PRODUCT LINE SELECTION, INVENTORY AND CONTRACTING FOR INTER-DEPENDENT PRODUCTS

HUA TAO

NATIONAL UNIVERSITY OF SINGAPORE

2009

PRODUCT LINE SELECTION, INVENTORY AND CONTRACTING FOR INTER-DEPENDENT PRODUCTS

HUA TAO

(*M.E., Management Science and Engineering, Tianjin University, China*

B.E., Project Management/ Commercial Laws, Tianjin University, China)

A THESIS SUBMITTED

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF DECISION SCIENCES

NATIONAL UNIVERSITY OF SINGAPORE

2009

ACKNOWLEDGMENT

First and foremost, I would like to express my sincere gratefulness to my supervisor, Dr. Mabel Chou for the mentorship and full support she has provided me throughout my graduate studies. I am also indebted to Professor Teo Chung-Piaw and Dr. Karthik Natarajam, who not only introduced me to the exciting research topic and techniques, but was also willing to take time out of their busy work to keep up with my progress. What I've learnt from them in the past five years, including passion and rigorous self-discipline for academic excellence and self-improvement, is a great benefit for my life.

I am also appreciative of Professor Sun Jie and Associate Professor Melvyn Sim for their patience and guidance, which led me through my first teaching experience. I am grateful to my thesis committee members, Professor Sun Jie and Dr. Lucy Chen for their valuable suggestions and comments, which make my study more complete and valuable.

My parents always gave me their unconditional love and support. And Kang Jian, you accompany me this long way, with continuous love, encouragement and care. You and Yiming gave me a sweet home in Singapore.

Last but not least, I would like to thank all my friends in NUS, who have made my life in NUS truly colorful and memorable.

CONTENTS

ABSTRACT

Product proliferation has become so common that most companies now offer hundreds, if not thousands, of stock keeping units (SKUs) in order to compete in the market place. High correlations may exist among the customers' utilities of these products due to the common attributes among them. These correlations may affect the demand for each product, which makes demand forecasts and production/inventory decisions even harder. Therefore, such correlations should be properly incorporated into the supply chain management to improve the profitability.

In the first part of this thesis, we develop a product line selection model in conjunction with a utility maximization model to describe the choice behavior of customers. Semi-definite Programming (SDP) is used to approximate the expected utility and the customer choice probabilities. The product line selection problem is then solved by incorporating the SDP approach with popular product swapping and greedy heuristics. With the ability to incorporate the correlation between products arising from common attributes in the choice behavioral model, this model successfully address the issue of Independence of Irrelevant Attributes (I.I.A.) property, which is an inherent limitation of the popular Multinomial Logit (MNL) model. We compare the performance of the new SDP model with the classic MNL based product line selection model in a simulated example. Our experimental results indicate that for both the buyer's welfare problem and seller's profit problem, our model can lead to better design of the product line, and can perform significantly better than MNL model, especially when the products share many common attributes.

In the second part, we extend the above work to include the inventory decisions. We embed our Cross Moment Model into the assortment and inventory joint decision problem for retailers, and focus on comparing the resulting offer set and inventory levels decision with those decision under classic MNL choice models. We also quantify the improvement of the total expected profits through Monte Carlo simulation. We found that under static substitution, less correlated products set can bring more profit. We also show that the total varieties of products can be reduced under dynamic substitution. And through simulation, considerable improvement in expected profits result from taking account of utilities' correlations.

The third part of this thesis analyzed how flexibility in order quantity created by using options in a supply contract affects the payoffs of the manufacturer and the retailer as well as their joint payoff. We examine the impact of reorder options in a single-product case and further compare the differences between pooled and non-pooled options in a multi-product case. While reorder options seem to offer the retailer more flexibility, we find that in some cases the retailer may end up with a lower payoff. For multi-product cases, we identify some conditions where pooled and non-pooled option contracts may provide the same payoff, and other conditions where one can be higher than the other.

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

Product proliferation has become so common that most companies now offer hundreds, if not thousands, of stock keeping units (SKUs) in order to compete in the market place. It has been identified as a critical strategy to compete in today's business world since it benefits the consumers by meeting diversified preferences, improving their satisfaction, and consequently stimulating the sales. On the other hand, product proliferation can also lead to negative consequences such as customer confusion, cost increases, inventory imbalances, product stock-outs, and cannibalization. For this reason, it is important for a company to understand consumer choices so that it can better predict customers' demand, which enables the company to better balance the breath and depth of the components of its product lines.

Besides product line decisions, controlling inventory costs is also important in managing such a multiple-product supply chain. These decisions can be very complicated since they involve allocating limited resources among various products, whose demand may be interdependent such as substitutable or complementary goods.

There is a vast literature on consumer choice models. As we will explain in Section 1.1 and Section 1.2, the existing models either do not model product interdependence or are computationally tedious. Therefore, in this thesis, we aim to propose a computationally-efficient model which captures consumer choices for interdependent products and incorporate this model into supply chain decisions including product line selection and inventory planning. We also study how contracts between manufacture and retailer will affect supply chain efficiency when we face multiple interdependent products.

1.1 Consumer Choice Models

In this section, we briefly discuss choice models in general followed by two widely adopted stochastic choice models in the literature: multinomial logit (MNL) model and locational model. Through this brief discussion, we will explain why we are motivated to propose a new method "Cross Moment Model (CMM)" which will be presented in Chapter 2. More detailed literature will be presented in Section 1.2.

1.1.1 General Choice Modelling Methods

According to Mahajan and van Ryzin [39], there are two generic approaches for modeling choices: (1) construct preference relations directly, or (2) construct utilities and then apply utility maximization. They showed that approach (1) is essentially equivalent to approach (2).

To construct preference relations directly, it typically consists of modelling attributes of each alternative and specifying a ranking rule. The key advantage of attribute models of choice is that consumer preferences can be linked directly to attributes of a firm's products. Therefore, this approach is well suited to operations management problems involving product design or product positioning, since the firm can have control over the design features of its products.

On the other hand, if the product design decisions are not so much concerned, the decision maker can directly focus on the utility values of each product. Utility models are more naturally suited to problems of product selection.

The distinction between attribute and utility models, however, is not entirely sharp. Indeed, one frequently used transportation choice model relate attributes to utilities directly. That is the *linear in attributes* utility model(Ben-Akiva and Lerman[5]), in which the utility is expressed as a linear function of a product's attributes. We will demonstrate later that our CMM model actually adopts this *linear in attributes* utility model to take into account of the utilities' correlations among products with certain common attributes.

1.1.2 Multinomial Logit Model

The multinomial logit model (MNL) is the most popular random utility model. Instead of assigning deterministic utilities for the products, MNL assumes a probability distribution for the consumer's utility on a specific product j. Specifically, for product j, its utility $\tilde{U}_j$ is equal to the utility mean V_j, plus a random error term ϵ_j, where the error terms are independent and identically distributed (iid) Gumbel random variables:

$$\tilde{U}_j = V_j + \epsilon_j$$

. Given an offer set Y, when the i.i.d. Gumbel error terms have mean zero and scale parameter β, the probability that a given individual chooses product j within set Y is given by:

$$P_Y(j) = \frac{e^{\beta V_j}}{\sum_{k \in Y} e^{\beta V_k}}. \tag{1.1}$$

Note that variance for a Gumbel distributed random variable is $\frac{\pi^2}{6\beta^2}$, where π is the ratio of circumference of a circle to its diameter. MNL model predicts customer choice based on (1.1). A well-known result related to the expected maximum utility that can be achieved under MNL model is given in the following:

$$E(max_j(\tilde{U}_j)) = \frac{1}{\beta} ln \sum_{j \in Y} e^{(\beta V_j)}$$

Note that the MNL model suffers from the Independence of Irrelevant Alternatives (IIA) property: the ratio of choice probabilities for any two alternatives is unaffected by the presence of other alternatives.

1.1.3 Locational Model

Locational model was studied in Lancaster [32]. Suppose there are n products located along the interval [0,1], which is called the “attribute space”. Denote the location of product j as l_j, and denote the customer t's “ideal point” as L^t, which can be a random variable. Then the utility of product j for customer t is given by: $U_j^t = a - b \parallel L^t - l_j \parallel$, where a specifies the utility of a product that exactly matches the customer's ideal point and b measures how fast the utility declines with the deviations from the ideal point. It uses a distributional assumption on the customer ideal points L_t to capture the randomness of the utility.

Certain correlations among utilities of different products in locational model can be captured, hence the Independence of Irrelevant Alternatives (IIA) property of MNL model can be tackled to some extent. However, we noticed that it is not easy to directly quantify and specify that correlation in high-dimension attributes' case. And also due to the difficulties from high-dimension integral, the current literature on assortment and inventory management is restricted to the study on one-dimension attribute locational model.

1.2 Literature Review

In this section we will review the related literature from three aspects: Section 1.2.1 focuses on product line selection articles; Section 1.2.2 reviews literature that integrated product line selection and inventory decisions; Section 1.2.3 is from the contract coordination perspective since we will study in Chapter 4 how contracts between manufacturer and retailer affect supply chain efficiency when we face multiple products which are interdependent.

1.2.1 Related Literature on Product line Selection and Pricing

The product line selection problem has been the focus of numerous research articles in the past two decades ([4, 2, 18]). A fundamental issue is the modeling of the random utility functions. The simplest model assumes a linear function to approximate the utility with:

$$U_i(\mathbf{z}) = \mathbf{x_i} \cdot \beta(\mathbf{z}) + \epsilon_{\mathbf{z},i}, \tag{1.2}$$

The vector $\mathbf{x_i}$ describes the observable attributes of product i and $\beta(\mathbf{z})$ is a vector of weights attached to each attribute of the product. The random term $\epsilon_{\mathbf{z},i}$ denotes the corresponding error term associated with this approximation. Each consumer is assumed to choose the product in the product line with the highest utility.

In the simple first choice approach, both $\beta(\mathbf{z})$ and $\epsilon_{\mathbf{z},i}$ are assumed to be completely deterministic. Each customer thus goes for the product with the highest deterministic utility (cf. Green and Krieger [23], McBride and Zufryden [41], Dobson and Kalish [17, 18], Kohli and Sukumar [29]). Product line selection models using the first choice assumption are shown to be NP-hard, and the research community has focussed on devising sophisticated heuristic approaches such as the Genetic Algorithm [23] or the Beam search heuristic [44]. Complete enumeration can serve as a benchmark to evaluate the performance of candidate heuristics. Recently, Camm et al. [12] proposed an exact branch-and-bound algorithm to solve the share-of-choice single product design problem to optimality. Wang et al. [61] extends that of Camm et al. [12] to obtain the optimal integer solution for the share-of-choice product line design problem.

Although it seems straightforward, the first choice assumption tends to exaggerate the market share of popular products and underestimate the share of unpopular products [55]. To rectify this bias, probabilistic choice models have been incorporated into the product line selection problem. These models typically satisfy Luce's Axiom (cf. [36]): the choice probability for product i is given by

$$\frac{v_{\mathbf{z},i}}{\sum_k v_{\mathbf{z},k}}, \tag{1.3}$$

where $v_{\mathbf{z},i}$ is the customer's ratio-scaled preference value or utility for product i. Among these models, the Multinomial Logit ($v_{\mathbf{z},i} = e^{\lambda \mathbf{x}_i \cdot \beta(\mathbf{z})}$ for some constant λ) is currently the most popular method used in modelling the consumer's choice probabilities (see Aydin and Ryan [2]).

Hanson and Martin [24] were arguably the first to systematically study the MNL based product line selection and pricing problem. They discussed the difficulty of the MNL based profit optimization problem in view of the fact that MNL converges to the first choice rule as the utility measurement errors go to zero. They proposed an efficient path-following heuristic to solve the non-concave seller's profit maximization problem. In their formulation, all the products are assumed to be offered and decisions are only made on the price vector. Chen and Hausman [13] discretized the product prices and relaxed the resulting mixed integer program into a quasi-concave nonlinear program based on the objective's special structure. They constrained the number of launched products within a certain range, so that the product line decisions can be made simultaneously with the pricing decisions. However, as noted by Kraus and Yano [31], their lower bound on the number of products is redundant and the problem can be viewed as choosing a fixed number (equal to the upper bound) of products and deciding their associated prices. Aydin and Ryan [2] built three basic models based on the MNL rule: new product offering choice and pricing model, optimal pricing of given products and eventually the pricing and product selection joint optimization problem. Hopp and Xu [25] incorporated the product development cost and focused on the value of modularity design. They used one-dimensional measurement "quality" to differentiate the products. The customers are further restricted to come from a homogenous population.

All the above choice models suffer from the Independence of Irrelevant Alternatives (IIA) property: the ratio of choice probabilities for any two alternatives is unaffected by the presence of other alternatives. These models, including the MNL model, tend to exaggerate the market share of similar products, or products with many common characteristics. The issue of correlation in utility evaluation can be addressed using the GEV (Generalized Extreme Value) models discussed in McFadden [42]. This family of models includes the (generalized) nested logit, pair-combinatorial logit, and various other models as special cases. These models have the advantage that the choice probabilities have a closed form expression (as in the MNL model), but suffers from the fact that the dependence structure in the error terms is extremely complex. The approach is also more suitable when the products are well specified, but not suitable when the product set is itself a decision (as in the product line selection problem).

The Probit model is another popular approach used in place of MNL. However, the computational burden associated with choice probability computation (involving multi-dimensional integrals or simulations) has limited its applicability in practice. Several authors have built on this model to propose choice models to capture the interdependency among the alternatives. Clark [14] and Daganzo et al. [15] proposed numerical approximations for the choice probabilities for normal variates, building on an approximation method for pairs of normal random variables. Kamakura and Srivastava [27] overcame this issue by approximating the covariance matrix using two parameters and a proximity measure, whereas Dalal and Klein [16] proposed the generalized logit model, and reduced the computational burden to one over a much smaller type space. However, due to computational complexity and implementation difficulties, to the best of our knowledge, none of these approaches have been applied into the product line selection problem.

Steenburgh [56] recently noticed that many of the popular consumer choice models described above (including MNL, GEV and Probit models) suffer from an additional limitation known as the Invariant Proportion of Substitution (IPS) property. Namely, the shares that product i draws from product k does not depend on which attribute in i is changed, but only on the net change in $\mathbf{x_i} \cdot \beta(\mathbf{z})$. More generally, he showed that if the utility $U_i(\mathbf{z})$ for product i can be decomposed as deterministic component $v(\mathbf{x_i}, \mathbf{z})$ and random noise $\epsilon_{\mathbf{z},i}$ which is independent of the attribute vector $\mathbf{x_i}$, and the choice probability for product i depends on the attribute only through the deterministic component $v(\mathbf{x_i}, \mathbf{z})$, then the IPS property holds. This property is undesirable as we expect that if a product k is more similar to product i in attribute a than attribute a', then the change in choice probability for product k will be more substantial if attribute a in product i is improved, compared to improvements in attribute a' in product i.

Due to the intrinsic limitation of these models, Sawtooth [55] suggested using a "randomized first choice" rule, with added random perturbations to the utilities of each feature and overall products for each individual customer. The market shares were estimated by running multiple iterations of simulation. This method avoids the drawbacks of MNL, but significantly increases the computational time.

In this thesis, we attempt to propose a new choice model (see Cross Moment Model (CMM) in Chapter 2) and apply it to the product line selection and inventory planning problem. The attractiveness of this stochastic choice model is its capability in correcting those undesirable properties of the MNL model and at the same time maintaining a reasonable computational complexity.

1.2.2 Related Literature on Product Line Selection and Inventory Control

Research on retailer's assortment planning and inventory management has advanced rapidly in recent years. One of the most prominent progress is the incorporation of individual-level consumer choice theory from the marketing literature into the modelling of substitution between products. Among them, the MNL model and locational choice model are the most commonly adopted consumer choice models. We will first summarize the literature on assortment planning and inventory management using the traditional exogenously specified model and these two consumer choice models.

Traditionally, exogenous modelling of the demand substitution is most commonly adopted in the literature on inventory management for substitutable products. See McGillivray and Silver [43], Parlar and Goyal [48], Noonan [46], Parlar [47], Wang and Parlar [60], Rajaram and Tang [51], Ernst and Kouvelis [21], Smith and Agrawal [54], and Netessine and Rudi [45]. In these models, distribution of random demand for each product is assumed to be exogenous, and when demand realization exceeds the stocking quantity of a particular product, the ratio of the excess demand to be re-allocated to other products is also assumed to be exogenous. Unsatisfied re-allocated demand is lost. It is also named as Markovian Second Choice in Mahajan and van Ryzin [38].

The advantage of exogenous substitution is in its ability to differentiate the substitution between different product categories by specifying different substitution rates for them. However, since there is no underlying consumer behavior such as a utility model to generate the demands or to explain the substitutions, for tractability, most of the exogenous model only allows one time substitution and need to stipulate a fixed substitution rate by the decision maker. It is also hard to incorporate marketing variables such as prices and promotions into this choice model.

Application of consumer choice model to capture the demand substitution has advanced rapidly in recent years. When first choice product is unavailable, certain degree of substitution can be implied by the consumer choice model through their parameters, instead of postulated by decision makers.

van Ryzin and Mahajan [53] were the first to study assortment planning and inventory decisions under the MNL model. They defined the so-called *static* substitution, where the customer's choice is affected by the set of variants offered in the assortment, but not by the current inventory levels. Static substitution assumption simplifies the resulting inventory and variety analysis, yet generates many interesting managerial insights. However, it is a somehow unsatisfying assumption, especially for those products such as grocery items, soft drinks, etc., where consumers substitute readily when products are out of stock. Aydin and Ryan [2] also apply the MNL model to study the joint assortment planning and pricing problem under static substitution. They built three basic models based on the MNL rule: new product offer choice and pricing model, optimal pricing of given products, and the pricing and product selection joint optimization problem. They found that optimal solutions have equal profit margins for all the offered products.

Dynamic substitution under MNL choice model is much more complicated and first studied in Mahajan and van Ryzin [38], where substitution times and orders are totally determined by the customers' utilities when stock out. They proved the non-concavity of the total expected profit in each product's inventory level, and proposed a sample path gradient algorithm to find the stationary points. They used the MNL and locational model in their numerical examples to predict the real demand. In Chapter 3, we will adopt a model setting which is similar with the one in Mahajan and van Ryzin [38], but imbed our Cross Moment Model (CMM) to characterize the consumer choice. We will also examine a pooled newsboy algorithm to quantify the effects from dynamic substitution.

Vishal and Honhon [22] also used the locational choice model in their paper. They incorporate the locational choice model (Hotelling [26], Lancaster [32]) to capture the dynamic substitution of customer demands. The locational model rectifies the Independence of Irrelevant Alternatives property inherited in the MNL model. However, to remain tractable, only one dimension of attribute was handled for locational model, whereas in our cross moment model (CMM), multi-dimension of attributes can be easily handled. Besides, the randomness of customer choice is limited in this paper for locational model to certain distribution. In contrast, we don't impose such assumption in our CMM model. Our CMM model actually is capable of handling multiple dimensions of differentiation in products' selection and factor in the utilities correlations among the products in the offer set.

Most recently, Maddah and Bish [37], Tang and Yin [57] and Dong et al.[19] incorporate both selling price and production quantity decisions into the product line selection framework. For further research on empirical and analytical models on assortment planning with consumer choice, we refer the readers to an extensive literature review by Mahajan and van Ryzin [39] and more recently by Kok et. al. [30].

1.2.3 Related Literature on Flexible Contracts

In recognition of channel coordination, many extensive studies have centered on the design of coordinating contracts in achieving system optimal performance. These include nonlinear pricing (e.g. two-part tariff pricing, quantity discounts) (Lee [35]), return policies (buy-backs) (Pasternack [49]), backup agreements (Eppen and Iyer [20]), quantity-flexible contracts (Tsay [59]), revenue sharing contracts (Cachon [11]) and pay-to-delay arrangements. Extensive reviews of the supply contracts literature include Anupindi and Bassok [1], Lariviere [34], Tsay et al. [58], and recently Cachon [9]. Compared as a benchmark, Lariviere and Porteus [33] also study price-only contracts, where they identify the coefficient of variation as the key element affecting channel efficiency. Cachon [10] shows the combined use of push and pull price-only contracts in achieving high channel performance.

Ritchken and Tapiero [52] were the first to introduce option contracts in inventory management, where they assumed a standard B-S formula for option pricing. Option models are explicitly modeled in recent works due to their attractiveness, especially in the context of high demand uncertainty. Barnes- Schuster et al. [3] show that backup, quantity flexible, and pay-to-delay contracts can all be viewed as special cases of option contracts that permit expedited orders, and they develop the sufficient conditions of the cost parameters for linear prices to coordinate the channel in its general option contracts. Kamrad and Siddique [28] employ real options methodologies to analyze supply contracts with quantity flexibility, supplier-switching options, and reaction options under exchange rate uncertainty. In commodity procurement studies, option contracts have been studied to find the optimal contracts under different market conditions. Martinez and Levi [40] focus on the design of an option portfolio in a multi-period environment with inventory holding costs, where a modified base-stock policy is derived as the optimal replenishment policy. Wu and Kleindorfer [62] characterize the price of capacity options, concentrating on the competition effects between sellers with heterogeneous technologies.

Burnetas and Ritchken [8] explicitly price call (put) supply chain options, which they map as the retailer's right to reorder (return) goods at a pre-determined price with the manufacturer. While most previous papers assume risk-neutral agents in the supply chain and use simple profit as the objective, Burnetas and Ritchken [8] relax this assumption by applying option pricing methodologies in finance theory to parameterize the risk preferences of the supply chain participants.

We adopt a similar approach to incorporate risk preference into the models. However, our problem assumes a different market structure, where retail price is determined exogenously and therefore is not affected by the behavior of a single retailer. Also, we assign a certain reservation value to the retailer. In reality, it is a common belief that the retailer will reject the contract offer if he cannot obtain more than his reservation. Most importantly, in this thesis we emphasize extending these studies to multi-product joint options.

To our knowledge, study on flexible contracts in a multi-product context has just started. Brown et al. [7] examine the return policies in multi-product cases, where they define a "pooled" return policy as one where the distributor can return any combination of the products up to R percent of the total purchases across all products, while a "non-pooled" policy only allows each product to be returned separately. They identify a counterintuitive result regarding the retailer's optimal order quantity under both pooled and non-pooled return policies. Our study differs from the above in the handling of risk preference; we also extend the analysis to discuss the implied requirements for the manufacturer in offering such flexibility.

1.3 Purpose and Structure of the Thesis

In chapter 2, we develop a new choice estimation model for the product line selection problem. The new approach will only require the mean and covariance matrix associated with the random utility evaluation. We refer to this new approach as the Cross Moment Model (CMM). The attractiveness of the CMM model is its capability to capture the correlation between the product candidates with little computation burden increased. The new approach will predict the choice probabilities more accurately and help to achieve the product line optimization more effectively. We demonstrate this in Chapter 2 with our computational results on the performance comparison between the CMM and the MNL model.

In a supply chain, since the retailers' ordering set and ordering quantity decisions affect the efficiency of the whole chain, he acts as the interface between the manufacturer and the end consumers. Therefore, in Chapter 3, we extend our CMM model to integrate product line selection and inventory decisions from the retailer's point of view.

Flexible contracts are usually used as supply chain coordination tools. In Chapter 4, we study the flexibility in supply contracts with the focus on multi-products reorder option contracts. In a multiple product environment, in addition to product quantity flexibility, product mix flexibility should also be considered, thus we study the impact of contract flexibility from both dimensions in such a multiple product environment.

We conclude our study in Chapter 5.

2. PRODUCT LINE SELECTION WITH INTER-DEPENDENT PRODUCTS

2.1 Introduction

We consider in this thesis a product line design problem of the following form: Let $\mathcal{N} = \{1, 2, \ldots, n\}$ denote a set of product options and $U_i(\mathbf{z})$ denote the random utility for product i for a customer with random attributes $\mathbf{z}$. We assume that $(U_1(\mathbf{z}), \ldots, U_n(\mathbf{z}), \mathbf{z})$ is drawn from a joint distribution F with the conditional density function $f(U_1(\mathbf{z}), \ldots, U_n(\mathbf{z})|\mathbf{z})$. Each customer picks the product that yields the greatest utility in the choice set. Our goal is to design a product line with exactly K products so as to maximize the expected utility:

$$(PLD) \qquad \max_{\mathcal{S}\subset\mathcal{N}:|\mathcal{S}|=K} \left(E_F \left[\max_{i\in\mathcal{S}} U_i(\mathbf{z}) \right] \right). \tag{2.1}$$

The utility functions $U_i(\mathbf{z})$ may be correlated across different products, due to the presence of common product attributes and the random customer attributes $\mathbf{z}$.

This class of problems is motivated by a practical problem faced by a local service parts supplier in Singapore. The company stores various standard boxes to pack and ship their products to different customer destinations. Unfortunately, due to the varying sizes and shapes of the products in an order, and the limitation on the number of standard boxes available, the company has to often use a large box to pack the few products in an order. Figure 2.1 illustrates a typical order with items packed inside the standard box. This box is the best available to ship this order, but the volume usage is quite low. The third party logistics provider, however, charges the company based on the larger of volumetric weight (defined as volume in cm^3 divided by 6000) and actual weight. An inefficient utilization of the volume in the standard boxes may thus lead to excessive shipping costs, which occasionally may be more than the value of the items shipped!

Fig. 2.1: An example of a box with low volume usage

The company would thus like to select a set of K standard boxes, to minimize the average shipping cost for the business. Note that the deterministic problem (with known input of the items in each order and their shape distribution) is already a notorious combinatorial packing problem. The complexity of the problem is exacerbated by the fact that item's shape distribution usually fluctuates with each order, and finding a set of standard sized boxes that work well for all orders is thus a daunting problem. We can encode the attributes of an order by a random tuple $\mathbf{z} = (j, R_j, \mathbf{s}_j)$[1], where j is the destination of the order, R_j is the revenue generated by the order, and $\mathbf{s}_j$ encodes the shape of each item (length, width and height) in the order. Product i can be described by the shape attributes of the box, say (L_i, W_i, H_i), denoting the length, width and height of box i. The utility of an order attached to product i is thus

$$U_i(\mathbf{z}) := \max\left(0, \left(R_j - c_j(L_i \times W_i \times H_i)\right)\chi\left(\mathbf{s}_j \text{ can be packed into box } i\right)\right), \tag{2.2}$$

where $\chi(\cdot)$ is an indicator function, and $c_j(V)$ is the cost of shipping a box with volumetric weight V to the destination j. Clearly, the utilities attached to the boxes are correlated, depending on the shape distribution of items in the order and the destinations and shapes of the boxes.

There are plenty of other examples in practice where the utility evaluation is not independent across products. In many consumer markets, hardware/software configuration problems, and even in airline network revenue management (cf. Bront et al. [6]), slight variation in features are often used to distinguish products. In general, in these problem settings, different resources are combined to provide for the configuration of different products (e.g. each resource corresponds to a single-leg flight, and a product is defined as an itinerary and fare-class combination). The sharing of common resources

[1] In most instances, the actual weight of items in each order is smaller than the volumetric weight, so that the shipping cost is dominated by volumetric weight alone.

can result in high correlations in utility evaluation among the products. In these circumstances, the product line design model using the assumption that the products are evaluated independently could be far from accurate, and thus the product line designed under such assumptions may be far from ideal.

In this thesis, we propose a parsimonious model (called the Cross Moment model or abbreviated CMM model) to obtain choice estimates, using only information on the mean and covariance of the utility evaluation across products. Surprisingly, despite using only the moments information, our numerical results suggest that CMM model can generate reasonable choice estimates, even for highly correlated products. A key advantage of the model is that there is no need for exhaustive simulations to generate the choice probability estimates. This allows the model to be embedded into a heuristic to search for a good set of products for the product line design problem.

Section 2.2 introduces the CMM discrete choice model for customer choice prediction, taking into account the interdependency of products due to common attributes. In section 2.3, we show that our proposed consumer choice model is also able to circumvent the issues associated with the IIA (Independence from Irrelevant Alternatives) and IPS (Invariant Proportion of Substitution) properties inherent in many existing consumer choice models. In Section 2.4, a detailed comparison of MNL and CMM models on a flexible packaging problem is provided.

2.2 Consumer Choice Model

In this section, we develop a new customer choice model using only the mean and covariance information for the utilities of the products. No assumption on the form of the utility function is made. All we assume is that the mean vector μ and the second moment matrix $\mathcal{Q}$ for the random utilities are known:

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \mathcal{Q} = \begin{pmatrix} \mathcal{Q}_{11} & \dots & \mathcal{Q}_{1n} \\ \vdots & \ddots & \vdots \\ \mathcal{Q}_{n1} & \dots & \mathcal{Q}_{nn} \end{pmatrix},$$

where $\mu_i = E[U_i(\mathbf{z})]$ and $\mathcal{Q}_{ij} = E[U_i(\mathbf{z})U_j(\mathbf{z})]$ and the moments satisfy the feasibility condition $\mathcal{Q} \succeq \mu\mu'$. We are interested in estimating the choice probability

$$P\bigg(U_i(\mathbf{z}) \geq \max_{k \in \mathcal{N}} U_k(\mathbf{z})\bigg). \tag{2.3}$$

2.2.1 Distribution of Random Utilities

In general, there are many possible distributions that satisfy the prescribed moment conditions. One such distribution is the multivariate normal distribution for which the choice probabilities can be accurately computed only through simulation. Instead, we look for a joint distribution where the choice estimates can be obtained easily through solving a tractable optimization problem.

Consider the following mixture distribution representation for the utilities:

$$
\begin{aligned}
\Big(U_1(\mathbf{z}), \ldots, U_n(\mathbf{z})\Big) &= \Big(Y_k + \beta_{1k}, \ldots, Y_k + \beta_{nk}\Big) \\
&\quad \text{with probability } y_k \text{ for } k = 1, \ldots, n,
\end{aligned} \tag{2.4}
$$

where $(Y_1, \ldots, Y_n)$ are independent random variables with zero means and β_{ik} are fixed numbers. Under scenario k, we have $\max\left(U_1(\mathbf{z}), \ldots, U_n(\mathbf{z})\right) = Y_k + \max\left(\beta_{1k}, \ldots, \beta_{nk}\right)$. Then, the product with the highest utility is known irrespective of the value of Y_k and the choice process is a simple deterministic problem. If we assume further that

$$
\beta_{kk} \geq \max_{i: i \neq k} \beta_{ik}, \text{ for } k = 1, \ldots, n,
$$

then the customer picks product k in scenario k and the choice probability is simply y_k.

Our model thus attempts to find β_{ik}, y_k, and independent random variables Y_k with zero means and variance δ_k^2, so that the moment conditions are satisfied. Over these class of mixture distributions, the values are chosen such that the expected utility of the customer is maximized. This reduces to solving the following nonlinear problem:

$$
\begin{aligned}
Z_1 = \max_{\beta_{ik}, y_k, \delta_k} \quad & \sum_{k \in \mathcal{N}} y_k \beta_{kk} \\
\text{s.t.} \quad & \sum_{k \in \mathcal{N}} y_k \beta_{ik} \beta_{jk} + \sum_{k \in \mathcal{N}} y_k \delta_k^2 = \mathcal{Q}_{ij}, \quad && i, j = 1, \ldots, n, \\
& \sum_{k \in \mathcal{N}} y_k \beta_{ik} = \mu_i, && i = 1, \ldots, n, \\
& \sum_{k \in \mathcal{N}} y_k = 1, \\
& \beta_{kk} \geq \max_{i: i \neq k} \beta_{ik}, && k = 1, \ldots, n, \\
& y_k, \delta_k \geq 0, && k = 1, \ldots, n.
\end{aligned}
\tag{2.5}
$$

Note that a priori, it is not clear why the choice of this distribution is appropriate. As it turns out, interestingly, this approach is a “good” way to approximate the choice process - the joint distribution obtained under this approach maximizes the expected utility over all joint distributions of the utilities with the given mean and covariance structure. More importantly, this nonlinear model can be recast into a convex semidefinite optimization problem in a higher dimensional space, and is therefore computationally tractable.

2.2.2 Cross Moment (CMM) model

The problem of maximizing the expected utility of the products selected by customers over all joint probability distributions F for the utility functions satisfying the moment conditions is formulated as:

$$Z := \max_{F} \left\{ E_F\left[\max_{i\in\mathcal{N}} U_i(\mathbf{z})\right] \;\middle|\; \begin{array}{l} E_F[U_i(\mathbf{z})] = \mu_i;\ i \in \mathcal{N}, \\ E_F[U_i(\mathbf{z})U_j(\mathbf{z})] = \mathcal{Q}_{ij};\ i,j \in \mathcal{N} \end{array} \right\}. \tag{2.6}$$

This problem can be reformulated as a semidefinite optimization problem and the choice probability estimates are obtained from the optimal value of the variables.

Proposition 1: Let e_k denote a vector of dimension n with 1 in the kth position and 0 otherwise. Problem (2.6) is solvable as the semidefinite optimization problem:

$$\begin{aligned} Z = \max\nolimits_{W_k, w_k, y_k} \quad & \sum_{k\in\mathcal{N}} e_k' w_k \\ \text{s.t.} \quad & \sum_{k\in\mathcal{N}} \begin{pmatrix} W_k & w_k \\ w_k' & y_k \end{pmatrix} = \begin{pmatrix} \mathcal{Q} & \mu \\ \mu' & 1 \end{pmatrix}, \\ & \begin{pmatrix} W_k & w_k \\ w_k' & y_k \end{pmatrix} \succeq 0, \qquad k = 1, \ldots, n, \end{aligned} \tag{2.7}$$

where the decision variables W_k are symmetric matrices of dimension $n \times n$, w_k are vectors of dimension n and y_k are scalars. The optimal y_k values are the choice probabilities under the optimal distribution in Problem (2.6).

Proof. We first show that the Formulation (2.7) provides an upper bound on Z. Consider a partition of space of the utility vector

$$U(\mathbf{z}) = \begin{pmatrix} U_1(\mathbf{z}) \\ \vdots \\ U_n(\mathbf{z}) \end{pmatrix},$$

into the sets:

$$\mathcal{T}_k = \left\{ U(\mathbf{z}) \in \Re^n \;\middle|\; U_k(\mathbf{z}) \geq \max_{i \in \mathcal{N}} U_i(\mathbf{z}) \right\}.$$

Define the decision variables as the scaled conditional moments over these sets:

$$\begin{pmatrix} W_k & w_k \\ w_k' & y_k \end{pmatrix} = \begin{pmatrix} E[U(\mathbf{z})U(\mathbf{z})' \big| \mathcal{T}_k] P(\mathcal{T}_k) & E[U(\mathbf{z}) \big| \mathcal{T}_k] P(\mathcal{T}_k) \\ E[U(\mathbf{z})' \big| \mathcal{T}_k] P(\mathcal{T}_k) & P(\mathcal{T}_k) \end{pmatrix}.$$

The expected utility objective in Problem (2.6) is then expressed as:

$$\begin{aligned} E\left[\max_{k \in \mathcal{N}} U_k(\mathbf{z})\right] &= E\left[\max_{k \in \mathcal{N}} e_k' U(\mathbf{z})\right] \\ &= \sum_{k \in \mathcal{N}} E\left[e_k' U(\mathbf{z}) \big| \mathcal{T}_k\right] P\left[\mathcal{T}_k\right] \\ &= \sum_{k \in \mathcal{N}} e_k' w_k. \end{aligned}$$

The first set of constraints in (2.7) are obtained by expressing the mean and the second moment matrix for the utility levels as the sum of the scaled conditional moments:

$$\begin{aligned}\begin{pmatrix} \mathcal{Q} & \mu \\ \mu' & 1 \end{pmatrix} &= \sum_{k\in\mathcal{N}} \begin{pmatrix} E[U(\mathbf{z})U(\mathbf{z})'\big|\mathcal{T}_k]P(\mathcal{T}_k) & E[U(\mathbf{z})\big|\mathcal{T}_k]P(\mathcal{T}_k) \\ E[U(\mathbf{z})'\big|\mathcal{T}_k]P(\mathcal{T}_k) & P(\mathcal{T}_k) \end{pmatrix} \\ &= \sum_{k\in\mathcal{N}} \begin{pmatrix} W_k & w_k \\ w_k' & y_k \end{pmatrix}.\end{aligned}$$

The second set of constraints in (2.7) arises from the standard positive semidefiniteness condition that the first moment vector and the second moment matrix must satisfy:

$$\begin{pmatrix} W_k & w_k \\ w_k' & y_k \end{pmatrix} \succeq 0.$$

The necessary conditions in Formulation (2.7) implies that it provides an upper bound on Z.

Next, we use the optimal variables (W_k^*, w_k^*, y_k^*) to generate the multivariate distribution that attains the bound. In the optimal solution, if there exists a product k such that $y_k^* = 0$, then the positive semidefiniteness condition implies that w_k^* must be a vector of zeros. We then perturb the solution by adding the matrix W_k^* to the matrix W_j^* for any product j with $y_j^* > 0$. This maintains feasibility and does not affect the objective value.

The distribution F^* is now generated as follow:

(a) Choose product $k \in \mathcal{N}$ with probability$y_k^* > 0$

(b) Generate normally distributed utilities with mean w_k^*/y_k^* and second moment matrix W_k^*/y_k^*.

The moment conditions for $U(\mathbf{z})$ are clearly satisfied from the feasibility conditions in (2.7). Furthermore, under this distribution F^* by simply looking at the utility for the kth product under the kth scenario, we have:

$$
\begin{aligned}
E_{F^*}\left[\max_{k \in \mathcal{N}} U_k(\mathbf{z})\right] \geq & \sum_{k \in \mathcal{N}: y_k^* > 0} y_k^* \left(\frac{e_k' w_k^*}{y_k^*}\right) \\
& = \sum_{k \in \mathcal{N}} e_k' w_k^*.
\end{aligned}
$$

This proves that the bound is attainable. ■

From the argument in Proposition 1, it is clear whenever product k is selected with probability y_k^*, the optimal distribution for the utilities must be perfectly correlated so that it will always attain the maximum utility. Note that for bivariate normal variables X and Y, $X > Y$ holds with probability 1 only when $Y = X - \beta$ for some $\beta > 0$. The joint distribution identified in

Proposition 1 hence has the form:

$$\Big(U_1(\mathbf{z}),\ldots,U_n(\mathbf{z})\Big) = \Big(Y_k + w^*_{1k},\ldots,Y_k + w^*_{nk}\Big) \qquad (2.8)$$
$$\text{with probability } y^*_k \text{ for } k = 1,\ldots,n$$

where Y_k is a normal random variable with mean 0 and variance δ_k^2. The variance δ_k^2 can be identified from the matrix equation:

$$\delta_k^2 ee' = W^*_k/y^*_k - (w^*_k/y^*_k)\,(w^*_k/y^*_k)',$$

where e is a vector of ones of dimension n. This distribution degenerates into the mixture distribution that we described at the start of this section and we obtain the following corollary:

Corollary 1: $Z_1 = Z$.

The equivalence of Formulations (2.5) and (2.7) can be used to generate numerous insights into the construction of the optimal distribution. In Formulation (2.5), let Δ^2 denote $\sum_k y_k\delta_k^2$. Using a change of variable, we define:

$$\begin{aligned} v_0 &= (\sqrt{y_1},\ldots,\sqrt{y_n},0), \\ v_i &= (\sqrt{y_1}\beta_{i1},\ldots,\sqrt{y_n}\beta_{in},\Delta). \end{aligned}$$

Let $I_k(v)$ denote the projection of v onto the k-th coordinate. We can then find Z_1 or Z by solving the reformulation:

$$\begin{aligned} Z_1 = Z = \max_{v_0, v_k} \quad & \sum_{k \in \mathcal{N}} v_0 \cdot I_k(v_k) \\ \text{s.t.} \quad & v_i \cdot v_j = \mathcal{Q}_{ij}, && i, j = 1, \ldots, n, \\ & v_0 \cdot v_i = \mu_i, && i = 1, \ldots, n \\ & v_0 \cdot v_0 = 1, \\ & v_0 \cdot e_{n+1} = 0, \\ & v_i \cdot e_{n+1} = v_j \cdot e_{n+1}, && i, j = 1, \ldots, n. \end{aligned} \tag{2.9}$$

Geometrically, the vectors $v_0, v_1, \ldots, v_n$ can be interpreted as finding a Cholesky factorization of the $(n+1) \times (n+1)$ moments matrix

$$\begin{pmatrix} \mathcal{Q} & \mu \\ \mu' & 1 \end{pmatrix},$$

oriented so that the $n+1$th coordinates satisfy the boundary conditions.

Now, consider the problem of scaling each of the utility functions by a constant λ:

$$Z(\lambda) := \max_F \left\{ E_F\left[\max_{i \in \mathcal{N}} U_i(\mathbf{z})\right] \;\middle|\; \begin{array}{cc} E_F[U_i(\mathbf{z})] = \lambda \mu_i; & i \in \mathcal{N} \\ E_F[U_i(\mathbf{z}) U_j(\mathbf{z})] = \lambda^2 \mathcal{Q}_{ij}; & i, j \in \mathcal{N} \end{array} \right\} \tag{2.10}$$

The argument to the proof in Proposition 1 shows that the choice probabilities obtained under the CMM model is scale invariant.

Corollary 2: Let $y_i(\lambda)$ be the choice probability for product i in the CMM model $Z(\lambda)$. Then for all $\lambda, \lambda' > 0$, we have:

$$y_i(\lambda) = y_i(\lambda').$$

In this thesis, we focus solely on utility maximization. There is an analogous formulation with min-objective function. More specifically, we have the following analogous proposition:

Proposition 2: Let e_k denote a vector of dimension n with 1 in the kth position and 0 otherwise. The problem

$$\min_F \left\{ E_F\left[\min_{i\in\mathcal{N}} U_i(\mathbf{z})\right] \;\middle|\; E_F[U_i(\mathbf{z})] = \mu_i;\ i \in \mathcal{N},\ E_F[U_i(\mathbf{z})U_j(\mathbf{z})] = \mathcal{Q}_{ij};\ i, j \in \mathcal{N} \right\}.$$

is solvable as the semidefinite optimization problem:

$$\begin{aligned}
\min_{W_k, w_k, y_k} \quad & \sum_{k\in\mathcal{N}} e_k' w_k \\
\text{s.t.} \quad & \sum_{k\in\mathcal{N}} \begin{pmatrix} W_k & w_k \\ w_k' & y_k \end{pmatrix} = \begin{pmatrix} \mathcal{Q} & \mu \\ \mu' & 1 \end{pmatrix}, \\
& \begin{pmatrix} W_k & w_k \\ w_k' & y_k \end{pmatrix} \succeq 0, \qquad\qquad k = 1, \ldots, n.
\end{aligned}$$

2.3 Performance of the CMM model

We provide three simple examples to validate the quality of the choice probability predictions obtained from the Cross Moment model. The first example deals with solving the two product case in closed form. The second example discusses how the model can be used to overcome the behavorial limitations observed in discrete choice models that display the IIA and IPS properties. The third example deals with the use of the Cross Moment model in retrieving a close approximation to the arcsine law observed in random walk models.

Example 1: Two Product Closed Form

Consider a set of uncorrelated products with means μ_i and variances σ_i^2. By choosing a different set of orthonormal basis, the optimal v_i in Formulation (2.9) can be rewritten as:

$$v_i = \mu_i v_0 + \sigma_i s_i,$$

where the vectors $(v_0, s_1, \ldots, s_n)$ forms a set of orthonormal basis, with $v_0 \cdot e_{n+1} = 0$, and $s_i \cdot e_{n+1} = \Delta/\sigma_i$ for all i. This can be solved in closed form for the case of two products.

Corollary 3: Consider two uncorrelated products, with means μ_1, μ_2 satisfying $\mu_1 \geq \mu_2$ and variances $\sigma_1^2, \sigma_2^2 > 0$. The choice probabilities obtained from

CMM have the following values:

$$y_1 = \frac{1}{2}\left(1 + \frac{\mu_1 - \mu_2}{\sqrt{(\mu_1 - \mu_2)^2 + \sigma_1^2 + \sigma_2^2}}\right) \text{ and } y_2 = \frac{1}{2}\left(1 - \frac{\mu_1 - \mu_2}{\sqrt{(\mu_1 - \mu_2)^2 + \sigma_1^2 + \sigma_2^2}}\right).$$

Proof. For the two product case, the set of orthonormal basis can be expressed as:

$$v_0 = \begin{pmatrix} \sqrt{y_1} \\ \sqrt{y_2} \\ 0 \end{pmatrix}, \; s_1 = \begin{pmatrix} \alpha_1\sqrt{y_2} \\ -\alpha_1\sqrt{y_1} \\ \Delta/\sigma_1 \end{pmatrix}, \; s_2 = \begin{pmatrix} -\alpha_2\sqrt{y_2} \\ \alpha_2\sqrt{y_1} \\ \Delta/\sigma_2 \end{pmatrix}.$$

From the orthonormality conditions, we have

$$y_1 + y_2 = 1,\; \alpha_1^2 + \Delta^2/\sigma_1^2 = 1,\; \alpha_2^2 + \Delta^2/\sigma_2^2 = 1,\; -\alpha_1\alpha_2 + \Delta^2/(\sigma_1\sigma_2) = 0.$$

Solving these equations, we obtain

$$\Delta = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}},\; \alpha_1 = \sqrt{\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}},\; \alpha_2 = \sqrt{\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}.$$

We obtain the optimal values for y_1 and y_2 by solving the following problem:

$$\begin{array}{rl} \max_{y_1,y_2} & \mu_1 y_1 + \sqrt{(\sigma_1^2 + \sigma_2^2) y_1 y_2} + \mu_2 y_2 \\ \text{s.t.} & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0. \end{array}$$

The optimality conditions yields the choice probabilities as:

$$y_1 = \frac{1}{2}\left(1 + \frac{\mu_1 - \mu_2}{\sqrt{(\mu_1 - \mu_2)^2 + \sigma_1^2 + \sigma_2^2}}\right) \text{ and } y_2 = 1 - y_1.$$

To evaluate the usefulness of the above results, we used it to approximate the probability that $X_1 \geq X_2$, when both X_1 and X_2 are normally distributed and independent of each other. Note that the exact solution to $P(X_1 \geq X_2)$ cannot be evaluated in closed form. The first plot in Figure 2.2 shows the solution obtained from numerical evaluations (in black dash line), and that obtained from the CMM model (in blue solid line), with $X_1 \sim N(1, 0.1^2)$, $X_2 \sim N(\mu_2, 0.1^2)$, as μ_2 varies from 0 to 1. The second plot shows the performance when the distribution of X_1 changes to $N(1, 0.5^2)$. Clearly the closed form solution provided by the CMM model tracks closely the actual performance for this range of parameters.

Example 2: IIA and IPS Properties

The second example demonstrates that CMM model can be used to overcome the counterintuitive behavior implied by the IIA and IPS properties that is observed in many discrete choice models. Consider the example taken from Steenburgh [56] where an customer faces a choice among laptop computers. The observable attributes of the computers are the weight and the processor speed (see Table 2.1).

In Choice Set I, an customer faces a choice between two laptop computers A and C. Laptop A is the lightest alternative but it runs at the slowest speed. Laptop C is the fastest alternative but is heaviest in weight. In Choice Set II, an additional Laptop B is added to the choice set with medium weight

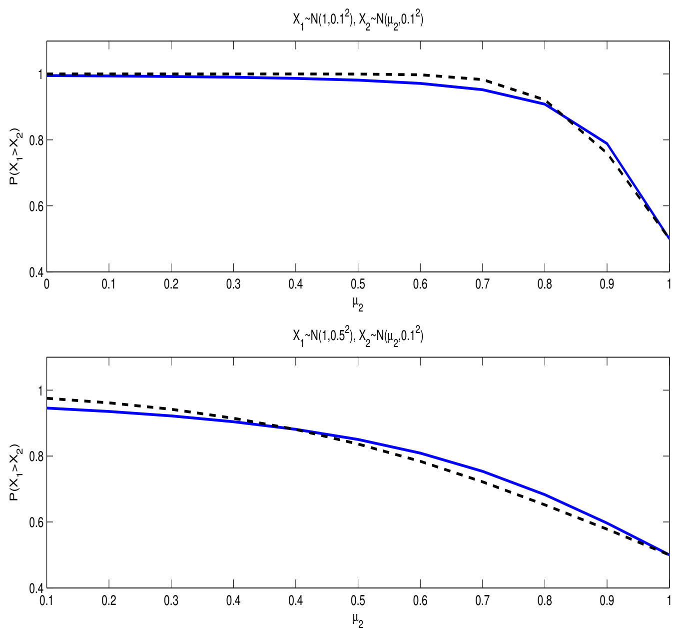

Fig. 2.2: Comparison of two normal variates

	Attributes		Choice Set	
Laptop	Weight	Speed	I	II
A	LIGHT (1)	SLOW (0)	YES	YES
C	HEAVY (0)	FAST (1)	YES	YES
B	MEDIUM (α_1)	MEDIUM (α_2)	NO	YES

Tab. 2.1: Laptop choice set

and speed. The laptop attributes denoted by the vector (Weight, Speed) are $\mathbf{x}_A = (1,0)$, $\mathbf{x}_C = (0,1)$ and $\mathbf{x}_B = (\alpha_1, \alpha_2)$ for products A, C and B respectively. We assume that the utility value of product P with weight α_W and speed α_S is represented by

$$U_P = (1+\epsilon_1)\alpha_W + (1+\epsilon_2)\alpha_S + \epsilon_P,$$

where ϵ_1 and ϵ_2 correspond to the random noise terms in utility evaluation relating to the weight and speed attributes of the laptops. We assume ϵ_1 and ϵ_2 have mean 0, variance 1 and are uncorrelated. We also let ϵ_P denote the noise term introduced by the unobserved attributes of product P in utility evaluation. ϵ_P is assumed to have mean 0, variance 1, for all products in this example, and is independent of ϵ_1 and ϵ_2. Thus

$$\begin{aligned} U_A &= 1 + \epsilon_1 + \epsilon_A \\ U_C &= 1 + \epsilon_2 + \epsilon_C \\ U_B &= \alpha_1 + \alpha_2 + \alpha_1\epsilon_1 + \alpha_2\epsilon_2 + \epsilon_B \end{aligned} \tag{2.11}$$

Furthermore, we assume ϵ_C is uncorrelated with ϵ_A and ϵ_B while the correlation factor between ϵ_A and ϵ_B is ρ. Computing the mean and second moment matrix for the utilities gives us:

$$\mu = \begin{pmatrix} 1 \\ 1 \\ \alpha_1 + \alpha_2 \end{pmatrix}, \mathcal{Q} = \begin{pmatrix} 3 & 1 & 2\alpha_1 + \alpha_2 + \rho \\ 1 & 3 & \alpha_1 + 2\alpha_2 \\ 2\alpha_1 + \alpha_2 + \rho & \alpha_1 + 2\alpha_2 & 2\alpha_1^2 + 2\alpha_2^2 + 2\alpha_1\alpha_2 + 1 \end{pmatrix}.$$

The corresponding correlation matrix for the utilities is

$$\begin{pmatrix} 1 & 0 & \dfrac{\alpha_1+\rho}{\sqrt{2(\alpha_1^2+\alpha_2^2+1)}} \\ 0 & 1 & \dfrac{\alpha_2}{\sqrt{2(\alpha_1^2+\alpha_2^2+1)}} \\ \dfrac{\alpha_1+\rho}{\sqrt{2(\alpha_1^2+\alpha_2^2+1)}} & \dfrac{\alpha_2}{\sqrt{2(\alpha_1^2+\alpha_2^2+1)}} & 1 \end{pmatrix}.$$

As the weight of laptop B is decreased (α_1 increases), the utilities for laptops A and B become more correlated. Likewise as the speed of laptop B increases (α_2 increases), the utilities for laptops C and B become more correlated.

We first consider the counterintuitive behavior implied by the IIA property. In Choice Set I, using the two product formula the choice probabilities are 1/2 under CMM. For Choice Set II, we set $(\alpha_1, \alpha_2) = (1, 0)$ and change $\rho \in [-1, 1]$. The correlation between the utilities of products C and B is thus set to 0 while the correlation between product B and A changes from 0 to 1. The choice probabilities obtained from CMM is plotted in Figure 2.3. Clearly the IIA property is absent in this model. For example if B is uncorrelated with A, then the choice probabilities for (A, B, C) are $(1/3, 1/3, 1/3)$ while if B is identical to A we get $(1/4, 1/4, 1/2)$

We next consider the counterintuitive behavior implied by the IPS property. To check for this property, we compute the change in choice probabilities as the attributes are modified. We set $\rho = 0$ and locate laptop B in the middle of the competing laptops $(\alpha_1, \alpha_2) = (1/2, 1/2)$. Holding the speed fixed, we make laptop B lighter and compute the new choice probabilities. Similarly, we hold the weight fixed and increase the speed of laptop B. The choice

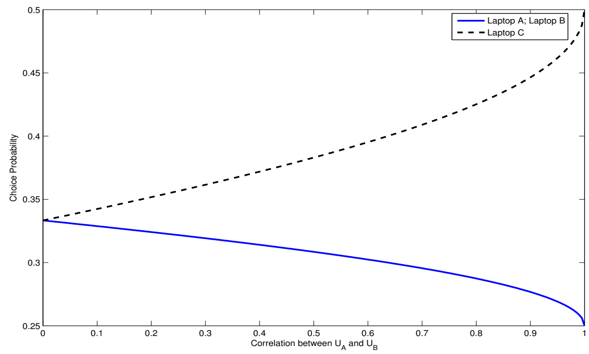

Fig. 2.3: Absence of IIA property in CMM

probabilities in Table 2.2 indicate that the IPS property does not hold under this model. As B becomes lighter, a greater proportion of the growth in it's probability is drawn from A (63.78%) as compared to C (36.22%). Similarly as B becomes faster, a greater proportion of the growth in it's probability is drawn from C (63.78%) as compared to A (36.22%). In fact, Steenburgh [56] suggests that the IPS property can be overcome by allowing the error terms to become more correlated as the alternatives become more similar. Using Eq. (2.11) and the CMM model, we provide a rigorous model to capture this behavior.

	(α_1,α_2)		
Laptop	(0.5,0.5)	(1,0.5) (B becomes lighter)	(0.5,1) (B becomes faster)
A	0.3586	0.2585	0.3018
C	0.3586	0.3018	0.2585
B	0.2829	0.4397	0.4397

Tab. 2.2: Absence of IPS property in CMM

Example 3: Random Walk model

The third example demonstrates the accuracy of CMM on a larger scale model. We test it's performance on a random walk problem, wherein the explicit solution on the choice probabilities is well known. Suppose X_i are random variables with mean μ_i and standard deviation σ_i. Let

$$S_k = X_1 + \ldots + X_k, \quad k = 1, \ldots, n,$$

with $S_0 = 0$. The goal is to estimate the probability that the random walk attains it maximum value at step k. i.e., find

$$P\left(S_k = \max_{0 \le j \le n} S_j\right).$$

If X_i's are iid, this probability can be rewritten as:

$$\begin{aligned} P\left(S_k = \max_{0 \le j \le n} S_j\right) &= P\left(X_k \ge 0, \sum_{j=k-1}^{k} X_j \ge 0, \ldots, \sum_{j=1}^{k} X_j \ge 0\right) P\left(X_{k+1} \le 0, \ldots, \sum_{j=k+1}^{n} X_j \le 0\right) \\ &= P\left(S_1 \ge 0, S_2 \ge 0, \ldots, S_k \ge 0\right) \times P\left(S_1 \le 0, \ldots, S_{n-k} \le 0\right). \end{aligned}$$

Let $\alpha = \sum_{i=1}^{n} P(S_i > 0)/n$. Then the classical arcsine law states that the probability

$$P\left(S_k = \max_{0 \le j \le n} S_j\right)$$

converges in distribution to

$$\frac{1}{n\pi} \sin\left(\frac{\pi}{\alpha}\right) \left(\frac{k}{n}\right)^{\alpha - 1} \left(1 - \frac{k}{n}\right)^{-\alpha}$$

Contrary to popular intuition, the two end points ($k = 0$ or $k = n$) have the highest probability of attaining the maximum.

The above is identical to a discrete choice problem, where the utility of product k is given by the summand $S_k = \sum_{j=1}^{k} X_k$. We can obtain the choice probability estimates using the CMM model, with

$$\mu = \begin{pmatrix} \vdots \\ E(S_k) = \sum_{j=1}^{k} \mu_j \\ \vdots \end{pmatrix},$$

$$\mathcal{Q} = \begin{pmatrix} & & \vdots & & \\ \cdots & E(S_i S_j) = \sum_{a \le i, b \le j, a \neq b} \mu_a \mu_b + \sum_{a \le i,j} (\mu_a^2 + \sigma_a^2) & \cdots \\ & & \vdots & & \end{pmatrix}.$$

Figure 2.4 shows the choice prediction of the above random walk model, based on the arcsine law and the Cross Moment model, for $n = 80$, using X_i with mean $\mu_i = 0$, standard deviation $\sigma_i = 1$, and $\alpha = 0.5$ (i.e. X_i's are symmetrical about the mean). Interestingly, the Cross Moment model is able to approximately return the arcsine law behaviour of the choice probabilities, with slight over-estimation for the popular products ($k = 0$ and $k = n$), and under-estimation for the less popular products ($k \approx n/2$).

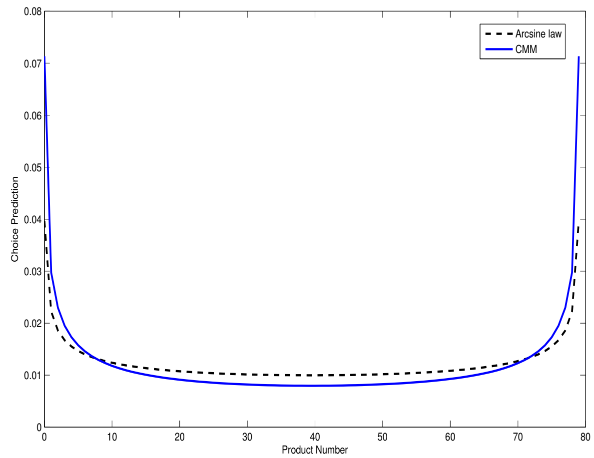

Fig. 2.4: Comparison of choice probabilities under Arcsine Law and CMM with $n = 80$

2.4 Application of Model: Flexible Packaging Design Problem

Using the Cross Moment model, the product line design problem is to choose a set of K products to maximize the expected utility:

$$\max_{\mathcal{S}\subset\mathcal{N}:|\mathcal{S}|=K} Z(\mathcal{S}),$$

where $Z(\mathcal{S})$ is obtained by solving the semidefinite optimization problem:

$$\begin{aligned} Z(\mathcal{S}) = \max_{W_k,w_k,y_k} \quad & \sum_{k\in\mathcal{S}} e_k' w_k \\ \text{s.t.} \quad & \sum_{k\in\mathcal{S}} \begin{pmatrix} W_k & w_k \\ w_k' & y_k \end{pmatrix} = \begin{pmatrix} \mathcal{Q}_{\mathcal{S}} & \mu_{\mathcal{S}} \\ \mu_{\mathcal{S}}' & 1 \end{pmatrix}, \\ & \begin{pmatrix} W_k & w_k \\ w_k' & y_k \end{pmatrix} \succeq 0, \qquad\qquad k \in \mathcal{S}. \end{aligned}$$

The second moment matrix $\mathcal{Q}_{\mathcal{S}}$ and mean vector $\mu_{\mathcal{S}}$ are obtained by looking at the corresponding subset $\mathcal{S} \subset \mathcal{N}$ from the original matrix $\mathcal{Q}$ and vector μ. The number of decision variables in solving $Z(\mathcal{S})$ is $O(K^3)$ which can be much smaller than $O(n^3)$ for $K << n$.

We augment the ease of choice estimation with a greedy-swapping heuristic (used in many product line design problems) to obtain a "good" product set. More specifically, we first use a standard greedy strategy to iteratively enlarge the set of products till we have K candidates, and then we look for the best option to swap a selected product with one that has not been selected by the heuristic. To determine the best swap to adopt each time, we iteratively replace the current candidates with those unselected ones and compare the resulting total utilities. The swapping heuristic thus maintain a set of K products throughout. This heuristic terminates when no improvement can be achieved through swapping.

With the MNL model, we solve the following problem directly to find optimal solutions:

$$\max_{\mathcal{S} \subset \mathcal{N}: |\mathcal{S}|=K} \ln\left(\sum_{i \in \mathcal{S}} e^{E[U_i(\mathbf{z})]}\right)$$

The objective herein is obtained using the well known fact that

$$E\left(\max_{i \in \mathcal{S}} U_i(\mathbf{z})\right) = \ln\left(\sum_{i \in \mathcal{S}} e^{E[U_i(\mathbf{z})]}\right)$$

for the standard MNL model. We solve the above model via complete enumeration to obtain the global optimal product line design.

Application of the above model could be found in many manufacturing and service industries. From the manufacturers' point of view, it may be purely to decide on its product set; While from the retailers' perspective, more of their concerns may focus on the integration of product selection with inventory planning decisions. We will extend our model to incorporate inventory decision in next chapter. Here we will constraint the application to a pure product selection problem. The following flexible packaging design problem can use our Cross Moment Model to model its underlying uncertainties.

We use the framework discussed above to address a problem faced by a local service parts supplier. The company deals with instruments and devices for electronics and communications, and has three distribution centers (DC) worldwide to coordinate the flow of service parts throughout the supply chain. The DC in Singapore was the latest addition and established to cater to the needs of the expanding Asian markets. It was responsible for satisfying customer orders mainly from the Asia-Pacific region, and providing replenishment support to other DCs. The customer orders arrived from six main destinations: China, Japan, Korea, United States, Malaysia and Taiwan. Other destinations included Germany, Thailand, France and some other countries but the number of orders from these countries were much smaller as compared to the six main locations. Customers were individual firms, company's regional offices and other warehouses (for replenishment). The observed variance in orders was quite high. While orders could be large, order with just one item were also not uncommon. In fact most of the orders from Malaysia and Taiwan usually had a small number of items.

Each order required the DC to ship the ordered items in a mother box through one of the freight services (e.g. DHL express, KWE, FEDEX etc.). With rare exceptions, the freight service used by the warehouse for a particular destination was fixed - FEDEX was used mainly for shipping to the US, KWE for shipping to China, Japan, and Korea etc. Each freight service had a different freight rate but they followed an international standard set by IATA (International Air Transport Association), which required shipments to be rated based on the larger of "actual weight" and "volume weight".

On receiving an order, each item is packed in a rectangular box (item-box), size and shape of which could vary considerably from item to item. All the item-boxes belonging to an order were then packed in a mother-box. Usually the items shipped by the warehouse have low density and it is the volume of the mother-box that determines the freight cost and not the actual weight. Shipping cost is based on the volumetric weight of the mother box. As per the IATA standards,

$$volumeweight = (l \times w \times h)/6000 \tag{2.12}$$

where l, w, and h are the length, width and height of the box respectively, in centimeters. The volume weight and actual weight are measured in kilograms.

The DC is faced with the task of maintaining an inventory of different sized mother-boxes, while keeping the shipping cost as low as possible. On average the shipping cost constitutes a high percentage (around 85%) of the total cost of the warehouse services per line item. The variations in size, shape and number of items per order makes the problem of selecting a manageable number of boxes in the box-set an extremely difficult one to solve. The DC explored the usage of a flexible packaging option, in which a box with one base can be adjusted to have more than one heights. One such box is shown in Figure 2.5. This box can be adjusted to have three different heights (more heights can be achieved easily). A flexible box costs a little more than the normal fixed height box, but gives the warehouse a degree of freedom in terms of height. The DC thus needs to select a fixed number of bases for the mother-boxes. We call this the *flexible packaging design problem*.

Fig. 2.5: A flexible box with 3 adjustable heights

This is a product line selection problem in a loose manner, where orders are analogous to customers with random utilities. We can think of the orders arriving from each destination as a different customer segment. Assuming utility of base (product) i for a order (customer) j is given by

$$U_{ij} = Max0, M - c_j((l_i \times w_i \times h_{ij})/6000), \tag{2.13}$$

where M is a constant which can be seen as a fixed revenue from each order (we used M=1001 for all orders, and assume $M == R_j \forall j$), and $c_j(x)$ is the cost of shipping x kg volume weight to destination j. The parameters l_i, w_i are the length and width (in cm) of base i, and h_{ij} is the height achieved by the order j when packed in a box with base $l_i \times w_i$. Details on the packing is provided in the next section.

2.4.1 Data

We were provided the data of a total of 101 orders. The data was collected during consecutive working days over a week. Note that there is no seasonality involved in the business of this DC. For each order, the dimension of the individual item-boxes, order destination, and dimensions of mother-box used were provided. Along with the order data, we were also provided the freight rates of various freight services used by the warehouse.

Figure 2.6 shows the dimensions (length, width, and height in cm) of some of the typical item-boxes shipped by the DC. It gives a fair idea about the shapes and sizes distribution of the item-boxes: some are elongated (like a rod), some are flat (like a pizza box) while some have regular cuboid shape.

Figure 2.7 shows the frequency distribution of the sum of the volume weights of all the item-boxes belonging to an order, for six major order-destinations. This statistics shows the per-order volume weight distribution. Orders from China and Japan cover a big range of volume weights and may have high values (upto 105 kg), whereas orders from Japan were quite uniformly distributed. Orders from US and Malaysia, on the other hand, have

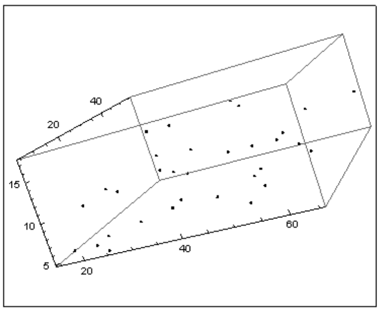

Fig. 2.6: Dimensions of various item-boxes

low volume weights (upto 25 kg). It is clear that orders do have some peculiar characteristics (in-terms of volume weights) destination-wise. Moreover we have different freight rates applicable for different destinations. There

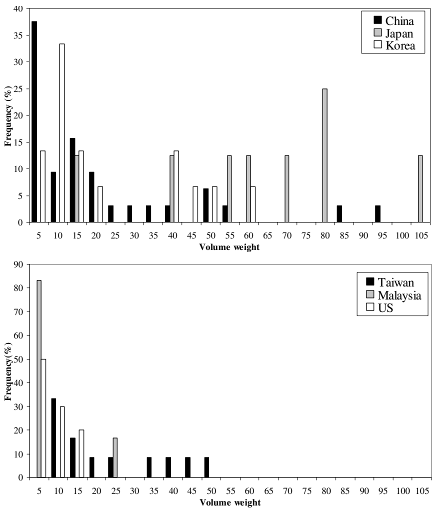

Fig. 2.7: Destination-wise volume weight distribution for orders

are mainly two types of freight services used by the DC - freight forwarders (such as KWE and DHL Global), and express services (such as DHL Express and FEDEX). A snapshot of the shipping cost structures for both types of providers are shown in Figure 2.8. The shipping cost structure affect U_{ij} through the term $c_j(\cdot)$.

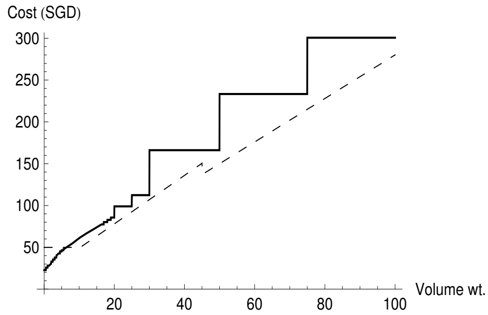

Fig. 2.8: A typical shipping cost curve for freight-forward services (dashed line) and express services (solid line)

Tracking both the smallest and largest item-boxes involved in all the orders, we identified the base candidates of the mother-box. We start the length/width from 15cm, and increase by 10cm interval each time, until we arrive at 85cm for both length and width. Therefore, totally we get 36 base candidates. Given these base candidates and the order information, our objective is to select K bases for mother-boxes. We solve this base selection problem using both MNL and CMM models to compare their efficacy.

We describe next how we obtain empirical estimates of the first and second moment matrix of the utility function U_{ij}.

- We randomly select 10 orders per destination from the available historical data. For each order to destination j and each base i, we find height h_{ij} by packing the items (belonging to the order) in the box with base i. We use a commercial 3D packing software, 3D load packer developed by Astrokettle Algorigthms, to find this height with rectangular packing (without orientation). The height h_{ij} is chosen so as to attain minimum volume using the base i so that all the items belonging to the order can be packed. In Figure 2.9, we show a snapshot of the packing using the 3D Load packer software, and the corresponding packing obtained is shown in Figure 2.10.

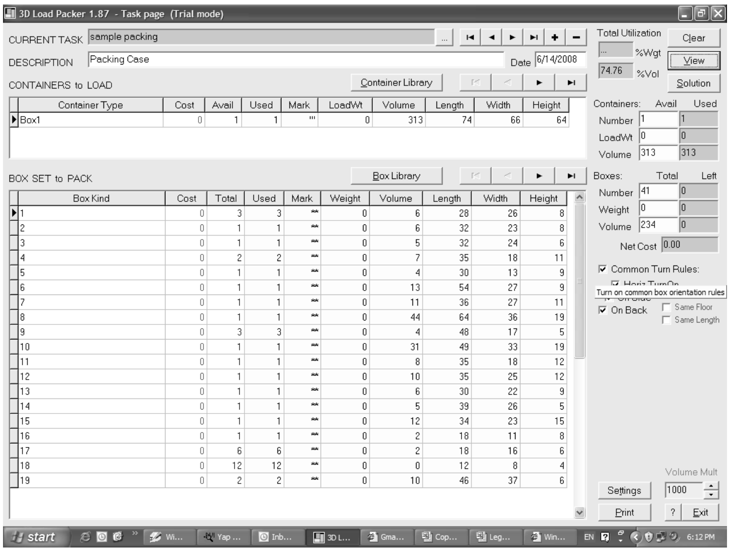

Fig. 2.9: A sample of packing using 3D loadpacker

We assume that packing crew at the warehouse, in practice, packs the items as efficiently as the 3D packing software. Given a small number of items per order this assumption is fairly justified. Given a base and an order, we were able to find the minimum possible volume (and hence maximum possible utility) that is achievable using the given base for the order.

- With this height h_{ij} and length l_i and width w_i of the base i, we find

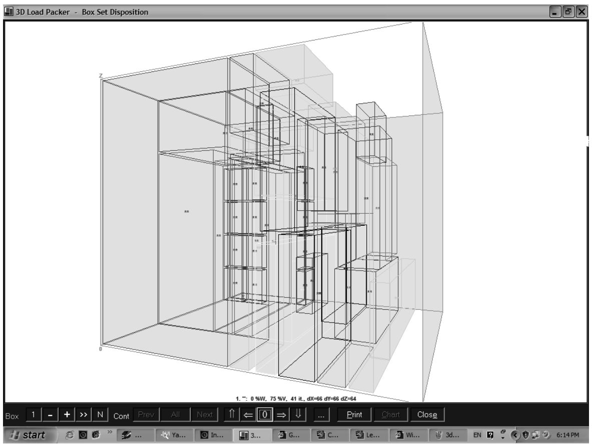

Fig. 2.10: View of packing generated in the sample of Figure 2.9

the volume weight of the mother-box with base i and order j.

- Given the freight rates of various services, we find the shipping cost $c_j(.)$ for each order. Note that the freight service used for a particular destination is known in advance.

- Once the $c_j(.)$ is found, we can find utilities U_{ij} using equation (2.13). If order j does not fit into any box with base i, we assign a small number as utility for this order and base combination.

- From these utilities we find the first and second moment matrices which are input to CMM. For MNL model, we use the mean utilities for different products and calculate the variance of all utilities as the estimated random error.

2.4.2 Computational Results

Table 2.3 provides the bases chosen for the mother-boxes using both MNL and CMM for varying values of K.

K	MNL selection
5	45×65 55×65 55×85 65×75 65×85
6	45×65 55×65 55×85 65×75 65×85 75×75
7	45×65 55×65 55×85 65×75 65×85 75×75 55×75
8	45×65 55×65 55×85 65×75 65×85 75×75 55×75 45×75
9	45×65 55×65 55×85 65×75 65×85 75×75 55×75 45×75 65×65
10	45×65 55×65 55×85 65×75 65×85 75×75 55×75 45×75 65×65 75×85
K	CMM selection
5	15×55 25×25 25×45 45×55 65×85
6	15×55 25×25 25×45 35×75 65×85 45×45
7	15×55 25×25 25×45 45×55 65×85 45×45 85×85
8	15×55 25×25 25×45 45×55 65×85 45×45 85×85 15×15
9	15×55 25×25 25×55 45×55 65×85 45×45 85×85 15×15 25×35
10	15×55 25×25 25×45 45×55 65×85 65×65 85×85 15×15 25×35 35×75

Tab. 2.3: Base sets selected by MNL and CMM

The results indicate that the product line selected from the CMM tends to span a broader range than the MNL model suggests. This can be explained by the fact that CMM captures the information about *common attributes* of bases in terms of their width and length by the utilities covariance. Therefore, the CMM model successfully avoids the tendency to focus only on bases with high utility means, instead it includes bases with more variant dimensions to satisfy divergent order needs. For example, we can find in CMM's offer set, base 65×75 is avoided throughout, whereas base 65×85 is kept in the product line. MNL model includes both bases in all the cases. In terms of utility values, these two bases have high correlation of 0.9779. Thus it often suffices to carry one of the two bases in the product line.

K	Avg. utilities from MNL	Avg utilities from CMM	Improvements in utilities	Avg. cost from MNL	Avg. cost from CMM	Savings in cost
5	5215.6	5571.2	6.8%	131.73	72.47	45.0%
6	5219.5	5635.2	8.0%	131.08	61.80	52.9%
7	5243.2	5719.4	9.1%	127.13	47.77	62.4%
8	5244.9	5721.8	9.1%	126.85	47.37	62.7%
9	5247.4	5754.8	9.7%	126.43	41.87	66.9%
10	5247.9	5794.1	10.4%	126.35	35.32	72.0%

Tab. 2.4: Simulated utilities and costs for MNL and CMM

Using simulation, we attempt to confirm and quantify the advantage from the CMM's results over MNL's results. Based on the utilities mean and covariance estimations, we simulate the performance of both product lines using 10,000 randomly generated utility values, with the multivariate normal distributions. Employment of the multivariate normal distribution for the utilities is the most reasonable and convenient approach to simulate the utilities, although both CMM and MNL hold under different assumptions regarding the distribution of random utilities. Table 2.4 lists the resulting average utilities and the corresponding average cost obtained by each base set.

From the above table, we can see that the CMM significantly improves the average resulting utilities compared to MNL model. The savings in the corresponding shipping cost is even more significant. From our numerical results, the extent of improvements in utility and savings in cost tend to increase as the number of bases selected increases.

We used the actual shipping cost incurred in shipping each order as a benchmark to test the performance of CMM. For each order, the dimension of the mother-box used was provided to us so we could find this benchmark cost for each order. A total of 7 standard sized boxes (without flexible heights) were being used at the time the data was collected. Dimensions of these mother-boxes were $74 \times 66 \times 64$, $76 \times 66 \times 41$, $70 \times 65 \times 30$, $55 \times 50 \times 35$, $41 \times 39 \times 21$, $45 \times 32.5 \times 15$, and $31 \times 23.5 \times 12$. We used our base-set with 7 bases (corresponding to bases 15×55, 25×45, 25×25, 45×45, 45×55, 65×85, 85×85) with flexible packaging option to pack each order and find the shipping cost using the box with this base-set. To calculate the shipping cost, the 3D packaging software was used and each order was assigned the base which achieves the minimum shipping cost.

The comparison of total cost with the existing box-set and that with the 7 base set generated by CMM shows a percentage reduction of 11.24 in the shipping cost. Another interesting finding is that the total shipping cost using only the 6-base set (corresponding to bases $85 \times 65, 45 \times 25, 75 \times 35, 25 \times 25, 55 \times 15, 45 \times 45$) already reduces the shipping cost by 8.53% as compared to the benchmark performance. The option of the flexible height can thus potentially allow the DC to use less bases to meet the needs of the packaging operation. It should be mentioned here that the cost reduction highlighted here is the result of solution suggested by CMM model as well as the flexible packaging option.

2.5 Conclusions

There are many real life situations in product line design where products are interdependent. In this thesis we proposed a model, called Cross Moment (CMM) model, to solve this class of problems. The consumer choice model proposed here is parsimonious in that, it uses only the mean and covariance information of utility evaluations across the products. Despite exploiting just the first and second moment information, our results suggest that the model generates reasonable choice estimates, even in the situations where product utilities are highly correlated. The key advantage of CMM is that it avoids the need of exhaustive simulation to generate the choice probability estimates. This allows CMM to be embedded into a heuristic to search for a good set of products for the product line design problem. Also CMM is computationally tractable as a convex semidefinite optimization problem, which makes it more attractive for practical purposes.

We used three examples from various settings to validate the quality of choice predictions obtained from CMM. We showed that CMM is able to circumvent the issues associated with IIA (Independence from Irrelevant Alternatives) and IPS (Invariant Proportion of Substitution) properties inherent in many popular consumer choice models.

We applied CMM to address a flexible packaging design problem faced by a local service part supplier and augmented the ease of choice estimation with a greedy-swapping heuristic. Finally we compared the performance of CMM and MNL in this flexible packaging problem to check the efficacy of CMM. The product set delivered by CMM indeed seems better than the one suggested by MNL model, even though the latter is obtained via complete enumeration.

In real world, many supply chains involved both manufacturers and retailers. Their respective decisions affect the efficiency of the whole supply chain. Obviously interdependent characteristics among multiple products will affect the retailer's stocking strategies. To extend our CMM model to incorporate the inventory decisions, we will study the product line selection and inventory joint decisions from a retailer's standpoint in the next chapter.

3. PRODUCT LINE SELECTION AND INVENTORY JOINT DECISIONS

3.1 Introduction

From a retailer's perspective, product proliferation causes complexity to the fundamental assortment planning and inventory decisions. For example, a retailer who sells hand phones often faces various models of hand phones with different brands and different attributes such as camera, video, radio, mp3, bluetooth and so on. The retailer who is a profit maximizer not only needs to consider which brands to carry and which models to put in display, but also needs to determine the inventory levels for each model in order to meet customers' demand while keeping the inventory cost low.

To achieve these goals, the retailer needs to predict and incorporates the customers' demand pattern into his product offering and inventory decisions. To a large extent, the retailer's ultimate profit depends on the accuracy and reasonability of her forecast of her customers' demand for each product model respectively. To make the forecast practical, recent literature in "random utility theory" acknowledged that the randomness not only exists among the utilities of different individuals, but also in the utility of the same individual due to its "unobservable" nature. We will examine different customer choice models under random utility framework in this chapter.

Based on random utility theory, multinomial logit model (MNL) has been applied widely in both marketing and operation management literature. Tractability and simplicity are the key advantages of this model. However, due to its simplified assumption of independence among products utilities, MNL suffers from the property of Independence of Irrelevant Alternatives (IIA). That is, the ratio of the probabilities of any two alternatives is entirely unaffected by the systematic utilities of any other alternatives. Consequently, MNL will exaggerate the market share of similar products. A classic example, the blue bus/red bus paradox, illustrates this prediction bias inherent in MNL model. In the setting where three kinds of transportation methods are provided, namely, train, red bus and blue bus, the customers have the same utility mean on either taking the train or the bus which can be blue or red. Then the MNL model will give a choice prediction as $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ for each vehicle, while the actual situation should be $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$.

Based on attribute model, the locational model to some extent tackles the IIA property. But this model is not tractable for high dimensions case when we consider many attributes of the products.

To address the above issue, a new consumer choice model based on both random utility and attribute model is proposed in Chapter 2, with the name Cross Moment Model (CMM). This CMM model relates the attributes to the customer utilities in a linear function, and satisfactorily models and factors in the complicated utility correlations among the different products. In Chapter 2, we showed that the CMM model can be applied to the product design issues and bring a significant improvement over the MNL model. However, we did not take inventory into consideration, therefore the complicated interaction between customer choice and retailer's inventory level resulting from substitution have not been modeled and studied.

Thus in Chapter 3, we will incorporate this CMM model into the assortment planning problem. We will examine the characteristics of its solutions and compare the performance of CMM model and MNL model when being applied to the retailer's assortment planning problem.

Additional complexity in retailer's assortment and inventory decision comes from the customers' substitution behavior. It is common for a customer to enter a store, looking for a particular product, but not able to find it thus settle for another similar product instead. This is called substitution. Substitution depends on product availability as well as product accessibility.

To illustrate product availability, we consider hand phones as an example again. Assume there are only 3 attributes: camera, color and radio. If two hand phones have both 2-megapixel camera and radio, but differ in color, and the third hand phone is totally different which is in quirky color and has neither camera nor radio, then we can say the first two hand phones are closer to each other. Close products provide the availability for substitution.

On the other hand, accessibility measures the ease of access for the customer to make the substitution. For example, in a supermarket, it is easy for customers to substitute the product in mind which is out of stock with other similar products which are nearby. This is not the case for catalogue shopping when customers have no idea about the product availability.

Therefore, substitution depends on both product availability and product accessibility. In this chapter, we will consider three different types of substitution: static substitution, perfect substitution, and dynamic substitution. We will examine the performance of our model for retailer assortment planning problem under each type.

Our contributions can be summarized as follows:

a) We build a framework and a tractable approach to factor product inter-dependence into retailers' assortment decisions through the embedded customer demand forecast process. This enables us to propose a method to deal with both the horizontal and vertical differentiated products in the joint product line selection and inventory problem. In contrast, the previous MNL and locational models only target at horizontal differentiation.

b) We examine the influence of the inter-dependence among product attributes on the retailer's assortment decisions. By taking correlation among products' utilities into account, we show that when the correlation increases, under our model, the optimal offer set should be smaller and the inventory level should be more concentrated, and the expected profit can be achieved at a higher level.

3.2 Retailer's Assortment Planning Model with Customer Choice Embedded

Consider a big retailer such as Walmart who faces a procurement problem whereby he needs to choose the products to carry (that is, to determine his offer set) from M alternatives as well as to decide the order quantity for each product. This product line selection and order quantity joint problem is also called the assortment planning problem. We consider a single period problem here.

Suppose there are M products in the market, and the retailer needs to select some products to carry and the inventory level q_j for each product j. The revenue of each product is denoted as r_j and procurement cost s_j. Excess demand at the end of the period is lost and excess inventory is salvaged at value v_j. We assume $r_j > s_j > v_j$. For simplicity of the expressions, we let $v_j = 0$ in our following presentation. Let N denote the total number of customers in the market.

To estimate the choice probability p_j, we use the consumer choice model developed in Chapter 2. Suppose the customers have only rough knowledge about what products are offered in the whole market before heading to a particular retailer. To avoid the complexity of incorporating competition modelling from other retailers, we propose a virtual product, numbered zero, to represent the customer's reservation utilities, say, for the not-purchase option from this particular retailer for the time being. And for simplicity, we assume this reservation utility is constant.

Since the customer can not observe the inventory until he enters the store, he can only buy his utility-maximization product at the store if it is available. If it is out of stock, the customer may or may not substitute the product in mind with other products at this store, depending on different nature of the products. For example, if the customer is looking for a specific medicine, he is unlikely to substitute and may choose to continue his search in other stores. On the other hand, substitution will "always" happen if the searching cost is extremely high. For example, when you invite some guests to a restaurant and realize the restaurant ran out of your favorite drinks after you have ordered the main dish, then you may switch to whatever drinks they offer. The first case permits no substitution and the second case allows perfect substitution.

In addition to the above two extreme cases, substitution can happen in a limited manner. For example, it is common for different models of an electronic product to have various appearances and functions while sharing certain key features. In this case, the customers may restrict the substitution within a small group. For example, some customer may only accept any hand phones with three-mega camera and blue-tooth technology.

Limited substitution is common in practice, however it is difficult to solve when it is factored into the assortment planning problem. Next, we will first examine the problem with no substitution or perfect substitution. We will show that these two extreme cases provide the lower bound and upper bound for limited substitution. We will also derive some heuristics for limited substitution and examine the properties of the optimal solution.

3.2.1 Static Substitution

The concept of *Static substitution* first appeared in van Ryzin and Mahajan[53]. Under static substitution, customers choices are based only on knowledge of the variants but not on inventory status of the offer set, so they do not substitute if the store has run out of their first choice product by the time of their arrival.

With static substitution, the demand for product j, is simply the number of customers that pick product j as their first choice, which is dependent on the offer set, but independent of the inventory levels. A typical case in life of static substitution is catalogue shopping without back-order, where customers are only aware of product variants on the timing of their purchase choice, assuming availability for all products offered.

An extreme case would be *No Substitution*. No substitution requires the consumers to possess perfect information of the product variants on the market, as well as the absolute loyalty to their first choice. Thus the demand for any particular product is quite inflexible, independent of both the offer set and the inventory levels. In fact, we can deem *No Substitution* as a special case of the *Static substitution*. Since in no substation case, the customer simply knows all the product variety information, and hence it is equivalent to the case when the retailer adopts the whole product spectrum as the offer set with static substitution conditions. [1]

Given any complete product set, the retailer's problem with product line selection and order quantity joint decisions can be formulated as follows:

[1] We augment with the Greedy and Swapping Heuristic to the following solution to find the optimal offer set in static substitution case.

$$
\begin{aligned}
\Pi = \max_{y_j, q_j} \quad & E\left[\sum_{j=1}^{M} r_j \min(\tilde{D}_j, q_j) - s_j q_j\right] \\
s.t. \quad & 0 \le q_j \le y_j N \quad \forall j = 1, 2, ..., M \\
& y_j \in \{0, 1\} \quad \forall j = 1, 2, ..., M.
\end{aligned}
\tag{3.1}
$$

The random variables $\tilde{D}_j$ above denote the demand for each product. Assume homogeneous consumer segment and from the choice probability, we can derive the demand $\tilde{D}_j$ as a binomial distribution with choice probability p_j and Poisson consumers arriving with rate λ. Let N denote the total number of customers arrived and N_j denote the number of customers with product j as their first choice. Then N is Poisson with rate λ, and N_j is Poisson with rate $\lambda * p_j$. Here y_j is the zero-one variable to decide whether product j will be selected into the offer set [2]. To derive the optimal expected profit in closed form, we use Normal approximation for Poisson distribution. That is, we assume that the demand for product j is normally distributed with mean $\lambda * p_j$, and standard deviation $\sqrt{\lambda * p_j}$. Indeed, inventory model in practice is quite common to adopt simple distribution assumptions of this sort. Readers can refer to [38] and [22]. If the market shares are predicted using MNL model, than we call the combined model as MNL-INV model, and if CMM model is used to produce the market shares, we call the whole method CMM-INV model.

[2] It is different from in Chapter 2, where we use y_j for choice probability.

Let K denote the number of products we would like to choose from M products' set. Let Y denote the product offer set as a vector $(y_1,y_2,y_3,...,y_M)$, where $\sum_{j=1}^{M} y_j = K, y_j \in \{0,1\}, j = 1,2,3,...,M$. Given an offer set Y, the expected total profit of this offer set can be written as follows:

$$\begin{aligned} \Pi(Y) = \max_{q_j} \quad & E\Bigg[\sum_{j=1}^{M} r_j \min(\tilde{D}_j, q_j) - s_j q_j\Bigg] \\ s.t. \quad & 0 \leq q_j \leq y_j\lambda \quad \forall j = 1,2,3,...,M. \end{aligned} \tag{3.2}$$

With static substitution condition, profits from different products will not affect each other, therefore problem(3.2) can be decomposed into M subproblems, with each problem as:

$$\Pi_j = \max_{q_j \geq 0} \quad E\Bigg[r_j \min(\tilde{D}_j, q_j) - s_j q_j\Bigg]$$

Proposition 3: Given any offer set Y, under static substitution, the choice probability is determined as $p_j, \forall j = 1,2,3,...,M$, then the optimal order quantity for each product can be obtained as $\bar{q}_j^* = F^{-1}(\theta_j) = \lambda p_j + z\sqrt{\lambda p_j}$. The optimal expected profit achieved is

$$\Pi_j^* = (r_j - s_j)p_j\lambda - r_j\sqrt{p_j\lambda}\phi(z). \tag{3.3}$$

where $z = \Phi^{-1}(\theta_j)$ is the service level, $\theta_j = \frac{r_j - s_j}{r_j}$ is the critical fractile, and $\phi(\cdot)$ and $\Phi(\cdot)$, respectively, denote the density function and cumulative distribution function of the standard normal distribution.

Proof. Newsboy model can be used to get the optimal order quantity and optimal expected profit for each product respectively. The advantage of the normal demand distribution is that it gives a close form expression for the optimal expected profit. Refer to Porteus ([50]:p13), and plug in our demand mean of $p_j\lambda$, and standard deviation of $\sqrt{p_j\lambda}$ to get the optimal expected profit for product j in formula (3.3). ■

Using the above proposition, we can propose a two level solution framework to the original problem (3.1). At the lower level, for any given offer set Y, inventory decision for each product will be determined based on the demand forecast derived from customer choice prediction, and expected profit objective Π can be obtained for each product correspondingly; At the higher level, offer set decision will be made by maximizing the total expected profit, which is the sum of the expected profit for each product as we derived in the lower level.

Lemma 1: Optimal expected profit for product j is strictly convex in choice probability p_j.

Proof. From Formula (3.3), we see first term $(r_j - s_j)p_j\lambda$ is linear in p_j, and second term $-r_j\sqrt{p_j\lambda}\phi(z)$ is strictly convex in p_j, therefore, optimal expected profit for product j Π_j is strictly convex in choice probability p_j. ■

From the above Proposition 3 and Lemma 1, we can construct the following solution for static problem (3.1) under the "MNL-INV" model.

Proposition 4: Given the complete product set, when all the products have same procurement costs and revenues, the optimal offer set of Problem (3.1) contains the products with highest market shares in the complete product set. That is, if we index the products according to their market share p_j in a descendant order, $p_1 \geq p_2 \geq p_3 ...$, and let $A_k = \{1, 2...k\}$ denote the set consisting of first k variants, then the optimal offer set using MNL choice model in static substitution will always belong to one of the sets A_k. And the offer set will cover more products as the customer population increase.

Proof. van Ryzin and Mahajan[53] proved the similar "nested set property" for MNL-INV assortment problems, although they used different demand variance in the product demand distributions. Our proof follows their proof methodology closely as shown in the following.

Let $w_j = e^{\beta V_j}, j \in Y$, $w_0 = e^{\beta V_0}$, where V_0 is the reservation utility from no purchase, then from Formula (1.1), we get the choice probability under MNL model: $p_j = \frac{w_j}{\sum_{j \in Y} w_j + w_0}$.

Now let us consider perturbing the offer set by adding one more product from the remaining set, which has expected utility V_n. Let $\delta = e^{\beta V_n}$, then the new choice probability with this new product incorporated to the offer set is: $p_j(\delta) = \frac{w_j}{\sum_{j \in Y} w_j + w_0 + \delta}$.

From Formula (3.3) and sum in products, we can get the total expected profit with the new offer set.

$\Pi(\delta) = \frac{f(\delta)}{g(\delta)}$, where

$$\begin{aligned} f(\delta) &= \textstyle\sum_{j \in Y} w_j + \delta + w_0 \\ g(\delta) &= \begin{array}{l} (r-s)\lambda(\sum_{j \in Y} w_j + \delta) \\ - \; r\sqrt{\lambda}\frac{e^{-z^2/2}}{\sqrt{2\pi}}(\sum_{j \in Y} \sqrt{w_j} + \sqrt{\delta})\sqrt{(\sum_{j \in Y} w_j + \delta + w_0)}. \end{array} \end{aligned} \tag{3.4}$$

Now using the following result from Mangasarian (1969): The function $g(\cdot)/f(\cdot)$ is quansiconvex on X if (1)$g(\cdot)$ is convex and $f(\cdot) > 0$ for all $v \in X$ and (2) $f(\cdot)$ is linear on X.

As we can see the function defined in $f(\delta)$ is linear in δ and greater than zero for all possible δ, and easy to get the function in $g(\delta)$ is convex in δ. Thus, the total expected profit $\Pi(\delta)$ is quasi convex in δ.

Because the function is quasi convex, it follows that the maximum profit is achieved at the end points of the interval for δ. Therefore the expected profit is maximized by either not adding any more products or by adding the product with highest w_j among those not included in Y.

To proof the structure of the optimal offer set, we consider any optimal set Y^*, the number of products in set Y^* is m, and their w_j^* is order as $w_1^* \geq w_2^* \geq ... \geq w_M^* > 0$. If $Y^* = A_m$, the proposition holds already. Otherwise, then there exists a w_j not belongs to Y^* such that $w_j > w_m^*$. Then from the quasi convexity of the total expected functions, it must also be true that we can either remove w_m^* or exchange it for $w_j > w_m^*$ without decreasing profits. Redefine Y^* to be this new set and repeat the procedure.

Eventually the optimal offer set arrives at $Y^* \in \{A_1, A_2, ..., A_M\}$ ■

For CMM-INV model, the market share of any new introduced products will affect the previous products depending on the complicated correlation matrix structure. Therefore, for any new subset, we need to recalculate the market share again. The above Proposition 4 may not apply to CMM-INV model. However, when all the products in the offer set have same utility variance and correlations, we adopt this "nested set heuristic" to derive offer set for CMM-INV model. To be more specific, we only consider sets $A_k = \{1, 2...k\}$, which only consists of first k products, according to their market share p_j in a descendant order. The whole solution algorithm framework is displayed as follows in Figure 3.1.

Proposition 5: Under static substitution, when utility variance of the two products are the same and small enough, the total inventory level from CMM-INV Model is lower than that from MNL-INV model. And the inventory levels are more concentrated on popular products. Such inventory gap and skew increase as the positive correlation among utilities of the two products increase.

Proof.

Compare MNL market share and CMM market share under correlations equal to zero for two product case. Suppose $\mu_1 > \mu_2$, let y_1^{CMM} denote product one's market share under CMM model, and y_1^{MNL} denote product one's market share under MNL model, then the difference Δy is as follows.

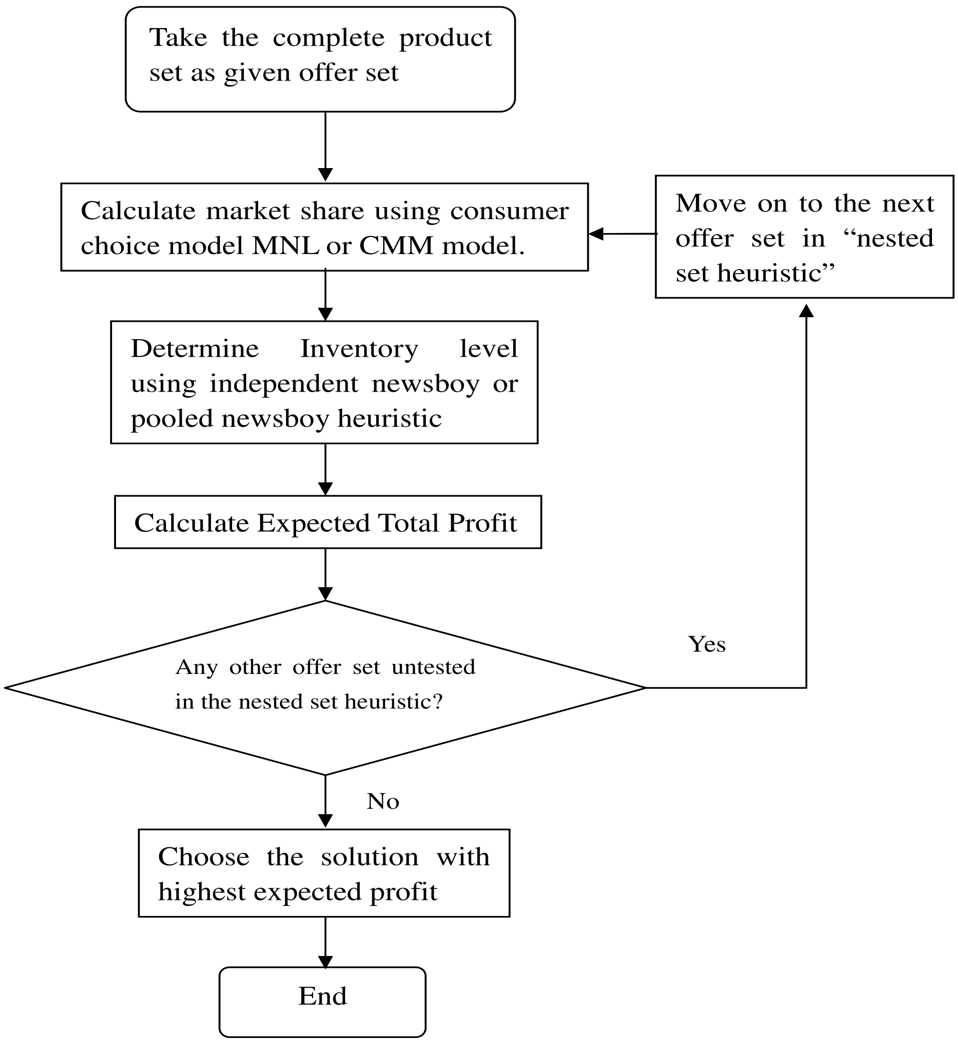

Fig. 3.1: Algorithm for assortment problems with CMM-INV model

$$\Delta y = y_1^{CMM} - y_2^{MNL} = \frac{1}{2}\left(\sqrt{\frac{(\mu_1-\mu_2)^2}{\frac{\pi^2}{3\beta^2}+(\mu_1-\mu_2)^2}} - \frac{e^{\beta\mu_1}-e^{\beta\mu_2}}{e^{\beta\mu_1}+e^{\beta\mu_2}}\right). \tag{3.5}$$

As β increase, above (3.5) also increase. Therefore, we can claim, there exist a $\underline{\beta}$, when $\beta > \underline{\beta}$, $\Delta y > 0$. Therefore, as the utility variance small enough, CMM model will produce more concentrated market share then MNL model.

From Proposition 3, summing the inventory for all products,

$$\sum_{j\in Y} q_j^* = \lambda + z\sum_{j\in Y}\sqrt{\lambda p_j}. \tag{3.6}$$

Then for two products, total inventory Q is: $Q = \lambda + z\sqrt{\lambda}(\sqrt{p}+\sqrt{1-p})$

Easy to see Q is concave in p, with maximum inventory at $p = \frac{1}{2}$. Then from Proposition 7 and 8, we prove that market share will become more concentrated as utility correlation increase. Therefore, total inventory will decrease with the utility correlation increase. As the market share of popular product under CMM model will be larger than under MNL model when utility variance is small enough, therefore the total inventory will be smaller under CMM model than MNL model when utility variance is small enough. ■

Proposition 6: Under static substitution, when utility variance of two products are same and small enough, the expected profit from CMM-INV Model is bigger than that from MNL-INV model. And the difference in total expected profit would increase as the positive correlation among product utilities increase.

Proof. From expected profit formula (3.3), we can sum up to get the total expected profits for the two products:

$$\Pi = (r - s)\lambda - r\sqrt{\lambda}\phi(z)(\sqrt{p} + \sqrt{1-p}). \tag{3.7}$$

As $\frac{d^2\Pi}{dp^2} = r\sqrt{\lambda}\phi(z)(\frac{1}{4}p^{\frac{-3}{2}} - \frac{1}{4}(1-p)^{-\frac{3}{2}}) > 0$ for $p > 0$ and optimal $p^* = \frac{1}{2}$. Therefore the total expected profit will increase as the market share of popular product increase. And from Proposition 7 and 8, we prove that CMM model will produce more concentrated market share as utility correlation increase. Therefore, expected profits will also increase with the utility correlation increase. From the Proof of Proposition 5, market share of popular product under CMM model will be larger than under MNL model when utility variance is small enough, therefore the expected profits will be larger under CMM model than MNL model when utility variance is small enough. ■

3.2.2 Perfect Substitution

Under perfect substitution, once the customer arrives at the retailer, he will definitely buy one product in the end. If his first choice product is out of stock, he will substitute with the second choice, and if the second choice is also out of stock, he then changes to the third, and so on. This extreme case represents an extreme loyal customer or the cost of switching to another retailer is extremely high (e.g. monopoly retailer).

Perfect substitution will provide an upper bound for the retailer in terms of the total expected profit. With perfect substitution assumption, the retailer will only stock the most profitable product. By doing so, he forces all the customer to purchase this product and achieves highest expected profit.

The profit under perfect substitution is simply: $(r_m - s_m)N$, where r_m and s_m are revenue and cost for product m, which produced highest profit margin.

3.2.3 Dynamic Substitution

Under dynamic substitution, the customer only substitutes when the utility from available products are higher than his reservation utility. That is to say, the customer will buy his first choice if available, and if not he will change to his second choice, he then change to third, and so on, until nothing is worth of buying. This is the most common practice in our life, where we face limited competition among the retailers and the customers retain certain level of reservation utilities.

Given the same product set and inventory level, we can expect such substitution will top on extra expected profit compared to the static substitution case, because it can actually satisfy part of the previously lost sale with alternative products. So we could use static substitution as its lower bound. And since perfect substitution can use up all the available inventory to satisfy the previously lost sale, it provides an upper bound for limited substitution problem.

Mahajan and van Ryzin(2001)[38] showed that assortment planning and inventory decisions under dynamic substitution is non-concave and generally hard to solve even with MNL choice model. Therefore, pooled newsboy model has been adopted as a heuristic to tackle this complicated problem. We will adopt the same *pooled newsboy* model (Details will be explained later in the thesis) to handle inventory decision here since, instead of finding an exact solving algorithm, our main purpose is to investigate the influence of correlation among product utilities by comparing the performance of two customer choice model: MNL-INV and CMM-INV under different situations. We will plug in this offer-set dependent inventory decisions to the nested set heuristic to derive the offer set as displayed in Figure 3.1.

Heuristic Policy: The "pooled newsboy" model calculates the inventory as if all the products in the offer set can be substituted freely. Therefore, demand is pooled and an aggregate quantity is determined to maximize the total profits, and then it allocates in a rough-cut fashion to individual variants depending on their choice probabilities.

Specifically, let $p(S) = \sum_{j \in S} p_j(S)$ denote the probability that a customer chooses at least one of the products in the offer set S. Then the optimal aggregate inventory level for the whole set, denoted $x(S)$, is computed using $x(S) = \lambda p(S) + z\sqrt{\lambda p(S)}$, where z is the newsboy fractile determined using a weighted average price and cost as follows: $z = \Phi^{-1}(1 - \frac{\overline{s}}{\overline{r}})$. Here $\overline{s}$ and $\overline{r}$ are averaged based on the choice probabilities: $\overline{r} = \frac{\sum_{j \in S} p_j(S) r_j}{p(S)}$, $\overline{s} = \frac{\sum_{j \in S} p_j(S) s_j}{p(S)}$, and $\Phi(z)$ denotes the c.d.f. of a standard normal random variable. The Pooled newsboy inventory for variant j, denoted x_j^p, is then determined by allocating the aggregate inventory proportional to choice probabilities $p_j(S)$

as follows: $x_j^p = x(S)\frac{p_j(S)}{p(S)} \quad j \in S$.

3.3 CMM Predictions for Two Product Case

In the last chapter, we have demonstrated the closeness between the market share predictions from the close form solution of CMM and the actual situations for two *independent* products. Whereas what we aim to investigate through CMM model here is the relationship between the utility correlation and the concentration degree of the choice probabilities.

3.3.1 Close Form Solution from CMM

Proposition 7: Consider $m = 2$, with means μ_1 and μ_2 ($\mu_1 \geq \mu_2$), and $Q_{12} = \mu_1\mu_2 + \rho_{1,2}\sigma_1\sigma_2$. The choice probabilities obtained from the CMM models have the following closed form solution:

$$\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{(\mu_1 - \mu_2)^2}{s^2 + (\mu_1 - \mu_2)^2}},$$

where

$$s = \begin{cases} \frac{|\sigma_1^2 - \sigma_2^2|}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}}, & 1 > \rho_{1,2} \geq min\{\frac{\sigma_2}{\sigma_1}, \frac{\sigma_1}{\sigma_2}\}; \\ \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}, & \rho_{1,2} < min(\frac{\sigma_1}{\sigma_2}, \frac{\sigma_2}{\sigma_1}); \\ 0, & \rho_{1,2} = 1, \sigma_1 = \sigma_2 \end{cases}$$

Proof. Consider a set of correlated products with means μ_i variances σ_i^2 and correlations $\rho_{i,j}$. By choosing a different set of orthonormal basis, the optimal v_i in Formulation (2.9) can be rewritten as:

$$v_i \quad = \quad \mu_i v_0 + \sigma_i s_i, \qquad i = 1, \ldots, m. \tag{3.8}$$

where the vectors $(v_0, s_1, \ldots, s_m)$ forms a set of orthonormal basis, with $v_0 \cdot e_{m+1} = 0$, and $s_i \cdot e_{m+1} = \Delta/\sigma_i$ for all i. This can be solved in closed form for the case of two products.

For two products case, the set of orthonormal basis corresponding to above formula (3.8) can be expressed as follows, with $\alpha_1, \alpha_2, \Delta$ as coefficients to be determined:

$$v_0 = \begin{pmatrix} \sqrt{y}_1 \\ \sqrt{y}_2 \\ 0 \end{pmatrix}, s_1 = \begin{pmatrix} \alpha_1 \sqrt{y}_2 \\ -\alpha_1 \sqrt{y}_1 \\ \Delta/\sigma_1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -\alpha_2 \sqrt{y}_2 \\ \alpha_2 \sqrt{y}_1 \\ \Delta/\sigma_2 \end{pmatrix}.$$

As the orthonormal basis, v_0, s_1, s_2 should satisfy $v_0 \cdot v_0 = 1, s_1 \cdot s_1 = 1, s_2 \cdot s_2 = 1$, which translates to:

$$y_1 + y_2 = 1, \; \alpha_1^2 + \Delta^2/\sigma_1^2 = 1, \; \alpha_2^2 + \Delta^2/\sigma_2^2 = 1.$$

Consider the constraint conditions in formulation (2.9). First constraint translates to:

$$-\alpha_1 \alpha_2 + \frac{\Delta^2}{\sigma_1 \sigma_2} = \rho_{1,2}.$$

Second to last constraints are satisfied as the orthonormal basis constructed deliberately.

We can solve to obtain, when $\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2 \neq 0$,

$$\Delta = \sigma_1\sigma_2\sqrt{\frac{1-\rho_{1,2}^2}{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}},\ \alpha_1 = \frac{|\sigma_1-\rho_{1,2}\sigma_2|}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}},\ \alpha_2 = \frac{|\sigma_2-\rho_{1,2}\sigma_1|}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}}.$$

Otherwise, when $\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2 = 0$, which leads to $\rho_{1,2} = 1, \sigma_1 = \sigma_2$, we can deduce $\Delta = \pm\sigma_1, \alpha_1 = \alpha_2 = 0$ accordingly.

Substitute the above solution for $\alpha_1, \alpha_2, \Delta$ to the objective in formulation (2.9), we obtain the optimal choice for y_1 and y_2 by solving the following quadratic problem:

$$\begin{aligned} &\max \quad \mu_1 y_1 + s\sqrt{y_1 y_2} + \mu_2 y_2 \\ &s.t. \quad y_1 + y_2 = 1 \end{aligned}$$

where

$$s = \begin{cases} \frac{|\sigma_1^2-\sigma_2^2|}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}}, & 1 > \rho_{1,2} \geq min\{\frac{\sigma_2}{\sigma_1}, \frac{\sigma_1}{\sigma_2}\}; \\ \sqrt{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}, & \rho_{1,2} < min(\frac{\sigma_1}{\sigma_2}, \frac{\sigma_2}{\sigma_1}); \\ 0, & \rho_{1,2} = 1, \sigma_1 = \sigma_2 \end{cases}$$

This gives,

$$y_1 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{(\mu_1-\mu_2)^2}{s^2+(\mu_1-\mu_2)^2}}$$

$$y_2 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{(\mu_1 - \mu_2)^2}{s^2 + (\mu_1 - \mu_2)^2}}.$$

Proposition 8: For two product case, when the variances are equal, the market share difference is monotonously increasing with the utility correlation increase.

Proof. We can see three different cases in terms of the relationship between market share difference and the utility correlation for the two products.

Case 1. When $\mu_1 = \mu_2$, then regardless of correlation $\rho_{1,2}$, we always get $y_1 = y_2 = \frac{1}{2}$, and hence the market share difference $|y_1 - y_2| = 0$. Otherwise, the market share difference depends on the correlation $\rho_{1,2}$. ■

.

Case 2. When $\mu_1 \neq \mu_2$ and $\sigma_1 = \sigma_2$, then when $\rho_{1,2} = 1$, we have $s = 0, y_1 = 1, y_2 = 0$ and market share difference $|y_1 - y_2| = 1$. This corresponds to the case when one of the products totally dominates the other. In this case, if the utility correlation $\rho_{1,2}$ decreases from 1, s will increase accordingly, which shrinks the market share difference. In sum, under Case 2, where two products have different utility means, but same utility variance, we can see the market share difference positively depends on the utility correlation.

Case 3. When $\mu_1 \neq \mu_2$ and $\sigma_1 \neq \sigma_2$, since $\rho_{1,2} \in [0, 1]$, then s will take two segments' expression. For $\rho_{1,2} \in [0, min(\frac{\sigma_1}{\sigma_2}, \frac{\sigma_2}{\sigma_1})]$, s decreases as the correlation increases; and for $\rho_{1,2} \in [min(\frac{\sigma_1}{\sigma_2}, \frac{\sigma_2}{\sigma_1}), 1]$, s will increase with the correlation. Consequently, the market share difference $|y_1 - y_2|$ will first increase with the correlation and then decrease with the correlation. ■

.

We present the dependence of market share gap on utility correlation in different scenarios in the following table and figure.

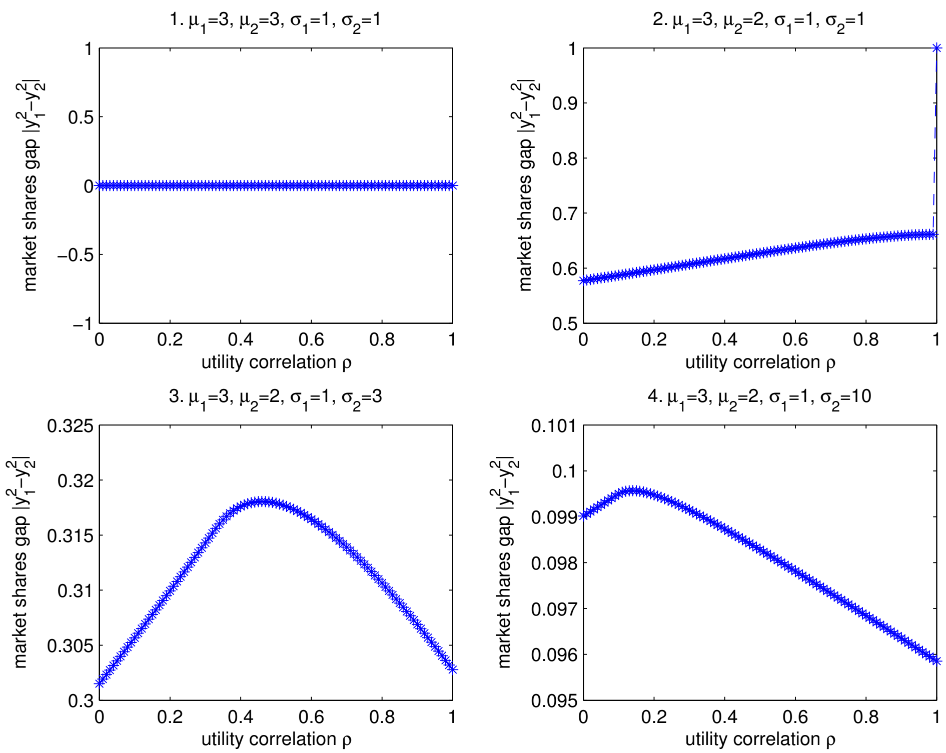

Fig. 3.2: Dependence of market shares gap on utility correlations predicted by CMM

Remarks: The insights we can get from the above analysis are:

1. As long as the utility means are equal, the market share will split evenly between two products, regardless of the correlation structure.

2. With different means, but the same variance, the market shares' difference will expand as the two products share more common attributes. This conforms to our intuition. As we expect the inferior (in mean value) product will lose more market share if it has less distinct attributes from the

Scenario No.	μ_1	μ_2	σ_1	σ_2	Curve Shape
1	3	3	1	1	Horizontal Line
2	3	2	1	1	Monotone Increasing
3	3	2	1	3	First Increasing, then Decreasing
4	3	2	1	10	First Increasing, then Decreasing

Tab. 3.1: Four scenarios to display dependence of market shares gap on utility correlations predicted by CMM

superior product. The extreme case is when the utility correlation reaches 1, all the market share goes to the superior product.

3. With different utility means and variances, the relationship between market shares' difference and utility correlation becomes less stable. Firstly, the absolute value of the market share difference shrank dramatically when one of the variances increased. And also the direction of the market share difference-correlation line changes at certain point.

Interestingly, using probit model[3], which assumes multivariate normal distribution for the utilities, we may get the similar results through 100,000 cases simulation.

Comparing the results from CMM and Probit, we can find more fluctuations in the market share difference-correlation dependence predicted by the Probit model than CMM model as the variances difference increased. But both models agree that less absolute value for the market share difference results as the variance difference increase. In other words, when two products dramatically differ in their utility variances, it is not safe to use the intuitive rule that predicts higher market share difference with higher corre-

[3] Probit model adopts the multi-variate normal distribution for the customer utilities, and derives the choice probabilities through multi-dimension integration. Hence it is difficult to be applied to high dimension problems.

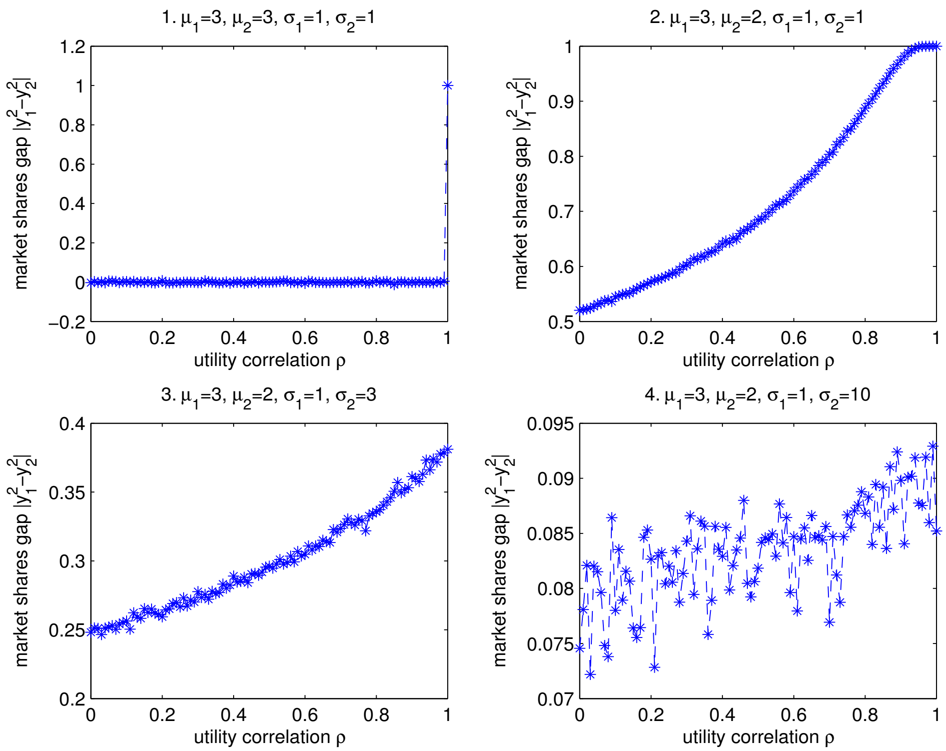

Fig. 3.3: Dependence of market shares gap on utility correlations predicted by Probit

R.V.	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Correlation with X	Mean	STD
X	0	5	7	9	N.A.	5.25	3.8622
Y_1	8	8	6	6	-0.8222		
Y_2	8	6	8	6	-0.5232		
Y_3	8	6	6	8	-0.2242	7	1.1547
Y_4	6	8	8	6	0.2242		
Y_5	6	8	6	8	0.5232		
Y_6	6	6	8	8	0.8222		

Tab. 3.2: Realization of random variables in four scenarios

lations. The intuitive rule only holds when two products have comparable utility variances.

3.3.2 Example

We illustrate the above counterintuitive findings with the following simple example. Consider two random variables: X and Y. For simplicity, we suppose the probability distribution as equal likelihood for 4 scenarios. We fix X's realizations in these four scenarios as 0,5,7 and 9. While we change the realizations of Y in six different cases to attain different correlations between X and Y. Y_i corresponds to case i for random variable Y. Their realizations and statistic parameters are showed in Table 3.2.

Through the straightforward simple comparison, we can get the market shares as displayed in Table 3.3, and we plot the market share difference against the correlations in Figure 3.4. Through this example, we can see the market share difference is not monotonously depend on the utility correlation. In other words, the market shares' difference may not always enlarge with the correlation increase. Note in this example, no predictive model is involved. The market shares are calculated directly from the actual utility realizations.

R.V.	Correlations with X	Market share Y	Market share X	Difference
Y_1	-0.8222	0.5	0.5	0
Y_2	-0.5232	0.75	0.25	0.5
Y_3	-0.2242	0.5	0.5	0
Y_4	0.2242	0.75	0.25	0.5
Y_5	0.5232	0.5	0.5	0
Y_6	0.8222	0.75	0.25	0.5

Tab. 3.3: Four scenarios to display dependence of market shares gap on utility correlations predicted by CMM

The actual results confirm our counterintuitive findings.

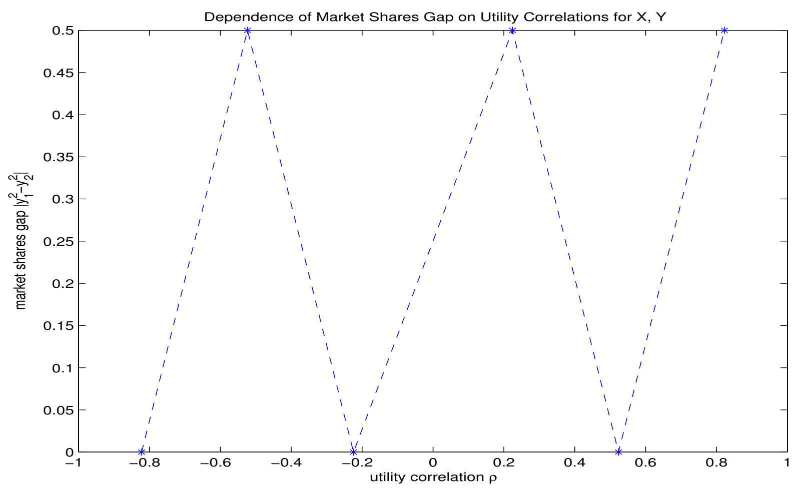

Fig. 3.4: Dependence of market shares gap on utility correlations for X and Y

3.4 Computational Results

As indicated in our analysis, inventory decision is significantly impacted by the market share prediction from the Choice Model. Therefore to get the sense how MNL and CMM model behave, we conducted some preliminary numerical experiments with the focus on comparison of the choice probabilities predicted by CMM and MNL. The insights we got from the experiments is summarized as following two cases.[4]

Case 1. Independent product utilities.

- For products with same utility means and variances, MNL and CMM both produce the evenly spread market shares.
- When the products have same variances, but different means, CMM model will assign bigger market shares than MNL to those products with higher means.
- When the products have same means, but different variances, CMM model will assign bigger market shares than MNL to those products with higher variances.

Case 2. Correlated product utilities. Positive correlation will accelerate CMM's skewing effect, namely, with higher positive correlation among product utilities, CMM will assign higher market shares to the popular products which have either higher utility means or higher variances.

In the rest of this section, we will incorporate the inventory level and offer set decisions into consideration. We will take example 1 from Mahajan and van Ryzin (2001)[38], and compare results from CMM-INV model with the MNL-INV model in the original paper. Consider 10 product candidates, n=10. The no-purchase option as a reservation utility level is denoted as $j = 0$. We associate this no-purchase utility to each customer to represent the reservation utility possibly from other channels or from the money saved.

[4] Since we will present more detailed study including inventory level later, the experiment data and results are omitted here. Though all the data and results are available on request.

Assume production costs s_j and prices r_j are same for all products, with $s_j = 3, r_j = 8, j = 1, ..., n$. The utilities means are of the form: $u_j = a_j - r_j, \forall j = 1, ..., n$, $u_0 = a_0$, where a_j is the quality indices. a_j is linearly decreasing: $a_j = 12.25 - 0.5(j-1), \forall j = 1, ...n, a_0 = 4$. The utilities' variances are constant, equal to 1.18, which translates to $\mu = 1.5$ in Gumbel distribution for MNL model.

We use independent newsboy model to compute the inventory levels for static substitution problem, and approximates the dynamic substitution with pooled newsboy solution. To find the offer set, for MNL-INV model, we employ Proposition 4; for CMM-INV model, we adopt the algorithm as shown in Figure 3.1.

3.4.1 Static Substitution

1. **Inventory Levels**

Figure 3.5 depicts the inventory levels computed by independent newsboy for all the ten products assuming all of them are offered. Observations are:

- When correlation=0, the total inventory under the CMM is lower than that suggested by MNL (29.7 vs 30.3)
- The gap of total inventory level became larger as the correlation increases (4.2 for correlation=0.5, 7.0 for correlation=0.8, and 11.2 for correlation=1).
- Compared to MNL solution with more evenly spread inventory across different products, CMM recommended higher inventory for the most

popular product (product 1), and lower inventory for the other products.

- CMM produced a more skewed inventory as the correlations among the products' utilities increase.

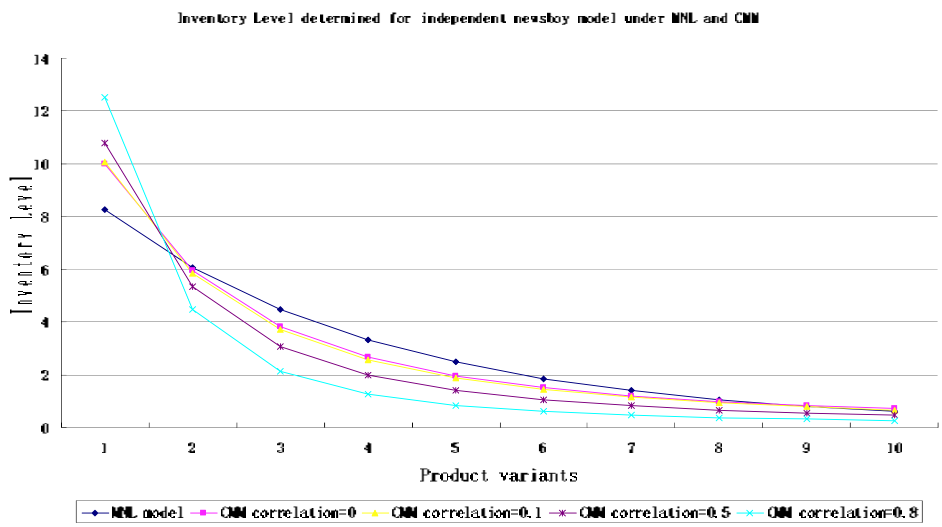

Fig. 3.5: Comparison of inventory levels for 10 products set under MNL and CMM

The prediction by CMM that higher correlation leads to more concentrated inventory is intuitive. As we know, with the correlation increase, it implies more common attributes shares by different products, then the uncertainty in customers' choice is less, which in turn results more concentration on the popular products. We can get in the extreme case, when all the products are perfectly correlated in their utilities, then inventory under CMM actually is only for product one at 19.1, and zero for all other products.

2. **Offer Set**

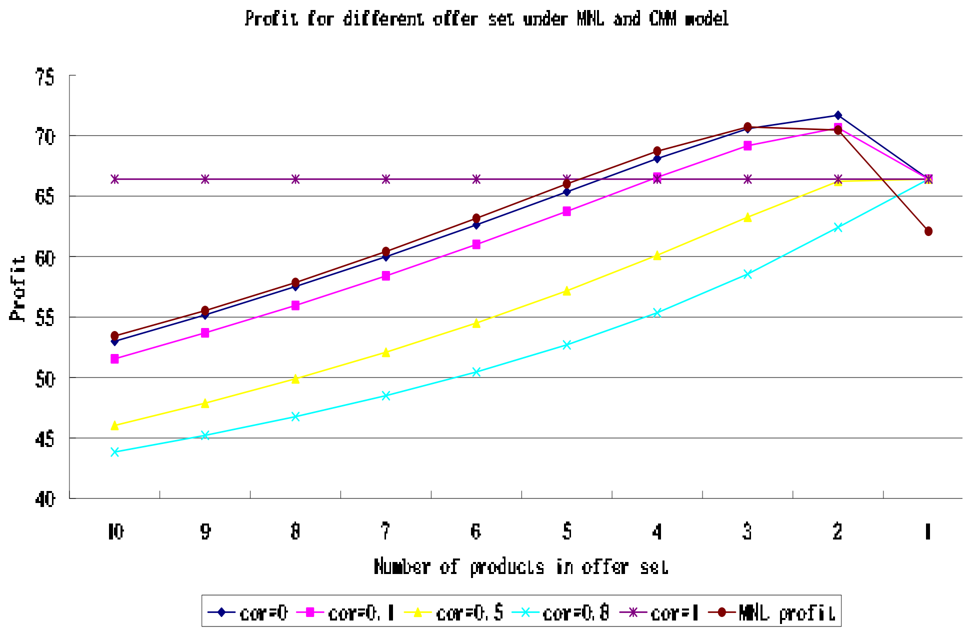

Fig. 3.6: Comparison of profit from different offer sets under MNL and CMM for N=30

Figure 3.6 shows the profit objective obtained from different offer sets under MNL and CMM model when customer number is 30. Observations are:

- The optimal offer set under MNL is {1,2,3}, with profit achieved at 70.72. It is 32% improvement from the results of 10 products' set assumed in the previous section. The reason is scale of economy resulting from eliminating those products with too small demand.

- The offer set under CMM model depends on the utility correlations, and generally tends to be narrower than under MNL model. Figure 3.6 shows for correlation 0.1 and independent case, optimal offer set is {1,2}, while for correlation 0.5 and 0.8, optimal offer set shrinks to {1} only.

- The profit objective achieved will decrease as the correlations among the utilities of products increase. The reason is that, according to CMM model, higher correlations among the products in the offer set will lead to more concentration of the choice on the popular products, and as well as the increase on market share of not-correlated product (no-purchase option). Therefore, as the utility correlations among offered products increase, the optimal offer set tend to downsize, and the covered market tend to shrink, resulting in lower total profit.

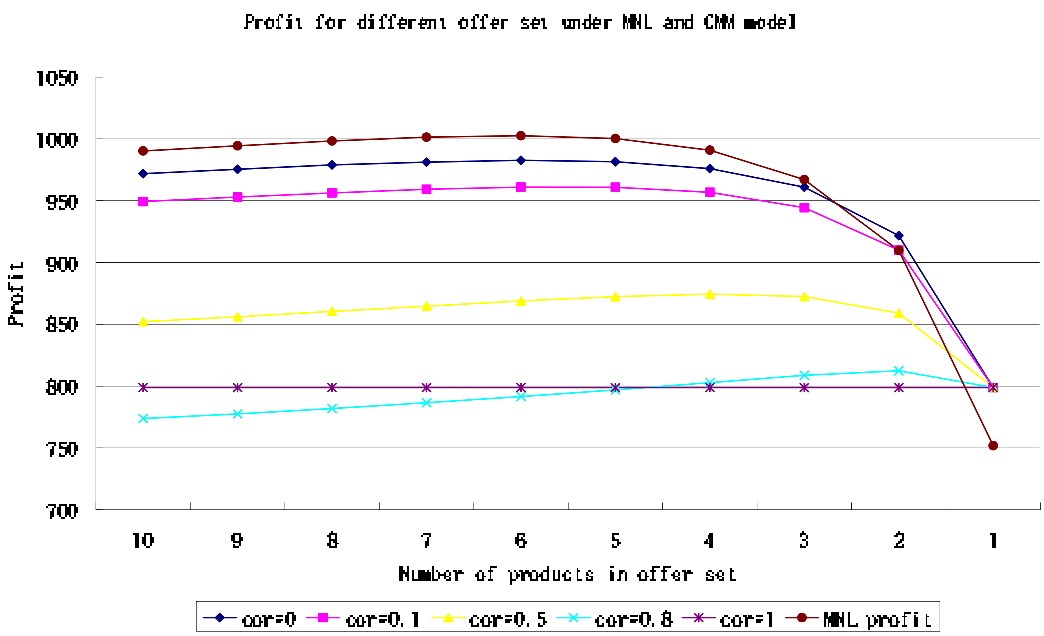

Fig. 3.7: Comparison of profit from different offer sets under MNL and CMM for N=300

Figure 3.7 depicts when customer number is 300, the profit generated from different offer sets under MNL and CMM model. Compared to Figure

3.6, we have observations:

- As the customer number increase, the optimal offer sets tend to expand. The optimal offer set under MNL is {1,2,3,4,5,6}, larger than {1,2,3} when customer number equals to 30.
- Size of the offer set under CMM model is decreasing with the correlations among the utilities of offered products. CMM recommends different offer sets, with set {1,2,3,4,5,6} for independent products and 0.1 correlations, set {1,2,3,4} for 0.5 correlation, set {1,2} for 0.8 correlation.
- Consistent with our previous result, CMM will produce an optimal offer set no bigger than MNL model.

Similarly, when the customer number increase to 30,000, we found the optimal offer set expands to including all the 10 products for both the MNL and CMM model (See Figure 3.8) .

We can depict the size of the optimal offer sets for different customer volume in Figure 3.9. First observation is positive relationship between customer volume and the size of offer set. The intuitive explanation is as the number of independent customer increase, we need to offer more variety to cover more market, and can take advantage of the scale economy at the same time.

Second observation is the size of offer set increases more rapidly with the customer volume for lower correlated products. The optimal offer size goes to 10 with 1500 customers for products with utility correlation less than 0.1;

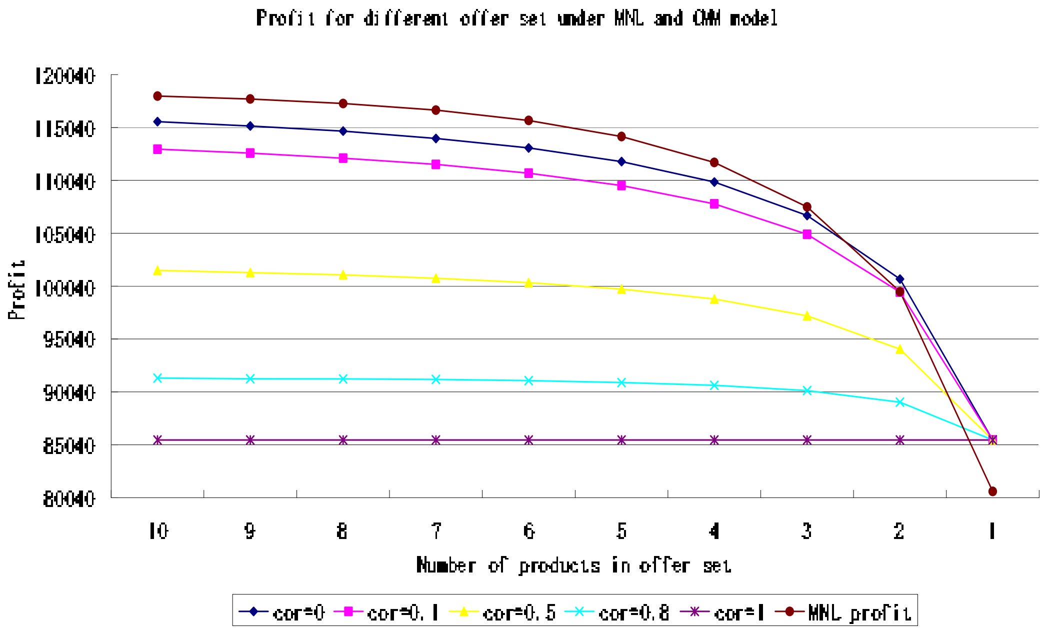

Fig. 3.8: Comparison of profit from different offer sets under MNL and CMM for N=30000

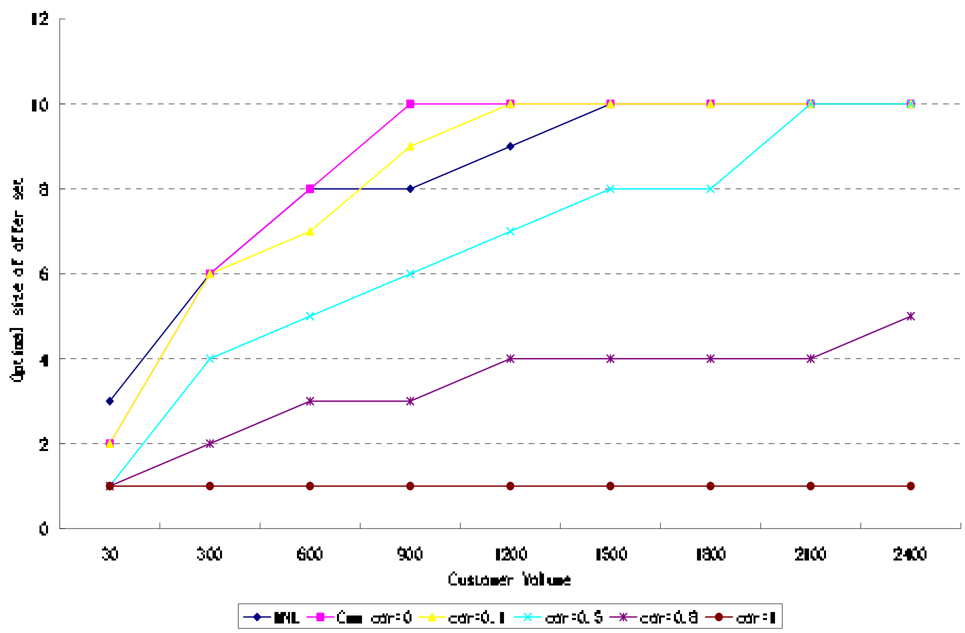

Fig. 3.9: Offer set size versus customer volume under MNL and CMM

but for products with 0.5 correlation, it needs 2100 customers to justify the full optimal offer set, and for 0.8 correlation, even when customers increase to 2400, the optimal offer set is restricted at 5 products. At extreme case with perfect correlated product candidates, optimal offer size remains only one product, irrespective to the customer volume.

3. **Comparison of the Simulated Profits**

For static substitution, the arrival sequence of the customer will not affect the total profit. So to simulate the customers' purchase process, we can first use multi-variate normal distribution to randomly generate the utilities for 10 products with given mean and correlations for all the customers. We get the 300 customers' utilities in a 300 by 11 matrix for 1000 times, with the first column contains the utility from no-purchase option and the rest 10 columns are the utilities for the 10 product candidates. We then adopt the first principle for each customer, and apply the offer sets and inventory levels computed through MNL-INV and CMM-INV model, thus we can calculate the expected total simulated profits. We summarize the results, optimal offer sets and inventory levels under MNL-INV and CMM-INV model as follows.

From Figure 3.10, we can see generally the profits improve from MNL to CMM model. And the profit improvement increase as the correlation among the products' utilities increase. The negative improvements in the bottom line is due to the negative profit from MNL model under extreme situation when utilities are perfect correlated.

This is because CMM model can make more accurate prediction about the market share by capturing the correlations among the products, and hence give a more profitable inventory arrangement through the following

N=300	MNL			CMM			
Correlations	Offer set	Inventory	Profit	Offer set	Inventory	Profit	Improvements
0	{1,2,3, 4,5,6}	{81,59,42, 31,22,16}	734	{1,2,3,4,5,6}	{99,57,36,24,17,13}	879	19.8%
0.1			693	{1,2,3,4,5,6}	{99,56,34,23,16,12}	853	23.1%
0.5			495	{1,2,3,4}	{112,53,29,18}	835	68.7%
0.8			247	{1,2}	{140,48}	829	235.6%
1			-105	{1}	{178}	817	-878.1%

Fig. 3.10: Comparison of profit from different offer sets under MNL and CMM

two ways: one is from concentrating more on popular products, and the other is from reducing the offer set. In the extreme case, when the products have perfect correlation among their utilities, expected profit got from MNL-INV solution is negative, while CMM-INV solution still maintain total expected profit of 817, which is less than 10% decrease from independent product case.

3.4.2 Dynamic Substitution

Under dynamic substitution situation, when the customers' demand could not be satisfied with the product he most desires, he will substitute with other products that are available at the moment as long as the utility gained is higher than no-purchase. So in the shop, each customer will evaluate the utilities for all the available products and pick the one with highest utility as long as it is better than reservation utility.

Under dynamic substitution, it is hard to write the total profit in a closed form, since the profits depend on the arriving sequence of customers.

Therefore, we solved by maximization of the simulated profits. We generate a 11 by 300 matrix with given mean and covariance matrix to represent a realization of arrival of 300 customers and their utilities on the 10 products plus their no-purchase utility. The profit is obtained by accounting the real sales of each product by allowing the customer to switch choice when his previous preference is stock-out. We then average the profits obtained from 50000 realizations and use it as our objective function.

1. **Inventory Levels**

We depict the inventory levels for 10 products' set under MNL-INV and CMM-INV in the following Figure 3.11.

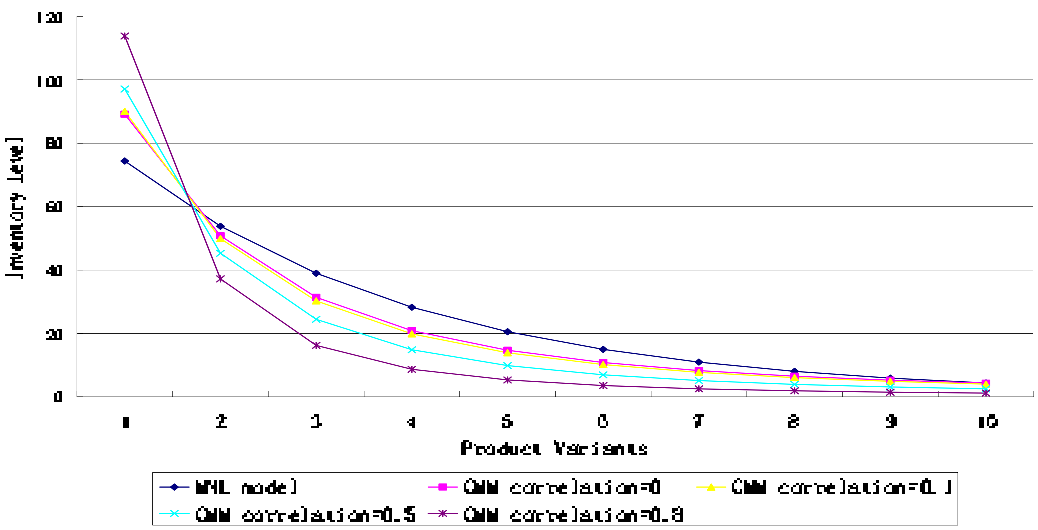

Fig. 3.11: Comparison of inventory levels for 10 products set under MNL and CMM

Similar to the situation under static substitution, the inventory level by CMM-INV model is less in total number and demonstrate a concentration on popular products, while the MNL model produce higher total inventory and more evenly inventory among different variants.

As the correlation among product utilities increase, the above phenomena become more significant. In other words, CMM model recommend lower total inventory and more concentrated inventory for more highly correlated products.

2. **Offer Set**

The optimal offer set under dynamic substitution is smaller than under static substitution for both MNL-INV and CMM-INV model. When customer volume is 300, for MNL, the optimal set is 3 products for dynamic substitution, but 6 products for static substitution. For CMM, the optimal offer set also shrink under dynamic substitution. (See Figure 3.12) This shrinkage is expected since by allowing stock-out substitution, less product variety is needed. The reason is that retailer can benefit from the economy of scale by reducing variety in the offer set with less hurt from shrinkage of market coverage.

And similar to the result from static substitution, the optimal offer set under CMM model is no bigger than the optimal set under MNL model. The size of the offer set increase with the customer volume. And for CMM model, the size of offer set decrease as the utilities' correlation increase. The offer set expands more quickly with the customer volume when the correlation among products is low.

3. **Comparison of the Simulated Profits**

We summarize the optimal solution and the simulated profits for MNL and CMM model under dynamic substitution in Figure 3.12. Compared to Figure 3.10 static substitution case, we can see higher profits under dynamic substitution than under static substitution in each circumstances. This profit improvement results from increased sales, which exactly comes from stock-out substitution.

Secondly, when considering the profit improvements between MNL model and CMM model, it is less significant under dynamic substitution than under static substitution. The possible reason may be that by allowing freely substitution, MNL model also shrink its offer set and adjust its inventory to a more concentrated style. Hence its solution get closer to that of CMM model, and hence reduce the profits gap.

N=300	MNL			CMM			
Correlations	Offer set	Inventory	Profit	Offer set	Inventory	Profit	Improvements
0	{1,2,3}	{100,72,52}	960	{1,2,3}	{117,66,40}	1011	5.31%
0.1			933	{1,2,3}	{117,64,38}	991	6.22%
0.5			803	{1,2,3}	{118,55,30}	910	13.33%
0.8			668	{1,2}	{139,47}	849	27.10%
1			556	{1}	{178}	817	46.94%

Fig. 3.12: Comparison of profit from different offer sets under MNL and CMM

3.5 Conclusions

In this chapter, we extend from the last chapter on product line selection for correlated products to include the inventory decisions. It is a practical problem especially for big retailers who need to decide on the variety of his assortment as well as inventory levels for each variety.

In the marketing and operations literature, MNL model has been adopted to handle such problems. But due to its IIA property, MNL model is incapable to take care of the utility correlations among the offer products. To factor in the products correlations, we incorporate our Cross Moment Model into this assortment and inventory joint decision problem, and compare the resulting offer set and inventory levels from these two different choice models. We found significant improvement of the total profits by CMM-INV model over MNL-INV model through Monte Carlo simulation. And such improvement increase with the utility correlation among offering products increase.

In sum, CMM model has demonstrated its advantage and potential to deal with the customer choice prediction in product selection and inventory decision problems.

In practice, the retailers decisions (offer set and inventory level) are not only based on the customer demand forecast, but also impacted by the contract terms between the manufacturer and retailer. The above chapter studied the problem based on a simple contract assumption. We will try to examine the effects of contract terms on the retailer's decisions in the next chapter.

4. MULTI-PRODUCT REORDER OPTION CONTRACTS

4.1 Introduction

Option contracts allow a buyer to postpone part of his order quantity decision until some or all demand uncertainties are resolved. By paying a reservation fee upfront, the buyer secures the right to buy the products at the predetermined execution price up to the specified level in the selling season. The buyer can also let the right expire and forgo the initial reservation fee. Although the total unit cost (reservation plus execution price) is typically higher than the simple fixed wholesale price, buyers with option contracts can enjoy the flexibility of adjusting order quantities according to the observed demand, and therefore largely mitigate their inventory overage and underage risk.

In a decentralized supply chain, parties with different interests tend to create an uncoordinated system, which leads to sub-optimal performance in the entire chain. In response to this "double- marginalization" issue, various supply contracts have been designed and applied; among them, the above-described contingent claims (option contracts) have received much attention.

Option contracts have been recently explored in the chemical industry, electricity market, and semi- conductor industry (Wu and Kleindorfer 2005).

In fact, we find the practice of option contracts in many industries faced with a long lead time, short selling season, and high demand uncertainties owing to the retailers' need for flexibility in their order quantities from the supplier. Indeed, in the fashion, sporting goods, toy, pharmaceutical, and textile garment industries, the retailer's ability to place more than one order in a season is becoming increasingly common (Fisher and Raman 1996; Eppen and Iyer 1988). A story from the Reebok company highlights the urgent need for flexible supply contracts. Reebok, who is the exclusive provider of NFL licensed merchandise, used to face a big challenge in the inventory control of replica jerseys (Parsons 2005):

"I have a warehouse full of jerseys out there and retailers are screaming for the teams and players I don't have." "I wish there was someway to plan inventory that would allow me to react faster to hot players and teams. But with player demand changing so much from year to year, I really can't increase inventory; in fact, I like to minimize inventory at year-end."

In the financial industry, the business of underwriting securities, like IPOs or corporate bonds, also uses similar sale contracts as options. For example, between 1997 and 1999, 47.7% of IPOs in Canada used a best efforts offering (Kooli et al. 2003). Namely, the total issue quantity is not fixed before hand, instead, the underwriter can return any unsold subscribed shares to the issuer unconditionally . Interestingly, as we will see later, such *"return"* policies can also be viewed as *"put options"* in the framework of supply chain contracts.

The previous literature has considered only option contracts on a single product. But sometimes the total demand may be less volatile, while high uncertainty exists in the demand for each specific type/model choice among a product family. Recall the above case of Reebok: the total demand for jerseys remains relatively stable from year to year. But the demand for each specific team's jerseys depends on the performance of each football team, which is highly unpredictable. Therefore, requirements with respect to contract flexibility arise not only in product quantities, but also in product choice among the product family. This precisely asks for a thorough study on multiple product option contracts.

Although option contracts can be beneficial through the flexibility they provide, what are their implied requirements for the manufacturer in offering such flexibility? In this chapter, we aim on the one hand to quantify the improvements that option contracts bring to each individual and the joint payoffs to the players, and on the other to identify the manufacturer's profile, in terms of his capacity constraints, that would favor an option contract.

We borrow the real option framework from finance to analyze flexibility in supply chain contracts; we also employ game theory to consider the interactive decision-making process between the manufacturer and the retailer.

This chapter is organized as follows. Section 4.2 builds up the basic analysis model and introduces the risk-neutral pricing approach. In Section 4.3, we quantify the impact of the reorder option on each party and the entire supply chain in a single product setting. In Section 4.4, we examine the reorder option in a multi-product setting and study the value of product choice flexibility by comparing pooled and non-pooled options. Finally, Section 4.5 summarizes the conclusions.

4.2 The Model

This section builds up the general settings and is applicable to both single-product and multi-product cases. We will elaborate on these two scenarios in detail in Sections 4 and 5, respectively.[1] We consider a two-echelon supply chain with one manufacturer and one retailer. The products have an exogenous retail price p, which is determined by the whole market and not affected by the individual retailer. The manufacturer incurs K^0 as the unit production cost in the normal production mode, and K^1 per unit cost in the emergency production mode. The manufacturer takes the *Stackelberg game* leader's position and offers the contract conditions to the retailer. In response, the retailer can either choose to take the conditions or leave the game.

4.2.1 Decision Sequence and Analysis Framework

We focus on a specific type of flexible contract, namely, the reorder option. At a certain expense, each unit of the reorder option gives the retailer the right to buy one unit of product from the manufacturer at a predetermined exercise price X in the selling season. The retailer can also let the right expire if the retail market is weak.

Before the selling season, the manufacturer decides on both the wholesale price and the option price and offers the contract to the retailer. [2]We let the

[1] In the multi-product section, we use subscripts to indicate each product. For example, p_1, p_2 are retail prices for product one and product two respectively. Here in this model section, for the ease of expression, we drop the subscripts.

[2] Exercise price X can also be a decision of the manufacturer. But for the purpose of analytical simplicity, we assume exercise price X is chosen within certain range according to the manufacturer's rule of thumb.

wholesale price be S and the option price be C, so the contract's form is (S, C). The retailer can either accept the offer or leave the game; if he chooses to accept it, he then decides his optimal inventory level I and option quantity U, based on his outlook of the future market status α. The retailer's objective is to maximize his present expected value of total profit R, and we impose a reservation value $\overline{R}$, below which the retailer will not participate in the business.

Therefore, the retailer needs to make a two-stage decision. In stage 1, the retailer decides on the desired inventory level and option quantity to purchase. Then in stage 2, without considering the sunk cost in stage 1, he decides v, the number of options to exercise according to the realized market status[3]. The typical decision/action sequence is as follows:

Stage 1: 1. Manufacturer offers contract (S, C);

2. Retailer builds up inventory I, purchases option U;

Stage 2: 3. Retailer decides on the number of options to exercise v;

This two-stage optimization model can be analyzed using a backward induction approach. At stage 2, the retailer's problem is to maximize his current value R' by choosing v within [0,U], given I, U, and realized α: $\max R'(v \mid I, U; \alpha),\ s.t.\ 0 \leqslant v \leqslant U$. At stage 1, the retailer's problem is to maximize his expected value at that point ER by choosing I and U, given the offered S and

[3] An additional assumption is that the manufacturer should not find it advantageous to build inventories as a contingency against possible orders arriving in stage 2.

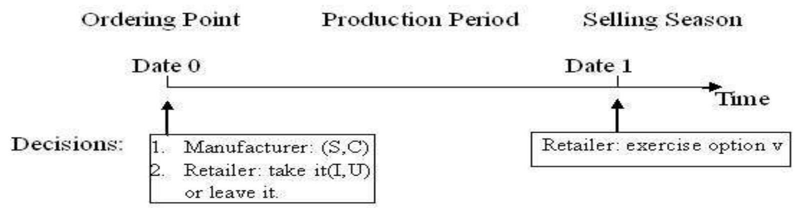

Fig. 4.1: The decision sequence and time span

C, where the expectation is based on his estimation of the future uncertainty and his own risk preference: $\max ER(I, U \mid S, C),\ s.t.\ 0 \leqslant I,\ 0 \leqslant U$. On the other hand, the manufacturer's problem at stage 1 is to maximize his current expected value EM by choosing S and C: $\max EM(S, C),\ s.t.\ 0 \leqslant S,\ 0 \leqslant C$.

4.2.2 Mechanism of the Reorder Option

As we described above, the mechanism of a single product reorder option is as follows: In the ordering season, for each unit of options the retailer purchased from the manufacturer, the retailer need to pay option price C, to gain the right to buy one unit of product from the manufacturer at an exercise price X in the selling season.

We illustrate the process with a simple example. In stage 1, the retailer builds up 800 units of inventory with a wholesale unit price of $10 and purchases 200 units of options at the option unit price of $1 and option exercise price of $10. When the actual demand turns out to be 1000 units in stage 2, the retailer should use up all his inventory, but it is still not enough to meet all the demand. He now can exercise all his options to meet the extra demand. Suppose the unit retail price is $15, then the retailer's total profit is simply \$$(15 - 10) \times 1000 - 200 =$ \$4800. In comparison, without the reorder option the retailer can satisfy only 800 units of demand, and he will incur 200 lost sale, which results \$1000 less in profit. After taking account of option price upfront payout, option contract still can make \$800 more in total profit. Similarly, if the actual demand turns out to be 900 units, the retailer can exercise 100 option rights to supplement the inventory and earn an extra \$300 in total profit. If the actual demand is equal to or less than 800 units, the retailer will choose not to exercise any rights and forgo his option cost of \$200. The mechanism of the reorder is simple, but its implications are profound. The retailer may actively use the reorder option to decrease inventory levels, or the manufacturer may manipulate either the retail price or the option price to induce the retailer to take actions that benefit the manufacturer.

In the multi-product case, we consider an added dimension of reorder flexibility, namely, pooled and non-pooled options. The pooled reorder option allows the retailer to request different products when exercising the option. This pooled arrangement is commonly used in distributing high-technology products (Brown et al., 2008). Let's look at an example: suppose on day 0, the retailer purchases 10 units of options for product 1, and 5 units of options for product 2. On day 1, it turns out that the realized demand for product 1 is 15 units more than product 1's inventory, while product 2's inventory is just enough to meet its real demand. In the non-pooled option scenario, only 10 units of options for product 1 will be exercised, leaving 5 units of demand unsatisfied. In the pooled option case, the additional demand of 15 units can be satisfied by exercising all 15 units of options.

4.2.3 Risk-neutral Pricing

Most papers usually adopt simple expected payoff as the objective, which is equal to assuming the decision maker to be risk-neutral. Our model targets to relax this assumption and accounts for the decision maker's risk preference. One way is to adopt the standard *expected utility* theory, which involves the cumbersome task of estimating each decision maker's utility function. Alternatively, we can adopt the risk-neutral pricing approach. It converts real world probability into risk-neutral world probability, through inclusion on the prices of certain assets. Risk neutral pricing actually incorporates the market-determined risk preference.[4]

Future demand is a random variable. With certan probability respectively, we assume it will result in either a high state α^H or a low state α^L. For the convenience of the analysis, we assume demand to be Bernoulli distributed, but with the potential to be extended to other general distributions.

$$\alpha = \begin{cases} \alpha^H & with \;\; probability: \;\; Pb^H \\ \alpha^L & with \;\; probability: \;\; Pb^L \end{cases} \tag{4.1}$$

Arrow-Debreu assets are constructed as a set of fundamental assets whose future payoff is 1 in a certain future state, and zero in all other states.

[4] A key assumption under risk-neutral pricing is the completeness of the market, which requires sufficient assets being traded. The basic idea lies in having a complete market where the current price of every cash flow is determined uniquely according to the market's risk preference. Such risk preference is reflected by the prices of a series of assets. The reason for ignoring an individual's risk preference is that the individual can hedge freely in the markets to change his risk profile.

The current values (or prices) of the whole set of Arrow-Debreu assets are also called *pricing kernels* in modern finance, since every future uncertain cash flow can be priced uniquely with these Arrow-Debreu state prices. Let e^H (e^L) be the Arrow-Debreu state price corresponding to 1 unit of payoff only in the high (low) state, and 0 otherwise.

To find out the value of the "pricing kernels", we need to make use of the price information of two assets. One is the riskless bond, which pays out 1 unit on day 1 regardless of the resultant state. The other is a traded security that pays out α^H in the high state, and α^L in the low state. Suppose we can observe the current price of the riskless bond as B_0 and the current price of that traded security as A_0. Then, to ensure no arbitrage, clearly,

$$\begin{aligned} A_0 &= \alpha^H e^H + \alpha^L e^L \\ B_0 &= e^H + e^L. \end{aligned}$$

We take the riskless bond price B_0 as a discount factor and rewrite the above pricing formula: $A_0 = (\alpha^H \times \frac{e^H}{B_0} + \alpha^L \times \frac{e^L}{B_0}) \times B_0$. By defining the risk-neutral probability of the high state and the low state as $q^j = \frac{e^j}{B_0}, j = H, L$, we can clearly interpret the present value (or price) of any cash flow as the discounted expected value of its future payoff under the risk-neutral probabilities.[5]

Similar to Burnetas and Ritchken(2005), we next can use the market-

[5] Note that risk-neutral probabilities do not have a direct relationship with real state probabilities. Rather, they are obtained by comparing the Arrow-Debrau state prices and the riskless assets' price. Therefore, even if we are not very sure about the actual demand distribution, we can price the option as long as we can get the information for the Arrow-Debrau state prices and the riskless assets' price.

observed prices A_0, B_0 to reparametrize the unknown $\alpha^H, \alpha^L, e^H, e^L$. Let μ and σ^2 represent the mean and variance of the intercept term of the demand curve under the risk-neutral measure. It is easy to find that

$$\begin{aligned}\mu &= \alpha^H \times q^H + \alpha^L \times q^L = \frac{A_0}{B_0} \\ \sigma^2 &= \frac{e^H e^L}{B_0^2}(\alpha^H - \alpha^L)^2\end{aligned}$$

Also, let $\rho = e^L/e^H$[6] and we can express $\alpha^L, \alpha^H, e^L, e^H$ in terms of μ, σ, B_0, ρ as follows.

$$\begin{aligned}\alpha^H &= \frac{A_0}{B_0} + \sqrt{\rho}\sigma \\ \alpha^L &= \frac{A_0}{B_0} - \frac{1}{\sqrt{\rho}}\sigma \\ e^H &= \frac{B_0}{1+\rho} \\ e^L &= \frac{B_0\rho}{1+\rho}.\end{aligned}$$

4.3 Single Product

In this section, we first analyze the single product supply chain. Through the comparison among centralized system, decentralized system under price-only contract and under reorder option, the conceptual and analytical framework will be built up and the value of re-order option will be discussed in detail.

[6] ρ generally reflects the relative risk preference of the market

4.3.1 Centralized System

With the market parameters introduced in Section 3, we identify the optimal decisions for a centralized system. Suppose I products will be produced before selling season, and U products' capacity will be reserved as emergency supply. If the future demand turns out to be high, then supply total number of $I+U$ products, otherwise, supply only I products. To maximize present value of the centralized system:

$$max_{I\in\{\alpha^L,\alpha^H\},U\in\{0,\alpha^H-I\}}(e^H p_1 - K_1^0)I + e^H(p_1 - K_1^1)U + \alpha^L e^L p_1.$$

Lemma 2: Under the assumption $K^0 \geq e^H K^1$, optimal decisions of the centralized system is:

$$I^* = \alpha^L, \quad U^* = \alpha^H - \alpha^L.$$

Proof: $e^H p_1 - K_1^0 - (e^H(p_1 - K_1^1)) = e^H K_1^1 - K_1^0 \leq 0$ ■

4.3.2 Price-only Contract

We examine the basic price-only contract as a benchmark. Let ER represent the retailer's present value.[7] The retailer decides on the inventory level: $\max_{\alpha^H \geq I \geq \alpha^L} ER = I(e^H p_1 - S) + e^L \alpha^L p_1$ The optimal decision is straight-

[7] Hereafter, the present value in the objectives are all calculated using the risk neutral pricing kernels.

forward:

$$I^* = \begin{cases} \alpha^H & if \;\; e^H p_1 \geq S \\ \alpha^L & if \;\; (e^H + e^L)p_1 \geq S \geq e^H p_1. \end{cases} \tag{4.2}$$

With the retailer's reservation payoff in mind, the manufacturer's objective is to maximize present value by offering the appropriate wholesale price:

$$\max M = I^* * (S - K^0) \;\; subject \;\; to : ER^* = I^*(e^H p_1 - S) + e^L \alpha^L p_1 \geq \overline{R}$$

Lemma 3: When price-only contracts are used, the manufacturer's optimal solution is piecewise linear depending on the value of the retailer's reservation.

$$S^* = \begin{cases} e^H p_1 + e^L p_1 - \frac{\overline{R}}{\alpha^L} & if \;\; \overline{R} \leq e^L \alpha^L p_1 - (\alpha^H - \alpha^L)(e^H p_1 - K_1^0) \\ e^H p_1 & if \;\; e^L \alpha^L p_1 - (\alpha^H - \alpha^L)(e^H p_1 - K_1^0) \leq \overline{R} \leq e^L \alpha^L p_1 \\ e^H p_1 + e^L p_1 - \frac{\overline{R}}{\alpha^L} & if \;\; \overline{R} \geq e^L \alpha^L p_1. \end{cases} \tag{4.3}$$

And the retailer's best response in these three regions are as follows.

$$I^* = \begin{cases} \alpha^L & if \ \ \overline{R} \leq e^L\alpha^L p_1 - (\alpha^H - \alpha^L)(e^H p_1 - K_1^0) \\ \alpha^H & if \ \ e^L\alpha^L p_1 - (\alpha^H - \alpha^L)(e^H p_1 - K_1^0) \leq \overline{R} \leq e^L\alpha^L p_1 \\ \alpha^H & if \ \ \overline{R} \geq e^L\alpha^L p_1. \end{cases} \tag{4.4}$$

Proof: Consider case 1: $\overline{R} \leq e^L\alpha^L p_1$. To satisfy the retailer's reservation constraint, we require $S \leq e^H p_1 + \frac{e^L\alpha^L p_1 - \overline{R}}{I^*}$. On the other hand, taking account of retailer's best response in (4.2), we get:

$$M^* = \begin{cases} \alpha^H(e^H p_1 - K_1^0) & if \ \ e^H p_1 \geq S > 0 \\ \alpha^L(e^H p_1 + e^L p_1 - \frac{\overline{R}}{\alpha^L} - K_1^0) & if \ \ e^H p_1 + \frac{e^L\alpha^L p_1 - \overline{R}}{I} \geq S > e^H p_1 \end{cases}$$

Comparing the two situations, we easily find that when $(\alpha^H - \alpha^L)(e^H p_1 - K_1^0) < e^L\alpha^L p_1 - \overline{R}, I^* = \alpha^L, S^* = (e^H + e^L)p_1 - \frac{\overline{R}}{\alpha^L}$. Otherwise, $I^* = \alpha^H, S^* = e^H p_1$. In case 2, $\overline{R} \geq e^L\alpha^L p_1$, we get $S \leq e^H p_1 + \frac{e^L\alpha^L p_1 - \overline{R}}{I^*} \leq e^H p_1$, so $I^* = \alpha_H, S^* = e_H p_1 + \frac{e^L\alpha^L p_1 - \overline{R}}{I^*}$; therefore, $R^* = \overline{R}, M^* = \alpha^H(e^H p_1 + \frac{e^L\alpha^L p_1 - \overline{R}}{I^*} - K_1^0)$. ■

To gain a clear view, we plot in Figure 4.2 the optimal order quantity I^*, optimal whole sale price S^*, and the corresponding optimal manufacturer's, retailer's and system's payoff (M^*, R^* and $M^* + R^*$), assuming $p_1 = 24, K_1^0 = 3, \alpha^H = 29.8, \alpha^L = 15, e^H = 0.27, e^L = 0.53$. Figure 4.2 shows that generally speaking, with other parameters fixed, as the retailer's reservation value increases, the manufacturer has to decrease his wholesale price, leading to a decrease in manufacturer's payoff. An interesting jump happens when the retailer's reservation value reaches the threshold, where the manufacturer would rather reduce the wholesale price further to induce more order quantity.

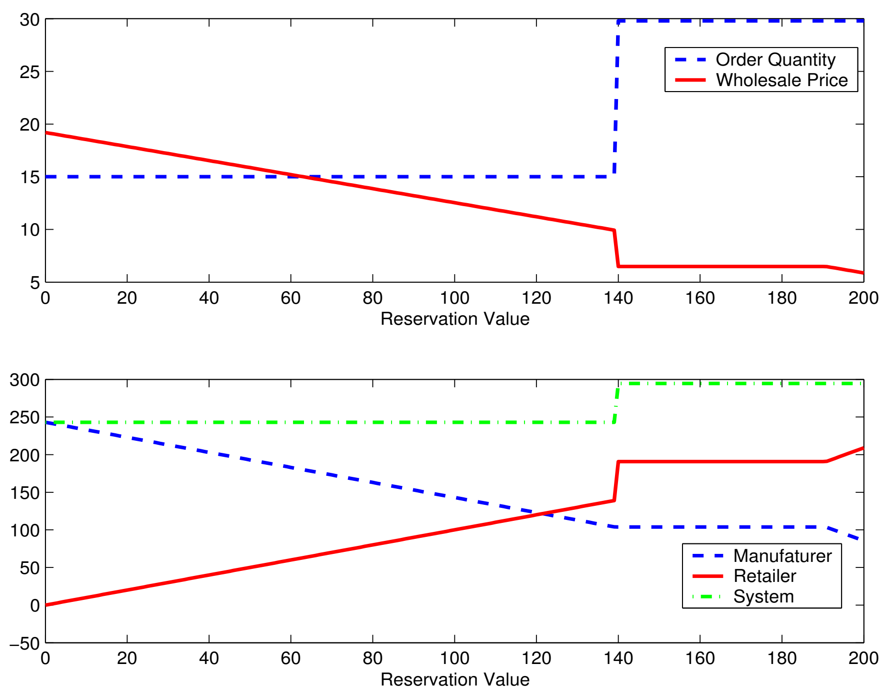

Fig. 4.2: Optimal decisions under price-only contract depending on various $\overline{R}$

Lemma 4: When price-only contracts are used, the manufacturer's optimal solution is piecewise non-linear depending on the value of the standard deviation of the demand distribution.

$$S^* = \begin{cases} e^H p_1 + e^L p_1 - \frac{\overline{R}}{\alpha^L} & if \;\; \sigma \leq \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{(1+2\rho)B_0 p_1 - (1+\rho)^2 K_1^0} \\ e^H p_1 & if \;\; \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{(1+2\rho)B_0 p_1 - (1+\rho)^2 K_1^0} \leq \sigma \leq \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{B_0 \rho p_1} \\ e^H p_1 + e^L p_1 - \frac{\overline{R}}{\alpha^L} & if \;\; \sigma \geq \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{B_0 \rho p_1}. \end{cases} \tag{4.5}$$

And the retailer's best response in these three regions are as follows.

$$I^* = \begin{cases} \alpha^L & if \;\; \sigma \leq \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{(1+2\rho)B_0 p_1 - (1+\rho)^2 K_1^0} \\ \alpha^H & if \;\; \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{(1+2\rho)B_0 p_1 - (1+\rho)^2 K_1^0} \leq \sigma \leq \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{B_0 \rho p_1} \\ \alpha^H & if \;\; \sigma \geq \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{B_0 \rho p_1}. \end{cases} \tag{4.6}$$

Proof: Based on lemma 3, we only need to substitute the parameters' expression for state price in (4.2). Then we can get the critical determinant conditions in terms of standard deviation of demand under the risk-neutral measure. ■

To gain a clear view, we plot the optimal order quantity I^*, optimal whole sale price S^*, and the corresponding optimal manufacturer's and retailer's payoff, M^* and R^* in Figure 4.3, assuming $p_1 = 24, K_1^0 = 3, A = 15.996, \overline{R} = 160, e^H = 0.27, e^L = 0.53$.

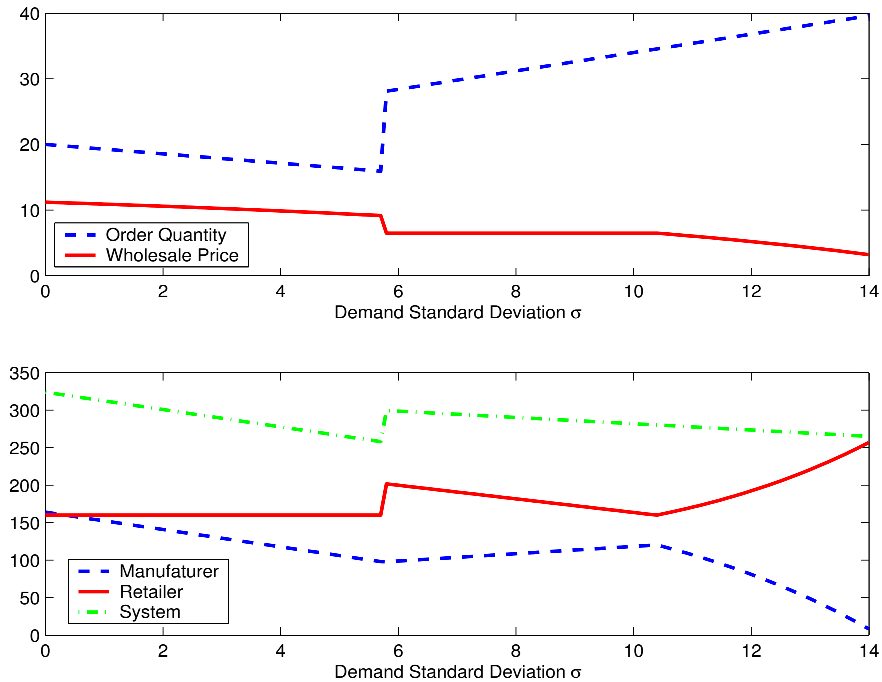

Fig. 4.3: Optimal decisions under price-only contract depending on Various σ

Figure 4.3 shows that, generally speaking, the system's total payoff suffers as the demand becomes more variant. However, there is a jump where manufacturer is willing to slash the wholesale price down to entice higher order quantity level. Before this threshold, the manufacturer will set the wholesale price as high as possible and leave the retailer with only the reservation payoff. In response, the retailer will choose only the lower inventory level. The supply chain thus loses those sales opportunities in high demand scenarios. But since the variation of sales is comparably small, the manufacturer is better off by charging the retailer a higher unit price. The retailer can get more payoff than his reservation value when the demand variance is above the threshold.

In next subsection, we will study how to improve the performance of the price-only contract. We show the conditions under which we can coordinate the decentralized system to the centralized optimal level by adding the reorder option scheme.

4.3.3 Reorder Option

The use of option contracts necessitates a two-stage decision process; we adopt backward induction to solve the problem. In stage 2, the retailer decides on the optimal quantity of options to exercise:

$$\begin{aligned} \max_{0 \le v_1 \le U_1} ER(v_1 \mid I_1, U_1; \alpha_1) &= \max_{0 \le v_1 \le U_1}[\min(\alpha_1, I_1 + v_1) * p_1 - v_1 * x_1] \\ &= \begin{cases} I_1 * p_1 + v_1 * (p_1 - X_1) & if \;\; \alpha_1 > I_1 + v_1 \\ \alpha_1 * p_1 - v_1 * X_1 & if \;\; I_1 + v_1 > \alpha_1 > 0. \end{cases} \end{aligned} \tag{4.7}$$

Lemma 5: At stage 2, the retailer's optimal policy takes the following form:

1. If $I_1 > \alpha_1$,then the optimal number of options to exercise is 0;

2. If $I_1 < \alpha_1$, then $v_1^* = \min(U_1, \alpha_1 - I_1)$.

Proof: This can be obtained by analyzing the piecewise objective function. Ensuring that the retailer behaves reasonably, we make the assumptions $X_1 > 0$, and $p_1 - X_1 > 0$. If $X_1 < 0$, the retailer will make a profit just by exercising his options as much as possible without even selling the product. If $p_1 - X_1 < 0$, the retailer will not use any options owing to their negative intrinsic value. Finally, the result follows as the number of exercised options should not be more than the number purchased at stage 1. ■

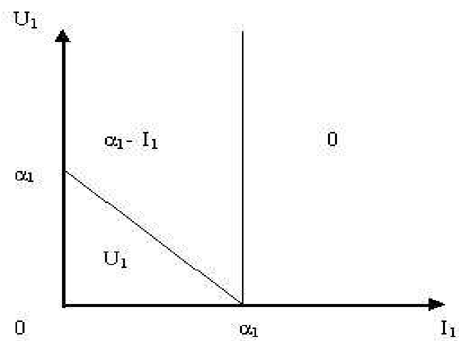

Fig. 4.4: The retailer's optimal exercise decisions at stage 2

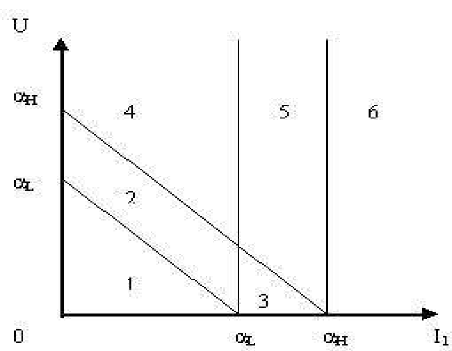

Fig. 4.5: The retailer's decisions at stage 2 in different scenarios

Figure 4.4 depicts the retailer's optimal exercise decision at stage 2. Since the demand is assumed to be Bernoulli distributed, we combine the two scenarios into the decision graph in Figure 4.5, which divides the whole decision space into six areas.

In stage 2 the retailer has a different optimal exercise decision in each area. We list all the decisions in these six areas in Table 4.1 for clarity.

Region	v_H^*	v_L^*
1	U	U
2	U	$\alpha^L - I$
3	U	0
4	$\alpha^H - I$	$\alpha^L - I$
5	$\alpha^H - I$	0
6	0	0

Tab. 4.1: The retailer's optimal exercise decision at stage 2 by region in Figure 4.5.

Lemma 6: At stage 1, if the option is offered and the retailer chooses to purchase it, then the retailer's optimal policy, in terms of inventory level and number of options purchased, should fall into area 3 in Figure 4.5. In particular, either $U^* = 0$, or $\alpha_1^H \geq I^* \geq \alpha_1^L$ and $\alpha_1^H - I^* \geq U^* \geq 0$.

If the retailer's decision falls within area 3, he faces the problem of deciding the desired inventory level and option quantity:

$$\begin{aligned} & \max_{\alpha_1^L \leq I_1, 0 \leq U_1 \leq \alpha_1^H - I_1} ER(I_1, U_1 \mid S_1, C_1) \\ = & \max_{\alpha_1^L \leq I_1, 0 \leq U_1 \leq \alpha_1^H - I_1} -I_1 * S_1 - U_1 * C_1 + e^H * [(I_1 + U_1)p_1 - U_1 * X_1] + e^L * \alpha_1^L * p_1 \end{aligned}$$

Rearranging the above expression in terms of I_1 and U_1, we get:

$$ER = \max_{\alpha_1^L \le I_1 \le \alpha_1^H, 0 \le U_1 \le \alpha_1^H - I_1} (e_1^H p_1 - S_1) I_1 + (e_1^H p_1 - e_1^H X_1 - C_1) U_1 + e_1^L * \alpha_1^L * p_1. \tag{4.8}$$

We requires that $(e_1^H + e_1^L) p_1 \ge S_1$. Otherwise,the retailer has no incentive to carry any inventory. On the other hand, if $e_1^H p_1 \ge S_1$, the retailer will choose to stock as much as possible and will not purchase any options. To induce the retailer into choosing a combination of stock and options at stage 1, the manufacturer must set the wholesale price S_1 between $e_1^H p_1$ and $(e_1^H + e_1^L) p_1$ and set the option price C_1 at less than $e_1^H (p_1 - X_1)$. With these settings, the retailer's stage 1 optimal decision is simply:

$$I^* = \alpha_1^L, U^* = \alpha_1^H - \alpha_1^L.$$

Keeping the above retailer's reaction in mind, the manufacturer can move on to the contract design. The problem faced by the manufacturer is as follows:

$$M = \max_{e_1^H p_1 \le S_1 \le (e_1^H + e_1^L) p_1, 0 \le C_1 \le e_1^H (p_1 - X_1)} (S_1 - K_1^0) * \alpha_1^L + (C_1 + e^H (X_1 - K_1^1)) * (\alpha_1^H - \alpha_1^L). \tag{4.9}$$

And the whole chain's payoff is:

$$R + M = (e^H p_1 - K_1^0) I_1 + e^H (p_1 - K_1^1) U_1 + \alpha^L e^L p_1. \tag{4.10}$$

If the reservation utility of the retailer is not considered in the model, the manufacturer, as the Stackelberg leader, will obviously choose to offer the upper bound of the wholesale price and the option price, which is $(e_1^H + e_1^L)p_1$ and $e_1^H(p_1 - X_1)$ respectively. However, this degenerate solution is not practical, since the retailer under such a contract condition will gain exactly zero payoff. To ensure that the retailer participates in the contract, the manufacturer has to assign at least the reservation payoff to the retailer $\overline{R}$.

$$R^* = [(e^H + e^L)p_1 - S_1]\alpha^L + (e^H(p_1 - X_1) - C_1)(\alpha^H - \alpha^L) \geq \overline{R}. \quad (4.11)$$

Solving his problem (4.9) with the constraint (4.11). It is not difficult to reach the following proposition:

Proposition 9: Using reorder option contract, the manufacturer's payoff will always be improved if $K_1^1 < p_1$. The manufacturer will maintain the reservation payoff for the retailer by adjusting the wholesale price and option price according to the following key condition 1 (KC1):

$$(KC1) \quad \begin{cases} \alpha^L S + (\alpha^H - \alpha^L)C = B_0 p \alpha^L + e^H(p - X)(\alpha^H - \alpha^L) - \overline{R} \\ (e^H + e^L)p_1 \geq S_1 \geq e^H p_1 \\ e^H(p_1 - X_1) \geq C \geq 0. \end{cases}$$

Therefore the manufacturer is indifferent to the specific combination of (S^*, C^*) as long as it conforms to the above specified relationship. Furthermore, the exercise price X_1 will not affect the results as long as it is within a certain reasonable interval, (K_1^1, p_1).

Proof: Substitute the retailer's reservation constraint (4.11) into the manufacturer's problem (4.9), we get:

$$M \leq (e^H p_1 - K_1^0)\alpha^L + e^H(p_1 - K_1^1)(\alpha^H - \alpha^L) + \alpha^L e^L p_1 - \overline{R}.$$

First, we see that the manufacturer will attain the upper bound M^* when the retailer's reservation constraint is a strict equation. Second, compare the manufacturer's payoff here with the one under price-only contract. Under reorder option, the manufacturer present payoff is:

$$M^* = [(e^H + e^L)p_1 - K_1^0] * \alpha^L + (\alpha^H - \alpha^L)e^H(p_1 - K_1^1) - \overline{R}.$$

Recalling the optimal payoff in price only contract in Lemma 4, we can find the increment of manufacturer's is:

$$\Delta M^* = \begin{cases} -\alpha^L(e^H p_1 + e^L p_1 - \frac{\overline{R}}{\alpha^L} - K_1^0) + ([(e^H + e^L)p_1 - K_1^0] * \alpha^L + (\alpha^H - \alpha^L)e^H(p_1 - K_1^1) - \overline{R}) \\ -\alpha^H * (e^H p_1 - K_1^0) + ([(e^H + e^L)p_1 - K_1^0] * \alpha^L + (\alpha^H - \alpha^L)e^H(p_1 - K_1^1) - \overline{R}) \\ -\alpha^H * (e^H p_1 + e^L p_1 - \frac{\overline{R}}{\alpha^L} - K_1^0) + ([(e^H + e^L)p_1 - K_1^0] * \alpha^L + (\alpha^H - \alpha^L)e^H(p_1 - K_1^1) - \overline{R}) \end{cases}$$

$$= \begin{cases} (\alpha^H - \alpha^L)e^H(p_1 - K_1^1) & if \ \ \sigma \leq \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{(1+2\rho)B_0 p_1 - (1+\rho)^2 K_1^0} \\ (\alpha^H - \alpha^L)(K_1^0 - e^H K_1^1) + \alpha^L e^L p_1 - \overline{R} & if \ \ \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{(1+2\rho)B_0 p_1 - (1+\rho)^2 K_1^0} \leq \sigma \leq \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{B_0 \rho p_1} \\ (\alpha^H - \alpha^L)(e^H p_1 - K_1^0) - (\alpha^H - \alpha^L)e^H(p_1 - K_1^1) & if \ \ \sigma \geq \frac{\sqrt{\rho}\rho A_0 p_1 - \sqrt{\rho}(1+\rho)\overline{R}}{B_0 \rho p_1} \end{cases}$$

This increment is greater than zero if $p_1 > K_1^1$ in case 1. And in case 2 and 3, due to the assumption $K^0 \geq K^1 e^H$, it is also greater than zero. ■

Proposition 1 established the conditions for the option contracts to be beneficial and attractive to the manufacturer. In summary, the manufacturer can assume these five steps to find out whether to adopt and how to set up a suitable contract menu:

Step 1: Evaluate demand standard deviation, if it falls in area 2 and 3 in Lemma 4, use option contract and skip Step 2.

Step 2: Evaluate its emergency produce cost K_1^1, and compare it to the market price p_1. If $p_1 > K_1^1$, then choose to offer contracts with options.

Step 3: For the option, arbitrarily choose the exercise price $X_1 \in (K_1^1, p_1)$.

Step 4: According to key relationship 1, work out the entire menu of (S, C) combinations, with $C_1 \in (0, e^H(p_1 - X_1))$ and $S_1 \in (e^H p_1, (e^H + e^L)p_1)$.

Step 5: If the implicit option contract is preferred, where the option price is incorporated into the wholesale price, the manufacturer can simply let $C^* = 0$, and find the corresponding S as the wholesale price.

Comparing the contracts with and without options, we see the manufacturer will always be better off if his emergency production cost is less than the retail price. However, all benefits will be absorbed by the manufacturer, while the retailer gets only the reservation payoff in his pocket. At first look, option scheme seems to benefit retailers by providing more flexibility in their

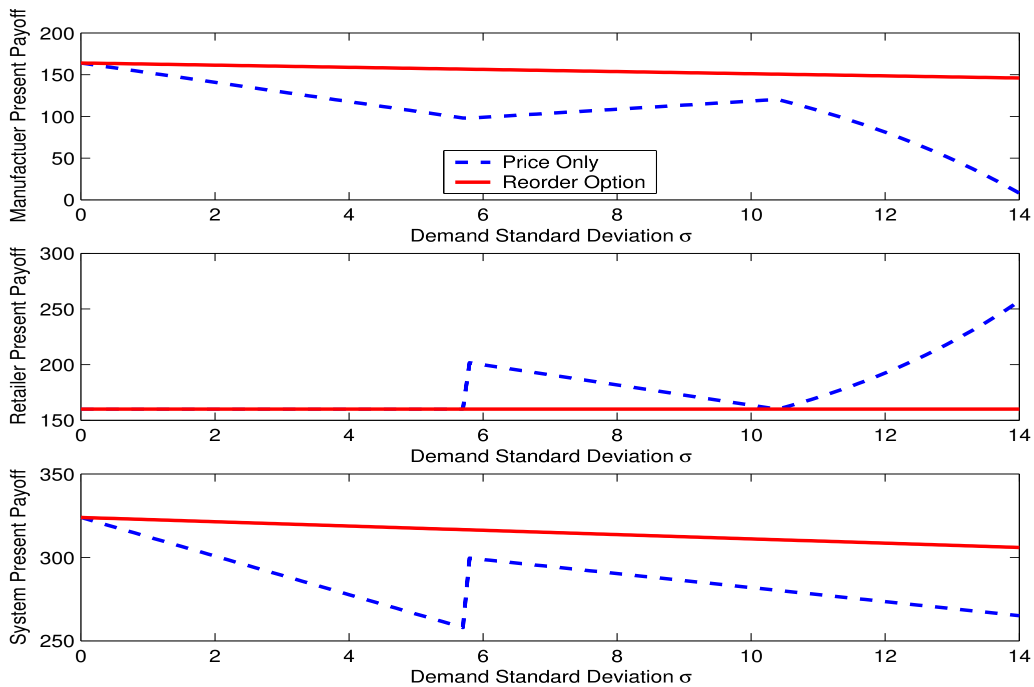

Fig. 4.6: Comparison between price-only and reorder option performance

order commitment. But as a Stackelberg follower, the retailer actually is also forced to give up upward profit potential. In other words, circumstances exist where the retailer gets a lower payoff under the option contract. For example, when the demand variance is beyond the threshold, the retailer may gain more under the price-only contract by keeping a higher inventory. But with an option contract, the retailer's payoff will be reduced to the reservation value.

With the option, the wholesale price will increase if the option price is not explicitly charged. Therefore if your supplier offer you the right to adjust your order quantity freely in the future. Please take note such freedom may have already be charged incorporated in the wholesale price. This also explains why commission fee is lower in the "best-efforts" offering than normal in the security underwriting business.

4.4 Multiple Products

A supply of product family, such as different designs of garments or different models of an electronic products, is likely to happen between the same supplier and retailers. An added dimension of flexibility, namely, pooling, can thus be incorporated into the contract. A pooled option allows interchanging the underlying assets on the exercise date, while a non-pooled option does not.

A non-pooled multiple products' option can be seen as separate options contracts, whose solution process is essentially the same as the single product contract. We thus focus on pooled options in this section and investigate the implications of this added flexibility by comparing the optimal decisions, objectives, and supply chain efficiencies under pooled and non-pooled options.

Similar to the single-product setting, we reparametrize the uncertainty in two demands as follows: Suppose a traded security exists that pays out $\alpha_1^H(\alpha_2^H)$ when the demand for product 1 (2) is in the high state, and $\alpha_1^L(\alpha_2^L)$ is in the low state. The price of this security at day 0 is $A_{10}(A_{20})$, and the price of the riskless bond is B_0. Let e^{HH} (e^{LH}, e^{HL}, e^{LL}) be Arrow-Debreu state prices corresponding to 1 unit of payoff only in the high (low) state, and 0 otherwise.[8] Then clearly:

$$
\begin{aligned}
A_{10} &= \alpha_1^H(e^{HH} + e^{HL}) + \alpha_1^L(e^{LH} + e^{LL}) \\
A_{20} &= \alpha_2^H(e^{HH} + e^{LH}) + \alpha_2^L(e^{LL} + e^{HL}) \\
B_0 &= e^{HH} + e^{LH} + e^{HL} + e^{LL}
\end{aligned}
$$

Now we have three market-observable parameters: A_{10}, A_{20} and B_0. Let σ_1^2, σ_2^2 represent the variances of the intercept term of the demand curves under the risk-neutral measure for the two products, ρ_{12} be the correlation between the two demands, and $\theta_1 = \frac{e^{HH}}{e^{HL}}, \theta_2 = \frac{e^{LH}}{e^{LL}}$ represent the decision maker's subjective risk attitude. $\alpha_1^L, \alpha_1^H, \alpha_2^L, \alpha_2^H, e^{LL}, e^{HL}, e^{LH}, e^{HH}$ can then be numerically expressed in terms of $A_{10}, A_{20}, B_0, \theta_1, \theta_2, \sigma_1, \sigma_2, \rho_{12}$ accordingly.

[8] $e_1^H = e^{HH} + e^{HL}, e_2^H = e^{HH} + e^{LH}, e_1^L = e^{LH} + e^{LL}, e_2^L = e^{HL} + e^{LL}$,and practically assume $e^{HH}, e^{LH}, e^{HL}, e^{LL}$ are all positive

Region	v_1^* Conditions	v_2^* Conditions	v_1^*	v_2^*
1	$\alpha_1 > I_1 + U$	$\alpha_2 > I_2$	U	0
2	$\alpha_1 < I_1 + U, \alpha_1 > I_1$	$I_1 + I_2 + U < \alpha_1 + \alpha_2$	$\alpha_1 - I_1$	$U - (\alpha_1 - I_1)$
3	$\alpha_1 < I_1 + U, \alpha_1 > I_1$	$I_1 + I_2 + U < \alpha_1 + \alpha_2, I_2 < \alpha_2$	$\alpha_1 - I_1$	$\alpha_2 - I_2$
4	$\alpha_1 < I_1$	$\alpha_2 < I_2 + U, I_2 < \alpha_2$	0	$\alpha_2 - I_2$
5	$\alpha_1 < I_1$	$\alpha_2 < I_2 + U$	0	U
6	$\alpha_1 > I_1, I_1 + U > \alpha_1$	$\alpha_2 < I_2$	$\alpha_1 - I_1$	0
7	$\alpha_1 < I_1$	$\alpha_2 < I_2$	0	0

Tab. 4.2: The retailer's optimal exercise decision on date 1 when the pooled option is adopted by different regions of I_1, I_2, U.

4.4.1 The Retailer's Problem

Throughout this section, we assume $p_1 - X_1 > p_2 - X_2$, such that it is more profitable to use product 1 as the underlying asset when exercising the pooled option. Using backward induction, the retailer faces this problem in stage 2:

$$\begin{aligned} & \max_{0 \leq v_1, 0 \leq v_2, 0 \leq v_1 + v_2 \leq U} R'(v_1, v_2 | I_1, I_2, U; \alpha_1, \alpha_2) \\ = & \max_{0 \leq v_1, 0 \leq v_2, 0 \leq v_1 + v_2 \leq U} [\min(\alpha_1, I_1 + v_1)p_1 - v_1 X_1 + min(\alpha_2, I_2 + v_2)p_2 - v_2 X_2]. \end{aligned} \tag{4.12}$$

Lemma 7: Given the retailer's stage 1 decision (I_1, I_2, U), and the revealed demand $\alpha \in \{\alpha_1, \alpha_2\}$ in stage 2, the retailer's stage 2 decision is piecewise linear as shown in Table 4.2.

Proof: According to different demand realization, we can list the retailer's exercise decision in Table 4.3. We then can map it into the seven regions in Table 4.2 conditional on the relationship between I_1, I_2 and U. To visualize the seven regions, we depict them in Figure 4.7. ∎

Conditions	v_1^*	v_2^*
$\alpha_1 > I_1 + v_1, \alpha_2 > I_2 + v_2$	$\max(0, \min(U, \alpha_1 - I_1))$	$\max(0, \min(U - v_1^*, \alpha_2 - I_2))$
$\alpha_1 > I_1 + v_1, \alpha_2 < I_2 + v_2$	$\max(0, \min(U, \alpha_1 - I_1))$	$\min(U - v_1^*, \max(0, \alpha_2 - I_2))$
$\alpha_1 < I_1 + v_1, \alpha_2 > I_2 + v_2$	$\max(0, \alpha_1 - I_1)$	$\max(0, \min(U - v_1^*, \alpha_2 - I_2))$
$\alpha_1 < I_1 + v_1, \alpha_2 < I_2 + v_2$	$\max(0, \alpha_1 - I_1)$	$\min(U - v_1^*, \max(0, \alpha_2 - I_2))$

Tab. 4.3: The retailer's optimal exercise decision on date 1 when the pooled option is adopted.

.

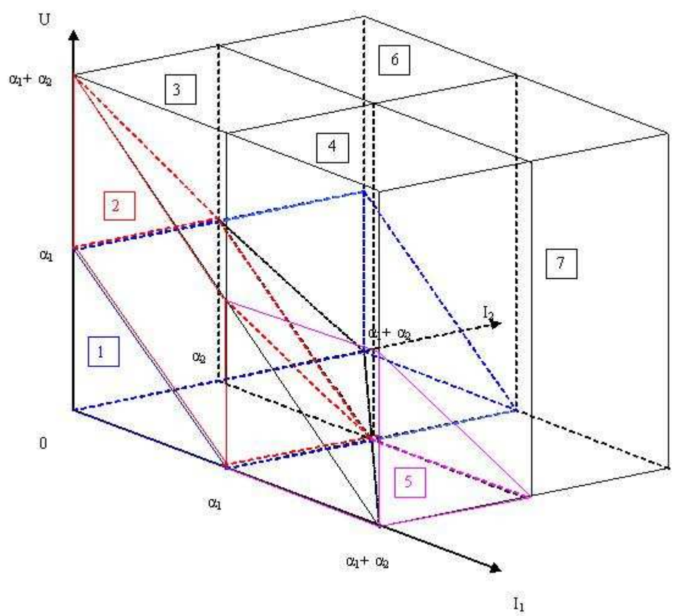

Fig. 4.7: The retailer's optimal exercise decisions in seven regions

Now consider the two possible future states, α^H and α^L.

Lemma 8: At stage 1, if the option is offered and the retailer chooses to purchase it, then the retailer's optimal policy, in terms of inventory level of each product and the number of options purchased, should fall into one of the four efficient regions in Figure 4.9.

Figure 4.9 identifies the only four efficient regions where the retailer will choose to use options in stage 1. Within these four regions, the retailer's stage 2 exercise decisions are shown in Figure 4.8.

Regions	Scenarios		Decisions	
	α_1	α_2	v_1^*	v_2^*
1	L	L	0	0
	L	H	0	$\alpha_{2H}-I_2$
	H	L	$\alpha_{1H}-I_1$	0
	H	H	$\alpha_{1H}-I_1$	$U-(\alpha_{1H}-I_1)$
2	L	L	0	0
	L	H	0	U
	H	L	$\alpha_{1H}-I_1$	0
	H	H	$\alpha_{1H}-I_1$	$U-(\alpha_{1H}-I_1)$
3	L	L	0	0
	L	H	0	$\alpha_{2H}-I_2$
	H	L	U	0
	H	H	U	0
4	L	L	0	0
	L	H	0	U
	H	L	U	0
	H	H	U	0

Fig. 4.8: The retailer's optimal exercise decisions in four efficient regions

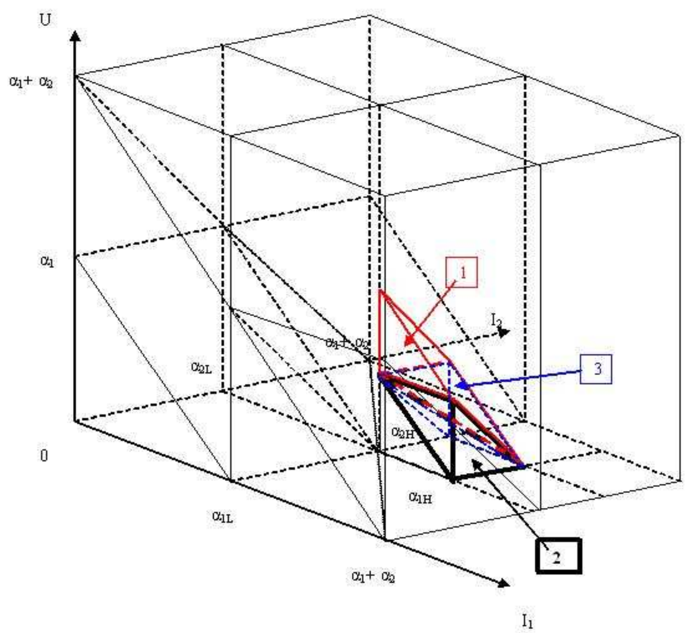

Fig. 4.9: The retailer's decisions in different scenarios by four efficient regions

We consider the retailer's stage 1 problem in these four regions. For example, in region 1, the problem can be written as

$$\begin{aligned}
& \max_{\alpha_1^L \le I_1 \le \alpha_1^H, \alpha_2^L \le I_2 \le \alpha_2^H, \max(\alpha_1^H - I_1, \alpha_2^H - I_2) \le U \le \alpha_1^H - I_1 + \alpha_2^H - I_2} R(I_1, I_2, U|S_1, S_2, C) \\
= \;& -I_1 S_1 - I_2 S_2 - U * C \\
+ \;& e^{HH}(\alpha_1^H p_1 - (\alpha_1^H - I_1)X_1 + (I_2 + (U - \alpha_1^H + I_1)p_2 - (U - \alpha_1^H + I_1)X_2) \\
+ \;& e^{HL}(\alpha_1^H p_1 - (\alpha_1^H - I_1)X_1 + \alpha_2^L p_2) \\
+ \;& e^{LH}(\alpha_1^L p_1 + \alpha_2^H p_2 - (\alpha_2^H - I_2)X_2) \\
+ \;& e^{LL}(\alpha_1^L p_1 + \alpha_2^L p_2)
\end{aligned} \tag{4.13}$$

Lemma 9: In region 1, the conditional optimal decision is: $I_1^* = \alpha_1^L, I_2^* = \alpha_2^L$,

$$U^* = \begin{cases} \alpha_1^H - \alpha_1^L + \alpha_2^H - \alpha_2^L & when \ \ C < (p_2 - X_2)e^{HH} \\ \max(\alpha_1^H - \alpha_1^L, \alpha_2^H - \alpha_2^L) & when \ \ C > (p_2 - X_2)e^{HH} \end{cases}$$

Proof: Rearrange the expression in terms of I_1, I_2 and U, and note $S_1 > p_1 e_1^H$, $S_2 > p_2 e_2^H$; the result then follows. ■

Similarly, we can find out the conditional optimal decisions in regions 2, 3, and 4.

Lemma 10: In region 2, $\alpha_1^H - I_1 < \alpha_2^H - I_2$, the conditional optimal decision is: $I_1^* = \alpha_1^L, I_2^* = \alpha_2^L, U^* = \begin{cases} \alpha_1^H - \alpha_1^L & when \ \ C > (p_2 - X_2)e_2^H \\ \alpha_2^H - \alpha_2^L & when \ \ C < (p_2 - X_2)e_2^H \end{cases}$

Lemma 11: In region 3, $\alpha_1^H - I_1 > \alpha_2^H - I_2$, the conditional optimal decision is: $I_1^* = \alpha_1^L, I_2^* = \alpha_2^L, U^* = \begin{cases} \alpha_2^H - \alpha_2^L & when \ \ C > (p_1 - X_1)e_1^H \\ \alpha_1^H - \alpha_1^L & when \ \ C < (p_1 - X_1)e_1^H \end{cases}$

Throughout all these four regions, the retailer's optimal decision can be summarized in the following proposition:

Proposition 10: When the pooled option policy is used, $I_1^* = \alpha_1^L, I_2^* = \alpha_2^L$, and if

$$\alpha_1^H - \alpha_1^L < \alpha_2^H - \alpha_2^L,$$

then

$$U^* = \begin{cases} \alpha_1^H - \alpha_1^L + \alpha_2^H - \alpha_2^L, & 0 < C < (p_2 - X_2)e^{HH}; \\ \alpha_2^H - \alpha_2^L, & (p_2 - X_2)e^{HH} < C < (p_2 - X_2)e_2^H; \\ \alpha_1^H - \alpha_1^L, & (p_2 - X_2)e_2^H < C < (p_1 - X_1)B_0. \end{cases}$$

Otherwise, if

$$\alpha_1^H - \alpha_1^L > \alpha_2^H - \alpha_2^L,$$

then

$$U^* = \begin{cases} \alpha_1^H - \alpha_1^L + \alpha_2^H - \alpha_2^L, & 0 < C < (p_2 - X_2)e^{HH}; \\ \alpha_1^H - \alpha_1^L, & (p_2 - X_2)e^{HH} < C < (p_1 - X_1)e_1^H; \\ \alpha_2^H - \alpha_2^L, & (p_1 - X_1)e_1^H < C < (p_1 - X_1)B_0. \end{cases}$$

From the above proposition, we see that when using pooled options, the retailer may have three types of optimal decisions depending on the option price charged by the manufacturer. Suppose the demand for two products is negatively correlated (with demand correlation$= -1$), then we have $e^{HH} = 0$, and the proposition implies that the retailer will never purchase $\alpha_1^H - \alpha_1^L + \alpha_2^H - \alpha_2^L$ options, which may likely happen in non-pooled option contracts without considering the demand correlation among different products.

Comparing the results between pooled and the non-pooled policy, since $e^{HH} < e_1^H, and \;\; e^{HH} < e_2^H$, the manufacturer may need to restrict to lower unit option price range to make the retailer keep the same order and option quantity as in non-pooled case. To further examine the impact of a pooled options policy on the supply chain, we move on to study the manufacturer's payoff in these three situations and generate optimal and applicable contract conditions.

4.4.2 The Manufacturer's Problem

In normal production mode, the manufacturer will usually arrange capacity for each product respectively. But in an emergency production mode, different products may need to compete for limited production capacity/resources. For example, if two products require the same equipment or staff to produce, the emergency production cost of one will rise if the other needs emergency production at the same time. The manufacturer needs to be able to tell whether his emergency production costs are independent or inter-dependent among different products.

Independent Emergency Production Costs

Under the pooled option policy, the manufacturer also anticipates the retailer's best response to his offered price menu (S_1, S_2, C), and as a Stackelberg leader, he will set a menu of combinations of (S_1, S_2, C) to maximize his gains while satisfying the retailer's reservation payoff. The retailer's reservation constraint can be expressed as follows:

$$
\begin{aligned}
R^* &= (B_0p_1 - S_1)I_1^* + (B_0p_2 - S_2)I_2^* - U^*C \\
&+ e^{HH}(\min(U^*, (\alpha_1^H - I_1^*))(p_1 - X_1) + (U^* - \max(0, (\alpha_1^H - I_1^*)))(p_2 - X_2)) \\
&+ e^{HL}\min(U^*, (\alpha_1^H - I_1^*))(p_1 - X_1) \\
&+ e^{LH}\min(U^*, (\alpha_2^H - I_2^*))(p_2 - X_2) \\
R^* &\geq \overline{R}
\end{aligned}
\tag{4.14}
$$

Apart from the retailer's reservation payoff, any excess payoff will be reaped by the manufacturer. Therefore, the manufacturer's problem is equivalent to maximizing the system's joint payoff. Consider the supply chain's payoff:

$$
\begin{aligned}
R^* + M^* &= (B_0p_1 - K_1^0)I_1^* + (B_0p_2 - K_2^0)I_2^* \\
&+ e^{HH}\min(U^*, ((\alpha_1^H - I_1^*))(p_1 - K_1^1) + (U^* - \max(0, (\alpha_1^H - I_1^*)))(p_2 - K_2^1)) \\
&+ e^{HL}\min(U^*, (\alpha_1^H - I_1^*))(p_1 - K_1^1) \\
&+ e^{LH}\min(U^*, (\alpha_2^H - I_2^*))(p_2 - K_2^1)
\end{aligned}
\tag{4.15}
$$

Proposition 11: When emergency production costs are independent among multiple products, under the condition of $K_1^1 < p_1$ and $K_2^1 < p_2$, the payoffs for the whole chain, the retailer, and the manufacturer are the same under both pooled and non-pooled settings. The manufacturer will maintain the least reservation payoff for the retailer by adjusting the wholesale price and option price according to the following key condition 2 (KC2):

$$(KC2) \quad \begin{cases} S_1\alpha_1^L + S_2\alpha_2^L + (\alpha_1^H - \alpha_1^L + \alpha_2^H - \alpha_2^L)C \\ = B_0p_1\alpha_1^L + B_0p_2\alpha_2^L + e_1^H(p_1 - X_1)(\alpha_1^H - \alpha_1^L) + e_2^H(p_2 - X_2)(\alpha_2^H - \alpha_2^L) - \overline{R} \\ B_0p_1 \geq S_1 \geq e_1^H p_1, B_0p_2 \geq S_2 \geq e_2^H p_2 \\ (p_2 - X_2)e^{HH} \geq C \geq 0 \end{cases}$$

Therefore, the manufacturer is indifferent to the specific combination of (S_1^*, S_2^*, C^*) as long as it conforms to the above specified relationship. Furthermore, the exercise price X_1, X_2 will not affect the results as long as it is within a certain reasonable interval, namely (K_1^1, p_1) and (K_2^1, p_2).

For the whole chain, the wholesale price S_1, S_2 and option price C are transfer payments. They do not affect the whole chain's payoff and do not appear in the above expressions. Consider the efficient area 1 in Figures 4.9 and 4.8, when $p_1 > K_1^1, p_2 > K_2^1$, the supplier would like U^* to be $\alpha_1^H - \alpha_1^L + \alpha_2^H - \alpha_2^L$ to maximize the joint payoff. The pooled policy is same as the non-pooled policy in terms of the resulting order quantity, option quantity, and retailer's and supplier's optimal payoff. However, the supplier's decision over the design of the price menu is changed. To solicit the retailer's desirable response, he needs to adjust the option price to a more restricted lower range and sets the new wholesale prices accordingly.

It seems counterintuitive that added flexibility does not increase the joint payoff. But if we look at the non-pooled option contract, we will see that the supplier, as the Stackelberg leader, has already taken full advantage of the flexible contract settings to reap all potential benefits. In the case of $p_1 > K_1^1, p_2 > K_2^1$, from non-pooled to pooled, added flexibility only affects the rule of internal transfer payment.

Meanwhile, when $p_1 > K_1^1, p_2 < K_2^1$, U^* should take $\alpha_1^H - \alpha_1^L$ regardless of the contract type. But we see that the whole chain's payoff decreases in the pooled case. This is because when $p_2 < K_2^1$, no option should be offered for product 2, but the pooled option grant such option and hence make things worse. It is similar for the cases of $p_1 < K_1^1, p_2 > K_2^1$ or $p_1 > K_1^1, p_2 > K_2^1$.

We show the comparison between pooled and non-pooled options in terms of their performance, each party's decisions and payoffs under different scenarios in Table 4.4

Conditions	Whole Chain Value	R	M	Contract Menu
$p_1 > K_1^1, p_2 > K_2^1$	unchanged	unchanged	unchanged	changed
$p_1 > K_1^1, p_2 < K_2^1$	$P < NP$	unchanged	$P < NP$	changed
$p_1 < K_1^1, p_2 > K_2^1$	$P < NP$	unchanged	$P < NP$	changed

Tab. 4.4: Comparison between pooled and non-pooled options with independent emergency production costs.

The relationship between retail price and emergency production price is the sole determinant for the manufacturer to determine the optimal option quantity that the retailer should buy. Although this is not affected by the option type, the latter does influence the specific price combination in the contract menu. The pooled options are usually priced in a tighter and lower range.

Inter-dependent Emergency Production Costs

The previous analysis assumes that the emergency production costs of different products are independent of one another. We now consider the dependency in such costs for a two-product case. In this case, we can show that the manufacturer may utilize the pooled policy to achieve better results than non-pooled option contracts.

Let us make a further assumption on the emergency production cost. We call K_1^1 and K_2^1 the normal emergency production costs when only one product is produced in emergency mode. And we denote K_1^2 and K_2^2 as the emergency production costs for the two products, respectively, when they are required to be produced together in emergency mode. [9] Since the retailer's problem does not involve these parameters, his decisions remain the same as before. For the manufacturer, we compare his actions and payoffs under the pooled and non-pooled cases.

Proposition 12: When all $p_i > K_i^1$ and at least one $p_i \in (K_i^1, K_i^2)$, occasions exist where

[9] $K_i^2 > K_i^1$, for $\forall i$, and $p_i \in (K_i^1, K_i^2)$ for at least one i. Otherwise, the problem can be reduced to the previous discussion with no inter-dependent emergency production costs.

(1) the option price will be set higher under a pooled policy, such that fewer options will be purchased under this policy than under the non-pooled policy,

(2) the manufacturer can gain a higher payoff under the pooled than the non-pooled policy.

Proof: Consider two products and use the pooled option. From Proposition 2, we know that the retailer may have one of three possible responses to the manufacturer's price menu:

1) $U^* = \alpha_1^H - \alpha_1^L + \alpha_2^H - \alpha_2^L$;

2) $U^* = \alpha_1^H - \alpha_1^L$;

3) $U^* = \alpha_2^H - \alpha_2^L$;

Now write down the payoffs of the whole chain under the non-pooled policy in these three cases:

1)

$$\begin{aligned}(R+M)_{NP}^1 &= (B_0p_1 - K_1^0)I_1^* + e^{HH}(\alpha_1^H - \alpha_1^L)(p_1 - K_1^2) + e^{HL}(\alpha_1^H - \alpha_1^L)(p_1 - K_1^1) \\ &+ (B_0p_2 - K_{02})I_2^* + e^{HH}(\alpha_2^H - \alpha_2^L)(p_2 - K_2^2) + e^{LH}(\alpha_2^H - \alpha_2^L)(p_2 - K_2^1)\end{aligned}$$

2)$(R+M)_{NP}^2 = (B_0p_1 - K_1^0)I_1^* + e_1^H(\alpha_1^H - \alpha_1^L)(p_1 - K_1^1) + (B_0p_2 - K_{02})I_2^*$;

3)$(R+M)_{NP}^3 = (B_0p_1 - K_1^0)I_1^* + e_2^H(\alpha_2^H - \alpha_2^L)(p_2 - K_2^1) + (B_0p_2 - K_{02})I_2^*$.

Compare them with the payoffs of the whole chain under the pooled policy in the above cases:

1)

$$
\begin{aligned}
(R+M)_P^1 &= (B_0p_1 - K_1^0)I_1^* + e^{HH}(\alpha_1^H - \alpha_1^L)(p_1 - K_1^2) + e^{HL}(\alpha_1^H - \alpha_1^L)(p_1 - K_1^1) \\
&+ (B_0p_2 - K_{02})I_2^* + e^{HH}(\alpha_2^H - \alpha_2^L)(p_2 - K_2^2) + e^{LH}(\alpha_2^H - \alpha_2^L)(p_2 - K_2^1)
\end{aligned}
$$

2) $(R+M)_P^2 = (B_0p_1 - K_1^0)I_1^* + (B_0p_2 - K_{02})I_2^* + e_1^H(\alpha_1^H - \alpha_1^L)(p_1 - K_1^1) + e^{LH}\min(\alpha_1^H - \alpha_1^L, \alpha_2^H - \alpha_2^L)(p_2 - K_2^1)$

3) $(R+M)_P^3 = (B_0p_1 - K_1^0)I_1^* + (B_0p_2 - K_{02})I_2^* + e_1^H(\alpha_2^H - \alpha_2^L)(p_1 - K_1^1) + e^{LH}(\alpha_2^H - \alpha_2^L)(p_2 - K_2^1)$, when $\alpha_1^H - \alpha_1^L > \alpha_2^H - \alpha_2^L$;

or $(B_0p_1 - K_1^0)I_1^* + (B_0p_2 - K_{02})I_2^* + e^{HH}[(\alpha_1^H - \alpha_1^L)(p_1 - K_1^2) + (\alpha_2^H - \alpha_2^L - (\alpha_1^H - \alpha_1^L))(p_2 - K_2^2)] + e^{HL}(\alpha_1^H - \alpha_1^L)(p_1 - K_1^1) + e^{LH}(\alpha_1^H - \alpha_1^L)(p_2 - K_2^1)$, when $\alpha_1^H - \alpha_1^L < \alpha_2^H - \alpha_2^L$;

Let M_P^j denote the manufacturer's payoff under a pooled policy in case j, $M_P^j = (R+M)_P^j - \overline{R}$. We adopt similar notation for non-pooled cases. We see that since all $p_i > K_i^1$, $M_P^1 = M_{NP}^1$, and $M_P^2 > M_{NP}^2$, the manufacturer will choose case 1 under the non-pooled policy and case 2 under the pooled policy under a certain criteria set,[10] namely, that an occasion exists where $M_P^2 > M_P^1 = M_{NP}^1 > M_{NP}^2$. In this situation, the manufacturer will prefer to lead the retailer to purchase fewer options by setting a higher option price in a pooled policy; at the same time, he can gain a higher payoff by applying a pooled policy than a non-pooled one. ■

.

[10] The conditions' set is: $M_{NP}^1 > M_{NP}^2, M_{NP}^1 > M_{NP}^3, M_P^2 > M_P^1, M_P^2 > M_P^3$

Summing up the analysis thus far, we see that the choice of a pooled or non-pooled policy depends on the emergency production cost structure of the manufacturer. When the costs are independent, a non-pooled policy will never be worse than the pooled one. On the other hand, under inter-dependent emergency production costs, a pooled policy can demonstrate substantial advantages in certain occasions. The manufacturer should analyze his own emergency production costs before choosing the option contract type. We summarize the results with interactive emergency production costs in the following table. We use "P" to represent pooled option and "NP" to represent non-pooled option.

$$Term1 = \frac{\alpha_1^H - \alpha_1^L}{\alpha_2^H - \alpha_2^L}(max(p_1 - K_1^1, p_2 - K_2^1) - (p_1 - K_1^2)).$$

Conditions	Whole Value	M	U	Contract
$p_1 > K_1^2, p_2 > K_2^2 + Term1$	unchanged	unchanged	unchanged	changed
$p_1 > K_1^2, K_2^2 + Term1 > p_2 > K_2^1$	$P > NP$	$P > NP$	$P < NP$	changed
$p_1 > K_1^2, p_2 < K_2^1$	$P < NP$	$P < NP$	unchanged	changed

Tab. 4.5: Comparison of pooled and non-pooled options with inter-dependent emergency production costs

4.4.3 Numerical Examples

We demonstrate the difference between pooled and non-pooled options using a numerical example. Suppose we have two products with correlated emergency production costs: $p_1 = 24, K_1^0 = 6, K_1^1 = 10, K_1^2 = 18, \alpha_1^L = 15, \alpha_1^H = 29.8, A_{10} = 15.99, \sigma_1^2 = 255.9, K_2^0 = 6, K_2^1 = 10, K_2^2 = 18, \alpha_2^L = 15, \alpha_2^H =$

$29.8, A_{20} = 17.00, B_0 = 0.8, \sigma_2^2 = 268.2, e^{LL} = 0.33, e^{HL} = 0.17, e^{LH} = 0.2, e^{HH} = 0.1$. We then find that when $K_2^1 > p_2$, the pooled option is worse,

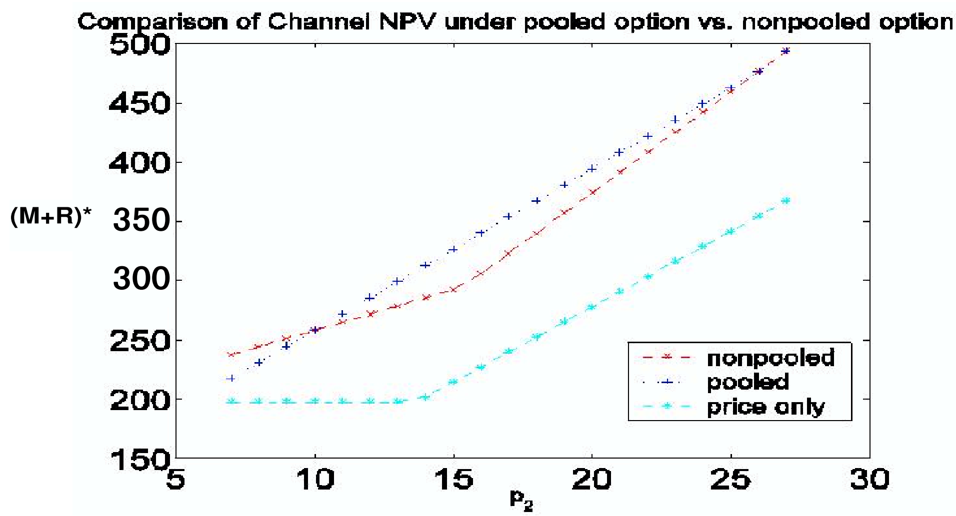

Fig. 4.10: Comparison of channel NPV under pooled and non-pooled options

but

when $K_2^1 < p_2 < K_2^2 + \frac{\alpha_1^H - \alpha_1^L}{\alpha_2^H - \alpha_2^L}(max(p_1 - K_1^1, p_2 - K_2^1) - (p_1 - K_1^2))$, then the pooled option is better.

When $p_2 > K_2^2 + \frac{\alpha_1^H - \alpha_1^L}{\alpha_2^H - \alpha_2^L}(max(p_1 - K_1^1, p_2 - K_2^1) - (p_1 - K_1^2))$, both pooled and non-pooled options have the same contribution.

Figure 4.10 compares the channel Net Present Value (NPV) under the three different contracts: price-only contract, non-pooled option contract, and pooled option contract. The channel NPV from the price-only contract is always the lowest among the three contracts within the price domain. This phenomenon is consistent with the well-documented "double-marginalization effect" in the supply chain contract literature. It also confirms our belief that flexibility can bring added value to the whole chain. We use this lowest

NPV from the price-only contract as the worst case benchmark for our other contracts.

Figure 3.10 also demonstrates that a pooled contract achieves more channel NPV than a non-pooled contract when the price of the second product is between 10 and 26 (region 2). This result indicates that product flexibility does add more value on top of volume flexibility in some circumstances. So we observe that contract flexibility is value-added within this region. The improvement in the channel NPV probably results from the exchangeability of the underlying assets when the pooled option is exercised. Such exchangeability allows the production greater flexibility and hence adds more value to the whole supply chain.

However, as the figure demonstrates, when the price is below 10 (region 1), the NPV under the pooled option is less than that under the non-pooled option. Additionally, as the price further decreases, the difference increases. This phenomenon that a pooled option can even worsen the situation is somehow counterintuitive because we usually expect more value from greater exchangeability. Nevertheless, this result is consistent with Goyal and Netessine (2005), who identify the conditions for the volume flexibility technology to be a better solution than both volume and product flexibility technology. of volume flexibility technology as the better solution than both volume and product flexibility technology. The probable reason for this counterintuitive phenomenon is the misuse of the pooled option in the contracts. Since the option value is negatively correlated with the spot price, it becomes too costly to offer the option for the second product when that product's spot price falls below 10. Under the non-pooled option contract, the manufacturer can easily choose not to offer the option contract for the second product, while under the pooled option contract, the options for the two products are combined as a whole; the retailer can thus easily take advantage of this and deliberately choose the strategy that benefits himself but harms the manufacturer and the whole chain. Therefore, our results show that a pooled option contract, which offers more flexibility, can still exacerbate the situation if used inappropriately.

In addition, Figure 3.10 shows that in region 3, where the price is beyond 26, both pooled and non- pooled contracts actually achieve the same NPV. This suggests that the benefits from product flexibility diminish as the second product's spot price goes up and finally disappears beyond a certain threshold. This can be explained by the mechanism whereby when the spot price of the second product rises, the benefits from its volume flexibility grow increasingly important compared with those from the product flexibility effect. Therefore, the difference in NPV between the pooled and non-pooled option contracts, which stands exactly for the benefits from product flexibility, increasingly declines. In the end, when the second product's spot price reaches the threshold, the benefits from volume flexibility begin to fully dominate those from product flexibility. Hence, the two contracts, both pooled and non-pooled, accomplish the same results when the spot price is high enough.

In sum, our results suggest that it is not necessarily correct that more flexibility will add more value to the whole chain. Depending on the product's different spot prices, pooled option contracts can do better, worse, or the same as non-pooled option contracts. In other words, although the benefits from volume flexibility are obvious, it is not as simple when we come to the benefits of product flexibility. The impact of product flexibility on the supply chain can be positive, negative, or fully dominated when product flexibility interacts with volume flexibility. Basically this result is analogous to what Goyal and Netessine (2005) find, although they assume a centralized supply chain in their models.

4.5 Conclusions

The main purpose of this study was to examine two specific supply contracts, namely, pooled and non- pooled reorder options. We attempted to determine the necessary and sufficient conditions for the reorder options to improve the efficiency of a distributed supply chain. We also investigated the potentials of pooled option contracts for a multi-product supply chain.

With a reorder options contract, a distributed supply chain achieves a higher expected profit than an ordinary newsboy inventory control method in a price-only contract setting. This benefit created by the option arrangement can be explained by its ability to make full use of valuable information. By holding back part of the initial investment, options allow decision makers to make appropriate adjustments to their initial production plan when information on the latest market environment becomes available.

Reorder option contracts also coordinate the objectives and risk profiles of the different parties in the supply chain. Using option contracts eliminates the double-marginalization effect, and the total profits of a distributed system approach that of a centralized system. We prove options portfolio contracts to have coordination capability. These coordination benefits may result from the profit and risk reallocation effects introduced by the option contracts. As we know, different parties in the supply chain may have different objectives and risk preferences. These differences can cause discrepancies and finally inefficiencies in the whole system. Interestingly, option contracts provide a possible channel for the different parties to negotiate and transfer their proceedings and risks so that they can coordinate their efforts towards a common objective, and system efficiency can be improved.

Extending volume flexibility to product flexibility, pooled option contracts demonstrate their specific advantages and limitations. Our results show that pooled options outperform non-pooled options only within a certain price region outside which pooled options might have the same performance as or even underperform non-pooled options. The limitation of the benefits from product flexibility can be attributed to the fact that ordinary non-pooled individual options have already made good use of the available information and provided a certain degree of transfer channel. Therefore, additional flexibility from product exchange can produce extra value only when the two products have a close price region and require a mutual transfer of their profits and risks.

This study has systematically studied for the first time the implications of pooled reorder options in distributed supply chains. The results challenge the commonly held notion that greater flexibility brings higher profits.

To simplify the model and analysis, we assumed the decision makers have access to a complete financial market. Therefore, commodity risk can be fully hedged off and risk preference can be addressed by using the risk-neutral probabilities to replace the actual probabilities. For an incomplete market, our model is still valid for risk-neutral market players. We expect that risk aversion on the buyer's or seller's side would lead to more pronounced benefits from option contracts. However, the behavior and strategies for risk averse companies in an incomplete market need to be further quantified.

Another simplification of our study is the exogenously determined retail price. This is suitable for those industries with intensive competition, for example, oil, electricity, or bank loans. However, for many other industries, retailers may have the power to influence market price. It would be very interesting to incorporate pricing strategy into the analysis and examine its interplay with options portfolio contracts.

Future studies could also elaborate the informational aspect of option contracts. We treated the symmetry information case in our study, which assumes that all market players have complete information about their opponents' situations. Such information includes cost structure in addition to parameters, objectives, demands, and risk preferences. Obviously, this is an ideal case. Asymmetric information about demand or cost structure may prevail in diverse circumstances, which would lead to changes in the optimal decisions and contract parameters. Recently, more and more research in game theory has concentrated on asymmetric information and the resulting principal-agent problem. It would be promising to extend our options portfolio contracts to situations with asymmetric information.

5. CONCLUSIONS AND FUTURE WORK

This study developed a product line selection model in conjunction with a utility maximization model to deal with the complicated choice behavior of customers. Semi-definite Programming (SDP) is used to approximate the expected utility and the customer choice probabilities. The product line selection problem is then solved by incorporating the SDP approach with product swapping and greedy heuristics.

Compared to the popular multinomial logit model (MNL), we showed that our new method is able to incorporate the correlation among products arising from common attributes in the choice behavioral model. Thus the inherent drawback of MNL and IIA property can be addressed nicely. From the computational results, we found that our new SDP model consistently outperformed other product line selection methods that are based on MNL model. This gap gets wider when the correlations of products increase, coefficients of variances of the attributes decrease, and the number of heterogenous customer segments decrease. In other words, when it comes to those highly correlated products, our methods would fit much better than the popular MNL model. Therefore, we expect our methods to have many useful applications including airline revenue management, software configuration, etc.

In the second part of the thesis, we extended our work on product line selection to include the inventory decisions. It is a practical problem especially for big retailers who need to decide on the variety of his assortment as well as inventory levels for each variety.

We incorporated our Cross Moment Model into the product line selection and inventory joint decision problem and focused on comparing the resulting offer set and inventory levels from these two different choice models. Several managerial insights have been gained through numerical examples. We showed that under static substitution, less correlated products set can generate more profit, which is in the similar spirit of the findings of "spaced out positioning" from the locational model [22]. We also showed that the total varieties of products can be reduced under dynamic substitution. And through the simulation, we demonstrated the considerable improvement in expected profits when the utilities' correlation is factored in.

The CMM choice model is useful to approximate market shares when products' utilities are random and largely depend on their various attributes. However, CMM requires to use mean and covariance estimations on products' utilities. How to get these estimation still remain a challenge in marketing research. Even if we can use the *linear in attributes* method to break down the products' utilities to their attributes level, the issue to estimate the mean and variance of each attribute's utility still need to be addressed.

In the last chapter, we analyze how flexibility in order quantity created by using options in a supply contract affects the payoffs of the manufacturer and the retailer as well as their joint payoff. We consider a Stackelberg game in which the manufacturer sets the contract and the retailer reacts to it. We examine the impact of reorder options in a single-product case and further compare the differences between pooled and non-pooled options in a multi- product case. While reorder options seem to offer the retailer more flexibility, we find that in some cases the retailer may end up with a lower payoff. For multi-product cases, we identify some conditions where pooled and non-pooled option contracts may provide the same payoff, and other conditions where one can be higher than the other.

BIBLIOGRAPHY

[1] R. Anupindi and Y. Bassok. *Supply contracts with quantity commitments and stochastic demand*. Kluwer Academic Publishers, 1999.

[2] G. Aydin and J. K. Ryan. Product line selection and pricing under the multinomial logit choice model. Working Paper, Purdue University, 2000.

[3] D. Barnes-Schuster, Y. Bassok, and R. Anupindi. Coordination and flexibility in supply contracts with options. *Manufacturing & Service Operations Management*, 4(3):171–207, 2002.

[4] A. Belloni, R. Freund, M. Selove, and D. Simester. Optimizing product line designs: Efficient methods and comparisons. *Management Science*, 54(9):1544–1552, 2008.

[5] M. Ben-Akiva and S. R. Lerman, editors. *Discrete Choice Analysis*. The MIT Press, Cambridge, Massachusettes.

[6] J. J. M. Bront, I. Mendez-Diaz, and G. Vulcano. A column generation algorithm for choice-based network revenue management. *Operations Research*, 57 (3):769–784, 2009.

[7] A. Brown, M.C. Chou, and C.S. Tang. The implications of pooled returns policies. *International Journal of Production Economics*, 111(1):129–146, 2008.

[8] A. Burnetas and P. Ritchken. Option pricing with downward-sloping demand curves: The case of supply chain options. *Management Science*, 51(4):566–580, 2005.

[9] G. P. Cachon. *Supply Chain Coordination with Contracts* The Handbook of Operations Research and Management Science: Supply Chain Management. Kluwer, Amsterdam, PA., 2003.

[10] G. P. Cachon. The allocation of inventory risk in a supply chain: Push, pull, and advance-purchase discount contracts. *Management Science*, 50(2):222–238, 2004.

[11] G. P. Cachon and M. A. Lariviere. Supply chain coordination with revenue-sharing contracts: Strengths and limitations. *Management Science*, 51(1):30–44, 2005.

[12] J. D. Camm, J. J. Cochran, D. J. Curry, and S. Kannan. Conjoint optimization: An exact algorithm for the share-of-choice problem. *Management Science*, 52(3):435–447, 2006.

[13] K. D. Chen and W. H. Hausman. Technical note: Mathematical properties of the optimal product line selection problem using choice-based conjoint analysis. *Management Science*, 46(2):327–332, 2000.

[14] C. Clark. The greatest of a finite set of random variables. *Operations Research*, 9, March:145–162, 1961.

[15] C. F. Daganzo, F. Bouthelier, and Y. Sheffi. An efficient approach to estimate and predict multinomial probit models. *56th Meeting Transportation research Board*, 1977.

[16] S. R. Dalal and R. W. Klein. A flexible class of discrete choice models. *Marketing Science*, Vol 7, No. 3, Summer:232–250, 1988.

[17] G. Dobson and S. Kalish. Positioning and pricing of a product-line: Formulation and heurisitics. *Marketing Science*, 7(2):107–125, 1988.

[18] G. Dobson and S. Kalish. Heuristics for pricing and positioning a product-line using conjoint and cost data. *Marketing Science*, 39(2):160–175, 1993.

[19] L. Dong, P. Kouvelis, and Z. Tian. Dynamic pricing and invenotry control of substitute products. *Manufacturing & Service Operations Management*, 11(2):317–339, 2009.

[20] G. D. Eppen and A. V. Iyer. Backup agreements in fashion buying-the value of upstream flexibility. *Management Science*, 43(11):1469–1484, 1998.

[21] R. Ernst and P. Kouvelis. The effects of selling packaged goods on inventory decisions. *Management Science*, 45(8):1142–1155, 1999.

[22] V. Gaur and D. Honhon. Assortment planning and inventory decisions

under a locational choice model. *Management Science*, 52:1528–1543, 2006.

[23] P. E. Green and A. M. Krieger. Models and heurisitics for product line selection. *Marketing Science*, 4:1–19, 1985.

[24] W. Hanson and K. Martin. Optimizing multinomial logit profit functions. *Management Science*, 42(7):992–1003, 1996.

[25] W. J. Hopp and X. Xu. Product line selection and pricing with modularity in design. *Manufacturing & Service Operations Management*, 7(3):172–187, 2005.

[26] H. Hotelling. Stability in competition. *Economic Journal*, 39:41–57, 1929.

[27] W. A. Kamakura and R. K. Srivastava. Predicting choice shares under conditions of brand interdependence. *Journal of Marketing Research*, Vol XXI, November:420–434, 1984.

[28] B. Kamrad and A. Siddique. Supply contracts, profit sharing, switching, and reaction options. *Management Science*, 50:64–82, 2004.

[29] R. Kohli and R. Sukumar. Heuristics for product-line selection using conjoint analysis. *Management Science*, 36(12):1464–1478, 1990.

[30] A. G. Kok, M. L. Fisher, and R. Vaidyanathan. *Assortment Planning: Review of Literature and Industry Practice*. Retail Supply Chain Management. Kluwer Publishers., 2008.

[31] U. G. Kraus and C. A. Yano. Product line selection and pricing under a share-of-surplus choice model. *European Journal of Operational Research*, 150:653–671, 2003.

[32] K. Lancaster. A new approach to consumer theory. *Journal of Political Economy*, 74:132–157, 1966.

[33] E.L. Porteus Lariviere, M.A. Selling to the newsvendor: An analysis of price-only contracts. *Manufacturing & Service Operations Management*, 3(4):293–305, 2001.

[34] M. A. Lariviere. *Supply Chain Contracting and Coordination with Stochastic Demand, Chapter 8 of* Quantitative Models for Supply Chain Management,. Kluwer Academic Publishers., 1999.

[35] H. L. Lee and M. J. Rosenblatt. A generalized quantity discount pricing model to increase supplier's profits. *Management Science*, 30:1179–1187, 1986.

[36] R. D. Luce. *Individual Choice Behavior.* John Wiley and Sons, New York, 1959.

[37] B. Maddah and E.K. Bish. Joint pricing, assortment, and inventory decisions for a retailers product line. *Naval Research Logistics*, 54:315330, 2007.

[38] S. Mahajan, G. van Ryzin, and G. J. Ryzin. Stocking retail assortments under dynamic consumer substitution. *Operations Research*, 49(3):334–351, 2001.

[39] S. Mahajan and G. J. van Ryzin. *Retail Inventories and Consumer Choice*, chapter 17 in Quantitative Models in Supply Chain Management. Kluwer, Amsterdam, 1998.

[40] V. Martínez-de Albéniz and D. Simchi-Levi. A portfolio approach to procurement contracts. *Production and Operations Management*, 14(1):90–114, 2005.

[41] R. D. McBride and F. S. Zufryden. An integer programming appraoch to optimal product-line selection. *Marketing Science*, 7:126–140, 1988.

[42] D. Mcfadden and K. Train. Mixed mnl models for discrete response. *Journal of Applied Econometrics*, 15:447–470, 2000.

[43] A. R. McGillivray and E. A. Silver. Some concepts for inventory control under substitutable demand. *INFOR*, 16(1):47–63, 1978.

[44] S.K. Nair, L.S.Thakur, and K.Wen. Near optimal solutions for product line design and selection: Beam search heuristics. *Management Science*, 41(5):767–785, 1995.

[45] S. Netessine and N. Rudi. Centralized and competitive inventory models with demand substitution. *Operations Research*, 51(2):329–336, 2003.

[46] P. S. Noonan. When customers choose: a multi-product, multi-location newsboy model with substitution. 1995.

[47] M. Parlar. Game theoretic analysis of the substitutable product inventory problem with random demands. *Naval Research Logistics*, 35:397–409, 1988.

[48] M. Parlar and S. K. Goyal. Optimal ordering decisions for two substitutable products with stochstic demand. *Opsearch*, 21(1):1–15, 1984.

[49] B. A Pasternack. Optimal pricing and returns policies for perishable commodities. *Marketing Science*, 4:166–176, 1985.

[50] E. L. Porteus. *Foundations of Stochastic Invenotry Theory*. Standford University Press, 2002.

[51] K. Rajaram and C. S. Tang. The impact of product substitution on retail merchandising. *European Journal of Operational Research*, 135:582–601, 2001.

[52] P. H. Ritchken and C. S. Tapiero. Contingent claims contracting for purchasing decision in inventory management. *Operations Research Society of America*, 34:864–870, 1986.

[53] G. Ryzin and S. Mahajan. On the relationship between inventory costs and variety benefits in retail assortments. *Management Science*, 45(11):1496–1509, 1999.

[54] S. A. Smith and N. Agrawal. Management of multi-item retail inventory systems with demand substitution. *Operations Research*, 48(1):50–64, 2000.

[55] Sawtooth Software. Advanced simulation module (asm) for product optimization. Technical report, Sawtooth Software Technical Paper Series, 2003.

[56] T. J. Steenburgh. The invariant proportion of substitution property (ips) of discrete-choice models. *Marketing Science*, 27:300–307, 2008.

[57] C. S. Tang and R. Yin. The implications of costs, capacity, and competition on product line selection. *European Journal of Operational Research*, doi:10.1016/j.ejor.2009.01.012, 2009.

[58] A. Tsay, S. Nahmias, and N. Agrawal. *Modeling supply chain contracts: A review*. Quantitative Models for Supply Chain Management, Kluwer Academic Publishers., 1999.

[59] A. A. Tsay. The quantity flexibility contract and supplier-customer incentives. *Management Science*, 45:1339–1358, 1999.

[60] Q. Wang and M. Parlar. A three-person game theory model arising in stochastic inventory control theory. *European Journal of Operations Research*, 76:83–97, 1994.

[61] X. F. Wang, J. Camm, and D. Curry. A branch-and-price approach to the share-of-choice product line design problem. *Management Science*, Forthcoming, 2009.

[62] D. J. Wu and P. R. Kleindorfer. Competitive options, supply contracting, and electronic markets. *Management Science*, 51(3):452–466, 2005.