$, it is clear that $(x_1(t), z_1(t), \bar{y}_{d_1}(t)) \in \Omega_1$, and thus, the stabilizing function $\alpha_1(x_1, z_1, \bar{y}_{d_1})$ in (4.7) is bounded since it is a smooth function. As a result, $\sup_{(x_1,\ z_1,\ \bar{y}_{d_1}) \in\ \Omega_1} |\alpha_1(x_1, z_1, \bar{y}_{d1})|$ exists, and an upper bound $A_1$ can be found. Then, from $|z_2(t)| \le D_{z_1} < k_{b_1}$, we infer that

$$|x_2(t)| \le D_{z_1} + |\alpha_1(t)| < k_{b_1} + |\alpha_1(t)| \tag{4.29}$$

Since $|\alpha_1(t)| \le A_1$, we conclude that $|x_2(t)| \le D_{z_1} + A_1 < k_{b_1} + A_1 < k_{c_2}$, $\forall\ t$.

We can progressively show that $|x_{i+1}(t)| \le k_{c_{i+1}}$, $i = 2, ..., n-1$, after verifying that there exist positive constants $A_i$ such that $|\alpha_i(t)| \le A_i$, $\forall\ t$. Since $|x_i(t)| \le D_{z_1} + A_{i-1}$, $|z_i(t)| \le D_{z_1}$, and $|y_d^{(i)}(t)| \le Y_i$, it is clear that $(\bar{x}_i(t), \bar{z}_i(t), \bar{y}_{d_i}(t)) \in \Omega_i$, and thus, the stabilizing function $\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})$ in (4.7) is bounded since it is a smooth function. As a result, we have that $\sup_{(\bar{x}_i,\ \bar{z}_i,\ \bar{y}_{d_i}) \in\ \Omega_i} |\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})|$ exists, and an upper bound $A_i$ can be found. Then, from $|z_{i+1}(t)| \le D_{z_1} < k_{b_1}$, we infer that

$$|x_{i+1}(t)| \le D_{z_1} + |\alpha_i(t)| < k_{b_1} + |\alpha_i(t)| \tag{4.30}$$

Since $|\alpha_i(t)| \le A_i$, we conclude that $|x_{i+1}(t)| \le D_{z_1} + A_i < k_{b_1} + A_i < k_{c_{i+1}}$, $\forall\ t$.

iii) By inspection of the stabilizing functions $\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i})$ and the control $u(\bar{x}_n, \bar{z}_n, \bar{y}_{d_n})$, it is clear that they are bounded, by virtue of the boundedness of $\bar{x}_n(t), \bar{z}_n(t), \bar{y}_{d_n}(t)$, and, in particular, by $|z_i(t)| \le D_{z_1} < k_{b_1}$, which prevents any term comprising $(k_{b_1}^2 - z_i^2)$ in the denominator from becoming unbounded.

iv) Finally, we show that $y(t) \to y_d(t)$ as $t \to \infty$. Based on (4.8), (4.14), and (4.19), we compute $\ddot{V}_n$ as follows:

$$\ddot{V}_n = 2\sum_{j=1}^{n}(k_{b_1}^2 - z_j^2)\kappa_j^2 z_j^2 + 2\sum_{j=2}^{n}\frac{k_{b_1}^2 - z_j^2}{k_{b_1}^2 - z_{j-1}^2}\kappa_j g_{j-1} z_{j-1} z_j - 2\sum_{j=1}^{n-1}\kappa_j g_j z_j z_{j+1}$$

From the fact that $|x_i(t)| \le k_{c_i}$, $|z_i(t)| \le D_{z_1}$, $i = 1, ..., n$, we infer that $\ddot{V}_n(t)$ is bounded. Thus, $\dot{V}_n(t)$ is uniformly continuous. Then, by Barbalat's Lemma, we obtain that $\dot{V}_n(t) \to 0$, and thus $z_i(t) \to 0$, as $t \to \infty$. Since $z_1(t) = x_1(t) - y_d(t)$ and $y(t) = x_1(t)$, it is clear that $y(t) \to y_d(t)$ as $t \to \infty$. ■

### 4.3.2 Full State Constraints: Uncertain Case

When the nonlinearities $f_i(\bar{x}_i)$ are uncertain, but can be linearly parameterized according to (4.2), the foregoing design methodology can be modified, based on the certainty equivalence approach, i.e. replacing instances of $\theta^T\psi_i(\bar{x}_i)$ in the controls with their estimates $\hat{\theta}^T\psi_i(\bar{x}_i)$, followed by the design of the adaptation law for $\hat{\theta}$ that guarantees closed loop stability. To be consistent with the output constraint case, we adopt the tuning functions approach [94] for stable design of an adaptation law.

Denote $z_1 = x_1 - y_d$ and $z_i = x_i - \alpha_{i-1}$, $i = 2, ..., n$. Consider the Lyapunov function candidate $V_n$ composed by:

$$V_1 = \frac{1}{2}\log\frac{k_{b_1}}{k_{b_1}^2 - z_1^2} + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{4.31}$$

$$V_i = V_{i-1} + \frac{1}{2}\log\frac{k_{b_1}}{k_{b_1}^2 - z_i^2}, \quad i = 2, ..., n \tag{4.32}$$

where $k_{b_1} = k_{c_1} - A_0$, $\Gamma := \mathrm{diag}(\gamma_1, \gamma_2, ..., \gamma_l) > 0$, and $\tilde{\theta} := \hat{\theta} - \theta$ is the error between $\theta$ and its estimate, $\hat{\theta}$. Note that $V_n$ is positive definite and continuously differentiable in the set $|z_i| < k_{b_1}$ for all $i = 1, 2, ..., n$. The adaptive backstepping control is designed as follows:

$$\alpha_1 = -\hat{\theta}^T w_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d \tag{4.33}$$

$$\alpha_2 = \frac{1}{g_2}\left(-\hat{\theta}^T w_2 - (k_{b_1}^2 - z_2^2)\kappa_2 z_2 - \frac{k_{b_1}^2 - z_2^2}{k_{b_1}^2 - z_1^2} g_1 z_1 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \sum_{j=0}^{1}\frac{\partial \alpha_1}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial \alpha_1}{\partial \hat{\theta}}\Gamma\tau_2\right) \tag{4.34}$$

$$\alpha_i = \frac{1}{g_i}\left(-\hat{\theta}^T w_i - (k_{b_1}^2 - z_i^2)\kappa_i z_i - \frac{k_{b_1}^2 - z_i^2}{k_{b_1} - z_{i-1}^2} g_{i-1} z_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{x_j} x_{j+1} \right.$$
$$\left. + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \sum_{j=2}^{i-1} \frac{z_j}{k_{b_1}^2 - z_j^2} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_i \right), \quad i = 3, ..., n \tag{4.35}$$

$$w_1 = \psi_1(x_1), \qquad w_i = \psi_i(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{x_j} \psi_j(\bar{x}_j), \quad i = 2, ..., n \tag{4.36}$$

$$\tau_1 = \frac{w_1 z_1}{k_{b_1}^2 - z_1^2}, \qquad \tau_i = \tau_{i-1} + \frac{w_i z_i}{k_{b_1}^2 - z_i^2} \tag{4.37}$$

$$u = \alpha_n \tag{4.38}$$

$$\dot{\hat{\theta}} = \Gamma \tau_n \tag{4.39}$$

which yields the closed loop system

$$\dot{z}_1 = -(k_{b_1}^2 - z_1^2)\kappa_1 z_1 + z_2 - \tilde{\theta}^T \psi_1(x_1) \tag{4.40}$$

$$\dot{z}_2 = -(k_{b_1}^2 - z_2^2)\kappa_2 z_2 - \frac{k_{b_1}^2 - z_2^2}{k_{b_1}^2 - z_1^2} g_1 z_1 + g_2 z_3 - \tilde{\theta}^T w_2 + \frac{\partial \alpha_1}{\partial \hat{\theta}}(\Gamma \tau_2 - \dot{\hat{\theta}}) \tag{4.41}$$

$$\dot{z}_i = -(k_{b_1}^2 - z_i^2)\kappa_i z_i - \frac{k_{b_1}^2 - z_i^2}{k_{b_1}^2 - z_{i-1}^2} g_{i-1} z_{i-1} + g_i z_{i+1} - \tilde{\theta}^T w_i + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}(\Gamma \tau_i - \dot{\hat{\theta}})$$
$$+ \sum_{j=2}^{i-1} \frac{z_j}{k_{b_1}^2 - z_j^2} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_i \tag{4.42}$$

$$\dot{z}_n = -(k_{b_1}^2 - z_n^2)\kappa_n z_n - \frac{k_{b_1}^2 - z_n^2}{k_{b_1}^2 - z_{n-1}^2} g_{n-1} z_{n-1} - \tilde{\theta}^T w_n + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}(\Gamma \tau_n - \dot{\hat{\theta}})$$
$$+ \sum_{j=2}^{n-1} \frac{z_j}{k_{b_1}^2 - z_j^2} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_n \tag{4.43}$$

$$\dot{\hat{\theta}} = \Gamma \tau_n \tag{4.44}$$

The derivative of $V_n$ along (4.40)-(4.44) can be written as:

$$\dot{V}_n = -\sum_{j=1}^{n} \kappa_j z_j^2 \tag{4.45}$$

Let the closed loop system (4.40)-(4.44) be written as $\dot{\eta} = h(t, \eta)$, where $\eta = [z^T,\ \tilde{\theta}^T]^T$. By inspection, $h(t, \eta)$ satisfies the conditions (2.17)-(2.20) in the open set $\eta \in \mathcal{Z} := \{z \in \mathbb{R}^n, \tilde{\theta} \in \mathbb{R}^l : |z_i| < k_{b_1}, i = 1, 2, ..., n\}$. Together with (4.45), we infer, from Lemma 2.4.2, $|z_i(t)| < k_{b_1}$, for all $t > 0$ and $i = 1, ..., n$, provided that $|z_i(0)| < k_{b_1}$.

**Theorem 4.3.2** *Consider the closed loop system (4.1), (4.38), (4.39) under Assumptions 4.2.1-4.2.2. Denote by $A_i$ an upper bound for $\alpha_i$ in the compact set $\Omega_i$, that is,*

$$A_i \geq \sup_{(\bar{x}_i,\ \bar{z}_i,\ \bar{y}_{d_i},\ \hat{\theta})\ \in\ \Omega_i} |\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\theta}; \bar{\kappa}_i, \Gamma)|, \qquad i = 1, ..., n-1 \tag{4.46}$$

*where $\alpha_i$ is parameterized by $\Gamma$ and $\bar{\kappa}_i := [\kappa_1, \kappa_2, ..., \kappa_i]^T$, and $\Omega_i$ is a compact set defined by:*

$$\begin{aligned}\Omega_i &:= \Big\{\bar{x}_i \in \mathbb{R}^i, \bar{z}_i \in \mathbb{R}^i, \bar{y}_{d_i} \in \mathbb{R}^i, \hat{\theta} \in \mathbb{R}^l : \\ &\qquad |x_j| \leq D_{z_1} + A_{j-1}, |z_j| \leq D_{z_1}, \|\hat{\theta}\| \leq D_{\hat{\theta}}, |y_d^{(j)}| \leq Y_j, j = 1, ..., i\Big\} \end{aligned} \tag{4.47}$$

$$D_{z_1} := k_{b_1}\sqrt{1 - \frac{\Pi_{i=1}^n (k_{b_1}^2 - z_i^2(0))}{k_{b_1}^{2n} e^{2\bar{V}_{\hat{\theta}}}}} \tag{4.48}$$

$$D_{\hat{\theta}} := \theta_M + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma^{-1})}} \tag{4.49}$$

$$\bar{V}_{\hat{\theta}} := \frac{1}{2}\lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}(0)\| + \theta_M)^2 \tag{4.50}$$

$$\bar{V}_n := \frac{1}{2}\sum_{i=1}^n \log\frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(0)} + \bar{V}_{\hat{\theta}} \tag{4.51}$$

*Given the constraints $k_{c_{i+1}} > 0$, $i = 1, ..., n-1$, and that*

*C1) there exist $A_i$, for all $i = 1, ..., n-1$, such that*

$$k_{c_{i+1}} > A_i + k_{b_1} \tag{4.52}$$

*C2) the initial conditions are such that*

$$\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_i| < k_{b_1}, i = 1, 2, ..., n\} \tag{4.53}$$

*then the following properties hold.*

*i) The signals $z_i(t)$ and $\hat{\theta}(t)$, $i = 1, 2, ..., n$, remain, for all $t > 0$, in the compact sets defined by*

$$\Omega_z = \{\bar{z}_n \in \mathbb{R}^n : |z_i| \leq D_{z_1}, i = 1, 2, ..., n\} \tag{4.54}$$

$$\Omega_{\hat{\theta}} = \Big\{\hat{\theta} \in \mathbb{R}^l : \|\hat{\theta}\| \leq D_{\hat{\theta}}\Big\} \tag{4.55}$$

*ii) Every state* $x_i(t)$ *remains in the set* $\Omega_x := \{\bar{x}_n \in \mathbb{R}^n : |x_i| \leq D_{z_1} + A_{i-1} < k_{c_i},\ i = 1, ..., n\}\ \forall t > 0$*, i.e. the full state constraint is never violated.*

*iii) All closed loop signals are bounded.*

*iv) The output tracking error* $z_1(t)$ *converges to zero asymptotically, i.e.,* $y(t) \rightarrow y_d(t)$ *as* $t \rightarrow \infty$*.*

**Proof:** The properties $(i) - (iv)$ are proved in sequence as follows.

i) Since $\|\theta\| \leq \theta_M$, and $|z_i(0)| < k_{b_1}$ from Condition C2, it can be shown that

$$\begin{aligned} V_n(0) &= \sum_{i=1}^{n} \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(0)} + \frac{1}{2}\tilde{\theta}(0)^T \Gamma^{-1} \tilde{\theta}(0) \\ &\leq \sum_{i=1}^{n} \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(0)} + \frac{1}{2}\lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}(0)\| + \theta_M)^2 \\ &= \bar{V}_n \end{aligned} \tag{4.56}$$

From the fact that $\dot{V}_n \leq 0$, it is clear that $V_n(t) \leq V_n(0) \leq \bar{V}_n$, from which we obtain

$$\frac{1}{2}\lambda_{\min}(\Gamma^{-1})\|\hat{\theta}(t) - \theta\|^2 \leq \bar{V}_n \tag{4.57}$$

and hence

$$\|\hat{\theta}(t)\| \leq \theta_M + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma^{-1})}} \tag{4.58}$$

Therefore, $\hat{\theta}$ remains in the compact set $\Omega_{\hat{\theta}}\ \forall\ t > 0$.

Furthermore, from $V_n(t) \leq \bar{V}_n$, we also have that

$$\frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(t)} \leq \frac{k_{b_1}^{2n} e^{2\bar{V}_{\hat{\theta}}}}{\Pi_{i=1}^{n}(k_{b_1}^2 - z_i^2(0))}, \quad i = 1, ..., n \tag{4.59}$$

Since $|z_i(0)| < k_{b_1}$, we know that $k_{b_1}^2 - z_i^2(t) > 0\ \forall\ t$ from Lemma 2.4.2. A simple rearrangement yields $|z_i(t)| \leq D_{z_1} < k_{b_1}$, and thus, $z_i(t)$ remains in the compact set $\Omega_z\ \forall\ t > 0$.

ii) The proof follows the a similar line of argument as that in Theorem 4.3.1, and is shown here for completeness. From $\dot{V}_n \leq 0$ and Lemma 2.4.2, we have established that $|z_i(t)| \leq D_{z_1} < k_{b_1}$, $i = 1, ..., n$, $\forall\ t$, and hence

$$|x_1(t)| \leq D_{z_1} + |y_d(t)| < k_{c_1} - A_0 + |y_d(t)| \tag{4.60}$$

Noting that $|y_d(t)| \leq A_0$ from Assumption 4.2.1, we therefore conclude that $|x_1(t)| \leq D_{z_1} + A_0 < k_{c_1}$, $\forall\ t > 0$.

We can progressively show that $|x_{i+1}(t)| \leq k_{c_{i+1}}$, $i = 2, ..., n-1$, after verifying that there exist positive constants $A_i$ such that $|\alpha_i(t)| \leq A_i$, $\forall\ t$. Since $\|\hat{\theta}(t)\| \leq D_{\hat{\theta}}$, $|x_i(t)| \leq D_{z_1} + A_{i-1}$, $|z_i(t)| \leq D_{z_1}$, and $|y_d^{(i)}(t)| \leq Y_i$, it is clear that

$$(\bar{x}_i(t), \bar{z}_i(t), \bar{y}_{d_i}(t), \hat{\theta}(t)) \in \Omega_i \tag{4.61}$$

and thus, the stabilizing function $\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\theta})$ in (4.7) is bounded since it is a continuous function. As a result, we have that $\sup_{(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\theta}) \in\ \Omega_i} |\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\theta})|$ exists, and an upper bound $A_i$ can be found. Then, since $|z_{i+1}(t)| \leq D_{z_1} < k_{b_1}$, we can show that

$$|x_{i+1}(t)| \leq D_{z_1} + |\alpha_i(t)| < k_{b_1} + |\alpha_i(t)| \tag{4.62}$$

From Condition C1 and the fact that $|\alpha_i(t)| \leq A_i$, we conclude that $|x_{i+1}(t)| \leq D_{z_1} + A_i < k_{c_{i+1}}$, $\forall\ t$.

iii) It is straightforward to prove that all closed loop signals are bounded, based on the results $|z_i(t)| \leq D_{z_1} < k_{b_1}$, $|x_i(t)| < k_{c_i}$, and $\|\hat{\theta}(t)\| \leq D_{\hat{\theta}}$, for $i = 1, ..., n$. By inspection of the stabilizing functions $\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\theta})$ and control $u(\bar{x}_n, \bar{z}_n, \bar{y}_{d_n}, \hat{\theta})$ , it is clear that they are bounded, by virtue of the boundedness of $\bar{x}_n(t), \bar{z}_n(t), \bar{y}_{d_n}(t), \hat{\theta}(t)$, and, in particular, by $|z_i(t)| < k_{b_1}$, which prevents any term comprising $(k_{b_1}^2 - z_i^2)$ in the denominator from becoming unbounded.

iv) Based on (4.40), (4.41), (4.42), and (4.43), we have that

$$\begin{aligned}
\ddot{V}_n \ = \ & 2\sum_{j=1}^{n}(k_{b_1}^2 - z_j^2)\kappa_j^2 z_j^2 + 2\sum_{j=2}^{n}\frac{k_{b_1}^2 - z_j^2}{k_{b_1}^2 - z_{j-1}^2}\kappa_j g_{j-1} z_{j-1} z_j \\
& -2\sum_{j=1}^{n-1}\kappa_j g_j z_j z_{j+1} + 2\sum_{j=1}^{n}\tilde{\theta}^T w_j \kappa_j z_j - 2\sum_{j=2}^{n}\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\Gamma(\tau_j - \tau_n)\kappa_j z_j \\
& -2\sum_{k=3}^{n}\sum_{j=2}^{k-1}\frac{z_j}{k_{b_1}^2 - z_j^2}\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\Gamma w_k \kappa_k z_k
\end{aligned} \tag{4.63}$$

From the fact that $|z_i(t)| \leq D_{z_1} < k_{b_1}$, $|x_i(t)| < k_{c_i}$, and $\|\hat{\theta}(t)\| \leq D_{\hat{\theta}}$, $i = 1, ..., n$, it can be shown that all right hand side terms are bounded. Thus, $\ddot{V}_n(t)$ is bounded, which implies that $\dot{V}_n(t)$ is uniformly continuous. Then, by Barbalat's Lemma, we obtain that $z_i(t) \to 0$ as $t \to \infty$. Since $z_1(t) = x_1(t) - y_d(t)$ and $y(t) = x_1(t)$, it is clear that $y(t) \to y_d(t)$ as $t \to \infty$. ∎

### 4.3.3 Full State Constraints: Feasibility Check

As mentioned earlier, the proposed method is unable to handle arbitrary state constraints. The state constraints $k_{c_i}$ need to satisfy the feasibility conditions C1 and C2 in Theorems 4.3.1-4.3.2, which depend on the initial conditions and the design parameters. Since initial conditions cannot be chosen, this amounts to a search for a set of design parameters $\bar{\kappa}_{n-1}$ and $\Gamma$ that satisfies C1 and C2.

We wish to choose the design parameters to be sufficiently large so as to achieve higher rates of error convergence and adaptation. However, the feasibility conditions impose an upper bound on the design parameters. This tradeoff can be formulated as a *static* nonlinear constrained optimization problem that can be solved offline prior to actual implementation, using state of the art numerical solvers such as the MATLAB function "fmincon.m".

When the system (4.1) is known, we check if there exists a solution $\bar{\kappa}_{n-1} := [\kappa_1, ..., \kappa_{n-1}]^T$ for the optimization problem:

$$
\begin{aligned}
&\max_{\kappa_1,...,\kappa_{n-1}>0} P(\bar{\kappa}_{n-1}) = \sum_{i=1}^{n-1} a_i \kappa_i \\
&\text{subject to:} \\
&\quad k_{c_{i+1}} > A_i(\bar{\kappa}_i) + k_{b_1} \\
&\quad k_{b_1} > |x_{i+1}(0) - \alpha_i(\bar{x}_i(0), \bar{z}_i(0), \bar{y}_{d_i}(0); \bar{\kappa}_i)| \\
&\qquad\qquad i = 1, ..., n-1
\end{aligned} \tag{4.64}
$$

where $P$ is the objective function, and $a_i$ are positive constants. If a solution $\bar{\kappa}^*_{n-1}$ to the above optimization problem exists, then C1 and C2 in Theorem 4.3.1 are satisfied, and the proposed control (4.18) with $\bar{\kappa}_{n-1} = \bar{\kappa}^*_{n-1}$ is feasible in ensuring output tracking for the system (4.1) with full state constraint.

When (4.1) is uncertain, the matrix of adaptation parameters, $\Gamma := \mathrm{diag}(\gamma_1, \gamma_2, ..., \gamma_l)$, is also considered in the optimization:

$$\begin{aligned}
&\max_{\kappa_1,...,\kappa_{n-1},\Gamma>0} P(\bar{\kappa}_{n-1}, \Gamma) = \sum_{i=1}^{n-1} a_i \kappa_i + \sum_{i=1}^{l} b_i \gamma_i \\
&\text{subject to:} \\
&\quad k_{c_{i+1}} > A_i(\bar{\kappa}_i, \Gamma) + k_{b_1} \\
&\quad k_{b_1} > |x_{i+1}(0) - \alpha_i(\bar{x}_i(0), \bar{z}_i(0), \bar{y}_{d_i}(0), \hat{\theta}(0); \bar{\kappa}_i, \Gamma)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad i = 1, ..., n-1
\end{aligned} \tag{4.65}$$

where $b_i$ are positive constants. If a solution $(\bar{\kappa}^*_{n-1}, \Gamma^*)$ to the above optimization problem is found, then the proposed adaptive control (4.38)-(4.39), with $\bar{\kappa}_{n-1} = \bar{\kappa}^*_{n-1}$ and $\Gamma = \Gamma^*$, is feasible in ensuring output tracking for the system (4.1) with full state constraint, according to Theorem 4.3.2.

**Remark 4.3.2** *The conditions C1 and C2, in Theorems 4.3.1 and 4.3.2, are sufficient conditions to achieve output tracking in the presence of state constraints. In particular, C1 ensures that the state constraints $|x_i(t)| < k_{c_i}$ are met, given that $|z_i(t)| < k_{b_1}$, for all $i = 1, 2, ..., n$. The condition $|z_i(t)| < k_{b_1}$, for all $i = 1, 2, ..., n$, is ensured by our proposed BLF-based control.*

**Remark 4.3.3** *The bounds $A_1, ..., A_{n-1}$ are computable for any set of control parameters $\kappa_1, ..., \kappa_n, \Gamma$ and initial conditions $x(0)$, and thus, these conditions can be checked before the control is implemented, provided that knowledge of the initial condition is available.*

## 4.4 Partial State Constraints

When all states need to be constrained, the feasibility conditions C1-C2, as described in Theorems 4.3.1-4.3.2, may become rather restrictive. In the case where only some, but not all, of the states have constraints, the design procedure is modified such that Barrier Lyapunov Functions are only used up to the step with the highest order state under constraint, and the feasibility conditions can be relaxed. Consider the partition of the full state $x = [x_1, ..., x_n]^T$ into free states $x_r = [x_{r_1}, x_{r_2}, ..., x_{r_{n_r}}]^T$ and constrained states $x_s = [x_{s_1}, x_{s_2}, ..., x_{s_{n_s}}]^T$, where $n_r + n_s = n$, and the number sequences, $\{r_1, r_2, ..., r_{n_r}\}$ and $\{s_1, s_2, ..., s_{n_s}\}$, are both ascending.

### 4.4.1 Partial State Constraints: Known Case

According to the backstepping methodology, we employ BLFs from steps 1 to $s_{n_s}$:

$$V_1 = \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} + V_\theta, \tag{4.66}$$

$$V_i = V_{i-1} + \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2}, \qquad i = 2, 3, ..., s_{n_s} \tag{4.67}$$

to design the corresponding stabilizing functions

$$\alpha_1 = \frac{1}{g_1}(-f_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d) \tag{4.68}$$

$$\alpha_i = \frac{1}{g_i}\left(-f_i + \dot{\alpha}_{i-1} - (k_{b_1}^2 - z_i^2)\kappa_i z_i - \frac{k_{b_1}^2 - z_i^2}{k_{b_1}^2 - z_{i-1}^2} g_{i-1} z_{i-1}\right), \quad i = 2, 3, ..., s_{n_s} \tag{4.69}$$

From step $(s_{n_s} + 1)$ onwards until the final step, quadratic Lyapunov functions are used for the design of the remaining stabilizing functions and final control law:

$$V_i = V_{i-1} + \frac{1}{2}z_i^2, \qquad i = s_{n_s} + 1,\ s_{n_s} + 2,\ \dots,\ n \tag{4.70}$$

$$\alpha_{s_{n_s}+1} = \frac{1}{g_{s_{n_s}+1}}\left(-f_{s_{n_s}+1} + \dot{\alpha}_{s_{n_s}} - \kappa_{s_{n_s}+1} z_{s_{n_s}+1} - \frac{g_{s_{n_s}} z_{s_{n_s}}}{k_{b_1}^2 - z_{s_{n_s}}^2}\right) \tag{4.71}$$

$$\alpha_j = \frac{1}{g_j}(-f_j + \dot{\alpha}_{j-1} - \kappa_j z_j - g_{j-1} z_{j-1}), \quad j = s_{n_s} + 2\ , s_{n_s} + 3\ ,\ \dots,\ n \tag{4.72}$$

$$u = \alpha_n \tag{4.73}$$

For $i \in \{s_1, s_2, ..., s_{n_s}\}$, the given constraints $k_{c_i}$ need to satisfy feasibility conditions similar to those in Theorems 4.3.1-4.3.2. However, for $i \in \{r_1, r_2, ..., r_{n_s}\}$, where $r_{n_s} < s_{n_s}$, the constraints $k_i$ are not explicitly specified as problem requirements, but rather, they are artificially imposed as part of the design procedure. As such, they can be chosen as design parameters, thus relaxing the feasibility conditions.

We state the results concisely for the known case in the following theorem.

**Theorem 4.4.1** *Consider known system (4.1) under Assumptions 4.2.1-4.2.2, stabilizing functions and control law (4.68)-(4.69),(4.71)-(4.73). Denote by $A_i$ an upper bound for $\alpha_i$ in the compact set $\Omega_i$, that is,*

$$A_i \geq \sup_{(\bar{x}_i,\ \bar{z}_i,\ \bar{y}_{d_i})\ \in\ \Omega_i} |\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}; \bar{\kappa}_i)|, \qquad i = 1, ..., s_{n_s} - 1 \tag{4.74}$$

*where $\alpha_i$ is parameterized by $\bar{\kappa}_i = [\kappa_1, \kappa_2, ..., \kappa_i]^T$, and $\Omega_i$ is a compact set defined by:*

$$\begin{aligned} \Omega_i &:= \Big\{\bar{x}_i \in \mathbb{R}^i, \bar{z}_i \in \mathbb{R}^i, \bar{y}_{d_i} \in \mathbb{R}^i : \\ &\qquad |x_j| \leq D_{z_1} + A_{j-1}, |z_j| \leq D_{z_1}, |y_d^{(j)}| \leq Y_j, j = 1, ..., i\Big\} \end{aligned} \tag{4.75}$$

$$D_{z_1} := k_{b_1}\sqrt{1 - \frac{\Pi_{i=1}^{s_{n_s}}(k_{b_1}^2 - z_i^2(0))}{k_{b_1}^{2s_{n_s}} e^{\sum_{i=s_{n_s}}^{n} z_i^2(0)}}} \tag{4.76}$$

*Given the constraints $\{k_{c_{s_2}}, k_{c_{s_3}}, ..., k_{c_{s_{n_s}}}\}$, and that*

*C1) there exist $A_i$ and $\{k_{c_i}\}_{i\in\mathcal{F}}$, where $\mathcal{F} := \{r_1, r_2, ..., r_{n_r}\} \cap \{1, 2, ..., s_{n_s}\}$, such that*

$$k_{c_{i+1}} > A_i + k_{b_1}, \qquad i = 1, ..., s_{n_s} - 1 \tag{4.77}$$

*C2) the initial conditions are such that*

$$\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_i| < k_{b_1}, i = 1, 2, ..., s_{n_s}\} \tag{4.78}$$

*then the following properties hold.*

*i) The signals $z_i(t)$, $i = 1, 2, ..., n$, remain, for all $t > 0$, in the compact set defined by $\Omega_z = \{\bar{z}_n \in \mathbb{R}^n : |z_i| \leq D_{z_1}, i = 1, 2, ..., s_{n_s}, \|z_{s_{n_s}+1:n}\| \leq \sqrt{2V_n(0)}\}$, where $z_{s_{n_s}+1:n} := [z_{s_{n_s}+1}, z_{s_{n_s}+2}, ..., z_n]^T$.*

*ii) The partial state $x_s(t)$ remains in the set $\Omega_{x_s} := \{x_s \in \mathbb{R}^{n_s} : |x_i| \leq D_{z_1} + A_{i-1} < k_{c_i},\ i = s_1, s_2, ..., s_{n_s}\}$ $\forall t > 0$, i.e. the partial state constraint is never violated.*

*iii) All closed loop signals are bounded.*

*iv) The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

**Proof:** The properties $(i) - (iv)$ will be proved in sequence as follows:

i) From the fact that $\dot{V}_n \leq 0$, it is clear that

$$V_n(t) \leq \sum_{i=1}^{s_{n_s}} \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(0)} + \sum_{i=s_{n_s}+1}^{n} \frac{1}{2} z_i^2(0)$$

Using the identity $\log a + \log b = \log ab$, we have that

$$\frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(t)} \leq \frac{k_{b_1}^{2s_{n_s}} e^{\sum_{i=s_{n_s}}^{n} z_i^2(0)}}{\Pi_{i=1}^{s_{n_s}}(k_{b_1}^2 - z_i^2(0))} \tag{4.79}$$

Since $|z_i(0)| < k_{b_1}$, we know that $k_{b_1}^2 - z_i^2(t) > 0 \; \forall \; t$ from Lemma 2.4.2. Simple rearrangement yields that $|z_i(t)| \leq D_{z_1}, i = 1, 2, ..., s_{n_s}$. Then, based on the fact that $\frac{1}{2} \sum_{i=s_{n_s}+1}^{n} z_i^2(t) \leq V_n(0)$, it is easy to see that $\|z_{s_{n_s}+1:n}(t)\| \leq \sqrt{2V_n(0)}$. Hence, $z_i(t)$ remains in the compact set $\Omega_z \; \forall \; t > 0$.

ii) This part of the proof is similar to that in Theorem 4.3.1, with a minor difference that $|x_i| \leq k_{c_i}$ for $i = 1, 2, ..., s_{n_s}$, instead of $i = 1, 2, ..., n$. Since the sequence $\{s_1, s_2, ..., s_{n_s}\} \subset \{1, 2, ..., s_{n_s}\}$, we can conclude that $x_s(t) \in \Omega_{x_s} \; \forall t > 0$.

iii) We have already established the boundedness results $|z_i(t)| \leq D_{z_1} < k_{b_1}$, $|x_i(t)| < k_{c_i}$, and $\alpha_i(t) \leq A_i$ for $i = 1, ..., s_{n_s}$. Together with the fact $\|z_{s_{n_s}+1:n}\| \leq \sqrt{2V_n(0)}$, we can progressively show, via the usual signal chasing, that the remaining $\alpha_i(t)$ and $x_i(t)$ are also bounded $(i = s_{n_s}+1, ..., n)$. Then, it is straightforward to show that the control $u(\bar{x}_n, \bar{z}_n, \bar{y}_{d_n})$ is bounded. Thus, all closed loop signals are bounded.

iv) The proof follows by showing that $\ddot{V}_n$ is bounded and then invoking Barbalat's Lemma to conclude asymptotic stability of $z_1$. ■

### 4.4.2 Partial State Constraints: Uncertain Case

When dealing with parametric uncertainty, we employ the same Lyapunov function candidates as described in (4.66) and (4.70), with the exception that $V_1$ is augmented with a quadratic term of the parameter estimation error:

$$V_1 = \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \tag{4.80}$$

From step 1 to step $s_{n_s}$, BLF candidates are considered, leading to the following stabilizing functions:

$$\begin{aligned}
\alpha_1 &= \frac{1}{g_1}(-\hat{\theta}^T w_1 - (k_{b_1}^2 - z_1^2)\kappa_1 z_1 + \dot{y}_d) \\
\alpha_2 &= \frac{1}{g_2}\Bigg(-\hat{\theta}^T w_2 - (k_{b_1}^2 - z_2^2)\kappa_2 z_2 - \frac{k_{b_1}^2 - z_2^2}{k_{b_1}^2 - z_1^2} g_1 z_1 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \sum_{j=0}^{1} \frac{\partial \alpha_1}{\partial y_d^{(j)}} y_d^{(j+1)} \\
&\quad + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2\Bigg) \\
\alpha_i &= \frac{1}{g_i}\Bigg(-\hat{\theta}^T w_i - (k_{b_1}^2 - z_i^2)\kappa_i z_i - \frac{k_{b_1}^2 - z_i^2}{k_{b_1} - z_{i-1}^2} g_{i-1} z_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{x_j} x_{j+1} \\
&\quad + \sum_{j=0}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \sum_{j=2}^{i-1} \frac{z_j}{k_{b_1}^2 - z_j^2} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_i\Bigg), \quad i = 3, 4, ..., s_{n_s}
\end{aligned} \tag{4.81}$$

From step $(s_{n_s} + 1)$ onwards until the final step, QLF candidates are used, and the following stabilizing functions are designed:

$$\begin{aligned}
\alpha_{s_{n_s}+1} &= \frac{1}{g_{s_{n_s}+1}}\Bigg(-\hat{\theta}^T w_{s_{n_s}+1} - \kappa_{s_{n_s}+1} z_{s_{n_s}+1} - \frac{g_{s_{n_s}} z_{s_{n_s}}}{k_{b_1}^2 - z_{s_{n_s}}^2} + \sum_{j=1}^{s_{n_s}} \frac{\partial \alpha_{s_{n_s}}}{x_j} x_{j+1} \\
&\quad + \sum_{j=0}^{s_{n_s}} \frac{\partial \alpha_{s_{n_s}}}{\partial y_d^{(j)}} y_d^{(j+1)} + \frac{\partial \alpha_{s_{n_s}}}{\partial \hat{\theta}} \Gamma \tau_{s_{n_s}+1} + \sum_{j=2}^{s_{n_s}} \frac{z_j}{k_{b_1}^2 - z_j^2} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_{s_{n_s}+1}\Bigg)
\end{aligned}$$

$$\begin{aligned}
\alpha_k &= \frac{1}{g_k}\Bigg[-\hat{\theta}^T w_k - \kappa_k z_k - g_{k-1} z_{k-1} + \sum_{j=1}^{k-1} \frac{\partial \alpha_{k-1}}{x_j} x_{j+1} + \sum_{j=0}^{k-1} \frac{\partial \alpha_{k-1}}{\partial y_d^{(j)}} y_d^{(j+1)} \\
&\quad + \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma \tau_k + \left(\sum_{j=2}^{s_{n_s}} \frac{z_j}{k_{b_1}^2 - z_j^2} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} + \sum_{j=s_{n_s}+1}^{k-1} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\right) \Gamma w_k\Bigg], \\
&\quad k = s_{n_s} + 2, s_{n_s} + 3, ..., n
\end{aligned} \tag{4.82}$$

The control and adaptation laws are chosen as follows:

$$u = \alpha_n \tag{4.83}$$

$$\dot{\hat{\theta}} = \Gamma \tau_n \tag{4.84}$$

where the intermediate functions $w_i$, tuning functions $\tau_i$, and adaptation law are given by

$$\begin{aligned} w_1 &= \psi_1(x_1), \qquad w_i = \psi_i(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{x_j} \psi_j(\bar{x}_j), \qquad i = 2, ..., n \\ \tau_1 &= \frac{w_1 z_1}{k_{b_1}^2 - z_1^2}, \qquad \tau_i = \begin{cases} \tau_{i-1} + \frac{w_i z_i}{k_{b_1}^2 - z_i^2}, & i = 2, ..., s_{n_s} \\ \tau_{i-1} + w_i z_i, & i = s_{n_s} + 1, ..., n \end{cases} \end{aligned} \tag{4.85}$$

The results for the uncertain case are summarized in the following theorem.

**Theorem 4.4.2** *Consider uncertain system (4.1), under Assumptions 4.2.1-4.2.2, stabilizing functions and control law (4.81)-(4.82), and adaptation law (4.84). Denote by $A_i$ an upper bound for $\alpha_i$ in the compact set $\Omega_i$, that is,*

$$A_i \geq \sup_{(\bar{x}_i,\ \bar{z}_i,\ \bar{y}_{d_i},\ \hat{\theta})\ \in\ \Omega_i} |\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\theta}; \bar{\kappa}_i, \Gamma)|, \quad i = 1, ..., s_{n_s} - 1 \tag{4.86}$$

*where $\alpha_i$ is parameterized by $\Gamma =$ and $\bar{\kappa}_i = [\kappa_1, \kappa_2, ..., \kappa_i]^T$, and $\Omega_i$ is a compact set defined by:*

$$\Omega_i := \Big\{ \bar{x}_i \in \mathbb{R}^i, \bar{z}_i \in \mathbb{R}^i, \bar{y}_{d_i} \in \mathbb{R}^i, \hat{\theta} \in \mathbb{R}^l : \\ |x_j| \leq D_{z_1} + A_{j-1}, |z_j| \leq D_{z_1}, |y_d^{(j)}|, \|\hat{\theta}\| \leq D_{\hat{\theta}} \leq Y_j, j = 1, ..., i \Big\} \tag{4.87}$$

$$D_{z_1} := k_{b_1} \sqrt{1 - \frac{\Pi_{i=1}^{s_{n_s}} (k_{b_1}^2 - z_i^2(0))}{k_{b_1}^{2 s_{n_s}} e^{2\bar{V}_\xi}}} \tag{4.88}$$

$$D_{\hat{\theta}} := \theta_M + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma^{-1})}} \tag{4.89}$$

$$\bar{V}_\xi := \frac{1}{2} \sum_{i=s_{n_s}+1}^{n} z_i^2(0) + \frac{1}{2} \lambda_{\max}(\Gamma^{-1}) (\|\hat{\theta}(0)\| + \theta_M)^2 \tag{4.90}$$

$$\bar{V}_n := \frac{1}{2} \sum_{i=1}^{s_{n_s}} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(0)} + \bar{V}_\xi \tag{4.91}$$

*Given the constraints $\{k_{c_{s_2}}, k_{c_{s_3}}, ..., k_{c_{s_{n_s}}}\}$, and that*

*C1) there exist $A_i$ and $\{k_{c_i}\}_{i \in \mathcal{F}}$, where $\mathcal{F} := \{r_1, r_2, ..., r_{n_r}\} \cap \{1, 2, ..., s_{n_s}\}$, such that*

$$k_{c_{i+1}} > A_i + k_{b_1}, \qquad i = 1, ..., s_{n_s} - 1 \tag{4.92}$$

*C2) the initial conditions are such that*

$$\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_i| < k_{b_1}, i = 1, 2, ..., s_{n_s}\} \tag{4.93}$$

*then the following properties hold.*

*i) The signals $z_i(t)$, $i = 1, 2, ..., n$, and $\hat{\theta}(t)$ remain in the compact sets defined by*

$$\Omega_z := \left\{\bar{z}_n \in \mathbb{R}^n : |z_i| \le D_{z_1}, i = 1, 2, ..., s_{n_s}, \|z_{s_{n_s}+1:n}\| \le \sqrt{2\bar{V}_n}\right\} \tag{4.94}$$

$$\Omega_{\hat{\theta}} := \left\{\hat{\theta} \in \mathbb{R}^l : \|\hat{\theta}\| \le D_{\hat{\theta}}\right\} \tag{4.95}$$

*ii) The partial state $x_s(t)$ remains in the set $\Omega_{x_s} := \{x_s \in \mathbb{R}^{n_s} : |x_i| \le D_{z_1} + A_{i-1} < k_{c_i},\ i = s_1, s_2, ..., s_{n_s}\}$ $\forall t > 0$, i.e. the partial state constraint is never violated.*

*iii) All closed loop signals are bounded.*

*iv) The output tracking error $z_1(t)$ converges to zero asymptotically, i.e., $y(t) \to y_d(t)$ as $t \to \infty$.*

**Proof:** The properties $(i) - (iv)$ will be proved in sequence as follows.

i) Since $\|\theta\| \le \theta_M$, we have

$$\begin{aligned} V_n(0) &\le \sum_{i=1}^{s_{n_s}} \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(0)} + \sum_{i=s_{n_s}+1}^{n} \frac{1}{2} z_i^2(0) + \frac{1}{2}\lambda_{\max}(\Gamma^{-1})(\|\hat{\theta}(0)\| + \theta_M)^2 \\ &= \bar{V}_n \end{aligned}$$

From the fact that $\dot{V}_n \le 0$, it is clear that $V_n(t) \le V_n(0) \le \bar{V}_n$, and hence $\frac{1}{2}\lambda_{\min}(\Gamma^{-1})\|\hat{\theta}(t) - \theta\|^2 \le \bar{V}_n$. It is straightforward to show that $\|\hat{\theta}(t)\| \le \theta_M + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma^{-1})}}$ such that $\hat{\theta}(t)$ remains in the compact set $\Omega_{\hat{\theta}}$ $\forall$ $t$.

Furthermore, from $V_n(t) \le \bar{V}_n$, we have that

$$\frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(t)} \le \frac{k_{b_1}^{2n} e^{2\bar{V}_\xi}}{\Pi_{i=1}^{s_{n_s}}(k_{b_1}^2 - z_i^2(0))} \tag{4.96}$$

Since $|z_i(0)| < k_{b_1}$, we know that $k_{b_1}^2 - z_i^2(t) > 0$ $\forall$ $t$ from Lemma 2.4.2. A simple rearrangement yields $|z_i| \le D_{z_1}, i = 1, 2, ..., s_{n_s}$. Then, based on the fact that $\frac{1}{2}\sum_{i=s_{n_s}+1}^{n} z_i^2(t) \le \bar{V}_n$, it is easy to see that $\|z_{s_{n_s}+1:n}(t)\| \le \sqrt{2\bar{V}_n}$. Hence, $z_i$ remains in the compact set $\Omega_z$ $\forall$ $t > 0$.

ii) Similar to the proof of Theorem 4.3.2(ii), we can show that $|x_i| \leq k_{c_i}$ for $i = 1, 2, ..., s_{n_s}$. Since the sequence $\{s_1, s_2, ..., s_{n_s}\} \subset \{1, 2, ..., s_{n_s}\}$, we can conclude that $x_s(t) \in \Omega_{x_s}$ $\forall t > 0$.

iii) We have already established the boundedness results $\|\hat{\theta}\| \leq D_{\hat{\theta}}$, $|z_i(t)| \leq D_{z_1} < k_{b_1}$, $|x_i(t)| < k_{c_i}$, and $\alpha_i \leq A_i$ for $i = 1, ..., s_{n_s}$. Together with the fact $\|z_{s_{n_s}+1:n}\| \leq \sqrt{2\bar{V}_n}$, we can progressively show, along the lines of the proof of Theorem 4.3.2(iii), that the remaining $\alpha_i$ and $x_i$, for $i = s_{n_s} + 1, ..., n$, and the control $u(\bar{x}_n, \bar{z}_n, \bar{y}_{d_n}, \hat{\theta})$, are all bounded.

iv) The proof follows by showing that $\ddot{V}_n$ is bounded and then invoking Barbalat's Lemma to conclude asymptotic stability of $z_1$. ∎

### 4.4.3 Partial State Constraints: Feasibility Check

With only a portion of the states to be constrained, the feasibility conditions are relaxed. Recall that the full state is partitioned into free states $x_r = [x_{r_1}, x_{r_2}, ..., x_{r_{n_r}}]^T$ and constrained states $x_s = [x_{s_1}, x_{s_2}, ..., x_{s_{n_s}}]^T$. Then, the parameters $k_{c_i}$, for $i \in \mathcal{F} := \{r_1, r_2, ..., r_{n_r}\} \cap \{1, 2, ..., s_{n_s}\}$, are no longer hard constraints imposed by the problem, but are now design constants at our disposal. Additionally, there are less conditions to satisfy, except for the special case when $s_{n_s} = n$, i.e. $x_n$ needs to be constrained.

Similar to the full state constraint problem, we check offline the feasibility conditions C1-C2 in Theorems 4.4.1-4.4.2 by solving a nonlinear constrained optimization problem. When the plant is known, we check if there exists a solution $\bar{\kappa}_{s_{n_s}-1} := [\kappa_1, ..., \kappa_{s_{n_s}-1}]^T$ for the optimization problem:

$$\begin{aligned}
&\max_{\kappa_1,...,\kappa_{s_{n_s}-1}>0} P(\bar{\kappa}_{s_{n_s}-1}, \{k_{c_i}\}_{i\in\mathcal{F}}) = \sum_{i=1}^{s_{n_s}-1} a_i\kappa_i - \sum_{i\in\mathcal{F}} d_i k_{c_i} \\
&\text{subject to:} \\
&\quad k_{c_{i+1}} > A_i(\bar{\kappa}_i) + k_{b_1} \\
&\quad k_{b_1} > |x_{i+1}(0) - \alpha_i(\bar{x}_i(0), \bar{z}_i(0), \bar{y}_{d_i}(0); \bar{\kappa}_i)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad i = 1, ..., s_{n_s} - 1
\end{aligned} \tag{4.97}$$

where $P$ is the objective function, and $a_i, d_i$ are positive constants. If a solution $(\bar{\kappa}^*_{s_{n_s}-1}, \{k^*_{c_i}\}_{i\in\mathcal{F}})$ to the above optimization problem exists, then C1 and C2 in Theorem 4.4.1 are satisfied, and the proposed control (4.73) with $\bar{\kappa}_{s_{n_s}-1} = \bar{\kappa}^*_{s_{n_s}-1}$ is feasible in ensuring output tracking for the system (4.1) with partial state constraint.

When the plant is uncertain, the matrix of adaptation parameters, $\Gamma := \mathrm{diag}(\gamma_1, \gamma_2, ..., \gamma_l)$, is also taken into consideration in the optimization problem:

$$\max_{\kappa_1,...,\kappa_{s_{n_s}-1},\Gamma>0} P(\bar{\kappa}_{s_{n_s}-1}, \{k_{c_i}\}_{i\in\mathcal{F}}, \Gamma) = \sum_{i=1}^{s_{n_s}-1} a_i\kappa_i + \sum_{i=1}^{l} b_i\gamma_i - \sum_{i\in\mathcal{F}} d_i k_{c_i}$$

subject to:

$$\begin{aligned} & k_{c_{i+1}} > A_i(\bar{\kappa}_i, \Gamma) + k_{b_1} \\ & k_{b_1} > |x_{i+1}(0) - \alpha_i(\bar{x}_i(0), \bar{z}_i(0), \bar{y}_{d_i}(0), \hat{\theta}(0); \bar{\kappa}_i, \Gamma)| \\ & \qquad\qquad i = 1, ..., s_{n_s} - 1 \end{aligned} \tag{4.98}$$

where $b_i$ are positive constants. If a solution $(\bar{\kappa}^*_{s_{n_s}-1}, \{k^*_{c_i}\}_{i\in\mathcal{F}}, \Gamma^*)$ to the above optimization problem exists, then C1 and C2 in Theorem 4.4.2 are satisfied, and the proposed adaptive control (4.83)-(4.84), with $\bar{\kappa}_{s_{n_s}-1} = \bar{\kappa}^*_{s_{n_s}-1}$ and $\Gamma = \Gamma^*$, is feasible in ensuring output tracking for the system (4.1) with partial state constraint.

Note that a penalty term $-\sum_{i\in\mathcal{F}} d_i k_{c_i}$ is appended in the above objective functions to limit the growth of the design constants $k_{c_i}$, $i \in \mathcal{F}$, during the optimization. For each $i \in \mathcal{F}$, ensuring $k_{c_i}$ to be as small as possible helps to ensure that $A_i$ is also small, thus increasing the possibility of satisfying the condition $k_{c_{i+1}} > A_i + k_{b_1}$.

## 4.5 Simulation

In this section, we present simulation studies to demonstrate the effectiveness of the proposed control, with and without uncertainty in the plant model. Consider the second-order nonlinear system

$$\begin{aligned} \dot{x}_1 &= \theta_1 x_1^2 + x_2 \\ \dot{x}_2 &= \theta_2 x_1 x_2 + \theta_3 x_1 + (1 + x_1^2)u \end{aligned} \tag{4.99}$$

where $\theta_1 = 0.1$, $\theta_2 = 0.1$, and $\theta_3 = -0.2$. The objective is for $x_1$ to track desired trajectory $y_d = 0.2 + 0.3\sin t$, subject to full state constraint $|x_1| < k_{c_1} = 0.8$ and

$|x_2| < k_{c_2} = 2.5$. Since $|y_d| \leq A_0 = 0.5$, we have that $k_{b_1} = 0.8 - 0.5 = 0.3$. Further, we have $|\dot{y}_d| \leq Y_1 = 0.3$. The initial conditions are $x_1(0) = 0.0$ and $x_2(0) = 0.5$.

For the known case, it can be verified that with control gain $\kappa_1 = 1.0$, we obtain that $z_1(0) = -0.2$, $z_2(0) = 0.19$, $D_{z_1} = 0.249$ from (4.23), and $A_1$, with the help of Lemma 2.4.4, as follows:

$$\sup_{(x_1,z_1,\dot{y}_d)\in\Omega_1} |\alpha_1| \ \leq \ \theta_1(D_{z_1} + A_0)^2 + Y_1 + \frac{2}{3\sqrt{3}} k_{b_1}^3 \kappa_1 = A_1 = 0.367 \qquad (4.100)$$

where $\Omega_1 = \{x_1 \in \mathbb{R}, z_1 \in \mathbb{R}, \dot{y}_d \in \mathbb{R} : |x_1| \leq D_{z_1} + A_0, |z_1| \leq D_{z_1}, |\dot{y}_d| \leq Y_1\}$. Therefore, the condition $k_{c_2} > A_1$ is satisfied. At the same time, we have $|z_2(0)| \leq |z_1(0)| < k_{b_1}$. Thus, the feasibility conditions C1-C2 in Theorems 4.3.1 are satisfied. Further, we obtain that $k_{b_1} = 2.5 - 0.367 = 2.133$, and choose $\kappa_2 = 1.0$.

For the adaptive case, it can be verified that with the control gain $\kappa_1 = 1.0$, adaptation parameters $\gamma_1 = \gamma_2 = \gamma_3 = 5.0$, and $\hat{\theta}(0) = 0.0$, we obtain that $z_1(0) = -0.2$, $z_2(0) = 0.19$, $D_{z_1} = 0.250$ from (4.48), $D_{\hat{\theta}} = 2.682$ from (4.49), and $A_1$, with the help of Lemma 2.4.4, as follows:

$$\sup_{(x_1,z_1,\dot{y}_d,\hat{\theta})\in\Omega_1} |\alpha_1| \ \leq \ D_{\hat{\theta}}(D_{z_1} + A_0)^2 + Y_1 + \frac{2}{3\sqrt{3}} k_{b_1}^3 \kappa_1 = A_1 = 1.819 \qquad (4.101)$$

where $\Omega_1 = \{x_1 \in \mathbb{R}, z_1 \in \mathbb{R}, \dot{y}_d \in \mathbb{R}, \hat{\theta} \in \mathbb{R} : |x_1| \leq D_{z_1} + A_0, |z_1| \leq D_{z_1}, |\dot{y}_d| \leq Y_1, \|\hat{\theta}\| \leq D_{\hat{\theta}}\}$. Therefore, the condition $k_{c_2} > A_1$ is satisfied. At the same time, we have $|z_2(0)| \leq |z_1(0)| < k_{b_1}$. Thus, the feasibility conditions C1-C2 in Theorems 4.3.2 are satisfied. Further, we obtain that $k_{b_1} = 2.5 - 1.819 = 0.681$, and choose $\kappa_2 = 1.0$.

Simulation results for full state constraint problem with and without uncertainty are shown in Figures 4.1-4.5. Good tracking performance is exhibited, and the state constraint requirements $|x_1| < k_{c_1}$ and $|x_2| < k_{c_2}$ are satisfied, as a result of enforcing constraints on error signals $|z_1| < k_{b_1}$ and $|z_2| < k_{b_1}$. The control signals and the parameter estimates are well behaved and bounded.

**Remark 4.5.1** *In this simulation, we have selected parameters for the controller based on trial and error out of simplicity. Alternatively, the parameters can be selected by solving the optimization problems described in (4.64) and (4.65). The optimization problem can be solved by using state of the art solvers such as the MATLAB function "fmincon.m" in the MATLAB Optimization Toolbox.*

## 4.6 Conclusions

In this chapter, we have presented control designs for strict feedback systems with constraints on the states, based on Barrier Lyapunov Functions. Besides the nominal case where the plant is known exactly, the presence of parametric uncertainties has also been handled. When dealing with full state constraints, asymptotic tracking is achieved without violation of constraints, and all closed loop signals remain bounded, under some feasibility conditions which involve the initial states and the control parameters. When handling only partial state constraints, the conditions can be relaxed. These feasibility conditions can be checked offline. The effectiveness of the proposed control has been demonstrated through a simulation example.

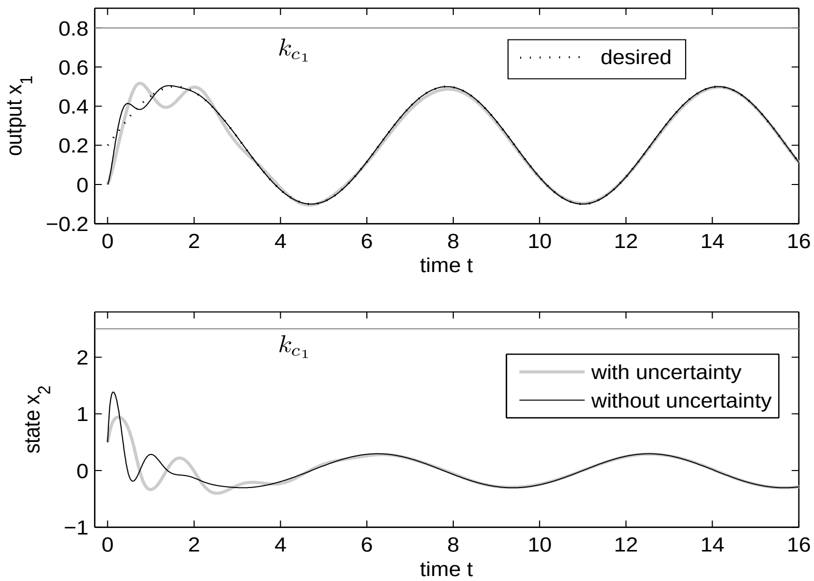


Figure 4.1: The output $x_1$ and the state $x_2$ for the full state constraint problem with and without uncertainty.

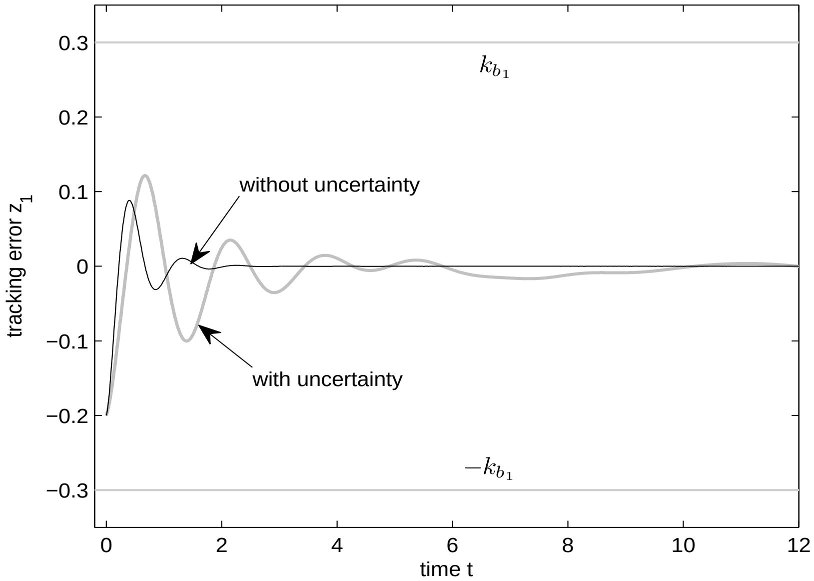


Figure 4.2: Tracking error $z_1$ for the full state constraint problem with and without uncertainty.

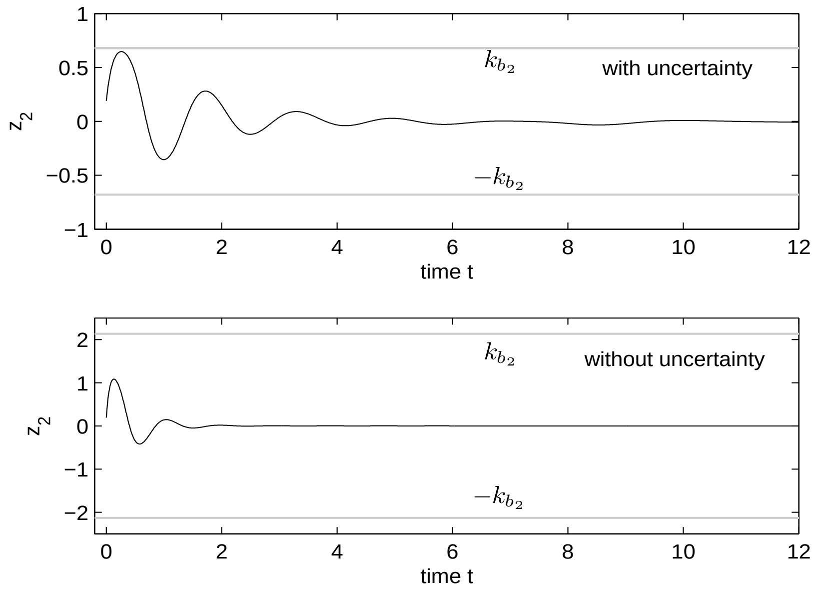


Figure 4.3: The error signal $z_2$ for the full state constraint problem with and without uncertainty.

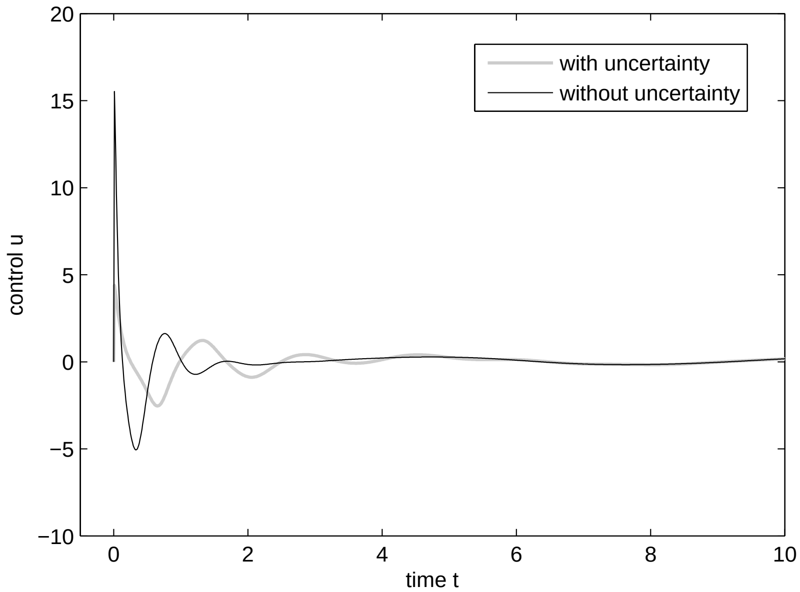


Figure 4.4: Control signal.

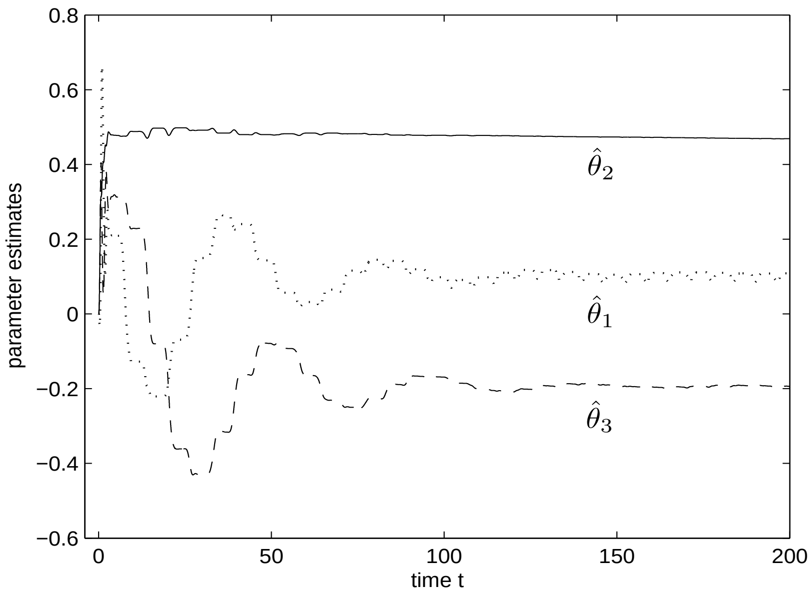


Figure 4.5: Parameter estimates.

# Chapter 5

# Control of Constrained Systems with Uncertain Control Gain Functions

## 5.1 Introduction

In this chapter, we extend our investigations to the adaptive control problem for SISO nonlinear strict feedback systems with uncertain control gain functions and constraints in the output and states. Methods for handling unknown virtual control gains include the use of Integral Lyapunov Functions [43] and quadratic-like Lyapunov functions with reciprocal of control gain function [44]. As these approaches are difficult to combine with Barrier Lyapunov Functions for handling of constraints, we adopt, in this chapter, the robust adaptive domination approach of handling unknown virtual control gains. In the adaptive domination approach, we do not try to cancel the nonlinearities as in feedback linearization, but instead dominate them by adaptively estimating constant bounds for the nonlinear functions within some local region. Then, with the help of BLFs, it can be shown that the state never leaves the said region, thus validating the control design and analysis.

Within this framework, conditions for practical stability with guaranteed non-violation of constraints are established, and both cases of full state constraint and output constraint are considered. For the case of full state constraints, we employ Barrier Lyapunov Functions for each step of the backstepping design. Feasibility conditions on the initial states and control parameters are provided, which can be checked *a priori*, to determine if the given problem can be solved with these approaches, and can generally be relaxed when handling only partial state constraints. For the special case of output constraint with linearly parameterized system nonlinearities, feasibility conditions are not required, and the design employs BLF only in the first step of backstepping, while the subsequent steps are all based on quadratic ones.

The organization of the remainder of this chapter is outlined as follows. In Section 5.2, the tracking control problem for nonlinear constrained systems in strict feedback form is formulated, where we pay special attention to the uncertainty of the control gain functions. Following that, the control design methodology is detailed in Section 5.3 for the case of full state constraint, along with the conditions that govern the feasibility of proposed control. Section 5.4 extends these results to the special case of output constraint with linearly parameterized system nonlinearities, for which feasibility conditions are not required. Finally, computer simulation results are presented in Section 5.5 to illustrate the performance of the control, before concluding remarks are made in Section 5.6.

## 5.2 Problem Formulation and Preliminaries

Consider the system in strict feedback form:

$$\begin{aligned}\dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, \quad i = 1, 2, ..., n-1 \\ \dot{x}_n &= f_n(\bar{x}_n) + g_n(\bar{x}_n)u \\ y &= x_1 \end{aligned} \tag{5.1}$$

where $x_1, x_2, ..., x_n$ are the states, $\bar{x}_i := [x_1, x_2, ..., x_i]^T \in \mathbb{R}^i$, $f_i$ and $g_i$ are uncertain smooth functions, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the input and output respectively, for $i = 1, 2, ..., n$. We consider the problems of output and state constraints. For the case of output constraint, the output is required to remain in the set $|y| \leq k_{c_1}$, with $k_{c_1}$ being a constant. For the case of state constraints, every state $x_i$ is required to remain in the set $|x_i| \leq k_{c_i}$, with $k_{c_i}$ being a constant, for $i = 1, ..., n$.

The control objective is to track a desired trajectory $y_d$ while ensuring that all closed loop signals are bounded and that *output or state constraints are not violated.* In this chapter, for convenience of notation, we group the derivatives of the desired trajectory in the vector $\bar{y}_{d_i} := [y_d^{(1)}, y_d^{(2)}, ..., y_d^{(i)}]^T$. The assumptions on the desired trajectory $y_d$, as well as the functions $g_i(\cdot)$, $i = 1, ..., n$, from (5.1), are stated as follows.

**Assumption 5.2.1** *For any $k_{c_1} > 0$, there exist positive constants $A_0$, $Y_1$, $Y_2$,..., $Y_n$ such that the desired trajectory $y_d(t)$ and its time derivatives satisfy*

$$|y_d(t)| \leq A_0 < k_{c_1}, \quad |\dot{y}_d(t)| < Y_1, \quad |\ddot{y}_d(t)| < Y_2, \quad \cdots, \quad |y_d^{(n)}(t)| < Y_n \tag{5.2}$$

*for all $t \geq 0$.*

**Assumption 5.2.2** *The control gain functions $g_i(\bar{x}_i)$ satisfy $|g_i(\bar{x}_i)| \geq g^* \geq g_{\min} > 0$ for $i = 1, 2, ..., n$ where $g^* := \min_{i=1,...,n}\{\inf_{\bar{x}_i \in \mathbb{R}^i} g_i(\bar{x}_i)\}$ is uncertain, while $g_{\min}$ is a known positive constant. Note that $g_{\min}$ can be a conservative estimate for $g^*$. We further assume that the $g_i(\bar{x}_i)$ are all positive.*

## 5.3 Control Design for State Constraints

In this section, we consider the case of full state constraints, and employ BLFs in every step of backstepping design, so as to keep each error signal $z_i = x_i - \alpha_{i-1}$ $(i = 2, ..., n)$ constrained. Provided that each stabilizing function $\alpha_{i-1}$ is bounded in the specified constrained region for $x_i$, we can ensure that $x_i$ remains in the constrained region, subject to feasibility conditions.

Unlike the previous chapters, which adopted a cancellation based approach, this chapter is based on a domination based approach, due to the presence of uncertain control gain functions. We first explain the technique of robust adaptive domination design with BLF using a simple first order nonlinear system as a motivating example. Subsequently, the design methodology is extended to the $n$-order system (5.1) with the use of backstepping techniques.

### 5.3.1 Robust Adaptive Domination Design

For clarity of presentation, we outline the method of employing Robust Adaptive Domination Design together with BLF to design a control that not only handles the uncertain control gain function, but also prevents the state constraint from being transgressed. Consider, as a motivating example, the first order nonlinear system:

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u \tag{5.3}$$

where the objective is to stabilize the origin while ensuring that $|x_1| < k_c$. Choose Lyapunov function candidate as:

$$V_x = \frac{1}{2}\log\frac{k_c^2}{k_c^2 - x_1^2} \tag{5.4}$$

which is positive definite and continuously differentiable in the region $|x_1| < k_c$. The derivative of $V_x$ along the solution of (5.3) is given by

$$\dot{V}_x \;=\; \frac{x_1}{k_c^2 - x_1^2}(f_1(x_1) + g_1(x_1)u) \tag{5.5}$$

In the adaptive domination approach, we do not try to cancel the nonlinearity $f_1(x_1)$, but instead dominate it by adaptively estimating a local constant bound for the non-linear function. Considering the set $|x_1| \leq k_c$, we know, by virtue of the smoothness of function $f_1(x_1)$, that $f_1(x_1) \leq F_1$ where $F_1 := \sup_{|x_1|\leq k_c}|f_1(x_1)|$, which yields the inequality:

$$\dot{V}_x \;\leq\; \left|\frac{x_1}{k_c^2 - x_1^2}\right| F_1 + \frac{x_1 g_1(x_1)u}{k_c^2 - x_1^2} \tag{5.6}$$

By completion of squares on the first term on the right hand side, it can be obtained that

$$\dot{V}_x \;\leq\; \frac{\lambda g^* x_1^2}{(k_c^2 - x_1^2)^2}\theta_1 + \frac{x_1 g_1(x_1)u}{k_c^2 - x_1^2} + \frac{1}{4\lambda} \tag{5.7}$$

where $\lambda$ is a positive constant, and $\theta_1 := F_1^2/g^*$ is an unknown parameter to be estimated adaptively. Denote by $\hat{\theta}_1$ an estimate for $\theta_1$, and design the control and adaptation laws as follows:

$$u \;=\; -\left(\frac{\lambda}{k_c^2 - x_1^2}\hat{\theta}_1 + (k_c^2 - x_1^2)\kappa_1\right)x_1 \tag{5.8}$$

$$\dot{\hat{\theta}}_1 \;=\; \frac{\Gamma_1 x_1^2}{(k_c^2 - x_1^2)^2} - \sigma\hat{\theta}_1, \quad \hat{\theta}_1(0) \geq 0 \tag{5.9}$$

where $\sigma$ is a positive constant, and $\kappa_1 > c/k_c^2$ with $c$ to be defined later. Then, the closed loop system consists of (5.9) and

$$\dot{x}_1 = f_1(x_1) - g_1(x_1)\left(\frac{\lambda}{k_c^2 - x_1^2}\hat{\theta}_1 - (k_c^2 - x_1^2)\kappa_1\right)x_1 \tag{5.10}$$

From (5.9), it is easy to see that $\hat{\theta}_1(t) \geq 0\ \forall t > 0$. At the same time, from Assumption 5.2.2, we know that $g_1(x_1) > g_{\min} > 0$. Therefore, it is clear that the following inequality holds:

$$-\frac{\lambda g_1(x_1)x_1^2}{(k_c^2 - x_1^2)^2}\hat{\theta}_1 \quad \leq \quad -\frac{\lambda g^* x_1^2}{(k_c^2 - x_1^2)^2}\hat{\theta}_1 \tag{5.11}$$

Substituting the control law into (5.7), and using the above inequality, the derivative of $V_x$ can be rewritten in the form:

$$\dot{V}_x \quad \leq \quad -\kappa_1 g_1(x_1)x_1^2 - \frac{\lambda g^* x_1^2}{(k_c^2 - x_1^2)^2}\tilde{\theta}_1 + \frac{1}{4\lambda} \tag{5.12}$$

To analyze closed loop stability due to online parameter adaptation, we augment $V_x$ with a quadratic term of the parameter estimation error $\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1$, which yields the new Lyapunov function candidate as:

$$V = V_x + \frac{\lambda g^*}{2}\Gamma_1^{-1}\tilde{\theta}_1^2 \tag{5.13}$$

where $\Gamma_1$ is a positive constant. Finally, the derivative of $V$ satisfies the inequality:

$$\dot{V} \quad \leq \quad -\kappa_1 g_1(x_1)x_1^2 - \frac{\lambda g^* \sigma}{2}\Gamma_1^{-1}\tilde{\theta}_1^2 + c \tag{5.14}$$

where

$$c := \frac{1}{4\lambda} + \frac{\lambda g^* \sigma}{2}\Gamma_1^{-1}\theta_1^2 \tag{5.15}$$

Let the closed loop system (5.9)-(5.10) be written as $\dot{\eta} = h(t,\eta)$, where $\eta = [x_1,\ \hat{\theta}]^T$. By inspection, $h(t,\eta)$ satisfies the conditions (2.17)-(2.20) in the open set $\eta \in \mathcal{Z} := \{x_1 \in \mathbb{R}, \hat{\theta} \in \mathbb{R} : |x_1| < k_c\}$. Together with (5.14) and $\kappa_1 > c/k_c^2$, Lemma 2.4.3 can be invoked to show that the state constraint is never violated, i.e. $|x_1(t)| < k_c\ \forall t > 0$, as long as $|x_1(0)| < k_c$.

Thus far, in the foregoing design and analysis, we have assumed that $f_1(x_1) \leq F_1$ in the set $|x_1| \leq k_c$. Since the proposed control indeed renders the set $|x_1| < k_c$ positively invariant, we can safely conclude that $f_1(x_1(t)) \leq F_1$ is valid $\forall t > 0$, such that the control design and analysis are valid.

### 5.3.2 Adaptive Backstepping Design

In this section, the foregoing control design methodology is extended to the $n$-order strict feedback system (5.1) via backstepping. For any $\delta \geq 0$, let

$$\Omega_c \;:=\; \{x \in R^n : |x_i| < k_{c_i},\ i = 1, ..., n\} \tag{5.16}$$

$$\Omega_x \;:=\; \{x \in R^n : |x_i| \leq k_{c_i} + \delta,\ i = 1, ..., n\} \tag{5.17}$$

For $x \in \Omega_x$, the uncertain functions $f_i(\bar{x}_i)$ and $g_i(\bar{x}_i)$ are bounded by known positive constants $\bar{F}_i$ and $\bar{G}_i$ respectively. Then, robust adaptive backstepping with Barrier Lyapunov Functions is employed to ensure that $x(t) \in \Omega_c \subset \Omega_x$, under certain initial conditions. The detailed design procedure is presented as follows.

**Step 1** Denote $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, where $\alpha_1$ is a stabilizing function to be designed. Choose Lyapunov function candidate as:

$$V_1 = \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} + \frac{\lambda g^*}{2}\Gamma_1^{-1}\tilde{\theta}_1^2 \tag{5.18}$$

where $k_{b_1} = k_{c_1} - A_0$, $\lambda$ and $\Gamma_1$ are positive constants, and $\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1$ is the estimation error, with $\theta_1$ an unknown positive parameter and $\hat{\theta}_1$ its estimate. The derivative of $V_1$ is given by

$$\dot{V}_1 \;\leq\; \left|\frac{z_1}{k_{b_1}^2 - z_1^2}\right|(F_1 + Y_1) + \frac{z_1}{k_{b_1}^2 - z_1^2}(g_1(x_1)z_2 + g_1(x_1)\alpha_1) + \frac{\lambda g^*}{\Gamma_1}\tilde{\theta}_1\dot{\hat{\theta}}_1 \tag{5.19}$$

where $F_1 := \sup_{x\in\Omega_x} f_1(x_1)$ and $\dot{y}_d \leq Y_1$. By completion of squares, we have that

$$\left|\frac{z_1}{k_{b_1}^2 - z_1^2}\right|(F_1 + Y_1) \;\leq\; \frac{\lambda g^* z_1^2}{(k_{b_1}^2 - z_1^2)^2}\theta_1 + \frac{1}{4\lambda} \tag{5.20}$$

where $\theta_1 := (F_1 + Y_1)^2/g^*$. Then, it can be shown that

$$\dot{V}_1 \;\leq\; \frac{\lambda g^* z_1^2}{(k_{b_1}^2 - z_1^2)^2}\theta_1 + \frac{g_1(x_1)z_1\alpha_1}{k_{b_1}^2 - z_1^2} + \frac{g_1(x_1)z_1 z_2}{k_{b_1}^2 - z_1^2} + \frac{\lambda g^*}{\Gamma_1}\tilde{\theta}_1\dot{\hat{\theta}}_1 + \frac{1}{4\lambda} \tag{5.21}$$

Design the stabilizing function $\alpha_1$ and adaptation law designed as:

$$\alpha_1 \;=\; -\left(\frac{\lambda}{k_{b_1}^2 - z_1^2}\hat{\theta}_1 + (k_{b_1}^2 - z_1^2)\kappa_1\right) z_1 \tag{5.22}$$

$$\dot{\hat{\theta}}_1 \;=\; \frac{\Gamma_1 z_1^2}{(k_{b_1}^2 - z_1^2)^2} - \sigma\hat{\theta}_1, \quad \hat{\theta}_1(0) \geq 0 \tag{5.23}$$

where $\kappa_1$ is a positive constant. Substituting (5.22) and (5.23) into (5.21) yields

$$\begin{aligned}\dot{V}_1 &\leq -\kappa_1 g_1(x_1) z_1^2 + \frac{\lambda g^* z_1^2}{(k_{b_1}^2 - z_1^2)^2}\theta_1 - \frac{\lambda g_1(x_1) z_1^2}{(k_{b_1}^2 - z_1^2)^2}\hat{\theta}_1 + \frac{g_1(x_1) z_1 z_2}{k_{b_1}^2 - z_1^2} \\ &\quad + \lambda g^* \tilde{\theta}_1 \left( \frac{z_1^2}{(k_{b_1}^2 - z_1^2)^2} - \Gamma_1^{-1}\sigma\hat{\theta}_1 \right) + \frac{1}{4\lambda} \end{aligned} \qquad (5.24)$$

From (5.23), we know that $\hat{\theta}_1 \geq 0$, and from Assumption 5.2.2, we know that $g_1(x_1) > g_{\min} > 0$. As a result, it is easy to obtain that

$$\begin{aligned}\dot{V}_1 &\leq -\kappa_1 g_1(x_1) z_1^2 - \frac{\lambda g^* z_1^2}{(k_{b_1}^2 - z_1^2)^2}\tilde{\theta}_1 + \lambda g^* \tilde{\theta}_1 \left( \frac{z_1^2}{(k_{b_1}^2 - z_1^2)^2} - \Gamma_1^{-1}\sigma\hat{\theta}_1 \right) \\ &\quad + \frac{g_1(x_1) z_1 z_2}{k_{b_1}^2 - z_1^2} + \frac{1}{4\lambda} \\ &\leq -\kappa_1 g_1(x_1) z_1^2 + \frac{g_1(x_1) z_1 z_2}{k_{b_1}^2 - z_1^2} - \lambda g^* \sigma \Gamma_1^{-1} \tilde{\theta}_1 \hat{\theta}_1 + \frac{1}{4\lambda} \end{aligned} \qquad (5.25)$$

Using the property that $-\tilde{\theta}_1\hat{\theta}_1 \leq \frac{1}{2}(-\tilde{\theta}_1^2 + \theta_1^2)$, we obtain

$$\dot{V}_1 \leq -\kappa_1 g_1(x_1) z_1^2 + \frac{g_1(x_1) z_1 z_2}{k_{b_1}^2 - z_1^2} - \frac{\lambda g^* \sigma}{2}\Gamma_1^{-1}\tilde{\theta}_1^2 + c_1 \qquad (5.26)$$

where

$$c_1 = \frac{1}{4\lambda} + \frac{\lambda g^* \sigma}{2}\Gamma_1^{-1}\theta_1^2 \qquad (5.27)$$

The coupling term $\frac{g_1(x_1) z_1 z_2}{k_{b_1}^2 - z_1^2}$ is dominated in the subsequent step.

**Step i** $(i = 2, ..., n)$

Denote $z_{i+1} = x_{i+1} - \alpha_i$, where $\alpha_i$ is a stabilizing function to be designed, and $z_{i+1} := 0$. Choose the Lyapunov function candidate:

$$V_i = V_{i-1} + \frac{1}{2}\log\frac{k_{b_1}}{k_{b_1}^2 - z_i^2} + \frac{\lambda g^*}{2}\tilde{\theta}_i^T \Gamma_i^{-1}\tilde{\theta}_i \qquad (5.28)$$

where $k_{b_1}$ is to defined later, $\Gamma_i := \text{diag}(\gamma_{i,1}, \gamma_{i,2}, ..., \gamma_{i,2i}) > 0$, and $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$ is the estimation error between $\theta_i$ and its estimate $\hat{\theta}_i$. The derivative of $\alpha_{i-1}$ can be described by the expression:

$$\dot{\alpha}_{i-1} = \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(f_j(\bar{x}_j) + g_j(\bar{x}_j)x_{j+1}) + w_{i-1} \qquad (5.29)$$

where $w_{i-1}$ is a computable quantity represented by:

$$w_{i-1} := \sum_{j=0}^{i-2} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}_j} \dot{\hat{\theta}}_j \tag{5.30}$$

The derivative of $V_i$ is given by

$$\dot{V}_i = \frac{z_i}{k_{b_1}^2 - z_i^2}[f_i(\bar{x}_i) - \dot{\alpha}_{i-1} + g_i(\bar{x}_i)z_{i+1} + g_i(\bar{x}_i)\alpha_i] + \lambda g^* \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\hat{\theta}}_i + \dot{V}_{i-1} \tag{5.31}$$

Substituting (5.29) into the above equation yields:

$$\begin{aligned}
\dot{V}_i \;\; \leq \;\; & \frac{z_i}{k_{b_1}^2 - z_i^2}[f_i(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(f_j(\bar{x}_j) + g_j(\bar{x}_j)x_{j+1}) - w_{i-1} + g_i(\bar{x}_i)(z_{i+1} + \alpha_i)] \\
& + \lambda g^* \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\hat{\theta}}_i - \sum_{j=1}^{i-1} \kappa_j g_j(\bar{x}_j) z_j^2 + \frac{g_{i-1}(\bar{x}_{i-1}) z_{i-1} z_i}{k_{b_1}^2 - z_{i-1}^2} - \sum_{j=1}^{i-1} \frac{\lambda g^* \sigma}{2} \tilde{\theta}_j^2 + c_{i-1} \\
\leq & \left| \frac{z_i}{k_{b_1}^2 - z_i^2} \right| \left( F_i + \sum_{j=1}^{i-1} \left| \frac{\partial \alpha_{i-1}}{\partial x_j} \right| (F_j + G_j |x_{j+1}|) + |w_{i-1}| \right) + \left| \frac{z_{i-1} z_i}{k_{b_1}^2 - z_{i-1}^2} \right| G_{i-1} \\
& + \lambda g^* \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\hat{\theta}}_i + \frac{z_i}{k_{b_1}^2 - z_i^2}(g_i(\bar{x}_i) z_{i+1} + g_i(\bar{x}_i)\alpha_i) - \sum_{j=1}^{i-1} \kappa_j g_j(\bar{x}_j) z_j^2 \\
& - \sum_{j=1}^{i-1} \frac{\lambda g^* \sigma}{2} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + c_{i-1}
\end{aligned} \tag{5.32}$$

where $F_i := \sup_{x \in \Omega_x} |f_i(\bar{x}_i)|$, and $G_{i-1} := \sup_{x \in \Omega_x} |g_{i-1}(\bar{x}_{i-1})|$.

**Remark 5.3.1** *Although it would appear more convenient to consider the bound* $\sup_{x \in \Omega_x}(f_i(\bar{x}_i) - \dot{\alpha}_{i-1})$ *in (5.31), this is not viable because* $\alpha_{i-1}$*, and thus* $\frac{\partial \alpha_{i-1}}{\partial x_j}$ *and* $w_{i-1}$*, are not continuous at the points* $|z_j| = k_{b_1}$*, due to the terms* $(k_{b_1}^2 - z_j^2)$*, in the denominator, for* $j = 1, ..., i-1$*. As a result,* $\sup_{x \in \Omega_x}(f_i(\bar{x}_i) - \dot{\alpha}_{i-1})$ *is not finite for all* $x \in \Omega_x$*. To circumvent this problem, we note, from (5.29)-(5.30), that the unknown parts of* $\dot{\alpha}_{i-1}$*, namely* $f_j(\bar{x}_j) + g_j(\bar{x}_j)x_{j+1}$*, are continuous, such that they are upper bounded by positive constants in* $\Omega_x$*. The splitting of* $\dot{\alpha}_{i-1}$ *into continuous and non-continuous parts result in the need for the multiple bounding constants* $F_j$ *and* $G_j$*.*

Using completion of squares, the following inequalities can be shown to hold:

$$\left|\frac{z_i}{k_{b_1}^2 - z_i^2}\right| F_i \leq \frac{\lambda z_i^2}{(k_{b_1}^2 - z_i^2)^2} F_i^2 + \frac{1}{4\lambda} \tag{5.33}$$

$$\left|\frac{z_i}{k_{b_1}^2 - z_i^2}\right| \left|\frac{\partial \alpha_{i-1}}{\partial x_j}\right| F_j \leq \frac{\lambda z_i^2}{(k_{b_1}^2 - z_i^2)^2} \left|\frac{\partial \alpha_{i-1}}{\partial x_j}\right|^2 F_j^2 + \frac{1}{4\lambda} \tag{5.34}$$

$$\left|\frac{z_i}{k_{b_1}^2 - z_i^2}\right| |w_{i-1}| \leq \frac{\lambda z_i^2}{(k_{b_1}^2 - z_i^2)^2} w_{i-1}^2 + \frac{1}{4\lambda} \tag{5.35}$$

$$\left|\frac{z_i}{k_{b_1}^2 - z_i^2}\right| \left|\frac{\partial \alpha_{i-1}}{\partial x_j}\right| |x_{j+1}| G_j \leq \frac{\lambda z_i^2 x_{j+1}^2}{(k_{b_1}^2 - z_i^2)^2} \left|\frac{\partial \alpha_{i-1}}{\partial x_j}\right|^2 G_j^2 + \frac{1}{4\lambda} \tag{5.36}$$

$$\left|\frac{z_{i-1} z_i}{k_{b_1}^2 - z_{i-1}^2}\right| G_{i-1} \leq \frac{\lambda z_{i-1}^2 z_i^2}{(k_{b_1}^2 - z_{i-1}^2)^2} G_{i-1}^2 + \frac{1}{4\lambda} \tag{5.37}$$

for $j = 1, ..., i-1$. Substituting the above inequalities into (5.32) yields

$$\begin{aligned} \dot{V}_i \leq\ & \frac{\lambda g^* z_i^2}{(k_{b_1}^2 - z_i^2)^2} \theta_i^T \Psi_i + \frac{z_i}{k_{b_1}^2 - z_i^2} (g_i(\bar{x}_i) z_{i+1} + g_i(\bar{x}_i)\alpha_i) \\ & + \lambda g^* \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\hat{\theta}}_i - \sum_{j=1}^{i-1} \kappa_j g_j(\bar{x}_j) z_j^2 - \sum_{j=1}^{i-1} \frac{\lambda g^* \sigma}{2} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + c_{i-1} + \frac{2i+1}{4\lambda} \end{aligned} \tag{5.38}$$

where $\theta_i = \frac{1}{g^*}[F_i^2, ..., F_1^2, 1, G_1^2, ..., G_{i-1}^2]^T$, and the regressor is given by

$$\begin{aligned} \Psi_i =\ & \left[1, \left|\frac{\partial \alpha_{i-1}}{\partial x_{i-1}}\right|^2, ..., \left|\frac{\partial \alpha_{i-1}}{\partial x_1}\right|^2, w_{i-1}^2, \left|\frac{\partial \alpha_{i-1}}{\partial x_1}\right|^2 x_2^2, ..., \left|\frac{\partial \alpha_{i-1}}{\partial x_{i-2}}\right|^2 x_{i-1}^2, \right. \\ & \left. \frac{z_{i-1}^2 (k_{b_1}^2 - z_i^2)^2}{(k_{b_1}^2 - z_{i-1}^2)^2} + \left|\frac{\partial \alpha_{i-1}}{\partial x_{i-1}}\right|^2 x_i^2 \right]^T \end{aligned} \tag{5.39}$$

Choose stabilizing function, control and adaptation laws as follows:

$$\alpha_i = -\left(\frac{\lambda}{k_{b_1}^2 - z_i^2} \hat{\theta}_i^T \Psi_i + (k_{b_1}^2 - z_i^2)\kappa_i\right) z_i \tag{5.40}$$

$$u = \alpha_n \tag{5.41}$$

$$\dot{\hat{\theta}}_i = \frac{z_i^2}{(k_{b_1}^2 - z_i^2)^2} \Gamma_i \Psi_i - \sigma \hat{\theta}_i, \quad \hat{\theta}_i(0) \geq 0 \tag{5.42}$$

The closed loop system is given by

$$\dot{z}_1 = f_1 - \dot{y}_d + g_1 z_2 - \frac{\lambda g_1 \hat{\theta}_1 z_1}{k_{b_1}^2 - z_1^2} - (k_{b_1}^2 - z_1^2)\kappa_1 g_1 z_1 \tag{5.43}$$

$$\dot{z}_i = f_i - \dot{\alpha}_{i-1} + g_i z_i - \frac{\lambda g_i \hat{\theta}_i^T \Psi_i z_i}{k_{b_1}^2 - z_i^2} - (k_{b_1}^2 - z_i^2)\kappa_i g_i z_i, \quad i = 2, ..., n \tag{5.44}$$

along with (5.23) and (5.42).

Due to the fact that $\hat{\theta}_i \geq 0$, and that $g_i(\bar{x}_i) \geq g^* > 0$, it is clear that (5.38) can be rewritten as

$$\begin{aligned} \dot{V}_i \leq & -\frac{\lambda g^* z_i^2}{(k_{b_1}^2 - z_i^2)^2}\tilde{\theta}_i^T \Psi_i + \frac{g_i(\bar{x}_i) z_i z_{i+1}}{k_{b_1}^2 - z_i^2} + \lambda g^* \tilde{\theta}_i^T \left( \frac{z_i^2}{(k_{b_1}^2 - z_i^2)^2}\Psi_i - \sigma \Gamma_i^{-1} \hat{\theta}_i \right) \\ & -\sum_{j=1}^{i} \kappa_j g_j(\bar{x}_j) z_j^2 - \sum_{j=1}^{i-1} \frac{\lambda g^* \sigma}{2} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + c_{i-1} + \frac{2i+1}{4\lambda} \end{aligned} \tag{5.45}$$

Using the property that $-\tilde{\theta}_i^T \Gamma_i^{-1} \hat{\theta}_i \leq \frac{1}{2}(-\tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i + \theta_i^T \Gamma_i^{-1} \theta_i)$, we obtain that

$$\dot{V}_i \leq -\sum_{j=1}^{i} \kappa_j g_j(\bar{x}_j) z_j^2 - \sum_{j=1}^{i} \frac{\lambda g^* \sigma}{2} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + \frac{g_i(\bar{x}_i) z_i z_{i+1}}{k_{b_1}^2 - z_i^2} + c_i \tag{5.46}$$

where

$$c_i = c_{i-1} + \frac{2i+1}{4\lambda} + \frac{\lambda g^* \sigma}{2} \theta_i^T \Gamma_i^{-1} \theta_i \tag{5.47}$$

and the coupling term $\frac{g_i(\bar{x}_i) z_i z_{i+1}}{k_{b_1}^2 - z_i^2}$ is dominated in the subsequent step. Particularly, in the final step, the derivative of $V_n$ can be expressed in the form:

$$\dot{V}_n \leq -\sum_{j=1}^{n} \kappa_j g_j(\bar{x}_j) z_j^2 - \frac{\lambda g^* \sigma}{2} \sum_{j=1}^{n} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + c_n \tag{5.48}$$

where the constant $c_n$ is given by:

$$\begin{aligned} c_n &= c_{n-1} + \frac{2n+1}{4\lambda} + \frac{\lambda g^* \sigma}{2} \theta_n^T \Gamma_n^{-1} \theta_n \\ &= \frac{2 + (n-1)(2n+6)}{8\lambda} + \frac{\lambda g^* \sigma}{2} \sum_{i=1}^{n} \theta_i^T \Gamma_i^{-1} \theta_i \end{aligned} \tag{5.49}$$

Based on the fact that $g_j(\bar{x}_j) \geq g^* \geq g_{\min}$, we can rewrite (5.48) into the form:

$$\dot{V}_n \leq -g^* \sum_{j=1}^{n} \kappa_j z_j^2 - \frac{\lambda g^* \sigma}{2} \sum_{j=1}^{n} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + g^* \bar{c}_n \tag{5.50}$$

where the computable constant $\bar{c}_n$ is defined by:

$$\bar{c}_n = \frac{2+(n-1)(2n+6)}{8\lambda g_{\min}} + \frac{\lambda\sigma}{2}\sum_{j=1}^{n}\lambda_{\max}(\Gamma_j^{-1})\bar{\theta}_j^2 \tag{5.51}$$

$$\bar{\theta}_1 := \frac{1}{g_{\min}}(\bar{F}_1+\bar{Y}_1)^2 \tag{5.52}$$

$$\bar{\theta}_i := \frac{1}{g_{\min}}\sqrt{\sum_{j=1}^{i}\bar{F}_j^4+\sum_{j=1}^{i-1}\bar{G}_j^4+1}, \quad i=2,...,n \tag{5.53}$$

with $\bar{F}_i, \bar{G}_i$ as known constants satisfying $f_i(\bar{x}_i) \le \bar{F}_i$, $g_i(\bar{x}_i) \le \bar{G}_i$, for $x \in \Omega_x$.

Let the closed loop system (5.23), (5.42)-(5.44) be written as $\dot{\eta} = h(t,\eta)$, where $\eta := [z, \hat{\Theta}]^T$ and $\hat{\Theta} := [\hat{\theta}_1, \hat{\theta}_2^T, ..., \hat{\theta}_n^T]^T$. The right hand side $h(t,\eta)$ satisfies the conditions (2.17)-(2.20) for $\eta \in \mathcal{Z} := \{z \in \mathbb{R}^n, \hat{\Theta} \in \mathbb{R}^l : |z_i| < k_{b_1}, i = 1,2,...,n\}$, where $l = 1 + \frac{(n-1)(n+6)}{2}$. Together with (5.50) and the condition

$$\sqrt{\bar{c}_n/\kappa_i} < k_{b_1}, \quad i=1,...,n \tag{5.54}$$

we invoke Lemma 2.4.3 to yield $|z_i(t)| < k_{b_1}$ for all $t > 0$ and $i = 1,...,n$, provided that $|z_i(0)| < k_{b_1}$.

Although we have shown that each error signal $z_i(t)$ is constrained in the set $|z_i| < k_{b_1}$, $\forall t > 0$, the question remains as to how we can ensure that $x(t) \in \Omega_c$ $\forall t > 0$, where $\Omega_c$ is defined in (5.16). In the control design, we considered the region $x \in \Omega_x$, where $\Omega_x$ is defined in (5.17), such that there exist constant upper bounds $F_i$ and $G_i$ for the uncertain smooth functions $f_i(\bar{x}_i)$ and $g_i(\bar{x}_i)$, respectively. By ensuring that $x(t) \in \Omega_c \subset \Omega_x$ in the closed loop, we verify that the assumptions $f_i(\bar{x}_i) \le F_i$ and $g_i(\bar{x}_i) \le G_i$ are valid. The details are explained in the following theorem.

**Theorem 5.3.1** *Consider the closed loop system (5.1), (5.42), (5.41) under Assumptions 5.2.1-5.2.2. Denote by $A_i$ an upper bound for $\alpha_i$ in the compact set $\Omega_i$, that is,*

$$A_i \ge \sup_{(\bar{x}_i,\ \bar{z}_i,\ \bar{y}_{d_i},\ \hat{\Theta}_i)\in\ \Omega_i} |\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\Theta}_i; \bar{\kappa}_i, \Gamma_1, ..., \Gamma_i)|, \quad i=1,...,n-1 \tag{5.55}$$

*where $\alpha_i$ is parameterized by $\Gamma_1, ..., \Gamma_i$ and $\bar{\kappa}_i := [\kappa_1, \kappa_2, ..., \kappa_i]^T$, and $\Omega_i$ is a compact* *set defined by:*

$$
\begin{aligned}
\Omega_i &:= \Big\{\bar{x}_i \in \mathbb{R}^i, \bar{z}_i \in \mathbb{R}^i, \bar{y}_{d_i} \in \mathbb{R}^i, \hat{\Theta}_i \in \mathbb{R}^l : \\
&\quad |x_j| \leq D_{z_1} + A_{j-1}, |z_j| \leq D_{z_1}, |y_d^{(j)}| \leq Y_j, \|\hat{\theta}_j\| \leq D_{\hat{\theta}_j}, j = 1, ..., i\Big\} && (5.56)\\
D_{z_1} &:= k_{b_1}\sqrt{1 - e^{-2\bar{V}_n}} && (5.57)\\
D_{\hat{\theta}_i} &:= \bar{\theta}_i + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma_i^{-1})\lambda g_{\min}}}, && (5.58)\\
V_a &:= \frac{1}{2}\sum_{i=1}^{n} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(0)} + \frac{\lambda \bar{G}}{2}\lambda_{\max}(\Gamma^{-1})(\|\hat{\Theta}(0)\| + \|\bar{\Theta}\|^2) && (5.59)\\
V_b &:= \sum_{i=1}^{n} \frac{1}{2}\log \frac{k_{b_1}^2}{k_{b_1}^2 - \frac{\bar{c}_n}{\kappa_i}} + \frac{\bar{G}\bar{c}_n}{\sigma} && (5.60)\\
\bar{V}_n &:= \max\{V_a, V_b\} && (5.61)
\end{aligned}
$$

*with* $\Gamma := \text{blockdiag}(\Gamma_1, ..., \Gamma_n)$, $\bar{\Theta} := [\bar{\theta}_1, \bar{\theta}_2^T, ..., \bar{\theta}_n^T]^T$, $\hat{\Theta}_i := [\hat{\theta}_1^T, ..., \hat{\theta}_i^T]^T$, $\tilde{\Theta} := [\tilde{\theta}_1, \tilde{\theta}_2^T, ..., \tilde{\theta}_n^T]^T$*, and* $\bar{G} = \max_{i=1,...,n} \bar{G}_i$*. Given the constraints* $k_{c_{i+1}} > 0$*,* $i = 1, ..., n-1$*, and that*

*C1) there exist positive constants* $\kappa_i$ *and* $A_i$ *such that*

$$
\begin{aligned}
k_{c_{i+1}} &> A_i + k_{b_1}, \qquad i = 1, ..., n-1 && (5.62)\\
\kappa_i &> \frac{\bar{c}_n}{k_{b_1}^2}, \qquad i = 1, ..., n && (5.63)
\end{aligned}
$$

*C2) the initial conditions are such that*

$$
\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_i| < k_{b_1}, i = 1, 2, ..., n\} \tag{5.64}
$$

*then the following properties hold.*

*i) The signals* $z_i(t)$ *and* $\hat{\theta}(t)$*,* $i = 1, 2, ..., n$*, remain, for all* $t > 0$*, in the compact sets defined by*

$$
\begin{aligned}
\Omega_z &= \{\bar{z}_n \in \mathbb{R}^n : |z_i| \leq D_{z_1}, i = 1, 2, ..., n\} && (5.65)\\
\Omega_{\hat{\theta}_i} &= \Big\{\hat{\theta}_i \in \mathbb{R}^{l_i} : \|\hat{\theta}_i\| \leq D_{\hat{\theta}_i}\Big\} && (5.66)
\end{aligned}
$$

*where* $l_1 = 1$ *and* $l_i = 2i$ *for* $i = 2, 3, ..., n$*.*

*ii)* *Every state* $x_i(t)$ *remains in the compact set* $\Omega_{cc} := \{\bar{x}_n \in \mathbb{R}^n : |x_i| \le D_{z_1} + A_{i-1} < k_{c_i},\ i = 1, ..., n\}$ $\forall t > 0$*, where* $\Omega_{cc} \subset \Omega_c$*, i.e. the full state constraint is never violated.*

*iii)* *All closed loop signals are bounded.*

*iv)* *The output tracking error* $z_1(t)$ *converges to the set* $|z_1| \le \sqrt{\bar{c}_n/\kappa_1}$*.*

**Proof:**

i) From (5.50), we have that $\dot{V}_n \le 0$ whenever

$$|z_i| \ \ge \ \sqrt{\frac{\bar{c}_n}{\kappa_i}}, \quad i = 1, ..., n \tag{5.67}$$

$$\tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta} \ \ge \ \frac{2\bar{c}_n}{\lambda\sigma} \tag{5.68}$$

As a result, provided that $\sqrt{\bar{c}_n/\kappa_i} < k_{b_1}$, $i = 1, ..., n$, an upper bound for $V_n$ is obtained:

$$V_n(t) \ \le \ \begin{cases} V_b, & \text{if } V_n(0) \le V_b \\ V_n(0), & \text{otherwise} \end{cases} \tag{5.69}$$

Since $V_n(0) \le V_a$, we infer that

$$V_n(t) \ \le \ \begin{cases} V_b, & \text{if } V_n(0) \le V_b \\ V_a, & \text{otherwise} \end{cases} \tag{5.70}$$

The upper bound for $V_n(t)$ depends on the initial condition $V_n(0)$. We take the maximum of $V_a$ and $V_b$ to obtain the overall bound $\bar{V}_n$, such that that $V_n(t) \le \bar{V}_n$ for all $V_n(0) \in \mathbb{R}$ and all $t > 0$.

Then, from the fact that $\frac{\lambda g^*}{2}\tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i \le \bar{V}_n$, and thus $\frac{\lambda g^*}{2}\lambda_{\min}(\Gamma_i^{-1})\|\hat{\theta}_i - \theta_i\|^2 \le \bar{V}_n$, it is straightforward to show that

$$\|\hat{\theta}_i\| \le \bar{\theta}_i + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma_i^{-1})\lambda g_{\min}}} \tag{5.71}$$

such that $\hat{\theta}_i$ remains in the compact set $\Omega_{\hat{\theta}_i}$ $\forall\ t$.

Similarly, it can be shown that $\frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(t)} \le \bar{V}_n$, which yields $\frac{k_{b_1}^2}{k_{b_1}^2 - z_i^2(t)} \le e^{2\bar{V}_n}$. Since $|z_i(0)| < k_{b_1}$, we know that $k_{b_1}^2 - z_i^2(t) > 0\ \forall\ t$ from Lemma 2.4.1.

Simple rearrangement yields $|z_i(t)| \le D_{z_1} < k_{b_1}$, and thus, $z_i(t)$ remains in the compact set $\Omega_z$ $\forall$ $t$.

ii) From $V_n(t) \le \bar{V}_n$ and Lemma 2.4.3, we have established that $|z_i(t)| \le D_{z_1} < k_{b_1}$, $i = 1, ..., n$, $\forall$ $t$. Then, $|x_1(t)| \le D_{z_1} + |y_d(t)| < k_{c_1} - A_0 + |y_d(t)|$. Noting that $|y_d(t)| \le A_0$ from Assumption 5.2.1, we conclude that $|x_1(t)| \le D_{z_1} + A_0 < k_{c_1}$, $\forall$ $t$.

We can progressively show that $|x_{i+1}(t)| < k_{c_{i+1}}$, $i = 2, ..., n-1$, after verifying that there exist positive constants $A_i$ such that $|\alpha_i(t)| \le A_i$, $\forall$ $t$. As a result of $\|\hat{\theta}_i\| \le D_{\hat{\theta}_i}$, $|x_i(t)| \le D_{z_1} + A_{i-1}$, $|z_i(t)| \le D_{z_1}$, and $|y_d^{(i)}(t)| \le Y_i$, it can be shown that $(\bar{x}_i(t), \bar{z}_i(t), \bar{y}_{d_i}(t), \hat{\Theta}_i(t)) \in \Omega_i$. Therefore, boundedness of the stabilizing function $\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\Theta}_i)$ in (5.22) is established, since it is a continuous function in the region $|z_j| < k_{b_1}$ for all $j = 1, ..., i$. Hence, we know that $\sup_{(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\Theta}_i) \in \Omega_i} |\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\Theta}_i)|$ exists, so it is possible to find an upper bound $A_i$. Following the fact that $|z_{i+1}(t)| \le D_{z_1}$ and $|\alpha_i(t)| \le A_i$, it is straightforward that $|x_{i+1}(t)| \le D_{z_1} + A_i < k_{c_{i+1}}$, $\forall$ $t$.

iii) Thus far, we have obtained the results $|z_i(t)| \le D_{z_1} < k_{b_1}$, $|x_i(t)| < k_{c_i}$, and $\|\hat{\theta}_i(t)\| \le D_{\hat{\theta}_i}$, for $i = 1, ..., n$. By inspecting the stabilizing functions $\alpha_i(\bar{x}_i, \bar{z}_i, \bar{y}_{d_i}, \hat{\Theta}_i)$ and the control $u(\bar{x}_n, \bar{z}_n, \bar{y}_{d_n}, \hat{\Theta}_n)$ , it is clear that they are also bounded. Therefore, all closed loop signals are bounded.

iv) First, note the property $V_n(z_a, \tilde{\Theta}_a) < V_n(z_b, \tilde{\Theta}_b)$ for $\|z_a\| < \|z_b\|$ and $\|\tilde{\Theta}_a\| < \|\tilde{\Theta}_b\|$. Together with the fact that $\dot{V}_n \le 0$ in the region $V_n(z, \tilde{\Theta}) \ge V_b^*$, where

$$V_b^* \quad := \quad V_n(z, \tilde{\Theta})|_{\left\{|z_i| = \sqrt{\frac{\bar{c}_n}{\kappa_i}},\ \tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta} = \frac{2\bar{c}_n}{\lambda\sigma}\right\}} \tag{5.72}$$

two cases ensue, depending on the initial condition $V_n(z(0), \tilde{\Theta}(0))$.

For the first case, where $V_n(z(0), \tilde{\Theta}(0)) \le V_b^*$, it is clear that $V_n(z(t), \tilde{\Theta}(t))$ cannot escape from the region $V_n(z, \tilde{\Theta}) \le V_b^*$ since $\dot{V}_n \le 0$ whenever $V_n(z(t), \tilde{\Theta}(t)) \ge V_b^*$. On the other hand, if we start from $V_n(z(0), \tilde{\Theta}(0)) \ge V_b^*$, then $\dot{V}_n \le 0$ whenever $V_n(z(t), \tilde{\Theta}(t)) \ge V_b^*$, so that there exists a positive constant $T$ where $V_n(z(T), \tilde{\Theta}(T)) \le V_b^*$, and $V_n(z(t), \tilde{\Theta}(t)) \le V_b^*$ for $t > T$.

Thus, $V_n(z, \tilde{\Theta}) \le V_b^*$ is a positively invariant set, and $(z(t), \tilde{\Theta}(t))$ remains in the interior of the level set

$$\Omega_b = \{z \in R^n, \tilde{\Theta} \in R^l \mid V_n(z, \tilde{\Theta}) = V_b^*\} \tag{5.73}$$

We obtain the bounds, albeit conservatively, as $|z_i(t)| \leq \sqrt{\frac{\bar{c}_n}{\kappa_i}}$, $i = 1, ..., n$ and $\tilde{\Theta}^T(t)\Gamma^{-1}\tilde{\Theta}(t) \leq \frac{2\bar{c}_n}{\lambda\sigma}$, for $t > T$. Therefore, we conclude that the output tracking error $z_1(t)$ converges to the set $|z_1| \leq \sqrt{\bar{c}_n/\kappa_1}$. ■

### 5.3.3 Feasibility Check

As mentioned earlier, the proposed method is unable to handle arbitrary state constraints. The state constraints $k_{c_i}$ need to satisfy feasibility conditions C1 and C2 in Theorem 5.3.1. These provide criteria to check if the backstepping induced stabilizing functions $\alpha_i$ are sufficient to achieve output tracking in the presence of state constraints. The bounds $A_1, ..., A_{n-1}$ are computable for any set of control parameters $\kappa_1, ..., \kappa_n, \Gamma$ and initial conditions $x(0)$, and thus, these conditions can be checked before the control is implemented.

Specifically, we check if there exists a solution $(\bar{\kappa}_{n-1}, \Gamma)$, where $\bar{\kappa}_{n-1} := [\kappa_1, ..., \kappa_{n-1}]^T$, $\Gamma :=$ diag($\gamma_1$ ,$\gamma_2$,...,$\gamma_l$), for the optimization problem:

$$\begin{aligned}
&\max_{\kappa_1,...,\kappa_{n-1},\Gamma>0} P(\bar{\kappa}_{n-1}, \Gamma) = \sum_{i=1}^{n-1} a_i\kappa_i + \sum_{i=1}^{l} b_i\gamma_i \\
&\text{subject to:} \\
&\qquad \kappa_i > \frac{\bar{c}_n}{k_{b_1}^2} && (5.74)\\
&\qquad k_{c_{i+1}} > A_i(\bar{\kappa}_i, \Gamma) + k_{b_1} \\
&\qquad k_{b_1} > |x_{i+1}(0) - \alpha_i(\bar{x}_i(0), \bar{z}_i(0), \bar{y}_{d_i}(0), \hat{\theta}(0); \bar{\kappa}_i, \Gamma)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad i = 1, ..., n-1 && (5.75)
\end{aligned}$$

where $b_i$ are positive constants. If a solution $(\bar{\kappa}_{n-1}^*, \Gamma^*)$ to the above optimization problem is found, then the proposed adaptive control (4.38)-(4.39), with $\bar{\kappa}_{n-1} = \bar{\kappa}_{n-1}^*$, $\Gamma = \Gamma^*$ and a choice of $\kappa_n > \frac{\bar{c}_n}{k_{b_1}^2}$, is feasible in ensuring output tracking for the system (5.1) with full state constraint, according to Theorem 5.3.1.

**Remark 5.3.2** *We require the constants $g_{\min}$, $\bar{F}_i$ and $\bar{G}_i$, which represent bounds for system nonlinearities, to be known so that $\bar{c}_n$, $D_{z_1}$ and $D_{\hat{\theta}_i}$ can be computed. The latter bounds are required to estimate $A_i$, the bound for the stabilizing function $\alpha_i$, so that the feasibility conditions C1 and C2 can be checked. Note that $g_{\min}$, $\bar{F}_i$ and $\bar{G}_i$ may be crude estimates for this purpose. In the adaptive control design, the maximal lower bound, given by $g^* := \min_{i=1,\ldots,n}\{\inf_{\bar{x}_i \in R^i} g_i(\bar{x}_i)\}$, and the minimal upper bounds, given by $F_i := \sup_{x\in\Omega_x} f_i(\bar{x}_i) \leq \bar{F}_i$ and $G_i := \sup_{x\in\Omega_x} g_i(\bar{x}_i) \leq \bar{G}_i$, are considered to be unknown and adaptively compensated for.*

**Remark 5.3.3** *Thus far, we have dealt with constraint on full state. For the case of partial state constraint, where not all states need to be bounded within any pre-specified constrained regions, we gain flexibility in design, since the constants $k_{c_{i+1}}$ for the unconstrained states can be freely chosen to bound $A_i$ in (5.62), instead of being imposed as a requirement. As a result, the feasibility conditions are relaxed.*

## 5.4 Control Design for Output Constraint

As seen in the previous section, the control design for the full state constraint case employs BLFs in every step of the design, and involves feasibility conditions that can be checked *a priori*. Under partial state constraint, these conditions can be relaxed to some extent. In this section, we consider the special case of output constraint with linearly parameterized system nonlinearities, for which feasibility conditions are not required.

According to system (5.1), we consider the class of linearly parameterizable nonlinear functions $f_i(\bar{x}_i) = \theta^T \psi_i(\bar{x}_i)$ and $g_i(\bar{x}_i) = \phi^T \varphi_i(\bar{x}_i) \geq g^* > 0$, $i = 1, ..., n$ where $\psi_i$ and $\varphi_i$ are known smooth functions, $\theta \in \mathbb{R}^l$ and $\phi \in \mathbb{R}^m$ are vectors of uncertain parameters satisfying $\|\theta\| \leq \theta_M$ and $\|\phi\| \leq \phi_M$ for some known positive constants $\theta_M$ and $\phi_M$. With the consideration of constraint in the output only, it follows that only the first step of backstepping employs a Barrier Lyapunov Function, while the subsequent steps are all based on quadratic ones, thus simplifying the design procedure and analysis.

The detailed procedure for adaptive backstepping design is outlined in the following.

**Step 1** Denote $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, where $\alpha_1$ is a stabilizing function to be designed. Choose Lyapunov function candidate as:

$$V_1 = \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} + \frac{\lambda g^*}{2}\Gamma_1^{-1}\tilde{\theta}_1^2 \tag{5.76}$$

where $k_{b_1} = k_{c_1} - A_0$, $\lambda$, and $\Gamma_1$ are positive constants, and $\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1$ is the estimation error, with $\theta_1$ as an unknown positive parameter and $\hat{\theta}_1$ its estimate.

The derivative of $V_1$ is given by

$$\dot{V}_1 \;=\; \frac{z_1}{k_{b_1}^2 - z_1^2}(\theta^T\psi_1(x_1) - \dot{y}_d + g_1(x_1)z_2 + g_1(x_1)\alpha_1) + \lambda g^*\Gamma_1^{-1}\tilde{\theta}_1\dot{\hat{\theta}}_1 \tag{5.77}$$

Based on completion of squares, we have that

$$\frac{z_1(\theta^T\psi_1(x_1) - \dot{y}_d)}{k_{b_1}^2 - z_1^2} \;\leq\; \frac{\lambda g^* z_1^2}{(k_{b_1}^2 - z_1^2)^2}\theta_1\Psi_1 + \frac{1}{4\lambda} \tag{5.78}$$

where $\theta_1 := \frac{1}{g^*}(\|\theta\|^2 + 1)$ and

$$\Psi_1 \;:=\; \|\psi_1\|^2 + (\dot{y}_d)^2 \tag{5.79}$$

Then, it can be shown that

$$\dot{V}_1 \;\leq\; \frac{\lambda g^* z_1^2}{(k_{b_1}^2 - z_1^2)^2}\theta_1\Psi_1 + \frac{g_1(x_1)z_1\alpha_1}{k_{b_1}^2 - z_1^2} + \frac{g_1(x_1)z_1z_2}{k_{b_1}^2 - z_1^2} + \lambda g^*\Gamma_1^{-1}\tilde{\theta}_1\dot{\hat{\theta}}_1 + \frac{1}{4\lambda} \tag{5.80}$$

Design stabilizing function and adaptation law as follows:

$$\alpha_1 \;=\; -\left(\frac{\lambda}{k_{b_1}^2 - z_1^2}\hat{\theta}_1\Psi_1 + \kappa_1(k_{b_1}^2 - z_1^2)\right)z_1 \tag{5.81}$$

$$\dot{\hat{\theta}}_1 \;=\; \frac{z_1^2}{(k_{b_1}^2 - z_1^2)^2}\Gamma_1\Psi_1 - \sigma\hat{\theta}_1, \quad \hat{\theta}_1(0) \geq 0 \tag{5.82}$$

it can be shown that

$$\begin{aligned}\dot{V}_1 \;\leq\; & -\kappa_1 g_1(x_1)z_1^2 + \frac{\lambda g^* z_1^2}{(k_{b_1}^2 - z_1^2)^2}\theta_1\Psi_1 - \frac{\lambda g_1(x_1)z_1^2}{(k_{b_1}^2 - z_1^2)^2}\hat{\theta}_1\Psi_1 + \frac{g_1(x_1)z_1z_2}{k_{b_1}^2 - z_1^2} \\ & + \lambda g^*\tilde{\theta}_1\left(\frac{z_1^2}{(k_{b_1}^2 - z_1^2)^2}\Psi_1 - \Gamma_1^{-1}\sigma\hat{\theta}_1\right) + \frac{1}{4\lambda}\end{aligned} \tag{5.83}$$

From (5.23), we know that $\hat{\theta}_1 \geq 0$, and from Assumption 5.2.2, we know that $g_1(x_1) > g_{\min} > 0$. Therefore, it is clear that the following inequalities holds:

$$-\frac{\lambda g_1(x_1) z_1^2}{(k_{b_1}^2 - z_1^2)^2}\hat{\theta}_1 \Psi_1 \quad \leq \quad -\frac{\lambda g^* z_1^2}{(k_{b_1}^2 - z_1^2)^2}\hat{\theta}_1 \Psi_1 \tag{5.84}$$

As a result, it is easy to obtain that

$$\begin{aligned}
\dot{V}_1 \quad \leq & \quad -\kappa_1 g_1(x_1) z_1^2 - \frac{\lambda g^* z_1^2}{(k_{b_1}^2 - z_1^2)^2}\tilde{\theta}_1 \Psi_1 + \lambda g^* \tilde{\theta}_1 \left( \frac{z_1^2}{(k_{b_1}^2 - z_1^2)^2}\Psi_1 - \Gamma_1^{-1}\sigma\hat{\theta}_1 \right) \\
& + \frac{g_1(x_1) z_1 z_2}{k_{b_1}^2 - z_1^2} + \frac{1}{4\lambda} \\
\leq & \quad -\kappa_1 g_1(x_1) z_1^2 + \frac{g_1(x_1) z_1 z_2}{k_{b_1}^2 - z_1^2} - \frac{\lambda g^* \sigma}{2}\Gamma_1^{-1}\tilde{\theta}_1^2 + c_1
\end{aligned} \tag{5.85}$$

where

$$c_1 = \frac{1}{4\lambda} + \frac{\lambda g^* \sigma}{2}\Gamma_1^{-1}\theta_1^2 \tag{5.86}$$

The coupling term $\frac{g_1(x_1) z_1 z_2}{k_{b_1}^2 - z_1^2}$ is dominated in the subsequent step.

**Step i** $(i = 2, ..., n)$

Denote $z_{i+1} = x_{i+1} - \alpha_i$, where $\alpha_i$ is a stabilizing function to be designed, and $z_{n+1} := 0$. Choose a Lyapunov function candidate as follows:

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{\lambda g^*}{2}\tilde{\theta}_i^T \Gamma_i^{-1}\tilde{\theta}_i \tag{5.87}$$

where $\Gamma_i > 0$ is a diagonal matrix, $\hat{\theta}_i$ is the estimate of $\theta_i$, and $\tilde{\theta}_i := \hat{\theta}_i - \theta_i$. The derivative of $V_i$ along the closed loop trajectories satisfies

$$\begin{aligned}
\dot{V}_i \quad \leq & \quad z_i \left( \theta^T \psi_i(\bar{x}_i) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(\theta^T \psi_j(\bar{x}_j) + \phi^T \varphi_j(\bar{x}_j) x_{j+1}) - w_{i-1} \right) \\
& + z_i g_i(\bar{x}_i)(z_{i+1} + \alpha_i) + \lambda g^* \tilde{\theta}_i^T \Gamma_i^{-1}\dot{\hat{\theta}}_i - \sum_{j=1}^{i-1}\left( \kappa_j g_j(\bar{x}_j) z_j^2 + \frac{\lambda g^* \sigma}{2}\tilde{\theta}_i^T \Gamma_i^{-1}\tilde{\theta}_i \right) \\
& + \phi^T \varphi_{i-1}(\bar{x}_{i-1}) z_{i-1} z_i + c_{i-1} \\
\leq & \quad \theta^T \left( \psi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\psi_j \right) z_i + \phi^T \left( \varphi_{i-1} z_{i-1} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\varphi_j x_{j+1} \right) z_i \\
& + w_{i-1} z_i + z_i \left( g_i(\bar{x}_i) z_{i+1} + g_i(\bar{x}_i)\alpha_i \right) + \lambda g^* \tilde{\theta}_i^T \Gamma_i^{-1}\dot{\hat{\theta}}_i \\
& - \sum_{j=1}^{i-1}\left( \kappa_j g_j(\bar{x}_j) z_j^2 + \frac{\lambda g^* \sigma}{2}\tilde{\theta}_i^T \Gamma_i^{-1}\tilde{\theta}_i \right) + c_{i-1}
\end{aligned} \tag{5.88}$$

For the first three terms on the right hand side of the above expression, we use completion of squares arguments to obtain

$$\theta^T \left( \psi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \psi_j \right) z_i \leq \lambda g^* z_i^2 \left\| \psi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \psi_j \right\|^2 \theta_{i1} + \frac{1}{4\lambda} \quad (5.89)$$

$$w_{i-1} z_i \leq \lambda g^* z_i^2 w_{i-1}^2 \theta_{i3} + \frac{1}{4\lambda} \quad (5.90)$$

$$\phi^T \left( \varphi_{i-1} z_{i-1} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j x_{j+1} \right) z_i \leq \lambda g^* z_i^2 \left\| \varphi_{i-1} z_{i-1} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j x_{j+1} \right\|^2 \theta_{i2} + \frac{1}{4\lambda} \quad (5.91)$$

where $\theta_{i1} := \frac{1}{g^*}\|\theta\|^2$, $\theta_{i2} := \frac{1}{g^*}\|\phi\|^2$, $\theta_{i3} := \frac{1}{g^*}$. Substituting the above inequalities into (5.88) yields

$$\begin{aligned} \dot{V}_i \leq\; & -\sum_{j=1}^{i-1} \left( \kappa_j g_j(\bar{x}_j) z_j^2 + \frac{\lambda g^* \sigma}{2} \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i \right) + \lambda g^* \theta_i^T \Psi_i z_i^2 + z_i \left( g_i(\bar{x}_i) z_{i+1} + g_i(\bar{x}_i) \alpha_i \right) \\ & + \lambda g^* \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\hat{\theta}}_i + c_{i-1} + \frac{3}{4\lambda} \end{aligned} \quad (5.92)$$

where $\theta_i = [\theta_{i1}, \theta_{i2}, \theta_{i3}]^T$ and

$$\Psi_i = \left[ \left\| \psi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \psi_j \right\|^2, \left\| \varphi_{i-1} z_{i-1} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \varphi_j x_{j+1} \right\|^2, w_{i-1}^2 \right]^T \quad (5.93)$$

Design the stabilizing function, control and adaptation laws as

$$\alpha_i = -(\lambda \hat{\theta}_i^T \Psi_i + \kappa_i) z_i \quad (5.94)$$

$$u = \alpha_n \quad (5.95)$$

$$\dot{\hat{\theta}}_i = \Gamma_i \Psi_i z_i^2 - \sigma \hat{\theta}_i, \quad \hat{\theta}_i(0) \geq 0 \quad (5.96)$$

From (5.96), we know that $\hat{\theta}_i \geq 0$, and from Assumption 5.2.2, we know that $g_i(\bar{x}_i) > g_{\min} > 0$. Therefore, (5.92) together with (5.94)-(5.96) can be rewritten as

$$\begin{aligned} \dot{V}_i \leq\; & -\sum_{j=1}^{i-1} \left( \kappa_j g_j(\bar{x}_j) z_j^2 + \frac{\lambda g^* \sigma}{2} \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i \right) - \lambda g^* \tilde{\theta}_i^T \Psi_i z_i^2 + g_i(\bar{x}_i) z_i z_{i+1} \\ & + \lambda g^* \tilde{\theta}_i^T \left( \Psi_i z_i^2 - \sigma \Gamma_i^{-1} \hat{\theta}_i \right) + c_{i-1} + \frac{3}{4\lambda} \\ \leq\; & -\sum_{j=1}^{i} \kappa_j g_j(\bar{x}_j) z_j^2 - \sum_{j=1}^{i} \frac{\lambda g^* \sigma}{2} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + g_i(\bar{x}_i) z_i z_{i+1} + c_i \end{aligned} \quad (5.97)$$

where

$$c_i = c_{i-1} + \frac{3}{4\lambda} + \frac{\lambda g^* \sigma}{2} \theta_i^T \Gamma_i^{-1} \theta_i \tag{5.98}$$

The final step of backstepping yields the derivative of $V_n$, along the closed loop trajectories, as:

$$\dot{V}_n \leq -\sum_{j=1}^{n} \kappa_j g_j(\bar{x}_j) z_j^2 - \frac{\lambda g^* \sigma}{2} \sum_{j=1}^{n} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + c_n \tag{5.99}$$

where the constant $c_n$ is given by:

$$\begin{aligned} c_n &= c_{n-1} + \frac{3}{4\lambda} + \frac{\lambda g^* \sigma}{2} \theta_n^T \Gamma_n^{-1} \theta_n \\ &= \frac{3n-2}{4\lambda} + \frac{\lambda g^* \sigma}{2} \sum_{i=1}^{n} \theta_i^T \Gamma_i^{-1} \theta_i \end{aligned} \tag{5.100}$$

Based on the fact that $g_j(\bar{x}_j) \geq g^* \geq g_{\min}$, we can rewrite (5.99) into

$$\dot{V}_n \leq -g^* \sum_{j=1}^{n} \kappa_j z_j^2 - \frac{\lambda g^* \sigma}{2} \sum_{j=1}^{n} \tilde{\theta}_j^T \Gamma_j^{-1} \tilde{\theta}_j + g^* \bar{c}_n \tag{5.101}$$

where the constant $\bar{c}_n$ is computable:

$$\bar{c}_n := \frac{3n-2}{4\lambda g_{\min}} + \frac{\lambda \sigma}{2} \sum_{j=1}^{n} \lambda_{\max}(\Gamma_j^{-1}) \bar{\theta}_j^2 \tag{5.102}$$

$$\bar{\theta}_1 := \frac{1}{g_{\min}}(\theta_M^2 + 1), \qquad \bar{\theta}_i := \frac{1}{g_{\min}} \sqrt{\theta_M^4 + \phi_M^4 + 1}, \quad i = 2, ..., n \tag{5.103}$$

and $\theta_M$ and $\phi_M$ are known positive constants satisfying $\|\theta\| \leq \theta_M$ and $\|\phi\| \leq \phi_M$ respectively.

Similar to the analysis presented after Step $n$ in Section 5.3.2, we can write the closed loop system (5.1), (5.96) and (5.95) as $\dot{\eta} = h(t, \eta)$, where $\eta := [z, \hat{\Theta}]^T$ and $\hat{\Theta} := [\hat{\theta}_1, \hat{\theta}_2^T, ..., \hat{\theta}_n^T]^T$. Then, it can be shown that $h(t, \eta)$ satisfies the conditions (2.17)-(2.20) for $\eta \in \mathcal{Z} := \{z \in \mathbb{R}^n, \hat{\Theta} \in \mathbb{R}^l : |z_1| < k_{b_1}\}$, where $l = 3n - 2$. Together with (5.101) and the condition

$$\sqrt{\bar{c}_n / \kappa_1} < k_{b_1} \tag{5.104}$$

we invoke Lemma 2.4.3 to yield $|z_1(t)| < k_{b_1}$ for all $t > 0$ and $i = 1, ..., n$, provided that $|z_1(0)| < k_{b_1}$.

**Remark 5.4.1** *Unlike the state constraint case, there is no need to first consider the compact set* $\Omega_x$ *satisfying* $\Omega_x \supset \Omega_c = \{x \in R^n : |x_i| < k_{c_i},\ i = 1, ..., n\}$*, in which* $f_i(\bar{x}_i)$ *and* $g_i(\bar{x}_i)$ *are upper bounded, and then ensure that* $x \in \Omega_c \subset \Omega_x$*. This is due to the fact that the nonlinearities* $f_i(\bar{x}_i)$ *and* $g_i(\bar{x}_i)$ *are linearly parameterized with known regressor functions* $\psi_i(\bar{x}_i)$ *and* $\varphi_i(\bar{x}_i)$*. Together with the fact that state constraints are not specified at the outset, it is clear that feasibility conditions, similar to C1 and C2 in Theorem 5.3.1 for the state constraint case, are no longer needed. Only mild conditions on the initial output* $y(0)$ *and the control parameter* $\kappa_1$ *are needed.*

**Theorem 5.4.1** *Consider the closed loop system (5.1), (5.96), (5.95) under Assumptions 5.2.1-5.2.2. Given the constraint* $k_{c_1} > 0$*, we define the following positive constants:*

$$D_{z_1} := k_{b_1}\sqrt{1 - e^{-2\bar{V}_n}} \tag{5.105}$$

$$D_{\hat{\theta}_i} := \bar{\theta}_i + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma_i^{-1})\lambda g^*}}, \quad i = 1, ..., n \tag{5.106}$$

$$V_a := \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2(0)} + \sum_{j=2}^{n}\frac{z_j^2(0)}{2} + \frac{\lambda g^*}{2}\lambda_{\max}(\Gamma^{-1})(\|\hat{\Theta}(0)\| + \|\bar{\Theta}\|^2) \tag{5.107}$$

$$V_b := \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - \frac{\bar{c}_n}{\kappa_1}} + \bar{c}_n\left(\sum_{j=2}^{n}\frac{1}{2\kappa_j} + \frac{g^*}{\sigma}\right) \tag{5.108}$$

$$\bar{V}_n := \max\{V_a, V_b\} \tag{5.109}$$

*where* $k_{b_1} = k_{c_1} - A_0$*,* $V_b$ *is defined in (5.108),* $\bar{c}_n$ *in (5.102),* $\bar{\Theta} := [\bar{\theta}_1, \bar{\theta}_2^T, ..., \bar{\theta}_n^T]^T$*, and* $\hat{\Theta} := [\hat{\theta}_1, \hat{\theta}_2^T, ..., \hat{\theta}_n^T]^T$*.*

*If the initial conditions are such that* $\bar{z}_n(0) \in \Omega_{z_0} := \{\bar{z}_n \in \mathbb{R}^n : |z_1| < k_{b_1}\}$*, and the control parameter* $\kappa_1$ *satisfies (5.104), then the following properties hold.*

*i) The signals* $z_i(t)$ *and* $\hat{\theta}$*,* $i = 1, 2, ..., n$*, remain in the compact sets defined by*

$$\Omega_z = \left\{\bar{z}_n \in \mathbb{R}^n : |z_1| \le D_{z_1}, \|z_{2:n}\| \le \sqrt{2\bar{V}_n}\right\} \tag{5.110}$$

$$\Omega_{\hat{\theta}_i} = \left\{\hat{\theta}_i \in \mathbb{R}^{l_i} : \|\hat{\theta}_i\| \le D_{\hat{\theta}_i}\right\} \tag{5.111}$$

*where* $z_{2:n} := [z_2, ..., z_n]^T$*,* $l_1 = 1$*, and* $l_i = 3$ *for* $i = 2, 3, ..., n$*.*

*ii) The output $y(t)$ remains in the compact set $\Omega_y := \{y \in \mathbb{R} : |y| \leq D_{z_1} + A_0 < k_{c_1}$ $\forall t > 0$, i.e. the output constraint is never violated.*

*iii) All closed loop signals are bounded.*

*iv) The output tracking error $z_1(t)$ converges to the set $|z_1| \leq \sqrt{\bar{c}_n/\kappa_1}$.*

**Proof:**

i) Similar to the proof of Theorem 5.3.1, we can show that $V_n(t)$ is bounded by

$$V_n(t) \quad \leq \quad \begin{cases} V_b, & \text{if } V_n(0) \leq V_b \\ V_a, & \text{otherwise} \end{cases} \tag{5.112}$$

as a result of (5.101) and (5.104). Then, $\bar{V}_n$ is obtained by taking the maximum of $V_a$ and $V_b$, such that $V_n(t) \leq \bar{V}_n$ for all $V_n(0)$.
Thus, from the fact that $\frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2(t)} \leq \bar{V}_n$, and that $|z_1(0)| < k_{b_1}$, we know that $k_{b_1}^2 - z_1^2(t) > 0 \,\forall\, t$ from Lemma 2.4.3. Simple rearrangement yields $|z_1(t)| \leq D_{z_1} < k_{b_1}$, and thus, $z_i(t)$ remains in the compact set $\Omega_z$, as described in (5.110). Since $\frac{\lambda g^*}{2}\tilde{\theta}_i^T\Gamma_i^{-1}\tilde{\theta}_i \leq \bar{V}_n$, and thus $\frac{\lambda g^*}{2}\lambda_{\min}(\Gamma_i^{-1})\|\hat{\theta}_i(t) - \theta_i\|^2 \leq \bar{V}_n$, it follows that $\|\hat{\theta}_i(t)\| \leq \bar{\theta}_i + \sqrt{\frac{2\bar{V}_n}{\lambda_{\min}(\Gamma_i^{-1})\lambda g^*}}$ such that $\hat{\theta}_i(t)$ remains in the compact set $\Omega_{\hat{\theta}_i}$, as described in (5.111).

ii) It is straightforward to show, from $y(t) = z_1(t) + y_d(t)$, $|z_1(t)| \leq D_{z_1} < k_{b_1}$, and $|y_d(t)| \leq A_0$, that $|y(t)| < k_{b_1} + A_0 = k_{c_1}$. Hence, we can conclude that $y(t) \in \Omega_y \; \forall t > 0$.

iii) From $V_n(t) \leq \bar{V}_n$, we know that the error signals $z_i(t)$ and $\tilde{\theta}_i(t)$, for $i = 1, ..., n$, are bounded. Since $\theta_i$ are constants, we have that $\hat{\theta}_i(t)$ are bounded. The boundedness of $z_1(t)$ and the reference trajectory $y_d(t)$ imply that the state $x_1(t)$ is bounded. Together with the fact that $\dot{y}_d(t)$ is bounded from Assumption 5.2.1, it is clear that $\alpha_1(t)$ is also bounded from (5.81). This leads to the boundedness of state $x_2(t) = z_2(t) + \alpha_1(t)$. It is also straightforward to show that $\alpha_2$ is a continuous function of the bounded signals $\bar{x}_2(t)$, $\bar{z}_2(t)$, $\bar{y}_{d_2}(t)$, $\hat{\theta}_1(t)$, $\hat{\theta}_2(t)$ in the set $z_1 \in (-k_{b_1}, k_{b_1})$. Together with the fact that $|z_1(t)| \leq D_{z_1} < k_{b_1}$, as established in item (i), we know that $\alpha_2(t)$ is bounded. Following this line of argument, we can progressively show that each $\alpha_i(t)$, for $i = 1, ..., n-1$, is bounded, since it is a continuous function of the bounded signals $\bar{x}_i(t)$, $\bar{z}_i(t)$, $\bar{y}_{d_i}(t)$, $\hat{\theta}_1(t), ..., \hat{\theta}_i(t)$ in the set $z_1 \in (-k_{b_1}, k_{b_1})$. Thus, the state $x_{i+1}(t) = z_{i+1}(t) + \alpha_i(t)$ is bounded. Since $\bar{x}_n(t)$, $\bar{z}_n(t)$, $\bar{y}_{d_n}(t)$, $\hat{\theta}_1(t), ..., \hat{\theta}_n(t)$ are bounded, and particularly with $|z_1(t)| < k_{b_1}$, we conclude that the control $u(t)$ is also bounded. Hence, all closed loop signals are bounded.

iv) Recall the definition of $V_b^*$ from (5.72). Similar to the proof Theorem 5.3.1(iv), we establish that the set $V_n(z, \tilde{\Theta}) \leq V_b^*$ is positively invariant, due to the fact that $\dot{V}_n \leq 0$ in the region $V_n(z, \tilde{\Theta}) \geq V_b^*$. As such, $(z(t), \tilde{\Theta}(t))$ remains in the interior of the level set $\Omega_b = \{z \in R^n, \tilde{\Theta} \in R^l \mid V_n(z, \tilde{\Theta}) = V_b^*\}$. Then, there exists a positive constant $T$ such that $|z_i(t)| \leq \sqrt{\frac{\bar{c}_n}{\kappa_i}}$ for $t > T$. It follows that the output tracking error $z_1(t)$ converges to the set $|z_1| \leq \sqrt{\bar{c}_n/\kappa_1}$. ■

**Remark 5.4.2** *The constants $g_{\min}$, $\theta_M$ and $\phi_M$ are to be known in order to obtain $\bar{c}_n$ in (5.102) only, so that the control parameter $\kappa_1$ can be chosen to satisfy $\kappa_1 > \bar{c}_n/k_{b_1}^2$. Note that $g_{\min}$, $\theta_M$ and $\phi_M$ may be crude estimates for this purpose. For a less conservative design, we do not use these bounds explicitly in the control, but consider the maximal lower bound, given by $g^* := \min_{i=1,...,n}\{\inf_{\bar{x}_i \in R^i} g_i(\bar{x}_i)\}$, and the actual norms $\|\theta\|$ and $\|\phi\|$, to be unknown and compensate for them using adaptive control.*

**Remark 5.4.3** *Unlike the case of state constraint, the output constraint case does not involve checking of feasibility conditions, and thus, $D_{z_1}$ and $D_{\hat{\theta}_i}$ are not necessary for control implementation, but rather, for analytical purposes. As such, they are described in terms of the unknown parameter $g^*$.*

**Remark 5.4.4** *To achieve greater flexibility in control design and to relax conditions on starting values of the output, the presented method can be extended to employ the asymmetric Barrier Lyapunov Function described in Section 3.4.*

## 5.5 Simulation Results

In this section, computer simulation studies are presented to demonstrate the effectiveness of the proposed control. We focus on the control for the output constraint case as described in Section 5.4. Consider the strict feedback system

$$\begin{aligned} \dot{x}_1 &= x_1 + (0.8 + 0.1e^{-x_1^2})x_2 \\ \dot{x}_2 &= x_1^2 + 0.1\tanh x_2 + (1 + 0.2\sin x_2)u \end{aligned} \tag{5.113}$$

When the nonlinearities are expressed in linearly parameterized forms, it can be obtained that $\theta = [1, 1, 0.1]^T$, $\phi = [0.8, 0.1, 1, 0.2]^T$, $\psi_1 = [x_1, 0, 0]^T$, $\psi_2 = [0, x_1^2, \tanh x_2]^T$, $\varphi_1 = [1, e^{-x_1^2}, 0, 0]^T$, and $\varphi_2 = [0, 0, 1, \sin x_2]^T$. The objective is for $x_1$ to track desired trajectory $y_d = 0.7\sin t$, subject to output constraint $|x_1| < 1.0$. Since $|y_d| \le A_0 = 0.7$, we have that $k_{b_1} = 1.0 - 0.7 = 0.3$, and that $|\dot{y}_d| \le Y_1 = 0.7$. It is straightforward to verify that Assumptions 5.2.1-5.2.2 are satisfied, with $g^* = 0.8$.

The initial conditions are $x_1(0) = 0.1$, $x_2(0) = 0.0$, and $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0.0$. For simplicity, the control gains and adaptation parameters are selected as $\kappa_1 = 15.0$, $\kappa_2 = 2.0$, $\sigma = 0.1$, $\Gamma_1 = 20.0$, and $\Gamma_2 = 20.0I$ and we set $g_{\min} = 0.8$, $\theta_M = \|\theta\|$, and $\phi_M = \|\phi\|$. From the choice of parameters, $\bar{c}_n = 1.316$ can be computed based on (5.102). Then, it can be verified that initial tracking error satisfies $|z_1(0)| \le k_{b_1}$ and the control parameter $\kappa_1$ satisfies $\kappa_1 > \bar{c}_n/k_{b_1}^2$, as required in Theorem 5.4.1.

We implement the following control and adaptation laws in computer simulation:

$$\begin{aligned} \alpha_1 &= -\left(\frac{\lambda}{k_{b_1}^2 - z_1^2}\hat{\theta}_1\Psi_1 + \kappa_1(k_{b_1}^2 - z_1^2)\right)z_1 \\ u &= -(\lambda\hat{\theta}_2^T\Psi_2 + \kappa_2)z_2 \\ \dot{\hat{\theta}}_1 &= \frac{z_1^2}{(k_{b_1}^2 - z_1^2)^2}\Gamma_1\Psi_1 - \sigma\hat{\theta}_1 \\ \dot{\hat{\theta}}_2 &= \Gamma_2\Psi_2 z_2^2 - \sigma\hat{\theta}_2 \end{aligned} \tag{5.114}$$

where $\Psi_1$ and $\Psi_2$ are defined in (5.79) and (5.93), respectively.

Figure 5.1 shows that good practical tracking performance is achieved, and the output constraint requirement $|x_1| < k_{c_1}$ is satisfied as a result of enforcing constraints on the tracking error signal $|z_1| < k_{b_1}$. From Figures 5.2 and 5.3, the state $x_2$, control signal $u$, and parameter estimates $\hat{\theta}_1$ and $\hat{\theta}_2$, are well behaved and bounded. Tracking performance for different values of $\Gamma_1$, $\Gamma_2$, and $\kappa_1$ are shown in Figures 5.4 and 5.5, where it is observed that increase of $\Gamma_1$, $\Gamma_2$, or $\kappa_1$ leads to decrease in steady state tracking error.

## 5.6 Conclusions

In this chapter, we have presented control design based on BLFs for nonlinear constrained systems in strict feedback form with uncertain (virtual) control gain functions. Conditions for practical stability with guaranteed non-violation of constraints have been established in Lemma 2.4.3. For the case of full state constraints, it has been shown that practical output tracking is achieved under certain feasibility conditions on the initial states and control parameters, which can generally be relaxed when handling only partial state constraints. Furthermore, we have shown that, for the special case of output constraint with linearly parameterized system nonlinearities, feasibility conditions are not required,and similar results of practical output tracking are achieved without violation of output constraint. Finally, the effectiveness of the proposed control has been demonstrated through a simulation example.

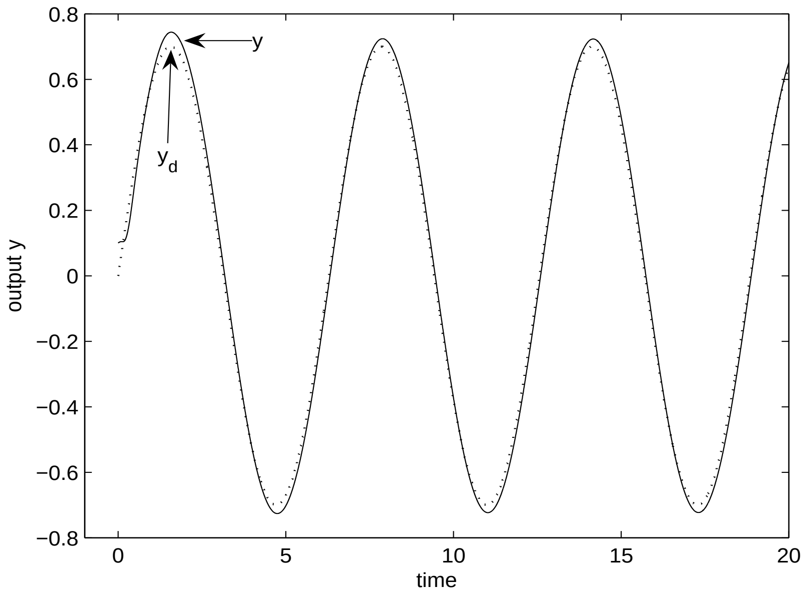


Figure 5.1: Tracking performance.

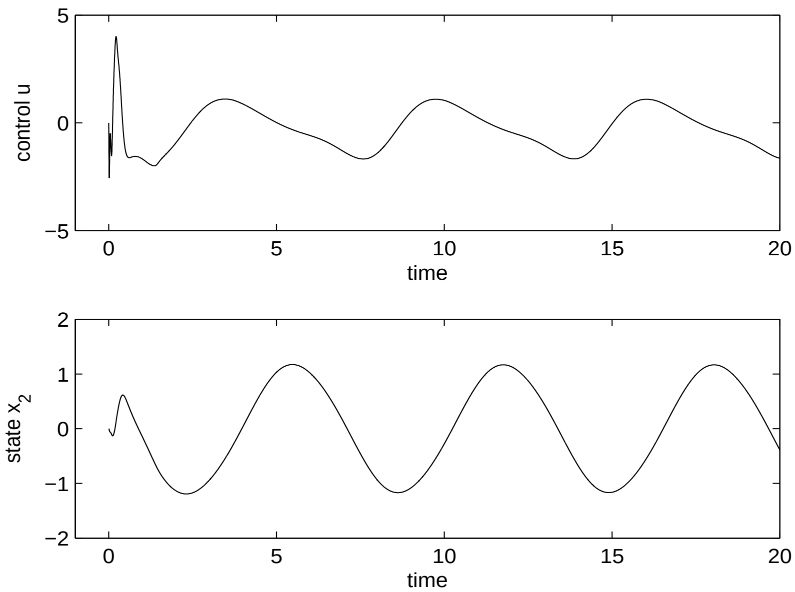


Figure 5.2: Control signal $u$ and state $x_2$.

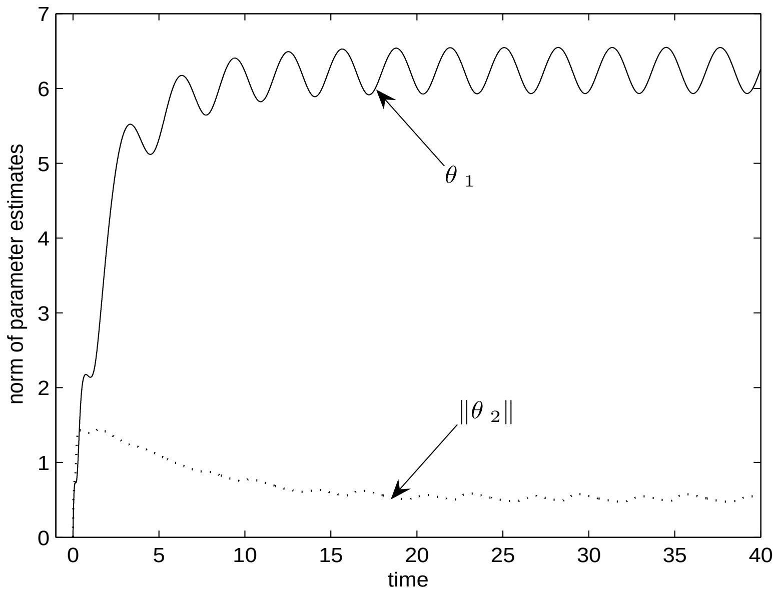


Figure 5.3: Norms of parameter estimates.

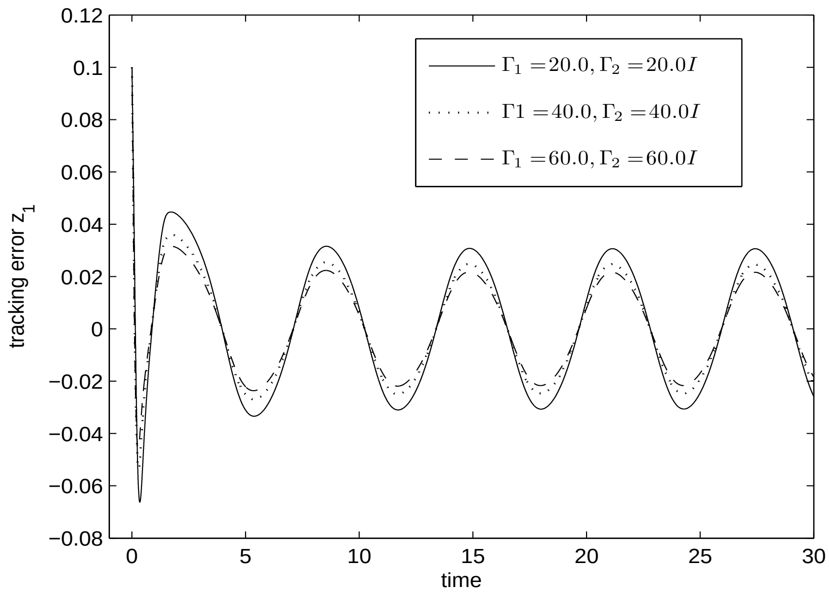


Figure 5.4: Tracking performance for different $\Gamma_1$ and $\Gamma_2$.

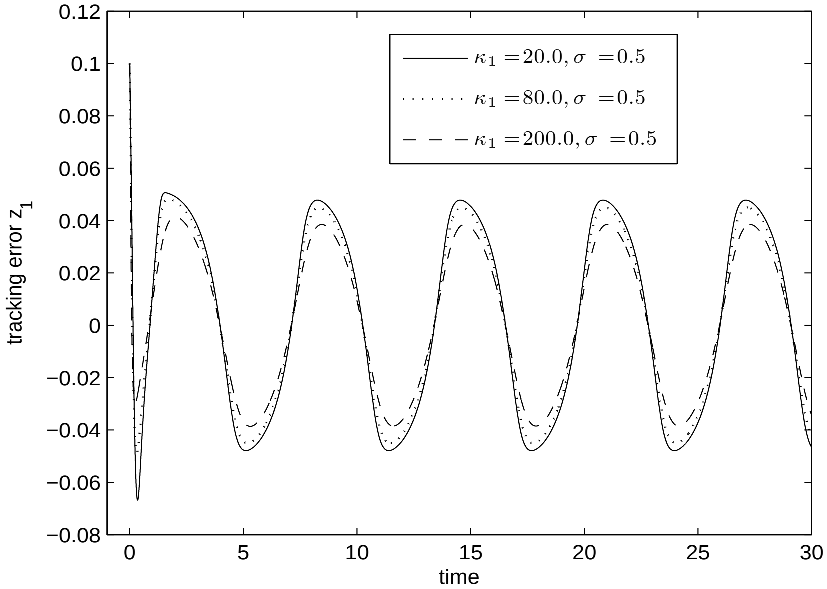


Figure 5.5: Tracking performance for different $\kappa_1$.

# Chapter 6

# Adaptive Control of Electrostatic Microactuators

## 6.1 Introduction

Electrostatic microactuators have gained widespread acceptance in MEMs applications, due to the simplicity of their structure, ease of fabrication, and the favorable scaling of electrostatic forces into the micro domain. This has ignited an interest in how to control these devices effectively to achieve greater precision and speed of response. In this chapter, we focus on the adaptive control of electrostatic microactuators with bi-directional drive, which are less prone to pull-in instability due to the fact that they can be actively controlled in both directions, unlike uni-directional drive actuators where only passive restoring force is provided by mechanical stiffness in one direction. Although less challenging as a theoretical control design problem, the study of micro-actuators with bi-directional drive is nevertheless important since its controllability is an advantage in high performance applications.

In most of the works on MEMs control, knowledge of model parameters is required and typically estimated through offline system identification methods. However, inconsistencies in bulk micromachining result in variation of parameters across pieces, and may require extensive efforts in parameter identification, with higher costs. Furthermore, some of the parameters, such as the damping constant, are usually difficult to identify accurately, so a viable alternative is to rely on intelligent feedback control for online compensation of parametric uncertainties.

There has been relatively few works in the literature on application of adaptive techniques in MEMs. Motivated by our previous works on intelligent control for general nonlinear systems [42] and robotic manipulators [48], we apply adaptive backstepping control for 1DOF electrostatic microactuators with bi-directional drive, based on rigorous Lyapunov synthesis, to force the movable plate to track a reference trajectory within the air gap without knowledge of plant parameters. When full-state information is available, adaptive backstepping is carried out following a suitable change of coordinates that transforms the system into parametric strict feedback form. When velocity feedback is unavailable, the plant is transformed into the parametric output feedback form and adaptive observer backstepping is employed to achieve asymptotic tracking without velocity measurement. We employ special barrier functions in Lyapunov synthesis so as to design a control ensuring that the movable plate and the electrodes do not come into contact. To the best of the authors' knowledge, the latter objective has not been tackled rigorously in published works on control of electrostatic microactuators, which usually base the control design on the unconstrained system and subsequently demonstrate by simulations that the constraints are not violated.

The organization of the remainder of this chapter is as follows. Section 6.2 presents a description of the electrostatic microactuator under study, the problem statement, and the related state transformations to facilitate the control design. This is followed by Sections 6.3 and 6.4, which provide full details on the use of barrier functions to enforce constraints on the output, as well as the control design and rigorous stability analyses for the full-state feedback and output feedback cases, based on adaptive backstepping and adaptive observer backstepping, respectively. Finally, detailed simulation results for both cases are shown in Section 6.5.

## 6.2 Problem Formulation and Preliminaries

Consider the dynamic model of the 1-DOF electrostatic microactuator with bidirectional drive, as illustrated in Figure 6.1. The capacitances $C_f$ and $C_b$, between the movable plate and the top and bottom electrodes respectively, are described by

$$C_f = \frac{\epsilon A}{l_0 - l}, \quad C_b = \frac{\epsilon A}{l_0 + l} \tag{6.1}$$

where $l \in \mathbb{R}$ denotes the air gap between the movable plate and the top electrode, and $l_0$ the gap when both input voltages $V_f$ and $V_b$ are zero. The corresponding electrostatic forces acting on the movable plate due to the input voltages $V_f$ and $V_b$ are:

$$\begin{aligned} F_f &= -\frac{1}{2}\frac{\partial C_f}{\partial l}V_f^2 = \frac{\epsilon A}{2(l_0 - l)^2}V_f^2, \\ F_b &= -\frac{1}{2}\frac{\partial C_b}{\partial l}V_b^2 = -\frac{\epsilon A}{2(l_0 + l)^2}V_b^2 \end{aligned} \tag{6.2}$$

Thus, the state space equations governing the dynamics of the electrostatic microactuator are given by:

$$m\ddot{l} + b(l)\dot{l} + kl = \frac{\epsilon A}{2}\left(\frac{V_f^2}{(l_0 - l)^2} - \frac{V_b^2}{(l_0 + l)^2}\right) =: \frac{\epsilon A}{2}\nu \tag{6.3}$$

where $m$ denotes the mass of the movable electrode, $\epsilon$ the permittivity of the gap, $A$ the plate area, $k$ the spring constant, and $b(l)$ the nonlinear squeeze film damping. A simplified form for $b(l)$ obtained from linearization of the compressible Reynolds gas-film equation [12]

$$b(l) = \frac{b_c}{g^3} \tag{6.4}$$

This function, exhibiting a cubic dependence on the air gap, $g$, in the denominator, has been described in several works [3, 96, 98, 173, 174], but with different values of the coefficient $b_c$. In this chapter, by averaging the effects of the two layers of squeeze films on both sides of the movable electrode, we arrive at the following modified model:

$$b(l) = \frac{b_c}{2}\left(\frac{1}{(l_0 - l)^3} + \frac{1}{(l_0 + l)^3}\right) \tag{6.5}$$

The constant parameters $m$, $\epsilon$, $A$, $b_c$ and $k$ may be difficult to identify accurately in practice, and are thus considered to be uncertain. For example, $m$, $k$ and $A$ can vary from unit to unit due to limitations in fabrication precision. The permittivity $\epsilon$ can change according to the ambient humidity. The coefficient $b_c$ in the damping model is composed of parameters such as fluid viscosity and plate dimensions, and is thus likely to vary according to ambient conditions and fabrication consistency. Nevertheless, it is reasonable to have good indication of the order of magnitudes of these parameters.

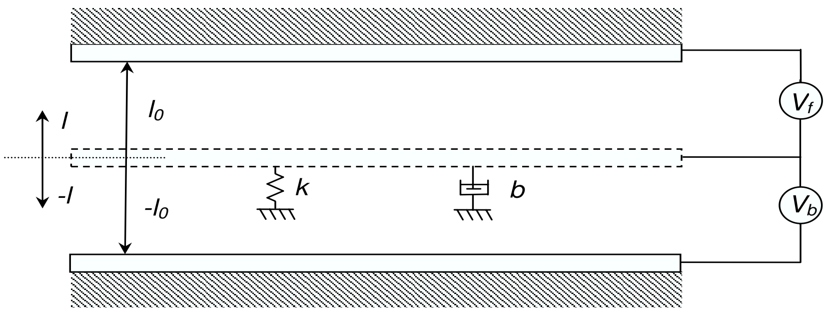


Figure 6.1: One-degree-of-freedom electrostatic microactuator with bi-directional drive.

**Remark 6.2.1** *While bi-directional parallel plate actuators, as shown in Figure 6.1, can be used for both out-of-plane and in-plane applications, out-of-plane bi-directional configurations involve complex fabrication processes, such that the derived benefits need to be weighed against the costs. On the other hand, lateral parallel plate microactuators are much more feasible, as they can be easily fabricated and configured for bi-directional actuation, such as that shown in [61] for optical moving-fibre switches, and that in [62] for positioning of disk drive sliders.*

**Remark 6.2.2** *The voltages $V_f$ and $V_b$ are independent inputs which collectively provide controllability of the movable plate in both directions. By lumping the two voltage terms into an aggregate control variable $\nu$ in (6.3), we can design it as an unconstrained input first, and subsequently apportion it to the actual voltage inputs.*

**Remark 6.2.3** *To prevent shorting of the electrical circuit, an insulating layer is present in each of the driving electrodes. This also helps to prevent singularity, which is evident from (6.3) whenever* $|l| = l_0$*, causing the input* $\nu$ *to be undefined. Hence, the state space of the system in constrained in the compact set* $\chi = \{(l, \dot{l}) \in \mathbb{R}^2 | \; |l| < l_0 - \delta\}$*, where* $0 < \delta < l_0$*.*

To obtain the same order of magnitude of the variables and thereby avoid numerical problems in simulation, we perform a change of time scale $\tau = \sigma t$ and a change of variables $x_1 = \frac{l}{l_0}$, $x_2 = \frac{1}{l_0}\frac{dl}{d\tau}$, $u = \frac{\nu}{\beta}$, for large constants $\sigma > 0$ and $\beta > 0$, thus yielding the strict-feedback form:

$$\begin{aligned} \frac{dx_1}{d\tau} &= x_2(\tau) \\ \frac{dx_2}{d\tau} &= -\frac{b_c}{2m\sigma l_0^3}\bar{b}(x_1(\tau))x_2(\tau) - \frac{k}{m\sigma^2}x_1(\tau) + \frac{\epsilon A\beta}{2m\sigma^2 l_0}u(\tau) \\ y &= x_1(\tau) \end{aligned} \tag{6.6}$$

where $y \in \mathbb{R}$ is the output and $\bar{b}(x_1)$ is described by:

$$\bar{b}(x_1) = \frac{1}{(1-x_1)^3} + \frac{1}{(1+x_1)^3} \tag{6.7}$$

For ease of notation, $\dot{x}_1$ and $\dot{x}_2$ are henceforth understood as $\frac{dx_1}{d\tau}$ and $\frac{dx_2}{d\tau}$ respectively, following the change of time scale.

The scaling constants $\sigma$ and $\beta$ condition the magnitude of the coefficients. For instance, the large constant $\sigma$ moderates the value of $\frac{k}{m\sigma^2}$, which is otherwise very large and may pose problems in numerical implementation. On the other hand, the coefficient $\frac{\epsilon A}{2m\sigma^2 l_0}$ in the second equation of (6.6) can be very small. By working with the scaled input $u = \frac{\nu}{\beta}$ instead of $\nu$, the large constant $\beta$ is introduced, which moderates the magnitude of the coefficient for easier simulation.

**Remark 6.2.4** *These scalings are introduced for analysis purposes only, and do not change the properties of the original plant (6.3). The choice of the scaling constants may be motivated by a priori knowledge of the order of magnitude of the uncertain parameters.*

The control objective is to force the movable electrode to track a reference trajectory $y_d(t)$ within the air gap, i.e. $|y(t) - y_d(t)| \to 0$ as $t \to \infty$. At the same time, all closed loop signals are to be kept bounded. To avoid complicated switched systems analysis, we aim to design a control scheme which ensures that the movable plate does not come into contact with the electrodes.

**Assumption 6.2.1** *The first and second order time-derivatives of the reference trajectory $y_d(t)$ are bounded, i.e. $|\dot{y}_d(t)| < Y_1$, $|\ddot{y}_d(t)| < Y_2$, where $Y_1$ and $Y_2$ are constants. In addition, the reference trajectory is bounded by $\underline{y}_d \leq y_d(t) \leq \overline{y}_d$, where $\underline{y}_d$ and $\overline{y}_d$ are constants that satisfy $\underline{y}_d > -1 + \frac{\delta}{l_0}$ and $\overline{y}_d < 1 - \frac{\delta}{l_0}$.*

## 6.3 Full-State Feedback Adaptive Control Design

In this section, we investigate full-state feedback adaptive control for 1DOF electrostatic microactuators described by (6.6), in the presence of parametric uncertainty. The control design follows the procedures detailed in Section 3.4 for $n = 2$.

<u>**Step 1**</u> Define error variables $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, where $\alpha_1$ is the stabilizing function to be designed. We consider the Lyapunov function candidate to facilitate the design of stabilizing function $\alpha_1$ that will ensure that the constraint on $x_1$ is respected:

$$V_1 = \frac{\kappa_0}{2} q(z_1) \log \frac{k_b^2}{k_b^2 - z_1^2} + \frac{\kappa_0}{2}(1 - q(z_1)) \log \frac{k_a^2}{k_a^2 - z_1^2} \tag{6.8}$$

where $\kappa_0$ is a positive design constant, the function $q(\cdot) : \mathbb{R} \to \{0, 1\}$ is defined by

$$q(\bullet) = \begin{cases} 1, & \text{if } \bullet > 0 \\ 0, & \text{if } \bullet \leq 0 \end{cases} \tag{6.9}$$

and

$$k_a = 1 - \frac{\delta}{l_0} - |\underline{y}_d|, \qquad k_b = 1 - \frac{\delta}{l_0} - |\overline{y}_d| \tag{6.10}$$

are positive constants representing the constraints in the $z_1$ state space, given by $-k_a < z_1 < k_b$, induced from the constraints in the $x_1$ state space, given by $|x_1| < 1 - \frac{\delta}{l_0}$. By invoking Lemma 3.4.1 with $p = 2$, we obtain that the Lyapunov function candidate $V_1(z_1)$ in (6.8) is positive definite, continuous and continuously differentiable in the open interval $z_1 \in (-k_a, k_b)$.

Choose the stabilizing function as

$$\alpha_1 = -\kappa_1 \left[ q(k_b^2 - z_1^2) + (1-q)(k_a^2 - z_1^2) \right] z_1^3 + \dot{y}_d \tag{6.11}$$

with $\kappa_1$ being a positive constant. This yields

$$\dot{V}_1 = -\kappa_0 \kappa_1 z_1^4 + \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 z_1 z_2 \tag{6.12}$$

where the first term is always non-positive and the second term is cancelled in the second step. According to Lemma 3.4.2, the stabilizing function $\alpha_1(z_1, \dot{y}_d)$ described in (6.11) is continuously differentiable with respect to $z_1$ in the open interval $z_1 \in (-k_a, k_b)$.

**Step 2** This is the step in which the actual control input will be designed. Consider the Lyapunov function candidate

$$V_2^* = V_1 + \frac{m\sigma^2 l_0}{\epsilon A \beta} z_2^2 \tag{6.13}$$

Ideally, we can design the control input as

$$u = u^* = -\kappa_2 z_2 - \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 z_1 + \theta^T \psi \tag{6.14}$$

where $\theta = \frac{2m\sigma^2 l_0}{\epsilon A \beta} \left[ \frac{k}{m\sigma^2},\ \frac{b_c}{2m\sigma l_0^3},\ 1 \right]^T$, $\psi = \left[ x_1,\ \bar{b}(x_1) x_2,\ \dot{\alpha}_1 \right]^T$, and $\kappa_2$ is a positive constant, which leads to the following equation:

$$\dot{V}_2^* = -\kappa_0 \kappa_1 z_1^4 - \kappa_2 z_2^2 \tag{6.15}$$

from which the asymptotic convergence of the error signals $z_1$ and $z_2$ to zero can be shown after some analysis.

However, the ideal control law (6.14) is not viable due to the fact that the parameters $m$, $\epsilon$, $A$, $b$ and $k$ in $\theta^*$ are not available. To deal with the parametric uncertainty, we employ the certainty-equivalent control law:

$$u = -\kappa_2 z_2 - \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 z_1 + \hat{\theta}^T \psi \tag{6.16}$$

$$\dot{\hat{\theta}} = -\Gamma \psi z_2 \tag{6.17}$$

where $\hat{\theta} \in \mathbb{R}^3$ is the estimate of $\theta$. Since $u$ is an aggregate control variable defined for ease of analysis, we still need to compute the actual voltage controls $V_f$ and $V_b$, which is performed with the following algorithm

$$\begin{aligned} V_f &= \sqrt{\beta l_0^2 q(u)(1-x_1)^2 u} \\ V_b &= \sqrt{-\beta l_0^2 (1-q(u))(1+x_1)^2 u} \end{aligned} \tag{6.18}$$

where the function $q(\cdot)$ is defined in (6.9). It can be checked that $\beta l_0^2 q(u)(1-x_1)^2 u$ and $-\beta l_0^2(1-q(u))(1+x_1)^2 u$, i.e., the terms within the square root operators, are always non-negative.

**Remark 6.3.1** *The algorithm in (6.18) minimizes the sum of $V_f^2$ and $V_b^2$ for a given $u$. From (6.3), it can be shown that*

$$V_f^2 + V_b^2 = \begin{cases} \left(1 + \frac{(1-x_1)^2}{(1+x_1)^2}\right) V_b^2 + \beta l_0^2 (1-x_1)^2 u & \text{if } u > 0 \\ \left(1 + \frac{(1+x_1)^2}{(1-x_1)^2}\right) V_f^2 - \beta l_0^2 (1+x_1)^2 u & \text{if } u \leq 0 \end{cases} \tag{6.19}$$

*It is clear that for $u > 0$, the minimum is obtained when $V_b = 0$ and for $u < 0$, the minimum is obtained when $V_f = 0$.*

For stability analysis and design of the adaptation law, we augment the Lyapunov function candidate with a quadratic term in the parameter estimation error as follows

$$V_2 = V_1 + \frac{m\sigma^2 l_0}{\epsilon A \beta} z_2^2 + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \tag{6.20}$$

where $\tilde{\theta} = \hat{\theta} - \theta$, and $\Gamma = \Gamma^T > 0$ is a constant matrix. The time derivative of $V_2$ along the closed loop trajectories is given by

$$\dot{V}_2 = -\kappa_0 \kappa_1 z_1^4 - \kappa_2 z_2^2 \tag{6.21}$$

With the above equation, we are ready to present our main results.

**Theorem 6.3.1** *Consider the uncertain 1DOF electrostatic microactuator system (6.6) under Assumption 6.2.1, full-state feedback control law (6.16), and adaptation law (6.17). If the initial conditions are such that $(x_1(0), x_2(0)) \in \bar{\Omega}$, where the latter set is described by:*

$$\bar{\Omega} := \{(x_1, x_2) \in \mathbb{R}^2 \mid -k_a < x_1(0) - y_d(0) < k_b\} \tag{6.22}$$

*with $k_a$ and $k_b$ defined in (6.10), then the output tracking error with respect to a reference trajectory within the air gap, i.e. $y_d(t) \in (-l_0 + \delta, l_0 - \delta)$, is asymptotically stabilized, i.e., $y(t) \to y_d(t)$ as $t \to \infty$, while keeping all closed loop signals bounded. Furthermore, the output $y(t)$ remains in the set $\Omega_y := \{y \in \mathbb{R} : |y| \leq 1 - \delta/l_0\}$ $\forall t > 0$, i.e. the output constraint is never violated.*

**Proof:** The proof follows along the lines of the proof of Theorem 3.4.1, and is briefly outlined here for the sake of completeness. First, $\dot{V}_2(t) \leq 0$ implies that for any bounded $V_2(0)$, we have that $V_2(t)$ remains bounded $\forall t > 0$. From (6.20), it follows that $V_1(t)$ is bounded $\forall t > 0$ and thus $-k_a < z_1(t) < k_b$. From (6.10) and $z_1 = y - y_d$, it can be shown that

$$-1 + \frac{\delta}{l_0} + y_d(t) + |\underline{y}_d| < \quad y(t) \quad < 1 - \frac{\delta}{l_0} + y_d(t) - |\overline{y}_d|$$

From Assumption 6.2.1, we know that $\underline{y}_d \leq y_d(t) \leq \overline{y}_d$, which yields fact that $y_d(t) + |\underline{y}_d| \geq 0$ and $y_d(t) - |\overline{y}_d| \leq 0$, leading to the following inequality

$$-1 + \frac{\delta}{l_0} < \quad y(t) \quad < 1 - \frac{\delta}{l_0}$$

Hence, we conclude that $y(t) \in \Omega_y$ $\forall t > 0$.

Next, we show that all closed loop signals are bounded. From (6.21), we know that $z_1(t)$, $z_2(t)$, and $\hat{\theta}(t)$ are bounded. The boundedness of $z_1(t)$ and the reference trajectory $y_d(t)$ imply that the state $x_1(t)$ is bounded. Given that $\dot{y}_d(t)$ is bounded, the stabilizing function $\alpha_1(t)$ is also bounded from (6.11). This leads to the boundedness of state $x_2(t) = z_2(t) + \alpha_1(t)$. Since $-k_a < z_1(t) < k_b$, $|\ddot{y}_d(t)| \leq Y_2$, and $|x_1(t)| < 1 - \delta/l_0$, we infer that the control $u(t)$ from (6.16) is bounded. Therefore, all closed loop signals are bounded.

Lastly, we show that $y(t) \to y_d(t)$ as $t \to \infty$. From the boundedness of the closed loop signals, it can be shown that

$$\ddot{V}_2 \quad = \quad -4\kappa_0\kappa_1 z_1^3(x_2 - \dot{y}_d) - \frac{\epsilon A\beta}{m\sigma^2 l_0}\kappa_2 z_2(-\kappa_2 z_2 + \tilde{\theta}^T\psi)$$

is bounded, thus implying that $\dot{V}_2(t)$ is uniformly continuous. Then, by Barbalat's Lemma, we obtain that $z_1(t), z_2(t) \to 0$ as $t \to \infty$. Since $z_1(t) = x_1(t) - y_d(t)$, it is clear that $y(t) \to y_d(t)$ as $t \to \infty$. ■

## 6.4 Output Feedback Adaptive Control Design

Full-state feedback control, as presented in the previous section, requires measurements of displacement $l$ and velocity $\dot{l}$. Among the various sensing methods in MEMs, which include capacitive, optical, electromagnetic, piezoelectric, and tunneling, one of the more successful types is capacitive sensing, due to the simplicity of the sensor, low power consumption, and good temperature stability. The displacement $l$ can be measured by state-of-the-art capacitive sensing methods (see e.g. [151]), which are suitable for BLF-based control since they are fast, reliable and low-noise.

However, it is generally difficult to measure the velocity $\dot{l}$ for feedback control. Thus, $x_1$ is available but $x_2$ is not. Furthermore, since the BLF-based control designs for general strict feedback systems presented in Chapter 3 dealt with full state feedback, they are not directly applicable to the output feedback problem. In this section, we provide a detailed exposition of the output feedback control design based on the BLF and adaptive observer backstepping [94].

### 6.4.1 State Transformation and Filter Design

To facilitate the design of the adaptive observer backstepping control, we first perform a change of coordinates:

$$\eta_1 = x_1 \tag{6.23}$$

$$\eta_2 = x_2 + \frac{b_c}{m\sigma l_0^3}\bar{\phi}(x_1) \tag{6.24}$$

where $\bar{\phi}(x_1)$ is defined by

$$\bar{\phi}(x_1) = \frac{1}{2}\left(\frac{1}{(1-x_1)^2} - \frac{1}{(1+x_1)^2}\right) \tag{6.25}$$

The time derivative of $\bar{\phi}$ is given by

$$\dot{\bar{\phi}} = \frac{\partial\bar{\phi}(x_1)}{\partial x_1}\dot{x}_1 = \left(\frac{1}{(1-x_1)^3} + \frac{1}{(1+x_1)^3}\right)x_2 = \bar{b}(x_1)x_2 \tag{6.26}$$

Substituting (6.23)-(6.26) into (6.6), we can rewrite the system dynamics in parametric output feedback form:

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 - \theta_1 \bar{\phi}(\eta_1) \\ \dot{\eta}_2 &= -\theta_2 \eta_1 + \vartheta u \\ y &= \eta_1 \end{aligned} \tag{6.27}$$

where $\theta_1 = \frac{b_c}{m\sigma l_0^3}$, $\theta_2 = \frac{k}{m\sigma^2}$, $\vartheta = \frac{\epsilon A \beta}{2m\sigma^2 l_0}$. This can be represented by the simplified form:

$$\begin{aligned} \dot{\eta} &= A\eta + \sum_{i=1}^{2} \theta_i \phi_i(y) + \vartheta e_2 u \qquad (6.28) \\ y &= \eta_1 \end{aligned}$$

where $e_2 := [0, 1]^T$, $\eta = [\eta_1, \eta_2]^T$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\phi_1(y) = \left[-\bar{\phi}(y)\ , 0\right]^T$, $\phi_2(y) = [0, -y]^T$.

Design the following filters:

$$\dot{\xi}_0 = A_0 \xi_0 + cy \tag{6.29}$$

$$\dot{\xi}_i = A_0 \xi_i + \phi_i(y), \quad i = 1, 2 \tag{6.30}$$

$$\dot{v} = A_0 v + e_2 u + \varphi \tag{6.31}$$

where $\xi_i \in \mathbb{R}^2$ $(i = 0, 1, 2)$, $v \in \mathbb{R}^2$, $\varphi(\cdot) = [\varphi_1, \varphi_2]^T \in \mathbb{R}^2$ is a correction function to be designed, and $c = [c_1, c_2]^T$ with positive constants $c_1$ and $c_2$ chosen such that the matrix $A_0 = \begin{bmatrix} -c_1 & 1 \\ -c_2 & 0 \end{bmatrix}$ satisfies

$$A_0^T P + P A_0 = -R \tag{6.32}$$

for some $P = P^T > 0$ and $R = R^T > 0$.

**Remark 6.4.1** *For systems in the parametric output feedback form, the regressors $\phi_i$ $(i = 1, 2)$ depend only on the output $y$, hence adaptive observer backstepping can be employed for stable output feedback control design [94].*

**Remark 6.4.2** *The filters (6.29)-(6.31) are similar to the K-filters presented in [94], but include the additional correction term $\varphi(\cdot)$, which will be designed to cancel the terms containing the observation error that appear in the Lyapunov derivative during backstepping design. It is an alternative to nonlinear damping techniques, which dominate, instead of cancelling, the terms [94].*

**Remark 6.4.3** *It is necessary to implement the filters (6.29)-(6.31) due to the problems associated with reconstructing the states using certainty equivalence methods, namely that the observation error dynamics will be corrupted by parameter estimation errors. As will be shown subsequently, the use of these filters renders the observation error dynamics almost autonomous, if not for the correction term $\varphi(\cdot)$, which will be systematically designed to guarantee closed loop stability.*

By constructing the state estimate as follows

$$\hat{\eta}(t) = \xi_0(t) + \sum_{i=1}^{2} \theta_i \xi_i(t) + \vartheta v(t) \tag{6.33}$$

it is easy to see that the dynamics of the observation error, $\tilde{\eta} = \hat{\eta} - \eta$, are given by

$$\begin{aligned}
\dot{\tilde{\eta}} = \dot{\hat{\eta}} - \dot{\eta} &= A_0\xi_0 + cy + \sum_{i=1}^{2} \theta_i(A_0\xi_i + \phi_i(y)) + \vartheta(A_0 v + e_2 u + \varphi) - A\eta \\
&\quad - \sum_{i=1}^{2} \theta_i \phi_i(y) - \vartheta e_2 u \\
&= A_0 \left( \xi_0 + \sum_{i=1}^{2} \theta_i \xi_i + \vartheta v \right) - A_0\eta + \vartheta\varphi \\
&= A_0\tilde{\eta} + \vartheta\varphi
\end{aligned} \tag{6.34}$$

The constructive procedure for adaptive observer backstepping design will be presented next.

### 6.4.2 Adaptive Observer Backstepping

The method presented in this section is similar to the backstepping procedure in Section 6.3, but the filter signal $v_2$ of (6.31) is used as the stabilizing function, instead of the state $x_2$, which is unavailable.

**Step 1** Define $z_1 = y - y_d$, whose derivative is given by

$$\dot{z}_1 = \xi_{02} + \sum_{i=1}^{2} \theta_i \xi_{i2} + \vartheta v_2 - \tilde{\eta}_2 - \theta_1 \bar{\phi}(y) - \dot{y}_d \tag{6.35}$$

where $\xi_{ij}$ and $v_j$ denote the $j$-th elements of $\xi_i$ and $v$, respectively. Denote $z_2 = v_2 - \alpha_1$, where $\alpha_1$ is a stabilizing function to be designed, and consider the Lyapunov function candidate:

$$V_1^* = \frac{\kappa_0}{2} q(z_1) \log \frac{k_b}{k_b^2 - z_1^2} + \frac{\kappa_0}{2}(1 - q(z_1)) \log \frac{k_a}{k_a^2 - z_1^2} \tag{6.36}$$

where $\kappa_0$ is a positive design constant, and $q(\cdot)$ is defined in (6.9). The derivative of $V_1^*$ is given by

$$\begin{aligned} \dot{V}_1^* &= \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 z_1 \left( \xi_{02} + \sum_{i=1}^{2} \theta_i \xi_{i2} + \vartheta v_2 - \tilde{\eta}_2 - \theta_1 \bar{\phi}(y) - \dot{y}_d \right) \\ &= \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 z_1 \left( \xi_{02} + \sum_{i=1}^{2} \theta_i \xi_{i2} + \vartheta (z_2 + \alpha_1) - \tilde{\eta}_2 - \theta_1 \bar{\phi}(y) - \dot{y}_d \right) \end{aligned} \tag{6.37}$$

Ideally, we can design the stabilizing function as

$$\alpha_1 = \alpha_1^* := \frac{1}{\vartheta} \left[ -\xi_{02} - \Theta_1^T \Psi_1 - \kappa_1 \left( q(k_b^2 - z_1^2) + (1-q)(k_a^2 - z_1^2) \right) z_1^3 + \dot{y}_d \right] \tag{6.38}$$

where the parameter and regressor vectors are respectively defined by:

$$\Theta_1 = [\theta_1, \ \theta_2]^T \tag{6.39}$$

$$\Psi_1 = [\xi_{12} - \bar{\phi}(y), \ \xi_{22}]^T \tag{6.40}$$

By substitution of the ideal stabilizing function $\alpha_1 = \alpha_1^*$ into (6.37), it can be obtained that

$$\dot{V}_1^* = -\kappa_0 \kappa_1 z_1^4 + \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 (\vartheta z_1 z_2 - \tilde{\eta}_2 z_1) \tag{6.41}$$

for which the first right-hand-side term is always negative and the second term can be eliminated in the subsequent step.

However, due to the fact that the parameters $\theta_1$, $\theta_2$ and $\vartheta$ are unknown, the ideal stabilizing function $\alpha_1^*$ is not admissible. To circumvent this problem, we augment the $V_1^*$ with quadratic terms of the parameter estimation errors to form $V_1$, the new Lyapunov function candidate:

$$V_1 = \frac{\kappa_0}{2} q(z_1) \log \frac{k_b}{k_b^2 - z_1^2} + \frac{\kappa_0}{2}(1 - q(z_1)) \log \frac{k_a}{k_a^2 - z_1^2} + \frac{1}{2}\tilde{\Theta}_1^T \Gamma_1^{-1} \tilde{\Theta}_1 + \frac{\vartheta}{2\gamma_\varrho}\tilde{\varrho}^2 \tag{6.42}$$

where $\tilde{\Theta}_1 = \hat{\Theta}_1 - \Theta_1$ is the estimation error for the unknown parameter vector $\Theta_1$. The derivative of $V_1$ is given by

$$\begin{aligned} \dot{V}_1 &= \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right) \kappa_0 z_1 \left[\xi_{02} + \Theta_1^T \Psi_1 + \vartheta(z_2 + \alpha_1) - \tilde{\eta}_2 - \dot{y}_d\right] \\ &\quad + \tilde{\Theta}_1^T \Gamma_1^{-1} \dot{\hat{\Theta}}_1 + \frac{\vartheta}{\gamma_\varrho} \tilde{\varrho}\dot{\hat{\varrho}} \end{aligned} \tag{6.43}$$

Denote $\hat{\varrho}$ as the estimate of $\varrho = 1/\vartheta$, with $\tilde{\varrho} = \hat{\varrho} - \varrho$ as the estimation error, and let the stabilizing function $\alpha_1 = \hat{\varrho}\bar{\alpha}_1$, where $\bar{\alpha}_1$ is to be defined shortly. Hence, the above equation can be rewritten as

$$\begin{aligned} \dot{V}_1 &= \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right) \kappa_0 z_1 \left[\xi_{02} + \Theta_1^T \Psi_1 + \vartheta(z_2 + \hat{\varrho}\bar{\alpha}_1) - \tilde{\eta}_2 - \dot{y}_d\right] \\ &\quad + \tilde{\Theta}_1^T \Gamma_1^{-1} \dot{\hat{\Theta}}_1 + \frac{\vartheta}{\gamma_\varrho} \tilde{\varrho}\dot{\hat{\varrho}} \\ &= \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right) \kappa_0 z_1 \left[\xi_{02} + \Theta_1^T \Psi_1 + (\hat{\vartheta} - \tilde{\vartheta}) z_2 + \vartheta(\tilde{\varrho} + \varrho)\bar{\alpha}_1 - \tilde{\eta}_2 - \dot{y}_d\right] \\ &\quad + \tilde{\Theta}_1^T \Gamma_1^{-1} \dot{\hat{\Theta}}_1 + \frac{\vartheta}{\gamma_\varrho} \tilde{\varrho}\dot{\hat{\varrho}} \end{aligned} \tag{6.44}$$

To facilitate the design of the stabilizing function and adaptation laws, we rearrange the above equation into the following form:

$$\begin{aligned} \dot{V}_1 &= \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right) \kappa_0 z_1 \left(\xi_{02} + \bar{\alpha}_1 + \Theta_1^T \Psi_1 + \hat{\vartheta} z_2 - \dot{y}_d\right) + \tilde{\Theta}_1^T \Gamma_1^{-1} \dot{\hat{\Theta}}_1 \\ &\quad + \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right) \kappa_0 (-z_1 \tilde{\eta}_2 - \tilde{\vartheta} z_1 z_2) \\ &\quad + \vartheta\tilde{\varrho}\left[\left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right) \kappa_0 \bar{\alpha}_1 z_1 + \frac{1}{\gamma_\varrho}\dot{\hat{\varrho}}\right] \end{aligned} \tag{6.45}$$

The stabilizing function is designed as

$$\alpha_1 = \hat{\varrho}\bar{\alpha}_1 \tag{6.46}$$

where

$$\bar{\alpha}_1 := -\xi_{02} - \hat{\Theta}_1^T \Psi_1 - \left[ q(k_b^2 - z_1^2) + (1-q)(k_a^2 - z_1^2) \right] \kappa_1 z_1^3 + \dot{y}_d \tag{6.47}$$

while the adaptation laws are given by

$$\dot{\hat{\Theta}}_1 = \Gamma_1 \Psi_1 \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 z_1 \tag{6.48}$$

$$\dot{\hat{\varrho}} = -\gamma_\varrho \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 \bar{\alpha}_1 z_1 \tag{6.49}$$

Substituting the stabilizing function and adaptation laws (6.46)-(6.49) into (6.45) yields the following

$$\begin{aligned} \dot{V}_1 &= -\kappa_0 \kappa_1 z_1^4 + \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 \hat{\vartheta} z_1 z_2 \\ &\quad + \vartheta \tilde{\varrho} \left[ \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 \bar{\alpha}_1 z_1 + \frac{1}{\gamma_\varrho} \dot{\hat{\varrho}} \right] \\ &\quad + \tilde{\Theta}_1^T \left[ \Gamma_1^{-1} \dot{\hat{\Theta}}_1 - \Psi_1 \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 z_1 \right] \\ &\quad + \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 (-z_1 \tilde{\eta}_2 - \tilde{\vartheta} z_1 z_2) \\ &= -\kappa_0 \kappa_1 z_1^4 + \left( \frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2} \right) \kappa_0 (\hat{\vartheta} z_1 z_2 - z_1 \tilde{\eta}_2 - \tilde{\vartheta} z_1 z_2) \end{aligned} \tag{6.50}$$

From the above equation, it can be seen that the first term is stabilizing, while the second term consisting of state and parameter estimation errors will be brought forward into the subsequent step to be handled by the actual control.

We assert that $\alpha_1(z_1, \cdot)$ is a $C^1$ function with the following lemma, which ensures that $\dot{\alpha}_1$ is well-defined.

**Lemma 6.4.1** *The stabilizing function $\alpha_1(z_1, \cdot)$ in (6.46) is continuously differentiable with respect to $z_1$ in the open interval $z_1 \in (-k_a, k_b)$.*

**Proof:** The stabilizing function $\alpha_1(z_1, \cdot)$ is piecewise $C^1$, with respect to $z_1$, over the two intervals $z_1 \in (-k_a, 0]$ and $z_1 \in (0, k_b)$. Thus, to show that $\alpha_1$ is a $C^1$ function for $-k_a < z_1 < k_b$, we need only to show that $\lim_{z_1 \to 0} \frac{\partial \alpha_1}{\partial z_1}$ is identical from both directions. For $0 < z_1 < k_b$, we have

$$\lim_{z_1 \to 0^+} \frac{\partial \alpha_1}{\partial z_1} = \lim_{z_1 \to 0^+} \hat{\varrho} \kappa_1 \left( -3k_b^2 + 5z_1^2 \right) z_1^2 = 0 \tag{6.51}$$

Similarly, for $-k_a < z_1 < 0$, we obtain that

$$\lim_{z_1 \to 0^-} \frac{\partial \alpha_1}{\partial z_1} = \lim_{z_1 \to 0^-} \hat{\varrho}\kappa_1 \left(-3k_a^2 + 5z_1^2\right) z_1^2 = 0 \tag{6.52}$$

Hence, $\lim_{z_1 \to 0^+} \frac{\partial \alpha_1}{\partial z_1} = \lim_{z_1 \to 0^-} \frac{\partial \alpha_1}{\partial z_1}$, and we conclude that $\alpha_1(z_1, \cdot)$ is $C^1$ with respect to $z_1$. ■

**Step 2** This is the second and final step of the backstepping procedure, in which the control input $u$ appears. According to Lemma 6.4.1, the derivative of the stabilizing function $\alpha_1(\xi_0, \xi_1, \xi_2, y, \hat{\Theta}_1, \hat{\varrho}, y_d, \dot{y}_d)$ is well-defined, and can be computed as:

$$\begin{aligned}
\dot{\alpha}_1 \;&= \frac{\partial \alpha_1}{\partial \xi_0}(A_0\xi_0 + cy) + \sum_{i=1}^{2} \frac{\partial \alpha_1}{\partial \xi_i}(A_0\xi_i + \phi_i) + \frac{\partial \alpha_1}{\partial \hat{\Theta}_1}\Gamma_1\Psi_1 \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1 \\
&+ \frac{\partial \alpha_1}{\partial z_1}\left(\xi_{02} + \sum_{i=1}^{2}\theta_i\xi_{i2} + \vartheta v_2 - \tilde{\eta}_2 - \theta_1\bar{\phi}(y) - \dot{y}_d\right) \\
&- \frac{\partial \alpha_1}{\partial \hat{\varrho}}\gamma_\varrho \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0\bar{\alpha}_1 z_1 + \sum_{i=0}^{1}\frac{\partial \alpha_1}{\partial y_d^{(i)}} y_d^{(i+1)}
\end{aligned} \tag{6.53}$$

where $y_d^{(i)} := \frac{d^i}{dt^i}(y_d)$, and the partial derivatives are obtained as:

$$\begin{aligned}
&\frac{\partial \alpha_1}{\partial \xi_0} = -e_2^T\hat{\varrho}, \quad \frac{\partial \alpha_1}{\partial \xi_1} = -e_2^T\hat{\varrho}\hat{\Theta}_{11}, \quad \frac{\partial \alpha_1}{\partial \xi_2} = -e_2^T\hat{\varrho}\hat{\Theta}_{12}, \quad \frac{\partial \alpha_1}{\partial y_d} = \hat{\varrho}\hat{\Theta}_{11}\bar{b}(y), \\
&\frac{\partial \alpha_1}{\partial \hat{\Theta}_1} = -\hat{\varrho}\Psi_1^T, \quad \frac{\partial \alpha_1}{\partial \hat{\varrho}} = \bar{\alpha}_1, \quad \frac{\partial \alpha_1}{\partial \dot{y}_d} = \hat{\varrho}, \\
&\frac{\partial \alpha_1}{\partial z_1} = \hat{\varrho}\left[\hat{\Theta}_{11}\bar{b}(y) - 3\left(qk_b^2 + (1-q)k_a^2\right)\kappa_1 z_1^2 + 5\kappa_1 z_1^4\right]
\end{aligned} \tag{6.54}$$

with $\Theta_{1i}$ denoting the $i$-th element of $\Theta_1$, for $i = 1, 2$, and $\bar{b}(\bullet)$ defined in (6.7). From Lemma 6.4.1, we deduce that $\dot{\alpha}_1$ is continuous. Note that (6.53) can be written as the sum of two parts $F(\cdot)$ and $G(\cdot)$:

$$\dot{\alpha}_1 = F\left(\xi_0, \xi_1, \xi_2, z_1, \hat{\Theta}_1, \hat{\varrho}, y_d, \dot{y}_d\right) + G\left(\theta_1, \theta_2, \vartheta, \tilde{\eta}\right) \tag{6.55}$$

in which $F(\cdot)$ is known and can be directly cancelled by the control $u$, while $G(\cdot)$

contains unknown elements. The functions $F(\cdot)$ and $G(\cdot)$ are defined as follows

$$\begin{aligned}
F &= \frac{\partial\alpha_1}{\partial\xi_0}(A_0\xi_0+cy)+\sum_{i=1}^{2}\frac{\partial\alpha_1}{\partial\xi_i}(A_0\xi_i+\phi_i)+\frac{\partial\alpha_1}{\partial z_1}(\xi_{02}-\dot{y}_d)+\sum_{i=0}^{1}\frac{\partial\alpha_1}{\partial y_d^{(i)}}y_d^{(i+1)} \\
&\quad +\frac{\partial\alpha_1}{\partial\hat{\Theta}_1}\Gamma_1\Psi_1\left(\frac{q}{k_b^2-z_1^2}+\frac{1-q}{k_a^2-z_1^2}\right)\kappa_0 z_1-\frac{\partial\alpha_1}{\partial\hat{\varrho}}\gamma_\varrho\left(\frac{q}{k_b^2-z_1^2}+\frac{1-q}{k_a^2-z_1^2}\right)\kappa_0\bar{\alpha}_1 z_1 \\
&= \hat{\varrho}\omega-\gamma_\varrho\left(\frac{q}{k_b^2-z_1^2}+\frac{1-q}{k_a^2-z_1^2}\right)\kappa_0\bar{\alpha}_1^2 z_1 \qquad (6.56) \\
G &= \frac{\partial\alpha_1}{\partial z_1}\left(\sum_{i=1}^{2}\theta_i\xi_{i2}+\vartheta v_2-\tilde{\eta}_2-\theta_1\bar{\phi}(y)\right) \\
&= \frac{\partial\alpha_1}{\partial z_1}(\Theta_2^T\Psi_2-\tilde{\eta}_2) \qquad (6.57)
\end{aligned}$$

where

$$\begin{aligned}
\Psi_{1,a} &= \begin{bmatrix} c_2\xi_{11}+\xi_{02}\bar{b}(y) \\ c_2\xi_{21}+y \end{bmatrix}, \quad \Theta_2=\begin{bmatrix}\theta_1\\ \theta_2\\ \vartheta\end{bmatrix}, \quad \Psi_2=\begin{bmatrix}\xi_{12}-\bar{\phi}(y)\\ \xi_{22}\\ v_2\end{bmatrix} \\
\omega &= c_2(\xi_{01}-y)+\ddot{y}_d-\Psi_1^T\Gamma_1\Psi_1\left(\frac{q}{k_b^2-z_1^2}+\frac{1-q}{k_a^2-z_1^2}\right)\kappa_0 z_1+\hat{\Theta}_1^T\Psi_{1,a} \\
&\quad +\left[-3(qk_b^2+(1-q)k_a^2)\kappa_1 z_1^2+5\kappa_1 z_1^4\right](\xi_{02}-\dot{y}_d) \qquad (6.58)
\end{aligned}$$

This yields the derivative of $z_2$ as

$$\begin{aligned}
\dot{z}_2 &= -c_2 v_1+u+\varphi_2-F(\cdot)+\hat{\varrho}\left[\hat{\Theta}_{11}\bar{b}(y)-3\left(qk_b^2+(1-q)k_a^2\right)\kappa_1 z_1^2+5\kappa_1 z_1^4\right] \\
&\quad \times(-\Theta_2^T\Psi_2+\tilde{\eta}_2) \qquad (6.59)
\end{aligned}$$

Consider the Lyapunov function candidate

$$V_2^*=V_1^*+\frac{1}{2}z_2^2+\frac{1}{2\vartheta}\tilde{\eta}^T P\tilde{\eta} \qquad (6.60)$$

for which the derivative can be written as

$$\begin{aligned}
\dot{V}_2^* &= -\kappa_0\kappa_1 z_1^4+\left(\frac{q}{k_b^2-z_1^2}+\frac{1-q}{k_a^2-z_1^2}\right)\kappa_0(\vartheta z_1 z_2-z_1\tilde{\eta}_2)+z_2\dot{z}_2 \\
&\quad +\frac{1}{2\vartheta}\tilde{\eta}^T(A_0^T P+PA_0)\tilde{\eta}+\tilde{\eta}^T P\varphi \qquad (6.61)
\end{aligned}$$

Substituting (6.59) into the above equation yields:

$$
\begin{aligned}
\dot{V}_2^* &= -\kappa_0\kappa_1 z_1^4 - \frac{1}{2\vartheta}\tilde{\eta}^T R\tilde{\eta} + \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0(\vartheta z_1 z_2 - z_1\tilde{\eta}_2) + \tilde{\eta}^T P\varphi \\
&\quad + z_2\left(-c_2 v_1 + u + \varphi_2 - F^* + \frac{\partial\alpha_1^*}{\partial z_1}\tilde{\eta}_2\right) \\
&= -\kappa_0\kappa_1 z_1^4 - \frac{1}{2\vartheta}\tilde{\eta}^T R\tilde{\eta} + \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0\vartheta z_1 z_2 + z_2(-c_2 v_1 + u \\
&\quad + \varphi_2 - F^*) + \tilde{\eta}^T\left[P\varphi + e_2\left(\frac{\partial\alpha_1^*}{\partial z_1}z_2 - \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1\right)\right] \quad (6.62)
\end{aligned}
$$

where

$$
\begin{aligned}
F^* &= \frac{\partial\alpha_1^*}{\partial\xi_0}(A_0\xi_0 + cy) + \sum_{i=1}^{2}\frac{\partial\alpha_1^*}{\partial\xi_i}(A_0\xi_i + \phi_i) + \frac{\partial\alpha_1^*}{\partial z_1}\left(\xi_{02} - \dot{y}_d + \Theta_2^T\Psi_2\right) \\
&\quad + \sum_{i=0}^{1}\frac{\partial\alpha_1^*}{\partial y_d^{(i)}}y_d^{(i+1)} \quad (6.63)
\end{aligned}
$$

If the parameters were known, then it would be a straightforward affair to design the control as

$$
u = u^* := -\kappa_2 z_2 + c_2 v_1 - \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0\vartheta z_1 - \varphi_2^* + F^* \quad (6.64)
$$

where the correction term is chosen as

$$
\varphi^* := -P^{-1}e_2\left[\frac{\partial\alpha_1^*}{\partial z_1}z_2 - \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1\right] \quad (6.65)
$$

to cancel out the last term in (6.61), thus yielding

$$
\dot{V}_2^* = -\kappa_0\kappa_1 z_1^4 - \frac{1}{2\vartheta}\tilde{\eta}^T R\tilde{\eta} - \kappa_2 z_2^2 \quad (6.66)
$$

from which it is possible to show that the error signals $z_1$ and $z_2$ converge asymptotically to zero.

However, since the parameters $\theta_1$, $\theta_2$ and $\vartheta$ are actually unknown, the ideal control $u^*$ is not implementable. To circumvent this problem, $V_2^*$ is augmented with quadratic terms of the parameter estimation errors, so that we obtain the new Lyapunov function candidate $V_2$ as follows:

$$
V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2\vartheta}\tilde{\eta}^T P\tilde{\eta} + \frac{1}{2\gamma_\vartheta}\tilde{\vartheta}^2 + \frac{1}{2}\tilde{\Theta}_2^T\Gamma_2^{-1}\tilde{\Theta}_2 \quad (6.67)
$$

where $\tilde{\Theta}_2 = \hat{\Theta}_2 - \Theta_2$ is the estimation error for the unknown parameter vector $\Theta_2$. The derivative of $V_2$ is given by

$$\begin{aligned}\dot{V}_2 &= -\kappa_0\kappa_1 z_1^4 + \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0(\hat{\vartheta} z_1 z_2 - z_1\tilde{\eta}_2) + z_2\dot{z}_2 + \tilde{\Theta}_2^T\Gamma_2^{-1}\dot{\hat{\Theta}}_2 \\ &\quad + \frac{1}{2\vartheta}\tilde{\eta}^T(A_0^T P + PA_0)\tilde{\eta} + \tilde{\eta}^T P\varphi + \tilde{\vartheta}\left[-\left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1 z_2 + \frac{1}{\gamma_\vartheta}\dot{\hat{\vartheta}}\right]\end{aligned} \tag{6.68}$$

Substituting (6.59) and (6.32) into (6.68) yields

$$\begin{aligned}\dot{V}_2 &= -\kappa_0\kappa_1 z_1^4 - \frac{1}{2\vartheta}\tilde{\eta}^T R\tilde{\eta} + \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0(\hat{\vartheta} z_1 z_2 - z_1\tilde{\eta}_2) + \tilde{\eta}^T P\varphi \\ &\quad + \tilde{\vartheta}\left[-\left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1 z_2 + \frac{1}{\gamma_\vartheta}\dot{\hat{\vartheta}}\right] + \tilde{\Theta}_2^T\Gamma_2^{-1}\dot{\hat{\Theta}}_2 \\ &\quad + z_2\left[-c_2 v_1 + u + \varphi_2 - F(\cdot) + \frac{\partial\alpha_1}{\partial z_1}(-\Theta_2^T\Psi_2 + \tilde{\eta}_2)\right]\end{aligned} \tag{6.69}$$

For ease of design of the adaptation laws and the correction term, we rearrange the above equation into the form:

$$\begin{aligned}\dot{V}_2 &= -\kappa_0\kappa_1 z_1^4 - \frac{1}{2\vartheta}\tilde{\eta}^T R\tilde{\eta} + \tilde{\vartheta}\left[-\left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1 z_2 + \frac{1}{\gamma_\vartheta}\dot{\hat{\vartheta}}\right] \\ &\quad + z_2\left[u + \hat{\vartheta}\left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1 - c_2 v_1 + \varphi_2 - F(\cdot) - \frac{\partial\alpha_1}{\partial z_1}\hat{\Theta}_2^T\Psi_2\right] \\ &\quad + \tilde{\eta}^T\left[P\varphi + e_2\left(\frac{\partial\alpha_1}{\partial z_1}z_2 - \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1\right)\right] \\ &\quad + \tilde{\Theta}_2^T\left(\Gamma_2^{-1}\dot{\hat{\Theta}}_2 + \frac{\partial\alpha_1}{\partial z_1}\Psi_2 z_2\right)\end{aligned} \tag{6.70}$$

From (6.70), it can be seen that the last term containing the observation error $\tilde{\eta}$ may be eliminated by choosing the correction term $\varphi$ as:

$$\varphi = -P^{-1}e_2\left[\frac{\partial\alpha_1}{\partial z_1}z_2 - \left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1\right] \tag{6.71}$$

By designing the control and adaptation laws as follows:

$$u = -\kappa_2 z_2 + c_2 v_1 - \hat{\vartheta}\left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1 + F(\cdot) - \varphi_2 + \frac{\partial\alpha_1}{\partial z_1}\hat{\Theta}_2^T\Psi_2 \tag{6.72}$$

$$\dot{\hat{\Theta}}_2 = -\Gamma_2\Psi_2\frac{\partial\alpha_1}{\partial z_1}z_2 \tag{6.73}$$

$$\dot{\hat{\vartheta}} = \gamma_\vartheta\left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1 z_2 \tag{6.74}$$

and substituting (6.72)-(6.74) into (6.70), it can be shown that

$$\dot{V}_2 = -\kappa_0\kappa_1 z_1^4 - \kappa_2 z_2^2 - \frac{1}{2\vartheta}\tilde{\eta}^T R\tilde{\eta} \tag{6.75}$$

in which all three terms on the right hand side are always non-positive.

Since $u$ is an aggregate control variable defined for ease of analysis, we compute the actual voltage controls $V_f$ and $V_b$ by using the algorithm in (6.18).

**Remark 6.4.4** *It can be checked that* $u = u(y, v, \xi_0, \xi_1, \xi_2, \hat{\Theta}_1, \hat{\Theta}_2, \hat{\varrho}, \hat{\vartheta}, y_d, \dot{y}_d, \ddot{y}_d)$, *where the filter signals* $\xi_0(t), \xi_1(t), \xi_2(t)$ *are generated from* $y(t)$, *the signal* $v(t)$ *from* $u(t)$, *the parameter estimates* $\hat{\Theta}_1, \hat{\Theta}_2, \hat{\varrho}, \hat{\vartheta}$ *from* $y, y_d, \dot{y}_d, \xi_0, \xi_1, \xi_2$. *Therefore, the control* $u$ *is feasible based on only output measurement, and does not require the feedback of the state* $x_2$, *which is difficult to measure.*

**Theorem 6.4.1** *Consider the uncertain 1DOF electrostatic microactuator system (6.6) under Assumption 6.2.1, output feedback control law (6.72), and adaptation laws (6.48), (6.49), (6.73), and (6.74). If the initial conditions satisfy* $(x_1(0), x_2(0)) \in \bar{\Omega}$, *where*

$$\bar{\Omega} := \{(x_1, x_2) \in \mathbb{R}^2 \mid -k_a < x_1(0) - y_d(0) < k_b\} \tag{6.76}$$

*with* $k_a$ *and* $k_b$ *defined in (6.10), then the output tracking error with respect to any reference trajectory within the air gap, i.e.* $y_d(t) \in (-l_0 + \delta, l_0 - \delta)$, *is asymptotically stabilized, i.e.,* $y(t) \to y_d(t)$ *as* $t \to \infty$, *and all closed loop signals are bounded. Furthermore, the output* $y(t)$ *remains in the set* $\Omega_y := \{y \in \mathbb{R} : |y| \leq 1 - \delta/l_0\}$ $\forall t > 0$, *i.e. the output constraint is never violated.*

**Proof:** The proof for $y(t) \in \Omega_y$ $\forall t > 0$ is similar to that presented in Theorem 6.3.1 and is omitted. Next, we show that all closed loop signals are bounded. From (6.75), we know that $\dot{V}_2(t) \leq 0$ $\forall t > 0$, and thus, the error signals $z_1(t)$, $z_2(t)$, $\tilde{\Theta}_1(t)$, $\tilde{\Theta}_2(t)$, $\tilde{\varrho}(t)$, $\tilde{\vartheta}(t)$, and $\tilde{\eta}(t)$ are bounded. Since $\Theta_1$, $\Theta_2$, $\varrho$, $\vartheta$ are constants, we have that $\hat{\Theta}_1(t)$, $\hat{\Theta}_2(t)$, $\hat{\varrho}(t)$, $\hat{\vartheta}(t)$ are bounded. Since $|x_1(t)| < 1 - \delta/l_0$, we know, from the filters (6.29)-(6.30), that $\xi_i(t)$ $(i = 0, 1, 2)$ are all bounded.

Given that $\dot{y}_d(t)$ is bounded, the stabilizing function $\alpha_1(t)$ is also bounded, as seen from (6.46). This leads to the boundedness of $v_2(t) = z_2(t) + \alpha_1(t)$. According to

Lemma 2.4.1, the tracking error $z_1(t)$ remains in the set $-k_a < z_1 < k_b$. As such, the adaptation rates $\dot{\hat{\Theta}}_1(t)$, $\dot{\hat{\varrho}}(t)$, $\dot{\hat{\Theta}}_2(t)$, $\dot{\hat{\vartheta}}(t)$ in (6.48), (6.49), (6.73), (6.74) respectively, are all bounded. Furthermore, we can deduce, from (6.71), that $\varphi(t)$ is bounded. From (6.31), we have that $\dot{v}_1 = -c_1 v_1 + v_2 + \varphi_1$, which implies that $v_1(t)$ is also bounded. Thus, we infer that the control $u(t)$ in (6.72) is bounded. At the same time, from (6.33), $\hat{\eta}(t)$ is bounded, and thus, $\eta_2(t)$ and $x_2(t)$ are bounded too. We conclude that all closed loop signals are bounded.

To prove that $y(t) \to y_d$ as $t \to \infty$, we first establish that

$$\ddot{V}_2 \quad = \quad -4\kappa_1 z_1^3 \dot{z}_1 - 2\kappa_2 z_2 \dot{z}_2 - 2\tilde{\eta}^T R \dot{\tilde{\eta}}$$

is bounded, since $\dot{z}_1$ is bounded from (6.35), $\dot{z}_2$ is bounded from (6.59), and $\dot{\tilde{\eta}}$ is bounded from (6.34). As a result, $\dot{V}(t)$ is uniformly continuous. According to Barbalat's Lemma, $z_1(t), z_2(t) \to 0$ as $t \to \infty$. Since $z_1(t) = x_1(t) - y_d(t)$, it is clear that $y(t) \to y_d(t)$ as $t \to \infty$. ■

**Remark 6.4.5** *Although the adaptive control scheme in this chapter is developed for parallel plate microactuators, the same approach can be used for comb drive microactuators, as show in Figure 6.2, with a minor modification of the capacitance model to* $C(x) = \frac{2n\epsilon T}{d}(l + \bar{l})$ *[15], where $n$ denotes the number of movable fingers, $T$ the thickness of the structure, $d$ the gap between the fingers, and $\bar{l}$ the initial overlap between the electrodes. Consequently, the input is given by* $u = \frac{2n\epsilon T\beta}{d}(V_f^2 - V_b^2)$.

**Remark 6.4.6** *In our control design, we utilized more parameter estimates than the actual number of uncertain parameters. This is carried out mainly to simplify the design procedure and analysis, since any uncertain parameters encountered in each step is handled by a new set of estimates, even though the parameters appearing in different steps may be common. To avoid over-parametrization, it is feasible to employ the tuning functions approach, in which the number of parameter estimates is the same as that of the uncertain parameters, and the design of the adaptation is postponed until the final step. However, the design procedure and analysis will become more involved.*

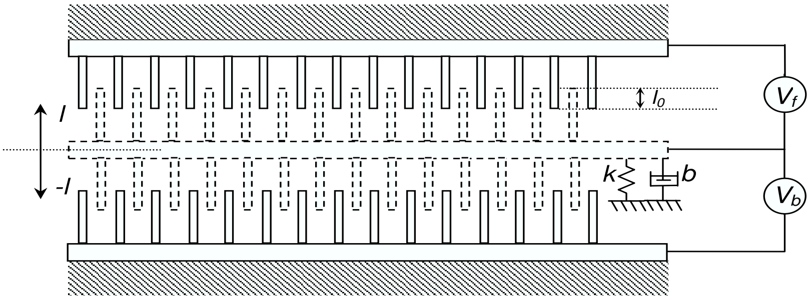


Figure 6.2: One-degree-of-freedom electrostatic comb drive

**Remark 6.4.7** *The possible rapid change of control voltages near the electrode surfaces can be viewed as a tradeoff from the ability of the controller to prevent electrode contacts in a relatively simple and robust way, particularly in face of model uncertainty and lack of velocity measurements. Since electrical dynamics are much faster than mechanical dynamics even in the micro scale, the plant model considered is still reasonable. If necessary, upper bounds for the rate of change of control voltages can always be computed for given design constants and initial conditions. From these estimates, the design constants and/or initial conditions can be appropriately selected to curb excessive rates.*

**Remark 6.4.8** *In practice, measurement noise may cause problems due to the high sensitivity near the barrier. A low pass filter can be employed to attenuate high frequency measurement noise. Furthermore, we propose to modify the barrier limits, $k_a$ and $k_b$, into the following:*

$$k_a' = \left(1 - \frac{\delta}{l_0} - |\underline{y}_d| - \Delta\right)^2, \qquad k_b' = \left(1 - \frac{\delta}{l_0} - |\overline{y}_d| - \Delta\right)^2 \tag{6.77}$$

*so as to provide for a safety margin $\Delta$, which accounts for measurement variance induced by noise. For small noise, we can reasonably expect that the filtered tracking error, denoted by $z_1'$ remains in the interval $(-k_a', -k_b')$. Then, for $|z_1'| \leq |z_1| + \Delta$, we expect that $z_1$ remains in the interval $(-k_a, k_b)$. In the subsequent section, we present simulation results to show that closed loop performance under these modifications are* *robust to small magnitude sensor noise.*

**Remark 6.4.9** *Although nonlinear squeeze film damping model (6.5) is considered in this chapter, the control design methodology is also applicable to linear damping models as a special case, for both full-state and output feedback cases.*

## 6.5 Simulation Results

To demonstrate the effectiveness of the control design, we perform simulations on plant (6.6), for both full-state feedback and output feedback cases, under the following choices of plant parameters values: $b_c = 2.659 \times 10^{-21} Nsm^2$, $k = 350.0 Nm^{-1}$, $m = 1.864 \times 10^{-11} kg$, $\epsilon = 8.859 \times 10^{-12} Fm^{-1}$, $A = 2.0 \times 10^{-8} m^2$, $l_0 = 1.0 \times 10^{-6} m$, $\delta = 2.0 \times 10^{-8} m$, and the scaling constants are chosen as $\sigma = 1.0 \times 10^6$ and $\beta = 2.0 \times 10^{17}$. The initial conditions are $x_1(0) = 0.0$, $x_2(0) = 0.0$, $\hat{\theta}_1(0) = 0.0$, and $\hat{\theta}_2(0) = 0.0$.

The performance of the proposed control is investigated for two types of tasks: set point regulation and trajectory tracking. For each task, the controller is required to ensure that the condition $-k_a < z_1 < k_b$ holds, thereby preventing electrode contact, i.e. $|x_1| < 0.98$.

For set point regulation, the movable plate is required to be stabilized at the specified set points $y_{si}$, i=1,2,3,4. Between the start position and each set point, the plate is to follow a reference trajectory $y_{di}(t)$ defined by:

$$y_{di}(t) = \begin{cases} y_0 + \left(6(\frac{t}{t_d})^5 - 15(\frac{t}{t_d})^4 + 10(\frac{t}{t_d})^3\right)(y_{si} - y_0) & \text{for } t \leq t_d \\ y_{si} & \text{for } t > t_d \end{cases} \tag{6.78}$$

where $y_0$ is the desired initial position, and $t_d$ is the time to reach $y_s$, starting from $y_0$. We simulate stabilization to four set points within the gap, namely $y_{s1} = -0.2$, $y_{s2} = 0.4$, $y_{s3} = -0.6$, and $y_{s4} = 0.8$, with each case starting from $y_0 = 0.0$. The duration is specified as $t_d = 100\ \mu s$. The bounds on $z_1$ corresponding to the set points can be computed as $\sqrt{k_{a1}} = 0.78$, $\sqrt{k_{b1}} = 0.98$, $\sqrt{k_{a2}} = 0.98$, $\sqrt{k_{b2}} = 0.58$, $\sqrt{k_{a3}} = 0.38$, $\sqrt{k_{b3}} = 0.98$, $\sqrt{k_{a4}} = 0.98$, and $\sqrt{k_{b4}} = 0.18$.

For trajectory tracking, the movable plate is required to follow the reference trajectory:

$$y_d(t) = 0.4(\sin(0.1t) + \sin(0.2t)) \tag{6.79}$$

from which it can be computed that $|\overline{y}_d|$=$|\underline{y}_d| = 0.705$. Thus, we have $k_a = k_b = 1 - \frac{0.02}{1.0} - 0.705 = 0.275$.

### 6.5.1 Full-State Feedback Control

According to (6.3), (6.16), and (6.18), the full-state feedback control law is given by

$$u = \frac{1}{\beta}\left(\frac{V_f^2}{l_0^2(1-x_1)^2} - \frac{V_b^2}{l_0^2(1+x_1)^2}\right)$$

where

$$\begin{aligned} V_f &= \sqrt{\beta l_0^2 q(\bar{u})(1-x_1)^2\bar{u}}\,, \qquad V_b = \sqrt{-\beta l_0^2(1-q(\bar{u}))(1+x_1)^2\bar{u}} \\ \bar{u} &= -\kappa_2 z_2 - \kappa_0\left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right) z_1 + \hat{\theta}^T\psi \end{aligned}$$

with the function $q(\bullet)$ is defined in (6.9). The control parameters are chosen as $\kappa_0 = 1.0\times 10^{-3}$, $\kappa_1 = 5.0\times 10^{7}$, $\kappa_2 = 1.0$, and $\Gamma$ =diag$\{70.0, 100.0, 50.0\}$.

For set point regulation, the simulation results are shown in Figures 6.3-6.5. From Figure 6.3, it can be seen that the movable electrode is successfully stabilized at each of the four set points, and does not come into contact with the fixed electrodes, whose positions are indicated by the grey lines. The tracking error for each case decays to a small value. From Figure 6.4, the boundedness and reciprocating action of the two control voltages are shown. Figure 6.5 shows that the velocity and parameter estimates are bounded.

Simulation results for the trajectory tracking are detailed in Figures 6.6-6.8. From Figure 6.6, it can be seen that the movable plate followed the sinusoidal trajectory closely, and successfully avoided contact with the electrodes. The tracking error $z_1(t) = x_1(t) - y_d(t)$ showed a trend of decreasing asymptotically to zero, while not violating the constraint $-0.275 < z_1 < 0.275$ during the transient response. From Figure 6.7, the boundedness and reciprocating action of the two control voltages are shown. Figure 6.8 shows that the velocity and parameter estimates are bounded.

### 6.5.2 Output Feedback Control

According to (6.3), (6.18), and (6.72), the output feedback control law is given by

$$u = \frac{1}{\beta}\left(\frac{V_f^2}{l_0^2(1-y)^2} - \frac{V_b^2}{l_0^2(1+y)^2}\right)$$

where

$$\begin{aligned} V_f &= \sqrt{\beta l_0^2 q(\bar{u})(1-y)^2\bar{u}}\ , \qquad V_b = \sqrt{-\beta l_0^2(1-q(\bar{u}))(1+y)^2\bar{u}} \\ \bar{u} &= -\kappa_2 z_2 + c_2 v_1 - \hat{\vartheta}\left(\frac{q}{k_b^2 - z_1^2} + \frac{1-q}{k_a^2 - z_1^2}\right)\kappa_0 z_1 + F(\cdot) - \varphi_2 \\ &\quad + \hat{\varrho}\left[\hat{\Theta}_{11} - 3\left(qk_b + (1-q)k_a\right)\kappa_1 z_1^2 + 5\kappa_1 z_1^4\right]\hat{\Theta}_2^T\Psi_2 \end{aligned}$$

with the function $q(\bullet)$ defined in (6.9), and the output $y = x_1$. The control parameters are chosen as $\kappa_0 = 1.0$, $\kappa_1 = \kappa_2 = 2.0$, $\Gamma_1 = \mathrm{diag}\{60.0, 10.0\}$, $\Gamma_2 = 10.0I$, $\gamma_\varrho = \gamma_\vartheta = 1.0$, $c_1 = 8.0$, $c_2 = 15.0$, and $R = I$.

For the task of set point regulation, the results are shown in Figures 6.9-6.11. From Figure 6.9, it can be seen that the movable electrode is successfully stabilized at each of the four set points without coming into contact with the electrodes. The boundedness of the control voltages, the velocity and parameter estimates are shown in Figures 6.10 and 6.11.

Results of simulation for sinusoidal tracking is shown in Figures 6.12-6.14. It can be seen in Figure 6.12 that the movable plate followed the sinusoidal trajectory closely without contacting the electrodes. The tracking error $z_1(t)$ decreased rapidly to a small value without violating the constraint $-0.275 < z_1 < 0.275$ during the transient response. From Figures 6.13 and 6.14 the boundedness of the control voltages, velocity and parameter estimates can be seen.

### 6.5.3 Measurement Noise

To test the effectiveness of the controller in the presence of sensor noise, we inject noise into the output, such that the measured signal is given by

$$y_m = y + n_a\mu(t) \tag{6.80}$$

where $\mu \in [-1, 1]$ is a random variable with uniform distribution, and $n_a$ is the noise magnitude. The raw signal $y_m$ is passed through a low pass filter $\frac{1}{1+2s}$, where $s$ is the Laplace variable, and the output of the filter, $y_f$, is then used in the estimation filters, adaptation laws, and control law. The barrier limits are modified according to (6.77) with $\Delta = 0.05$ so as to provide a safety margin that accounts for measurement variance induced by noise. This yields $\sqrt{k_a'} = \sqrt{k_b'} = 0.27$.

Simulation results for the output feedback tracking control are shown in Figures 6.15-6.17 for $n_a = 0.03$, $n_a = 0.06$, and $n_a = 0.1$, respectively. It can be seen that the effect of the controller is to minimize the filtered tracking error, $z_1' = y_f - y_d$, instead of the actual tracking error, $z_1 = y_- y_d$. As a result, the actual trajectory $y(t)$ fluctuates about the desired trajectory $y_d(t)$. As noise magnitude, $n_a$, increases, the actual tracking error also increases. The fact that $z_1'(t) \in (-0.27, 0.27)$ ensures that $z_1(t) \in (-0.275, 0.275)$, since $|z_1'| \leq |z_1| + 0.05$. This in turn ensures that the true position does not violate constraints, i.e. $|y(t)| < 0.98$.

**Remark 6.5.1** *From our simulation study, we found that the selection of design parameters affects the performance quite significantly. Trial and error tuning is needed to find a set of parameters that yield good performance in the two tracking scenarios studied in our simulation. For the full-state feedback case, there are six design parameters, namely* $\kappa_0$, $\kappa_1$, $\kappa_2$, *and* $\Gamma =$*diag* $\{\gamma_1, \gamma_2, \gamma_3\}$. *These are tuned by trial and error. For the output feedback case, there are considerably more design parameters, 16 in total, namely* $\kappa_0$, $\kappa_1$, $\kappa_2$, $\Gamma_1 =$*diag*$\{\gamma_{11}, \gamma_{12}\}$, $\Gamma_2 =$*diag*$\{\gamma_{21}, \gamma_{22}, \gamma_{23}\}$, $\gamma_\varrho$, $\gamma_\vartheta$, $c_1$, $c_2$, *and* $R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$. *To simplify the selection procedure, we first set* $R = I$ *and determine* $c_1$, $c_2$, *that give reasonable responses in filters (6.29)-(6.31). The remaining 10 parameters* $\kappa_0$, $\kappa_1$, $\kappa_2$, $\Gamma_1 =$*diag*$\{\gamma_{11}, \gamma_{12}\}$, $\Gamma_2 =$*diag*$\{\gamma_{21}, \gamma_{22}, \gamma_{23}\}$, $\gamma_\varrho$, $\gamma_\vartheta$ *are then tuned by trial and error.*

## 6.6 Conclusions

We have presented adaptive control for a class of single-degree-of-freedom (1DOF) electrostatic microactuator systems, such that the movable plate is able to track a reference trajectory within the air gap without knowledge of the plant parameters. Both full-state feedback and output feedback schemes have been developed, with guaranteed asymptotic output tracking. Simulation results show that the proposed adaptive control is effective for both set point regulation and trajectory tracking tasks. It can be seen from the control design that, in the adaptive setting, the output feedback treatment, which required the implementation of additional filters, became much more involved as compared to the full-state feedback case. If velocity measurements can be used, then full-state feedback control can be implemented with relative ease.

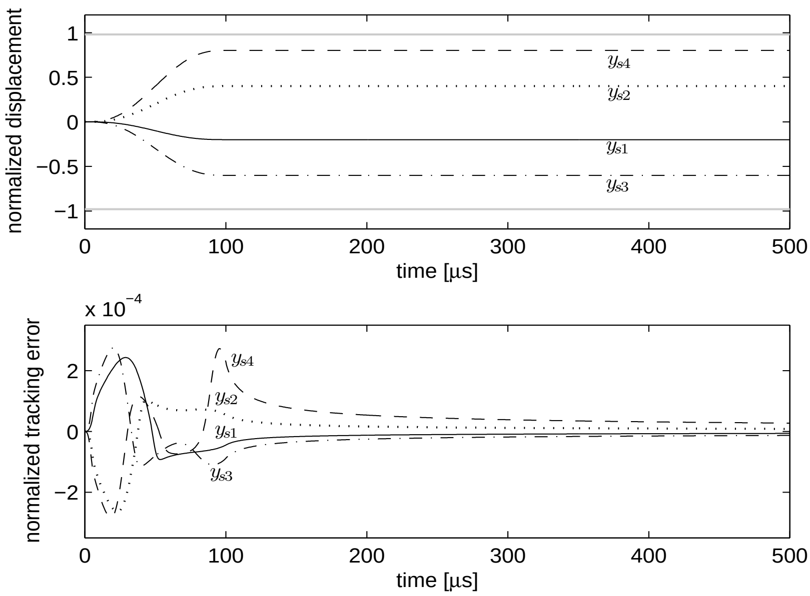


Figure 6.3: Normalized displacement $x_1$ and tracking error $z_1$

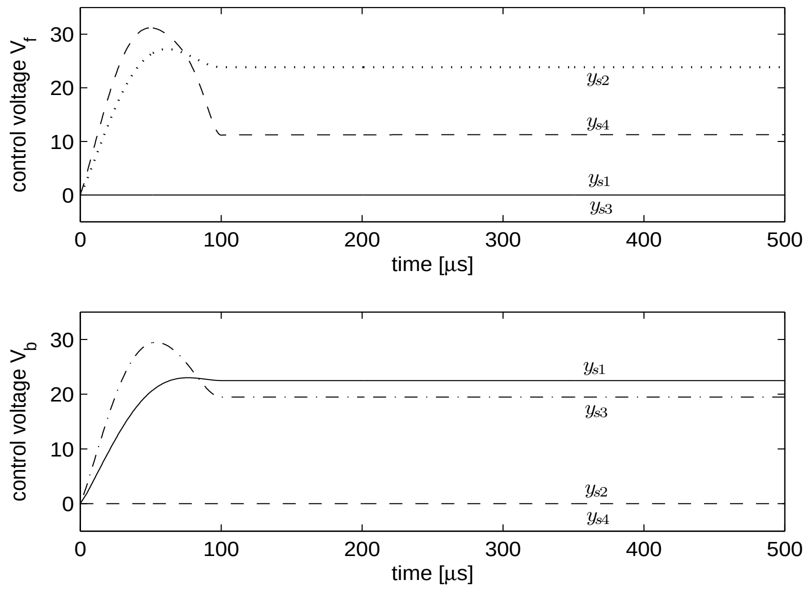


Figure 6.4: Control inputs $V_f$ and $V_b$

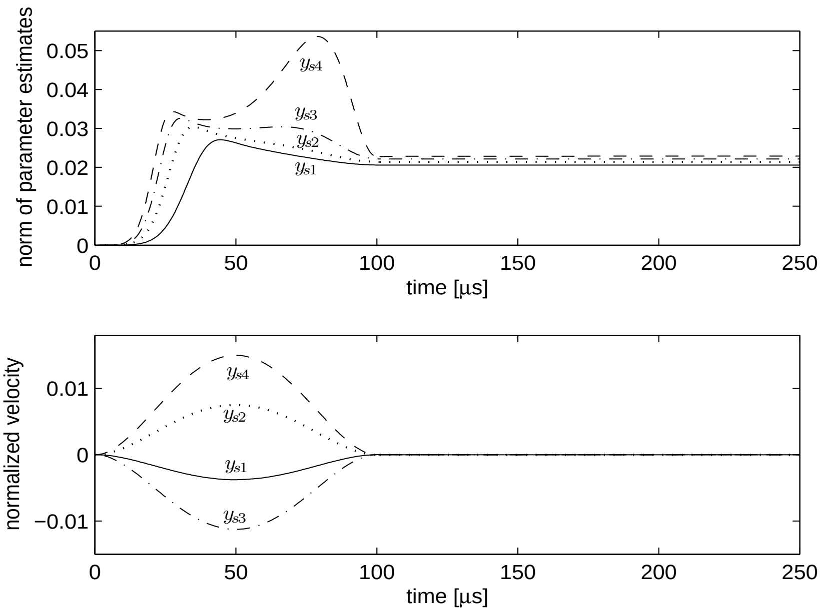


Figure 6.5: Norm of parameter estimates $\|\theta\|$ and normalized velocity $x_2$

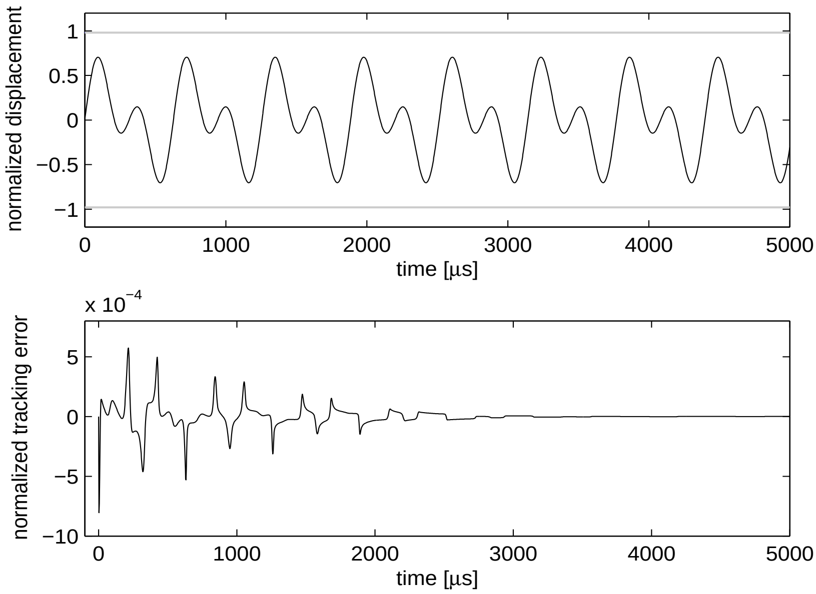


Figure 6.6: Normalized displacement $x_1$ and tracking error $z_1$

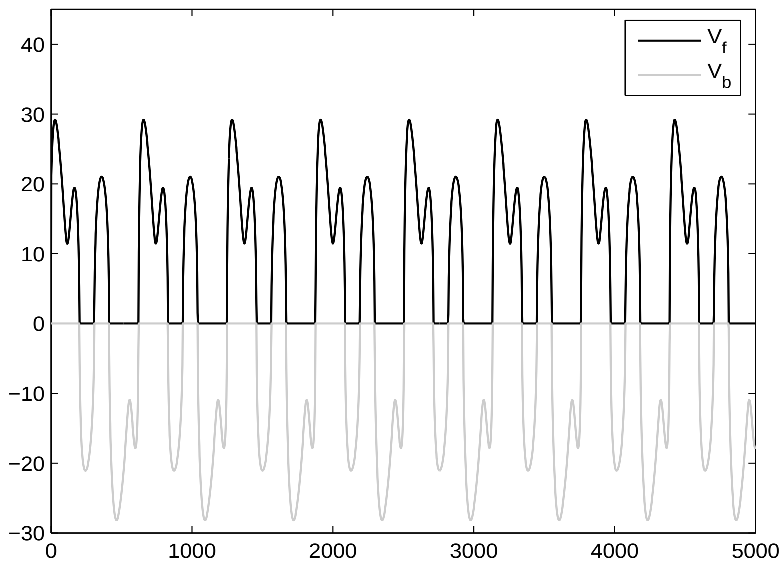


Figure 6.7: Control inputs $V_f$ and $V_b$

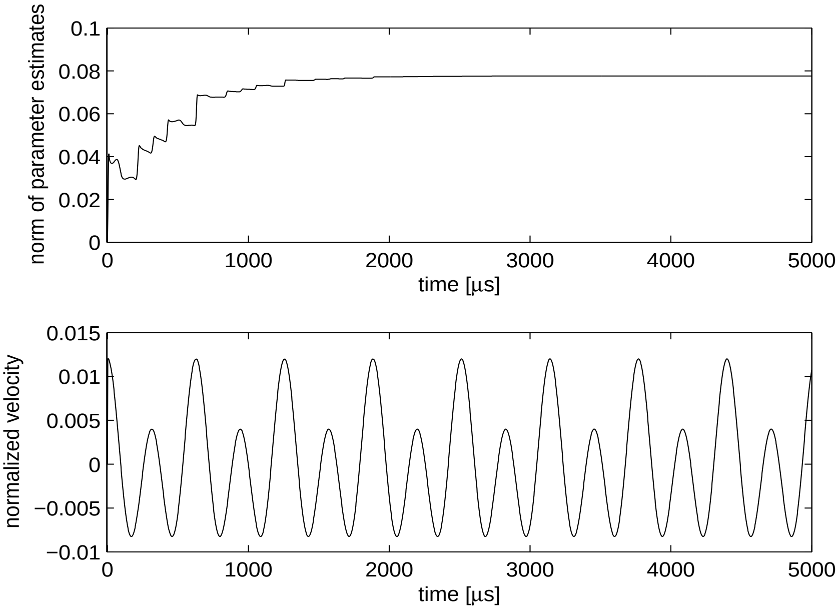


Figure 6.8: Norm of parameter estimates $\|\theta\|$ and normalized velocity $x_2$

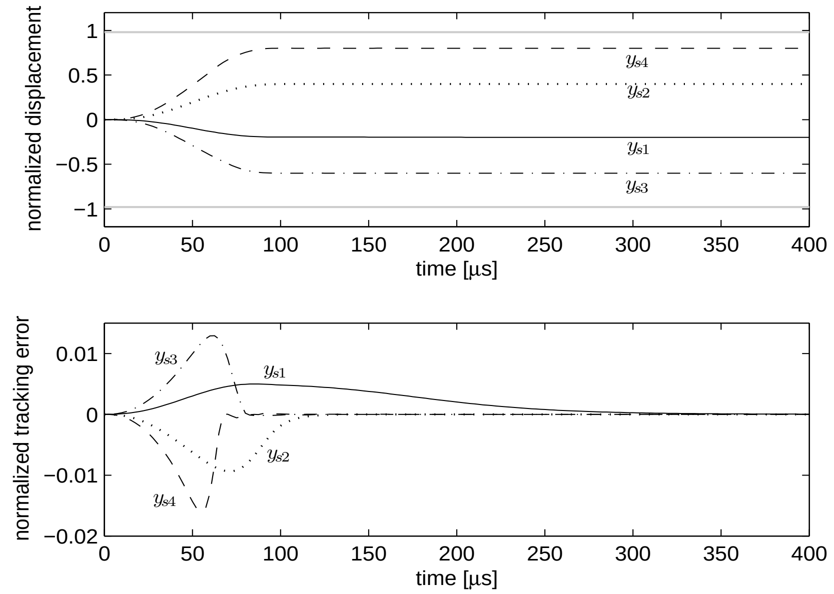


Figure 6.9: Normalized displacement $x_1$ and tracking error $z_1$

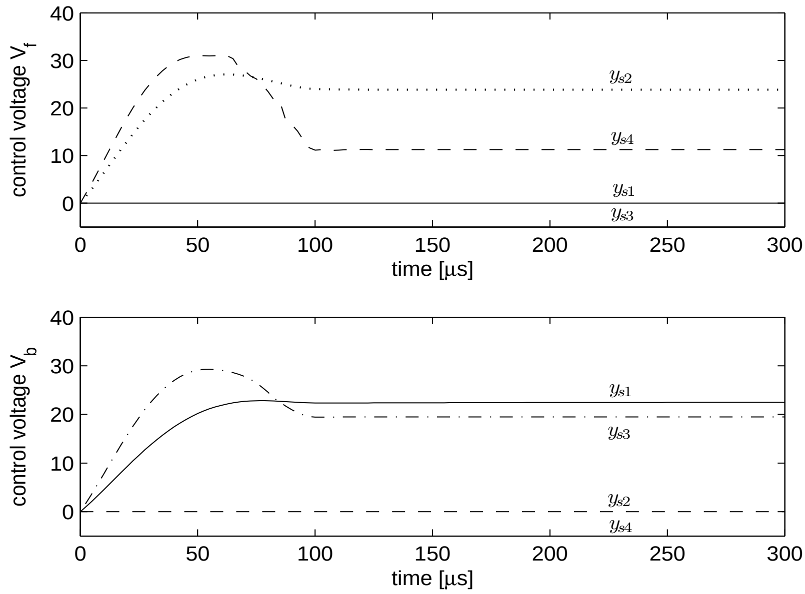


Figure 6.10: Control inputs $V_f$ and $V_b$

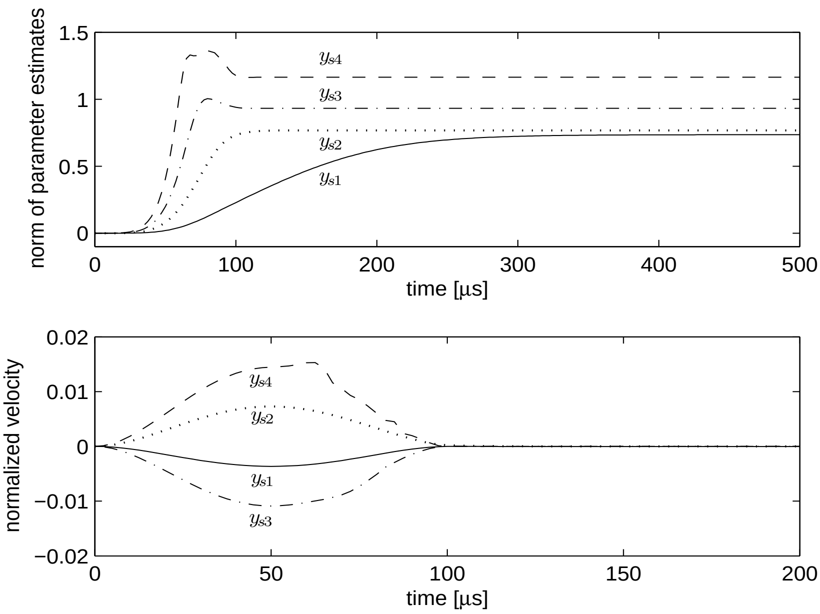


Figure 6.11: Norm of parameter estimates and normalized velocity $x_2$

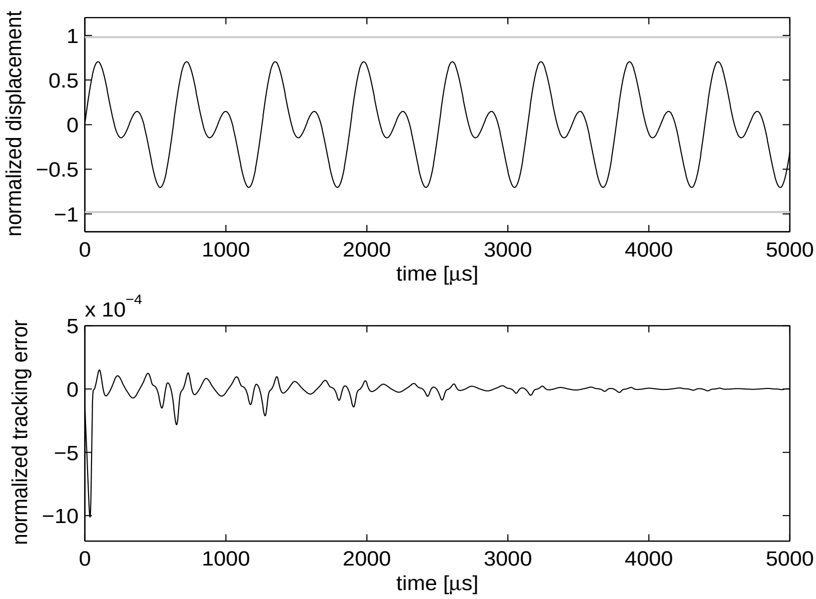


Figure 6.12: Normalized displacement $x_1$ and tracking error $z_1$

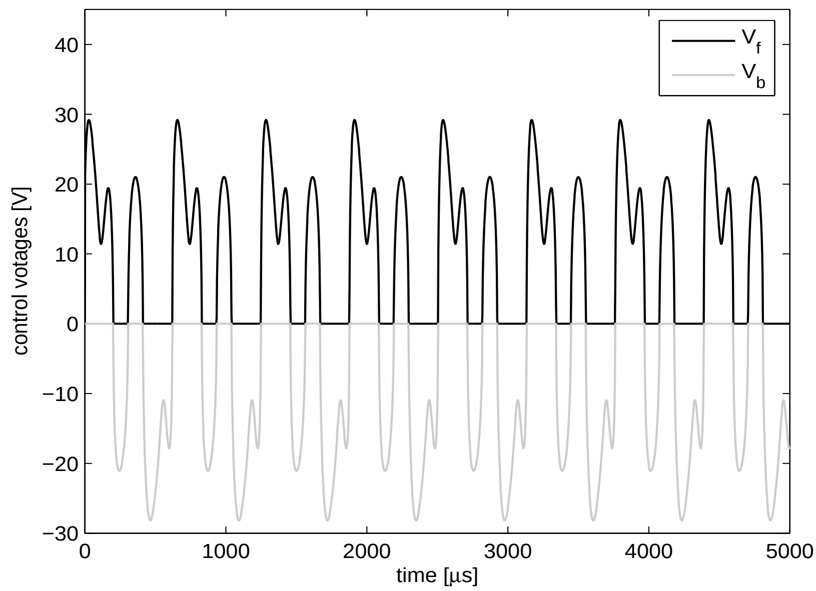


Figure 6.13: Control inputs $V_f$ and $V_b$

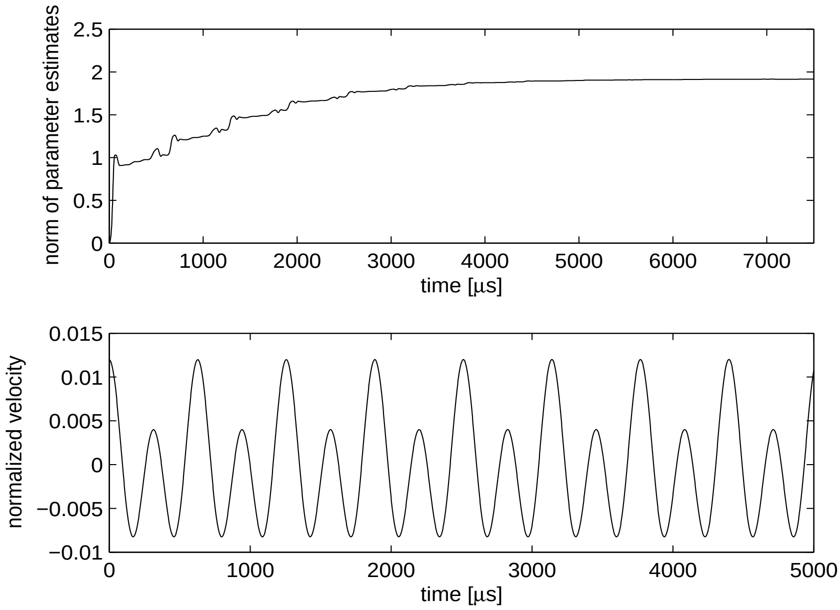


Figure 6.14: Norm of parameter estimates and normalized velocity $x_2$

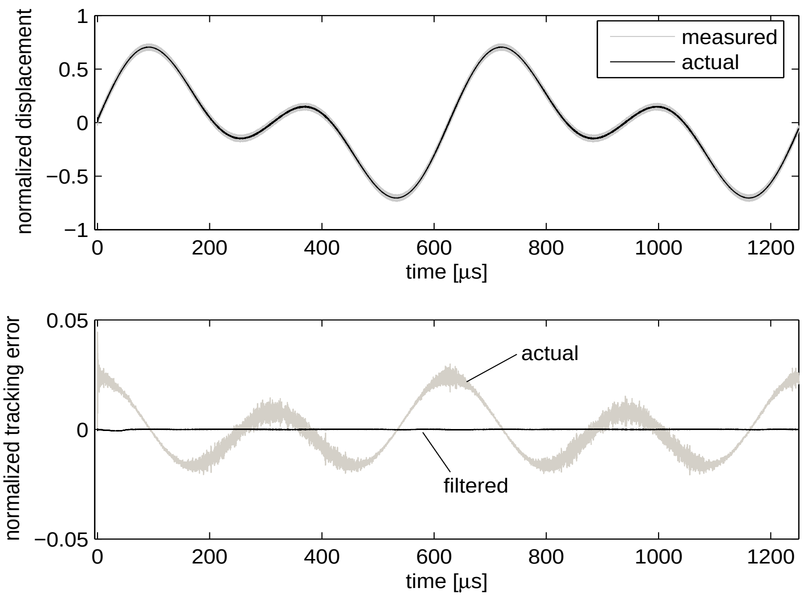


Figure 6.15: Normalized displacement and tracking error in presence of measurement noise with $n_a = 0.03$.

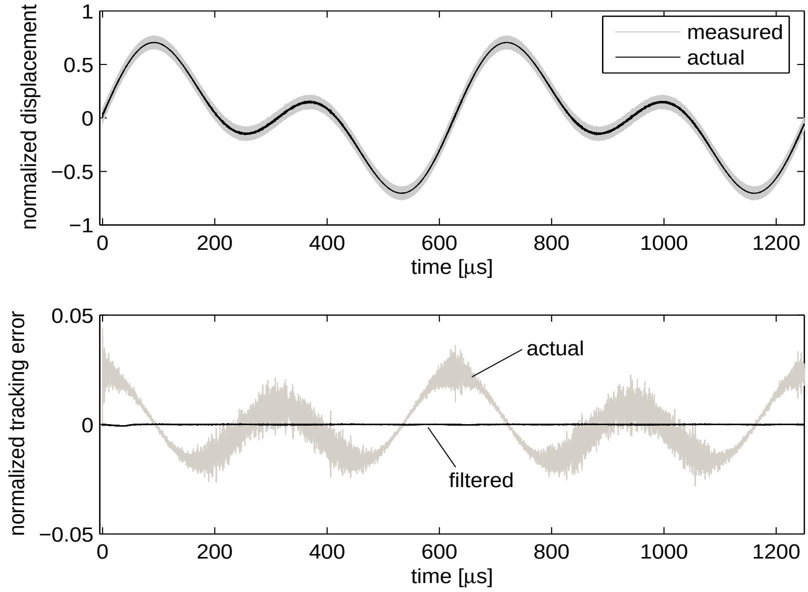


Figure 6.16: Normalized displacement and tracking error in presence of measurement noise with $n_a = 0.06$.

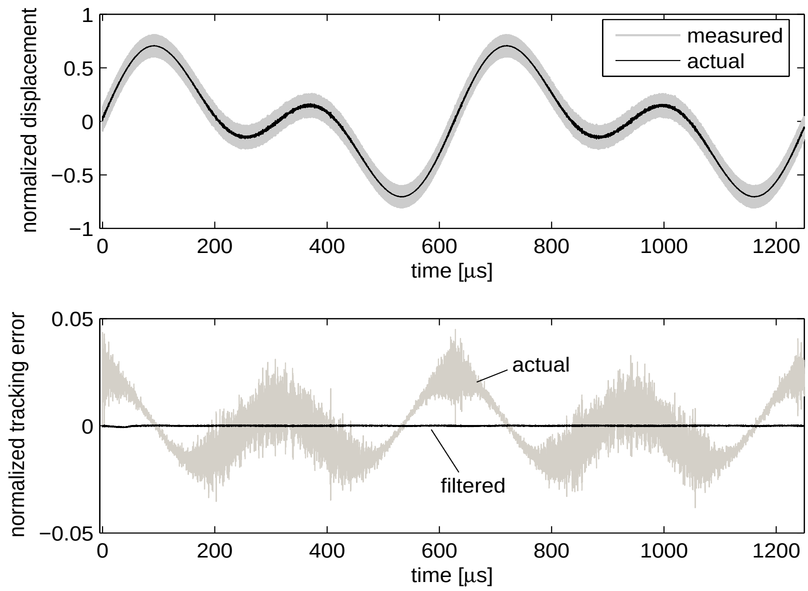


Figure 6.17: Normalized displacement and tracking error in presence of measurement noise with $n_a = 0.1$.

# Chapter 7

# Conclusions and Future Work

This thesis investigated the use of Barrier Lyapunov Functions for the control of SISO nonlinear systems in strict feedback form with constraints in the output and states. To begin with, the notion of Barrier Lyapunov Functions has been formally introduced to pave the way for a systematic and technically rigorous framework for control design that ensures non-transgression of constraints in nonlinear systems. Key technicalities underlying the use of BLFs for constraint satisfaction are exposed, and motivating examples based on low order systems are shown to elucidate the design methodology. While the idea of barrier functions as a means of preventing excursions of variables from a region of interest is not new, as noted by their applications in constrained optimization problems and collision avoidance algorithms, a formal treatment of barrier functions in Lyapunov synthesis is currently lacking in the control literature, and we endeavor to partly fill this gap in this thesis.

Following the preliminaries and motivating examples, tracking control design was presented for strict feedback systems with constraints on the output, and in the presence of parametric uncertainties. Both symmetric and asymmetric BLFs have been investigated, with the latter being a more generalized approach that can provide greater design flexibility and relax the starting conditions. We have shown that asymptotic tracking is achieved without violation of constraint, and all closed loop signals remain bounded, under a mild condition on the initial output. The use of QLFs in handling output constraint has been investigated, and it is shown that more conservative initial conditions are required as compared with using BLFs.

The BLF based control design is then extended to strict feedback systems with constraints on the states, with adaptive versions of the controllers designed to handle the presence of uncertainties. Unlike the output constraint case, some feasibility conditions which involve the initial states and selection of control parameters are required when dealing with full state constraints. Although they can be restrictive, due to the fact that they are based on conservative bound estimates, the good thing is that they can be checked offline prior to control implementation. When handling only partial state constraints, the conditions can be relaxed. It has been shown that asymptotic tracking is achieved without violation of constraint, and all closed loop signals remain bounded, provided that the feasibility conditions are fulfilled.

Subsequently, the thesis tackled the adaptive control problem for nonlinear constrained systems in strict feedback form with uncertain control gain functions, the latter being notorious for causing difficulties in adaptive control design. Although there are good methods in the literature for handling unknown control gains in the absence of constraints, such as Integral Lyapunov Functions [43] and quadratic-like Lyapunov functions with reciprocal of control gain function [44], these approaches are difficult to combine with BLFs for handling of constraints. In this thesis, we have adopted the robust adaptive domination approach of handling unknown virtual control gains. Based on the conditions for practical stability with guaranteed non-violation of constraints, which we have established in Lemma 2.4.3, it has been shown that practical output tracking is achieved without violation of output constraint. For the case of full state constraints, feasibility conditions on the initial states and control parameters are needed, which can generally be relaxed when handling only partial state constraints, and obviated for the special case of output constraint with linearly parameterized system nonlinearities.

To demonstrate the effectiveness of the proposed method of adaptive control design for constrained nonlinear systems, we have chosen, as an application study, a class of single-degree-of-freedom (1DOF) electrostatic microactuator systems, which is constrained in the sense that the movable electrode is to track a reference trajectory within the air gap without touching any of the fixed driving electrodes. Both full-state feedback and output feedback schemes have been developed, with guaranteed asymptotic output tracking. Computer simulation results show that the proposed adaptive control is effective for both set point regulation and trajectory tracking tasks. It can be seen from the control design that, in the adaptive setting, the output feedback treatment, which required the implementation of additional filters, became much more involved as compared to the full-state feedback case. If velocity measurements can be realized, then full-state feedback control can be implemented with considerable ease.

In light of existing methods in the literature for dealing with constraints in nonlinear systems, particularly Model Predictive Control, our proposed method has pros and cons. The main advantages are that there is no issue related to computational tractability since there is no need to solve optimization problem online, and that the feasibility of the control with respect to state constraints can be evaluated a priori. Shortcomings of the method include conservative feasibility conditions that limit the class of applicable systems, as well as the difficulty of handling an input constraint due to the high-gain nature of the control that uses the gradient of a barrier function.

**Recommendations For Future Work:**
Despite the existing applications of barrier functions in constrained optimization problems and multi-agent collision avoidance algorithms, the investigations of barrier functions in Lyapunov synthesis, in the form of BLFs, is relatively new in the context of providing a systematic framework of control design for general nonlinear systems. The focus of this thesis, on uncertain strict feedback systems with state and output constraints, is but a part of the wider scope consisting of numerous interesting and meaningful open research topics. In the following, we outline several possible topics for future investigations:

- **Constrained Input.** In this thesis, we have focused on output and state constraints, but neglected any consideration of constraints on the input. The reason is that provision for a potentially large control effort is key to safeguarding against any constraint transgression. This is an inevitable consequence of the design methodology, stemming from the use of BLFs that grow to infinity when the states approach the boundaries of the constrained region, and can be viewed as a drawback of the proposed method, although we have established, in

Theorem 3.3.1, the fact that the control signal remains bounded for all time. By careful selection of the control parameters, it is possible to limit the growth of the control signal within a desirable operating range. In fact, a straightforward extension will be to add one more condition to the full state constraint problem, namely $k_{c_{n+1}} > A_n \geq \sup_{\Omega_n} |u(\cdot)|$, where $k_{c_{n+1}}$ is the input constraint. This condition needs to be checked to assess feasibility of the proposed method, albeit with extra conservatism. More investigations are needed to relax the feasibility conditions and to find more effective ways to deal with input constraints.

- **Different Classes of Systems.** As a starting point in our research of BLFs, we have dealt with only strict feedback nonlinear systems in this thesis, which are sufficiently rich to elucidate the main principles and some for the problems associated with BLF based control design for constraint handling. Many more classes of systems in the presence of constraints, including pure feedback systems, time delay systems, mechanical systems, general MIMO systems, among others, carry with them unique and meaningful problems, and await to be investigated under the proposed BLF control design framework. For analytical purposes and performance assessment, BLF based control design can be applied to linear systems with constraints, and the results compared with existing results, such as those based on positively invariant sets.

- **Different Choices of BLFs.** The choice of BLF, as for any control Lyapunov function, is not unique, and different selections of BLFs can lead to different transient performance and stability properties. For an open region $\mathcal{D}$, any positive definite and continuously differentiable function, $V_1 : \mathcal{D} \to \mathbb{R}$, which satisfies the condition $V_1(z_1) \to \infty$ as $z_1 \to \pm k_{b_1}$ is a valid candidate. An alternative to (3.14) which satisfies these conditions is the barrier function $V_1 = \frac{k_{b_1}}{\pi} \tan^2(\frac{\pi z_1}{2k_{b_1}})$, which can be shown to yield very different stabilizing functions as well as control and adaptation laws. More classes of BLFs with desirable properties need to be proposed, and investigations and comparisons of control performances, induced by different classes of BLFs, are welcome.

- **Output Feedback Designs.** Although we have presented an output feedback design based on adaptive observer backstepping for the MEMs system in the application study of Chapter 6, this is a special case where the problem is tractable due to the low order dynamics and simplified system nonlinearities. Designing an output feedback control for nonlinear systems with guarantee of constraint satisfaction is very much an open and challenging problem. A natural and promising point to start is with nonlinear systems in the output feedback form under state and output constraints, since such systems are amenable to adaptive observer backstepping techniques that can be fused with BLFs.

- **Practical Applications.** As mentioned, constraints are ubiquitous in practical applications. There is plenty of scope for more in-depth application studies to be performed, including robotics manipulators in constrained workspace, ocean vessels moving in constrained channels, mechanical systems with saturated actuators, as well as process control applications with state constraints. Both computer simulations and experimental work need to be carried out extensively to verify the effectiveness and expose the limitations of the controllers, especially in the face of unmodelled dynamics, process disturbances, and measurement noise.

- **Approximation-Based Control.** Approximation-based control rely on universal approximation property in a compact set in order to approximate unknown nonlinearities in the plant dynamics. As long as the arguments of the unknown function remain within the set, stable tracking with guaranteed performance bounds can be achieved. One method of ensuring that the approximation condition holds is by careful selection of the control parameters, via rigorous transient performance analysis, so that the system states do not transgress the compact set of approximation [42, 44]. Another method is to rely on a sliding mode control mechanism operating in parallel to the approximation-based control, such that the compact set is rendered positively invariant [35, 178]. The BLF based control design methodology presented in this thesis appears very promising in providing yet another means of tackling the approximation-based control problem, by actively constraining the states of the system to remain within the compact set of approximation.

# List of Publications

The contents of this thesis are based on the following papers that have been published, accepted, or submitted to peer-reviewed journals and conferences.

1. K.P. Tee, S.S. Ge, and E.H. Tay, "Adaptive Control of a Class of Uncertain Electrostatic Microactuators", In: Proceedings of American Control Conference, New York, USA, July 11-13, 2007, pp. 3186-3191.

2. K.P. Tee, S.S. Ge, and E.H. Tay, "Adaptive Control of Electrostatic Microactuators with Bidirectional Drive", *IEEE Transactions on Control Systems Technology*, to appear in early 2009.

3. K. P. Tee, S. S. Ge, and E. H. Tay, "Barrier Lyapunov Functions for the Control of Output-Constrained Nonlinear Systems," *Automatica*, accepted in Oct 2008.

4. S. S. Ge, K. P. Tee, and E. H. Tay, "Control of State-Constrained Nonlinear Strict Feedback Systems Using Barrier Lyapunov Functions," *IEEE Trans. Automatic Control*, submitted.

5. K. P. Tee, S. S. Ge, and E.H. Tay, "Robust Adaptive Control of Uncertain Constrained Nonlinear Systems in Strict Feedback Form," *Automatica*, submitted.

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