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ADAPTIVE CONTROL AND NEURAL NETWORK CONTROL OF NONLINEAR DISCRETE-TIME SYSTEMS

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Acknowledgements

First of all, I would like to thank my main supervisor, Professor Shuzhi Sam Ge, for his advice and guidance on shaping my research direction and goals, and the research philosophy he imparted to me. I would also like to express my thanks to my co-supervisor, Professor Tong Heng Lee. His experience and knowledge always provide me most needed help on research work. I am of heartfelt gratitude to my supervisors for their remarkable passion and painstaking efforts in training me, without which I would not have honed my research skills and capabilities as well as I did in my Ph.D studies. My appreciation goes to Professor Jianxin Xu and Professor Ai-Poh Loh in my thesis committee, for their kind help and advice in my Ph.D studies. My thanks also go to Professor Cheng Xiang and Professor Hai Lin, for their interesting and inspiring group discussions from which I benefit much.

To my fellows in the X-1 team for TechX Ground Robot Challenge, in particular, Dr Pey Yuen Tao, Mr Aswin Thomas Abraham, Dr Brice Rebsamen, Dr Bingbing Liu, Dr Qinghua Xia, Ms Bahareh Ghotbi, Mr Dong Huang and many others that have been part of the team, for the stressful but exciting time spent together. Special thanks to Dr Hongbin Ma, who has always been willing to provide me help with his excellent mathematical skills. To Dr Keng Peng Tee, Dr Cheng Heng Fua, Dr Xuecheng Lai, Dr Han Thanh Trung, Dr Zhuping Wang, Dr Fan Hong, Dr Feng Guan and Mr. Yong Yang, my seniors, for their generous help since the first day I joined the research team. To my collaborators, Dr Shilu Dai and Dr Lianfei Zhai, for the endless hours of useful discussions that are always filled with creativity and inspiration. Special thanks to Ms Beibei Ren and Ms Yaozhang Pan, my fellow adventurers in the research course, for their encourage and friendship. To Dr Rongxin Cui, Dr Mou Chen, Mr Voon Ee How, Mr Deqing Huang, Dr Zhijun Li and Dr Yu Kang for the many enlightening discussions and help they have provided in my research.

I would also like to thank Mr Qun Zhang, Mr Yanan Li, Mr Hongsheng He, Mr Wei He, Mr Kun Yang, Mr Zhengcheng Zhang, Mr Hewei Lim, Mr Sie Chyuan Law, Mr Feng Lin, Mr Chow Yin Lai, Ms Lingling Cao, Mr Han Yan and many other fellow colleagues and researchers for their friendship, help and the happy time we have enjoyed together.

To my girl friend Ms Ning Wang, for her unquestioning trust, support and encouragement. To my family, for they have always been there for me, stood by me through the good times and the bad. Finally, I am very grateful to the National University of Singapore for providing me with the research scholarship to undertake the PhD study.

Contents

Summary

Nowadays nearly all the control algorithms are implemented digitally and consequently discrete-time systems have been receiving ever increasing attention. However, for the development of nonlinear adaptive control and neural network (NN) control, which are generally regarded as smart ways to deal with system uncertainties, most researches are conducted in continuous-time, such that many well developed methods are not directly applied in discrete-time, due to fundament difference between differential and difference equations for modeling continuous-time and discrete-time systems, respectively. Therefore, nonlinear adaptive control and neural network control of discrete-time systems need to be further investigated.

In the first part of the thesis, a framework of future states prediction based adaptive control is developed to avoid possible noncausal problems in high order systems control design. Based on the framework, a novel adaptive compensation approach for nonparametric model uncertainties in both matched and unmatched condition is constructed such that asymptotic tracking performance can be achieved. By proper incorporating discrete Nussbaum gain, the adaptive control becomes insensitive to system control directions and the bounds of control gain become not necessary for control design. The adaptive control is also studied with incorporation of discrete-time Prandtl-Ishlinskii (PI) model to deal with hysteresis type input constraint. Furthermore, adaptive control is designed for block-triangular nonlinear multi-input-multi-output (MIMO) systems with strict-feedback subsystems coupled together. By exploiting block triangular structure properties and construction of uncertainties compensations, the design difficulties caused by the couplings among various inputs and states, as well as the uncertainties in the couplings are solved.

In the second part of the thesis, it is established that for single-input-single-output (SISO) case, under certain conditions both pure-feedback systems and nonlinear autoregressive-moving-average-with-exogenous-inputs (NARMAX) systems are transformable into a suitable input-output form and adaptive NN control design for both systems can be carried

out in a unified approach without noncausal problem. To overcome the difficulty associated with nonaffine appearance of control variables, implicit function theorem is exploited to assert the existence of a desired implicit control. In the control design, discrete Nussbaum gain is further extended to deal with time varying control gains. The adaptive NN control constructed for nonaffine SISO systems is also extended to nonaffine MIMO systems in block triangular form and NARMAX form.

The research work conducted in this thesis is meant to push the boundary of academic results further beyond. The systems considered in this thesis represent several general classes of discrete-time nonlinear systems. Numerical simulations are extensively carried out to illustrate the effectiveness of the proposed controls.

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Throughout this thesis, the following notations and conventions have been adopted:

$A \stackrel{def}{=} B$	A is defined as B
$A := B$	B is defined as A
$\mathbf{R}$	the set of all real numbers
$\mathbf{Z}$	the set of all integers
$\mathbf{Z_t^+}$	the set of all integers which are not less than integer t
$\|A\|$	the Euclidean norm of vector A or the induced norm of matrix A
A^T	the transpose of vector or matrix A
$\mathbf{0}_{[p]}$	p-dimension zero vector
$\det A$	the determinant of matrix A
$\lambda(A)$	the set of eigenvalues of A
$\lambda_{max}(A)$	the maximum eigenvalue of real symmetric matrix A
$\lambda_{min}(A)$	the minimum eigenvalue of real symmetric matrix A
$|a|$	the absolute value of number a
$\arg\max S$	the index of maximum element of ordered set S
$\arg\min S$	the index of minimum element of ordered set S
$[a, b)$	the real number set $\{t \in \mathbf{R} : \mathbf{a} \le \mathbf{t} < \mathbf{b}\}$ or
	the integer set $\{t \in \mathbf{Z} : \mathbf{a} \le \mathbf{t} < \mathbf{b}\}$
$[a, b]$	the real number set $\{t \in \mathbf{R} : \mathbf{a} \le \mathbf{t} \le \mathbf{b}\}$ or
	the integer set $\{t \in \mathbf{Z} : \mathbf{a} \le \mathbf{t} \le \mathbf{b}\}$
$u(k)$	control input(s) of the system to be controlled
$y(k)$	output(s) of the system to be controlled
$\xi_j(k)$	the jth state variable of the system to be controlled
$\bar{\xi}_j(k)$	the vector of states defined as $\bar{\xi}_j(k) = [\xi_1(k), \xi_2(k), \ldots, \xi_j(k)]^T$
$y_d(k)$	reference signal(s) to be tracked by system output(s)
$e(k)$	output(s) tracking error(s) defined as $e(k) = y(k) - y_d(k)$
$\hat{\Theta}(k)$	estimate of vector parameter Θ at step k
$\tilde{\Theta}(k)$	estimate error of vector parameter Θ at step k

In this thesis, the time steps are assumed in the set of $\mathbf{Z_{-n}^+}$, where n is the order of the system, unless specified otherwise.

Chapter 1

Introduction

It is well known that the control design is critical to the performance of the closed-loop controlled system while an accurate system model is essential for a good control design. But for modeling of practical systems, there are always inevitable uncertainties. These modeling uncertainties may result in poor performance and may even lead to instability of the closed-loop systems. To improve control performance, many control strategies have been developed to consider these uncertainties in the control design stage. Adaptive control has been developed with particular attention paid to parametric uncertainties. Over the years of progress from linear systems to nonlinear systems, rigorous stability analysis of the closed-loop adaptive system has been well established.

The advantage of adaptive control lies in its ability to estimate and compensate for parametric uncertainties in a large range, but towards the increasingly complex systems with complicated nonlinear functional uncertainties, it is necessary to develop more powerful control design methodologies. Therefore, neural network (NN) control along with other intelligent controls has been introduced in the early 90's. In NN control methodology, NN has been extensively studied for functions approximation to compensate for the system uncertain nonlinearities in control design. In the last two decades, NN control has been proved to be very useful for controlling highly uncertain nonlinear systems and has demonstrated superiority over traditional controls. Especially, the marriage of adaptive control theories and NN techniques give birth to adaptive NN control, which guarantees stability, robustness and convergence of the closed-loop NN control systems without beforehand offline NN training.

In the past decades, many significant progresses in adaptive control and NN control made for nonlinear continuous-time systems and there is considerable lag in the development

for nonlinear discrete-time systems. While nowadays nearly all the control algorithms are implemented digitally such that the process data are typically available only at discrete-time instants, and it is sometimes more convenient to model processes in discrete-time for ease of control design. Thus, adaptive control and NN control of nonlinear discrete-time system deserve deeply further investigation.

The remainder of this Chapter is organized as follows. In Section 1.1, brief introduction of the development of adaptive control, especially for nonlinear discrete-time systems, is presented. Some research problems to be studied in this thesis are highlighted, such as robust issue and unknown control direction problem in adaptive control, which are both theoretically challenging and practically meaningful. In Section 1.2, NN control is briefly reviewed. Background knowledge of NN is given first, and then the recent researches on NN control of nonaffine systems and multi-variable systems are discussed. Finally, in Section 1.3, the motivation, objectives, scope, as well as the structure of the thesis are presented.

1.1 Adaptive Control of Nonlinear Systems

Adaptive control has been developed more than half a century with intense research activities involving rigorous problem formulation, stability proof, robustness design, performance analysis and applications [1]. Originally adaptive control was proposed for aircraft autopilots to deal with parameter variations during changing flight conditions. In the 1960's, the advances in stability theory and the progress of control theory improved the understanding of adaptive control and by the early 1980's, several adaptive approaches have been proven to provide stable operation and asymptotic tracking. The adaptive control problem since then, was rigorously formulated and the theoretical foundations have been laid.

The early adaptive controls were mainly designed for the linear systems. The solid theoretical foundations of general solution to the linear adaptive control problem were laid in simultaneous publications [2–5], in which the global stability of linear adaptive systems was analyzed. The success of adaptive control of linear systems has motivated the rapid growing interest in nonlinear adaptive control from the end of 1980's. In particular, adaptive control of nonlinear systems using feedback linearization techniques has been developed in [6–9], based on the differential geometric theory of nonlinear feedback control [10]. It is noted in these results that global stability cannot be established without some restrictions on the plants, such as the matching condition [7], extended matching condition [11], and growth conditions on system nonlinearities [12]. The technique of backstepping, rooted in the independent works of [13–15], and further developed in [16–18], heralded an important

breakthrough for adaptive control that overcame the structural and growth restrictions. The combination of adaptive control and backstepping technique, i.e. adaptive backstepping, yields a means of applying adaptive control to parametric-uncertain systems with non-matching conditions [19, 20]. As a result, adaptive backstepping can be applied to a large class of nonlinear systems in lower triangular form with parametric uncertainties.

For most nonlinear adaptive control designs, Lyapunov's direct method has been adopted as a primary tool for control design, stability and performance analysis. Lyapunov's direct method is a mathematical interpretation of the physical property that if a system's total energy is dissipating, then the states of the system will ultimately reach an equilibrium point. The direct method provides a means of determining stability without the need for explicit knowledge of system solutions. The basic idea to apply Lyapunov's method in control design is to design a feedback control law that renders the derivative of a specified Lyapunov function candidate negative definite or negative semi-definite [21, 22]. The task of selecting a Lyapunov function candidate is in general non-trivial. For ease of manipulation, a significant portion of the literature on Lyapunov based control synthesis employ quadratic Lyapunov functions, which are often sufficient to solve a large variety of control problems. But sometimes more sophisticated forms of Lyapunov functions are needed for certain difficult problems. To avoid controller singularity problem in feedback linearization based adaptive control of continuous-time nonlinear systems, integral Lyapunov functions have been developed in [23]. For stability analysis for time-delay systems, a class of special Lyapunov functionals, Lyapunov-Krasovskii functionals, can be employed such that when the derivative of the Lyapunov function/functional is taken, the terms containing the delayed states can be matched and canceled [24–26].

In practice, there may be some nonsmooth, nonlinear input constraint, such as dead zone, backlash and hysteresis, which are common in actuator and sensors such as mechanical connections, hydraulic actuators and electric servomotors. The existence of these constraints in control input can result in undesirable inaccuracies or oscillations, which severely limits the closed-loop control system's performance and can even lead to instability [27]. Therefore, the studies of these constraints have been drawing much interest in the adaptive control community for a long time [28–32]. To handle systems with unknown dead zones, adaptive dead-zone inverses were proposed [28, 30]. Robust adaptive control was developed for a class of special nonlinear systems without constructing the inverse of the dead zone [31]. Smooth inverse function of the dead zone together with backstepping has been proposed for output feedback control design in [32]. To control systems with hysteresis input constraint, an inverse operator was constructed to eliminate the effects of the hysteresis in [29]. In the

literature, various models have been proposed to describe the hysteresis, such as Preisach model [33], Prandtl-Ishlinskii (PI) model [34, 35], and Krasnosel'skii-Pokrovskii model [36].

Many practical systems are of multi-variable characteristics, thus an ever increasing attention in control community has been paid to MIMO nonlinear systems in recent years. However, compared with myriad researches conducted for SISO nonlinear systems, adaptive control theory for multi-input-multi-output (MIMO) nonlinear systems has been less investigated. It is noted that it is generally non-trivial to extend the control designs from single-input-single-output (SISO) systems to MIMO systems, due to the interactions among various inputs, outputs and states. Several algorithms have been proposed in the literature for solving the problem of exact decoupling for nonlinear MIMO systems [10, 37–39]. In [40], global diffeomorphism is studied for square invertible nonlinear systems such that backstepping design can be applied. In [41], the problem of semi-global robust stabilization was investigated for a class of MIMO uncertain nonlinear system, which cannot be transformed into lower dimensional zero dynamics representation, via change of coordinates or state feedback. All the above mentioned designs need the determination of the so-called decoupling matrix, i.e., the system interconnections are known functions. As a matter of fact, when there are uncertain couplings, the closed-loop stability analysis becomes much more complex.

For nonlinear MIMO systems that are feedback linearizable, a variety of adaptive controls have been proposed based on feedback linearization techniques [6, 42], in which an invertible estimated decoupling matrix is also required during parameter adaptation such that couplings among system inputs can be decoupled. Backstepping design has also been investigated for adaptive control of some classes of MIMO systems that are not feedback linearizable. In [20], adaptive backstepping control has been studied for parametric strict-feedback MIMO nonlinear systems, in which it is assumed that no parametric uncertainties appear in the input matrix. As an extension, robust adaptive control has been studied for a class of MIMO nonlinear systems transformable to two semi-strict feedback forms in [43], where the parametric uncertainty is considered in the coupling matrix, and uncertain system interconnections are assumed to be bounded by known nonlinear functions.

1.1.1   Discrete-time adaptive control

Discrete-time systems are of ever increasing importance with the advance of computer technology. Even at the very early stage of adaptive control development, discrete-time systems received great attention. In fact, one foundational research work of adaptive control,

the self tuning regulator (STR), was presented in discrete-time [44]. In the development of linear adaptive control, many advances in discrete-time have been achieved in parallel to those in continuous-time. Rigorous global stability of adaptive control was established in [2, 4] for continuous-time linear systems and in [3, 5] for discrete-time linear systems. The adaptive control design without a priori knowledge of control direction was proposed in [45] for continuous-time linear system while the counterpart result in discrete-time was obtained in [46]. Robust adaptive control using persistent excitation of the reference input was proposed in [47] for continuous-time linear systems while the work for discrete-time linear systems was made in [48]. It is worth to mention that the Key Technical Lemma developed in [5] has been a major stability analysis tool in discrete-time adaptive control.

Though for adaptive control of linear continuous-time systems, there are lots of counterpart results for linear discrete-time systems, adaptive control of nonlinear discrete-time systems have been considerately less studied than their counterparts in discrete-time. As a matter of fact, many techniques developed for continuous-time systems cannot be applied in discrete-time, especially when the systems to be controlled are nonlinear. Discrete-time systems are described by difference equations, which in great contrast to the differential equations of continuous-time systems, involve states at different time steps. Due to the different nature of difference equation and differential equation, even some concepts in discrete-time have very different meaning from those in continuous-time, e.g., the "relative degrees" defined for continuous-time and discrete-time systems have totally different physical explanations [49].

Generally, adaptive control design for nonlinear systems in discrete-time is much more difficult than for those in continuous-time. The stability analysis techniques become much more intractable for difference equations than those for differential equations, e.g., the linearity property of the derivative of a Lyapunov function in continuous-time is not present in the difference of a Lyapunov function in discrete-time [50]. Thus, many nice Lyapunov adaptive control design methodologies developed in continuous-time are not applicable to discrete-time systems. Sometimes the noncausal problem may arise when continuous-time control design is directly applied to discrete-time counterpart systems, such that the conventional backstepping design proposed in continuous-time, a crucial ingredient for the development of adaptive control of nonlinear systems in lower triangular form, is not directly applicable to counterpart discrete-time systems [51]. To illustrate, let us consider a

second order discrete-time systems in strict-feedback form as follows:

$$
\begin{aligned}
\xi_1(k+1) &= f_1(\xi_1(k)) + \xi_2(k) \\
\xi_2(k+1) &= f_2(\xi_1(k), \xi_2(k)) + u(k)
\end{aligned}
\tag{1.1}
$$

The first state variable at the $(k+1)$th step, $\xi_1(k+1)$, is driven by the second state variable at the kth step, $\xi_2(k)$, while the second state variable at the $(k+1)$th step, $\xi_2(k+1)$, is driven by the control input at the kth step, $u(k)$. If we treat the second state variable at kth step, $\xi_2(k)$, as virtual control variable as in the procedure of conventional backstepping design, the control input at the kth step, $u(k)$, will involve first state variable at $(k+1)$th step, $\xi_1(k+1)$, which is not available at current step, the kth step.

To extend the conventional backstepping design procedure from continuous-time to discrete-time, a coordinate transformation for strict-feedback systems was proposed in [52] such that adaptive control can be designed to "looks ahead" and choose the control law to force the states to acquire their desired values. From the perspective of parameter identification for strict feedback system, a novel parameter estimation was proposed [53], in which the convergence of parameter estimates to the true values in finite steps is guaranteed if there is no other nonparametric uncertainties. To robustify the discrete-time backsteping proposed in [52], projection method has been incorporated into the parameter update law [54–56] to deal with nonparametric model uncertainties. However, it is noted that all these methods were developed for special strict-feedback systems with known control gains and are not directly applicable to more general strict-feedback systems with unknown control gains. To explain clearly, let us consider a simple plant $y(k+1) = \theta y(k) + g u(k)$. If g is known, then we are able to calculate the value of $\theta y(k-1)$ by $\theta y(k-1) = y(k) - g u(k-1)$, but if g is unknown we are not able to obtain the value of $\theta y(k-1)$. In the discrete-time backstepping in [52,54–56], the coordinate transformation involves the similar problem as in the example above, and thus, the control gains are assumed to be simply ones in these work. When the control gains are unknown, the discrete-time backstepping developed in [52, 54–56] will be not directly applicable.

On the other hand side, there are no general discrete-time adaptive nonlinear controls by now that allow the nonlinearity in systems to grow faster than linear. When the known nonlinear functions are of growth rates larger than linear, most existing design methods become not valid because the Key Technical Lemma [57], a main stability analysis tool in discrete-time adaptive control, is not applicable for the unknown parameters multiplying nonlinearities that are not sector bounded. As revealed in [58, 59], there are considerable

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limitations of feedback mechanism for discrete-time adaptive control, such that it is impossible to have global stability results for noised adaptive controlled systems when the known nonlinear system functions are of general high growth order or when the size of the uncertain nonlinearity is larger than a certain number. In an early work [60] on discrete-time adaptive systems, it is also pointed out that when there is large parameter time-variation, it may be impossible to construct a global stable control even for a first order system.

1.1.2 Robust issue in adaptive control

The early developed adaptive controls were mainly concerning on the parametric uncertainties, i.e., unknown system parameters, such that the designed controls have limited robustness properties, where minute disturbances and the presence of nonparametric model uncertainties can lead to poor performance and even instability of the closed-loop systems [61, 62]. Subsequently, robustness in adaptive control has been the subject of much research attention in both continuous-time and discrete-time.

Some researches suggested that the persistently exciting reference inputs with a sufficient degree of persistent excitation can be used to achieve robustness for system perturbed by bounded disturbances and certain classes of unmodeled dynamics [47, 48]. To enhance the robustness, many modification techniques were proposed in the control parameter update law of the adaptive controlled systems, such as normalization [62, 63] where a normalization term is employed; deadzone method [61, 64] which stops the adaptation when the error signal is smaller than a threshold; projection method [54, 56, 65] which projects the parameter estimates into a limited range; σ-modification [66] which incorporate an additional term; and e-modification [21] where the constant σ in σ-modification is replaced by the absolute value of the output tracking error. These methods make the adaptive closed-loop system robust in the presence of external disturbance or model uncertainties but sacrifice the tracking performance.

In addition, sliding mode as one of the most popular robust control methods that results in invariance properties to uncertainties [67–69], e.g., modeling uncertainty or external disturbance, has also been incorporated into adaptive control design to offer robustness. Extensive studies of adaptive control using sliding mode has been made in continuous-time for the recent decades. To guarantee the smoothness of the control law, $tanh(\cdot)$ function instead of the saturation function $sat(\cdot)$ have been employed in the adaptive control design [70–72].

However, due to the above mentioned difficulties associated with uncertain nonlinear

discrete-time system model, there are not many researches on robust adaptive control in discrete-time to deal with nonparametric nonlinear model uncertainties. As mentioned above, parameter projection method has also been studied in [54–56] to guarantee boundedness of parameter estimates. The sliding mode method has also been incorporated into discrete-time adaptive control [73–76]. However, in contrast to continuous-time systems for which a sliding mode control can be constructed to eliminate the effect of the general uncertain model nonlinearity, in discrete-time the uncertain nonlinearity is required to be of small growth rate or globally bounded, but sliding mode control is not able to completely compensate for the effect of nonlinear uncertainties in discrete-time. As a matter of fact, unlike in continuous-time, it is much more difficulty in discrete-time to deal with nonlinear uncertainties. As mentioned above, when the size of the uncertain nonlinearity is larger than a certain level, even a simple first-order discrete-time system cannot be globally stabilized [59]. Mover, in discrete-time most existing robust approaches only guarantee the closed-loop stability in the presence of the nonparametric model uncertainties but are not able to improve control performance by completely compensation for the effect of uncertainties.

1.1.3 Unknown control direction problem in adaptive control

As observed by the early researchers that one challenge of adaptive control design lies in the unknown signs of the control gains [45, 77], which are normally required to be known a priori in the adaptive control literature. These signs, called control directions in [78], represent motion directions of the system under any control. When the signs of control gains are unknown, the adaptive control problem becomes much more difficult, since we cannot decide the direction along which the control operates. Moveover, in discrete-time adaptive control the control directions are usually required to avoid controller singularity when the estimate of control gains appear in the denominator. The unknown control directions problem in adaptive control had remained open till the Nussbaum gain was first introduced in [77] for adaptive control of first order continuous-time systems. In [45], adaptive control of high order linear continuous-time systems with unknown control directions has been constructed using Nussbaum gain. Thereafter, the problem of adaptive control of systems with unknown control directions has received a great deal of attention for the continuous-time systems [78–82]. In [80], the Nussbaum gain was adopted in the adaptive control of linear systems with nonlinear uncertainties to counteract the lack of a prior knowledge of control directions. Toward high order nonlinear systems, backstepping with

Nussbaum function was then developed for general nonlinear systems in lower triangular structure, with constant control gains [81], and time-varying control gains [82]. Alternative approaches to deal with the unknown control directions can also be found in the literature. In [83], the projected parameter approach has been used for adaptive control of first-order nonlinear systems with unknown control directions. In [78], online identification of the unknown control directions was proposed for a class of second-order nonlinear systems. But not as general as Nussbaum gain, the application of these methods are restricted to certain classes of systems.

It is mentioned in Section 1.1.1 that it is generally not easy to extend successful continuous-time control methods to discrete-time. It is also true for the control design using continuous-time Nussbaum gain. It is pointed in [84] that simply sampling the continuous-time Nussbaum gain may not results in a discrete-time Nussbaum gain. To solve the unknown control direction problem, a two-step adaptation law was proposed for a first-order discrete-time system [85]. But this procedure is limited to first-order linear system. In order for stable adaptive control of high order linear systems, the first Nussbaum type gain in discrete-time was developed in [46]. The discrete Nussbaum gain is more intractable compared to its continuous-time counterpart, and hence, the control design using discrete Nussbaum gain for discrete-time systems is more difficult than control design using continuous-time Nussbaum gain for continuous-time systems.

1.2 Adaptive Neural Network Control

Adaptive control design has been elegantly developed for nonlinear systems with parametric uncertainties, but as a matter of fact, most of the nonlinear adaptive control techniques rely on the key assumption of linear parameterization, i.e., nonlinearities of the studied plants can be represented in the linear-in-parameters (LIPs) form in which the regression functions are known. Though there is much effort dedicated to adaptive control of nonlinear systems in nonlinear-in-parameters (NIPs) form [86–90], usually the form of the system models and the nonlinear functions in the model are required to be known a priori in adaptive control design. Thus, we call traditional adaptive control as model based adaptive control. Recognizing the fact that model building itself might be very difficult for complex practical systems and it is not easy to identify the general nonlinear functions in the models, many researchers have been devoted to function approximation based control such as neural network (NN) control [1, 91–99].

The universal approximation ability of NN makes it an effective tool in approximation

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based control of highly uncertain, nonlinear and complex systems. NN's approximation ability has been developed based on the Stone-Weierstrass theorem, which states that a universal approximator can approximate, to an arbitrary degree of accuracy, any real continuous function on a compact set [100–105]. Besides the universal approximation abilities, NN also shows its excellence in parallel distributed processing abilities, learning, adaptation abilities, natural fault tolerance and feasibility for hardware implementation. These advantages make NN particularly attractive and promising for applications to modelling and control of nonlinear systems. NN has been successfully applied to robot manipulators control [97, 98, 106–108], distillation column control [109], spark ignition engines control [110, 111], chemical processes identification [112–114], etc. In addition, sometimes NN has also been combined with fuzzy logic for control design [108, 115].

In the early stage, backpropagation (BP) algorithm [116] greatly boosted the development of NN control [91, 92, 117, 118]. It is noted that in the early NN control results, the control performances were demonstrated through simulation or by particular experimental examples, and consequently there were shortage of analytical analysis. In addition, an offline identification procedure was essential for achieving a stable NN control system. Thereafter, the emergence of Lyapunov-based NN design makes it possible to use the available adaptive control theories to rigorously guarantee stability, robustness and convergence of the closed-loop NN control systems [1, 93, 94, 97–99]. We call the control design combining adaptive control theories and NN techniques adaptive NN control, in comparison with model based adaptive control.

1.2.1 Background of neural network

Inspired by the biological NN that consist of a number of simple processing neurons interconnected to each other, McCulloch and Pitts introduced the idea to study the computational abilities of networks composed of simple models of neurons in the 1940s [119]. Neural network, like human's brain, consists of massive simple processing units which correspond to biological neurons. With the highly parallel structure, NN is of powerful computing ability and learning ability to emulate various systems dynamics. It is well established that NN is capable of universally approximating any unknown function to arbitrary precision [100–105]. In addition to system modeling and control, NN has been successfully applied in many other fields such as learning, pattern recognition, and signal processing.

Based on the feedback link connection architecture, NN can be classified into two types, i.e., recurrent NN (e.g., Hopfield NN, cellular NN), and non-recurrent NN or feedforward

NN. For feedforward NN, there are generally two basic types: (i) linearly parametrized neural network (LPNN) in which the adjustable parameters appear linearly, and (ii) multilayer neural networks (MNN) in which the adjustable parameters appear nonlinearly [1]. In this thesis, two kinds of LPNN will be studied for NN control design, i.e., High Order Neural Network (HONN) and Radial Basis Function (RBFNN). The structure of HONN is an expansion of the first order Hopfield [120] and Cohen-Grossberg [121] models that allow higher-order interactions between neurons. HONN is of strong storage capacity, approximation and learning capability. It is pointed in [122] that by utilizing *a priori* information, HONN is very efficient in solving problems because the order or structure of HONN can be tailored to the order or structure of a given problem. RBFNN can be considered as a two-layer network in which the hidden layer performs a fixed nonlinear transformation with no adjustable parameters, i.e., the input space is mapped into a new space. The output layer then combines the outputs in the latter space linearly. The detailed structure and properties of HONN and RBFNN will be discussed in Section 2.2.

1.2.2 Adaptive NN control of nonaffine systems

As mentioned above, adaptive NN control design combines adaptive control theories with NN techniques. It updates NN weight online and the stability of the closed-loop system is well guaranteed. In both continuous-time and discrete-time, adaptive NN control has been extensively studied for affine nonlinear systems through feedback linearization. In continuous-time, MNN based control has been studied for nonlinear system in normal form with functional control gain [123], in which a special switching action is designed to avoid controller singularity problem because NN approximated control gain function appearing in the denominator. Adaptive NN control of normal form affine nonlinear system has also been studied in [124], where the controller singularity problem is solved by introducing control gain function as denominator of Lyapunov function in the design stage. Using high-gain observer, output feedback adaptive NN control has been further studied in [125] for nonlinear system in normal form. In [126], constant time delays have been considered in states measurement for controlling normal form nonlinear system with known constant control gains, with employment of a modified Smith predictor and recurrent NN. For strict-feedback systems with unknown constant control gains, adaptive NN control was designed in [127] via backstepping design. For strict-feedback systems with functional control gains, adaptive NN control based on backstepping has been proposed in [128], where integral Lyapunov functions are used to overcome the controller singularity problem. In [129,130], time delayed

states in strict-feedback systems have been considered. Adaptive NN control has been designed with help of Lyapunov-Krasovskii functionals, and the method in [124] was used to avoid controller singularity problem. Adaptive NN control designed via backstepping has also been studied for general affine nonlinear systems of minimum phase and known relative degree in [131]. In discrete-time, for high order affine nonlinear system in normal form, adaptive NN controls using LPNN and MNN have been developed in [132, 133] using filtered tracking error. The control design has been extended in [110, 134] combining with reinforcement learning technique to improve control performance. A critic NN has been introduced to approximate a strategic utility function which is considered as the long-term system performance measure. For discrete-time systems in strict-feedback form, after system transformation, adaptive NN control via backstepping design has been developed in [51]. In [135], adaptive NN control has been investigated for discrete-time system in affine NARMAX form.

In the above mentioned results, the adaptive NN control designs are carried out through either feedback linearization or backstepping. But these approaches are not applicable to nonaffine systems, especially feedback linearization based methods, which greatly depends the affine appearance of control variables. As a matter of fact, adaptive NN control for nonaffine systems have been less studied in comparison with large amount of researches on affine nonlinear systems, because the difficulty of control design caused by the nonaffine form of control input. To overcome the difficulty, linearization based NN controls have been put forward. In [136], the nonaffine discrete-time system has been decomposed into a linear part and a nonlinear part, and consequently a liner adaptive controller and a nonlinear adaptive NN controller have been designed, with a switching rule specifying when the nonlinear NN controller should be invoked. Similarly, nonaffine systems have been linearized in [137], where a generalized minimum variance linear controller has been designed for the linear part. In [138], control has been designed based on the online linearization of the offline identified NN model with restriction on the control growth. This design approach has been further studied in [139] using internal mode control.

To control nonaffine systems with finite relative degree, some researchers have suggested the idea that NN control can be designed based on the "inverse" of the nonlinear system. Pseudo inverse (approximated inverse) NN control method have been developed in [140,141]. In [140], NN is used to approximate the error between pseudo inverse control signal and the ideal inverse control signal. Similar pseudo inverse NN control has been studied in [141], where the pseudo inverse control consists of a linear dynamic compensator and an adaptive NN compensator. The pseudo inverse NN control has also been studied using a

self structuring NN with online variation neurons number in [142]. The idea is to create more neurons when the plant nonlinearity is complex such that control performance can be guaranteed.

In [143], it is investigated to directly utilize NN as emulator of the "inverse" of the nonlinear discrete-time systems. Furthermore, the study in [144] for discrete-time systems paved the way for adaptive NN control using implicit function to assert the existence of an ideal inverse control. Thereafter, the implicit function based adaptive NN control has been widely studied in both discrete-time [145,146] and continuous-time [125,147,148]. Based on implicit function theory, adaptive NN control using backstepping was constructed for two special classes of nonaffine pure-feedback systems which are affine in control input [147]. But to extend the control design to more general nonaffine pure-feedback systems that are nonaffine in all the control variables, one technical difficulty arise when NN is used to approximate the control u in backstepping design, u and $\dot{u}$ will be involved as inputs to NN. This will lead to a circular construction of the practical control as indicated in [148], in which the difficulty was solved by proposing a ISS-modular approach with implicit function theory used to ensure the existence of desired virtual controls.

It is noted that in adaptive NN control design for both affine and nonaffine systems, the control directions, which is defined as the signs of control gain functions in the affine systems or the signs of partial derivatives over control variables in the nonaffine systems, are normally assumed to be known. Though there are some NN control designs in continuous-time [149,150] using Nussbaum gain to overcome unknown control directions problem, there are little study of unknown control direction problem in discrete-time adaptive NN control so far. One may note that in [144], the control direction is not assumed to be known. But the stability is proved using NN weights convergence results, which cannot be guaranteed without the persistent exciting condition.

1.2.3 Adaptive NN control of multi-variable systems

As mentioned in the beginning of Section 1.1, practically most systems are of nonlinear and multi-variable characteristics, but the control problem of MIMO nonlinear systems is very complicated. It it is generally non-trivial to extend the control designs of SISO systems to MIMO systems, due to the interactions among various inputs, outputs and states. Similar to model based adaptive control, there are fewer results on MIMO systems compared with SISO system in adaptive NN control literature.

In continuous-time, block triangular form systems with subsystems in normal form has

been studied in [23]. This class of systems covers a large class of plants including the decentralized systems studied in [151, 152]. Block triangular form systems with normal form subsystems have also been studied in [150, 153] with particular attention paid to time delayed states, deadzone input constraint and unknown control gains. More general block triangular form systems with strict-feedback subsystems have been investigated in [154]. For general MIMO system in affine form, adaptive NN control based on linearization has been proposed in [155].

In discrete-time, block triangular systems with normal form subsystems have been studied in [132, 133, 156]. For block triangular systems with strict-feedback subsystems, state feedback and output feedback adaptive NN control have been developed in [157, 158] by extending the systems transformation based backstepping technique proposed for SISO case in [51]. In [155], adaptive NN control has been developed for sampled-data nonlinear MIMO systems in general affine form based on linearization. The control scheme is an integration of an NN approach and the variable structure method. For MIMO systems in affine NAR-MAX form, adaptive NN control design has been performed in [159]. The existence of an orthogonal matrix is required to construct the NN weights update law, which as indicated in [159], is generally still an open problem when there exists unknown strong inter connections between subsystems. The aforemention adaptive NN controls for MIMO systems, especially in discrete-time, are all carried out for affine systems.

1.3 Objectives, Scope, and Structure of the Thesis

The general objectives of the thesis are to develop constructive and systematic methods of designing adaptive controls and NN controls for discrete-time nonlinear systems with guaranteed stability. For adaptive control, we will study SISO/MIMO systems in strict-feedback forms. While for adaptive NN control, we will study SISO/MIMO systems in both pure-feedback and NARMAX forms. The control design objective focuses on the output tracking problem.

A framework of adaptive control based on predicted future states will be first established for general strict-feedback systems. The framework provides a novel approach in nonlinear discrete-time control and is expandable to deal with more general uncertainties. In particular, nonparametric model uncertainties are considered. The adaptive control design aims at asymptotic tracking performance in the presence of the nonparametric model uncertainties. A compensation scheme is devised and incorporated into the prediction law and control law, such that the effect of the uncertainties can be eliminated ultimately by using past states

information. Additionally, unknown control directions are accommodated in the adaptive control design via proper introduction of discrete Nussbaum gain into the control parameter update law.

The adaptive control with fully compensation of nonparametric model uncertainties developed in this thesis achieves asymptotic output tracking performance for high order nonlinear strict-feedback system. The proper incorporation of discrete Nussbaum make the adaptive closed-loop insensitive to control directions without loss of asymptotic tracking performance. In order to enlarge the class of systems under the designed adaptive control, input constraint of hysteresis type will also be considered as well as systems with multi-variable. The nonparametric model uncertainty compensation technique has been further developed to compensate for the uncertain coupling terms among subsystems in the MIMO systems. The adaptive control designed in this thesis provide a constructive structure of prediction based adaptive control design approach that may also lead to more useful results and inspire new control design approach.

On the other hand, for adaptive NN control design, the research conducted in the thesis combines implicit function control and future state/outputs prediction together to form a unified approach for SISO systems in both pure-feedback and NARMAX forms. It solves the difficulty caused by the nonaffine appearance of control input and possible noncausal problem in the control design. The study also extends the discrete Nussbaum gain and adopts it for adaptive NN control of nonlinear systems with unknown time varying control gains. The research in adaptive NN control simplifies the previous results using backstepping design and provide a new design approach for adaptive NN control of high order nonlinear systems in nonaffine form.

The adaptive NN control designed will also be extended to control nonlinear MIMO system, both in block triangular form with nonaffine pure-feedback subsystems and in nonaffine NARMAX form. By fully exploit the properties of block-triangular structure, the recursive design method in [157, 158] is extended such that the interaction among each subsystems are considered not only appear in the control range, namely in the last equation of each subsystem, but also appear in every equation of each subsystem. The assumption of known control direction and the assumption that each subsystems are of equal order [158] in output feedback control design will be completely removed. By exploiting discrete Nussbaum gain in NN weights update law, the stringent assumption on control gain matrix of NARMAX system in [159] is relaxed.

The work presented in this thesis is problem oriented and dedicated to the fundamental academic exploration of adaptive and NN control of discrete-time nonlinear systems. Thus,

the focus is given to control theory development. In addition, our studies are focused on the nonlinear systems in lower triangular and NARMAX forms, which cover large classes of nonlinear systems in discrete-time. It would be a future research topic to extend our control design methods to nonlinear systems in other forms.

The thesis is organized as follows. After the introduction in Chapter 1, some necessary mathematical preliminaries and control design tools are give in Chapter 2, in which we will also discuss some nice properties of systems in general lower triangular form and detail the structure and properties of HONN and RBFNN to be used in this thesis.

In Chapter 3, we start with the study of adaptive control of strict-feedback systems with nonparametric model uncertainties. In the first place, the simple case when uncertainties appear in the control range (matched condition) is considered. Asymptotical tracking performance will be obtained by compensation for the uncertain nonlinearities. Then, by further development of future states prediction with incorporation of elimination of the effect of unmatched uncertainties, asymptotic tracking adaptive control is designed for systems with uncertainties outside control range (unmatched condition).

Chapter 4 studies adaptive control of strict-feedback systems with unknown control directions with exploit of discrete Nussbaum gain in the nonlinear control design. Using future states prediction developed in Chapter 3, we first study systems without nonparametric uncertainties. After further investigation of the uncertainties compensation and property of discrete Nussbaum gain, essential modifications are made such that marriage between discrete Nussbaum gain and the nonparametric uncertainties compensation techniques is made for systems with both unknown control directions and nonparametric model uncertainties in matched and unmatched manner. The proposed adaptive control design guarantee the asymptotic tracking performance when the system is in the absence of external disturbance.

Chapter 5 extends the adaptive control designed in previous two Chapters for systems with hysteresis input constraint and systems with multi-inputs and multi-outputs. Discrete-time Prandtl-Ishlinskii (PI) model is utilized to construct the hysteresis constraint and to facilitate the adaptive control design. Uncertain nonlinearities compensation technique has been explored to deal with uncertain couplings among each subsystems. The properties of the block-triangular structure has been well exploited in order for a decoupling recursive control design.

In Chapter 6, NN control of SISO systems in pure-feedback form has been studied. The design difficulty associated with the nonaffine appearance of control variables have been solved by seeking an implicit control using implicit function theorem. Using prediction functions the system has been transformed into a compact form for states feedback design

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employing only a single NN. It greatly reduces the complexity of tedious backstepping design [51]. The system is then further transformed into an input-output form so that output NN control is carried out. In addition, discrete Nussbaum gain is also extended to deal with time varying control gains in adaptive NN control design.

Chapter 7 studies NN control of nonaffine MIMO systems in block-triangular form and NARMAX form. Using properties of block-triangular structure, output feedback NN control has been synthesized without any assumption on subsystem orders [158]. For nonaffine system in NARMAX form, discrete time Nussbaum gain is studied in the NN weights update law to relax assumptions on the control gain matrix [159].

Finally, Chapter 8 concludes the contributions of the thesis and makes recommendation on the future research works.

Chapter 2

Preliminaries

In this Chapter, we will describe in detail the mathematical preliminaries, useful technical lemmas, and control design tools, which will be extensively used throughout this thesis. The properties of general lower triangular SISO nonlinear systems and block-triangular MIMO nonlinear systems will be studied. For completeness, the structure and properties of two kinds of LPNNs, HONN and RBFNN, will be discussed.

2.1 Useful Definitions and Lemmas

Definition 2.1. *A square matrix $A \in \mathbf{R^{n \times n}}$ is said to be*

- *positive definite (denoted by $A > 0$) if $x^T A x > 0$, $\forall x \in \mathbf{R^n}$, $x \neq 0$, or if for some $\epsilon > 0$, $x^T A x \geq \epsilon \|x\|^2$, $\forall x$;*

- *positive semi-definite (denoted by $A \geq 0$) if $x^T A x \geq 0$, $\forall x \in R^n$;*

- *negative semi-definite if $-A$ is positive semi-definite;*

- *negative definite if $-A$ is positive definite;*

- *symmetric if $A^T = A$;*

- *skew-symmetric if $A^T = -A$; and*

- *symmetric positive definite (semi-definite) if $A > 0 (\geq 0)$ and $A = A^T$*

Definition 2.2. *A function $f(x_1, x_2, \ldots, x_n) : \mathbf{R^n} \to \mathbf{R}$ is said to be of class C^k if all its partial derivatives $\frac{\partial^k f}{\partial x_{i_1}, x_{i_2} \ldots x_{i_k}}$ exist, and are continuous, where each of $i_1, i_2, \ldots, i_k$ is an integer between 1 and n, for any $k \in [1, \infty)$.*

Lemma 2.1. *[160] (Implicit Function Theorem) Consider a C^r function $f : R^{k+n} \to R^n$ with $f(a,b) = \boldsymbol{0}_{[n]}$ and $rank(D_f(a,b)) = n$ where $D_f(a,b) = \frac{\partial f(x,y)}{\partial y}|_{(x,y)=(a,b)} \in R^{n \times n}$. Then, there exists a neighborhood A of a in $\mathbf{R}^k$ and a unique C^r function $g : A \to R^n$ such that $g(a) = b$ and $f(x, g(x)) = \boldsymbol{0}_{[n]}, \forall x \in A$.*

Definition 2.3. *[136] Let $x_1(k)$ and $x_2(k)$ be two discrete-time scalar or vector signals, $\forall k \in Z_t^+$, for any t.*

- *We denote $x_1(k) = O[x_2(k)]$, if there exist positive constants m_1, m_2 and k_0 such that $\|x_1(k)\| \leq m_1 \max_{k' \leq k} \|x_2(k')\| + m_2, \quad \forall k > k_0$.*

- *We denote $x_1(k) = o[x_2(k)]$, if there exists a discrete-time function $\alpha(k)$ satisfying $\lim_{k \to \infty} \alpha(k) \to 0$ and a constant k_0 such that $\|x_1(k)\| \leq \alpha(k) \max_{k' \leq k} \|x_2(k')\|, \forall k > k_0$.*

- *We denote $x_1(k) \sim x_2(k)$ if they satisfy $x_1(k) = O[x_2(k)]$ and $x_2(k) = O[x_1(k)]$.*

For the convenience, in the followings we use $O[1]$ and $o[1]$ to denote bounded sequences and sequences converging to zero, respectively. In addition, if sequence $y(k)$ satisfies $y(k) = O[x(k)]$ or $y(k) = o[x(k)]$, then we may directly use $O[x(k)]$ or $o[x(k)]$ to denote sequence $y(k)$ for convenience.

According to Definition 2.3, we have the following proposition.

Proposition 2.1. *According to the definition on signal orders in Definition 2.3, we have following properties:*

(i) $O[x_1(k + \tau)] + O[x_1(k)] \sim O[x_1(k + \tau)], \forall \tau \geq 0$.

(ii) $x_1(k + \tau) + o[x_1(k)] \sim x_1(k + \tau), \forall \tau \geq 0$.

(iii) $o[x_1(k + \tau)] + o[x_1(k)] \sim o[x_1(k + \tau)], \forall \tau \geq 0$.

(iv) $o[x_1(k)] + o[x_2(k)] \sim o[|x_1(k)| + |x_2(k)|]$.

(v) $o[O[x_1(k)]] \sim o[x_1(k)] + O[1]$.

(vi) if $x_1(k) \sim x_2(k)$ and $\lim_{k \to \infty} \|x_2(k)\| = 0$, then $\lim_{k \to \infty} \|x_1(k)\| = 0$.

(vii) If $x_1(k) = o[x_1(k)] + o[1]$, then $\lim_{k \to \infty} \|x_1(k)\| = 0$.

(viii) Let $x_2(k) = x_1(k) + o[x_1(k)]$. If $x_2(k) = o[1]$, then $\lim_{k \to \infty} \|x_1(k)\| = 0$.

Proof. See Appendix 2.1. ■

Lemma 2.2. *Given a bounded sequence $X(k) \in R^m$. Define*

$$l_k = \arg \min_{l \leq k-n} \|X(k) - X(l)\| \tag{2.1}$$

Then, we have

$$\lim_{k \to \infty} \|X(k) - X(l_k)\| = 0$$

Proof. See Appendix 2.2. ■

Lemma 2.3. *[5] (Key Technical Lemma) For some given real scalar sequences $s(k)$, $b_1(k)$, $b_2(k)$ and vector sequence $\sigma(k)$, if the following conditions hold:*

(i) $\lim_{k \to \infty} \frac{s^2(k)}{b_1(k) + b_2(k)\sigma^T(k)\sigma(k)} = 0$,

(ii) $b_1(k) = O[1]$ and $b_2(k) = O[1]$,

(iii) $\sigma(k) = O[s(k)]$.

Then, we have
a) $\lim_{k \to \infty} s(k) = 0$, and b) $\sigma(k)$ is bounded.

Definition 2.4. *Let U be an open subset of R^{i+1}. A mapping $f(\omega) : U \to R$ is said to be Lipschitz on U, if there exists a positive constant L such that*

$$|f(\omega_a) - f(\omega_b)| \leq L\|\omega_a - \omega_b\|$$

for all $(\omega_a, \omega_b) \in U$.

Lemma 2.4. *If functions $f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot)$ are Lipschitz functions with Lipschitz coefficient $L_1, L_2, \ldots, L_n$, respectively. Then their composite function $f_1 \circ f_2 \circ \ldots f_n(\cdot)$ is still a Lipschitz function with Lipschitz coefficient $L = L_1 L_2 \ldots Ł_n$.*
Proof. By the definition of Lipschitz function,

$$
\begin{aligned}
|f_1 \circ f_2 \circ \ldots f_n(\omega_a) - f_1 \circ f_2 \circ \ldots f_n(\omega_b)| &\leq L_1\|f_2 \circ \ldots f_n(\omega_a) - f_2 \circ \ldots f_n(\omega_b)\| \\
&\leq \ldots \leq L_1 L_2 \ldots L_n\|\omega_a - \omega_b\| \tag{2.2}
\end{aligned}
$$

where ω_a and ω_b are arguments of $f_n(\cdot)$ and $L_1, L_2, \ldots, L_n$ are some constants. Let $L = L_1 L_2 \ldots Ł_n$ and it completes the proof. ■

Definition 2.5. *[161] The future state variables of a discrete-time system is said to be semi-determined future states (SDFS) at time instant k, if it can be determined based on the available system information up to time instant k, and controls up to time instant $k-1$ under the assumption that the dynamics of the plant and the disturbance are known.*

Definition 2.6. *[135] The future output of a discrete-time control system is said semi-determined future output (SDFO) at time instant k, if it can be predicted based on the available system information up to time instant k and controls up to time instant $k-1$ without considering the unknown uncertainties.*

Let us consider a class of general lower-triangular nonlinear systems described as

$$\begin{cases} \xi_i(k+1) = f_i(\bar{\xi}_i(k), \xi_{i+1}(k)), \ i = 1, 2, \ldots, n-1 \\ \xi_n(k+1) = f_n(\bar{\xi}_n(k), u(k), d(k)) + O[\bar{\xi}_n(k)] \\ y(k) = \xi_1(k) \end{cases} \tag{2.3}$$

with Lipschitz functions $f_i(\cdot)$ differentiable with respect to the second argument and bounded external disturbance $d(k) \in \mathbf{R}$.

Definition 2.7. *The partial derivatives $g_{1,i}(\cdot) = \frac{\partial f_i(\bar{\xi}_i(k), \xi_{i+1}(k))}{\partial \xi_{i+1}(k)}$, $i = 1, 2, \ldots, n-1$, and $g_{1,n}(\cdot) = \frac{\partial f_n(\bar{\xi}_n(k), u(k), d(k))}{\partial u(k)}$ are defined as control gain functions of system (2.3).*

Assuming that there exist constants $\bar{g}_j > \underline{g}_j > 0$ such that the control gain functions satisfy $\underline{g}_j \leq |g_{1,j}(\cdot)| \leq \bar{g}_j$, $j = 1, 2, \ldots, n$. Then, we have the following lemmas:

Lemma 2.5. *In system (2.3), the future states $\bar{\xi}_i(k+j)$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, n-i$, are SDFSs, and there exist prediction functions $P_{j,i}(\cdot)$ such that*

$$\bar{\xi}_i(k+j) = P_{j,i}(\bar{\xi}_{i+j}(k))$$

In addition, the prediction functions $P_{j,i}(\cdot)$ are also Lipschitz functions.
Proof. See Appendix 2.3. ∎

Lemma 2.6. *In system (2.3), the states and input of the system satisfy*

$$\bar{\xi}_i(k) \sim y(k+i-1), \ i = 1, 2, \ldots, n, \ u(k) = O[y(k+n)]$$

In particular, we have

$$\bar{\xi}_n(k) \sim y(k+n-1) \sim e(k+n-1), \quad u(k) = O[y(k+n)] = O[e(k+n)] \tag{2.4}$$

where $e(k) = y(k) - y_d(k)$ and $y_d(k)$ is bounded reference signal such that $y(k) \sim e(k)$.
Proof. See Appendix 2.4. ∎

Let us consider a class of general block-triangular MIMO nonlinear systems with pure-feedback subsystems described as

$$\Sigma : \begin{cases} \Sigma_1 \begin{cases} \xi_{1,i_1}(k+1) = f_{1,i_1}(\bar{\xi}_{1,i_1-m_{11}}(k), \bar{\xi}_{2,i_1-m_{12}}(k), \ldots, \bar{\xi}_{n,i_1-m_{1n}}(k), \\ \qquad\qquad \xi_{1,i_1+1}(k)), \ i_1 = 1, 2, \ldots, n_1 - 1 \\ \xi_{1,n_1}(k+1) = f_{1,n_1}(\Xi(k), u_1(k), d_1(k)) + O[\Xi(k)] \\ y_1(k) = \xi_{1,1}(k) \end{cases} \\ \vdots \\ \Sigma_j \begin{cases} \xi_{j,i_j}(k+1) = f_{j,i_j}(\bar{\xi}_{1,i_j-m_{j1}}(k), \bar{\xi}_{2,i_j-m_{j2}}(k), \ldots, \bar{\xi}_{n,i_j-m_{jn}}(k), \\ \qquad\qquad \xi_{j,i_j+1}(k)), \ i_j = 1, 2, \ldots, n_j - 1 \\ \xi_{j,n_j}(k+1) = f_{j,n_j}(\Xi(k), \bar{u}_j(k), d_j(k)) + O[\Xi(k)] \\ y_j(k) = \xi_{j,1}(k) \end{cases} \\ \vdots \\ \Sigma_n \begin{cases} \xi_{n,i_n}(k+1) = f_{n,i_n}(\bar{\xi}_{1,i_n-m_{n1}}(k), \bar{\xi}_{2,i_n-m_{n2}}(k), \ldots, \bar{\xi}_{n,i_n-m_{nn}}(k), \\ \qquad\qquad \xi_{n,i_n+1}(k)), \ i_n = 1, 2, \ldots, n_n - 1 \\ \xi_{n,n_n}(k+1) = f_{n,n_n}(\Xi(k), \bar{u}_n(k), d_n(k)) + O[\Xi(k)] \\ y_n(k) = \xi_{n,1}(k) \end{cases} \end{cases} \quad (2.5)$$

where $f_{j,i_j}(\cdot)$ are Lipschitz functions and

$$\bar{\xi}_{j,i_j}(k) = [\xi_{j,1}(k), \xi_{j,2}(k), \ldots, \xi_{j,i_j}(k)]^T \qquad (2.6)$$

is a vector of the first to the i_jth state variables of subsystem Σ_j, $i_j = 1, 2, \ldots, n_j$, and

$$\Xi(k) = [\bar{\xi}_{1,n_1}(k), \bar{\xi}_{2,n_2}(k), \ldots, \bar{\xi}_{n,n_n}(k)]^T \qquad (2.7)$$

is a vector of all the states in the whole system, which is assumed to be measurable.

Definition 2.8. *[154] The notation $m_{jl} = n_j - n_l$ used in (2.5) represents the order differ-ence between the jth and the lth subsystem. When $i_j - m_{jl} \leq 0$, states vectors $\bar{\xi}_{j,i_j-m_{jl}}(k)$ do not exist and are thus not included in the i_jth equation of subsystem Σ_j in (5.18). It is noted that when $l = j$, we have $m_{jl} = 0$ and $\bar{\xi}_{j,i_j-m_{jl}}(k) = \bar{\xi}_{j,i_j}(k)$, and when $i_j = n_j$, $j = 1, 2, \ldots, n$, we have $[\bar{\xi}_{1,n_j-m_{j1}}(k), \bar{\xi}_{2,n_j-m_{j2}}(k), \ldots, \bar{\xi}_{n,n_j-m_{jn}}(k)] = \Xi(k)$. This is the reason we use notation $\Xi(k)$ in the last equations of every subsystem Σ_j (2.5).*

Remark 2.1. *For a given subsystem Σ_j, the n_jth equation includes state vectors $\bar{\xi}_{n_l}(k)$ of all the subsystems Σ_l, $l = 1, 2, \ldots, n$. The $(n_j - 1)$th equation includes state vectors $\bar{\xi}_{n_l-1}(k)$ (because $n_j - 1 - m_{jl} = n_l - 1$) of all the subsystems Σ_l that are of order $n_l > 1$; the $(n_j - 2)$th*

equation includes state vectors $\bar{\xi}_{n_l-2}(k)$ (because $n_j - 2 - m_{jl} = n_l - 2$) of all the subsystems Σ_l that are of order $n_l > 2$; and so on and so forth. Besides, in the last equation of the dynamics of subsystem Σ_j, inputs from the first subsystem to the jth subsystem, $\bar{u}_j(k)$, are included.

Definition 2.9. *Denote $\bar{n} = \max\limits_{j=1}^{n}\{n_j\}$ and define a set $s_i = \{j|n_j = \bar{n}+1-i\}, i = 1, 2, \ldots, \bar{n}$, such that all the subsystems can be divided into $\bar{n}$ groups, with each group defined by a set $S_i = \{\Sigma_i|i \in s_i\}, i = 1, 2, \ldots, \bar{n}$. The set S_i may be an empty set if there is no subsystem of order $(\bar{n} + 1 - i)$. Furthermore, we assume that the number of the elements in S_i is m_i.*

Definition 2.10. *Define $g_{j,i_j}(\cdot) = \frac{\partial f_{j,i_j}(\cdot,\cdot)}{\partial \xi_{j,i_j+1}(k)}$, and $g_{j,n_j}(\cdot) = \frac{\partial f_{j,n_j}(\cdot,\cdot,\cdot)}{\partial u_j(k)}$ as control gain functions of system (2.5).*

Assume that there exist constants $\bar{g}_{j,i_j} > \underline{g}_{j,i_j} > 0$ such that $0 \leq \underline{g}_{j,i_j} \leq |g_{j,i_j}(\cdot)| \leq \bar{g}_{j,i_j}$, $j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, n_j$. Then, we have the following lemma:

Lemma 2.7. *Let $\bar{\xi}_{l,i_j-m_{jl}}(k) = 0$ and $y_l(k + i_j - m_{jl} - 1) = 0$, if $i_j - m_{jl} \leq 0$. The states and inputs of system (5.18) satisfy*

$$\sum_{l=1}^{n} O[\bar{\xi}_{l,i_j-m_{jl}}(k)] \sim \sum_{l=1}^{n} O[y_l(k + i_j - m_{jl} - 1)], \quad u_j(k) = O[\Xi(k+1)]$$

for $j = 1, 2, \ldots, n$ and $i_j = 1, 2, \ldots, n_j$. In particular, when $i_j = n_j$ we have $\sum\limits_{j=1}^{n} O[\bar{\xi}_{j,n_j}(k)] \sim \sum\limits_{j=1}^{n} O[y_j(k + n_j - 1)]$.

Proof. See Appendix 2.5. ■

Remark 2.2. *Lemma 2.7 for MIMO systems can be regarded as a counterpart of Lemma 2.6 for SISO systems.*

Lemma 2.8. *Consider sequences $x_j(k)$, $j = 1, 2, \ldots, n$, satisfy*

$$x_j(k) = \sum_{i=1}^{n} o[x_i(k - m_{ji})] + o[1]$$

Then, we have $\lim\limits_{k\to\infty} x_j(k) = 0$, $j = 1, 2, \ldots, n$.

Proof. For a given l, $l = 1, 2, \ldots, n$ from $x_j(k) = \sum\limits_{i=1}^{n} o[x_i(k - m_{ji})] + o[1]$, we have

$$x_j(k + n_j - n_l) = \sum_{i=1}^{n} o[x_i(k + n_i - n_l)] + o[1] \sim o[\sum_{i=1}^{n} |x_i(k + n_i - n_l)|] + o[1]$$

which further leads to

$$\sum_{j=1}^{n} |x_j(k + n_j - n_l)| + o[n] \sim o[\sum_{i=1}^{n} |x_i(k + n_i - n_l)|] + o[1]$$

from which we can obtain $\sum_{j=1}^{n} |x_j(k + n_j - n_l)| \sim o[1] \to 0$ which completes the proof. ∎

2.2 Preliminaries for NN Control

In this thesis, the following two kinds of LPNNs are used for approximation of general nonlinear functions to facilitate adaptive NN control design.

High Order Neural Networks: [1] The structure of HONN is expressed as followings:

$$\phi(W, z) = W^T S(z) \qquad W, \ S(z) \in R^l$$

$$S(z) = [s_1(z), \ s_2(z), \ \ldots, \ s_l(z)]^T, \tag{2.8}$$

$$s_i(z) = \prod_{j \in I_i} [s(z_j)]^{d_j(i)}, \quad i = 1, 2, ..., l \tag{2.9}$$

where $z \in \Omega_z \subset \mathbf{R}^m$ is the input to HONN, l the NN nodes number, $\{I_1, I_2, ..., I_l\}$ a collection of l not-ordered subsets of $\{1, 2, ..., m\}$, e.g., $I_1 = \{1, \ 3, \ m\}$, $I_2 = \{2, \ 4, \ m\}$, $d_j(i)$'s nonnegative integers, W an adjustable synaptic weight vector, and $s(z_j)$ a monotonically increasing and differentiable sigmoidal function. In this thesis, it is chosen as a hyperbolic tangent function, i.e., $s(z_j) = \frac{e^{z_j} - e^{-z_j}}{e^{z_j} + e^{-z_j}}$.

For a smooth function $\varphi(z)$ over a compact set $\Omega_z \subset \mathbf{R}^m$, given a small constant real number $\mu^* > 0$, if l is sufficiently large, there exist a set of ideal bounded weights vector W^* such that

$$\max |\varphi(z) - \phi(W^*, z)| < \mu(z), \ \ |\mu(z)| < \mu^* \tag{2.10}$$

From the universal approximation results for neural networks [162], it is known that the constant μ^* can be made arbitrarily small by increasing the NN nodes number l.

Lemma 2.9. *[1] Consider the basis functions of HONN (2.8) with z being the input vector. The following properties of HONN will be used in the proof of closed-loop system stability.*

$$\lambda_{max}[S(z)S^T(z)] < 1, \ \ S^T(z)S(z) < l \tag{2.11}$$

where $\lambda_{max}(M)$ denotes the max eigenvalue of M.

Radial Basis Function Neural Networks: [98] Considering the following RBF NN used to approximate a function $h(z) : R^m \rightarrow R$,

$$\phi(W, z) = W^T S(z) \tag{2.12}$$

where the input vector $z \in \Omega_z \subset \mathbf{R}^m$ is of NN input dimension m. Weight vector $W = [w_1, w_2, \cdots, w_l]^T \in R^l$, the NN node number $l > 1$, and $S(z) = [s_1(z), \cdots, s_l(z)]^T$, with $s_i(z)$ chosen as Gaussian functions as follows:

$$s_i(z) = \exp\left[\frac{-(z - \mu_i)^T (z - \mu_i)}{\eta_i^2}\right], i = 1, 2, ..., l \tag{2.13}$$

where $\mu_i = [\mu_{i1}, \mu_{i2}, \cdots, \mu_{iq}]^T$ is the center of the receptive field and η_i is the width of the Gaussian function.

It has been proven that the RBFNN (2.12) can approximate any continuous function over a compact set $\Omega_z \subset \mathbf{R}^q$ to arbitrary accuracy as

$$\phi(z) = W^{*T} S(z) + \epsilon_z, \ \forall z \in \Omega_z \tag{2.14}$$

where W^* is ideal constant weights, and ϵ_z is the approximation error.

Lemma 2.10. *[1] For the Gaussin RBFNN, if $\hat{z} = z - \epsilon\bar{\psi}$ where $\bar{\psi}$ is a bounded vector and constant $\epsilon > 0$, then*

$$S(\hat{z}) = S(z) + \epsilon S_t \tag{2.15}$$

where S_t is a bounded function vector.

Definition 2.11. *[157] A trajectory $x(k)$ of the closed-loop system is said to be semi-globally-uniformly-ultimately-bounded (SGUUB), if for any a priori given compact set, there exists a feedback control, a bound $\mu \geq 0$, and a number $N(\mu, x_0)$, such that the trajectory of the closed-loop system starting from the compact satisfy $\|x(k)\| \leq \mu$ for all $k \geq k_0 + N$.*

Remark 2.3. *The concept of SGUUB can be illustrated by three compact sets, namely, the initial compact set Ω_0, the bounding compact set Ω, and the steady state compact set Ω_s within Ω. If given any initial condition Ω_0, there is a corresponding control law valid on the bounding compact set Ω such that the states in the closed-loop system will never go beyond the bounding compact set Ω and will eventually be bounded in the steady state compact set Ω_s, then the closed-loop system is of SGUUB stability. Normally, the size of Ω_0 only affects the bounding compact set Ω but not affects the steady state compact set Ω_s.*

Part I

Adaptive Control Design

Chapter 3

Systems with Nonparametric Model Uncertainties

3.1   Introduction

As introduced in Section 1.1.1, adaptive backstepping in discrete-time was developed in [52] for strict-feedback system with unit control gains. The design approach has been further robustified to deal with nonparametric model uncertainties in [54–56], where projection operation was utilized in the control parameter update law to guarantee the boundedness of parameter estimates. The control design approach in these existing work depend on the knowledge of control gains and are not directly applicable to more general strict-feedback systems with unknown control gains. In this thesis, we will study adaptive control design for strict-feedback nonlinear systems with unknown control gains. In this Chapter, we start from the case that the control gains are partially unknown, i.e., the absolute values of the gains are unknown while the signs of the control gains are known. In the consequent Chapter 4, we will further remove the assumption on control directions.

The robust technique using projection operation in [54–56] guarantee the global stability of the adaptive closed-loop system in the presence of nonparametric model uncertainties. But this robustification method together with most other existing methods in discrete-time (refer to Section 1.1.2) is not able to achieve asymptotical tracking performance. However, it is interesting and challenging in discrete-time adaptive control to fully compensate for the effect of nonparametric nonlinear model uncertainties for exact tracking performance. There are some recent successful attempts to completely eliminate a class of nonparametric nonlinear uncertainty made in [163,164], but the designs greatly rely on the system structure

of first order and only scalar unknown parameter. In this Chapter we carry forward the study on full compensation of the effect of nonparametric nonlinear model uncertainties in discrete-time adaptive control of strict-feedback systems.

In Section 3.2, we start from compensation of matched nonparametric uncertainty. First, an auxiliary output which includes future states as well as both parametric and nonparametric uncertainties is introduced. Then, the prediction of the auxiliary output is constructed using the predicted states and estimated parameters, and is used to facilitate adaptive control design. In Section 3.3, we consider more complicated case of unmatched uncertainties. Auxiliary states including both parametric and nonparametric uncertainties are introduced to facilitate unmatched uncertainties compensation at the future states prediction stage. Auxiliary output is also introduced at the control stage for compensation of the uncertainty in the control range. For system with both matched and unmatched uncertainties, the adaptive control designed guarantee not only closed-loop stability but also asymptotic output tracking performance.

The uncertainty compensation technique requires the the nonlinearity satisfying Lipschitz condition, which is a common assumption for nonlinearity in the control community [163, 165–167]. Another requirement is the small Lipschitz coefficient of the uncertain nonlinearity, which is also usual in discrete-time control [56,75,137,168]. When the Lipschitz coefficient is large, discrete-time uncertain systems are not stabilizable as indicated in [59]. Actually, if the discrete-time models are derived from continuous-time models, the growth rate of nonlinear uncertainty can always be made sufficient small by choosing sufficient small sampling time. For example, let us consider a discrete-time system model derived from continuous-time model $\dot{x} = f_c(x) + \nu_c(x)$ with unknown function $\nu_c(\cdot)$ satisfying Lipschitz condition. Then the discrete-time model would be $x(k+1) = f_d(x(k)) + \nu_d(x(k))$ where $f_d(x(k)) = \int_{kT}^{(k+1)T} f_c(x)dx + x(k)$ and $\nu_d(x(k)) = \int_{kT}^{(k+1)T} \nu_c(x)dx$, where T is the sampling time. Then, it is always possible to make the Lipschitz coefficient of $\nu_d(x(k))$ arbitrarily small by choosing sufficiently small sampling time T.

The contributions in this Chapter lies in

(i) A systematic adaptive control design framework based on the predicted future states is developed for nonlinear discrete-time systems in strict-feedback form.

(ii) A novel deadzone with threshold converging to zero is proposed in the estimated parameter update law to handle the effect of uncertain nonlinearities.

(iii) A novel uncertain nonlinearities compensation technique is devised to eliminate the

effects of both matched and unmatched nonparametric uncertainties such that asymptotical tracking performance is obtained.

3.2 Systems with Matched Uncertainties

In this Section, we consider the simple case that the nonparametric model uncertainties only appear in the control range, i.e., matched condition. In discrete-time, sliding mode has been well studied to deal with matched uncertainty and offer robustness [69, 75, 76], but unlike in continuous-time, sliding mode in discrete-time is not able to eliminate the effect of uncertain nonlinearities in the output tracking performance. In this Section, adaptive control is constructed for strict-feedback systems using predicted future states on the base of the transformed systems, and a novel uncertain nonlinearity compensation mechanism is embedded into the control. There are parameter estimates update laws for both predictor and controller. The update laws for predictor are driven by the prediction errors of one step ahead predicted states, while the update law for controller is driven by an augmented error that combines both prediction errors and output tracking error.

In this Section, we will also consider time delayed states in the uncertain nonlinearity. Time-delay is an active topic of research because it is frequently encountered in engineering systems to be controlled [169]. Of great concern is the effect of time delay on stability and asymptotic performance. In continuous-time, some of the useful tools in robust stability analysis for time delays systems are based on the Lyapunov's second method, the Lyapunov-Krasovskii theorem and the Lyapunov-Razumikhin theorem. Following its success in stability analysis, the utility of Lyapunov-Krasovskii functionals were subsequently explored in adaptive control designs for continuous-time time delayed systems [149, 153, 170–172]. However, in the discrete-time there is not a counterpart of Lyapunov-Krasovskii functional. To solve the difficulties associated with delayed states in the nonparametric nonlinear uncertainties, an augmented states vector is introduced such that the effect of time delays can be canceled at the same time when the effect of nonlinear uncertainties are compensated.

3.2.1 Problem formulation

Let us consider strict-feedback nonlinear systems with both parametric uncertainties and matched nonparametric uncertainties as follows:

$$
\begin{cases}
\xi_i(k+1) = \Theta_i^T \Phi_i(\bar{\xi}_i(k)) + g_i \xi_{i+1}(k), \\
\quad i = 1, 2, \ldots, n-1 \\
\xi_n(k+1) = \Theta_n^T \Phi_n(\bar{\xi}_n(k)) + g_n u(k) + \nu(\bar{\xi}_n(k-\tau)) \\
y(k) = \xi_1(k)
\end{cases}
\tag{3.1}
$$

where state vectors $\bar{\xi}_j(k)$, control input $u(k)$ and system output $y(k)$ are are measurable, $\Theta_j \in \mathbf{R}^{\mathbf{p_j}}$, $g_j \in \mathbf{R}$, $j = 1, 2, \ldots, n$, are unknown parameters (p_j's are positive integers), $\Phi_j(\bar{\xi}_j(k)) : \mathbf{R}^{\mathbf{j}} \to \mathbf{R}^{\mathbf{p_j}}$ are known vector-valued functions, and $\nu(\bar{\xi}_n(k-\tau))$ is unknown nonlinear function which is regarded as nonparametric nonlinear model uncertainties. The unknown time delay τ satisfies $0 \leq \tau_{\min} \leq \tau \leq \tau_{\max}$ with known $\tau_{\min}$ and $\tau_{\max}$. The control objective is to make output exactly track a given bounded reference trajectory $y^*(k)$ and to guarantee the boundedness of all the closed-loop signals.

Assumption 3.1. *The functional uncertainty $\nu(\cdot)$, satisfies Lipschitz condition, i.e., $\|\nu(\varepsilon_1) - \nu(\varepsilon_2)\| \leq L_\nu \|\varepsilon_1 - \varepsilon_2\|$, $\forall \varepsilon_1, \varepsilon_2 \in \mathbf{R}^{\mathbf{n}}$, where $L_\nu < \lambda^*$ with λ^* is a small number defined in (3.52). The system functions $\Phi_j(\cdot)$, $j = 1, 2, \ldots, n$, are also Lipschitz functions with Lipschitz coefficients L_j.*

Remark 3.1. *Any continuously derivable function is Lipschitz on a compact set [173] and any function with bounded derivative is global Lipschitz. As our objective is to achieve global asymptotical stability, it is not stringent to assume that the nonlinearity is global Lipschitz.*

Remark 3.2. *As pointed in [59], it is impossible to obtain global stability results for discrete-time controlled system when the nonlinear uncertainties are of large growth rates. Thus, it is usual to assume that the nonparametric nonlinear uncertainties are of small growth rates [54, 56, 75, 76, 168] or even globally bounded [62, 136] and their growth rates can be guaranteed to be smaller than a specified constant. In the case that the discrete-time model is derived from a continuous-time model, the growth rates of the nonlinear uncertainties can be made small enough by choosing sufficient small sampling time T.*

Assumption 3.2. *The control directions, signs of control gains g_j, $(j = 1, 2, \ldots, n)$ are known. Without loss of generality, it is assumed that g_j are positive constants with known lower bounds $\underline{g}_j > 0$, i.e., $g_j \geq \underline{g}_j > 0$.*

In Chapter 4, the assumption of knowledge of control directions and lower bounds will be removed for adaptive control design.

3.2.2 Future states prediction

As mentioned , the discrete-time backstepping in [52,54–56] involves coordinate transformation than relies on know gains. In this Chapter and next Chapter, we develop an alternative control design approach without any knowledge of control gains. The key component in the control design is future states prediction and it will be constructed in this Section. According to the structure of system (3.1), the future states $\bar{\xi}_i(k+n-i)$, $i = 1, 2, \ldots, n-1$, are deterministic at the kth step because they are not dependent of control input, in other words, the systems in (3.1) are of relative degree n.

Let us consider predicting these future states at the kth step despite the presence of the unknown parameters. Denote $\hat{\Theta}_i(k)$ and $\hat{g}_i(k)$ as the estimates of Θ_i and g_i at the kth step, respectively, and further let us denote

$$\bar{\hat{\Theta}}_i(k) = [\hat{\Theta}_i^T(k), \hat{g}_i(k)]^T \in \mathbf{R}^{p_i+1}, \qquad \bar{\tilde{\Theta}}_i(k) = [\tilde{\Theta}_i^T(k), \tilde{g}_i(k)]^T \tag{3.2}$$

where $\tilde{\Theta}_i(k) = \hat{\Theta}_i(k) - \Theta_i$ and $\tilde{g}_i(k) = \hat{g}_i(k) - g_i$ are parameter estimates errors.

Using the estimated parameters, we define one-step ahead predictions $\hat{\xi}_i(k+1|k)$, as prediction of one-step future states $\xi_i(k+1)$ as follows:

$$\hat{\xi}_i(k+1|k) = \bar{\hat{\Theta}}_i^T(k-n+2)\Psi_i(k), \quad i = 1, 2, \ldots, n-1 \tag{3.3}$$

where

$$\Psi_i(k) = [\Phi_i^T(\bar{\xi}_i(k)), \xi_{i+1}(k)]^T \in \mathbf{R}^{p_i+1} \tag{3.4}$$

It is noted that the prediction is only proceeded for the first $(n-1)$ states because the nth state $\xi_n(k+1|k)$ involves control input and thus is not predictable at the kth step.

Moving one step ahead in the equations of system (3.1), we see that the two-step ahead predictions can be constructed by substituting the one-step future states with one-step predicted states. But because there is no prediction for $\bar{\xi}_n(k+1|k)$, the two-step ahead prediction can only be proceeded up to the $(n-2)$th state, i.e., $\bar{\xi}_i(k+2)$, $i = 1, 2, \ldots, n-2$. Let us defined two-step ahead predictions $\hat{\xi}_i(k+2|k)$, as prediction of two-step future states $\xi_i(k+2)$ as follows:

$$\hat{\xi}_i(k+2|k) = \bar{\hat{\Theta}}_i^T(k-n+3)\hat{\Psi}_i(k+1|k), \quad i = 1, 2, \ldots, n-2 \tag{3.5}$$

where

$$\begin{aligned}
\hat{\Psi}_i(k+1|k) &= [\Phi_i^T(\bar{\hat{\xi}}_i(k+1|k)), \hat{\xi}_{i+1}(k+1|k)]^T \in \mathbf{R}^{p_i+1} \\
\bar{\hat{\xi}}_i(k+1|k) &= [\hat{\xi}_1(k+1|k), \hat{\xi}_2(k+1|k), \ldots, \hat{\xi}_i(k+1|k)]^T
\end{aligned} \tag{3.6}$$

In the same manner, with parameter estimates and predicted future states at previous steps, we define the j-step ($j = 3, 4, \ldots, n - 1$) prediction, $\hat{\xi}_i(k+j|k)$, as predict of $\xi_i(k+j)$ as follows:

$$\hat{\xi}_i(k+j|k) = \bar{\hat{\Theta}}_i^T(k - n + j + 1)\hat{\Psi}_i(k+j-1|k), \quad i = 1, 2, \ldots, n - j \tag{3.7}$$

where

$$\hat{\Psi}_i(k+j) = [\Phi_i^T(\hat{\bar{\xi}}_i(k+j-1|k)), \quad \hat{\xi}_{i+1}(k+j-1|k)]^T$$
$$\hat{\bar{\xi}}_i(k+j-1|k) = [\hat{\xi}_1(k+j-1|k), \hat{\xi}_2(k+j-1|k), \ldots, \hat{\xi}_i(k+j-1|k)]^T \tag{3.8}$$

Remark 3.3. *It is noted in (3.3), (3.5) and (3.7) that estimated parameters at previous steps rather than at the kth step are used in the predictions. The estimated parameters at which step are to be utilized depend on how many steps ahead the predictions are carried out. The advantage of arranging estimated parameters in this way in the predictions can be seen in the proof of Lemma 3.2.*

The parameter estimates used above are calculated from the following update law:

$$\bar{\hat{\Theta}}_i(k+1) = \bar{\hat{\Theta}}_i(k - n + 2) - \frac{\tilde{\xi}_i(k+1|k)\Psi_i(k)}{D_i(k)}$$
$$\tilde{\xi}_i(k+1|k) = \hat{\xi}_i(k+1|k) - \xi_i(k+1), \quad i = 1, 2, \ldots, n - 1$$
$$D_i(k) = 1 + \Psi_i^T(k)\Psi_i(k) \tag{3.9}$$

Lemma 3.1. *The parameter estimates $\bar{\hat{\Theta}}_i(k)$, $i = 1, 2, \ldots, n - 1$, in (3.9) are bounded and the prediction errors satisfy*

$$\tilde{\bar{\xi}}_i(k + n - i|k) = o[O[y(k + n - 1)]]$$

where

$$\tilde{\bar{\xi}}_i(k + n - i|k) = \hat{\bar{\xi}}_i(k + n - i|k) - \bar{\xi}_i(k + n - i)$$
$$\hat{\bar{\xi}}_i(k + n - i|k) = [\hat{\xi}_1(k + n - 1|k), \hat{\xi}_2(k + n - 2|k), \ldots, \hat{\xi}_i(k + n - i|k)]^T \tag{3.10}$$

Proof. In the beginning, let us start to analyze the one-step prediction error,

$$\tilde{\xi}_i(k + 1|k) = \hat{\xi}_i(k + 1|k) - \xi_i(k + 1), \quad i = 1, 2, \ldots, n - 1$$

Note that

$$\tilde{\xi}_i(k + 1|k) = \bar{\hat{\Theta}}_i^T(k - n + 2)\Psi_i(k)$$

and consider a Lyapunov function

$$V_i(k) = \sum_{j=k-n+2}^{k} \|\bar{\tilde{\Theta}}_i(j)\|^2$$

It is easy for us to follow the analysis of projection algorithm in [57] and conclude from (3.9) that $\bar{\tilde{\Theta}}_i(k)$ is bounded and

$$\frac{\tilde{\xi}_i(k+1|k)}{D_i^{\frac{1}{2}}(k)} \quad := \quad \alpha(k) \in L^2[0,\infty) \tag{3.11}$$

According the definition of $D_i(k)$ in (3.9), Lemma 2.6 and the Lipschitz condition of $\Psi_i(\cdot)$, we have

$$D_i^{\frac{1}{2}}(k) = O[y(k+i)] \tag{3.12}$$

Then, from (3.11) and Proposition 2.1, we can see

$$\tilde{\xi}_i(k+1|k) = o[O[y(k+i)]], \quad i = 1, 2, \ldots, n-1$$
$$\bar{\tilde{\xi}}_i(k+1|k) = [\tilde{\xi}_1(k+1|k), \tilde{\xi}_2(k+1|k), \ldots, \tilde{\xi}_i(k+1|k)] = o[O[y(k+i)]] \tag{3.13}$$

Next, let us analyze the prediction errors of two-step ahead predictions:

$$\tilde{\xi}_i(k+2|k) = \hat{\xi}_i(k+2|k) - \xi_i(k+2), \quad i = 1, 2, \ldots, n-2$$

which can be written as

$$\tilde{\xi}_i(k+2|k) = \tilde{\xi}_i(k+2|k+1) + \check{\xi}_i(k+2|k)$$

where

$$\tilde{\xi}_i(k+2|k+1) \stackrel{def}{=} \hat{\xi}_i(k+2|k+1) - \xi_i(k+2) = o[O[y(k+i+1)]] \tag{3.14}$$
$$\check{\xi}_i(k+2|k) \stackrel{def}{=} \hat{\xi}_i(k+2|k) - \hat{\xi}_i(k+2|k+1)$$
$$= \bar{\tilde{\Theta}}_i^T(k-n+3)[\hat{\Psi}_i(k+1|k) - \Psi_i(k+1)] \tag{3.15}$$

Remark 3.4. *In (3.15), it can be seen that if there is not a common factor $\bar{\tilde{\Theta}}_i^T(k-n+3)$ in the expressions of $\hat{\xi}_i(k+2|k)$ and $\hat{\xi}_i(k+2|k+1)$, the expression of $\check{\xi}_i(k+2|k)$ will involve the difference of estimated parameters at different steps and will become very complicated. This demonstrate the advantage of using estimated parameters at different steps stated in Remark 3.3*

From the Lipschitz condition of $\Psi_i(\cdot)$ and (3.13), we have

$$\|\hat{\Psi}_i(k+1|k) - \Psi_i(k+1)\| \leq L_i\|\bar{\tilde{\xi}}_{i+1}(k+1|k)\| = o[O[y(k+i+1)]]$$

Consider the boundedness of $\bar{\hat{\Theta}}_i^T(k-n+3)$, from (3.15) we have

$$\check{\xi}_i(k+2|k) = o[O(y(k+i+1))]$$

Consequently, we have

$$
\begin{aligned}
\tilde{\xi}_i(k+2|k) &= o[O[y(k+i+1)] \quad i = 1,2,\ldots,n-2 & (3.16)\\
\bar{\tilde{\xi}}_i(k+2|k) &= [\tilde{\xi}_1(k+2|k), \tilde{\xi}_2(k+2|k), \ldots, \tilde{\xi}_i(k+2|k)]\\
&= o[O[y(k+i+1)] & (3.17)
\end{aligned}
$$

Let us analyze the prediction errors of the j step ahead predictions:

$$\tilde{\xi}_i(k+j|k) = \hat{\xi}_i(k+j|k) - \xi_i(k+j), \quad i = 1,2,\ldots,n-j, \quad j = 3,4,\ldots,n-1$$

In the similar way, it can be written as

$$\tilde{\xi}_i(k+j|k) = \tilde{\xi}_i(k+j|k+1) + \check{\xi}_i(k+j|k)$$

where

$$
\begin{aligned}
\tilde{\xi}_i(k+j|k+1) &= \hat{\xi}_i(k+j|k+1) - \xi_i(k+j) = o[O(y(k+i+j-1))] & (3.18)\\
\check{\xi}_i(k+j|k) &= \hat{\xi}_i(k+j|k) - \hat{\xi}_i(k+j|k+1) & (3.19)\\
&= \bar{\hat{\Theta}}_i^T(k-n+j+1)[\hat{\Psi}_i(k+j-1|k) - \Psi_i(k+j-1|k+1)]
\end{aligned}
$$

Consider the Lipschitz condition of $\Psi_i(\cdot)$, we have

$$\|\hat{\Psi}_i(k+j-1|k) - \Psi_i(k+j-1|k+1)\| \leq L_i\|\bar{\tilde{\xi}}_{i+1}(k+j-1|k)\| = o[O[y(k+i+j-1)]]$$

where

$$\bar{\check{\xi}}_{i+1}(k+j|k) = [\check{\xi}_1(k+j|k), \check{\xi}_2(k+j|k), \ldots, \check{\xi}_{i+1}(k+j|k)]$$

It together with the boundedness of $\bar{\hat{\Theta}}_i^T(k-n+j-1)$ leads to

$$\tilde{\xi}_i(k+j|k) = o[O[y(k+i+j-1)]\}]c\check{\tilde{\xi}}_i(k+j|k) = o[O[y(k+i+j-1)]\}] \quad (3.20)$$

Let $j = n - i$, we have the following result

$$\bar{\tilde{\xi}}_i(k+n-i|k) = o[O[y(k+n-1)] \quad i = 1,2,\ldots,n-1$$

This completes the proof. ∎

3.2.3 Adaptive control design

Using the predicted future states in Section 3.2.2, adaptive control is synthesized in this Section. To begin with, let us rewrite 3.1 as follows:

$$\begin{cases} \xi_i(k+n-i+1) = \Theta_i^T \Phi_i(\bar{\xi}_i(k+n-i)) + g_i \xi_{i+1}(k+n-i) \\ \qquad i = 1, 2, \ldots, n-1 \\ \xi_n(k+1) = \Theta_n^T \Phi_n(\bar{\xi}_n(k)) + g_n u(k) + \nu(\bar{\xi}_n(k-\tau)) \\ y(k) = \xi_1(k) \end{cases} \tag{3.21}$$

Then, combining the n equations in (3.21) together by iterative substitutions, we have

$$y(k+n) = \Theta_f^T \Phi(k+n-1) + g u(k) + \nu(\bar{\xi}_n(k-\tau)) \tag{3.22}$$

where

$$\Theta_f = [\Theta_{f1}^T, \ldots, \Theta_{fn}^T]^T \in \mathbf{R}^p, \quad g = \prod_{j=1}^n g_j$$

$$\Theta_{f_1} = \Theta_1, \ \Theta_{f_i} = \Theta_i \prod_{j=1}^{i-1} g_j, \quad i = 2, 3, \ldots, n \tag{3.23}$$

$$\Phi(k+n-1) = [\Phi_1^T(\xi_1(k+n-1)), \Phi_2^T(\bar{\xi}_2(k+n-2)), \ldots, \Phi_n^T(\bar{\xi}_n(k))]^T \in \mathbf{R}^p$$

with $p = \sum_{j=1}^n p_j$. It is easy to check that $g \geq \prod_{j=1}^n \underline{g}_j := \underline{g}$.

It is noted that in (3.22) and (3.23)that function $\Phi(k+n-1)$ involves states at future steps such that the noncausal problem will occur if adaptive control is directly designed based (4.7). To solve the noncausal problem, the predicted future states in Section 3.2.2 can be used to construct a prediction of $\Phi(k+n-1)$ in the following manner:

$$\hat{\Phi}(k+n-1|k) = [\Phi_1^T(\hat{\xi}_1(k+n-1|k)), \Phi_2^T(\hat{\bar{\xi}}_2(k+n-2|k)), \ldots, \Phi_n^T(\bar{\xi}_n(k))]^T \tag{3.24}$$

with predicted future state vectors

$$\hat{\bar{\xi}}_i(k+n-i|k) = [\hat{\xi}_1(k+n-i|k), \hat{\xi}_2(k+n-i|k), \ldots, \hat{\xi}_i(k+n-i|k)]^T, i = 1, 2, \ldots, n-1$$

obtained from Section 3.2.2.

Lemma 3.2. : *Denote* $\tilde{\Phi}(k+n-1|k) = \hat{\Phi}(k+n-1|k) - \Phi(k+n-1)$, *where* $\hat{\Phi}(k+n-1|k)$ *and* $\Phi(k+n-1)$ *are defined in (3.24) and (3.23). Then, we have*

$$\tilde{\Phi}(k+n-1|k) = o[O[y(k+n-1)]]$$

Proof. Noting the Lipschitz condition of $\Phi_i(\cdot)$, $i = 1, 2, \ldots, n$, one can easily derive it from the result that $\bar{\bar{\xi}}_i(k + n - 1|k) = o[O[y(k + n - 1)]]$ in Lemma 3.2. ∎

In order for compensation for the effect of time delays in the uncertain term $\nu(\bar{\xi}_n(k - \tau))$, we introduce a vector of delayed states as follows

$$X(k) = [\bar{\xi}_n^T(k - \tau_{\min}), \ldots, \bar{\xi}_n^T(k - \tau), \ldots, \bar{\xi}_n^T(k - \tau_{\max})] \tag{3.25}$$

According to Lemma 2.2, we define

$$l_k = \arg \min_{l \leq k - n} \|X(k) - X(l)\| \tag{3.26}$$

From Lemma 2.2, one sees that if $X(k)$ is bounded, then $\|X(k) - X(l_k)\| \to 0$, where

$$X(l_k) = [\bar{\xi}_n^T(l_k - \tau_{\min}), \ldots, \bar{\xi}_n^T(l_k - \tau), \ldots, \bar{\xi}_n^T(l_k - \tau_{\max})] \tag{3.27}$$

Next, let us define an auxiliary output $y_a(k + n - 1)$ as

$$y_a(k + n - 1) = \Theta_f^T \Phi(k + n - 1) + \nu(\bar{\xi}_n(k - \tau)) \tag{3.28}$$

which includes both unknown parameter vector Θ_f and nonparametric uncertainty $\nu(\cdot)$. Then, system (3.22) can be rewritten as

$$y(k + n) = y_a(k + n - 1) + gu(k) \tag{3.29}$$

From (3.28) and (3.29), it is easy to derive that

$$
\begin{aligned}
y_a(k + n - 1) &= y_a(k + n - 1) - y_a(l_k + n - 1) + y_a(l_k + n - 1) \\
&= \Theta_f^T[\Phi(k + n - 1) - \Phi(l_k + n - 1)] + \nu(\bar{\xi}_n(k - \tau)) - \nu(\bar{\xi}_n(l_k - \tau)) \\
&\quad + y(l_k + n) - gu(l_k)
\end{aligned}
\tag{3.30}
$$

According to Assumption 3.1, if

$$\|\bar{\xi}_n(k - \tau) - \bar{\xi}_n(l_k - \tau)\| \to 0$$

then

$$\|\nu(\bar{\xi}_n(k - \tau)) - \nu(\bar{\xi}_n(l_k - \tau))\| \to 0$$

so that the effect of the uncertain function $\nu_j(\cdot)$ will be eliminated in (3.30). Thus, we predict $\hat{y}_j^a(k + n_j - 1|k)$ based on (3.30) in a straightforward manner by ignoring the nonlinear uncertainty terms of $\nu_j(\cdot)$ and only dealing with the parametric uncertainty.

Define $\hat{\Theta}_f(k)$ and $\hat{g}(k)$ as the estimates of unknown parameters Θ_f and g and they will be calculated from (3.37). Then, let us define the following prediction of $y_a(k+n-1)$

$$\hat{y}_a(k+n-1|k) = \hat{\Theta}_f^T(k)[\hat{\Phi}(k+n-1|k) - \Phi(l_k+n-1)] + y(l_k+n) - \hat{g}(k)u(l_k) \quad (3.31)$$

where l_k is defined in (3.26), satisfying $l_k \leq k-n$, and $\hat{\Phi}(k+n-1|k)$

Define parameter estimate errors $\tilde{\Theta}_f(k) = \hat{\Theta}_f(k) - \Theta_f$ and $\tilde{g}(k) = \hat{g}(k) - g$, and then from (3.30) and (3.31), we have the prediction error of auxiliary output as

$$
\begin{aligned}
\tilde{y}_a(k+n-1|k) &= \hat{y}_a(k+n-1|k) - y_a(k+n-1) \\
&= \tilde{\Theta}_f^T(k)[\Phi(k+n-1) - \Phi(l_k+n-1)] + \beta(k+n-1) - \tilde{g}(k)u(l_k) \\
&\quad -[\nu(\bar{\xi}_n(k-\tau)) - \nu(\bar{\xi}_n(l_k-\tau))]
\end{aligned}
\quad (3.32)
$$

where

$$\beta(k+n-1) = \hat{\Theta}_f^T(k)[\hat{\Phi}(k+n-1|k) - \Phi(k+n-1)] \quad (3.33)$$

can be regarded as a measure of future states prediction error.

Using the estimated auxiliary output, the adaptive control law is designed as

$$u(k) = -\frac{1}{\hat{g}(k)}(\hat{y}_a(k+n-1|k) - y^*(k+n)) \quad (3.34)$$

Remark 3.5. *It will be shown later in Lemma 3.3 that $\hat{g}(k)$ obtained from (3.37) is guaranteed to be bounded away from zero such that the adaptive control defined in (3.34) is well defined without singularity problem.*

Combining equations (3.29), (3.32) and (3.34) together, we obtain the error dynamics as

$$
\begin{aligned}
e(k+n) &= y_a(k+n-1) + \hat{g}(k)u(k) - \tilde{g}(k)u(k) - y^*(k+n) \\
&= -\tilde{y}_a(k+n-1|k) - \tilde{g}(k)u(k) \\
&= -\tilde{\Theta}_f^T(k)[\Phi(k+n-1) - \Phi(l_k+n-1)] - \tilde{g}(k)[u(k) - u(l_k)] \\
&\quad -\beta(k+n-1) + \nu(\bar{\xi}_n(k-\tau)) - \nu(\bar{\xi}_n(l_k-\tau))
\end{aligned}
\quad (3.35)
$$

According to Lipschitz condition of $\nu(\cdot)$ in Assumption 3.1 and the definition of $X(k)$ in (3.25), we have

$$
\begin{aligned}
|\nu(\bar{\xi}_n(k-\tau)) - \nu(\bar{\xi}_n(l_k-\tau))| &\leq L_\nu\|\bar{\xi}_n(k-\tau) - \bar{\xi}_n(l_k-\tau)\| \\
&\leq \lambda\|X(k) - X(l_k)\|
\end{aligned}
\quad (3.36)
$$

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where λ can be any constant satisfying $L_\nu \leq \lambda < \lambda^*$, with λ^* defined later in (3.52).

The estimated parameters in the auxiliary output estimation (3.31) and adaptive controller (3.34) are calculated by the following update law

$$
\begin{aligned}
\epsilon(k) &= e(k) + \beta(k-1) \\
\hat{\Theta}_f(k) &= \hat{\Theta}_f(k-n) + \gamma \frac{a(k)\epsilon(k)[\Phi(k-1) - \Phi(l_{k-n} + n - 1)]}{D(k-n)} \\
\hat{g}(k) &= \begin{cases} \hat{g}'(k) & \text{if } \hat{g}'(k) > \underline{g} \\ \underline{g} & \text{otherwise} \end{cases} \\
\hat{g}'(k) &= \hat{g}(k-n) + \frac{\gamma a(k)\epsilon(k)}{D(k-n)}[u(k-n) - u(l_{k-n})] \\
D(k-n) &= 1 + \|\Phi(k-1) - \Phi(l_{k-n} + n - 1)\|^2 + [u(k-n) - u(l_{k-n})]^2
\end{aligned}
\tag{3.37}
$$

where $0 < \gamma < 2$, $\epsilon(k)$ is introduced as an augmented tracking error and the deadzone $a(k)$ is defined as

$$
a(k) = \begin{cases} 1 - \frac{\lambda\|X(k-n) - X(l_{k-n})\|}{|\epsilon(k)|} & \text{if } |\epsilon(k)| > \lambda\|X(k-n) - X(l_{k-n})\| \\ 0 & \text{otherwise} \end{cases}
\tag{3.38}
$$

Remark 3.6. *It should be noted that $\beta(k-1)$ and $\Phi(k-1)$ used in the update law are available at the kth step because they involve no future states, such that there is no noncausal problem in the control parameter update law defined in (3.37).*

From definition of $a(k)$ in (3.38), we have the following equality and inequality

$$
a^2(k)\epsilon^2(k) = a(k)\epsilon^2(k) - a(k)\lambda|\epsilon(k)|\|X(k-n) - X(l_{k-n})\|
\tag{3.39}
$$

$$
0 \leq |\epsilon(k)| \leq a(k)|\epsilon(k)| + \lambda\|X(k-n) - X(l_{k-n})\|
\tag{3.40}
$$

which will be used for stability analysis later.

Remark 3.7. *It will be shown later that the threshold of the deadzone converges to zero because $\|X(k-n) - X(l_{k-n})\|$ will be made to vanish ultimately. At the same time, it will be shown that the augmented tracking error will also converge to zero.*

Lemma 3.3. *Consider the parameter estimates $\hat{g}(k)$ and $\hat{g}'(k)$ defined in (3.37), we have $\tilde{g}'^2(k) \geq \tilde{g}^2(k)$, where $\tilde{g}'(k) = \hat{g}'(k) - g$ and $\tilde{g}(k) = \hat{g}(k) - g$.*

Proof. According to (3.37), we see that $\tilde{g}'(k) = \tilde{g}(k)$ when $\hat{g}'(k) > \underline{g}$. Now, consider when $\hat{g}'(k) \leq \underline{g}$, we have $\tilde{g}(k) = \underline{g} - g$, so that

$$
\tilde{g}'^2(k) = [(\hat{g}'(k) - \underline{g}) + (\underline{g} - g)]^2 \geq (\underline{g} - g)^2 = \tilde{g}^2(k)
$$

where $\underline{g} \leq g$ is used. This completes the proof of $\tilde{g}'^2(k) \geq \tilde{g}^2(k)$. ∎

The main result of the control performance is summarized in the following theorem.

Theorem 3.1. *Consider the adaptive closed-loop system consisting of system (3.1), control law (3.34) and parameter adaptation law (3.37). All the signals in the closed-loop system are bounded and furthermore, the tracking error $e(k)$ converges to zero eventually.*

Proof. Substituting the error dynamics (3.35) into the augmented error $\epsilon(k)$ defined in (3.37), we have

$$
\begin{aligned}
\epsilon(k) &= -\tilde{\Theta}_f^T(k-n)[\Phi(k-1) - \Phi(l_{k-n} + n - 1)] - \tilde{g}(k-n)[u(k-n) - u(l_{k-n})] \\
&\quad + \nu(\bar{\xi}_n(k-n-\tau)) - \nu(\bar{\xi}_n(l_{k-n} - \tau))
\end{aligned} \tag{3.41}
$$

Choose a Lyapunov candidate as

$$
V(k) = \sum_{j=1}^{n} \|\tilde{\Theta}_f(k-n+j)\|^2 + \sum_{j=1}^{n} \tilde{g}^2(k-n+j) \tag{3.42}
$$

From (3.37), it is easy to derive that the difference of $V(k)$ is

$$
\begin{aligned}
\Delta V(k) &= V(k) - V(k-1) \\
&\leq \tilde{\Theta}_f^T(k)\tilde{\Theta}_f(k) - \tilde{\Theta}_f^T(k-n)\tilde{\Theta}_f(k-n) + \tilde{g}'^2(k) - \tilde{g}^2(k-n) \\
&= \frac{a^2(k)\gamma^2\epsilon^2(k)\{\|\Phi(k-1) - \Phi(l_{k-n} + n - 1)\|^2 + [u(k-n) - u(l_{k-n})]^2\}}{D^2(k-n)} \\
&\quad + \tilde{\Theta}_f^T(k-n)[\Phi(k-1) - \Phi(l_{k-n} + n - 1)]\epsilon(k)\frac{2a(k)\gamma}{D(k-n)} \\
&\quad + \tilde{g}(k-n)[u(k-n) - u(l_{k-n})]\epsilon(k)\frac{2a(k)\gamma}{D(k-n)}
\end{aligned} \tag{3.43}
$$

where inequality $\tilde{g}^2(k) \leq \tilde{g}'^2(k)$ in Lemma 3.3 is used.

From inequality (3.36) and the augmented error equation (3.41), it is easy to obtain

$$
\begin{aligned}
&a(k)\{\tilde{\Theta}_f^T(k-n)[\Phi(k-1) - \Phi(l_{k-n} + n - 1)] + \tilde{g}(k-n)[u(k-n) - u(l_{k-n})]\}\epsilon(k) \\
&= a(k)\epsilon(k)[\nu(\bar{\xi}_n(k-n-\tau)) - \nu(\bar{\xi}_n(l_{k-n} - \tau))] - a(k)\epsilon^2(k) \\
&\leq a(k)[\lambda|\epsilon(k)|\|X(k-n) - X(l_{k-n})\| - \epsilon^2(k)] = -a^2(k)\epsilon^2(k)
\end{aligned} \tag{3.44}
$$

where the last equality is established in (3.39). According to inequality (3.44), and the definition of $D(k-n)$ in (3.37), the difference of $V(k)$ in (3.43) can be written as

$$
\Delta V(k) \leq \frac{a^2(k)\gamma^2\epsilon^2(k)}{D(k-n)} - \frac{2a^2(k)\gamma\epsilon^2(k)}{D(k-n)} = -\frac{\gamma(2-\gamma)a^2(k)\epsilon^2(k)}{D(k-n)} \tag{3.45}
$$

Noting that $0 < \gamma < 2$ and $\Delta V(k)$ is nonpositive, the boundedness of $V(k)$ and thus the boundedness of $\hat{\Theta}_f(k)$ and $\hat{g}(k)$ are guaranteed. Taking summation on both hand sides of the equation above, we obtain $\sum_{k=0}^{\infty} \gamma(2-\gamma)\frac{a^2(k)\epsilon^2(k)}{D(k-n)} \leq V(0) - V(\infty)$ which implies

$$
\lim_{k\to\infty} \frac{a^2(k)\epsilon^2(k)}{D(k-n)} = 0 \tag{3.46}
$$

Now, consider the definitions of $\beta(k)$ in (3.33), $\Phi(k+n-1)$ in (3.23), and $\hat{\Phi}(k+n-1|k)$ in (3.24). Following Lemma 3.2 and Lipschitz condition of $\Phi(\cdot)$ in Assumption 3.1, we have $\beta(k+n-1) = o[O[y(k+n-1)]]$. Due to $y(k) \sim e(k)$, we have

$$\beta(k-1) = o[O[e(k)]] \tag{3.47}$$

Thus, the augmented error $\epsilon(k)$ in (3.37) can be written as

$$\epsilon(k) = e(k) + o[O[e(k)]] \tag{3.48}$$

According to Proposition 2.1, we have $\epsilon(k) \sim e(k) \sim y(k)$. Furthermore, from Lemma 2.6, we have $\bar{\xi}_n(k-n+1) = O[y(k)]$, which yields

$$\|\bar{\xi}_n(k-n+1)\| \le C_1 \max_{k' \le k}\{|\epsilon(k')|\} + C_2 \tag{3.49}$$

where C_1 and C_2 are some constants. From the definition of deadzone in (3.38), when

$$|\epsilon(k)| > \lambda \|X(k-n) - X(l_{k-n})\|$$

we have

$$a(k)|\epsilon(k)| = |\epsilon(k)| - \lambda\|X(k-n) - X(l_{k-n})\| > 0$$

while when

$$|\epsilon(k)| \le \lambda \|X(k-n) - X(l_{k-n})\|$$

we have

$$a(k)|\epsilon(k)| = 0 \ge |\epsilon(k)| - \lambda\|X(k-n) - X(l_{k-n})\|$$

Therefore, we have

$$|\epsilon(k)| - \lambda\|X(k-n) - X(l_{k-n})\| \le a(k)|\epsilon(k)|$$

Thus, inequality (3.49) becomes

$$
\begin{aligned}
\|\bar{\xi}_n(k-n+1)\| \quad \le \quad & C_1 \max_{k' \le k}\{|\epsilon(k')| - \lambda\|X(k'-n) - X(l_{k'-n})\| \\
& + \lambda\|X(k'-n) - X(l_{k'-n})\|\} + C_2 \\
\le \quad & C_1 \max_{k' \le k}\{a(k')|\epsilon(k')|\} + \lambda C_1 \max_{k' \le k-n}\{\|X(k') - X(l_{k'})\|\} + C_2 \quad (3.50)
\end{aligned}
$$

Considering $X(k)$ and $X(l_k)$ defined in (3.25) and (3.27), it is clear that

$$\max_{k' \le k-n}\|X(k') - X(l_{k'})\| \le 2(\tau_{\max} - \tau_{\min} + 1)\max_{k' \le k-n}\|\bar{\xi}_n(k')\|$$

Together with inequality (3.50), we have

$$
\begin{aligned}
\max_{k' \leq k-n+1} \{\|\bar{\xi}_n(k')\|\} \leq & \; C_1 \max_{k' \leq k}\{a(k')|\epsilon(k')|\} + 2\lambda C_1(\tau_{\max} - \tau_{\min} + 1) \\
& \times \max_{k' \leq k-n+1} \{\|\bar{\xi}_n(k')\|\} + C_2
\end{aligned} \tag{3.51}
$$

Then, we see that there exists a small positive constant

$$
\lambda^* = \frac{1}{2C_1(\tau_{\max} - \tau_{\min} + 1)} \tag{3.52}
$$

such that

$$
\begin{aligned}
\max_{k' \leq k-n+1} \{\|\bar{\xi}_n(k')\|\} \leq & \; \frac{C_1}{1 - 2\lambda C_1(\tau_{\max} - \tau_{\min} + 1)} \max_{k' \leq k}\{a(k')|\epsilon(k')|\} \\
& + \frac{C_3}{1 - 2\lambda C_1(\tau_{\max} - \tau_{\min} + 1)}, \quad \forall \lambda < \lambda^*
\end{aligned} \tag{3.53}
$$

Note that inequality (3.53) implies

$$
\bar{\xi}_n(k - n + 1) = O[a(k)\epsilon(k)]
$$

From definition of $\Phi(k + n - 1)$ in (3.23), Lemma 2.6, and Assumption 3.1, it can be seen that

$$
\Phi(k - 1) = O[\bar{\xi}_n(k - n)], \quad u(k - n) = O[y(k)] = O[\bar{\xi}_n(k - n + 1)]
$$

Then, according to the definition of $D(k - n)$ in (3.37), we have

$$
\begin{aligned}
D^{\frac{1}{2}}(k - n) \leq & \; 1 + \|\Phi(k - 1) - \Phi(l_{k-n} + n - 1)\| + |u(k - n) - u(l_{k-n})| \\
= & \; O[\bar{\xi}_n(k - n + 1)] = O[a(k)\epsilon(k)]
\end{aligned}
$$

Applying the Lemma 2.3 to (3.46), we have

$$
\lim_{k \to \infty} a(k)\epsilon(k) = 0 \tag{3.54}
$$

According to (3.53), it is easy to see that the boundedness of $\bar{\xi}_n(k)$ is guaranteed. It follows that the output $y(k)$ and the tracking error $e(k)$ are bounded, as well as the control input $u(k)$, according to Lemma 2.6. The boundedness of $\bar{\xi}_n(k)$ immidiately leads to the boundedness of $X(k)$ defined in (3.25). Therefore, using Lemma 2.2, we have

$$
\lim_{k \to \infty} \|X(k) - X(l_k)\| = 0 \tag{3.55}
$$

Combining equations (3.55), (3.54) and inequality (3.40) resulted from the deadzone, we conclude that $\lim_{k \to \infty} \epsilon(k) = 0$. Therefore, we have $\lim_{k \to \infty} e(k) = 0$ according to Proposition 2.1 and equation (3.48). This completes the proof. ∎

Remark 3.8. *The underlying reason that the asymptotic tracking performance is achieved can be seen in (3.35), in which it is clear that under the proposed adaptive control, the effect of the uncertain function $\nu(\cdot)$ will ultimately vanish due to $|\nu(\bar{\xi}_n(k-\tau)) - \nu(\bar{\xi}_n(l_k-\tau))| \to 0$, which is guaranteed by $\|X(k) - X(l_k)\| \to 0$.*

Remark 3.9. *The control law in (3.34) requires the computation of l_k in (3.26) and the computation may cost infinite memory as time increase. In practice, however, finite memory control can be obtained by computing l_k not from range $[0, k-n]$ but from $[k-M-n, k-n]$, where $M > 0$ can be chosen as a large integer. In this case, the stability will not be affected and the magnitude of ultimate tracking error can be made sufficiently small by increasing M.*

3.3 Systems with Unmatched Uncertainties

In Section 3.2, we have studied adaptive control with compensation of nonparametric uncertainty that appear in the last equation of system, i.e., in the control range (matching condition). In this Section, we study more complicated case that the uncertainties appear out of control range, i.e., in unmatched manner. Unmatched uncertainties have been studied in continuous-time for linear systems [174] using sliding mode control, and have also been studied for nonlinear strict-feedback systems using nonlinear damping method which can be regarded as a modified sliding mode [70,71,175]. But like high gain control, this method is not applicable to discrete-time systems, even for the matched uncertainties. In discrete-time, there are only a few researches on adaptive control for systems with unmatched uncertainties [54–56], but there is no consideration of compensation of the uncertainties.

In this Section, we will consider extending the design approach in Section 3.3 to deal with uncertainties in unmatched manner. Like the auxiliary output introduced in Section 3.2.3, we will introduce auxiliary states in Section 3.3.2 and utilize these auxiliary states to compensate for unmatched nonparametric uncertainties in the future states prediction stage. The structure of this Section is similar to Section 3.2, but the techniques involved are much more complicated. Future state prediction is carried out first in Section 3.3.2, and then system transformation and control design is presented in 3.3.3.

3.3.1 System presentation

The strict-feedback systems with both matched and unmatched uncertainties to be studied are described as follows:

$$
\begin{cases}
\xi_i(k+1) = \Theta_i^T \Phi_i(\bar{\xi}_i(k)) + g_i \xi_{i+1}(k) + \upsilon_i(\bar{\xi}_i(k)) \\
\qquad i = 1, 2, \ldots, n-1 \\
\xi_n(k+1) = \Theta_n^T \Phi_n(\bar{\xi}_n(k)) + g_n u(k) + \upsilon_n(\bar{\xi}_n(k)) \\
y(k) = \xi_1(k)
\end{cases}
\tag{3.56}
$$

where as same as notations in system (3.1), $\bar{\xi}_j(k)$ are measurable system state vectors, $\Theta_j \in \mathbf{R}^{p_j}$, $g_j \in \mathbf{R}$, $j = 1, 2, \ldots, n$, are unknown parameters (p_j's are positive integers), $\Phi_j(\bar{\xi}_j(k)) : \mathbf{R}^j \to \mathbf{R}^{p_j}$ are known vector-valued functions.

Remark 3.10. *It should be highlighted that the compensation technique for time delays in the uncertain nonlinearities developed in Section 3.2 can be easily implemented in this Section, thus, for conciseness, time delays in the uncertainties will not be considered in this Section.*

The control objective is to make the output $y(k)$ exactly track a bounded reference trajectory $y^*(k)$ and to guarantee the boundedness of all the closed-loop signals. It is noted that the uncertain nonlinearities $\upsilon_i(\bar{\xi}_j(k))$ appear in every equation of system (3.56) (out of control range) such that it is not easy to compensate for their effects and accomplish asymptotic tracking performance.

Assumption 3.3. *The nonparametric uncertain functions $\upsilon_i(\cdot)$, are Lipschitz functions with Lipschitz coefficients L_{υ_i} satisfying $\max_{1 \leq i \leq n} L_{\upsilon_i} < \lambda^*$ and λ^* is a small number defined in (3.110). The system functions, $\Phi_i(\cdot)$, $i = 1, 2, \ldots, n$, are also Lipschitz functions with Lipschitz coefficients L_i.*

Assumption 3.4. *The signs of control gains g_j, $(j = 1, 2, \ldots, n)$ are known. Without loss of generality, it is assumed that g_j are positive with known lower bounds $\underline{g}_j > 0$, i.e., $g_j \geq \underline{g}_j > 0$.*

3.3.2 Future states prediction

According to Lemma 2.5, there exist prediction functions $P_{n-i,i}(\cdot)$ for system (3.56) with Lipschitz coefficients L_{pi} such that $\bar{\xi}_i(k) = P_{n-i,i}(\bar{\xi}_n(k-n+i))$. Then, system (3.1) can be

rewritten as follows:

$$
\begin{cases}
\xi_i(k+1) = \Theta_i^T \Phi_i(\bar{\xi}_i(k)) + g_i \xi_{i+1}(k) + \nu_i(\bar{\xi}_n(k-n+i)) \\
\quad\quad i = 1, 2, \ldots, n-1 \\
\xi_n(k+1) = \Theta_n^T \Phi_n(\bar{\xi}_n(k)) + g_n u(k) + \nu_n(\bar{\xi}_n(k)) \\
y(k) = \xi_1(k)
\end{cases}
\tag{3.57}
$$

where

$$
\nu_i(\bar{\xi}_n(k-n+i)) \quad = \quad \upsilon_i(P_{n-i,i}(\bar{\xi}_n(k-n+i))) = \upsilon_i(\bar{\xi}_i(k))
\tag{3.58}
$$

are unknown composite functions satisfying Lipschitz condition.

According to Lemma 2.2, we define

$$
l_k = \arg \min_{l \leq k-n} \|\bar{\xi}_n(k) - \bar{\xi}_n(l)\|
\tag{3.59}
$$

from which, it is obvious that $l_k \leq k - n$. Further, let us define

$$
\Delta \bar{\xi}_n(k) = \bar{\xi}_n(k) - \bar{\xi}_n(l_k).
\tag{3.60}
$$

Remark 3.11. *If there is time delayed states in the nonparametric uncertainties $\nu_i(\cdot)$, then similar to equations (3.25) and (3.26), we can introduce augmented states vector and define index l_k accordingly and then utilize them to compensate the effect of time delayed states in the uncertainties.*

Similar to Section 3.2.2, in the next step we consider predicting future states $\xi_i(k+j)$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, n-i$, to facilitate the adaptive control design. But due to the existence of the unmatched uncertainties, we consider incorporating the compensation technique into the prediction method developed in Section 3.2.2 such that the effect of the unmatched uncertainties will be eliminated for the predicted future states.

First, let us define auxiliary states $\xi_i^a(k)$ as follows:

$$
\xi_i^a(k) = \Theta_i^T \Phi_i(\bar{\xi}_i(k)) + \nu_i(\bar{\xi}_n(k-n+i)), \quad i = 1, 2, \ldots, n-1
\tag{3.61}
$$

which include both uncertain parameters Θ_i and uncertain nonlinearities $\nu_i(\cdot)$. From (3.57) and (3.61), we have

$$
\xi_i(k+1) = \xi_i^a(k) + g_i \xi_{i+1}(k), \quad i = 1, 2, \ldots, n-1
\tag{3.62}
$$

and it is easy to derive that

$$
\begin{aligned}
\xi_i^a(k) \quad = \quad & \xi_i^a(k) + \xi_i^a(l_{k-n+i} + n - i) - \xi_i^a(l_{k-n+i} + n - i) \\
= \quad & \Theta_i^T [\Phi_i(\bar{\xi}_i(k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i} + n - i))] \\
& + \xi_i(l_{k-n+i} + n - i + 1) - g_i \xi_{i+1}(l_{k-n+i} + n - i) \\
& + \nu_i(\bar{\xi}_n(k-n+i)) - \nu_i(\bar{\xi}_n(l_{k-n+i}))
\end{aligned}
\tag{3.63}
$$

where l_{k-n+i} is defined in (3.59) and it satisfies $l_{k-n+i} + n - i + 1 \le k - n + 1$.

Let $\hat{\Theta}_i(k)$ and $\hat{g}_i(k)$ be the estimates of Θ_i and g_i at the kth step, respectively. Then, let us define

$$
\begin{aligned}
\hat{\xi}_i^a(k) &= \hat{\Theta}_i^T(k - n + 2)[\Phi_i(\bar{\xi}_i(k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i} + n - i))] \\
&\quad + \xi_i(l_{k-n+i} + n - i + 1) \\
&\quad - \hat{g}_i(k - n + 2)\xi_{i+1}(l_{k-n+i} + n - i)
\end{aligned}
\tag{3.64}
$$

as the estimate of the auxiliary state $\xi_i^a(k)$ defined in (3.61).

According to (3.62), we define one-step ahead prediction $\hat{\xi}_i(k+1|k)$, $i = 1, 2, \ldots, n-1$, as the prediction of the one-step future states $\xi_i(k+1)$ as follows:

$$
\hat{\xi}_i(k+1|k) = \hat{\xi}_i^a(k) + \hat{g}_i(k - n + 2)\xi_{i+1}(k)
\tag{3.65}
$$

From (3.63), we see that one-step future auxiliary state $\xi_i^a(k+1)$, $i = 1, 2, \ldots, n-2$, can be expressed as

$$
\begin{aligned}
\xi_i^a(k+1) &= \Theta_i^T[\Phi_i(\bar{\xi}_i(k+1)) - \Phi_i(\bar{\xi}_i(l_{k-n+i+1} + n - i))] \\
&\quad + \xi_i(l_{k-n+i+1} + n - i + 1) - g_i\xi_{i+1}(l_{k-n+i+1} + n - i) \\
&\quad + \nu_i(\bar{\xi}_n(k - n + i + 1)) - \nu_i(\bar{\xi}_n(l_{k-n+i+1}))
\end{aligned}
\tag{3.66}
$$

and then we take

$$
\begin{aligned}
\hat{\xi}_i^a(k+1|k) &= \hat{\Theta}_i^T(k - n + 3)[\Phi_i(\bar{\hat{\xi}}_i(k+1|k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i+1} + n - i))] \\
&\quad + \xi_i(l_{k-n+i+1} + n - i + 1) - \hat{g}_i(k - n + 3)\xi_{i+1}(l_{k-n+i+1} + n - i)
\end{aligned}
\tag{3.67}
$$

as the prediction of the one step future auxiliary states $\xi_i^a(k+1)$, where

$$
\bar{\hat{\xi}}_i(k+1|k) = [\hat{\xi}_1(k+1|k), \hat{\xi}_2(k+1|k), \ldots, \hat{\xi}_i(k+1|k)]^T
$$

is a vector of one-step ahead future states predictions defined in (3.65) and $l_{k-n+i+1} + n - i + 1 \le k - n + 2$ according to (3.59).

Define two-step ahead prediction $\hat{\xi}_i(k+2|k)$, $i = 1, 2, \ldots, n-2$, as the prediction of two-step ahead future states $\xi_i(k+2)$

$$
\hat{\xi}_i(k+2|k) = \hat{\xi}_i^a(k+1|k) + \hat{g}_i(k - n + 3)\hat{\xi}_{i+1}(k+1|k)
\tag{3.68}
$$

Similarly to (3.64), the $(j-1)$-step future auxiliary state $\xi_i^a(k+j)$, $i = 1, 2, \ldots, n-1$, $j = 2, 3, \ldots, n-i$, can be predicted as

$$
\begin{aligned}
&\hat{\xi}_i^a(k+j-1|k) \\
=\ & \hat{\Theta}_i^T(k-n+j+1)[\Phi_i(\hat{\bar{\xi}}_i(k+j-1|k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i+j-1}+n-i))] \\
& + \xi_i(l_{k-n+i+j-1}+n-i+1) - \hat{g}_i(k-n+j+1)\xi_{i+1}(l_{k-n+i+j-1}+n-i)
\end{aligned}
$$

$$(3.69)$$

where $l_{k-n+i+j-1} + n - i + 1 \leq k - n + j$ holds according to (3.59) and

$$
\hat{\bar{\xi}}_i(k+j-1|k) = [\hat{\xi}_1(k+j-1|k), \hat{\xi}_2(k+j-1|k), \ldots, \hat{\xi}_i(k+j-1|k)]^T
$$

are vectors of predicted states at previous steps.

Then, let us define j-step ahead prediction $\hat{\xi}_i(k+j|k)$, $i = 1, 2, \ldots, n-1$, $j = 2, 3, \ldots, n-j$, as the estimate of j-step ahead future states $\xi_i(k+j)$

$$
\hat{\xi}_i(k+j|k) = \hat{\xi}_i^a(k+j-1|k) + \hat{g}_i(k-n+j+1)\hat{\xi}_{i+1}(k+j-1|k) \qquad (3.70)
$$

Remark 3.12. *Compared with the future states prediction in the absence of nonparametric uncertainties developed in Section 3.2.2, we have introduced additional auxiliary states and their predictions, in which the states information at previous steps has been utilized to compensate for the effect of nonparametric uncertainties at current step, as shown in (3.63) and (3.64).*

According to the definition of $\nu_i(\bar{\xi}_n(k-n+i))$ in (3.58), Assumption 3.3, Lemma 2.5 and definition of $\Delta\bar{\xi}_n(k)$ in (3.60), we have

$$
\|\nu_i(\bar{\xi}_n(k-n+i)) - \nu_i(\bar{\xi}_n(l_{k-n+i}))\| \leq L_{pi}L_{v_i}\|\Delta\bar{\xi}_n(k-n+i)\| \qquad (3.71)
$$

where L_{pi} and L_{v_i} are Lipschitz coefficients of prediction functions $P_i(\cdot)$ and nonparametric uncertainty functions $\upsilon_i(\cdot)$, respectively.

Let us denote $\hat{c}_i(k)$ as the estimate of L_{pi}. The update laws for $\hat{\Theta}_i(k)$, $\hat{g}_i(k)$, $\hat{c}_i(k)$, $i = 1, 2, \ldots, n-1$, are given as follows:

$$
\begin{aligned}
\hat{\Theta}_i(k+1) &= \hat{\Theta}_i(k-n+2) - \frac{a_i(k)\gamma[\Phi_i(\bar{\xi}_i(k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i}+n-i))]\tilde{\xi}_i(k+1|k)}{D_i(k)} \\
\hat{g}_i(k+1) &= \hat{g}_i(k-n+2) - \frac{a_i(k)\gamma[\xi_{i+1}(k) - \xi_{i+1}(l_{k-n+i}+n-i)]\tilde{\xi}_i(k+1|k)}{D_i(k)} \\
\hat{c}_i(k+1) &= \hat{c}_i(k-n+2) + \frac{a_i(k)\gamma\lambda|\tilde{\xi}_i(k+1|k)|\|\Delta\bar{\xi}_n(k-n+i)\|}{D_i(k)}
\end{aligned}
$$

$$(3.72)$$

with

$$\tilde{\xi}_i(k+1|k) = \hat{\xi}_i(k+1|k) - \xi_i(k+1|k)$$

$$D_i(k) = 1 + \|\Phi_i(\bar{\xi}_i(k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i}+n-i))\|^2$$

$$+ |\xi_{i+1}(k) - \xi_{i+1}(l_{k-n+i}+n-i)|^2 + \lambda^2 \|\Delta\bar{\xi}_n(k-n+i)\|^2 \quad (3.73)$$

$$a_i(k) = \begin{cases} 1 - \frac{\lambda \hat{c}_i(k-n+2)\|\Delta\bar{\xi}_n(k-n+i)\|}{|\tilde{\xi}_i(k+1|k)|} \\ \quad \text{if } |\tilde{\xi}_i(k+1|k)| > \lambda\hat{c}_i(k-n+2)\|\Delta\bar{\xi}_n(k-n+i)\| \quad (3.74) \\ 0, \quad \text{otherwise} \end{cases}$$

$$\hat{\Theta}_i(0) = \mathbf{0}_{[n]}, \ \hat{g}_i(0) = 0, \ \hat{c}_i(0) = 0$$

where $0 < \gamma < 2$ and λ can be chosen as any constant satisfying $\max_{1\leq i\leq n} L_{v_i} \leq \lambda < \lambda^*$, with λ^* defined later in (3.110).

According to the deadzone defined in (3.74), we have

$$-a_i^2(k)\tilde{\xi}_i^2(k+1|k) = -a_i(k)\tilde{\xi}_i^2(k+1|k) + \lambda a_i(k)$$
$$\times \hat{c}_i(k-n+2)|\tilde{\xi}_i(k+1|k)|\|\Delta\bar{\xi}_n(k-n+i)\| \quad (3.75)$$

Lemma 3.4. *Consider the future states prediction laws defined in (3.65), (3.68) and (3.70), in which the estimated parameters are calculated from update law (3.72). The estimated parameters $\hat{\Theta}_i(k)$, $\hat{g}_i(k)$ and $\hat{c}_i(k)$, $i = 1, 2, \ldots, n-1$, are bounded and there exist constants $\bar{c}_{n-i}$ such that the future prediction errors satisfy*

$$\|\bar{\tilde{\xi}}_i(k+n-i|k)\| \leq o[O[y(k+n-1)]] + \lambda\bar{c}_{n-i}\Delta_s(k,n-1) \quad (3.76)$$

where

$$\bar{\tilde{\xi}}_i(k+n-i) = [\tilde{\xi}_1(k+n-i), \ldots, \tilde{\xi}_i(k+n-i)]^T \quad (3.77)$$

$$\Delta_s(k,m) = \max_{1\leq j\leq m} \{\|\Delta\bar{\xi}_n(k-n+j)\|\} \quad (3.78)$$

with

$$\tilde{\xi}_i(k+n-i) = \hat{\xi}_i(k+n-i) - \xi_i(k+n-i)$$

and $\Delta\bar{\xi}_n(k)$ defined in (3.60).

Proof. See Appendix 3.1. ∎

3.3.3 System transformation and adaptive control design

In the similar manner as system transformation conducted in Section 3.2.3, let us rewrite system (3.57) into a compact form as follows by iterative substitution:

$$y(k+n) = \Theta_f^T \Phi(k+n-1) + gu(k) + \Theta_g^T \bar{\nu}(k) \quad (3.79)$$

where Θ_f, g and $\Phi(k + n - 1)$ are defined in the same way as in (3.23), and

$$g_{f1} = 1, \quad g_{fi} = \prod_{j=1}^{i-1} g_j, \quad i = 2, \ldots, n$$

$$\Theta_g = [g_{f1}, \ldots, g_{fn}]^T \in \mathbf{R}^p$$

$$\bar{\nu}(k) = [\nu_1(\bar{\xi}_n(k)), \ldots, \nu_n(\bar{\xi}_n(k))]^T \in \mathbf{R}^n \tag{3.80}$$

Let us introduce an auxiliary output $y_a(k)$ as

$$y_a(k + n - 1) = \Theta_f^T \Phi(k + n - 1) + \Theta_g^T \bar{\nu}(k). \tag{3.81}$$

Then, equation (3.79) can be rewritten as

$$y(k + n) = y_a(k + n - 1) + gu(k) \tag{3.82}$$

From (3.81) and (3.82), it is easy to derive that

$$\begin{aligned} y_a(k + n - 1) &= y_a(k + n - 1) - y_a(l_k + n - 1) + y_a(l_k + n - 1) \\ &= \Theta_f^T[\Phi(k + n - 1) - \Phi(l_k + n - 1)] \\ &\quad + \Theta_g^T[\bar{\nu}(k) - \bar{\nu}(l_k)] + y(l_k + n) - gu(l_k) \end{aligned} \tag{3.83}$$

Denote $\hat{\Theta}_f(k)$ and $\hat{g}(k)$ as the estimates of unknown parameters Θ_f and g defined in (3.80). The parameter estimates will be calculated from (3.92). Define the estimate of $y_a(k+n-1)$ as follows:

$$\begin{aligned} \hat{y}_a(k + n - 1|k) &= \hat{\Theta}_f^T(k)[\hat{\Phi}(k + n - 1|k) - \Phi(l_k + n - 1)] \\ &\quad + y(l_k + n) - \hat{g}(k)u(l_k) \end{aligned} \tag{3.84}$$

where l_k is defined in (3.59) satisfying $l_k + n \leq k$, and

$$\hat{\Phi}(k + n - 1|k) = [\Phi_1^T(\hat{\xi}_1(k + n - 1|k)), \ldots, \Phi_n^T(\bar{\xi}_n(k))]^T \tag{3.85}$$

with

$$\bar{\hat{\xi}}_i(k + n - i|k) = [\hat{\xi}_1(k + n - i), \ldots, \hat{\xi}_i(k + n - i)]^T, \quad i = 1, 2, \ldots, n - 1$$

which is defined in Section 3.3.2.

Define parameter estimate errors

$$\tilde{\Theta}_f(k) = \hat{\Theta}_f(k) - \Theta_f, \quad \tilde{g}(k) = \hat{g}(k) - g$$

Then from (3.83) and (3.84), we have the estimate error of auxiliary output as

$$
\begin{aligned}
\tilde{y}_a(k+n-1|k) &= \hat{y}_a(k+n-1|k) - y_a(k+n-1) \\
&= \tilde{\Theta}_f^T(k)[\Phi(k+n-1) - \Phi(l_k+n-1)] + \beta(k+n-1) - \tilde{g}(k)u(l_k) \\
&\quad - \Theta_g^T[\bar{\nu}(k) - \bar{\nu}(l_k)]
\end{aligned}
\tag{3.86}
$$

where

$$
\beta(k+n-1) = \hat{\Theta}_f^T(k)[\hat{\Phi}(k+n-1|k) - \Phi(k+n-1)]
\tag{3.87}
$$

Using the estimated auxiliary output, the adaptive control law is constructed as

$$
u(k) = -\frac{1}{\hat{g}(k)}(\hat{y}_a(k+n-1|k) - y^*(k+n))
\tag{3.88}
$$

where the parameter estimate $\hat{g}(k)$ will be guaranteed to be bounded away from zero such that above control law (3.88) is well defined.

Considering adaptive control law in (3.88), the estimation error of auxiliary output in (3.86), and system described in (3.82), we obtain the closed-loop error dynamics as follows:

$$
\begin{aligned}
e(k) &= y_a(k-1) + \hat{g}(k-n)u(k-n) - \tilde{g}(k-n)u(k-n) - y^*(k) \\
&= -\tilde{y}_a(k-1|k-n) - \tilde{g}(k-n)u(k-n) \\
&= -\tilde{\Theta}_f^T(k-n)[\Phi(k-1) - \Phi(l_{k-n}+n-1)] \\
&\quad - \tilde{g}(k-n)[u(k-n) - u(l_{k-n})] \\
&\quad - \beta(k-1) + \Theta_g^T[\bar{\nu}(k-n) - \bar{\nu}(l_{k-n})]
\end{aligned}
\tag{3.89}
$$

According to the definition of $\bar{\nu}(k)$ in (3.80) and equation (3.71), we have

$$
\|\Theta_g^T[\bar{\nu}(k-n) - \bar{\nu}(l_{k-n})]\| \leq \lambda \theta_g \|\Delta \bar{\xi}_n(k-n)\|
\tag{3.90}
$$

where

$$
\theta_g = \sum_{i=1}^{n} g_{fi} L_{p_i}
\tag{3.91}
$$

is an unknown constant and λ can be any constant satisfying $\max_{1 \leq i \leq n} L_{v_i} \leq \lambda < \lambda^*$, with λ^* defined later in (3.110).

Denote $\hat{\theta}_g(k)$ as the estimate of θ_g and define the estimate error as

$$
\tilde{\theta}_g(k) = \hat{\theta}_g(k) - \theta_g
$$

The parameter estimates used in control law (3.88) are calculated by the following update law

$$\hat{\Theta}_f(k) = \hat{\Theta}_f(k-n) + \gamma \frac{a(k)e(k)[\Phi(k-1) - \Phi(l_{k-n}+n-1)]}{D(k-n)}$$

$$\hat{g}(k) = \begin{cases} \hat{g}'(k), & \text{if } \hat{g}'(k) > \underline{g} \\ \underline{g}, & \text{otherwise} \end{cases} \quad (3.92)$$

$$\hat{g}'(k) = \hat{g}(k-n) + \frac{\gamma a(k)e(k)}{D(k-n)}[u(k-n) - u(l_{k-n})]$$

$$\hat{\theta}_g(k) = \hat{\theta}_g(k-n) + \frac{a(k)\gamma\lambda|e(k)|\|\Delta\bar{\xi}_n(k-n)\|}{D(k-n)}$$

$$D(k-n) = 1 + \|\Phi(k-1) - \Phi(l_{k-n}+n-1)\|^2$$

$$+ [u(k-n) - u(l_{k-n})]^2 + \lambda^2\|\Delta\bar{\xi}_n(k-n)\|^2$$

where $0 < \gamma < 2$ and $\max_{1 \le i \le n} L_{v_i} \le \lambda < \lambda^*$ with λ^* defined in (3.110) can be chosen as the same value as used in (3.72)-(3.74), and the deadzone indicator $a(k)$ is defined as

$$a(k) = \begin{cases} 1 - \frac{\lambda\hat{\theta}_g(k-n)\|\Delta\bar{\xi}_n(k-n)\|+|\beta(k-1)|}{|e(k)|}, & \text{if } |e(k)| > \\ \quad \lambda\hat{\theta}_g(k-n)\|\Delta\bar{\xi}_n(k-n)\| + |\beta(k-1)| \\ 0, & \text{otherwise} \end{cases} \quad (3.93)$$

and from the definition of $a(k)$ above, it is guaranteed that

$$a(k)|e(k)| \ge |e(k)| - \lambda\hat{\theta}_g(k-n)\|\Delta\bar{\xi}_n(k-n)\| - |\beta(k-1)| \quad (3.94)$$

Remark 3.13. *In comparison with control parameter update law (3.37), it is noted that in (3.92) that the update law is directly driven by tracking error $e(k)$ instead of augmented tracking error $\epsilon(k)$, while the the effect of $\beta(k)$ caused by prediction error is handled by the deadzone.*

3.3.4 Stability analysis and asymptotic tracking performance

The main result of the control performance is presented in the following theorem.

Theorem 3.2. *Consider the adaptive closed-loop system consisting of system (3.1), future states prediction laws defined in (3.65) and (3.70) using parameter update law (3.72), control law (3.88) using parameter update law (3.92). All the signals in the closed-loop system are bounded and furthermore, the tracking error $e(k)$ converges to zero.*

Proof. Choose a Lyapunov candidate function as follows:

$$V(k) = \sum_{j=1}^{n} \|\tilde{\Theta}_f^T(k-n+j)\|^2 + \sum_{j=1}^{n} \tilde{g}^2(k-n+j) + \sum_{j=1}^{n} \tilde{\theta}_g^2(k-n+j) \quad (3.95)$$

It follows that the difference of $V(k)$ is

$$
\begin{aligned}
\Delta V(k) &= V(k) - V(k-1) \\
&\leq \tilde{\Theta}_f^T(k)\tilde{\Theta}_f(k) - \tilde{\Theta}_f^T(k-n)\tilde{\Theta}_f(k-n) + \tilde{g}'^2(k) - \tilde{g}^2(k-n) + \tilde{\theta}_g^2(k) - \tilde{\theta}_g^2(k-n) \\
&= \{\|\Phi(k-1) - \Phi(l_{k-n} + n - 1)\|^2 + [u(k-n) - u(l_{k-n})]^2 + \lambda^2\|\Delta\bar{\xi}_n(k-n)\|^2\} \\
&\quad \times \frac{a^2(k)\gamma^2 e^2(k)}{D^2(k-n)} \\
&\quad + \{\tilde{\Theta}_f^T(k-n)[\Phi(k-1) - \Phi(l_{k-n} + n - 1)] + \tilde{g}(k-n)[u(k-n) - u(l_{k-n})]\} \\
&\quad \times e(k)\frac{2a(k)\gamma}{D(k-n)} + \lambda\tilde{\theta}_g(k-n)\|\Delta\bar{\xi}_n(k-n)\|\,|e(k)|\frac{2a(k)\gamma}{D(k-n)}
\end{aligned} \tag{3.96}
$$

where the inequality $\tilde{g}^2(k) \leq \tilde{g}'^2(k)$ established in Lemma 3.3 is used.

From inequality (3.90) and the error equation (3.89), it is easy to obtain that

$$
\begin{aligned}
&\{\tilde{\Theta}_f^T(k-n)[\Phi(k-1) - \Phi(l_{k-n} + n - 1)] \\
&\quad + \tilde{g}(k-n)[u(k-n) - u(l_{k-n})]\}e(k) \\
&= -e^2(k) - \beta(k-1)e(k) + \Theta_g^T[\bar{\nu}(k-n) - \bar{\nu}(l_{k-n})]e(k) \\
&\leq -e^2(k) + \lambda\theta_g|e(k)|\|\Delta\bar{\xi}_n(k-n)\| + |e(k)||\beta(k-1)|
\end{aligned} \tag{3.97}
$$

From deadzone $a(k)$ defined in (3.93), it follows that

$$
a(k)[-e^2(k) + \lambda\hat{\theta}_g(k-n)|e(k)|\|\Delta\bar{\xi}_n(k-n)\| + |e(k)||\beta(k-1)|] = -a^2(k)e^2(k) \tag{3.98}
$$

According to inequality (A.26), equality (3.98), and the definition of $D(k-n)$ in (3.92), the difference of $V(k)$ in (3.96) can be written as

$$
\Delta V(k) \leq \frac{a^2(k)\gamma^2 e^2(k)}{D(k-n)} - \frac{2a^2(k)\gamma e^2(k)}{D(k-n)} = -\frac{\gamma(2-\gamma)a^2(k)e^2(k)}{D(k-n)} \tag{3.99}
$$

where $\theta_g + \tilde{\theta}_g(k-n) = \hat{\theta}_g(k-n)$ is used.

Noting that $0 < \gamma < 2$ and $\Delta V(k)$ is nonpositive, the boundedness of $V(k)$ and thus the boundedness of $\hat{\Theta}_f(k)$, $\hat{g}(k)$, and $\hat{\theta}_g(k)$ are guaranteed. Furthermore, we have

$$
\lim_{k\to\infty} \frac{a^2(k)e^2(k)}{D(k-n)} = 0 \tag{3.100}
$$

$$
|e(k)| - \lambda\bar{\theta}_g\|\Delta\bar{\xi}_n(k-n)\| - |\beta(k-1)| \leq a(k)|e(k)| \tag{3.101}
$$

where (3.101) is obtained from (3.94) with a constant $\bar{\theta}_g$ satisfying $\hat{\theta}_g(k) \leq \bar{\theta}_g$.

Further, according to the definition of $\beta(k+n-1)$ in (3.87), Lemma 3.4 and Assumption 3.3, there exits a constant c_β such that

$$
|\beta(k+n-1)| \leq o[O[y(k+n-1)]] + \lambda c_\beta \Delta_s(k, n-1) \tag{3.102}
$$

Considering $\Delta_s(k, n-1)$ defined in (3.78) and $\Delta\bar{\xi}_n(k)$ defined in (3.60) and noting the fact that $l_k \le k - n$, it follows

$$
\begin{aligned}
\Delta_s(k, n-1) &= \max_{1 \le j \le n-1} \{\|\bar{\xi}_n(k - n + j) - \bar{\xi}_n(l_k - n + j)\|\} \\
&\le 2 \max_{k' \le k} \{\|\bar{\xi}_n(k')\|\} \\
\Delta\bar{\xi}_n(k) &\le 2 \max_{k' \le k} \{\|\bar{\xi}_n(k')\|\}
\end{aligned}
\tag{3.103}
$$

$$\tag{3.104}$$

From Lemma 2.6, definition of $o[\cdot]$ in Definition 2.3, and inequality (3.103), it is clear that

$$
\begin{aligned}
|\beta(k + n - 1)| &\le o[O[\bar{\xi}_n(k)]] + \lambda c_\beta \Delta_s(k, n-1) \\
&\le (\alpha(k) + \lambda) c_{\beta,1} \max_{k' \le k} \{\|\bar{\xi}_n(k')\|\} + \alpha(k) c_{\beta,2}
\end{aligned}
\tag{3.105}
$$

where $\alpha(k)$ is a sequence that converges to zero, and $c_{\beta,1}$ and $c_{\beta,2}$ are finite constants. Since $\lim_{k \to \infty} \alpha(k) \to 0$, for any given arbitrary small positive constant ϵ_1, there exists a k_1 such that $\alpha(k) \le \epsilon_1$, $\forall k > k_1$. Thus, it is clear that

$$
|\beta(k + n - 1)| \le (\epsilon_1 + \lambda) c_{\beta,1} \max_{k' \le k} \{\|\bar{\xi}_n(k')\|\} + \epsilon_1 c_{\beta,2}, \ \forall k > k_1
\tag{3.106}
$$

From Lemma 2.6, we have $\bar{\xi}_n(k - n + 1) = O[y(k)]$, which yields

$$
\|\bar{\xi}_n(k - n + 1)\| \le C_1 \max_{k' \le k} \{|e(k')|\} + C_2
\tag{3.107}
$$

where $y(k) \sim e(k)$ is used and C_1 and C_2 are finite constants. Hence, inequality (3.107) can be expressed as

$$
\begin{aligned}
\|\bar{\xi}_n(k - n + 1)\| &\le C_1 \max_{k' \le k} \{|e(k')| - \lambda\bar{\theta}_g\|\Delta\bar{\xi}_n(k' - n)\| - |\beta(k' - 1)| \\
&\quad + \lambda\bar{\theta}_g\|\Delta\bar{\xi}_n(k' - n)\| + |\beta(k' - 1)|\} + C_2 \\
&\le C_1 \max_{k' \le k} \{a(k')|e(k')|\} + \lambda\bar{\theta}_g C_1 \max_{k' \le k-n} \{\|\Delta\bar{\xi}_n(k')\|\} \\
&\quad + C_1 \max_{k' \le k-n} \{|\beta(k' + n - 1)|\} + C_2
\end{aligned}
\tag{3.108}
$$

From inequalities (3.104), (3.106), and (3.108), we have $C_3 = (2\bar{\theta}_g + c_{\beta,1})C_1$, $\epsilon_2 = c_{\beta,1}\epsilon_1 C_1$ and $C_4 = C_2 + \epsilon_1 c_{\beta,2} C_1$ such that

$$
\begin{aligned}
\max_{k' \le k-n+1} \{\|\bar{\xi}_n(k')\|\} &\le C_1 \max_{k' \le k} \{a(k')|e(k')|\} \\
&\quad + (\lambda C_3 + \epsilon_2) \max_{k' \le k-n+1} \{\|\bar{\xi}_n(k')\|\} + C_4, \ k > k_1
\end{aligned}
\tag{3.109}
$$

which implies the existence of a small positive constant

$$
\lambda^* = \frac{1 - \epsilon_2}{C_3}
\tag{3.110}
$$

where ϵ_2 can be arbitrarily small. It further implies that $\forall k > k_1, \lambda < \lambda^*$

$$\max_{k' \leq k-n+1} \{\|\bar{\xi}_n(k')\|\} \leq \frac{C_1}{1 - \lambda C_3 - \epsilon_2} \max_{k' \leq k} \{a(k')|e(k')|\} + \frac{C_4}{1 - \lambda C_3 - \epsilon_2} \quad (3.111)$$

Note that inequality (3.111) implies

$$\bar{\xi}_n(k - n + 1) = O[a(k)e(k)]$$

From $\Phi(k + n - 1)$ in defined (3.80), Lemma 2.6, and Assumption 3.3, it can be seen that

$$\Phi(k - 1) = O[\bar{\xi}_n(k - n)], \quad u(k - n) = O[y(k)] = O[\bar{\xi}_n(k - n + 1)]$$

According to the definition of $D(k - n)$ in (3.92) and inequality (3.104), we have

$$
\begin{aligned}
D^{\frac{1}{2}}(k - n) &\leq& 1 + \|\Phi(k - 1) - \Phi(l_{k-n} + n - 1)\| \\
&& + |u(k - n) - u(l_{k-n})| + \lambda\|\Delta\bar{\xi}_n(k - n)\| \\
&=& O[\bar{\xi}_n(k - n + 1)] = O[a(k)e(k)]
\end{aligned}
\quad (3.112)
$$

Then, applying the Lemma 2.3 to equation (3.100) yields

$$\lim_{k \to \infty} a(k)e(k) = 0. \quad (3.113)$$

From inequality (3.111), we see that the boundedness of $\bar{\xi}_n(k)$ is guaranteed. It follows that the output $y(k)$ and tracking error $e(k)$ are bounded, as well as the the control input $u(k)$, according to Lemma 2.6. Next, from Lemma 2.2, we have

$$\lim_{k \to \infty} \|\Delta\bar{\xi}_n(k)\| = 0 \quad (3.114)$$

which further leads to

$$\lim_{k \to \infty} \|\Delta_s(k, n - 1)\| = 0 \quad (3.115)$$

Additionally, considering (3.102) and noting that $y(k) \sim e(k)$, it follows

$$|\beta(k - 1)| \leq o[O[e(k)]] + \lambda c_\beta \Delta_s(k - n, n - 1) \quad (3.116)$$

which yields

$$
\begin{aligned}
&|e(k)| - |\beta(k - 1)| + \lambda c_\beta \Delta_s(k - n, n - 1) \\
\geq\ & |e(k)| - o[O[e(k)]] \geq (1 - \alpha(k)m_1)|e(k)| - \alpha(k)m_2
\end{aligned}
\quad (3.117)
$$

according to Definition 2.3, where m_1 and m_2 are positive constants, and

$$\lim_{k \to \infty} \alpha(k) \to 0$$

such that there exists constant k_3 such that $\alpha(k) \leq 1/m_1$, $\forall k > k_3$. Therefore, it can be seen from (3.117) that

$$|e(k)| - |\beta(k-1)| + \lambda c_\beta \Delta_s(k-n, n-1) + \alpha(k)m_2 \geq (1 - \alpha(k)m_1)|e(k)| \geq 0 \quad (3.118)$$

On the other hand, note that (3.101) implies

$$
\begin{aligned}
& |e(k)| - |\beta(k-1)| + \lambda c_\beta \Delta_s(k-n, n-1) + \alpha(k)m_2 \\
\leq \quad & a(k)|e(k)| + \lambda c_\beta \Delta_s(k-n, n-1) + \lambda \bar{\theta}_g \|\Delta \bar{\xi}_n(k-n)\| + \alpha(k)m_2 \quad (3.119)
\end{aligned}
$$

From (3.118) and (3.119), we have $\forall k > k_3$

$$
\begin{aligned}
0 \quad \leq \quad & (1 - \alpha(k)m_1)|e(k)| \\
\leq \quad & a(k)|e(k)| + \lambda c_\beta \Delta_s(k-n, n-1) + \lambda \bar{\theta}_g \|\Delta \bar{\xi}_n(k-n)\| + \alpha(k)m_2 \quad (3.120)
\end{aligned}
$$

which implies that $\lim_{k \to \infty} e(k) = 0$ according to (3.113), (3.114), (3.115), and $\lim_{k \to \infty} \alpha(k) \to 0$. This completes the proof.

Remark 3.14. *From (3.90) and (3.116), it can be seen that the last two terms in (3.89), $\beta(k)$ caused by prediction error and $\bar{\nu}(k)$ caused by nonlinear model uncertainties will ultimately vanish due to $\|\Delta \bar{\xi}_n(k-n)\| \to 0$. This illustrates the underlying mechanism of our control design: to use states information at previous steps to compensate for the uncertainties at current step. It is in great contrast to the continuous-time counterpart results presented in [70, 175], where nonlinear damping is used to compensate for the effect of nonlinear uncertainties.*

3.4 Simulation Studies

In this Section, simulation studies are carried out to verify the developed adaptive controller. Consider that control design in Section 3.2 can be regarded as a special case of the control design in Section 3.3. In this Section, we only study controller developed in Section 3.3 for system with both matched and unmatched nonparametric nonlinear model uncertainties. To show the effectiveness of the proposed adaptive control, we compare it with the adaptive control designed without compensation, i.e., adaptive control designed without consideration of nonparametric uncertainties (it can be easily designed following

the procedure in Section 3.2 but ignoring nonparametric uncertainties in the control design stage). The nonparametric uncertainties are proper chosen such that for adaptive control without compensation, the closed-loop system is still stable. Then, performance comparison can be focused on tracking performance.

The system used for simulation is given below:

$$
\begin{cases}
\xi_1(k+1) = a_1\xi_1(k)\cos(\xi_1(k)) + a_2\xi_1(k)\sin(\xi_1(k)) + g_1\xi_2(k) + \upsilon_1(\xi_1(k)) \\
\xi_2(k+1) = b_1\xi_2(k)\frac{\xi_1(k)}{1+\xi_1^2(k)} + b_2\frac{\xi_2^3(k)}{2+\xi_2^2(k)} + g_2u(k) + \upsilon_2(\bar{\xi}_2(k)) \\
y(k) = \xi_1(k)
\end{cases}
\tag{3.121}
$$

where $a_1 = 0.2$, $a_2 = 0.3$, $g_1 = 0.4$, $b_1 = 0.5$, $b_2 = 0.5$, $g_2 = 0.8$, and

$$
\upsilon_1(\xi_1(k)) = 0.04(\sin(0.05k))\xi_1(k), \quad \upsilon_2(\bar{\xi}_2(k)) = 0.04(\cos(0.05k))(\xi_1(k) + \xi_2(k))
$$

The desired reference trajectory is chosen as $y^*(k) = 1.5\sin(\frac{\pi}{5}kT) + 1.5\cos(\frac{\pi}{10}kT)$, $T = 0.1$.

For both adaptive controls with and without compensation, the parameters are chosen exactly same. The control parameters are chosen as $\underline{g} = 0.32$, $\gamma = 0.08$, and and $\lambda = 0.05$. The initial system states are also chosen same as $\bar{\xi}_2(0) = [0.1, \ 0.1]^T$. The advantage of the adaptive control developed in this Chapter is clearly demonstrated in the comparisons plotted in Figures 3.1(a), 3.2(a),3.3(a), and 3.4(a) (with compensation) and Figures 3.1(b), 3.2(b),3.3(b), and 3.4(b) (without compensation). It can be seen from Figures 3.1(a) and 3.1(b) that with compensation the output tracking performance is much improved compared with that without compensation and the tracking error nearly goes to zero within the simulation steps. From Figures 3.3(a), 3.4(a), 3.4(b) and 3.4(b), it is seen with compensation the parameter estimates are much smoother than those without compensation.

3.5 Summary

In this Chapter, adaptive control with complete compensation of the effect of nonparametric model uncertainties in output tracking performance has been studied for nonlinear discrete-time systems in strict-feedback form. Matched nonparametric uncertainties are studied in Section 3.2, by constructing compensation in controller design stage, while unmatched nonparametric uncertainties are studied in 3.3 by constructing compensation in both future states prediction stage and control design stage. It has been rigorously established that besides the boundedness of all the closed-loop signals, the developed the adaptive control guarantees that the effect of the nonparametric uncertainties is eventually eliminated such that the tracking error converges to zero ultimately.
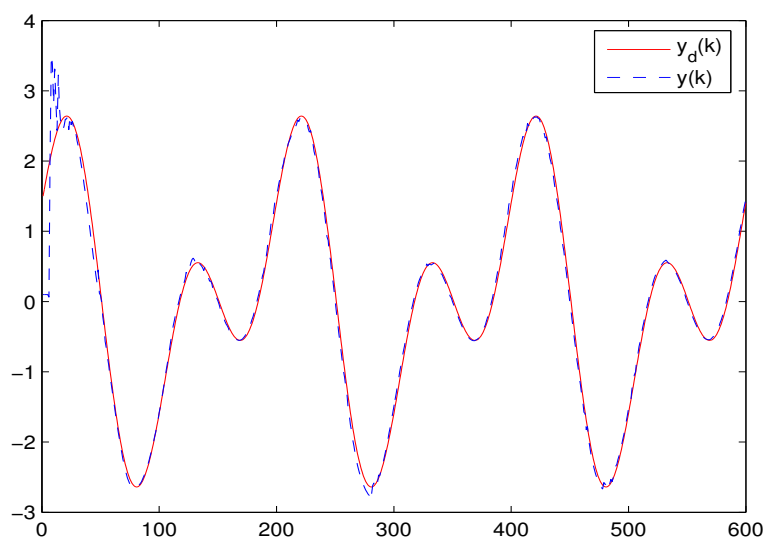
(a) with compensation
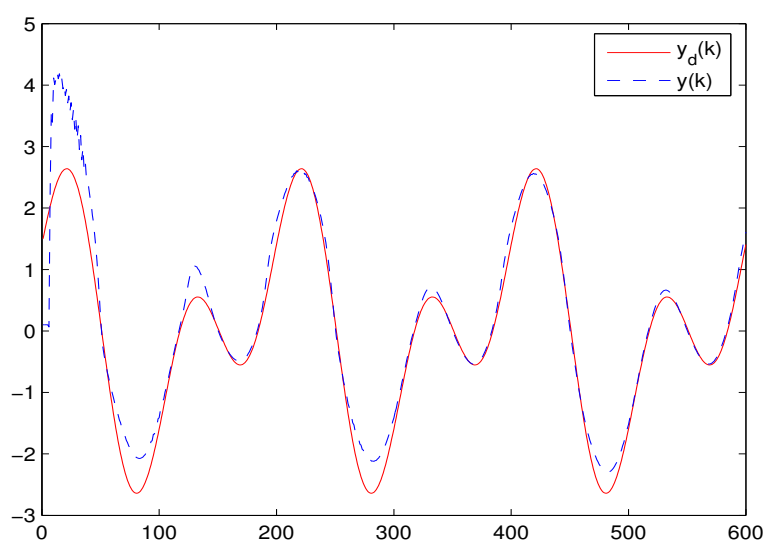
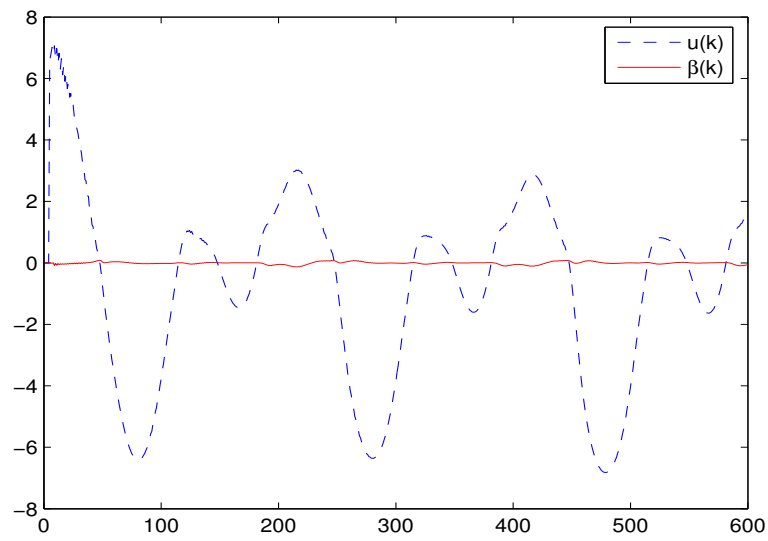(b) without compensation

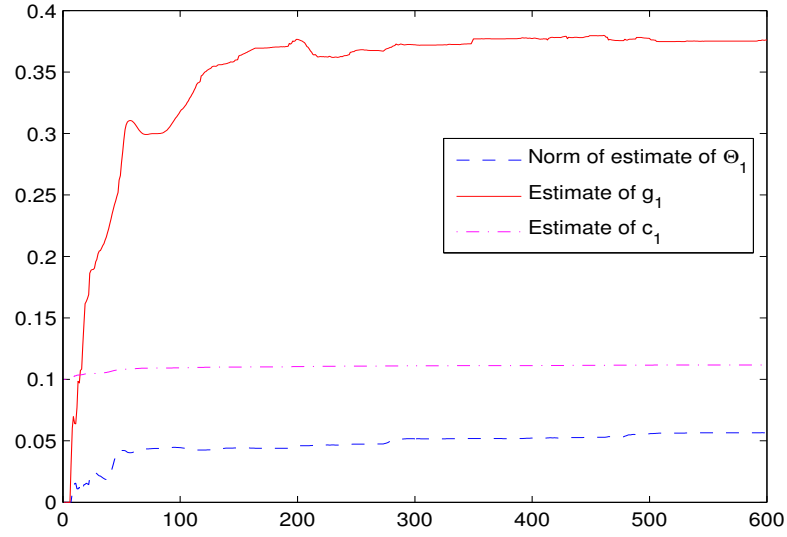Figure 3.1: Reference signal and system output
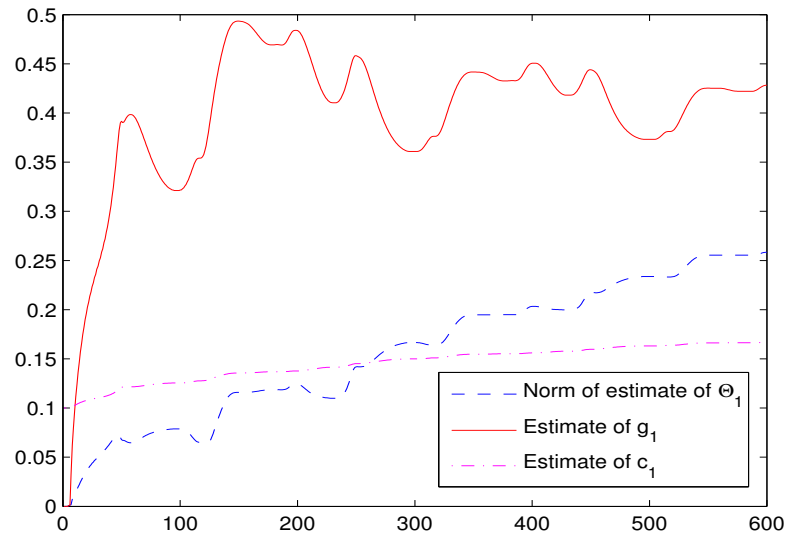
56
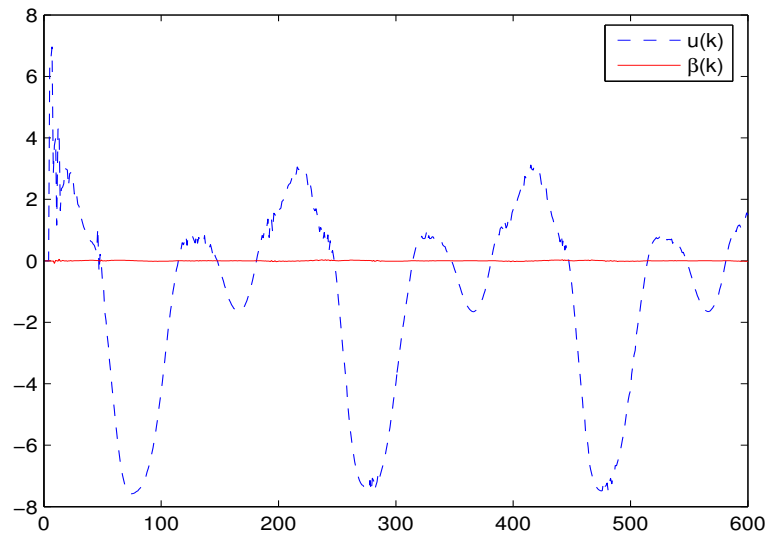
(a) with compensation

(b) without compensation

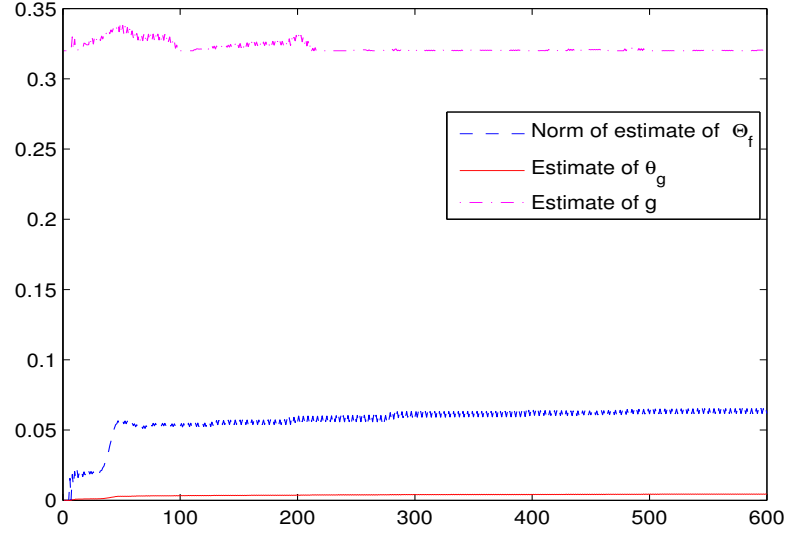Figure 3.2: Control input and signal $\beta(k)$
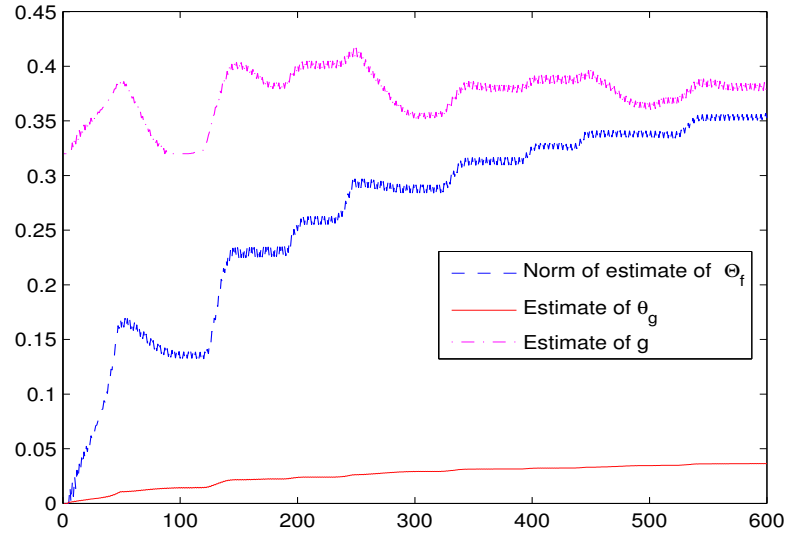
57

(a) with compensation

(b) without compensation

Figure 3.3: Norms of estimated parameters in prediction law

58

(a) with compensation

(b) without compensation

Figure 3.4: Norms of estimated parameters in control law

59

Chapter 4

Systems with Unknown Control Directions

4.1 Introduction

In Chapter 3, adaptive control with complete compensation of nonparametric uncertainties has been successfully developed for strict-feedback systems with nonparametric uncertainties. It is noted in the adaptive control design, the control directions (the signs of control gains g_i) are assumed to be known as well as lower bounds of control gains. But these a priori information may not be obtained easily and it is worth to study adaptive control design without these a priori information. As mentioned in Section 1.1.3, unknown control directions problem is a research topic that has received much attention in adaptive control community for decades. As the control directions represent motion directions of the system under any control, the adaptive control problem becomes much more difficult when the signs of control gains are unknown because we cannot decide the direction along which the control operates. The breakthrough for unknown control directions problem was made in continuous-time [77] by introducing a powerful tool of so called Nussbaum gain, which has been thereafter extensively studied in continuous-time adaptive control [78–82, 149].

Counterpart of the Nussbaum gain in discrete-time, namely the discrete Nussbaum gain, has been first proposed in [46], in which a digital algorithm has been developed to construct a discrete Nussbaum gain and consequently, a general framework for adaptive control of high order linear discrete-time systems with unknown control directions has been established. But due to the nature of the discrete Nussbaum gain, which is quite different from its continuous-time counterpart, e.g., there is no restriction of the growth rate of

the argument x of the Nussbaum gain $N(x)$ in continuous-time but in discrete-time the argument is required to grow with bounded increments, it is generally intractable to design control using discrete Nussbaum gain.

In this Chapter, we will explore discrete Nussbaum gain to design adaptive control for the strict-feedback nonlinear systems without assumption on control gains as in Chapter 3. The definition and properties of the discrete Nussbaum gain is discussed in Section 4.2. In order to better illustrate the control design procedure and keep focused on the unknown control directions problem, in Section 4.3 we start from design for systems without nonparametric uncertainties. After a clearly demonstration of the control design approach in the ideal case with only parametric uncertainties in Section 4.4.2 where asymptotical tracking performance is obtained, we will show in Section 4.4.3 that by slight modification of the control parameter update law, the developed adaptive control is robust to external disturbance in the control range. Simulation studies are provided to show the efficiency of the proposed adaptive control in Sections 4.5. In Section 4.6, we study combination of the control approaches developed in this Chapter and in Chapter 3, in order to design adaptive control for strict-feedback systems with both nonparametric uncertainties and unknown control directions.

The contributions in this Chapter lies in

(i) Discrete Nussbaum gain has been successfully incorporated into the adaptive control of high order nonlinear discrete-time systems, such that control directions and bounds of control gains are not required to be known in the adaptive control design.

(ii) By exploiting the properties of discrete Nussbaum gain, which not only adapts its sign but also change its amplitude, a novel deadzone has been developed to deal with external disturbance without knowledge on the disturbance amplitude.

(iii) The nonparametric uncertainties compensation technique has been well combined with discrete Nussbaum gain technique such that adaptive control for systems with both nonparametric uncertainties and unknown control gains has been developed.

4.2 The Discrete Nussbaum Gain

Definition 4.1. *Consider a discrete nonlinear function $N(x(k))$ defined on a sequence $x(k)$ with $x_s(k) = \sup_{k' \leq k}\{x(k')\}$. $N(x(k))$ is a discrete Nussbaum gain if and only if it satisfies the following two properties:*

(i) If $x_s(k)$ increases without bound, then

$$\sup_{x_s(k) \geq \delta_0} \frac{1}{x_s(k)} S_N(x(k)) = +\infty, \qquad \inf_{x_s(k) \geq \delta_0} \frac{1}{x_s(k)} S_N(x(k)) = -\infty$$

(ii) If $x_s(k) \leq \delta_1$, then $|S_N(x(k))| \leq \delta_2$ with some positive constants δ_0, δ_1 and δ_2.

where $S_N(x(k))$ is defined as

$$S_N(x(k)) = \sum_{k'=0}^{k'=k} N(x(k'))\Delta x(k') \tag{4.1}$$

with $\Delta x(k) = x(k+1) - x(k)$.

In Definition 4.1, we see that similar to Nussbaum gain in continuous-time, for a discrete Nussbaum gain, if $x_s(k)$ is unbounded then $S_N(x(k))$ oscillates between positive infinity and negative infinity, but if $x_s(k)$ is bounded, then $S_N(x(k))$ is bounded as well.

The first algorithm to build a discrete Nussbaum gain was proposed in [46], in which it is pointed that it is essential for the discrete sequence $x(k)$ to satisfy

$$x(0) = 0, \quad x(k) > 0, \quad |\Delta x(k)| \leq \delta_0 \tag{4.2}$$

Then, the discrete Nussbaum gain proposed in [46] is defined on the sequence $x(k)$ as

$$N(x(k)) = x_s(k)s_N(x(k)) \tag{4.3}$$

where $s_N(x(k))$ is the sign function of the discrete Nussbaum gain, i.e., $s_N(x(k)) = \pm 1$. The initial value is set as $s_N(x(0)) = +1$. Thereafter, the sign function $s_N(x(k))$ will be chosen by comparing the summation $S_N(x(k))$ with a pair of switching curves defined by $f(x_s) = \pm x_s^{\frac{3}{2}}(k)$. The detail follows:

Step (a): At $k = k_1$, measure the output $y(k_1)$ and compute $\Delta x(k_1)$ and $x(k_1 + 1) = x(k_1) + \Delta x(k_1)$ and $S_N(x(k_1)) = S_N(x(k_1 - 1)) + N(x(k_1))\Delta x(k_1)$.

$$\text{Case } (s_N(x(k_1)) = +1): \begin{cases} \text{If} \quad S_N(x(k_1)) \leq x_s^{\frac{3}{2}}(k_1), & \text{then go to Step (b)} \\ \text{If} \quad S_N(x(k_1)) > x_s^{\frac{3}{2}}(k_1), & \text{then go to Step (c)} \end{cases}$$

$$\text{Case } (s_N(x(k_1)) = -1): \begin{cases} \text{If} \quad S_N(x(k_1)) < -x_s^{\frac{3}{2}}(k_1), & \text{then go to Step (b)} \\ \text{If} \quad S_N(x(k_1)) \geq -x_s^{\frac{3}{2}}(k_1), & \text{then go to Step (c)} \end{cases}$$

Step (b): Set $s_N(x(k_1 + 1)) = 1$, go to step (d).

Step (c): Set $s_N(x(k_1 + 1)) = -1$, go to step (d).

Step (d): Return to Step (a) and wait for the measurement of output.

Remark 4.1. *It should be emphasized here that in contrast to continuous-time Nusbaum gain, there is a strong restriction on the argument of the discrete Nussbuam gain, $x(k)$, i.e., (i) it is a non-negative sequence, and (ii) the magnitude of the increment, $|\Delta x(k)|$, is bounded by some constant. These constraints make the design based on the discrete Nussbaum gain more challenging than the continuous-time case.*

Lemma 4.1. *: Let $V(k)$ be a positive definite function, $\forall k$, $N(x(k))$ be the discrete Nussbaum gain defined in Definition 4.1, and g be a nonzero constant. If the following inequality holds:*

$$V(k) \le \sum_{k'=k_1}^{k} (c_1 + gN(x(k')))\Delta x(k') + c_2 x(k) + c_3, \quad \forall k \tag{4.4}$$

where c_1, c_2 and c_3 are some constants, k_1 is a positive integer, then $V(k)$, $x(k)$ and $N(x(k))$ must be bounded, $\forall k$.

Proof. Suppose that $x(k)$ is unbounded, then, because $x(k) \ge 0$, $\forall k$, $x_s(k)$ must increase without upper bound. Therefore, there must exist a k_0 such that $x_s(k) \ge \delta_0 \ge |\Delta x(k)|$, $\forall k \ge k_0$.

Noting that $x(k+1) \le x_s(k) + \delta_0$, we have the following inequality from (4.4), $\forall k \ge k_0$.

$$
\begin{aligned}
0 \;\le\; & \frac{V(k)}{x_s(k)} \le \frac{g}{x_s(k)} \sum_{k'=0}^{k} N(x(k'))\Delta x(k') + c_1 \frac{x(k+1)}{x_s(k)} + c_2 \frac{x(k)}{x_s(k)} + \frac{c_3}{x_s(k)} \\
& - \frac{1}{x_s(k)} \sum_{k'=0}^{k_1-1} (c_1 + \theta N(x(k')))\Delta x(k') \\
\le\; & \frac{g}{x_s(k)} S_N(x(k)) + 2c_1 + c_2 + \frac{c_3}{\delta_0} + c_4 \tag{4.5}
\end{aligned}
$$

where $c_4 = \frac{1}{\delta_0}|\sum_{k'=0}^{k_1-1}(c_1+\theta N(x(k')))\Delta x(k')|$ is some finite constant. According to property (i) in Definition 4.1, it yields a contradiction if $x(k)$ is unbounded, no matter $\theta > 0$ or $\theta < 0$. Therefore, $x(k)$ is bounded, as well as $x_s(k)$, $\forall k$. According to Property (ii) in Definition 4.1, $\sum_{k'=0}^{k}(c_1 + \theta N(x(k')))\Delta x(k') + c_2 x(k) + c_3$ and $V(k)$ are also bounded. ∎

4.3 System Presentation

For clearly demonstration of the key techniques involved in the nonlinear adaptive control design using discrete Nussbaum gain, let us first focus only on the unknown control directions problem without consideration of nonparametric uncertainties. The system to be

controlled is described as follows:

$$
\begin{cases}
\xi_1(k+1) = \Theta_1^T \Phi_1(\bar{\xi}_1(k)) + g_1 \xi_2(k) \\
\xi_2(k+1) = \Theta_2^T \Phi_2(\bar{\xi}_2(k)) + g_2 \xi_3(k) \\
\quad\vdots \\
\xi_n(k+1) = \Theta_n^T \Phi_n(\bar{\xi}_n(k)) + g_n u(k) + d(k) \\
y(k) = \xi_1(k)
\end{cases}
\tag{4.6}
$$

where $d(k)$ is external disturbance and other notations are as same as those in Chapter 3. In the following parts, for better illustration, we first do not consider the disturbance $d(k)$ and focus on the design with discrete Nussbaum gain in Section 4.4.2. Later in Section 4.4.3, we consider to deal with the effect of disturbance in the control design by modification of control parameter update law.

Similarly as in Chapter 3, we assume Lipschitz condition of the nonlinear regression functions, but for the control gains, we only assume they are not naughts.

Assumption 4.1. *The system functions $\Phi_i(\bar{\xi}_i(k))$ are Lipschitz functions with Lipschitz coefficient L_i. The control gains $g_i \neq 0$, $i = 1, 2, \ldots, n$. In addition, the external disturbance is bounded by an unknown constant $\bar{d}$, i.e., $|d(k)| \leq \bar{d}$.*

4.4 Adaptive Control Design

4.4.1 Singularity problem

Following the transformation from (3.21) to (3.22), all the equations in (4.6) can be combined together by iterative substitution such that we have the following:

$$
y(k+n) = \Theta_f^T \Phi(k+n-1) + g u(k) + d_o(k)
\tag{4.7}
$$

where Θ_f^T and $\Phi(k+n-1)$ are defined in (3.23) and

$$
d_o(k) = \frac{g}{g_n} d(k)
\tag{4.8}
$$

Similar as control design in Chapter 3, the control can be designed by certainty equivalence principal as follows:

$$
u(k) = \frac{1}{\hat{g}(k)} (-\hat{\Theta}_f^T(k) \hat{\Phi}(k+n-1|k) + y^*(k+n))
\tag{4.9}
$$

where $\hat{\Phi}(k+n-1|k)$ is the prediction of $\Phi(k+n-1)$ defined in (3.24). But as there is no a priori information of the sign of g and the lower bound of g, we cannot devise a projection as

in (3.37) and (3.92) to guarantee that the estimate of g be bounded away from zero. Then the controller (4.9) runs risk of singularity, i.e., $\hat{g}(k)$ may fall into a small neighborhood of zero. As indicated in [123], this problem is far more from trivial because in order to avoid singularity, the existing solutions to the control problem are usually given locally or assume a priori knowledge of the system, i.e., the sign and upper bound of the control gain g.

We will seek an alternative approach to avoid singularity problem. Consider estimating $\Theta_{fg} = g^{-1}\Theta_f$ and g^{-1} instead of Θ_f and g and thus, we have the resultant control well defined as follows:

$$u(k) = -\hat{\Theta}_{fg}^T(k)\hat{\Phi}(k+n-1|k) + \hat{g}_I(k)y^*(k+n) \tag{4.10}$$

where $\hat{\Theta}_{fg}^T(k)$ and $\hat{g}_I(k)$ are the estimates of $\Theta_{fg} = g^{-1}\Theta_f$ and g^{-1}.

4.4.2 Update law without disturbance

In this section, we consider the adaptive control in the disturbance free case, i.e., $d_o(k) = 0$. Substituting the adaptive control (4.10) into (4.7) and subtracting $y^*(k+n)$ on both hand sides, we obtain the following error dynamics when $d_o(k) = 0$

$$
\begin{aligned}
e(k+n) &= y(k+n) - y^*(k+n) \\
&= \Theta_f^T\Phi(k+n-1) - g\hat{\Theta}_{fg}^T(k)\hat{\Phi}(k+n-1|k) + g\hat{g}_I(k)y^*(k+n) - y^*(k+n) \\
&= -g\tilde{\Theta}_{fg}^T(k)\Phi(k+n-1) + g\tilde{g}_I(k)y^*(k+n) - g\beta_g(k+n-1) \tag{4.11}
\end{aligned}
$$

where

$$\tilde{\Theta}_{fg}(k) = \hat{\Theta}_{fg}(k) - \Theta_{fg}, \quad \tilde{g}_I(k) = \hat{g}_I(k) - g^{-1} \tag{4.12}$$

and $\beta_g(k+n-1)$ is defined as

$$\beta_g(k+n-1) = \hat{\Theta}_{fg}^T(k)\tilde{\Phi}(k+n-1|k) \tag{4.13}$$

It should be mentioned that $\beta_g(k)$ defined above is slightly different from $\beta(k)$ defined previously in (3.33).

We see from (4.11) that unknown control gain g appear in the expression of tracking error $e(k)$, such that the sign of control gain g, the control direction, will be required to determine to which direction the estimation proceed. To overcome the difficulty caused by

unknown control direction, the discrete Nussbaum gain is used in the update law as follows:

$$
\begin{aligned}
\epsilon(k) &= \frac{\gamma e(k) + N(x(k))\psi(k)\beta_g(k-1)}{G(k)} \\
\hat{\Theta}_{fg}(k) &= \hat{\Theta}_{fg}(k-n) + \gamma\frac{N(x(k))}{D(k)}\Phi(k-1)\epsilon(k), \quad \hat{\Theta}_{fg}(j) = \mathbf{0}_{[p_j]} \\
\hat{g}_I(k) &= \hat{g}_I(k-n) - \gamma\frac{N(x(k))}{D(k)}y^*(k)\epsilon(k) \\
\hat{g}_I(j) &= 0, \; j = 0, -1, \ldots, -n+1
\end{aligned}
\tag{4.14}
$$

with

$$
\begin{aligned}
\Delta\psi(k) &= \psi(k+1) - \psi(k) = \frac{-N(x(k))\beta_g(k-1)\epsilon(k)}{D(k)} \\
\Delta z(k) &= z(k+1) - z(k) = \frac{G(k)\epsilon^2(k)}{D(k)}, \qquad z(0) = \psi(0) = 0 \\
\beta_g(k-1) &= \hat{\Theta}_{fg}^T(k-n)\tilde{\Phi}(k-1|k-n) \\
x(k) &= z(k) + \frac{\psi^2(k)}{2} \\
G(k) &= 1 + |N(x(k))| \\
D(k) &= (1 + |\psi(k)|)(1 + |N^3(x(k))|) \\
&\quad \times (1 + \|\Phi(k-1)\|^2 + y_d^2(k) + \beta_g^2(k-1) + \epsilon^2(k))
\end{aligned}
\tag{4.15}
$$

where $\epsilon(k)$ is introduced as an augmented error and the tuning parameter $\gamma > 0$ can be arbitrary constant specified by the designer. It should be mentioned that the requirement on sequence $x(k)$ in (4.2) is satisfied.

Theorem 4.1. *Consider the adaptive closed-loop system consisting of system (4.6) under Assumption 4.1, adaptive control (4.10) with parameters update law (4.14), predicted future states defined in Section 3.2.2. All the signals in the closed-loop system are guaranteed to be bounded and the tracking error $e(k)$ will converge to zero, if there is no external disturbance.*

Proof. Substituting the error dynamics (4.11) into the augmented error $\epsilon(k)$, one obtains

$$
\begin{aligned}
&\gamma\tilde{\Theta}_{fg}^T(k-n)\Phi(k-1) - \gamma\tilde{g}_I(k-n)y^*(k) \\
&= -\frac{1}{g}G(k)\epsilon(k) - \gamma\beta_g(k-1) + \frac{1}{g}N(x(k))\psi(k)\beta_g(k-1)
\end{aligned}
\tag{4.16}
$$

Consider a positive definite function $V(k)$ as

$$
V(k) = \sum_{j=1}^{n}\|\tilde{\Theta}_{fg}(k-n+j)\|^2 + \sum_{j=1}^{n}\tilde{g}_I{}^2(k-n+j)
\tag{4.17}
$$

The difference equation of $V(k)$ is given as

$$
\begin{aligned}
\Delta V(k) &= V(k) - V(k-1) \\
&= \tilde{\Theta}_{fg}^T(k)\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}^T(k-n)\tilde{\Theta}_{fg}(k-n) + \tilde{g}_I^2(k) - \tilde{g}_I^2(k-n) \\
&= (\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n))^T(\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n)) \\
&\quad + 2\tilde{\Theta}_{fg}^T(k-n)(\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n)) \\
&\quad + (\tilde{g}_I(k) - \tilde{g}_I(k-n))^2 + 2\tilde{g}_I(k-n)(\tilde{g}_I(k) - \tilde{g}_I(k-n)) \\
&= \gamma^2 \frac{N^2(x(k))(\Phi^T(k-1)\Phi(k-1) + y_d^2(k))}{D^2(k)}\epsilon^2(k) \\
&\quad + 2N(x(k))\frac{\gamma\tilde{\Theta}_{fg}^T(k-n)\Phi(k-1)}{D(k)}\epsilon(k) - 2N(x(k))\frac{\gamma\tilde{g}_I(k-n)y^*(k)}{D(k)}\epsilon(k)
\end{aligned}
$$

and note that

$$
\Delta x(k) = \Delta z(k) + \psi(k)\Delta\psi(k) + \frac{[\Delta\psi(k)]^2}{2}
$$
$$
0 \le \Delta z(k) \le 1, \quad 0 \le |\Delta\psi(k)| \le 1
$$
$$
|N(x(k))|[\Delta\psi(k)]^2 \le \Delta z(k)
$$

we have

$$
\begin{aligned}
\Delta V(k) &\le \gamma^2\frac{G(k)\epsilon^2(k)}{D(k)} - 2\gamma\frac{N(x(k))\beta_g(k-1)\epsilon(k)}{D(k)} - \frac{2}{g}N(x(k))\frac{G(k)\epsilon^2(k)}{D(k)} \\
&\quad + \frac{2}{g}N(x(k))\frac{N(x(k))\psi(k)\beta_g(k-1)\epsilon(k)}{D(k)} \\
&\le \gamma^2\Delta z(k) + 2\gamma\Delta\psi(k) - \frac{2}{g}N(x(k))(\Delta z(k) + \psi(k)\Delta\psi(k) + \frac{[\Delta\psi(k)]^2}{2}) \\
&\quad + \frac{1}{|g|}|N(x(k))|[\Delta\psi(k)]^2 \\
&\le c_1\Delta z(k) + 2\gamma\Delta\psi(k) - \frac{2}{g}N(x(k))\Delta x(k) \tag{4.18}
\end{aligned}
$$

where $c_1 = \gamma^2 + \frac{1}{|g|}$. Taking summation of the above equation results

$$
\begin{aligned}
V(k) &\le -\frac{2}{g}\sum_{\sigma=0}^{k'=k} N(x(k'))\Delta x(k') + c_1 z(k) + c_1 + 2\gamma\psi(k) + 2\gamma \\
&\le -\frac{2}{g}\sum_{\sigma=0}^{k'=k} N(x(k'))\Delta x(k') + c_1(z(k) + \frac{\psi^2(k)}{2}) + c_1 + \frac{2\gamma^2}{c_1} + 2\gamma \\
&\le -\frac{2}{g}\sum_{k'=0}^{k'=k} N(x(k'))\Delta x(k') + c_1 x(k) + c_2 \tag{4.19}
\end{aligned}
$$

where $c_2 = c_1 + \frac{2\gamma^2}{c_1} + 2$. Applying Lemma 4.1 to (4.46) results the boundedness of $V(k)$ and $x(k)$. Considering that $z(k)$ is an nondecreasing sequence satisfying $0 \le z(k) \le x(k)$, thus the boundedness of $x(k)$ means that $z(k)$ and $\psi(k)$ are bounded. Further, this result implies the following conclusions:

(a) $\hat{\Theta}_{fg}(k)$, $\hat{g}_I(k)$, $G(k)$, and $N(x(k))$ are bounded, and

(b) $\sqrt{\Delta z(k)} \in L^2[0, \infty)$.

According to Lemma 2.6, one can easily obtain $\Phi(k - 1) = O[e(k - 1)]$ from the Lipschitz condition of system functions $\Phi_i(\cdot)$, $i = 1, 2, \ldots, n$.

Similar to (3.47), it is easy to establish that

$$\beta_g(k - 1) = o[O[e(k - 1)]]$$

Then, from the boundedness of $N(x(k))$, $\psi(k)$, and $G(k)$, one sees that $\epsilon(k) \sim e(k)$. Furthermore, from the definition of $D(k)$ in (4.14), we have $D(k) = O[\epsilon^2(k)]$. The conclusion (b) implies that

$$\Delta z(k) = \frac{G(k)\epsilon^2(k)}{D(k)} \to 0$$

Applying Lemma 2.3 and noting the boundedness of $G(k)$, we conclude that $\epsilon(k) \to 0$ and thus $e(k) \to 0$ and then the boundedness of states $\bar{\xi}_n(k)$ and control input is obvious according to Lemma 2.6. According to Lemma 3.2, we have the boundedness of the future states prediction and parameters estimates used in the prediction law. This complete the proof of the ultimately boundedness of all the closed-loop signals. ∎

4.4.3 Update law with disturbance

In this section, we consider to deal with the effect of the external disturbance $d(k)$ by adding a dead zone in the control parameter update law, while the control law still assume the form in (4.10).

The control parameter update law with a deadzone is described as follows:

$$
\begin{aligned}
\epsilon(k) &= \frac{\gamma e(k) + N(x(k))\psi(k)\beta_g(k - 1)}{G(k)} \\
\hat{\Theta}_{fg}(k) &= \hat{\Theta}_{fg}(k - n) + \gamma\frac{a(k)N(x(k))}{D(k)}\Phi(k - 1)\epsilon(k), \quad \hat{\Theta}_{fg}(j) = \mathbf{0}_{[p_j]} \\
\hat{g}_I(k) &= \hat{g}_I(k - n) - \gamma\frac{a(k)N(x(k))}{D(k)}y^*(k)\epsilon(k), \quad \hat{g}_I(j) = 0, j = 0, -1, \ldots, -n + 1 \\
\Delta\psi(k) &= \psi(k + 1) - \psi(k) = \frac{-a(k)N(x(k))\beta_g(k - 1)\epsilon(k)}{D(k)}
\end{aligned}
\tag{4.20}
$$

with

$$
\Delta z(k) = z(k+1) - z(k) = \frac{a(k)G(k)\epsilon^2(k)}{D(k)}, \qquad z(0) = \psi(0) = 0
$$

$$
\beta_g(k-1) = \hat{\Theta}_{fg}^T(k-n)\tilde{\Phi}(k-1|k-n)
$$

$$
x(k) = z(k) + \frac{\psi^2(k)}{2}
$$

$$
G(k) = 1 + |N(x(k))|
$$

$$
D(k) = (1 + |\psi(k)|)(1 + |N(x(k))|^3) \times
$$

$$
(1 + \|\Phi(k-1)\|^2 + y_d^2(k) + \beta_g^2(k-1) + \epsilon^2(k))
$$

$$
a(k) = \begin{cases} 1 & \text{if } |\epsilon(k)| > \chi \\ 0 & \text{others} \end{cases} \tag{4.21}
$$

where the tuning factor $\gamma > 0$ and threshold value $\chi > 0$ can be arbitrary positive constants specified by the designer. In addition, it is obvious that requirement on sequence $x(k)$ in (4.2) is still satisfied.

Remark 4.2. *It should be mentioned that the proposed deadzone method does not require a priori knowledge of the upper bound of the disturbance, which is necessary in building the adaptive laws with dead-zones traditionally. The reason lies in the use of the discrete Nussbaum gain, which by itself will oscillate between infinity and minus infinity if the augmented tracking error becomes unbounded. Because the discrete Nussbaum gain not only adapt its sign but also change its amplitude according to the tracking error, we do not need to known the bounds of the control gains in the update law. In addition, as long as the amplitude of the tracking error is of some value larger than zero, the discrete Nussbaum gain is able to adapt to overcome the effect of the external disturbance in the closed-loop system.*

Theorem 4.2. *Consider the adaptive closed-loop system consisting of system (4.6), control (4.10) with parameters update law (4.20), predicted future state defined in Section 3.2.2. Under Assumption 4.1, all the signals in the closed-loop system are bounded and $G(k) = 1 + |N(x(k))|$ will converge to a constant. Denote $C = \lim_{k \to \infty} G(k)$, then the tracking error satisfy $\lim_{k \to \infty} \sup |e(k)| < \frac{C\chi}{\gamma}$, where γ and χ are the tuning factor and the threshold value specified by the designer.*

Proof. Substituting the error dynamics (4.11) into the augmented error $\epsilon(k)$ and considering $d_o(k) \neq 0$, one obtains

$$
\gamma\tilde{\Theta}_{fg}^T(k-n)\Phi(k-1) - \gamma\tilde{g}_I(k-n)y^*(k)
$$

$$
= -\frac{1}{g}G(k)\epsilon(k) - \gamma\beta_g(k-1) + \frac{1}{g}\gamma d_o(k-n) + \frac{1}{g}N(x(k))\psi(k)\beta_g(k-1) \tag{4.22}
$$

Consider the positive definite function $V(k)$ same as in Section 4.4.2

$$V(k) = \sum_{j=1}^{n} \|\tilde{\Theta}_{fg}(k-n+j)\|^2 + \sum_{j=1}^{n} \tilde{g}_I^2(k-n+j) \tag{4.23}$$

According to the definition of $a(k)$ in (4.20), we have

$$\frac{2}{g}a(k)N(x(k))d_o(k-n)\epsilon(k) \leq a(k)|\frac{2\bar{d}}{g_n\chi}\|N(x(k))|\epsilon^2(k) \tag{4.24}$$

which serves as a key inequality in the consequent stability analysis. Now, following the similar techniques in Section 4.4.2, we have the difference equation of $V(k)$ as followings:

$$\Delta V(k) = V(k) - V(k-1)$$
$$= \frac{\gamma^2 a^2(k)N^2(x(k))(\Phi^T(k-1)\Phi(k-1)+y_d^2(k))}{D^2(k)}G(k)\epsilon^2(k)$$
$$+2N(x(k))\frac{a(k)\gamma\tilde{\Theta}_{fg}^T(k-n)\Phi(k-1)}{D(k)}\epsilon(k) - 2N(x(k))\frac{a(k)\gamma\tilde{g}_I(k-n)y^*(k)}{D(k)}\epsilon(k)$$
$$\leq \gamma^2\frac{a(k)G(k)\epsilon^2(k)}{D(k)} + |\frac{2\bar{d}}{g_n\chi}|\frac{a(k)|N(x(k))|\epsilon^2(k)}{D(k)} - 2\gamma\frac{a(k)N(x(k))\beta_g(k-1)\epsilon(k)}{D(k)}$$
$$-\frac{2}{g}N(x(k))\frac{a(k)G(k)\epsilon^2(k)}{D(k)} + \frac{2}{g}N(x(k))\frac{a(k)N(x(k))\psi(k)\beta_g(k-1)\epsilon(k)}{D(k)} \tag{4.25}$$

Note that

$$\frac{a(k)|N(x(k))|\epsilon^2(k)}{D(k)} \leq \Delta z(k), \quad |N(x(k))|[\Delta\psi(k)]^2 \leq \Delta z(k)$$

Then, we have

$$\Delta V(k) \leq (\gamma^2 + |\frac{2\bar{d}}{g_n\chi}|)\Delta z(k) + 2\gamma\Delta\psi(k) + \frac{1}{|g|}|N(x(k))|[\Delta\psi(k)]^2$$
$$-\frac{2}{g}N(x(k))(\Delta z(k) + \psi(k)\Delta\psi(k) + \frac{[\Delta\psi(k)]^2}{2}) \tag{4.26}$$

which leads to

$$V(k) \leq -\frac{2}{g}\sum_{k'=0}^{k'=k} N(x(k'))\Delta x(k') + c_3 x(k) + c_4 \tag{4.27}$$

where c_3 and c_4 are some finite constants.

Then, using the same analysis as in Section 4.4.2, we conclude the boundedness of $\hat{\Theta}_{fg}(k)$, $\hat{g}_I(k)$, $G(k)$, $N(x(k))$ and $\psi(k)$. In addition, we have $\Delta z(k) \to 0$. Let us define a time interval as $Z_{a=1} = \{k|a(k) = 1\}$ and suppose that $Z_{a=1}$ is an infinite set. Then, applying Lemma 2.3 we have

$$\lim_{k\to\infty, k\in Z_1} \epsilon(k) = \lim_{k\to\infty, k\in Z_{a=1}} a(k)\epsilon(k) = 0$$

which conflicts with $a(k) = 1$, $k \in Z_{a=1}$, because $|\epsilon(k)| \geq \chi$ when $a(k) = 1$. Therefore, $Z_{a=1}$ must be a finite set and then, we have

$$\lim_{k\to\infty} a(k) = 0, \quad \lim_{k\to\infty} \sup |\epsilon(k)| \leq \chi$$

and there must be a constant such that

$$\lim_{k\to\infty} G(k) = C$$

Noting that $\beta_g(k-1) = o[O[e(k-1)]] \to 0$, we derive from the definition of $\epsilon(k)$ in (4.20) that

$$
\begin{aligned}
\lim_{k\to\infty} \sup |\epsilon(k)| &= \lim_{k\to\infty} \sup\{|\frac{\gamma e(k) + N(x(k))\psi(k)\beta_g(k-1)}{G(k)}|\} \\
&= \lim_{k\to\infty} \sup\{|\frac{\gamma e(k)}{G(k)}|\} \leq \chi
\end{aligned}
$$

which implies

$$\lim_{k\to\infty} \sup |e(k)| \leq \lim_{k\to\infty} \frac{G(k)\chi}{\gamma} = \frac{C\chi}{\gamma} \tag{4.28}$$

Then, following the same procedure as in the previous section, the boundedness of other closed-loop signals can be concluded. This complete the proof of the boundedness of all the closed-loop signals.

4.5 Simulation Studies

The following second order nonlinear plant is used for simulation.

$$
\begin{cases}
\xi_1(k+1) &= a_1\xi_1(k)\cos(\xi_1(k)) + a_2\xi_1(k)\sin(\xi_1(k)) + g_1\xi_2(k) \\
\xi_2(k+1) &= b_1\xi_2(k)\frac{\xi_1(k)}{1+\xi_1^2(k)} + b_2\frac{\xi_2^3(k)}{2+\xi_2^2(k)} + g_2u(k) + d(k) \\
y(k) &= \xi_1(k)
\end{cases} \tag{4.29}
$$

where $d(k) = 0.2\cos(0.05k)\cos(y(k))$ and system parameters are $a_1 = 0.2$, $a_2 = 0.1$, $g_1 = 3$, $b_1 = 0.3$, $b_2 = -0.6$, and $g_2 = \pm 0.2$. The control objective is to make the output $y(k)$ track a desired reference trajectory

$$y^*(k) = 1.5\sin(\frac{\pi}{5}kT) + 1.5\cos(\frac{\pi}{10}kT), T = 0.05$$

The initial system states are $\bar{\xi}_2(j) = [1, \ 1]^T$, $j = -1, 0$. The tuning rate and the threshold value are chosen as $\gamma = 6$ and $\chi = 0.1$.

The simulation is carried out for twice for comparison and in both simulations, the control law, the prediction law and all the parameters except g_2 are of same values. For the first time, the control gain g_2 is chosen to be negative while in the second time the control gain g_2 is chosen to be positive.

The results are presented in Figures 4.1, 4.2, 4.3 and 4.4. Figure 4.1 shows the output $y(k)$ and reference trajectory $y^*(k)$. We see that when g_2 is negative, though initially the tracking performance is not good (the output goes to a reverse direction), but after the discrete Nussbaum gain $N(x(k))$ turns to negative (see Figure 4.4), the tracking becomes better and better. Figure 4.2 illustrates the boundedness of the control input $u(k)$, the estimated parameters $\hat{g}_I(k)$ and $\|\hat{\Theta}_{fg}(k)\|$ used in the control law. Figure 4.3 shows the signal $\beta_g(k)$ caused by prediction error and $\bar{\hat{\Theta}}_1(k)$ used in the the prediction. Figure 4.4 shows the discrete sequence $x(k)$, $\psi(k)$ and discrete Nussbaum gain $N(x(k))$. The discrete Nussbaum gain $N(x(k))$ adapts by searching alternately in the two directions such that it can been see that it turns from positive to negative in Figure 4.4(a).

In summary, the adaptive NN control with discrete Nussbaum gain adapts by searching alternately in the two directions. The adaptive NN control will be able to reverse its direction of adaptation if initially the adaptation is in the wrong direction. However, we also noted that while the boundeness of all the signals in the adaptive system was maintained, during those intervals when the adaptation is in the wrong direction, the bounds may be very large. This appears to be a limitation of the proposed control. Actually, when the control direction is unknown, no matter what approach is used, if the adaptive NN control is initialized to start in the bad regime where it adapts in the wrong direction, it must at least remain in that regime until the errors become correspondingly large. Only then can the adaptive NN control determine that the direction of adaptation is wrong so that it reverse its direction of adaptation.

4.6 System with Nonparametric Uncertainties

In this section, we design control for systems with nonparametric uncertainties in addition to unknown control directions. Now let us consider system (3.56) studied in Section 3.3 with completely unknown control gains g_i, $i = 1, 2, \ldots, n$, i.e., with Assumption 3.4 removed.

4.6.1 Adaptive control design

Following the similar steps in Section 3.3, we transform the system into a compact form as in (3.79):

$$y(k+n) = \Theta_f^T \Phi(k+n-1) + gu(k) + \Theta_g^T \bar{\nu}(k)$$

Instead of (3.81), we introduce the following auxiliary output

$$y_{ag}(k+n-1) = \Theta_{fg}^T \Phi(k+n-1) + \Theta_{gI}^T \bar{\nu}(k) \tag{4.30}$$

where same as in Section 4.4.2, $\Theta_{fg} = g^{-1}\Theta_f$ and $\Theta_{gI} = g^{-1}\Theta_g$. Then, system (3.79) can be rewritten as

$$y(k+n) = g[y_{ag}(k+n-1) + u(k)] \tag{4.31}$$

From (4.30) and (4.31), it is easy to derive

$$\begin{aligned}
y_{ag}(k+n-1) &= y_{ag}(k+n-1) - y_{ag}(l_k+n-1) + y_{ag}(l_k+n-1) \\
&= \Theta_{fg}^T[\Phi(k+n-1) - \Phi(l_k+n-1)] + g^{-1}y(l_k+n) - u(l_k) \\
&\quad + \Theta_{gI}^T[\bar{\nu}(k) - \bar{\nu}(l_k)]
\end{aligned} \tag{4.32}$$

Let us introduce the following prediction of $y_{ag}(k+n-1)$:

$$\begin{aligned}
\hat{y}_{ag}(k+n-1|k) &= \hat{\Theta}_{fg}^T(k)[\Phi(\hat{y}(k+n-1)) - \Phi(y(l_k+n-1))] \\
&\quad + \hat{g}_I(k)y(l_k+n) - u(l_k)
\end{aligned} \tag{4.33}$$

where $\hat{\Theta}_{fg}(k)$ and $\hat{g}_I(k)$ are the estimates of Θ_{fg} and g^{-1}, and $\Phi(l_k+n-1)$, $y(l_k+n)$ are available at the kth step since $l_k + n \leq k$ according to (3.59).

From (4.32) and (4.33), we have

$$\begin{aligned}
\tilde{y}_{ag}(k+n-1|k) &= \hat{y}_{ag}(k+n-1|k) - y_{ag}(k+n-1|k) \\
&= \tilde{\Theta}_{fg}^T(k)[\Phi(k+n-1) - \Phi(y(l_k+n-1)] + \beta_g(k+n-1) + \tilde{g}_I(k)y(l_k+n) \\
&\quad - \Theta_{gI}^T[\bar{\nu}(k) - \bar{\nu}(l_k)]
\end{aligned} \tag{4.34}$$

where $\tilde{\Theta}_{fg}(k)$, $\tilde{g}_I(k)$ are defined in (4.12) and $\beta_g(k+n-1)$ defined in (4.13). Using the above estimated auxiliary output $\hat{y}_{ag}(k+n-1|k)$, the adaptive control law is constructed as

$$u(k) = -\hat{y}_{ag}(k+n-1|k) + \hat{g}_I(k)y^*(k+n) \tag{4.35}$$

Considering system (4.31), adaptive control law in (4.35), and the estimation error of auxiliary output in (4.34), we obtain the error dynamics as

$$
\begin{aligned}
e(k+n) &= gy_{ag}(k+n-1) - g\hat{y}_{ag}(k+n-1|k) + g\hat{g}_I(k)y^*(k+n) - y^*(k+n) \\
&= -g\tilde{y}_{ag}(k+n-1|k) + g\tilde{g}_I(k)y^*(k+n) \\
&= -g\tilde{\Theta}_{fg}^T(k)[\Phi(y(k+n-1)) - \Phi(y(l_k+n-1))] - g\beta_g(k+n-1) \\
&\quad -g\tilde{g}_I(k)y(l_k+n) + g\tilde{g}_I(k)y^*(k+n) + g\Theta_{gI}^T[\bar{\nu}(k) - \bar{\nu}(l_k)]
\end{aligned} \tag{4.36}
$$

which leads to

$$
\begin{aligned}
e(k) &= -g\tilde{\Theta}_{fg}^T(k-n)[\Phi(y(k-1)) - \Phi(y(l_{k-n}+n-1))] \\
&\quad +g\tilde{g}_I(k-n)[y^*(k) - y(l_{k-n}+n)] - g\beta_g(k-1) \\
&\quad +g\Theta_{gI}^T[\bar{\nu}(k-n) - \bar{\nu}(l_{k-n}]
\end{aligned} \tag{4.37}
$$

Similar to (3.90), it is easy to establish the following inequality

$$
\|\Theta_{gI}^T[\bar{\nu}(k-n) - \bar{\nu}(l_{k-n})]\| \leq \lambda\theta_{gI}\|\Delta\bar{\xi}_n(k-n)\| \tag{4.38}
$$

where $\theta_{gI} = g^{-1}\theta_g$ with θ_g defined in (3.91) and λ is a parameter chosen to satisfy $\max_{1\leq i\leq n} L_{v_i} \leq \lambda \leq \lambda^*$ and the existence of a parameter λ^* will be established in a similar was as in Section 3.3.4 (Refer to λ^* defined in (3.110)).

In the following, let us denote $\hat{\theta}_g(k)$ as the estimate of the unknown parameter θ_g. Then, the parameter estimates in control law (4.35) are calculated by the following update law:

$$
\begin{aligned}
e'(k) &= \gamma\frac{e(k)}{G(k)} \\
\hat{\Theta}_{fg}(k) &= \hat{\Theta}_{fg}(k-n) - \frac{\gamma a_g(k)N(x(k))[\Phi(y(k-1)) - \Phi(y(l_{k-n}+n-1))]}{D(k-n)}e'(k) \\
\hat{g}_I(k) &= \hat{g}_I(k-n) + \frac{\gamma a_g(k)N(x(k))[y(l_{k-n}+n) - y^*(k)]}{D(k-n)}e'(k) \\
\hat{\theta}_{gI}(k) &= \hat{\theta}_{gI}(k) + \frac{\gamma\lambda a_g(k)|N(x(k))|\|\Delta\bar{\xi}_n(k-n)\|}{D(k-n)}|e'(k)|
\end{aligned} \tag{4.39}
$$

with

$$
\begin{aligned}
G(k) &= 1 + |N(x(k))| \\
D(k-n) &= (1 + N^2(x(k)))\{1 + \|\Phi(y(k-1)) - \Phi(y(l_{k-n}+n-1))\|^2 \\
&\quad + [y(l_{k-n}+n) - y^*(k)]^2 + \lambda^2 \|\Delta\bar{\xi}_n(k-n)\|^2 + e'^2(k)\} \\
\Delta x(k) &= x(k+1) - x(k) = \frac{a_g(k)G(k)e'^2(k)}{D(k)} \\
a_g(k) &= \begin{cases} 1 - \frac{\lambda\hat{c}(k-n)\|\Delta\bar{\xi}_n(k-n)\| + |\beta_g(k-1)|}{|e'(k)|} \\ \qquad \text{if } |e'(k)| > \lambda\hat{\theta}_{gI}(k)\|\Delta\bar{\xi}_n(k-n)\| + |\beta_g(k-1)| \\ 0 \qquad \text{otherwise} \end{cases} \qquad (4.40) \\
\hat{\Theta}_{fg}(0) &= \mathbf{0}_{[n]}, \ \hat{g}_I(0) = 0, \ \hat{c}_c(0) = 0
\end{aligned}
$$

where $N(x(k))$ is the discrete Nussbaum gain defined in Definition 4.1, and $\gamma > 0$ is the tuning rate to be specified by the designer.

Remark 4.3. *It should be mentioned that the sequence $x(k)$ defined above in (4.40) is different from that defined in (4.15) or (4.21), but the requirement on sequence $x(k)$ in (4.2) is also well satisfied.*

4.6.2 Stability analysis

Let us state the main result of this Section in the following theorem.

Theorem 4.3. *Consider the adaptive closed-loop system consisting of system (3.56) without Assumption 3.2, predicted states in Section 3.3.2, control law (4.35) and parameter update law (4.39). All the signals in the closed-loop system are bounded and furthermore, the tracking error $e(k)$ converges to zero.*

Proof. Substituting the error dynamics (4.37) into the augmented error $e'(k)$ in (4.39) gives

$$
\begin{aligned}
e'(k)G(k) &= -\gamma g\tilde{\Theta}_{fg}^T(k-n)[\Phi(y(k-1)) - \Phi(y(l_{k-n}+n-1))] - \gamma g\beta_g(k-1) \\
&\quad - \gamma g\tilde{g}_I(k-n)[y(l_{k-n}+n) - y^*(k)] + \gamma g\Theta_{gI}^T[\bar{\nu}(k-n) - \bar{\nu}(l_{k-n})] \quad (4.41)
\end{aligned}
$$

which together with (4.38) yields

$$
\begin{aligned}
&\gamma N(x(k))e'(k)\{\tilde{\Theta}_{fg}^T(k-n)[\Phi(k-1) - \Phi(l_{k-n}+n-1)] \\
&\quad + \tilde{g}_I(k-n)[y(l_{k-n}+n) - y^*(k)]\} \\
&= \{-\frac{1}{g}G(k)e'(k) - \gamma\beta_g(k-1) + \gamma\Theta_{gI}^T[\bar{\nu}(k-n) - \bar{\nu}(l_{k-n})]\}e'(k)N(x(k)) \\
&\leq -\frac{1}{g}N(x(k))G(k)e'^2(k) + \gamma\theta_g|N(x(k))||e'(k)|\|\Delta\bar{\xi}_n(k-n)\| \qquad (4.42)
\end{aligned}
$$

Choose the Lyapunov candidate function as

$$V(k) = \sum_{j=1}^{n} \|\tilde{\Theta}_{fg}(k-n+j)\|^2 + \sum_{j=1}^{n} \tilde{g}_I^2(k-n+j) + \sum_{j=1}^{n} \tilde{\theta}_{gI}^2(k-n+j) \quad (4.43)$$

From (4.39) and (4.42), it is easy to derive that the difference of $V(k)$ is

$$\Delta V(k) = V(k) - V(k-1)$$
$$\leq \|\tilde{\Theta}_{fg}(k)\|^2 - \|\tilde{\Theta}_{fg}(k-n)\|^2 + \tilde{g}_I^2(k) - \tilde{g}_I^2(k-n) + \tilde{\theta}_{gI}^2(k) - \tilde{\theta}_{gI}^2(k-n)$$
$$= \{\|\Phi(y(k-1)) - \Phi(y(l_{k-n}+n-1))\|^2 + [y(l_{k-n}+n) - y^*(k)]^2$$
$$\quad + \lambda^2 \|\Delta\bar{\xi}_n(k-n)\|^2\} \times \frac{a_g^2(k)\gamma^2 e'^2(k)N^2(x(k))}{D^2(k-n)}$$
$$\quad + \{\tilde{\Theta}_{fg}^T(k-n)[\Phi(y(k-1)) - \Phi(y(l_{k-n}+n-1))] + \tilde{g}_I(k-n)$$
$$\quad \times [y(l_{k-n}+n) - y^*(k)]\}e'(k)\frac{2a_g(k)\gamma N(x(k))}{D(k-n)}$$
$$\quad + \lambda\tilde{\theta}_{gI}(k-n)\|\Delta\bar{\xi}_n(k-n)\|\,|e'(k)|\frac{2a_g(k)\gamma|N(x(k))|}{D(k-n)}$$

which together with update law (4.39) leads to

$$\Delta V(k) \leq \gamma^2 \frac{a_c^2(k)G(k)e'^2(k)}{D(k-n)} - \frac{2a_g(k)N(x(k))G(k)e'^2(k)}{gD(k-n)}$$
$$\quad - \frac{2\gamma a_g(k)N(x(k))e'(k)\beta_g(k-1)}{D(k-n)}$$
$$\quad + \frac{2\lambda\gamma a_g(k)|N(x(k))|\,|e'(k)|\hat{\theta}_{gI}(k)\|\Delta\bar{\xi}_n(k-n)\|}{D(k-n)}$$
$$\leq \gamma^2 \frac{a_c^2(k)G(k)e'^2(k)}{D(k-n)} - \frac{2a_g(k)N(x(k))G(k)e'^2(k)}{gD(k-n)}$$
$$\quad + \frac{2\gamma a_g(k)|N(x(k))|\,|e'(k)|(|\beta_g(k-1)| + \hat{\theta}_{gI}(k)\lambda\|\Delta\bar{\xi}_n(k-n)\|)}{D(k-n)}$$
$$\leq \gamma^2 \frac{a_g(k)G(k)e'^2(k)}{D(k-n)} - \frac{2a_g(k)N(x(k))G(k)e'^2(k)}{gD(k-n)}$$
$$\quad + \frac{2\gamma a_g(k)|N(x(k))|e'^2(k)}{D(k-n)} \quad (4.44)$$

where the definition of deadzone $a_g(k)$ in (4.40) and inequality $a_g^2(k) \leq a_g(k)$ were used in the last inequality. According to $\Delta x(k) = x(k+1) - x(k) = \frac{a_g(k)G(k)e'^2(k)}{D(k)}$ in (4.40) and

$G(k) = 1 + |N(x(k))|$, we can rewrite inequality (4.44) as

$$
\begin{aligned}
\Delta V(k) &\leq \gamma^2 \frac{a_g(k)G(k)e'^2(k)}{D(k)} - 2a_g(k)N(x(k))\frac{G(k)e'^2(k)}{gD(k)} + 2a_g(k)\frac{\gamma G(k)e'^2(k)}{|g|D(k)} \\
&\leq \gamma^2 \Delta x(k) - 2\frac{N(x(k))\Delta x(k)}{g} + 2\frac{\gamma}{|g|}\Delta x(k) \\
&= c_1 \Delta x(k) - c_2 N(x(k))\Delta x(k)
\end{aligned}
\tag{4.45}
$$

where $c_1 = \gamma^2 + 2\frac{\gamma}{|g|}$ and $c_2 = \frac{2}{g} \neq 0$. Taking summation on both hand sides of (4.45) results in

$$
V(k) \leq -c_2 \sum_{k'=0}^{k} N(x(k'))\Delta x(k') + c_1 x(k) + V(-1).
\tag{4.46}
$$

Applying the same techniques in Section 4.4.2, we can prove the boundedness of $V(k)$, $N(x(k))$, $x(k)$, $\hat{\theta}_{fg}(k)$, $\hat{g}_I(k)$ and $G(k)$ as well as

$$
\lim_{k\to\infty} \Delta x(k) = \lim_{k\to\infty} \frac{a_g(k)G(k)e'^2(k)}{D(k)} = 0
\tag{4.47}
$$

Using $a_g^2(k) \leq a_g(k)$ and $G(k) = 1 + |N(x(k))| \geq 1$, we have

$$
\frac{a_g(k)G(k)e'^2(k)}{D(k)} \geq \frac{a_g^2(k)G(k)e'^2(k)}{D(k)} \geq \frac{a_g^2(k)e'^2(k)}{D(k)} \geq 0
$$

which together with (4.47) yields

$$
\lim_{k\to\infty} \frac{a_g^2(k)e'^2(k)}{D(k)} = 0
\tag{4.48}
$$

From the definition of deadzone $a_g(k)$ in (4.40), when $|e'(k)| > \lambda\hat{\theta}_{gI}(k)\|\Delta\bar{\xi}_n(k-n)\| + |\beta_g(k-1)|$, we have

$$
a_g(k)|e'(k)| = |e'(k)| - \lambda\hat{c}(k-n)\|\Delta\bar{\xi}_n(k-n)\| - |\beta_g(k-1)| > 0
$$

and when $|e'(k)| \leq \lambda\hat{c}(k-n)\|\Delta\bar{\xi}_n(k-n)\| + |\beta_g(k-1)|$, we have

$$
a_g(k)|e'(k)| = 0 \geq |e'(k)| - \lambda\hat{c}(k-n)\|\Delta\bar{\xi}_n(k-n)\| - |\beta_g(k-1)|
$$

Noting that $\hat{\theta}_{gI}(k)$ is bounded and set $\hat{\theta}_{gI}(k) \leq \bar{c}_c, \forall k \in Z_{-n}^+$ we have

$$
|e'(k)| - \lambda\bar{c}_c\|\Delta\bar{\xi}_n(k-n)\| - |\beta_g(k-1)| \leq a_g(k)|e'(k)|
\tag{4.49}
$$

Refer to Section 3.3.4, we see that equations (4.48) and (4.49) correspond to equations (3.100) and (3.101), then applying the same techniques following equations (3.100) and (3.101) in Section 3.3.4 we can complete the proof. ∎

4.7 Summary

In this Chapter, we have exploited discrete Nussbaum gain to counter the lack of knowledge of control directions for adaptive control design of nonlinear discrete-time systems. The class of systems with only external disturbance but not nonparametric uncertainties has been studied first. Under the framework of future states prediction based adaptive control design, we have successfully incorporated discrete Nussbaum into the control parameter update law such that the adaptive control is insensitive to the control directions. In the adaptive control structure, the reciprocal of the control gain instead of control gain is used such that controller singularity problem is avoided. Thereafter, adaptive control designed is extended to systems with both unknown control directions and nonparametric uncertainties are studied by constructing a more complicated control parameter update law. All the signals in the closed-loop system are guaranteed bounded and the output tracking error is made to be zero ultimately in the absence of external disturbance. The efficiency of the designed adaptive control are demonstrated in the simulation studies.

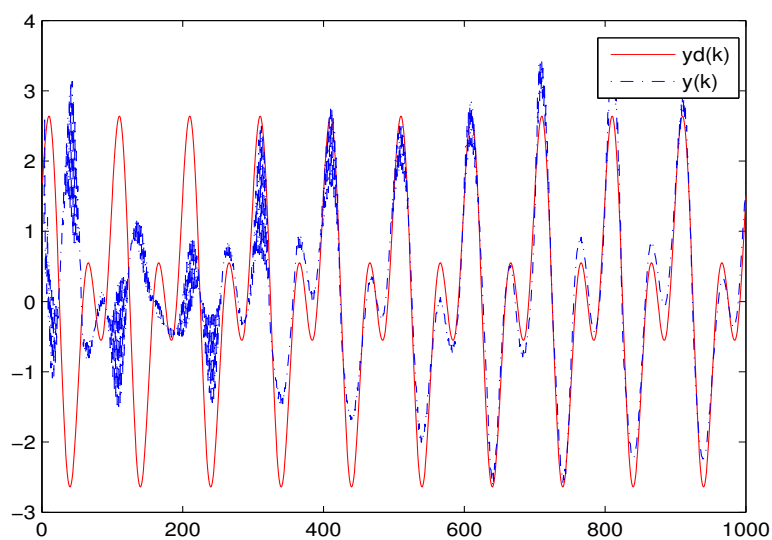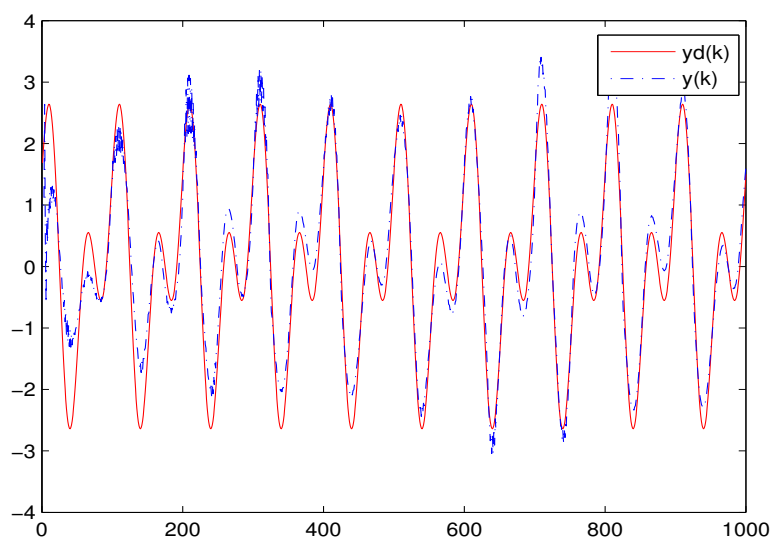(a) with negative g_2

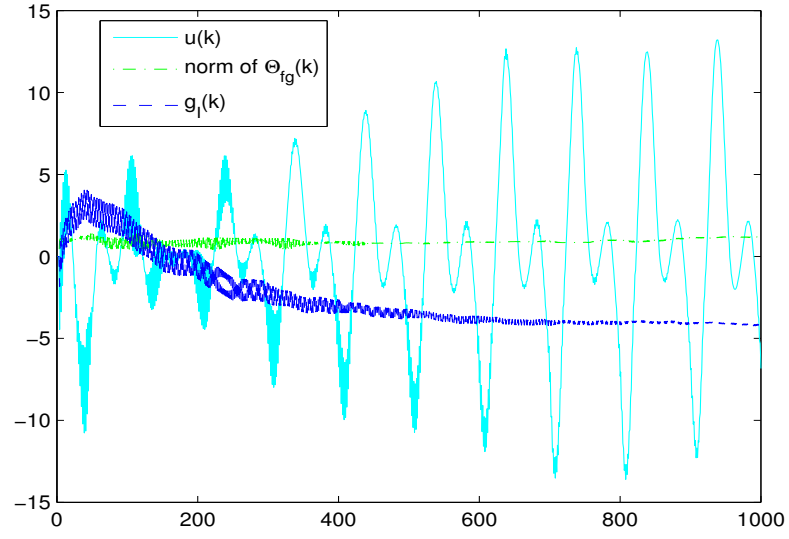(b) with positive g_2

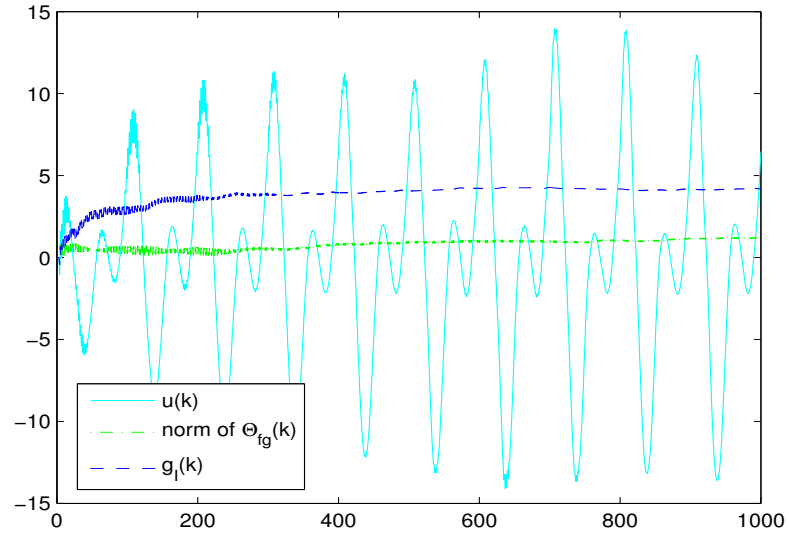Figure 4.1: Output and reference

(a) with negative g_2

(b) with positive g_2

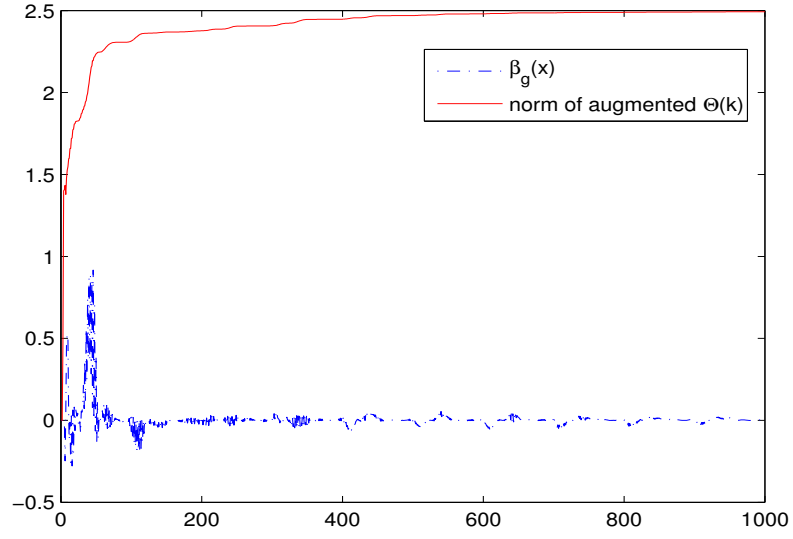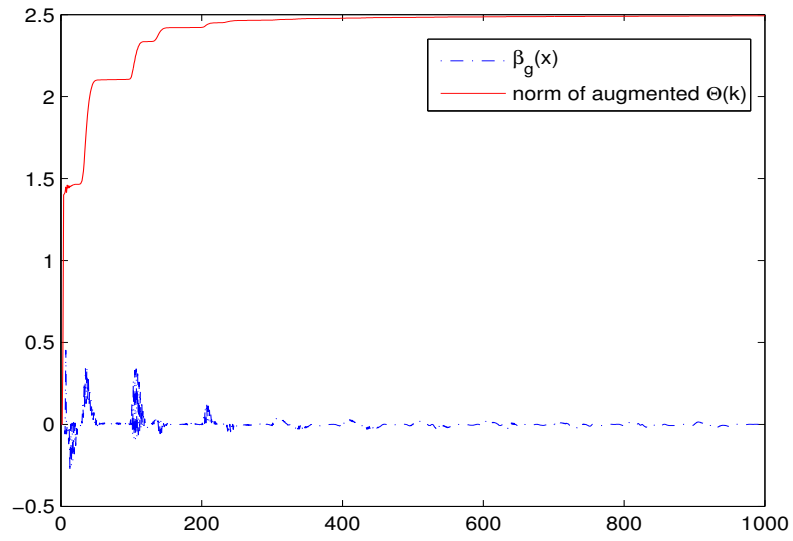Figure 4.2: Control input and estimated parameters in controller

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(a) with negative g_2

(b) with positive g_2

Figure 4.3: Signals in prediction law
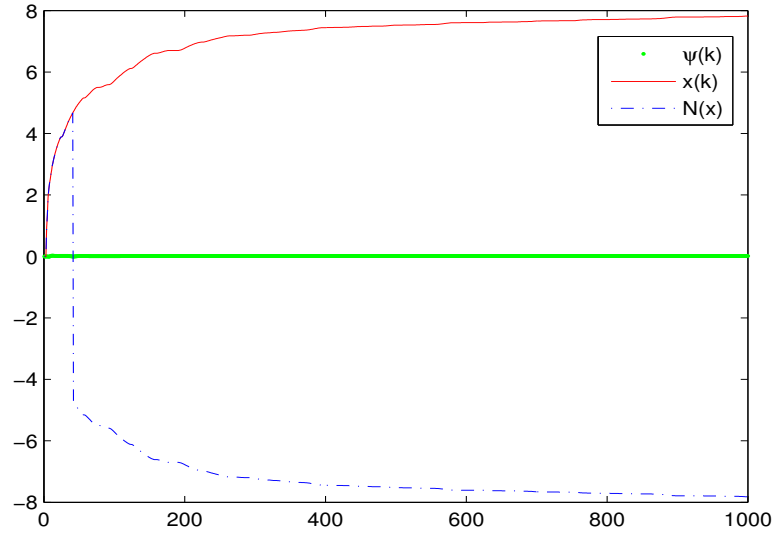
(a) with negative g_2
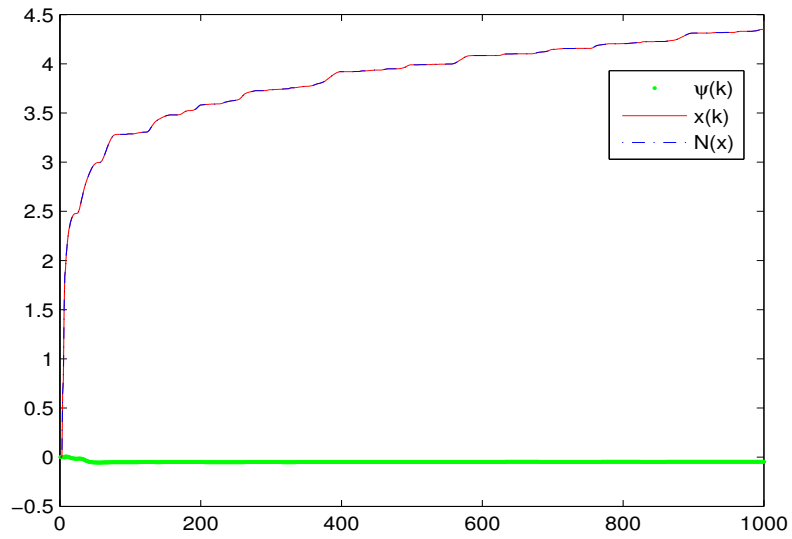
(b) with positive g_2

Figure 4.4: Signals in Discrete Nussbaum gain

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Chapter 5

Systems with Hysteresis Constraint and Multi-variable

5.1 Introduction

In the foregoing Chapters, we have developed a framework of adaptive control based on predicted future states for SISO discrete-time systems in strict-feedback form. Under the proposed framework, we have studied nonparametric model uncertainties compensation and have exploited discrete Nussbaum gain to deal with the lack of knowledge of control directions. In Chapters 3 and 4, we assumed that the control input directly enter into the system and the system is SISO. In this Chapter, we extend the adaptive control developed in last two Chapters by further investigating systems with hysteresis input constraint and systems with multi-variable in block triangular structure.

In Section 5.2, we study adaptive control of strict-feedback systems with unknown control directions, which is proceeded by hysteresis type input constraint. As mentioned in Section 1.1, there may be nonlinear input constraints caused by characteristics of actuator and sensors. In recent years, some research effort has been made to exploit Prandtl-Ishlinskii (PI) model in adaptive control of linear systems with hysteresis input constraint [176], in which the control directions are assumed to be known. One recent attempt to control nonlinear system with unknown control directions using PI model has been made in in continuous-time [177]. However, as mentioned in Section 1.1, due to the inherent difficulties in discrete-time models many controls designed for continuous-time systems are not applicable for discrete-time systems, and in most cases, adaptive control design for discrete-time systems is much more difficult.

In Section 5.3, we study adaptive control of block-triangular MIMO nonlinear discrete-time systems, which is composed of a number of strict-feedback subsystems coupled with each others. It is well known that many practical systems are multi-variable systems. But the control problems for MIMO systems are very difficult and are very different from those for SISO systems. Adaptive control design of MIMO nonlinear systems becomes extremely difficult when there are nonlinear uncertain couplings. The MIMO systems to be studied are of interconnections in every equation of each subsystem rather than only in the last equation of each subsystem as in [157]. Since the state variables of one subsystem are embedded into system functions of another subsystem in an unmatched manner, and even in functional uncertain nonlinearities, we need to establish the relation among various states, inputs and outputs before hand. Later, using the established relation among system states, inputs and outputs, we are able to sort the growth rate of various closed-loop signals, then the closed-loop stability can be proved with resort to Lyapunov approach.

Similar to Section 3.2, delayed states in the uncertain couplings in the last equations of each subsystem are considered in the MIMO nonlinear systems to be studied. For a class of uncertain MIMO nonlinear systems in block-triangular forms with unknown time delays, adaptive NN control design based on Lyapunov-Krasovskii functional has been proposed in [172]. However, there is not a counterpart of Lyapunov-Krasovskii functional in discrete-time. The technique developed in Section 3.2 will be further exploited in this Chapter to deal with time delayed states in the uncertain coupling terms. By using Lyapunov method and ordering signals growth rate, it is rigourously proved that all the signals in the whole closed-loop systems are globally bounded and the output tracking errors asymptotically converge to zeros.

The contributions in this Chapter lies in

(i) To tackle the difficulty caused by hysteresis input constraint, discrete-time Prandtl-Ishlinskii (PI) model is exploited in the adaptive control design.

(ii) Future states predictions for each subsystem of the block triangular MIMO systems have be developed and the growth rate of the prediction errors has been established.

(iii) With exploration of the properties of block-triangular structure of the MIMO system, adaptive control has been developed to decouple the interactions of states and inputs among all the subsystems.

5.2 Systems Proceeded by Hysteresis Input

In this Section, PI model is used to describe the hysteresis. Based on the future states prediction method developed in Section 3.2.2, the adaptive control is designed with employment of the discrete Nussbaum gain.

5.2.1 Problem formulation

Consider strict-feedback nonlinear discrete-time systems with hysteresis input constraint described as follows:

$$
\begin{cases}
\xi_i(k+1) = \Theta_1^T \Phi_i(\bar{\xi}_i(k)) + g_i \xi_{i+1}(k) \\
\quad i = 1, 2, \ldots, n-1 \\
\xi_n(k+1) = \Theta_n^T \Phi_n(\bar{\xi}_n(k)) + g_n u(k) + d(k) \\
u(k) = H[v](k) \\
y(k) = \xi_1(k)
\end{cases}
\tag{5.1}
$$

in which hysteresis is denoted by the operator $u(k) = H[v](k)$, where $v(k)$ is the input and $u(k)$ is the output of the hysteresis and the input to the systems. Other notations and control objective are same in those in Section 4.3 and the system is also subject to Assumption 4.1 in Section 4.3. The hysteresis operator is represented by discrete-time PI model as follows [176]:

$$
u(k) \quad = \quad \int_0^\infty p(r) E_r[v](k) dr
\tag{5.2}
$$

where

$$
E_r(k) = e_r[v(k) - v(k_i) + E_r[v](k_i)], \quad e_r(v) = \min(r, \max(-r, v))
\tag{5.3}
$$

with $E_r(0) = e_r(v(0) - u(-1))$ and $p(r)$ is an unknown density function satisfying $p(r) \geq 0$ with $\int_0^\infty r p(r) dr < \infty$, and $E_r(\cdot)$ is called as stop operator. When the value r is large enough, the density function $p(r)$ will vanishes, i.e., there exists a constant $\bar{R}$ such that $p(r) = 0, \forall r > \bar{R}$, and thus the integral $\int_0^\infty p(r) E_r[v](k) dr$ is replaced by $\int_0^{\bar{R}} p(r) E_r[v](k) dr$ in the sequel.

Figure 5.1 illustrates the input (v) and output (u) relationship of the PI model in (5.2). The density function used is $p(r) = e^{-0.07(r-1)^2}$ with $\bar{R} = 10$. The input is chosen as $v(k) = 12.0 \sin(0.0524k)/(1 + 0.0175k)$ with $k = 1, 2, \ldots, 360$.
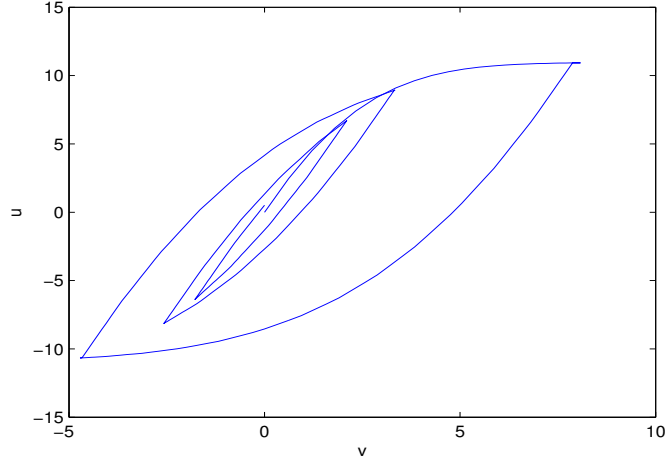
Figure 5.1: Hysteresis curve give by the PI model

5.2.2 Adaptive control design

Let us perform similar techniques that transforming (3.1) into (3.22) in Section 3.2.3 to system (5.1) such that the equations in (5.1) can be combined together by iterative substitution as follows:

$$y(k+n) = \Theta_f^T \Phi(k+n-1) + g \int_0^{\bar{R}} p(r) E_r[v](k) dr + d_o(k) \tag{5.4}$$

where Θ_f, g, $\Phi(k+n-1)$ are defined in (3.23) and $d_o(k)$ is defined in (4.8)

Similarly as in Section 4.4.2, let us denote $\hat{\Theta}_{fg}(k)$ and $\hat{g}_I(k)$ as the estimates of $g^{-1}\Theta_f$ and g^{-1}, respectively. Using the predicted function $\hat{\Phi}(k+n-1|k)$ defined in (3.24), let us define

$$u'(k) = -\hat{\Theta}_{fg}^T(k) \hat{\Phi}(k+n-1|k) + \hat{g}_I(k) y_d(k+n) \tag{5.5}$$

Let $[v_{\min}, \ v_{\max}]$ be the practical input range to the hysteresis operator, which is a strict subset of $[-\bar{R}, \ \bar{R}]$, and the saturation output of $\int_0^{\bar{R}} \hat{p}(r,k) E_r[v^*](k) dr$ be $\hat{u}'_{\mathrm{sat}}(k)$, in which these notations are borrowed from [176] and $v^*(k)$ is derived following the techniques in [176]. If $u'(k) < -\hat{u}'_{\mathrm{sat}}(k)$, then $v^*(k) = v_{\min}$; if $u'(k) > \hat{u}'_{\mathrm{sat}}(k)$, then $v^*(k) = v_{\max}$; otherwise, following the algorithm proposed in Section C of [176], a $v^*(k)$ can be obtained such that

$$\mu(k) = \int^{\bar{R}_0} \hat{p}(r,k) E_r[v^*](k) dr - u'(k), \ \ |\mu(k)| \le \bar{\mu} \tag{5.6}$$

where $\bar{\mu}$ is an assigned admissible error, $\hat{p}(r,k)$ is the estimate of $p(r)$ defined later in (5.11) and it is guaranteed to be nonnegative.

The adaptive control input is chosen as

$$v(k) = v^*(k) \tag{5.7}$$

Substituting the adaptive control (5.7) into (5.4), we have

$$
\begin{aligned}
e(k+n) &= y(k+n) - y_d(k+n) \\
&= -g\tilde{\Theta}_{fg}^T(k)\Phi(k+n-1) + g\tilde{g}_I(k)y_d(k+n) \\
&\quad -g\int_0^{\bar{R}} \tilde{p}(r,k)E_r[v^*](k)dr - g\beta_g(k+n-1) + g\mu(k) + d_0(k) \tag{5.8}
\end{aligned}
$$

where $\tilde{\Theta}_{fg}(k)$ is defined in (4.12), $\beta_g(k)$ is defined in (4.13), and $\tilde{p}(r,k)$ and $\mu(k)$ are defined as

$$
\begin{aligned}
\tilde{p}(r,k) &= \hat{p}(r,k) - p(r)nb \tag{5.9} \\
\mu(k) &= \int_0^{\bar{R}} \hat{p}(r,k)E_r[v^*](k)dr - u'(k) \tag{5.10}
\end{aligned}
$$

The parameters estimates in the control law are updated by the following adaptation law

$$
\begin{aligned}
\epsilon(k) &= \frac{\gamma e(k) + N(x(k))\psi(k)\beta_g(k-1)}{G(k)} \\
\hat{\Theta}_{fg}(k) &= \hat{\Theta}_{fg}(k-n) + \gamma\frac{a(k)N(x(k))}{D(k)}\Phi(k-1)\epsilon(k) \\
\hat{g}_I(k) &= \hat{g}_I(k-n) - \gamma\frac{a(k)N(x(k))}{D(k)}y_d(k)\epsilon(k) \\
\hat{p}'(r,k) &= \hat{p}(r,k-n) + \gamma\frac{a(k)N(x(k))}{D(k)}E_r[v^*](k-n)\epsilon(k) \\
\hat{p}(r,k) &= |\hat{p}'(r,k)| \tag{5.11}
\end{aligned}
$$

where

$$
\begin{aligned}
G(k) &= 1 + |N(x(k))| \\
D(k) &= (1+|\psi(k)|)(1+|N(x(k))|^3)(1+\|\Phi(k-1)\|^2 + y_d^2(k) + \beta_g^2(k-1) \\
&\quad + \epsilon^2(k) + \int_0^{\bar{R}} E_r^2[v^*](k-n)dr) \\
a(k) &= \begin{cases} 1 & \text{if } |\epsilon(k)| > \chi \\ 0 & \text{others} \end{cases}
\end{aligned}
$$

where the tuning rate $\gamma > 0$ and threshold $\chi > 0$ can be any positive numbers specified by the designer. $N(x(k))$ is a discrete Nussbaum gain defined in Definition 4.1 with

$$
\begin{aligned}
\Delta\psi(k) &= \psi(k+1) - \psi(k) = \frac{-a(k)N(x(k))\beta_g(k-1)\epsilon(k)}{D(k)} \\
\Delta z(k) &= z(k+1) - z(k) = \frac{a(k)G(k)\epsilon^2(k)}{D(k)}, \quad z(0) = \psi(0) = 0 \\
x(k) &= z(k) + \frac{\psi^2(k)}{2}
\end{aligned}
\tag{5.12}
$$

Remark 5.1. *It can be shown later that the estimate $\hat{p}(r,k)$ is guaranteed to be nonnegative such that the algorithm solving for $v^*(k)$ from (5.6) developed in [176] can be applied.*

Lemma 5.1. *Consider the parameters $\hat{p}(r,k)$ and $\hat{p}'(r,k)$ in (5.11), we have*

$$
\int_0^{\bar{R}} \tilde{p}'^2(r,k)dr \geq \int_0^{\bar{R}} \tilde{p}^2(r,k)dr
$$

where

$$
\tilde{p}'(r,k) = \hat{p}'(r,k) - p(r), \quad \tilde{p}(r,k) = \hat{p}(r,k) - p(r)
$$

Proof. According to (5.11), we can see that $|\tilde{p}'(r,k)| = |\tilde{p}(r,k)|$ when $\hat{p}'(r,k) \geq 0$. Now, considering the case that $\hat{p}'(k) < 0$ and noting that $p(r) > 0$ defined in (5.2), thus we have

$$
|\tilde{p}(r,k)| = |-\hat{p}'(r,k) - p(k)| \leq -\hat{p}'(r,k) + p(r) = |\tilde{p}'(r,k)|
$$

In summary, we always have $|\tilde{p}'(r,k)| \geq |\tilde{p}(r,k)|$, which implies $\int_0^{\bar{R}} \tilde{p}'^2(r,k)dr \geq \int_0^{\bar{R}} \tilde{p}^2(r,k)dr$. This completes the proof. ■

5.2.3 Stability analysis

Theorem 5.1. *Consider the adaptive closed-loop system consisting of system 5.1 under Assumption 4.1. If there exists an integer $k_1 > 0$ such that $|u'(k)| \leq \hat{u}'_{\text{sat}}(k)$, $\forall k > k_1$, then all the signals in the closed-loop system are bounded and $G(k) = 1 + |N(x(k))|$ will converge to a constant. Denote $C = \lim_{k\to\infty} G(k)$, then the tracking error satisfies $\lim_{k\to\infty} \sup |e(k)| < \frac{C\chi}{\gamma}$, where γ and χ are the tuning factor and the threshold value specified by the designer.*

Proof. In the proof, we assume that $|u'(k)| \leq \hat{u}'_{\text{sat}}(k)$ [176]. Substituting the error dynamics (5.8) into the augmented error $\epsilon(k)$, it can be obtained that

$$
\begin{aligned}
&\gamma \tilde{\Theta}_{fg}^T(k-n)\Phi(k-1) - \gamma \tilde{g}_I(k-n)y_d(k) + \gamma \int_0^{\bar{R}} \tilde{p}(r,k-n)E_r[v^*](k-n)dr \\
=\ & -\frac{1}{g}G(k)\epsilon(k) - \gamma \beta_g(k-1) + \gamma \mu(k-n) \\
&+ \gamma \frac{1}{g}d_0(k-n) + \frac{1}{g}N(x(k))\psi(k)\beta_g(k-1)
\end{aligned}
\tag{5.13}
$$

Denote $d_b = \frac{1}{\chi}(\frac{\bar{d}}{|g_n|} + \bar{\mu})$ and then, from the update law (5.11), we have

$$
a(k)N(x(k))(\mu(k-n) + \frac{1}{g}d_o(k-n))\epsilon(k) \leq a(k)d_b|N(x(k))|\epsilon^2(k)
\tag{5.14}
$$

Choose a positive definite function $V(k)$ as

$$
\begin{aligned}
V(k) &= \sum_{j=1}^n \|\tilde{\Theta}_{fg}(k-n+j)\|^2 + \sum_{j=1}^n \tilde{g}_I^2(k-n+j) \\
&\quad + \sum_{j=1}^n \int_0^{\bar{R}} \tilde{p}^2(r,k-n+j)dr
\end{aligned}
\tag{5.15}
$$

Then, together with (5.11) the difference equation of $V(k)$ is written as:

$$
\begin{aligned}
&\Delta V(k) = V(k) - V(k-1) \\
=\ & (\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n))^T(\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n)) \\
&+ 2\tilde{\Theta}_{fg}^T(k-n)(\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n)) \\
&+ (\tilde{g}_I(k) - \tilde{g}_I(k-n))^2 + 2\tilde{g}_I(k-n)(\tilde{g}_I(k) - \tilde{g}_I(k-n)) \\
&+ \int_0^{\bar{R}} (\tilde{p}'(r,k) - \tilde{p}(r,k-n))^2 dr + 2\int_0^{\bar{R}} \tilde{p}(r,k-n)(\tilde{p}'(r,k) - \tilde{p}(r,k-n))dr \\
=\ & \gamma^2 a^2(k)\frac{N^2(x(k))\epsilon^2(k)}{D^2(k)}(\|\Phi(k-1)\|^2 + y_d^2(k) + \int_0^{\bar{R}} E_r^2[v^*](k-n)dr) \\
&+ 2\gamma a(k)N(x(k))\frac{\tilde{\Theta}_{fg}^T(k-n)\Phi(k-1)}{D(k)}\epsilon(k) - 2\gamma a(k)N(x(k))\frac{\tilde{g}_I(k-n)y_d(k)}{D(k)}\epsilon(k) \\
&+ 2\gamma a(k)N(x(k))\frac{\int_0^{\bar{R}} \tilde{p}(r,k-n)E_r[v^*](k-n)dr}{D(k)}\epsilon(k)
\end{aligned}
$$

Considering Lemma 5.1, equality (5.13), inequality (5.14) and referring the derivation in

Section 4.4.2, we can obtain:

$$
\begin{aligned}
\Delta V(k) &\leq \gamma^2 \frac{a(k)G(k)\epsilon^2(k)}{D(k)} + 2\gamma \frac{a(k)d_b|N(x(k))|\epsilon^2(k)}{D(k)} \\
&\quad - \frac{2}{g}a(k)N(x(k))\frac{G(k)\epsilon^2(k)}{D(k)} - 2\gamma\frac{a(k)N(x(k))\beta_g(k-1)\epsilon(k)}{D(k)} \\
&\quad + \frac{2}{g}a(k)N(x(k))\frac{N(x(k))\psi(k)\beta_g(k-1)\epsilon(k)}{D(k)} \\
&\leq (\gamma^2 + 2\gamma d_b)\Delta z(k) + 2\gamma\Delta\psi(k) - \frac{2}{g}N(x(k))(\Delta z(k) + \psi(k)\Delta\psi(k) \\
&\quad + \frac{[\Delta\psi(k)]^2}{2}) + \frac{1}{|g|}|N(x(k))|[\Delta\psi(k)]^2 \\
&\leq c_1\Delta z(k) + 2\gamma\Delta\psi(k) - \frac{2}{g}N(x(k))\Delta x(k) \qquad (5.16)
\end{aligned}
$$

where $c_1 = \gamma^2 + 2\gamma d_b$. Noting that $x(k) = z(k) + \frac{\psi^2(k)}{2}$ and taking summation on both hand sides of (5.16) results in

$$
V(k) \leq -\frac{2}{g}\sum_{k'=0}^{k}N(x(k'))\Delta x(k') + c_1 z(k) + 2\gamma\psi(k) + c_2
$$

$$
\leq -\frac{2}{g}\sum_{k'=0}^{k}N(x(k'))\Delta x(k') + c_1 x(k) + c_3 \qquad (5.17)
$$

where $c_2 = V(-1)$ and $c_3 = c_2 + \frac{2\gamma^2}{c_1}$ are some finite constants. Then, performing the similar analysis as in Section 4.4.2, we have the boundedness of $x(k)$, $N(x(k))$, $G(k)$, $\hat{\Theta}_{fg}(k)$, $\hat{g}_I(k)$, $\int_0^{\bar{R}}\hat{p}(r,k)dr$, $\hat{p}(r,k)$ and $\lim_{k\to\infty}\sup\{|e(k)|\} \leq \frac{C_\chi}{\gamma}$. This implies the boundedness of $y(k)$. From Lemma 2.6, it is clear that the boundedness of control input $u(k)$ and states $\bar{\xi}_n(k)$ is guaranteed. This completes the proof. ■

5.2.4 Simulation studies

The following second order nonlinear plant is used for simulation.

$$
\begin{cases}
\xi_1(k+1) &= a_1\xi_1(k)\cos(\xi_1(k)) + a_2\xi_1(k)\sin(\xi_1(k)) + g_1\xi_2(k) \\
\xi_2(k+1) &= b_1\xi_2(k)\frac{\xi_1(k)}{1+\xi_1^2(k)} + b_2\frac{\xi_2^3(k)}{2+\xi_2^2(k)} + g_2 u(k) + d(k) \\
y(k) &= \xi_1(k) \\
u(k) &= \int_0^{\bar{R}} p(r)E_r[v](k)dr
\end{cases}
$$

where disturbance $d(k) = 0.2\cos(0.05k\cos(y(k)))$, system parameters are chosen as $a_1 = 0.1$, $a_2 = 0.1$, $g_1 = 2$, $b_1 = 0.3$, $b_2 = -0.4$, $g_2 = -0.1$, reference trajectory $y_d(k) = $

$1.5\sin(\pi/5kT) + 1.5\cos(\pi/10kT)$, with $T = 0.2$. The initial condition is $\bar{\xi}_2(0) = [1, 1]^T$. The tuning rate $\gamma = 4$ and the threshold value $\chi = 0.1$. The density function is selected as $p(r) = e^{-0.07(r-1)^2}$ with $\bar{R} = 10$. The simulation results are showed in Figures 5.2, 5.3 and 5.4. Figure 5.2 depicts the output $y(k)$ and the reference signal $y_d(k)$. Figure 5.3 illustrates the boundedness of the control input $u(k)$, the estimated parameters $\hat{g}_I(k)$, $\hat{\Theta}_{fg}(k)$, and $\hat{p}(r, k)$. Figure 5.4 demonstrates the discrete Nussbaum gain $N(x(k))$ and the sequences $x(k)$ and $\beta_g(k)$. As control gain g_2 is chosen to be negative, we see in Figure 5.4 that the discrete Nussbaum gain turn to be negative.

5.3 Block-triangular MIMO Systems

In this Section, adaptive control is investigated for block-triangular MIMO nonlinear systems with uncertain couplings of delayed states among subsystems. Future states prediction for SISO system developed in Section 3.2.2 is extended to each subsystems. Nonparametric uncertainties compensation technique in Chapter 3 has also been extended in Section 5.3.3 to compensate for the effect of the uncertain nonlinear couplings.

5.3.1 Problem formulation

Consider a MIMO system with each subsystem Σ_j, $j = 1, 2, \ldots, n$, in strict-feedback form and interacting with each other in the follow mannter:

$$\Sigma : \begin{cases} \Sigma_1 \begin{cases} \xi_{1,i_1}(k+1) = \Theta_{1,i_1}^T \Phi_{1,i_1}(\bar{\xi}_{1,i_1-m_{11}}(k), \bar{\xi}_{2,i_1-m_{12}}(k), \ldots, \bar{\xi}_{n,i_1-m_{1n}}(k)) \\ \qquad + g_{1,i_1}\xi_{1,i_1+1}(k), \qquad\qquad i_1 = 1, 2, \ldots, n_1 - 1 \\ \xi_{1,n_1}(k+1) = \Theta_{1,n_1}^T \Phi_{1,n_1}(\Xi(k)) + g_{1,n_1}u_1(k) + \nu_1(\Xi_{\tau_1}(k)) \\ y_1(k) = \xi_{1,1}(k) \end{cases} \\ \vdots \\ \Sigma_j \begin{cases} \xi_{j,i_j}(k+1) = \Theta_{j,i_j}^T \Phi_{j,i_j}(\bar{\xi}_{1,i_j-m_{j1}}(k), \bar{\xi}_{2,i_j-m_{j2}}(k), \ldots, \bar{\xi}_{n,i_j-m_{jn}}(k)) \\ \qquad + g_{j,i_j}\xi_{j,i_j+1}(k), \qquad\qquad i_j = 1, 2, \ldots, n_j - 1 \\ \xi_{j,n_j}(k+1) = \Theta_{j,n_j}^T \Phi_{j,n_j}(\Xi(k), \bar{u}_{j-1}(k)) + g_{j,n_j}u_j(k) + \nu_j(\Xi_{\tau_j}(k)) \\ y_j(k) = \xi_{j,1}(k) \end{cases} \\ \vdots \\ \Sigma_n \begin{cases} \xi_{n,i_n}(k+1) = \Theta_{n,i_n}^T \Phi_{n,i_n}(\bar{\xi}_{1,i_n-m_{n1}}(k), \bar{\xi}_{2,i_n-m_{n2}}(k), \ldots, \bar{\xi}_{n,i_n-m_{nn}}(k)) \\ \qquad + g_{n,i_n}\xi_{n,i_n+1}(k), \qquad\qquad i_n = 1, 2, \ldots, n_n - 1 \\ \xi_{n,n_n}(k+1) = \Theta_{n,n_n}^T \Phi_{n,n_n}(\Xi(k), \bar{u}_{n-1}(k)) + g_{n,n_n}u_n(k) + \nu_n(\Xi_{\tau_n}(k)) \\ y_n(k) = \xi_{n,1}(k) \end{cases} \end{cases} \quad (5.18)$$

where $\xi_{j,i_j}(k)$ and $\Xi(k)$ are defined in Section 2.1 and the delayed state vectors $\Xi_{\tau_j}(k)$, are defined as

$$\Xi_{\tau_j}(k) = [\bar{\xi}_{1,n_1}(k - \tau_{j,1}), \bar{\xi}_{2,n_2}(k - \tau_{j,2}), \ldots, \bar{\xi}_{n,n_n}(k - \tau_{j,n})]^T, \quad j = 1, 2, \ldots, n \qquad (5.19)$$

where the unknown delays $\tau_{j,l}$ satisfy $\tau_{\min} \leq \tau_{j,l} \leq \tau_{\max}$, $l = 1, 2, \ldots, n$.

Similar to previous Chapters, the system functions $\Phi_{j,i_j}(\cdot)$, $j = 1, 2, \ldots, n$, are known, but system parameters $\Theta_{j,i_j}^T \in R^{p_{j,i_j}}$ and $g_{j,i_j} \in R$ are unknown as well as the uncertain coupling terms $\nu_j(\Xi_{\tau_j}(k))$. The notation $u_j(k)$ and $y_j(k)$ represent input and output of subsystem Σ_j, $j = 1, 2, \ldots, n$. The control objective is also to drive the outputs, $y_j(k)$, to follow given desired reference trajectories $y_j^*(k)$, respectively, and guarantee the boundedness of all the closed-loop signals.

We make the following assumptions that are similar to previous Chapters.

Assumption 5.1. *The uncertain nonlinear coupling terms $\nu_j(\cdot)$, are Lipschitz functions with Lipschitz coefficient L_j^ν satisfy $L_j^\nu < \lambda^*$, where λ^* is defined later in (5.62). The system functions, $\Phi_{j,i_j}(\cdot)$, $1 \leq j \leq n$, $1 \leq i_j \leq n_j$, are also Lipschitz functions with Lipschitz coefficients L_{j,i_j}.*

Assumption 5.2. *The signs of control gains g_{j,i_j}, $(1 \leq j \leq n)$ are known and satisfy $|g_{j,i_j}| \geq \underline{g}_{j,i_j} > 0$. Without loss of generality, it is assumed that g_{j,i_j} are positive.*

Remark 5.2. *The discrete Nussbaum gain techniques developed in Chapter 4 can be easily extended in this Section to deal with the unknown control directions problem. But for conciseness and focus on the control design of multi-variable systems, control directions are assumed to be known in this Section. Later, in Chapter 7 we will consider unknown control directions in more general block-triangular MIMO systems.*

5.3.2 Future states prediction

By utilizing the block-triangular structure property of system (5.18), future states up to $(k + n_j - 1)$ step ahead for subsystem Σ_j are to be predicted at the kth step. To proceed, let us denote the estimates of Θ_{j,i_j} and g_{j,i_j} at the kth step as $\hat{\Theta}_{j,i_j}(k)$ and $\hat{g}_{j,i_j}(k)$, respectively, and

$$\tilde{\Theta}_{j,i_j}(k) = \hat{\Theta}_{j,i_j}(k) - \Theta_{j,i_j}, \tilde{g}_{j,i_j}(k) = \hat{g}_{j,i_j}(k) - g_{j,i_j} \qquad (5.20)$$

as estimate errors. For convenience, the following notations will be used in the later discussion.

$$\bar{\hat{\Theta}}_{j,i_j}(k) \quad = \quad [\hat{\Theta}^T_{j,i_j}(k), \hat{g}_{j,i_j}(k)]^T \in R^{p_{j,i_j}+1} \tag{5.21}$$

$$\bar{\tilde{\Theta}}_{j,i_j}(k) \quad = \quad [\tilde{\Theta}^T_{j,i_j}(k), \tilde{g}_{j,i_j}(k)]^T \in R^{p_{j,i_j}+1} \tag{5.22}$$

Based on the states prediction for SISO system in Section 3.2.2, we propose the following states prediction for MIMO system (5.18) as follows.

By using the estimates of unknown system parameters, the one-step ahead future states of subsystem Σ_j can be straightforwardly predicted in the following manner:

$$\hat{\xi}_{j,i_j}(k+1|k) \quad = \quad \bar{\hat{\Theta}}^T_{j,i_j}(k-n_j+2)\Psi_{j,i_j}(k), \quad i_j = 1,2,\ldots,n_j-1, \ j=1,2,\ldots,n$$

$$\Psi_{j,i_j}(k) \quad = \quad [\Phi^T_{j,i_j}(\bar{\xi}_{1,i_j-m_{j1}}(k),\ldots,\bar{\xi}_{j,i_j}(k),\ldots,\bar{\xi}_{n,i_j-m_{jn}}(k)), \xi_{j,i_j+1}(k)]^T \tag{5.23}$$

Similar to Section 3.2.2, the l-step ahead states prediction, $\xi_{j,i_j}(k+l|k)$, $l=2,3,\ldots,n_j-1$, can be constructed in the following manner.

$$\hat{\xi}_{j,i_j}(k+l|k) \quad = \quad \bar{\hat{\Theta}}^T_{j,i_j}(k-n_j+l+1)\hat{\Psi}_{j,i_j}(k+l-1|k)$$

$$\hat{\Psi}_{j,i_j}(k+l-1|k) \quad = \quad [\Phi^T_{j,i_j}(\bar{\hat{\xi}}_{1,i_j-m_{j1}}(k+l-1|k)),\ldots,\bar{\hat{\xi}}_{j,i_j}(k+l-1|k)),\ldots,$$

$$\bar{\hat{\xi}}_{n,i_j-m_{jn}}(k+l-1|k)), \ \hat{\xi}_{j,i_j+1}(k+l-1|k)]^T \tag{5.24}$$

$$\bar{\hat{\xi}}_{j,i_j}(k+l-1|k) \quad = \quad [\hat{\xi}_{j,1}(k+l-1|k),\hat{\xi}_{j,2}(k+l-1|k),\ldots,\hat{\xi}_{j,i_j}(k+l-1|k)]^T$$

$$i_j \quad = \quad 1,2,\ldots,n_j-l \tag{5.25}$$

Remark 5.3. *Unlike the prediction of SISO system developed in Section 3.2.2, for MIMO systems, the prediction of future states of subsystem Σ_j involves the predicted future states of other systems. For one-step ahead predicted state vectors of subsystem Σ_j, $\bar{\xi}_{j,i_j}(k+1|k)$, $i_j = 1,2,\ldots,n_j-1$, they involve state vectors of subsystem Σ_l, $\bar{\xi}_{l,i_j-m_{jl}}(k)$, $l=1,2,\ldots,n$, and $\xi_{j,i_j+1}(k)$, which are available at kth step. For two-step ahead predicted state vectors $\bar{\hat{\xi}}_{j,i_j}(k+2|k)$, $i_j = 1,2,\ldots,n_j-2$, they involve one-step ahead predicted state vectors of subsystem Σ_l, $\bar{\hat{\xi}}_{l,i_j-m_{jl}}(k+1|k)$, $l=1,2,\ldots,n$ and $\hat{\xi}_{j,i_j+1}(k+1|k)$, which are also available at kth step because $i_j - m_{jl} \le n_l-2$ and $i_j+1 \le n_j-1$ and for each subsystem the one-step prediction is proceeded up to the (n_l-1)th state. Continuing the analysis, we see that the prediction method developed above is well defined without any noncausal problem.*

The parameter estimates are obtained by the following update law:

$$
\begin{aligned}
\bar{\bar{\Theta}}_{j,i_j}(k+1) &= \bar{\bar{\Theta}}_{j,i_j}(k-n_j+2) - \frac{\tilde{\bar{\xi}}_{j,i_j}(k+1|k)\Psi_{j,i_j}(k)}{D_{j,i_j}(k)} \\
D_{j,i_j}(k) &= 1 + \Psi_{j,i_j}^T(k)\Psi_{j,i_j}(k) \\
\tilde{\xi}_{j,i_j}(k+1|k) &= \hat{\xi}_{j,i_j}(k+1|k) - \xi_{j,i_j}(k+1) \\
j &= 1,2,\ldots,n, \quad i_j = 1,2,\ldots,n_j-1 \quad\quad (5.26)
\end{aligned}
$$

Lemma 5.2. *The parameter estimates* $\bar{\bar{\Theta}}_{j,i_j}(k)$, $j = 1,2,\ldots,n$, $i_j = 1,2,\ldots,n_j-1$, *in* (5.26) *are bounded and the prediction errors satisfy*

$$
\bar{\bar{\xi}}_{j,i_j}(k+n_j-i_j|k) = \sum_{l=1}^{n} o[O[y_l(k+n_j-m_{jl}-1)]] \quad\quad (5.27)
$$

with

$$
\begin{aligned}
\tilde{\bar{\xi}}_{j,i_j}(k+n_j-i_j|k) &= \bar{\hat{\xi}}_{j,i_j}(k+n_j-i_j|k) - \bar{\xi}_{j,i_j}(k+n_j-i_j) \quad\quad (5.28) \\
\bar{\hat{\xi}}_{j,i_j}(k+n_j-i_j|k) &= [\hat{\xi}_{j,1}(k+n_j-1|k),\hat{\xi}_{j,2}(k+n_j-2|k),\ldots,\hat{\xi}_{j,i_j}(k+n_j-i_j|k)]^T
\end{aligned}
$$

Proof. See Appendix 5.1. ∎

5.3.3 Adaptive control design

First, let us perform the similar technique in Section 3.2.3 and transform each subsystem Σ_j into a compact form. For the first subsystem, we have

$$
y_1(k+n_1) = \Theta_1^T\Phi_1(k+n_1-1) + \Theta_{1,n_1}^{gT}\Phi_{1,n_1}(\Xi(k)) + \nu_1(\Xi_{\tau_1}(k)) + g_1 u_1(k) \quad\quad (5.29)
$$

Similarly, for subsystems Σ_j, $j = 2,3,\ldots,n$, we have

$$
\begin{aligned}
y_j(k+n_j) &= \Theta_j^T\Phi_j(k+n_j-1) + \Theta_{j,n_j}^{gT}\Phi_{j,n_j}(\Xi(k),\bar{u}_{j-1}(k)) \\
&\quad + \nu_j(\Xi_{\tau_j}(k)) + g_j u_j(k) \quad\quad (5.30)
\end{aligned}
$$

where for all the subsystems Σ_j, $j = 1,2,\ldots,n$, we have

$$
\Phi_j(k+n_j-1) = [\Phi_{j,1}^T(k+n_j-1),\Phi_{j,2}^T(k+n_j-2),\ldots,\Phi_{j,n_j-1}^T(k)]^T \quad\quad (5.31)
$$

where $\Phi_{j,i_j}^T(k+n_j-i_j)$, $i_j = 1,2,\ldots,n_j-1$, is the abbreviation of $\Phi_{j,i_j}^T(\bar{\xi}_{1,i_j-m_{j1}}(k+n_j-i_j),\ldots,\bar{\xi}_{n_j,i_j-m_{jn}}(k+n_j-i_j))$ and

$$
\Theta_j = [\Theta_{j,1}^{gT},\Theta_{j,2}^{gT},\ldots,\Theta_{j,n_j-1}^{gT}], \; \Theta_{j,i_j}^g = (\prod_{l=1}^{i_j-1} g_{j,l})\Theta_{j,i_j} \in R^{p_{j,i_j}}, \; g_j = \prod_{i_j=1}^{n_j} g_{j,i_j} \geq \underline{g}_j \quad (5.32)
$$

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In the following, we extend the nonparametric uncertainties compensation technique in Chapter 3 to compensate the effect of $\nu_j(\cdot)$ including states with unknown time delays. To start with, let us introduce the following notations.

Similar to the definition of $X(k)$ in 3.25, we introduce an augmented state vector as follows:

$$\bar{\Xi}(k) = [\Xi^T(k - \tau_{min}), \dots, \Xi^T(k - \tau_j), \dots, \Xi^T(k - \tau_{max})]^T \tag{5.33}$$

which includes $\Xi^T(k - \tau_j)$ as a subvector, for any j satisfying $1 \leq j \leq n_j$. But it is noted $\bar{\Xi}(k)$ is independent of subindex j.

According to Lemma 2.2, we define

$$l_k = \arg \min_{l \leq k - \max_{1 \leq j \leq n} \{n_j\}} \|\bar{\Xi}(k) - \bar{\Xi}(l)\| \tag{5.34}$$

such that $l_k + n_j \leq k$ and

$$\bar{\Xi}(l_k) = [\Xi^T(l_k - \tau_{min}), \dots, \Xi^T(l_k - \tau_j), \dots, \Xi^T(l_k - \tau_{max})]^T \tag{5.35}$$

In the following part, we use notation $\Phi_{1,n_1}(\Xi(k), \bar{u}_0(k))$ to denote $\Phi_{1,n_1}(\Xi(k))$ for convenience without any confusion. Let us introduce an auxiliary output $y_j^a(k)$ for each subsystem Σ_j, $j = 1, 2, \dots, n$, defined as follows:

$$y_j^a(k + n_j - 1) = \Theta_j^T \Phi_j(k + n_j - 1) + \Theta_{j,n_j}^{gT} \Phi_{j,n_j}(\Xi(k), \bar{u}_{j-1}(k)) + \nu_j(\Xi_{\tau_j}(k)) \tag{5.36}$$

such that (5.30) can be rewritten as

$$y_j(k + n_j) = y_j^a(k + n_j - 1) + g_j u_j(k) \tag{5.37}$$

From (5.36) and (5.37), the following equality can be obtained

$$
\begin{aligned}
y_j^a(k + n_j - 1) &= y_j^a(k + n_j - 1) - y_j^a(l_k + n_j - 1) + y_j^a(l_k + n_j - 1) \\
&= \Theta_j^T [\Phi_j(k + n_j - 1) - \Phi_j(l_k + n_j - 1)] \\
&\quad + \Theta_{j,n_j}^{gT} [\Phi_{j,n_j}(\Xi(k), \bar{u}_{j-1}(k)) - \Phi_{j,n_j}(\Xi(l_k), \bar{u}_{j-1}(l_k))] \\
&\quad + y_j(l_k + n_j) - g_j u_j(l_k) + \nu_j(\Xi_{\tau_j}(k)) - \nu_j(\Xi_{\tau_j}(l_k))
\end{aligned} \tag{5.38}
$$

Denote $\hat{\Theta}_j(k)$, $\hat{\Theta}_{j,n_j}^g(k)$ and $\hat{g}_j(k)$ as the estimates of unknown parameter Θ_j, Θ_{j,n_j}^g and g_j, respectively. Then, let us predict $y_j^a(k + n_j - 1)$ as follows:

$$
\begin{aligned}
\hat{y}_j^a(k + n_j - 1 | k) &= \hat{\Theta}_j^T(k)[\hat{\Phi}_j(k + n_j - 1 | k) - \Phi_j(l_k + n_j - 1)] \\
&\quad + \hat{\Theta}_{j,n_j}^{gT}(k)[\Phi_{j,n_j}(\Xi(k), \bar{u}_{j-1}(k)) - \Phi_{j,n_j}(\Xi(l_k), \bar{u}_{j-1}(l_k))] \\
&\quad + y_j(l_k + n_j) - \hat{g}_j(k) u_j(l_k)
\end{aligned}
$$

where $l_k \leq k - n_j$ is defined in (5.34) and $\hat{\Phi}_j(k + n_j - 1|k)$ is defined as

$$\hat{\Phi}_j(k + n_j - 1|k) = [\hat{\Phi}_{j,1}^T(k + n_j - 1|k), \hat{\Phi}_{j,2}^T(k + n_j - 2|k), \ldots, \hat{\Phi}_{j,n_j-1}^T(k + 1|k)]^T \quad (5.39)$$

with

$$\hat{\Phi}_{j,i_j}(k + n_j - i_j|k) = \Phi_{j,i_j}(\bar{\bar{\xi}}_{1,i_j-m_{j1}}(k + n_j - i_j|k), \ldots, \bar{\bar{\xi}}_{n_j,i_j-m_{jn}}(k + n_j - i_j|k))$$

for $i_j = 1, 2, \ldots, n_j - 1$ and the predicted future states are obtained from Section 5.3.2.

Based on equation (5.37), the adaptive control is designed using certainty equivalence principal as follows:

$$u_j(k) = -\frac{1}{\hat{g}_j(k)}(\hat{y}_j^a(k + n_j - 1|k) - y_j^*(k + n_j)) \quad (5.40)$$

where $\hat{g}_j(k)$ is the estimate of g_j at the kth step defined later in (5.47) and will be guaranteed to be bounded away from naughty such that control law (5.40) is well defined.

Next, the task is to design a proper parameter estimate law for the adaptive control. Let us consider the following augmented tracking error

$$\epsilon_j(k) = e_j(k) + \beta_j(k - 1) \quad (5.41)$$

where output tracking error $e_j(k)$ is defined as $e_j(k) = y_j(k) - y_j^*(k)$ and $\beta_j(k)$ is defined as

$$\beta_j(k) = \hat{\Theta}_j^T(k - n_j + 1)[\hat{\Phi}_j(k|k - n_j + 1) - \Phi_j(k)] \quad (5.42)$$

According to Assumption 5.1 and the definition of $\bar{\Xi}(k)$ in (5.33), we have

$$|\nu_j(\Xi_{\tau_j}(k)) - \nu_j(\Xi_{\tau_j}(l_k))| \leq \lambda_j \|\Xi_{\tau_j}(k) - \Xi_{\tau_j}(l_k)\| \leq \lambda_j \|\bar{\Xi}(k) - \bar{\Xi}(l_k)\| \quad (5.43)$$

where λ_j can be any constant satisfying $L_j^\nu \leq \lambda_j < \lambda^*$, with λ^* defined later in (5.62).

To tackle the effect of nonlinear uncertainties $\nu_j(\cdot)$ in parameter estimation, we use the following deadzone defined as

$$a_j(k) = \begin{cases} 1 - \frac{\lambda_j \|\bar{\Xi}(k-n_j) - \bar{\Xi}(l_{k-n_j})\|}{|\epsilon_j(k)|}, & \text{if } |\epsilon_j(k)| > \lambda_j \|\bar{\Xi}(k - n_j) - \bar{\Xi}(l_{k-n_j})\| \\ 0, & \text{otherwise} \end{cases} \quad (5.44)$$

For convenience, let us define an auxiliary tracking error as

$$\epsilon_j^a(k) = a_j(k)\epsilon_j(k) \quad (5.45)$$

According to the definition in (5.44), it is easy to obtain the following inequality

$$|\epsilon_j(k)| \leq |\epsilon_j^a(k)| + \lambda_j \|\bar{\Xi}(k - n_j) - \bar{\Xi}(l_{k-n_j})\| \quad (5.46)$$

The estimated parameters in the auxiliary output estimate (5.39) are obtained from the following update law, $j = 1, 2, \ldots, n$

$$
\begin{aligned}
\hat{\Theta}_j(k) &= \hat{\Theta}_j(k - n_j) + \gamma_j \frac{\epsilon_j^a(k)[\Phi_j(k-1) - \Phi_j(l_{k-n_j} + n_j - 1)]}{D_j(k - n_j)} \\
\hat{\Theta}_{j,n_j}^g(k) &= \hat{\Theta}_{j,n_j}^g(k - n_j) + \gamma_j \frac{\epsilon_j^a(k)[\Phi_{j,n_j}(k - n_j) - \Phi_{j,n_j}(l_{k-n_j})]}{D_j(k - n_j)} \\
\hat{g}_j(k) &= \begin{cases} \hat{g}_j'(k), & \text{if } \hat{g}_j'(k) > \underline{g}_j \\ \underline{g}_j, & \text{otherwise} \end{cases} \\
\hat{g}_j'(k) &= \hat{g}_j(k - n_j) + \frac{\gamma_j \epsilon_j^a(k)}{D_j(k - n_j)}[u_j(k - n_j) - u_j(l_{k-n_j})] \\
D_j(k - n_j) &= 1 + \|\Phi_j(k-1) - \Phi_j(l_{k-n_j} + n_j - 1)\|^2 + \|\bar{\bar{\Xi}}(k - n_j) - \bar{\bar{\Xi}}(l_{k-n_j})\|^2 \\
&\quad + \|\Phi_{j,n_j}(k - n_j) - \Phi_{j,n_j}(l_{k-n_j})\|^2 + [u_j(k - n_j) - u_j(l_{k-n_j})]^2
\end{aligned}
\tag{5.47}
$$

where $\Phi_{j,n_j}(k)$ is used to denote $\Phi_{j,n_j}(\Xi(k), \bar{u}_j(k))$ and $0 < \gamma_j < 2$.

5.3.4 Control performance analysis

The main result in this Section is summarized in the following theorem.

Theorem 5.2. *Consider the whole closed-loop adaptive system that combines all the n coupled closed-loop subsystems, with each closed-loop subsystem consisting of subsystem Σ_j described in (5.18), adaptive control input (5.40), and parameter update law (5.47). All the signals in the whole closed-loop adaptive system are bounded. Furthermore, the output of each subsystem Σ_j asymptotically tracks the desired reference trajectory $y_j^*(k)$, $j = 1, 2, \ldots, n$*

Proof. In the following, we use $\Phi_{j,n_j}(k)$ to denote $\Phi_{j,n_j}(\Xi(k), \bar{u}_j(k))$ for convenience. By comparing (5.38) and (5.39), the prediction error of the auxiliary output $y_j^a(k + n_j - 1)$ can be written as

$$
\begin{aligned}
\tilde{y}_j^a(k + n_j - 1|k) &= \hat{y}_j^a(k + n_j - 1|k) - y_j^a(k + n_j - 1) \\
&= \tilde{\Theta}_j^T(k)[\Phi_j(k + n_j - 1) - \Phi_j(l_k + n_j - 1)] + \tilde{\Theta}_{j,n_j}^{gT}(k)[\Phi_{j,n_j}(k) - \Phi_{j,n_j}(l_k)] \\
&\quad - [\nu_j(\Xi_{\tau_j}(k)) - \nu_j(\Xi_{\tau_j}(l_k))] + \beta_j(k + n_j - 1) - \tilde{g}_j(k)u_j(l_k)
\end{aligned}
\tag{5.48}
$$

where $\tilde{\Theta}_j(k) = \hat{\Theta}_j(k) - \Theta_j$, $\tilde{\Theta}_{j,n_j}^g(k) = \hat{\Theta}_{j,n_j}^g(k) - \Theta_{j,n_j}^g(k)$, $\tilde{g}_j(k) = \hat{g}_j(k) - g_j$.

Now, by combining (5.37), (5.40) and (5.48), the output tracking error can be written

as

$$e_j(k + n_j) = y_j^a(k + n_j - 1) + \hat{g}_j(k)u(k) - \tilde{g}_j(k)u_j(k) - y_j^*(k + n_j)$$
$$= -\tilde{y}_j^a(k + n - 1|k) - \tilde{g}_j(k)u_j(k)$$
$$= -\tilde{\Theta}_j^T(k)[\Phi_j(k + n_j - 1) - \Phi_j(l_k + n_j - 1)] - \tilde{\Theta}_{j,n_j}^{gT}(k)[\Phi_{j,n_j}(k) - \Phi_{j,n_j}(l_k)]$$
$$- \tilde{g}_j(k)[u_j(k) - u_j(l_k)] - \beta_j(k + n_j - 1) + \nu_j(\Xi_{\tau_j}(k)) - \nu_j(\Xi_{\tau_j}(l_k)) \tag{5.49}$$

which according to (5.41) immediately leads to

$$\epsilon_j(k) = -\tilde{\Theta}_j^T(k - n_j)[\Phi_j(k - 1) - \Phi_j(l_{k-n_j} + n_j - 1)] \tag{5.50}$$
$$- \tilde{\Theta}_{j,n_j}^T(k - n_j)[\Phi_{j,n_j}(k - n_j) - \Phi_{j,n_j}(l_{k-n_j})]$$
$$- \tilde{g}_j(k - n_j)[u_j(k - n_j) - u_j(l_{k-n_j})] + \nu_j(\Xi_{\tau_j}(k - n_j)) - \nu_j(\Xi_{\tau_j}(l_{k-n_j}))$$

Then, we consider a Lyapunov function candidate

$$V_j(k) = \sum_{l=1}^{n_j} \|\tilde{\Theta}_j(k - n_j + l)\|^2 + \sum_{l=1}^{n_j} \|\tilde{\Theta}_{j,n_j}(k - n_j + l)\|^2$$
$$+ \sum_{l=1}^{n_j} \tilde{g}_j^2(k - n_j + l) \tag{5.51}$$

Since $\tilde{g}_j'^2(k) \geq \tilde{g}_j^2(k)$ is guaranteed according to (5.47), we see that

$$\Delta V_j(k) = V_j(k) - V_j(k - 1)$$
$$= \|\tilde{\Theta}_j(k)\|^2 - \|\tilde{\Theta}_j(k - n_j)\|^2 + \|\tilde{\Theta}_{j,n_j}(k)\|^2 - \|\tilde{\Theta}_{j,n_j}(k - n_j)\|^2 + \tilde{g}_j'^2(k) - \tilde{g}_j^2(k - n_j)$$

which together with (5.43), (5.44) and (5.50) leads to the difference of Lyapunov function $V_j(k)$ as follows:

$$\Delta V_j(k) = V_j(k) - V_j(k - 1)$$
$$= \{\|\Phi_j(k - 1) - \Phi_j(l_{k-n_j} + n_j - 1)\|^2 + [u_j(k - n_j) - u_j(l_{k-n_j})]^2$$
$$+ \|\Phi_{j,n_j}(k - n_j) - \Phi_{j,n_j}(l_{k-n_j})\|^2\} \frac{\gamma_j^2 \epsilon_j^{a2}(k)}{D_j^2(k - n_j)}$$
$$+ \tilde{\Theta}_j^T(k - n_j)[\Phi_j(k - 1) - \Phi_j(l_{k-n_j} + n_j - 1)] \frac{2\epsilon_j^a(k)\gamma_j}{D_j(k - n_j)}$$
$$+ \tilde{g}_j(k - n_j)[u_j(k - n_j) - u_j(l_{k-n_j})] \frac{2\epsilon_j^a(k)\gamma_j}{D_j(k - n_j)}$$
$$+ \tilde{\Theta}_{j,n_j}^T[\Phi_{j,n_j}(k - n_j) - \Phi_{j,n_j}(l_{k-n_j})]$$
$$\leq \frac{\gamma_j^2 \epsilon_j^{a2}(k)}{D_j(k - n_j)} - \frac{2\gamma_j \epsilon_j^{a2}(k)}{D_j(k - n_j)} = -\frac{\gamma_j(2 - \gamma_j)\epsilon_j^{a2}(k)}{D_j(k - n_j)} \tag{5.52}$$

Noting that $0 < \gamma_j < 2$ in (5.47), we can conclude from (5.52) that $\Delta V_j(k)$ is non-positive, such that the boundedness of $V_j(k)$ is obvious, and immediately the boundedness of $\hat{\Theta}_j(k)$, $\hat{\Theta}_{j,n_j}^g(k)$ and $\hat{g}_j(k)$ is guaranteed. Furthermore, we can derive from (5.52) that

$$\lim_{k \to \infty} \frac{\epsilon_j^{a2}(k)}{D_j(k - n_j)} = 0, \quad \text{or} \quad \epsilon_j^a(k) = o[D_j^{\frac{1}{2}}(k - n_j)] \tag{5.53}$$

Let us order the growth rates of signals with respect to each other in the adaptive closed-loop systems. First, consider $\beta_j(k)$ defined in (5.42). Due to the boundedness of $\hat{\Theta}_j(k)$ proved above, there exists a constant C_{β_j} such that

$$
\begin{aligned}
|\beta_j(k + n_j - 1)| &\leq C_{\beta_j}\|\hat{\Phi}_j(k + n_j - 1|k) - \Phi_j(k + n_j - 1)\| \\
&= \sum_{l=1}^{n} o[O[y_l(k + n_j - m_{jl} - 1)]] \tag{5.54}
\end{aligned}
$$

where Lemma 5.2 and Assumption 5.1 are used to establish the equality. Considering $y_j(k) \sim e_j(k)$ because $y_j^*(k)$ is bounded, we are ready to show that

$$\beta_j(k + n_j - 1) = \sum_{l=1}^{n} o[O[e_l(k + n_j - m_{jl} - 1)]] \tag{5.55}$$

which together with the definition of augmented error in (5.41) implies

$$|e_j(k + n_j - 1)| \sim |\epsilon_j(k + n_j - 1)| + \sum_{l=1}^{n} o[O[e_l(k + n_l - 2)]] \tag{5.56}$$

Taking summation on both hand sides of (5.56) and using Proposition 2.1, we have

$$\sum_{j=1}^{n} |e_j(k + n_j - 1)| \sim \sum_{j=1}^{n} |\epsilon_j(k + n_j - 1)| \tag{5.57}$$

From Lemma 2.7, it is easy to derive

$$\sum_{j=1}^{n} O[\bar{\xi}_{j,n_j}(k)] \sim \sum_{j=1}^{n} O[y_j(k + n_j - 1)] \sim \sum_{l=1}^{n} O[\bar{\xi}_{l,i_j-m_{jl}}(k + n_j - i_j)] \tag{5.58}$$

which together with (5.57), $y_j(k) \sim e_j(k)$ and Proposition 2.1 leads to

$$
\begin{aligned}
\max_{k' \leq k} \|\Xi(k')\| &\leq \sum_{j=1}^{n} \max_{k' \leq k} \|\bar{\xi}_{j,n_j}(k')\| = \sum_{j=1}^{n} O[e_j(k + n_j - 1)] \\
&= \sum_{j=1}^{n} O[\epsilon_j(k + n_j - 1)] \tag{5.59}
\end{aligned}
$$

Then, we have

$$\max_{k'\le k}\|\bar{\Xi}(k')\| \le \sum_{\tau=\tau_{min}}^{\tau_{max}} \max_{k'\le k}\|\Xi(k'-\tau)\| = \sum_{j=1}^{n} O[\epsilon_j(k+n_j-1)] \qquad (5.60)$$

which asserts the existence of constants $C_{j,1}$ and $C_{j,2}$ such that

$$\max_{k'\le k}\|\bar{\Xi}(k')\| \le \sum_{j=1}^{n}\{C_{j,1}\max_{k'\le k+n_j-1}\{|\epsilon_j(k')|\} + C_{j,2}\}$$

$$\le \sum_{j=1}^{n} C_{j,1}\max_{k'\le k+n_j-1}|\epsilon_j^a(k')| + \sum_{j=1}^{n} C_{j,1}\lambda_j \max_{k'\le k-1}\|\bar{\Xi}(k')-\bar{\Xi}(l_{k'})\| + \sum_{j=1}^{n} C_{j,2} \quad (5.61)$$

where the last inequality is established in (5.46). It together with

$$\max_{k'\le k-1}\|\bar{\Xi}(k')-\bar{\Xi}(l_{k'})\| \le 2\max_{k'\le k}\|\bar{\Xi}(k')\|$$

implies that there exists a constant

$$\lambda^* = 1/\sum_{j=1}^{n}(2C_{j,1}) \qquad (5.62)$$

such that $\forall \lambda_j < \lambda^*$, $j = 1, 2, \ldots, n$, we have

$$\max_{k'\le k}\|\bar{\Xi}(k')\| \le \frac{\sum_{j=1}^{n} C_{j,1}}{\sum_{j=1}^{n}(2C_{j,1})}\max_{k'\le k+n_j-1}\epsilon_j^a(k') + \frac{\sum_{j=1}^{n} C_{j,2}}{1-\sum_{j=1}^{n}\lambda_j C_{j,1}}$$

which leads to

$$\Xi(k-n_l+1) = O[\bar{\Xi}(k-n_l+1)] = \sum_{j=1}^{n} O[\epsilon_j^a(k+n_j-n_l)] \qquad (5.63)$$

From definition of $\Phi_j(k+n_j-1)$ in (5.31), we derive the following equation according to Lemma 2.7, equation (5.58) and Lipschitz condition of $\Phi_j(\cdot)$

$$\Phi_j(k+n_j-1) = \sum_{j=1}^{n} O[\bar{\xi}_{j,n_j}(k)] = O[\Xi(k)] = O[\bar{\Xi}(k)] \qquad (5.64)$$

From Lemma 2.7 we also have $u_j(k-n_j) = O[\Xi(k-n_j+1)]$. According to the definition of $D_j(k-n_j)$ in (5.47), we have

$$D_j^{\frac{1}{2}}(k-n_j) \le 1 + \|\Phi_j(k-1)-\Phi_j(l_{k-n_j}+n_j-1)\| + \|\bar{\Xi}(k-n_j)-\bar{\Xi}(l_{k-n_j})\|$$

$$+L_{j,n_j}\|\Xi(k)-\Xi(l_k)\| + L_{j,n_j}\|\bar{u}_{j-1}(k)-\bar{u}_{j-1}(l_k)\| + |u_j(k-n_j)-u_j(l_{k-n_j})|$$

$$= O[\bar{\Xi}(k-n_j)] + O[\bar{\Xi}(k-n_j+1)] \sim O[\bar{\Xi}(k-n_j+1)] \qquad (5.65)$$

From (5.53), (5.63) and (5.65), we have the following equalities,

$$\epsilon_j^a(k) = o[O[\Xi(k - n_j + 1)]] = \sum_{l=1}^n o[\epsilon_j^a(k - m_{jl})] + o[1] \tag{5.66}$$

Applying Lemma 2.8 to equation (5.57), we have $\lim_{k \to \infty} \epsilon_j^a(k) = 0$, which combined with (5.63) implies the boundedness of states vectors, $\Xi(k)$ and $\bar{\Xi}(k)$. Using Lemma 2.2, we have $\lim_{k \to \infty} \|\bar{\Xi}(k) - \bar{\Xi}(l_k)\| = 0$, then we obtain $\lim_{k \to \infty} \epsilon_j(k) = 0$ from (5.46). It together with (5.57) leads to $\lim_{k \to \infty} e_j(k) = 0$. Then, the boundedness of outputs $y_j(k)$ is established. According to Lemma 2.7, the boundedness of inputs $u_j(k)$ of all the subsystems are guaranteed. This completes the proof. ∎

5.3.5 Simulation studies

The following MIMO nonlinear system with three subsystems is used for simulation.

$$\Sigma : \begin{cases} \Sigma_1 \begin{cases} \xi_{1,1}(k+1) = \Theta_{1,1}^T \Phi_{1,1}(\xi_{1,1}(k)) + 0.2\xi_{1,2}(k) \\ \xi_{1,2}(k+1) = \Theta_{1,2}^T \Phi_{1,2}(\bar{\xi}_{1,2}(k), \xi_{2,1}(k)) + 0.8\xi_{1,3}(k) \\ \xi_{1,3}(k+1) = \Theta_{1,3}^T \Phi_{1,3}(\bar{\xi}_{1,3}(k), \bar{\xi}_{2,2}(k), \xi_{3,1}(k)) \\ \qquad\qquad + u_1(k) + \nu_1(\Xi_{\tau_1}(k)) \\ y_1(k) = \xi_{1,1}(k) \end{cases} \\ \Sigma_2 \begin{cases} \xi_{2,1}(k+1) = \Theta_{2,1}^T \Phi_{2,1}(\bar{\xi}_{1,2}(k), \xi_{2,1}(k)) + 0.3\xi_{2,2}(k) \\ \xi_{2,2}(k+1) = \Theta_{2,2}^T \Phi_{2,2}(\bar{\xi}_{1,3}(k), \bar{\xi}_{2,2}(k), \xi_{3,1}(k), u_1(k)) \\ \qquad\qquad + 1.2u_2(k) + \nu_2(\Xi_{\tau_2}(k)) \\ y_2(k) = \xi_{2,1}(k) \end{cases} \\ \Sigma_3 \begin{cases} \xi_{3,1}(k+1) = \Theta_{3,1}^T \Phi_{3,1}(\bar{\xi}_{1,3}(k), \bar{\xi}_{2,2}(k), \xi_{3,1}(k), u_1(k), u_2(k)) \\ \qquad\qquad + u_3(k) + \nu_3(\Xi_{\tau_3}(k)) \\ y_3(k) = \xi_{3,1}(k) \end{cases} \end{cases} \tag{5.67}$$

in which we see that in each equation of each subsystem there are states from the other subsystem and there are uncertain coupling terms in the last equations. The system parameters are

$$\Theta_{1,1} = 0.2, \Theta_{1,2}^T = [0, 0.3], \Theta_{1,3}^T = [0.5, 0.4],$$
$$\Theta_{2,1}^T = [0, 0.01], \Theta_{2,2}^T = [0.05, 0.1], , \Theta_{3,1}^T = 0.1,$$

and the system functions are

$$\Phi_{1,1}^T(\xi_{1,1}(k)) = \xi_{1,1}(k)\cos(\xi_{1,1}(k)), \quad \Phi_{1,2}^T(\bar{\xi}_{1,2}(k), \xi_{2,1}(k)) = [0, \frac{1}{1+\xi_{1,2}^2(k)}]$$

$$\Phi_{1,3}^T(\bar{\xi}_{1,3}(k), \bar{\xi}_{2,2}(k), \xi_{3,1}(k)) = [\frac{\xi_{1,1}(k)\xi_{1,2}(k)}{1+\xi_{1,1}^2(k)+\xi_{1,3}^2(k)}, \frac{\xi_{2,2}(k)}{2+\xi_{2,1}^2(k)}]$$

$$\Phi_{2,1}^T(\bar{\xi}_{1,2}(k), \xi_{2,1}(k)) = [0, \xi_{1,2}(k)\sin(\xi_{2,1}(k))]$$

$$\Phi_{2,2}^T(\bar{\xi}_{1,3}(k), \bar{\xi}_{2,2}(k), \xi_{3,1}(k), u_1(k)) = [\frac{u_1(k)\xi_{2,2}(k)}{1+\xi_{2,1}^2(k)+\xi_{2,1}^2(k)}(1+e^{-\xi_{1,1}^2(k)}), \sin(\xi_{1,3}(k))\xi_{3,1}(k)]$$

$$\Phi_{3,1}^T(\bar{\xi}_{1,3}(k), \bar{\xi}_{2,2}(k), \xi_{3,1}(k), u_1(k), u_2(k)) = \frac{\sin(u_2(k))\xi_{3,1}(k)}{1+\xi_{1,2}^2(k)}$$

and uncertain terms are

$$\nu_1(\Xi_{\tau_1}(k)) = 0.01\cos(\xi_{1,1}(k-2))\xi_{1,3}(k-2) + 0.03\xi_{2,1}(k-1)\log(1+\xi_{3,1}^2(k-1)),$$

$$\nu_2(\Xi_{\tau_2}(k)) = 0.3(\xi_{1,1}(k-1) + \xi_{1,2}(k-1)) + 0.1\cos(\xi_{2,1}(k-2)\xi_{2,2}(k-2)),$$

$$\nu_3(\Xi_{\tau_3}(k)) = 0.01\cos(\xi_{2,1}(k-1))\xi_{2,2}(k-2)$$

The reference trajectories are $y_1^*(k) = 2.5\sin(\frac{\pi}{2}kT) + 1.5\cos(\frac{\pi}{4}kT)$, $y_2^*(k) = 2.5\cos(\frac{\pi}{2}kT) + 1.5\sin(\frac{\pi}{4}kT)$, and $y_3^*(k) = 1.5\cos(\frac{\pi}{2}kT) + 2.5\sin(\frac{\pi}{4}kT)$, where $T = 0.02$. The initial system states are $\Xi(0) = [0.1, 0.1, 0.1, 0.1, 0.1, 0.1]^T$. The control parameters are chosen as $\underline{g}_1 = 0.16, \underline{g}_2 = 0.15, \underline{g}_3 = 0.1$, $\gamma_1 = 0.2, \gamma_2 = 0.03, \gamma_3 = 0.05$, and $\lambda_1 = 0.001, \lambda_2 = 0.001, \lambda_3 = 0.001$.

The simulation results are presented in Figures 5.5, 5.6, 5.7 and 5.8. The tracking performances are shown in Figure 5.5. The boundedness of estimated parameters in control and prediction is shown in Figures 5.6 and 5.7, respectively. The boundedness of control signals and signals $\beta_1(k)$ and $\beta_2(k)$ caused by prediction errors are presented in Figure 5.8.

5.4 Summary

In this Chapter, we have studied to extend the adaptive control designed in Chapters 3 and 4 to more general classes of systems. In Section 5.2, we have extended the control designed in Section 4.4 to systems with input constraint by using PI model to represent the hysteresis. In Section 5.3, we have extended the control designed in Section 3.2 to MIMO system with uncertain couplings among each subsystems. By utilizing the structure properties of the MIMO systems, the effect of interactions among subsystems has been decoupled, and by uncertain couplings compensation, the designed adaptive control achieves asymptotic tracking performance.
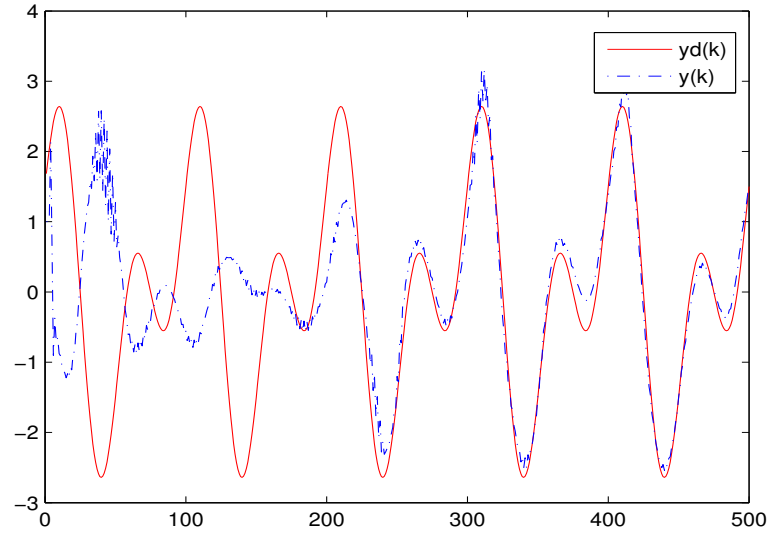
Figure 5.2: Reference signal and system output
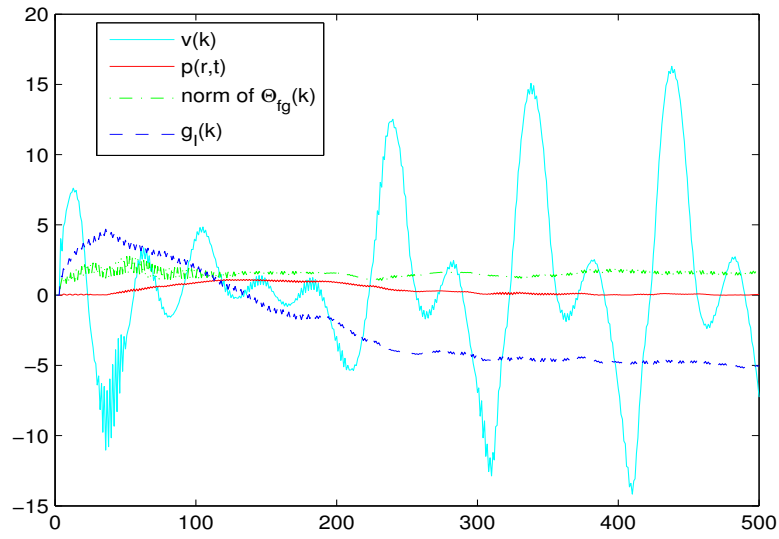
Figure 5.3: Control signal and estimated parameters, $r = 1$ for $\hat{p}(r, t)$
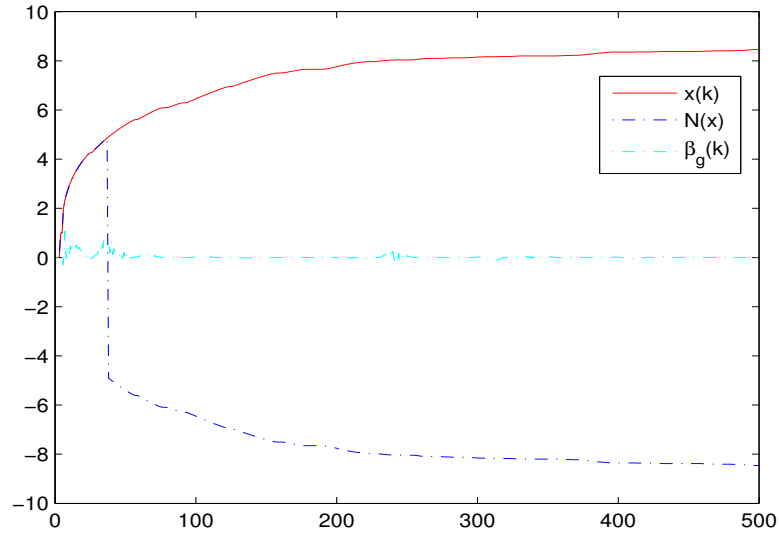
Figure 5.4: Nussbaum gain $N(x(k))$ and its argument $x(k)$ and $\beta_g(k)$
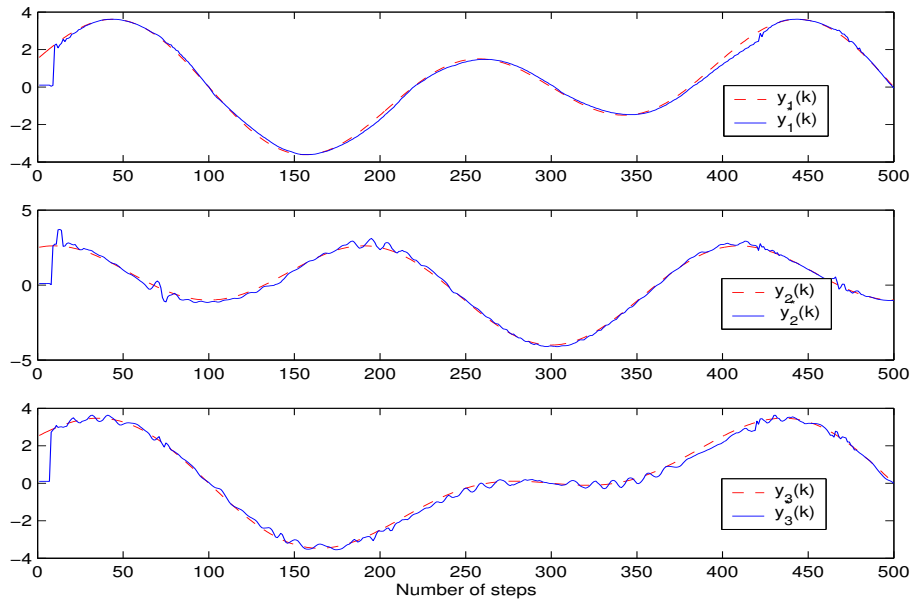
Figure 5.5: System outputs and reference trajectories
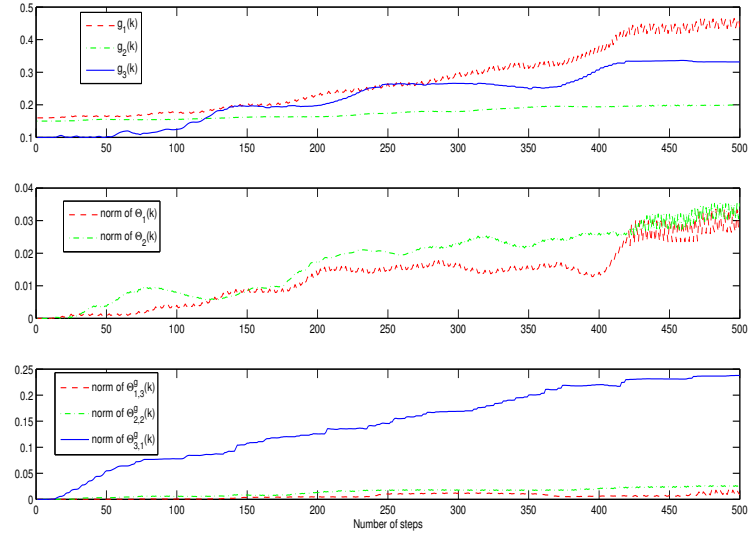
Figure 5.6: Estimated parameters in control
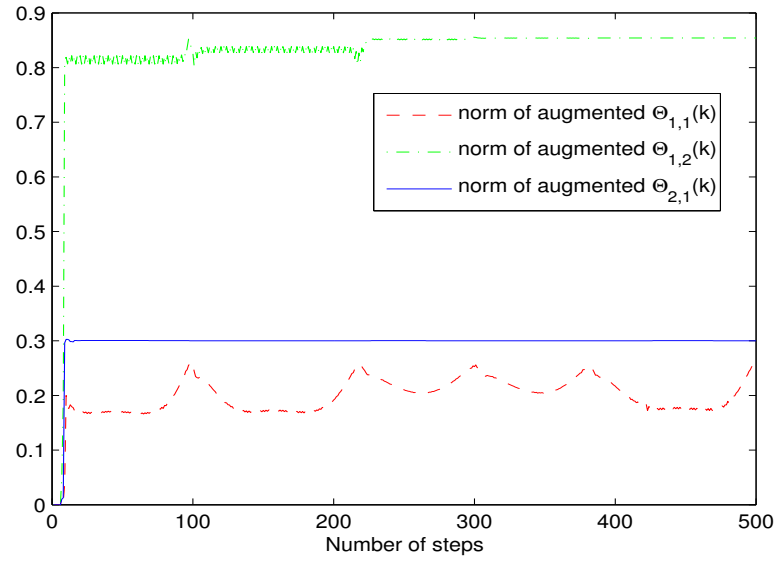
Figure 5.7: Estimated parameters in prediction

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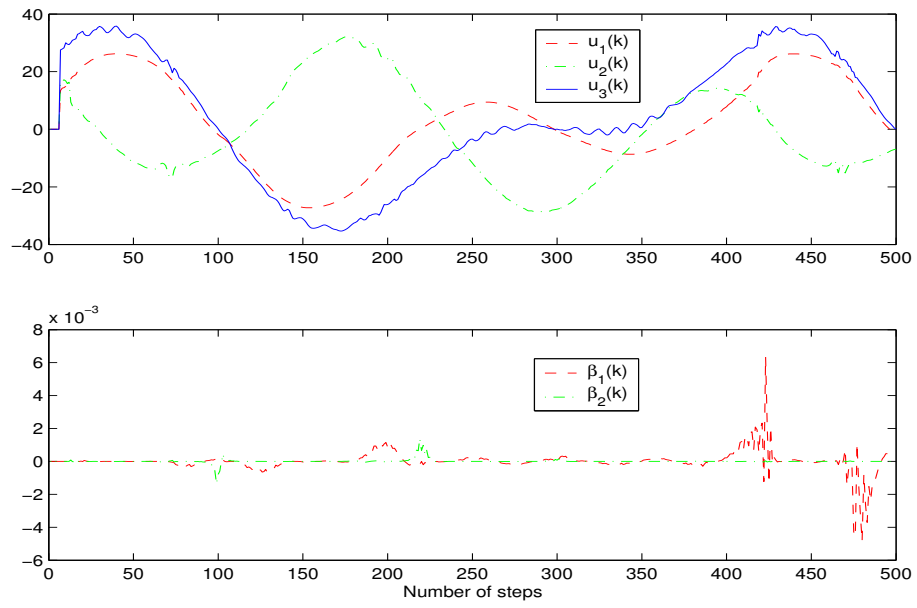
Figure 5.8: Control inputs and signals $\beta_1(k)$ and $\beta_2(k)$

Part II

Neural Network Control Design

Chapter 6

SISO Nonaffine systems

6.1 Introduction

As mentioned in Section 1.2.2, most of the existing adaptive NN control in discrete-time are focused on affine systems. In this Chapter, we will study adaptive NN control of two classes of nonlinear discrete-time systems in nonaffine form: (1) nonlinear pure-feedback systems, and (2) NARMAX systems, to further develop implicit adaptive NN control in discrete-time [144, 146]. The pure-feedback systems to be studied assumes in the general form of lower triangular structure, such that it covers the nonlinear strict-feedback systems in LIPs form studied in Part I of the thesis. The NARMAX model also comprises a general nonlinear discrete-time model structure [178] , and it has received much attention in the literature of discrete-time control.

In this Chapter, implicit function theorem is exploited to assert the existence of a desired control input such that the difficulty of the nonaffine appearance of the control input can be solved. The high-order-neural-network (HONN) is employed to approximate the unknown desired control. The control directions for nonaffine systems are defined as the partial derivatives of the nonlinear system functions over the control variables. It should be mentioned that control directions play same important role in adaptive NN control design as in model based adaptive control design. When the control directions are unknown, the design becomes much more intractable. In this Chapter, we extend the technique of discrete Nussbaum gain to deal with unknown control directions problem in adaptive NN control design. It may be noted that for adaptive NN control in [144], the control direction was not assumed to be known, but the stability result was proved using NN weights convergence results, which cannot be guaranteed without persistent exciting condition.

The contributions in this Chapter lies in

(i) Implicit function theory has been exploited to solve the difficulty of nonaffine appearance of control input in NN control design of nonlinear systems.

(ii) Discrete Nussbaum gain technique developed in Chapter 4 has been extended to facilitate NN control design of nonlinear systems with time varying control gains of unknown signs.

(iii) By states and outputs prediction, a unified output feedback adaptive NN control scheme has been constructed for controlling SISO systems in both pure-feedback and NARMAX form.

6.2 Problem Formulation and Preliminaries

Similar to Part I, the control objective in Part II is to synthesize a control input $u(k)$ for the systems to be controlled such that that all signals in the closed-loop systems are bounded and the output $y(k)$ tracks a bounded reference trajectory $y_d(k)$.

6.2.1 Pure-feedback systems

Consider the following SISO discrete-time systems in pure-feedback form

$$\begin{cases} \xi_i(k+1) = f_i(\bar{\xi}_i(k), \xi_{i+1}(k)) \\ \quad i = 1, 2, \ldots, n-1 \\ \xi_n(k+1) = f_n(\bar{\xi}_n(k), u(k), d(k)) \\ y(k) = \xi_1(k) \end{cases} \tag{6.1}$$

where $f_i(\cdot, \cdot)$ and $f_n(\cdot, \cdot, \cdot)$ are unknown nonlinear functions and $d(k)$ denotes the external disturbance, which is bounded by an unknown constant $\bar{d}$ so that $|d(k)| \leq \bar{d}$. Similar to Part I, $\bar{\xi}_j(k)$, $j = 1, 2, \ldots, n$, are system states, and $u(k)$ and $y(k)$ are system input and output, respectively.

Assumption 6.1. *System functions $f_i(\cdot, \cdot)$ and $f_n(\cdot, \cdot, 0)$ in (6.1) are continuous with respect to all the arguments and continuously differentiable with respect to the second argument.*

Assumption 6.2. *There exist constants $\bar{g}_i > \underline{g}_i > 0$ such that $0 < \underline{g}_i \leq |g_{1,i}(\cdot)| \leq \bar{g}_i$, $i = 1, 2, \ldots, n$, where the control gain functions $g_{1,i}(\cdot)$ are defined in Definition 2.7.*

For convenience, let us introduce the notations $\underline{g}' = \Pi_{i=1}^n \underline{g}_i$ and $\bar{g}' = \Pi_{i=1}^n \bar{g}_i$. It should be noted that the constants $\bar{g}'$ and $\underline{g}'$ are only used for analysis and are not required to be known in the control design.

Assumption 6.3. *The system functions $f_i(\cdot, 0)$ and $f_n(\cdot, 0, \cdot)$ are Lipschitz functions.*

6.2.2 NARMAX systems

Consider the following SISO discrete-time systems in NARMAX form

$$y(k + \tau) \quad = \quad f(y(k + \tau - 1), \cdots, y(k + \tau - n), u(k), \cdots, u(k - m + 1), d(k)) \quad (6.2)$$

where $\tau \geq 1$, $m \geq 1$, $f(\cdot) : R^{n+m+1} \rightarrow R$ is an unknown nonlinear function, and similarly, $d(k)$ denotes the external disturbance, which is bounded by an unknown constant $\bar{d}$, i.e. $|d(k)| \leq \bar{d}$.

Assumption 6.4. *The system function $f(\cdot) : R^{m+n+1} \rightarrow R$ in (6.2) is continuous with respect to all the arguments and continuously differentiable with respect to $u(k)$.*

Assumption 6.5. *There exist constants $\bar{g} > \underline{g} > 0$ such that $0 < \underline{g} \leq |g(\cdot)| \leq \bar{g}$, where control gain $g(\cdot) = \frac{\partial f(\cdot)}{\partial u(k)}$.*

Assumption 6.6. *System (6.2) is inverse stable, i.e., system (6.2) is bounded-output-bounded-input (BOBI). In addition, the function $f(y(k + \tau - 1), y(k + \tau - 2), y(k + \tau - n), \mathbf{0}_{[m]}, d(k))$ is a Lipschitz function.*

Remark 6.1. *Without loss of generality, we shall assume that $\bar{g}' = \bar{g}$ and $\underline{g}' = \underline{g}$ in the following parts.*

Remark 6.2. *Assumptions 6.2 and 6.5 imply that the control directions are unknown, i.e., the control gains can be either positive or negative. But in Sections 6.3 and 6.4, the control directions are first assumed known such that we focus on the key techniques in adaptive NN control design. While Section 6.6 is dedicated to the design in the presence of unknown control directions.*

6.2.3 Preliminaries

In this Section, first some useful lemmas for stability analysis and control design are introduced and then, the generalization of discrete Nussbaum gain is made to cope with time varying control gains.

Lemma 6.1. *Let $V(k) = \sum_{i=1}^{m} V_i(k)$, where $V_i(k) \geq 0$, $i = 1, 2, \ldots, m$. If the following inequality holds*

$$V(k+1) \leq \sum_{i=1}^{m} c_i(k) V_i(k) + b(k) \tag{6.3}$$

where $|c_i(k)| \leq \bar{c} < 1$, and $|b(k)| \leq \bar{b}$.
Then, we have

$$
\begin{aligned}
V(k) &\leq V(0) + \frac{\bar{b}}{1 - \bar{c}} \\
\lim_{k \to \infty} \sup\{V(k)\} &\leq \frac{\bar{b}}{1 - \bar{c}}
\end{aligned}
\tag{6.4}
$$

Proof. See Appendix 6.1 ∎

Corollary 6.1. *Let $V(k) = \sum_{i=1}^{m} V_i(k)$, where $V_i(k) \geq 0$. If the following inequality holds*

$$
\begin{aligned}
V(k+1) &\leq \sum_{i=1}^{m} c_i(k_1) V_i(k_1) + b(k_1) \\
k_1 &= k - n + 1, \quad k \geq n - 1, \quad n \geq 1
\end{aligned}
$$

where $|c_i(k)| \leq \bar{c} < 1$, and $|b(k)| \leq \bar{b}$.
Then, we have

$$
\begin{aligned}
V(k) &\leq \bar{V}(0) + \frac{\bar{b}}{1 - \bar{c}}, \quad k \geq n - 1 \\
\lim_{k \to \infty} \sup\{V(k)\} &\leq \frac{\bar{b}}{1 - \bar{c}}
\end{aligned}
\tag{6.5}
$$

where $\bar{V}(0) = \max_{-n \leq j \leq -1}\{V(j)\}$.
Proof. See Appendix 6.2. ∎

Lemma 6.2. *Define a positive definite function $V(k) = V_1(k) + V_2(k)$, with $V_1(k)$ and $V_2(k)$ are given by*

$$
\begin{aligned}
V_1 &= a_e e^2(k) \\
V_2 &= a_W \tilde{W}^T(k) \tilde{W}(k)
\end{aligned}
$$

where $e(k) = y(k) - y_d(k)$, $y_d(k) \in \Omega_{yd}$, is output tracking error, $W^* \in R^l$ and $\hat{W}(k) \in R^l$ are ideal NN weights vector and its estimate, $\tilde{W}(k) = \hat{W}(k) - W^*$ is the estimate error, a_e and a_W are some positive constants. If the following inequality holds

$$
\begin{aligned}
V(k+1) &\leq c_1(k) V_1(k_1) + c_2(k) V_2(k_1) + b(k) \\
k_1 &= k - n + 1, \quad k \geq n - 1
\end{aligned}
\tag{6.6}
$$

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where $|c_i(k)| < \bar{c} < 1$, $i = 1, 2$, and $|b(k)| < \bar{b}$. Then, given any initial compact set defined by

$$
\begin{aligned}
\Omega_0 &= \Omega_{\xi_0} \times \Omega_{\hat{W}_0} \\
&= \{\bar{\xi}_n(0) \mid \|\bar{\xi}_n(0)\| \leq C_1 C_{e0} + C_1 \max\{|y_d(i)|\} + C_2\} \\
&\quad \times \{\hat{W}(i) \mid \|\hat{W}(i)\| \leq \|W^*\| + C_{\tilde{W}0}\} \\
i &= 0, 1, \ldots, n - 1
\end{aligned}
$$

where C_1 and C_2 are finite coefficients, C_{e0} and $C_{\tilde{W}0}$ are defined as

$$
C_{e0} = \max_{0 \leq i \leq n-1}\{|e(i)|\}, \quad C_{\tilde{W}0} = \max_{0 \leq i \leq n-1}\{\|\tilde{W}(i)\|\} \tag{6.7}
$$

Then, it can be concluded that

(i) The states $\bar{\xi}_n(k)$ and the NN weights vector $\hat{W}(k)$ remain in the compact set defined by

$$
\begin{aligned}
\Omega &= \Omega_\xi \times \Omega_{\hat{W}} \\
&= \{\bar{\xi}_n(k) \mid \|\bar{\xi}_n(k)\| \leq C_1 \sup_{y_d(k) \in \Omega_{yd}}\{|y_d(k)|\} + C_1 c_{e\,\max} + C_2\} \\
&\quad \times \{\hat{W}(k) \mid \|\hat{W}(k)\| \leq \|W^*\| + c_{\tilde{W}\,\max}\}
\end{aligned}
$$

(ii) The states $\bar{\xi}_n(k)$ and the NN weights vector $\hat{W}(k)$ will eventually converge to the compact set defined by

$$
\begin{aligned}
\Omega_s &= \Omega_{\xi_s} \times \Omega_{\hat{W}_s} \\
&= \{\bar{\xi}_n(k) \mid \|\bar{\xi}_n(k)\| \leq C_1 \sup_{y_d(k) \in \Omega_{yd}}\{|y_d(k)|\} + C_1 c_{es} + C_2\} \\
&\quad \times \{\hat{W}(k) \mid \|\hat{W}(k)\| \leq \|W^*\| + c_{\tilde{W}s}\}
\end{aligned}
$$

where constants

$$
\begin{aligned}
c_{e\,\max} &= \sqrt{\frac{1}{a_e}\left(C_0 + \frac{\bar{b}}{1 - \bar{c}}\right)} \\
c_{\tilde{W}\,\max} &= \sqrt{\frac{1}{a_W}\left(C_0 + \frac{\bar{b}}{1 - \bar{c}}\right)} \tag{6.8} \\
c_{es} &= \sqrt{\frac{\bar{b}}{a_e(1 - \bar{c})}} \\
c_{\tilde{W}s} &= \sqrt{\frac{\bar{b}}{a_W(1 - \bar{c})}} \\
C_0 &= a_e C_{e0}^2 + a_W C_{\tilde{W}0}^2 \tag{6.9}
\end{aligned}
$$

Proof. See Appendix 6.3 ∎

Lemma 6.3. *Consider the algorithm to construct a discrete Nussbaum gain $N(x(k))$ detailed in Section 4.2.*

(i) Given an arbitrary bounded function $g(k) : R \to R$, and $g_1 \le |g(k)| \le g_2$, where g_1 and g_2 are unknown positive constants, then $N'(x(k)) = g(k)N(x(k))$ is also a discrete Nussbaum gain if $\Delta x(k) \ge 0$.

(ii) Given an arbitrary function $-\epsilon_0 \le C(k) \le \epsilon_0$, then $N'(x(k)) = N(x(k)) + C(k)$ is still a discrete Nussbaum gain if $\Delta x(k) \ge 0$.

Proof. See Appendix 6.4. ∎

Remark 6.3. *Compared with discrete Nussbaum gain $N(x(k))$ in Section 4.2, the discrete Nussbaum gain $N'(x(k))$ obtained from Lemma 6.3 has more restriction, $\Delta x(k) \ge 0$.*

6.3 State Feedback NN Control

In this section, let us design adaptive NN control with state feedback for the pure-feedback system (6.1), for which in this Section we assume the control directions are known.

6.3.1 Pure-feedback system transformation

From Lemma 2.5 we know that the future states $\bar{\xi}_i(k + n - i)$, $i = 1, 2, \ldots, n - 1$, are SDFS and can be predicted by the prediction functions $P_{n-i,i}(\bar{\xi}_n(k))$, which are functions of current states.

Substituting the prediction functions in Lemma 2.5 into system (6.1), we obtain

$$
\begin{cases}
\xi_1(k + n) = \phi_{1,1}(P_{n-1,1}(\bar{\xi}_n(k)), \xi_2(k + n - 1)) \\
\xi_2(k + n - 1) = \phi_{1,2}(P_{n-2,2}(\bar{\xi}_n(k)), \xi_3(k + n - 2)) \\
\qquad\qquad \vdots \\
\xi_n(k + 1) = \phi_{1,n}(\bar{\xi}_n(k), u(k), d(k)) \\
y(k + n) = \xi_1(k + n)
\end{cases}
\tag{6.10}
$$

where $\phi_{1,n}(\bar{\xi}_n(k), u(k), d(k))$ is defined in the following for consistency:

$$
\phi_{1,n}(\bar{\xi}_n(k), u(k), d(k)) = f_n(\bar{\xi}_n(k), u(k), d(k))
\tag{6.11}
$$

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Remark 6.4. *As same as in control design in Part I, we consider transform system (6.10) into a compact form by combining the n equations in (6.10) together. It will be noted that by using the prediction functions in the transformation, the system can be transformed into an n-step ahead predictor form such that the n-step ahead output can be predicted by the current states. In this way, the consequent control design avoid the complicated backstepping [51] and only a single NN is employed to generate a control input.*

Replacing $\xi_2(k+n-1)$ in the first equation of (6.10) with the right hand side of the second equation yields

$$\xi_1(k+n) = \phi_{1,1}(P_{n-1,1}(\bar{\xi}_n(k)), \phi_{1,2}(P_{n-2,2}(\bar{\xi}_n(k)), \xi_3(k+n-2)))$$

Continuing to iteratively replace $\xi_j(k+n-j+1)$ in the above equation with the right hand side of the jth equation in (6.10), $j = 3, 4, \ldots, n-1$, until $u(k)$ appears at the last step, we obtain

$$y(k+n) = \xi_1(k+n) = \phi(\bar{\xi}_n(k), u(k), d(k)) \tag{6.12}$$

where

$$\phi(\bar{\xi}_n(k), u(k), d(k))$$
$$= \phi_{1,1}(P_{n-1,1}(\bar{\xi}_n(k)), \phi_{1,2}(P_{n-2,2}(\bar{\xi}_n(k)), \phi_{1,3}(\ldots, \phi_{1,n}(\bar{\xi}_n(k), u(k), d(k))\ldots))) \tag{6.13}$$

Now the original pure-feedback system (6.1) is transformed into an n-step ahead predictor (6.12).

6.3.2 Adapgtive NN control design

The n-step ahead predictor function (6.12) can be written as

$$y(k+n) = \phi(\bar{\xi}_n(k), u(k), d(k)) = \phi_s(\bar{\xi}_n(k), u(k)) + d_s(k) \tag{6.14}$$

where

$$\phi_s(\bar{\xi}_n(k), u(k)) = \phi(\bar{\xi}_n(k), u(k), 0)$$
$$d_s(k) = \phi(\bar{\xi}_n(k), u(k), d(k)) - \phi(\bar{\xi}_n(k), u(k), 0)$$

According to Assumption 6.3, there exists a finite constant L_d such that

$$|d_s(k)| \leq L_d|d(k)| \leq L_d\bar{d} := \bar{d}_s \tag{6.15}$$

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For $\phi_s(\bar{\xi}_n(k), u(k))$, from (A.3) and (6.10), it is easy to show that

$$0 < \underline{g} < \frac{\partial \phi_s(\bar{\xi}_n(k), u(k))}{\partial u(k)} = g_{1,1}(\cdot)g_{1,2}(\cdot)\ldots g_{1,n}(\cdot) := g_s(\cdot) < \bar{g} \tag{6.16}$$

Denote $e(k) = y(k) - y_d(k)$ and then, we have

$$e(k+n) = \phi_s(\bar{\xi}_n(k), u(k)) - y_d(k+n) + d_s(k) \tag{6.17}$$

From (6.16), it is clear that

$$\frac{\partial \phi_s(\bar{\xi}_n(k), u(k)) - y_d(k+n)}{\partial u(k)} = g_s(\cdot) > 0$$

According to Lemma 2.1, there exists a continuous ideal control input $u_s^*(z(k))$ such that

$$\phi_s(\bar{\xi}_n(k), u_s^*(z(k))) - y_d(k+n) = 0$$
$$z(k) = [\bar{\xi}_n^T(k), y_d(k+n)]^T \in \Omega_z \in R^{n+1} \tag{6.18}$$

Substituting this ideal control $u_s^*(z(k))$ into (6.17) results in $e(k+n) = d_s(k)$. This means that the ideal control $u_s^*(z(k))$ is an n-step deadbeat control because after n steps, we have $y(k+n) = y_d(k+n)$, if $d_s(k) = 0$. It is known that $d_s(k)$ is bounded, then $y(k)$ must be bounded. According to Lemma 2.6, the ideal control input $u_s^*(z(k))$ is bounded.

From Section 2.2, there exists a HONN with an ideal weight vector $W_s^* \in R^{l_s}$ such that $u_s^*(z(k))$ can be approximated in the following manner:

$$u_{nn}^*(z(k)) = W_s^{*T}S(z(k)), \ S(z(k)) \in R^{l_s}$$
$$u_s^*(z(k)) = u_{nn}^*(z(k)) + \mu(z(k)), \ \forall z(k) \in \Omega_z \tag{6.19}$$

where $\Omega_z = \Omega_\xi \times \Omega_{yd}$ and $\mu(z(k))$ is the NN function approximation error that can be made arbitrary small by increasing NN nodes number l_s.

Consider the following control with an adaptive HONN to approximate $u_s^*(z(k))$:

$$u(k) = \frac{\eta_s(k)}{\bar{g}}e(k) + \hat{u}_s(z(k)) \tag{6.20}$$
$$\hat{u}_s(z(k)) = \hat{W}_s^T(k)S(z(k))$$

where $|\eta_s(k)| \leq \bar{\eta}_s < 1$ is a scaling factor, $\hat{W}_s(k)$ is the estimate of ideal NN weight W_s^* and it is updated by the adaptation law

$$\hat{W}_s(k+1) = \hat{W}_s(k_1) - \gamma_s S(z(k_1))e(k+1) - \sigma_s \hat{W}_s(k_1)$$
$$k_1 = k - n + 1 \tag{6.21}$$

where $0 < \sigma_s < 1$ and $\gamma_s > 0$ are NN tuning parameters to be chosen.

Theorem 6.1. *The closed-loop adaptive system consisting of the plant (6.1), the adaptive NN control (6.20) and the NN adaptation law (6.21) achieves SGUUB stability, provided that Assumptions 6.1, 6.2 and 6.3 hold, and the design parameters $0 < \sigma_s < 1$, $0 < \bar{\eta}_s < 1$ and γ_s are suitably chosen such that*

$$2\gamma_s \bar{g} l_s + \bar{\eta}_s \bar{g} + \bar{\eta}_s < 1 \tag{6.22}$$

Furthermore, the tracking error and the NN weight estimation error are ultimately bounded as

$$\lim_{k \to \infty} \sup \{|e(k)|^2 + \frac{\bar{g}}{\gamma_s}\|\tilde{W}_s(k)\|^2\} \leq \frac{\bar{b}}{1 - \bar{c}}$$

where

$$\begin{aligned}
\bar{b} &= \frac{\bar{g}}{\bar{\eta}_s}\mu_s^{*2} + 2\frac{\bar{g}}{\gamma_s}\sigma_s\|W_s^*\|^2 \\
\bar{c} &= \max\{\bar{\eta}_s, (1 - 2\sigma_s)\} \\
\mu_s^* &= \mu^* + \frac{\bar{d}_s}{\underline{g}}
\end{aligned} \tag{6.23}$$

and μ^* is the NN approximation error bound defined in (2.10).

Proof. Adding and subtracting $\phi_s(\bar{\xi}_n(k), u_s^*(k))$ on the right hand side of (6.17) leads to

$$\begin{aligned}
e(k+n) &= \phi_s(\bar{\xi}_n(k), u(k)) - \phi_s(\bar{\xi}_n(k), u_s^*(z(k))) + d_s(k) \\
&= g_s(\bar{\xi}_n(k), u^c(k))(u(k) - u_s^*(z(k))) + d_s(k)
\end{aligned} \tag{6.24}$$

where $u^c(k) \in [\min\{u_s^*(z(k)), u(k)\}, \max\{u_s^*(z(k)), u(k)\}]$ and the last equality is obtained by using Mean Value Theorem. For convenience, denote

$$g_s(k) = g_s(\bar{\xi}_n(k), u^c(k))$$
$$S(k) = S(z(k))$$

Combining (6.19), (6.20), and (6.24) yields

$$\begin{aligned}
e(k+1) &= \eta_s(k)\frac{g_s(k_1)}{\bar{g}}e(k_1) + g_s(k_1)\tilde{W}_s(k_1)S(k_1) \\
&\quad - g_s(k_1)\mu(z(k_1)) + d_s(k_1)
\end{aligned} \tag{6.25}$$

where $\tilde{W}_s(k) = \hat{W}_s(k) - W_s^*$ is the NN weight estimation error.

First, let us assume that the NN is constructed to cover a large enough compact set Ω such that the NN approximation ability is never violated and equation (6.25) always holds.

While later we will show that it is indeed the case, if we initially construct the NN with approximation range covering a prescribed compact set, and the so-called circular argument does not apply here in this very proof.

Consider a positive definite function $V(k)$ as

$$
\begin{aligned}
V(k) &= V_1(k) + V_2(k) \\
V_1(k) &= e^2(k) \\
V_2(k) &= \frac{\bar{g}}{\gamma_s} \tilde{W}_s^T(k) \tilde{W}_s(k)
\end{aligned}
\tag{6.26}
$$

It can be derived from (6.21) that

$$
\tilde{W}_s(k+1) = \tilde{W}_s(k_1) - \gamma_s S(k_1) e(k+1) - \sigma_s \hat{W}_s(k_1)
\tag{6.27}
$$

From (6.25), it can be shown that

$$
\tilde{W}^T(k_1) S(k_1) e(k+1) = \frac{e^2(k+1)}{g_s(k_1)} - \frac{\eta_s(k)}{\bar{g}} e(k_1) e(k+1) + e(k+1) \mu_s(k_1)
$$

where

$$
\mu_s(k_1) = \mu(z(k_1)) - \frac{d_s(k_1)}{g_s(k_1)}
\tag{6.28}
$$

Noting the following facts

$$
\begin{aligned}
& 0 < g_s(k_1) < \bar{g} \\
& S^T(k) S(k) \le l_s \\
& |\mu_s(k_1)| \le \mu_s^* \\
& 2 \tilde{W}_s(k_1) \hat{W}_s(k_1) = \tilde{W}_s^T(k_1) \tilde{W}_s(k_1) + \|\hat{W}_s(k_1)\|^2 - \|W_s^*\|^2 \\
& 2 \sigma_s \hat{W}_s(k_1) S(k_1) e(k+1) \le \gamma_s l_s e^2(k+1) + \frac{\sigma_s^2}{\gamma_s} \|\hat{W}_s(k_1)\|^2 \\
& -2 e(k+1) \mu_s(k_1) \le \bar{\eta}_s e^2(k+1) + \frac{\mu_s^2(k_1)}{\bar{\eta}_s} \\
& 2 \eta_s(k) e(k_1) e(k+1) \le \bar{\eta}_s e^2(k_1) + \bar{\eta}_s e^2(k+1)
\end{aligned}
$$

we have the following inequality from (6.27),

$$V_2(k+1) = \frac{\bar{g}}{\gamma_s}\tilde{W}_s^T(k+1)\tilde{W}_s(k+1)$$

$$= \frac{\bar{g}}{\gamma_s}[\tilde{W}_s^T(k_1)\tilde{W}_s(k_1) + \gamma_s^2 S^T(k_1)S(k_1)e^2(k+1)$$

$$+ \sigma_s^2\|\hat{W}_s(k_1)\|^2 - 2\gamma_s\tilde{W}_s^T(k_1)S(k_1)e(k+1) - 2\sigma_s\tilde{W}_s^T(k_1)\hat{W}_s(k_1)$$

$$+ 2\gamma_s\sigma_s\hat{W}_s^T(k_1)S(k_1)e(k+1)]$$

$$\leq \frac{\bar{g}}{\gamma_s}\tilde{W}_s^T(k_1)\tilde{W}_s(k_1) + \gamma_s l_s\bar{g}e^2(k+1) + \frac{\bar{g}}{\gamma_s}\sigma_s^2\|\hat{W}_s(k_1)\|^2 - 2\frac{\bar{g}}{g_s(k_1)}e^2(k+1)$$

$$- 2\eta_s(k)e(k_1)e(k+1) - 2\bar{g}\mu_s(k_1)e(k+1)$$

$$- 2\frac{\bar{g}}{\gamma_s}\sigma_s(\tilde{W}_s^T(k_1)\tilde{W}_s(k_1) + \|\hat{W}_s(k_1)\|^2 - \|W_s^*\|^2)$$

$$+ 2\bar{g}\sigma_s\hat{W}_s^T(k_1)S(k_1)e(k+1)$$

$$\leq \frac{\bar{g}}{\gamma_s}(1 - 2\sigma_s)\tilde{W}_s^T(k_1)\tilde{W}_s(k_1) + \bar{\eta}_s e^2(k_1) + \frac{\bar{g}}{\bar{\eta}_s}\mu_s^{*2} + 2\frac{\bar{g}}{\gamma_s}\sigma_s\|W_s^*\|^2$$

$$+ (2\gamma_s\bar{g}l_s + \bar{\eta}_s\bar{g} + \bar{\eta}_s - 2)e^2(k+1) - 2\frac{\bar{g}}{\gamma_s}\sigma_s(1 - \sigma_s)\|\hat{W}_s(k_1)\|^2 \qquad (6.29)$$

Further, combining $V_2(k+1)$ with

$$V_1(k+1) = e^2(k+1) \qquad (6.30)$$

yields

$$V(k+1) = V_1(k+1) + V_2(k+1)$$

$$\leq \frac{\bar{g}}{\gamma_s}(1 - 2\sigma_s)\tilde{W}_s^T(k_1)\tilde{W}_s(k_1) + \bar{\eta}_s e^2(k_1) + \frac{\bar{g}}{\bar{\eta}_s}\mu_s^{*2} + 2\frac{\bar{g}}{\gamma_s}\sigma_s\|W_s^*\|^2$$

$$+ (2\gamma_s\bar{g}l_s + \bar{\eta}_s\bar{g} + \bar{\eta}_s - 1)e^2(k+1)$$

$$= \bar{\eta}_s V_1(k_1) + (1 - 2\sigma_s)V_2(k_1) + \bar{b} + (2\gamma_s\bar{g}l_s + \bar{\eta}_s\bar{g} + \bar{\eta}_s - 1)e^2(k+1) \qquad (6.31)$$

where

$$\bar{b} = \frac{\bar{g}}{\bar{\eta}_s}\mu_s^{*2} + 2\frac{\bar{g}}{\gamma_s}\sigma_s\|W_s^*\|^2 \qquad (6.32)$$

If the parameters are chosen such that the following inequality holds

$$2\gamma_s\bar{g}l_s + \bar{\eta}_s\bar{g} + \bar{\eta}_s < 1$$

then equation (6.31) becomes

$$V(k+1) \leq \bar{\eta}_s V_1(k_1) + (1 - 2\sigma_s)V_2(k_1) + \bar{b} \qquad (6.33)$$

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Let $a_e = 1$, $a_W = \frac{\bar{g}}{\gamma_s}$, $\bar{c} = \max\{\bar{\eta}_s, (1 - 2\sigma_s)\}$. Noting that $0 < \bar{\eta}_s < 1$, $0 < \sigma_s < 1$ and applying Lemma 6.2, we obtain the bounds on states and NN weights vector. According to Lemma 2.6, the control input is also bounded.

Now we show the validness of NN approximation indeed holds given any initial condition Ω_0, if the NN used in (6.20) is pre-designed with approximation range covering a specified compact set. From Remark 6.5, we see that given any initial condition, Ω_0, because the bounding compact set $\Omega = \Omega_\xi \times \Omega_{\hat{W}}$ is determined, if NN is constructed such that its approximation range covers the determinant compact set $\Omega_z = \Omega_\xi \times \Omega_{yd}$, then NN approximation ability always holds. It implies that given any initial condition Ω_0, with employment of an NN whose approximation range is over the corresponding Ω_z, the NN control (6.20) guarantees the boundedness of closed-loop signals. According to Definition 2.11, the closed-loop signals are SGUUB.

In addition, according to Corollary 6.1, it can be seen that the tracking error and the NN weight estimation error are ultimately bounded as

$$\lim_{k\to\infty} \sup\{|e(k)|^2 + \frac{\bar{g}}{\gamma_s}\|\tilde{W}_s(k)\|^2\} = \lim_{k\to\infty} \sup V(k) \leq \frac{\bar{b}}{1 - \bar{c}}$$

where $\bar{b}$ and $\bar{c}$ are defined in Theorem 6.1. This completes the proof. ∎

In the theoretical analysis of stability above, we can see that the larger the Ω_0 is, the larger the Ω is. As the actual size of initial condition, Ω_0, may not be specified in advance, the NN should be chosen to cover a compact set of sufficiently large size such that Ω is within NN approximation range.

6.4 Output Feedback NN Control

In this Section, we design adaptive NN control with output feedback for both pure-feedback system (6.1) and NARMAX system (6.2) in a unified approach. In this Section, we still assume that the control directions are known.

6.4.1 From pure-feedback form to NARMAX form

First, it will be shown that system (6.1) under Assumptions 6.1, 6.2 and 6.3 is transformable to system (6.2) under Assumptions 6.4, 6.5 and 6.6. For convenience, we define

$$\underline{y}(k) = [y(k), y(k-1), \ldots, y(k-n+1)]^T \tag{6.34}$$

Let us rewrite the first equation of (6.1) as

$$\xi_1(k+1) - f_1(\xi_1(k), \xi_2(k)) = 0$$

According to Assumption 6.2, the derivative of the left hand side of the above equation over $\xi_2(k)$ is not zero, thus, according to Lemma 2.1, there exists an implicit function $p_2'(\cdot)$ asserted by Lemma 2.1 such that $\xi_2(k)$ can be seen as a function of $\xi_1(k+1)$ and $\xi_1(k)$ as follows

$$\xi_2(k) = p_2'(\xi_1(k+1), \xi_1(k)) := p_2(y(k+1), y(k)) \tag{6.35}$$

In the same manner, from the second equation of (6.1), we see that there exists an implicit function $p_3'(\cdot)$ of $\xi_2(k+1)$, $\xi_2(k)$ and $\xi_1(k)$ such that $\xi_3(k)$ can be expressed as

$$
\begin{aligned}
\xi_3(k) &= p_3'(\xi_2(k+1), \xi_2(k), \xi_1(k)) \\
&= p_3'(p_2(y_1(k+2), y(k+1)), p_2(y(k+1), y(k)), y(k)) \\
&:= p_3(y(k+2), y(k+1), y(k)) \tag{6.36}
\end{aligned}
$$

Continuing the same procedure, we can see that $\xi_i(k)$, $i = 2, 3, \cdots, n$, can be expressed as

$$
\begin{aligned}
\xi_i(k) &= p_i'(\xi_{i-1}(k+1), \xi_{i-1}(k), \xi_{i-2}(k), \cdots, \xi_1(k)) \\
&= p_i'(p_{i-1}(y(k+i-1), \cdots, y(k+1)), p_{i-1}(y(k+i-2), \cdots, y(k)), \\
&\quad p_{i-2}(y(k+i-3), \cdots, y(k)), \cdots, y(k)) \\
&:= p_i(y(k+i-1), y(k+i-2), \cdots, y(k)) \tag{6.37}
\end{aligned}
$$

where $p_i'(\cdot)$ is the implicit function asserted by Lemma 2.1 and $p_i(\cdot)$, $i = 2, 3, \ldots, n$, are defined recursively. Then, it is easy to derive a vector function only dependent on outputs to express $\bar{\xi}_i(k)$ as follows

$$
P_i(y(k+i-1), y(k+i-2), \cdots, y(k))
$$
$$
\overset{def}{=} \begin{bmatrix} y(k) \\ p_2(y(k+1), y(k)) \\ \vdots \\ p_i(y(k+i-1), y(k+i-2), \cdots, y(k)) \end{bmatrix}
$$

which leads to

$$
\begin{aligned}
\bar{\xi}_i(k) &= P_i(y(k+i-1), y(k+i-2), \cdots, y(k)) \\
i &= 1, 2, \ldots, n \tag{6.38}
\end{aligned}
$$

Now, let us rewrite the equations in system (6.1) as follows:

$$
\begin{cases}
\xi_1(k+n) = f_1(\bar{\xi}_1(k+n-1), \xi_2(k+n-1)) \\
\xi_2(k+n-1) = f_2(\bar{\xi}_2(k+n-2), \xi_3(k+n-2)) \\
\quad\quad\quad\vdots \\
\xi_{n-1}(k+2) = f_{n-1}(\bar{\xi}_{n-1}(k+1), \xi_n(k+1)) \\
\xi_n(k+1) = f_n(\bar{\xi}_n(k), u(k), d(k)) \\
y(k) = \xi_1(k)
\end{cases}
\tag{6.39}
$$

Combining with $\bar{\xi}_i(k+n-i) = P_i(\underline{y}(k+n-1))$ derived from (6.38), we obtain

$$
\begin{aligned}
\xi_i(k+n+1-i) &= f_i(P_i(\underline{y}(k+n-1), \xi_{i+1}(k+n-i)) \\
i &= 1, 2, \ldots, n-1 \\
\xi_n(k+1) &= f_n(P_n(\underline{y}(k+n-1), u(k), d(k))
\end{aligned}
$$

Then, let us substitute the $(i+1)$-th equation into the i-th equation, $i = 1, 2, \ldots, n-1$, so that we can obtain equation (7.17).

$$
\begin{aligned}
y(k+n) &= f_1(y(k+n-1), f_2(P_2(\underline{y}(k+n-1)), \xi_3(k+n-2))) \\
&= f_1(y(k+n-1), f_2(P_2(\underline{y}(k+n-1)), f_3(P_3(\underline{y}(k+n-1)), \xi_4(k+n-3)))) \\
&= f_1(y(k+n-1), f_2(P_2(\underline{y}(k+n-1)), f_3(P_3(\underline{y}(k+n-1)), \\
&\quad \ldots, f_{n-1}(P_{n-1}(\underline{y}(k+n-1)), f_n(P_n(\underline{y}(k+n-1)), u(k), d(k))) \ldots))) \\
&:= f(\underline{y}(k+n-1), u(k), d(k))
\end{aligned}
\tag{6.40}
$$

Using the chain rule of derivative, we will have

$$
\frac{\partial \psi_{2,1}(\cdot)}{\partial \xi_3(k+n-2)} = g_{1,1}(\cdot) g_{1,2}(\cdot) := g_{2,1}(\cdot)
\tag{6.41}
$$

Continuing to iteratively replace $\xi_j(k+n-j+1)$ in the above equations with the right hand side of the jth equation in (6.39), $j = 3, 4, \ldots, n-1$, we have

$$
\begin{aligned}
\xi_1(k+n) &= \psi_{j-1,1}(\bar{\xi}_1(k+n-1), \bar{\xi}_2(k+n-2), \cdots, \bar{\xi}_{j-1}(k+n-j), \\
&\quad f_j(\bar{\xi}_j(k+n-j), \xi_{j+1}(k+n-j))) \\
&:= \psi_{j,1}(\bar{\xi}_1(k+n-1), \bar{\xi}_2(k+n-2), \cdots, \bar{\xi}_j(k+n-j), \xi_{j+1}(k+n-j))
\end{aligned}
\tag{6.42}
$$

where $\psi_{j,1}(\cdot)$, $j = 3, 4, \ldots, n-1$, are defined recursively. Similarly, we have

$$
\frac{\partial \psi_{j,1}(\cdot)}{\partial \xi_{j+1}(k+n-j)} = g_{j-1,1}(\cdot) g_{1,j}(\cdot) := g_{j,1}(\cdot)
\tag{6.43}
$$

where $g_{j,1}(\cdot)$, $j = 3, 4, \ldots, n-1$, are also defined recursively. Continuing the substitution until control $u(k)$ appears on the right hand side of equation (6.42), we have

$$
\begin{aligned}
y(k+n) & = \psi_{n-1,1}(\bar{\xi}_1(k+n-1), \bar{\xi}_2(k+n-2), \\
& \quad \cdots, \bar{\xi}_{n-1}(k+1), f_n(\bar{\xi}_n(k), u(k), d(k))) \\
& := \psi_{n,1}(\bar{\xi}_1(k+n-1), \bar{\xi}_2(k+n-2), \cdots, \bar{\xi}_n(k), u(k), d(k)) \qquad (6.44)
\end{aligned}
$$

In the same manner, we have

$$
\frac{\partial \psi_{n,1}(\cdot)}{\partial u(k)} = g_{n-1,1}(\cdot)g_{1,n}(\cdot) := g_{n,1}(\cdot) \qquad (6.45)
$$

From the definition of vector functions $P_i(\cdot)$ in (6.38), equation (6.44) can be further written as

$$
\begin{aligned}
y(k+n) & = \psi_{n,1}(P_1(y(k+n-1)), P_2(y(k+n-1), \\
& \quad y(k+n-2)), \cdots, P_{n-1}(y(k+n-1), \\
& \quad \cdots, y(k+1)), P_n(y(k+n-1), \cdots, y(k)), u(k), d(k)) \\
& := f(y(k+n-1), y(k+n-2), \cdots, y(k), u(k), d(k)) \qquad (6.46)
\end{aligned}
$$

Accordingly, we have

$$
\frac{\partial f(\cdot)}{\partial u(k)} = g_{n,1}(\cdot) = \Pi_{i=1}^{n} g_{1,i}(\cdot) := g_o(\cdot), \underline{g} \leq |g(\cdot)| \leq \bar{g} \qquad (6.47)
$$

It is easy to check that

$$
\frac{\partial f(\cdot)}{\partial u(k)} = \Pi_{i=1}^{n} g_{1,i}(\cdot) := g(\cdot), \underline{g}' \leq |g(\cdot)| \leq \bar{g}' \qquad (6.48)
$$

According to Assumption 6.1, it is easy to show that the system function $f(\cdot)$ in (6.46) is continuous with respect to all the arguments and continuously differentiable with respect to $u(k)$.

Remark 6.5. *Assume that the output $y(k)$ is bounded, then according to (6.46), $u(k)$ must also be bounded because $\underline{g} \leq |g(\cdot)| \leq \bar{g}$. According to Lemma 2.6, the output boundedness guarantees the states boundedness for system (6.1). Then, it is easy to check that after transformation from original system (6.1) under Assumptions 6.1, 6.2 and 6.3, the transformed system (6.46) satisfies Assumptions 6.4, 6.5 and 6.6.*

At this stage, the pure-feedback system (6.1) is transformed to the NARMAX system (6.2) with $\tau = n$ and $m = 1$, and the control objective for both systems (6.1) and (6.2) becomes unified.

6.4.2 NARMAX systems transformation

The difficulty in controlling system (6.2) lies in the existence of future outputs $y(k+1), \ldots,$ $y(k + \tau - 1)$, which are not available at the current step. However, by carefully examining equation (6.2), it can be seen that control input $u(k)$ only affects future output $y(k + \tau)$ and those beyond, which means that the future outputs, $y(k + 1), \ldots, y(k + \tau - 1)$, are independent of $u(k)$. When the external disturbance $d(k)$ is ignored, the future outputs on the right hand side of equation (6.2) can be predicted at the current step.

Hence, let us consider applying output prediction approach in [135]. For convenience, we define

$$
\underline{y}(k) = [y(k), y(k-1), \ldots, y(k-n+1)]^T \tag{6.49}
$$

$$
\underline{u}(k) = [u(k), \ldots, u(k-n+2)]^T \tag{6.50}
$$

Moving back $(\tau - 1)$ steps in equation (6.2), we obtain

$$
\begin{aligned}
y(k+1) &= f(y(k), \cdots, y(k-n+1), u(k-\tau+1), \\
&\quad \cdots, u(k-m-\tau+2), d(k-\tau+1)) \\
&:= F_1(\underline{y}(k), u(k-\tau+1), \cdots, u(k-m-\tau+2), d(k-\tau+1)) \tag{6.51}
\end{aligned}
$$

It implies that the output $y(k + 1)$ is a SDFO according to Definition 2.6. Assuming that $\tau \geq 2$, by moving a step forward we obtain the following equation from (6.51)

$$
y(k+2) = F_1(\underline{y}(k+1), u(k-\tau+2), \cdots, u(k-m-\tau+3), d(k-\tau+2)) \tag{6.52}
$$

Substituting (6.51) into (6.53), we see that there exists a function $F_2(\cdot)$ such that

$$
y(k+2) = F_2(\underline{y}(k), u(k-\tau+2), \cdots, u(k-m-\tau+2), d(k-\tau+2), d(k-\tau+1)) \tag{6.53}
$$

which implies that $y(k + 2)$ is also a SDFO. Continuing the substituting recursively, it is easy to show $y(k + j)$, $j = 1, 2, \ldots, \tau - 1$, are all SDFOs, such that at the $(\tau - 1)$th step, we see that $y(k + \tau - 1)$ is a function of $\underline{y}(k)$, $\underline{u}(k-1)$ and $d(k-1), \cdots, d(k-n+1)$ as expressed below:

$$
y(k+n-1) = F_{\tau-1}(\underline{y}(k), \underline{u}(k-1), d(k-1), \cdots, d(k-n+1)) \tag{6.54}
$$

Moving one step ahead in equation (6.54), we see that there must exist a function $F_\tau(\cdot)$ such that

$$y(k + \tau) = F_\tau(\underline{z}(k), u(k), \underline{d}(k)) \tag{6.55}$$

where

$$\begin{align} \underline{z}(k) &= [\underline{y}^T(k), \underline{u}^T(k-1)]^T \tag{6.56} \\ \underline{d}(k) &= [d(k), d(k-1), \ldots, d(k-\tau+1)]^T \tag{6.57} \end{align}$$

if $\tau + m > 2$ and if $\tau + m = 2$, $\underline{z}(k) = \underline{y}(k)$. It can be easily shown that function $F_\tau(\cdot)$ is continuous and continuously differentiable with respect to $u(k)$ according to Assumption 6.4. Rewrite system (6.55) as

$$y(k + \tau) = \phi_o(\underline{z}(k), u(k)) + d_o(k) \tag{6.58}$$

where

$$\begin{align} \phi_o(\underline{z}(k), u(k)) &= F_\tau(\underline{z}(k), u(k), \mathbf{0}_{[\tau]}) \\ d_o(k) &= F_\tau(\underline{z}(k), u(k), \underline{d}(k)) - F_\tau(\underline{z}(k), u(k), \mathbf{0}_{[\tau]}) \tag{6.59} \end{align}$$

Note that $F_\tau(\cdot)$ is obtained by iteratively substitution of system function $f(\cdot)$ which satisfies Lipschitz condition in Assumption 6.6. According to Assumption 6.3, there exist a constant L_m such that

$$\begin{align} |d_o(k)| &= |F_n(\underline{z}(k), u(k), \underline{d}(k)) - F_n(\underline{z}(k), u(k), \mathbf{0}_{[n]})| \\ &\leq L_m|d(k)| + L_m|d(k-1)| + \ldots + L_m|d(k-n+1)| \\ &\leq nL_m\bar{d} := \bar{d}_o \tag{6.60} \end{align}$$

6.4.3 Adaptive NN control design

The dynamics of the tracking error $e(k) = y(k) - y_d(k)$ is given by

$$e(k + n) = \phi_o(\underline{z}(k), u(k)) - y_d(k + n) + d_o(k) \tag{6.61}$$

It is trivial to show that

$$\frac{\partial(\phi_o(\underline{z}(k), u(k)) - y_d(k + n))}{\partial u(k)} > 0$$

Therefore, there exists an ideal control input $u_o^*(\bar{z}(k))$ satisfying that

$$\phi_o(\underline{z}(k), u_o^*(\bar{z}(k))) - y_d(k+n) = 0 \tag{6.62}$$

$$\bar{z}(k) = [\underline{z}^T(k), y_d(k+n)]^T \in \Omega_{\bar{z}} \subset R^{2n} \tag{6.63}$$

where $\Omega_{\bar{z}}$ is a compact set corresponding to Ω_ξ and Ω_{yd}. Using the ideal control $u_o^*(\bar{z}(k))$, we will have $e(k) = 0$ after n steps if $d_o(k) = 0$. It implies that the ideal control $u_o^*(\bar{z}(k))$ is an n-step deadbeat control. According to Lemma 2.6, the ideal control $u_o^*(\bar{z}(k))$ is bounded.

As mentioned in Section 2.2, there exist an ideal NN weights vector $W_o^* \in R^{l_o}$, such that $u_o^*(\bar{z}(k))$ can be approximated by HONN as follows

$$
\begin{aligned}
u_{nn}^*(\bar{z}(k)) &= W_o^{*T} S(\bar{z}(k)), \quad S(\bar{z}(k)) \in R^{l_o} \\
u_o^*(\bar{z}(k)) &= u_{nn}^*(\bar{z}(k)) + \mu(\bar{z}(k)), \quad \forall \bar{z} \in \Omega_{\bar{z}}
\end{aligned}
\tag{6.64}
$$

where $\mu(\bar{z}(k))$ is the NN approximation error.

Remark 6.6. *Since system (6.58) is transformed from the original system (6.2), Assumption 6.6 still holds for (6.58). Considering that we input $u_o^*(\bar{z}(k))$ to system (6.58), then the output $y(k)$ catches up $y_d(k)$ in τ steps. This implies the boundedness of output $y(k)$ because the reference signal $y_d(k)$ is bounded. Then, from the BOBI property in Assumption 6.6, the boundedness of $u_o^*(\bar{z}(k))$ is guaranteed.*

Consider using a online adaptive HONN as to approximate $u_o^*(\bar{z}(k))$. Then, the output feedback adaptive NN control is given as

$$
\begin{aligned}
u(k) &= \frac{\eta_o(k)}{\bar{g}} e(k) + \hat{u}_o(k) \\
\hat{u}_o(k) &= \hat{W}_o(k) S(\bar{z}(k))
\end{aligned}
\tag{6.65}
$$

where $|\eta_o(k)| \le \bar{\eta}_o < 1$ is a scaling parameter to be specified and the NN weights vector is updated by the following adaptation law

$$
\begin{aligned}
\hat{W}_o(k+1) &= \hat{W}_o(k_1) - \gamma_o S(\bar{z}(k_1)) e(k+1) - \sigma_o \hat{W}_o(k_1) \\
k_1 &= k - n + 1
\end{aligned}
\tag{6.66}
$$

where $0 < \sigma_o < 1$ and $\gamma_o > 0$ are NN tuning parameters to be chosen.

Theorem 6.2. *Consider the adaptive closed-loop system consisting of the system (6.1), adaptive NN control (6.65) and NN adaptation law (6.66). Under Assumptions 6.1, 6.2 and 6.3, and with design parameters $0 < \sigma_o < 1$, $0 < \bar{\eta}_o < 1$ and γ_o satisfying*

$$2\gamma_o \bar{g} l_o + \eta_o \bar{g} + \bar{\eta}_o < 1 \tag{6.67}$$

all the closed-loop signals are SGUUB and the tracking error and NN weight estimation error will eventually be bounded as

$$\lim_{k\to\infty}\sup\{|e(k)|^2 + \frac{\bar{g}}{\gamma_o}\|\tilde{W}_o(k)\|^2\} \leq \frac{\bar{b}}{1-\bar{c}}$$

where

$$
\begin{aligned}
\bar{b} &= \frac{\bar{g}}{\eta_o}\mu_o^{*2} + 2\frac{\bar{g}}{\gamma_o}\sigma_o\|W_o^*\|^2 \\
\bar{c} &= \max\{\bar{\eta}_o, (1-2\sigma_o)\} \\
\mu_o^* &= \mu^* + \frac{\bar{d}_o}{\underline{g}}
\end{aligned}
\tag{6.68}
$$

and μ^* is the NN approximation error bound defined in (2.10).

Proof.   It is similar to the proof of Theorem 6.1 and is thus omitted. ∎

6.5   Simulation Studies I

For conciseness, simulation studies for state feedback and output feedback adaptive NN control are only carried out for systems in pure-feedback form, while for systems in NARMAX form simulation studied will be conducted in Section 6.7. To demonstrate the effectiveness of the proposed NN control, the following continuous stirred tank reactor (CSTR) system in [179] is used for simulation.

$$
\begin{cases}
\dot{x}_1 = -x_1 + D_a(1-x_1)e^{\frac{x_2}{1+\frac{x_2}{\gamma}}} \\
\dot{x}_2 = -x_2 + BD_a(1-x_1)e^{\frac{x_2}{1+\frac{x_2}{\gamma}}} - \beta(x_2 - u) + d \\
y = x_1
\end{cases}
\tag{6.69}
$$

where x_1 is the concentration and x_2 is the temperature, $B = 21.5$, $\gamma = 28.5$, $D_a = 0.036$, and $\beta = 25.2$ are scalar parameters [179], and $d = \cos(t)\cos(\xi_1)$ is unmeasured disturbance. It is noted that in system (6.69), the state variable x_2 appear in nonaffine appearance. The control objective is to make the output y track a smooth reference signal y_d, which is generated by passing a discontinuous set-point step signal r with amplitude 0.4 ± 0.2 into the following linear model:

$$\frac{y_d(s)}{r(s)} = \frac{\omega_n^2}{s^2 + 2\zeta_n\omega_n s + \omega_n^2}\tag{6.70}$$

where the natural frequency $\omega_n = 5.0$ and the damping ration $\zeta_n = 1.0$.

Denoting $\xi_1 = x_1$ and $\xi_2 = x_2$ and by using first order Taylor expansion, the CSTR system (6.69) can be approximated by a discrete-time model as

$$\begin{cases} \xi_1(k+1) = f_1(\xi_1(k), \xi_2(k)) \\ \xi_2(k+1) = f_2(\xi_1(k), \xi_2(k), u(k)) + d(k) \\ y(k) = \xi_1(k) \end{cases} \tag{6.71}$$

where

$$\begin{aligned} f_1(\cdot) &= \xi_1(k) + [-\xi_1(k) + D_a(1 - \xi_1(k))e^{\frac{\xi_2(k)}{1 + \frac{\xi_2(k)}{\gamma}}}]T \\ f_2(\cdot) &= \xi_2(k) + [-\xi_2(k) + BD_a(1 - \xi_1(k))e^{\frac{\xi_2(k)}{1 + \frac{\xi_2(k)}{\gamma}}} - \beta(\xi_2(k) - u(k))]T \end{aligned}$$

with sampling period $T = 0.05$ and $d(k) = 0.05\cos(0.05k)\cos(\xi_1(k))$.

For system (6.71), it is obvious that Assumption 6.1 holds. Assumptions 6.2 and 6.3 are not strictly satisfied, but it is seen in the simulation results that practically the proposed controls still work well. Consider an operation range $0.02 < \xi_1(k) < 0.8$ and $0 < \xi_2(k) < 5$. It is easy to check that $0.18 < g_{1,1}(\cdot) < 0.13$ and $g_{2,1}(\cdot) = 1.26$ and the partial directives $\frac{\partial f_1}{\partial \xi_1}$, $\frac{\partial f_2}{\partial \xi_1}$ and $\frac{\partial f_2}{\partial \xi_2}$ are upper bounded in the operation range. In this operation range, we have $\bar{g} = 0.17$ such that $g_{1,1}g_{2,1} < \bar{g}$.

It should be noted that the discretized model (6.71) is only used for analysis. The simulation is carried out on original system (6.69).

State feedback control

The NN employed in the controller is constructed according to equations (2.8) and (2.9) with $l_s = 18$ NN nodes. The parameters in the control law are chosen as $\gamma_s = 0.1$, $\sigma_s = 0.01$ and $\bar{\eta}_s = 0.04$ according to the criteria (6.22). The gain $\eta_s(k)$ is simply chosen as a constant $\eta_s(k) = \bar{\eta}_s$.

In the simulation, the initial states are $\bar{\xi}_2(0) = [0.1, \ 0.1]^T$, and for the initial weights vector $\hat{W}_s(j) \in R^{l_s}$, $j = -1, 0$, each element is selected as a standard uniform distributed random number divided by 10. The results are presented in Figures 6.1(a) and 6.2(a). Figure 6.1(a) shows the output $y(k)$ and the reference signal $y_d(k)$. Figure 6.2(a) illustrates the boundedness of the control input $u(k)$ and the norm of NN weights. It can be seen that all the signals are bounded in the operation range.

Output feedback control

The NN is constructed according to equations (2.8) and (2.9) with $l_s = 30$ NN nodes. The initial system states are $\bar{\xi}_2(0) = [0.1, \ 0.1]^T$. The initial NN weights vector $\hat{W}_o(j)$, $j = -1, 0$, is chosen in the same manner as that for state feedback control design. The

design parameters are chosen as $\bar{\eta}_o = 0.04$, $\gamma_o = 0.07$ and $\sigma_o = 0.01$, which satisfy the criterion in (6.67) The simulation results are presented in Figures 6.1(b), and 6.2(b). Figure 6.1(b) shows the output $y(k)$ and the reference signal $y_d(k)$. Figure 6.2(b) illustrates the boundedness of the control input $u(k)$ and the norm of NN weight $\|\hat{W}_o(k)\|$.

NN learning performance

Let us define the following mean square error (MSE) as a measurement of NN learning performance

$$e_s(k) = \frac{1}{k} \sum_{k'=1}^{k} [\phi_s(\bar{\xi}_n(k'), \hat{u}_s(z(k'))) - y_d(k'+n)]^2$$

$$e_o(k) = \frac{1}{k} \sum_{k'=1}^{k} [\phi_o(\underline{z}(k'), \hat{u}_o(\bar{z}(k'))) - y_d(k'+n)]^2 \tag{6.72}$$

According to (6.18) and (6.62), the smaller the NN approximation error $\hat{u}_s(k) - u_s^*(k)$ and $\hat{u}_o(k) - u_o^*(k)$ are, the smaller $e_s(k)$ and $e_o(k)$ are. If $\hat{u}_s(k) - u_s^*(k) = 0$ and $\hat{u}_o(k) - u_o^*(k) = 0$, we have $e_s(k) = 0$ and $e_o(k) = 0$.

The values of $\phi_s(\cdot)$ and $\phi_o(\cdot)$ are not available from the real plant but they can be obtained in the simulation. The mean square errors of state feedback and output feedback NN learning are demonstrated in Figure 6.3(a) and 6.3(b). It is noted that the NN learning performance is satisfactory, i.e., the defined mean square errors $e_s(k)$ and $e_o(k)$ are made to be bounded around zero.

It is obvious that all the signals in the closed-loop system are bounded in the operation range as seen from the simulation results above. From Figure 6.1, we see that the transient tracking performance is not very good. However, as the simulation time increases, the output tracking becomes much better. This is because the initial NN weights vector is set randomly and after a period of online learning, the NN is able to well approximate the unknown function.

Comparison with PID control To demonstrate the superiority over PID control, we compare the proposed output feedback NN control (6.65) with a standard PID control. In the simulation, the system initial condition is set to be $\bar{\xi}_2(0) = [0.1, \ 0.1]^T$ and the PID control is given in discretized manner as

$$\begin{aligned} u(k) &= u(k-1) + K_p[e(k) - e(k-1)] \\ &\quad + K_i e(k) + K_d[e(k) - 2e(k-1) + e(k-2)] \end{aligned}$$

where the parameters $K_P = 4$, $K_I = -0.2$ and $K_D = 1$ were found by trial and error to minimize the sum of squared output tracking errors.

The proposed output feedback adaptive NN control is further compared with the linear error observer based NN inverse control constructed in [141], which is a continuous-time design for nonaffine system. The system initial condition is also set to be $\bar{\xi}_2(0) = [0.1,\ 0.1]^T$. The dynamic compensator parameters used in the control are set to be $A_c = -0.86$, $B_c = -1.4$, $C_c = 0.1$ and $D_c = -0.75$. HONN with 45 neurons is used with the same initial condition as that for our proposed output feedback control. The design parameters are $\gamma_W = 35$, $Q_2 = I$, $\lambda_W = 0.01$, $\lambda_\Phi = 0.01$ and $\gamma_\Phi = 0.001$. The poles of the observer have been set to be five times faster than those of the closed-loop error system [141].

The comparison results are shown in Figure 6.4, where it is very clear that the two NN based controls perform much better than the PID control with respect to either tracking error or control effort, though NN based controls response not as quick as PID control in the initial steps. This is because the two NN controls are based on online NN learning. From the tracking performance of the two NN based controls in Figure 6.4(a), it is seen that the inverse NN control has an obvious steady state error while the steady state error for our proposed output-feedback adaptive NN control is very small.

6.6 Unknown Control Direction Case

In this Section, we assume that the control directions of systems in (6.1) and (6.2) are unknown. We will carry out adaptive NN control based on the transformed systems (6.58) in Section 6.4.2. Consider that the signs of control gains are unknown, from the derivation of $F_\tau(\cdot)$ we see that

$$\frac{\partial \phi_o(\underline{z}(k), u(k))}{\partial u(k)} = \frac{\partial F_\tau(\cdot)}{\partial u(k)} = \frac{\partial f(\cdot)}{\partial u(k)} = g(\cdot) \neq 0$$

The dynamics of the tracking error $e(k) = y(k) - y_d(k)$ is given by

$$e(k+\tau) = \phi_o(\underline{z}(k), u(k)) - y_d(k+n) + d_o(k) \tag{6.73}$$

It is easy to show that
$$\frac{\partial(\phi_o(\underline{z}(k), u(k)) - y_d(k+n))}{\partial u(k)} \neq 0$$

Then, similar as in Section 6.4.2, Lemma 2.1 asserts the existence of an ideal control input $u^*(\bar{z}(k))$ such that

$$\phi_o(\underline{z}(k), u^*(\bar{z}(k))) - y_d(k+n) = 0$$
$$\bar{z}(k) = [\underline{z}^T(k), y_d(k+n)]^T \tag{6.74}$$

In addition, there exists an ideal constant weights vector $W^* \in R^l$, such that

$$
\begin{aligned}
u_{nn}^*(\bar{z}(k)) &= W^{*T}S(\bar{z}(k)), \quad S(\bar{z}(k)) \in R^l \\
u^*(\bar{z}(k)) &= u_{nn}^*(\bar{z}(k)) + \mu(\bar{z}(k)), \quad \forall \bar{z} \in \Omega_{\bar{z}}
\end{aligned}
\tag{6.75}
$$

where $\mu(\bar{z}(k))$ is the NN approximation error and $\Omega_{\bar{z}}$ is a sufficient large compact set.

Using HONN as an approximator of $u^*(\bar{z}(k))$ and then, the output feedback adaptive NN control is given as

$$
u(k) = \hat{W}^T(k)S(\bar{z}(k))
\tag{6.76}
$$

Remark 6.7. *For ease of technical derivation for incorporation of discrete Nussbaum gain, the scaling tracking error term is not considered in the NN control (6.76). It will be noted later that the deadzone method instead of σ-modification will be used in the NN weights update law to dead with NN approximation error and external disturbance.*

Adding and subtracting $\phi_o(\bar{z}(k), u^*(\bar{z}(k)))$ on the right hand side of (6.73) leads to

$$
\begin{aligned}
e(k+\tau) &= \phi_o(\underline{z}(k), u(k)) - \phi_o(\underline{z}(k), u^*(\bar{z}(k))) + d_o(k) \\
&= g(\underline{z}(k), u^c(k))(u(k) - u^*(\bar{z}(k))) + d_o(k)
\end{aligned}
\tag{6.77}
$$

where

$$
g(\underline{z}(k), u^c(k)) = \frac{\partial \phi_o(\underline{z}(k), u^c(k))}{\partial u^c(k)}
$$

with $u^c(k) \in [\min\{u^*(\bar{z}(k)), u(k)\}, \max\{u^*(\bar{z}(k)), u(k)\}]$ according to the mean value theorem. For convenience, let us introduce the following notations:

$$
g(k) = g(\underline{z}(k), u^c(k)), S(k) = S(\bar{z}(k)), \mu(k) = \mu(\bar{z}(k))
\tag{6.78}
$$

and it is obvious that $\underline{g} \leq g(k) \leq \bar{g}$. Substituting (6.75) into (6.77) and noting that $\tilde{W}(k) = \hat{W}(k) - W^*$, we obtain

$$
e(k+\tau) = g(k)\tilde{W}^T(k)S(k) + d^*(k)
\tag{6.79}
$$

where

$$
d^*(k) = -g(k)\mu(k) + d_o(k)
$$

and it is trivial to show that $|d^*(k)| \leq \bar{g}\mu^* + \bar{d}_0 := d_0^*$.

Consider the following adaptation law for the NN weights:

$$
\begin{aligned}
e'(k) &= \gamma \frac{e(k)}{G(k)} \\
\hat{W}(k) &= \hat{W}(k-\tau) - \gamma N(x(k))S(k-\tau)\frac{a(k)e'(k)}{D(k)} \\
\Delta x(k) &= x(k+1) - x(k) = \frac{a(k)G(k)e'^2(k)(k)}{D(k)}, \quad x(0) = 0 \\
G(k) &= 1 + |N(x(k))| \\
D(k) &= 1 + \|S(k-\tau)\|^2 + |N(x(k))| + e'^2(k) \\
a(k) &= \begin{cases} 1 & \text{if } |e'(k)| > \chi \\ 0 & \text{others} \end{cases} \\
\hat{W}(j) &= \mathbf{0}_{[l]}, \quad j = -\tau + 1, \ldots, 0
\end{aligned}
\tag{6.80}
$$

where $N(x(k))$ is the discrete Nussbaum gain defined in Section 4.2, the tuning rate $\gamma > 0$ and deadzeon threshold $\chi > 0$ can be an arbitrary positive constants to be specified by the designer. It should be mentioned that the requirement on the sequence $x(k)$ in (4.2) is satisfied and furthermore, $\Delta x(k) \geq 0$.

Theorem 6.3. *Consider the adaptive closed-loop system consisting of system (6.1) under Assumptions 6.1, 6.2 and 6.3 or system (6.2) under Assumptions 6.4, 6.5 and 6.6, NN control (6.76) with NN weights adaptation law (6.80). All the signals in the closed-loop system are SGUUB and the discrete Nussbaum gain $N(x(k))$ will converge to a constant ultimately. Denote $C = \lim_{k\to\infty} G(k)$, then the tracking error satisfies $\lim_{k\to\infty} \sup\{|e(k)|\} < \frac{C\chi}{\gamma}$, where the tuning rate $\gamma > 0$ and the threshold value $\chi > 0$ can be arbitrary constants to be specified by the designer.*

Proof. Similar to the proof of Theorem 6.1, the proof is carried out in two parts. First, we assume that inputs $u(k)$ and outputs $y(k)$ are within the NN approximation range $\Omega_{\bar{z}}$ while later we will show that if initially the NN approximation range covers this set then the inputs and outputs are guaranteed to be within $\Omega_{\bar{z}}$ without a priori assumption in the first step. From (6.79), we have

$$
\gamma \tilde{W}^T(k-\tau)S(k-\tau) = \frac{1}{g(k-\tau)}G(k)e'(k) - \frac{1}{g(k-\tau)}\gamma d^*(k-\tau)
\tag{6.81}
$$

Choose a positive definite function $V(k)$ as

$$
V(k) = \sum_{j=1}^{\tau} \tilde{W}^T(k-\tau+j)\tilde{W}(k-\tau+j)
\tag{6.82}
$$

and note that

$$\frac{2a(k)}{g(k-\tau)}N(x(k))d^*(k-\tau)e'(k) \leq a(k)|\frac{2d_0^*}{\underline{g}\chi}|G(k)e'^2(k) \tag{6.83}$$

and $a^2(k) = a(k)$. Using (6.81), the first difference equation of $V(k)$ can be written as

$$
\begin{aligned}
\Delta V(k) &= V(k) - V(k-1) \\
&= \tilde{W}^T(k)\tilde{W}(k) - \tilde{W}^T(k-\tau)\tilde{W}(k-\tau) \\
&= (\tilde{W}(k) - \tilde{W}(k-\tau))^T(\tilde{W}(k) - \tilde{W}(k-\tau)) + 2\tilde{W}^T(k-\tau)(\tilde{W}(k) - \tilde{W}(k-\tau)) \\
&= \frac{\gamma^2 a^2(k)N^2(x(k))S^T(k-\tau)S(k-\tau)}{D^2(k)}e'^2(k) \\
&\quad -2N(x(k))\frac{a(k)\gamma\tilde{W}^T(k-\tau)S(k-\tau)}{D(k)}e'(k) \\
&\leq \gamma^2 \frac{a(k)G(k)e'^2(k)}{D(k)} + |\frac{2d_0^*}{\underline{g}\chi}|\frac{a(k)G(k)e'^2(k)}{D(k)} \\
&\quad -\frac{2}{g(k-\tau)}N(x(k))\frac{a(k)G(k)e'^2(k)}{D(k)} \tag{6.84}
\end{aligned}
$$

Denote $N'(x(k)) = \frac{1}{g(k-\tau)}N(x(k))$ and then, noting $\frac{1}{\bar{g}} \leq \frac{1}{g(k-\tau)} \leq \frac{1}{\underline{g}}$ and according to Lemma 6.3, we can see that $N'(x(k))$ is still a discrete Nussbaum gain. Taking summation on both hand sides of (6.84) and noting $0 \leq \Delta x(k) \leq 1$, Note that

$$\frac{a(k)G(k)e'^2(k)}{D(k)} = \Delta x(k)$$

Then, by denoting $N'(x(k)) = \frac{1}{g(k-\tau)}N(x(k))$, we have

$$\Delta V(k) \leq c_1\Delta x(k) - 2N'(x(k))\Delta x(k) \tag{6.85}$$

where $c_1 = \gamma^2 + |\frac{2d_0^*}{\underline{g}\chi}|$. The following inequality follows immediately

$$V(k) \leq -2\sum_{k'=0}^{k} N'(x(k'))\Delta x(k') + c_1 x(k) + c_1 \tag{6.86}$$

Applying Lemma 4.1 to (6.86) results in the boundedness of $V(k)$ and $x(k)$. Noting the definition of $V(k)$, we obtain the boundedness of $\hat{W}(k)$ immediately. From the definition of $N(x(k))$, it is seen that $|N(x(k))| = |x_s(k)|$. Thus, the boundedness of $x(k)$ implies the boundedness of $N(x(k))$ and $G(k) = 1 + |N(x(k))|$. In (6.80), we see that $\Delta x(k) = \frac{G(k)e'^2(k)}{D(k)} \geq 0$ and therefore $x(k)$ is a nondecreasing sequence. Thus, the boundedness of $x(k)$ results in

$$\lim_{k\to\infty}\Delta x(k) = \lim_{k\to\infty}\frac{a(k)G(k)e'^2(k)}{D(k)} = 0 \tag{6.87}$$

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Applying the similar techniques in Section 4.4.3, we see $G(k)$ will converge to a constant C. Then, it can be derived from the definition of $e'(k)$ in (6.80) that

$$\lim_{k \to \infty} \sup\{|e(k)|\} \leq \frac{C\chi}{\gamma}$$

Then, the boundedness of output $y(k)$ is obvious. According to Remark 6.5, the boundedness of control $u(k)$ and states of system (6.1) is guaranteed. So far, we have proved that given any initial condition $\bar{z}(0) \in \Omega_0$, there is a corresponding bounding compact set $\Omega_{\bar{z}}$ so that $\bar{z}(k) \in \Omega_{\bar{z}}$, $\forall k$, if the NN approximation range is initialized to cover $\Omega_{\bar{z}}$.

Next, let us consider that the initial condition Ω_0 and control parameters to be chosen are known at the beginning. It implies the bounding set $\Omega_{\bar{z}}$ is determined. Then, if initially the NN approximation range Ω is constructed to cover the bounding set $\Omega_{\bar{z}}$, the boundedness of all the closed-loop signals is guaranteed. According to Definition 2.11 (given any initial condition, there is a corresponding control such that the all the closed-loop signals are bounded), the proposed adaptive NN control achieves SGUUB stability. This completes the proof. ■

6.7 Simulation Studies II

In this section, simulation is carried out with the following NARMAX system studied in [92].

$$y(k+1) \;\; = \;\; \frac{y(k)y(k-1)[y(k)+2.5])}{1+y^2(k)+y^2(k-1)} + gu(k-2) + d(k) \qquad (6.88)$$

where the control gain is chosen to be $g = \pm 1.25$ and the disturbance is

$$d(k) = 0.1\cos(0.05k)\cos(\xi_1(k))$$

The reference trajectory is chosen as

$$y_d(k) = \frac{1}{2}\sin(\frac{\pi}{5}kT) + \frac{1}{2}\cos(\frac{\pi}{10}kT), T = 0.05$$

The initial condition is $y(-1) = y(-2) = y(0) = 0.1$. The tuning rate and the threshold value are chosen as $\gamma = 0.9$ and $\chi = 0.02$. The simulation results for $g = -1.25$ are presented in Figures 6.5, 6.6 and 6.7. Figure 6.5 shows the reference signal $y_d(k)$ and system output $y(k)$. Figure 6.6 illustrates the boundedness of the control input $u(k)$ and the NN weights vector estimate $\hat{W}(k)$. Figure 6.7 shows the discrete sequence $x(k)$ and discrete Nussbaum gain $N(x(k))$.

When $g = -1.25$, we have $g(\cdot) = \frac{\partial f(\cdot)}{\partial u(k)} = g < 0$. Therefore, it is seen in Figure 6.7 that the Nussbaum gain changes to be negative after step 150 and remains to be so. Accordingly, the output and the control signal go to a wrong direction at initial stage as shown in Figures 6.5 and 6.6. After the discrete Nussbaum gain turns to be negative, the output tracking performance improves to be much better. Next, let us change $g = -1.25$ to $g = 1.25$. The simulation results by the same control law and NN weights adaptation law are shown in Figures 6.8, 6.9 and 6.10.

It is noted in Figure 6.10 that $N(x(k))$ always keeps positive while in Figure 6.7 it turns to negative and remains so. This is because $N'(x(k))$ must turn to be positive to make $\Delta V(k)$ negative.

NN learning performance

To demonstrate the NN learning performance, we define the following NN learning error

$$e_{nn}(k) = \phi_o(\underline{z}(k), u(\bar{z}(k))) - y_d(k + n) \tag{6.89}$$

as measurement of NN learning performance. According to (6.74) and (6.75), the better the NN approximation is (the smaller the NN approximation error $u(k) - u^*(k)$ is), the smaller $e_{nn}(k)$ is. If $u(k) - u^*(k) = 0$, we have $e_{nn}(k) = 0$

The NN learning errors are demonstrated in Figure 6.11. It is noted that the NN learning performance is satisfactory, i.e., the defined NN learning error $e_{nn}(k)$ is ultimately bounded in a neighborhood of zero.

6.8 Summary

In this Chapter, it has been shown that under certain conditions, nonlinear discrete-time pure-feedback systems are transformable to a class of inverse stable NARMAX system, and the output-feedback adaptive NN control design for both systems can be synthesized in a unified framework. By prediction approach, the original NARMAX system is transformed to a suitable form to avoid noncausal problem in the control design. Implicit Function Theorem has been exploited to identify the existence of an ideal deadbeat control, while HONN has been used to approximate the ideal control and discrete Nussbaum gain has been further studied to handle the lack of knowledge on control gain. The resulted adaptive NN control guarantees the SGUUB of all the closed-loop signals.
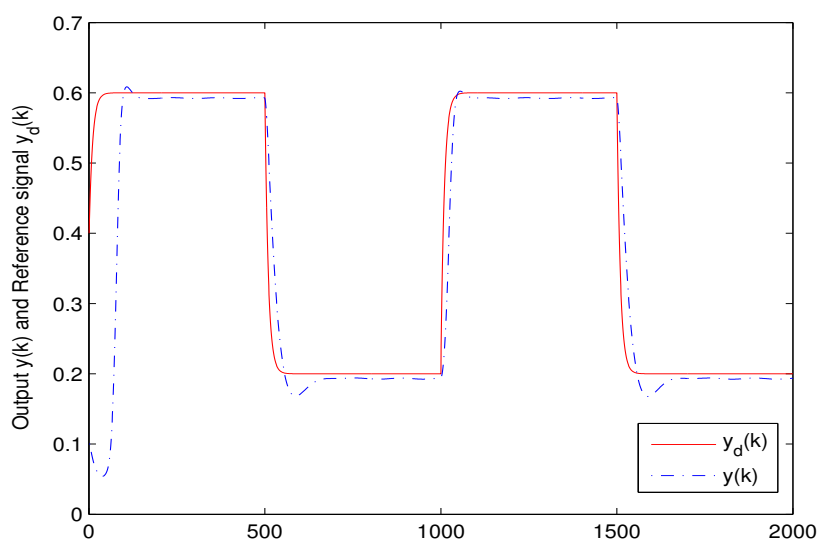
(a) State feedback NN control
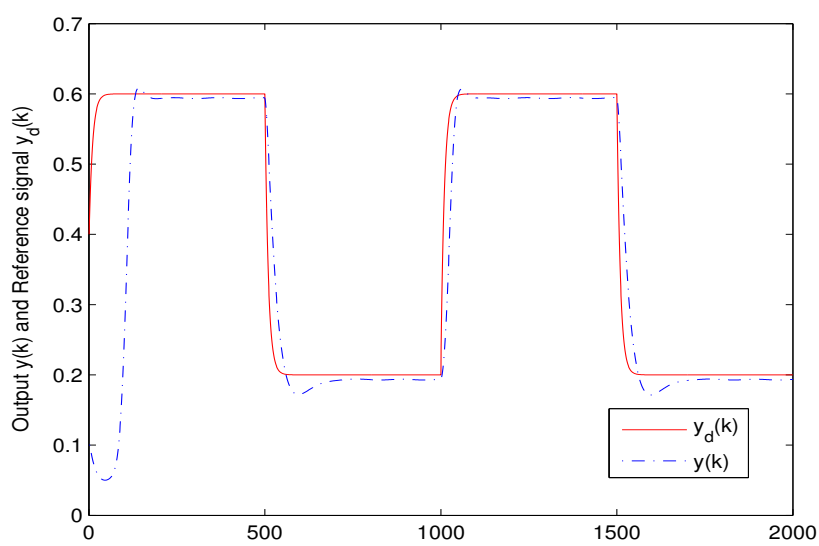
(b) Output feedback NN control

Figure 6.1: System output and reference trajectory

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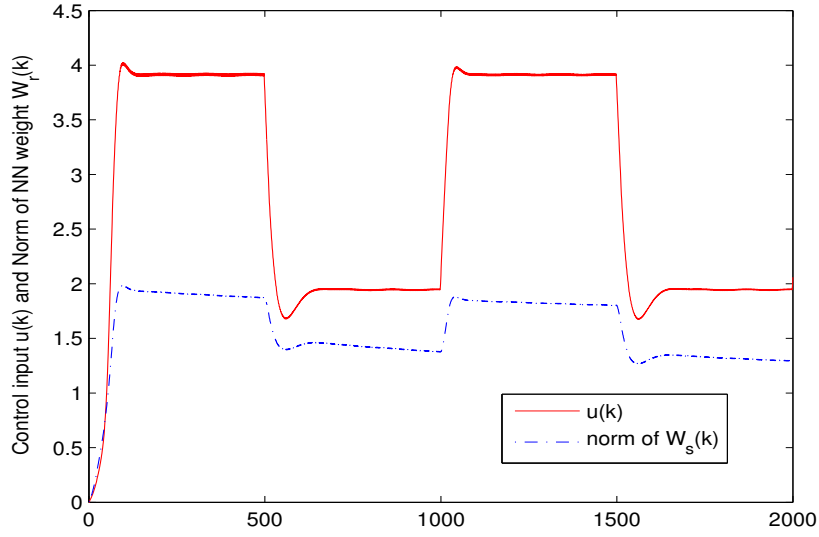
(a) State feedback NN control
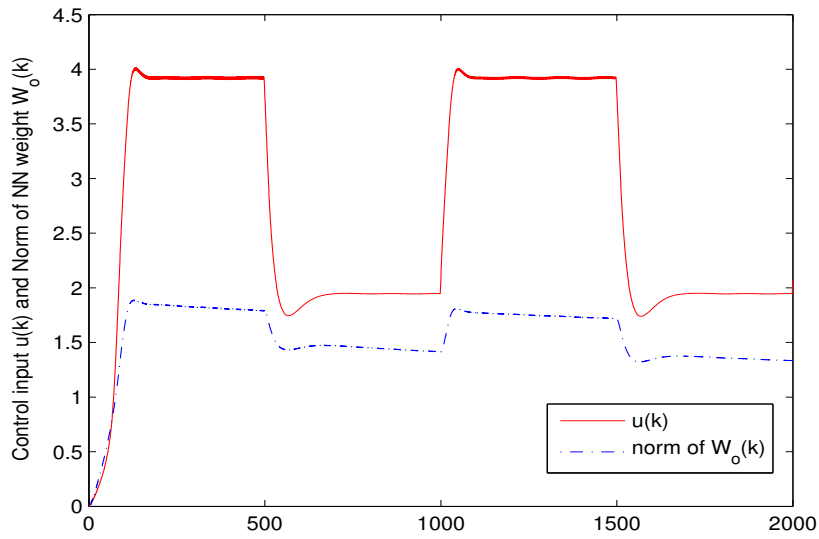
(b) Output feedback NN control

Figure 6.2: Boundedness of control signal and NN weights

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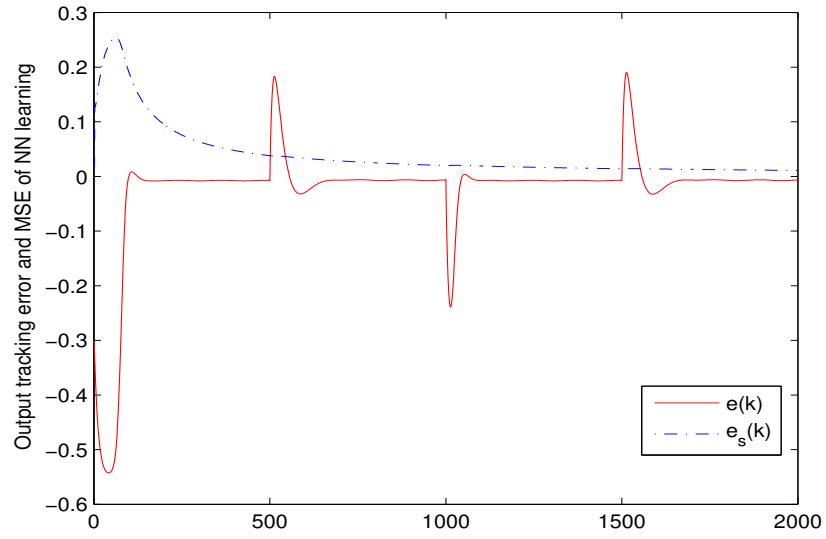
(a) State feedback NN control
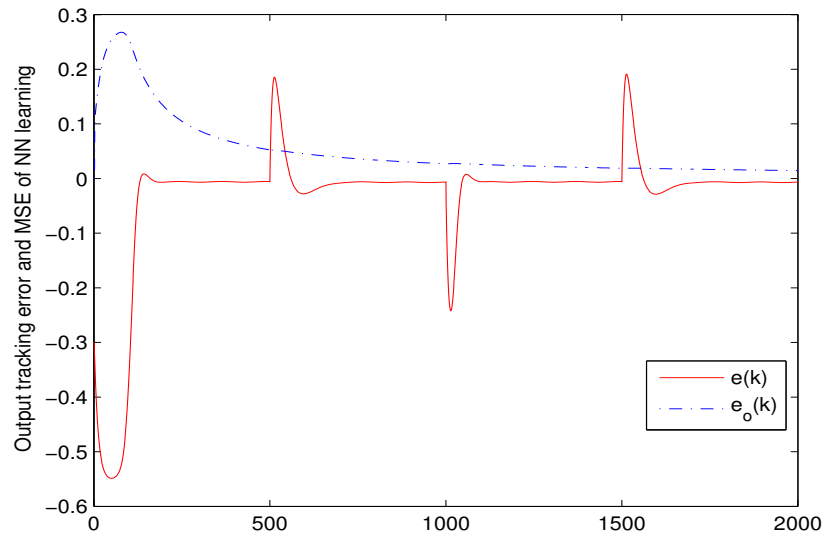
(b) Output feedback NN control

Figure 6.3: Output tracking error and MSE of NN learning
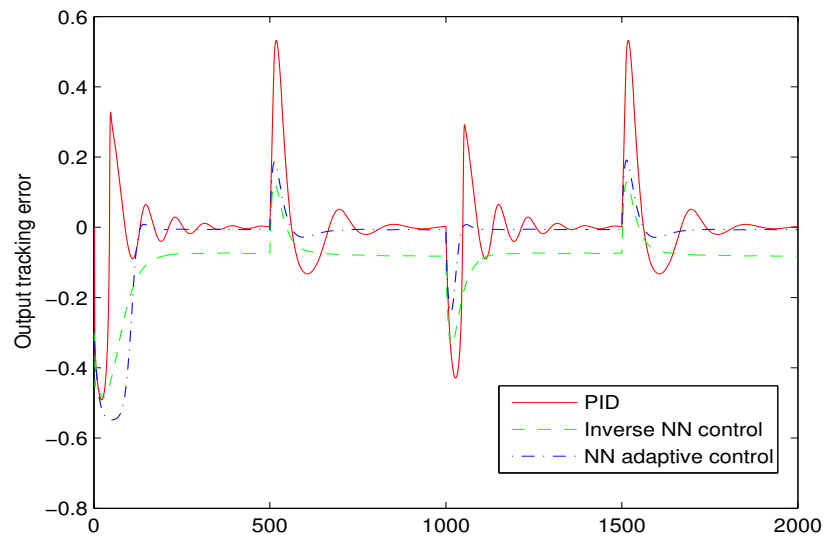
(a) Comparison on tracking errors
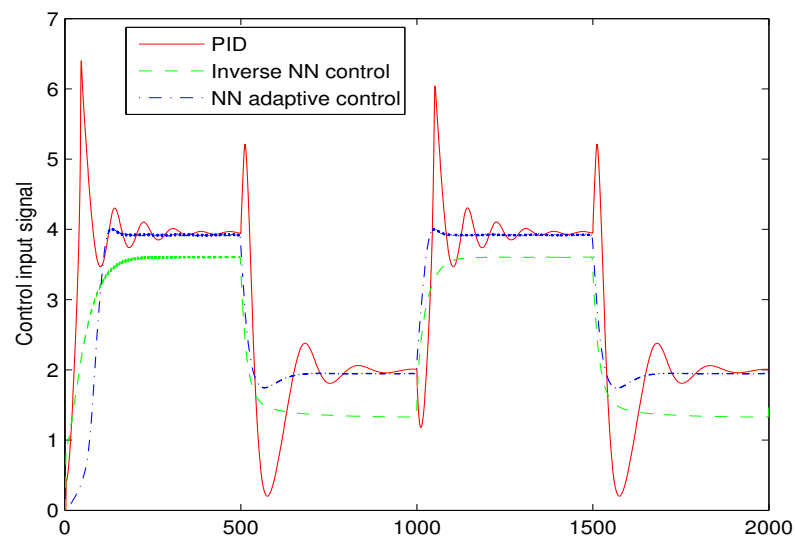
(b) Comparison on control signals

Figure 6.4: Comparison of PID, NN Inverse and adaptive NN control

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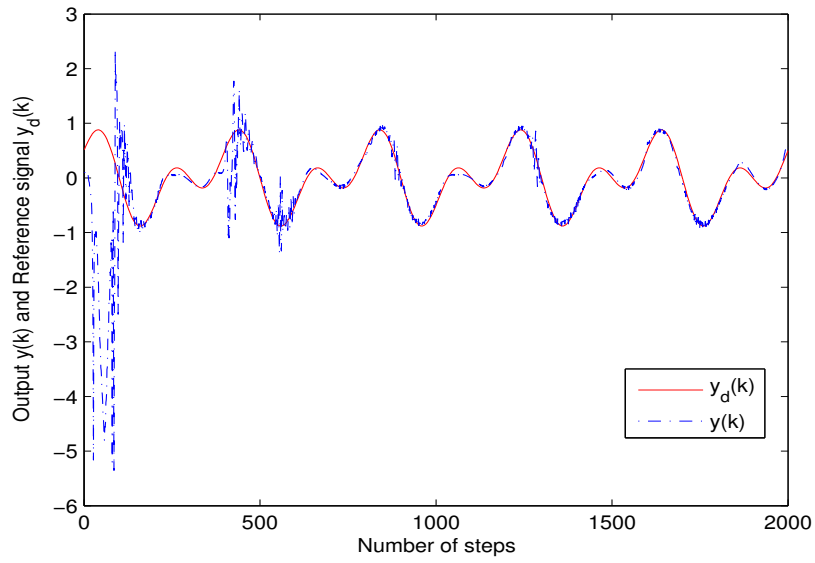
Figure 6.5: Reference signal and system output
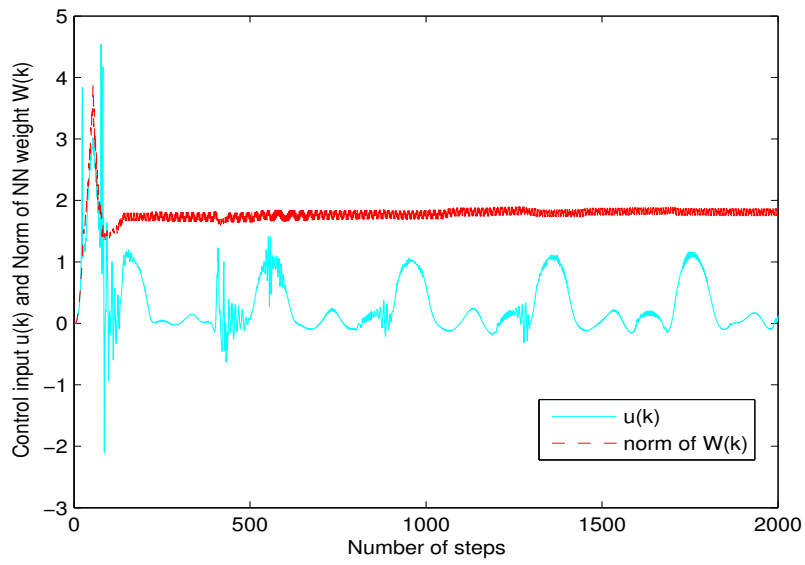
Figure 6.6: Control signal and NN weights norm

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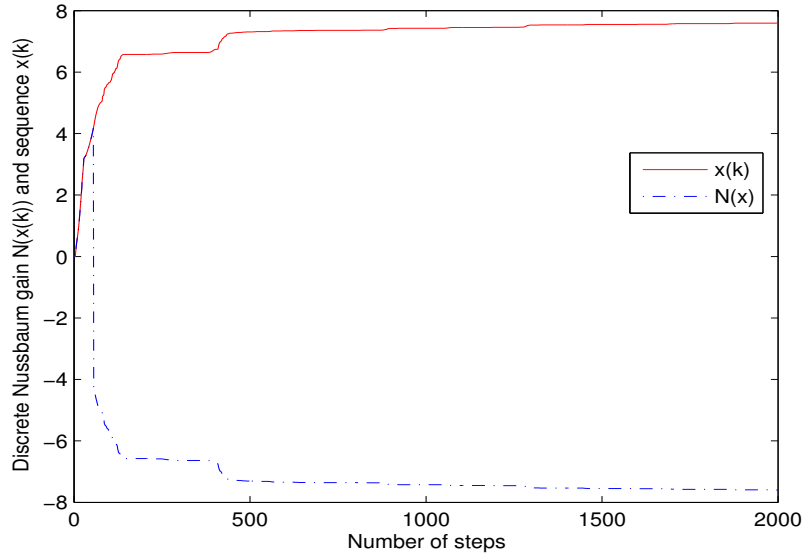
Figure 6.7: Discrete Nussbaum gain $N(x(k))$ and its argument $x(k)$
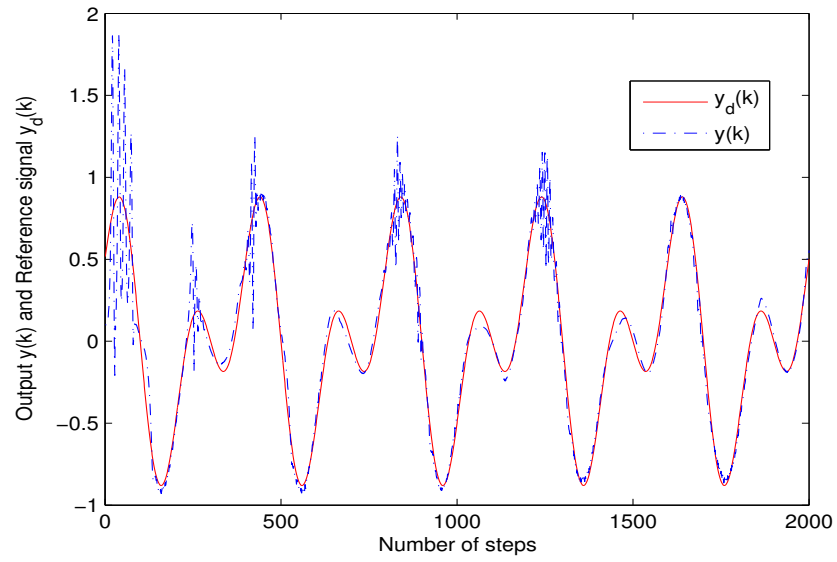
Figure 6.8: Reference signal and system output
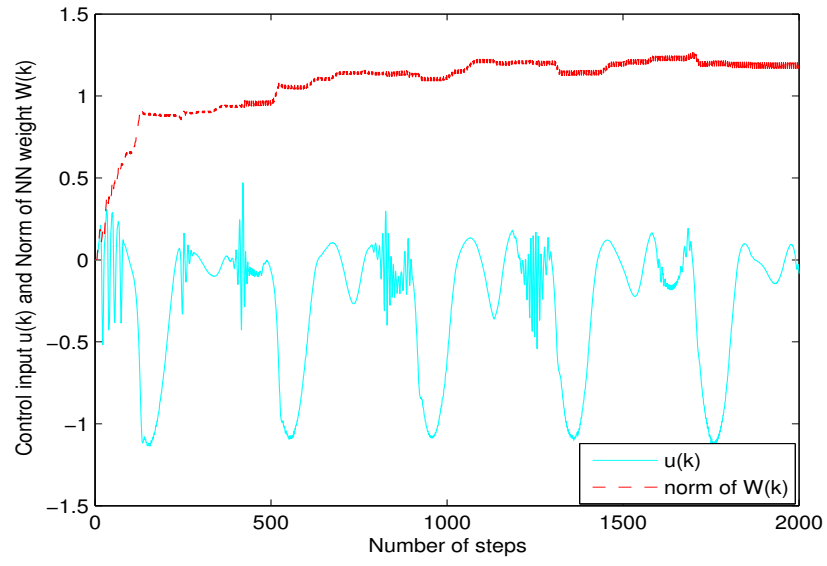
Figure 6.9: Control signal and NN weights norm
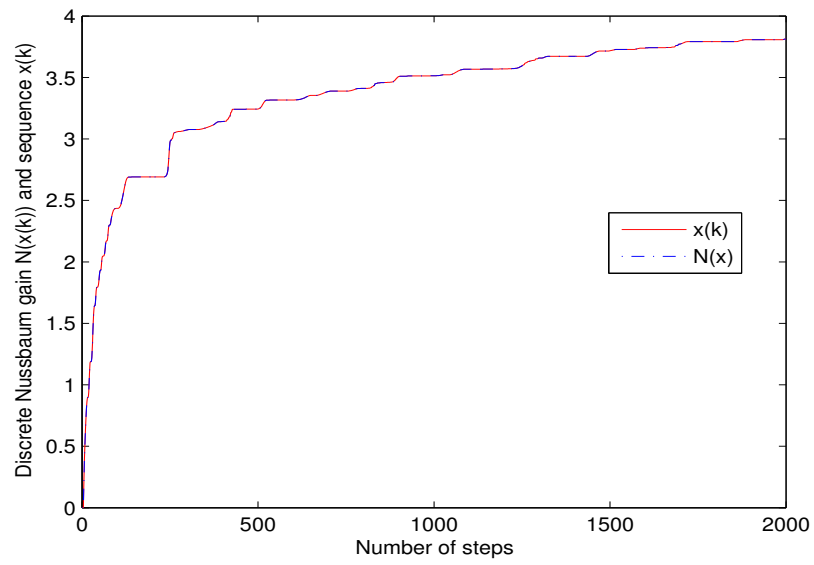
Figure 6.10: Discrete Nussbaum gain

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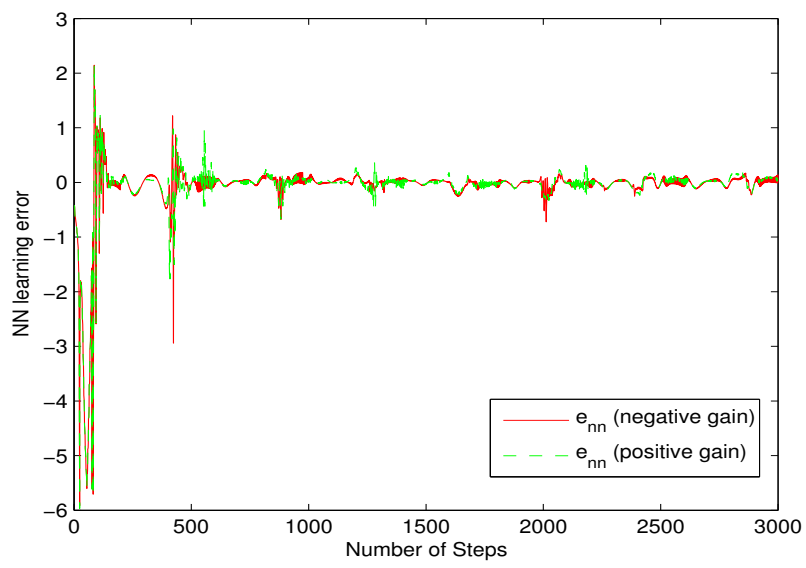
Figure 6.11: NN learning error

Chapter 7

MIMO Nonaffine systems

7.1 Introduction

In Chapter 6, implicit function theorem and discrete Nussbaum gain have been exploited to solve the nonaffine problem and unknown control direction problem, in order to facilitate adaptive NN control of nonlinear SISO systems in nonaffine form. In this Chapter, we are going to further investigate adaptive NN control of MIMO nonlinear system in nonaffine form.

A series of excellent research work has been carried out in [132, 133, 156] for block-triangular discrete-time MIMO nonlinear systems with subsystems in normal form and of same order. LPNN and MNN are employed for control design in [132] and [133], respectively. Cerebellar Model Articulation NN is investigated in [156]. In [157, 158, 180], adaptive NN control is investigated for block-triangular discrete-time MIMO systems with strict-feedback subsystems. To deal with uncertain couplings of both states and inputs, the nice properties of the block-triangular structure have been well exploited in the adaptive NN control design. In [157] and [180], the backstepping adaptive NN control design developed for SISO systems in [51] has been extended to MIMO systems using state feedback design. Furthermore, output-feedback adaptive NN control design has been performed in [158]. But these results are limited for affine systems. In addition, it is noted that in the output feedback design [158] all the subsystems are required to be of same order and the couplings only appear in the last equations of each subsystem.

On the other hand, MIMO systems in NARMAX form have also received much research attention. Affine NARMAX systems have been studied in [159], in which it is pointed out that it is generally very hard to construct the NN weights update law due to the

couplings and thus an orthogonal matrix is assumed to be available for ease of construction of the update law. Nonaffine NARMAX systems have been studied by many researchers using linearization based method. In [138], based on the NN identified model, a novel linearization method at each step was proposed to deal with the difficulty of nonaffine input and the method was further used in [181] to construct an inverse NN control. But the linearization based method requires an NN identified system model beforehand. Lyapunov based adaptive NN control has been studied in [137] by representing the nonaffine systems by a linear part plus a nonlinear part and using NN to design a control for compensation of the nonlinear part.

In this Chapter, nonaffine block-triangular MIMO systems will be studied in Section 7.2. The block-triangular MIMO systems to be studied are of interconnections in every equation of each subsystem rather than only in the last equation of each subsystem [157, 158]. In addition, the assumption of equal subsystem orders for output feedback design [158] is not imposed. The block-triangular systems in this Section actually cover the systems (5.18) of LIPs form studied in Section 5.3. Adaptive NN control of systems of similar structure in continuous-time have been studied using state feedback in [154], where it is indicated that the complicated interactions make it impossible to conclude the stability of the whole system by stability analysis of individual subsystem separately. In Section 7.3, nonaffine NARMAX MIMO systems will be studied. In Section 7.3, we design adaptive NN control using implicit function theory and extend the work in [135, 146] by introducing discrete Nussbaum gain into the NN weights update law to relax the stringent assumption on control gain matrix.

The contributions in this Chapter lies in

(i) By rearranging the subsystems of the block-triangular MIMO system according to their orders in the control design, the assumption of equal orders of each subsystem for output feedback design in [158] has been removed.

(ii) Each subsystem in the block-triangular MIMO system has been transformed into a input-output model despite the presence of interactions among each subsystems.

(iii) The restriction on the control gain matrix of MIMO NARMAX system for NN weight tuning assumed in [159] has been relaxed by exploring discrete Nussbaum gain in the NN weight update law.

7.2 Nonlinear MIMO Block-Triangular Systems

7.2.1 Problem formulation

Consider the following MIMO discrete-time system with each subsystem in the nonaffine pure-feedback form

$$
\Sigma : \begin{cases}
\Sigma_1 \begin{cases}
\xi_{1,i_1}(k+1) = f_{1,i_1}(\bar{\xi}_{1,i_1-m_{11}}(k), \bar{\xi}_{2,i_1-m_{12}}(k), \ldots, \bar{\xi}_{n,i_1-m_{1n}}(k), \\
\qquad \xi_{1,i_1+1}(k)), \ i_1 = 1, 2, \ldots, n_1 - 1 \\
\xi_{1,n_1}(k+1) = f_{1,n_1}(\Xi(k), u_1(k), d_1(k)) \\
y_1(k) = \xi_{1,1}(k)
\end{cases} \\
\vdots \\
\Sigma_j \begin{cases}
\xi_{j,i_j}(k+1) = f_{j,i_j}(\bar{\xi}_{1,i_j-m_{j1}}(k), \bar{\xi}_{2,i_j-m_{j2}}(k), \ldots, \bar{\xi}_{n,i_j-m_{jn}}(k), \\
\qquad \xi_{j,i_j+1}(k)), \ i_j = 1, 2, \ldots, n_j - 1 \\
\xi_{j,n_j}(k+1) = f_{j,n_j}(\Xi(k), \bar{u}_j(k), d_j(k)) \\
y_j(k) = \xi_{j,1}(k)
\end{cases} \\
\vdots \\
\Sigma_n \begin{cases}
\xi_{n,i_n}(k+1) = f_{n,i_n}(\bar{\xi}_{1,i_n-m_{n1}}(k), \bar{\xi}_{2,i_n-m_{n2}}(k), \ldots, \bar{\xi}_{n,i_n-m_{nn}}(k), \\
\qquad \xi_{n,i_n+1}(k)), \ i_n = 1, 2, \ldots, n_n - 1 \\
\xi_{n,n_n}(k+1) = f_{n,n_n}(\Xi(k), \bar{u}_n(k), d_n(k)) \\
y_n(k) = \xi_{n,1}(k)
\end{cases}
\end{cases}
\tag{7.1}
$$

where the notations used are defined as same as those in Section 5.3. It is assumed that the external disturbance $d_j(k)$ is bounded by an unknown constant $\bar{d}_j$, i.e. $|d_j(k)| \leq \bar{d}_j$.

Assumption 7.1. *Functions $f_{j,i_j}(\cdot,\cdot)$ and $f_{j,n_j}(\cdot,\cdot,0)$, $j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, n_j - 1$, in (7.1) are continuous with respect to all the arguments and continuously differentiable with respect to the second argument.*

It should be mentioned that systems described in (7.1) are more general than and cover the block-triangular systems in LIPs form studied in Section 5.3. Moveover, in this Section we assume that the control directions are unknown, i.e., the control gains $g_{j,i_j}(\cdot)$ defined in Definition 2.10 are strictly either positive or negative, but their signs are unknown.

Assumption 7.2. *There exist constants $\bar{g}_{j,i_j} > \underline{g}_{j,i_j} > 0$ so that $0 \leq \underline{g}_{j,i_j} \leq |g_{j,i_j}(\cdot)| \leq \bar{g}_{j,i_j}$, where the control gains are defined in Definition 2.10.*

For the convenience, we introduce the notations $\underline{g}_j = \prod\limits_{i_j=1}^{n_j} \underline{g}_{j,i_j}$ and $\bar{g}_j = \prod\limits_{i_j=1}^{n_j} \bar{g}_{j,i_j}$.

Assumption 7.3. *The system functions $f_{j,i_j}(\cdot,0)$ and $f_{j,n_j}(\cdot,0,\cdot)$ are Lipschitz functions.*

7.2.2 Transformation of pure-feedback systems

To facilitate control design, in this Section we will perform system transformation and will show that each subsystem Σ_j can be transformed into an input-output model similar to that for the SISO pure-feedback system in Section 6.4. It should be mentioned that the transformation procedure in Section 6.4 for SISO pure-feedback systems cannot be applied straightforwardly to MIMO system as there are states interactions in every equation of each subsystem.

Consider the definition of largest subsystem order $\bar{n}$ and set s_i in Definition 2.9. In the following, we perform system transformation for (7.1) in the sequence according the orders of subsystems.

Transformation to State-Output Model

Step 1: Consider all the subsystems $\Sigma_{l_{1,t_1}}$ with $l_{1,t_1} \in s_1$. Because only states from subsystems $\Sigma_{l_{1,t_1}}$ appear in the first equations, the first equations of $\Sigma_{l_{1,t_1}}$ can be rewritten as

$$\xi_{l_{1,t_1},1}(k+1) - f_{l_{1,t_1},1}(\xi_{l_{1,1},1}(k),\ldots,\xi_{l_{1,m_1},1}(k),\xi_{l_{1,t_1},2}(k)) = 0, \ t_1 = 1,\ldots m_1 \qquad (7.2)$$

According to Assumption 7.2, the derivative of left hand side of the above equation over $\xi_{l_{1,t_1},2}(k)$ is not zero, so it is asserted by Lemma 2.1 that there exists an implicit function $p_{l_{1,t_1},2}(\cdot)$ such that

$$
\begin{aligned}
\xi_{l_{1,t_1},2}(k) &= p_{l_{1,t_1},2}(\xi_{l_{1,t_1},1}(k+1),\xi_{l_{1,1},1}(k),\ldots,\xi_{l_{1,m_1},1}(k)) \\
&= p_{l_{1,t_1},2}(y_{l_{1,t_1}}(k+1),y_{l_{1,1}}(k),\ldots,y_{l_{1,m_1}}(k)), \ t_1 = 1,\ldots m_1 \qquad (7.3)
\end{aligned}
$$

Step 2: Consider all the subsystems $\Sigma_{l_{1,t_1}}$ with $l_{1,t_1} \in s_1$, $t_1 = 1,\ldots m_1$ and $\Sigma_{l_{2,t_2}}$ with $l_{2,t_2} \in s_2$, $t_2 = 1,\ldots m_2$.

substep 1: In the similar way as (7.3) is derived, from the second equations of subsystems $\Sigma_{l_{1,t_1}}$, $\xi_{l_{1,t_1},3}(k)$ can be expressed as

$$
\begin{aligned}
\xi_{l_{1,t_1},3}(k) &= p'_{l_{1,t_1},3}(\xi_{l_{1,t_1},2}(k+1),\bar{\xi}_{l_{1,1},2}(k),\ldots,\bar{\xi}_{l_{1,m_1},2}(k),\xi_{l_{2,1},1}(k),\ldots,\xi_{l_{2,m_2},1}(k)) \\
t_1 &= 1,\ldots m_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.4)
\end{aligned}
$$

where the existence of the implicit functions $p'_{l_{1,t_1},3}(\cdot)$ is asserted by Lemma 2.1. By substituting (7.3) into (7.4), we obtain a new function $p_{l_{1,t_1},3}(\cdot)$ such that

$$
\begin{aligned}
\xi_{l_{1,t_1},3}(k) = \ & p_{l_{1,t_1},3}(y_{l_{1,t_1}}(k+2),y_{l_{1,1}}(k+1),\ldots,y_{l_{1,m_1}}(k+1),y_{l_{1,1}}(k),\ldots,y_{l_{1,m_1}}(k), \\
& y_{l_{2,1}}(k),\ldots,y_{l_{2,m_2}}(k)), \ t_1 = 1,\ldots m_1 \qquad\qquad\qquad\qquad (7.5)
\end{aligned}
$$

substep 2: Similar to Step 1, we can rewrite the first equations of subsystems $\Sigma_{l_{2,t_2}}$ as

$$\xi_{l_{2,t_2},2}(k) = p'_{l_{2,t_2},2}(\xi_{l_{2,t_2},1}(k+1), \bar{\xi}_{l_{1,1},2}(k), \dots, \bar{\xi}_{l_{1,m_1},2}(k), \xi_{l_{2,1},1}(k), \dots, \xi_{l_{2,m_2},1}(k))$$
$$t_2 = 1, \dots m_2 \tag{7.6}$$

where the existence of an implicit function $p'_{l_{2,t_2},2}(\cdot)$ is asserted by Lemma 2.1. We substitute (7.3) into (7.6) and obtain a new function $p_{l_{2,t_2},2}(\cdot)$ such that

$$\xi_{l_{2,t_2},2}(k) = p_{l_{2,t_2},2}(y_{l_{2,t_2}}(k+1), y_{l_{1,1}}(k+1), \dots, y_{l_{1,m_1}}(k+1), y_{l_{1,1}}(k), \dots, y_{l_{1,m_1}}(k),$$
$$y_{l_{2,1}}(k), \dots, y_{l_{2,m_2}}(k)), \ t_2 = 1, \dots m_2 \tag{7.7}$$

Step r $(3 \le r \le \bar{n} - 1)$: Consider all the subsystems $\Sigma_{l_{1,t_1}}$ with $l_{1,t_1} \in s_1, \ t = 1, \dots m_1$, $\Sigma_{l_{2,t_2}}$ with $l_{2,t_2} \in s_2, \ t = 1, \dots m_2$, until $\Sigma_{l_{r,t_r}} \ l_{r,t_r} \in s_r, \ t_r = 1, \dots m_r$.

substep 1: From the rth equations of subsystems $\Sigma_{l_{1,t_1}}$, $\xi_{l_{1,t_1},r+1}(k)$ can be expressed as

$$\xi_{l_{1,t_1},r+1}(k) = p'_{l_{1,t_1},r+1}(\xi_{l_{1,t_1},r}(k+1), \bar{\xi}_{l_{1,1},r}(k), \dots, \bar{\xi}_{l_{1,m_1},r}(k),$$
$$\dots, \xi_{l_{r,1},1}(k), \dots, \xi_{l_{r,m_r},1}(k)), t_1 = 1, \dots m_1 \tag{7.8}$$

where the existence of the implicit functions $p'_{l_{1,t_1},r+1}(\cdot)$ is asserted by Lemma 2.1. By substituting the results in the first $(r-1)$ steps into (7.8), we obtain a new function $p_{l_{1,t_1},r+1}(\cdot)$ such that

$$\xi_{l_{1,t_1},r+1}(k) = p_{l_{1,t_1},r+1}(y_{l_{1,t_1}}(k+r), y_{l_{1,1}}(k+r-1), \dots, y_{l_{1,m_1}}(k+r-1), \dots,$$
$$y_{l_{1,1}}(k), \dots, y_{l_{1,m_1}}(k), \dots, y_{l_{r,1}}(k), \dots, y_{l_{r,m_r}}(k)), \ t_1 = 1, \dots m_1 \tag{7.9}$$

substep q $(2 \le q \le r)$: We rewrite the $(r+1-q)$th equations of subsystems $\Sigma_{l_{q,t_q}}$, $t_q = 1, \dots m_q$ as

$$\xi_{l_{q,t},r+2-q}(k) = p'_{l_{q,t},r+2-q}(\xi_{l_{q,t},r+1-q}(k+1), \bar{\xi}_{l_{1,1},r}(k), \dots, \bar{\xi}_{l_{1,m_1},r}(k),$$
$$\dots, \xi_{l_{r,1},1}(k), \dots, \xi_{l_{r,m_r},1}(k)) \tag{7.10}$$

where the existence of the implicit function $p'_{l_{q,t},r+2-q}(\cdot)$ is asserted by Lemma 2.1. By substituting the results in the first $(r-1)$ steps into (7.10), we obtain a new function $p_{l_{q,t},r+2-q}(\cdot)$ such that

$$\xi_{l_{q,t_q},r+2-q}(k) = p_{l_{q,t_q},r+2-q}(y_{l_{q,t_q}}(k+r-q+1), y_{l_{1,1}}(k+r-1), \dots,$$
$$y_{l_{1,m_1}}(k+r-1), \dots, y_{l_{1,1}}(k), \dots, y_{l_{1,m_1}}(k), \dots, y_{l_{r,1}}(k), \dots, y_{l_{r,m_r}}(k))$$
$$t_q = 1, \dots m_q \tag{7.11}$$

Then, let us introduce a vector function $P_{l_{q,t_q},r+2-q}(\cdot)$

$$P_{l_{q,t_q},r+2-q}(y_{l_{q,t_q}}(k+r-q+1), y_{l_{1,1}}(k+r-1), \ldots, y_{l_{1,m_1}}(k+r-1), \ldots,$$

$$y_{l_{1,1}}(k), \ldots, y_{l_{1,m_1}}(k), \ldots, y_{l_{r,1}}(k), \ldots, y_{l_{r,m_r}}(k))$$

$$\overset{def}{=} \begin{bmatrix} y_{l_{q,t_q}}(k) \\ \vdots \\ p_{l_{q,t_q},r+2-q}(y_{l_{q,t_q}}(k+r-q+1), y_{l_{1,1}}(k+r-1), \ldots, y_{l_{1,m_1}}(k+r-1), \ldots, \\ y_{l_{1,1}}(k), \ldots, y_{l_{1,m_1}}(k), \ldots, y_{l_{r,1}}(k), \ldots, y_{l_{r,m_r}}(k)) \end{bmatrix}$$

for $1 \le r \le \bar{n}$, $1 \le t_q \le m_q$, $1 \le q \le r$, which leads to the following equation

$$\bar{\xi}_{l_{q,t_q},r+2-q}(k) = P_{l_{q,t_q},r+2-q}(y_{l_{q,t_q}}(k+r-q+1), y_{l_{1,1}}(k+r-1), \ldots,$$

$$y_{l_{1,m_1}}(k+r-1), \ldots, y_{l_{1,1}}(k), \ldots, y_{l_{1,m_1}}(k), \ldots, y_{l_{r,1}}(k), \ldots, y_{l_{r,m_r}}(k)) \quad (7.12)$$

The above equations reveal the relationship between system states and outputs, which will be used later in the next part.

Transformation to Input-Output Model

Let us consider rewriting the jth subsystem in system (7.1) as follows:

$$\begin{cases} \xi_{j,1}(k+n_j) = f_{j,1}(\bar{\xi}_{1,1-m_{j1}}(k+n_j-1), \bar{\xi}_{2,1-m_{j2}}(k+n_j-1), \ldots, \\ \qquad\qquad \bar{\xi}_{n,1-m_{jn}}(k+n_j-1), \xi_{j,2}(k+n_j-1)) \\ \xi_{j,2}(k+n_j-1) = f_{j,2}(\bar{\xi}_{1,2-m_{j1}}(k+n_j-2), \bar{\xi}_{2,2-m_{j2}}(k+n_j-2), \ldots, \\ \qquad\qquad \bar{\xi}_{n,2-m_{jn}}(k+n_j-2), \xi_{j,3}(k+n_j-2)) \\ \qquad\qquad\qquad \vdots \\ \xi_{j,n_j-1}(k+2) = f_{j,n_j-1}(\bar{\xi}_{1,n_j-1-m_{j1}}(k+1), \bar{\xi}_{2,n_j-1-m_{j2}}(k+1), \ldots, \\ \qquad\qquad \bar{\xi}_{n,n_j-1-m_{jn}}(k+1), \xi_{j,n_j}(k+1)) \\ \xi_{j,n_j}(k+1) = f_{j,n_j}(\Xi(k), \bar{u}_j(k), d_j(k)) \\ y_j(k) = \xi_{j,1}(k) \end{cases} \quad (7.13)$$

Replacing $\xi_{j,2}(k+n_j-1)$ in the first equation of (7.13) with the right hand side of the second equation yields

$$\xi_{j,1}(k+n_j) = f_{j,1}(\bar{\xi}_{1,1-m_{j1}}(k+n_j-1), \bar{\xi}_{2,1-m_{j2}}(k+n_j-1), \ldots,$$

$$\bar{\xi}_{n,1-m_{jn}}(k+n_j-1), f_{j,2}(\bar{\xi}_{1,2-m_{j1}}(k+n_j-2),$$

$$\bar{\xi}_{2,2-m_{j2}}(k+n_j-2), \bar{\xi}_{n,2-m_{jn}}(k+n_j-2), \ldots,$$

$$\xi_{j,3}(k+n_j-2))) \quad (7.14)$$

Then replacing $\xi_{j,3}(k + n_j - 2)$ in (7.14) with the right hand side of the third equation of (7.13) yields

$$
\begin{aligned}
\xi_{j,1}(k + n_j) \quad = \quad & f_{j,1}(\bar{\xi}_{1,1-m_{j1}}(k + n_j - 1), \bar{\xi}_{2,1-m_{j2}}(k + n_j - 1), \ldots, \\
& \bar{\xi}_{n,1-m_{jn}}(k + n_j - 1), f_{j,2}(\bar{\xi}_{1,2-m_{j1}}(k + n_j - 2), \\
& \bar{\xi}_{2,2-m_{j2}}(k + n_j - 2), \ldots, \bar{\xi}_{n,2-m_{jn}}(k + n_j - 2), \\
& f_{j,3}(\bar{\xi}_{1,3-m_{j1}}(k + n_j - 3), \bar{\xi}_{2,3-m_{j2}}(k + n_j - 3), \ldots, \\
& \bar{\xi}_{n,3-m_{jn}}(k + n_j - 3), \xi_{j,4}(k + n_j - 3))))
\end{aligned}
\tag{7.15}
$$

By continuing to replace $\xi_{j,i_j}(k + n_j - i_j + 1), i_j = 4, 5, \ldots, n_j$, iteratively, we have a function $F'_j(\cdot)$ such that

$$
\begin{aligned}
\xi_{j,1}(k + n_j) \quad = \quad & F'_j(\bar{\xi}_{1,1-m_{j1}}(k + n_j - 1), \bar{\xi}_{2,1-m_{j2}}(k + n_j - 1), \ldots, \\
& \bar{\xi}_{n,1-m_{jn}}(k + n_j - 1), \ldots, \bar{\xi}_{1,n_j-1-m_{j1}}(k + 1), \\
& \bar{\xi}_{2,n_j-1-m_{j2}}(k + 1), \ldots, \bar{\xi}_{n,n_j-1-m_{jn}}(k + 1), \\
& \Xi(k), \bar{u}_j(k), d_j(k))
\end{aligned}
\tag{7.16}
$$

By subtracting (7.12) into the above equations, we obtain a function $F_j(\cdot)$ such that

$$
y_j(k + n_j) \quad = \quad F_j(\underline{y}_1(k + n_1 - 1), \ldots, \underline{y}_n(k + n_n - 1), \bar{u}_j(k), d_j(k))
\tag{7.17}
$$

where

$$
\underline{y}_j(k) = [y_j(k), y_j(k - 1), \ldots, y_j(k - n_j + 1)]^T.
\tag{7.18}
$$

It is easy to check that

$$
\frac{\partial F_j(\cdot)}{\partial u_j(k)} = \prod_{i_j=1}^{n_j} g_{j,i_j}(\cdot) := g_j(\cdot), \quad \underline{g}_j \le |g_j(\cdot)| \le \bar{g}_j
\tag{7.19}
$$

According to Assumption 7.1, it is easy to find that the system function $F_j(\cdot)$ is continuous with respect to all the arguments and continuously differentiable with respect to $u_j(k)$.

So far, each subsystem Σ_j has been transformed into an input-output model, as indicated by (7.17). However, since there are future states on the right handside of (7.17), it is necessary to further transform (7.17) in order to design an causal control input $u_j(k)$. The future outputs prediction procedure is given in the next section.

Future Outputs Prediction

For the convenience of further analysis, we denote

$$
\begin{aligned}
Y_i(k) &= [\underline{y}_{l_{i,1}}(k), \ldots, \underline{y}_{l_{i,m_i}}(k)] \\
U_i(k) &= [\bar{u}_{l_{i,1}}(k), \ldots, \bar{u}_{l_{i,m_i}}(k)] \\
D_i(k) &= [d_{l_{i,1}}(k), \ldots, d_{l_{i,m_i}}(k)]
\end{aligned}
\tag{7.20}
$$

where $i = 1, 2, \ldots, \bar{n}$ and m_i is defined in Section 7.2.2. Then (7.17) can be rewritten as

$$
\begin{aligned}
y_{l_{i,t_i}}(k + \bar{n} - i + 1) &= F_{l_{i,t_i}}(Y_1(k + \bar{n} - 1), Y_2(k + \bar{n} - 2), \ldots, Y_{\bar{n}}(k), \\
&\quad \bar{u}_{l_{i,t_i}}(k + \bar{n} - i), d_{l_{i,t_i}}(k + \bar{n} - i))
\end{aligned}
\tag{7.21}
$$

Step 1: Consider all the subsystems $\Sigma_{l_{1,t_1}}$ with $l_{1,t_1} \in s_1$. Moving back $(\bar{n} - 1)$ steps in (7.21), we obtain

$$
y_{l_{1,t_1}}(k + 1) = F_{l_{1,t_1},1}(Y_1(k), \ldots, Y_{\bar{n}}(k - \bar{n} + 1), \bar{u}_{l_{1,t_1}}(k - \bar{n} + 1), d_{l_{1,t_1}}(k - \bar{n} + 1)) \tag{7.22}
$$

where $F_{l_{1,t_1},1} = F_{l_{1,t_1}}$.

Step 2: *substep 1:* Consider all the subsystems $\Sigma_{l_{1,t_1}}$ with $l_{1,t_1} \in s_1$ again. Moving one step forward in (7.22), we obtain

$$
\begin{aligned}
y_{l_{1,t_1}}(k + 2) &= F_{l_{1,t_1},1}(Y_1(k + 1), \ldots, \\
&\quad Y_{\bar{n}}(k - \bar{n} + 2), \bar{u}_{l_{1,t_1}}(k - \bar{n} + 2), d_{l_{1,t_1}}(k - \bar{n} + 2))
\end{aligned}
\tag{7.23}
$$

In (7.23), we note that the one step future outputs $Y_1(k + 1)$ are all from subsystems $\Sigma_{l_{1,t_1}}$ with $l_{1,t_1} \in s_1$. Substituting (7.22) into (7.23), we obtain a new function $F_{l_{1,t_1},2}$ such that

$$
\begin{aligned}
y_{l_{1,t_1}}(k + 2) &= F_{l_{1,t_1},2}(Y_1(k), Y_2(k), \ldots, Y_{\bar{n}}(k - \bar{n} + 2), U_{l_{1,t_1}}(k - \bar{n} + 1), \\
&\quad D_{l_{1,t_1}}(k - \bar{n} + 1), \bar{u}_{l_{1,t_1}}(k - \bar{n} + 2), d_{l_{1,t_1}}(k - \bar{n} + 2))
\end{aligned}
\tag{7.24}
$$

substep 2: Consider all the subsystems $\Sigma_{l_{2,t_2}}$ with $l_{2,t_2} \in s_2$. Moving back $(\bar{n} - 2)$ steps in (7.21), we obtain

$$
\begin{aligned}
y_{l_{2,t_2}}(k + 1) &= F_{l_{2,t_2},1}(Y_1(k + 1), \ldots, \\
&\quad Y_{\bar{n}}(k - \bar{n} + 2), \bar{u}_{l_{2,t_2}}(k - \bar{n} + 2), d_{l_{2,t_2}}(k - \bar{n} + 2))
\end{aligned}
\tag{7.25}
$$

where $F_{l_{2,t_2},1} = F_{l_{2,t_2}}$. Similarly as in substep 1, we substitute (7.22) into (7.25) and obtain a new function $F_{l_{2,t_2},2}$ such that

$$
\begin{aligned}
y_{l_{2,t_2}}(k + 1) &= F_{l_{2,t_2},2}(Y_1(k), Y_2(k), \ldots, Y_{\bar{n}}(k - \bar{n} + 2), U_{l_{1,t_1}}(k - \bar{n} + 1), \\
&\quad D_{l_{1,t_1}}(k - \bar{n} + 1), \bar{u}_{l_{2,t_2}}(k - \bar{n} + 2), d_{l_{2,t_2}}(k - \bar{n} + 2))
\end{aligned}
\tag{7.26}
$$

Continuing the procedure as above iteratively, at step $\bar{n}$ we obtain

$$y_{l_{i,t_i}}(k + n_{l_{i,t_i}}) = F_{l_{i,t_i}, n_{l_{i,t_i}}}(\underline{z}_{l_{i,t_i}}(k), u_{l_{i,t_i}}(k), \underline{d}_{l_{i,t_i}}(k)), \ i = 1, 2, \ldots, \bar{n} \tag{7.27}$$

where

$$
\begin{array}{rcl}
\underline{z}_{l_{i,t_i}}(k) & = & [Y_1(k), \ldots, Y_{\bar{n}}(k), \underline{U}_{l_1}(k-1), \ldots, \underline{U}_{l_{\bar{n}}}(k-1), \bar{u}_{l_{i,t_i}-1}(k)] \\
\underline{U}_{l_{i,t_i}}(k-1) & = & [U_{l_{i,t_i}}(k-1), \ldots, U_{l_{i,t_i}}(k - n_{l_{i,t_i}} + 1)] \\
\underline{d}_{l_{i,t_i}}(k) & = & [\underline{D}_{l_1}(k-1), \ldots, \underline{D}_{l_{\bar{n}}}(k-1), d_{l_{i,t_i}}(k)] \\
\underline{D}_{l_{i,t_i}}(k-1) & = & [D_{l_{i,t_i}}(k-1), \ldots, D_{l_{i,t_i}}(k - n_{l_{i,t_i}} + 1)]
\end{array}
\tag{7.28}
$$

Rewrite (7.27) as

$$y_j(k + n_j) = \phi_j(\underline{z}_j(k), u_j(k)) + d_j'(k) \tag{7.29}$$

where

$$
\begin{array}{rcl}
\phi_j(\underline{z}(k), u_j(k)) & = & F_{j,n_j}(\underline{z}_j(k), \bar{u}_j(k), \mathbf{0}_{[\mathbf{n_j}]}) \\
d_j'(k) & = & F_{j,n_j}(\underline{z}_j(k), \bar{u}_j(k), \underline{d}_j(k)) - F_{j,n_j}(\underline{z}_j(k), \bar{u}_j(k), \mathbf{0}_{[\mathbf{n_j}]})
\end{array}
\tag{7.30}
$$

Since $F_{j,n_j}(\cdot)$ is obtained by iteratively substitution of system function $f_{j,i_j}(\cdot)$ which satisfies Lipschitz condition in Assumption 7.3, the function $F_{j,n_j}(\cdot)$ still satisfies Lipschitz condition by Lemma 2.4. Therefore, there exists a finite constant $\bar{d}_j$ such that $|d_j'(k)| \leq \bar{d}_j$.

So far, since it has been shown that each subsystem Σ_j can be transformed into an input-output model without the future states, we are ready to consider the control design based on this model.

7.2.3 Adaptive NN control design

Let us extend the control design in Chapter 6 to the MIMO systems under study, in which which there are states and inputs couplings.

First, we consider the tracking error $e_j(k) = y_j(k) - y_{j,d}(k)$, which is given by

$$e_j(k + n_j) = \phi_j(\underline{z}_j(k), u_j(k)) - y_{j,d}(k + n_j) + d_j'(k) \tag{7.31}$$

It is easy to show that

$$\frac{\partial(\phi_j(\cdot) - y_{j,d}(k + n))}{\partial u_j(k)} = \frac{\partial F_j(\cdot)}{\partial u_j(k)} = g_j(\cdot) \neq 0 \tag{7.32}$$

Therefore, according to Lemma 2.1, there exists an ideal control input $u_j^*(z_j(k))$ such that

$$\phi_j(\underline{z}_j(k), u_j^*(z_j(k))) - y_{j,d}(k + n_j) = 0, \ z_j(k) = [\underline{z}_j(k), y_{j,d}(k + n_j)] \tag{7.33}$$

Using the ideal control $u_j^*(z_j(k))$, we have $e_j(k) = 0$ after n_j steps if $d_j'(k) = 0$. Consider using RBFNN to approximate the ideal control. As mentioned in Section 2.2, there exists an ideal constant weight vector $W_j^* \in R^{l_j}$ as follows

$$
\begin{aligned}
u_{j,nn}^*(z_j(k)) &= W_j^{*T} S_j(z_j(k)), \quad S_j(z_j(k)) \in R^{l_j} \\
u_j^*(z_j(k)) &= u_{j,nn}^*(z_j(k)) + \mu_j(z_j(k)), \forall z_j \in \Omega_{z_j}
\end{aligned} \tag{7.34}
$$

where $\mu_j(z_j(k))$ is the NN weight estimation error and Ω_{z_j} is a sufficiently large compact set.

Using RBFNN as an approximator of $u_j^*(z_j(k))$, the control is given as

$$u_j(k) = \hat{W}_j(k) S_j(z_j(k)) \tag{7.35}$$

Substituting $\phi_j(z_j(k), u_j^*(z_j(k)))$ into (7.31), we obtain

$$
\begin{aligned}
e_j(k + n_j) &= \phi_j(\underline{z}_j(k), u_j(k)) - \phi_j(\underline{z}_j(k), u_j^*(z_j(k))) + d_j'(k) \\
&= g_j(\underline{z}_j(k), u_j^c(k))(u_j(k) - u_j^*(z_j(k))) + d_j'(k)
\end{aligned} \tag{7.36}
$$

where

$$g_j(\underline{z}_j(k), u_j^c(k)) = \frac{\partial \phi_j(\underline{z}_j(k), u_j^c(k))}{\partial u_j^c(k)}$$

with $u_j^c(k) \in [\min\{u_j^*(z_j(k)), u_j(k)\}, \max\{u_j^*(z(k)), u_j(k)\}]$. For the convenience, we introduce the following notations

$$g_j(k) = g_j(\underline{z}_j(k), u_j^c(k)), \ S_j(k) = S_j(z_j(k)), \ \mu_j(k) = \mu_j(z_j(k)) \tag{7.37}$$

Substituting (7.34) into (7.36) with $\tilde{W}_j(k) = \hat{W}_j(k) - W_j^*$, we obtain

$$e_j(k + n_j) = g_j(k) \tilde{W}_j(k) S_j(k) + d_j^*(k) \tag{7.38}$$

where

$$d_j^*(k) = -g_j(k) \mu_j(k) + d_j'(k) \tag{7.39}$$

and it is trivial to show that

$$|d_j^*(k)| \le \bar{g}_j \mu_j^* + \bar{d}_j := \bar{d}_j^* \tag{7.40}$$

From (7.38) we can find that the tracking error constitutes of two parts: the external disturbance and NN approximation error. The adaptation law of NN weights is presented as follows where a deadzone is used.

$$
\begin{aligned}
e'_j(k) &= \gamma_j \frac{e_j(k)}{G_j(k)} \\
\hat{W}_j(k) &= \hat{W}_j(k - n_j) - \gamma_j N_j(x_j(k)) S_j(k - n_j) \frac{a_j(k) e'_j(k)}{D_j(k)} \\
\Delta x_j(k) &= x_j(k+1) - x_j(k) = \frac{a_j(k) G_j(k) e'^2_j(k)}{D_j(k)} \\
x_j(0) &= 0, \quad G_j(k) = 1 + |N_j(x_j(k))| \\
D_j(k) &= 1 + \|S_j(k - n_j)\|^2 |N_j(x_j(k))| + e'^2_j(k) \\
a_j(k) &= \begin{cases} 1 & \text{if } |e'_j(k)| > \chi_j \\ 0 & \text{others} \end{cases} \\
\hat{W}_j(t_j) &= \mathbf{0}_{[n_j]}, \quad t_j = -n_j + 1, \dots, 0
\end{aligned}
\tag{7.41}
$$

where $N_j(x_j(k))$ is the discrete Nussbaum gain.

Theorem 7.1. *Consider the adaptive closed-loop system consisting of system (7.1) under Assumptions 7.1, 7.2 and 7.3, control (7.35) with NN weights adaptation law (7.41). All the signals in the closed-loop system are SGUUB and the discrete Nussbaum gain $N_j(x_j(k))$ will converge to a constant ultimately. Denote*

$$
C_j = \lim_{k \to \infty} G_j(k)
\tag{7.42}
$$

then the tracking error satisfies

$$
\lim_{k \to \infty} \sup\{|e_j(k)|\} < \frac{C_j \chi_j}{\gamma_j}
\tag{7.43}
$$

where γ_j and χ_j are the tuning factor and the threshold value specified by the designer.

Proof. First, we assume the NN is constructed to cover a large enough compact set Ω_j such that the inputs $u_j(k)$ and outputs $y_j(k)$ are within the NN approximation range Ω_j.

Substituting the error equation (7.38) into the augmented error $e'_j(k)$, we obtain

$$
\gamma_j \tilde{W}_j(k - n_j) S_j(k - n_j) = \frac{1}{g_j(k - n_j)} G_j(k) e'_j(k) - \frac{1}{g_j(k - n_j)} \gamma_j d^*_j(k - n_j)
\tag{7.44}
$$

Choose a positive definite function $V_j(k)$ as

$$
V_j(k) = \sum_{t=1}^{n_j} \tilde{W}_j^T(k - n_j + t) \tilde{W}_j(k - n_j + t)
\tag{7.45}
$$

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Then, the difference equation of $V_j(k)$ becomes

$$
\begin{aligned}
\Delta V_j(k) &= V_j(k) - V_j(k-1) \\
&= \frac{\gamma_j^2 a_j^2(k) N_j^2(x_j(k)) S_j^T(k-n_j) S_j(k-n_j)}{D_j^2(k)} e_j'^2(k) \\
&\quad + 2N_j(x_j(k)) \frac{a_j(k)\gamma_j \tilde{W}_j^T(k-n_j) S_j(k-n_j)}{D_j(k)} e_j'(k) \\
&\leq \gamma_j^2 \frac{a_j(k) G_j(k) e_j'^2(k)}{D_j(k)} + |\frac{2\bar{d}_j^*}{\underline{g}_j \chi_j}| \frac{a_j(k) G_j(k) e_j'^2(k)}{D_j(k)} \\
&\quad - \frac{2}{g_j(k-n_j)} N_j(x_j(k)) \frac{a_j(k) G_j(k) e_j'^2(k)}{D_j(k)}
\end{aligned}
\tag{7.46}
$$

By denoting

$$
N_j'(x_j(k)) = \frac{1}{g_j(k-n_j)} N_j(x_j(k))
\tag{7.47}
$$

we have

$$
\Delta V_j(k) \leq c_{j,1} \Delta x_j(k) - 2N_j'(x_j(k)) \Delta x_j(k)
\tag{7.48}
$$

where

$$
c_{j,1} = \gamma_j^2 + |\frac{2\bar{d}_j^*}{\underline{g}_j \chi_j}|
\tag{7.49}
$$

Then we have

$$
V_j(k) \leq -2 \sum_{k'=0}^{k'=k} N_j'(x_j(k')) \Delta x_j(k') + c_{j,1} x_j(k) + c_{j,1}
\tag{7.50}
$$

Applying Lemma 4.1 to (7.50) we have the boundedness of $V_j(k)$ and $x_j(k)$. Noting the definition of $V_j(k)$, we can conclude the boundedness of $\|\hat{W}_j(k)\|$. Since $|N_j(x_j(k))| = |\sup_{k' \leq k}\{x_j(k')\}|$, the boundedness of $N_j(x_j(k))$ and $G_j(k) = 1 + |N_j(x_j(k))|$ is guaranteed. In addition, from $\Delta x_j(k) \geq 0$ in (7.41), we can find $x_j(k)$ is a nondecreasing sequence. Thus, we have

$$
\lim_{k \to \inf} \Delta x_j(k) = \lim_{k \to 0} \frac{a_j(k) G_j(k) e_j'^2(k)}{D_j(k)} = 0
\tag{7.51}
$$

Applying similar techniques in Section 4.4.3, we see that there exist constants C_j such that $\lim_{k \to \inf} G_j(k) = C_j$ and from the definition of $e_j'(k)$ in (7.41) that

$$
\lim_{k \to \infty} \sup\{|e_j(k)|\} \leq \frac{C_j \chi_j}{\gamma_j}
\tag{7.52}
$$

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Then, the boundedness of output $y_j(k)$ is obvious and according to Lemma 2.7, the boundedness of states $\Xi(k)$ and control inputs $u_j(k)$ is guaranteed.

We have proved that given any initial condition $z_j(0) \in \Omega_{0_j}$, there is a corresponding bounding compact set Ω_{z_j} so that $z_j(k) \in \Omega_{z_j}, \forall k$, if the NN approximation range is initialized to cover Ω_{z_j}. Suppose that the initial condition Ω_{0_j} and control parameters to be chosen are known at the beginning, then the bounding set Ω_{z_j} is determined. Then, if initially the NN approximation range Ω_j is constructed to cover the bounding set Ω_{z_j}, the boundedness of all the closed-loop signals is guaranteed. According to Definition 2.11, the proposed adaptive NN control achieves SGUUB stability. This completes the proof. ∎

7.2.4 Simulation studies

In this section, the following three-input three-output nonlinear plant is used for simulation.

$$\Sigma : \begin{cases} \Sigma_1 \begin{cases} \xi_{1,1}(k+1) = f_{1,1}(\xi_{1,1}(k), \xi_{1,2}(k)) \\ \xi_{1,2}(k+1) = f_{1,2}(\xi_{1,1}(k), \xi_{1,2}(k), \xi_{2,1}(k), \xi_{1,3}(k)) \\ \xi_{1,3}(k+1) = f_{1,3}(\xi_{1,1}(k), \xi_{1,2}(k), \xi_{1,3}(k), \xi_{2,1}(k), \xi_{2,2}(k), \xi_{3,1}(k), \\ \qquad u_1(k), d_1(k)) \\ y_1(k) = \xi_{1,1}(k) \end{cases} \\ \Sigma_2 \begin{cases} \xi_{2,1}(k+1) = f_{2,1}(\xi_{1,1}(k), \xi_{1,2}(k), \xi_{2,1}(k), \xi_{2,2}(k)) \\ \xi_{2,2}(k+1) = f_{2,2}(\xi_{1,1}(k), \xi_{1,2}(k), \xi_{1,3}(k), \xi_{2,1}(k), \xi_{2,2}(k), \xi_{3,1}(k), \\ \qquad \bar{u}_2(k), d_2(k)) \\ y_2(k) = \xi_{2,1}(k) \end{cases} \\ \Sigma_3 \begin{cases} \xi_{3,1}(k+1) = f_{3,1}(\xi_{1,1}(k), \xi_{1,2}(k), \xi_{1,3}(k), \xi_{2,1}(k), \xi_{2,2}(k), \xi_{3,1}(k), \\ \qquad \bar{u}_3(k), d_3(k)) \\ y_3(k) = \xi_{3,1}(k) \end{cases} \end{cases} \tag{7.53}$$

where system functions are

$$f_{1,1}(\cdot) = \frac{\xi_{1,1}^2(k)}{1 + \xi_{1,1}^2(k)} + 0.3\xi_{1,2}(k)$$

$$f_{1,2}(\cdot) = \frac{\xi_{1,1}^2(k)}{1 + \xi_{1,2}^2(k) + \xi_{2,1}^2(k)} + 0.1\xi_{1,3}(k)$$

$$f_{1,3}(\cdot) = \frac{\xi_{1,3}^2(k)}{1 + \xi_{1,2}^2(k) + \xi_{1,3}^2(k) + \xi_{2,1}^2(k) + \xi_{2,2}^2(k)} + g_1(u_1(k) + 0.5\sin(u_1(k))) + d_1(k)$$

$$f_{2,1}(\cdot) = \frac{\xi_{2,1}^2(k)}{1 + \xi_{1,2}^2(k) + \xi_{2,1}^2(k)} + 0.2\xi_{2,2}(k)$$

$$f_{2,2}(\cdot) = \frac{\xi_{2,1}^2(k)}{1 + \xi_{1,1}^2(k) + \xi_{1,2}^2(k) + \xi_{1,3}^2(k) + \xi_{2,1}^2(k) + \xi_{2,2}^2(k) + \xi_{3,1}^2(k)}$$
$$+ g_2(u_2(k) + 0.5\sin(u_1(k))) + d_2(k)$$

$$f_{3,1}(\cdot) = \frac{\xi_{2,1}^2(k)}{1 + \xi_{1,1}^2(k) + \xi_{1,2}^2(k) + \xi_{1,3}^2(k) + \xi_{2,1}^2(k) + \xi_{2,2}^2(k) + \xi_{3,1}^2(k)}$$
$$+ g_2(u_2(k) + 0.5\sin(u_1(k))\cos(u_2(k))) + d_3(k) \tag{7.54}$$

and $d_i(k) = 0.1\cos(0.01k)\cos(\xi_{i,1}(k))$, $i = 1, 2, 3$, and $g_1 = 10$, $g_2 = \pm 10$, $g_3 = 10$.

The desired reference trajectories are:

$$y_{d,1}(k) = 0.05 + 0.25\cos(\frac{\pi}{4}kT) + 0.25\sin(\frac{\pi}{2}kT)$$

$$y_{d,2}(k) = 0.05 + 0.25\sin(\frac{\pi}{4}kT) + 0.25\sin(\frac{\pi}{2}kT)$$

$$y_{d,3}(k) = 0.05 + 0.25\sin(\frac{\pi}{4}kT) + 0.25\cos(\frac{\pi}{2}kT) \tag{7.55}$$

with $T = 0.01$. The initial system states are $\xi_{1,1}(0) = 0$, $\xi_{1,2}(0) = 0$, $\xi_{1,3}(0) = 0$, $\xi_{2,1}(0) = 0$, $\xi_{2,2}(0) = 0$, $\xi_{3,1}(0) = 0$. Three RBFNNs are constructed with $l_1 = 10$, $l_2 = 11$, $l_3 = 12$ neurons. The initial NN weight estimates $\hat{W}_1(0)$, $\hat{W}_2(0)$, $\hat{W}_3(0)$ are chosen to be zero vectors and $S_1(0)$, $S_2(0)$, $S_3(0)$ are chosen with each element being a random number with amplitude less than 0.2. The tuning factors and the threshold values are chosen as $\gamma_1 = 1$, $\gamma_2 = 0.5$, $\gamma_3 = 0.2$ and $\lambda_1 = 0.001$, $\lambda_2 = 0.001$, $\lambda_3 = 0.001$.

To demonstrate the designed NN control is insensitive to the control direction, the simulation is carried out twice with both negative and positive g_2. Similar to simulation in Chapter 6, we will see that the discrete Nussbaum gain reverse its direction if initially the NN weights adaptation is in the wrong direction. First, let us choose $g_1 = 10$, $g_2 = -10$, $g_3 = 10$, for which the simulation results are presented in Figures 7.1(a), 7.2(a) and 7.3(a). For subsystem Σ_2, it can be seen that initially the output goes to an opposite direction compared with the reference signal, and after the discrete Nussbaum gain turns to

be negative at about the $50th$ step, the output tracking performance becomes to be much better. Second, we change the value of g_2 from -10 to 10 and carry out the simulation again. With employment of the same control law and same NN weights adaptation law, the simulation results are shown in Figures 7.1(b),7.2(b) and 7.3(b). It can be seen that the discrete Nussbaum gains are always positive and the initial tracking performance is better than the results in Figure 7.1(a). For both cases, we see that after the initial stage, good tracking performance is guaranteed though there there are couplings of states and inputs among subsystems.

Furthermore, to demonstrate the NN learning performance, we define the following NN learning error:

$$e_{j,nn}(k) = \phi_j(\underline{z}_j(k), u_j(k)) - y_{j,d}(k + n_j) \tag{7.56}$$

as the measurement of NN learning performance. According to (7.33) and (7.34), the better the NN approximation is (the smaller the NN approximation error $u_j(k) - u_j^*(k)$ is), the smaller $e_{j,nn}(k)$ is. If $u_j(k) - u_j^*(k) = 0$, we have $e_{j,nn}(k) = 0$. The NN learning errors are demonstrated in Figure 7.4. It can be found that the defined NN learning error $e_{j,nn}(k)$ is ultimately bounded in a neighborhood of zero.

7.3 MIMO Nonlinear NARMAX Systems

In Section 7.2, we have studied adaptive NN control of block triangular nonaffine discrete-time MIMO systems. In this Section, we investigate general nonaffine NARMAX MIMO systems. By assuming the inverse control gain matrix has an either positive definite or negative definite symmetric part, the adaptive tuning of NN weights for the NARMAX MIMO system can be simplified to as similar as that for SISO system with unknown control direction. Based on this observation, we only restrict on the inverse control gain matrix of the system instead of assuming the existence of an orthogonal matrix [159] for tuning.

7.3.1 Problem formulation

Consider p-input and p-output nonlinear discrete-time systems described in the NARMAX model as follows

$$y(k + \tau) = F(Y(k), U_{k-1}(k), u(k), D_{k-1}(k), \bar{d}(k)) + d(k + \tau - 1) \tag{7.57}$$

where τ is the system delay, $F(\cdot) \in R^p$ is unknown smooth vector valued system function, $u(k) = [u_1(k), \cdots, u_p(k)]^T \in R^p$ and $y(k) = [y_1(k), \cdots, y_p(k)]^T \in R^p$ are the system inputs

and outputs, respectively, $d(k) = [d_1(k), \cdots, d_p(k)]^T \in R^p$ denotes the external disturbance which is bounded by an unknown constant $d_b > 0$, i.e., $\|d(k)\| \leq d_b$, and the vectors $Y(k)$, $U_{k-1}(k)$, $D_{k-1}(k)$, and $\bar{d}(k)$ are defined as

$$Y(k) = [y_1(k), \cdots, y_1(k - n_1 + 1), y_2(k), \cdots,$$
$$y_2(k - n_2 + 1), \cdots, y_p(k), \cdots, y_p(k - n_p + 1)]^T$$
$$U_{k-1}(k) = [u_1(k - 1), \cdots, u_1(k - m_1), u_2(k - 1), \cdots,$$
$$u_2(k - m_2), \cdots, u_p(k - 1), \cdots, u_p(k - m_p)]^T$$
$$D_{k-1}(k) = [d_1(k - 1), \cdots, d_1(k - t_1 + 1), d_2(k - 1), \cdots,$$
$$d_2(k - t_2 + 1), \cdots, d_p(k - 1), \cdots, d_p(k - t_p + 1)]^T$$
$$\bar{d}(k) = [d(k + \tau - 2), \cdots, d(k)]^T, \quad \text{if} \quad \tau \geq 2$$

with n_i denotes the length of the ith outputs, m_i the length of the ith inputs, and t_i the length of the ith disturbance, $i = 1, \cdots, p$.

Assumption 7.4. *The vector valued system function $F(Y(k), U_{k-1}(k), D_{k-1}(k), \bar{d}(k))$ satisfies Lipschitz condition w.r.t. $D_{k-1}(k)$ and $\bar{d}(k)$, i.e., there exists Lipschitz constants L_1 and L_2 such that*

$$\|F(Y(k), U_{k-1}(k), u(k), D_{k-1}(k), \bar{d}(k)) - F(Y(k), U_{k-1}(k), u(k), 0, 0)\|$$
$$\leq \quad L_1 \|D_{k-1}(k)\| + L_2 \|\bar{d}(k)\|$$

Assumption 7.5. *The control gain matrix $G(k) = \frac{\partial F(\cdot)}{\partial u(k)}, \forall k \geq 0$, is a full rank matrix, and its inverse, $G^{-1}(k)$, has an either positive definite or negative definite symmetric part, $G_{IS}(k) = \frac{G^{-1}(k) + G^{-T}(k)}{2}$. In addition, the eigenvalues of $G_{IS}(k)$ are assumed to be bounded.*

Remark 7.1. *It should be pointed that matrices $G(k)$ and $G^{-1}(k)$ are general real matrices and they are not required to be symmetric.*

Remark 7.2. *Assumption 7.5 is quite looser than Assumption 4 in [159], which requires existence of an orthogonal matrix $Q(k)$ multiplying $G^{-1}(k)$ to guarantee the eigenvalues of the product matrix are all positive.*

Assumption 7.6. *System (7.57) is bounded-output-bounded-input (BOBI).*

7.3.2 Control design and stability analysis

Define error vector $e(k) = y(k) - y_d(k) = [e_1(k), e_2(k), \ldots, e_p(k)]^T$. From (7.57)) the error dynamics is

$$
\begin{aligned}
e(k+\tau) &= F(Y(k), U_{k-1}(k), u(k), D_{k-1}(k), \bar{d}(k)) - y_d(k+\tau) + d(k+\tau-1) \\
&= F(Y(k), U_{k-1}(k), u(k), 0, 0) - y_d(k+\tau) \\
&\quad + \Delta F(k) + d(k+\tau-1)
\end{aligned}
\tag{7.58}
$$

where

$$
\Delta F(k) = F(Y(k), U_{k-1}(k), u(k), D_{k-1}(k), \bar{d}(k)) - F(Y(k), U_{k-1}(k), u(k), 0, 0)
$$

According to the boundedness of disturbance $D_{k-1}(k)$ and $\bar{d}(k)$, and Assumption 7.4, $\Delta F(k)$ is also bounded. From Assumption 7.5, the control gain matrix $G(k)$ is nonsingular, $\forall k \geq 0$. According to implicit function theorem, there exists a unique and smooth desired control $u^*(k) = \alpha^c(Y(k), U_{k-1}(k), y_d(k+\tau))$ such that

$$
F(Y(k), U_{k-1}(k), u^*(k), 0, 0) - y_d(k+\tau) = 0
\tag{7.59}
$$

where $\alpha^c(\cdot)$ is an implicit function asserted by Lemma 2.1.

Consider employing HONN in Section 2.2 to approximate the ideal $u^*(k)$ as follows

$$
u^*(k) = W^{*T} S(\bar{z}(k)) + \mu(k)
\tag{7.60}
$$

where $\bar{z}(k) = [Y^T(k), U_{k-1}^T(k), y_d^T(k+\tau)]^T \in \Omega_z \subset R^q$ with $q = \sum_{i=1}^{p}(n_i + m_i + 1)$ and $\mu(k)$ is the bounded NN approximation error vector satisfying $\|\mu(k)\| \leq \mu^*$, which can be reduced by increasing the number of NN nodes. Then the adaptive NN control $u(k)$ is constructed as

$$
u(k) = \hat{W}^T(k) S(\bar{z}(k))
\tag{7.61}
$$

where $\hat{W}(k) \in R^{l \times q}$ and $S(\bar{z}(k)) \in R^l$. The NN weight adaptation law is given as

$$
\begin{aligned}
&\hat{W}(k) = \hat{W}(k-\tau) - \gamma N(x(k)) S(\bar{z}(k-\tau)) a(k) e^T(k)/D(k) \\
&\Delta x(k) = a(k)\gamma e^T(k) e(k)/D(k), \, x(0) = 0 \\
&D(k) = (1 + |N(x(k))|^2)(1 + \|S(\bar{z}(k-\tau))\|^2 + \|e(k)\|^2)
\end{aligned}
\tag{7.62}
$$

$$
a(k) = \begin{cases} 1, & \text{if } \|e(k)\|/(1 + |\text{N}(\text{x}(k))|) > \chi \\ 0, & \text{otherwise} \end{cases}
\tag{7.63}
$$

where $\gamma > 0$ and $\chi > 0$ can be arbitrary positive constants, and $N(\cdot)$ is discrete Nussbaum gain defined in Section 4.2.

Remark 7.3. *Like in Section 6.6, deadzone (7.63) is introduced in the NN weight adaptation law (7.62) to deal with external disturbance and NN approximation error.*

Theorem 7.2. *Consider the closed-loop system consisting of system (7.57), adaptive NN control (7.61), and NN weights adaptation law (7.62)-(7.63). All signals in the closed-loop system are SGUUB, the discrete Nussbaum gain $N(x(k))$ will converge to a constant ultimately, and the tracking error satisfies $\lim_{k\to\infty} \|e(k)\| < C\chi$, with $C = \lim_{k\to\infty}(1 + |N(x(k))|)$.*

Proof. The proof is proceeded in two parts: Firstly, we assume inputs and outputs are within Ω_z such that NN approximation holds; Secondly, given any initial condition, we show that there exists a determined compact set such that if initially the NN approximation range covers this set then the inputs and outputs are guaranteed to be within Ω_z without priori assumption in the first step.

Using mean value theorem, (7.58) can be written as

$$
\begin{aligned}
e(k+\tau) &= F(Y(k), U_{k-1}(k), u^*(k), 0, 0) - y_d(k+\tau) \\
&\quad + \Delta F(k) + G_\xi(k)[u(k) - u^*(k)] + d(k+\tau-1)
\end{aligned}
\tag{7.64}
$$

where $G_\xi(k) = \left.\frac{\partial F(\cdot)}{\partial u(k)}\right|_{u_\xi(k)}$ and $u_\xi(k)$ is a point of line $L(u(k), u^*(k)) = \{\xi \mid \xi = \theta u(k) + (1-\theta)u^*(k),\ 0 \le \theta \le 1\}$. Considering (7.59)-(7.61) and (7.64), we obtain

$$
e(k+\tau) = G_\xi(k)[\tilde{W}^T(k)S(\bar{z}(k)) - \mu(k)] + \Delta F(k) + d(k+\tau-1)
\tag{7.65}
$$

where $\tilde{W}(k) = \hat{W}(k) - W^*$ is the NN weights estimation error.

According to Assumption 7.5, there exist two positive constants $\bar{g}$ and $\underline{g}$ such that

$$
\underline{g}I \le \frac{1}{2}(G_\xi^{-1}(k) + G_\xi^{-T}(k)) \le \bar{g}I, \qquad \text{or} \qquad -\bar{g}I \le \frac{1}{2}(G_\xi^{-1}(k) + G_\xi^{-T}(k)) \le -\underline{g}I
\tag{7.66}
$$

where I is the identity matrix. It implies there exists a sequence $g(k)$ satisfying $\underline{g} \le |g(k)| \le \bar{g}$ such that

$$
e^T(k)G_\xi^{-1}(k-\tau)e(k) = e^T(k)\frac{G_\xi^{-1}(k-\tau) + G_\xi^{-T}(k-\tau)}{2}e(k) = g(k)e^T(k)e(k)
\tag{7.67}
$$

From (7.65), we have

$$
\tilde{W}^T(k-\tau)S(\bar{z}(k-\tau)) = G_\xi^{-1}(k-\tau)e(k) + d^*(k-1)
\tag{7.68}
$$

where

$$
d^*(k-1) = -G_\xi^{-1}(k-\tau)[\Delta F(k-\tau) + d(k-1)] + \mu(k)
\tag{7.69}
$$

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According to the boundedness of $d(k)$, $\Delta F(k-\tau)$ and $\mu(k)$, and Assumption 7.5, $d^*(k-1)$ is bounded, i.e., $\|d^*(k-1)\| \leq d_b^*$, where d_b^* an unknown constant.

Choose a positive definite Lyapunov function as follows:

$$V(k) = \sum_{j=1}^{\tau} \text{tr}\{\tilde{W}^T(k-\tau+j)\tilde{W}(k-\tau+j)\} \tag{7.70}$$

Considering (7.68) and (7.67), we have

$$\begin{aligned}
&\text{tr}\{2a(k)\gamma N(x(k))\frac{\tilde{W}^T(k-\tau)S(\bar{z}(k-\tau))e^T(k)}{D(k)}\} \\
&= 2a(k)\gamma N(x(k))\left[g(k)\frac{e^T(k)e(k)}{D(k)} + \frac{e^T(k)d^*(k-1)}{D(k)}\right]
\end{aligned} \tag{7.71}$$

Then, the difference of $V(k)$ along (7.68) is

$$\begin{aligned}
\Delta V(k) &= V(k) - V(k-1) \\
&= \text{tr}\{\tilde{W}^T(k)\tilde{W}(k) - \tilde{W}^T(k-\tau)\tilde{W}(k-\tau)\} \\
&= \text{tr}\{[\tilde{W}(k) - \tilde{W}(k-\tau)]^T[\tilde{W}(k) - \tilde{W}(k-\tau)] + 2\tilde{W}^T(k-\tau)[\tilde{W}(k) - \tilde{W}(k-\tau)]\}
\end{aligned}$$

which together with NN weights update law (7.62) leads to

$$\begin{aligned}
\Delta V(k) &= a(k)\gamma^2 N^2(x(k))\frac{S^T(\bar{z}(k-\tau))S(\bar{z}(k-\tau))e^T(k)e(k)}{D^2(k)} \\
&\quad -\text{tr}\{2a(k)\gamma N(x(k))\frac{\tilde{W}^T(k-\tau)S(\bar{z}(k-\tau))e^T(k)}{D(k)}\} \\
&= a(k)\gamma^2 N^2(x(k))\frac{S^T(\bar{z}(k-\tau))S(\bar{z}(k-\tau))e^T(k)e(k)}{D^2(k)} \\
&\quad -2\gamma a(k)\left[g(k)N(x(k))\frac{e^T(k)e(k)}{D(k)} + \frac{N(x(k))e^T(k)d^*(k-1)}{D(k)}\right]
\end{aligned} \tag{7.72}$$

From (7.63), we know $a(k)\|d^*(k-1)\| \leq a(k)\frac{\|e(k)\|}{(1+|N(x(k))|)\chi}d_b^*$, which implies that

$$|a(k)N(x(k))e^T(k)d^*(k-1)| \leq a(k)\frac{d_b^*}{\chi}e^T(k)e(k) \tag{7.73}$$

Considering $N^2(x(k))S^T(\bar{z}(k-\tau))S(\bar{z}(k-\tau)) \leq D(k)$ and noting (7.73), we have

$$\Delta V(k) \leq c_1 \frac{a(k)\gamma e^T(k)e(k)}{D(k)} - 2g(k)N(x(k))\frac{\gamma a(k)e^T(k)e(k)}{D(k)}$$

with $c_1 = \gamma + 2d_b^*/\chi$. Considering $\Delta x(k)$ defined in (7.62), we obtain

$$\Delta V(k) \leq c_1 \Delta x(k) - 2g(k)N(x(k))\Delta x(k) \tag{7.74}$$

Performing similar techniques used in Section 6.6 (proof of Theorem 7.2) and according to Lemma 4.1, we conclude the boundedness $x(k)$ and $V(k)$ and furthermore the boundedness of $\|\hat{W}(k)\|$ and $|N(x(k))|$. In addition, we have

$$\lim_{k\to\infty}\sup\{\frac{\|e(k)\|}{1+|N(x(k))|}\} \leq \chi \tag{7.75}$$

If we denote $C = \lim_{k\to\infty}(1+|N(x(k))|)$, the tracking error $e(k)$ satisfies $\lim_{k\to\infty}\|e(k)\| < C\chi$. Then, the boundedness of outputs $y(k)$ is obvious. The boundedness of $u(k)$ is obtained from Assumption 7.6.

For discrete-time system, the boundedness of $y(k)$ and $u(k)$ implies there is a largest bounding set depending on initial condition such that it includes $y(k)$ and $u(k)$. If initially the NN approximate range Ω_z is constructed to cover this set, then NN approximation will always hold, such that a priori assumption the NN approximation range is large enough can be replaced by that NN approximation range covers a specified set depending on initial condition. According to the definition of SGUUB (given any initial condition, there is a corresponding control that can guarantee the closed-loop stability), the proof is completed. ∎

7.4 Summary

In this Chapter, the control designs in Chapter 6 has been extended to nonaffine MIMO system in block-triangular form and NARMAX form. In the output feedback control design for block triangular systems, the assumption of equal subsystem orders [158] has been removed and the coupling terms are assumed in every equation of each subsystem rather than only in the last equation [157, 158]. For MIMO systems in NARAMX form, we have relaxed the assumption on the control gain matrix [159] by incorporate discrete Nussbaum gain into the control design. SGUUB stability is achieved for the closed-loop adaptive NN controlled systems.

(a) with negative g_2

(b) with positive g_2
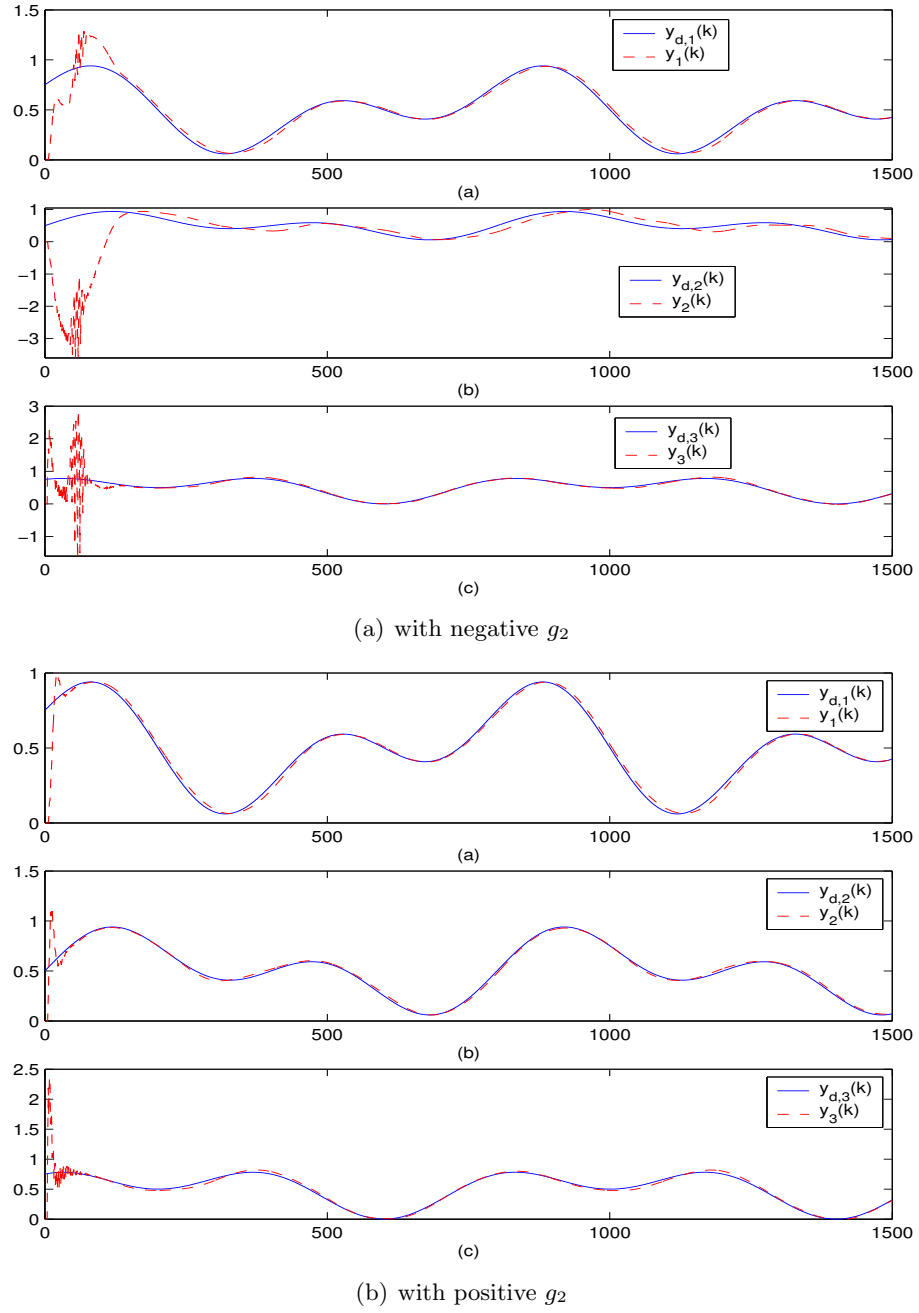
Figure 7.1: System output and reference trajectory

(a) with negative g_2

(b) with positive g_2
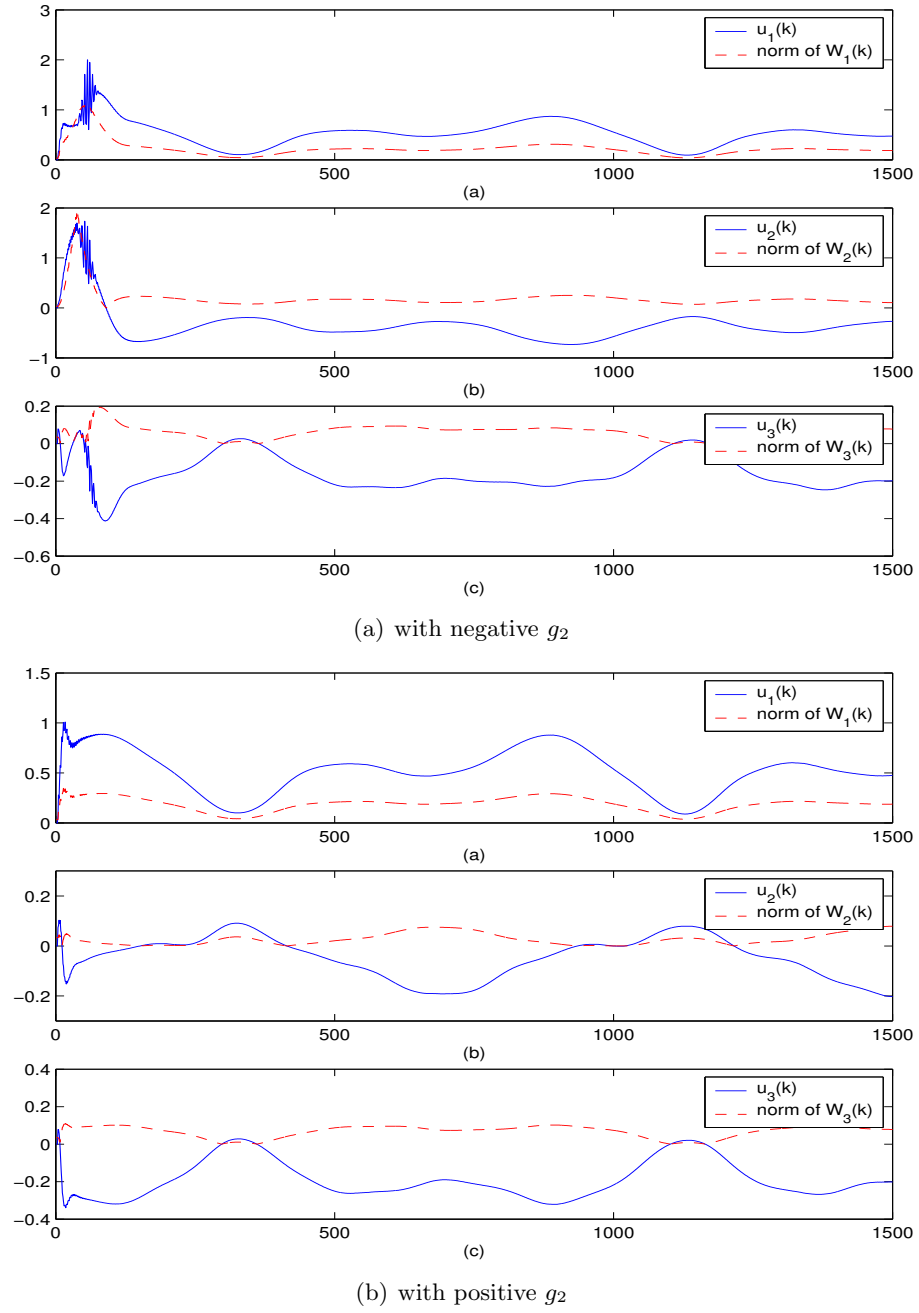
Figure 7.2: Control signal and NN weight

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(a) with negative g_2

(b) with positive g_2

Figure 7.3: Discrete Nussbaum gain and its argument

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Figure 7.4: NN learning errors

Chapter 8

Conclusions and Future Work

8.1 Conclusion

Part I of the thesis has been dedicated to model based adaptive control of SISO/MIMO strict-feedback nonlinear systems in LIPs form. Part II of the thesis has been dedicated to adaptive NN control of SISO/MIMO systems with general unknown nonlinearities in pure-feedback and NARMAX forms. In this Chapter, the results of the research work conducted in this thesis are summarized and the contributions made are reviewed. Suggestions for future work are also presented.

In Chapter 3, a framework of adaptive control design using predicted future states has been developed for nonlinear LIPs systems in strict-feedback form. Then, the study focused on how to completely compensate for the effect of nonparametric uncertainties in adaptive control design such that asymptotical tracking performance can be achieved. First, we studied the matched nonparametric uncertainties which appear in the control range. An auxiliary output including both parametric and nonparametric uncertainties as well as predicted future states has been defined. Then, its estimate has been constructed by using states information in previous steps such that the effect of the nonparametric uncertainti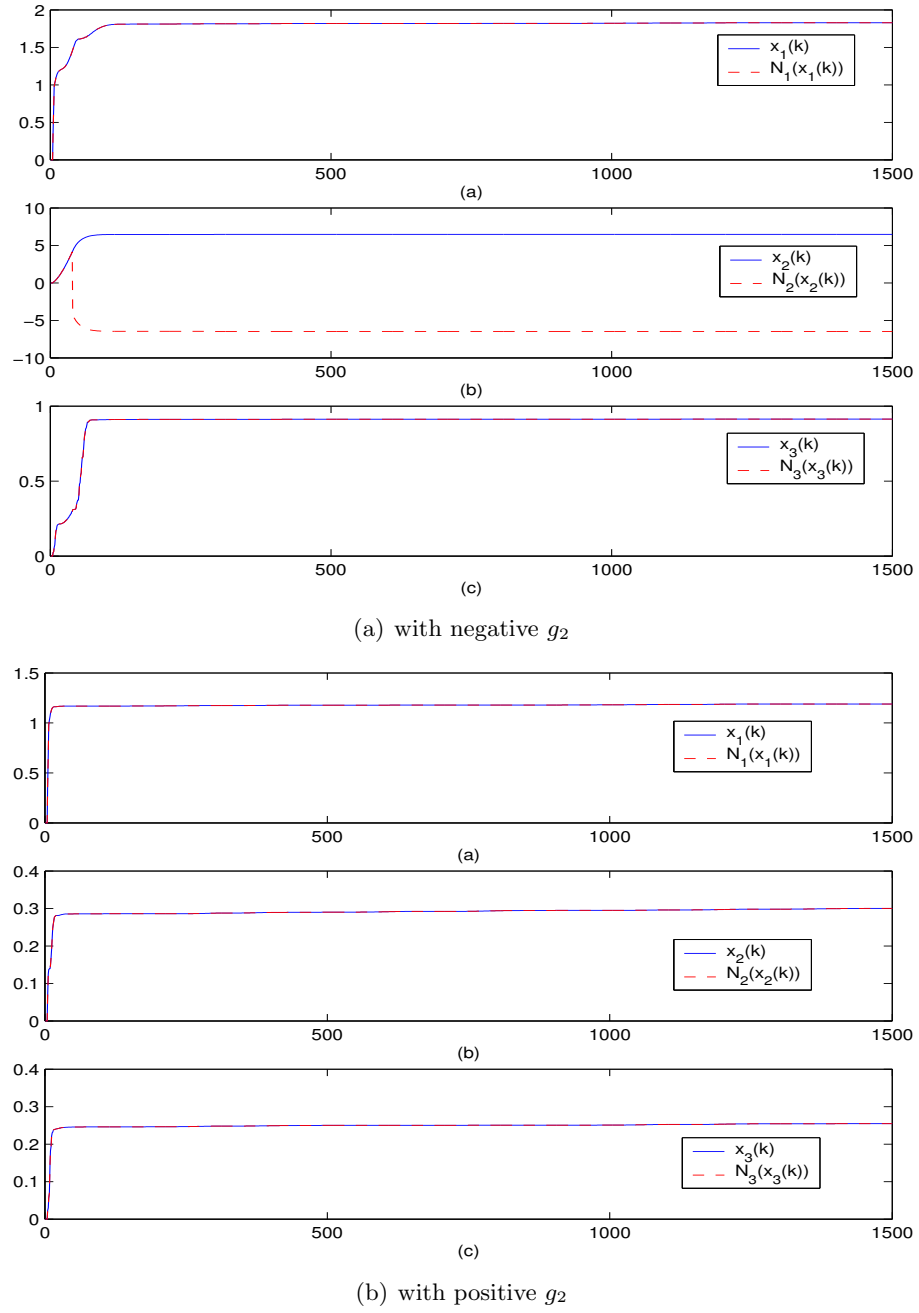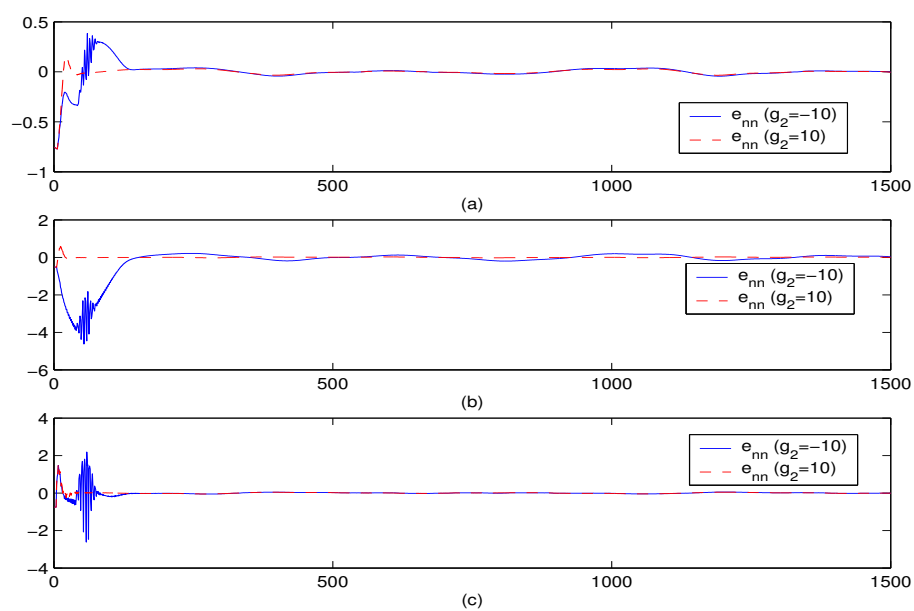es can be ultimately canceled. Next, compensation technique for unmatched nonparametric uncertainties out of control range has been studied in the future states prediction stage by introducing auxiliary states and their estimates in the future states prediction stage. Deadzone technique has been used in the parameter estimates update laws to make them robust to uncertain nonlinearities and the threshold of the deadzone is made to converge to zero. The synthesized adaptive control guarantees the boundedness of all the closed-loop signals. As the estimates of the auxiliary output and auxiliary states go to their true values

ultimately, asymptotical output tracking is achieved.

In Chapter 4, we have studied how to remove the a priori assumption on the knowledge of the control directions, namely, the signs of the control gains, in adaptive control design for strict-feedback systems. By incorporating discrete Nussbaum gain, the adaptive control become insensitive to the control directions. In addition, a priori requirement on the lower and upper bounds of the control gains has also been removed. First, the ideal case when the systems only subject to parametric uncertainties has been studied. It has been rigourously proved that the proposed control guarantees the boundedness of all the closed-loop signals and the output tracking error converge to zero. Next, to make the closed-loop system robust in the presence of external disturbance in the control range, deadzone technique has been used in the control parameter update law such that the boundedness of all the the closed-loop signals still hold and the output tracking error is bounded in a neighborhood of zero. There is no requirement of the amplitude of the external disturbance for construction of the deadzone. At last, adaptive control design for strict-feedback systems with both unknown control directions and nonparametric uncertainties has been studied. The developed adaptive control is able to completely compensate for the nonparametric uncertainty while at same time to deal with unknown control directions.

In Chapter 5, we have extended adaptive control designs developed in Chapters 3 and 4 to more general systems with input constraint and multivariable. In the first part, we have studied systems with both unknown control directions and hysteresis type input constraint. Discrete-time Prandtl-Ishlinskii (PI) model is used to describe the hysteresis. By combining discrete Nussbaum gain and PI model, adaptive control has been developed to achieve closed-loop global stability and to make output tracking error within a neighborhood around zero ultimately. In the second part, adaptive control has been investigated for block-triangular MIMO nonlinear systems with uncertain couplings of delayed states among subsystems in strict-feedback form. Future states prediction for each subsystem is carried out to facilitate adaptive control design and auxiliary outputs are introduced for compensation of the uncertain nonlinear couplings. By using Lyapunov method and ordering signals growth rate, it is rigorously established that all the signals in the whole closed-loop systems are bounded and the output tracking errors asymptotically converge to zeros.

In Chapters 6, adaptive NN control has been studied for SISO nonaffine systems in both pure-feedback and NARMAX forms. To solve the difficulty of nonaffine appearance of control input, implicit function theory has been utilized to assert the existence of an ideal control. Discrete Nussbaum gain has been further extended to deal with the time

varying unknown control gains. Based on the future states prediction functions established in Lemma 2.5, state feedback NN control has been designed for pure-feedback systems which is transformed into a state-output form such that only a single NN is required for the control design. Thereafter, it is established that the pure-feedback system is transformable to a class of NARMAX system. Then, a unified output feedback NN control has been developed for both pure-feedback systems and NARMAX system based output prediction approach.

In Chapter 7, we have studied adaptive NN control of nonaffine MIMO system in block-triangular form and NARMAX form. The block triangular systems studied are of couplings in every equation of each subsystem in pure-feedback form rather than only in the last equation of each subsystem [157,158]. By further exploring the properties of block-triangular form, the couplings of inputs and states among subsystems have been decoupled. For MIMO systems in NARAMX form, our adaptive NN control incorporate discrete Nussbaum gain technique to relax the requirement on the control gain matrix.

8.2 Future Research

In this section, some research topics are proposed for further investigation:

- *Discrete-time adaptive control of nonlinear systems with varying parameters.*

The discrete-time adaptive control presented in this thesis studies nonlinear strict-feedback systems with constant unknown parameters. In practice, system parameter may change under different conditions such that it is worth to study discrete-time adaptive control in the presence of varying parameters. Some researches have been carried out in discrete-time for slowly time varying case [56,60]. Adaptive control for systems with time periodic varying parameters was studied in continuous-time [182] and recently, it has been investigated in discrete-time [183] using lifting approach. It may be not possible to design discrete-time adaptive control for arbitrary time varying parameters as pointed in [60], but it is worthwhile to explore more general conditions of varying parameters for discrete-time adaptive control design, such as spatial periodic varying parameters [184].

- *Discrete-time adaptive control of systems with nonlinear parameterizations.*

The systems studied in this thesis are assumed to be in LIPs form for adaptive control design. Unlike in continuous-time, there are very few results in discrete-time studying nonlinear parameterized systems. Recently, convex/concave nonlinear parametrization has

been studied in [185, 186] by introducing the min-max strategy developed in continuous-time [90] for adaptive control of simple first order nonlinear discrete-time systems. It is meaningful to explore alternative adaptive control approach for high order nonlinear systems with more general types of nonlinear parameterizations.

- *Enhancement of NN learning ability to improve control performance.*

In this thesis, much effort has been spent to guarantee the stability of the adaptive NN controlled system, but few discussion is made on the possible way to enhance NN learning ability for better control performance. In the literature, there are some explorations to improve NN control performance, such as employment of self structuring NN [142], incorporation of reinforcement learning into NN control design [99]. It is an interesting and challenging problem to design smarter NN control approaches for uncertain nonlinear systems such that control performance can be further enhanced.

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Appendix A

Long Proofs

Appendix 2.1: Proof of Proposition 2.1

Proof. Only proof of properties (ii) and (viii) are given below. Proofs of other properties are easy and are thus omitted here.

(ii) From Definition 2.3, we see that $\|o[x(k)]\| \leq \alpha(k) \max_{k' \leq k+\tau} \|x(k')\|$, $\forall k > k_0, \tau \geq 0$, where $\lim_{k \to \infty} \alpha(k) \to 0$. It implies that there exist constants k_1 and $\bar{\alpha}_1$ such that $\alpha(k) \leq \bar{\alpha}_1 < 1$, $\forall k > k_1$. Then, we have

$$\|x(k+\tau) + o[x(k)]\| \leq \|x(k+\tau)\| + \|o[x(k)]\| \leq (1 + \bar{\alpha}_1) \max_{k' \leq k+\tau} \|x(k')\|, \forall k > k_1$$

which leads to $x(k+\tau) + o[x(k)] = O[x(k+\tau)]$. On the other hand, we have

$$
\begin{aligned}
\max_{k_1 < k' \leq k+\tau} \|x(k')\| &\leq \| \max_{k_1 < k' \leq k+\tau} x(k') + o[x(k)]\| + \|o[x(k)]\| \\
&\leq \| \max_{k_1 < k' \leq k+\tau} x(k') + o[x(k)]\| + \bar{\alpha}_1 \max_{k_1 < k' \leq k+\tau} \{\|x(k')\}
\end{aligned}
$$

and

$$\max_{k_1 < k' \leq k+\tau} \|x(k')\| \leq \frac{1}{1 - \bar{\alpha}_1} \| \max_{k_1 < k' \leq k} x(k') + o[x(k')]\|, \forall k > k_1$$

which implies $x(k+\tau) = O[x(k) + o[x(k)]]$. Then, it is obvious that $x(k+\tau) + o[x(k)] \sim x(k)$.

(viii) First, let us suppose that $x_1(k)$ is unbounded and define $i_k = \arg\max_{i \leq k} \|x_1(i)\|$. Then, it is easy to see that $i_k \to \infty$ as $k \to \infty$. Due to $\lim_{k \to \infty} \alpha(k) \to 0$, there exist a k_2 such that $\alpha(i_k) \leq \frac{1}{2}$ and $\|o[x_1(k)]\| \leq \frac{1}{2} \max_{k' \leq k} \|x_1(k')\|$, $\forall k > k_2$. Considering $x_2(k) = x_1(k) + o[x_1(k)]$, we have $\|x_2(i_k)\| = \|x_1(i_k) + o[x_1(i_k)]\| \geq \|x_1(i_k)\| - \|o[x_1(i_k)]\| \geq \frac{1}{2}\|x_1(i_k)\|$, $\forall k > k_2$ which leads to $\|x_1(i_k)\| \leq 2\|x_2(i_k)\|$, $\forall k \geq k_2$. Then, the unboundedness of $x_1(k)$ conflicts with $\lim_{k \to \infty} \|x_2(k)\| = 0$. Therefore, $x_1(k)$ must be bounded. Considering that $\alpha(k) \to 0$, we have

$$0 \leq \|x_1(k)\| \leq \|x_1(k) + o[x_1(k)]\| + \|o[x_1(k)]\| \leq \|x_2(k)\| + \alpha(k) \max_{k' \leq k} \|x_1(k')\| \to 0$$

which implies $\lim_{k\to\infty}\|x_1(k)\| = 0$. ∎

Appendix 2.2: Proof of Lemma 2.2

Proof. The proof has been given in [59] for $m = 1$ and $n = 1$ and it is easy to extend the proof when m and n are larger than one at follows:

We will prove it by seeking a contradiction in a similar way as in [59]. Firstly, let us suppose that

$$\overline{\lim}_{k\to\infty}\|X(k) - X(l_k)\| = \epsilon > 0 \tag{A.1}$$

where $\overline{\lim}$ denote the upper limit. Then we can take from $X(k)$ a subsequence $\{X(k_j), j \geq 1\}$ such that

$$\|X(k_j) - X(l_{k_j})\| > \frac{\epsilon}{2}, \quad k_j - l_{k_j} \geq \tau$$

According to the definition in (2.1), we have

$$\|X(k_j) - X(k')\| > \frac{\epsilon}{2}, \quad \forall 0 \leq k' \leq k_j - \tau$$

Noting that $k_i \leq k_j - \tau$, $i < j$, we have $\|X(k_j) - X(k_i)\| > \frac{\epsilon}{2}$, or equivalently

$$\|X(k_j) - X(k_i)\| > \frac{\epsilon}{2}, \quad i \neq j$$

which means that $\{X(k_j), j \geq 1\}$ is unbounded. This contradicts to $\sup\{\|X(k)\|\} < \infty$. Consequently (A.1) cannot hold and thus we have

$$\underline{\lim}_{k\to\infty}\|X(k) - X(l_k)\| = \overline{\lim}_{k\to\infty}\|X(k) - X(l_k)\| = 0$$

where $\underline{\lim}$ denotes the lower limit. Then, we have

$$\lim_{k\to\infty}\|X(k) - X(l_k)\| = 0$$

This completes the proof. ∎

Appendix 2.3: Proof of Lemma 2.5

Proof. It is noted in system (2.3) that among the future states at the $(k+1)$th step, only the last state $\xi_n(k+1)$ depends on the control input, while other $(n-1)$ states are independent of $u(k)$. Therefore, they can be predicted at the kth step provided that the system dynamics is known exactly. This implies that these states are SDFSs. The prediction functions of one step ahead states are as follows:

$$\bar{\xi}_i(k+1) = \begin{bmatrix} \xi_1(k+1) \\ \vdots \\ \xi_i(k+1) \end{bmatrix} = \begin{bmatrix} p_{1,1}(\bar{\xi}_2(k)) \\ \vdots \\ p_{1,i}(\bar{\xi}_{i+1}(k)) \end{bmatrix} := P_{1,i}(\bar{\xi}_{i+1}(k)), i = 1, 2, \ldots, n-1 \quad (A.2)$$

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where

$$p_{1,i}(\bar{\xi}_{i+1}(k)) \stackrel{def}{=} f_i(\bar{\xi}_i(k), \xi_{i+1}(k)), \quad i = 1, 2, \ldots, n-1$$

According to Assumption 6.2, it can be checked that

$$\frac{\partial p_{1,i}(\bar{\xi}_{i+1}(k))}{\partial \xi_{i+1}(k)} = g_{1,i}(\cdot), \ |g_{1,i}(\cdot)| > 0 \tag{A.3}$$

Moving one step forward in equation (A.2) and using the predicted states vector in (A.2), we see that the first $(n-2)$ states at the $(k+2)$th step are still independent of control $u(k)$ and thus, they are SDFSs.

$$\begin{aligned}
\bar{\xi}_i(k+2) &= \begin{bmatrix} \xi_1(k+2) \\ \vdots \\ \xi_i(k+2) \end{bmatrix} = \begin{bmatrix} p_{1,1}(\bar{\xi}_2(k+1)) \\ \vdots \\ p_{1,i}(\bar{\xi}_{i+1}(k+1)) \end{bmatrix} \\
&= \begin{bmatrix} p_{1,1}(P_{1,2}(\bar{\xi}_3(k))) \\ \vdots \\ p_{1,i}(P_{1,i+1}(\bar{\xi}_{i+2}(k))) \end{bmatrix} = \begin{bmatrix} p_{2,1}(\bar{\xi}_3(k)) \\ \vdots \\ p_{2,i}(\bar{\xi}_{i+2}(k)) \end{bmatrix} := P_{2,i}(\bar{\xi}_{i+2}(k)) \\
i &= 1, 2, \ldots, n-2
\end{aligned} \tag{A.4}$$

where

$$p_{2,i}(\bar{\xi}_{i+2}(k)) \stackrel{def}{=} p_{1,i}(P_{1,i+1}(\xi_{i+2}(k))), i = 1, 2, \ldots, n-2 \tag{A.5}$$

Continuing the procedure above iteratively, after $(n-2)$ steps, we note that the first state at the $(k+n-1)$-th step can be predicted by the states at the kth step as follows:

$$\xi_1(k+n-1) = p_{1,1}(P_{n-2,2}(\bar{\xi}_n(k))) := p_{n-1,1}(\bar{\xi}_n(k)) \tag{A.6}$$

where vector valued function $P_{j,i}(\bar{\xi}_{j+i}(k))$, $j = 3, 4, \ldots, n-2$, $i = 1, 2, \ldots n-j$, are defined consistently via the above procedure. Then, we see that $\xi_1(k+n-1)$ is still a SDFS.

For consistency, we denote

$$\bar{\xi}_1(k+n-1) = p_{n-1,1}(\bar{\xi}_n(k)) := P_{n-1,1}(\bar{\xi}_n(k)) \tag{A.7}$$

In addition, according to Lemma 2.4, we see that the composite functions $P_{j,i}(\cdot)$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, n-i$, are still Lipschitz functions. This completes the proof. $\blacksquare$

Appendix 2.4: Proof of Lemma 2.6

Proof. The first equation of system (2.3) can be written as follows according to the mean

value theorem

$$y(k+1) = f_1(\xi_1(k), \xi_2(k))$$
$$= f_1(y(k), 0) + g_{1,1}(y(k), \xi_2^c(k))\xi_2(k) \tag{A.8}$$

where $\xi_2^c(k) \in [\min\{0, \xi_2(k)\}, \max\{0, \xi_2(k)\}]$ and the control gain functions

$$g_{1,1}(\cdot) = \frac{\partial f_1(\xi_1(k), \xi_2(k))}{\partial \xi_2(k)}$$

has been assumed to be bounded. Due to function $f_1(\cdot)$ satisfies Lipschitz condition, we have

$$\bar{\xi}_2(k) = O[y(k+1)], \quad y(k+1) = O[\bar{\xi}_2(k)] \tag{A.9}$$

Similarly, the second equation of system (2.3) can be written as

$$\xi_2(k+1) = f_2(y(k), \xi_2(k), \xi_3(k)) = f_2(y(k), \xi_2(k), 0)$$
$$+ g_{1,2}(y(k), \xi_2(k), \xi_3^c(k))\xi_3(k) \tag{A.10}$$

where $\xi_3^c(k) \in [\min\{0, \xi_3(k)\}, \max\{0, \xi_3(k)\}]$ and $g_{1,2}(\cdot) = \frac{\partial f_2(y(k), \xi_2(k), \xi_3(k))}{\partial \xi_3(k)}$ has also been assumed to be bounded. Substituting equation (A.10) into (A.8) yields

$$y(k+2) = f_1(y(k+1), 0) + g_{1,1}(y(k+1), \xi_2^c(k+1))$$
$$\times [f_2(y(k), \xi_2(k), 0) + g_{1,2}(y(k), \xi_2(k), \xi_3^c(k))\xi_3(k)] \tag{A.11}$$

Noting the boundedness of $g_{1,1}(\cdot)$ and $g_{1,2}(\cdot)$, the Lipschitz condition of functions $f_1(\cdot)$ and $f_2(\cdot)$, equations (A.9) and (A.11), we have

$$\bar{\xi}_3(k) = O[y(k+2)], \ y(k+2) = O[\bar{\xi}_3(k)]. \tag{A.12}$$

Continuing the above procedure, we have

$$\bar{\xi}_i(k) = O[y(k+i-1)], \ y(k+i-1) = O[\bar{\xi}_i(k)] \tag{A.13}$$

which results in $\bar{\xi}_i(k) \sim y(k+i-1)$. From the last equation in (2.3), one has

$$|u(k)| = |\frac{\xi_n(k+1) - f_n(\bar{\xi}_n(k), 0, d(k)) - O[\bar{\xi}_n(k)]}{g_{1,n}(\bar{\xi}_n(k), u^c(k), d(k))}|$$
$$= O[\xi_n(k+1)] + O[\bar{\xi}_n(k)]$$
$$= O[y(k+n)] \tag{A.14}$$

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where $u^c(k) \in [\min\{0, u(k)\}, \max\{0, u(k)\}]$ and $g_{1,n}(\cdot)$ has been assumed to be bounded. This completes the proof. ∎

Appendix 2.5: Proof of Lemma 2.7

Proof. According to Definition 2.9, all the subsystems Σ_l, $l = 1, 2, \ldots, n$, are divided into $\bar{n}$ groups, with each group denoted by a set S_i, $1 \leq i \leq \bar{n}$. Considering Lipschitz properties of systems functions and bounded control gains in system (2.5), we apply similar techniques used for the proof of Lemma 2.6 in Appendix 2.4 to analyze signal orders in the followings.

Step 1: Consider the first equations of subsystems $\Sigma_{j_1} \in S_1$ ($j_1 \in s_1$), i.e., $i_{j_1} = 1$. Because $i_{j_1} - m_{j_1 l} = 1 + n_l - \bar{n} \leq 0$, $\forall l \notin s_1$, only states vectors $\bar{\xi}_{j_1,1}(k)$ from subsystems $\Sigma_{j_1} \in S_1$ ($j_1 \in s_1$), are included in the first equations ($i_{j_1} = 1$) of subsystem Σ_{j_1}. Then, it is easy to show that

$$y_{j_1}(k+1) = \sum_{j_1 \in s_1} O[y_{j_1}(k)] + O[\xi_{j_1,2}(k)] \text{ and } \xi_{j_1,2}(k) = O[y_{j_1}(k+1)] + \sum_{l \in s_1} O[y_l(k)] \quad \text{(A.15)}$$

Together with Proposition 2.1 and $\bar{\xi}_{j_1,2}(k) \sim O[\xi_{j_1,1}(k)] + O[\xi_{j_1,2}(k)]$, we have

$$\sum_{j_1 \in s_1} O[\xi_{j_1,1}(k)] \sim \sum_{j_1 \in s_1} O[y_{j_1}(k)], \quad \sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,2}(k)] \sim \sum_{j_1 \in s_1} O[y_{j_1}(k+1)] \quad \text{(A.16)}$$

Step 2: *sub-step 1*-Consider the second equations of subsystems $\Sigma_{j_1} \in S_1$ ($j_1 \in s_1$), i.e., $i_{j_1} = 2$. Because $i_{j_1} - m_{j_1 l} = 2 + n_l - \bar{n} \leq 0$, $\forall l \notin s_1 \cup s_2$, only states vector $\bar{\xi}_{j_1,2}(k)$ from subsystems $\Sigma_{j_1} \in S_1$ ($j_1 \in s_1$) and $\xi_{j_2,1}(k)$ from subsystems $\Sigma_{j_1} \in S_2$ ($j_1 \in s_2$), are included in the second equations ($i_{j_1} = 2$) of subsystems $\Sigma_{j_1} \in S_1$. Thus, using (A.15) we have

$$\begin{aligned} \xi_{j_1,2}(k+1) &= \sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,2}(k)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)] + O[\xi_{j_1,3}(k)] \\ &= \sum_{j_1 \in s_1} O[y_{j_1}(k+1)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)] + O[\xi_{j_1,3}(k)] \text{ and} \\ \xi_{j_1,3}(k) &= \sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,2}(k)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)] + O[\xi_{j_1,2}(k+1)] \\ &= \sum_{j_1 \in s_1} O[y_{j_1}(k+1)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)] + O[\xi_{j_1,2}(k+1)] \quad \text{(A.17)} \end{aligned}$$

which together with (A.16), Proposition 2.1 and $\bar{\xi}_{j_1,3}(k) \sim O[\xi_{j_1,3}(k)] + O[\bar{\xi}_{j_1,2}(k)]$ leads to

$$\sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,3}(k)] \sim \sum_{j_1 \in s_1} O[y_{j_1}(k+2)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)] \quad \text{(A.18)}$$

sub-step 2-Consider the first equations of subsystems $\Sigma_{j_2} \in S_2$ ($j_2 \in s_2$), i.e., $i_{j_2} = 1$. Because $i_{j_2} - m_{j_2 l} = 2 + n_l - \bar{n} \leq 0$ for $l \notin s_1 \cup s_2$, only state vectors $\bar{\xi}_{j_1,2}(k)$ from subsystems

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$\Sigma_{j_1} \in S_1$ $(j_1 \in s_1)$, and $\bar{\xi}_{j_2,1}(k)$ from subsystems $\Sigma_{j_2} \in S_2$ $(j_2 \in s_2)$, are included in the first equations $(i_{j_2} = 1)$ of subsystems $\Sigma_{j_2} \in S_2$. Thus, we have

$$y_{j_2}(k+1) = \sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,2}(k)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)] + O[\xi_{j_2,2}(k)] \quad \text{and}$$

$$\xi_{j_2,2}(k) = \sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,2}(k)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)] + O[y_{j_2}(k+1)] \tag{A.19}$$

which together with (A.16), Proposition 2.1 and

$$\bar{\xi}_{j_2,2}(k) = O[\xi_{j_2,2}(k)] + O[\xi_{j_2,1}(k)], \quad \sum_{j_2 \in s_2} O[\bar{\xi}_{j_2,1}(k)] \sim \sum_{j_2 \in s_2} O[y_{j_2}(k)]$$

implies

$$\sum_{j_2 \in s_2} O[\bar{\xi}_{j_2,2}(k)] \sim \sum_{j_1 \in s_1} O[y_{j_1}(k+1)] + \sum_{j_2 \in s_2} O[y_{j_2}(k+1)] \tag{A.20}$$

Step l, $3 \le l \le \bar{n} - 1$: Consider the lth equations of $\Sigma_{j_1} \in S_1$ $(j_1 \in s_1)$, the $(l-1)$th equations of $\Sigma_{j_2} \in S_2$ $(j_2 \in s_2)$, ..., and the first equations of $\Sigma_{j_l} \in S_l$, $(j_l \in s_l)$. Following the procedure above and considering $\sum_{j_l \in s_l} O[\bar{\xi}_{j_l,l}(k)] \sim \sum_{j_l \in s_l} O[y_{j_l}(k)]$, we have

$$\sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,l+1}(k)] \sim \sum_{j_1 \in s_1} O[y_{j_1}(k+l)] + \sum_{j_2 \in s_2} O[y_{j_2}(k+l-2)] \ldots + \sum_{j_l \in s_l} O[y_{j_l}(k)]$$

$$\sum_{j_2 \in s_2} O[\bar{\xi}_{j_2,l}(k)] \sim \sum_{j_2 \in s_2} O[y_{j_2}(k+l-1)] + \sum_{j_1 \in s_1} O[y_{j_1}(k+l-1)]$$

$$+ \sum_{j_3 \in s_3} O[y_{j_3}(k+l-3)] + \ldots + \sum_{j_l \in s_l} O[y_{j_l}(k)]$$

$$\vdots$$

$$\sum_{j_l \in s_l} O[\bar{\xi}_{j_l,2}(k)] \sim \sum_{j_l \in s_l} O[y_{j_l}(k+1)] + \sum_{j_1 \in s_1} O[y_{j_1}(k+l-1)]$$

$$+ \sum_{j_2 \in s_2} O[y_{j_2}(k+l-2)] + \ldots \sum_{j_{l-1} \in s_3} O[y_{j_{l-1}}(k+1)] \tag{A.21}$$

For subsystems $\Sigma_{j_{\bar{n}}} \in S_{\bar{n}}$ $(j_{\bar{n}} \in s_{\bar{n}})$, the system order is one $(n_{j_{\bar{n}}} = 1)$ and obviously we have $\bar{\xi}_{j_{\bar{n}},1}(k) = O[y_{j_{\bar{n}}}(k)]$. In summary of the analysis above and using Proposition 2.1, we have

$$\sum_{l=1}^{\bar{n}} \sum_{j_l \in s_l} O[\bar{\xi}_{j_l,n_{j_l}-i+1}(k)] \sim \sum_{l=1}^{\bar{n}} \sum_{j_l \in s_l} O[y_{j_l}(k+n_{j_l}-i)], \ 1 \le i \le \bar{n} \tag{A.22}$$

where we let $\bar{\xi}_{j_l,n_{j_l}-i+1}(k) = y_{j_l}(k+n_{j_l}-i) = 0$, if $n_{j_l}-i+1 \leq 0$. The equation above is equivalent to

$$\sum_{l=1}^{n} O[\bar{\xi}_{l,i_j-m_{jl}}(k)] \sim \sum_{l=1}^{n} O[y_l(k+i_j-m_{jl}-1)], 1 \leq i_j \leq n_j, 1 \leq j \leq n \qquad (A.23)$$

where $\bar{\xi}_{l,i_j-m_{jl}}(k) = y_l(k+i_j-m_{jl}-1) = 0$, if $i_j - m_{jl} \leq 0$.

Considering the last equation of the jth subsystem, we have

$$\begin{aligned} |u_j(k)| &= |\frac{\xi_{j,n_j}(k+1) - f_{j,n_j}(\Xi(k), \bar{u}_{j-1}(k), d_j(k)) - O[\Xi(k)]}{g_{j,n_j}(k)(\Xi(k), u_j^{c'}(k)), d_j(k))}| \\ &= O[\Xi(k+1)] + O[\bar{u}_{j-1}(k)] \end{aligned} \qquad (A.24)$$

for $j = 2, 3, \ldots, n$, where $u_j^{c'}(k)) \in [\min\{0, u_j(k)\}, \max\{0, u_j(k)\}]$. From (A.24), it is obvious that $u_1(k) = O[\Xi(k+1)]$. Next, we can deduce that $u_2(k) = O[\Xi(k+1)]$ and consequently $u_j(k) = O[\Xi(k+1)]$. This completes the proof. ■

Appendix 3.1: Proof of Lemma 3.4

Proof. Denote $\tilde{\Theta}_i(k) = \hat{\Theta}_i(k) - \Theta_i$, $\tilde{g}_i(k) = \hat{g}_i(k) - g_i$, and $\tilde{c}_i(k) = \hat{c}_i(k) - L_{pi}$. It follows from (3.62)-(3.65) that

$$\begin{aligned} \tilde{\xi}_i(k+1|k) &= \hat{\xi}_i(k+1|k) - \xi_i(k+1) \\ &= \hat{\xi}_i^a(k) - \xi_i^a(k) + \tilde{g}_i(k-n+2)\xi_{i+1}(k) \\ &= \tilde{\Theta}_i^T(k-n+2)[\Phi_i(\bar{\xi}_i(k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i} + n - i)] \\ &\quad + \tilde{g}_i(k-n+2)[\xi_{i+1}(k) - \xi_{i+1}(l_{k-n+i} + n - i)] \\ &\quad - [\nu_i(\bar{\xi}_n(k-n+i)) - \nu_i(\bar{\xi}_n(l_{k-n+i}))] \end{aligned} \qquad (A.25)$$

which yields

$$\begin{aligned} &-\{\tilde{\Theta}_i^T(k-n+2)[\Phi_i(\bar{\xi}_i(k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i} + n - i))] \\ &\quad + \tilde{g}_i(k-n+2)[\xi_{i+1}(k) - \xi_{i+1}(l_{k-n+i} + n - i)]\}\tilde{\xi}_i(k+1|k) \\ &= -\tilde{\xi}_i^2(k+1|k) - [\nu_i(\bar{\xi}_n(k-n+i)) - \nu_i(\bar{\xi}_n(l_{k-n+i}))]\tilde{\xi}_i(k+1|k) \\ &\leq -\tilde{\xi}_i^2(k+1|k) + \lambda L_{pi}|\tilde{\xi}_i(k+1|k)|\|\Delta\bar{\xi}_n(k-n+i)\| \end{aligned} \qquad (A.26)$$

where the last inequality is established by (3.71) and $\max_{1 \leq i \leq n} L_{\nu_i} \leq \lambda$.

To prove the boundedness of all the estimated parameters, we choose the Lyapunov candidate function as follows:

$$V_i(k) = \sum_{j=k-n+2}^{k} [\|\tilde{\Theta}\|^2 + \tilde{g}_i^2(j) + \tilde{c}_i^2(j)] \qquad (A.27)$$

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From (3.72), the difference of $V_i(k)$ is given by

$$
\begin{aligned}
\Delta V_i(k) &= V_i(k+1) - V_i(k) \\
&= \tilde{\Theta}_i^T(k+1)\tilde{\Theta}_i(k+1) - \tilde{\Theta}_i^T(k-n+2)\tilde{\Theta}_i(k-n+2) \\
&\quad + \tilde{g}_i^2(k+1) - \tilde{g}_i^2(k-n+2) + \tilde{c}_i^2(k+1) - \tilde{c}_i^2(k-n+2) \\
&= \{\|\Phi_i(\bar{\xi}_i(k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i}+n-i))\|^2 + |\xi_{i+1}(k) - \xi_{i+1}(l_{k-n+i}+n-i)|^2 \\
&\quad + \lambda^2\|\Delta\bar{\xi}_n(k-n+i)\|^2\}\frac{a_i^2(k)\gamma^2\tilde{\xi}_i^2(k+1|k)}{D_i^2(k)} \\
&\quad - \{\tilde{\Theta}_i^T(k-n+2)[\Phi_i(\bar{\xi}_i(k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i}+n-i))] \\
&\quad + \tilde{g}_i(k-n+2)[\xi_{i+1}(k) - \xi_{i+1}(l_{k-n+i}+n-i)]\}\tilde{\xi}_i(k+1|k)\frac{2a_i(k)\gamma}{D_i(k)} \\
&\quad + \lambda\tilde{c}_i(k-n+2)|\tilde{\xi}_i(k+1|k)|\|\Delta\bar{\xi}_n(k-n+i)\|\frac{2a_i(k)\gamma}{D_i(k)}. \quad\text{(A.28)}
\end{aligned}
$$

According to the definition of $D_i(k)$ in (3.73) and inequality (A.26), the difference of $V_i(k)$ in (A.28) can be written as

$$
\begin{aligned}
\Delta V_i(k) &\leq \frac{a_i^2(k)\gamma^2\tilde{\xi}_i^2(k+1|k)}{D_i(k)} - \frac{2a_i(k)\gamma\tilde{\xi}_i^2(k+1|k)}{D_i(k)} \\
&\quad + \frac{2a_i(k)\gamma\lambda\hat{c}_i(k-n+2)|\tilde{\xi}_i(k+1|k)|\|\Delta\bar{\xi}_n(k-n+i)\|}{D_i(k)} \\
&= \frac{a_i^2(k)\gamma^2\tilde{\xi}_i^2(k+1|k)}{D_i(k)} - \frac{2a_i^2(k)\gamma\tilde{\xi}_i^2(k+1|k)}{D_i(k)} \\
&= -\frac{a_i^2(k)\gamma(2-\gamma)\tilde{\xi}_i^2(k+1|k)}{D_i(k)} \quad\text{(A.29)}
\end{aligned}
$$

where $L_{pi} + \tilde{c}_i(k-n+2) = \hat{c}_i(k-n+2)$ and equality (3.75) are used.

Noting that $0 < \gamma < 2$, we can see from (A.29) that the difference of Lyapunov function $V_i(k)$, $\Delta V_i(k)$, is nonpositive and thus, the boundedness of $V_i(k)$ is guaranteed. It further implies the boundedness of $\hat{\Theta}_i(k)$, $\hat{g}_i(k)$, and $\hat{c}_i(k)$. Thus, there exist finite constants $\bar{\Theta}$, $\bar{g}$, and $\bar{c}$, such that

$$
\|\hat{\Theta}_i(k)\| \leq \bar{\Theta}, \ \hat{g}_i(k) \leq \bar{g}, \ \hat{c}_i(k) \leq \bar{c}, \ i = 1, 2, \ldots, n-1 \quad\text{(A.30)}
$$

Taking summation on both hand sides of (A.29), we obtain

$$
\sum_{k=0}^{\infty} \frac{a_i^2(k)\gamma(2-\gamma)\tilde{\xi}_i^2(k+1|k)}{D_i(k)} \leq V_i(0) - V_i(\infty)
$$

which together with the boundedness of $V_i(k)$ implies

$$
\frac{a_i^2(k)\tilde{\xi}_i^2(k+1|k)}{D_i(k)} := \alpha_i(k) \to 0, \ i = 1, 2, \ldots, n-1 \quad\text{(A.31)}
$$

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From Assumption 3.3, Lemma 2.6, and the definition of $D_i(k)$ in (3.73), it can been seen that

$$
\begin{aligned}
D_i^{\frac{1}{2}}(k) \quad \leq \quad & 1 + \|\Phi_i(\bar{\xi}_i(k)) - \Phi_i(\bar{\xi}_i(l_{k-n+i} + n - i))\| + |\xi_{i+1}(k) - \xi_{i+1}(l_{k-n+i} + n - i)| \\
& + \lambda \|\Delta \bar{\xi}_n(k - n + i)\| = O[y(k + i)], \ i = 1, 2, \ldots, n - 1
\end{aligned} \tag{A.32}
$$

From equation (A.31), for $i = 1, 2, \ldots, n - 1$, we have

$$
a_i(k)|\tilde{\xi}_i(k + 1|k)| = \alpha_i^{\frac{1}{2}}(k) D_i^{\frac{1}{2}}(k) = o[D_i^{\frac{1}{2}}(k)] = o[O[y(k + i)]] \tag{A.33}
$$

Further, we have

$$
\begin{aligned}
a_i(k)\|\tilde{\bar{\xi}}_i(k + 1|k)\| &\sim a_i(k)|\tilde{\xi}_i(k + 1|k)| = o[O[y(k + i)]] \\
& i = 1, 2, \ldots, n - 1
\end{aligned} \tag{A.34}
$$

From the definition of deadzone in (3.74), we have

$$
|\tilde{\xi}_i(k + 1|k)| \leq a_i(k)|\tilde{\xi}_i(k + 1|k)| + \lambda \hat{c}_i(k - n + 2)\|\Delta \bar{\xi}_n(k - n + i)\| \tag{A.35}
$$

which together with (A.30), (A.33) and the definition of $\Delta_s(k, i)$ in (3.78) yields

$$
|\tilde{\xi}_i(k + 1|k)| \leq o[O[y(k + i)]] + \lambda c_1 \Delta_s(k, i) \tag{A.36}
$$

where $c_1 = \bar{c}$. Denote $\bar{c}_1 = n c_1$, we further have

$$
\|\tilde{\bar{\xi}}_i(k + 1|k)\| \leq \sum_{j=1}^{i} |\tilde{\xi}_j(k + 1|k)| \leq o[O[y(k + i)]] + \lambda \bar{c}_1 \Delta_s(k, i) \tag{A.37}
$$

Continuing the analysis above, for j-step estimation error $\tilde{\xi}_i(k + j|k)$, $i = 1, 2, \ldots, n - 1$, $j = 2, 3, \ldots, n - i$, we have

$$
\begin{aligned}
\tilde{\xi}_i(k + j|k) \quad &= \quad \hat{\xi}_i(k + j|k) - \xi_i(k + j) \\
&= \quad \breve{\xi}_i(k + j|k) + \tilde{\xi}_i(k + j|k + 1)
\end{aligned} \tag{A.38}
$$

where

$$
\begin{aligned}
\tilde{\xi}_i(k + j|k + 1) \quad &\overset{def}{=} \quad \hat{\xi}_i(k + j|k + 1) - \xi_i(k + j) \\
\breve{\xi}_i(k + j|k) \quad &\overset{def}{=} nm \quad \hat{\xi}_i(k + j|k) - \hat{\xi}_i(k + j|k + 1)
\end{aligned} \tag{A.39}
$$

Similar as the proof of Lemma 3.2 in Section 3.2.2, based on the result in previous steps, for j-step estimation error $\tilde{\xi}_i(k + j|k)$, $j = 2, 3, \ldots, n - i$, $i = 1, 2, \ldots, n - 1$, we see that there exist constants c_{j-1} and $\breve{c}_{j-1}$ such that

$$
\begin{aligned}
|\tilde{\xi}_i(k + j - 1|k)| \quad &\leq \quad o[O[y(k + i + j - 2)]] + \lambda c_{j-1} \Delta_s(k, i + j - 2) \\
|\breve{\xi}_i(k + j - 1|k)| \quad &\leq \quad o[O[y(k + i + j - 2)]] + \lambda \breve{c}_{j-1} \Delta_s(k, i + j - 2)
\end{aligned} \tag{A.40}
$$

From (3.69) and (3.70), it is clear that $\breve{\xi}_i(k+j|k)$ can be expressed as

$$
\begin{aligned}
\breve{\xi}_i(k+j|k) &= \hat{\xi}_i(k+j|k) - \hat{\xi}_i(k+j|k+1) \\
&= \hat{\xi}_i^a(k+j-1|k) + \hat{g}_i(k-n+j+1)\hat{\xi}_{i+1}(k+j-1|k) - \hat{\xi}_i^a(k+j-1|k+1) \\
&\quad -\hat{g}_i(k-n+j+1)\hat{\xi}_{i+1}(k+j-1|k+1) \\
&= \hat{\Theta}_i^T(k-n+j+1)[\Phi_i(\bar{\hat{\xi}}_i(k+j-1|k)) - \Phi_i(\bar{\hat{\xi}}_i(k+j-1|k+1))] \\
&\quad +\hat{g}_i(k-n+j+1)[\hat{\xi}_{i+1}(k+j-1|k) - \hat{\xi}_{i+1}(k+j-1|k+1)] \\
&= \hat{\Theta}_i^T(k-n+j+1)[\Phi_i(\bar{\hat{\xi}}_i(k+j-1|k)) - \Phi_i(\bar{\hat{\xi}}_i(k+j-1|k+1))] \\
&\quad +\hat{g}_i(k-n+j+1)\breve{\xi}_{i+1}(k+j-1|k) \quad\quad\quad\quad\quad\quad\quad\quad\quad (A.41)
\end{aligned}
$$

According to the Lipschitz condition of $\Phi_i(\cdot)$ and (A.39), the following equality holds:

$$
\|\Phi_i(\bar{\hat{\xi}}_i(k+j-1|k)) - \Phi_i(\bar{\hat{\xi}}_i(k+j-1|k+1))\| \leq L_i \|\bar{\breve{\xi}}_i(k+j-1|k)\| \quad\quad (A.42)
$$

From (A.38)-(A.42), it follows that there exist constants c_j such that

$$
|\tilde{\xi}_i(k+j|k)| \leq o[O[y(k+i+j-1)]] + \lambda c_j \Delta_s(k, i+j-1)
$$

Denote $\bar{c}_j = n c_j$, then we have

$$
\begin{aligned}
\|\bar{\tilde{\xi}}_i(k+j|k)\| &\leq \sum_{j=1}^{i} |\tilde{\xi}_j(k+j|k)| \\
&\leq o[O[y(k+i+j-1)]] + \lambda \bar{c}_j \Delta_s(k, i+j-1) \quad\quad (A.43)
\end{aligned}
$$

Let $j = n - i$, $i = 1, 2 \ldots, n - 1$, then we see (A.43) leads to (3.76) and it completes the proof. ∎

Appendix 5.1: Proof of Lemma 5.2

Proof. Consider one-step prediction error of a given subsystem Σ_j,

$$
\tilde{\xi}_{j,i_j}(k+1|k) = \hat{\xi}_{j,i_j}(k+1|k) - \xi_{j,i_j}(k+1), \ i_j = 1, 2, \ldots, n_j - 1
$$

Performing the similar technique in Section 3.2.2 (Proof of Lemma 3.2), we obtain

$$
\tilde{\xi}_{j,i_j}(k+1|k) = o[D_{j,i_j}(k)] \quad\quad\quad\quad (A.44)
$$

From the definition of $D_{j,i_j}(k)$ in (5.26) and Lemma 2.7, we have

$$
D_{j,i_j}(k) = \sum_{l=1}^{n} O[\xi_{l,i_j - m_{jl}}(k)] + O[\xi_{j,i_j+1}(k)] = \sum_{l=1}^{n} O[y_l(k+i_j - m_{jl})] \quad\quad (A.45)
$$

Combining (A.44) and (A.45), we have

$$\tilde{\xi}_{j,i_j}(k+1|k) = \sum_{l=1}^{n} o[O[y_l(k+i_j-m_{jl})]], \quad i_j = 1, 2, \ldots, n_j - 1 \tag{A.46}$$

Next, let us analyze the two-step prediction error, $\tilde{\xi}_{j,i_j}(k+2|k) = \hat{\xi}_{j,i_j}(k+2|k) - \xi_{j,i_j}(k+2)$, $i_j = 1, 2, \ldots, n_j - 2$.

$$\begin{aligned} \tilde{\xi}_{j,i_j}(k+2|k) &= \tilde{\xi}_{j,i_j}(k+2|k+1) + \check{\xi}_{j,i_j}(k+2|k), \quad \text{with} \\ \tilde{\xi}_{j,i_j}(k+2|k+1) &= \hat{\xi}_{j,i_j}(k+2|k+1) - \xi_{j,i_j}(k+2) = \sum_{l=1}^{n} o[O[y_l(k+i_j-m_{jl}+1)]] \\ \check{\xi}_{j,i_j}(k+2|k) &= \hat{\xi}_{j,i_j}(k+2|k) - \hat{\xi}_{j,i_j}(k+2|k+1) \\ &= \bar{\hat{\Theta}}_j^T(k-n_j+3)[\hat{\Psi}_{j,i_j}(k+1|k) - \Psi_{j,i_j}(k+1)] \end{aligned} \tag{A.47}$$

Because the Lipschitz condition of $\Psi_{j,i_j}(\cdot)$, we have

$$\begin{aligned} \|\hat{\Psi}_{j,i_j}(k+1|k) - \Psi_{j,i_j}(k+1)\| &\leq L_{j,i_j}[\sum_{t=1}^{n} \|\bar{\tilde{\xi}}_{t,i_j-m_{jt}}(k+1|k)\|] + |\tilde{\xi}_{j,i_j+1}(k+1|k)| \\ &= \sum_{t=1}^{n}\sum_{l=1}^{n} o[O[y_l(k+i_j-m_{jt}-m_{tl})]] + \sum_{l=1}^{n} o[O[y_j(k+i_j+1-m_{jl})]] \\ &= \sum_{l=1}^{n} o[O[y_j(k+i_j+1-m_{jl})]] \end{aligned} \tag{A.48}$$

Considering the boundedness of $\bar{\hat{\Theta}}_j^T(k-n_j+3)$, we have

$$\begin{aligned} \tilde{\xi}_{j,i_j}(k+2|k) &= \sum_{l=1}^{n} o[O[y_l(k+i_j+1-m_{jl})]] \\ \bar{\tilde{\xi}}_{j,i_j}(k+2|k) &= \sum_{l=1}^{n} o[O[y_l(k+i_j+1-m_{jl})]], \quad i_j = 1, 2, \ldots, n_j - 2 \end{aligned} \tag{A.49}$$

Similarly, for the tth step prediction error $\tilde{\xi}_{j,i_j}(k+t|k) = \hat{\xi}_{j,i_j}(k+t|k) - \xi_i(k+t)$, $i_j = 1, 2, \ldots, n_j - t$, $t = 3, 4, \ldots, n_j - 1$, we have

$$\tilde{\xi}_{j,i_j}(k+t|k) = \sum_{l=1}^{n} o[O[y_l(k+i_j+t-1-m_{jl})]] \tag{A.50}$$

Let $t = n_j - i_j$, we complete the proof with $\bar{\tilde{\xi}}_{j,i_j}(k+n_j-i_j|k) = \sum_{l=1}^{n} o[O[y_l(k+n_j-m_{jl}-1)]]$. ∎

Appendix 6.1: Proof of Lemma 6.1

Proof. Firstly, let us consider the following inequality of $V(k) \geq 0$

$$V(k+1) \leq c(k)V(k) + b(k) \tag{A.51}$$

where $|c(k)| \le \bar{c} < 1$ and $|b(k)| \le \bar{b}$. It is straightforward to show that

$$
\begin{aligned}
V(1) &\le \bar{c}V(0) + \bar{b} \\
V(2) &\le \bar{c}V(1) + \bar{b} \le \bar{c}^2 V(0) + (\bar{c}+1)b \\
&\vdots \\
V(k) &\le \bar{c}^k V(0) + \frac{1-\bar{c}^k}{1-\bar{c}}\bar{b} \le V(0) + \frac{\bar{b}}{1-\bar{c}}
\end{aligned}
$$

and furthermore,

$$
\lim_{k\to\infty}\sup\{V(k)\} \le \lim_{k\to\infty}\bar{c}^k V(0) + \lim_{k\to\infty}\frac{1-\bar{c}^k}{1-\bar{c}}\bar{b} = \frac{\bar{b}}{1-\bar{c}}
$$

Now, if we choose $c(k) = \max\{c_i(k)\}$, $i = 1, 2, \ldots, m$, then, the inequality (6.3) in Lemma 6.1 becomes (A.51) . It is easy to see that equation (6.4) holds. ∎

Appendix 6.2: Proof of Corollary 6.1

Proof. Define $V_i^j(l) = V_i(ln+j)$ and $V^j(l) = \sum_{i=1}^m V_i^j(l)$, where $l \in Z_{-n}^+$, $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, n-1$. It is obvious that $V^j(0) \le \bar{V}(0)$. Then, from the definition of $V^j(l)$, we have

$$
\begin{aligned}
V^j(l+1) &= \sum_{i=1}^m V_i^j(l+1) = \sum_{i=1}^m V_i((l+1)n+j) \\
&= V(ln+n+j) \tag{A.52}
\end{aligned}
$$

According to equation (6.5), it is easy to obtain

$$
\begin{aligned}
V(ln+n+j) &\le \sum_{i=1}^m c_i(ln+j)V_i(ln+j) + b(ln+j) \\
&= \sum_{i=1}^m c_i^j(l)V_i^j(l) + b^j(l) \tag{A.53}
\end{aligned}
$$

where $c_i^j(l) = c_i(ln+j)$ and $b^j(l) = b(ln+j)$. Combining equation (A.52) and (A.53) results

$$
V^j(l+1) \le \sum_{i=1}^m c_i^j(l)V_i^j(l) + b^j(l) \tag{A.54}
$$

Noting that $|c_i^j(l)| \le \bar{c}$ and $|b^j(l)| \le \bar{b}$, we apply Lemma 6.1 to equation (A.54) and it results

$$
\begin{aligned}
V^j(l) &\le V^j(0) + \frac{\bar{b}}{1-\bar{c}} \\
&\le \bar{V}(0) + \frac{\bar{b}}{1-\bar{c}} \\
\lim_{l\to\infty}\sup\{V^j(l)\} &\le \frac{\bar{b}}{1-\bar{c}} \tag{A.55}
\end{aligned}
$$

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It is obvious that $\forall k$, $k \geq n-1$, there exist $j = k(\mathrm{mod}\ n)$, $j \in \{0, 1, \ldots, n-1\}$, and $l = \frac{k-j}{n}$, such that we can obtain

$$
\begin{aligned}
V(k) &= \sum_{i=1}^{m} V_i(ln+j) = \sum_{i=1}^{m} V_i^j(l) \\
&= V^j(l) \leq \bar{V}(0) + \frac{\bar{b}}{1-\bar{c}}, \quad k \geq n-1 \\
\lim_{k\to\infty} \sup\{V(k)\} &\leq \frac{\bar{b}}{1-\bar{c}}
\end{aligned}
\tag{A.56}
$$

This completes the proof. ■

Appendix 6.3: Proof of Lemma 6.2

Proof. Noting that $\max_{0 \leq i \leq n-1}\{V(i)\} \leq C_0$, we have the following inequality from Corollary 6.1

$$
V(k) \leq C_0 + \frac{\bar{b}}{1-\bar{c}}, \quad \lim_{k\to\infty} \sup\{V(k)\} \leq \frac{\bar{b}}{1-\bar{c}}
\tag{A.57}
$$

From the definition of $V(k)$, we have

$$
\begin{aligned}
e^2(k) &\leq \frac{1}{a_e} V(k) \\
\tilde{W}^T(k)\tilde{W}(k) &\leq \frac{1}{a_W} V(k)
\end{aligned}
\tag{A.58}
$$

Combining equation (A.57) and (A.58), we have that

$$
\begin{aligned}
|e(k)| &\leq \sqrt{\frac{1}{a_e}\left(C_0 + \frac{\bar{b}}{1-\bar{c}}\right)} := c_{e\,\max} \\
\lim_{k\to\infty} \sup |e(k)| &\leq \sqrt{\frac{\bar{b}}{a_e(1-\bar{c})}} := c_{es} \\
\|\tilde{W}(k)\| &\leq \sqrt{\frac{1}{a_W}\left(C_0 + \frac{\bar{b}}{1-\bar{c}}\right)} := c_{\tilde{W}\,\max} \\
\lim_{k\to\infty} \sup \|\tilde{W}(k)\| &\leq \sqrt{\frac{\bar{b}}{a_W(1-\bar{c})}} := c_{\tilde{W}s}
\end{aligned}
\tag{A.59}
$$

Then, it is easy to show that

$$
\begin{aligned}
\|\bar{\xi}_n(k)\| &\leq C_1 \max_{k \leq i \leq k+n-1}\{|y(i)|\} + C_2 \\
&\leq C_1 \sup_{y_d \in \Omega_{yd}}\{|y_d(k)|\} + C_1 c_{e\,\max} + C_2 \\
\|\hat{W}(k)\| &\leq \|W^*\| + \|\tilde{W}(k)\| \leq \|W^*\| + c_{\tilde{W}}
\end{aligned}
$$

and

$$\lim_{k\to\infty} \sup \|\bar{\xi}_n(k)\| \quad \leq \quad C_1 \sup_{y_d\in\Omega_{yd}} \{|y_d(k)|\} + C_1 c_{es} + C_2$$

$$\lim_{k\to\infty} \sup \|\hat{W}(k)\| \quad \leq \quad \|W^*\| + \|\tilde{W}(k)\| \leq \|W^*\| + c_{\tilde{W}s}$$

This completes the proof. ■

Appendix 6.4: Proof of Lemma 6.3

Proof. Case (i)

According to the prerequisite that $g_1 \leq |g(k)| \leq g_2$, $g(k)$ is either strict positive or negative. Only proof with positive $g(k)$ is given here and the proof with negative $g(k)$ is omitted because they are quite similar. It should be noted that because $\Delta x(k)$ is nonnegative, we have $x(k) = x_s(k)$ and $f(x_s(k)) = \pm x^{\frac{3}{2}}(k)$.

Firstly, let us consider that $x(k)$ grows without bound. If the sign of $s_N(x(k))$ changes infinite times, then the switching curve $f(x_s(k)) = \pm x^{\frac{3}{2}}(k)$ will be crossed infinite number of times. Then, the first properties in Definition 4.1 is satisfied. In the following, we prove that $s_N(x(k))$ definitely change its sign for infinite number of times if $x(k)$ grows without bound. Suppose that $s_N(x(k)) = 1$ remains positive in an interval $\{l_1 \leq k \leq l_2\}$, where $x(l_1) > \delta_0$ and noting that $x(k) \geq 0$, we have

$$
\begin{aligned}
S'_N(x(l_2)) &= \sum_{k=0}^{l_2} N'(x(k))\Delta x(k) \\
&= c_1 + \sum_{k=l_1}^{l_2} x(k)g(k)\Delta x(k) \geq c_1 + g_1 \sum_{k=l_1}^{l_2} x(k)\Delta x(k) \quad (A.60)
\end{aligned}
$$

where $c_1 = \sum_{k=0}^{l_1-1} N'(x(k))\Delta x(k)$. It is noted that in equation (A.60), the inequality cannot be obtained without $\Delta x(k) \geq 0$. This is why the restriction $\Delta x(k) \geq 0$ is indispensable.

Since $x(k) > \delta_0 \geq \Delta x(k)$, $\forall k \in \{k|l_1 \leq k \leq l_2\}$, we have

$$
\begin{aligned}
\Delta\{x(k)\}^2 &= x^2(k+1) - x^2(k) \\
&= 2x(k)\Delta x(k) + \{\Delta x(k)\}^2 \\
&\leq 2x(k)\Delta x(k) + x(k)\Delta x(k) = 3x(k)\Delta x(k) \quad (A.61)
\end{aligned}
$$

Substituting equation (A.61) into (A.60), we have

$$S'_N(x(l_2)) \geq \tfrac{g_1}{3}x^2(l_2+1) - \tfrac{g_1}{3}x^2(l_1) + c_1 \quad (A.62)$$

which implies that when $s_N(x(k)) = 1$, $S'_N(x(l_2))$ increases at least as fast as $\frac{g_1}{3}x^2(l_2+1)$ as l_2 increases. Therefore, it is obvious that the switching curve $f(x_s(k)) = x^{\frac{3}{2}}(k)$ will be crossed as l_2 increases if $x(k)$ is unbounded.

On the other hand, suppose that $s_N(x(k)) = -1$ remains on the interval $\{k | l_1 \leq k \leq l_2\}$, then, by the similarly approach we have

$$
\begin{aligned}
S'_N(x(l_2)) &= \sum_{k=0}^{l_2} N'(x(k))\Delta x(k) = c_1 - \sum_{k=l_1}^{l_2} x(k)g(k)\Delta x(k) \\
&\leq c_1 - g_1 \sum_{k=l_1}^{l_2} x(k)\Delta x(k) = -\frac{g_1}{3}x^2(l_2+1) + \frac{g_1}{3}x^2(l_1) + c_1 \quad \text{(A.63)}
\end{aligned}
$$

It implies $S'_N(x(k))$ decreases at least as fast as $-\frac{g_1}{3}x^2(l_2+1)$ when l_2 increases so that the switching curve of $f(x_s(k)) = -x^{\frac{3}{2}}(k)$ will always be crossed as l_2 increases if $x(k)$ is unbounded.

According to the above analysis, it is impossible for $s_N(x(k))$ to keep its sign unchanged as $x(k)$ grows unbounded. Therefore, $s_N(x(k))$ will change infinite times as $k \to \infty$. It is equivalent to that $S_N(x(k))$ grows unbounded in both positive direction and negative direction as $x(k)$ grows unbounded. By now, it is proved that the first property in Definition 4.1 is satisfied.

Secondly, let us consider that $x(k)$ is bounded, i.e., $x(k) \leq \delta_1$. Let us denote

$$
\lim_{k \to \infty} \sup\{x(k)\} = \bar{x}
$$

Note that $x(k)$ is a monotonic nondecreasing sequence, we have $x(k) \leq \bar{x}$. According to the definition of $N(x(k))$, we have $\lim_{k \to \infty} |N(x(k))| = \bar{x}$ and $|N(x(k))| \leq \bar{x}, \forall k$.

Then, it is easy to derive

$$
\begin{aligned}
|S'(x(k))| &= |\sum_{k'=0}^{k} g(k')N(x(k'))\Delta x(k')| \\
&\leq \sum_{k'=0}^{k} |g(k')||N(x(k'))|\Delta x(k') \leq g_2 \bar{x} \sum_{k'=0}^{k} \Delta x \leq g_2 \bar{x}^2 \quad \text{(A.64)}
\end{aligned}
$$

Since the two properties in the definition of discrete Nussbaum gain are satisfied, it is concluded that $g(k)N(x(k))$ is also a discrete Nussbaum gain.

Case (ii)

Noting that $-\epsilon_0 \leq C(k) \leq \epsilon_0$ and $\Delta x(k) \geq 0$, then, we have

$$
S_N(x(k)) - \epsilon_0 x(k) - \epsilon_0 \leq \sum_{k'=0}^{k} N'(x(k'))\Delta x(k') \leq S_N(x(k)) + \epsilon_0 x(k) + \epsilon_0 \quad \text{(A.65)}
$$

where $S_N(x(k))$ is defined in (4.1). It is noted in (A.65) that the inequality will not hold without $\Delta x(k) \geq 0$. This is the reason why the restriction $\Delta x(k) \geq 0$ is indispensable.

According to the properties of discrete Nussbaum gain $N(x(k))$, when $x(k)$ increase without bound, it is easy to obtain the following

$$\lim_{k\to\infty} \sup_{x(k)\geq\delta_0} \{S_N(x(k)) \pm \epsilon_0 x(k) \pm \epsilon_0\}$$

$$= \lim_{k\to\infty} \sup_{x(k)\geq\delta_0} [x(k)\{\pm\epsilon_0 \pm \frac{\epsilon_0}{x(k)} + \frac{1}{x(k)} S_N(x(k))\}] = +\infty \qquad (A.66)$$

and similarly,

$$\lim_{k\to\infty} \inf_{x(k)\geq\delta_0} \{S_N(x(k)) \pm \epsilon_0 x(k) \pm \epsilon_0\} = -\infty \qquad (A.67)$$

Then, from (A.65) we conclude that $N'(x(k))$ satisfies the first property in Definition 4.1. When $x(k)$ is bounded, from the property of $N(x(k))$, it is obvious $S_N(x(k))$ is bounded. Therefore, it is easy to see from (A.65) that $N'(x(k))$ also satisfies the second property in Definition 4.1. This completes the proof. ∎

Author's Publications

Journal Publications

1. C. Yang, S. S. Ge, Y. Li and T. H. Lee, Adaptive Control of a Class of Discrete-Time MIMO Nonlinear Systems with Uncertain Couplings, submitted to International Journal of Control.

2. Y. Li, C. Yang, S. S. Ge, and T. H. Lee, Adaptive Output Feedback NN Control of a Class of Discrete-Time MIMO Nonlinear Systems with Unknown Control Directions, submitted to IEEE Transactions on System, Man and Cybernetics, Part B

3. S.-L. Dai, C. Yang, S. S. Ge, and T. H. Lee, Robust Adaptive Output Feedback Control of a Class of Discrete-Time Nonlinear Systems with Nonlinear Uncertainties and Unknown Control Directions, submitted to System & Control Letters

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