RELIABILITY MODELING AND ANALYSIS WITH MEAN RESIDUAL LIFE

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SUMMARY

Mean residual life (MRL), representing how much longer components will work for from a certain point of time, is an important measure in reliability analysis and modeling. It offers condensed information for various decision-making problems, such as optimizing burn-in test, planning accelerated life test, establishing warranty policy, and making maintenance decision. Realizing the importance of the mean residual life, this thesis focuses on the modeling (Chapter 3 and Chapter 4) and analysis (Chapter 5 and Chapter 6) based on this characteristic.

This thesis studies both parametric models and nonparametric methods, which are the two common ways in reliability modeling. In Chapter 3, a parametric model is developed for a simple, closed-formed upside-down bathtub-shaped mean residual life (UBMRL). This model is derived from the derivative function of MRL, instead of reliability function and failure rate function that are often used in model construction. We first characterize the derivative function and develop a general form for the model. Based on the general form, a suitable function is selected as a starting point of the derivation of the new UBMRL model. The MRL function and the failure rate function

are further studied. Numerical examples and comparisons indicate that the new model performs well in modeling lifetime data with bathtub-shaped failure rate function and UBMRL function.

Besides the parametric model, we propose a nonparametric method for the estimation of decreasing MRL (DMRL) with Type II censored data (Chapter 4). This method is based on the comparison between two estimators of the reliability function, the Kaplan-Meier estimator and an estimator derived from the empirical MRL function. Based on data generated from Weibull and gamma distributions, simulation results indicate that the new approach is able to give good performance and can outperform some existing parametric methods when censoring is heavy.

Moreover, the analysis of the relationship between MRL and other reliability measures is another important issue. Hence, Chapter 5 focuses on the relations between MRL and the failure rate function by studying the effect of the change of one characteristic on the other characteristic. The range that the MRL will decrease (increase) if the associated failure rate function is increased (decreased) to a certain level is investigated. On the other hand, the difference of two failure rate function is also studied in the case that their corresponding MRL functions are ordered. Some inequalities are established to indicate upper or lower bound on the extent of change. The application of the inequalities is also discussed.

As an extension of the MRL of single items that is discussed in foregoing chapters, the MRL of systems is investigated in Chapter 6. We discuss MRL of series and parallel systems with independent and identically distributed components; and obtain the relationships between the change points of MRL functions for systems and

for components. Compared with the change point for single components assuming that it exists, the change point for a series system occurs later. For a parallel system, its change point is located before that for the components, if it exists at all. Moreover, for both types of systems, the distance between the change points for systems and for components increases with the component number. In addition, the MRL of a parallel system with two non-identical components is briefly discussed in a graphic way.

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LIST OF NOTATIONS

MRL   Mean Residual Life

IMRL   Increasing Mean Residual Life

DMRL   Decreasing Mean Residual Life

IDMRL   Increasing and then Decreasing Mean Residual Life

UBMRL   Upside-down Bathtub-shaped Mean Residual Life

BMRL   Bathtub-shaped Mean Residual Life

NBUE   New Better than Used in Expectation

NWUE   New Worse than Used in Expectation

NWBUE   New Worse then Better than Used in Expectation

NBWUE   New Better then Worse than Used in Expectation

IFR   Increasing Failure Rate Function

DFR   Decreasing Failure Rate Function

DIFR   Decreasing and then Increasing Failure Rate Function

BFR   Bathtub-shaped Failure Rate Function

UBFR   Upside-down Bathtub-shaped Failure Rate Function

NBUFR   New Better than Used in Failure Rate (Function)

NWUFR   New Worse than Used in Failure Rate (Function)

TTT   Total Time Test

MLE   Maximum Likelihood Estimation/Estimator

LSE   Least Square Estimation/Estimator

MSE   Mean Squared Error

AIC   Akaike Information Criterion

i.i.d.   Independent and Identically Distributed

CHAPTER 1   INTRODUCTION

This thesis contributes to some methodological and analytical issues concerning Mean Residual Life (MRL) in reliability analysis. In this introductory chapter, some background information is provided, which is followed by motivations of the research on MRL. We then give the scope and objective of our study. Finally, a summary of the contents of this thesis and its structure are presented.

1.1   Background information

The study of lifetimes is a prevailing and important topic for researchers. Actuaries may be interested in the lifetime of a person to determine the amount of premium he should pay for his annuity. Biostatisticians may investigate the lifetimes of cancer patients who are subject to different therapies. Reliability engineers may be concerned about the lifetime of a light bulb or a private computer so that proper warranties or maintenance can be planned.

However, the lifetimes of either humans or products always differ from one to

another. Leibniz, a famous mathematician and philosopher, said "no leaves are ever exactly alike". Even for the products made from the same materials and under the same process, their lifetimes vary from each other due to some uncontrollable factors, such as the float of temperature and moisture. One way to deal with this kind of uncertainty is to measure it in terms of probability, which is the essence of reliability analysis. Reliability, regarded as quality over time, was textually defined in Leemis (1995) as follows,

"*The reliability of an item is the probability that it will adequately perform its specified purpose for a specified period of time under specified environmental conditions.*"

In reliability analysis, lifetimes are treated as random variables subject to probability distributions, either continuous or discrete distributions. The most famous distribution used in reliability analysis is exponential distribution, which is the simplest model in describing lifetimes. But the application of this distribution is limited in practice, because few components have the property of lack of memory. Compared to exponential distribution, Weibull, lognormal, inverse Gaussian and other distributions are more flexible in modeling different types of failure mechanisms.

As lifetimes are assumed to follow probability distributions, the reliability is usually measured by a function of time that can represent the distributions. There are five main characteristics used to measure reliability: the reliability function, the probability density function, the failure rate function, the cumulative failure rate function, and the MRL function. Although these five functions are actually equivalent in the sense of probability (i.e. knowing any one of them, the other four functions can

be uniquely determined), each of them provides different descriptions for the lifetimes of products. The reliability function interprets the possibility that items will last for a certain period of time. The probability density function describes how frequently products fail at each time point. The failure rate function indicates the instantaneous risk an item faces. The cumulated failure rate function gives the information about the expected number of failures that will occur by some time point. The MRL presents how much longer components will work from a certain point of time. According to their different statistical meanings, these five characteristics are often used to make various decisions with different focuses. In this thesis, the MRL will be extensively discussed and studied in the aspects of reliability modeling, analysis, and application.

Conceptually, the MRL function is derived from residual life, a conditional random variable. For an item that has survived a period of time, its residual life is defined as a random variable conditioning on the time it has experienced. This measure contains two aspects of information, the lifetime of an item and the fact that this item has been working for some time period without failure. Because of its dual characters, residual life is widely applied in reliability engineering.

In engineering reliability tests, we often consider the residual life of a device. For example, in a step-stress accelerated life test, the life of a specimen corresponding to current stress is actually the residual life of this specimen after the previous testing steps. (Tang et al., 1996). Another instance is burn-in test, which eliminates weak components before releasing strong components. The lifetimes of the passed components are residual lives as well.

The applications of residual life in maintenance have also drawn much

attention. An aircraft whose mileage is ten thousand miles may need a properly scheduled maintenance plan for its engines to ensure its next 1000-mile flight. In industries, an accurate prediction of the residual life of machines will conduce proactive maintenance processes that would help to minimize downtime of machinery and production (Yan et al., 2004).

However, for the decision making in maintenance and tests, such as the determination of optimal time for stopping a burn-in procedure or executing a repair, it is inconvenient to define and analyze a series of residual lives according to different survival times. Hence the MRL function, generated as the expectation of residual life, is helpful in making such decisions. An extensive literature review on the property and modeling of the MRL will be presented in Chapter 2 in order to discuss the extensive research on the MRL function in literature and to demonstrate the importance of the MRL in reliability analysis.

1.2 Research motivation

In practice, before analyzing the MRL function and making decisions, we always need to estimate the MRL function from the data of failure times. The two common ways in modeling are parametric modeling and nonparametric method.

For the parametric modeling, an underlying distribution needs to be predetermined before the analysis of failure data. One important distribution family is Weibull family that is developed based on Weibull distribution (Weibull, 1951), including exponentiated Weibull distribution (Mudholkar & Srivastava, 1993),

additive Weibull distribution (Xie & Lai, 1996), generalized Weibull model (Lai et al., 2003) and extended Weibull distribution (Chen, 2000 ; Xie et al., 2002). Besides, there are also other distributions, such as generalized gamma distribution (Gupta & Lvin, 2005a) and generalized lognormal distribution (Gupta & Lvin, 2005b). All these models, with different parameters, have both monotonic and non-monotonic MRL functions; thus they are able to model lifetimes exhibiting different types of MRL. However, for these existing models, the MRL functions are of complicated forms, which usually involve an integral of a reliability function that is not of a closed form. This problem motivates the formulation of a new class of life distribution, which has different characteristics from the existing models, such as a new model with some form of the MRL function. Obviously, this kind of new model will make the analysis based on MRL easier.

Compared to the parametric modeling assuming underlying distributions, nonparametric methods use only failure data to estimate the MRL function regardless of the forms of models and thus introduce less bias. Yang (1978) proposed the empirical MRL function for complete data, which is the first and basic nonparametric estimation for the MRL function. Based on this estimator, several other MRL estimators for complete data were also constructed (Zhao & Qin, 2006; Kochar et al., 2000). Moreover, the case of random right censorship was considered in the estimation of the MRL function. Li (1997) presented a confidence bound for the MRL. Statistical inference for the MRL under random right censoring was provided by Na & Kim (1999) and Qin & Zhao (2007). In contrast to numbers of studies on the randomly right censoring, only a few papers in literature focused on the estimation of the MRL under extreme right censorship. This is because that it is more difficult to

deal with the lost information in extreme right censoring than in other types of censoring. In Guess & Park (1991), only conservative confidence intervals were presented under extreme right censorship. Hence, other feasible methods are expected and required for the estimation of the MRL function.

Besides studying only the MRL function itself, the relationship between the MRL and the failure rate function is another important issue in reliability analysis. This is because these two characteristics are closely related to each other and the comparison between them is helpful in decision-making and estimation. In literature, many works have found that the MRL function is closely related to the failure rate function. Bryson & Siddiqui (1969), Gupta & Akman (1995a), and Tang et al. (1999) proved that the shape of the MRL function depends on the shape of the failure rate function for both monotonic and non-monotonic cases. Also, Gupta & Kirmani (1987) proved that the failure rate ordering dominates the MRL ordering and proposed a sufficient condition under which the reverse also holds. These studies discussed the relationship between the MRL and the failure rate function mainly from a qualitative point of view. Only a few papers tried to quantify the relations of the two functions. Finkelstein (2003a) gave a quantitative analysis on how the MRL changes with increased failure rate, but unfortunately he did not give any concrete result on the extent of change. Hence, more quantitative analysis on the relationship would be useful and meaningful in reliability for both theory and application. Moreover, most discussions focused on the effect that the failure rate function has on the corresponding MRL, such as limiting property and shape, as the failure rate function usually can be explicitly expressed. But, sometimes, it may be easier to start from the MRL function. For example, it is more convenient to get the empirical estimation of

the MRL than of the failure rate function, because the failure rate function encounters derivative function, whose estimation is hard to be obtained. Therefore, some quantitative studies on the relationship between MRL and failure rate function are required as a complement to those existing works.

Compared to the MRL of single items considered in all the previous studies, the MRL function of systems also plays a significant role. The system reliability is often studied at either system level or component level. If a system is analyzed at system level, then it is treated as a whole without considering its inner structure and thus can be similarly discussed as a single item. For component level, the structure of a system always needs to be clearly defined, because in this case, the reliability of a system is determined by the allocation and the properties of components; see Leemis (1995) for a systematic definition and an annotated overview. In literature, several papers discussed the properties of MRL that series and parallel systems can preserve from their components. Abouammoh & El-Neweihi (1986) showed that parallel systems inherit the DMRL from components. The reversed preservation ageing properties for series and parallel systems were discussed in Li & Yam (2005), Belzunce et al. (2007a), and Li & Xu (2008). These works made great contributions to the preservation behaviors of series and parallel systems, but they did not investigate the shape of the MRL function. Hence, the study of the MRL's shape, especially the non-monotonic shape, is needed for series and parallel systems, as such analysis would help to determine whether application decisions should be made at system level or component level.

1.3   Research scope and objective

The aim of this research was to make a comprehensive study on reliability modeling and analysis based on mean residual life. The specific aims of this research were:

- To propose a parametric model with relatively simple and closed-form upside-down bathtub-shaped MRL (UBMRL) from the starting point of the derivative function of the MRL; to study the general form of the proposed model so that a new way for the definition of probability distributions could be established.

- To develop a nonparametric method to estimate DMRL under extreme right censorship by comparing two estimators of the reliability function, the Kaplan-Meier estimator (Kaplan & Meier, 1958) and an estimator derived from empirical MRL function (Yang, 1978).

- To quantitatively study the relationships between the MRL function and the failure rate function by establishing some inequalities; to utilize the inequalities to construct bounds for one characteristic based on the other characteristic.

- To study the MRL functions for series and parallel systems that are composed of components with UBMRL; to compare the MRL of systems with the MRL of components in terms of changing point.

Results of the present study would enhance our understanding of the properties, modeling, and applications of the MRL function. The proposed model with relatively simple and closed-form UBMRL may provide more accurate description for the lifetime of items and also may be of great importance in decision making based on the

MRL function in terms of its shape; and the general form of the proposed model could shed light on a new definition of probability distributions. The nonparametric estimation of the MRL under extreme right censorship may provide an innovative method to deal with information loss due to censoring. The study of the relationship between the MRL and the failure rate function may provide guidelines on how to control the deterioration of products more efficiently. The results on change point of the MRL function for series and parallel systems may lead to a better understanding of the role of redundancy that is usually built into systems.

In this thesis, the MRL function refers to continuous, differentiable and univariate MRL function, which is most commonly used in reliability analysis compared to discrete and multivariate MRL. The same assumptions are also applied to other probability characteristics, such as the reliability function and the failure rate function etc. Moreover, in most parts of our research on the MRL, only DMRL and UBMRL are considered, because they are two most natural and simplest shapes in real life application and other more complex curves, if needed, can be easily obtained by combining the DMRL and UBMRL. Additionally, all the calculation and simulation experiments are based on the platform provided by the software "Mathematica".

1.4   Organization of the thesis

This thesis consists of seven chapters and focuses on the study of the MRL in two aspects, reliability modeling and reliability analysis. For the modeling issue, a parametric model with UBMRL and the general form are proposed and studied in Chapter 3; in Chapter 4, a nonparametric method is developed to estimate the MRL

function with Type II censored data. In the analysis part, Chapter 5 analyzes the relationships between the MRL and the failure rate function and applies the results for the estimation of bounds for the two functions. In Chapter 6, change points of the MRL functions for series and parallel systems are discussed and compared with change point of the MRL for single components in terms of location. Finally, a conclusion of the entire work as well as some potential future research topics is given in Chapter 7. A graphic summary of the content of each chapter is shown in Figure 1.1.
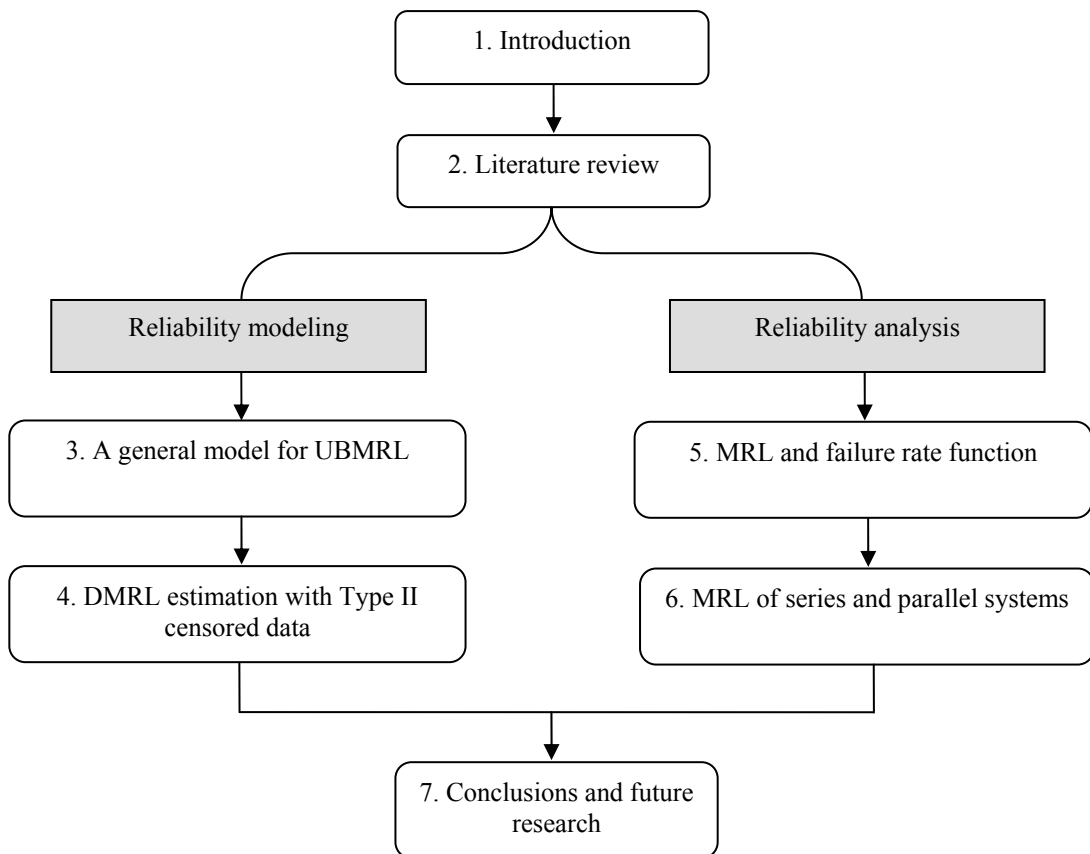
Figure 1.1   Structure of the thesis.

In the next chapter, the papers on MRL in literature will be extensively reviewed so that a better comprehension of how our research was originated and what

our results will contribute can be achieved, as well as the challenges associated with various aspects of the research on the MRL. The topics covered in the next chapter include definitions and properties of the MRL and other characteristics, the parametric and nonparametric modeling of the MRL, the MRL of systems, and also some common   uses   in   practice.

CHAPTER 2   LITERATURE REVIEW

The MRL function has been widely used in fields of reliability, statistics, and insurance. In literature, many useful results have been derived in various aspects of the MRL, such as the properties, the shape, the estimation, and the application etc. A recent and detailed review of the MRL in reliability analysis was presented in Chapter 4 of Lai & Xie (2006). In this chapter, we focus on the existing works most related to this thesis and give a focused but informative survey in support of our research. Definitions and properties of the MRL and other related reliability measures are first presented in Section 2.1. Section 2.2 presents a comprehensive review on parametric models and nonparametric estimation for the MRL. Section 2.3 discusses the MRL for coherent systems. In Section 2.4, some applications of the MRL are given.

2.1   Definitions and properties

The MRL function, the failure rate function, and the reliability function are mathematically defined and explained in this section. Also, according to different shapes of the MRL and the failure rate function, various classes of life distributions

are defined and categorized. After that, the properties of the MRL are studied and compared to that of the failure rate function, which is considered reciprocal with the MRL function.

2.1.1 Basic definitions and concepts

Suppose T is a continuous non-negative random variable with cumulative distribution function (CDF) $F(t)$, probability density function (PDF) $f(t)$, and reliability function $R(t) = 1 - F(t)$. Define the residual life random variable at age t by $T_t = T - t \,|\, T > t$; see Banjevic (2008) for discussion. If $E[T] < \infty$, then the MRL function exists and is defined as the expectation of the residual life

$$m(t) = E(T - t \,|\, T > t) = \frac{1}{R(t)} \int_t^\infty R(x) dx, \ t \geq 0. \tag{2.1}$$

It is easy to show that MRL determines distributions uniquely; see Langford (1983) and Wesolowski & Gupta (2001) for example. The reason for this fact is that the MRL function $m(t)$ is equivalent to the reliability function $R(t)$ in the sense of probability; and the reliability function is known to be able to determine probability distributions. In (2.1), the MRL $m(t)$ is mathematically defined as a function of $R(t)$. On the other hand, we can also express $R(t)$ in terms of $m(t)$

$$R(t) = \frac{m(0)}{m(t)} \exp\left\{ -\int_0^t \frac{1}{m(x)} dx \right\}, \ t \geq 0. \tag{2.2}$$

Another characteristic that is closely related to the MRL, as mentioned frequently in previous chapter, is the failure rate function $r(t)$.

$$r(t) = \frac{f(t)}{R(t)}, \ t \geq 0 . \qquad (2.3)$$

Since $m(t)$ is assumed differentiable in this thesis, it can be shown that

$$m'(t) = m(t) r(t) - 1 . \qquad (2.4)$$

As $m(t) \geq 0$ and $r(t) \geq 0$, we have $m'(t) \geq -1$, which means the slope of the MRL should be always no less than -1. Equation (2.4) also implies that the shape of $m(t)$ depends on both $m(t)$ and $r(t)$.

Also, the failure rate function can be used to define the reliability function

$$R(t) = \exp\left(-\int_0^t r(x) dx \right) . \qquad (2.5)$$

Together with (2.1) – (2.5), it can be concluded that, the MRL function $m(t)$, the failure rate function $r(t)$, and the reliability function $R(t)$ are equivalent in the sense that all of them are able to uniquely determine the distribution; and knowing any one of them, the other two could be obtained given that they exist. In addition, the transform and the combination of these measures are also found to be able to characterize distributions; see Roy (1993), Ruiz & Navarro (1994), Navarro et al. (1998), Navarro & Ruiz (2004), Sankaran & Sunoj (2004), Gupta & Kirmani (2004), and Xekalaki & Dimaki (2005) for discussion.

2.1.2   Mean residual life classes

Different MRL classes describe different aging properties. In general, the MRL classes can be divided into two groups based on the behavior of the MRL function: monotonic and non-monotonic. The monotonic aging classes for the MRL function include distributions with decreasing mean residual life (DMRL) and with increasing mean residual life (IMRL). The non-monotonic MRL classes have much more types of distributions. Some known classes are upside-down bathtub-shaped MRL (UBMRL), bathtub-shaped MRL (BMRL) and new better than used in expectation (NBUE), etc. As the MRL function is closely related to the failure rate function, the MRL classes are also linked to the classes defined via the failure rate function, such as increasing failure rate (IFR), decreasing failure rate (DFR), bathtub-shaped failure rate (BFR) and upside-down bathtub-shaped failure rate (UBFR), etc. Next, mathematical definitions of different distribution classes are presented and a chain of implication used to indicate the connection between some of these classes is also given.

Definition 2.1 A distribution is said to be DMRL (IMRL) if the mean residual life function $m(t)$ is decreasing (increasing) in t, i.e. $m'(t) < 0$ for $t \geq 0$ or $m'(t) > 0$ for $t \geq 0$.

   As explained in Lai & Xie (2006), DMRL means that, the older an item is, the smaller is its MRL, and IMRL implies that an older item has longer MRL. Similar to monotonic MRL class, the definition for the class with increasing (decreasing) failure rate function (IFR, DFR) is

Definition 2.2 A distribution is said to be IFR (DFR) if the failure rate function $r(t)$ is increasing (decreasing) in t, i.e. $r'(t) > 0$ for $t \geq 0$ or $r'(t) < 0$ for $t \geq 0$.

Bryson & Siddiqui (1969) showed that IFR (DFR) implies DMRL (IMRL) and claimed that DMRL does not imply IFR by giving a counter example. Similar problem was also studied in Lillo (2000). Sufficient conditions under which MRL also dominates the failure rate function will be presented in Section 2.1.3.

There are also several aging notions representing the non-monotonic behavior of the MRL $m(t)$. One of the most popular classes is UBMRL, which is developed on the basis of the corresponding failure rate class, BFR. These bathtub distribution classes plays an important role in reliability, because this type of distribution classes usually could be observed in the lifetime of a population containing both normal and inferior products (Lawless, 1982; Kao, 1959; Bebbington et al., 2007a). An intuitive explanation is that, due to the initial quick die-out of inferior products, the overall reliability of the population improves exhibiting a DFR and an IMRL, and then enters a stable period with relatively constant MRL and failure rate before finally wears out with an IFR and a DMRL, as the normal products start to deteriorate. In literature, there are several definitions for UBMRL and BFR. Mi (1995) defined a bathtub curve by three segments: increasing (decreasing), constant and decreasing (increasing).

Definition 2.3 A real valued function $g(t)$ with support $[0, \infty)$ has a bathtub (upside-down bathtub) shape if there exists $0 \leq t_1 \leq t_2 \leq \infty$ such that

(a) $g(t)$ is strictly decreasing (increasing) if $0 \leq t \leq t_1$;

(a) $g(t)$ is constant if $t_1 \leq t \leq t_2$; and

(a) $g(t)$ is strictly increasing (decreasing) if $t \geq t_2$.

In Definition 2.3, if $t_1 = t_2 = 0$, $g(t)$ becomes a strictly increasing function; if $t_1 = t_2 = \infty$, $g(t)$ is strictly decreasing; in general, if $t_1 = t_2$, the interval with constant degenerates to a single point. Obviously, strictly monotonic function is a special case in this definition.

To avoid ambiguity and distinguish monotonic case from bathtub classes, we adopt the definition in Glaser (1980) to describe the UBMRL and BFR in this thesis, which defined the bathtub shape via the derivative functions and assumed that the constant period in Definition 2.3 degenerates to a point. In fact, the UBMRL and the BFR defined in Glaser (1980) can also refer to increasing initially then decreasing mean residual life (IDMRL) and decreasing initially then increasing failure rate (DIFR). Typical curves of UBMRL and BFR are displayed in Figure 2.1.

Definition 2.4 A distribution is said to be UBMRL if there exists a t_0 such that the MRL function $m(t)$ is increasing for $0 \leq t < t_0$ and then decreasing for $t > t_0$, i.e. $m'(t) > 0$ for $0 \leq t < t_0$, $m'(t) = 0$ for $t = t_0$ and $m'(t) < 0$ for $t > t_0$. And t_0 is called change point of $m(t)$.

Definition 2.5 A distribution is said to be BFR if there exists a v_0 such that the failure rate function $r(t)$ is decreasing for $0 \leq t < v_0$ and then decreasing for $t > v_0$, i.e. $r'(t) < 0$ for $0 \leq t < v_0$, $r'(t) = 0$ for $t = v_0$ and $r'(t) > 0$ for $t > v_0$. And v_0 is called

change point or critical point of $r(t)$.
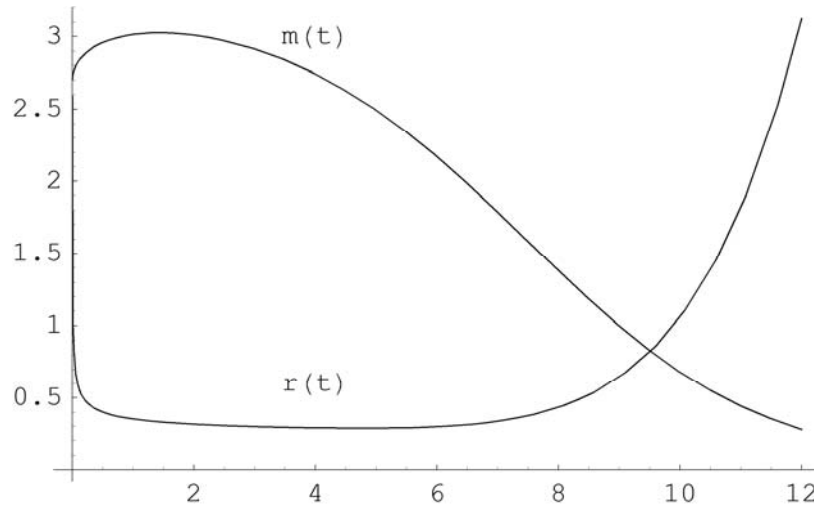
Figure 2.1   Typical curves of UBMRL and BFR.

Since the MRL $m(t)$ and the failure rate function $r(t)$ are assumed to be continues and differentiable in this thesis, it is reasonable to define the life classes in terms of the behaviors of their derivatives. Thus without being specific, it is understood that the acronyms UBMRL and BFR that appear throughout the rest of this thesis are defined by Definition 2.4 and 2.5 respectively. In addition, the distribution classes with BMRL and UBFR also can be analogically defined, but these two classes are seldom encountered in practical reliability engineering, because it is unrealistic the case that the older an time is, the better is its performance.

Other definitions of bathtub classes were also presented in Deshpande & Suresh (1990), Mitra & Basu (1995), and Haupt & Schabe (1997). More general MRL classes, which extend monotonic MRL and UBMRL, were also considered and defined based on either mean time to failure, i.e. $m(0)$, such as new better than used in expectation (NBUE), new worse used in expectation (NWUE) in Barlow &

Proschan (1981a). Then the new better than used in failure rate (NBUFR), new worse than used in failure rate (NWUFR) distribution classes can be correspondingly defined in the failure rate function (Deshpande et al., 1986).

Definition 2.6 A distribution with mean $\mu = m(0)$ is said to be NBUE if $m(t) \le \mu$ for all $t \ge 0$; similarly, a NWUE distribution is such a distribution that $m(t) \ge \mu$ for all $t \ge 0$.

Definition 2.7 A distribution is said to be NBUFR if $r(0) \le r(t)$ for all $t \ge 0$; similarly, a NWUFR distribution is such a distribution that $r(0) \ge r(t)$ for all $t \ge 0$.

The classes in a group often can be connected by some chains of implications (Deshpande et al., 1986; Kochar & Wiens, 1987) and also closely linked to the failure rate classes with similar monotonicity.

$$
\begin{array}{ccc}
\text{IFR (DFR)} & \Longrightarrow & \text{NBUFR (NWUFR)} \\
\Downarrow & & \\
\text{DMRL (IMRL)} & \Longrightarrow & \text{NBUE (NWUE)}
\end{array}
$$
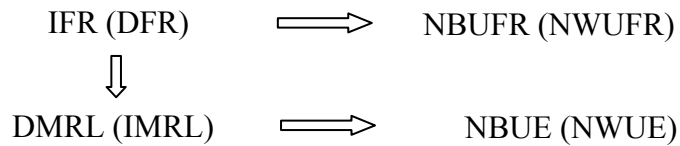
In a similar manner, we can define more general classes based on the non-monotonic behavior of MRL. Mitra & Basu (1995) proposed new worse then better than used in expectation (NWBUE) and new better then worse than used in expectation (NBWUE) distributions; and also showed that {UBMRL} $\subset$ {NWBUE} and {BMRL} $\subset$ {NBWUE}.

Definition 2.8 A lifetime distribution with mean $\mu = m(0)$ is said to be NWBUE if

there exists a point $0 < \tau < \infty$ such that

$$m(t) \begin{cases} \geq \mu, & \text{for } t < \tau \\ \leq \mu, & \text{for } t \geq \tau \end{cases};$$

Similarly, a NBWUE distribution satisfies that there exists a point $0 < \tau < \infty$ such that

$$m(t) \begin{cases} \leq \mu, & \text{for } t < \tau \\ \geq \mu, & \text{for } t \geq \tau \end{cases}.$$

Besides, there are also other classes, which are developed by comparing $m(t)$ with the MRL at a specified time point, like better MRL at t_0 in Kulasekera & Park (1987). Further and recent discussions of distribution classes can refer to Belzunce et al. (2004), Ahmad et al. (2005), and Al-Zahrani & Stoyanov (2008). In Sun & Zhang (2009), a class of transformed mean residual life models was proposed for fitting survival data under right censoring.

2.1.3 Properties and relations with failure rate function

Since the definition of MRL was proposed, the associated properties have been studied for over a half century. An early extensive discussion on theoretical properties of the MRL might date back to Cox (1962). Limiting properties of the MRL were provided in Meilijson (1972) and Beirlant et al. (1992). Recently, Bradley & Gupta (2003) investigated limiting behaviors of the MRL and derived an asymptotic expansion which could give a good approximation for the MRL when time variable is large. Another study on the similar topic was also conducted in their work Gupta & Bradley (2003).

Compared to the studies considering only the MRL function itself, many more works in literature focus on the relationship with other characteristics, especially the failure rate function. This is because the failure rate function, to some extent, could be treated as the reciprocal of the MRL. In Calabria & Pulcini (1987), an asymptotic relationship between the MRL and the failure rate function was derived by applying the L'Hospital's rule to (2.1)

$$\lim_{t \to \infty} m(t) = \lim_{t \to \infty} \frac{1}{r(t)},$$

given that the latter limit exists and is finite. Besides limiting behaviors, this reciprocity between the MRL and the failure rate function exists through the entire time period. Finkelstein (2003a) generally discussed the reduction in MRL due to an extra risk represented by increased failure rate. Based on it, Bebbington et al. (2008) assumed that the extra risk could be modeled by a constant failure rate and further discussed in details the effect of such a risk on the change of MRL.

In addition to above works, the other two issues reflecting the reciprocity are curve shape and partial ordering, which have attracted much attention in literature. Thus, in the following, the papers on the relationships between the MRL and the failure rate function are surveyed in the aspects of these two issues.

• **Shapes of the MRL and the failure rate function**

Bryson & Siddiqui (1969) proved that IFR (DFR) implies DMRL (IMRL) and claimed that the converse proposition is not true by giving a counter example. In Ghai & Mi (1999), a sufficient condition is provided for a DMRL (IMRL) distribution that

is also IFR (DFR).

Theorem 2.1 *Let $m(t)$ be a MRL and $r(t)$ be the corresponding failure rate function.*

Then

(1) If $m(t)$ is increasing and concave, then $r(t)$ is decreasing;

(1) If $m(t)$ is decreasing and convex, then $r(t)$ is increasing.

Motivated by the relationship between the monotonic MRL and failure rate classes, one may conjecture that the UBMRL distribution class is also closely related to the BFR class. As early as two decades ago, Rajarshi & Rajarshi (1988) pointed out that the relationship between UBMRL and BFR could be empirically observed "from the life tables of human and animal populations". A general result on the relation between the non-monotonic classes of $m(t)$ and $r(t)$ was given in both Gupta and Akman (1995a) and Mi (1995).

Theorem 2.2 *Suppose $r(t)$ is of bathtub shape (BFR), then*

(1) $m(t)$ is DMRL if $r(0)m(0) \leq 1$;

(2) $m(t)$ is UBMRL if $r(0)m(0) > 1$.

On the other hand, if $r(t)$ is of upside-down bathtub shape (UBFR), then

(1) $m(t)$ is IMRL if $r(0)m(0) \geq 1$;

(2) $m(t)$ is BMRL if $r(0)m(0) < 1$.

Based on Theorem 2.2, some useful results were derived for the change points of non-monotonic MRL and failure rate functions that are defined in Definition 2.4

and 2.5. Detailed proofs of the following theorem can refer to Mi (1995), Guess et al. (1998), and Tang et al. (1999).

Theorem 2.3 *Suppose $r(t)$ is of bathtub shape (BFR) with a change point v_0. If $r(0)m(0) > 1$, then $m(t)$ is UBMRL with a unique change point $t_0 \in (0, v_0)$, i.e. $t_0 < v_0$. Otherwise, $m(t)$ is DMRL.*

A similar conclusion also can be made for the UBFR and BMRL classes. These results indicate that the change point of a non-monotonic MRL $m(t)$ always occurs prior to the change point of its related failure rate function $r(t)$. They also show that the shape of a non-monotonic $m(t)$ depends on the shape of the corresponding $r(t)$, which is also true for monotonic MRL and failure rate function discussed before.

On the other hand, the shape of a non-monotonic MRL, the same as the monotonic case, cannot determine the shape of the failure rate function. In Ghai & Mi (1999), a sufficient condition under which UBMRL implies BFR was developed, as shown in Theorem 2.4. Although the bathtub curve defined in their paper follows Definition 2.3 which includes a constant period, the results still can be applied to our interested UBMRL and BFR in Definition 2.4 and 2.5 by some modifications.

Theorem 2.4 *Let $m(t)$ is UBMRL with a change point t_0. Suppose there exist $t_1 \in [t_0, \infty)$ such that $m(t)$ is concave on $[0, t_1)$ and convex on $[t_1, \infty)$. If $m'(t)$ is convex on $[t_0, t_1)$, then one of the following alternatives is true for $r(t)$:*

(1) $r(t)$ exhibits a bathtub shape (Definition 2.2) that has two change points, say

$v_1 < v_2$ *; where $t_0 \leq v_1 < v_2 \leq t_1$;*

(2) $r(t)$ exhibits a bathtub shape that has a unique change points, say v_0; where

$t_0 \leq v_0 \leq t_1$.

Furthermore, Tang et al. (1999) and Bekker & Mi (2003) studied change points of a roller-coaster shaped MRL function, a more complex curve with increasing and decreasing segments appearing alternately, and found that the number and location of the change points are determined by the corresponding failure rate function and the derivative function of MRL. Similar results were obtained in Bekker & Mi (2003) and Gupta & Gupta (2000) when considering the crossings of two MRL functions as well as two failure rate functions. They found that the crossing points of two MRL functions also depend on those of the corresponding failure rate functions in terms of number and location. More general discussions on the shape of the MRL can be found in Finkelstein (2002) and Mi (2004).

- **MRL ordering and failure rate ordering**

As pointed out in the forgoing sections, the MRL, the failure rate function, and the reliability function are equivalent to each other. Moreover, the shape of the failure rate function often could determine the shape of the MRL. An interesting question is whether a partial ordering with respect to one characteristic would imply the same partial ordering with respect to another characteristic.

Definition 2.9 Consider two life time random variables X and Y with reliability

function $R_X(t)$ and $R_Y(t)$, respectively,

(1) Stochastic ordering: $X \geq_{\text{ST}} Y$ if $R_X(t) \geq R_Y(t)$ for all $t \geq 0$;

(2) Failure rate ordering: $X \geq_{\text{FR}} Y$ if $r_X(t) \leq r_Y(t)$ for all $t \geq 0$;

(2) Mean residual life ordering: $X \geq_{\text{MRL}} Y$ if $m_X(t) \geq m_Y(t)$ for all $t \geq 0$.

Gupta & Kirmani (1987) proved that failure rate ordering implies MRL ordering and under some sufficient condition, two random variables that are subject to a MRL ordering also have the same failure rate ordering.

Theorem 2.5 *Let X and Y be two life time random variables with reliability function* $R_X(t)$ *and* $R_Y(t)$. *Then*

(1) $r_X(t) \leq r_Y(t) \Rightarrow m_X(t) \geq m_Y(t)$;

(2) Suppose $m_X(t) \geq m_Y(t)$ *and* $m_Y(t)/m_X(t)$ *is a non-decreasing function for all* $t \geq 0$, *then* $r_X(t) \leq r_Y(t)$.

Due to the close dependence between the MRL and the failure rate function , Finkelstein (2006) reviewed some common relationships between MRL ordering and failure rate ordering and showed that failure rate ordering leads to the corresponding ultimate mean residual life ordering with some assumptions. Hu et al. (2004) did a unified study on the two orderings from a new point of view of likelihood ratio ordering with different degrees. Frostig (2006) studied the sum of several dependent Bernoulli distributed random variables in MRL ordering. Also, the relationship between MRL ordering and other partial orderings was dealt with in other works. Kochar & Wiens (1987) extensively studied MRL ordering, NBUE ordering and

NBUFR order ordering, etc. Belzunce et al. (1999) discussed Laplace order and ordering of residual lives. Reversed MRL ordering, also called mean inactive time ordering was considered in Nanda et al. (2006) and Li & Xu (2006).

2.2   Reliability modeling

Reliability modeling, which in fact refers to lifetime data modeling, aims to find out underlying failure mechanisms of items and describe their failure behaviors in a proper way, so that accurate predictions and correct decisions could be made. This thesis focuses on how to model and analyze lifetime data in terms of the MRL function. There are two main methods in modeling: parametric modeling and nonparametric estimation. In parametric modeling, different probability distributions with various parameters are used to fit diverse lifetime data. Maximum likelihood estimation (MLE) and least square estimation (LSE) are the two most common approaches in estimating distribution parameters. For nonparametric estimation, as its name shows, this type of estimation does not assume any distribution and obtain the MRL function from data empirically.

- **Test of exponentiality versus different MRL classes**

To assure the effective fitness of data, statistical tests are always conducted before the modeling to identify different aging classes defined on MRL. Commonly, the tests are proposed for exponentiality, i.e. constant MRL, against other MRL class alternative. One graphic and intuitive test method is derived from total time on test (TTT) test (Bergman & Klefsjo, 1984). Formally, the TTT transform of a distribution with CDF

$F(t)$ and reliability function $R(t)$ is defined as

$$H_F^{-1}(x) = \int_0^{F^{-1}(x)} R(u)du, \quad 0 \le x \le 1.$$

And scaled TTT transform is

$$\phi_F(x) = H_F^{-1}(x) \big/ H_F^{-1}(1).$$

By studying the derivative function of $\phi_F(x)$ with respect to x, we have that $\phi_F(x)$ is concave for IFR and convex for DFR, and has an *s*-shape for bathtub shape distribution. The following Figure 2.2 summarizes shapes of $\phi_F(x)$ for different classes.
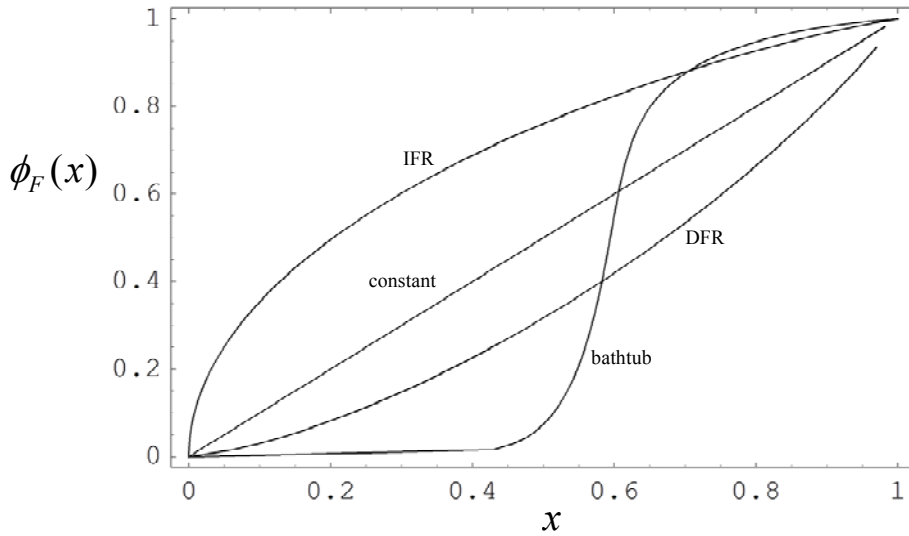
Figure 2.2   Scaled TTT transform for different distribution classes.

There are also plenty of other statistical tests established in literature. Bandyopadhyay & Basu (1990), Ahmad (1992), El-Bassiouny & Alwasel (2003), and Li et al. (2006) proposed tests for constant MRL against DMRL. Chen et al. (1983), Aly (1990), Lim & Koh (1996), and Abu-Youssef (2002) dealt with the problem of

monotonic MRL classes. Tests for distinguishing exponentiality from non-monotonic MRL classes were presented in Hawkins et al. (1992), Na & Lee (2003), and Anis & Mitra (2005). In Dauxois (2003), a more general test was established for exponential distribution versus non-exponential's. Lim & Park (1998) used a test to detect trend change in MRL. A detailed discussion and comparison of classical and recent tests can be found in Henze & Meintanis (2005).

2.2.1 Parametric models

Parametric models are regarded as useful tools in extracting and summarizing the information from failure data. The two common statistical techniques are MLE and LSE. In literature, numerous parametric distributions have been proposed to model different types of failure data that have either a monotone MRL or UBMRL. The most famous distribution in reliability is exponential distribution, which is the simplest model and the only continuous distribution with constant MRL. But the application of this distribution is limited, because few items in practice that have not failed may be statistically as good as new. Although the applicability is restricted, the exponential distribution still plays an important role in lifetime modeling; and its reliability function, failure rate function and MRL function are

$$R(t) = \exp(-\lambda t) \quad r(t) = \lambda \quad m(t) = \frac{1}{\lambda} \quad \text{for } \lambda > 0, t \geq 0 .$$

- **Weibull family**

In practice, mechanical items typically deteriorate over time and hence are more likely to have an increasing failure rate instead of constant. In this case, the exponential

distribution is inappropriate to be used for modeling. As a generalization of the exponential distribution, Weibull distribution, proposed by Weibull (1951), has constant, strictly increasing, and strictly decreasing failure rate. For $\alpha, \beta > 0$, $t \geq 0$,

$$R(t) = e^{-(t/\alpha)^{\beta}} \ ,$$

$$r(t) = \beta \alpha^{-\beta} t^{\beta-1} e^{-(t/\alpha)^{\beta}} \ ,$$

$$m(t) = \alpha \beta^{-1} e^{(t/\alpha)^{\beta}} \Gamma\left(\frac{1}{\beta}\right)\left(1 - I\left(\frac{1}{\beta}, \left(\frac{t}{\alpha}\right)^{\beta}\right)\right),$$

where $I(y,x) = \Gamma(y)^{-1} \int_{0}^{x} u^{y-1} e^{-u} du$ is the incomplete gamma function with $y > 0$ and $x > 0$. When $\beta = 1$, it degenerates to the exponential distribution; when $\beta > 1$, the failure rate function $r(t)$ is increasing and the MRL $m(t)$ decreases and approaches 0; when $\beta < 1$, $r(t)$ is decreasing to 0 and $m(t)$ is increasing to infinity.

However, when the failure rate function and the MRL function are non-monotonic such as having a bathtub shape, the Weibull distribution would be unable to give goodness-of-fit. In literature, several modified Weibull models were developed to deal with the modeling of non-monotonic MRL, especially UBMRL.

o *Exponentiated Weibull distribution*

One of the modified Weibull distributions is exponentiated Weibull distribution proposed by Mudholkar & Srivastava (1993).

$$R(t) = 1 - \left(1 - \exp\left(-(t/\alpha)^{\beta}\right)\right)^{\nu}, \ \alpha, \beta, \nu > 0 \ .$$

This model was derived from a parallel system with its components following the

Weibull distribution by generalizing the parameter that represents component number. When $\beta v < 1$, this model has BFR and UBMRL. The reliability measures of this distribution were summarized in Nassar & Eissa (2003).

o *Additive Weibull distribution*

Compared to the exponentiated Weibull distribution, the additive Weibull distribution proposed by Xie & Lai (1996) is based on series systems composed of two components.

$$R(t) = \exp\!\left(-\left(a\,t\right)^{b} - \left(c\,t\right)^{d}\right),\ a,b,c,d > 0 \,.$$

It is easy to find that the reliability function of this distribution is the product of two Weibull's with parameters a,b and c,d respectively. For $b > 1$ and $d < 1$, a BFR and a UBMRL would occur. An intuitive explanation is that a BFR could be produced by paralleling an IFR distribution and a DFR distribution.

Lai et al. (2004) generalized the additive model by adding a constant to the failure rate function, so that a MRL function with comparatively longer stable period could be obtained. The reliability function of this generalized additive model is

$$R(t) = \exp\!\left(-ct - \left(a\,t\right)^{b} - \left(c\,t\right)^{d}\right),\ a,b,c,d > 0 \,.$$

o *Modified Weibull distribution*

Moreover, Lai et al. (2003) considered a modified Weibull distribution having the reliability function given by

$$R(t) = \exp\!\left(-a\,t^{b}e^{\lambda t}\right),\ \lambda > 0,\ a > 0,\ b \geq 0 \,.$$

When $0 \leq b < 1$, this distribution has a BFR and a UBMRL. One distinct advantage of this model is that the change point of its failure rate function has a closed form, which is

$$\tau_0 = \frac{\sqrt{b} - b}{\lambda}.$$

It is clear that this change point τ_0 depends on only b and λ.

o *Weibull extension distribution*

In addition, Chen (2000) introduced a new two-parameter Weibull extension capable of describing bathtub curves.

$$R(t) = \exp\left(\lambda\left(1 - e^{t^{\beta}}\right)\right), \lambda, \beta > 0.$$

This model is also called the exponential power model in literature. To enhance the applicability of this distribution, Xie et al. (2002) modified and improved this distribution by adding an extra scale parameter. The reliability function of the improved model with an additional scale parameter is

$$R(t) = \exp\left(\lambda\alpha\left(1 - e^{(t/\alpha)^{\beta}}\right)\right), \lambda, \alpha, \beta > 0.$$

Direct deduction reveals that this distribution could model BFR and UBMRL failure data if $\beta < 1$.

o *Sectional models with two or more Weibull distributions*

Another kind of Weibull extensions with UBMRL are sectional distributions, which combines two or more Weibull distribution. Murthy & Jiang (1997) constructed

a sectional model with two Weibull distributions.

$$R(t) = \begin{cases} \exp\!\left(-\left(t/\eta_1\right)^{\beta_1}\right), & 0 \le t \le t^* \\[2mm] \exp\!\left(-\left(\dfrac{t-\gamma}{\eta_2}\right)^{\beta_2}\right), & t^* \le t < \infty \end{cases}$$

where $t^* = \left(\eta_1^{\beta_1}\left(\beta/\eta_2\right)^{\beta_2}\right)^{1/(\beta_1-\beta_2)}$, $\beta = \beta_1/\beta_2$, and $\gamma = (1-\beta)t^*$.

When $\beta_1 < 1$, $\beta_2 > 1$ and the change point of the failure rate function is set at t^*, the model would have a BFR and a UBMRL. A generalized sectional model involving three Weibull distributions can be found in Jiang & Murthy (1997). It is straight to use the sectional models to model non-monotonic MRL, as for example a UBMRL distribution can be composed of an IMRL Weibull and a DMRL Weibull. But the disadvantage of sectional models is that in these models, too many parameters are involved and thus more data are always needed for estimation.

o *Other Weibull related distributions*

There are also other Weibull related models that can display BFR and UBMRL, for example, a three-parameter modified Weibull distribution in Marshall & Olkin (1997) and the recent odd Weibull distribution in Cooray (2006). For MRL with complex shape, one construction technique is to mix two or more Weibull distributions. A graphic representation of a two mixed-Weibull distribution was given in Jiang & Kececioglu (1992). Gupta & Gupta (1996) presented a general approach to study the Weibull mixture in terms of the failure rate and MRL functions.

Because of the definition of MRL involving an integral of the reliability

function, most of these modified Weibull models do not have their MRL in an explicit analytical expression. Hence, the analysis of the MRL functions is often carried out numerically. A graphical study on the behavior of the MRL functions for different Weibll extensions was given in Lai et al. (2004). More detailed discussion of Weibull family can refer to Murthy et al. (2004).

- **Other distribution families**

  o *Gamma distribution*

  Gamma distribution is the second important generalization of the exponential distribution. But the gamma distribution is less popular than the Weibull distribution in lifetime modeling, because its reliability function is intractable (expressed as transcendental function). The probability density function is given by

  $$f(t) = \frac{\lambda}{\Gamma(\kappa)} (\lambda t)^{\kappa-1} e^{-\lambda t} , \ \lambda, \kappa > 0 .$$

  The distribution is IFR and DMRL if $\kappa > 1$, DFR and IMRL if $\kappa < 1$, and exponential if $\kappa = 1$. Different from the Weibull distribution of which the MRL goes to either infinity or 0, the MRL of the gamma distribution will approaches a positive constant, $1/\lambda$. This fact indicates a lifetime with a gamma distribution has an exponential tail.

  o *Inverse Gaussian distribution*

  Another distribution also with constant MRL and failure rate tail is the inverse Gaussian distribution.

$$f(t) = \sqrt{\frac{\lambda}{2\pi t^3}} \exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\right), \lambda, \mu > 0.$$

This distribution has UBFR and BMRL for all values of λ and μ. As $t \to \infty$, its MRL function approaches $2\mu^2/\lambda$. Unfortunately, the inverse Gaussian cannot describe the lifetime with UBMRL and BFR. A comprehensive study of the inverse Gaussian distribution was presented in Chhikara & Folks (1989).

o *Lognormal distribution*

Similar to the inverse Gaussian, the lognormal distribution also has only UBFR and BMRL. Its reliability function is expressed via the CDF for a standard normal random variable Φ.

$$R(t) = 1 - \Phi\left(\frac{\log t - \mu}{\sigma}\right), \mu \geq 0, \sigma > 0.$$

Change points of the failure rate function and the MRL for the lognormal distribution were studied in Gupta et al. (1997). In practical application, the lognormal distribution may be less popular than the inverse Gaussian distribution, because as $t \to \infty$, its failure rate function goes to 0 and the MRL increases to infinity. This implies that almost no failure will occur for items with long operating hours, which is not realistic in practice.

o *Other distributions*

The expressions of the MRL functions for normal, gamma and lognormal distributions were studied in Govil & Aggarwal (1983). To make these distributions capable of modeling different types of lifetime data, the expansion of distribution

falimiy is always needed. Gupta & Lvin (2005a, b) discussed a gamma-type model and a generalized lognormal model by introducing extra parameters. Ghitany (1998) and Agarwal & Al-Saleh (2001) also generalized the gamma distribution and analyzed its statistical properties. Weighted and mixed inverse Gaussian was studied in Gupta & Akman (1995b) and Gupta & Akman (1997) respectively.

Interests also arise for other models, such as the Pareto distribution, Pearson family and other distributions. Xekalaki & Dimaki (2005) identified Pareto and Yule distribution by their reliability measures. Generalized Pareto distributions were studied by Asadi (2004), Tavangar & Asadi (2005), and Tavangar & Asadi (2008). Besides, Asadi (1998) and Sankaran & Nair (2000) discussed Pearson family of distributions. The characterizations of beta distribution were reviewed in Nadarajah & Gupta (2004). Moreover, Gupta et al. (1996), Gupta et al. (1999), Ghitany et al. (2005), and Gupta et al. (2008) studied Burr type XII, log-logistic, Topp-Leone, and Hurwitz-Lerch Zeta distributions respectively. A distribution family with linear MRL was characterized in Korwar (1992). In Joseph & Kumaran (2008), generalized lambda distribution family was used to derive single unified expressions for the MRL of all non negative univariate continuous distributions.

2.2.2 Nonparametric estimation

Nonparametric estimation, without any preliminary assumption of distribution, depends only on lifetime data and provides estimates of characteristics empirically. Compared to parametric method that has good performance for small sample size, nonparametric method usually gives satisfactory estimation when the size of available

data is large. In the last several decades, various nonparametric studies have been conducted for the MRL function with different focuses.

- **Empirical estimation for MRL function**

Based on the well known Kaplan-Meier estimator (Kaplan & Meier, 1958), Yang (1978) proposed the empirical estimation for MRL function and proved its asymptotic uniformity in the sense of probability. Let $t_{(1)}, t_{(2)}, ..., t_{(i)}, ...t_{(r)}, t_{(r+1)}, ..., t_{(n)}$ be ordered potential failure times of n independent and identically distributed items, then the empirical MRL function is

$$\hat{m}(t) = \frac{1}{n - l(t)} \sum_{i=l(t)+1}^{n} t_{(i)} - t, \ 0 \le t \le t_{(n)} \ , \tag{2.6}$$

where $l(t) = \max \{i : t_{(i)} \le t\}$. Based on this estimator, many efforts have been contributed to the study and the modification of the empirical MRL estimator in order to deal with different types of lifetime data.

One important technique is to utilize the kernel method, which is derived from functional analysis. In Kulasekera (1991) and Na & Kim (1999), smooth nonparametric estimations of the MRL were presented for complete and right censored data. Ruiz & Guillamon (1996) and Guillamon et al. (1998) proposed nonparametric recursive estimators for MRL based on kernel density estimator and kernel reliability estimator under mixing dependence conditions. Chaubey & Sen (1999) modified the weighted scheme in kernel reliability function and proposed a new estimator for MRL by integrating the kernelled reliability function. A special kernel reliability estimator was also presented by Swanepoel & Van Graan (2005).

Under the assumption that samples come from a same but unkown distribution, Abdous & Berred (2005) applied the local linear fitting technique as well as kernel methods on the empirical function to produce a smooth estimator.

There are also some works with focus on specified aspects of the MRL. Ahmad (1982) discussed the estimation for the MRL of multi-component systems. Lahiri & Park (1991) was interested in the tail estimation of MRL and used Dirichlet process prior to derive empirical Bayes estimators. For some special MRL distribution classes, relatively simple estimations were studied and proposed. Ebrahimi (1998) estimated the MRL with finite population. In Kochar et al. (2000), a very simple estimation for monotone MRL class was developed, which utilizes the order sample just before a specified time point. Hu et al. (2002) dealt with the MRL estimation of two ordered MRL functions. In Ghebremichael (2009), the estimation of the MRL was studied assuming that the MRL was banded by two other MRL functions.

- **Confidence bounds for MRL function**

Guess & Park (1991) constructed conservative confidence intervals for MRL with censored data. For large samples, Li (1997) developed nonparametric confidence intervals for components as well as for systems and renewal processes. In Zhao & Qin (2006) and Qin & Zhao (2007), empirical likelihood ratio was used to estimate confidence bands for the MRL with both complete and right randomly censored data. Korczak (2001) proposed simple upper bounds for DMRL and NBUE distribution classes.

- **Estimations for changing points**

The estimation of change point for non-monotonic MRL was also considered in literature. Ebrahimi (1991) obtained an estimator for change point of MRL function for the distributions that have a "general" increasing MRL, which has a constant period after its initial increasing. Mitra & Basu (1995) considered change-point estimation for UBMRL distribution classes.

In addition to parametric models and nonparametric estimation, some semi-parametric models were also proposed and studied. One popular model that has gained many extensions is proportional MRL model, which was originally proposed by Oakes & Dasu (1990). As a extension of the proportional MRL model, a regression model was developed in Maguluri & Zhang (1994). Recently, Chen et al. (2005) used counting process theory to establish a inference procedure.

2.3 Mean residual life of systems

The previous sections are concerned with the MRL function for a single item or a system that is studied at system level. Most often, systems need to be investigated at system level by incorporating the information of their components, as this would lead to a better understanding of the underlying failure mechanism of complex systems. If a system is analyzed at component level, its inner structure needs to be clearly defined, because the structure, which mainly refers to the allocation and the properties of components, determines the reliability of the system; see Leemis (1995) for a systematic definition and an annotated overview.

As a main reliability measure, the MRL function has gained much attention for a better understanding of the reliability of systems. This section will give a brief summary on the MRL for systems, practically for parallel systems, series systems, and k-out-of-n systems, which are common structures in industry.

- **Different definitions for the MRL function of systems**

A k-out-of-n system is such a system that consists of n independent and identically distributed components and functions as long as least k components are working. In other words, a k-out-of-n system will stop working if $n - k + 1$ components fail, which indicates that the lifetime of a k-out-of-n system equals to the lifetime of the $(n - k + 1)$ th failed component. Particularly, the series systems and parallel systems are 1-out-of-n and n-out-of-n systems, respectively.

Suppose that a system consists of n $(n > 1)$ independent and identically distributed components. Let $T_1, T_2, ..., T_n$ be the lifetimes of n components and assume that $T_i, i = 1, ..., n$, are continuous and non-negative random variables with reliability function $R(t)$. Denote by $T_{1:n}, T_{2:n}, ..., T_{n:n}$ ordered lifetimes of n components. Then, the lifetime of a k-out-of-n system can be represented by that of the $(n - k + 1)$ th order statistic $T_{n-k+1:n}$. And the lifetimes of a series system and a parallel system are $T_{1:n}$ and $T_{n:n}$.

Because of the concern of system structure, different definitions for the MRL of systems have been proposed in order to accurately describe the lifetimes of systems with various operating conditions. A nature definition of the MRL function for

systems directly comes from that for single component in (2.1). Denote by $M_{k:n}(t)$ the MRL function of k-out-of-n systems, then we have

$$M_{k,n}(t) = E(T_{n-k+1:n} - t \mid T_{n-k+1:n} > t).$$

Conditioning on the $(n-k)$th failure time, Belzunce et al. (1999) defined the residual life and the MRL of k-out-of-n systems.

$$RLS_{k,n,t} = (T_{n-k+1:n} - T_{n-k:n} \mid T_{n-k:n} = t),$$

$$M_{k,n}(t) = E(T_{n-k+1:n} - T_{n-k:n} \mid T_{n-k:n} = t).$$

In their work, the MRL function could be interpreted as the additional lifetime to be gained by using a $(k-1)$-out-of-n system instead of a k-out-of-n system.

In Bairamov et al. (2002), a new definition of MRL was proposed for parallel systems with the condition that no failure occurs at a specified time point.

$$M_{n,n}^{1}(t) = E(T_{n:n} - t \mid T_{1:n} > t).$$

Motivated by this idea, Asadi & Bayramoglu (2005) extended the definition in Bairamov et al. (2002) and gave a series of MRL functions for parallel systems by conditioning on the $(n-r+1)$th lifetime of components.

$$M_{n,n}^{r}(t) = E(T_{n:n} - t \mid T_{r:n} > t).$$

It is obvious that the above definition method could provide n different MRL functions for a parallel system. Similar results also can be found in Asadi & Bayramoglu (2006) for k-out-of-n systems.

$$M_{k,n}^{r}(t) = E(T_{n-k+1:n} - t \mid T_{r:n} > t), 1 \le r \le n - k + 1.$$

In addition, Eryilmaz (2008) studied the lifetime of a combined *k*-out-of-*n* & consecutive *k*-out-of-*n* system.

- **Properties of systems**

For certain definition of the MRL of systems, its distributional properties are of great interest for further reliability analysis. The comparison between the MRL of systems and of components was investigated in Lim & Koh (1997). Khaledi & Shaked (2007) studied the systems with warning lights that come up when failures of components occur, and derived upper and lower bound on the MRL of systems. In Li & Zhao (2006), the general residual life of *k*-out-of-*n* systems, which is similar to the definition in Asadi & Bayramoglu (2006), was discussed in terms partial orderings. Navarro et al. (2006) and Navarro et al. (2008) talked about basic reliability properties of systems with exchangeable components following exponential and pareto distributions respectively. In Sadegh (2008), a parallel system with independent but non-identical components was studied based on generalized MRL. Gurler & Bairamov (2009) also considered systems with non-identical components and evaluated the relationship between the MRL of systems and that of its components. Furthermore, both Li & Zhang (2008) and Li & Zhao (2008) took into consideration residual life and inactivity time of systems.

Furthermore, several papers discussed the properties of MRL that systems could preserves from their components. Abouammoh & El-Neweihi (1986) showed that parallel systems inherit DMRL from components. Asadi & Goliforushani (2008)

proved that k-out-of-n systems also have this preservation property. The reversed preservation ageing properties for series and parallel systems were discussed in Li & Yam (2005), Belzunce et al. (2007a) and Li & Xu (2008). Considering partial orderings for components and systems, Singh & Vijayasree (1991) and Eryilmaz (2007) found that usual ordering is preserved under the formation of consecutive k-out-of-n systems but the MRL ordering does not hold. When comparing two parallel systems, Kochar & Xu (2007) showed that the system with more reliable components has longer lifetime.

2.4   Some applications

Two main applications of MRL in reliability engineering are the determination of optimal burn-in time in burn-in tests and the planning of optimal policy in maintenance. In this section, we want to review some decision making policies for the two application areas based the MRL.

Burn-in is a testing procedure to eliminate inferior products from product population before they are released and shipped to customers or factories. In general, burn-in tests is considered to be expensive and the cost increases with the increase of testing time. So the determination of optimal burn-in time, the best time to stop burn-in procedure, is crucial in production and sales, as the trade-off between high product quality and low burn-in cost need to be balanced.

The product population, which is composed of inferior and normal products, is often considered to have a BFR or a UBMRL. This is rational, because after infant

period caused by inferior products, only normal products survive and the population enters a stable period before the normal products begin to wear out. Hence, in fact, burn-in tests deal with the beginning infant period and the optimal burn-in time often refers to the time epoch that marks the end of the infant period. By far, various criteria related to the MRL have been established for burn-in tests, see Park (1985) and Block & Savits (1997) for a summary. There are mainly three kinds of criteria:

C1) To maximize the MRL function;

C2) To balance other functions derived from the MRL;

C3) To minimize a cost function related to the MRL.

Based on C1, the optimal burn-in time is actually the change point of the MRL, which has been extensively discussed in Section 2.1.3. A discussion also can be found in Block et al. (1999). Besides the criterion of maximizing MRL function, other characteristics related to the MRL are utilized to determine the optimal burn-in time. Block et al. (2002) determined the optimal burn-in time by balancing the MRL and residual variance. Bebbington et al. (2006) tried to utilize curvature to obtain the time as well as the useful period. There are also some cost related criteria based on the MRL. This type of models often assumes that the profit from selling products is proportional to MRL. Example cost functions were presented in Chang (2000) and Cha et al. (2008). In addition, the problem of burn-in on the population with a generalized bathtub curve was considered; see Bebbington et al. (2007a) and Cha (2006).

As pointed out in Guess et al. (1992), the optimal burn-in plan obtained from one reliability measure does not necessarily imply the optimal for the other measure.

In Section 2.1.3, we have known that the time at which the MRL reaches its maximum always occurs before the time that corresponds to the smallest failure rate. Xie et al. (2004) discussed the difference between change points of the MRL and the failure rate function and tried to use it as an index for the length of the useful\stable period. A statistical inferential theory was given by Bebbington et al. (2007b).

Burn-in tests are conducted before the release of products while maintenance and repair are carried out either during production process or when products have been deployed and put into use. Lee & Lee (1999) studied the optimal proportion of perfect repair when systems have DMRL. Mi (2002) considered age-replacement problem with computers and determined their optimal work size. Yue & Cao (2001) compared replacement policy with stochastic orders under shock models. Moreover, Cha et al. (2004) investigated optimal burn-in plan assuming that failed components during burn-in can be restored by a repair.

Besides the burn-in and maintenance problems discussed above, there are many other studies on MRL function in practice. Some aspects with the real application of the MRL and the residual life are broadly listed.

- **Product technology**

Wiklund (1998) discussed on-line prediction of cutting tool life by two statistical approaches. Herzog et al. (2009) predicted the residual life of machines and components based on data from hydraulic systems.

- **Economics and social studies**

In Sohn & Lee (2008), a competing risk model was proposed to estimate customers' MRL under a new phone system. They studied three competing causes that might affect the MRL of customers: pricing policy, quality of communication, and usefulness of service, and also different groups of customers with respect to their sex and age.

- **Survival analysis**

In survival analysis, bathtub curves are often used to describe the mortality of diseases. Often, the mortality of a disease is considered to reach a peak after some finite period and then declines gradually. We can conjecture that, the patients who suffer from this disease probably have their MRL initially decreasing due to the abrupt attack of the disease, and then entering a stable period due to body's resistance before finally recovering to a normal level when the disease ultimately disappears.

Besides, the MRL function also has wide applications in many other areas, such as life insurance, demography, management science, and even financial market.

CHAPTER 3   A GENERAL MODEL FOR UPSIDE-DOWN BATHTUB-SHAPED MEAN RESIDUAL LIFE[1]

3.1   Introduction

For many mechanical and electronic components, the failure rate function has a bathtub shape; the failure rate function is decreasing initially, and then flattens out before it increases again. The corresponding MRL function is usually of upside-down bathtub shape. Many papers deal with models for BFR and UBMRL. For example, Mudholkar & Srivastava (1993) proposed an exponentiated Weibull model. Xie & Lai (1996) proposed an additive Weibull model with BFR. Another three-parameter generalized Weibull distribution was presented by Xie et al. (2002), which modified the model introduced by Chen (2000). Recently Cooray (2006) derived the odd Weibull family. Besides these generalized Weibull distributions, other useful models with UBMRL and BFR were also discussed; see Ghitany et al. (2005) and Gupta &

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[1] Part of the work in this chapter is published in *IEEE Transactions on Reliability*.

"A model for upside-down bathtub-shaped mean residual life and its properties", *IEEE Transactions on Reliability*, 2009, 58(3), pp. 425-431.

Lvin (2005b) for examples. A recent text by Lai et al. (2004) contains several chapters dealing with the general issues.

However, for these existing models, the MRL functions are of complicated forms, which usually involve an integral of a reliability function that is not of a closed form. This problem motivates the formulation of a new class of life distribution, which has different characteristics from the existing models, such as a new model with some form of the MRL function. This new model type will make analyses based on MRL easier to conduct, including activities such as the determination of useful periods based on the curvature of MRL (Bebbington et al., 2006), the study of proportional MRL models (Zhao & Elsayed, 2005), the analysis of the distance between the change points of MRL and failure rate functions (Xie et al., 2004), and the study of MRL for coherent systems (Asadi & Goliforushani, 2008).

In this chapter, a model with UBMRL is proposed with the starting point of the derivative function of MRL. A general framework is studied in Section 3.2. Section 3.3 is devoted to propose the new model. We also investigate its distributional properties and parameter estimation. We use two sets of lifetime data in Section 3.4 to illustrate the application of the new model, along with some comparative studies. The model application in decision making is discussed in Section 3.5. In section 3.6, a nonlinear regression method based on MRL function is proposed for the estimation of parameters and also compared with MLE. Finally, Section 3.7 is a conclusion part.

3.2   A general framework

The derivative of the MRL function, when it exists, measures how the MRL function changes as the value of the input changes, and then determines the behavior and shape of the MRL. It could be simpler than the MRL itself. For example, if the derivative of an MRL has one crossing of zero from above, then the MRL is of upside-down bathtub shape. Hence we could find a simple function for the derivative of the UBMRL function rather than for the MRL function itself.

As shown in Definition 2.4, a differentiable MRL function $m(t)$ is said to be of upside-down bathtub shape if for some t_0 its derivative function $m'(t)$ satisfies $m'(t) > 0$ if $0 \le t < t_0$, $m'(t) = 0$ if $t = t_0$, and $m'(t) < 0$ if $t > t_0$.

Moreover the derivative of UBMRL should approach 0 as t goes to infinity, i.e. $m'(t) \to 0$, as $t \to \infty$; otherwise, the MRL would become negative. A typical curve of such a derivative function is depicted in Figure 3.1.
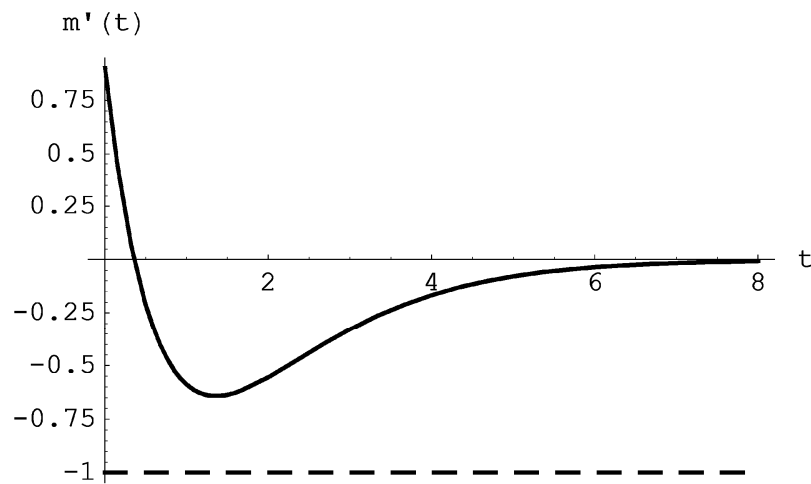
Figure 3.1   A desired shape of the derivative of MRL function $m'(t)$.

Here the aim is to find a function capable of representing the derivative function of MRL with the desired shape. A general way to construct a suitable function is to multiply a continuous function, which has positive values before some time point and after that goes to negative, by a function of which the limit as t approaches infinity is zero. The idea behind this general way is that to find a fast convergent function so that the tail of the positive and then negative function could be lifted up to 0. Following this idea, such a function can be expected to be the derivative of the MRL function.

$l(t) = g_1(t) \cdot g_2(t)$ is a multiplicative function in which the following requirements are true.

Requirement 1. $g_1(t)$ is a continuous function that satisfies that for some t^*, $g_1(t) > 0$ for $0 \leq t < t^*$, $g_1(t) = 0$ for $t = t^*$ and $g_1(t) < 0$ for $t > t^*$.

Requirement 2. $g_2(t)$ is a continuous, positive function with $g_2(t) \to 0$ as $t \to \infty$, and $g_1(t) \cdot g_2(t) \to 0$ as $t \to \infty$.

Requirement 3. The general integral of $l(t)$ should be positive and exists at any time, and also finite at time 0; i.e. $\int l(t)dt > 0$ for all $t \geq 0$, and $\left. \int l(t)dt \right|_{t=0} < \infty$.

Requirement 4. $l(t) \geq -1$.

Proposition 3.1 *A smooth function is the derivative of UBMRL if and only if it can be written in the form of the multiplicative function $l(t)$.*

Proof: "If part": A function that satisfies Requirement 1 and 2 must decrease initially to negative values before it increases and approaches to 0 asymptotically. Requirement 3 and 4 guarantee that the general integral of $l(t)$ is a MRL function, which has the following two properties.

P1. $m(t) = \int m'(t) > 0$, for all $t \geq 0$, and $m(0) < \infty$.

P2. $m'(t) = m(t)r(t) - 1 \geq -1$, for all $t \geq 0$.

A function that fulfills the four requirements can be the derivative of UBMRL.

"Only if part": For an arbitrarily chosen UBMRL $m(t)$, we denote by t_0 its change point. We have $m'(t) > 0$ for $t < t_0$, $m'(t) = 0$ for $t = t_0$ and $m'(t) < 0$ when $t > t_0$; also $m'(t) \to 0$ as $t \to \infty$. As $m'(t)$ is continuous, $m'(t)$ has lower bounds. Denote by τ_0 the time at which $m'(t)$ reaches its smallest value. We have $m'(\tau_0) < 0$. Then two continuous functions could be constructed,

$$g_1(t) = \begin{cases} -\dfrac{m'(t)}{m'(\tau_0)}, & t \leq \tau_0 \\ -\dfrac{m'(\tau_0)}{m'(t)}, & t > \tau_0 \end{cases}, \quad g_2(t) = \begin{cases} -m'(\tau_0), & t \leq \tau_0 \\ -\dfrac{(m'(t))^2}{m'(\tau_0)}, & t > \tau_0 \end{cases}.$$

It is easy to verify that $m'(t) = g_1(t) \cdot g_2(t)$. Let $t^* = t_0$. We also can find that $g_1(t)$ is a continuous function, and positive before t^* and negative after t^*. Moreover, $g_2(t)$ is positive and continuously approaches 0 when time t goes to infinity. As $m'(t)$ is a derivative function of MRL, Requirement 3 and 4 for $l(t)$ are satisfied. Here completes the proof. ∎

The proposition implies that the derivative of UBMRL is equivalent to the multiplicative function $l(t)$. Hence, the function $l(t)$ can be regarded as a reliability characteristic that could be utilized to determine probability distributions. In other words, a distribution could be developed based on any function in the form of $l(t)$. In next section, a proper candidate from the multiplicative functions will be used as a starting point of the derivation of the new UBMRL model.

3.3   The UBMRL model

3.3.1   Construction of the model

Based on the general form of the multiplicative function in the previous section, the function considered in this paper is a linear function multiplied with an exponential function

$$l(t) = (a - bt)\exp(-ct), \ a,b,c > 0 , \ t \geq 0 , \tag{3.1}$$

with conditions

$$\frac{ac - b}{c^2} < 0 , \text{ and} \tag{3.2}$$

$$-1 \leq -\frac{b}{c}\exp\left(-\frac{ac}{b} - 1\right). \tag{3.3}$$

For (3.1), both $(a - bt)$, and $\exp(-ct)$ are simple functions for $g_1(t)$ and $g_2(t)$. Their multiplication can be integrated easily, and the general integral of $l(t)$ preserves the same form as $l(t)$ itself. Equations (3.2) and (3.3) are derived from Requirement 3 and 4 for the multiplicative function respectively to ensure the feasibility of $l(t)$ as

the derivative function of MRL. See next subsection for the derivations of (3.2) and (3.3).

From the above discussion, a model with UBMRL is derived in terms of the MRL function

$$m(t) = \left(\frac{b}{c}t - \frac{ac-b}{c^2}\right)\exp(-ct), \ a,b,c > 0, \ t \geq 0,$$ (3.4)

with constraints (3.2) and (3.3).

Let $\alpha = b/c$, $\beta = -(ac-b)/c^2$, $\gamma = c$. Then the model in (3.4) with the constraints (3.2) and (3.3) can be reduced to

$$m(t) = (\alpha t + \beta)\exp(-\gamma t), \ \alpha,\beta,\gamma > 0, \ t \geq 0,$$ (3.5)

where

$$\alpha\exp\left(\frac{\beta\gamma}{\alpha} - 2\right) \leq 1.$$ (3.6)

The derivative function of the MRL is

$$m'(t) = (-\alpha\gamma t + \alpha - \beta\gamma)\exp(-\gamma t).$$ (3.7)

3.3.2 Derivation of (3.2) and (3.3)

- **Derivation of (3.2)**

Requirement 3 implies that the general integral of $l(t)$ should be greater than 0, which is

$$\int l(t)dt = -\frac{1}{c}(a-bt)\exp(-ct)+\frac{b}{c^2}\exp(-ct)+\Delta \geq 0,$$

where Δ is an integration constant. Because the general integral increases first and then decreases, its smallest values will be attained at $t=0$ or $t=\infty$. So the above formula is equivalent to the two inequalities

$$\int l(t)dt\bigg|_{t=0} = \frac{-ac+b}{c^2}+\Delta > 0, \text{ and } \int l(t)dt\bigg|_{t=\infty} = \Delta \geq 0. \tag{3.8}$$

By applying to (3.8) the fact that the MRL function is usually assumed to achieve 0 as $t \to \infty$, i.e. $\Delta = 0$, we can obtain (3.2). ∎

- **Derivation of (3.3)**

Formula (3.3) is derived from Requirement 4 which indicates that $l(t) \geq -1$ for all $t \geq 0$, which is equivalent to $\min_{t \geq 0} l(t) \geq -1$. Solving the equation $dl(t)/dt = 0$ yields the time point corresponding to the minimum of $l(t)$, denoted by τ_0

$$\tau_0 = \frac{a}{b}+\frac{1}{c}.$$

Hence, we obtain the inequality

$$l(\frac{a}{b}+\frac{1}{c}) = \left(a-b\left(\frac{a}{b}+\frac{1}{c}\right)\right)\exp\left(-c\left(\frac{a}{b}+\frac{1}{c}\right)\right) = -\frac{b}{c}\exp\left(-\frac{ac}{b}-1\right) \geq -1.$$

which is (3.3). ∎

3.3.3 Failure rate function and other functions

The corresponding failure rate function $r(t)$ is

$$r(t) = \frac{m'(t)+1}{m(t)} = \frac{(-\alpha\gamma t + \alpha - \beta\gamma)\exp(-\gamma t)+1}{(\alpha t + \beta)\exp(-\gamma t)} . \qquad (3.9)$$

The reliability function is then given by

$$
\begin{aligned}
R(t) &= \frac{m(0)}{m(t)}\exp\left(-\int_0^t \frac{1}{m(x)}dx\right) \\
&= \frac{\beta\exp(\gamma t)}{(\alpha t + \beta)}\exp\left(-\int_0^t \frac{\exp(\gamma x)}{\alpha x + \beta}dx\right) \qquad (3.10) \\
&= \frac{\beta\exp(\gamma t)}{(\alpha t + \beta)}\exp\left(-\frac{1}{\alpha}\exp\left(-\frac{\beta\gamma}{\alpha}\right)\left[\Gamma\left(0,-\frac{\beta\gamma}{\alpha}\right)-\Gamma\left(0,-\gamma t-\frac{\beta\gamma}{\alpha}\right)\right]\right)
\end{aligned}
$$

where $\Gamma(a,x) = \int_x^\infty u^{a-1}e^{-u}du$ is the upper incomplete gamma function.

Remark 3.1: Suppose that the MRL function ultimately approaches a positive constant instead of 0, and then (3.2) should be replaced by (3.8) in Section 3.3.2. This change will introduce a more general model with UBMRL, which will approach a constant instead of 0 as time t goes to infinity.

Remark 3.2: The parameters a, b, and c of (3.1) are set positive in this paper. In fact, the expansion of their domains will enable the distribution to present different types of MRL. Assume $c > 0$, as it is a scale parameter. There are four cases: 1) $a > 0$ and $b > 0$, UBMRL; 2) $a > 0$ and $b < 0$, increasing MRL; 3) $a < 0$ and $b > 0$, decreasing MRL; and 4) $a < 0$ and $b < 0$, bathtub-shaped MRL. This is because,

according to (3.1), the signs of parameters a and b determine the sign of $(a - bt)$ and thus further determine the positive or negative of the derivative function $l(t)$.

3.3.4   Shapes and changing points of MRL and failure rate functions

Figure 3.2 and Figure 3.3 show respectively the MRL and the failure rate functions of the new model. They are plotted with parameters $\alpha = 4$, $\beta = 0.5$, and $\gamma = 1$ for the solid line; and $\alpha = 2$, $\beta = 1$, and $\gamma = 1$ for the dashed line.

From the derivative of MRL (3.1) or (3.7), it is easy to obtain the change point of the MRL function

$$t_0 = \frac{a}{b} = \frac{1}{\gamma} - \frac{\beta}{\alpha}. \tag{3.11}$$

Based on (3.5), solving $r'(t) = 0$ yields the change point of the failure rate function, denoted by v_0.

$$v_0 = \frac{a}{b} + \frac{z_0}{c} = \frac{1 + z_0}{\gamma} - \frac{\beta}{\alpha}, \tag{3.12}$$

where z_0 is the solution to the equation $z e^z = b \exp(-ac/b)/c = \alpha \exp(\beta\gamma/\alpha - 1)$.
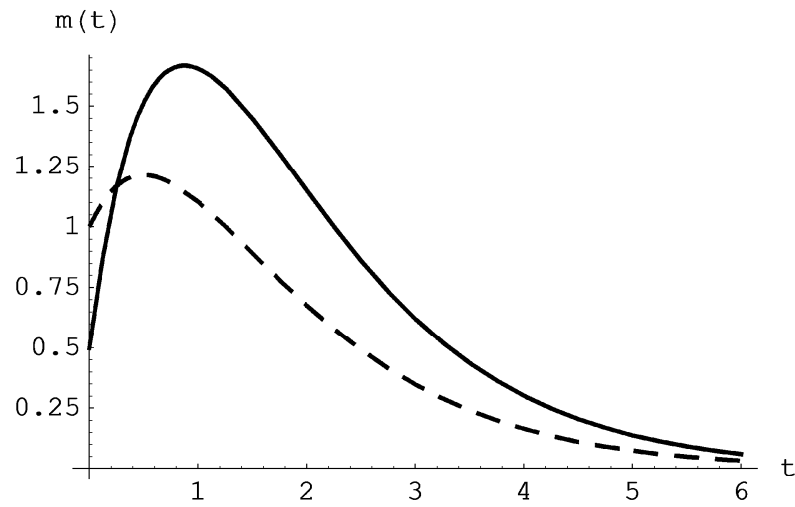
Figure 3.2   Example UBMRL functions.
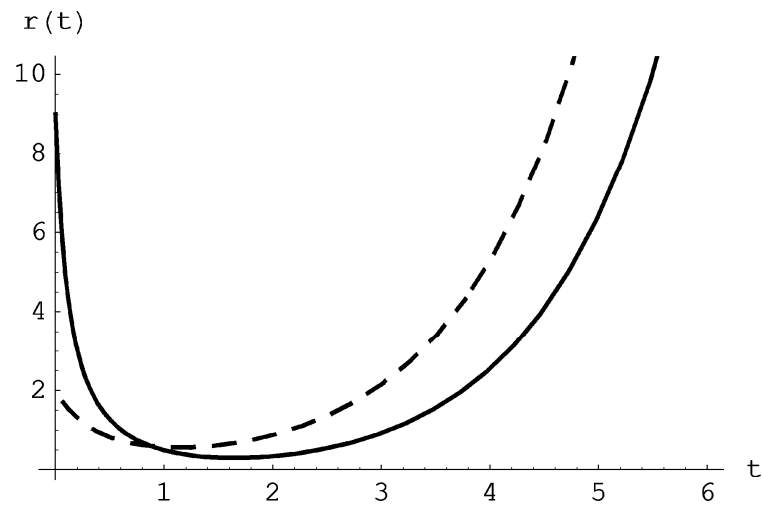
Figure 3.3   Example BFR functions.

The result of $t_0 < v_0$ validates the theorem proved by Gupta and Akman (1995a) that the change point of MRL function always precedes the change point of failure rate function. Furthermore, (3.11) suggests that the changing point of the MRL function t_0 approaches 0 as the parameter a gets close to 0. But in this case, the change point of the corresponding failure rate function v_0 remains strictly greater

than 0 because $v_0 > t_0$. Hence, when $a = 0$, the new model will degenerate into the one with a decreasing MRL and a BFR function. This phenomenon gives an example for the fact that BFR does not necessarily imply UBMRL and also supports the theorem in Gupta and Akman (1995a).

An underlying explanation to this unusual instant is that the failure rate function at time 0 is finite for this degenerative model, which is different from the other existing models with infinite initial failure rates. More similar cases with BFR and decreasing MRL may refer to the distributions with $a < 0$ mentioned in Remark 3.2. From (3.11), we find that as $a < 0$, the change point of $m(t)$ is also less than 0. This implies that $m(t)$ is strictly DMRL with $m'(0) < 0$. But the failure rate function is still of bathtub shape with a change point v_0.

It is also of interest to study the property of the change point of the derivative of the MRL function $m'(t)$, denoted by τ_0. From (3.6), we get an inequality

$$0 < z_0 e^{z_0} = \alpha \exp\left(\frac{\beta\gamma}{\alpha} - 1\right) \le \exp(1),\tag{3.13}$$

which implies $0 < z_0 \le 1$. Then we obtain

$$v_0 = \frac{1 + z_0}{\gamma} - \frac{\beta}{\alpha} \le \frac{2}{\gamma} - \frac{\beta}{\alpha} = \frac{a}{b} + \frac{1}{c} = \tau_0,\tag{3.14}$$

which means that in this model the change point of $m'(t)$, denoted as τ_0, is positioned after the change point of the failure rate function v_0. So τ_0 can be used as the time epoch that marks the start of the useful life period during which the failure

rate function is considered to be stable.

3.3.5   Parameter estimation

Parameter estimation is an important issue in lifetime data analysis. Maximum likelihood estimation (MLE) can be used to estimate the parameters of the new model, for either complete data, or censored data. For the case of complete data, let $t_{(1)} \le t_{(2)} \le ... \le t_{(n)}$ be the exact ordered failure times from a sample of size n. The underlying log-likelihood function for these complete data is given by

$$l(a,b,c) = \Lambda = \ln L(t_{(1)}, t_{(2)}, ..., t_{(n)}; \alpha, \beta, \gamma) = \sum_{i=1}^{n} \ln f(t_{(i)}) = \sum_{i=1}^{n} (\ln r(t_{(i)}) + \ln R(t_{(i)}))$$

$$
\begin{aligned}
= \sum_{i=1}^{n} \Big\{ &\ln\big[(-\alpha\gamma\, t_{(i)} + \alpha - \beta\gamma)\exp(-\gamma\, t_{(i)}) + 1\big] - 2\ln(\alpha t_{(i)} + \beta) + 2\gamma\, t_{(i)} + \ln(\beta) \\
&- \frac{1}{\alpha}\exp\left(-\frac{\beta\gamma}{\alpha}\right)\left[\Gamma\left(0, -\frac{\beta\gamma}{\alpha}\right) - \Gamma\left(0, -\gamma\, t - \frac{\beta\gamma}{\alpha}\right)\right]\Big\}
\end{aligned}
$$
.   (3.15)

By taking the partial derivatives of the above function with respect to α, β, and γ, and equating them to zero, three equations can be obtained. Solving the equations will yield the MLE for the three parameters α, β, and γ. For cases involving censoring, the log-likelihood functions in (3.15) will be added with the logarithm of reliability functions corresponding to the censored time points. Moreover, based on the second partial derivatives of the log-likelihood function, we can obtain the Fisher information matrix and use it to estimate confidence intervals for the three parameters.

Based on the second partial derivatives of the log-likelihood function, the 3×3 observed information matrix, from Fisher information matrix, is given by

$$O = - \begin{bmatrix} -\dfrac{\partial^2 \Lambda}{\partial \alpha^2} & -\dfrac{\partial^2 \Lambda}{\partial \alpha \partial \beta} & -\dfrac{\partial^2 \Lambda}{\partial \alpha \partial \gamma} \\[2ex] -\dfrac{\partial^2 \Lambda}{\partial \beta \partial \alpha} & -\dfrac{\partial^2 \Lambda}{\partial \beta^2} & -\dfrac{\partial^2 \Lambda}{\partial \beta \partial \gamma} \\[2ex] -\dfrac{\partial^2 \Lambda}{\partial \gamma \partial \alpha} & -\dfrac{\partial^2 \Lambda}{\partial \gamma \partial \beta} & -\dfrac{\partial^2 \Lambda}{\partial \gamma^2} \end{bmatrix}_{\alpha=\hat{\alpha}, \beta=\hat{\beta}, \gamma=\hat{\gamma}} ,$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ are the estimated parameters from MLE.

By inverting the matrix, we can obtain the local estimates of the covariance matrix

$$\begin{bmatrix} \hat{Var}(\hat{\alpha}) & \hat{Cov}(\hat{\alpha},\hat{\beta}) & \hat{Cov}(\hat{\alpha},\hat{\gamma}) \\[1ex] \hat{Cov}(\hat{\alpha},\hat{\beta}) & \hat{Var}(\hat{\beta}) & \hat{Cov}(\hat{\beta},\hat{\gamma}) \\[1ex] \hat{Cov}(\hat{\alpha},\hat{\gamma}) & \hat{Cov}(\hat{\beta},\hat{\gamma}) & \hat{Var}(\hat{\gamma}) \end{bmatrix} = O^{-1} .$$

The standard errors of the parameter estimators can be computed from the diagonal of the above matrix. Since the parameters of this model must be positive, the logarithm of them, i.e. $\ln \alpha$, $\ln \beta$, $\ln \gamma$, can be treated as normally distributed. Hence the two-sided approximate confidence intervals for these three parameters, at confidence level δ, are constructed as follows, where $z_{1-\frac{\delta}{2}}$ is the $1-\delta/2$ percentile of the standard normal distribution.

$$\alpha_U = \hat{\alpha}\, e^{z_{1-\frac{\delta}{2}}\sqrt{Var(\hat{\alpha})}\big/\hat{\alpha}}, \qquad \alpha_L = \dfrac{\hat{\alpha}}{e^{z_{1-\frac{\delta}{2}}\sqrt{Var(\hat{\alpha})}\big/\hat{\alpha}}}$$

$$\beta_U = \hat{\beta}\, e^{z_{1-\frac{\delta}{2}}\sqrt{Var(\hat{\beta})}\big/\hat{\beta}}, \qquad \beta_L = \dfrac{\hat{\beta}}{e^{z_{1-\frac{\delta}{2}}\sqrt{Var(\hat{\beta})}\big/\hat{\beta}}} .$$

$$\gamma_U = \hat{\gamma}\, e^{z_{1-\frac{\delta}{2}}\sqrt{Var(\hat{\gamma})}\big/\hat{\gamma}}, \qquad \gamma_L = \dfrac{\hat{\gamma}}{e^{z_{1-\frac{\delta}{2}}\sqrt{Var(\hat{\gamma})}\big/\hat{\gamma}}}$$

In this model, both the parameter estimates and the confidence intervals should be computed numerically, because the equations and the matrices can not be analytically solved.

3.4 Two application examples

Two data sets are used in this section to illustrate the modeling and estimation procedure. One is the widely used data set of 50 failure times from Aarset (1987); the other data set with the 18 exact failure times is taken from Example 2 in Wang (2000).

3.4.1 Example 3.1

In this example, the lifetime data from the testing of 50 devices reported in Aarset (1987) are used. The data are shown in Table 3.1.

Table 3.1   Lifetimes of 50 devices from Aarset (1987).

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18	18
21	32	36	40	45	46	47	50	55	60	63	63	67	67	67	67	72	75
79	82	82	83	84	84	84	85	85	85	85	85	86	86				

Plotting empirical failure rate function (Klein & Moeschberger, 2003) based on the data shows that the failure rate function could be approximately of bathtub shape (Figure 3.4).

The result in Figure 3.4 indicates that it may be reasonable to consider our new distribution to model the failure data. Using the maximum likelihood method, and the log-likelihood function in (3.15), we get the estimates for the parameters of the new

model: $\hat{\alpha} = 3.3097$, $\hat{\beta} = 44.1388$, and $\hat{\gamma} = 0.04428$. The estimated parameters satisfy

the constraint required by our model shown in (3.6). From Figure 3.4, we can see that

the fitted model has its failure rate function rather close to the empirical failure rate

function.

Figure 3.4   The failure rate function for the model (bold line) and the empirical failure rate function (jagged line).

Some generalized Weibull models are also used to fit the data, compared to the

new model in terms of Akaike Information Criterion (AIC) values (Akaike, 1974).

Their estimated parameters and the AIC values are listed in Table 3.2. The plots of the

failure rate functions of different models are given in Figure 3.5. Moreover, the MRL

function plot of the new model is presented in Figure 3.6.

The results in Table 3.2 indicate that the AIC value of the new model is

smaller than the other distributions except for the additive Weibull model. But it

should be noted that the additive Weibull model contains more parameters than others,

though it has the smallest AIC value. Hence our new model still gives a satisfying

goodness-of-fit to the lifetime data.

Table 3.2   The estimated parameters and AIC values of different models.

Model	Estimated parameters (MLE)	AIC
The exponentiated Weibull (Mudholkar & Srivastava, 1993)	$\hat{\alpha} = 5.15$, $\hat{\theta} = 0.134$, $\hat{\sigma} = 90$	463.13
The modified Weibull extension (Xie et al., 2002)	$\hat{\alpha} = 13.747$, $\hat{\beta} = 0.5877$, $\hat{\lambda} = 0.00876$	469.29
The modified Weibull distribution (Lai et al., 2003)	$\hat{a} = 0.0624$, $\hat{b} = 0.3548$, $\hat{\lambda} = 0.02332$	460.31
The additive Weibull model (Xie & Lai, 1996)	$\hat{a} = 0.01178$, $\hat{b} = 82$, $\hat{c} = 0.016$, $\hat{d} = 0.7$	420.38
New model	$\hat{\alpha} = 3.3097$, $\hat{\beta} = 44.1388$, $\hat{\gamma} = 0.04428$	447.76

Moreover, see in Figure 3.5 that the failure rate functions of the five models behave differently from each other. We find it interesting that the new model seems to be an integration of the additive Weibull model, and the other three distributions in terms of the shape, and the change point of the failure rate function. The failure rate function of the new model performs a moderate behavior compared to the flat additive Weibull, and the other three steep Weibull models; and its change point locates between the change points of the additive Weibull, and of the others. So it may suggest that the new model is likely to provide all-sided information compared to the other four distributions.

In addition, the graphical description for the MRL function can be easily offered by our new model. By observing Figure 3.6, we find that the location of the change point is around 10, which means that the burn-in test for such product types could be terminated around that time. Hence, the overall performances suggest that the new model is the most reasonable one to be chosen.
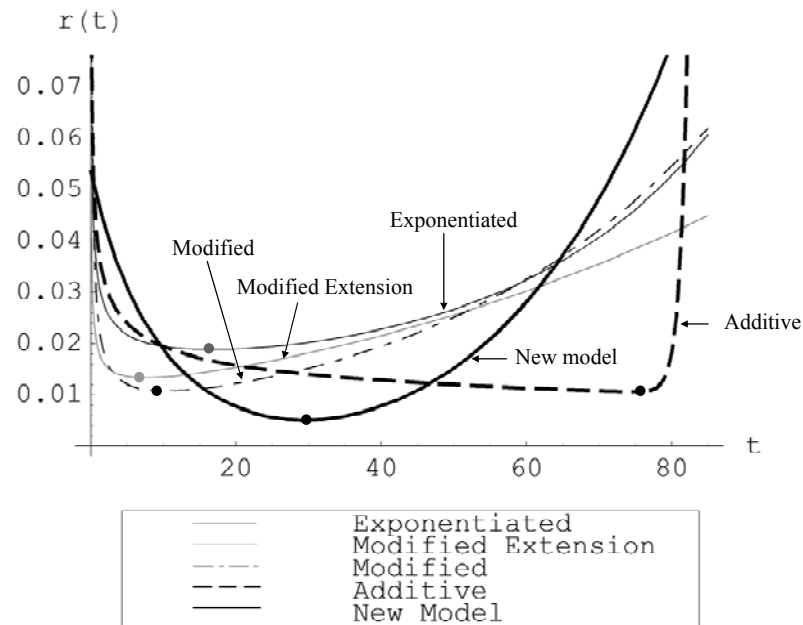
Figure 3.5   The failure rate function plot based on different models in Table 3.2.
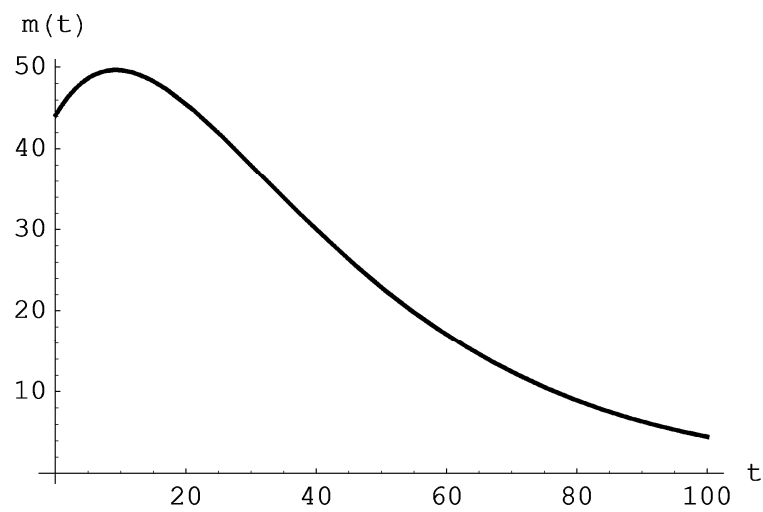
Figure 3.6   The MRL function for the new model in Table 3.2.

3.4.2   Example 3.2

This example adopts the data for the time to failure of 18 electronic devices reported by Wang (2000). The data are shown in Table 3.3.

Table 3.3   Time to failure of 18 electronic devices from Wang (2000).

5	11	21	31	46	75	98	122	145	165	195	224	245	293	321	330	350	420

Similar to the previous example, the data also suggest that the failure rate function could be bathtub-shaped (Figure 3.7). Fitting these data to the new model by MLE produces estimates for the three parameters: $\hat{\alpha} = 1.7694$, $\hat{\beta} = 171.564$, and $\hat{\gamma} = 0.00823$, which also satisfy constraint (3.6). In Figure 3.7, the graphic comparison of the failure rate function of the new model to the empirical failure rate function indicates good performance of the new distribution.
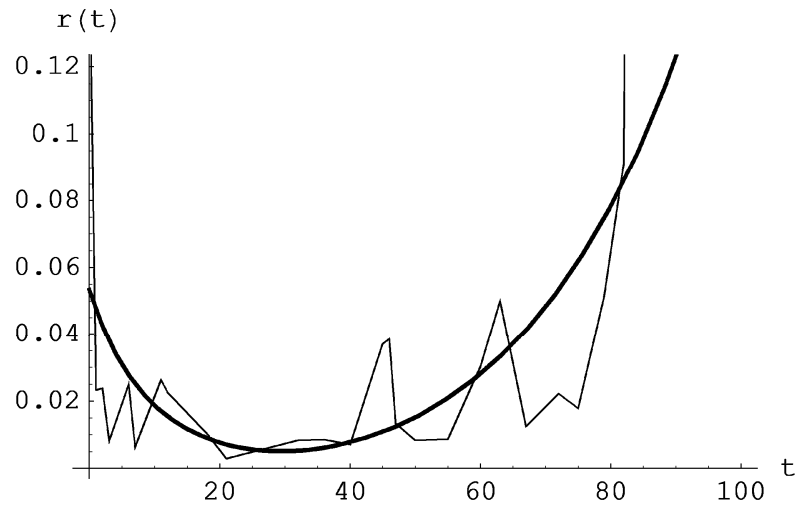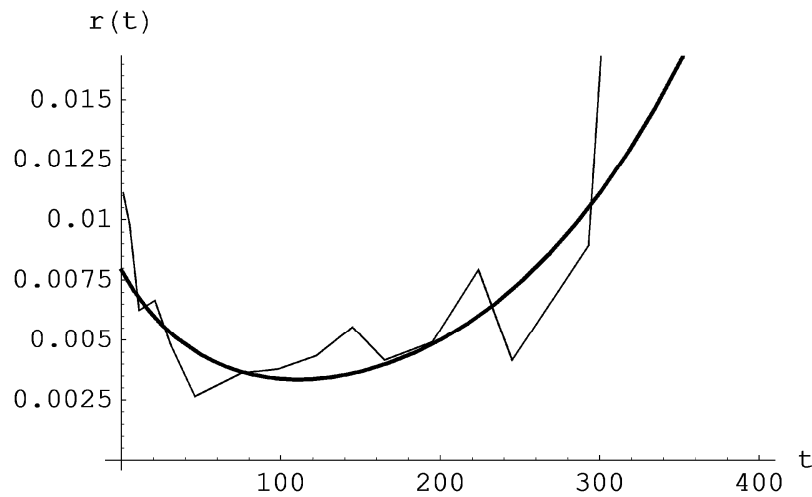

Figure 3.7   The failure rate function for the model (bold line) and the empirical failure rate function (jagged line).

For the generalized Weibull distributions, the estimated parameters, and AIC values are listed in Table 3.4. The plots of the failure rate functions of the different models are presented in Figure 3.8, while the MRL plot of the new model is in Figure 3.9. The results in Table 3.4 show that the new model has the smallest AIC value among the distributions employed in this example. Hence this new distribution is

competitive for modeling the lifetime data with BFR or UBMRL.

Table 3.4   The estimated parameters and AIC values of different models.

Model	Estimated parameters (MLE)	AIC
The exponentiated Weibull (Mudholkar & Srivastava, 1993)	$\hat{\alpha} = 7.8522$, $\hat{\theta} = 0.09286$, $\hat{\sigma} = 391.19$	222.494
The modified Weibull extension (Xie et al., 2002)	$\hat{\alpha} = 134.049$, $\hat{\beta} = 0.75226$, $\hat{\lambda} = 0.00255$	224.234
The modified Weibull distribution (Lai et al., 2003)	$\hat{a} = 0.01493$, $\hat{b} = 0.6468$, $\hat{\lambda} = 0.003612$	223.866
The additive Weibull model (Xie & Lai, 1996)	$\hat{a} = 0.0043$, $\hat{b} = 0.8612$, $\hat{c} = 0.002747$, $\hat{d} = 6.3505$	224.154
New model	$\hat{\alpha} = 1.7694$, $\hat{\beta} = 171.564$, $\hat{\gamma} = 0.00823$	222.108
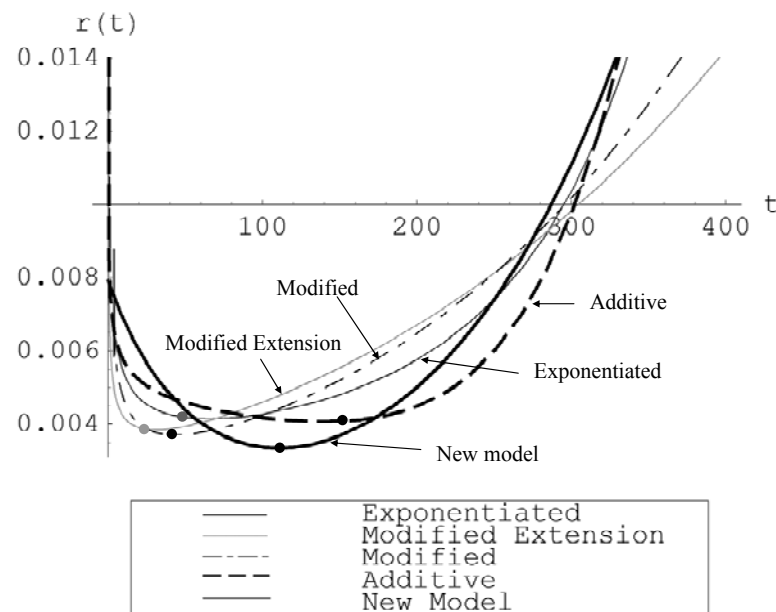
Figure 3.8   The failure rate function plot based on different models in Table 3.4.

Figure 3.8 depicts the different shapes of the failure rate functions of the different models. Similar to the previous example, the new model behaves in a moderate manner compared to the Weibull related distributions. The failure rate function of the new model is steeper than that of the additive Weibull distribution, but

flatter than the failure rate functions of other three Weibull models; also the change point of the new model locates between the change points of the additive Weibull, and of the others. All these show that the new model is able to provide the results that combine the information provided by other models. Based on the plot in Figure 3.9, we can obtain the approximate optimum time for burn-in tests or the near optimum replacement time. Our model compares favorably with other models.
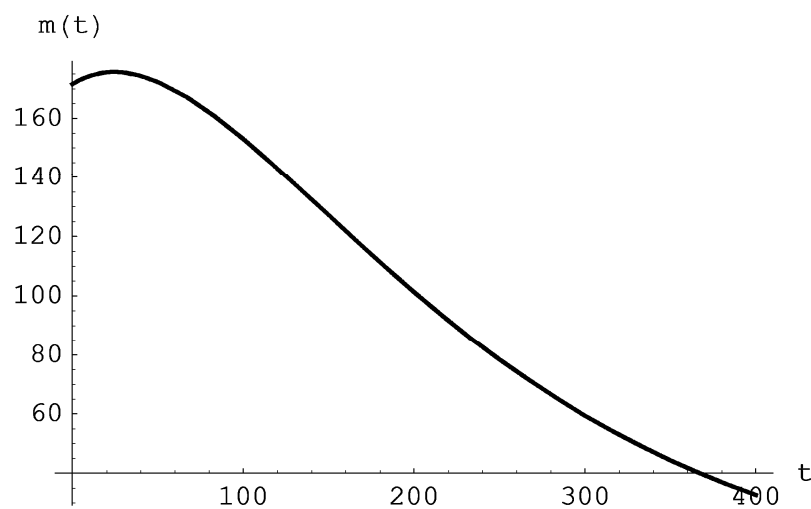
Figure 3.9   The MRL function for the new models in Table 3.4.

3.5   Model application in decision making

Bathtub curves are useful in reliability related decision making. One of the uses is to determine the optimum burn-in time in the case when the products have extremely high failure rate during the infant mortality period. After the useful period, replacement should be carried out to prevent failures from occurring when the products enter the wear-out period.

For those components that survive burn-in tests, it is the remaining lifetime

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that will be of interest to most people. The mean of the remaining lifetime is the MRL function $m(t)$. Therefore, the most common criterion to select the optimum burn-in time b^* is by maximizing the MRL function (Block & Savits, 1997)

$$m(b^*) = \max_{t \geq 0} m(t). \qquad (3.16)$$

Because our model has a UBMRL, the optimum burn-in time b^* is in fact the change point of the MRL function, i.e. $b^* = t_0 = 1/\gamma - \beta/\alpha$.

The burn-in time b^* can also be obtained by minimizing the failure rate function, which means $b^* = v_0$. In practical applications, the former decision of $b^* = t_0$ is usually preferred, because it is popularly believed that the period between t_0 and v_0 should be flat enough to be considered part of the useful life period. Thus, stopping the burn-in test when the MRL reaches its maximum should be more economical than when the failure rate function reaches its minimum.

There are also other criteria under which the optimum burn-in time could be determined. Suppose the product is considered acceptable when its MRL is more than m_b; then the optimum burn-in time can be determined by

$$m(t) = (\alpha t + \beta)\exp(-\gamma t) = m_b. \qquad (3.17)$$

If the product can only be released when the failure rate function falls below r_b, then the optimum burn-in time can be obtained by solving

$$r(t) = \frac{(-\alpha\gamma t + \alpha - \beta\gamma)\exp(-\gamma t) + 1}{(\alpha t + \beta)\exp(-\gamma t)} = r_b. \qquad (3.18)$$

It is worth noting that, for (3.17) and (3.18), the optimum burn-in time should be the smaller of the two solutions.

Similar criteria can be used in determining the optimum replacement time when the product enters the worn-out period. If the product should be replaced by a new one when the MRL is less than m_c, then the optimum time can be obtained by solving

$$m(t) = (\alpha t + \beta)\exp(-\gamma t) = m_c. \tag{3.19}$$

If the product is considered risky when the failure rate function becomes higher than r_c, then the optimum replacement time can be determined by

$$r(t) = \frac{(-\alpha\gamma t + \alpha - \beta\gamma)\exp(-\gamma t) + 1}{(\alpha t + \beta)\exp(-\gamma t)} = r_c. \tag{3.20}$$

Because the MRL function is of upside-down bathtub shape, there are also two solutions to (3.19). But the optimum replacement time should be the solution with the higher value. Also for (3.20), the larger root should be chosen.

Example: Consider the fitted model in Section 3.4.1 with the estimated parameters $\hat{\alpha} = 3.3097$, $\hat{\beta} = 44.1388$, and $\hat{\gamma} = 0.04428$. From (3.11), obtain the change point of the MRL function as

$$t_0 = 1/\hat{\gamma} - \hat{\beta}/\hat{\alpha} = 9.2463. \tag{3.21}$$

Hence, under (3.16), a burn-in test can be terminated at the time point of $b^* = 9.2463$. The formula (3.21) indicates that the optimum burn-in time b^* has a closed form.

Thus, further analysis for b^* can be carried out, such as confidence bound.

If this product is considered acceptable when its MRL is higher than 48, then under the criterion (3.17) we may stop burn-in test at the time of 3.9024, which is the smaller root to $(3.3097t + 44.1388)\exp(-0.04428t) = 48$.

If the product should be replaced when its MRL falls below 36, then the optimum time for replacement under the criterion (3.19) can be computed as 32.4664, which is the larger solution of $(3.3097t + 44.1388)\exp(-0.04428t) = 36$.

Similarly, other decisions can be similarly made under the criteria related to the failure rate function, as shown in (3.18) and (3.20).

3.6   Nonlinear regression method based on the MRL

As the MRL of the new model is in closed form, nonlinear regression on the MRL is another feasible approach for parameter estimation. The regression data are composed of failure times and their corresponding MRL. The MRL at each failure time is calculated by the empirical MRL function proposed by Yang (1978). For ordered failure times $t_{(1)}, t_{(2)}, ..., t_{(n)}$, the empirical MRL function $\hat{m}(t)$ is given by

$$\hat{m}(t) = \frac{1}{n-i} \sum\nolimits_{k=i+1}^{n} (t_{(k)} - t) \quad \text{for } t_{(i)} \le t < t_{(i+1)},$$

where $i = 0, 1, ..., n$ and $t_{(0)} = 0$. And $\hat{m}(t) = 0$ for all $t \ge t_{(n)}$.

In this nonlinear regression, the basis function is $m(t) = (\alpha t + \beta)\exp(-\gamma t)$

with parameters α, β, γ and a variable t. The data consist of a variable vector $\left(t_{(0)}, t_{(1)}, t_{(2)}, ..., t_{(n-1)}\right)$ and a response vector $\left(\hat{m}(t_{(0)}), \hat{m}(t_{(1)}), ..., \hat{m}(t_{(n-1)})\right)$. Then the estimates of the parameters will be chosen to minimize the sum of squared residuals $\sum_{i=0}^{n-1} e_i^2$, where e_i is the difference between the true value and the regressed value of responses.

Remark 3.3: Due to the large variation in the tail of the empirical MRL, it maybe suggested to drop the last several time points in the time vector. For example, it could be better to use $\left(t_{(0)}, t_{(1)}, t_{(2)}, ..., t_{(n-3)}\right)$ for regression rather than $\left(t_{(0)}, t_{(1)}, t_{(2)}, ..., t_{(n-1)}\right)$. And so as to the response vector.

For parallel comparison, the data set of Example 3.2 in Section 3.4 is used to do nonlinear regression. The reason to choose Example 3.2 is that no tie for failure times needs to be dealt with in this example. Denote the regression estimates for the three parameters as $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ respectively. The estimation results are given below, as well as the plotting of the fitted MRL functions from both nonlinear regression and MLE in Figure 3.10. We can find that, both the numerical and the graphic comparisons show that the nonlinear regression could produce goodness-of-fit results that are comparable to the results obtained from MLE.

- **Example 3.2**

For Nonlinear regression, the estimates for parameters are $\tilde{\alpha} = 1.9074$, $\tilde{\beta} = 175.144$, $\tilde{\gamma} = 0.00814$. And the fitted MRL $\tilde{m}(t)$ is

$$\widetilde{m}(t) = (175.144 + 1.9074t)\exp(-0.00814t).$$

The estimation results obtained from MLE, as shown in Section 3.4, are $\hat{\alpha} = 1.7694$,

$\hat{\beta} = 171.564$, $\hat{\gamma} = 0.00823$. And the fitted MRL $\hat{m}(t)$ is

$$\hat{m}(t) = (171.564 + 1.7694t)\exp(-0.00823t).$$
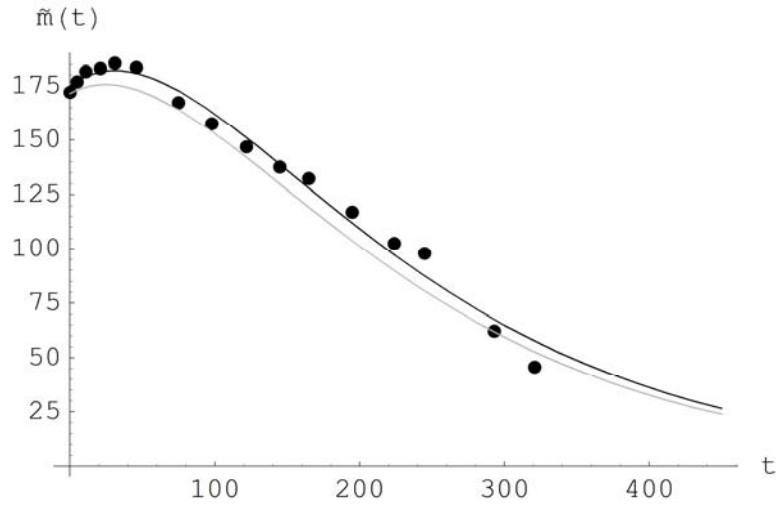
Figure 3.10 The fitted MRL by nonlinear regression (solid line) and MLE (grey line).

To evaluate the accuracy of the nonlinear regression estimation and compare it with MLE, we conduct a simulation experiment based on data generated from this new UBMRL model with $m(t) = (2t + 1)\exp(-t)$. The simulation results are computed from 500 replications with sample size $n = 50$. The accuracy of methods is measured by mean squared errors of the estimated values for three parameters. For parameter α, the mean squared errors for nonlinear regression estimation and MLE are defined by

$$\text{mse}_\alpha^{\text{reg}} = \frac{1}{500}\sum_{i=1}^{500}(\widetilde{\alpha}_i - \alpha)^2,$$
$$\text{mse}_\alpha^{\text{MLE}} = \frac{1}{500}\sum_{i=1}^{500}(\hat{\alpha}_i - \alpha)^2.$$

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The errors for parameter β and γ can be similarly defined. Table 3.5 lists the mean squared errors computed from simulation replications for the two methods respectively. The results shown in the table indicate that nonlinear regression fitting based on the MRL function performs well in the estimation of parameters. For β, the regression method produces much smaller error compared to the MLE. For α and γ, the accuracy of the regression can be considered to be comparable to that of the MLE. Hence, it is suitable to apply the nonlinear regression for estimating the parameters of this new UBMRL model.

Table 3.5   Mean squared errors for nonlinear regression estimation and MLE.

	$\mathrm{mse}_\bullet^{\mathrm{reg}}$	$\mathrm{mse}_\bullet^{\mathrm{MLE}}$
α	0.563888	0.455305
β	0.022219	0.11361
γ	0.045588	0.027294

3.7   Conclusion

In this chapter, a new distribution capable of modeling UBMRL and BFR is presented and studied. Compared to other existing distributions, this new model is derived from the derivative function of MRL, instead of reliability function and failure rate function that are often used in model construction; and has the MRL function in a simple, closed form. Hence the analysis and the application of the MRL function in further reliability analysis can be easily carried out. The parameters of this new model are estimated by MLE. Numerical examples and comparisons indicate that the new model performs well in modeling lifetime data with bathtub-shaped failure rate function and UBMRL function. Hence this new model serves as a good alternative when a bathtub

shaped function should be prescribed. In addition, data fitting based on the MRL function shows great feasibility of being an alternative to parameter estimation. However, the empirical MRL for larger failure time would suffer greater variance due to fewer available data, and thus may significantly influence the regression result and introduce more error. Hence, more endeavors should be made to control the effect of large variance that occurs at large failure time; and further investigation may be needed for the use of the regression method in parameter estimation.

CHAPTER 4   DECREASING MEAN RESIDUAL LIFE ESTIMATION WITH TYPE II CENSORED DATA[2]

Compared to the parametric modeling assuming underlying distributions, nonparametric methods use only failure data to estimate the MRL function regardless of the forms of models and thus introduce less bias. In literature, various methods have been proposed to estimate reliability measures empirically. Given a set of failure data, how to estimate the MRL in a nonparametric way is a challenging problem.

4.1   Introduction

Yang (1978) proposed the empirical MRL function for complete data, which is the first and basic nonparametric estimation for the MRL function. Based on this estimation, several other MRL estimations were constructed, such as confidence bands established in Zhao & Qin (2006) and a simple estimator for the monotone

MRL class proposed in Kochar et al. (2000).

Besides the estimation for complete data, the estimation of the MRL function in censoring has gained much attention, because censoring occurs frequently as it is often impractical to obtain the lifetimes of all the items on test. For example, highly reliable components used in an aircraft usually produce zero failure during testing stage. In literature, many works concerned with the MRL estimation under random censorship. Li (1997) established confidence bounds for the MRL function using randomly right censored data. The statistical inference for the MRL in random censoring was also presented in Na & Kim (1999) and Qin & Zhao (2007).

In contrast to the considerable studies of the estimation of the MRL for complete and randomly censored data, only a few papers focused on the estimation in extreme right censoring. This may be due to the fact that the data collected under such censorship cannot provide enough information over the whole time period. Suppose that the last failure data we get is the time point of 10. This means that the data collected before 10 are the only information we have. Hence, for the empirical reliability function, only the segment before 10 can be constructed by the Kaplan-Meier estimator (Kaplan & Meier, 1958). The values beyond the censoring time can be either constant or decreasing because of the lack of further data. Figure 4.1 presents the three alternatives for behaviors of the reliability function beyond the largest observation. This kind of diversity makes the problem of estimating the MRL complicated and unforeseen.

In Guess & Park (1991), confidence intervals of monotonic MRL in extreme right censoring were presented, but these bounds cannot give good performances

when the MRL increases or decreases rapidly. Hence other feasible methods are expected to serve as alternatives to estimate monotonic MRL function.

In this chapter, we introduce a new method for the estimation of DMRL with Type II censored data. The method estimates the MRL without assuming any distribution. The main idea is to estimate mean time to failure by comparing two estimators for the reliability function. One estimator is the Kaplan-Meier estimator and the other is derived from the empirical MRL function in Yang (1978). Theoretically, this approach is also applicable to IMRL. But the estimation of IMRL is not discussed in this work, because we find that the empirical function for IMRL is not stable in heavy Type II censoring. Fortunately, IMRL is less common than DMRL in real life. The organization of this chapter is as follows. In Section 4.2, we propose the method and the estimation procedure. Section 4.3 presents simulation results and compares the new approach to some parametric methods. Finally, some concluding remarks are given in Section 4.4.
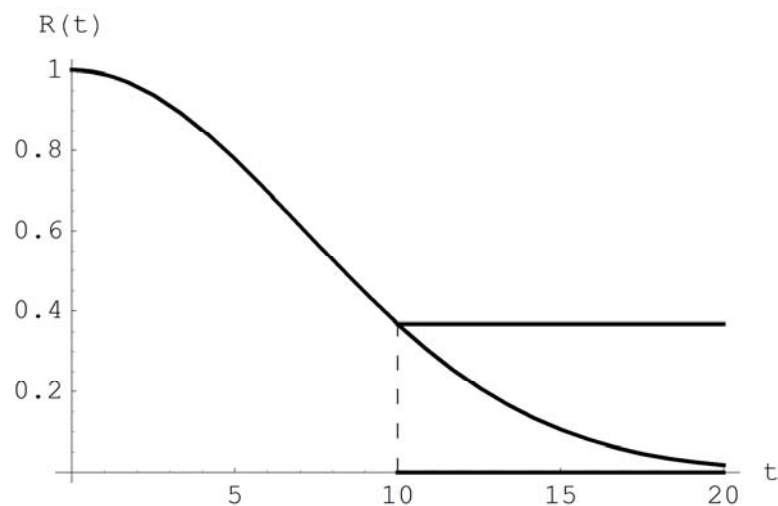
Figure 4.1   The curves of three possible reliability functions under censorship.

4.2   A methodology based on empirical functions

4.2.1   The empirical MRL function

Suppose the lifetime of items T is a continuous non-negative random variable with the reliability function $R(t)$ and the MRL function $m(t)$. Let $t_{(1)}, t_{(2)}, ..., t_{(i)}, ...t_{(r)}, t_{(r+1)}, ..., t_{(n)}$ be ordered potential failure times of n independent and identically distributed items. If the data is complete, i.e. all of the n exact failure times are known, the empirical MRL function can be easily estimated by

$$\hat{m}(t) = \frac{1}{n - l(t)} \sum_{i=l(t)+1}^{n} t_{(i)} - t \ , \tag{4.1}$$

$$= \frac{1}{n - l(t)} \left(n\hat{\mu} - \sum_{i=1}^{s(t)} t_{(i)} \right) - t, \quad 0 \le t < t_{(n)} \tag{4.2}$$

where $s(t) = \max\{i : t_{(i)} \le t\}$ and $\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} t_{(i)}$.

In Type II censoring, because the information of some failure times is lost due to the termination of tests, we have only the first $r(r < n)$ lifetimes, where r is a predetermined number. Hence using (4.1) and (4.2) to get estimation becomes infeasible, unless the mean time to failure μ is provided or can be estimated from another way instead of $\hat{\mu}$. If μ is known, then we can estimate the MRL function by

$$\hat{m}(t_{(i)}) = \frac{n\mu - \sum_{k=1}^{i} t_{(k)}}{n - i} - t_{(i)}, \quad i = 1, ..., r \ . \tag{4.3}$$

But at most time, μ is unknown. Hence, in order to complete (4.3), we need to obtain the information about μ by a method rather than $\hat{\mu}$.

4.2.2 Two estimators of the reliability function

In this work, we want to calculate a good estimate for the mean time to failure μ by controlling the difference between two estimators of the reliability function according to some criteria. One is the Kaplan-Meier estimator, while the other is obtained from (2.2) and (4.3). For convenience, we quote (2.2) here.

$$R(t) = \frac{m(0)}{m(t)} \exp\left\{-\int_0^t \frac{1}{m(x)} dx\right\}, \ t \geq 0. \tag{2.2}$$

The Kaplan-Meier estimator, the common estimation for the reliability function, is given by

$$\hat{R}(t_{(i)}) = \frac{n-i}{n}, i = 1,...,r. \tag{4.4}$$

The second estimator, denoted by $\widetilde{R}(t)$, is obtained by substituting (4.3) into (2.2)

$$\widetilde{R}(t_{(i)}) = \frac{\hat{m}(0)}{\hat{m}(t_{(i)})} \exp\left\{-\sum_{k=1}^{i} \frac{1}{\hat{m}(t_{(k-1)})} \cdot (t_{(k)} - t_{(k-1)})\right\}, \tag{4.5}$$

where $t_{(0)} = 0$ and $\hat{m}(0) = \mu$.

In order to make (4.5) close to (2.2) enough, we adjust each term in the sum by averaging the MRL at two adjacent failure times

$$\widetilde{R}(t_{(i)}) = \frac{\hat{m}(0)}{\hat{m}(t_{(i)})} \exp\left\{ -\sum_{k=1}^{i} \frac{2}{\hat{m}(t_{(k-1)}) + \hat{m}(t_{(k)})} \cdot (t_{(k)} - t_{(k-1)}) \right\}. \tag{4.6}$$

Theoretically, the two estimators, $\hat{R}(t)$ and $\widetilde{R}(t)$, must be equal to each other. However, due to the limited data and the approximation of integral by summation, they become different, which is shown in Figure 4.2. From Figure 4.2, by treating μ as a variable, we observe that $\widetilde{R}(t_{(r)})$ irregularly oscillates at the beginning, and after a certain μ, it starts to decline and finally approximates to $\hat{R}(t_{(r)})$ as μ goes to infinity. (This observation is supported in theory by Proposition 4.1 and 4.2.) An explanation to the early oscillation is that too small μ cannot guarantee positive empirical MRL $\hat{m}(t_{(i)})$ at the beginning and thus make $\widetilde{R}(t_{(i)})$ unstable. Thus, the true value of μ for all failure data should be able to make empirical MRL positive at all $t_{(i)}, i = 1,...,r$. It is rational because the empirical MRL for complete data at each point in time are always positive. Hence, we can say that mean time to failure μ should be at the stable period shown in Figure 4.2.

Moreover, as $\widetilde{R}(t_{(i)})$ declines to $\hat{R}(t_{(i)})$ at the stable period, the distance of $\widetilde{R}(t_{(i)})$ and $\hat{R}(t_{(i)})$ is in fact a decreasing function of mean time to failure μ. This monotonicity implies that it is a one-to-one correspondence between the distance of $\widetilde{R}(t_{(i)})$ and $\hat{R}(t_{(i)})$, and μ. Every possible distance between $\widetilde{R}(t_{(i)})$ and $\hat{R}(t_{(i)})$ can be mapped by exactly one μ. On the other hand, once the distance is determined, we can map it to a unique μ. Thus, this indicates that the true μ should be at some point

at which the distance between $\widetilde{R}(t_{(i)})$ and $\hat{R}(t_{(i)})$ is equal to a certain small value. As long as the distance at the true value of μ is determined, then the true μ can be easily obtained. Hence, we aim to utilize failure data to calculate a suitable value for the distance between $\widetilde{R}(t_{(i)})$ and $\hat{R}(t_{(i)})$ to get μ.
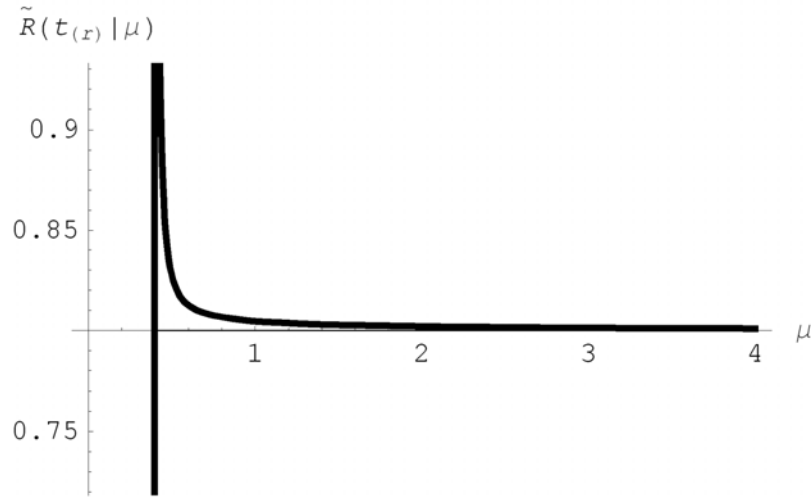
Figure 4.2   The behaviors of $\widetilde{R}(t_{(r)})$ with respect to μ $(r = 10, n = 50)$.

However, for different failure data, $\widetilde{R}(t_{(i)})$ has different shapes, and the μ after which $\widetilde{R}(t_{(i)})$ becomes stable also differs. This variety also happens in the decreasing trend of the distance between $\widetilde{R}(t_{(i)})$ and $\hat{R}(t_{(i)})$. Therefore, it would be better to incorporate the information contained in failure data into the choosing of a proper value for the distance between $\widetilde{R}(t_{(i)})$ and $\hat{R}(t_{(i)})$. A feasible way is to establish a suitable quantitative relationship between failure data and the distance of $\widetilde{R}(t_{(i)})$ and $\hat{R}(t_{(i)})$, so that the choosing procedure could be carried out via the relationship. But before that, a measure needs to be selected to characterize and

summarize the information that failure data have.

Proposition 4.1 $\widetilde{R}(t_{(i)})$, $i = 1,...,r$, is a decreasing function of μ, for sufficiently large μ.

Proposition 4.2 $\widetilde{R}(t_{(i)}) \rightarrow \hat{R}(t_{(i)})$ as $\mu \rightarrow \infty$, for $i = 1,...,r$.

See Section 4.2.3 for proofs.

4.2.3 Proofs of Proposition 4.1 and 4.2

- **Proof of Proposition 4.1**

Note that $\widetilde{R}(t_{(i)}), i = 1,...,r$ is a strictly positive function. Hence to prove Proposition 4.1 is equivalent to prove $\log(\widetilde{R}(t_{(i)}))$ is a decreasing function of μ for sufficiently large μ. Based on the formula (4.6) that is the definition of $\widetilde{R}(t_{(i)})$, we have that

$$
\begin{aligned}
\log(\widetilde{R}(t_{(i)})) &= \log(\hat{m}(0)) - \log(\hat{m}(t_{(i)})) - \sum_{k=1}^{i} \frac{2(t_{(k)} - t_{(k-1)})}{\hat{m}(t_{(k-1)}) + \hat{m}(t_{(k)})} \\
&= \sum_{k=1}^{i} \left(\log(\hat{m}(t_{(k-1)})) - \log(\hat{m}(t_{(k)})) - \frac{2(t_{(k)} - t_{(k-1)})}{\hat{m}(t_{(k-1)}) + \hat{m}(t_{(k)})} \right)
\end{aligned}
\tag{4.7}
$$

where $t_{(0)} = 0$ and $i = 1,...,r$.

Now the aim is to prove that every term in the summation is a decreasing function of μ for sufficiently large μ. Since $\frac{\partial}{\partial \mu} \hat{m}(t_{(i)}) = \frac{n}{n-i}$, the partial derivative

of the summation term in (4.7) with respect to μ is as follows

$$\frac{\partial}{\partial \hat{\mu}}\left(\log(\hat{m}(t_{(k-1)})) - \log(\hat{m}(t_{(k)})) - \frac{2(t_{(k)} - t_{(k-1)})}{\hat{m}(t_{(k-1)}) + \hat{m}(t_{(k)})} \right)$$

$$= \frac{1}{\hat{m}(t_{(k-1)})} \cdot \frac{n}{n-k+1} - \frac{1}{\hat{m}(t_{(k)})} \cdot \frac{n}{n-k} + \frac{2(t_{(k)} - t_{(k-1)})}{(\hat{m}(t_{(k-1)}) + \hat{m}(t_{(k)}))^2} \cdot \left(\frac{n}{n-k+1} + \frac{n}{n-k} \right).$$

$$(4.8)$$

For convenience, we denote

$$\hat{m}(t_{(k-1)}) = a_1 \mu - b_1,$$

$$\hat{m}(t_{(k)}) = a_2 \mu - b_2,$$

$$\hat{m}(t_{(k-1)}) + \hat{m}(t_{(k)}) = a_3 \mu - b_3, \qquad (4.9)$$

where

$$a_1 = \frac{n}{n-k+1}, \quad a_2 = \frac{n}{n-k}, \quad b_1 = \frac{\sum_{j=1}^{k-1} t_{(j)}}{n-k+1} + t_{(k-1)}, \quad b_2 = \frac{\sum_{j=1}^{k} t_{(j)}}{n-k} + t_{(k)}, \quad a_3 = a_1 + a_2,$$

$$b_3 = b_1 + b_2. \qquad (4.10)$$

Then together with (4.9) and (4.10), (4.8) can be re-written in terms of $a_i, b_i, i = 1, 2, 3$,

$$\frac{a_1(a_2\mu - b_2)(a_3\mu - b_3)^2 - a_2(a_1\mu - b_1)(a_3\mu - b_3)^2 + 2(t_{(k)} - t_{(k-1)})a_3(a_2\mu - b_2)(a_3\mu - b_3)^2}{(a_1\mu - b_1)(a_2\mu - b_2)(a_3\mu - b_3)^2}$$

$$(4.11)$$

Since the denominator is strictly positive, the sign of (4.11) depends only on its numerator. It is easy to see that the numerator is a quadratic function and the

coefficient of the quadratic term, denoted by C_2, is given by

$$C_2 = a_1(-2a_2 a_3 b_3 - a_3{}^2 b_2) + a_2(2a_1 a_3 b_3 + a_3{}^2 b_1) + 2(t_{(k)} - t_{(k-1)})a_1 a_2 a_3$$

$$= a_3\left((-a_1 b_2 + a_2 b_1)a_3 + 2(t_{(k)} - t_{(k-1)})a_1 a_2\right) \qquad (4.12)$$

From (4.10) and (4.12), we can get

$$\frac{C_2}{a_3} = \left[-\frac{n}{n-k+1}\left(\frac{\sum_{j=1}^{k} t_{(j)}}{n-k} + t_{(k)}\right) + \frac{n}{n-k}\left(\frac{\sum_{j=1}^{k-1} t_{(j)}}{n-k+1} + t_{(k-1)}\right)\right] \cdot \left(\frac{n}{n-k+1} + \frac{n}{n-k}\right)$$

$$+ 2(t_{(k)} - t_{(k-1)}) \cdot \frac{n}{n-k+1} \cdot \frac{n}{n-k}$$

$$= \frac{n^2}{(n-k+1)(n-k)}\left\{\left[-\sum_{j=1}^{k} t_{(j)} - (n-k)t_{(k)} + \sum_{j=1}^{k-1} t_{(j)} + (n-k+1)t_{(k-1)}\right]\right.$$

$$\left. \times\left(\frac{1}{n-k+1} + \frac{1}{n-k}\right) + 2(t_{(k)} - t_{(k-1)})\right\}$$

$$= \frac{n^2(t_{(k)} - t_{(k-1)})}{(n-k+1)(n-k)}\left[-(n-k+1)\times\left(\frac{1}{n-k+1} + \frac{1}{n-k}\right) + 2\right]$$

$$< \frac{n^2(t_{(k)} - t_{(k-1)})}{(n-k+1)(n-k)}\left[-(n-k+1)\times\left(\frac{2}{n-k+1}\right) + 2\right]$$

$$= 0$$

$$(4.13)$$

Since $a_3 > 0$, we have $C_2 < 0$. Thus the quadratic function that is the numerator of (4.11) is less than 0 for sufficiently large μ, which implies (4.11) is strictly negative. So is the formula (4.8). Now it is obtained that

$$\frac{\partial}{\partial \mu}\left(\log(m(t_{(k-1)})) - \log(m(t_{(k)})) - \frac{2(t_{(k)} - t_{(k-1)})}{m(t_{(k-1)}) + m(t_{(k)})}\right) < 0. \qquad (4.14)$$

By (4.7) and (4.14), the result shown in Proposition 4.1 is obtained. ■

- **Proof of Proposition 2**

From Equation (4.3), it is easy to see that $\hat{m}(t_{(i)}) \to \infty$, $i = 1,...,r$ as $\mu \to \infty$. Hence, it is obtained that as $\mu \to \infty$

$$\exp\left\{ -\sum_{k=1}^{i} \frac{2}{\hat{m}(t_{(k-1)}) + \hat{m}(t_{(k)})} \cdot (t_{(k)} - t_{(k-1)}) \right\} \to 1.$$

Moreover, as $\mu \to \infty$

$$\frac{\hat{m}(0)}{\hat{m}(t_{(i)})} = \frac{\mu}{\left(n\mu - \sum_{k=1}^{i} t_{(k)} \right) \Big/ (n-i) - t_{(i)}} \to \frac{n-i}{n}.$$

Therefore, based on the formula (4.4) and (4.6), we can get as $\mu \to \infty$

$$\widetilde{R}(t_{(i)}) \to \hat{R}(t_{(i)}), \text{ for } i = 1,...,r.$$

Hence the proof of Proposition 4.2 is complete. ∎

4.2.4   A estimation procedure to estimate mean time to failure and the MRL

Many characteristics could be served as the measure for different failure data, such as the sum of all failure data, etc. In this work, we aim to find a suitable measure, which could be used to establish some relationship with the difference of $\widetilde{R}(t)$ and $\hat{R}(t)$ at the true μ, so that given failure data, we could utilize the relationship to calculate the distance of $\widetilde{R}(t)$ and $\hat{R}(t)$ and then the true value of μ. The search of this kind of measures could be conducted by simulation based on some probability distributions. A brief description of the search procedure is as follows. For convenience, denote the

difference of $\widetilde{R}(t)$ and $\hat{R}(t)$ at μ by D_μ.

Search procedure

Step 1: Choose a characteristic C as the measure;

Step 2: Generate several data sets of different sample sizes from a predefined distribution with different known parameters;

Step 3: For each data set, compute the value of C and their true mean μ, i.e. μ_0 from the distribution;

Step 4: Compute vector (C, D_{μ_0}) for each data set and plot the vectors;

Step 5: Observe whether the plotted points have some trend or randomly spread; if trends exist, fit it with a suitable basis function, otherwise, go back to Step 1.

One simple measure is directly derived from $\widetilde{R}(t_{(i)}) - \hat{R}(t_{(i)})$, $i = 1,...,r$. For convenience, we treat $\widetilde{R}(t_{(i)}) - \hat{R}(t_{(i)})$ as a function of μ and denote it by

$$d_i(\mu) = \widetilde{R}(t_{(i)}) - \hat{R}(t_{(i)}), \ i = 1,...,r.$$

Also denote by μ_1 the smallest value of the following set Λ, as $\widetilde{R}(t)$ must be positive and non-increasing.

$$\Lambda = \{\mu \in R^+ : \widetilde{R}(t_{(i)} \mid \mu) > 0 \ \& \ \widetilde{R}(t_{(i)} \mid \mu) > \widetilde{R}(t_{(i+1)} \mid \mu), \ i = 1,...,r\}, \quad (4.15)$$

$$\mu_1 = \min \Lambda. \quad (4.16)$$

Denote by $g(\cdot)$ a transformation of $d_i(\mu), i = 1, ..., r$. Let the characteristic C summarizing failure data be $g(d_1(\mu_1), ..., d_r(\mu_1))$, and the distance of $\widetilde{R}(t)$ and $\hat{R}(t)$ be $D_\mu = g(d_1(\mu), ..., d_r(\mu))$. The relationship between C and D_μ is represented by $h(\cdot)$.

$$g(d_1(\mu), ..., d_r(\mu)) = h\big(g(d_1(\mu_1), ..., d_r(\mu_1))\big), \qquad (4.17)$$

where $h(\cdot)$ is a fitted function obtained from simulations to represent the relationship.

In this work, we let $g(\cdot)$ be the average on $d_i(\mu), i = 1, ..., r$

$$g(d_1(\mu), ..., d_r(\mu)) = \frac{1}{r}\sum_{i=1}^{r} d_i(\mu). \qquad (4.18)$$

Then (4.17) comes to

$$\frac{1}{r}\sum_{i=1}^{r} d_i(\mu) = h\left(\frac{1}{r}\sum_{i=1}^{r} d_i(\mu_1)\right). \qquad (4.19)$$

The function $h(\cdot)$'s for different censor degrees defined as $1 - r/n$ are listed in Table 4.1. These $h(\cdot)$'s are obtained by fitting the data sets $\left(\sum_{i=1}^{r} d_i(\mu_1)\big/r, \ \sum_{i=1}^{r} d_i(\mu)\big/r\right)$, which are generated from Weibull distribution with different parameters and different sample size n. Although these $h(\cdot)$'s are based on Weibull distribution, the simulation results show that they are also applicable for gamma distribution.

Given a group of censored failure data, solving (4.19) would produce a value

for the mean time to failure μ. Denote by $\tilde{\mu}$ the root of (4.19). By substituting $\tilde{\mu}$ into (4.3), we can obtain the estimate for the MRL function. The procedure is summarized as follows.

Estimation procedure

Step 1: Based on failure data, construct $\hat{m}(t)$, $\hat{R}(t)$ and $\tilde{R}(t)$ as a function of μ by (4.3), (4.4) and (4.6) respectively ;

Step 2: Find out μ_1 by (4.15) and (4.16) and compute $\sum_{i=1}^{r} d_i(\mu_1)\big/r$;

Step 3: Choose a proper $h(\cdot)$ from Table 4.1 and compute $\tilde{\mu}$ by (4.19);

Step 4: Estimate the MRL function by $\hat{m}(t\,|\,\mu = \tilde{\mu})$.

Table 4.1   The function $h(\cdot)$ for different censor degrees.

Censor degree $=1-r/n$	$h(x)$
0.1	$\exp(0.116463)x^{1.051466}$
0.2	$\exp(0.021660)x^{1.064074}$
0.3	$\exp(-0.008806)x^{1.088188}$
0.4	$\exp(-0.101985)x^{1.102799}$
0.5	$\exp(-0.165095)x^{1.128530}$
0.6	$\exp(-0.196420)x^{1.169706}$
0.7	$\exp(-0.351714)x^{1.199348}$
0.8	$\exp(-0.693084)x^{1.211148}$
0.9	$\exp(-1.251287)x^{1.221111}$

Remark 4.1: This work gives a possible way as shown in (4.19) for estimation. The function $g(\cdot)$ can be other transformations rather than (4.17). Also, $h(\cdot)$ can be other goodness-of-fit functions. The μ_1 can be other baseline index rather than the smallest value of the set Λ.

4.3   Simulation Study

Some simulation studies are presented to show the performance of the proposed procedure. We estimate the MRL based on data obtained from Weibull distribution and gamma distribution, and also compare this new approach to some common parametric methods with respect to the accuracy of estimation. The results indicate that our new approach is able to give good performance and can surpass the parametric methods when censoring is heavy.

4.3.1   Estimation results

The following two figures display some trials of the estimation for DMRL and compare them with the true MRL. In both figures, the bold line refers to the true MRL while the others are the plotting of different trials of the estimation. Figure 4.3 shows the fitting results of Weibull distribution with shape parameter 2 and scale parameter 1, Weibull (2, 1). The results for gamma distribution with shape parameter 4 and scale parameter 10, gamma (4, 10), are shown in Figure 4.4. The comparisons between the fitted and the real MRL functions suggest that in general, the new method could provide nice performances on the estimation of DMRL, because all the estimated curves are located not far away from the original one and have their decreasing trend

similar to that of the true curve.
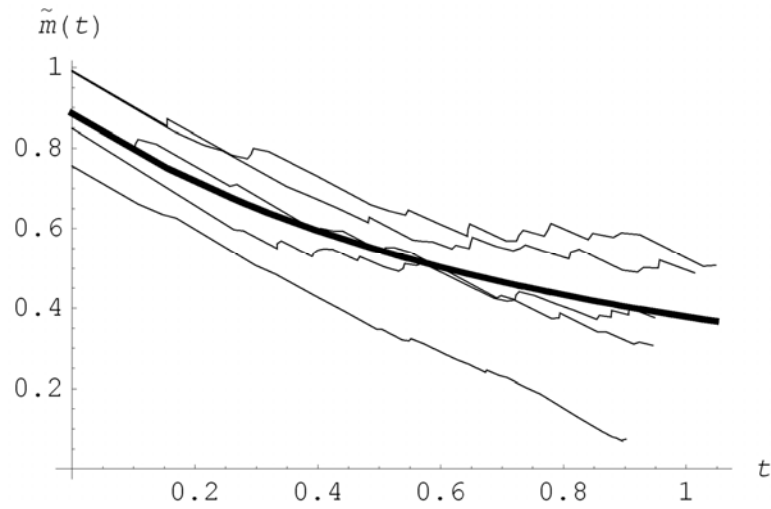
Figure 4.3   Comparison between real MRL (bold line) and estimated MRL function: Weibull distribution (2, 1) with censor degree of 0.4.
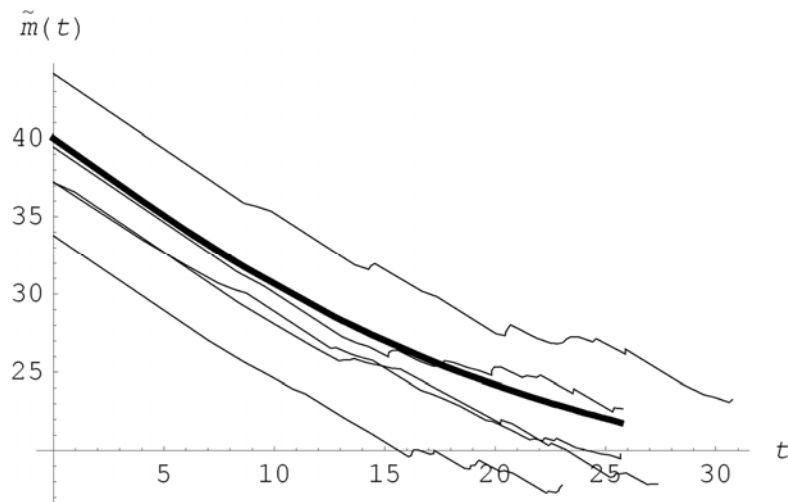
Figure 4.4   Comparison between real MRL (bold line) and estimated MRL function: gamma distribution (4, 10) with censor degree of 0.7.

4.3.2   Comparisons between the new and some parametric methods

To compare our new approach to the two main parametric methods - Maximum

Likelihood Estimation (MLE) and Least Square Estimation (LSE), the simulations are conducted for Weibull distribution and gamma distribution. We assume that the underlying distribution for the parametric methods is the Weibull model because of its popularity in reliability engineering. This means that, the underlying distribution is true when failure data comes from a Weibull distribution, and is misspecified if failure data comes from a gamma distribution.

Table 4.2 and Table 4.3 list the simulation results, which are computed from 100 replications with sample size $n = 50$, 100 and censor degrees of 0.4, 0.8. The accuracy of each estimation method is accessed by the following error, which is calculated as the average on squared errors of MRL at each failure data points.

$$\text{error} = \frac{\sum_{i=1}^{r} \left(\hat{m}(t_{(i)}) - m(t_{(i)}) \right)^2}{r}, \tag{4.20}$$

where $m(t_{(i)})$ and $\hat{m}(t_{(i)})$ are the real and the estimated MRL at $t_{(i)}$ respectively. The three approaches are compared in terms of the average value of their errors and the related variance, which are obtained from replications.

The results in both tables suggest that the new method performs well in the MRL estimation. From Table 4.2, we find that when the underlying distribution is correctly set, the new method is rather comparable to the MLE although not outperforming it, and can become slightly better when the available data size is small.

Table 4.2   **Simulation results for Weibull distribution with shape and scale parameters** (β, α).

Censor degree = 0.8

n=50	E(error) (2, 1)	V(error)	E(error) (2, 10)	V(error)	E(error) (4, 1)	V(error)	E(error) (4, 10)	V(error)
MLE	0.12020	0.08546	8.49816	243.087	0.00577	0.00019	1.45244	2.29518
LSE	2.31773	114.053	134.256	304758	0.09201	0.02635	11.8314	2885.79
New	0.06153	0.00969	6.96069	138.004	0.05205	0.02180	5.4182	130.368
n=100	(2, 1)		(2, 10)		(4, 1)		(4, 10)	
MLE	0.03250	175.92	5.16342	175.92	0.00459	0.00019	0.95798	2.27858
LSE	0.30707	99395.4	74.2731	99395.4	0.04978	0.02635	5.96174	878.98
New	0.08377	230.502	8.31186	230.502	0.08575	0.02180	6.84384	201.102

Censor degree = 0.4

n=50	E(error) (2, 1)	V(error)	E(error) (2, 10)	V(error)	E(error) (4, 1)	V(error)	E(error) (4, 10)	V(error)
MLE	0.01105	0.00017	0.73780	1.22898	0.00093	4.79E-06	0.18087	0.05136
LSE	0.03432	0.00336	7.17336	426.079	0.01005	0.00028	0.51578	1.34699
New	0.01265	0.00023	1.1846	1.80366	0.00492	3.98E-05	0.83706	6.63227
n=100	(2, 1)		(2, 10)		(4, 1)		(4, 10)	
MLE	0.00561	5.58E-05	0.49964	0.49052	0.00054	1.53E-06	0.09579	0.01450
LSE	0.02880	0.01104	1.74981	9.38191	0.00427	9.34E-05	0.36001	0.41472
New	0.00928	0.00023	0.81377	1.03232	0.01173	0.00013	1.26495	2.19568

The results in Table 4.3 show that, if the underlying distribution is wrongly chosen, the new method is still able to produce accurate and stable estimation for the MRL function and has better performances than the MLE most of time, especially when the censoring is heavy. Moreover, we find that our new procedure always gives more favorable estimation than the LSE regardless of censor degree, parameter and sample size. It is worth noting that the new method is a nonparametric approach, and thus much easier to be used in computation compared to the MLE and the LSE, which may need starting values. So it is a good choice to utilize this new approach to estimate the MRL function when the data is Type II censored.

Table 4.3   Simulation results for gamma distribution with shape and scale parameters (b, a).

Censor degree = 0.8

	E(error)	V(error)	E(error)	V(error)	E(error)	V(error)	E(error)	V(error)
n=50	(2, 1)		(2, 10)		(4, 1)		(4, 10)	
MLE	0.72308	1.52889	49.304	2516.8	1.15582	0.97629	245.158	1.71E+6
LSE	7.34984	3783.09	762.888	1.01E+7	2.92106	54.5256	164.471	91396.2
New	0.45416	0.14771	49.2319	2218.71	1.17507	2.90153	69.9264	5256.86
n=100	(2, 1)		(2, 10)		(4, 1)		(4, 10)	
MLE	0.36442	0.16574	41.7359	2550.8	0.663986	0.408652	316.79	1.13E+7
LSE	0.92661	14.5942	505.796	4.10E+6	0.836309	1.67657	261.396	1.49E+6
New	0.29363	0.09623	32.6164	1101.96	0.915245	4.53443	79.1259	23393.9

Censor degree = 0.4

	E(error)	V(error)	E(error)	V(error)	E(error)	V(error)	E(error)	V(error)
n=50	(2, 1)		(2, 10)		(4, 1)		(4, 10)	
MLE	0.13415	0.02018	18.3118	4476.89	0.282167	0.055064	65.6198	138549
LSE	0.37964	0.04597	36.4742	16758.4	0.33516	0.105586	31.764	1069.76
New	0.14089	0.03615	20.996	306.921	0.289309	0.060875	29.7066	705.183
n=100	(2, 1)		(2, 10)		(4, 1)		(4, 10)	
MLE	0.07421	0.00489	8.62614	420.743	0.198012	0.030083	287.337	434006
LSE	0.11412	0.01572	16.0043	739.35	0.277023	0.056159	25.4535	642.129
New	0.11199	0.01139	9.31751	105.047	0.135272	0.030934	17.1018	538.297

4.4   Conclusion

In this chapter, a new estimation procedure for the DMRL function has been introduced by comparing two estimations of the reliability function. Simulation results indicate the good performance of the new method for the distributions with DMRL, especially when censor degree is relatively high and data size is small. Further research may try to improve the proposed method and extend the idea so that more DMRL can be estimated in a nonparametric way, as well as IMRL or UBMRL, since the current method cannot provide satisfactory estimations for these two MRL classes.

CHAPTER 5 RELATIONSHIP BETWEEN MEAN RESIDUAL LIFE AND FAILURE RATE FUNCTION

After studying the MRL itself in the previous chapters, this chapter will focus on the relationship between the MRL and the failure rate function, and discuss the effect of the change of one characteristic on the other characteristic, as these two characteristics are closely related to each other. This type of study would give comprehensive descriptions for aging behaviors of products, and also provide guidelines on how to control the deterioration of products more efficiently.

5.1 Introduction

The MRL and the failure rate function are two important measures used to describe failure times. Although these two functions depict aging behaviors in different ways, both of them are in fact equivalent to the reliability function in the sense of probability; moreover, the characteristic of one function is always related to that of the other. Therefore, much attention has been addressed to the relationship between these two functions, for better decision-makings in reliability engineering, such as shock process,

burn-in, maintenance schedule, etc.

In literature, the MRL and the failure rate functions have been extensively studied and compared from various aspects, such as shape, change point and partial ordering. For shape, Bryson and Siddiqui (1969) proved that the IFR (DFR) implies the DMRL (IMRL), while Gupta and Akman (1995a) showed that the characteristic of the non-monotonic MRL depends on its mean and the failure rate at time zero. The properties of the change points for the MRL and the failure rate functions with roller-coaster shape were discussed in Bekker and Mi (2003). Also Tang et al. (2004) investigated the distance between the change points for MRL and failure rate functions. Furthermore, the partial orderings of these two characteristic play key roles in the field of reliability. Analogical to the fact that the failure rate function determines the trend of the MRL function, Gupta and Kirmani (1987) proved that the failure rate ordering dominates the MRL ordering and proposed a sufficient condition under with the MRL ordering also implies the failure rate ordering. A systematic review on the relationships between the MRL and the failure rate function is presented in Section 2.1.3.

The studies mentioned before discussed the relationship between the two functions mainly from a qualitative point of view. Hence, as a complement, doing some quantitative analysis would be useful and meaningful in reliability for both theory and applications. In this chapter, we aim to study the effect of the change of one characteristic on the other characteristic. Some inequalities are established to indicate upper or lower bound on the extent of change. The application of the inequalities is also discussed. In Section 5.2, we study the range that the MRL will

decrease (increase) if the associated failure rate is increased (decreased) to a certain level. Two examples are used for illustration. In section 5.3, the difference of two failure rate functions is investigated when their corresponding MRL functions are ordered. The result is shown to be useful in estimating failure rate function based on MRL that can be empirically estimated. Finally, conclusions are given in Section 5.4

5.2   From failure rate function to MRL

Finkelstein (2003a) introduced characteristics to measure the difference between two MRL functions and discussed how the MRL changes from a baseline to a more risky environment with increased failure rate function. One characteristic used to quantitatively measure the difference between two MRL functions is called D_{MRL}-distance

$$D_{\mathrm{MRL}}(t) = m_1(t) - m_2(t), \ \ t \geq 0.$$

The other is relative characteristic that is defined based on the ratio of two MRL functions.

$$RD_{\mathrm{MRL}}(t) = 1 - \frac{m_2(t)}{m_1(t)}, \ \ t \geq 0.$$

In this section, we want to extend the study in Finkelstein (2003a) and further study the extent that the MRL will be affected by the change of the corresponding failure rate function.

5.2.1 Some results on MRL due to the change of failure rate function

In reliability, failure rate function is a characteristic that measures the instantaneous risk an item faces at a certain time. As risks change in diverse environments, the failure rate function might also vary. The failure rate function may be increased due to an extra risk; or it will become smaller if a production process is improved. There are two main ways to model the change of the failure rate function. One is to use the additive failure rate models and the other is via the proportional failure rate model.

The independent additive model is describe in the following way

$$r_A(t) = r(t) + \lambda(t), \qquad t \geq 0, \tag{5.1}$$

where $\lambda(t)$ is a failure rate function representing additional independent risks.

The proportional hazards model is given by

$$r_P(t) = z\, r(t), \qquad t \geq 0, \tag{5.2}$$

where z is a constant or some parameter.

In this section, we are interested in how the MRL function would respond when the associated failure rate function varies via the above two models. Let T_1, T_2 be two lifetime random variables with the MRL functions $m_1(t), m_2(t)$ and the failure rate functions $r_1(t), r_2(t)$. Without loss of generalization, we assume that

$$r_1(t) \leq r_2(t), \quad t \geq 0.$$

Because the failure rate ordering implies the MRL ordering, it is easy to see that

$$m_1(t) \geq m_2(t), \quad t \geq 0.$$

In the following Theorem 5.1 and Theorem 5.2, some inequalities are established to show the difference of two MRL functions, $m_1(t)$ and $m_2(t)$, if the failure rate functions, $r_1(t)$ and $r_2(t)$, are related through the additive model and the proportional model.

Theorem 5.1: *Suppose that the MRL functions $m_1(t)$ and $m_2(t)$ are bounded. Denote constant, positive upper bound and lower bound for $m_1(t)$ and $m_2(t)$ by $c_{1U}, c_{1L}, c_{2U}, c_{2L}$ respectively. z is a constant and $z \geq 1$. Then*

(1) If $1 \leq \dfrac{r_2(t)}{r_1(t)} \leq z$, we have

$$0 \leq m_1(t) - m_2(t) \leq (z-1)c_{2U};$$

If the constant z satisfies that $1 \leq z \leq \dfrac{c_{1U}}{c_{1U} - c_{1L}}$, we also have

$$0 \leq m_1(t) - m_2(t) \leq \left(\frac{z-1}{z}\right)c_{1U}.$$

(2) If $\dfrac{r_2(t)}{r_1(t)} \geq z \geq 1$, we have

$$m_1(t) - m_2(t) \geq (z-1)c_{2L} \geq 0 \text{ and } m_1(t) - m_2(t) \geq \left(\frac{z-1}{z}\right)c_{1L} \geq 0.$$

Proof: For part (1), denote $m_3(t) = m_2(t) + (z-1)c_{2U}$ as the MRL function of another

lifetime random variable. The corresponding failure rate $r_3(t)$ is given by

$$r_3(t) = \frac{m_3'(t)+1}{m_3(t)} = \frac{m_2'(t)+1}{m_3(t)} = r_2(t)\frac{m_2(t)}{m_2(t)+(z-1)c_{2U}}.$$

Since $g(x) = \dfrac{x}{x+a}$, $a > 0$ is an increasing function of x, we can get

$$r_3(t) \le r_2(t)\frac{c_{2U}}{c_{2U}+(z-1)c_{2U}} = \frac{1}{z}r_2(t).$$

As $1 \le \dfrac{r_2(t)}{r_1(t)} \le z$, we obtain that $r_3(t) \le r_2(t)/z \le r_1(t) \le r_2(t)$. This inequality implies

that $m_3(t) = m_2(t)+(z-1)c_{2U} \ge m_1(t) \ge m_2(t)$, which is equivalent to

$$0 < m_1(t) - m_2(t) \le (z-1)c_{2U}.$$

Let $m_4(t) = m_1(t) - \left(\dfrac{z-1}{z}\right)c_{1U}$. As $1 \le z \le \dfrac{c_{1U}}{c_{1U}-c_{1L}}$, $m_4(t) \ge 0$ and so $m_4(t)$

is an MRL function. The corresponding failure rate $r_4(t)$ is

$$r_4(t) = \frac{m_4'(t)+1}{m_4(t)} = \frac{m_1'(t)+1}{m_4(t)} = r_1(t)\frac{m_1(t)}{m_1(t)-(z-1)c_{1U}/z}.$$

Since $\dfrac{x}{x-a}$, $a > 0$ is an decreasing function for $x > a$, we have

$$r_4(t) \ge r_1(t)\frac{c_{1U}}{c_{1U}-(z-1)c_{1U}/z} = zr_1(t).$$

As $1 \le \dfrac{r_2(t)}{r_1(t)} \le z$, then $r_4(t) \ge zr_1(t) \ge r_2(t) \ge r_1(t)$. From this inequality, we can get

$$m_4(t) = m_1(t) - \left(\frac{z-1}{z}\right)c_{1U} \leq m_2(t) \leq m_1(t),$$ which is equivalent to

$$0 < m_1(t) - m_2(t) \leq \left(\frac{z-1}{z}\right)c_{1U}.$$

The proof of Part (2) is similar. ∎

Theorem 5.2: *Suppose that the MRL functions $m_1(t)$ and $m_2(t)$ are bounded. Denote constant, positive upper bound and lower bound for $m_1(t)$ and $m_2(t)$ by $c_{1U}, c_{1L}, c_{2U}, c_{2L}$ respectively. l is a constant and $l \geq 0$. Then*

(1) If $0 \leq r_2(t) - r_1(t) \leq l$, we have

$$1 \leq \frac{m_1(t)}{m_2(t)} \leq 1 + l \cdot c_{1U};$$

If the constant l satisfies that $l < \min\left\{\dfrac{1}{c_{2U}}, \dfrac{c_{2L}r_2(t)}{c_{2U}}\right\}$, we also have

$$1 \leq \frac{m_1(t)}{m_2(t)} \leq \frac{1}{1 - l \cdot c_{2U}}.$$

(2) If $r_2(t) - r_1(t) > l$, we have

$$\frac{m_1(t)}{m_2(t)} \geq 1 + l \cdot c_{1L} \geq 1 \quad \text{and} \quad \frac{m_1(t)}{m_2(t)} \geq \frac{1}{1 - l \cdot c_{2L}} \geq 1.$$

Proof: For part (1), denote $m_3(t) = \dfrac{1}{1 + l \cdot c_{1U}} m_1(t)$ is the MRL function of another lifetime random variable. The associated failure rate $r_3(t)$ is given by

$$r_3(t) = \frac{m_3'(t)+1}{m_3(t)} = \frac{m_1'(t)+1}{m_1(t)} + \frac{l\,c_{1U}}{m_1(t)} = r_1(t) + \frac{l\,c_{1U}}{m_1(t)}.$$

Since $g(x) = \dfrac{a}{x}$, $a > 0$ is a decreasing function of x, we can get

$$r_3(t) \geq r_1(t) + \frac{l\,c_U}{c_U} = r_1(t) + l.$$

As $0 \leq r_2(t) - r_1(t) \leq l$, we have that $r_3(t) \geq r_1(t) + l > r_2(t) \geq r_1(t)$. This inequality

implies that $m_1(t) \geq m_2(t) \geq m_3(t) = \dfrac{1}{1+l\cdot c_U} m_1(t)$, which is

$$1 \leq \frac{m_1(t)}{m_2(t)} \leq 1 + l\cdot c_{1U}.$$

Denote $m_4(t) = \dfrac{1}{1-l\cdot c_{2U}} m_2(t)$. Because $l < \min\left\{\dfrac{1}{c_{2U}}, \dfrac{c_{2L}\,r_2(t)}{c_{2U}}\right\}$, $m_4(t) \geq 0$

and $m_4' \geq -1$. So $m_4(t)$ is a MRL function. The associated failure rate $r_4(t)$ is

$$r_4(t) = \frac{m_4'(t)+1}{m_4(t)} = \frac{m_2'(t)+1}{m_2(t)} - \frac{l\,c_{2U}}{m_2(t)} = r_2(t) - \frac{l\,c_{2U}}{m_2(t)}.$$

Since $g(x) = \dfrac{a}{x}$, $a < 0$ is an increasing function of x, we can get

$$r_4(t) \leq r_2(t) - \frac{l\,c_{2U}}{c_{2U}} = r_2(t) - l.$$

Because $0 \leq r_2(t) - r_1(t) \leq l$, it can be obtained that $r_2(t) \geq r_1(t) \geq r_2(t) - l \geq r_4(t)$.

This inequality implies that $m_2(t) \leq m_1(t) \leq m_3(t) = \dfrac{1}{1-l\cdot c_{2U}} m_2(t)$, which can be

rewritten as $1 \leq \dfrac{m_1(t)}{m_2(t)} \leq \dfrac{1}{1 - l \cdot c_{2U}}$.

Part (2) can be similarly proved. ■

Consider the exponential distribution as a special case. Let T_1, T_2 be random variables subject to the exponential distribution with the failure rate λ_1 and λ_2 and $\lambda_1 \leq \lambda_2$. Then their associated MRL function are $\mu_1 = 1/\lambda_1$ and $\mu_2 = 1/\lambda_2$ respectively. For convenience, we choose $c_{1U} = c_{1L} = \mu_1$, $c_{2U} = c_{2L} = \mu_2$.

Case 1: If $1 \leq \lambda_2 / \lambda_1 \leq z$, then we have $\dfrac{1}{\lambda_2} \geq \dfrac{1}{z}\dfrac{1}{\lambda_1}$, which is in fact $\mu_2 \geq \dfrac{1}{z}\mu_1$. Then it is easy to get the difference between μ_1 and μ_2 satisfying that

$\mu_1 - \mu_2 \leq \dfrac{z-1}{z}\mu_1 = \dfrac{z-1}{z}c_{1U}$. For $\lambda_2 / \lambda_1 \geq z$, we can obtain $\dfrac{1}{\lambda_2} \leq \dfrac{1}{z}\dfrac{1}{\lambda_1}$, which is in fact $\mu_2 \leq \dfrac{1}{z}\mu_1$. Then $\mu_1 - \mu_2 \geq \dfrac{z-1}{z}\mu_1 = \dfrac{z-1}{z}c_{1L}$. This is for Theorem 5.1.

Case 2: If $\lambda_2 - \lambda_1 \leq l$, we have $\dfrac{\lambda_2}{\lambda_1} \leq 1 + \dfrac{l}{\lambda_1}$, which is equivalent with

$\dfrac{\mu_1}{\mu_2} \leq 1 + l\,\mu_1 = 1 + l\,c_{1U}$. For $\lambda_2 - \lambda_1 \geq l$, we can obtain $\dfrac{\lambda_2}{\lambda_1} \geq 1 + \dfrac{l}{\lambda_1}$, which is in fact

$\dfrac{\mu_1}{\mu_2} \geq 1 + l\,\mu_1 = 1 + l\,c_{1L}$. This is an example for Theorem 5.2.

5.2.2   Numerical examples and practical implication

The results presented in the previous section can be easily interpreted in common

probability distributions such as the Weibull distribution as well as the extended Weibull model and the lognormal distribution.

Example 5.1: Suppose random variable T_2 follows Weibull distribution with $R_2(t) = \exp(-t^3)$ and $r_2(t) = 3t^2$. The MRL function of T_2, $m_2(t)$, is a decreasing function of t, so it is easy to see that $m_2(t) \le E[T_2] = m_2(0)$ for all $t \ge 0$. If there is another random variable T_1 with its failure rate function $r_1(t)$ such that $0.7r_2(t) < r_1(t) < r_2(t)$, then according to Theorem 5.1, the MRL of T_1, $m_1(t)$, falls in to the band composed of $m_2(t)$ and $m_2(t) + (1/0.7 - 1)E[T_2]$, i.e. $m_2(t) < m_1(t) < m_2(t) + (1/0.7 - 1)E[T_2]$. Figure 5.1 shows the plots of the MRL and the failure rate functions of T_1 and T_2, as well as $0.7r_2(t)$ and $m_2(t) + (1/0.7 - 1)E[T_2]$.

Example 5.2: Suppose random variable T_1 follows a distribution with $r_1(t) = 0.24t^2$. The corresponding MRL of T_1, $m_1(t)$, is a decreasing function of t, so we have $m_1(t) \le E[T_1] = m_1(0)$ for all $t \ge 0$. If there is another random variable T_2 with its failure rate function $r_2(t)$ such that $r_1(t) < r_2(t) < r_1(t) + 0.4$, then according to Theorem 5.2, the MRL function of T_2, $m_2(t)$, is bounded by $m_1(t)/(1 + 0.4E[T_1])$ and $m_1(t)$, i.e. $m_1(t)/(1 + 0.4E[T_1]) < m_2(t) < m_1(t)$. The corresponding MRL and failure rate functions are plotted in Figure 5.2.
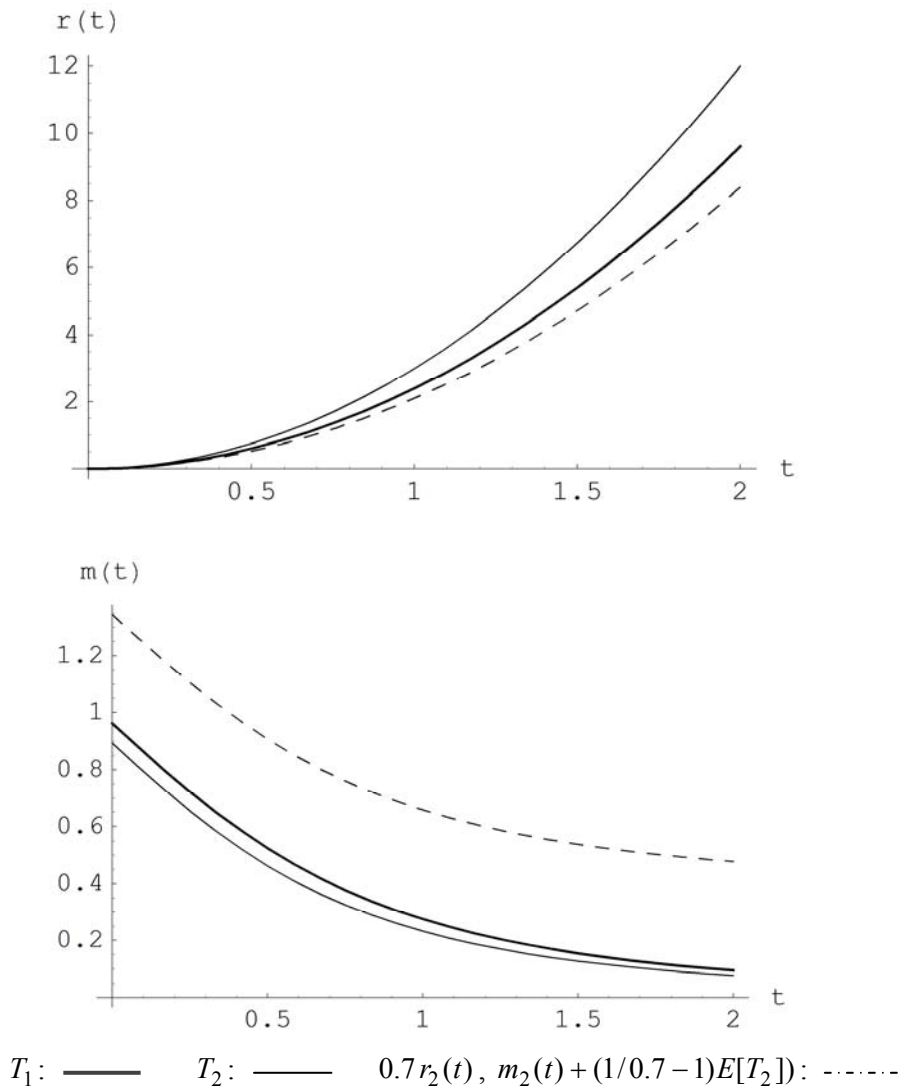
Figure 5.1   The plots of the MRL and the failure rate function for T_1, T_2 in Example 5.1.

- **Practical implication**

One important application is related to point process of recurrent event, particularly a shock process that acts on an object (Finkelstein, 2003a). Suppose the lifetime of an object T is a random variable with reliability function $R(t)$ and failure rate function $r(t)$. Let $\{P_t, t \geq 0\}$ denote a non-homogeneous Poisson process of an additional
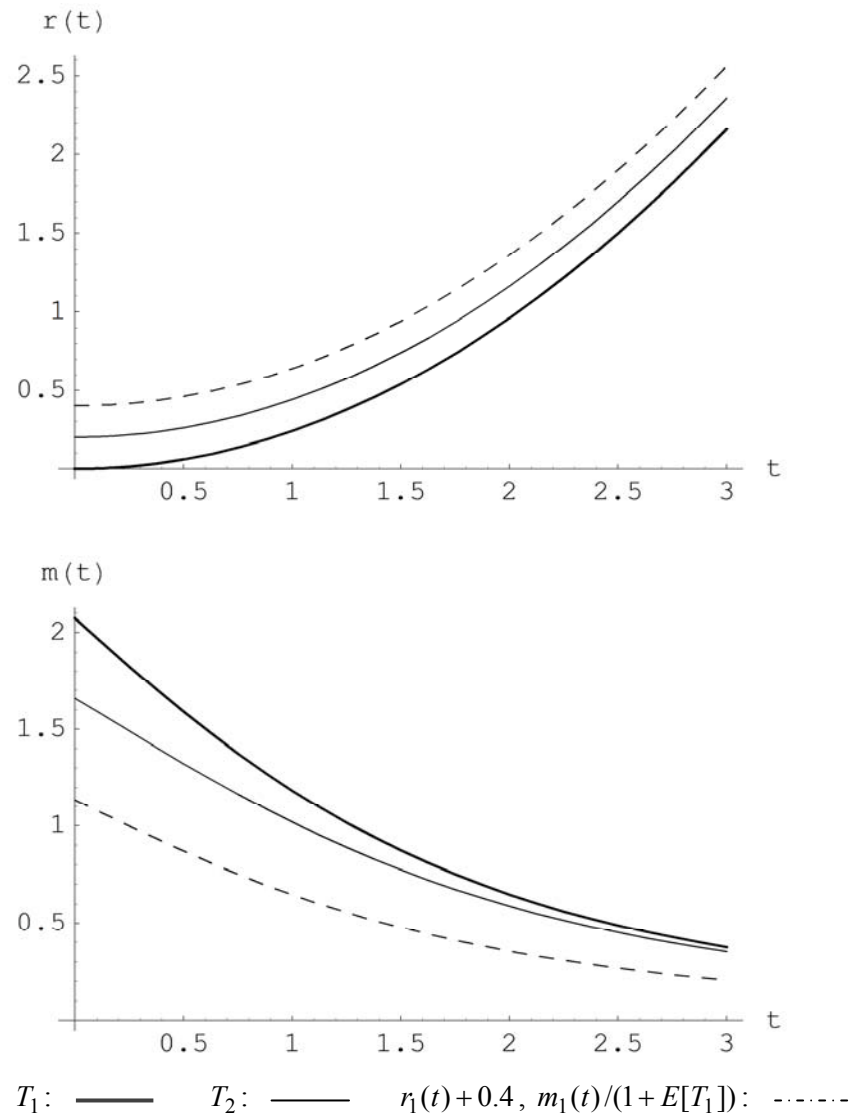
Figure 5.2   The plots of the MRL and the failure rate funcitons for T_1, T_2 in Example 5.2.

harmful shock, which is independent of lifetime T, but with a certain probability would cause failure of the object. Assume that the rate of the shock process is $\lambda(t)$ and the probability that the occurrence of the shock results in failure rate is $\theta(t)$. Block et al. (1985) and Finkelstein (2003b) showed that, under this only shock process, the survival probability of the object at time t, denoted by $G(t)$, comes to

$$G(t) = R(t)R_\lambda(t),\qquad(5.3)$$

where $R_\lambda(t) = \exp\left(-\int_0^t \theta(u)\lambda(u)du\right)$.

Obviously, this setting can be described by the independent additive model in (5.1). It can be seen that, with an additional risk from the shock process, the failure rate function that the object faces, denoted by $r_G(t)$, is a sum of its original failure rate function and the one introduced by the process.

$$r_G(t) = r(t) + \theta(t)\lambda(t). \tag{5.4}$$

Based on either (5.3) or (5.4), the loss in MRL due to the shock could be calculated, given that the information of $\theta(t)$ and $\lambda(t)$ is exactly known. But sometimes it is difficult to obtain full knowledge of the shock process, and thus maybe only part of information is available, such as upper or lower bounds. Then, in this case, Theorem 5.2 can be applied to get a range of the MRL loss.

On the other hand, the proportional hazards model is widely used in both reliability engineering and biostatistics to analyze the effect that some time-independent covariates have on the failure rate (hazard rate) of an object. Often the z in (5.2) is expressed as an exponential function, $z = \exp\left(\mathbf{\beta}^T\mathbf{Z}\right)$, where $\mathbf{Z} = (Z_1,...,Z_m)$ is a vector of fixed covariates and $\mathbf{\beta} = (\beta_1,...,\beta_m)$ is a unknown parameter vector. For example, we may be interested to assess the risk of smokers being exposed to lung cancer, and want to find out attributes that have effects on the risk. Based on the proportional hazards model, we may choose four time-independent covariates like sex, weight, blood pressure, race, and also a suitable baseline failure rate function with some unknown parameters. Then the following model can be established.

$$r_p(t) = \exp(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{Z})r(t,\boldsymbol{\alpha}) . \tag{5.5}$$

where $r(t,\boldsymbol{\alpha})$ is a baseline failure rate function. The unknown parameters can be estimated with data collected from patients, and estimated model can further be used to make statistical inferences. Given confidence intervals of parameters, the lost expected lifetime due to an increase in blood pressure can be obtained by Theorem 5.1.

5.3   From MRL to failure rate function

The foregoing section studies the change of the MRL due to the change of the corresponding failure rate function. This section focuses on the comparison between two failure rate function when their related MRL are ordered. This study would benefit the estimation of bounds on failure rate function, as the estimation could be conducted in the following way.

Step 1: Find two MRL functions such that the band between them covers the empirical MRL function estimated from failure data;

Step 2: Use the two failure rate functions associated with the two MRL to construct the bounds for the desired failure rate function.

5.3.1   Some results on failure rate function for ordered MRL

It is known that MRL ordering does not necessarily imply failure rate ordering. To make the implication valid, a sufficient condition, which was proposed in Gupta & Kirmani (1987) and is also shown as Theorem 2.5 in Chapter 2, is that the ratio of

smaller MRL over larger MRL should be a non-decreasing function. With this condition, a larger MRL function corresponds to a smaller failure rate function. In other words, for estimation, an upper (lower) bound of MRL function would produce a lower (upper) bound for the associated failure rate function.

Although this sufficient condition is simple, the disadvantage is that the behavior of ratio cannot be directly observed from the plotting of two functions. Also, the estimation is sensitive to the sufficient condition, which is not easy to fulfill when the empirical MRL is encountered. This is because, the empirical MRL is not a smooth function, and thus the ratio of the empirical MRL to its upper bound MRL may not be a non-decreasing function all the time, so as to the ratio of the lower bound MRL to the empirical MRL function. This possibility would result in the crossing of the two failure rate functions inverted from the MRL bounds, as well as the crossings of these two failure rate functions and the interested failure rate function.

Hence, more intuitive characteristics are expected as alternatives for the sufficient condition, so that bounds for the failure rate function can be easily obtained by graphically analyzing the related MRL functions. Also more robust estimation methods are needed.

Under the condition that the difference of two MRL function is monotonic, an inequality for two failure rate functions is established as shown in Theorem 5.3. This inequality can be further used to estimate bounds on failure rate function.

Theorem 5.3 *Let T_1 and T_2 be two life time random variables with reliability function $R_1(t)$ and $R_2(t)$. Then*

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(1) DMRL Class

Suppose $m_1(t)$ and $m_2(t)$ are decreasing functions, $m_1(t) \geq m_2(t)$, and $m_1(t) - m_2(t)$ is a non-decreasing function of t, then

$$r_1(t) - \frac{1}{m_1(t)} \geq r_2(t) - \frac{1}{m_2(t)} \; ;$$

(2) IMRL class

Suppose $m_1(t)$ and $m_2(t)$ are increasing functions, $m_1(t) \geq m_2(t)$, and $m_1(t) - m_2(t)$ is a non-increasing function of t, then

$$r_1(t) - \frac{1}{m_1(t)} \leq r_2(t) - \frac{1}{m_2(t)} \; .$$

Proof: For part (1), as $m_1(t) - m_2(t)$ is a non-decreasing function of t, it follows that for $t \geq 0$

$$m_1'(t) - m_2'(t) \geq 0 \; .$$

Since $m_1(t)$ and $m_2(t)$ are DMRL, we have

$$m_1'(t) < 0, \, m_2'(t) < 0 \; .$$

Because $m_1(t) > m_2(t) \geq 0$, by simple deduction, we have

$$\frac{m_2'(t)}{m_2(t)} \leq \frac{m_1'(t)}{m_1(t)} \; .$$

Based on (2.4), we have

$$r_2(t) = \frac{m_2'(t) + 1}{m_2(t)} \leq \frac{m_1'(t)}{m_1(t)} + \frac{1}{m_2(t)} = r_1(t) - \frac{1}{m_1(t)} + \frac{1}{m_2(t)} \; .$$

Rewrite the above inequality as

$$r_1(t) - \frac{1}{m_1(t)} \geq r_2(t) - \frac{1}{m_2(t)} .$$

So Part (1) is proved. Part (2) can be similarly proved. ∎

5.3.2   The application in estimating bounds for failure rate function

As a direct application of Theorem 5.3, the estimation for bounds on failure rate function is summarized in the following corollary. The methodology presented is applied to monotonic MRL class. But more complicated MRL function also could be handled by treating it as a combination of monotonic MRL functions.

Corollary 5.1 *Suppose T, T_1 and T_2 are nonnegative random variables with the failure rate functions $r(t)$, $r_1(t)$, $r_2(t)$ and the MRL functions $m(t)$, $m_1(t)$, $m_2(t)$. Without loss of generality, assume that $m_1(t) > m(t) > m_2(t)$.*

(1) DMRL class

If $m(t)$, $m_1(t)$, $m_2(t)$ are decreasing MRL functions and satisfy $m_1(t) - m(t)$ and $m(t) - m_2(t)$ are increasing functions of t, then we have

$$r_2(t) - \frac{1}{m_2(t)} + \frac{1}{m(t)} < r(t) < r_1(t) - \frac{1}{m_1(t)} + \frac{1}{m(t)} . \tag{5.6}$$

2) IMRL class

If $m(t)$, $m_1(t)$, $m_2(t)$ are increasing MRL functions and satisfy $m_1(t) - m(t)$ and

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$m(t) - m_2(t)$ are decreasing functions of t, then we have

$$r_2(t) - \frac{1}{m_2(t)} + \frac{1}{m(t)} > r(t) > r_1(t) - \frac{1}{m_1(t)} + \frac{1}{m(t)}. \tag{5.7}$$

Let $t_{(1)}, t_{(2)}, ..., t_{(i)}, ..., t_{(n)}$ be ordered failure times of n identical and independent items with the MRL function $m(t)$ and the failure rate function $r(t)$. Based on Yang (1978), the empirical MRL function can be estimated by (2.6). We quote (2.6) here for reference.

$$\hat{m}(t) = \frac{1}{n - l(t)} \sum_{i=l(t)+1}^{n} t_{(i)} - t, \text{ for } 0 \le t < t_n. \tag{2.6}$$

where $l(t) = \max\{i : t_i \le t\}$.

Based on the plot of $\hat{m}(t_{(i)})$, $i = 1, ..., n$, we can graphically find two random variables T_1 and T_2, which follow parametric distributions such that the related MRL functions $m_1(t)$ and $m_2(t)$ satisfy the conditions described in Corollary 5.1 at least in the interval $[0, t_{(n)}]$. Then together with (2.6) and (5.6) or (5.7) would yield the bounds for the failure rate function $r(t)$. The procedure is summarized as follows.

Estimation procedure

Step 1: Use the data to compute the empirical MRL function $\hat{m}(t)$ by (2.6);

Step 2: Determine the trend of $\hat{m}(t)$;

Step 3: Find T_1 and T_2 such that the conditions in Corollary 5.1 are satisfied;

Step 4: Compute the lower and upper bounds by (5.6) or (5.7).

Remark 5.1: Because $\hat{m}(t)$ is not a smooth curve but $m_1(t)$ and $m_2(t)$ are smooth everywhere, $m_1(t) - \hat{m}(t)$ and $\hat{m}(t) - m_2(t)$ cannot be increasing over time. Hence the choosing of T_1 and T_2 can be empirically done. Alternatively, the trend of $m_1(t) - \hat{m}(t)$ and $\hat{m}(t) - m_2(t)$ can be determined by some statistical tests.

5.3.3 Simulation results and sensitivity analysis

- **Simulation results**

Simulations are conducted on a data set of sample size 50 generated from Weibull distribution with scale parameter 1 and shape parameter 3.6, i.e. Weibull ($\alpha = 1$, $\beta = 3.6$). Let T_1 and T_2 follow Weibull ($\alpha = 0.8$, $\beta = 1.9$) and Weibull ($\alpha = 2$, $\beta = 7$). The plots of the empirical MRL function as well as the MRL function of T_1 and T_2 are shown in Figure 5.3. By (5.1), we can compute the upper and lower bounds of $r(t)$ for the target Weibull ($\alpha = 1, \beta = 3.6$), which are plotted in Figure 5.4.

The figures show that the new method does well in estimating the bounds for the failure rate function, except the later part of the lower bound. From Figure 5.4, it is found that the lower bound for DMRL may be inaccurate when t is large (refer to the dashed line when $t > 1.05$). This is because that $\hat{m}(t)$ tends to 0 as $t \to \infty$ in most situations and thus it becomes very difficult to find a random variable T_2 such that $\hat{m}(t) - m_2(t)$ is an increasing function for large t.

empirical $m(t)$: ———   upper bound: ———   lower bound: - ·· - ·· -
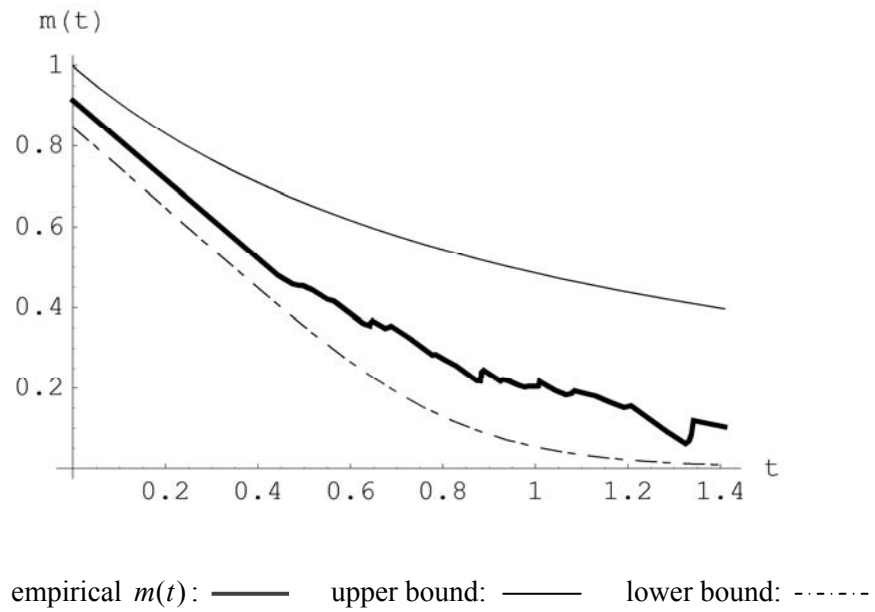
Figure 5.3   **The plotting of the empirical MRL function and the MRL functions of T_1 and T_2 for DMRL class.**

real $r(t)$: ———   upper bound: ———   lower bound: - · - ·· -
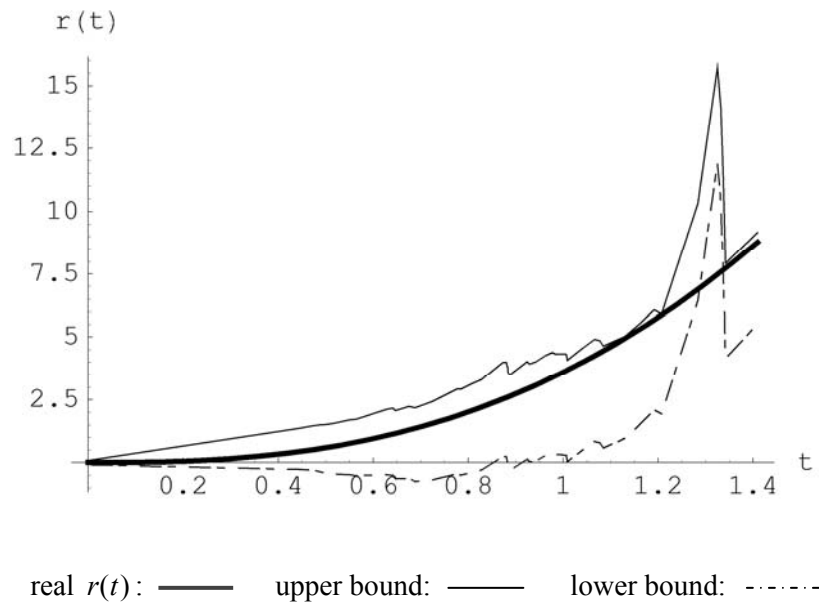
Figure 5.4   **The plotting of the real failure rate function and the estimated upper bound and lower bound for DMRL class.**

It is also worth noting that $m_1(t) - \hat{m}(t)$ and $\hat{m}(t) - m_2(t)$ are not increasing everywhere, as shown in Figure 5.3, but the final result is not affected. A possible

explanation is that $m_1(t) - m(t)$ and $m(t) - m_2(t)$ really increase with time t, as the

real MRL function $m(t)$ is a smooth function.

Simulations are also conducted on data obtained from an IMRL distribution,

Weibull ($\alpha = 1$, $\beta = 0.6$). And T_1 and T_2 are Weibull ($\alpha = 0.28$, $\beta = 0.8$) and

Weibull ($\alpha = 4.5$, $\beta = 0.3$) respectively. The corresponding plots are shown in

Figure 5.5 and Figure 5.6.

These two figures also indicate that the new method could provide good

performance in estimation of bounds for IMRL class, except the later part of the upper

bound. The reason for this phenomenon may be due to the large variation of the tail of

the empirical MRL function for IMRL, which is caused by too few data points that

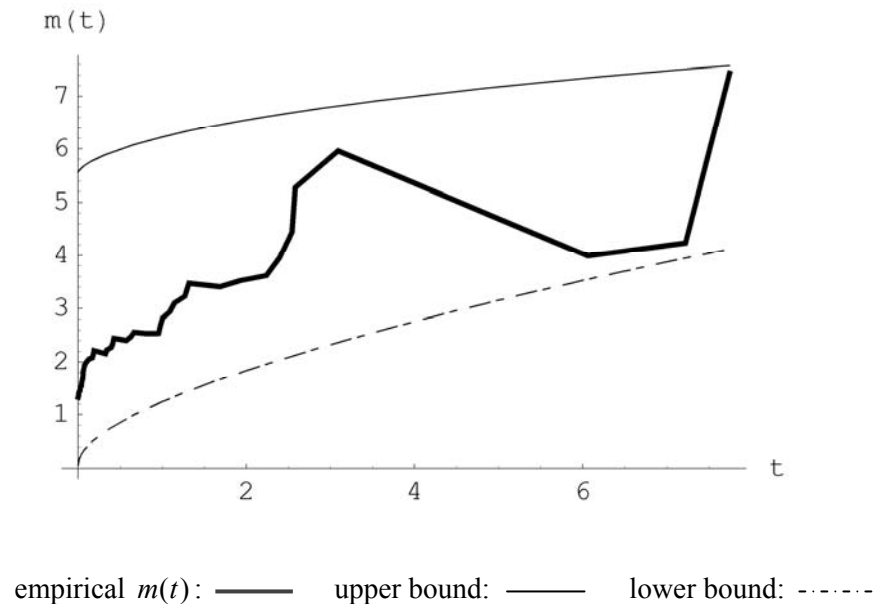can be collected for large t.

empirical $m(t)$: ⸺ upper bound: ⸺ lower bound: - ·· - ·· -

Figure 5.5 **The plotting of the empirical MRL function and the MRL functions of T_1 and T_2 for IMRL class.**

real $r(t)$: ———   lower bound: ———   upper bound: - - ·- - ·-
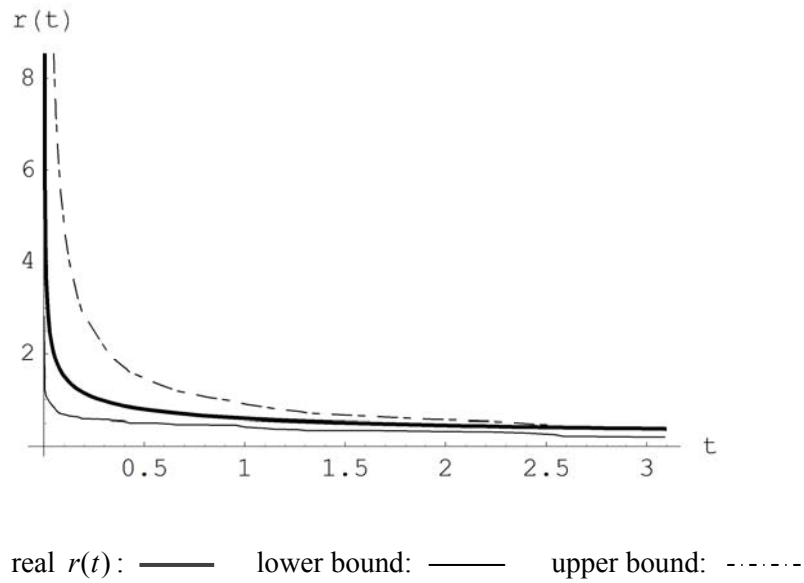
Figure 5.6   The plotting of the real failure rate function and the estimated upper bound and lower bound for IMRL class.

• **Sensitivity analysis**

In fact, the two random variables T_1 and T_2 can be arbitrarily chosen and thus infinite pairs of bounds can be constructed. So it is helpful and useful to analyze the influence of the choices of T_1 and T_2 on the estimation results. The following three sets of figures graphically show how the choosing of T_1 and T_2 affects the estimation of the bounds for the failure rate function. In these figures, two gray lines are added, compared to Figure 5.3 and Figure 5.4, to represent another choice of T_1 and T_2. For convenience, we denote the new T_1 and T_2 by T_1^* and T_2^* respectively, to distinguish from $T_1 =$ Weibull ($\alpha = 0.8$, $\beta = 1.9$) and $T_2 =$ Weibull ($\alpha = 2$, $\beta = 7$) defined in the previous. Also denote by $m_1^*(t)$ and $m_2^*(t)$ the related MRL functions.

In Figure 5.7, the MRL functions of T_1 and T_1^* are almost parallel and

$m_1(t) < m_1^*(t)$. Also it can be seen that the corresponding upper bound estimated from $m_1(t)$ is closer to the real failure rate function than the bound estimated from $m_1^*(t)$. This observation indicates that the distance between $m_1(t)$ and $\hat{m}(t)$ has an effect on the width of the bounds: the nearer $m_1(t)$ is to $\hat{m}(t)$, the narrower the bound will be. Similar conclusion also can be drawn from the lower bounds related to T_2 and T_2^*.
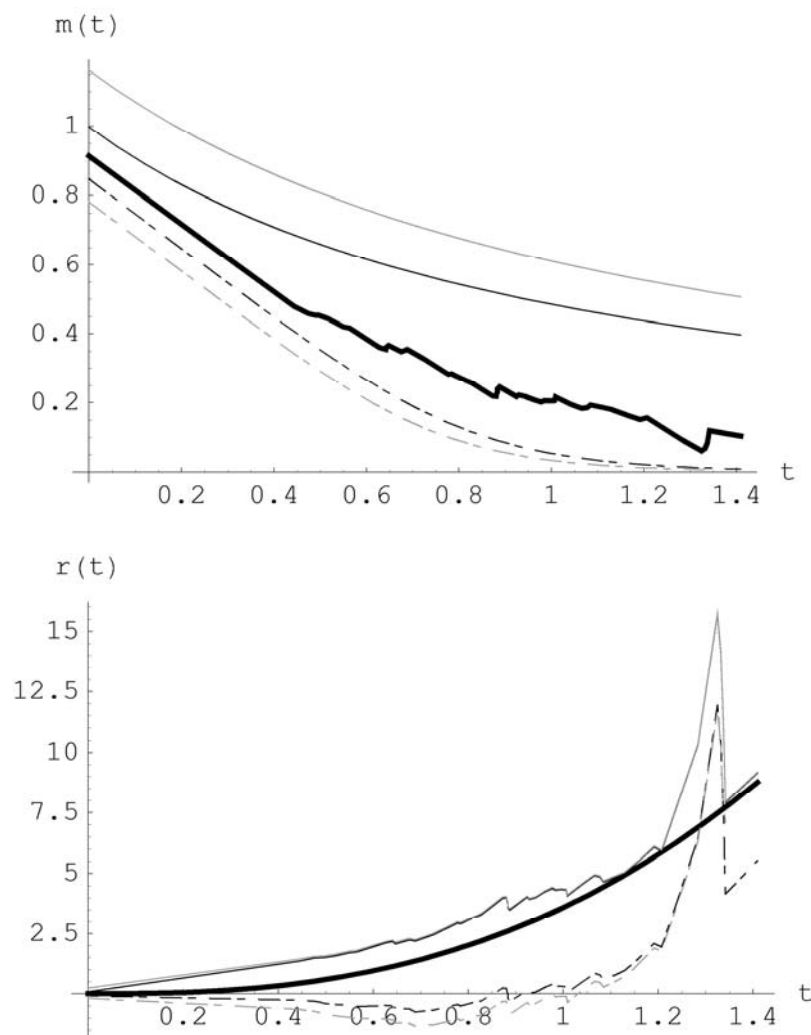
Figure 5.7   Parallel MRL and the associated failure rate functions.

Figure 5.8 and Figure 5.9 show the relation between the slope of the MRL function and the width of the bounds. In both figures, $m_1(t)$ is less than $m_1^*(t)$, i.e.

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$m_1(t) < m_1^*(t)$. From Figure 5.8, we can find that $m_1(t) - \hat{m}(t)$ increases less rapidly

than $m_1^*(t) - \hat{m}(t)$ and the bound estimated from $m_1(t)$ is narrower than that

estimated from $m_1^*(t)$.
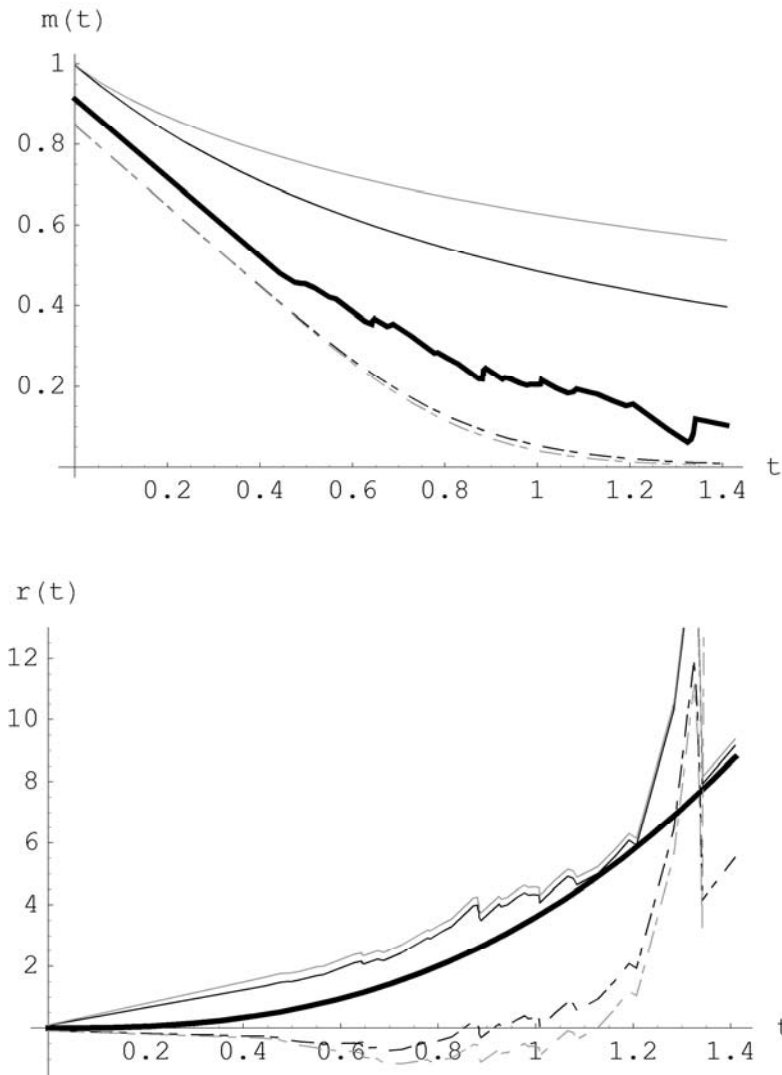
Figure 5.8   MRL with the same value at time 0 and the associated failure rate functions.

A similar situation is also shown in Figure 5.9: $m_1(t) - \hat{m}(t)$ increases more rapidly

but the bound estimated from $m_1(t)$ is wider. Upon these findings, we may conclude

that a lower increase speed of $m_1(t) - \hat{m}(t)$ implies a narrower bound. The above

analysis suggests that it is better to choose such T_1 and T_2 that their MRL functions are close to $\hat{m}(t)$ and the distances, $m_1(t) - \hat{m}(t)$ and $\hat{m}(t) - m_2(t)$, increase relatively slowly with time t.

Figure 5.9   **MRL with the same value at time** $t_{(n)}$ **and the associated failure rate functions.**

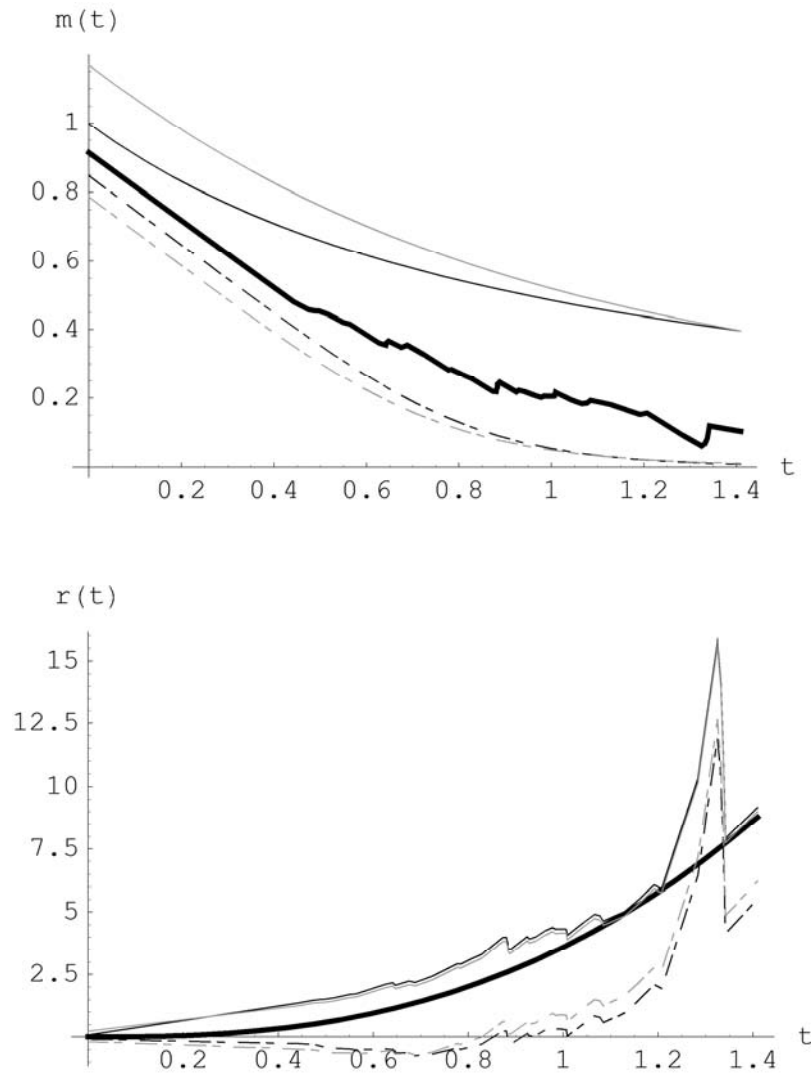Moreover, it is worth noting that the differences of the bounds, which are constructed by different T_1 and T_2, are very small. This means that to some extent, the new method is robust. Hence this new method is favorable in the inference of the

failure rate function.

5.4   Conclusion

This chapter focuses on the relationship between the MRL and the failure rate functions as these two characteristics are closely related to each other. We discuss the effect that the change of one characteristic has on the other characteristic and proposes some inequalities to quantify the range of change. The results show that (1) the change of MRL can be related to the extreme value of the MRL function and the failure rate function; (2) the range that failure rate function varies has a link with the MRL function and the derivative of MRL function. Based on these results, when exact information of the two functions is unavailable, upper and lower bounds for MRL function and failure rate function could be obtained based the inequalities that are shown in theorems proposed in this chapter.

In particular, based on the inequalities, an estimation method was introduced to estimate bounds for the failure rate function based on empirical MRL function. The simulation results indicate good and robust performance of this new approach. However, errors caused by arbitrarily choosing of the two random variables are unavoidable. Therefore, further research may focus on how to examine these errors from the statistical point of view.

CHAPTER 6   CHANGE POINT OF MEAN RESIDUAL LIFE OF SERIES AND PARALLEL SYSTEMS[3]

The foregoing chapters are mainly concerned with the MRL of single items or a system that is treated as a whole. However, because a system is often complex and composed of several components, its reliability is in fact determined by its inner structure, which refers to the allocation and the properties of components. Therefore, it is of great importance to analyze how a particular structure impacts the reliability of a system. In this chapter, we will focus on the change point of the MRL for series and parallel systems, and investigate the effect that system structure has on the location of change point. The study of the change point, at which the MRL changes the trend, is important, as its location provides useful information on the most reliable time of an item.

6.1   Introduction

In coherent systems (Triantafyllou & Koutras, 2008), series systems and parallel systems are two most important structures, as they are fundamental in the design of most complex systems that are common in real life. These two basic structures have been extensively investigated in reliability evaluation. In this chapter, we focus on the MRL of series systems and parallel systems, as the MRL is a key measure in reliability analysis since it represents how much longer an item will work for (Lai & Xie, 2006; Navarro & Hernandez, 2008).

Because systems are composed of components, an interesting question is how a system is related to its components in terms of MRL. Several papers discussed this problem from the angle of preservation properties for series and parallel systems. Abouammoh & El-Neweihi (1986) showed that parallel systems inherit DMRL from components, and the reversed preservation ageing properties for series and parallel systems were discussed in Li & Yam (2005), Belzunce et al. (2007a), and Li & Xu (2008). The current work focuses on the systems with independent and identically distributed (i.i.d.) components. We investigate the relationship between the systems and their components by comparing the change points of their respective MRLs. The change point, at which the MRL reaches its maxima or minima and begins to change the trend, provides useful information on the most reliable time of an item (Block et al., 1999), i.e., the time at which the item has maximum reliability.

In literature, there are many papers on the location of the change point of MRL. Gupta and Akman (1995a) showed that the change point of the MRL is located before that of the failure rate function. This issue was also discussed in Tang et al. (1999),

Finkelstein (2002), Bradley & Gupta (2003), and Mi (2004). Recently, Belzunce et al. (2007b) found that an extra constant risk will postpone the change point of MRL, and some numerical examples were provided in Bebbington et al. (2008). Motivated by these works, the purpose of this chapter is to extend their research to system structures and to investigate the effect of the series and parallel structures on the position of the change point of the MRL function.

Assume that all components in a system are i.i.d. and have UBMRL and BFR. This chapter explores the relation between the change points of the systems' MRL and of the components' MRL with respect to the location. After presenting the concepts and general results on MRL for series and parallel systems in Section 6.2, it is shown in Section 6.3 that the change point of the MRL for series systems is located after the change point for single components; but for parallel systems, the situation is opposite – the change point of a parallel system occurs prior to that for its components. Furthermore, we find that the difference in the positions of the change points between systems and components increases with the number of components. For illustration, an example is given in Section 6.4. In addition, a brief graphical study on a parallel system with two independent but non-identically distributed components is executed in Section 6.5. Finally, Section 6.6 gives some concluding remarks.

6.2   Definitions and background

Suppose that a system consists of n ($n > 1$) independent and identically distributed components. Let $T_1, T_2, ..., T_n$ be the lifetimes of n components and assume that they

are continuous and non-negative random variables with reliability function $R(t)$. If $E(T_i) < \infty$, the MRL for components, denoted by $m(t)$, is given by (2.1).

$$m(t) = E(T_i - t \mid T_i > t) = \frac{\int_t^\infty R(x)dx}{R(t)}.$$

The associated failure rate function (2.3), given the density function $f(t)$, is

$$r(t) = \frac{f(t)}{R(t)}.$$

Suppose that both $m(t)$ and $r(t)$ are differentiable, then we have the equation (2.4).

$$m'(t) = m(t)r(t) - 1.$$

It is well-known that the MRL and the failure rate function are equivalent to each other and also to the reliability function. Hence, both the MRL and the failure rate functions are able to uniquely determine the distribution of the lifetime of items. However, these two functions usually have opposite monotonic trends and represent the aging behavior of a component from different points of view. For example, an increasing failure rate function implies a decreasing MRL function.

6.2.1 MRL of series system

Let $T_{1:n}, T_{2:n}, ..., T_{n:n}$ be ordered lifetimes of n components. As a series system is defined as a system which functions if and only if all components function, then its lifetime can be represented by the first ordered statistic $T_{1:n}$. So the reliability function of the series system, denoted by $R_S(t)$, is

$$R_S(t) = \Pr(T_{1:n} > t) = (R(t))^n, \quad t \geq 0. \tag{6.1}$$

The MRL function and the failure rate function, denoted by $M_S(t)$ and $r_S(t)$ respectively, are given by

$$M_S(t) = E(T_{1:n} - t \mid T_{1:n} > t) = \frac{\int_t^\infty R_S(x)dx}{R_S(t)}$$

$$= \frac{\int_t^\infty (R(x))^n \, dx}{(R(t))^n}, \tag{6.2}$$

and

$$r_S(t) = n\frac{f(t)}{R(t)} = nr(t). \tag{6.3}$$

6.2.2   MRL of parallel system

A parallel system is a collection of components which works if and only if at least one component works. Hence, the lifetime of parallel systems is actually equal to the nth ordered component failure time, which is $T_{n:n}$. Thus, the corresponding reliability function, denoted by $R_P(t)$, is

$$R_P(t) = \Pr(T_{n:n} > t) = 1 - (1 - R(t))^n. \tag{6.4}$$

The related MRL function of parallel systems, denoted by $M_P(t)$, is represented by

$$M_{\mathrm{P}}(t) = E(T_{n:n} - t \mid T_{n:n} > t) = \frac{\int_t^\infty R_{\mathrm{P}}(x)\,dx}{R_{\mathrm{P}}(t)}$$

$$= \frac{\int_t^\infty \left(1 - (1 - R(x))^n\right)dx}{1 - (1 - R(t))^n}. \tag{6.5}$$

It can be seen that $M_{\mathrm{P}}(t)$ is defined under the condition that the whole system does not fail at time t. If we consider the condition that none of the components fails before time t, another definition of the MRL function of parallel systems can be proposed. This new MRL function, denoted by $M_{\mathrm{P}}^1(t)$, is defined as the expectation of remaining life of a parallel system given no failed component by time t. Mathematically, this MRL function is represented as

$$M_{\mathrm{P}}^1(t) = E(T_{n:n} - t \mid T_{1:n} > t)$$

$$= \int_t^\infty \left(1 - \left(1 - \frac{R(x)}{R(t)}\right)^n\right)dx. \tag{6.6}$$

For detailed discussion of $M_{\mathrm{P}}^1(t)$, see Bairamov et al. (2002) and Asadi & Bayramoglu (2005).

6.3 The change points of mean residual life of systems

Suppose the lifetimes T_i, $i = 1,...,n$, of components have an UBMRL with the change point t_0, and a bathtub-shaped failure rate with the critical point v_0. This means, for $t_0 > 0$ and $v_0 > 0$, the derivatives of $m(t)$ and $r(t)$ satisfy

$$m'(t) > 0, \text{ for } t \in [0, t_0), \ m'(t) = 0, \ t = t_0, \ m'(t) < 0, \text{ for } t > t_0, \tag{6.7}$$

$$r'(t) < 0, \text{ for } t \in [0, v_0), \ r'(t) = 0, \ t = v_0, r'(t) > 0, \text{ for } t > v_0.$$

Because the change point of the MRL must be smaller than the critical point of the failure rate function (Gupta & Akman, 1995a), we have $t_0 < v_0$.

As systems are composed of components, it is natural to believe that the properties of the MRL function of systems should be associated to the properties of MRL of components. In this section, we explore how the redundancy makes an impact on the position of the change point of the MRL. In the following, the MRL of series systems, $M_S(t)$, and the MRL of parallel systems, $M_P(t)$ and $M_P^1(t)$, will be investigated and compared to t_0, which is the change point of $m(t)$.

6.3.1   The change point of the MRL for series systems

The MRL and the failure rate function of series systems are defined in (6.2) and (6.3). As shown in (6.3), the failure rate function of series systems $r_S(t)$ is proportional to $r(t)$; hence $r_S(t)$ is also of bathtub shape and with the same critical point of $r(t)$, namely v_0. Because the change point of MRL must be before the critical point of failure rate, the change point of $M_S(t)$ must be less than v_0. A question is where the change point of $M_S(t)$ is compared to the location of t_0. The following theorem proves that the change point of the MRL for series systems is located between t_0 and v_0.

Theorem 6.1 *Suppose that the MRL of components $m(t)$ is of upside-down bathtub*

shape with change point t_0, and the failure rate function, $r(t)$, is of bathtub shape with critical point v_0. Then the change point of the MRL of series systems $M_S(t)$ exists and is larger than t_0.

Proof: First, we prove that the derivative of $M_S(t)$ is positive at t_0. From (6.2), we have

$$M_S(t) \cdot R(t)^n = \int_t^\infty R(x)^n \, dx = \int_t^\infty R(x)^{n-1} d\left(-\int_x^\infty R(u) \, du\right)$$

$$= R(x)^{n-1}\left(-\int_x^\infty R(u) \, du\right)\bigg|_t^\infty - \int_t^\infty (n-1)R(x)^{n-2}(-f(x))\left(-\int_x^\infty R(u) \, du\right) dx \quad (6.8)$$

$$= R(t)^{n-1}\int_t^\infty R(u) \, du - \int_t^\infty (n-1)R(x)^n r(x)m(x) \, dx.$$

Dividing both sides of (6.8) by $R(t)^n$, we obtain

$$M_S(t) = m(t) - \int_t^\infty (n-1)\left(\frac{R(x)}{R(t)}\right)^n m(x)r(x) \, dx. \quad (6.9)$$

Taking the derivative of (6.9) with respect to t yields

$$M_S'(t) = M_S(t) \cdot nr(t) + m'(t) - m(t)r(t). \quad (6.10)$$

Because t_0 is the change point of $m(t)$, that is, $m'(t_0) = 0$, we have

$$M_S'(t_0) = M_S(t_0) \cdot nr(t_0) - m(t_0)r(t_0)$$
$$= (n \cdot M_S(t_0) - m(t_0)) \cdot r(t_0). \quad (6.11)$$

Substituting (6.9) into (6.11) produces

$$M'_{S}(t_0) = (n-1)\left(M_{S}(t_0) - \int_{t_0}^{\infty}\left(\frac{R(x)}{R(t_0)}\right)^{n} m(x)r(x)dx \right)\cdot r(t_0)$$

$$= (n-1)\left(\int_{t_0}^{\infty}\left(\frac{R(x)}{R(t_0)}\right)^{n}(1-m(x)r(x))dx \right)\cdot r(t_0) \qquad (6.12)$$

$$= -(n-1)\left(\int_{t_0}^{\infty}\left(\frac{R(x)}{R(t_0)}\right)^{n} m'(x)dx \right)\cdot r(t_0).$$

The MRL $m(t)$ is decreasing for $t > t_0$; that is, $m'(t) < 0$ for $t > t_0$. So we obtain

$$M'_{S}(t_0) > 0. \qquad (6.13)$$

Next, we prove the existence of the change point. It is assumed that the failure rate function of single components, $r(t)$, is bathtub-shaped with critical point v_0. Then we have that the failure rate of systems $r_{S}(t)$ is also bathtub-shaped, decreasing from time 0 to v_0 and then increasing for $t > v_0$. From Tang et al. (1999), we have that $M_{S}(t)$ is a decreasing function on (v_0, ∞) and has at most one change point.

From this together with (6.13), we can conclude that there must be a unique change point of $M_{S}(t)$ between t_0 and v_0. ∎

Theorem 6.1 provides the upper and lower bounds for the change point of the MRL for series systems, which are v_0 and t_0 respectively. It also shows that the series structure postpones the change point of MRL. This means that the change point of the MRL for multi-component series systems must be located after the change point for a single component. Considering a single component as a degenerate series

system, we can find that the increase of the number of components will make the change point of the MRL occur later. This is shown in the next corollary.

Corollary 6.1 *The change point of $M_S(t)$ increases with n.*

Proof: The proof of Theorem 6.1 is in fact valid not only for integers $n > 1$, but also for any real numbers, n, greater than 1. Hence, this proposition can be considered in this way: for arbitrarily chosen positive integers n and m ($m > n > 1$), if we treat the system consisting of n components as a "new" component, then the system composed of m components can be regarded as a system consisting m/n the "new" components. Applying Theorem 6.1 yields the results. ∎

6.3.2 The change point of the MRL for parallel systems

We now consider the MRL functions of parallel systems, $M_P(t)$ and $M_P^1(t)$. The following Theorem 6.2 proves that the change point of the MRL for parallel systems $M_P(t)$, if it exists, is located prior to t_0.

Theorem 6.2 *Suppose that $m(t)$ is of upside-down bathtub shape with the change point t_0, and $r(t)$ is of bathtub-shape with critical point v_0. Then the MRL of parallel systems, $M_P(t)$, is strictly decreasing on $[t_0, \infty)$. This means that the change point of $M_P(t)$, if it exists, is smaller than t_0.*

Proof: See the Section 6.3.3 for the proof. ∎

As shown in Corollary 6.2, the change point of the MRL for parallel systems occurs earlier with an increase of the number of components. This result, together with Corollary 6.1, indicates that series and parallel structures have opposite effects on the location of the change point.

Corollary 6.2 The change point of $M_P(t)$, if it exists, decreases with n.

Proof: The proof is similar to that of Corollary 6.1. ∎

As shown in the following Theorem 6.3, the generalized MRL of a parallel system, $M_P^1(t)$, also has its change point located before t_0. This result is consistent with Theorem 6.2 and hence supports the fact that the parallel structure can bring forward the change point of the MRL.

Theorem 6.3 *Suppose that $m(t)$ is of upside-down bathtub shape with change point t_0, and $r(t)$ is of bathtub-shape with critical point v_0. Then the generalized MRL of parallel systems, $M_P^1(t)$, is strictly decreasing on $[t_0, \infty)$, that is, the change point of $M_P^1(t)$, if it exists, is smaller than t_0.*

Proof: From (2.9)-(2.11) in Asadi & Bayramoglu (2005), we have

$$\frac{dM_P^1(t)}{dt} = r(t)\frac{\int_t^\infty n(R(t) - R(x))^{n-1}R(x)dx}{R(t)^n} - 1$$

$$= m'(t) + r(t)\left(\frac{\int_t^\infty n(R(t) - R(x))^{n-1}R(x)dx}{R(t)^n} - m(t)\right)$$

(6.14)

As $m'(t) \leq 0$ for $t \in [t_0, \infty)$, the above equation can be written as

$$\frac{dM_P^1(t)}{dt} \cdot \frac{1}{r(t)} \leq \int_t^\infty n\left(1 - \frac{R(x)}{R(t)}\right)^{n-1}\frac{R(x)}{R(t)}dx - m(t) \qquad (6.15)$$

The failure rate function is defined to be positive; that is, $r(t) > 0$. Hence, to prove that $M_P^1(t)$ is strictly decreasing on $[t_0, \infty)$, we only need to prove that the right side of (6.15) is less than 0 for any t within $\{t : t_0 < t < \infty\}$.

From (6.6), we have

$$M_P^1(t) = \int_t^\infty \left(1 - \left(1 - \frac{R(x)}{R(t)}\right)^n\right)dx = \int\left(1 - \left(1 - \frac{R(x)}{R(t)}\right)^n\left(-\frac{1}{R(x)}\right)\right)d\left(\int_x^\infty R(u)du\right)$$

$$= \left(1 - \left(1 - \frac{R(x)}{R(t)}\right)^n\right)\left(-\frac{1}{R(x)}\right)\int_x^\infty R(u)du\Big|_t^\infty$$

$$- \int_t^\infty \left(-n\left(1 - \frac{R(x)}{R(t)}\right)^{n-1}\frac{f(x)}{R(t)}\left(\frac{-1}{R(x)}\right) + \left(1 - \left(1 - \frac{R(x)}{R(t)}\right)^n\right)\frac{-f(x)}{R(x)^2}\right)\int_x^\infty R(u)du\,dx$$

$$\Rightarrow$$

$$M_P^1(t) = m(t) - \int_t^\infty \left(n\left(1 - \frac{R(x)}{R(t)}\right)^{n-1}\frac{R(x)}{R(t)} - \left(1 - \left(1 - \frac{R(x)}{R(t)}\right)^n\right)\right)r(x)m(x)\,dx. \quad (6.16)$$

By using the expression for $m(t)$ from (6.16), the right-hand side of (6.15) becomes

$$\int_t^\infty n\left(1-\frac{R(x)}{R(t)}\right)^{n-1}\frac{R(x)}{R(t)}dx - m(t)$$

$$= \int_t^\infty \left(n\left(1-\frac{R(x)}{R(t)}\right)^{n-1}\frac{R(x)}{R(t)}-\left(1-\left(1-\frac{R(x)}{R(t)}\right)^n\right)\right)(1-r(x)m(x))dx \qquad (6.17)$$

$$= \int_t^\infty \left(\left(1-\frac{R(x)}{R(t)}\right)^n + n\left(1-\frac{R(x)}{R(t)}\right)^{n-1}\frac{R(x)}{R(t)}-1\right)\cdot(-m'(x))dx.$$

Note that $0 < R(x)/R(t) \le 1$ for $t \le x < \infty$. So applying Lemma 1 yields (see Section 6.3.3)

$$\left(1-\frac{R(x)}{R(t)}\right)^n + n\left(1-\frac{R(x)}{R(t)}\right)^{n-1}\frac{R(x)}{R(t)}-1 < 0, \ t \le x < \infty.$$

Furthermore, as t_0 is the change point of $m(t)$, we have $-m'(x) > 0$ for $t_0 < t \le x < \infty$. Therefore, it follows that the last line of (6.17) is less than 0. Equivalently, the right-hand side of (6.15) is also less than 0 for $t > t_0$; that is

$$\int_t^\infty n\left(1-\frac{R(x)}{R(t)}\right)^{n-1}\frac{R(x)}{R(t)}dx - m(t) < 0, \ t_0 \le t < \infty. \qquad \blacksquare$$

Remark 6.1: The change point of either $M_P(t)$ or $M_P^1(t)$ may be non-existent. The non-existence of the change point implies that the MRL function decreases with time t along the entire time axis; that is, DMRL. Moreover, the change point may not be unique. This means that the MRL function may be of roller-coaster shape, with increasing and decreasing segments appearing alternately.

Remark 6.2: Corollary 6.1 and 6.2 show that, for both systems, the distance between

the change points of the MRL for systems and for components increases with an increase in the number of components.

6.3.3 Proof of Theorem 6.2

Lemma 6.1 *For* $\alpha > 1$,

$$(1-x)^{\alpha} + \alpha(1-x)^{\alpha-1}x - 1 < 0, \ \ for \ x \in (0,1]$$.

Proof: Let $g(x) = (1-x)^{\alpha} + \alpha(1-x)^{\alpha-1}x - 1$. Taking derivative on $g(x)$ with respect to x yields,

$$g'(x) = -\alpha(\alpha-1)(1-x)^{\alpha-2}x.$$

Because $0 \le x \le 1$, we have $g'(x) < 0$. That is, $g(x)$ is a decreasing function on $[0, 1]$. Note that $g(0) = 0$. So we get $g(x) < 0$ for $0 < x \le 1$. ∎

Lemma 6.2 *For* $\alpha > 1$,

$$(\alpha-1)(1-x)^{\alpha-2}x - (1-x)^{\alpha-1} + \frac{\alpha(1-x)^{2\alpha-2}x}{1-(1-x)^{\alpha}} > 0, \ \ for \ x \in (0,1).$$

Proof: Note that, as $0 < 1-x < 1$, we have

$$(\alpha - 1)(1 - x)^{\alpha - 2} x - (1 - x)^{\alpha - 1} + \frac{\alpha(1 - x)^{2\alpha - 2} x}{1 - (1 - x)^{\alpha}} > 0$$

$$\Leftrightarrow (\alpha - 1)(1 - x)^{\alpha - 2} x \left(1 - (1 - x)^{\alpha} \right) - (1 - x)^{\alpha - 1} \left(1 - (1 - x)^{\alpha} \right) + \alpha(1 - x)^{2\alpha - 2} x > 0$$

$$\Leftrightarrow (\alpha - 1)(1 - x)^{\alpha - 2} x + (1 - x)^{2\alpha - 2} x - (1 - x)^{\alpha - 1} + (1 - x)^{2\alpha - 1} > 0$$

$$\Leftrightarrow (\alpha - 1)(1 - x)^{\alpha} x + (1 - x)^{2\alpha} x - (1 - x)^{\alpha + 1} + (1 - x)^{2\alpha + 1} > 0$$

$$\Leftrightarrow (\alpha - 1) x + (1 - x)^{\alpha} x - (1 - x) + (1 - x)^{\alpha + 1} > 0$$

$$\Leftrightarrow \alpha x - 1 + (1 - x)^{\alpha} > 0$$

Let $h(x) = \alpha x - 1 + (1 - x)^{\alpha}$. Taking derivative on $h(x)$ with respect to x yields,

$$h'(x) = \alpha - \alpha(1 - x)^{\alpha - 1}.$$

Because $0 < x < 1$ and $\alpha > 1$, we have $0 < (1 - x)^{\alpha - 1} < 1$. So $h'(x) > 0$. That is, $h(x)$ is an increasing function on $(0, 1)$. As $h(0) = 0$, $h(x) > 0$ holds for $0 < x < 1$. ∎

Proof of Theorem 6.2

Taking derivative of (6.5) with respect to t, we get

$$\frac{dM_P(t)}{dt} = \frac{\int_t^{\infty} \left(1 - (1 - R(x))^n \right) dx}{1 - (1 - R(t))^n} \cdot \frac{n(1 - R(t))^{n-1} R(t)}{1 - (1 - R(t))^n} r(t) - 1$$

$$= m'(t) + r(t) \left(\frac{\int_t^{\infty} \left(1 - (1 - R(x))^n \right) dx}{1 - (1 - R(t))^n} \cdot \frac{n(1 - R(t))^{n-1} R(t)}{1 - (1 - R(t))^n} - m(t) \right). \qquad (6.18)$$

As $m'(t) \leq 0$ for $t \in [t_0, \infty)$, the above equation can be written as,

$$\frac{dM_P(t)}{dt} \cdot \frac{1}{r(t)} \leq \frac{\int_t^{\infty} \left(1 - (1 - R(x))^n \right) dx}{1 - (1 - R(t))^n} \cdot \frac{n(1 - R(t))^{n-1} R(t)}{1 - (1 - R(t))^n} - m(t). \qquad (6.19)$$

Note that $r(t) > 0$. Hence, to prove that $M_P(t)$ is strictly decreasing on $[t_0, \infty)$ is equivalent to prove the right hand side of (6.19) < 0 for any t in $\{t : t_0 < t < \infty\}$.

Based on (6.5), we have

$$
\begin{aligned}
M_P(t) \cdot \left(1 - (1 - R(t))^n\right) &= \int_t^\infty 1 - (1 - R(x))^n \, dx \\
&= \left(1 - (1 - R(t))^n\right) \frac{-1}{R(x)} \int_x^\infty R(u) du \Big|_t^\infty \\
&\quad - \int_t^\infty \frac{-n(1 - R(x))^{n-1}(- R(x))f(x) - \left(1 - (1 - R(x))^n\right)f(x)}{(R(x))^2} \int_x^\infty R(u) du \, dx \\
&= \left(1 - (1 - R(t))^n\right) m(t) - \int_t^\infty \left(n(1 - R(x))^{n-1} R(x) - \left(1 - (1 - R(x))^n\right)\right) m(x) r(x) \, dx.
\end{aligned}
$$

Dividing both sides of the above equation by $1 - (1 - R(t))^n$, we obtain

$$
M_P(t) = m(t) - \frac{\int_t^\infty \left(n(1 - R(x))^{n-1} R(x) - \left(1 - (1 - R(x))^n\right)\right) m(x) r(x) \, dx}{1 - (1 - R(t))^n} \tag{6.20}
$$

According to Lemma 6.1, because $0 < R(x) \leq 1$,

$$
n(1 - R(x))^{n-1} R(x) - \left(1 - (1 - R(x))^n\right) < 0. \tag{6.21}
$$

Also, as t_0 is the change point of $m(t)$, we have

$$
0 < m(x)r(x) < 1 \quad \text{for } t_0 < t \leq x < \infty. \tag{6.22}
$$

From (6.20)-(6.22), we can get

$$m(t) = M_P(t) + \frac{\int_t^\infty \left(n(1-R(x))^{n-1}R(x) - \left(1-(1-R(x))^n\right)\right)m(x)r(x)\,dx}{1-(1-R(t))^n}$$

$$> M_P(t) + \frac{\int_t^\infty \left(n(1-R(x))^{n-1}R(x) - \left(1-(1-R(x))^n\right)\right)dx}{1-(1-R(t))^n} \qquad (6.23)$$

$$= \frac{\int_t^\infty n(1-R(x))^{n-1}R(x)\,dx}{1-(1-R(t))^n}.$$

Substituting the inequality of (6.23) into the right hand side of (6.19) yields

$$\frac{\int_t^\infty \left(1-(1-R(x))^n\right)dx}{1-(1-R(t))^n} \cdot \frac{n(1-R(t))^{n-1}R(t)}{1-(1-R(t))^n} - m(t)$$

$$< \frac{\int_t^\infty \left(1-(1-R(x))^n\right)dx}{1-(1-R(t))^n} \cdot \frac{n(1-R(t))^{n-1}R(t)}{1-(1-R(t))^n} - \frac{\int_t^\infty n(1-R(x))^{n-1}R(x)\,dx}{1-(1-R(t))^n} \qquad . (6.24)$$

Now we need to prove the right hand side of (6.24) is less than 0, i.e.

$$\frac{\int_t^\infty \left(1-(1-R(x))^n\right)dx}{1-(1-R(t))^n} \cdot n(1-R(t))^{n-1}R(t) - \int_t^\infty n(1-R(x))^{n-1}R(x)\,dx \le 0. \quad (6.25)$$

Note that

$$\int_t^\infty n(1-R(x))^{n-1}R(x)\,dx$$

$$= \int_t^\infty n(1-R(x))^{n-1}R(x)\frac{1}{1-(1-R(x))^n}\,d\left(-\int_x^\infty \left(1-(1-R(u))^n\right)du\right) \qquad (6.26)$$

$$= \frac{\int_t^\infty \left(1-(1-R(x))^n\right)dx}{1-(1-R(t))^n} \cdot n(1-R(t))^{n-1}R(t) - Z,$$

where

$$Z = \int_t^\infty \left[\frac{d}{dx}\left(\frac{n(1-R(x))^{n-1}R(x)}{1-(1-R(x))^n} \right) \right] \times \left(-\int_x^\infty 1-(1-R(u))^n \, du \right) dx \cdot$$

From (6.26), the inequality (6.25) is equivalent to $Z < 0$.

As $0 < R(u) < 1$, a sufficient condition for $Z < 0$ is

$$\frac{d}{dx}\left(\frac{(1-R(x))^{n-1}R(x)}{1-(1-R(x))^n} \right) > 0, \quad \text{for } t_0 < t < x < \infty. \tag{6.27}$$

By extending the derivative, we get

$$\frac{d}{dx}\left(\frac{(1-R(x))^{n-1}R(x)}{1-(1-R(x))^n} \right)$$
$$= \frac{(n-1)(1-R(x))^{n-2}R(x)f(x)-(1-R(x))^{n-1}f(x)}{1-(1-R(x))^n}$$
$$- \frac{(1-R(x))^{n-1}R(x)\left(-n(1-R(x))^{n-1}f(x)\right)}{(1-(1-R(x))^n)^2}$$
$$= \frac{f(x)}{1-(1-R(x))^n}\left((n-1)(1-R(x))^{n-2}R(x)-(1-R(x))^{n-1} + \frac{n(1-R(x))^{2n-2}R(x)}{1-(1-R(x))^n} \right).$$

From Lemma 6.2 and $0 < R(x) < 1$, we have

$$n(n-1)(1-R(x))^{n-2}R(x) - n(1-R(x))^{n-1} + \frac{n^2(1-R(x))^{2n-2}R(x)}{1-(1-R(x))^n} > 0.$$

Together with the fact that $f(x) > 0$ and $0 < R(u) < 1$, we can get (6.27) and thus

prove the theorem.　■

6.4   An illustrative example and application

In this section, an example is utilized to illustrate the theorems and their applications in burn-in test.

6.4.1   An example

Suppose that the lifetime of each component follows the modified Weibull distribution (Lai et al., 2003) with the reliability function $R(t) = \exp(-t^{0.25}e^{t})$. The corresponding MRL function for components is of upside-down bathtub shape, shown by the bold lines in the following three figures. From (6.2), (6.5) and (6.6), we obtain the MRL for series systems $M_S(t)$, the MRL for parallel systems $M_P(t)$ and the generalized MRL $M_P^1(t)$

$$M_S(t) = \frac{\int_t^{\infty} \exp(-n\,x^{0.25}e^{x})dx}{\exp(-n\,t^{0.25}e^{t})},$$

$$M_P(t) = \frac{\int_t^{\infty} \left(1 - \left(1 - \exp(-x^{0.25}e^{x})\right)^{n}\right)dx}{1 - \left(1 - \exp(-t^{0.25}e^{t})\right)^{n}},$$

$$M_P^1(t) = \int_t^{\infty} \left(1 - \left(1 - \exp(t^{0.25}e^{t} - x^{0.25}e^{x})\right)^{n}\right)dx.$$

These three MRL functions, with different number of components, are plotted in Figure 6.1, Figure 6.2, and Figure 6.3 respectively.
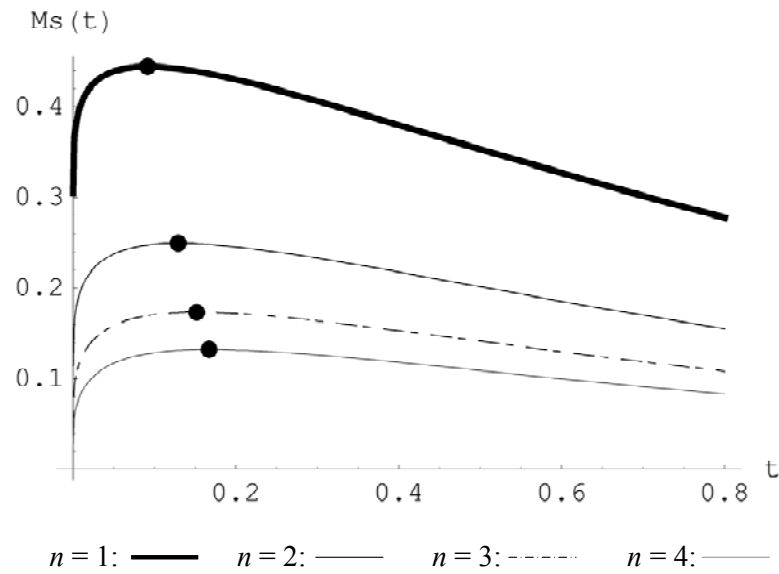
Figure 6.1   The plotting of the MRL of components (bold line) and the MRL function of series system with *n* components. The solid circles mark the locations of the change points.
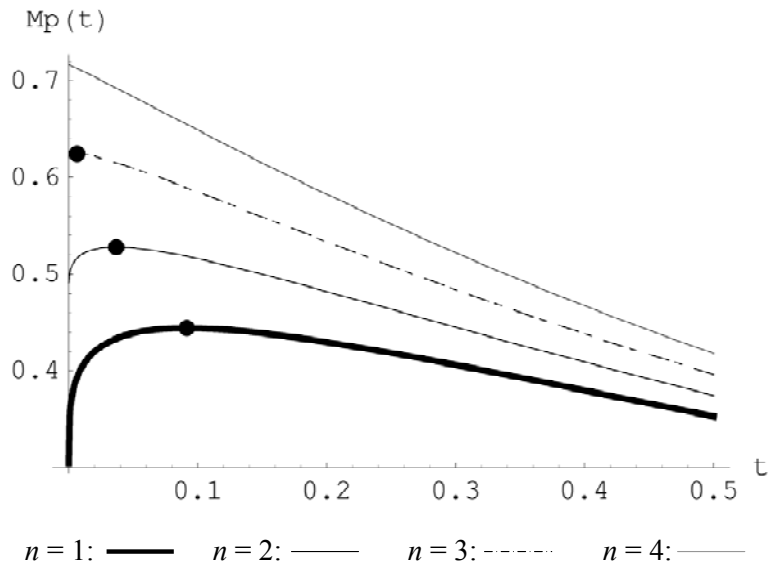

Figure 6.2   The plotting of the MRL of components (bold line) and the MRL function of parallel system with *n* components. The solid circles mark the locations of the change points.
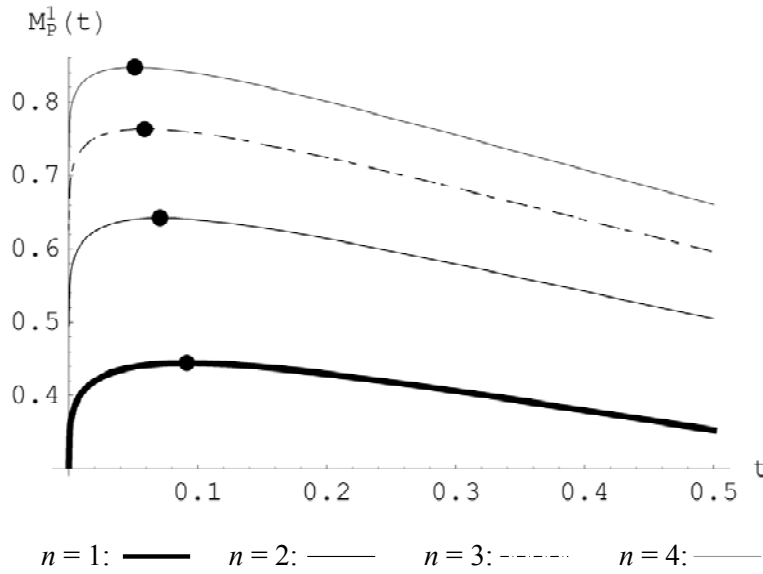
Figure 6.3 **The plotting of the MRL of components (bold line) and the generalized MRL of parallel system with *n* components. The solid circles mark the locations of the change points.**

From Figure 6.1, we see that the change point of $M_S(t)$ is located after $t_0 = 0.0914$, which is the change point of $m(t)$ (the solid circle on the bold line). Figure 6.2 and 6.3 show that the change points of both $M_P(t)$ and $M_P^1(t)$ are located prior to t_0.

These observations are consistent with the results presented in the three theorems. Moreover, as shown in Figure 6.1, the change point for series system is an increasing function of n, which is proved by Corollary 6.1. We also find from Figure 6.2 that the change point may not exist for $M_S(t)$ (with $n = 4$); but, if it exists, it will decrease with the number of components n, as stated in Corollary 6.2.

6.4.2   Some practical applications

These results are useful in decision making in reliability analysis, such as burn-in tests for a coherent system (Block & Savits, 1997). For example, consider a parallel system composed of individual components. If a system is considered to be a survivor only

when none of the components fails during the burn-in test, then $M_P^1(t)$ can be used to determine the optimal burn-in time. This is rational because no customer wants to buy a system with any faulty component. So we may suggest that by the criteria of maximizing the MRL function, the burn-in test can be terminated at the change point of $M_P^1(t)$.

Another important issue for systems in burn-in test is at which stage burn-in is most effective. For the above case, our conclusion is that burn-in at system level is better, because the change point for the system is located prior to t_0, which is the change point for a single component. Hence for the burn-in test for this system, the optimal time can be set at the change time of $M_P^1(t)$. This means that the test can be terminated if all components of this system are still functioning at the change time. Furthermore, if consider the MRL functions $M_S(t)$ and $M_P(t)$, we may suggest that burn-in tests should be carried out at component level for a series system and at system level for a parallel system.

Moreover, the optimization of component number in parallel systems is also of great interest in reliability analysis. As shown in Theorem 6.2, the greater the number of components in a parallel system is, the smaller is the change point of the MRL for the system. In other words, more components would shorten optimal burn-in time for a parallel system by the criteria of maximizing the MRL function, and also increase the reliability of the system. However, more components always mean more expense. Hence, it would be necessary and helpful to determine the optimal number of components that should be allocated in a parallel system.

Denote by c the cost of one component, c_0 the set-up cost, c_b the burn-in cost per unit burn-in time, p the profit obtained from per unit working time, and b_n^* the optimal burn-in time for an n - component parallel system. Then the net profit that an n - component parallel system could produce can be expressed by the following cost function.

$$\text{Profit}(n) = p \cdot M_{P,n}(b_n^*) - c_0 - c \cdot n - c_b \cdot b_n^*, \tag{6.28}$$

where $M_{P,n}(t)$ is the MRL function of a parallel system with n components in (6.5). Clearly, the optimal component number could be obtained by maximizing (6.28). Denote by n^* the optimal number. We have

$$n^* = \arg_n \left(\max_{n \in N} \text{Profit}(n) \right).$$

Following the example in Section 6.4.1, let $c = 0.9$, $c_0 = 2$, $c_b = 1.2$, and $p = 9.5$. Based on (6.28), a profit table corresponding to different numbers of components can be established, as shown in Table 6.1.

Table 6.1   Net profits for different component numbers.

n	b_n^*	$M_{P,n}(b_n^*)$	$\text{Profit}(n)$
1	0.0914	0.4444	1.2123
2	0.0368	0.528	1.1718
3	**0.0064**	**0.6245**	**1.2254**
4	0	0.7169	1.2107
5	0	0.7904	1.0091
6	0	0.8492	0.6674
7	0	0.8976	0.2271
8	0	0.9384	-0.2853

From the table, it can be found that the profit achieves the maximum of 1.2254 when $n = 3$. So we may conclude that under this scenario, a parallel system is suggested to be assembled with three components, and be put into a burn-in test for 0.0064 unit time before it is shipped out and sold to customers.

6.5 Parallel system with two different components

Previous discussions are based on the assumption that all components in a system are independent and identical. A straight extension is to study systems with independent but non-identical components (Zhao & Balakrishnan, 2009a; Zhao & Balakrishnan, 2009b). However, due to the different properties from component to component, it is difficult to investigate the change point of the MRL of this type of systems analytically. To obtain explicit results is almost an infeasible task, although it does play an important role in system reliability analysis. Hence, in this section, we focus on the MRL of parallel systems with only two different components and carry out a brief analysis on change point of the MRL in a graphic way.

6.5.1 Exponential distributed component

Consider a parallel system of two independent but unnecessarily identical components having respective life distributions as exponential distribution, $R_1(t) = \exp(-\lambda_1 t)$, $R_2(t) = \exp(-\lambda_2 t)$. Then the reliability function of the system is

$$R(t) = 1 - \left(1 - e^{-\lambda_1 t}\right)\left(1 - e^{-\lambda_2 t}\right).$$

The failure rate function $r(t)$ and the MRL function $m(t)$ are

$$r(t) = \frac{\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1 + \lambda_2) e^{-\lambda_1 t - \lambda_2 t}}{e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-\lambda_1 t - \lambda_2 t}},$$

$$m(t) = \frac{\frac{1}{\lambda_1} e^{-\lambda_1 t} + \frac{1}{\lambda_2} e^{-\lambda_2 t} - \frac{1}{\lambda_1 + \lambda_2} e^{-\lambda_1 t - \lambda_2 t}}{e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-\lambda_1 t - \lambda_2 t}},$$

so that the derivative of the MRL function is

$$m'(t) = m(t)r(t) - 1$$

$$= \frac{-\lambda_1^2 \lambda_2 - \lambda_1 \lambda_2^2 + \lambda_1^3 (1 - e^{-\lambda_2 t}) + \lambda_2^3 (1 - e^{-\lambda_1 t})}{\lambda_1 \lambda_2 (\lambda_1 + \lambda_2)(e^{\lambda_1 t - \lambda_2 t} + e^{-\lambda_1 t + \lambda_2 t} + e^{-\lambda_1 t - \lambda_2 t} + 2 - 2e^{-\lambda_1 t} - 2e^{-\lambda_2 t})}.$$

Barlow & Proschan (1981b) (Page 83) showed that the failure rate function of this parallel systems is of upside-down bathtub shape (UBFR) for $\lambda_1 \neq \lambda_2$. According to Theorem 2.2, because $m'(0) = -1 < 0$ or $m(0)r(0) = 0 < 1$, the MRL function $m(t)$ is of bathtub shape, i.e. BMRL. Denote by t_0 the change point of $m(t)$. Figure 6.4 plots the location of t_0 for different combinations of λ_1 and λ_2. From Figure 6.4, we can find that, the location of t_0 is small when there is large difference between λ_1 and λ_2; if λ_1 and λ_2 become close to each other, then the change point t_0 tends to infinity. This is because, when $\lambda_1 = \lambda_2$, $r(t)$ is an increasing function and $m(t)$ is a decreasing function, which means no change point exists. Moreover, the figure shows that for $\lambda_1 > \lambda_2$, the change point t_0 decreases with the increase of λ_1, and increases as λ_2 becomes larger. Similar phenomenon also can be observed in case of $\lambda_1 < \lambda_2$.
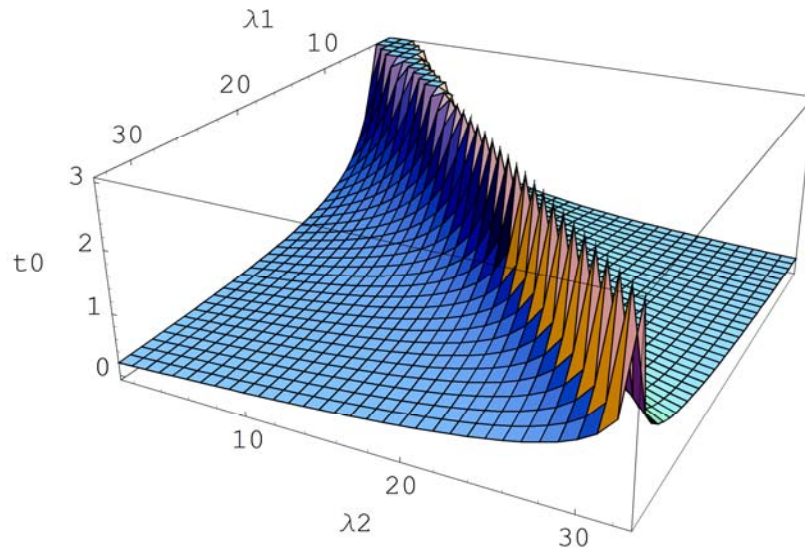
Figure 6.4   Locations of t_0 for different λ_1 and λ_2.

6.5.2   UBMRL type component

As proved in Theorem 6.2, parallel structure would make the change point of MRL occur earlier. Hence, it is of great interest to generally investigate the effect that parallel structure has on the change point of MRL of components, when the components have different lifetime distributions. Here, following Section 6.5.1, we further graphically study parallel systems composed of two non-identical UBMRL components, whose lifetimes follow same distribution but with different parameters. The three-parameter UBMRL distribution studied in this section is also modified Weibull distribution (Lai et al., 2003) $R(t) = \exp\left(- at^b e^{\lambda t}\right)$; see Section 2.2.1 for discussion.

In this study, to evaluate the effect of each parameter on the location of change point, we let two parameters of both components be the same, and fix the third

parameter of one component by treating the other one as a variable.

In a two-component parallel system, let one component have reliability function $R_1(t) = \exp\left(-t^{0.25}e^{0.25t}\right)$ and the other component have its reliability function with unknown parameters, i.e. $R_2(t \mid a) = \exp\left(-at^{0.25}e^{0.25t}\right)$, $R_2(t \mid b) = \exp\left(-t^b e^{0.25t}\right)$, $R_2(t \mid \lambda) = \exp\left(-t^{0.25}e^{\lambda t}\right)$. Figure 6.5, 6.6 and 6.7 describe the behaviors of change point of the MRL for the parallel system with respect to a, b and λ respectively.

In each plot, the ▲ line depicts the behavior of the change point of MRL for the parallel system, the ♦ horizontal line corresponds to the change point of MRL for $R_1(t)$, and the ■ curve represents the location of the change point for $R_2(t \mid \cdot)$. From the figures, we can find that, for all three parameters, the change point for systems seems to be always smaller than the larger one of the two change points for components. Also, Figure 6.5 and Figure 6.7 show that, as the values of parameter a and λ increase, the change point of MRL for systems approaches the smaller change point for two components. As larger values of a and λ corresponds to lower reliability, the phenomenon may imply that, the large difference in reliability of two components would make the change point of MRL of systems close to the smallest change point of components.
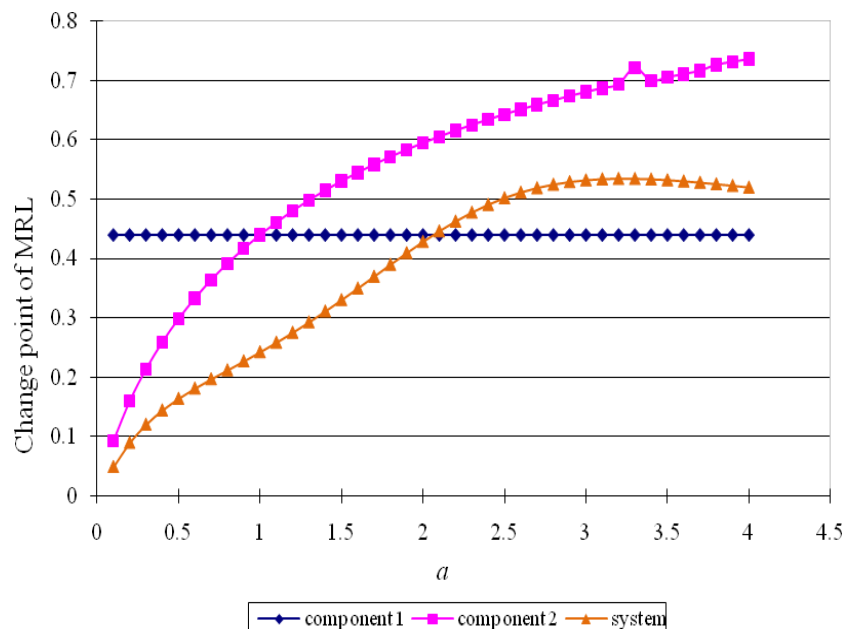
Figure 6.5   Change point for parallel systems with parameter *a* – modified Weibull distribution.
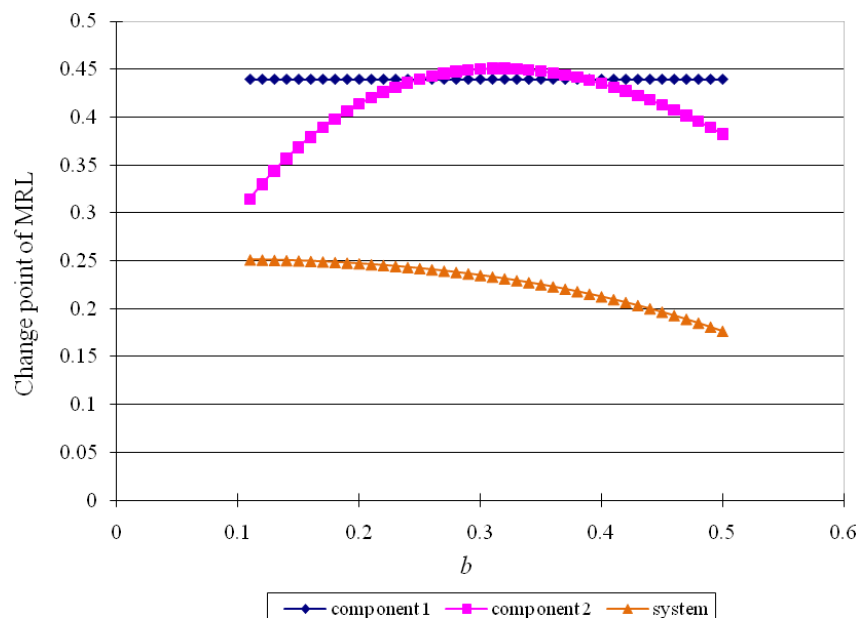

Figure 6.6   Change point for parallel systems with parameter *b* – modified Weibull distribution.
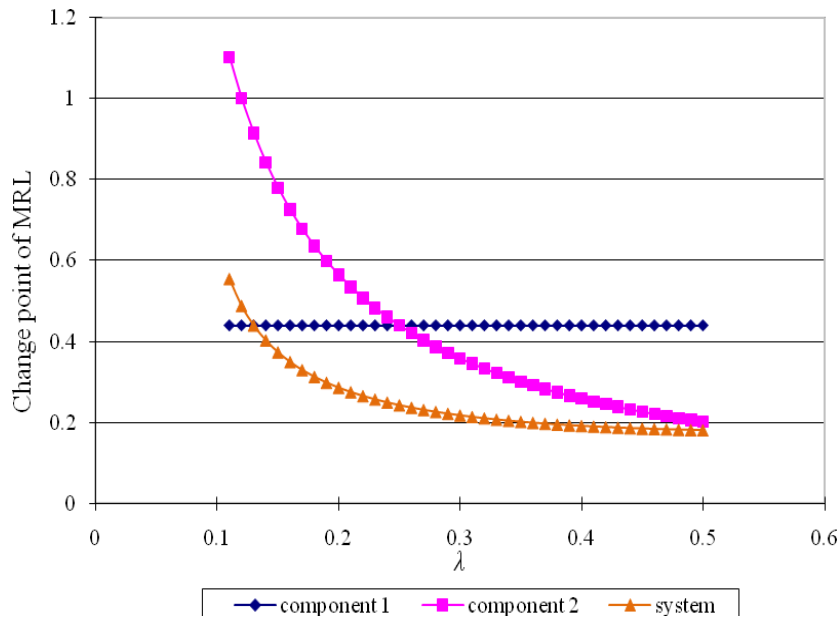
Figure 6.7   Change point for parallel systems with parameter λ – modified Weibull distribution.

6.6   Conclusion

In this chapter, we first investigate the MRL functions of series systems and parallel systems assuming that both systems are composed of i.i.d. components with UBMRL and bathtub-shaped failure rate. It is shown that that the change point of the MRL for a series system is located after the change point for single components while the change point for a parallel system occurs earlier than that for components. Also, a brief graphic study on parallel systems with two non-identical components is carried out. We find that the change point of the MRL for such parallel systems tends to be smaller than the larger change point for components, and seems to approach the smaller change point for components when two components are greatly different from each other in reliability. Further research may focus on the change point of the MRL of k-out-of-n systems (Beutner, 2008; Gurler & Bairamov, 2008) and other complex

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systems. The MRL of the systems with independent but not necessary identical components (Hu et al., 2001; Sadegh, 2008) is also worth of further investigation. We might expect some results similar to the work in this chapter.

CHAPTER 7   CONCLUSIONS AND FUTURE RESEARCH

7.1   Summary of results

This thesis made a study on analysis and reliability modeling based on continuous and univariate mean residual life.

This research proposed a model with relatively simple upside-down bathtub-shaped MRL. To ensure the MRL function has a closed form, the model was constructed by choosing a suitable function for the derivative function of the MRL, instead of the reliability function and the failure rate function. This is because the existing models derived from the latter two functions usually involve an integral of the reliability function that is not a closed form. The study also compared the new model to some existing distributions and showed that the new model is capable of integrating the characteristics of other distributions to provide all-sided information. The study also showed that the new model can provide accurate descriptions of the lifetime of products. Moreover, based on the simple MRL of the new model, the optimum time for both burn-in test and replacement can be easily determined. Hence

the research indicates that the model with simple MRL is a good choice in modeling failure data when the data exhibit an upside-down bathtub shape.

Besides the proposed parametric model, this research also developed a nonparametric estimator of the decreasing MRL function in Type II censoring. The estimation procedure was established by comparing two estimators of the reliability function, of which one is the Kaplan-Meier estimator, and the other is obtained from the empirical MRL function. The comparison of this new method and two common parametric methods showed that this new approach provides comparable even better performances especially when censor degree is high. Therefore this new method is recommended for construction of the estimator of the MRL function when the information of the underlying distribution is limited and the censored data are large.

Moreover, the relationship between the MRL and the failure rate function is of great importance in reliability analysis. This work studied how the change of one characteristic affects the other characteristic and proposed some inequalities to quantify the range of change. The results are useful and helpful in calculating upper and lower bounds for MRL and failure rate functions. More specifically, based on the inequalities, an estimation method was introduced to estimate bounds for the failure rate function based on empirical MRL function. Simulation results and analysis showed that this method could provide good performance in estimation, and to some extent, it could be considered as a robust approach.

As an extension of the MRL of single items that form the focus in all the previous studies, the MRL of systems were investigated in the research. The MRL functions of series and parallel systems were compared to that of their components in

terms of change point, under the assumption that components have UBMRL. It was shown that the change point of the MRL for series systems is located after the change point for single components; but for parallel systems, the change point for systems precedes that for its components. Furthermore, it was found that the difference in the positions of the change points between systems and components increases with the number of components. In addition, the MRL of parallel systems with two non-identical components was briefly and graphically studied. We found that the change point of the MRL for such parallel systems tends to be prior to the larger change point for components, and seems to approach the smaller change point for components when the difference between the reliability of two components is relatively great.

7.2   Possible future research

As pointed out in the previous section, a suitable and simple function was chosen for the derivative of the UBMRL in constructing parametric models. To extend the model, further research should be carried out to study the derivation of more complicated MRL, such as roller-coaster shaped MRL, because the study of this type of MRL is another interesting and meaningful topic. Similar to the general framework discussed in this work, it should be also a worthwhile endeavor to study reliability models with more complicated MRL by considering the derivative of the MRL in a general way.

Also, the regression method used to estimate model parameters based on the MRL function needs further investigation. The empirical MRL at a certain point of time depends on only the failure data larger than the time point. So this indicates that the empirical MRL for larger failure time would suffer greater variance due to fewer

available data, and thus may significantly influence the regression result and introduce more error. More endeavors should be made to control the affect of large variance that occurs at large failure time. One possible way to reduce the affect of variance is to directly remove the empirical MRL at last several data points. The optimization of the number and the procedure of dropping data points need an extensive study. Another possible solution is to take into account the empirical variance defined in Yang (1978), which is related to process theory. We may expect that confidence band could be established to give more information on the accuracy of the regression results.

In the nonparametric estimation in Type II censoring, this research focused on only the DMRL functions and did not provide any results on other types of the MRL, such as the increasing MRL. This is because the DMRL is the most common one in practice. But it will be helpful if the IMRL functions can be estimated in a nonparametric way, since the combined work, including the estimation of both the decreasing and the increasing MRL, would benefit the estimation of the MRL with complex shapes. Another useful extension is to discuss other censoring types, such as Type I censoring, or to consider truncation cases. For Type I censoring, the main idea could be similar and may need only some small modification. To deal with left truncation, we may artificially generate some failure data before the truncated time based on those observed failure times, so that the empirical MRL function could be directly apply to the data set that are composed of the truncated real data and the artificially generated data.

Another estimation approach for the MRL regardless of underlying distributions is to utilize data transformation method. Many transformation functions

has been proposed in literature, and the most famous one is the Box-Cox power transformation

$$h(y,\lambda) = \begin{cases} (y^{\lambda} - 1)/\lambda, & \lambda \neq 0 \\ \log y, & \lambda = 0 \end{cases}.$$

It is known that the Box-Cox transformation could approximate data to normal distribution. Hence, we could estimate the MRL function by using the transformation function and the normal distribution. Given failure data, a feasible method may be as follows: (1) first transform failure data by the transformation function above, and treat λ as unknown parameters as well as mean and variance for normal distribution, i.e. μ and σ; (2) use failure data to estimate the three parameters by MLE; (3) calculate and the MRL function for the normal distribution and transform it back to the MRL for original distribution. As shown in Yang & Tsui (2004), extra variance would be introduced due to the unknown parameter λ. So accuracy analysis of this method may be needed, as well as a comparison study between this method and other existing methods.

For the bounds of the failure rate function, only monotonic MRL class is considered in this thesis. A further research may extend the results to more general MRL classes by properly combining the DMRL and IMRL cases. Moreover, since the proposed method is shown to be robust by the graphical sensitivity analysis, it would be helpful to explore the underlying mechanism of the robustness of the method.

Another interesting topic is mixture, as this is an important phenomenon in reliability engineering. For example, a population that involves normal and inferior

products can be described by a mixture. In literature, there are known results concerning the behaviors of the failure rate functions under mixture. Because of the importance of the MRL, it will be helpful to study the effect of mixture on the MRL, as well as the relationship between mixtures on the failure rate function and on the MRL function.

This thesis is mainly about analysis and reliability modeling based on MRL. The studies presented in the first few chapters all deal with the MRL function of single items, and only Chapter 6 proved some results on the MRL of series and parallel systems. Since a system is a collection of components, we can treat the MRL of systems as a concept extended from the MRL of single items. Therefore, it should be very meaningful to conduct an in-depth study on the MRL of systems by applying the theories about the MRL of single items in this research to systems.

Following the topic in Chapter 6, it may be of great interest to study different kinds of systems that are composed of different kinds of components. We could generalize series and parallel systems studies to k-out-of-n systems. The "independent and identically distributed" assumptions for components also could be released by introducing dependence between components or assuming each component subject to different probability distributions. Furthermore, crossing properties of the MRL could be discussed. Suppose that the MRL functions of two components are crossed at a certain point in time. We may be curious about whether series and parallel structures would change the crossing point by some regulations, which might be related to the theorems proved in Chapter 6.

In addition, as shown in the scope, this thesis focuses on univariate and

continuous MRL function, because the lifetime of an item, either human beings or components, is often considered to be continuous. But in practice, when measurements are taken at discrete time, discrete distribution will give better modeling, analysis and interpretation. For example, the crack of a dam is always measured for every fixed duration of operation. The reliability of software may be indexed by the number of failures during a certain period of time. In these cases, discrete distributions are more useful. Therefore, to conduct an analysis and study on discrete MRL function would also of great importance in reliability field.

Recently, residual life distribution is defined and predicted based on degradation models that utilizes degradation-based sensory signals; see Gebraeel et al. (2009) for example. The fast development of this research topic attributes to the immense technology improvement, which results in the production of highly reliable items whose failure is often hardly to be observed. Hence, in order to obtain information on the failure time of such types of items, some degradation-based sensory signals are selected and used to detect the degradation process of the highly reliable items and measure their failure times indirectly. For example, the fatigue crack-size of the alloy and the lumen could be used as signals to describe the degradation of a fluorescent light bulb. In literature, most papers assume Wiener process as the underlying process. But some real study implies that gamma process seems to be able to provide a better description. So this gives a guideline for a further topic of the development of residual life and MRL based on gamma process.

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