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DYNAMIC INVENTORY RATIONING FOR SYSTEMS WITH MULTIPLE DEMAND CLASSES

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Contents

ABSTRACT

Inventory rationing among different demand classes is popular and critical for firms in many industries. In the literature most researchers consider the static rationing policies for the problems of inventory rationing in general are extremely difficult to analyze. Motivated by the wide application of inventory rationing and the potential of dynamic rationing policies in cost saving, this dissertation studies the dynamic inventory rationing in different circumstances.

The first part of the dissertation studies the dynamic inventory rationing in systems with Poisson demand and backordering, using dynamic programming. For a multiperiod system with zero lead time, we show that the optimal rationing policy in each period is a dynamic critical level policy and the optimal ordering policy is a base stock policy. We then extend the analysis to a multiperiod system with positive lead time. For the problem is very difficult to solve and the structure of the optimal rationing and ordering policies may be very complicated, we develop a near-optimal solution using the dynamic critical level rationing policy. A tight lower bound on optimal costs is also established. Numerical results show that the costs of our policy are very close to the optimal costs and that our

dynamic rationing policy can significantly reduce cost, comparing with current state-of-art static rationing policies: in many cases the cost reduction can be more than 10%.

The second part extends the first part by changing Poisson demand to general demand processes. The rejected demands are also backordered. Assuming the system adopts the dynamic critical level rationing policy, optimization models for both single period and multiperiod systems are developed. A method is proposed to obtain near-optimal parameters for the dynamic rationing and ordering policies. Some important characteristics of the rationing policy are also obtained. The numerical results show that the costs under the near-optimal dynamic rationing policy are quite close to the optimal costs in the examples.

The third part of the dissertation analyzes dynamic inventory rationing in systems with Poisson demand and lost sales. We first consider a multiperiod system with finite horizon under a periodic review ordering policy in which the ordering amount per period is fixed. A dynamic programming model is developed. Important characteristics of the optimal rationing policy, the optimal cost function and the optimal ordering amount are obtained. The model is then extended to the case of infinite horizon. Some important characteristics of the optimal rationing policy, cost function and ordering amount are also obtained. A numerical study is also conducted to obtain some important managerial insights.

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List of Symbols

i	Class index.
K	Number of classes.
λ_i	Arrival rate of class i of Poisson demands.
λ	Total arrival rate of all classes.
π_i	Penalty cost of shortage per unit for class i.
$\pi\breve{E}_i$	Penalty cost of shortage per unit per unit time for class i.
h	Holding cost per unit per unit time.
c	Variable ordering cost per unit.
u	Length of a selling period.
N	Number of intervals in a period which is divided into many intervals.
n	Index of intervals.
L	Lead time.
x_m	Net inventory at the beginning of period m before ordering.
y_m	Net inventory at the beginning of period m after ordering (for the case of zero lead time), or net inventory at the beginning of replenishment period m (for the case of positive lead time).

$\boldsymbol{B}_m$ Backorder vector at the beginning of period *m* before ordering (for the case of zero lead time), or backorder vector at the beginning of replenishment period *m* (for the case of positive lead time).

IP_m Inventory position at the beginning of ordering period *m* after ordering.

b_i Backorders of class *i*.

p_i Probability that there is one demand of class *i* during one interval.

p_0 Probability that there is no demand of any demand class during one interval.

x, s On-hand inventory.

S Base stock level.

ν ,υ Rationing Policy.

ω Ordering policy.

D_L Total demand of all demand classes during the lead time *L*.

t_c Remaining time before the end of a period.

M Number of periods.

k, m Index of periods.

Q The fixed ordering amount in an ordering policy.

α Discount factor.

Chapter 1

Introduction

In many inventory systems customers who demand a common product may have different characteristics in terms of penalty cost of shortage, service level requirement and so on. It is a very important strategy for firms in many industries to segment customers according to their characteristics into several demand classes and differentiate the service for different demand classes to reduce cost, and/or increase profit, and/or improve customer satisfaction and so on. When inventory is not enough to satisfy demands from all demand classes, it is obvious that the inventory system should reject demands from some classes to reserve stock for possible future demands from more important classes. How to satisfy or reject demands from different classes is referred to as an inventory rationing policy, which is the key decision problem in these inventory systems with multiple demand classes. When inventory systems have multiple ordering opportunities to replenish stock, the ordering policy will interact with the inventory rationing policy. In these cases how to replenish inventory is also a key decision problem.

There are many examples of inventory rationing. One example is that a warehouse sells a kind of product to depots and the depots can place two kinds of orders to the warehouse: ordinary orders and emergent orders. These two kinds of orders can bring different profits to the warehouse. So the warehouse can divide the demands into two classes. When the on-hand inventory in the warehouse is low and not enough to satisfy all demands, the warehouse may reject some ordinary orders to reserve stock for possible future emergent orders.

Another example is a kind of repair part that is consumed by airplanes from different airlines. These airlines have contracts with a company, which provide these repair parts to the airlines. Different airlines have different service level requirements, say, some airlines need a service level of 95% and some need a service level of 99% and so on. So the company that provides the repair part can classify demands according to the service level requirement and adopt an appropriate inventory rationing policy to increase its profit while satisfying its customersô requirements. There are also many other examples of inventory rationing in industries such as automobile, computer, handphone and so on.

It is obvious that an inventory rationing policy can reduce much cost, comparing with the first-come-first-served policy (i.e., without inventory rationing). Because of the competitive pressure and thin profit margin in many industries, inventory rationing among demand classes has become a necessary strategic tool to firms rather than a competitive advantage. If a new rationing policy can reduce cost even about 1%, comparing with current inventory rationing policies, it will be a very good improvement in performance, as the profit margin is quite thin in many industries. For firms with very large annual costs, a cost reduction of 1% means saving a very large amount of money. So an effective rationing policy is extremely important for firms in many industries.

Though inventory rationing has many application areas in industries, the theory of inventory rationing is relative limited. Tsay et al. (1999) have explained that inventory rationing problems are extremely difficult to solve and generally considered intractable. So some papers consider only two demand classes and simple rationing policies. Inventory rationing has attracted more and more attention from researchers and practitioners in recent years. For the theory about inventory rationing is quite limited, in practice the application of inventory rationing is quite primitive. People often use simple rationing policies which in general are not optimal, but easier to implement than the optimal policies. For example, in a system with only two demand classes, when the stock drops to a certain constant value, then reject demands of the less important class. Even for these simple rationing policies, it is not unusual that the parameter values of these policies often are set according to practitionersô experience, because obtaining the optimal parameter values of these simple policies also needs some complicated calculation.

There are two types of rationing policies in the literature: static critical level policies and dynamic critical level policies. In these policies there is a critical level of on-hand inventory for each demand class at any time point such that if the on-hand inventory is above the critical level of a certain class at a certain time, then the demand of this class is satisfied, otherwise it is rejected. In the static rationing policy, the critical levels do not change with time, while in the dynamic rationing policy, the critical levels change with time. The *critical level* is also called *threshold* in some papers. In the literature most researchers consider the static rationing policies, which are not optimal in many cases for the inventory managers may use such information as the arrival times of replenishments to dynamically ration stock to reduce cost. The inventory problems under dynamic rationing policies are much more complicated than those under the static rationing policies.

It is obvious that dynamic rationing policies are better than static rationing policies in many cases, but little is known about how much the benefit of implementing dynamic rationing policies is and how to find optimal or near-optimal parameters of the dynamic rationing policies and the ordering policies in typical practical settings. Recently Deshpande et al. (2003) consider the static critical level rationing policy for a typical practical setting with positive lead time. They have developed a lower bound on optimal costs under dynamic rationing policies. They do not provide particular dynamic rationing policies. They show that the gap between the lower bound and the cost under the static critical level policy is very large. In many cases it is more than 10% and in some other cases it can be more than 20%. People do not know whether their lower bound is tight and whether there truly exist such dynamic rationing policies that indeed can reduce cost significantly. Anyway, this gap brings interesting questions: Can the dynamic rationing policies significantly reduce cost, comparing with the static critical level policies? If yes, in what conditions?

Motivated by the wide application of inventory rationing in industries and the potential of dynamic rationing policies in cost saving, comparing with static critical level policies, we explore dynamic inventory rationing in different problem settings. Our main objective is to develop models to characterize optimal dynamic rationing policies, obtain effective dynamic rationing policies to reduce cost and derive managerial insights for inventory management. This research is divided into three parts according to properties of demand processes and whether the rejected demands are backordered or lost. For some problem settings, we obtain the optimal dynamic rationing policies. For other problem settings where the structure of the optimal rationing policies may be extremely complex, we obtain near-optimal solutions assuming a dynamic critical level rationing policy. The numerical results show that our dynamic critical level policy can indeed significantly reduce cost in many cases, comparing with the static critical level policy: in many cases the relative cost difference can be more than 10%. The costs under our dynamic rationing policy are also very close to the optimal costs. So Deshpande et al. (2003) show such a possibility in cost saving, while we find particular dynamic rationing policies which indeed can significantly reduce cost. Moreover, we characterize the structure of optimal rationing and ordering policies in some cases.

The remaining of this chapter is organized as follows. In Section 1.1, we present more applications of inventory rationing in industries. In Section 1.2, characteristics of inventory rationing problems and relevant research in the literature are summarized. In Section 1.3, an overview of the research in this dissertation is provided.

1.1 Application of Inventory Rationing

Inventory rationing has a very wide application in industries. Here we present more examples of its application besides those referred to in the previous section.

Standardization of components is a very important strategy and a wide practice in every industry and inventory rationing problems arise with it. A kind of standard part may be used in the production of a family of products and different products in general bring different profits to the firm. When the inventory of the common standard component is low, how to allocate the inventory to produce different products is an inventory rationing problem. In industries such as automobile, printer, computer and handphone and so on, we can see many examples that a certain standard component is used in different products. Inventory rationing problems also appear in the course of machine maintenance. When a spare part is used by different machines and the breakdowns of different machines bring different loss to the firm, the firm needs to decide how to allocate the inventory of the spare part to repair these machines. It is also an inventory rationing problem (Dekker et al., 1998, have provided such an example in an oil factory).

Many inventory rationing problems appear in the supply chain environment. In a supply chain it is not unusual for the downstream stage to place ordinary orders and emergent orders and it is analyzed by many researchers in different conditions (Rosenshine and Obee 1976; Chiang and Gutierrez 1996, 1998; Tagaras and Vlachos 2001; Teunter and Vlachos 2001). These two kinds of orders can be regards as different demand classes by the upstream stage. So the upstream stage faces an inventory rationing problem about how to satisfy these two demand classes. Another example is about the distribution of a product from a distribution center to many customer zones. There are different transportation costs from the distribution center to different customer zones. When the product is sold at the same price in the whole nation, then customers at different customer zones bring different benefit to the firm. How to send products from the distribution center to customer zones is also a rationing problem.

Many inventory rationing problems come out with supply contracts which are a popular practice in industries. A certain firm provides products or services to its customers and different customers may have different contracts with this firm to require different service levels (Urban 2000; Bassok et al. 1997; Anupindi and Bassok 1999). In the previous section we have already shown an example in which a firm provides to different airlines different service levels for a spare part that is consumed by airplanes.

Military material management is also an area in which inventory rationing problems often appear. For example, Kaplan (1968) presents a rationing problem faced by the Army Material Command. As it notes: ñStock is in short supply, but at some known date in the future stock levels will be replenished. Before that time two types of demand must be satisfied, low priority and high priority.ò Deshpande et al. (2003) present another example, managing the consumable service parts that are consumed by U.S. Army and Navy and they have different service level requirements.

In the above examples, the systems often have multiple ordering opportunities and the inventory has a holding cost. There are also other examples of inventory rationing without multiple ordering opportunities, i.e., some initial inventory is sold over a finite horizon. One example is the airline seat control in which the same seats are sold over a fixed finite horizon at different prices to different customers. However, there are some fundamental differences between the airline seats control and the general inventory rationing problems considered here. First, in the airline seat control problems, there are no ordering opportunities, while in the general inventory rationing problems there is an ordering policy. The ordering policy interacts with the inventory rationing policy and it makes the problem very complicated, especially when the lead time is positive. Second, there is no holding cost in the airline seat control, while in the general inventory rationing problems the holding cost exists and it affects the decisions of ordering and inventory rationing. Third, when there are multiple legs in the airline seat control, the problems are also very complicated. It is somehow similar to the general inventory rationing problems with multiple products (with some substitutions) or multiple echelons, but without holding cost. While here we focus on the inventory rationing problems at one place with one product. So, from the above, we can see that inventory rationing indeed has very wide application in industries. Of course, there are other application areas besides the above ones.

1.2 Characteristics of Inventory Rationing Problems and Relevant Research

When different demand classes have different service level requirements, one natural method of inventory rationing is to maintain independent stocks for different classes. But this method has a large disadvantage: it may lose the benefits of inventory pooling such as reducing safety stock and so on (Eppen 1979; Schwatz 1989; Baker et al. 1986; Kim 2002). So in practice this method of separating inventory for different classes may not be used frequently and the popular practice is to maintain a common stock to satisfy different customers using a rationing policy. In some cases the inventory may not be able to separate, for example, the airline seats inventory. So in this research we consider only the cases using a common stock to serve different demand classes.

The inventory rationing problems with multiple demand classes are significantly different from the classic inventory problems in which all customers are treated in the same way. These inventory rationing problems are very difficult to solve, many of which are regarded as intractable. When there exist multiple replenishment opportunities, the inventory rationing policy interacts with the ordering policy. It is often extremely difficult to find the optimal rationing and ordering policies simultaneously. Even given an ordering policy, in most cases it is also very difficult to find the optimal rationing policies under such an ordering policy. Researchers often consider the rationing problems assuming a certain ordering policy, sometime even assuming a certain type of rationing policy. For the great difficulty of rationing problems some researchers consider the cases with only 2 demand classes.

As noted in the previous section, there exist two types of rationing policies in the current literature: static critical level policies and dynamic critical level policies. Critical levels in the static rationing policy do not change with time, while in the dynamic critical level policy they change with time. The critical level is sometimes termed as *threshold*. Obviously, the dynamic rationing policy is more complicated and more difficult to analyze than the static rationing policy, and the dynamic rationing policy can save cost comparing with the static rationing policy in many cases. The static critical level policy can be regarded as a special case of the dynamic critical level policy.

In the literature most researchers consider the static critical level rationing policy and have made notable progress, while the literature considering dynamic rationing policies is quite limited. Researchers often analyze service levels or find appropriate parameters of the policies to minimize cost, assuming a static rationing policy and a certain ordering policy. Chapter 2 provides a detailed literature review.

1.3 Overview of This Research

In this research we study dynamic inventory rationing for different system settings. Analytic models to minimize cost are developed. For some inventory systems, important structural characteristic of the optimal dynamic rationing policy and ordering policy are obtained. For other inventory systems, optimization models are developed and near optimal solutions with dynamic rationing policies are obtained. Many important managerial insights are also obtained. These dynamic rationing policies will provide a finer level of service differentiation and lower costs than current state-of-art rationing policies.

The research is divided into three parts according to the type of demand processes and whether to backorder the rejected demands. The first part considers dynamic inventory rationing in systems with Poisson demands and backordering, the second part analyzes systems with general demand processes and backordering, i.e., extending the first part from Poisson demand process to general demand processes, and the third part studies systems with Poisson demands and lost sales.

Part 1: Inventory Rationing with Poisson Demands and backordering

In this part we analyze dynamic inventory rationing in multiperiod systems, assuming Poisson demands and backordering. We first consider a multiperiod system with zero lead time. Dynamic programming models are developed. We show that the optimal rationing policy in each period is a dynamic critical level policy and the optimal ordering policy is a base stock policy. Some other important characteristics of the optimal rationing policy and the optimal cost function are also obtained.

We then investigate a multiperiod system with positive lead time and develop an optimization model to minimize average cost. In the case with positive lead time, the structure of optimal rationing and ordering policies may be very complicated and there is no closed-form expression for the average cost under many rationing policies, so we develop a near-optimal solution for it: applying the dynamic critical level rationing policy of the model with zero lead time to ration stock in each period and adopting a base stock ordering policy. Some important properties of such policy are obtained. A lower bound on the optimal costs under optimal rationing and ordering policies is also developed.

The numerical results show the cost under the near-optimal solution is very close to the optimal cost for a practical range of parameters and also for poor service level conditions. It also shows that our dynamic rationing policy can significantly save cost, comparing with current state-of-art static critical level policies. In many cases the cost saving can be more than 10%. From the numerical results, we also obtain many important managerial insights.

Part 2: Inventory Rationing with General Demand Processes and backordering

In this part we extend the research of Part 1 by changing demand process from Poisson process to very general ones, for example, the customer arrival process can be other non-Poisson process and a customer can require more than one unit of the product. Under very general demand processes and positive lead time, little is known about the structure of optimal rationing policies. From Part 1 of this research we have known that the dynamic critical level policy can save much cost comparing with the static rationing policy. So we analyze dynamic inventory rationing, assuming a dynamic critical level policy in these systems. We develop models for both single period and multiperiod systems.

We first consider a single period system assuming a dynamic critical level rationing policy. A method is proposed to obtain near-optimal parameters for the dynamic rationing policy, and approximate expressions for the cost function are also developed. Then we use these results to analyze inventory rationing in a multiperiod system with the periodic review, base stock ordering policy, denoted as (R, S) policy, and positive lead time, following a similar procedure to in Part 1. A near-optimal solution with a dynamic critical level rationing policy to the optimization problem is obtained.

A numerical study is conducted to investigate effectiveness of the proposed method, assuming Poisson demands (for we can obtain optimal solutions for the Poisson demand, we can compare them with those of the proposed method). The results show that the costs under the near-optimal dynamic rationing policy are very close to optimal costs in these examples.

Part 3: Inventory Rationing with Poisson Demands and Lost Sales

In the previous two parts, the rejected demands are backordered, while in the third part they are lost. We consider both finite and infinite horizon multiperiod systems with Poisson demands. For the finite horizon M-period system with a periodic review, fixed ordering amount ordering policy, denoted as (R, Q) policy, a dynamic programming model is developed to minimize total discounted cost, dividing each period into many small intervals. The optimal rationing policy in each period is shown to be the dynamic critical level policy.

We then extend the model to infinite horizon. Important characteristics of the optimal rationing policy, cost function and ordering amount are also obtained. We show there is such an optimal dynamic rationing policy on the whole horizon in which the dynamic critical levels will not change from one period to another period, though dynamic critical levels in each period change with the remaining time before the end of the period. In other words the dynamic critical levels are independent on the index of the periods. A numerical study is conducted to obtain some important managerial insights.

The remainder of the dissertation is organized as follows. In Chapter 2, we provide a literature review about inventory rationing and a comparison between our research and relevant literature. In Chapter 3, we consider dynamic rationing for systems with Poisson demand and backordering. Chapter 4 studies dynamic rationing for systems with general demand processes and backordering. Chapter 5 analyzes dynamic rationing for systems with Poisson demand and lost sales. Finally, Chapter 6 concludes the thesis providing a summary of the research and a discussion of possible future research.

Chapter 2

Literature Review

Research about multiple class inventory problems can be traced back to Veinott (1965). It shows that the base stock ordering policy is optimal for the periodic review inventory systems under some conditions, in which there is no inventory rationing during each period and backorders are fulfilled according to demand class priority at the ends of periods when replenishments arrive. Inventory rationing among multiple demand classes is first analyzed by Topkis (1968) which shows the optimal rationing policy in a period is a dynamic critical level policy under some general demand processes. Since then there are many researchers to explore inventory rationing in different problem settings.

The literature can be categorized by different criteria, for example, number of demand classes that a model can apply to (many papers address the cases with only 2 demand classes), whether customers would like to wait for later fulfillment when shortage occurs (backordering or lost sales), property of the ordering policy (periodic review or continuous review, single or multiple ordering opportunities), and type of rationing policies (static or dynamic rationing policies). We organize the literature according to the type of rationing policies, for how to ration inventory is the key decision in these inventory problems and problems considering dynamic rationing policies are significantly different from those considering static rationing policies.

As noted in the previous chapter, in the literature there exist two types of rationing policies: static critical level policies and dynamic critical level policies. The critical level is sometimes termed as *threshold*. In the dynamic critical level rationing policy, the critical levels change with time, while in the static critical level policy, they are constants.

Though the first paper (Topkis 1968) addressing inventory rationing considers the dynamic rationing policy, it is surprising that since then most of later papers consider the static rationing policy and have made notable progress, and at the same time quite limited progress is made about the dynamic rationing policy. One main reason is that the dynamic rationing policies are extremely difficult to analyze. Another reason is that the static critical level policy is easy to understand and implement by inventory practitioners.

The remainder of this chapter is organized as follows. We first summarize papers considering static rationing policies, then papers considering dynamic rationing policies. In Section 2.1, we compare our work with relevant literature.

Research Considering Static Critical Level Policies

Inventory rationing with static rationing policies is considered by many researchers and they have made notable progress. Though in most cases the static rationing policy is not optimal for the system managers may dynamically ration stock to save cost using such information as the arrival time of the next replenishment and so on, in some special cases the static critical level policy is indeed optimal and some researchers have shown it.

Some people analyze the service levels of different classes and obtain expressions for them, assuming the static critical level rationing policy and a certain ordering policy. Nahmias et al. (1981) consider inventory rationing in both periodic review and continuous review (*r*, *Q*) systems with 2 demand classes and backordering, and obtain approximate expressions for service levels. Moon et al. (1998) extend the work of Nahmias et al. (1981). They extend the single period model in Nahmias et al. (1981) from 2 demand classes to multiple demand classes, and develop a single period model assuming the demands are deterministic and constant. They also develop two simulation models. Dekker et al. (1998) consider inventory rationing for a system with continuous review (*S*-1, *S*) ordering policy, 2 demand classes and backordering and approximate expressions for service levels are also developed.

Some authors develop optimization models to minimize cost, assuming the static critical level rationing policy and a certain ordering policy, trying to obtain optimal or near-optimal parameters of rationing and ordering policies. Cohen et al. (1988) consider inventory rationing in a system with periodic review (s, S) ordering policy, deterministic lead time, 2 demand classes and lost sales. An approximate, renewal-based model is derived and a greedy heuristic is developed to minimize expected cost subject to a fill rate service constraint. In this model, inventory is issued at the end of each period according to priority of demand classes, in other words, it assumes the static critical level for any class is 0. In some other optimization models the shortage cost is explicitly included in the total cost expressions.

Melchiors et al. (2000) analyze inventory rationing in a continuous review (s, Q) inventory system with 2 demand classes and lost sales, deterministic lead time and at most one outstanding order, assuming the static critical level rationing policy. For Poisson demand and deterministic lead times, the paper presents an expression for the average inventory cost and a simple optimization procedure based on enumeration and bounds.

Like Melchiors et al. (2000), Deshpande et al. (2003) also consider a (s, Q) inventory system with 2 demand classes, but the shortages are backordered. It assumes a threshold clearing mechanism for backorder clearing when a replenishment arrives. This backorder clearing mechanism is assumed to make the problem tractable. Under these assumptions exact expressions for average cost are obtained, and an efficient solution algorithm for computing stock control and rationing parameters is established.

Arslan et al. (2005) extend the model in Deshpande et al. (2003) by allowing more than 2 demand classes. This paper shows the equivalence between the considered inventory system and a serial inventory system. Based on this equivalence, a model for cost evaluation and optimization is developed. It proposes a computationally efficient heuristic and develops a bound on its performance. Unlike Deshpande et al. (2003) this model is to minimize holding cost subject to service level requirement.

In the above papers the inventory supply is exogenous. Some people consider inventory rationing in make-to-stock production systems where the supply is modeled explicitly as a production facility. In some cases, the static critical level rationing policy is indeed optimal. Ha (1997a) considers inventory rationing in a make-to-stock production system with exponential production time, Poisson demands, multiple demand classes and lost sales. It shows the optimal rationing policy is a static critical level policy and the optimal production policy is a base stock policy. For the property of memoryless of the exponential production time and the Poisson demand, it is quite intuitive that the optimal rationing policy is a static critical level policy. This production policy is equal to (S-1, S) ordering policy with exponential lead time.

Ha (1997b) extends Ha (1997a) by allowing backorders, but the model can apply to cases with only 2 demand classes. It shows that the optimal production policy is of base stock type and the optimal rationing policy has a monotone switching curve structure.

Véricourt et al. (2002) extend the model of Ha (1997b) by allowing more than 2 demand classes. It shows that the optimal rationing policy is a static critical level policy. It is also quite intuitive for the inventory system is memoryless. It has also developed an efficient algorithm to compute optimal parameters of the rationing policy.

Ha (2000) considers an inventory system that is the almost the same as Ha (1997a) expect the production time is modeled as an Erlang-k distribution. The work storage level (inventory level plus the finished stages for a job) is used to capture information regarding inventory level and the status of current production. The optimal rationing policy is characterized by a sequence of critical work storage levels. The optimal production policy is also characterized by a sequence of critical work storage levels.

Like Ha (2000), Gayon et al. (2004) consider inventory rationing in a make-to-stock system with the information about the production status. The production time is also an Erlang-k distribution, but the shortages are backordered in this model while shortages are lost in Ha (2000). It shows the optimal rationing policy is also the static critical level (of work storage) policy.

Lee and Hong (2003) analyze inventory rationing in a (s, S) controlled production system with 2-phase Coxian process times, Poisson demands and lost sales. Assuming a static critical level rationing policy, expressions for the steady state probability distribution of the system are obtained.

Research Considering Dynamic Critical Level Policies

The literature on dynamic critical level rationing policies is quite limited comparing with those on static rationing policies, though the earliest paper (Topkis 1968) about inventory rationing considers the dynamic rationing policy. Topkis (1968) first analyzes dynamic rationing in a single period system with general demand processes. By dividing the single period into some small intervals, a dynamic programming model is developed which can apply to cases of backordering, lost sales and partial backordering. It shows the optimal rationing policy is a dynamic critical level policy. Then he embeds the single period model into a multiperiod inventory system with zero lead time. Independent of Topkis (1968), Evans (1968) and Kaplan (1969) present some results similar to Topkis (1968) for the case with 2 demand classes.

Melchiors (2003) considers dynamic inventory rationing in an inventory system with lost sales, Poisson demands, deterministic lead time, continuous review (s, Q) ordering policy and at most one outstanding order, assuming a restricted dynamic critical level policy (called *restricted time-remembering policy* in the paper) which has some constraints for critical levels. In this rationing policy the lead time is divided into some intervals. It assumes the critical levels in each interval are constant and the critical levels when there is an outstanding order are the same as those in the first interval of the lead time. Expressions for the expected average cost are developed, given parameters of ordering and rationing policies. Based on some empirical observations a neighbor searching heuristic is developed to find appropriate values for policy parameters.

Teunter and Haneveld (2008) consider dyanmic inventory rationing in a single period system with backordering, Poisson demand and two demand classes. They first assume the system adopts a dynamic critical level policy and the critical level at the end of the period of the less important class is zero. Let T_i denote the time when the critical level rises from $i-1$ to i. They then develop a heuristic to find the times T_i through finding the lengths $L_i = T_i - T_{i-1}$. The expressions for L_i are complicated and long and their method is not appropriate for large values of critical levels. In fact, in the paper they just show the expressions of L_i for $i \le 5$.

Dynamic rationing policies have also been studied in the airline seat control in which a pool of identical seats is sold at different prices to different customers. The paper most relevant to our research is Lee and Hersh (1993). They consider the dynamic seats rationing over a finite horizon, assuming demands of each class follow a Poisson process. They show that the dynamic critical level rationing policy is optimal. It is a single period problem with no holding cost. While in our research, we consider inventory rationing in the multiperiod systems with ordering policies and holding cost, where the multiple ordering opportunities and holding cost make the problems much more complicated than the single period airline seat control problems. For more information about airline seat control or airline revenue management, see McGill et al. (1999), which presents a good review on airline revenue management.

2.1 Comparison between Our Work and Literature

From the above we can see that notable progress has been made about the static rationing policy and theory about the dynamic rationing policy is quite limited. Motivated by possible significant potential of dynamic rationing policies in cost saving, this dissertation considers dynamic inventory rationing in different situations with typical practical problem settings, for example, the lead time in some multiperiod systems is positive.

This research is divided into three parts. The first part considers the dynamic rationing for multiperiod systems with Poisson demand and backordering. We first consider dynamic inventory rationing in a multiperiod system with zero lead time. Characteristics of optimal ordering and rationing policies are obtained. Then we consider a multiperiod system with positive lead time and infinite horizon. An optimization model to minimize average cost is developed and a near-optimal solution is obtained. A lower bound on the optimal cost under optimal ordering and rationing policies is also established. In the literature most researchers consider inventory rationing assuming static rationing policies and only Topkis (1968) characterizes the optimal dynamic rationing policy in a single period system and then embeds the single period model into a multiperiod system with zero lead time. In this dissertation we also characterize the optimal rationing and ordering polices for a multiperiod system with zero lead time. There are some notable differences between Topkis (1968) and our work. One main difference between our work and Topkis is that the single period model with backordering in Topkis (1968) is a multi-dimensional dynamic programming one, while our model for dynamic inventory rationing during each period is a one-dimensional dynamic programming one without the curse of dimensionality which Topkisȏs model suffers. Another difference is that the penalty cost in our model includes a part of penalty per unit and a part of penalty per unit per unit time, which is more accurate than that in Topkis. In addition we also consider the case with positive lead time. In this case a near optimal solution is obtained and a lower bound on optimal cost is developed. Our multiperiod model with positive lead time is quite practical and the dynamic critical level policy is easy to calculate and implement in practice.

The second part of this research studies dynamic rationing for systems with very general demand processes and backordering. Under such general demand processes, one customer may demand a random amount of product and the arrivals of customers may not follow Poisson process. When the demand processes is very general and the lead time is positive in multiperiod systems, little is known about the structure of the optimal rationing policy. Currently we have not found other papers to address dynamic rationing in such cases. By assuming the dynamic critical level policy, we develop methods to obtain near optimal parameters for the dynamic rationing policy and ordering policy. Our work places a benchmark for relevant future research.

The third part of the research analyzes dynamic rationing for multiperiod systems with lost sales and Poisson demand. We first consider a multiperiod system with finite horizon, assuming a periodic review, fixed ordering amount ordering policy, and a dynamic programming model is developed. Characteristics of the optimal dynamic rationing policy, optimal cost function and optimal parameter of the ordering policy are obtained. We then extend it to the case with infinite horizon. In the literature there are a few papers consider dynamic rationing policies with lost sale. Lee and Hersh (1993) has considered a single period model with dynamic rationing policy and lost sales for airline seats management. It is in fact a special case of our multiperiod model by assuming there is only one period, no holding cost, and no salvage value of remaining stock at the end of the period. Melchiors (2003) considers a restricted dynamic inventory rationing in an inventory system with continuous review (*s*, *Q*) ordering policy and develops expressions for average cost under given rationing policy, while our model is a periodic review ordering policy and the optimal dynamic rationing policy is obtained.

Chapter 3

Inventory Rationing for Systems with Poisson Demands and Backordering

3.1 Introduction

From previous chapters, we have seen that inventory rationing among different demand classes has a very wide application in industries and currently there exist two types of rationing policies: static critical level policies and dynamic critical level policies. Most relevant papers consider inventory rationing with static critical level policies, which is not optimal in most cases, and people have made notable progress on it. On the other hand, the theory about dynamic rationing policies is very limited.

Deshpande et al. (2003) have analyzed inventory rationing for a military logistics system with two demand classes, assuming a static critical level rationing policy and a continuous review ordering policy. It shows that the unknown optimal dynamic rationing policies may significantly reduce the cost, comparing with the static critical level rationing policy (in many cases the lower bound on optimal costs is more than 10% less than the costs under the static rationing policy). Motivated by the possible significant potential of the dynamic rationing policy in cost saving, in this chapter we explore dynamic inventory rationing in multiperiod systems assuming Poisson process and backordering. In these inventory systems, how to ration inventory and how to replenish inventory are key decisions.

We first consider inventory rationing in a multiperiod system with zero lead time. Dynamic programming models are developed to characterize the optimal ordering and rationing policies. We show that the optimal dynamic rationing policy in each period is a dynamic critical level policy and the optimal ordering policy is a base stock policy. We then analyze a multiperiod system with positive lead time. For the structure of optimal rationing and ordering policies may be very complex, we develop a near-optimal solution to the optimization problem of minimizing average cost by applying the dynamic critical level policy of the model with zero lead time and assuming a base stock ordering policy. A lower bound on the optimal cost under the optimal rationing and ordering policies is also obtained. A numerical study is then conducted to compare the dynamic rationing policy with current state-of-art static critical level policies. Results show that the dynamic rationing policy can save more than 10% of the cost in many cases, comparing with the static critical level policy. Results also show the costs under the dynamic rationing policy are also very close to the optimal costs.

In the above inventory rationing problem with backordering, we assume such a backorder clearing mechanism at the ends of the periods when replenishments arrive: fulfill backorders as much as possible according to class priority, i.e., first fulfill backorders of the most important class until all backorders of this class are fulfilled or there is no remaining on hand inventory, and if there are remaining stock after fulfilling all backorders of the most important class, then fulfill backorders of the second most important class and so on. We also investigate another backorder clearing mechanism in which it is possible to reserve stock for next periods by not fulfilling some backorders. The numerical results show this backorder clearing mechanism is better than the previous mechanism, but the difference of costs is very small in all studied examples.

Topkis (1968) has also considered dynamic inventory rationing in both a single period system and a multiperiod system with zero lead time. For the single period system, he assumes that the period can be divided into many intervals and demands in different intervals are independent. Then he develops a general dynamic programming model which can deal with very general demand processes. For the backorder case, his model is a multi-dimensional dynamic programming one and has the curse of dimensionality of dynamic program. He shows that the dynamic critical level rationing policy is optimal, and for multiperiod system with zero lead time, the optimal ordering policy is a base stock policy. It is the first and very significant result for the dynamic inventory rationing. However, his models have some limitations and some important questions remain. For example, for the curse of dimensionality of his dynamic programming model, in general it will be difficult to obtain optimal policies in reasonable time or a short time. For the backorder case, the state space of dynamic program in general is infinite and it needs to truncate, hence incurring some calculation errors. The truncated state space also increases exponentially with the number of dimensionality and the calculation time also exponentially increases. For the model is so complicated, Topkis does not show how the critical levels changes during the period for the backorder case (however, he shows that for the case of lost sales, under some strict conditions the critical levels decrease towards the end of the period). In addition, from the viewpoint of practitioners and researchers, there are some important questions outstanding: for the more practical conditions such as positive lead time, how to order and ration stock to minimize cost? If we can not find the optimal solution, can we find an effective near optimal solution? How is the near optimal solution close to the optimal costs? Our work will answer these questions, besides showing some important properties of the dynamic rationing policy, for example, the dynamic critical level rationing policy is optimal, the critical levels decrease towards the end of the period and so on.

In the following we summarize some fundamental differences between Topkis and our work:

(a) The single period model with backordering in Topkis (1968) is a multiple dimensional dynamic programming one, while our model for dynamic inventory rationing during one period is one-dimensional dynamic programming one, assuming unmet demands during a period can be fulfilled only at the ends of the periods. This assumption is quite intuitive and practical, and brings significant benefit of eliminating the curse of dimensionality that Topkis (1968) suffers. So our model can deal with any number of demand classes.

(b) The demand process in Topkis is very general, while the demand process in our models is Poisson process. For the demand process is so general, Topkis has not shown some important properties of critical levels of the rationing policy. While in our models, by assuming Poisson process we have shown more structural properties of the optimal dynamic rationing policy.

(c) The third difference is about the penalty cost of backorders. Models in Topkis (1968) apply only to the case of penalty cost of delay, i.e., penalty cost per unit per unit time, while our models have a general penalty cost for backorders which includes both delay and stockout parts, i.e., includes the penalty cost per unit and the penalty cost per unit per unit time. This general penalty cost can more precisely represent the practical cases, but it is more difficult to trace (note that Deshpande et al. (2003) also consider this general penalty cost).

(d) For the multiperiod systems, the lead time in Topkis is zero, while we consider not only the case of zero lead time, but also the case of positive lead time. For the positive lead time case, we find a near optimal solution including both the ordering policy and the rationing policy. A lower bound on the optimal costs is also developed.

Our work is also significantly different from other literature. Lee and Hersh (1993) considers a single period model with lost sales and Poisson demand and without holding cost and without ordering policy, while we consider multiperiod systems with backordering, Poisson demand, holding cost and with an ordering policy. The holding cost is an important factor during the decision making about inventory rationing and ordering. Teunter and Haneveld (2008) consider dyanmic inventory rationing in a single period system with backordering, Poisson demand and two demand classes. Assuming the dyanmic critical level rationing policy, they have developed a heuristic to find the times when the critical level changes. Instead of asssuming a certian type of rationnig policy as in their model, we show that the dynamic critical level rationing policy is optimal and many important properties of the optimal policy are also found. In addition, our model can deal with any number of classes and any large values of critical levels, while their model can deal with only two demand classes and small values of critical levels. We also address multiple period systems, while they consider a single period problem.

Deshpande et al. (2003) consider the static rationing policy for a system with continuous review ordering policy, while we consider the dynamic rationing policy for a system with periodic review ordering policy. Moreover, the model in Deshpande et al. (2003) can apply to the cases with only two demand classes, while our models can apply to cases with any number of classes. The numerical results show that our dynamic inventory rationing policy can significantly reduce costs comparing with the static policies: in many cases, the cost reduction can be more than 10%.

The remainder of this chapter is organized as follows. In Section 3.2, we consider dynamic inventory rationing in a multiperiod system with zero lead time. Section 3.3 extends the analysis to a multiperiod system with positive lead time and infinite horizon. In Section 3.4, a numerical study is conducted to compare the dynamic rationing policy with the static rationing policy. We also compare the costs under our dynamic rationing policy with the lower bound on optimal costs. Section 3.5 compares two backorder clearing mechanisms. In Section 3.6, conclusions are given. Proofs of lemmas and theorems of this chapter are provided in Appendix A.

3.2 Dynamic Rationing for a Multiperiod System with Zero Lead Time

3.2.1 Model Formulation

Consider a multiperiod inventory system with M periods in which K independent demand classes request for the same product. Demand classes are characterized by different penalty costs of shortage. The periods are numbered as $1,2,\ldots,M$. At the beginning of period 1 there is some initial on-hand inventory. The inventory is reviewed at the beginning of each period and the system manager makes a decision about how much to order. Assume the lead time of orders is zero. During each period, demands from K classes continuously arrive. When a demand arrives, the system manager needs to make a decision immediately about whether or not to satisfy it, i.e., how to ration stock. Rejected demands are backlogged.

Assume backorders can be fulfilled only at the ends of periods by remaining on-hand inventory at the ends of periods or purchased new stock at the beginnings of periods. That is, a rejected demand can not be fulfilled at the middle of a period. Such assumption is quite practical and reasonable, though in some cases it is better to fulfill some outstanding backorders in the middle of a period. For example, if the remaining time before the end of the current period is very short and the future demand is small and there is a very large on-hand inventory, then it would be better for the system to fulfill some outstanding backorders at this time immediately than continuously waiting until the end of the period to fulfill backorders. Though there are such cases, they will be rare. More importantly, this assumption makes dynamic inventory rationing problems tractable as we will see later in this section.

Assume demands and cost factors are stationary. We are trying to find the optimal ordering and rationing policies to minimize expected total cost including ordering cost, holding cost and penalty cost of shortage on the whole horizon.

Assume demand of class i, $i \in \{1,...,K\}$, follows a Poisson process with arrival rate λ_i. If a demand of class i is rejected, then the demand is backordered and the penalty cost $\pi_i + \breve{\pi}_i \cdot t$ is incurred, where π_i is the penalty cost per unit, $\breve{\pi}_i$ is the penalty cost per unit per unit time and t is the length from the time when the demand is rejected to the time when the backorder is fulfilled. We assume that if $\pi_i \geq \pi_j$, then $\breve{\pi}_i \geq \breve{\pi}_j$ as

Deshpande et al. (2003) did. In general it is valid. But there may be some opposite cases in practice such that $\pi_i \geq \pi_j$, but $\hat{\pi}_i \leq \hat{\pi}_j$. In these cases, the order of demand classes is not regular, and the problem becomes more complicated and the structure of the optimal policy may be different from what we will show in later sections. We assume that if $i < j$, then $\pi_i \geq \pi_j$ and $\hat{\pi}_i \geq \hat{\pi}_j$, i.e., class 1 has the highest priority and class K has the lowest priority.

Assume the ordering cost is linear with the price c per unit. That is, there is no fixed setup cost. At the end of a period, the system uses the remaining stock to fulfill backorders. The remaining stock may not be enough to fulfill all backorders, so there are some remaining backorders unfulfilled at the end of a period. Let x_m, $m \in \{1,...,M+1\}$, denote the net inventory, which is the on-hand inventory minus total backorders, at the beginning of period m before ordering (i.e., at the end of period $m-1$) and $\boldsymbol{B}_m = \{b_1,...,b_K\}$ denote the vector of backorders of each demand class at the beginning of period m before ordering, where b_i is the backorder of class i. So if there are some backorders unfulfilled, then $x_m = -\sum_{i=1}^{K} b_i < 0$. Assume the terminal cost function at the end of horizon is $C_T(x_{M+1}) = -cx_{M+1}$, where c is the variable ordering cost. So if there is some remaining stock after fulfilling all backorders at the end of period M, the remaining stock has a salvage value of c per unit, and if there are remaining unfulfilled backorders, then the system has a cost of c per unit unfulfilled backorder.

At the beginning of period m, the system first observes the net inventory x_m and the backorder vector $\boldsymbol{B}_m$, and then makes a decision about how much to order. Let y_m denote the inventory position at the beginning of period m after ordering. Inventory position is the on-hand inventory plus outstanding orders minus backorders. For the lead time is zero, we have no outstanding orders. We have the following lemma for y_m under optimal ordering policies.

Lemma 3.1. *Under optimal ordering policies, there are no outstanding backorders at the beginning of period m,* $m \in \{1,...,M\}$*, after ordering, and hence* $y_m \geq 0$.

The above lemma shows that all rejected demands in a period will be fulfilled by the remaining on-hand inventory at the end of the period and the new purchased stock at the beginning of the next period. So the backorders in a period will never be forwarded to the next period. For there are no outstanding backorders at the beginning of a period after ordering, the inventory position $y_m \geq 0$, which is the on-hand inventory plus outstanding orders minus backorders. For the lead time is zero, there are no outstanding orders and y_m in fact is the on-hand inventory. The above result is quite intuitive. If a rejected demand is fulfilled after a longer time, then it will incur a larger penalty cost. On the other hand, the cost of purchasing stock to fulfill a backorder is stationary. So it is not economic to delay fulfilling backorders.

Let D_m denote the total demand of all demand classes in period m. For the demands are stationary, D_m has the same probability distribution for all index m. We also have $x_{m+1} = y_m - D_m$. In the following we develop a dynamic programming model for the optimization problem of minimizing expected total cost on the whole horizon, in which each period is a stage.

For all possible backorders at the end of a certain period should be fulfilled at the beginning of its next period according to Lemma 3.1, we consider net inventory x_m as the state variable in the dynamic programming model, ignoring $\boldsymbol{B}_m$. Let $C_m(y_m), y_m \geq 0,$ denote the expected holding and penalty cost in period m, given the on-hand inventory level y_m at the beginning of period m after ordering. $C_m(y_m)$ is dependent on the rationing policy in period m. Let $\upsilon = \{\upsilon_1,...,\upsilon_M\}$ denote the dynamic rationing policy on the whole horizon, where υ_m, $m \in \{1,...,M\}$, denote the dynamic rationing policy in period m.

Let $V_m(x_m)$ denote the optimal expected cost from period m to the end of the horizon, given net inventory x_m at the beginning of period m before ordering. The optimization problem is to choose the ordering and rationing policies to minimize the total cost, i.e.,

$$V_m(x_m) = \min_{y_m \geq x_m, \upsilon_m} c(y_m - x_m) + C_m(y_m) + E[V_{m+1}(y_m - D_m)], m \in \{1,...,M\}, \quad (3.1)$$

where $V_{M+1}(x_m) = -cx_{M+1}$ and $c(y_m - x_m)$ is the ordering cost.

In the above expression, υ_m is the dynamic rationing policy in period *m*. The following lemma shows that under the optimal ordering policy, the optimal rationing policy in period *m* is independent on the rationing policies in other periods and parameter settings in other periods.

Lemma 3.2. *Under the optimal ordering policies, the optimal rationing policy in period m, $m \in \{1,...,M\}$, is determined solely by the parameter setting in period m.*

The above lemma comes from Lemma 3.1. For a given inventory position y_m at the beginning of period m after ordering, the rationing policy in period *m* will affect the backorder vector $\boldsymbol{B}_{m+1}$, but will not affect the net inventory x_{m+1}, which is determined by y_m and the total demand D_m of all classes in period *m*. After ordering at the beginning of period $m+1$, all backorders in $\boldsymbol{B}_{m+1}$ will be fulfilled, so different rationing policies in period *m* make no difference for the costs of later periods. Thus the optimal rationing policy in period *m* can be determined solely by the parameter settings in period *m*.

The above property is very useful. Based on this property, we can separately consider the ordering and rationing policies for problem (3.1). In the following we first obtain the optimal dynamic rationing policy in each period and properties of $C_m(y_m)$ under the optimal rationing policy in subsection 3.2.1, and then based on these properties of $C_m(y_m)$ we obtain the optimal ordering policy by induction in subsection 3.2.2. For the cost factors and demand process are stationary, we drop the subscript *m* in $C_m(y_m)$ since here.

3.2.2 Characterization of the Optimal Dynamic Rationing Policy

In this section we consider how to dynamically ration stock to minimize expected total holding and penalty cost in a certain period, given initial on-hand inventory y at the beginning of this period, i.e., $\min_{\upsilon} C(y)$, given $y \geq 0$.

According to previous assumptions, there are K independent demand classes requesting for the same product during the period, and demand of each class follows a Poisson process with arrival rate λ_i, $i \in \{1,...,K\}$. During the period, when a demand of class i arrives, the system needs to immediately make a decision about whether to satisfy it or reject it. If it is rejected, then the demand is backordered. For we have assumed backorders can be fulfilled only at the ends of periods, the penalty cost of shortage incurred is $\pi_i + \breve{\pi}_i \cdot t$, where t is the remaining time before the end of the period. The inventory during the period has a holding cost of h per unit per unit time.

We formulate the dynamic inventory rationing problem in this period as a discrete-time Markov decision problem. Divide the period into N equal intervals such that the intervals are so small that the probability of more than 2 demands arriving in each interval is very small and it can be ignored. Let u denote the length of the period. So the length of each interval is $\Delta t = u / N$. Then the probability that a demand of class i arrives during an interval, p_i, is $\lambda_i \Delta t + o(\Delta t)$, and the probability that no demand of any class arrives during an interval, p_0, is $1 - \lambda \Delta t + o(\Delta t)$, where $\lambda = \sum_1^K \lambda_i$. Let the time points separating the intervals are indexed as $N, N-1, ..., 0$, i.e., the beginning of the period is time point N and the end of the period is time point 0. We also index the interval which begins at time point n and ends at time point $n-1$ as n. If a demand from class i arrives during interval n, we assume the system delays the decision about whether to satisfy or reject it until the end of the interval. If we reject this demand, penalty cost $\pi_i + \hat{\pi}_i \cdot (n-1) \cdot \Delta t$ incurs. Hence, these time points are the times when the decisions are made on whether to satisfy or reject the demands.

Let $x(n)$, $0 < n \le N$, denote the on-hand inventory at the beginning of interval n, which is just after the decision at time point n. Let $H(n,x)$, $0 \le n \le N$, denote the optimal expected holding and shortage cost from the beginning of interval n to the end of the period, given on-hand inventory $x(n)$. When $n = 0$, $H(0,x) \equiv 0$ for we consider only holding and shortage cost now. In the following development about $H(n,x)$, if we say "cost" and do not explicitly state *holding and penalty cost*, it means the holding and penalty cost.

Now consider interval n. Assume a demand of class i has arrived during interval n. When $x(n) > 0$, if we satisfy it, then the total expected cost from the beginning of interval n to the end of the period is $C_{sat}(n,x) = x \cdot \Delta t \cdot h + H(n-1, x-1)$. If we reject the demand, the demand is backordered and the total expected cost from the beginning of interval n to the end of the period is $C_{rej}(n,x) = x \cdot \Delta t \cdot h + H(n-1, x) + e_i(n-1)$, where $e_i(n-1) = \pi_i + \hat{\pi}_i \cdot (n-1) \cdot \Delta t$. Let $\Delta_x(n,x) = H(n,x) - H(n,x-1)$. We should satisfy the demand if and only if $C_{rej}(n,x) \ge C_{sat}(n,x)$, i.e.,

$$\Delta_x(n-1,x)+e_i(n-1)\geq 0.$$

In the case when $x(n)=0$, if a demand of class i arrives during interval n, then it should always be rejected and the expected total cost from the beginning of interval n to the end of the period is $e_i(n-1)+H(n-1,0)$. When $x(n)>0$ and there is no demand that arrives during interval n, the total expected cost from the beginning of interval n to the end of the period is $x\cdot\Delta t\cdot h+H(n-1,x)$. When $x(n)=0$ and there is no demand during interval n, the total expected cost is $H(n-1,0)$. So, given the on-hand inventory $x(n)$, the optimal expected holding and shortage cost $H(n,x)$ is

$$H(n,x)=\begin{cases} x\cdot\Delta t\cdot h+p_0\cdot H(n-1,x)+\sum_{i=1}^{K}p_i\cdot\min[H(n-1,x-1), \\ H(n-1,x)+e_i(n-1)]; & \text{for } x>0, n>0 \\ p_0\cdot H(n-1,0)+\sum_{i=1}^{K}p_i\cdot[e_i(n-1)+H(n-1,0)]; & \text{for } x=0, n>0 \\ 0. & \text{for } x\geq 0, n=0 \end{cases} \qquad (3.2)$$

The above model is a one-dimensional dynamic programming one in which the state variable is the on-hand inventory x, so it is easy to solve.

For a given time point n, the optimal cost function $H(n,x)$ is a discrete function defined on nonnegative integers, i.e., the on-hand inventory x is a nonnegative integer. In later parts, sometimes we may say ñincreasingò (for a function) which means ñnondecreasingò, i.e., we use the weak sense of ñincreasingò. We have the following lemma to show an important property of $H(n,x)$.

Lemma 3.3. *For a given time point* n, $0 < n \leq N$, *the first difference of the function* $H(n,x)$ *is nondecreasing in* x, *i.e.,* $\Delta_x(n,x) = H(n,x) - H(n,x-1)$ *is nondecreasing in* x.

The above lemma shows that the marginal cost (holding and penalty cost) of having one more inventory is nondecreasing. It is worth to note that the condition $H(0,x) = 0$ is not the necessary condition for the above lemma. From the proof of Lemma 3.3, we can see that we use only the condition that the first difference of $H(0,x)$ is nondecreasing in x (no matter what the exact expression is) to prove the above lemma.

From Lemma 3.3, we can obtain the optimal rationing policy. The system manager determines whether or not to satisfy a demand of class i at time n by comparing the cost of rejecting it with that of satisfying it. Given the current on-hand inventory x at time n, the system should satisfy the demand of class i, if and only if $H(n,x) + e_i(n) \geq H(n,x-1)$, i.e., $\Delta_x(n,x) + e_i(n) \geq 0$. For $\Delta_x(n,x) = H(n,x) - H(n,x-1)$ is nondecreasing in x, there exists a critical on-hand inventory level such that when x is larger than it, then $\Delta_x(n,x) + e_i(n) \geq 0$, hence the demand of class i should be satisfied, and when x is at or below this critical level, then $\Delta_x(n,x) + e_i(n) < 0$ and the demand should be rejected. Hence the optimal rationing policy is a critical level policy, and the critical level for class i at time point n can be obtained by

$$x_i^*(n) = \min\{x \mid \Delta_x(n,x) + e_i(n) \geq 0\} - 1. \quad (3.3)$$

So the optimal rationing policy is the dynamic critical level policy with critical level $x_i^*(n)$. We have the following theorem for the optimal rationing policy.

Theorem 3.1.

(a) *The optimal rationing policy is a dynamic critical level policy with critical levels* $x_i^*(n)$, $i \in \{1,...,K\}$ *and* $n \in \{0,...,N\}$.

(b) *If* $i < j$, *then the dynamic critical level of class* i *is below the critical level of class* j, *i.e.,* $x_i^*(n) \le x_j^*(n)$.

(c) *The dynamic critical level of class* i *is nondecreasing in* n *and the critical level for class* i *at time point 0 is 0, i.e.,* $x_i^*(n+1) \ge x_i^*(n)$ *and* $x_i^*(0) = 0$.

Thus we have obtained the optimal dynamic rationing policy in the period. Part (b) of the above theorem comes from the property of the function $H(n,x)$: the first difference of $H(n,x)$ is nondecreasing in x. It states that if a demand of a certain class should be rejected at a certain time, then a demand from less important classes should always be rejected at this time. Part (c) states the trend of critical levels. It means: when the remaining time is longer, then the system should have more stock reserved for more important classes. The critical levels decrease towards the end of the period. At the end of the period, the system does not need to reserve any stock for more important classes, for a new replenishment will come immediately. So if there is on-hand inventory at the end of the horizon, a demand of any class should be satisfied. Hence the critical level of any demand class at this time is 0, i.e., $x_i^*(0)=0$. The results in parts (b) and (c) are quite intuitive.

It is worth to note that the critical levels are determined by equation (3.3), i.e., they are completely determined by parameters of demand process, holding and penalty costs, and the length of the period. In other words, these critical levels are independent on the initial on-hand inventory at the beginning of the period. It is a nice property. So the dynamic rationing policy is easy to implement in practice.

The above dynamic critical level rationing policy is the rationing policy in one period. For the above inventory rationing model is for any period of the multiperiod system, we can see that the optimal rationing policy for the multiperiod system is to apply the above dynamic critical level rationing policy in each period. From the above we also have obtained $\min_{\upsilon} C(y) = H(N, y)$ and an important property of the function $H(N, y)$: the first difference of $H(N, y)$ is nondecreasing in y.

3.2.3 Characterization of the Optimal Ordering Policy

Now consider the ordering policy for optimization problem (3.1). For we have found the optimal rationing policy and the corresponding cost in the previous section, the optimization problem (3.1) now becomes:

$$V_m(x_m) = \min_{y_m \geq x_m} c(y_m - x_m) + H(N, y_m) + E[V_{m+1}(y_m - D_m)],\ m \in \{1,\ldots,M\}, \quad (3.4)$$

where $c(y_m - x_m)$ is the ordering cost and $H(N, y_m)$ is the holding and penalty cost in current period (i.e., period *m*). In other words, the optimization problem (3.1) is: given the net inventory x_m at the beginning of period *m,* determine the optimal ordering amount $(y_m - x_m)$ to minimize the expected total cost from period *m* to the end of the horizon. So $V_m(x_m)$ is the optimal cost from period *m* to the end of the horizon, given the inventory x_m. The above optimization problem (3.4) needs to be solved for any given x_m. To determine an optimal ordering policy, we must solve problem (3.4) for every value of x_m.

Define

$$W_m(y_m) = c \cdot y_m + H(N, y_m) + E[V_{m+1}(y_m - D_m)],\ m \in \{1,...,M\} \tag{3.5}$$

So Equation (3.4) can be rewritten as

$$V_m(x_m) = -cx_m + \min\{W_m(y_m \mid y_m \geq x_m)\}. \tag{3.6}$$

Let $\bar{y}_m = \arg \min_{y_m \geq 0} W_m(y_m)$.

$V_m(x_m)$ is a function of the net inventory x_m, that is, for different values of x_m, the optimal cost from period *m* to the end of the period may be different. We have the following Lemma for the functions $W_m(y_m)$ and $V_m(x_m)$.

Lemma 3.4.

(a) *The first difference of the function* $V_m(x_m)$ *(i.e.,* $V_m(x_m+1)-V_m(x_m)$ *),* $m \in \{1,...,M\}$*, is nondecreasing in* x_m.

(b) *The first difference of the function* $W_m(y_m)$ *(i.e.,* $W_m(y_m+1)-W_m(y_m)$ *),* $m \in \{1,...,M\}$*, is nondecreasing in* y_m.

Parts (a) and (b) of the above lemma shows a property of the optimal cost function $V_m(x_m)$ and the function $W_m(y_m)$. From Part (b), we can obtain the optimal ordering policy. In many inventory systems, people have shown that a base stock ordering policy is optimal. A base stock ordering policy with the base stock level s^* is such a policy: when the initial inventory x is less than the base stock level s^*, then order (s^*-x) so that the inventory after ordering reaches the base stock level s^*, and if the initial inventory x is greater than the base stock level s^*, then do not order anything. In the following we show that the base stock ordering policy is also optimal in our inventory system.

Consider Equation 3.6. The optimal solution y_m^* to the minimization problem $\min\{W_m(y_m \mid y_m \geq x_m)\}$ depends on the relation between x_m and $\bar{y}_m$. If $x_m \leq \bar{y}_m$, then the optimal solution $y_m^* = \bar{y}_m$, so the system should order inventory up to $\bar{y}_m$. If $x_m > \bar{y}_m$, then the optimal solution $y_m^* = x_m$, because $W_m(y_m)$ is nondecreasing in y_m over the range $y_m \geq \bar{y}_m$ (which comes from the fact: $W_m(y_m)$ arrives at its minimum at $\bar{y}_m$ and the first difference of $W_m(y_m)$ is nondecreasing in y_m). So when $x_m > \bar{y}_m$, then the system should not order to increase inventory, for increasing inventory just incurs more cost. Hence the ordering policy at the beginning of period *m* is precisely a base stock policy with base stock level $\bar{y}_m$. So we have the following theorem for the optimal ordering policy.

Theorem 3.2.

The optimal ordering policy in period m, $m \in \{1,...,M\}$, *is a base stock policy with base stock level* $\bar{y}_m$.

The above theorem shows the optimal ordering policy under the optimal rationing policy is a base stock policy. So at the beginning of period *m*, the system manager first observes the inventory x_m, if x_m is less than the base stock level $\bar{y}_m$, then order up to the base stock level $\bar{y}_m$, otherwise, do not order any inventory.

In this optimal ordering policy, we need to calculate a series of base stock level $\bar{y}_m$, $m \in \{1,...,M\}$. In the following we show a myopic ordering policy with a constant base stock level is also optimal (assuming the above optimal rationing policy is always used). In this way, we need to calculate only one base stock level.

Let $G_m(y_m) = c \cdot y_m + H(N, y_m) - E[c \cdot (y_m - D_m)]$, $m \in \{1,\cdots,M\}$. For the first difference of the function $H(N, y_m)$ is nondecreasing in y_m, we can see the first difference of $G_m(y_m)$ is also nondecreasing in y_m. Define $V'_m(x_m) = V_m(x_m) + cx_m$. The recursive Equation (3.4) is equal to the following equation:

$$
\begin{aligned}
V'_m(x_m) &= \min_{y_m \ge x_m} cy_m + H(N, y_m) + E[-c(y_m - D_m) + V'_{m+1}(y_m - D_m)] \\
&= \min_{y_m \ge x_m} \{G_m(y_m) + E[V'_{m+1}(y_m - D_m)]\},
\end{aligned}
\tag{3.7}
$$

where $y_m - D_m = x_{m+1}$ and $V'_{M+1}(x_{M+1}) = 0$.

From the above equation, we can obtain a myopic ordering policy, i.e., to obtain an approximate ordering amount by minimizing $G_m(y_m)$, ignoring its effect on later periods. Let

$$
\bar{y} = \arg\min_{y_m \ge 0} G_m(y_m). \tag{3.8}
$$

We can see that $\bar{y}$ is the same for all *m*, i.e., it is independent on *m*. $G_m(y_m)$ can be regarded as the expected total cost of a single period system in which the terminal cost function is $C_T(x) = -c \cdot x$. For the first difference of $G_m(y_m)$ is nondecreasing, we obtain a myopic base stock ordering policy with base stock level $\bar{y}$: at the beginning of period *m*, if the net inventory x_m is less than $\bar{y}$, then order up to $\bar{y}$, otherwise do not order. The following theorem shows this myopic ordering policy is optimal.

Theorem 3.3. *The myopic ordering policy with base stock level $\bar{y}$ is optimal.*

According to this theorem, we can easily obtain the parameters of the optimal ordering policy. We can see that $\bar{y}$ is quite easy to calculate. Thus we have found the optimal ordering and rationing policies for a multiperiod system with zero lead time.

However, in practice, the lead time sometimes can be positive. So in the following we will consider dynamic inventory rationing in multiperiod systems with positive lead time.

3.2.4 A Dynamic Programming Model with Positive Lead Time

In this subsection we develop a dynamic programming model for the case with positive lead time and finite horizon to show the structure of optimal ordering and optimal rationing policies may be extremely complex and very difficult to obtain. For these reasons in the following section we consider minimizing average cost of a multiperiod system with infinite horizon and positive lead time and will develop a near-optimal solution.

Consider a multiperiod system with $M+L$ periods. Assume the lead time is positive, which is an integer times L of the length of a period. The demands and cost factors are the same as the previous case with zero lead time. The last ordering opportunity is at the beginning of period M, which will arrive at the beginning of period $M+L$, i.e., at the end of period $M+L-1$.

At the beginning of each period, the system first receives the order which arrives at this time and uses the on-hand inventory to fulfill backorders, then decides how much to order. For it is possible that the stock after receiving an order is not enough to fulfill all backorders, we need an assumption about how to fulfill backorders. Assume such a backorder clearing mechanism: fulfill backorders as much as possible from the most important class to the least important class. So if there is some remaining on-hand inventory, then no backorders are unfulfilled, and if there are some backorders unfulfilled, then there is no remaining on-hand inventory. We call this mechanism as *full-priority backorder clearing mechanism* (denoted as *mechanism M* in the later numerical study).

Let x_m and $\boldsymbol{B}_m = \{b_1,...,b_K\}$ denote the net inventory and outstanding backorders respectively at the beginning of period m just before placing an order, where b_i is the number of backorders of class i. Let z_m denote the ordering amount at the beginning of period m. Note that the last ordering opportunity is at the beginning of period M, so $z_m = 0$, $m \in (M+1,...,M+L\}$. Let $\boldsymbol{Z}_m = [z_m^1, z_m^2,...,z_m^{L-1}]$, where z_m^l, $l \in \{1,...,L-1\}$, is the order placed l periods ago based on the beginning of period m. The system state is the vector $[x_m, \boldsymbol{B}_m, \boldsymbol{Z}_m]$. Still let D_m denote total demands of all classes in period m. So

$$x_{m+1} = x_m + z_m^{L-1} - D_m , \tag{3.9a}$$

and $$\boldsymbol{Z}_{m+1} = [z_m, z_m^1,...,z_m^{L-2}] . \tag{3.9b}$$

Let υ_m denote the dynamic inventory rationing policy in period m. Let $\tilde{D}_m$ denote the realization of demands of each class during period m (note that D_m is the total demand of all classes). $\boldsymbol{B}_{m+1}$ is affected by many factors such as υ_m, x_m, $\boldsymbol{B}_m$, and z_m^{L-2} (the order will arrive at the end of period m) and the demand $\tilde{D}_m$ in period m, i.e.,

$$\boldsymbol{B}_{m+1} = f_m(\upsilon_m, x_m, \boldsymbol{B}_m, z_m^{L-2}, \tilde{D}_m) . \tag{3.9c}$$

The state of the system changes according to Equation (3.9), which is quite complicated.

When a demand is rejected, penalty cost $\pi_i + \hat{\pi}_i \cdot t_d$ is incurred, where t_d is the time length from the time when the demand is rejected to the time when the backorder is fulfilled. t_d may be greater than the length u of a period. Let t_r denote the time from the arrival of the demand which is rejected to the end of the current period. So $t_d = t_r + j \cdot u$, $\mathrm{j} \in \{0,1,...\}$. We adopt such a way to account the penalty cost: account $\pi_i + \hat{\pi}_i \cdot t_r$ as the cost incurred in the period during which the demand is rejected; and if a backorder of class i is outstanding at the beginning of a period, then a penalty cost $\hat{\pi}_i \cdot u$ is accounted at this period.

Let $C_m(x, \boldsymbol{B} \,|\, \upsilon_m)$ denote the expected holding and penalty cost incurred in period m under the rationing policy υ_m, given $(x_m, \boldsymbol{B}_m)$. The ordering amount z_m and outstanding orders $\boldsymbol{Z}_m$ will not affect the cost $C_m(x, \boldsymbol{B} \,|\, \upsilon_m)$. Let $V_m(x_m, \boldsymbol{B}_m, \boldsymbol{Z}_m)$ denote the optimal expected cost from period m to the end of the horizon, starting from period m with state $(x_m, \boldsymbol{B}_m, \boldsymbol{Z}_m)$. Assume the terminal cost function at the end of period $M + L$ is a certain function $J(x_{M+L+1}, \boldsymbol{B}_{M+L+1}, \boldsymbol{Z}_{M+L+1})$, so

$$V_m(x_m, \boldsymbol{B}_m, \boldsymbol{Z}_m) = \min_{z_m \geq 0, \upsilon_m} cz_m + C_m(x_m, \boldsymbol{B}_m \,|\, \upsilon_m) + E[V_{m+1}(x_{m+1}, \boldsymbol{B}_{m+1}, \boldsymbol{Z}_{m+1})], \qquad (3.10)$$

where $V_{M+L+1}(x_{M+L+1}, \boldsymbol{B}_{M+L+1}, \boldsymbol{Z}_{M+L+1}) = J(x_{M+L+1}, \boldsymbol{B}_{M+L+1}, \boldsymbol{Z}_{M+L+1})$,

and $z_m = 0$, $m \in \{M+1, ..., M+L\}$.

From the above functional equation, we can see that it is a multiple dimensional dynamic programming one, which is extremely difficult to solve. The structure of optimal ordering and rationing policies may be very complicated for the following reasons:

(a) The ordering amount z_m is dependent not only on the inventory position as in the general inventory problems, but also on other factors such as backorder vector $\boldsymbol{B}_m$ and rationing policy υ_m, i.e., $z_m = g_m(x_m, \boldsymbol{B}_m, \boldsymbol{Z}_m, \upsilon_m)$. By intuition, when the total backorders and other conditions are the same, it would be better for the system to order more when all backorders are of the most important class than when all backorders are of the least important class. In addition, for different rationing policies applied in period *m*, the ordering amount z_m may not be the same. So the optimal ordering policy may not have such nice property (a base stock policy) as that in the case with lead time zero.

(b) Optimal rationing policy υ_m^* in any period *m* is affected by many factors such as the outstanding orders $\boldsymbol{Z}_m$ in this period (especially the order that will arrive at the end of this period) and x_m and so on. For it is possible that there are remaining backorders unfulfilled at the end of period *m* after an order arrives at the end of this period, different rationing policies in period *m* will affect the distribution of backorders of each class in $\boldsymbol{B}_{m+1}$ and hence affect the cost incurred in the next period. So the rationing policy in one period can not be determined solely by the parameters in current period as in the case with zero lead time. Thus for different ordering amount that will arrive at the end of period *m*, the optimal rationing policy in this period may be different.

(c) The ordering policy interacts with the rationing policy. The optimal rationing policy in period *m* can not be determined solely by the parameters in period *m* as in the case with zero lead time. So we need to consider the inventory rationing and ordering policies at the same time. While in the case with zero lead time, we can separately obtain optimal ordering and rationing policies (first obtain the rationing policy, then the ordering policy). The complex interaction between the ordering policy and rationing policy may make the structure of optimal ordering and rationing policies very complex and difficult to analyze. So if the optimal rationing policy in one period is the critical level policy, such critical levels should be a function of many factors such as outstanding orders, backorders and so on.

For the curse of dimensionality of the dynamic programming model and we can not separately optimize the ordering and rationing policies, problem (3.10) is extremely difficult to solve. In the following section we consider minimizing average cost for the systems with infinite horizon and positive lead time.

3.3 Dynamic Rationing for a Multiperiod System with Positive Lead Time

In this section we consider dynamic inventory rationing in a multiperiod system with infinite horizon and positive lead time. An optimization model to minimize average cost is developed and a near optimal solution is obtained. A lower bound on the optimal cost under optimal ordering and rationing policies is also developed.

3.3.1 Model Formulation

Consider a multiperiod system with infinite horizon and positive lead time L, in which the demand process and cost factors are the same as the previous multiperiod system with zero lead time. Note that here L can be any positive value, not necessarily be integer times of the period length. We call the time between two successive order opportunities as an *ordering period* and the time between two successive arrivals of orders as a *replenishment period*. The periods are indexed as $0,1,...$, i.e., the first period is period 0. Each period has the same length u. Assume the time at the beginning of ordering period 0 is 0. The order placed at the beginning of ordering period m, $m \in \{0,1,...\}$, will arrive at time $l_m = mu + L$. The time from l_m to l_{m+1} is the replenishment period m.

The change of inventory position and inventory level in the system is shown in Figure 3.1. In the figure, the solid line represents the inventory level and the dash line represents the inventory position.

During a period, when a demand of class i arrives, the system needs immediately make a decision about whether to satisfy or to reject it. The rejected demands are backordered. Again assume backorders can be fulfilled only at the ends of replenishment periods. Also assume a *full-priority* backorder clearing mechanism (also denoted as *mechanism M* for short) to fulfill backorders: to fulfill backorders as much as possible from the most important class to the least important class when a replenishment arrives. This mechanism is quite reasonable, for it first fulfills the most important backorders, then less important ones and so on. It also makes the problem tractable. This backorder clearing mechanism is not optimal in some cases, for example, when the on-hand inventory is low at the end of a period and there are many outstanding backorders, it may be better for the system not to fulfill a backorder of the least important class to reserve stock for the important demands in the next period. We will later consider another backorder clearing mechanism and compare it with this *full-priority* backorder clearing mechanism in the section of numerical study.

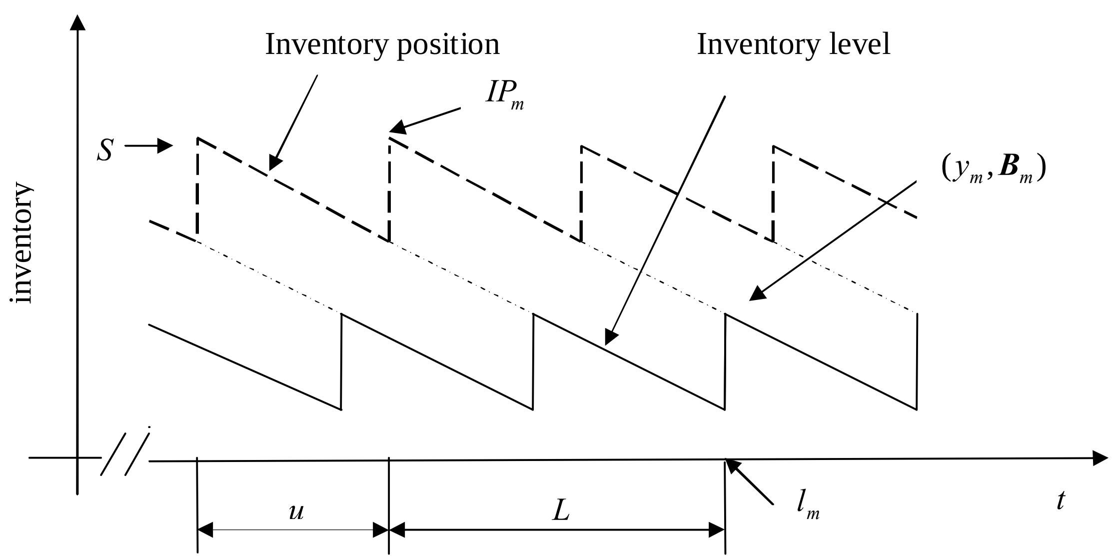

Figure 3.1 Inventory position and inventory level vs. time

When a demand is rejected and backordered, the penalty cost $\pi_i + \breve{\pi}_i \cdot t_d$ is incurred, where t_d is the time span from the time when the demand is rejected to the time when the backorder is fulfilled. Let t_r denote the time from the arrival of the rejected demand to the end of the current replenishment period. So $t_d = t_r + j \cdot u$, $\mathrm{j} \in \{0,1,...\}$. Assume the same method to count the penalty cost as in the previous dynamic programming model with positive lead time: count $\pi_i + \breve{\pi}_i \cdot t_r$ as the cost incurred in the replenishment period during which the demand is rejected, and if a demand of class i is outstanding at the beginning of a period, then a penalty cost $\breve{\pi}_i \cdot u$ is counted at this period.

We regard arrivals of replenishment and fulfilling backorders as events happened at the ends of replenishment periods. Let y_m denote the net inventory at the beginning of replenishment period *m* just after the replenishment arrives. Let $\boldsymbol{B}_m = \{b_1,...,b_K\}$ denote the outstanding backorders at the beginning of replenishment period *m*, where b_i is the number of backorders of class i. $(y_m, \boldsymbol{B}_m)$ describes the state of the system at the beginning of replenishment period *m*. According to the *full-priority* backorder clearing mechanism, if y_m is positive, which means all backorders are fulfilled, then y_m is the remaining stock after fulfilling all backorders; If y_m is negative, which means there are some backorders unfulfilled, then the sum of unfulfilled backorders equals to $-y$. So we have: if $y_m \geq 0$, then $b_i = 0$, $i \in \{1,...,K\}$, and if $y_m < 0$, then $y_m = -\sum_{i=1}^{K} b_i$.

Let ω denote the ordering policy applied on the horizon and $\omega \in \Psi$, where Ψ is the set of periodic-review ordering policies. Obviously the base stock ordering policy is one element of Ψ . Let ν denote a rationing policy applied on the horizon and $\nu \in \Phi$, where Φ is the set of rationing policies which assume that once a demand is rejected, it can be fulfilled only at the ends of replenishment periods and the backorder clearing mechanism is the full-priority mechanism. Let $\boldsymbol{X}_0$ denote the initial state (on-hand inventory, outstanding orders, and outstanding backorders of each demand class) of the system at the beginning of replenishment period 0.

Given an initial state X_0 of the system, an ordering policy ω and a rationing policy ν , for different realizations of the demand process which is a Poisson process for each demand class, there are different values of the net inventory y_m and backorder vector $\boldsymbol{B}_m$ at the beginning of period *m*. That is, $(y_m, \boldsymbol{B}_m)$, for a fixed *m*, is a random variable. Let $P_m(y, \boldsymbol{B})$ denote the probability of $(y_m, \boldsymbol{B}_m)$. The probability $P_m(y, \boldsymbol{B})$ is under the given ordering policy ω , rationing policy ν and initial state $\boldsymbol{X}_0$ of the system. In other words, for different ordering policies, rationing policies and initial states, $(y_m, \boldsymbol{B}_m)$ may have different probability distributions.

In the above we have shown: if y_m is positive, then there is no outstanding backorders, i.e., $b_i = 0$, and if y_m is negative, then $y_m = -\sum_{i=1}^{K} b_i$. So if ($y, \boldsymbol{B}$) does not satisfy this requirement, then $P_m(y, \boldsymbol{B}) = 0$ for all *m*. We have known that $(y_m, \boldsymbol{B}_m)$ is a random variable. Let Ω denote the set of possible values of $(y_m, \boldsymbol{B}_m)$ for all *m*.

For all rejected demands are backordered, the average ordering cost is a constant and it is ignored in calculating the average cost. Let $C_m(y,\boldsymbol{B})$ be the expected holding and penalty cost incurred in the replenishment period *m*, given $(y_m,\boldsymbol{B}_m)$. This cost depends on $(y_m,\boldsymbol{B}_m)$ as well as the rationing policy used in replenishment period m. Let $AC(\omega,\nu \mid \boldsymbol{X}_0)$ denote the expected average cost per period over the whole horizon when the ordering policy is ω and the rationing policy is ν and the initial state is $\boldsymbol{X}_0$.

The optimization problem is to minimize the expected average cost $AC(\omega,\nu \mid \boldsymbol{X}_0)$ by choosing the optimal ordering policy in Ψ and optimal rationing policy in Φ, given the initial state $\boldsymbol{X}_0$, i.e.,

$$\begin{aligned}\min_{\omega\in\Psi,\nu\in\Phi} \mathrm{AC}(\omega,\nu \mid \boldsymbol{X}_0) &= \min_{\omega,\nu}\lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M-1}\mathrm{E}[C_m(y,\boldsymbol{B})] \\ &= \min_{\omega,\nu}\lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M-1}\left[\sum_{\Omega} C_m(y,\boldsymbol{B})P_m(y,\boldsymbol{B})\right].\end{aligned} \tag{3.11}$$

When the above limit is not known to exist for some policies, we may use the definition $\mathrm{AC}(\omega,\nu \mid \boldsymbol{X}_0) = \lim_{M\to\infty}\sup\frac{1}{M}\sum_{m=0}^{M-1}\mathrm{E}[C_m(y,\boldsymbol{B})]$. It will not affect our later analysis and results, except that we may need to put ñsupò into the relevant expressions in the proofs of the lemmas and theorems. For general reasonable policies, this limit will exist. In the expression (3.11), there is a symbol of limiting. Note that in the later sections we will directly address the infinite horizon problem, not in this way: first consider a finite horizon *M*-period problem, then increase *M* to infinity (in Chapter 5, we use this way).

If under a certain ordering policy ω and a certain rationing policy ν , there exists a limiting distribution for $P_m(y,\boldsymbol{B})$, then the expression for the average cost $\mathrm{AC}(\omega,\nu \mid \boldsymbol{X}_0)$ under this policy can be simplified. Let $(y_\infty,\boldsymbol{B}_\infty)$ denote the random variable with the limiting distribution and $P_\infty(y,\boldsymbol{B})$ denote the limiting distribution. Let $C_\infty(y,\boldsymbol{B})$ denote the expected holding and penalty cost in a period under the given rationing policy, given the state $(y_\infty,\boldsymbol{B}_\infty)$ at the beginning of the period. So the average cost $\mathrm{AC}(\omega,\nu \mid \boldsymbol{X}_0)$ can be written as

$$\mathrm{AC}(\omega,\nu \mid \boldsymbol{X}_0) = \mathrm{E}[C_\infty(y,\boldsymbol{B})] = \sum_{\Omega} C_\infty(y,\boldsymbol{B})P_\infty(y,\boldsymbol{B}) .$$

The above average cost is independent on the initial state $\boldsymbol{X}_0$, for the existence of the limiting distribution $P_\infty(y,\boldsymbol{B})$. In later sections we will develop a near optimal solution to the problem (3.11). Under the policy of the near optimal solution, there exists a limiting distribution $P_\infty(y,\boldsymbol{B})$. In general inventory systems, under reasonable policies, the average cost over infinite horizon is independent on the initial state. For simplifying the notation, we remove the initial state $\boldsymbol{X}_0$ from later expressions of the average cost over the infinite horizon.

It is very difficult to find optimal ordering and rationing policies for the above optimization problem (3.11). One reason is that we are considering the dynamic inventory rationing policies and the number/type of elements in the rationing policy set Φ is huge and under many rationing polices there is no closed-form expression for the expected cost.

In addition, the ordering policy interacts with the rationing policies. These make the problem very complicated. In the previous section we have already shown that the structure of optimal ordering and rationing policies may be very complicated.

In the following section we develop a near-optimal solution to the optimization problem (3.11) and a lower bound on the optimal cost under the optimal ordering policy in Ψ and optimal rationing policy in Φ.

3.3.2 Analysis of a Near-Optimal Solution with a Dynamic Rationing Policy

In the previous multiperiod model with zero lead time, the backorders can be completely fulfilled at the ends of periods so that the inventory rationing in one period does not affect the cost after this period. While in the case with positive lead time, if there are remaining backorders unfulfilled at the end of period m, the rationing policy in period m will affect the distribution of each class in a given total remaining backorders, hence affect the cost after period m.

In the general inventory problems under an appropriate ordering policy, we can see that even if the probability of stockout before a replenishment arrives at the end of a period is large, for example 15%, the probability that the net inventory at the beginning of a period after a replenishment arrives is negative is small, for the penalty cost is larger than holding cost and it is not economic for the system to have a notable probability of no on-hand inventory at the beginning of a certain period. This should also be true in this inventory rationing problem, for the inventory rationing policy will not affect the net inventory at the beginning of a period. Let $P_m(y<0)$ denote the probability of $y_m<0$. So under an appropriate ordering policy, $P_m(y<0)$ should be small.

When $P_m(y<0)$ is small, i.e., in most cases the backorders in one period are completely fulfilled at the end of the period, then we may ration stock ignoring its effect on the cost after this period as a heuristic. Thus we obtain a myopic dynamic rationing policy for the multiperiod system with positive lead time: in each period the system rations stock by minimizing the penalty cost and holding cost in current period, given the initial inventory at the beginning of the period, ignoring the effect of the rationing policy on later costs. According to the model with zero lead time, we can see that such a myopic dynamic rationing policy for the case with positive lead time is exactly the dynamic rationing policy in the model with zero lead time (more detailed explanation will be shown later). Let dy denote such a dynamic rationing policy. For the base stock policy is optimal for the case with zero lead time and the demand process is stationary, we assume a stationary base stock ordering policy, denoted as (*R*, *S*) policy, for our solution to the optimization problem (3.11), where *S* is the base stock level and *R* represents periodic-review. Thus, the problem now is to find the optimal base stock level, assuming a (*R*, *S*) ordering policy and the rationing policy dy. Before proceeding to develop a method to obtain an appropriate base stock level, we describe the relation among this assumed policy and other policies.

Let Π_0 denote the set of vector (ω, v) of ordering and rationing policies with $\omega \in \Psi$ and $v \in \Phi_0$, where Ψ is the set of periodic-review ordering policies and Φ_0 is the set of any rationing policies. In previous analysis, we consider the rationing policies $\nu \in \Phi(\subset \Phi_0)$. Φ has two assumptions: backorders fulfilled only at the ends of periods and using the *full-priority* backorder clearing mechanism. Let Π_1 denote the set of (ω, v) with $\omega \in \Psi$ and $v \in \Phi$. The optimization problem (3.11) considers the policies in Π_1. Let Π_2 denote the set of (ω, v) where ω is a base stock ordering policy with a certain value for base stock and v is the above myopic dynamic rationing policy dy. So $\Pi_2 \subset \Pi_1 \subset \Pi_0$. In the following we consider the policies in Π_2. Let (S, dy) denote the policy in Π_2 with base stock S and $AC(S, dy)$ denote the average cost under policy (S, dy). We are trying to find the optimal base stock level to minimize average cost. For it is very difficult to find closed-form expression for the average cost $AC(S, dy)$, we first develop an upper bound and a lower bound for $AC(S, dy)$, then develop an approximate expression for $AC(S, dy)$ to find an appropriate base stock level. We will also develop a lower bound on the optimal cost under the optimal policies in Π_1.

Now consider how to calculate $AC(S, dy)$, given base stock level S and rationing policy dy and initial state $\boldsymbol{X}_0$. Let D_L denote total demands of all demand classes during the lead time L, which is independent on the rationing policy, base stock level and the index of periods. Let IP_m denote the inventory position just after ordering at ordering period m (see Figure 3.1). According to the base stock ordering policy, if the initial inventory position (in $\boldsymbol{X}_0$) at the first ordering period (i.e. ordering period 0) before ordering is below the base stock S, then the system should order to increase inventory position to the base stock level at ordering period 0. The inventory position at ordering period 1 before ordering will be less than S and the system should order up to S. Continue the process and the inventory position just after ordering at each period will always be equal to S, i.e., $IP_m = S$ for all m. If the initial inventory position at the ordering period 0 before ordering is above the base stock S, then according to the base stock policy, the system will not order until at a certain period when the inventory position drops below S. Since this period, the system should order and the inventory position just after ordering at each period will be equal to S. So no matter the initial state $\boldsymbol{X}_0$ is, after some transitional periods, the inventory position just after ordering will always be equal to S. For the costs during these transitional periods will not affect the average cost over infinite horizon, we assume that the initial inventory position at the first ordering period before ordering is below the base stock S in order to simplify the presentation. Hence, the inventory position just after ordering at each ordering period is always equal to S, i.e., $IP_m = S$ for all m.

The net inventory y_m is equal to the inventory position after ordering at the ordering period m minus the total demands during the lead time (see Figure 3.1), i.e., $y_m = IP_m - D_L$. Under the base stock policy, the inventory position IP_m just after ordering at each period is always equal to S, so we have: $y_m = S - D_L$ for all m. So y_m is completely determined by the base stock S and the total demand in the lead time, and it is independent on the rationing policy, index m of periods, the backorder clearing mechanism, and the initial state $\boldsymbol{X}_0$. Let $P(y_m)$ denote the probability of y_m. Under the assumed backorder clearing mechanism, if $y_m \geq 0$, then $\boldsymbol{B}_m = \mathbf{0}$, where the bold 0 represents the K-dimensional vector of zeros. When $y_m < 0$, the sum of probabilities $P_m(y, \boldsymbol{B})$ with the same y_m but different $\boldsymbol{B}_m$, is equal to the probability $P(y_m)$ for the given y_m. Let $\Omega(y_m)$ denote the set of values of $(y_m, \boldsymbol{B}_m)$ with the same y_m and different $\boldsymbol{B}_m$. So we have:

$$P_m(y, \boldsymbol{B}) = P(y_m) = P(D_L = S - y_m) \quad \text{when } y_m \geq 0 \text{, and}$$

$$\sum_{\Omega(y_m)} P_m(y, \boldsymbol{B}) = P(y_m) = P(D_L = S - y_m) \quad \text{when } y_m < 0 \text{.}$$

Under the assumed counting method about penalty cost, if a demand of class i is rejected in a period, then a penalty cost $\pi_i + \hat{\pi}_i \cdot t_r$ is incurred in current period, no matter whether the demand is fulfilled at the end of current period or it is fulfilled at the end of a later period, where t_r is the remaining time before the end of current period. So the counting of penalty cost and holding cost in a period is the same as that in the model with zero lead time, given an on-hand inventory at the beginning of a period. For the dynamic critical level rationing policy minimizes expected total holding and penalty cost in a single period, we have: when $y_m \geq 0$ ($\boldsymbol{B}_m = \mathbf{0}$ in this case), the expected holding and penalty cost $C_m(y, \boldsymbol{B} \mid dy)$ during period m under the rationing policy dy is $H(N, y)$, which is also the minimum of the expected cost $C_m(y, \boldsymbol{B})$ for all rationing policies in Φ, i.e., given $y_m \geq 0$ ($\boldsymbol{B}_m = \mathbf{0}$ in this case),

$$C_m(y, \boldsymbol{B} \mid dy) = H(N, y) = \min_{v \in \Phi} C_m(y, \boldsymbol{B}) \text{.} \qquad (3.12)$$

When $y_m < 0$ ($\boldsymbol{B}_m \neq 0$ in this case), the outstanding backorder of class i will incur a penalty cost of $\pi_i \cdot u$ and all demands in replenishment period m are backordered. In this case, the expected cost $C_m(y, \boldsymbol{B})$ in period m is the same under all rationing policies. So, given $(y_m, \boldsymbol{B}_m)$ with $y_m < 0$,

$$C_m(y, \boldsymbol{B} \mid dy) = H(N,0) + \sum_{i=1}^{K} b_i \pi_i u = C_m(y, \boldsymbol{B} \mid v \in \Phi). \qquad (3.13)$$

Even under the base stock ordering policy and rationing policy dy, we still can not obtain the closed-form expression for $AC(S, dy)$, because it is difficult to obtain the closed-form expression for the probability distribution of backorders of each class in the outstanding backorders $\boldsymbol{B}_m$. In the following we develop bounds for $AC(S, dy)$, given base stock S.

Under policy (S, dy), we have known that $y_m = S - D_L$ and the probability distribution of y_m is independent on the rationing policy, index m of periods and initial state $\boldsymbol{X}_0$. When $y_m < 0$, the expected cost $C_m(y, \boldsymbol{B} \mid dy)$ should not be less than the cost assuming all backorders in $\boldsymbol{B}_m$ come from the least important class K. Based on this fact, we can obtain a lower bound for $AC(S, dy)$. Let $AC_{LB}(S, dy)$ denote this lower bound on $AC(S, dy)$. Note that this lower bound is dependent on the given *S*. In other words, the lower bound $AC_{LB}(S, dy)$ is a function of the base stock *S*. Define

$U_{LB}(y) = H(N, y)$ when $y \geq 0$,

and $$U_{LB}(y) = H(N,0) - y \cdot \pi_K \cdot u \text{ when } y < 0 . \tag{3.14}$$

We have the following lemma to show a lower bound on $AC(S,dy)$.

Lemma 3.5. $AC(S,dy) \geq E[U_{LB}(S - D_L)] = AC_{LB}(S,dy)$.

Now we develop an upper bound on $AC(S,dy)$. Consider the distribution of backorders of each class in a given total backorder $-y$ at the beginning of a certain period. Assume there is no inventory rationing during periods and the backorders are fulfilled at the ends of periods based on a first-come-first-served rule. In this case, for a given amount $-y$ of total backorders at the beginning of a certain period, the expected proportion of backorders of class i in the total backorders $-y$ equals to the proportion of the arrival rate of this class in the total arrival rate, i.e., $E[b_i \mid -y]/(-y) = \lambda_i / \lambda$. This proportion is independent on the value of $-y$. Let $pr_i = \lambda_i / \lambda$. Given $-y$, the proportion $E[b_i \mid -y]/(-y)$ under policy (S,dy) will be different from the above value in the case without inventory rationing. Let $pr_i^{dy}(S,-y)$ denote the expected proportion of backorders of class i in the given total backorder $-y$ under policy (S,dy). Under policy (S,dy), there exist two factors to increase the proportion of more important classes and decrease the proportion of less important classes to reduce cost. The first factor is the dynamic critical level rationing policy dy which makes the system to reject demands of less important classes to reserve stock for more important classes. The other factor is the full-priority backorder clearing mechanism. Under this mechanism, the system first fulfills backorders of the most important class, then less important classes. Thus given the total backorder $-y$, the expected penalty cost resulting from the outstanding backorders at the beginning of the period under policy (S, dy) is not greater than that under the policy without inventory rationing during the period and fulfilling backorders based on first-come-first-served rule, i.e.,

$$-y \cdot \sum_{i=1}^{K} pr_i^{dy}(S,-y) \cdot \breve{\pi}_i \cdot u \leq -y \cdot \sum_{i=1}^{K} pr_i \cdot \breve{\pi}_i \cdot u \,. \tag{3.15}$$

According to the above relation, we can obtain an upper bound on $AC(S, dy)$. Let $AC_{UB}(S, dy)$ denote this upper bound. Define

$$U_{UB}(y) = H(N, y) \text{ when } y \geq 0 ,$$

and

$$U_{UB}(y) = H(N,0) - y\sum_{i=1}^{K} pr_i \cdot \breve{\pi}_i \cdot u \text{ when } y < 0 \,. \tag{3.16}$$

According to (3.15), we have the following lemma about an upper bound on $AC(S, dy)$ (for detailed proof, see Appendix A).

Lemma 3.6. $AC(S, dy) \leq E[U_{UB}(S - D_L)] = AC_{UB}(S, dy)$.

Under policy (S, dy), there exists a limiting distribution for $(y_m, \boldsymbol{B}_m)$, for the system is stationary: demand process is stationary, the base stock S is a constant, and the critical levels of the rationing policy do not change from one period to another period. Let $(y_\infty, \boldsymbol{B}_\infty)$ denote the variable with such limiting distribution and $P(y_\infty < 0)$ denote the probability of $y_\infty < 0$. When S increases, then $P(y_\infty < 0)$ will decrease. Hence from the definitions of $U_{UB}(S-D_L)$ and $U_{LB}(S-D_L)$ we can see that the gap between the upper and lower bounds of $AC(S,dy)$ will decrease and will infinitely approach to 0 when $S \to \infty$. Based on the upper and lower bounds for $AC(S,dy)$, we may develop approximate expressions for $AC(S,dy)$ to find an appropriate base stock level. For simplicity, we use the lower bound $AC_{LB}(S,dy)$ of $AC(S,dy)$ to approximate $AC(S,dy)$ to find an appropriate base stock. Let

$$S^* = \arg\min E[U_{LB}(S-D_L)]. \tag{3.17}$$

Thus we have found a solution to optimization problem (3.11): the dynamic rationing policy dy and the base stock ordering policy with base stock S^*.

In the following we develop a lower bound on the optimal cost under the optimal policies in Π_1 so that we can measure how the cost of our solution is close to the optimal costs. Let (ω_o, ν_o) denote the optimal policy in Π_1 and the corresponding optimal cost is $AC(\omega_o, \nu_o)$. Let $AC_{LB}(\omega_o, \nu_o)$ denote the lower bound on the optimal cost $AC(\omega_o, \nu_o)$. We have the following lemma for a lower bound on the optimal cost.

Lemma 3.7. *Under the optimal policies in* Π_1*, the optimal cost*

$$AC(\omega_o, \nu_o) \ge E[U_{LB}(S^* - D_L)] = AC_{LB}(\omega_o, \nu_o).$$

In the above analysis, we have found a solution to the optimization problem (3.11): the rationing policy dy and a base stock ordering policy with base stock S^*. We can not calculate the exact value of the average cost under such a policy, but its upper and lower bounds. Now we define a percentage to measure how the cost $AC(S^*, dy)$ of our solution is close to the lower bound on the optimal cost under polices in Π_1. Note that $AC_{UB}(S^*, dy)$ is the upper bound of $AC(S^*, dy)$, and $AC_{LB}(\omega_o, \nu_o)$ is the lower bound on the optimal cost under the optimal policies in Π_1. Define

$$\begin{aligned} CR_{LB} &= \frac{AC_{UB}(S^*, dy) - AC_{LB}(\omega_o, \nu_o)}{AC_{LB}(\omega_o, \nu_o)} \\ &= \frac{E[U_{UB}(S^* - D_L)] - E[U_{LB}(S^* - D_L)]}{E[U_{LB}(S^* - D_L)]}. \end{aligned} \tag{3.18}$$

We can see that the relative difference between the cost under policy (S^*, dy) and the optimal cost under the optimal policy (ω_o, ν_o) should not be greater than CR_{LB}, i.e.,

$$\frac{AC(S^*, dy) - AC(\omega_o, \nu_o)}{AC(\omega_o, \nu_o)} \le CR_{LB}. \tag{3.19}$$

So CR_{LB} can measure how the cost $AC(S^*, dy)$ of our solution is close to the optimal cost.

3.4 Comparing Performance of Rationing Policies

In the previous sections we have developed dynamic rationing policies. In practice, people often use the static critical level rationing policy for they can not find appropriate dynamic rationing policies for typical problem settings. In this section we conduct a numerical study to compare the dynamic rationing policy with the static rationing policy in multiperiod systems with infinite horizon under the full-priority backorder clearing mechanism (denote it as *mechanism M* for short). We also compare the cost under our solution with the lower bound on the optimal costs. In the next section we will compare the backorder clearing mechanism *M* with another mechanism called *mechanism T*.

We first investigate the cases with two demand classes under different operating conditions, and then look at those with three demand classes.

3.4.1 Numerical Study for Systems with Two Demand Classes

3.4.1.1 The Numerical Study

In some later symbols there are subscripts dyM and cnM, where dy means the dynamic critical level rationing policy and cn means the static critical level policy and M means backorder clearing mechanism M.

We use simulation to obtain the average costs under both dynamic and static rationing policies in different cases. Let $AC_{dyM}(S^*_{dyM})$ denote the average cost under the dynamic rationing policy dy and the base stock level S^*_{dyM}, where S^*_{dyM} is the base stock S^* obtained in the previous section using the lower bound of $AC(S,dy)$ to approximate it.

The average cost under the static critical level policy is a function of the base stock S and the vector $\boldsymbol{R} = (r_1,...,r_K)$ of critical levels, where r_i is the static critical level of class i. We run simulation for a wide range of $(S, \boldsymbol{R})$ parameters and the values of parameters which have the least average cost are identified as optimal. As there are only two demand classes and the critical level of class 1 is 0, we only need to perform an exhaustive search via simulation for the critical level of class 2, given a base stock level. Let $AC_{cnM}(S^*_{cnM}, \boldsymbol{R}^*_{cnM})$ denote the optimal average cost under the static critical level policy, where S^*_{cnM} is the optimal base stock, $\boldsymbol{R}^*_{cnM}$ is the vector of optimal critical levels.

Define the following percentage to measure the benefit of implementing the dynamic critical level rationing policy comparing with the static critical level policy:

$$CR_{cnM-dyM} = \frac{AC_{cnM}(S^*_{cnM}, \boldsymbol{R}^*_{cnM}) - AC_{dyM}(S^*_{dyM})}{AC_{dyM}(S^*_{dyM})} \cdot 100\% . \tag{3.20}$$

In the above definition, we use the cost under the dynamic rationing policy as the benchmark, so $CR_{cnM-dyM}$ is the percentage of cost that will increase if the system changes rationing policy from the dynamic rationing policy to the static rationing policy. Besides $CR_{cnM-dyM}$ we use CR_{LB}, which is defined in (3.18), to measure how the cost of our solution is close to the optimal costs. It is worth to note that $CR_{cnM-dyM}$ is obtained using the costs by simulation, while CR_{LB} does not need simulation and is obtained by direct calculation.

During the calculation of dynamic critical levels of the dynamic rationing policy, we divide one period into many small intervals such that the probability that more than one demand arrive in one interval is less than 4.9e-5. In order to ensure the value reaches the steady state in simulation, we run 10,000 replenishment periods for each scenario.

In the numerical study, the parameter settings are as follows. Set $u = 0.1$, $h = 1$, $\pi_1 = \pi_2 = 0$, and vary λ, λ_1, λ_2, $\breve{\pi}_1$, $\breve{\pi}_2$ and L. We create two problem sets to vary these parameters. In the first problem set, we have $\lambda_1 / \lambda_2 = 1$, while in the other set, we vary the ratio λ_1 / λ_2. In the first set, we have: $\lambda_1 = \lambda_2$, $\lambda \in \{300, 600, 900\}$, $\breve{\pi}_1 / \breve{\pi}_2 \in \{3, 100, 1000\}$, $\breve{\pi}_2 \in \{1.5, 5, 10\}$, $L/u \in \{0, 1, 2, 3, 4\}$. So in the first problem set, there are 135 different combinations. In the second set, $\lambda = 600$, $\breve{\pi}_2 = 5$, $\breve{\pi}_1 / \breve{\pi}_2 = 100$, $L/u \in \{0.5, 1, 1.5\}$, $\lambda_1 / \lambda_2 \in \{1/5, 1/3, 1/2, 1/1, 2/1, 3/1, 5/1\}$. So there are 21 combinations in the second set. For each problem we obtain percentages $CR_{cnM-dyM}$ and CR_{LB}. Part of the results is shown in Tables 3.1 and 3.2, and Figures 3.2 and 3.3, and the other results are shown in Appendix B.

3.4.1.2 Interpretation of Results

Table 3.1 shows a subset of the runs for the first problem set. We observe that the gap CR_{LB} is extremely small in all cases. So the costs under our solution with dynamic rationing policy *dy* are very close to the optimal costs. Moreover, we can see that the dynamic rationing policy performs significantly better than the static critical level policy, especially when ratio $\breve{\pi}_1 / \breve{\pi}_2$ is large. This result is intuitive. When the penalty costs of two demand classes differ more, a wrong decision in rationing inventory can incur a larger cost increasing. So the dynamic critical level policy can reduce more cost when $\breve{\pi}_1 / \breve{\pi}_2$ is large. Figure 3.2 shows an example of the critical levels of class 2 in one period under both rationing policies. The critical levels of class 1 under both policies are always 0 and are not shown in the figure. The critical levels are the same for different periods. The period is divided into 6000 intervals and interval 1 is at the end of the period. From the figure we can see that the critical level under the dynamic rationing policy decreases towards the end of the period and reaches 0 at the end of the period. Under the static rationing policy, the critical level is a constant regardless of the system state. The dynamic rationing policy reduces the cost by dynamically adjusting the critical level according to the remaining time before a new replenishment arrives. Other results also show the same trend. Moreover, the results show that when $\breve{\pi}_1 / \breve{\pi}_2$ becomes larger (while other settings are the same), then the dynamic critical level of class 2 becomes steeper, i.e., the critical level at the beginning of the period becomes larger and the critical level decreases more rapidly toward the end of the period. It is quite intuitive. When $\breve{\pi}_1 / \breve{\pi}_2$ becomes larger, the system needs to reserve more stock for class 1 at a certain time, for a shortage of class 1 will incur a larger penalty cost.

Table 3.1 Comparison of rationing policies when $\lambda_1 = \lambda_2 = 300, \breve{\pi}_2 = 5$

$\breve{\pi}_1 / \breve{\pi}_2$	L/u	CR_{LB}	$CR_{cnM-dyM}$
	0	0	0.91%
	1	1.47E-11	1.21%

3	2	1.82E-07	1.41%
	3	5.58E-06	1.49%
	4	4.53E-05	1.58%
100	0	0	6.29%
	1	2.53E-12	7.95%
	2	1.41E-07	9.88%
	3	5.84E-06	9.95%
	4	4.04E-05	8.92%
1000	0	0	8.18%
	1	8.15E-12	10.04%
	2	7.38E-08	11.71%
	3	4.09E-06	11.17%
	4	2.31E-05	7.54%

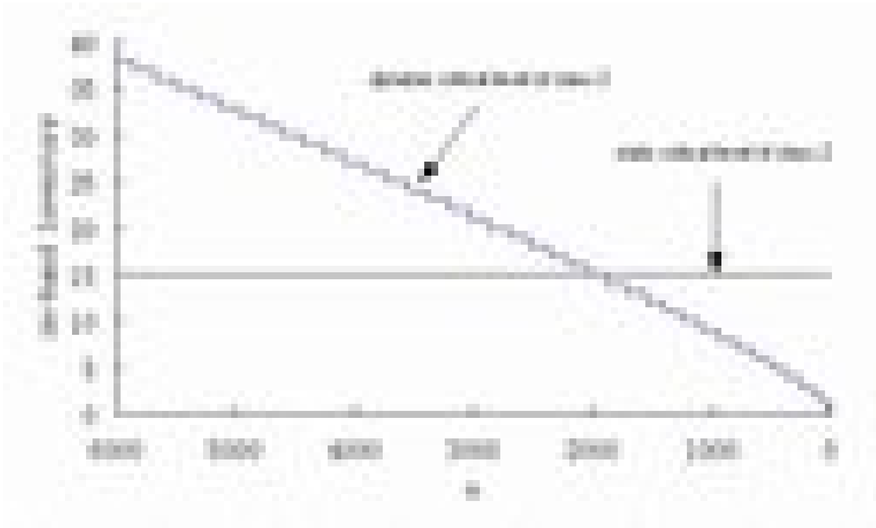

Figure 3.2 Critical levels under both rationing policies
($\lambda_1 = \lambda_2 = 300, \pi_2 = 5, \pi_1 / \pi_2 = 100, L/u = 1$)

From Table 3.1, it is interesting to note that the relative cost difference $CR_{cnM-dyM}$, which measures the benefit of implementing the dynamic rationing policy, does not necessarily monotonically increase with L/u. When L/u increases from 0, the cost difference $CR_{cnM-dyM}$ increases. When L/u arrives at a certain value and continues to increase, $CR_{cnM-dyM}$ decreases. The other results in the first problem set, which are not shown here (please see Appendix B), also show a similar trend.

The dynamic rationing policy can reduce cost comparing with the static rationing policy is because the critical levels in the dynamic policy decrease when the remaining time decreases. Consider a certain period, if the on-hand inventory at the beginning of the period is very high, the benefit of the dynamic critical level policy will be small, for there are almost enough stock to satisfy demands of all classes, hence rationing is not important. When the on-hand inventory at the beginning of the period is very small, the system under the static critical level policy will immediately reject demands of all less important classes and the benefit of the dynamic critical level policy is also small. When the on-hand inventory is some middle value, dynamic critical level policy can bring significant benefit. In the multiple period system, when L/u increases, the distribution of y_∞, which is a random variable with the limiting distribution of y_m under the rationing policy dy and a given base stock, will have larger variance, the probability mass function will be more flat, and the expected value of y_∞ will also shift. So when L/u increases from 0, there are more probability that the initial on-hand inventory of a certain period is in the region that the dynamic critical level policy can bring significant cost saving, thus the gap $CR_{cnM-dyM}$ increases. When L/u increases to a certain value and continues to increase, the probability mass function of y_∞ is very flat and the probability that the initial on-hand inventory of a certain period is in the significant benefit region will decrease, so the gap $CR_{cnM-dyM}$ decreases when L/u is larger than a certain value.

Now consider Table 3.2 which shows results when $L/u=1$, $\lambda_1=\lambda_2$ and we change penalty costs and arrival rates. We can see that changing values of λ and $\breve{\pi}_2$ while keeping $\breve{\pi}_1/\breve{\pi}_2$ fixed will not significantly affect the performance of the dynamic critical level policy. The most important factor that affects the performance of dynamic critical level policy is still the ratio $\breve{\pi}_1/\breve{\pi}_2$.

Table 3.2 Comparison of rationing policies when $L/u=1, \lambda_1=\lambda_2$

$\breve{\pi}_1/\breve{\pi}_2$	$\breve{\pi}_2$	λ	CR_{LB}	$CR_{cnM-dyM}$

3	1.5	300	2.19E-05	1.31%
	1.5	600	2.29E-08	1.14%
	1.5	900	3.19E-11	0.93%
	5	300	4.57E-07	1.08%
	5	600	1.47E-11	1.21%
	5	900	4.35E-14	1.11%
100	1.5	300	3.10E-06	8.51%
	1.5	600	8.60E-10	9.90%
	1.5	900	3.92E-13	10.02%
	5	300	7.43E-08	7.44%
	5	600	2.53E-12	7.95%
	5	900	1.97E-12	8.68%

Finally, consider Figure 3.3 which shows the performance of the dynamic critical level rationing policy over the static critical level policy when the ratio of demand rates, λ_1/λ_2, changes. From this graph, $CR_{cnM-dyM}$ is not monotonic with respect to λ_1/λ_2. When λ_1/λ_2 diverges from 1, one demand class dominates the other in arrivals. When the ratio gets very large or very small, arrivals of one class become rare as compared to those of the other class, thus the benefit of inventory rationing decreases.

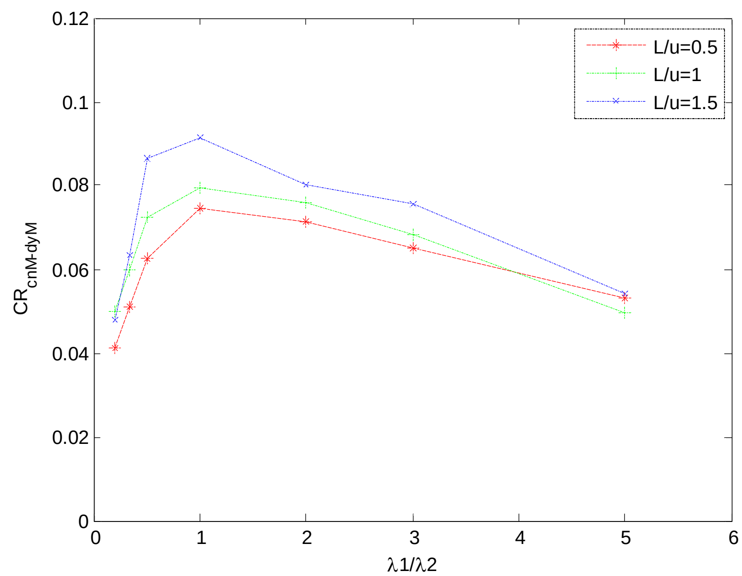

Figure 3.3 Relative cost difference $CR_{cnM-dyM}$ vs. λ_1/λ_2

In summary, the gap between the cost of our solution with a dynamic critical level policy and the optimal cost is extremely small in a wide range of parameter settings. Whether the dynamic critical level rationing policy can bring significant benefit relative to the static rationing policy depends on the operational conditions. In many cases the relative cost difference can be more than 10%, especially when the penalty costs differ very much and the demand rates of class 1 and class 2 are similar.

3.4.2 Numerical Study for Systems with Three Demand Classes

In this section we examine the performance of the dynamic rationing policies in systems with three demand classes and other very poor service level situations. We want to check whether the previous method is robust. We will compare the cost under the dynamic rationing policy with the lower bound on optimal costs, and will not compare the cost under the dynamic rationing policy with that under the static policy since the searching time for the optimal parameters of the static rationing policy is prohibitive when there are three demand classes. The numerical results are shown in Tables 3.3 and 3.4.

Table 3.3 shows the results when the number of demand classes is changed from 2 to 3. Let *case A* denote the case with two demand classes, and *case B* denotes the case with three classes. The parameters in this table are set according to the values in previous 2-class systems, making sure that the total arrival rates and weighted penalty costs in two cases are equal, i.e., $\sum_{i=1}^{2}\lambda_i^A = \sum_{i=1}^{3}\lambda_i^B$ and $\sum_{i=1}^{2}\lambda_i^A \cdot \pi_i^A = \sum_{i=1}^{3}\lambda_i^B \cdot \pi_i^B$. From Table 3.3 we can see that when the system changes from 2 demand classes to 3 demand classes, the gap CR_{LB} does not change notably.

Table 3.3 CR_{LB} for systems with two and three demand classes

L/u	Case A			Case B		
	π_1^A/π_2^A	λ_1^A,λ_2^A	CR_{LB}	π_1^B,π_2^B,π_3^B	$\lambda_1^B,\lambda_2^B,\lambda_3^B$	CR_{LB}
1	3	300,300	1.47E-11	200,200,200	17,8,5	1.46E-11
1	3	300,300	1.47E-11	100,200,300	25,10,5	2.92E-11
1	3	300,300	1.47E-11	300,200,100	13,8,5	7.16E-12
2	3	300,300	1.82E-07	200,200,200	17,8,5	1.19E-07
3	3	300,300	5.58E-06	200,200,200	17,8,5	5.54E-06

1	100	300,300	2.53E-12	200,200,200	600,152,5	8.18E-14
1	100	300,300	2.53E-12	100,200,300	900,300,5	1.74E-12
1	100	300,300	2.53E-12	300,200,100	400,155,5	2.25E-13
2	100	300,300	1.41E-07	200,200,200	600,152,5	1.98E-08
3	100	300,300	5.84E-06	200,200,200	600,152,5	1.29E-06

The parameters in Table 3.4 are chosen to check whether our method is robust under poor-service conditions. In the previous parameter settings, $P_\infty(y<0)$, which is the probability $y_\infty < 0$, is not large (typically less than 1.0e-3) under the appropriate base stock S^*. This is practical in most settings, since the system should have on-hand inventory at the beginning of a period at most of time because of the effect of penalty cost of shortage. In this study, we purposely choose the parameters such that $P_\infty(y<0)$ can be as high as 0.11, which means that the probability that the on-hand inventory at the beginning of a period after a replenishment arrives is 0 is as high as 11%. In this case the service level for the least important class is very poor, for the stock-out probability at the end of a period is much higher than 11%.

Table 3.4 CR_{LB} in some extreme cases

L/u	$\hat{\pi}_1^B, \hat{\pi}_2^B, \hat{\pi}_3^B$	$\lambda_1^B, \lambda_2^B, \lambda_3^B$	$P_\infty(y<0)$	CR_{LB}
1	1.5, 1.3, 1.1	100,100,100	0.00148	6.96E-5

2	1.5, 1.3, 1.1	100,100,100	0.01457	0.00092
3	1.5, 1.3, 1.1	100,100,100	0.03543	0.00263
4	1.5, 1.3, 1.1	100,100,100	0.05754	0.00478
5	1.5, 1.3, 1.1	100,100,100	0.07842	0.00706
6	1.5, 1.3, 1.1	100,100,100	0.09743	0.00932
7	1.5, 1.3, 1.1	100,100,100	0.11453	0.01151

From Table 3.4 we can see that the gap between the cost of our solution and the optimal cost remains very small. Even for the worst case when $P_\infty(y<0)\approx 0.11$ (case *L/u* = 7), the gap is still only 1.15%. Hence the results do indicate that the proposed method is robust in the quality of solution for larger $P_\infty(y<0)$ and the quality of solution does not suffer from rapid deterioration when the number of demand classes increases.

3.5 Comparing Backorder Clearing Mechanisms

In the previous sections we have obtained near-optimal solutions to the optimization problem of minimizing average cost, assuming the full-priority backorder clearing mechanism, i.e., *mechanism M,* and found that the dynamic rationing policy indeed can significantly reduce cost comparing the static rationing policies. In this section we explore further opportunities to reduce cost, in particular, considering another backorder clearing mechanism denoted as *mechanism T* and investigating its effect on the average cost by comparing the costs under mechanisms *T* and *M*.

Backorder clearing mechanism *T* comes from an observation: under mechanism *M*, the system fulfills backorders at the ends of periods as much as possible from the most important class to the least important class. Sometimes all on-hand inventory is used to fulfill these backorders and even the most important demands in the next period can not be satisfied. So it would be better for the system not to fulfill some least important backorders (hence forward them to the next period) to reserve some stock for demands of important classes in the next period. Hence, we propose another backorder clearing mechanism denoted as *T*, which works in this way: the system fulfills backorders also according to class priority as mechanism *M*, but there is a backorder clearing threshold for each demand class such that when the on-hand inventory decreases to the clearing threshold of a certain class, then the system forwards the remaining outstanding backorders of this class to the next period. We use $x_i^*(N)$ as the backorder clearing threshold of class i, where $x_i^*(N)$ is the critical level of class i of the dynamic critical level rationing policy at the beginning of a period where the period is divided into N intervals.

We compare the costs of the two cases by simulation: case dyM and case dyT, where case dyM is the system applies the dynamic rationing policy dy and backorder clearing mechanism *M*, and case dyT is the system applies the dynamic rationing policy dy and mechanism T. The base stock policy and dynamic rationing policy in case dyT

are the same as those in case dyM. Let $AC_{dyT}(S^*_{dyM})$ denote the average cost in case dyT. Define the following percentage to measure the relative cost difference under two mechanisms:

$$CR_{dyT-dyM} = \frac{AC_{dyT}(S^*_{dyM}) - AC_{dyM}(S^*_{dyM})}{AC_{dyM}(S^*_{dyM})} \cdot 100\%.$$

A negative value of $CR_{dyT-dyM}$ means mechanism T can reduce cost comparing with mechanism *M*. We use the same problem sets as when comparing rationing policies to compare mechanisms M and T. The results are shown in Tables 3.5 and 3.6.

From Table 3.5 we can see that in these typical problem settings (lead time equals one period) mechanism T is better than mechanism M, but the relative cost difference is very small. Other tables for other settings with $L/u = 1, \lambda_1 = \lambda_2$, which are not shown here, also show similar results. The minimum of $CR_{dyT-dyM}$ in these cases is -3.35%. In all these examples, results show that mechanism T is better than mechanism M, under the same rationing policy dy and the same base stock level.

We conjecture mechanism T is always better than mechanism M for all cases. We can not provide a rigorous proof for it. In the following we provide some insights for it, so that we believe that it is reasonable to expect that mechanism T is always better than mechanism M. Assume at the beginning of period m just before the arrival of the replenishment, there is a vector of backorders $\boldsymbol{B}_m^-$ and a remaining on-hand inventory (i.e., the remaining on-hand inventory at the end of period $m-1$). When the replenishment arrives, it will be added to the remaining on-hand inventory. Let x_m^- denote the total on-hand inventory. Then the system will use the total on-hand inventory x_m^- to fulfill $\boldsymbol{B}_m^-$ according to a backorder clearing mechanism *M* or *T,* and the resulting remaining backorder vector is $\boldsymbol{B}_m$.

(a) Given $\boldsymbol{B}_m^-$ and x_m^-, the cost incurred in period *m* under mechanism T is less than that under mechanism M.

Now we focus on only period *m*. We may regard the outstanding backorders $\boldsymbol{B}_m^-$ as the demands arrived at the beginning of period *m,* and if we reject a demand of class *i*, then a penalty cost $\pi_i \cdot u$ will incur. According to the previous model, when the on-hand inventory at the beginning of the period is below the critical level of class *i*, then the total cost in the period *m* of satisfying a demand class *i* is larger than the cost of rejecting it, and the system should reject it. Under the mechanism *T*, when the on-hand inventory has dropped to or below the critical level of class i, then the demand of class *i* will be rejected, i.e., not to fulfill the backorder of class *i* in $\boldsymbol{B}_m^-$. However, under mechanism *M*, the demands (i.e., the backorders) will be fulfilled if there is on-hand inventory. So from the single-period point of view, mechanism *T* is better than mechanism M.

(b) Under mechanism T, there are less backorders of more important classes forwarded to next periods than under mechanism M.

These two mechanisms have different effect on later periods, besides the above different effect on period *m*. Mechanism T has reserved more on-hand inventory at the beginning of period m for period m than mechanism M, so under mechanism T less demands of more important classes will be rejected in period m than under mechanism M, i.e., less backorders of more important classes in $\boldsymbol{B}_{m+1}^{-}$ than under mechanism M. Note that the total backorders under both mechanisms are the same, for it is independent on the rationing policy and backorder clearing mechanism. So mechanism T decreases the proportion of the backorders of more important classes in the total backorders of $\boldsymbol{B}_{m+1}^{-}$, which will decrease cost over later periods.

So based on the above (a) and (b), it is reasonable to believe that mechanism T is always better than mechanism M.

Mechanism T fulfills backorders by looking forward and considering its effect on later periods, while mechanism M considers only the previous period, trying to fulfill backorders from the previous period as much as possible. So mechanism T can reduce some cost, comparing with mechanism M. When the base stock level is near or above the optimal value, the initial on-hand inventory at the beginning of a period in general is positive under mechanism M, i.e., there is very small probability to forward backorders to the next period. So if we use mechanism T instead of M, the cost difference will be small.

Table 3.5 Comparison of mechanisms when $L/u = 1, \lambda_1 = \lambda_2$

		$CR_{dyT-dyM}$		
$\breve{\pi}_2$	λ	$\breve{\pi}_1 / \breve{\pi}_2 = 3$	$\breve{\pi}_1 / \breve{\pi}_2 = 100$	$\breve{\pi}_1 / \breve{\pi}_2 = 1000$
1.5	300	-0.04%	-1.98%	-0.61%

1.5	600	-0.01%	-0.26%	-0.76%
1.5	900	0.00%	-0.02%	0.00%
5	300	0.00%	-0.20%	-0.61%
5	600	0.00%	0.00%	0.00%
5	900	0.00%	0.00%	0.00%

Now consider Table 3.6 which shows the effect of lead time on the cost difference. We can see that relative cost difference $CR_{dyT-dyM}$ is still very small when lead time is significantly larger than the length of a period, for example $L/u = 4$. We can also see that $CR_{dyT-dyM}$ is not monotonic with L/u. Other results also show this trend. When L/u increases, the variance of total demands in lead time increases and the probability of forwarding some backorders to the next period in dyM increases, and mechanism T has more chance to reserve stock for later periods by unfulfilling some backorders of less important classes to reduce cost. So when L/u increases from 0 to a certain small value, $CR_{dyT-dyM}$ decreases. When L/u arrives at a certain value and continuously increases, for the randomness of lead time demands the average cost of case dyM notably increases, then the ratio of reduced cost by mechanism T to the average cost may decrease. Thus $CR_{dyT-dyM}$ is not monotonic with L/u.

Table 3.6 Comparison of mechanisms when $\lambda_1 = \lambda_2 = 300, \pi_2 = 5$

L/u	$CR_{dyT-dyM}$		
	$\breve{\pi}_1 / \breve{\pi}_2 = 3$	$\breve{\pi}_1 / \breve{\pi}_2 = 100$	$\breve{\pi}_1 / \breve{\pi}_2 = 1000$
0	0.00%	0.00%	0.00%
1	0.00%	0.00%	0.00%
2	0.00%	-0.36%	-0.62%
3	-0.02%	-1.13%	-2.36%
4	-0.07%	-1.79%	-2.24%

In summary, mechanism T can reduce cost, comparing with mechanism M, when the system is under the same dynamic rationing policy dy and the same appropriate base stock level, but the relative cost difference is very small in typical problem settings.

3.6 Conclusions

This chapter considered dynamic inventory rationing in multiperiod systems with Poisson demands, backordering and multiple demand classes. A multiperiod system with zero lead time was first analyzed and dynamic programming models were developed. We showed that the optimal rationing policy is the dynamic critical level policy and the optimal ordering policy is a base stock policy. Important properties of the optimal rationing policy, for example, the critical levels decrease towards the end of the period, and properties of optimal cost functions were also shown. In general, dynamic inventory rationing models with backordering are multi-dimensional dynamic programming ones. Hence, they suffer the curse of dimensionality of multi-dimensional dynamic programming, while we used the assumption that backorders can be fulfilled only at the ends of periods, which is quite reasonable, to develop a one-dimensional dynamic programming one to eliminate the curse of dimensionality.

We then considered a multiperiod system with positive lead time. An optimization model of minimizing expected average cost was developed, assuming the *full-priority* backorder clearing mechanism (i.e., mechanism M). In the case with positive lead time, the structure of optimal rationing and ordering policies may be extremely complicated, for the decision about whether or not to satisfy a demand is affected by not only on-hand inventory and the remaining time before the end of the period, but also by outstanding orders and so on. We developed a near-optimal solution for it: using the dynamic critical level rationing policy and the base stock ordering policy. Some important properties of such a policy were obtained. A lower bound on the optimal cost under optimal rationing and ordering policies was also developed. We also analyzed another backorder clearing mechanism, mechanism T.

Besides the above analytical results, the research also provides the following important managerial insights:

(a) An evaluation of the potential cost savings if we change the rationing policy from the static critical level policy to the dynamic critical level policy. The results show that the dynamic rationing policy can significantly reduce cost in many cases, comparing the state-of-art static rationing policy. In many cases with typical problem settings, the cost difference can be more than 10%. So our analysis provides very important information to the managers: changing from the static rationing policy to the dynamic rationing policy is well justified. It is worth for the system manager to pay more effort to obtain a dynamic rationing policy.

(b) An understanding of the situations where the dynamic rationing policy is most useful. The results show that the more the penalty costs of different classes differ, the larger the cost saving of the dynamic rationing policy is. The benefit of implementing the dynamic rationing policy will become smaller when the demands of one class dominate those of the other class, for in these cases the system is similar to a system with only one demand class. The benefit of dynamic rationing policy is also non-monotone with the lead time.

(c) A bound on the how well our solution with a dynamic rationing policy performs relative to the unknown, optimal policies. The results show that the cost of our policy is very close to the optimal costs for a practical range of parameter setting and for poor service level conditions. In most cases, the cost difference is less than 1%.

(d) An understanding of the effect of the backorder clearing mechanisms on the average cost. We have considered two backorder clearing mechanisms: mechanisms M and T. Mechanism M is easier to understand and implement. The results show that mechanism T is better than mechanism M in terms of average cost, but the relative cost difference is very small for a wide range of parameter setting. So it is justified to use mechanism M in practice.

(e) An understanding on how the dynamic critical levels change and the difference between the dynamic critical level policy and the static critical level policy. We have shown that the dynamic critical level of a certain class decreases toward the end of a period and will reach 0 at the end of the period, while under the static critical level policy, the critical level is a constant. The numerical results also show the critical level of a certain class under the dynamic rationing policy will become steeper when the penalty costs of different classes differ more. That means the system needs to reserve more stock at a certain time for the more important classes in these cases.

The dynamic inventory rationing models can apply to the cases with any number of demand classes, though in the numerical study we consider only 2 or 3 demand classes for searching for the optimal parameters of the static rationing policies are time-consuming. As our models consider typical problem settings such as positive lead time, and the dynamic rationing policy can significantly reduce cost in many cases comparing with current state-of-art static rationing policies, and the dynamic rationing policy is easy to calculate (one-dimensional dynamic programming) and implement, it is reasonable to believe that our dynamic rationing policy can have a wide application in practice.

Deshpande et al. have shown that the unknown optimal dynamic rationing policies may significantly reduce cost, comparing with the static rationing policy. Here we have found such a particular dynamic rationing policy which indeed can significantly reduce cost in many cases and the cost of this policy is very close to the optimal costs. These results show dynamic rationing policies in other problem settings are worth to explore, though dynamic rationing problems are very complicated and difficult to analyze.

The above multiperiod models can be easily extended to the case with a setup cost in the ordering policy. In this case the system may use (R, s, S) ordering policy as in the general inventory problems without inventory rationing and we can develop a method to find s and S, based on our expressions for the average cost. In the next chapter we extend the work of this chapter to the cases with general demand processes.

Chapter 4

Inventory Rationing for Systems with General Demand Processes and Backordering

4.1 Introduction

In the previous chapter we have studied dynamic inventory rationing in systems with Poisson demands and backordering. In this chapter we extend it by changing the demands from Poisson process to general demand processes in which the arrivals of customers may follow a Non-Poisson process and one customer may require more than one unit of the product. In practice such general demand processes are not unusual. For example, the demand of spare parts may be dependent on the life of the spare parts on the machines, and sometimes one machine has installed multiple units of the same kind of spare part.

In the relevant literature most people consider inventory rationing assuming a certain demand process such as Poisson process. When demand process is very general ones, the rationing problems become much more difficult to solve, and little is known in the literature about the structure of optimal dynamic rationing policies, which is especially true when having a positive lead time. Chapter 3 and Deshpande et al. (2003) have shown that the dynamic rationing policy can significantly reduce cost in many cases, comparing with the static critical level policy, so in this chapter we assume the systems adopt a dynamic critical level rationing policy to develop optimization models, trying to obtain optimal or near-optimal parameters for the dynamic critical level rationing policy and ordering policy.

We first consider a single period system and develop a method to obtain near-optimal parameter values of the dynamic critical level policy and approximate expressions for the expected total cost. Then we develop an optimization model for a multiperiod system with positive lead time and obtain parameter values of dynamic rationing and ordering policies using a similar method as in Chapter 3.

Topkis (1968) has considered dynamic inventory rationing for systems with general demand processes, though in the literature most people consider Poisson demands. There are some notable differences between our work and Topkis (1968). First, Topkis assumes the demand process in a period can be divided into independent demands in some small intervals, while we have no such assumption, so the demand process in this chapter is more general. Second, for the multiperiod systems, the lead time in Topkis is zero, while in our model it is positive and the positive lead time makes problems more difficult to analyze. Third, Topkis develops a dynamic programming model to get the optimal dynamic rationing policy, while in this chapter we assume the system adopts a dynamic critical level rationing policy (structure of the optimal rationing policy may be extremely complicated and we have shown it in Chapter 3 for the case with Poisson demands) and develop methods to find appropriate parameters for the rationing and ordering policies. Finally, the penalty cost in our models is a general one which includes penalty cost per unit and penalty cost per unit per unit time, while the penalty cost in Topkis is penalty cost per unit per unit time. So our model can more accurately count for costs.

Other papers in relevant literature assume a particular demand process such as Poisson process, while in this chapter we consider general demand processes. For example, Teunter and Haneveld (2008) consider dyanmic inventory rationing in a single period system with Poisson demand, backordering and two demand classes. Assuming the dyanmic critical level rationing policy, they have developed a heuristic to find the times when the critical levels change. As stated previously, the inventory rationing problems with general demand processes are much more complicated than those with Piosson demands. The model in Teunter and Haneveld (2008) is a single period one and there is only 2 demand classes, while in this chapter we consider both single period and mutiperiod systems and there are multiple demand classses. In the literature many other papers consider the static rationing policies, while here we develop dynamic rationing policies.

The remainder of this chapter is organized as follows. Section 4.2 considers inventory rationing in a single period system, assuming a dynamic critical level rationing policy. Section 4.3 analyzes dynamic inventory rationing in a multiperiod system with a periodic review base stock (R, S) ordering policy with positive lead time. Near-optimal parameters for rationing and ordering policies are obtained. In Section 4.4, a numerical study is conducted to examine the performance of the proposed solution for the multiperiod systems. Section 4.5 summarizes the results. Proofs of lemmas and theorems of this chapter are given in Appendix C.

4.2 Inventory Rationing for a Single Period System

In this section we consider inventory rationing in a single period system, assuming a dynamic critical level rationing policy. A method is developed to obtain near-optimal parameters for the rationing policy and approximate expressions for the expected cost. Some important properties of the rationing policy are also obtained. The single period model is the building block for multiperiod systems and these results will be used in a later multiperiod model in this chapter.

We have studied in Chapter 3 dynamic inventory rationing in a multiperiod system with zero lead time and Poisson demands. We found that the rationing policy in each period is determined solely by parameters in each period, for lead time is zero and the system can purchase enough stock at the beginning of a period to fulfill all backorders so that backorders in a period will never be forwarded to its next period. When the demand process is changed to general processes and other situations (such as zero lead time, stationary cost factors and so on) remain unchanged, it is still true that backorders in a period will never be forwarded to the next period, so the dynamic rationing policy in a period can also be determined solely by parameters in this period as in the case with Poisson demand. The obtained dynamic rationing policy in the case with zero lead time can also be used in the development of a heuristic for the multiperiod systems with positive lead time. So in this section we consider how to dynamically ration stock in a single period to minimize expected total holding and penalty cost when the demand processes are general ones.

4.2.1 Model Formulation

In the following we brief the problem setting. Consider a single period inventory system that carries a product to satisfy demands from K demand classes. Let u denote the length of the period. At the beginning of the period, the system has some initial stock. Assume the system adopts a dynamic critical level rationing policy. During the period, when a demand arrives, it is either satisfied or rejected according to the rationing policy. The rejected demand is backlogged and a penalty cost is incurred. Assume the backorders can be fulfilled only at the end of the period, which is the same as that in the case with Poisson demand and zero lead time.

Demands of each customer class follow a certain general demand process. In the case of compound demand process in which a customer may require a random amount of product, we assume that the system can partially satisfy the demand, and the remaining part is backordered. Assume the demand of each customer is discrete and demands of different classes are independent.

Let t_c denote the remaining time to the end of the period. In the remaining part of this section we simply say *time* t_c to mean the time point when remaining time is t_c. Let $s_i(t_c)$, $t_c \in [0,u]$, denote the dynamic critical level of class i at time t_c. If a demand of class i is rejected at time t_c, then a penalty cost $\pi_i + \hat{\pi}_i \cdot t_c$ incurs. As in Chapter 3, we assume that if $i < j$, then $\pi_i \ge \pi_j$ and $\hat{\pi}_i \ge \hat{\pi}_j$, i.e., class 1 has the highest priority. The holding cost is h per unit per unit time.

Let ν denote a dynamic critical level rationing policy in the period and Φ denote the set of dynamic critical level policies. Let $H_{dy}(u,s)$ denote the expected holding and penalty cost over the whole period under the rationing policy ν, given initial inventory s at the beginning of the period. The rationing policy ν is completely determined by its critical levels $s_i(t_c)$, $t_c \in [0,u]$, $i \in \{1,...,K\}$. We are trying to find the optimal rationing policy $\nu \in \Phi$ to minimize the expected cost $H_{dy}(u,s)$. Let ν^* denote the optimal critical level rationing policy with the critical levels $s_i^*(t_c)$, and $H_{dy}^*(u,s)$ denote the optimal expected total cost. So the optimization problem is:

$$H_{dy}^*(u,s) = \min_{\nu \in \Phi} H_{dy}(u,s). \tag{4.1}$$

As class 1 is the most important one, we can see that if there is any on-hand inventory, then the system should satisfy the demand of this class. So the optimal critical level of class 1 is always 0, i.e., $s_1^*(t_c) = 0$, $t_c \in [0,u]$. In the following we develop a method to obtain near-optimal critical levels of class i, $1 < i \le K$. We first consider a system with only two demand classes, and then extend the method to the case with $K(K > 2)$ demand classes.

4.2.2 Calculation of Dynamic Critical Levels in Case of Two Demand Classes

In this subsection we consider a system with only two classes and develop a method to obtain near-optimal dynamic critical levels for class 2 (the optimal critical level of class 1 is already known).

Let $X(t_c)$, $t_c \in [0, u]$, denote the on-hand inventory at time t_c. Suppose a customer of class 2 demanding one unit of the product arrives at a certain time t_c^-, where time t_c^- is just before time t_c, and the system needs to make a decision about whether or not to satisfy it at time t_c. Assume $X(t_c^-) = s$.

According to the definition of critical levels, if the inventory s is larger the optimal critical level $s_i^*(t_c)$, then satisfy it, otherwise reject it. Under the optimal critical level rationing policy, if a demand of class i is rejected at time t_c, it is possible that a later demand of this class may be satisfied at some later time t_c' (for example, there is no demand since time t_c and the on-hand inventory at time t_c' is above the critical level of this class at this time for the critical levels decrease towards the end of the period). But such events will be rare. Thus if a demand of a certain class is rejected at time t_c, then the demands of this class will almost always be rejected later. Based on this observation, we develop a method to obtain near-optimal critical levels.

Let $H_{dy}^{*}(t_c, s)$ denote the optimal expected holding and penalty cost from time t_c to the end of the period, given $X(t_c) = s$ and under the optimal critical levels. Let $J^2(t_c, s)$ denote the expected holding and penalty cost from time t_c to the end of the period, given $X(t_c) = s$ and assuming the system rejects all demands of class 2 since time t_c, where the superscript 2 means always rejecting demands of class 2. So when $s \le s_2^*(t_c)$, $J^2(t_c, s)$ can be used to approximate $H_{dy}^{*}(t_c, s)$.

If the demand of class 2 that arrives at time t_c^- is satisfied, then total expected cost from time t_c to the end of the period is $C_{sat} = H_{dy}^{*}(t_c, s-1)$. If it is rejected, then the total expected cost is $C_{rej} = H_{dy}^{*}(t_c, s) + e_2(t_c)$, where $e_2(t_c) = \pi_2 + \hat{\pi}_2 \cdot t_c$ is the penalty cost. When $C_{rej} \ge C_{sat}$, we should satisfy the demand. For $J^2(t_c, s)$ can be used to approximate $H_{dy}^{*}(t_c, s)$ when $s \le s_2^*(t_c)$, we obtain a near-optimal critical level $s_2^a(t_c)$ of class 2 in this way:

$$
\begin{aligned}
s_2^a(t_c) &= \min\left\{s \mid J^2(t_c, s) + e_2(t_c) \ge J^2(t_c, s-1)\right\} - 1 \\
&= \min\left\{s \mid \Delta J_X^2(t_c, s) + e_2(t_c) \ge 0\right\} - 1,
\end{aligned}
\tag{4.2}
$$

where $\Delta J_X^2(t_c, s) = J^2(t_c, s) - J^2(t_c, s-1)$. $\Delta J_X^2(t_c, s)$ is the marginal cost, given $X(t_c) = s$ and assuming the system rejects all demands of class 2 since time t_c.

Now consider the calculation of $\Delta J_X^2(t_c, s)$. Figure 4.1 shows the change of inventory with the remaining time for a realization of demands in two cases: case A with

$X(t_c) = s$ represented by solid line and case B with $X(t_c) = s-1$ represented by the dashed line. As all demands of class 2 are rejected, the inventory decreases only whenever a demand of class 1 arrives. When the lines are above the horizontal axis, they represent the on-hand inventory and when they are below the axis, they represent the backorders of class 1.

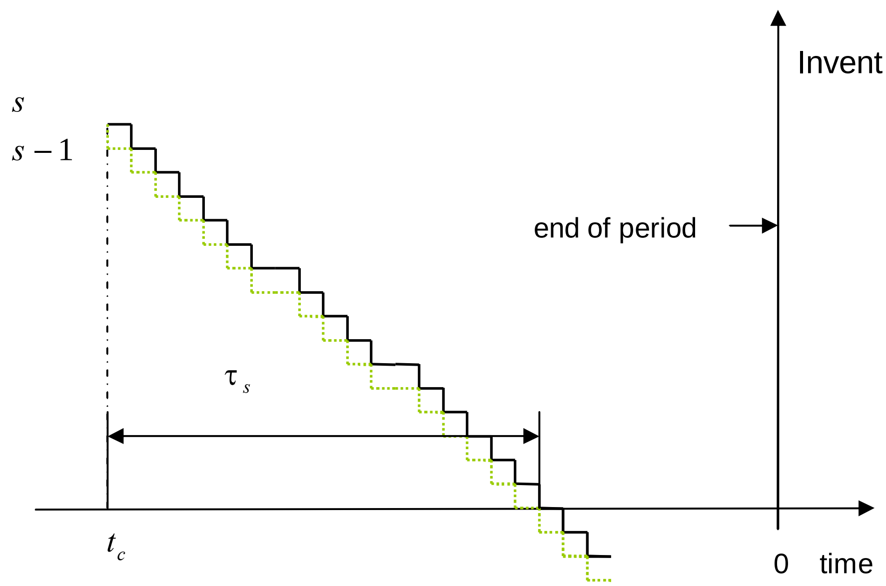

Figure 4.1 Inventory vs. remaining time with 2 classes

Let $D_1^{t_c}$ denote the total demand of class 1 from time t_c to the end of the period, and $P(D_1^{t_c} < s)$ denote the probability of $D_1^{t_c} < s$. Let t_s^1 denote the time when the s-th demand of class 1 arrives and τ_s denote the time difference between t_c and t_s^1. If the demand process is a compound process in which one customer may require more than 1 unit of product, the demand at time t_s^1 may be partially satisfied according to the assumption.

From Figure 4.1 we can see that case A has more holding cost than case B. Given a demand realization, if $D_1^{t_c} < s$, then the difference of holding costs of the two cases is $h \cdot t_c$. If $D_1^{t_c} \geq s$, then the difference of holding cost is $\tau_s \cdot h$. On the other hand, case A has less penalty cost than case B. If $D_1^{t_c} \geq s$, the s-th demand of class 1 can be satisfied in case A, while in case B it will be rejected, incurring a penalty cost $\pi_1 + \tilde{\pi}_1(t_c - \tau_s)$. If $D_1^{t_c} < s$, the difference of penalty costs is zero. Hence,

$$
\begin{aligned}
\Delta J_X^2(t_c, s) &= t_c \cdot h \cdot P(D_1^{t_c} < s) + h \cdot E(\tau_s \mid D_1^{t_c} \geq s) \cdot P(D_1^{t_c} \geq s) \\
&\quad - \{\pi_1 + \tilde{\pi}_1 \cdot E[(t_c - \tau_s) \mid D_1^{t_c} \geq s]\} \cdot P(D_1^{t_c} \geq s) \\
&= h \cdot t_c - [t_c \cdot (\tilde{\pi}_1 + h) + \pi_1] \cdot P(D_1^{t_c} \geq s) \\
&\quad + (\tilde{\pi}_1 + h) \cdot E[\tau_s \mid D_1^{t_c} \geq s] \cdot P(D_1^{t_c} \geq s).
\end{aligned} \tag{4.3}
$$

We have the following lemma for the marginal cost $\Delta J_X^2(t_c, s)$.

Lemma 4.1. *For a given remaining time* t_c, the first difference of the discrete function $J^2(t_c, s),\ s \geq 0$, *is nondecreasing in* s, *i.e.,* $\Delta J_X^2(t_c, s)$ *is nondecreasing in* s.

From Lemma 4.1, we immediate have the following theorem about a property of the critical level $s_2^a(t_c)$ of class 2.

Theorem 4.1. *For a given remaining time* t_c, *there exists a unique on-hand inventory* $s_2^a(t_c)$ *for class 2 such that if the on-hand inventory* $s > s_2^a(t_c)$, *then* $\Delta J_X^2(t_c, s) + e_2(t_c) \geq 0$, *otherwise* $\Delta J_X^2(t_c, s) + e_2(t_c) < 0$.

For the property of $s_2^a(t_c)$ and $\Delta J_X^2(t_c, s)$ shown in Lemma 4.1 and Theorem 4.1, we can easily search for $s_2^a(t_c)$ using equations (4.2) and (4.3). τ_s is the time difference between t_c and the epoch when the *s*-th demand of class 1 arrives and it is a continuous random variable. Assuming under the demand process of class 1 there is a continuous probability density function for τ_s, we have the following theorem which shows how the dynamic critical level $s_2^a(t_c)$ changes with the remaining time.

Theorem 4.2.

(a) $s_2^a(t_c) = 0$ *when* $t_c = 0$.

(b) *When* $\pi_1 = \pi_2 = 0$, $s_2^a(t_c)$ *is non-decreasing in remaining time* t_c, *i.e., if the remaining time* $t_1 \geq t_2$, *then* $s_2^a(t_1) \geq s_2^a(t_2)$.

Part (a) shows that if there is on-hand inventory at the end of the period, then the system should satisfy the demand of class 2. It is quite intuitive. At the end of the period the system does not need to reserve stock for class 1, for the new replenishment of next period (in the multiperiod system) will come immediately. Part (b) shows that when the penalty cost is per unit per unit time, then the dynamic critical level of class 2 decreases towards the end of the period. This result is also intuitive, since the system may need to reserve more stock for class 1 when the remaining time becomes longer. When π_1 and π_2 are positive, we cannot prove the above property, but the critical level also decreases towards the end of the period in the numerical examples which are not shown here.

4.2.3 Calculation of Dynamic Critical Levels in Case of $K(>2)$ Demand Classes

Now we extend the previous method to the case with more than 2 demand classes to obtain near-optimal dynamic critical levels. We sequentially determine the critical levels of class m, $m \in \{2,...,K\}$, i.e., first calculating the critical level of class 2, then that of class 3 and so on.

Let $J^m(t_c, s)$ denote the expected cost from any given time t_c to the end of the period, given $X(t_c) = s$ and the critical levels $s_i(t)$, $t \in [0,u]$, of class i, $i \in \{1,...,m-1\}$, and assuming all demands of class j, $j \in \{m,...,K\}$, are rejected from time t_c. Let $\Delta J_X^m(t_c, s) = J^m(t_c, s) - J^m(t_c, s-1)$. Similar to equation (4.2) we obtain a near-optimal dynamic critical level of class m, $m \in \{2,...,K\}$, by

$$s_m^a(t_c) = \min\left\{s \mid \Delta J_X^m(t_c, s) + e_m(t_c) \ge 0,\ s \ge s_{m-1}^a(t_c)\right\} - 1, \tag{4.4}$$

where $e_m(t_c) = \pi_m + \hat{\pi}_m \cdot t_c$ and $s_1^a(t_c) = s_1^*(t_c) = 0$. In reasonable critical level rationing policies, the critical level of a certain classes is lower than that of less important classes so that if a demand of a certain class should be rejected at time t_c, then the less important classes should always be rejected at this time, i.e., if $i < j$, then $s_i(t_c) \le s_j(t_c)$. So when searching for $s_m^a(t_c)$, we can increase s starting from $s_{m-1}^a(t_c)$.

$J^m(t_c, s)$ is the cost under the assumption that all demands of class $i, i \in \{m,...,K\}$, will be rejected since time t_c. We can ignore the demands of these classes during calculating $\Delta J_X^m(t_c,s)$. $J^m(t_c,s)$ is dependent on the dynamic critical levels of class i, $i \in \{2,...,m-1\}$. We have known that the optimal dynamic critical level of class 1 is $s_i^*(t_c)=0$. We can obtain near-optimal dynamic critical levels of all classes in the following procedure:

(a) First consider the demand process of class 1, ignoring demands of class j, $j \in \{2,...,K\}$, and use the method in the previous section to obtain the critical level of class 2, i.e., $s_2^a(t_c)$, $t_c \in [0,u]$.

(b) Assume we have obtained near-optimal dynamic critical levels of class i (i.e., $s_i^a(t_c)$), $i \in \{2,...,m-1\}$, where $2 \le m-1 < K$. Consider the demand processes of classes $1, 2,...,$ and $m-1$, ignoring demands of class j, $m \le j \le K$, and calculate $s_m^a(t_c)$, $t_c \in [0,u]$, using equation (4.4).

(c) Repeat Step b until the near-optimal dynamic critical level of class K is obtained.

In the following we show Step b of the above procedure, i.e., consider how to calculate $\Delta J_X^m(t_c,s)$ to obtain critical level $s_m^a(t_c)$ of class *m*, assuming the critical levels $s_i^a(t_c)$, $t_c \in [0,u]$, $i \in \{2,...,m-1\}$, are already obtained.

Figure 4.2 shows the change of inventory with remaining time for both case A with $X(t_c)=s$ and case B with $X(t_c)=s-1$. Case A is represented by the solid line and case B by the dashed line. The demands of class j, $j \in \{m, \cdots, K\}$, have no effect on the change of the on-hand inventory. For we are developing dynamic critical level of class m and the critical level of class m is not below that of class $m-1$, we assume the on-hand inventory s at time t_c is above $s_{m-1}^{a}(t_c)$. So in case A or B, the inventory system first satisfies demands of class i, $i \in \{1, ..., m-1\}$, since time t_c, and the inventory decreases. At a certain time after satisfying a demand or part of a demand, the on-hand inventory may be equal to the dynamic critical level of class $m-1$ at this time. We call this time *touch time* on the critical level of class $m-1$. Let t_{s-1}^{m-1} denote the touch time on the dynamic critical level of class $m-1$ in case B where $X(t_c) = s-1$. In case A, the touch time is t_{s}^{m-1}. It is possible that the inventory in case A or B does not reach the critical level of class $m-1$ until the end of the period. In this case we define the touch time is less than 0. The change of on-hand inventory after time t_{s-1}^{m-1} is not shown in Figure 4.2.

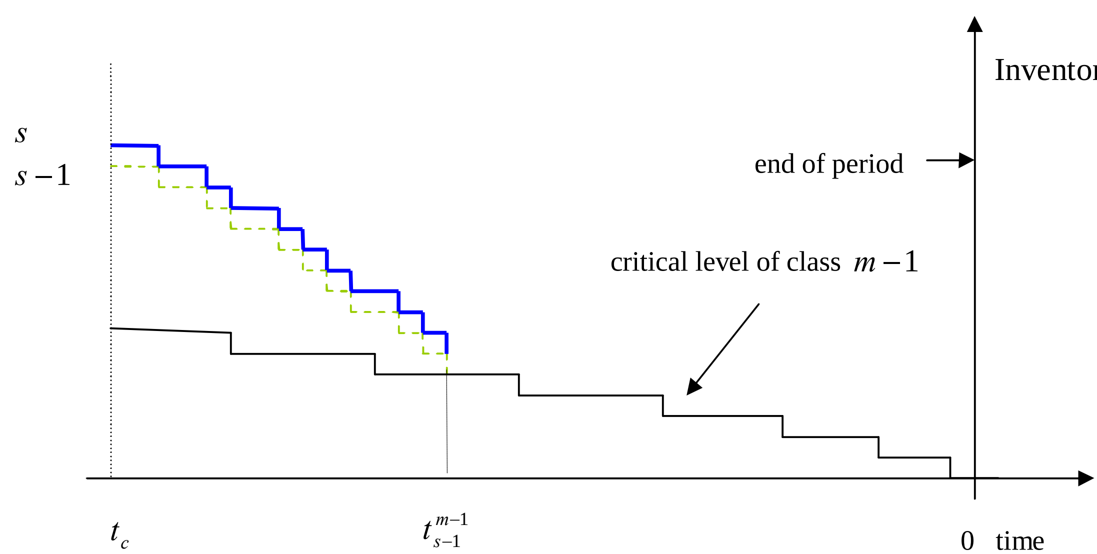

Figure 4.2 Inventory vs. remaining time with K (>2) classes

Now consider the difference of costs in cases A and B, i.e., $\Delta J_X^m(t_c, s)$. First consider the situation when $t_{s-1}^{m-1} > 0$. For a given realization of demand process with $t_{s-1}^{m-1} > 0$, the cost difference of cases A and B consists of two parts. The first part is the cost difference from time t_c to time t_{s-1}^{m-1} and the other part is from t_{s-1}^{m-1} to the end of period.

(i) **Cost difference from t_c to time t_{s-1}^{m-1}**

The cost difference between cases A and B from t_c to t_{s-1}^{m-1} is the holding cost $h \cdot (t_c - t_{s-1}^{m-1})$.

(ii) **Cost difference from time t_{s-1}^{m-1} to the end of the period**

In case B the inventory hits the critical level of class $m-1$ at time t_{s-1}^{m-1}, so the inventory at time t_{s-1}^{m-1} is equal to $s_{m-1}^a(t_{s-1}^{m-1})$. The inventory at this time in case A is $s_{m-1}^a(t_{s-1}^{m-1})+1$. According to the definition of critical level of class $m-1$, the cost difference of both cases from t_{s-1}^{m-1} to the end of period can be approximated by $\Delta J^{m-1}(t_{s-1}^{m-1}, s_{m-1}^a(t_{s-1}^{m-1}))$, which is already obtained during calculation of critical level of class $m-1$. In fact, $\Delta J^{m-1}(t_{s-1}^{m-1}, s_{m-1}^a(t_{s-1}^{m-1}))$ can be further approximated by $-e_{m-1}(t_{s-1}^{m-1})$, for according to the definition of critical level, $\Delta J^{m-1}(t_{s-1}^{m-1}, s_{m-1}^a(t_{s-1}^{m-1})) + e_{m-1}(t_{s-1}^{m-1})$ is close to zero.

So when $t_{s-1}^{m-1} > 0$, the total cost difference between two cases from t_c to the end of the period is approximated by $h \cdot (t_c - t_{s-1}^{m-1}) - e_{m-1}(t_{s-1}^{m-1})$. When $t_{s-1}^{m-1} \le 0$, the cost difference between cases A and B is the holding cost $t_c \cdot h$. Let $p(t_{s-1}^{m-1})$ denote the probability density function for random variable t_{s-1}^{m-1}, and $P_0^{m-1} = P(t_{s-1}^{m-1} \le 0)$ denote the probability of $t_{s-1}^{m-1} \le 0$. The probability density function $p(t_s^{m-1})$ depends on inventory s, critical levels $s_{m-1}^a(t_c)$, $t_c \in [0,u]$, and the demand process. We have

$$\int_0^{t_c} p(t_{s-1}^{m-1})dt_{s-1}^{m-1} + P_0^{m-1} = 1.$$

So

$$\begin{aligned}&\Delta J_X^m(t_c,s) = J^m(t_c,s) - J^m(t_c,s-1)\\&\approx \int_0^{t_c}[h\cdot(t_c - t_{s-1}^{m-1}) - e_{m-1}(t_{s-1}^{m-1})]\cdot p(t_{s-1}^{m-1})dt_{s-1}^{m-1} + P(t_{s-1}^{m-1} \le 0)\cdot h \cdot t_c\end{aligned} \tag{4.5}$$

Thus based on equations (4.4) and (4.5), we can obtain a near-optimal dynamic critical level $s_m^a(t_c)$ of class m, given $s_i^a(t_c)$, $i \in \{2,\cdots,m-1\}$, $t_c \in [0,u]$. Continue this procedure and we can obtain dynamic critical levels for all K classes.

4.2.4 Expected Total Cost

Now we develop expressions for the expected total cost $H_{dy}(u,s)$ over the whole period under the above dynamic rationing policy, given initial on-hand inventory $X(u) = s$.

These expressions will be used in later sections for the multiperiod systems with positive lead time.

When $s=0$, all demands of any class will be rejected. Let $\Pi_i(u)$ denote the expected penalty cost of class i in this period. So

$$H_{dy}(u,s)=\sum_{i=1}^{K}\Pi_i(u), \qquad \text{when } s=0 \; . \tag{4.6a}$$

In the previous section we have obtained the marginal costs $\Delta J_X^m(u,s)$, $m\in\{2,\cdots,K\}$. From these marginal costs and $H_{dy}(u,0)$, we can obtain the following approximate expressions for $H_{dy}(u,s)$, $s\in[0,s_K^a(u)]$:

$$H_{dy}(u,s)\approx H_{dy}(u,0)+\sum_{j=1}^{s}\Delta J_X^2(u,j), \qquad s\in(0,s_2^a(u)] \text{ , and} \tag{4.6b}$$

$$H_{dy}(u,s)\approx H_{dy}(u,s_m^a(u))+\sum_{j=s_m^a(u)+1}^{s}\Delta J_X^{m+1}(u,j),\; s\in(s_m^a(u),s_{m+1}^a(u)],\; 2\le m<K\,. \tag{4.6c}$$

Now consider the case when $s>s_K^a(u)$. For the convenience of denotation, let $\Delta J_X^{K+1}(t_c,s)=H_{dy}(t_c,s)-H_{dy}(t_c,s-1)$, where $s\in(s_K^a(t_c),\infty)$. When $s>s_K^a(t_c)$, demands of all classes are satisfied at time t_c. We can use the same method as that of obtaining $\Delta J_X^K(t_c,s)$ to obtain $\Delta J_X^{K+1}(t_c,s)$. Once we obtain $\Delta J_X^K(t_c,s)$, we have

$$H_{dy}(u,s) \approx H_{dy}(u,s_K^a(u)) + \sum_{j=s_K^a(u)+1}^{s} \Delta J_X^{K+1}(u,j), \quad s \in (s_K^a(u),\infty). \tag{4.6d}$$

Now consider calculating $\Delta J_X^{K+1}(t_c,s)$. Assume we have obtained the dynamic critical levels $s_i^a(t_c)$, $t_c \in [0,u]$, $i \in \{2,\cdots,K\}$. Consider a certain time t_c. Assume the on-hand inventory $X(t_c)$ at a given time t_c is greater than $s_K^a(t_c)$. The system first satisfies demands of all classes until the inventory reaches the critical curve of class K at a certain time (*touch time*) and since this time the system starts to reject demands of class K. Let t_{s-1}^K denote the touch time on the critical level of class K, given $X(t_c) = s-1$. So, similar to calculating $\Delta J_X^K(t_c,s)$, the marginal cost $\Delta J_X^{K+1}(t_c,s)$ is

$$\begin{aligned} &\Delta J_X^{K+1}(t_c,s) = H_{dy}(t_c,s) - H_{dy}(t_c,s-1) \\ &\approx \int_0^{t_c} [h \cdot (t_c - t_{s-1}^K) - e_K(t_{s-1}^K)] \cdot p(t_{s-1}^K) dt_{s-1}^K + P(t_{s-1}^K \le 0) \cdot h \cdot t_c \end{aligned} \tag{4.7}$$

Thus we have obtained approximate expressions (equation 4.6) for the expected holding and penalty cost $H_{dy}(u,s)$ under the above dynamic critical level policy.

After having obtained the dynamic critical level rationing policy in the single period system and the expressions for the expected cost, we can use these results to consider multiperiod systems. In the following section we consider dynamic inventory rationing in a multiperiod system with positive lead time and infinite horizon using a similar method to Chapter 3.

4.3 Inventory Rationing for a Multiperiod System with Positive Lead Time

This section considers dynamic inventory rationing in a multiperiod system with positive lead time, general demand processes and backordering. We develop an optimization model of minimizing average cost and obtain a near-optimal solution.

4.3.1 Model Formulation

Consider a multiperiod inventory system with infinite horizon that carries one product to satisfy the demands of K demand classes. The demand process and cost factors are the same as the previous single period model. The rejected demands are backordered. Assume the system adopts a periodic-review, base stock (R, S) ordering policy. The orders have a deterministic replenishment lead time L, $L \geq 0$. Under the (R, S) ordering policy, the ordering opportunities occur at fixed intervals of time. We call the time between two successive order opportunities as an *ordering period* and the time between two successive arrivals of orders as a *replenishment period*. The periods are indexed as 0,1,..., i.e., the first period is period 0. Each period has the same length u. Assume the time at the beginning of ordering period 0 is 0. The order placed at the beginning of ordering period m, $m \in \{0,1,...\}$, will arrive at time $l_m = mu + L$. The time from l_m to l_{m+1} is the replenishment period m.

Assume the system adopts a dynamic critical level rationing policy v. Let Φ denote the set of these dynamic critical level rationing policies. Again assume the backorders can be fulfilled only at the ends of replenishment periods and use the *full-priority* backorder clearing mechanism (i.e., mechanism M): at the end of a replenishment period the system first adds the arrived replenishment to stock, then fulfills outstanding backorders as much as possible from the most important class to the least important class. The above problem setting is similar to the multiperiod system in Chapter 3. The difference is that the demand process in Chapter 3 is Poisson process, while here we consider very general demand processes. We follow similar procedure to deal with this problem.

Let y_m denote the net inventory at the beginning of replenishment period m just after the replenishment arrives. For net inventory is the on-hand inventory minus backorders, y_m can be negative. Let $\boldsymbol{B}_m = \{b_1,...,b_K\}$ denote the outstanding backorders at the beginning of replenishment period m, where b_i is the number of backorders of class i. $(y_m, \boldsymbol{B}_m)$ describes the state of the system at the beginning of replenishment period m. According to the *full-priority* backorder clearing mechanism, we have: if y_m is positive, then there is no outstanding backorders, i.e., $b_i = 0$, and if y_m is negative, then $y_m = -\sum_{i=1}^{K} b_i$.

Let X_0 denote the initial state (on-hand inventory, outstanding orders, and outstanding backorders of each demand class) of the system at the beginning of replenishment period 0. We can see that $(y_m, \boldsymbol{B}_m)$, for a fixed m, is a random variable and let $P_m(y, \boldsymbol{B})$ denote the probability of $(y_m, \boldsymbol{B}_m)$. The probability $P_m(y, \boldsymbol{B})$ is under the given base stock (R, S) ordering policy, rationing policy ν and initial state X_0 of the system. In other words, for different base stock level of the ordering policy, rationing policy and initial state, $P_m(y, \boldsymbol{B})$ may have different probability distributions. Under the given ordering and rationing policies and the initial state X_0, when the system has arrived at the stable situations from the initial state X_0 after the transitional stages, then $P_m(y, \boldsymbol{B})$ may be independent on the initial state X_0. So when m is very large, $P_m(y, \boldsymbol{B})$ may be independent on the initial state. In the above we have shown that: if y_m is positive, then there is no outstanding backorders, i.e., $b_i = 0$, and if y_m is negative, then $y_m = -\sum_{i=1}^{K} b_i$. So when ($y, \boldsymbol{B}$) does not satisfy this requirement, then $P_m(y, \boldsymbol{B}) = 0$ for all m. We have known that $(y_m, \boldsymbol{B}_m)$ is a random variable. Let Ω denote the set of possible values of $(y_m, \boldsymbol{B}_m)$ for all m.

Assume the system adopts the same cost counting method as in Chapter 3. So if a demand is rejected, then the penalty cost $\pi_i + \hat{\pi}_i \cdot t_r$ (where t_r is the remaining time from the time when it is rejected to the end of current period) is counted in the period during which the demand is rejected, which is the same as that in the previous single period model. If this backorder is still outstanding at the beginning of a later period, then penalty cost $\hat{\pi}_i \cdot u$ is incurred in this period. As all rejected demands are backlogged, we do not consider the average ordering cost, for it is a constant.

Let $C_m(y, \boldsymbol{B})$ be the expected cost (penalty and holding cost) incurred in the replenishment period m, given $(y_m, \boldsymbol{B}_m)$. This cost depends on $(y_m, \boldsymbol{B}_m)$ as well as the rationing policy ν used in replenishment period m. Let $AC(S,\nu \mid X_0)$ denote the expected average cost per period when the base stock level is S and the system uses a dynamic critical level rationing policy ν in Φ, given the initial state X_0. Again, for the initial state in general does not affect the average cost over the infinite horizon, we remove the initial state X_0 from later expressions of the average cost. Then the optimization problem is to find an appropriate base stock level and a dynamic critical level rationing policy to minimize expected average cost, i.e.,

$$\begin{aligned}\min_{S,\nu} \mathrm{AC}(S,\nu) &= \min_{S,\nu} \lim_{M\to\infty} \frac{1}{M}\sum_{m=0}^{M-1} \mathrm{E}[C_m(y,\boldsymbol{B})] \\ &= \min_{S,\nu} \lim_{M\to\infty} \frac{1}{M}\sum_{m=0}^{M-1}\left[\sum_{\Omega} C_m(y,\boldsymbol{B})P_m(y,\boldsymbol{B})\right].\end{aligned} \tag{4.8}$$

The expected average cost in equation (4.8) consists of holding and penalty cost. If the above limit is not known to exist for some policies, we may use supremum for the item in right-hand side of (4.8) as in Chapter 3 and it will not affect our analysis. Under the base stock ordering policy and the dynamic critical level rationing policy, the limit in general exists.

4.3.2 A Solution to the Optimization Problem

One main difficulty in solving problem (4.8) is that probability of backorder vector $\boldsymbol{B}_m$ depends on the rationing policy ν and base stock S, and it is very difficult to find explicit expressions for the probability of $\boldsymbol{B}_m$. Another difficulty is that, given a rationing policy v

and the state $(y_m, \boldsymbol{B}_m)$, in general there is no closed-form expression for the expected cost $C_m(y, \boldsymbol{B} \mid v)$.

Based on the dynamic critical levels in the single period model, we obtain a myopic dynamic rationing policy for the multiperiod system: ration stock in each period using the dynamic critical levels of the single period model. Let dy denote this rationing policy for the multiperiod system and the corresponding expected average cost be $AC(S, dy)$. This rationing policy is trying to locally minimize the cost in each period, not considering its effect on later periods.

In the following we develop an approximate expression for the cost $AC(S, dy)$ under the above dynamic rationing policy dy and a given base stock S, and then use this approximate expression to obtain appropriate base stock level by minimizing the average cost, thus we obtain a solution to the optimization problem (4.8).

Let $P(y_m)$ denote the probability of y_m. If $y_m \geq 0$, then $\boldsymbol{B}_m = \mathbf{0}$. When $y_m < 0$, the sum of probabilities $P_m(y, \boldsymbol{B})$ with the same y_m but different $\boldsymbol{B}_m$, is equal to the probability $P(y_m)$ for the given y_m. Let $\Omega(y_m)$ denote the set of values of $(y_m, \boldsymbol{B}_m)$ with the same y_m and different $\boldsymbol{B}_m$. So when rationing policy dy is used and the base stock level is *S*, we have:

$$P_m(y, \boldsymbol{B}) = P(y_m) = P(D_L = S - y_m) \quad \text{when } y_m \geq 0 \text{, and}$$

$$\sum_{\Omega(y_m)} P_m(y, \boldsymbol{B}) = P(y_m) = P(D_L = S - y_m) \text{ when } y_m < 0 . \qquad (4.9)$$

where D_L is the total demand of all classes that occurs during lead time *L*. Note that the distribution of demand D_L is independent on the index *m*, rationing policy *v* and base stock level *S*.

$C_m(y, \boldsymbol{B} \mid dy)$ is the expected cost incurred in the replenishment period *m* under the rationing policy *dy* and this cost depends on the state variable $(y_m, \boldsymbol{B}_m)$. When $y_m \geq 0$, we can see that $C_m(y, \boldsymbol{B} \mid dy)$ is equal to the cost $H_{dy}(u, y)$ of the single period model, i.e.,

$$C_m(y, \boldsymbol{B} \mid dy) = H_{dy}(u, y) \text{ when } y \geq 0. \qquad (4.10)$$

When the net inventory after the replenishment arrives is negative, i.e., $y_m < 0$, the inventory cost in the period consists of the penalty cost of the outstanding backorders at the beginning of the period, if any, and the backorder cost for any new shortages that may occur in this period. Hence

$$C_m(y, B \mid dy) = H_{dy}(u, 0) + \sum_i b_i \pi \breve{E}_i u , \quad \text{when } y < 0 . \qquad (4.11)$$

As in the multiperiod model in Chapter 3, two factors will increase the expected proportion of backorders of less important classes in a given total backorders: the system reserves stock for important classes by rejecting demands of less important classes during each period, and the system uses the backorder clearing mechanism *M* under which the system fulfills backorders according to class priority. So there are more backorders of less important classes in a given total backorders. Moreover, the probability of $y < 0$ in general is not large in inventory systems. So when $y < 0$, we can approximate $C_m(y, \boldsymbol{B} \mid dy)$ by assuming all backorders in $\boldsymbol{B}_m$ are from the least important class *K*, i.e.,

$$C_m(y, \boldsymbol{B} \mid dy) \approx H_{dy}(u,0) - y\pi_K u, \quad \text{when } y < 0 . \tag{4.12}$$

Based on equations (4.9), (4.10) and (4.12), we can obtain the following approximate expression for the average cost $AC(S, dy)$:

$$AC(S,dy) \approx \sum_{y=0}^{S} P(D_L = S - y) H_{dy}(u, y) + \sum_{y=-\infty}^{-1} \left(H_{dy}(u,0) - y\pi_K u\right) P(D_L = S - y) .$$

Once having obtained the expression of average cost under a given base stock *S* and the rationing policy *dy*, the remaining task is to find an appropriate base stock level to minimize average cost. So the problem of finding the appropriate base stock level is

$$\min_S \sum_{y=0}^{S} P(D_L = S - y) H_{dy}(u, y) + \sum_{y=-\infty}^{-1} \left(H_{dy}(u,0) - y \cdot \pi_K \cdot u\right) P(D_L = S - y) . \tag{4.13}$$

Thus we have found a solution to the optimization problem (4.8): using the dynamic critical level rationing policy of the single period model to ration stock in each period of the multiperiod system, and an appropriate base stock is obtained by the above (4.13).

4.4 Numerical Study

4.4.1 The Numerical Study

In previous sections we have developed methods to obtain near-optimal parameters for the dynamic critical level rationing policy and ordering policy. Now we conduct a numerical study to investigate the effectiveness of the proposed method. Currently there is no method to obtain optimal solutions and/or tight lower bound on the optimal costs when the demands follow general non-Poisson processes. However, Chapter 3 has developed a method to obtain the optimal dynamic rationing policy for inventory rationing in each period of a multiperiod system with Poisson demands and zero lead time, and a tight lower bound on optimal costs for multiperiod systems with positive lead time, so we here assume Poisson demands to examine the above method, by comparing the results of the method in this chapter with those of the method in Chapter 3.

In this numerical study we mainly consider the single period systems to compare the near-optimal dynamic critical levels with the optimal critical levels, and to compare the costs under the near-optimal dynamic rationing policy with the optimal costs in different operational conditions. The model framework of the multiperiod systems with infinite horizon in this chapter is almost the same as that in Chapter 3 and the information about cost difference of these policies in the multiperiod systems can be obtained from the results of comparison in single period systems.

Let dya denote the near-optimal dynamic critical level policy obtained by the method in this chapter and dy^* denote the optimal dynamic critical level policy obtained by the method in Chapter 3. Let $H^*_{dy}(u,x)$ denote the expected holding and penalty cost in the single period under the optimal policy $dy*$, given the initial inventory x at the beginning of the period, where u is the length of period. Let $H^a_{dy}(u,x)$ denote the expected holding and penalty cost in the period under policy dya. Define the following percentage to measure the cost difference under the above two policies:

$$\Delta H^{a*}(x) = \frac{H^a_{dy}(u,x) - H^*_{dy}(u,x)}{H^*_{dy}(u,x)} \cdot 100\% .$$

The relative difference $\Delta H^{a*}(x)$ is a function of initial on-hand inventory x. We will check $\Delta H^{a*}(x)$ under different operational conditions and different x. Let $x^{a*} = \arg\max_{x \geq 0} \Delta H^{a*}(x)$.

In the numerical study there are 3 demand classes. We choose a parameter setting as a base case, and then change one factor at a time that can affect the relative difference of costs. We fix hold cost $h = 1$ and the penalty cost per unit $\pi_i = 0$. So the penalty cost is per unit per unit time. The input data for the base case is: $\lambda_1 = \lambda_2 = \lambda_3 = 300$, $\hat{\pi}_1 : \hat{\pi}_2 : \hat{\pi}_3 = 20:5:1.5$, $\hat{\pi}_3 = 1.5$ and $u = 0.1$. Then we change one of these factors at a time: ratio of arrival rates, ratio of penalty costs, and length of the period. When changing ratio of penalty costs, we fix penalty cost $\hat{\pi}_3 = 1.5$. When varying the ratio of arrival rates, we remain the total arrival rate unchanged.

In the numerical study, for each parameter setting we first obtain optimal and near-optimal dynamic critical levels, and then use simulation to obtain the expected costs under both rationing policies. For we obtained only the approximate expressions for the cost under policy dya, we use simulation to obtain the expected costs under both policies. In order to ensure enough accuracy in simulation, we repeat 10,000 times for the stochastic demand process in the single period systems for each scenario. The results are shown in Figures 4.3 and 4.4 and Table 4.1.

4.4.2 Interpretation of Results

First consider the base case. Figure 4.3 shows the optimal and near-optimal dynamic critical levels of class 2 and 3 in base case. We have known that the optimal dynamic critical level of class 1 is always 0. Curves *dy**_2* and *dy**_3* are the optimal dynamic critical levels of class 2 and 3 respectively. Curves *dya_2* and *dya_3* are the approximate optimal dynamic critical levels of class 2 and 3, respectively. The period is divided into 900 intervals. From Figure 4.3 we can see that the near-optimal critical levels are very close to the optimal critical levels respectively.

The relative cost difference $\Delta H^{a^*}(x)$ under two policies is a function of initial on-hand inventory x. From Figure 4.4 we can see that $\Delta H^{a^*}(x)$ is very small and is not monotonic with x. When x is some intermediate value, $\Delta H^{a^*}(x)$ is larger than those when x is very large or very small. When x is very large or near 0, the cost difference is very close to 0. When x is some intermediate value, the system needs to ration stock and the difference of critical levels in two policies can bring some notable difference of costs.

When x increases from these intermediate values and becomes very large, the system has almost enough stock to satisfy demands of all classes, so the difference of costs under two policies is very small. When initial on-hand inventory decreases from these intermediate values to near 0, most demands can not be satisfied and the total cost is large, so the relative cost difference is very small.

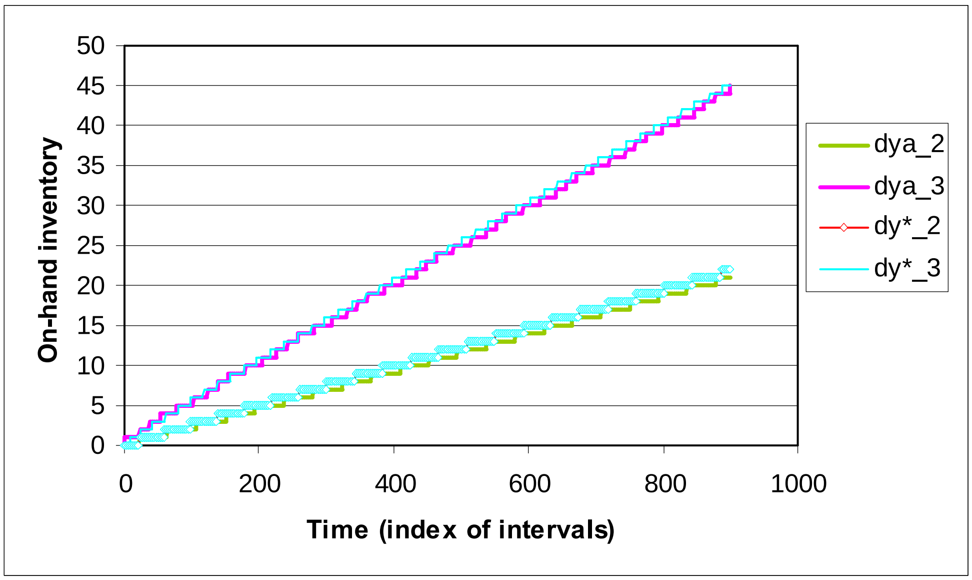

Figure 4.3 Optimal and approximate optimal dynamic critical levels in base case

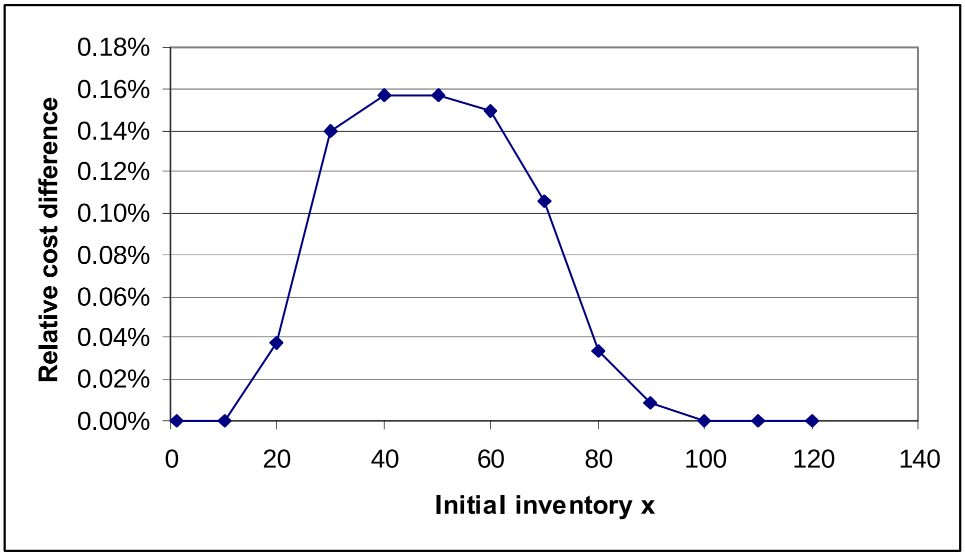

Figure 4.4 Relative cost difference $\Delta H^{a^*}(x)$ vs. initial inventory in base case

Now consider Table 4.1 which includes results for cases when changing a factor at a time. $\Delta H^{a^*}(x^{a^*})$ is the maximal cost difference for different initial on-hand inventory in a parameter setting. From this table we can see that $\Delta H^{a^*}(x^{a^*})$ is small in all these cases. The factor that has notable effect on cost difference is the ratio of penalty costs. When we increase the penalty costs of class 1 and 2 while remaining penalty cost of class 3 unchanged, the relative difference $\Delta H^{a^*}(x^{a^*})$ increases. The near-optimal dynamic critical levels are obtained by assuming all demands of some less important classes are rejected since a certain time. While under the optimal dynamic critical level policy, when a demand of less important classes is rejected at a certain time, it is possible that very small amount of demands from these less important classes are satisfied later. When we increase the penalty costs of important classes, the optimal dynamic critical levels at a certain time will increase, for the system should reserve more stock for them. In this case, under the optimal dynamic critical level policy the chance of satisfying later demands from less important classes will increase, hence the relative cost difference of two policies will increase. Other factors do not have significant effect on cost difference for they do not affect very much the above chance of satisfying later demands from less important classes.

From the above we can see that the critical levels obtained by the method in this chapter are very close to the optimal critical levels and the cost differences are also very small, so the above near-optimal dynamic critical level policy is a very good approximation to the optimal dynamic critical level policy. The above numerical study assumes there are 3 demand classes. When the number of demand classes increases, the relative difference $\Delta H^{a*}(x^{a*})$ may increase. In the model we use approximate expressions to obtain dynamic critical level of class i, based on dynamic critical level of class $i-1$ and so on. When the number of demand classes increases, the cumulative error will increase.

Table 4.1 Relative cost difference $\Delta H^{a*}(x^{a*})$ under different conditions

Factors	Parameters			$\Delta H^{a*}(x^{a*})$, x^{a*}
	u	$\lambda_1 : \lambda_2 : \lambda_3$	$\pi_1 : \pi_2 : \pi_3$	
Base case	0.1	1:1:1	20:5:1.5	0.16%, 60
			5:2:1.5	0.03%, 63
Ratio of penalty costs	0.1	1:1:1	10:3:1.5	0.08%, 45
			40:8:1.5	0.50%, 53
			100:10:1.5	1.31%, 55

Ratio of arrival rates	0.1	1:2:3	20:5:1.5	0.24%, 45
		1:3:6		0.24%, 38
		3:2:1		0.10%, 58
		6:3:1		0.08%, 58
Length of Period	0.05	1:1:1	20:5:1.5	0.85%, 32
	0.15			0.13%, 80
	0.2			0.10%, 118
	0.3			0.06%, 160

The above comparison is for dynamic inventory rationing in single period systems (i.e., rationing in each period of a multiperiod system). For the infinite horizon systems with the base stock (R, S) ordering policy and positive lead time, we can also compare the difference of average costs under the two dynamic critical level rationing policies: policy dya developed by the method in this chapter and policy $dy*$ developed in Chapter 3. Let $AC_{dya}(S)$ and $AC_{dy*}(S)$ denote the average cost under policies dya and $dy*$, respectively, where S is the base stock. Define

$$\Delta AC(S) = \frac{AC_{dya}(S) - AC_{dy*}(S)}{AC_{dya}(S)},$$

which can be used to measure the cost difference under both policies.

Under the base stock ordering policy, the net inventory at the beginning of a certain period is a random variable. We can see that under an appropriate base stock level, the probability that the net inventory at the beginning of a period is negative is very small, which is also shown in Chapter 3. So $\Delta AC(S)$ is the weighted sum of cost differences $\Delta H^{a^*}(x)$ for different initial on-hand inventory x in the above single period systems, ignoring the probability that the net inventory at the beginning of a period is negative. So $\Delta AC(S)$ is less than the maximal cost difference $\Delta H^{a^*}(x^{a^*})$ in the above single period systems. The cost difference $\Delta AC(S)$ under the appropriate base stock S is shown in Table 4.2.

Table 4.2 Relative difference of average costs for infinite horizon systems

Factors	Parameters			ΔAC
	u	$\lambda_1 : \lambda_2 : \lambda_3$	$\pi_1 : \pi_2 : \pi_3$	
Base case	0.1	1:1:1	20:5:1.5	< 0.16%
Ratio of penalty costs	0.1	1:1:1	5:2:1.5	< 0.03%
			10:3:1.5	< 0.08%
			40:8:1.5	< 0.50%
			100:10:1.5	< 1.31%
Ratio of arrival		1:2:3		< 0.24%

rates	0.1	1:3:6	20:5:1.5	< 0.24%
		3:2:1		< 0.10%
		6:3:1		< 0.08%
Length of period	0.05	1:1:1	20:5:1.5	< 0.85%
	0.15			< 0.13%
	0.2			< 0.10%
	0.3			< 0.06%

From the above table we can see that the cost difference under both policies for infinite horizon case is also very small. For the above parameter settings, we can also calculate the lower bound on the optimal cost and the gap CR_B between the lower bound and the average cost $AC_{dy*}(S)$ under optimal policy $dy*$ using the method in Chapter 3. The results show that the gap CR_B is very small, less than 1.0e-6 for cases when the ratio of lead time to the length of a period is 1 and 2. For the average cost under policy *dya* developed in this chapter is very close to the optimal cost under policy $dy*$ in Chapter 3, the cost of policy *dya* is also very close to the lower bound.

4.5 Conclusions

In this chapter we considered dynamic inventory rationing in systems with general demand processes and backordering. We first investigated dynamic inventory rationing in single period systems. Assuming a dynamic critical level rationing policy, a method was developed to obtain near-optimal parameters for the rationing policy and approximate expressions for the expected cost under this rationing policy. Some important properties of the rationing policy were also obtained. Then we studied dynamic rationing in a multiperiod system with a (R, S) ordering policy and positive lead time. An optimization model was developed and a solution was provided: embedding the dynamic critical level rationing policy of the single period model into each period of the multiperiod system to ration stock and a method was developed to obtain an appropriate base stock for the ordering policy.

A numerical study was conducted to investigate the effectiveness of the proposed method. For we can obtain the optimal dynamic critical level rationing policy and the optimal expected cost in the single period systems when demands follow Poisson processes, we assumed Poisson demands to compare the results from the proposed method in this chapter with the optimal results. The results show the critical levels obtained by the proposed method are close to the optimal ones. The cost difference under both rationing policies is small for a wide range of parameter values.

The proposed method has a few noticeable characteristics. One is that the dynamic critical levels of different demand classes are obtained sequentially, so it avoids the curse of dimensionality in dynamic programming models which existed in most dynamic inventory rationing problems with backorders. Another advantage of the proposed method is that the demand process can be very general. Third, from this method people may obtain a new rationing policy: whenever inventory drops to the critical level of a certain class, then reject all demands of this class until the end of the period. Under the dynamic critical level policy, it is possible that a customer of a certain class is rejected and a later customer of this class is satisfied. This may be unfair to the customers. The new rationing policy can avoid this problem. This new policy is also easy to understand and implement. Moreover, for the assumption (once a demand of a certain class is rejected, then all demands of this class will be rejected until the end of the period) is a good approximation to the practical cases, the cost penalty of implementing this new rationing policy will be small, comparing with the dynamic critical level policy, and this is justified in the numerical results.

For inventory rationing problems with very general demand processes and positive lead time, in the literature little is known about the optimal rationing policy. The dynamic critical level rationing policy in this chapter provides a benchmark for these cases. So one of possible future research directions is to investigate the structure of the optimal rationing policy for systems with such general demand processes, or develop other methods to obtain better rationing policies. In the next chapter, we study dynamic inventory rationing for systems with lost sales and Poisson demand.

Chapter 5

Inventory Rationing for Systems with Poisson Demands and Lost Sales

5.1 Introduction

Previous chapters have analyzed dynamic inventory rationing for systems with backordering, assuming Poisson demands or other general demand processes. In this chapter we consider dynamic inventory rationing in multiperiod systems with lost sale, assuming Poisson demand.

In the multiperiod systems under typical settings such as positive lead time, the rationing policy will interact with the ordering policy and it is very difficult to obtain both optimal rationing and ordering policies simultaneously. People often consider inventory rationing assuming a given ordering policy. Here we consider a multiperiod system with positive lead time, assuming a periodic review ordering policy with fixed ordering amount per period in which the system orders a fixed amount Q of stock at each ordering opportunity. Letôs denote it as (R, Q) policy, where R represents periodic-review and Q is the fixed ordering amount per period. This kind of ordering policy often appears as a supply contract. In some supply contracts, the customers may have some limited flexibility in adjusting their ordering amount, beside the fixed amount stock per period. For example, besides this fixed amount stock, the customer may place other orders with variable amount of the product. Of course, the price of these temporary orders may be different (often higher) from that of the fixed amount per period. We consider this fixed amount ordering policy as a starting point for the more complex problems.

This fixed ordering amount ordering policy and its many variations are widely used in practice and have been researched by many people. See, for example, Rosenshine and Obee (1976), Urban (2000), Anupindi and Bassok (1999), Bassok et al. (1997) and Johansen et al. (2000). Note that the above papers consider only one demand classes, i.e., no inventory rationing. Recently, Frank et al. (2003) consider a system with two demand classes: customers with deterministic, fixed ordering amount per period and customers with stochastic ordering. This fixed ordering amount policy from the customers appears as a supply contract. This type of ordering policy has some advantages. For the supplier has known the exact information about the future demands, it can have a better schedule of production and transportation so that the supplier can reduce cost and provide customers with a higher discount for the price. It can also reduce the lead time and make replenishments on time. In a supply chain environment, this ordering policy can also reduce the bullwhip effect. Lee et al. (1997) provide a quite complete analysis for the causes of bullwhip effect. Caplin (1985) and Blinder (1982, 1986) have shown that the use of (s, S) type of ordering policy results in the variance of replenishment orders exceeding the variance of demands, which is in fact the bullwhip effect. Note that the above papers consider general inventory problems without inventory rationing among demand classes. For these advantages of this type of ordering policy, we assume this type of ordering policy for the multiperiod system with multiple demand classes.

We first consider dynamic inventory rationing in a finite horizon M-period system. A dynamic programming model is developed for it. Important characteristics of the optimal rationing policy and optimal ordering amount Q are obtained. The optimal rationing policy is shown to be the dynamic critical level policy in each period. We then extend the M-period model to the case with infinite horizon. The optimal rationing policy is still a dynamic critical level rationing. We also provide a range for the optimal ordering amount so that we can search for the optimal ordering amount effectively.

Melchiors (2003) and Lee and Hersh (1993) have considered dynamic inventory rationing for systems with lost sales and Poisson demand. There are some notable differences from them. First, the ordering policy in this chapter is periodic review, while the ordering policy in Melchiors (2003) is continuous review. Second, Melchiors (2003) first assumes a particular dynamic critical level policy, then develop expressions for the average cost, while we develop a dynamic programming model to find the characteristics of the optimal rationing policy and so on. Third, Lee and Hersh (1993) is a single period model without holding cost, while we consider multiperiod systems with ordering policy and holding cost. Lee and Hersh (1993) is a special case of our model when there is only one period and no holding cost.

Topkis (1968) has developed a general dynamic programming model which can deal with both backordering and lost sales cases. For the lost sales case, his model is also a one-dimensional dynamic programming one. He considers both single period systems and multiperiod systems with zero lead time. There are some differences between our work and Topkis. First, the demand process is different. Topkis assumes a period is divided into some intervals and the demands in different intervals are independent and the demand is continuous, while we assume the demands follow a Poisson process. For the state variable is discrete, the analysis is difficult. Second, Topkis has shown the critical levels of the rationing policy decrease towards the end of the period under some conditions. We also show such a trend for the dynamic critical levels, but under more loose conditions. Third, for the multiperiod systems, the ordering policy is different. He assumes a zero lead time and finds the optimal ordering policy, while our model assumes a certain ordering policy which may appear as a supply contract.

The work in this chapter also has notable differences from the work in previous chapters. First, Chapters 3 and 4 consider the backordering case, while this chapter is for the lost sales case. Second, the ordering policy is different. In Chapters 3 and 4, we characterize the optimal ordering policies for some problems and in other cases we assume a periodic review, base stock (R, S) policy, while in this chapter we assume a periodic review, fixed ordering amount ordering policy to analyze the dynamic inventory rationing problems.

The remainder of this chapter is organized as follows. Section 5.2 develops a model for dynamic inventory rationing in an *M*-period system with finite horizon. In Section 5.3, the optimal rationing policy and optimal cost function of the *M*-period system are characterized. In Section 5.4, the *M*-period model is extended to infinite horizon. In Section 5.5, a numerical study is conducted to obtain some managerial insights. Finally, in Section 5.6, conclusions are provided and possible extensions are discussed. Proofs of lemmas and theorems in this chapter are presented in Appendix D.

5.2 Model Formulation for an *M*-period System with Finite Horizon

In this section we study dynamic inventory rationing in a multiperiod system with finite horizon, lost sale and Poisson demands, and a dynamic programming model is developed.

Consider an *M*-period inventory system, where *M* is a positive integer, in which a single product is stored to satisfy demands from K demand classes. Each period has the same length of u time unit. The periods are indexed as $M,\cdots,1$, i.e., the first period is period *M*.

At the beginning of period *M* the system has some initial inventory. Assume the system uses a supply contract to replenish stock under which an amount of Q units will arrive at the end of each period except the last period, in other words, Q units will arrive at the beginning of each period except the first period. For the system does not need stock any more at the end of the horizon, there is no replenishment to arrive at the end of the last period. Denote such a supply contract as (*R*, *Q*) ordering policy, where *R* presents periodic review. Let c denote the variable ordering cost per unit. We ignore the fixed ordering cost, for it is a constant under this kind of ordering policy and will not affect any decision of ordering and rationing policies.

Let x_k, $k \in \{1,...,M\}$, denote the on-hand inventory at the beginning of period *k*. So the initial inventory at the beginning of the horizon is x_M. For the convenience, let x_0 denote the remaining stock at the end of period 1. The remaining stock at the end of the horizon has a salvage value. The salvage value $S_0(x)$ is a function of remaining stock *x*. Assume $S_0(x)$ is nondecreasing with x, $S_0(0) = 0$, and the first difference of $S_0(x)$ is nonincreasing in x.

Demands from class i follow a Poisson process with rate λ_i and the demands of different classes are independent. The system adopts a dynamic rationing policy. Any rejected demand is lost and the penalty cost of rejecting a demand of class i is π_i. Without loss of generality, assume class 1 has the highest priority, so if $i < j$, then $\pi_i \geq \pi_j$.

The inventory has a holding cost of h per unit per unit time. Assume cost is accumulated in each period without discount and the cost in one period is discounted into its previous period by a discount factor α, $0 < \alpha < 1$.

Let $\upsilon = \{\upsilon_M, ..., \upsilon_1\}$, where u_i is the rationing policy in period *i*. So υ is an inventory rationing policy for the system on the whole horizon, *M*-period. Let Φ denote the set of all admissible rationing policies υ.

Given ordering amount *Q* of the ordering policy, the total discounted variable ordering cost $OC_{Q,M} = Q \cdot c \dfrac{1-\alpha^M}{1-\alpha}$, which is a constant under a given *Q* and independent on rationing policy υ.

Let g_k denote the penalty and holding cost occurred in period *k*, which is dependent on the realization of demands in this period, initial on-hand inventory x_k at the beginning of this period, and the rationing policy υ_k on this period. Let $J^{\upsilon}_{Q,M}(x_M)$ denote the expected discounted holding and penalty cost over the whole horizon, given initial inventory x_M at the beginning of the horizon, the ordering amount *Q*, and the inventory rationing policy υ. i.e.,

$$J^{\upsilon}_{Q,M}(x_M) = E\left[\sum_{k=1}^{M} \alpha^{M-k} g_k - \alpha^M S_0(x_0)\right]. \tag{5.1}$$

Let $TC_M(x_M, \upsilon, Q)$ denote the expected total discounted cost which is equal to the ordering cost $OC_{Q,M}$ plus the holding and penalty cost $J^{\upsilon}_{Q,M}(x_M)$, given x_M, υ and Q. The optimization problem is to find an inventory rationing policy υ and ordering amount *Q* to minimize the expected total discounted cost, i.e.,

$$\min_{\upsilon,Q} TC_M(x_M,\upsilon,Q) = \min_{\upsilon,Q} OC_{Q,M} + J^{\upsilon}_{Q,M}(x_M)\,. \tag{5.2}$$

For the above optimization problem is very complicated, in the following we first consider the dynamic inventory rationing under a given *Q*, then change *Q* to find global optimal solution. Now we fix *Q*.

Under a given *Q*, minimizing the total cost $TC_M(x_M,\upsilon,Q)$ is equal to minimizing $J^{\upsilon}_{Q,M}(x_M)$, for the total discounted ordering cost $OC_{Q,M}$ is a constant and will not affect the rationing policy. So in the following we consider minimizing the holding and penalty cost $J^{\upsilon}_{Q,M}(x_M)$. The optimal cost function $J^{*}_{Q,M}(x_M)$ under a given *Q* is defined as

$$J^{*}_{Q,M}(x_M) = \min_{\upsilon\in\Phi} J^{\upsilon}_{Q,M}(x_M)\,. \tag{5.3}$$

The optimal cost function $J^{*}_{Q,M}(x_M)$ represents the optimal costs for different initial on-hand inventory x_M, given ordering amount Q.

Chapter 3 has considered the dynamic inventory rationing in a single period in which the unmet demands are backordered, while here we assume unmet demands are lost. We use a similar procedure to that in Chapter 3 to model the problem as a discrete-time Markov decision problem to obtain characteristics of the optimal rationing policy and optimal cost function.

Divide each period into N equal intervals. The intervals are so small that the probability that more than 2 demands arrive in an interval is very small and it can be ignored. So the length of each interval is $\Delta t = u / N$, where u is the length of a period. The time points separating the intervals from the beginning of the period to the end are indexed as $N, N-1,...,0$, i.e. the beginning of the period is indexed as time point N and the end of the period is 0. The interval which begins at time point n, and ends at time point $n-1$ is given the index n. Let (k,n), $k \in \{1,...,M\}$ and $n \in \{0,1,...,N\}$, denote the time point n in period k. So the time point $(k,0)$ represents the end of period k and the time point $(k-1,N)$ represents the beginning of period k-1. Time points $(k,0)$ and $(k-1,N)$ approach to each other infinitely, but they are two different time points (for there is a discount factor α) and belong to different periods.

Let $\lambda = \sum_{i=1}^{K} \lambda_i$. Since the demand process is a Poisson process and the intervals are so small, the probability p_i that a demand of class i arrives during an interval is $\lambda_i \Delta t + o(\Delta t)$, the probability p_0 that no demand of any class arrives during an interval is $1 - \lambda \Delta t + o(\Delta t)$, and the probability that more than 1 demand from all classes arrive during an interval is ignored. If a demand from class i arrives during an interval of a certain period, we assume the system delays the decision about whether to satisfy or reject it until the end of the interval. We regard the decision at time point (k,n) is the behavior during interval $n+1$ which is from time point $(k,n+1)$ to (k,n).

At the ends of periods except period 1, there is an order of Q to arrive, i.e., arriving at time points $(k,0)$, $k \in \{M,...,2\}$. Assume the events happen as follows: at time point

$(k,0)$, $k \in \{M,...,2\}$, the system first makes a decision about whether to satisfy the demand arriving during interval 1 of period *k*, then an order of *Q* arrives and is added to stock. The total remaining stock becomes the initial on-hand inventory of next period.

Let $x(k,n)$, $n \in \{0,...,N-1\}$, denote the on-hand inventory at the time point (k,n) just after the decision about whether or not to satisfy the arrived demand in interval $n+1$. Time point (k,N) is just after the arrival of an order of Q at time point $(k+1,0)$ and it is not a decision point. Let $x(k,N)$ denote the initial on-hand inventory of period *k*. Thus $x(k,N) = x(k+1,0) + Q$, $k \in \{1,...,M-1\}$. $x(k,n)$, $n \in \{1,...,N\}$, is the on-hand inventory at the beginning of interval n which starts from time point (k,n) .

Let $H_T(k,n,x)$ denote the optimal expected total discounted cost (not including ordering cost, for *Q* is fixed we account it outsides) from the beginning of interval *n* of period *k* to the end of the horizon, i.e., from time point (k,n) to the end of horizon, given inventory $x(k,n)$. In the following we consider how to obtain $H_T(k,n,x)$ for different cases when k,n and x have different values. The following four cases are considered:

Case a: $k=1, n=0$ and $x(k,n) \geq 0$.

This case is about the cost at the end of period 1, i.e., at the end of the horizon. According to our assumption, the remaining stock has a salvage value and $S_0(x)$ is the salvage value function. For there is a discount factor from one period to its previous period, we have: $H_T(1,0,x) = -\alpha \cdot S_0(x)$. For the first difference of $S_0(x)$ is assumed to be nonincreasing in the remaining stock *x*, we can see the first difference of $H_T(1,0,x)$ is nondecreasing in *x*.

Case b: $k \in \{2,...,M\}, n = 0$ and $x(k,n) \geq 0$.

This case is about the cost since the end of period k, $k \in \{2,...,M\}$, to the end of the horizon. For an order of Q will arrive at time point $(k,0)$, the on-hand inventory $x(k-1,N) = x(k,0) + Q$. The cost $H_T(k-1,N,x)$ will be discounted to period *k*, so we have: $H_T(k,0,x) = \alpha \cdot H_T(k-1,N,x+Q)$.

Case c: $k \in \{1,...,M\}, n > 0$ and $x(k,n) = 0$.

In this case, there is no on-hand inventory at the beginning of interval *n*, $n \in \{1,\cdots,N\}$, of period *k*. If a demand of class *i* arrives during interval *n*, then the system needs to make a decision at the end of the interval, i.e., time point $(k,n-1)$, about whether to satisfy or reject it. For $x(k,n) = 0$, it is obvious that the system should reject the demand for there is no on-hand inventory, so we have: $H_T(k,n,0) = p_0 \cdot H_T(k,n-1,0) + \sum_{i=1}^{K} p_i \cdot [\pi_i + H_T(k,n-1,0)]$, where the holding cost in interval *n* is 0.

Case d: $k \in \{1,...,M\}, n > 0$ and $x(k,n) > 0$.

In this case there are some on-hand inventory at the beginning of interval *n*, $n \in \{1,...,N\}$, of period *k*. Again if there is a demand of class *i* arriving during interval *n*, then the system need to make a decision at time point $(k, n-1)$ about whether or not to satisfy it. If satisfy the demand, then the total discounted cost from time point (k, n) to the end of the horizon is $x \cdot \Delta t \cdot h + H_T(k, n-1, x-1)$, where $x \cdot \Delta t \cdot h$ is the holding cost in the interval n. If reject the demand, then the total discounted cost from time point (k, n) to the end of the horizon is $x \cdot \Delta t \cdot h + H_T(k, n-1, x) + \pi_i$. So the demand of class i should be satisfied if and only if the total cost of rejecting the demand is larger than or equal to the cost of satisfying it, i.e.,

$$H_T(k, n-1, x) + \pi_i \geq H_T(k, n-1, x-1). \qquad (5.4)$$

When there is no demand in an interval which has a probability p_0, then the system does not need to make a decision about whether to satisfy a demand. Given the on-hand inventory $x(k, n)$, the holding cost in the interval n is $x \cdot \Delta t \cdot h$. So in this case we have:

$$\begin{aligned} H_T(k, n, x) &= x \cdot \Delta t \cdot h + p_0[H_T(k, n-1, x)] \\ &\quad + \sum_{i=1}^{K} p_i \cdot \min[H_T(k, n-1, x-1), H_T(k, n-1, x) + \pi_i]. \end{aligned}$$

In summary, we have the following formula (after some adjustment) for the optimal expected total discounted cost $H_T(k, n, x)$ from the beginning of interval n of period k to the end of the period, given $x(k, n)$:

$$H_T(k,n,x) = \begin{cases} x \cdot \Delta t \cdot h + p_0[H_T(k,n-1,x)] \\ + \sum_{i=1}^{K} p_i \cdot \min[H_T(k,n-1,x-1), \\ H_T(k,n-1,x) + \pi_i]; & \text{for } x > 0, n > 0 \\ \\ p_0 \cdot H_T(k,n-1,0) \\ + \sum_{i=1}^{K} p_i \cdot [\pi_i + H_T(k,n-1,0)]; & \text{for } x = 0, n > 0 \\ \\ R_k(x). & \text{for } x \geq 0, n = 0. \end{cases} \tag{5.5}$$

where

$$R_k(x) = \begin{cases} -\alpha \cdot S_0(x), & \text{for } k = 1 \\ \alpha \cdot H_T(k-1, N, x+Q), & \text{for } M \geq k > 1. \end{cases}$$

From the above equation, we can see that $H_T(k,n,x)$ can be regarded as the total cost from the beginning of interval n to the end of the current period, given the on-hand inventory $x(k,n)$ and the terminal cost function $R_k(x)$ at the end of the period. So the formula is similar to the dynamic rationing model in a single period in Chapter 3.

5.3 Characterization of Optimal Cost Function and Optimal Rationing Policy

The above formula (5.5) is similar to the single period model in Chapter 3. In both models one interval (a period is divided into many small intervals) is a stage of the dynamic programming models. But there are a few notable differences. One main difference is that unmet demand is backordered in Chapter 3, while in this paper unmet demand is lost. The item π_i in the above model is corresponding to the item $e_i(n-1)$ in Chapter 3 where $e_i(n-1)$ is dependent on the decision time point, while π_i in the above model is independent on the decision time point $n-1$. Another difference is that there is only one period in Chapter 3. While this model is a multiple period one in which the costs accumulate during each period without discount and the cost of a period is discounted into its previous period. In the multiperiod model, we also need to consider the ordering policy and ordering cost.

Now consider the properties of optimal cost functions $H_T(k,n,x)$. Define $\Delta_x(k,n,x) = H_T(k,n,x) - H_T(k,n,x-1)$. We first consider the last period, i.e., when $k = 1$. We have the following lemma for the optimal cost function $H_T(k,n,x)$ when $k = 1$.

Lemma 5.1. *When $k = 1$, for a given n, $0 \le n \le N$, the first difference of the discrete function $H_T(k,n,x)$ is nondecreasing in x, i.e., $\Delta_x(k,n,x) = H_T(k,n,x) - H_T(k,n,x-1)$ is nondecreasing in x .*

When *k*=1, the above $H_T(k,n,x)$ is the optimal cost functions of a single period model. In the proof of the above lemma, we have shown that: given that the first difference of the terminal cost function $-\alpha \cdot S_0(x)$ is nondecreasing in *x*, the optimal cost functions $H_T(k,n,x)$ when *k*=1, $n \in \{1,...,N\}$, remain such a property for each *n*: the first difference of the function is nondecreasing in *x*. Now consider the optimal cost functions

$H_T(k,n,x)$ when $k>1$. The following lemma shows such a property remains for the case when $k>1$.

Lemma 5.2. *For a given k and n,* $1 \le k \le M$, $0 \le n \le N$ *, the first difference of the optimal cost function* $H_T(k,n,x)$ *is nondecreasing in x.*

The above lemma shows the important property (*the first difference of a function is nondecreasing in its variable*) of the optimal cost function remains for any *k* and *n*. We can see that the optimal total discounted cost $J^*_{Q,M}(x_M)$ of the *M*-period system is equal to $H_T(M,N,x)$ and the first difference of $J^*_{Q,M}(x_M)$ is nondecreasing in the initial on-hand inventory x_M at the beginning of the horizon. From the above lemma, we can obtain the structure of the optimal rationing policy. Let

$$x_i^*(k,n) = \min\{x \mid \Delta_x(k,n,x) + \pi_i \ge 0,\ x \ge 1\} - 1. \tag{5.6}$$

We have the following theorem for the optimal rationing policy.

Theorem 5.1. *The optimal rationing in each period of the whole horizon is a dynamic critical level policy with critical levels* $x_i^*(k,n)$.

The above theorem comes from the important property of $H_T(k,n,x)$: its first difference (under fixed *k* and *n*) is nondecreasing in *x*. The optimal dynamic critical levels in a certain period may change with remaining time before the end of period. It is worth to note that these dynamic critical levels are independent on the initial on-hand inventory at the beginning of each period. It is a nice property. So no matter how much the initial on-hand inventory of a certain period is, the system uses the same dynamic critical levels to ration stock in the period.

Let $\upsilon^*_{Q,M} = \{\upsilon^*_M, ..., \upsilon^*_1\}$ denote the optimal rationing policy on the whole horizon, under a given *Q*, where υ^*_i is the optimal rationing policy in period *i* and it is determined by how to ration stock at each time point in this period. For the *M*-period system, it is worth to note that policy υ^*_i may be different from υ^*_j when $i \neq j$. In the *M*-period system, *M* is a finite number. We may consider only finite state space, for the initial on-hand inventory is a finite number in practice and the state space at the end of the horizon is also finite. So it is easy to solve the above finite horizon dynamic programming model.

From the above analysis we can see that the optimal cost function under a given *Q* is $J^*_{Q,M}(x_M) = H_T(M, N, x)$. According to Lemma 5.1, we can see that the first difference of the optimal cost function $J^*_{Q,M}(x_M)$ is nondecreasing in the initial inventory x_M at the beginning of the horizon. The optimal rationing policy is the dynamic critical level policy. Thus we have solved the optimization (5.3) under a given *Q*. Then we can change *Q* and repeat the above procedure to get optimal cost functions under different *Q* and choose the optimal ordering amount *Q*. It is worth to note that optimal *Q* may depend on the initial on-hand inventory. Let $Q^*(x)$ denote the optimal ordering amount that minimizes the total cost $TC_M(x_M, \upsilon^*_{Q,M}, Q)$, given the initial inventory x_M.

We can see that the ordering cost OC_Q is linear with the ordering amount Q, but it is difficult to find exact expressions about how holding and penalty cost $J^*_{Q,M}(x_M)$ changes with Q for a fixed initial inventory. It is reasonable to believe that when Q increases, the holding cost and ordering cost will increase, but the penalty cost of shortage will decrease. We are trying to find a balance of the effects of increasing Q to minimize total cost. So we may expect that when Q increases from 0 and until a certain value the total cost will decrease, and if we continue to increase Q, the cost will continuously increase. The later numerical examples show that it is indeed true.

In fact we can see that the optimal Q can not be infinitely large. When Q becomes very large, then the stock may be almost enough to satisfy all demands. If the system increases Q by one more unit, the reduced penalty cost of shortage will be less than the increased ordering cost and holding cost. So we should not increase Q infinitely. Hence, given an initial inventory, we may search for optimal Q from 0 until $TC(x, \upsilon^*_{Q,M}, Q)$ continuously increases for a few values of Q to stop the searching.

5.4 Extension of the *M*-period Model to Infinite Horizon

In the above we have analyzed dynamic inventory rationing in an M-period system with finite horizon. It is quite natural to extend it to infinite horizon case. Now we consider extending it to the case with infinite horizon. It is just some mathematical operations, and the solving method is the same.

When the horizon becomes infinite, i.e., there are infinite number of periods, the optimization problem is still to find the optimal ordering amount Q and dynamic rationing policy to minimize total discounted cost. We still first consider the case under a fixed Q, and then we change Q to find optimal holding and penalty cost under other Qs. Again, we do not consider the ordering cost when considering optimal holding and penalty cost under a fixed Q, for it is a constant under a fixed Q.

For the infinite horizon problem, we can see that when Q is greater than λu, where λu is the expected demand in a period, the total discount cost under any rationing policy will become infinite (consider the policy without inventory rationing). So in the following we consider only Qs in the range $[0, \lambda u]$. When we do not explicitly say ranges for Q, we are referring to the range $[0, \lambda u]$.

Let $J_Q^*(x)$ denote the optimal expected discounted holding and penalty cost over the infinite horizon, given the ordering amount Q. Let $J_{Q,\infty}^*(x) = \lim_{M \to \infty} J_{Q,M}^*(x)$, with the assumption that the terminal cost function for the M-period system is $R(x) = 0$, for all $x \in \{0,1,...\}$. We can see that the limit $\lim_{M \to \infty} J_{Q,M}^*(x)$ exists when $Q \in [0, \lambda u]$, for $J_{Q,M}^*(x)$ increases with M and it is easy to find an upper bound for it, for example, the cost under a first-come-first-served policy. We have the following theorem for the optimal cost $J_Q^*(x)$ over the infinite horizon.

Theorem 5.2. *When* $Q \in [0, \lambda u]$, $J_Q^*(x) = J_{Q,\infty}^*(x)$.

The above theorem states that the optimal expected discounted holding and penalty cost over the infinite horizon is equal to the limit of the optimal cost of the *M*-period problem. It is a very important property. Form this, we can infer properties of the unknown function $J_Q^*(x)$ from the properties of the *M*-period optimal cost function $J_{Q,M}^*(x)$. We can increase M and infinitely approach to $+\infty$ to obtain the properties of the optimal cost function $J_Q^*(x)$ and optimal rationing policy for the infinite horizon problem.

Let $\upsilon_{Q,\infty}^* = \{\upsilon_1^*, \upsilon_2^*, ...\}$ denote the optimal rationing policy over the infinite horizon, given the ordering amount *Q*, where υ_i^* is the optimal rationing policy in period *i* (note that here we use period 1 to denote the first period of the horizon). Let $\{\upsilon, \upsilon, ...\}$ denote a stationary rationing policy, i.e., the rationing policy in a period does not change from period to period. We have the following theorem for the optimal rationing policy and optimal cost function over the infinite horizon.

Theorem 5.3.

(a) *The first difference of the optimal cost function* $J_{Q,\infty}^*(x)$ *under a given* $Q \in [0, \lambda u]$ *is nondecreasing in the initial inventory x.*

(b) *The optimal rationing policy under a given Q in each period is the dynamic critical level rationing policy.*

(c) *There exists an optimal stationary rationing policy.*

The properties (Parts (a) and (b) in the above theorem) of the optimal cost function and the optimal rationing policy over infinite horizon is inferred from those of the *M*-period systems. Part (c) states that these exists an optimal rationing policy in which the dynamic critical levels are independent on the index of periods. So the dynamic critical levels change with the remaining time before the end of the period, but do not change from one period to another period. Note that in the finite horizon *M*-period model, the dynamic critical levels in period i may be different from those in period j when $i \neq j$.

Now consider the calculation of the optimal cost function $J^*_{Q,\infty}(x)$. When M increases to infinite, it is impossible to obtain the exact value of the optimal cost function $J^*_{Q,\infty}(x)$ by value iteration, for the state space is infinite for this infinite horizon problem and the memory of computers is finite. From Theorem 5.2, we have known that the optimal cost function $J^*_{Q,\infty}(x)$ over infinite horizon is equal to the limit of the optimal cost function of the *M*-period problem, so we can use a finite horizon *M*-period problem with enough large *M* to approximate the infinite horizon problem. For the finite horizon *M*-period problem, we can obtain exact values of the optimal cost function $J^*_{Q,M}(x)$.

5.5 Numerical Study

5.5.1 The Numerical Study

In this section a numerical study is conducted for the dynamic inventory rationing on the infinite horizon to obtain some managerial insights, in particular, to examine how ordering amount Q and initial on-hand inventory x affect the total discounted cost and how the dynamic critical levels change with other factors.

The optimal cost over the infinite horizon $TC_\infty(x, \upsilon_Q^*, Q)$ consists of the total ordering cost OC_∞ and the holding and penalty costs $J^*_{Q,\infty}(x)$. We can see that when M approaches to infinite, then the ordering cost OC_M and $J^*_{Q,M}(x)$ of the M-period system approaches to OC_∞ and $J^*_{Q,\infty}(x)$, respectively. So given Q and initial inventory x, when $[J^*_{Q,M}(x) - J^*_{Q,M-1}(x)]/J^*_{Q,M}(x) < \delta$, where δ is a certain very small value, then we can regard $J^*_{Q,M}(x)$ is enough close to the cost $J^*_{Q,\infty}(x)$ over infinite horizon. For we are considering the costs under different Q and different initial on-hand inventory and will compare these costs, we set a common value of M for different cases. Let $M_{\max}$ denote such a value. Set $M_{\max} = \arg\min_M M$, s.t. $(OC_\infty - OC_M)/OC_\infty \le e_1$, where e_1 is a very small value. In the numerical study, we set $e_1 = 1.0e-4$ and get $M_{\max} = 926$.

We also set a range for the initial on-hand inventory. Let $x_{\max}$ denote the largest initial on-hand inventory. Set $x_{\max} = 100 \cdot \lambda \cdot u$, i.e., 100 times of the expected demand per period, which is enough large for an initial on-hand inventory. Define d_J as follows:

$$d_J = \frac{\| J^*_{Q,M}(x) - J^*_{Q,M-1}(x) \|}{\min\limits_{x \in [0, x_{\max}]} J^*_{Q,M}(x)}.$$

d_J is a measure of the convergence of the sequence of $J^*_{Q,M}(x)$ for different M. In the numerical study we also check d_J and find it is always less than $1.0e-5$.

An example is considered in the numerical study. In this example there are 2 demand classes. The parameter setting is as follows: $\lambda_1 = \lambda_2 = 100$/year, period length $u = 0.1$ year, the variable ordering cost $c = \$5/\text{unit}$, penalty cost $\pi_1 = \$8/\text{unit}$ and $\pi_2 = \$6.5/\text{unit}$, holding cost $h = \$1/\text{year/unit}$, and discount factor $\alpha = 0.99$.

Given the above parameter setting, we find the optimal cost functions $J^*_{Q,\infty}(x)$, $x \in [0, x_{\max}]$, under different Qs using the formula (5.5), i.e., following the general DP algorithm. We change Q in the whole range $[0, \lambda u]$ to investigate how Q affects the total cost.

In the numerical study, we also investigate how the initial on-hand inventory affects the total cost. Let $x^*(Q) = \arg\min\limits_x J^*_{Q,\infty}(x)$, i.e., $x^*(Q)$ is the initial on-hand inventory with minimal cost under a given Q. In addition, we also look into the dynamic critical level rationing policy. The results are shown in Figures 5.1, 5.2, 5.3 and 5.4.

5.5.2 Results and Discussion

Figure 5.1 shows the costs under different Qs for a fixed initial inventory x=18. From the figure we can see that when Q increases from 0, the sum of holding and penalty cost decreases, while the ordering cost increases. The total cost obtains its minimum when Q=19. When Q increases from 0, the total cost decreases fast, then decreases slow. When Q continues to increase, the total cost arrives at its minimum and then increases. It is quite fit to our intuition. When $Q = 0$, there are many demands lost and the penalty cost is very large. When we increase Q from 0, the increased stock can significantly decrease the penalty cost. When Q is near $\lambda \cdot u$, the expected penalty cost of shortage is far less than those when Q is very small. If we increase Q by one unit from a large Q, then the increased stock can decrease the penalty cost, but the effect is not as much as in the cases with very small Qs where many demands are lost. From the figure, the ordering amount Q has a significantly effect on the total cost. The minimal cost when Q=19 is 72.3% of the cost when Q=0 where the cost is maximal. For other initial on-hand inventory, the costs also have the same trend as shown in Figure 5.2.

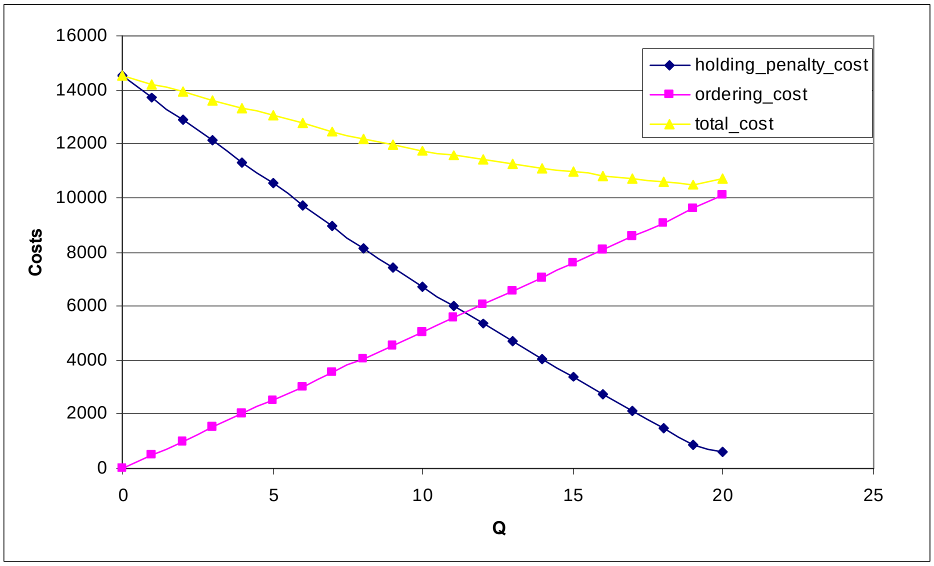

Figure 5.1 Costs vs. Q when initial inventory x=18

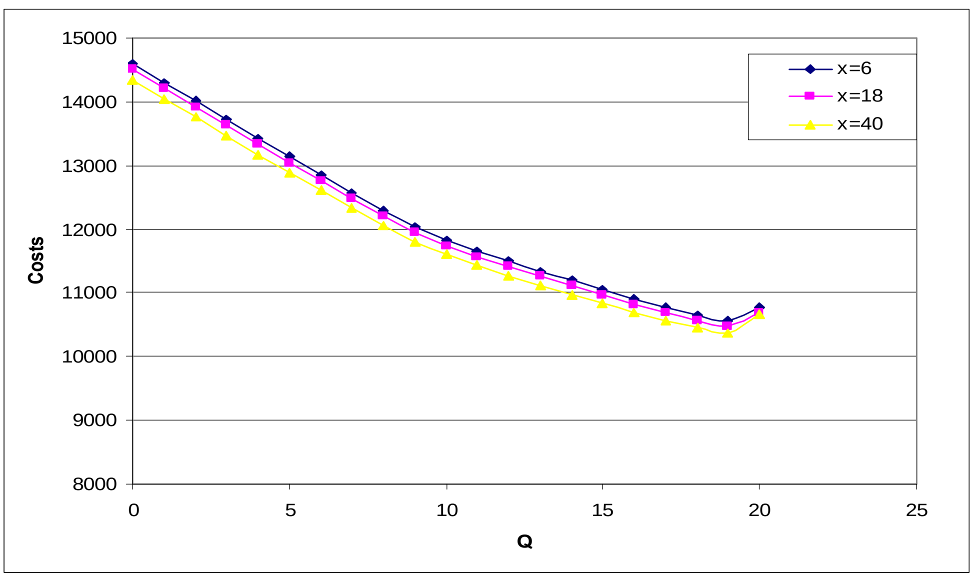

Figure 5.2 Costs vs. Q for other initial inventory

Now consider Figure 5.3 which shows the costs under different initial on-hand inventory. From the figure, the first difference of the cost function under a given Q is indeed nondecreasing in the initial inventory. So there is a value of initial inventory with the minimal cost, i.e., $x^*(Q) = \arg\min_x J^*_{Q,\infty}(x)$. When Q=20, $x^*(Q) = 34$, when Q=19, $x^*(Q) = 73$, and when Q=18, $x^*(Q) = 120$. When Q decreases from 20, then the optimal initial on-hand inventory decreases. It is also quite intuitive.

From the graph, we can see that the optimal Q is dependent on the initial on-hand inventory. When the initial inventory $x \le 73$, the cost under Q=19 is less than those under two other Qs. When $x > 71$, the cost under Q=18 is less than those under two other Qs. In fact the results under all Qs show: when $0 \le x \le 73$, $Q^*(x) = 19$ and when $73 < x \le 160$, $Q^*(x) = 18$. So though the optimal ordering amount is a function of initial inventory, it is insensitive to it. From the graph we observe that the initial on-hand inventory has some effect, but not very much on the total discount cost. When the initial inventory changes from 20 (the expected demand in one period) to 40 or 0, the cost change is less than 2%.

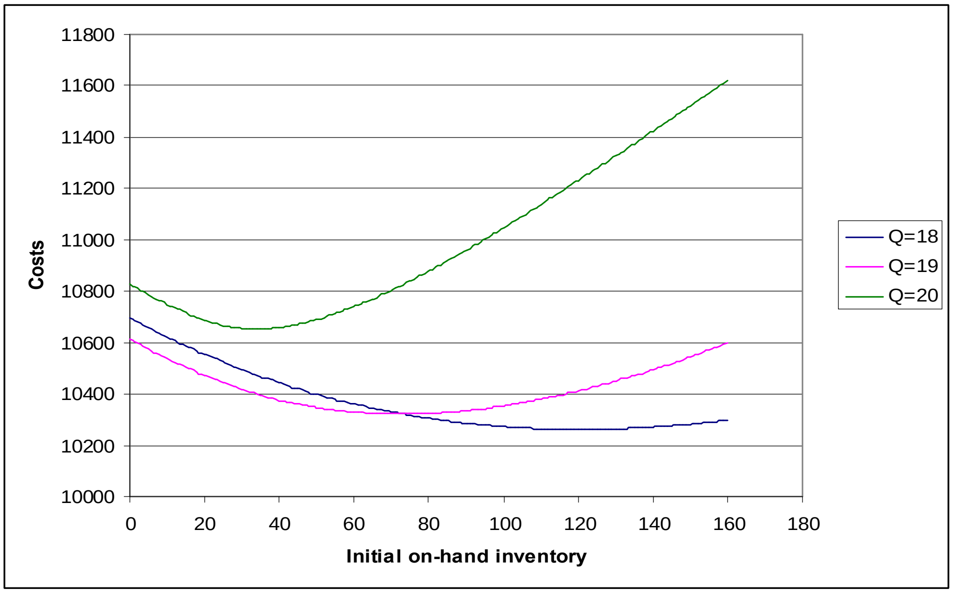

Figure 5.3 Costs vs. initial inventory

During calculating the optimal cost functions under different *Q*s, we simultaneously obtain the optimal dynamic rationing policy for a given *Q*. Given an *M*-period problem with a fixed *Q*, the numerical study shows that there exists an integer N_Q such that when $i, j \geq N_Q$, the dynamic critical levels in period *i* are the same as those in period j, i.e., the dynamic critical levels become stationary. When *M* is enough large, we can use the dynamic critical levels in period *M* to approximate the stationary rationing policy of the infinite horizon problem. Figure 5.4 shows the stationary dynamic critical level of class 2 (more accurately, the dynamic critical levels in period 926) for different *Q*s. For there are only two classes and the critical level of class 1 is always 0, the figure shows only critical level of class 2.

From the figure we can see that for a given *Q*, the critical level decreases towards the end of the period (in this example, a period is divided into 400 intervals and interval 1 is at the end of the period). It is quite intuitive. When there is less remaining time, the system needs less stock reserved for more important demands. We can also see that when Q increases, then the critical level decreases. When there are more units of the product to arrive at each period, then the system needs less stock reserved for more important demands.

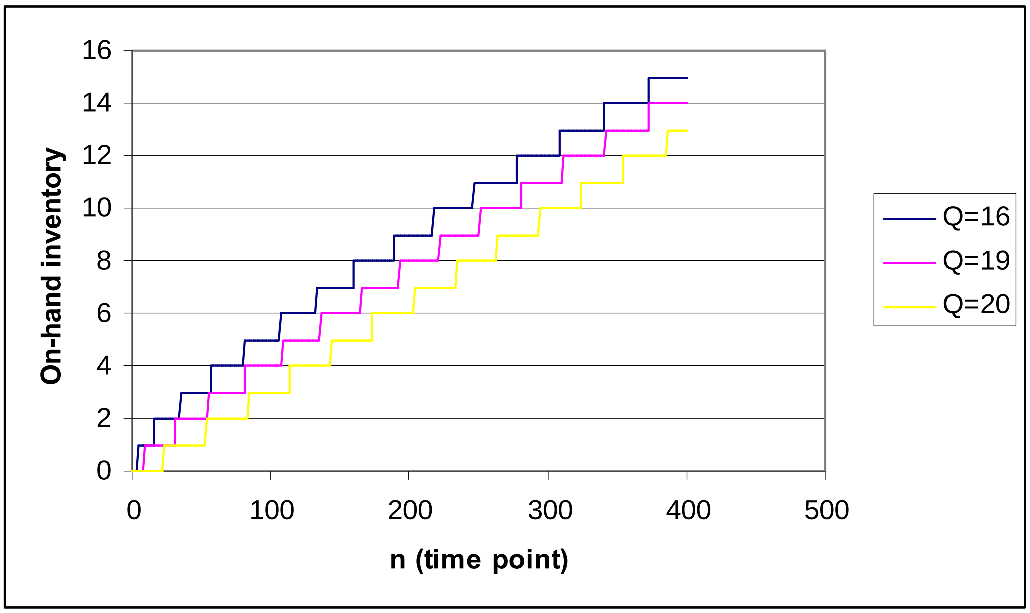

Figure 5.4 Dynamic critical levels

5.6 Conclusions

We analyzed dynamic inventory rationing in multiperiod systems with Poisson demands, lost sales and multiple demand classes. We first considered a multiperiod system with finite horizon, and then extended to infinite horizon. A dynamic programming model was developed for the finite horizon M-period system. Important characteristics of the optimal rationing policy, optimal cost function and optimal ordering amount were obtained. We showed that the optimal rationing policy is the dynamic critical level policy. We then extended the *M*-period model to the case with infinite horizon. The optimal rationing policy is still a dynamic critical level rationing policy. We also provided a range for the optimal ordering amount so that we can search for the optimal ordering amount effectively.

A numerical study was conducted to obtain some important managerial insights. The ordering amount Q is the most important factor affecting the total discounted expected cost. The initial on-hand inventory affects the total discounted cost, but the cost is not very sensitive to it. We also observed that when Q increases, then the critical level at a certain time decreases, i.e., when there are more stock to arrive at the end of each period, then the system needs to reserve less stock for more important classes in current period.

The above work can be extended in some directions. Note that the assumed ordering policy with fixed ordering amount per period is not an optimal ordering policy, especially in the case with zero lead time. This ordering policy is suitable to the cases with positive lead time. Of course, for the cases with positive lead time we may assume other ordering policies to analyze dynamic inventory rationing. In this chapter we have developed the dynamic inventory rationing in single period systems (a special case of *M*-period model) and it can be used as building blocks in developing heuristics under other ordering policies.

Chapter 6

Conclusion

Inventory rationing among demand classes is popular and critical in many industries. The competitive pressure and thin profit in many industries make it a necessary competitive strategy rather than a competitive advantage. How to ration inventory and how to replenish inventory are key management decisions. Currently most literature considers the static rationing policies and researchers have made notable progresses in this area, but the theory about dynamic inventory rationing is quite limited for such problems are extremely difficult to solve. Motivated by the possible significant potential in cost saving of dynamic inventory rationing policies shown in Deshpande et al. (2003), this dissertation studies the dynamic inventory rationing in different circumstances: Poisson demand process or general demand processes, backordering or lost sales. The objective is to develop analytical models to derive qualitative and managerial insights into the dynamic rationing policies and ordering policies, and to develop methods to obtain optimal or near-optimal parameters of dynamic rationing and ordering policies.

The first part of our research considered dynamic inventory rationing in systems with Poisson demand and backordering. For the multiperiod system with zero lead time, dynamic programming models were developed. The optimal dynamic rationing policy was shown to be the dynamic critical level policy in each period and the optimal ordering policy is a base stock policy. Important properties of the optimal rationing policy and optimal cost function were also obtained. The dynamic rationing in each period was modeled as a one-dimensional dynamic programming model by dividing the period into many small intervals and assuming rejected demands can be fulfilled only at the ends of periods. So our model eliminated the curse of dimensionality that general dynamic inventory rationing models with backordering suffer.

We next considered a multiperiod system with positive deterministic lead time. An optimization model of minimizing expected average cost was developed, assuming the full-priority backorder clearing mechanism. In the case with positive lead time, the structure of optimal dynamic rationing policies may be very complicated and there is no closed-form expression for the average cost under many rationing policies, so we developed a near-optimal solution for it: applying the dynamic critical level rationing policy of the multiperiod model with zero lead time to ration stock and adopting a base stock ordering policy. Some important properties of such a policy were obtained and a method was developed to obtain an appropriate base stock level. A lower bound on the optimal cost under optimal rationing and ordering policies was also established. The numerical results show the cost under our policy is very close to the optimal costs for a practical range of parameters and also for poor service level conditions. Moreover, the results also show that our dynamic rationing policy can significantly reduce cost comparing with current state-of-art static rationing policies: in many cases the cost saving can be more than 10%.

For the above models consider typical problem settings such as positive lead time, periodic review ordering policy and so on, the dynamic rationing policy can significantly reduce cost and this policy is easy to understand and implement, our dynamic rationing policy can have a wide application in practice. Deshpande et al. (2003) developed a lower bound on the optimal costs and showed that the gap between the cost under their static rationing policy and the lower bound is very large (in many cases the gap is more than 10%), here we have developed a particular dynamic rationing policy which indeed significantly reduces cost and the cost under our policy is very close to the optimal costs and it is easy to implement.

The second part of the research extended the work in the first part by changing the demand process from Poisson demand to general demand processes. Unmet demands are also backordered. In practice some demand processes are non-Poisson ones, but they are seldom analyzed in the literature about inventory rationing for its complexity and little is known about the structure of optimal rationing and ordering policies under typical conditions such as positive lead time. We considered dynamic inventory rationing in both single period and multiperiod systems, assuming the systems adopt a dynamic critical level rationing policy. For the single period system, a method was developed to obtain near-optimal parameters for the dynamic critical level policy. Approximate expressions for the expected total cost under this dynamic rationing policy were also obtained. These results were used to develop a later multiperiod model.

We then considered dynamic rationing in a multiperiod system with positive lead time and a base stock ordering policy. An optimization model was developed. Using a similar procedure to the previous multiperiod model with Poisson demand, a near-optimal solution to the optimization problem was obtained. A numerical study was conducted and the results show that the cost under the near-optimal rationing policy is quite close to the optimal cost in the examples. As for dynamic inventory rationing under very general demand processes and positive lead time, little is known in the literature about the structure of the optimal rationing policy or how to obtain optimal or near-optimal parameters of a certain dynamic rationing policy. Our models assuming dynamic critical level rationing policies provided a benchmark for future effective policies.

Finally, in the third part, we studied dynamic inventory rationing in multiperiod systems with Poisson demands and lost sales, assuming a supply contract with fixed ordering amount per period. For a finite horizon M-period system, we developed a dynamic programming model of minimizing total discounted cost by dividing each period into many intervals. Important characteristics of the optimal rationing policy, optimal cost function and optimal ordering amount were obtained. The optimal rationing policy was shown to be the dynamic critical level policy. We then extended the finite horizon model to the infinite horizon case. Important characteristics of the optimal cost function, rationing policy and ordering amount were also obtained. A numerical study was also conducted and some important managerial insights were obtained.

6.1 Directions for Future Research

The numerical results have shown that the dynamic inventory rationing policy can indeed significantly reduce cost in many cases, comparing with the static rationing policies. One main reason that the dynamic rationing policy can reduce cost is that it uses such information as the arrival times of next replenishments to dynamically adjust the critical levels to ration stock, while the critical levels in the static rationing policy are constants. So such information as the arrival times of next replenishments is valuable and can not be ignored simply during inventory rationing. We also showed that when the penalty costs of shortage of different demand classes differ more, then the benefit of implementing the dynamic rationing policy will be larger. It is quite intuitive. So when the penalty costs of different classes differ very much, the system manager needs to pay more attention to the dynamic rationing policies and should be very careful if they are using the static rationing policies. The above results were obtained assuming Poisson demand. We conjecture that the above insights can be carried over to problem settings with other demand processes and other ordering policies. For the large potential of dynamic rationing policies in cost saving shown in this dissertation, we think it is very valuable and interesting to explore the dynamic inventory rationing in other problem settings and develop effective policies for them. For in the literature most people consider the static rationing policies and little progress is made for the dynamic inventory rationing, there are many possible directions for future research about the dynamic inventory rationing. In the following we present some of them.

In the models in this dissertation, the demand processes of different classes are independent. In some practical cases, the demand processes of different classes may be correlated. Thus, one direction for future research is to consider the correlated demand processes in the modeling. In this case, we do not know whether the optimal rationing policy is still a dynamic critical level policy or it has a recognizable structure. However, we think it is possible to develop some effective heuristic policies for it.

The dynamic inventory rationing policy can reduce cost, comparing with the static rationing policy, for it uses such information as the arrival times of next replenishments. The information about the times of next replenishments is based on the assumption of deterministic lead time. In practice, sometimes the lead time may be stochastic. So it will be useful to consider inventory rationing with stochastic lead times, even explicitly modeling the supply of products as a production facility. Ha (1997a) has shown that the static rationing policy is optimal for a make-to-stock system with Poisson demand and exponential production times. It is intuitive, for the system is memoryless. This system is equal to an inventory system with exponential lead time for orders. When the lead time is stochastic and has other distribution, we conjecture that the optimal rationing policy will not be the static rationing policy, for the system is not memoryless again. We also think that in these cases the benefit of dynamic rationing policy will be between that with exponential lead times and that with deterministic lead times. Nevertheless, we feel that it would be useful to explore the dynamic rationing for the cases with stochastic lead times and evaluate its benefit.

We have considered the dynamic inventory rationing for a multiperiod system with lost sales and Poisson demand in Chapter 5, in which the ordering policy is a supply contract such that a fixed amount of replenishment will arrive at each period. In some supply contracts, there is some flexibility, besides this fixed amount of replenishment per period. For example, the system may place an additional order (or an emergency order) each period, except receiving this fixed amount of products each period. Of course, the price of these addition orders may be different from those of the fixed amount of products. There are other types of supply contracts. This additional flexibility of ordering policies provides a further opportunity to reduce cost or increase profit. Hence, one direction for future research is to consider the inventory rationing under these more flexible ordering policies. In these cases, the problems will become more complex and are very difficult to analyze. However, effective heuristics may be possible to develop. It is also interesting to evaluate the benefit of these more flexible ordering policies.

Another possible direction for future research is to consider inventory rationing for systems with continuous review ordering policies. Under the continuous review ordering policies, the dynamic rationing policy may be more complex and difficult to implement, for the dynamic rationing policy is dependent on the arrival times of next replenishments and these times are not as regular as in the systems with periodic review ordering policies. Nevertheless, it will be quite useful to consider inventory rationing in continuous review systems. Moreover, if the system has an option to choose between a periodic review and continuous review ordering policy, then it is very interesting to compare costs of dynamic inventory rationing in these two types of systems.

Inventory rationing in multi-echelon systems is also an interesting area for future research. For example, consider a retailer that sells a product to different demand classes and purchases stock from a distribution center using two types of orders: emergency orders and ordinary orders. These two types of orders can be regarded as two demand classes from the distribution center point of view. Both the retailer and the distribution center hold inventory. So it is a two-echelon inventory system with multiple demand classes. How to replenish inventory and ration inventory is very important in these systems. There are many interesting problems in this area. Again, the problems become more complex.

Finally, dynamic inventory rationing with dynamic pricing is another very rich area for future research. In our models, the penalty cost of rejecting a demand of a certain class is known and determined, i.e., it is not a variable. Currently the dynamic pricing has attracted intense attention of researchers and practitioners. People can dynamically change price to ration stock among classes, i.e., using price to adjust the demands to ration inventory. So it will be useful to develop dynamic inventory rationing models with dynamic pricing.

Bibliography

Anupindi, R. and Y. Bassok. 1999. Supply contracts with quantity commitments and stochastic demand. In *Quantitative Models for Supply Chain Management,* S. Tayur, M. Magazine, and R. Ganeshan (eds.), Kluwer Academic Publishers, Boston, MA.

Arrow, K., S. Karlin and H. Scarf. 1958. *Studies in the mathematical theory of inventory and production*. Stanford University. Stanford. CA.

Arrow, K., T. Harris and J. Marschak. 1951. Optimal inventory policy. *Econometrica*, **19** 250-272.

Arslan, H., S. C. Graves, and T. Roemer. 2005. A single-product inventory model for multiple demand classes. Technical report, MIT, Cambridge MA.

Baganha, M.P. and M.A. Cohen. 1998. The stabilizing effect of inventory in supply chains. *Operations Research*, **46**(3) S72-S83.

Baker, K. R., M. J. Magazine, H. L. Nuttle. 1986. The effect of commonality on safety stock in a simple inventory model. *Management Science*, **32** 982-988.

Barnes-Schuster D.C. 1996. Supply chain management and the use of long term supply contracts. Ph.D. Thesis, Northwestern University. Evanston, Illinois.

Bassok, Y., A. Bixby, R. Srinivasan, and H.Z. Wiesel. 1997. Design of component-supply contracts with commitment-revison flexibility. *IBM Journal of Research and Development*, **41**(6) 693-703.

Belobaba, P.P. 1987. Air travel demand and airline seat inventory management. Ph.D. Dissertation. MIT, Cambridge, Mass.

Bertsekas D.P. 1995. *Dynamic Programming and Stochastic Control*, Vol. 2. Athena Scientific, New York.

Bithan, G.R., E.A. Haas and H. Matsuo. 1986. Production planning of style goods with high setup costs and forecast revisions. *Operations Research*, **34** 226-236.

Blinder, A.S. 1982. Inventories and sticky prices. *American Econom. Rev.,* **72** 334-349.

Blinder, A.S. 1986. Can the production smoothing model of inventory behavior be saved? *Quart. J. Econom.*, **101** 431-454.

Brumelle, S. and D. Walczak. 1997. Dynamic allocation of airline seat inventory with batch arrivals. In *1997 Air Transport Research Group of the WCTR Society Proc.*, Faculty of Commerce and Business Administration, University of British Columbia, Vancouver, BC.

Cachon, G.P. and M. Fisher. 2000. Supply chain inventory management and the value of shared information. *Management Science*, **46**(8) 1032-1048.

Caplin, A.S. 1985. The variability of aggregate demand with (s,S) inventory policies. *Econometrica*, **53** 1395-1409.

Chiang, C. and G.J. Gutierrez. 1998. Optimal control policies for a periodic review inventory system with emergency orders. *Naval Research Logistics*, **45** 187-204.

Chiang, C. and G.J. Gutierrez. 1996. A periodic review inventory system with two supply modes. *European Journal of Operational Research*, **94** 527-547.

Cohen, M.A., P. Kleindorfer and H.L. Lee. 1988. Service Constrained (s, S) inventory systems with priority demand classes and lost sales. *Management Science*, **34** 482-499.

Curry, R.E. 1988. Optimum seat allocation with fare classes nested on segments and legs. Technical note 88-1, Aeronomics Incorporated, Fayetteville, Ga.

Curry, R.E. 1990. Optimal airline seat allocation with fare classes nested by origins and destinations. *Transportation Science*, **24** 193-203.

Daganzo, C.F. 2004. On the stability of supply chains. *Operations Research*, **52**(6) 909-921.

Dekker, R., R.M. Hill, M.J. Kleijn and R.H. Teunter. 2002. On the (*S*-1, *S*) lost sales inventory model with priority demand classes. *Naval Research Logistics*, **49** 539-610.

Dekker, R., M.J. Kleijn and P.J.D. Rooij. 1998. A spare parts stocking system based on equipment criticality. *Int. J. Prod. Econ.*, **56-57** 69-77.

Deshpande, V., M.A. Cohen and K. Donohue. 2003. A threshold inventory rationing policy for service-differentiated demand classes. *Management Science*, **49** 683-703.

Eppen, G. 1979. Effects of centralization on expected costs in multilocation newsboy problems. *Management Science*, **25** 498-501.

Evans, R.V. 1968. Sales and restocking policies in a single item inventory system. *Management Science*, **14** 463-472.

Forrester, J. 1961. *Industrial Dynamics*. MIT press, and John Wiley & Sons, Inc., New York.

Frank K.C., R.Q. Zhang and I. Duenyas. 2003. Optimal policies for inventory systems with priority demand classes. *Operations Research*, **51**(6) 993-1002.

Gayon J.P., S. Benjaafar and F.D. Véricourt. 2004. Stock rationing in a multi-class make-to-stock production system. MSOM, Eindhoven, July 2004.

Gayon, J.P., F.D. Véricourt, F. Karaesmen and Y. Dallery. 2004. Stock rationing in a multi-class make-to-stock queue with information on the production status. Technical report, Laboratoire Génie Industriel, Paris.

Ge Y., J.-B. Yang, N. Proudlove and M. Spring. 2004. System dynamics modeling for supply chain managementð a case study on a supermarket chain in the UK . *International Transactions on Operational Research*, **11**(5) 495-509.

Ha, A.Y. 1997a. Inventory rationing in a make-to-stock production system with several demand classes and lost sales. *Management Science*, **43** 1093-1103.

Ha, A.Y. 1997b. Stock rationing policy for a make-to-stock production system with two priority classes and backordering. *Naval Research Logistics*, **44** 457-472.

Ha, A.Y. 2000. Stock rationing in an $M/E_k/1$ make-to-stock queue. *Management Science*, **46** 77-87.

Hartung, P.H. 1973. A simple style goods inventory model. *Management Science*, **19** 1452-1458.

Henig, M., Y. Gerchak, R. Ernst and D. F. Pyke. 1997. An inventory model embedded in designing a supply contract. *Management Science*, **43**(2) 184-189.

Hertz, D.B., and K.H. Schaffir. 1960. A forecasting method for management of seasonal style-goods inventories. *Operations Research*, **8** 45-52.

Hill, R.M. 1999. On the suboptimality of (*S*-1, *S*) lost sales inventory policies. *International Journal of Production Economics*, **59** 387-393.

Janakiraman, G. and R.O. Roundy. 2004. Lost-sales problems with stochastic lead times: convexity results for base-stock policies. *Operations Research*, **52**(5) 795-803.

Janssen, F., R. Heuts and T.D. Kok. 1998. On the (*R*,*s*,*Q*) inventory model when demand is modeled as a compound Bernoulli process. *European Journal of Operations Research*, **104** 423-436.

Johansen, S.G. and R.M. Hill. 2000. The (*r*, *Q*) control of a periodic-review inventory system with continuous demand and lost sales. *International Journal of Production Economics*, **68** 279-286.

Johansen, S.G. and A. Thorstenson. 1993. Optimal and approximate (*Q*, *r*) inventory policies with lost sales and gamma-distributed lead time. *International Journal of Production Economics*, **30-31** 179-194.

Johansen, S.G. 2001. Pure and modified base-stock policies for the lost sales inventory system with negligible set-up costs and constant lead times. *International Journal of Production Economics*, **71** 391-399.

Kaplan, A. 1969. Stock rationing. *Management Science*, **15** 260-267.

Kim J.-S. 2002. On the benefits of inventory-pooling in production-inventory systems. *Manufacturing & Service Operations Management*, **4**(1) 12-16.

Kleijn, M.J. and R. Dekker. 1998. An overview of inventory systems with several demand classes. Technical Report 9838/A, Econometric Institute, Erasmus University, Rotterdam, The Netherlands.

Lee, H.L., V. Padmanabhan and S. Whang. 2004. Comments on ñinformation distortion in a supply chain: the bullwhip effectò. *Management Science*, **50**(12) 1887-1893.

Lee, H.L., V. Padmanabhan and S. Whang. 1997a. Information distortion in a supply chain: the bullwhip effect. *Management Science*, **43**(4) 546-558.

Lee, H.L., V. Padmanabhan and S. Whang. 1997b. The bullwhip effect in supply chains. *Sloan Management Rev*. **38**(3, Spring) 93-102.

Lee, T.C. and M. Hersh. 1993. A model for dynamic airline seat inventory control with multiple seat bookings. *Transportation Science*, **27** 252-265.

Littlewood, K. 1972. Forecasting and control of passenger bookings. *AGIFORS Symposium Proc*. 12, Nathanya, Israel.

Mahajan, V., E. Muller and F.M. Bass. 1990. New product diffusion models in marketing: a review and directions for research. *Journal of Marketing*, **54** 1-26.

Mayer, M. 1976. Seat allocation, or a simple model of seat allocation via sophisticated ones. In *Proceedings 16th AGIFORS Symposium*, 103-135.

Mcgill, J.I. 1988. Airline Multiple fare class seat allocation. Presented at Fall ORSA/TIMS Joint National Conference, Denver, Colo.

Mcgill, J.I. and G.J.V. Ryzin. 1999. Revenue Management: research overview and prospects. *Transportation Science*, **33** 233-256.

Melchiors, P.M., R. Dekker and M.J. Kleijn. 2000. Inventory rationing in an (s, Q) inventory model with lost sales and two demand classes. *J. Oper. Res. Soc.*, **51**(1) 111-122.

Melchiors, P.M. 2003. Restricted time-remembering policies for the inventory rationing problem. *Int. J. Prod. Econ.*, **81-82** 461-468.

Moon, I and S. Kang. 1998. Rationing policies for some inventory systems. *J. Oper. Res. Soc.,* **49**(5) 509-518.

Morton, T.E. 1971. The near-myopic nature of the lagged-proportional-cost inventory problem with lost sales. *Operations Research*, **19** 1708-1716.

Morton, T.E. 1969. Bounds on the solution of the lagged optimal inventory equation with no demand backlogging and proportional costs. *SIAM Review*, **11**(4) 572-596.

Nahmias, S. and W. Demmy. 1981. Operating characteristics of an inventory system with rationing. *Management Science*, **27** 1236-1245.

Robinson, L.W. 1990. Optimal and approximate control policies for airline booking with sequential fare classes. Working paper 90-93, Jonson Graduate School of Management, Cornell University, Ithaca, N.Y.

Roger, D. and H. Warburton. 2004. An analytical investigation of the bullwhip effect. *Production and operations management*, **13**(2) 150-160.

Rosenshine, M. and D. Obee. 1976. Analysis of a standing order inventory system with emergency orders. *Operations Research*, **24**(6) 1143-1155.

Sterman, J.D. 1989. Modeling managerial behavior: Misperceptions of feedback in a dynamic decision making experiment. *Management Science*, **35**(3) 321-339.

Nahmias, S. 1978. Simple approximations for a variety of dynamic leadtime lost-sales inventory models. *Operations Research*, **27**(5) 904-924.

Seaton, A.V. and M.M. Bennett. 1996. *The marketing of tourism products: concepts, issues, and cases*. International Thomson Business Press, London.

Tagaras, G. and D. Vlachos. 2001. A periodic review inventory system with emergency replenishments. *Management Science*, **47**(3) 415-429.

Teunter, R. and D. Vlachos. 2001. An inventory system with periodic regular review and flexible emergency review. *IIE Transactions*, **33** 625-635.

Teunter, R.H., and W. K.K. Haneveld. 2008. Dynamic inventory rationing strategies for inventory systems with two demand classes, Poisson demand and backordering. *European Journal of Operational Research*, **190** 156-178.

Topkis, D.M. 1968. Optimal ordering and rationing policies in a non-stationary dynamic inventory model with *n* demand classes. *Management Science*, **15** 160-176.

Tsay, A.A., S. Nahmias and N. Agrawal. 1999. Modeling supply chain contracts: a review. In *Quantitiative Models for Supply Chain Management*, S. Tayur, M. Magazine, and R. Ganeshan (eds.), Kluwer Academic Publishers, Boston, MA.

Urban, T.L. 2000. Supply contracts with periodic, stationary commitment. *Production and Operations Management*, **9**(4) 400-413.

Véricourt, D.F., F. Karaesmen and Y. Dallery. 2002. Optimal stock allocation for a capacitated supply system. *Management Science*, **48**(11) 1486-1501.

Wang J., J.D. Jia and K. Takahashi. 2005. A study on the impact of uncertain factors on information distortion in supply chains. *Production planning and control*, **16**(1) 2-11.

Wikner, J., D. Towill, and M. Naim. 1991. Smoothing supply chain dynamics. *International Journal of Production Economics*, **22** 231-248.

Wollmer, R.D. 1988. A seat management model for a single leg route when lower fare classes book first. Presented at Fall ORSA/TIMS Joint National Conference, Denver, Colo.

Zhao W. and Y.S. Zheng. 1998. General dynamic models for airline seat allocation. Working Paper. The Wharton School of the University of Pennsylvania, Philadelphia, PA.

Zhao, W. 1999. Dynamic and static yield management models. Ph.D. thesis. The Wharton School, Operations and Information Management Department, University of Pennsylvania, Philadelphia, PA.

Zipkin, P.H. 2000. *Foundations of inventory management*. McGraw-Hill, New York.

Appendix A

Proofs in Chapter 3

Proof of Lemma 3.1

Now assume there are outstanding backorders at the beginning of period *m* before ordering, i.e., $\boldsymbol{B}_m \neq 0$, and the system manager is making decision about how much to order. For the cost factors are stationary and lead time is zero, the cost of buying one unit stock at the beginning of period *m* to fulfill one unit backorder in $\boldsymbol{B}_m$ is the same as that of buying one unit at the beginning of any later period or consuming one unit of remaining stock at the end of the horizon to fulfill one unit in $\boldsymbol{B}_m$. But purchasing stock to fulfill one unit backorder in $\boldsymbol{B}_m$ at the beginning of period *m* will have less penalty cost than fulfilling one unit backorder in $\boldsymbol{B}_m$ at later periods, because the penalty cost is dependent on the lasting time of backorders. So it is obvious that the system should purchase enough stock to fulfill all backorders in $\boldsymbol{B}_m$ at the beginning of period *m,* if there are some outstanding backorders in $\boldsymbol{B}_m$. So after ordering at the beginning of period *m* and using the new purchased stock to fulfill backorders, there are no remaining backorders unfilled, i.e., after ordering there are no outstanding backorders. For lead time is zero and there is no backorder after ordering, the inventory position $y_m \geq 0$. Ĭ

Proof of Lemma 3.2

Given inventory position y_m at the beginning of period m after ordering, if the total demand D_m in this period is larger than y_m, then there exist outstanding backorders at the beginning of period $m+1$ before ordering, i.e., $x_{m+1} < 0$. The rationing policy υ_m in period m will not affect the value of x_{m+1}, but will affect the distribution of backorders of each class in the total backorders $-x_{m+1}$, i.e., will affect $\boldsymbol{B}_{m+1}$. For any backorder in $\boldsymbol{B}_{m+1}$ will be fulfilled at the beginning of period $m+1$ according to Lemma 3.1, how the backorders of each demand class distribute in the total backorder $-x_{m+1}$ does not affect the costs incurred in period $m+1$ and later periods. Thus the rationing policy υ_m in period *m* does not affect the cost after period *m*. So the optimal rationing policy in period *m* is determined solely by the parameter settings in period *m*. Ĭ

Proof of Lemma 3.3

We prove it by induction. When $n = 0$, we have: $H(n,x) = 0$. So the first difference of $H(n,x)$ is nondecreasing in x. Now assume the first difference of $H(n-1,x), n \ge 1$, is nondecreasing in x, i.e., $\Delta_x(n-1,x)$ is nondecreasing in x.

For a given on-hand inventory x at time point n, there exists a critical class k_x^{n-1} at time point $n-1$ such that when $i \ge k_x^{n-1}$, $\Delta_x(n-1,x) + e_i(n-1) < 0$ (reject the demand from class i), and when $i < k_x^{n-1}$, $\Delta_x(n-1,x) + e_i(n-1) \ge 0$ (accept the demand from class i). Since $\Delta_x(n-1,x)$ is non-decreasing in x, and $\pi_i \ge \pi_j$ and $\tilde{\pi}_i \ge \tilde{\pi}_j$, $i < j$, we have $k_x^{n-1} \ge k_{x-1}^{n-1} \ge k_{x-2}^{n-1}$.

From equation (3.2) in Chapter 3, we can know that when $x \ge 1$,

$$\begin{aligned} H(n,x) &= x \cdot \Delta t \cdot h + p_0 \cdot H(n-1,x) + (1-p_0) \cdot H(n-1,x-1) \\ &+ \sum_{i=1}^{K} p_i \cdot \min[0, \Delta_x(n-1,x) + e_i(n-1)] \\ &= x \cdot \Delta t \cdot h + p_0 \cdot H(n-1,x) + (1-p_0) \cdot H(n-1,x-1) \\ &+ \sum_{i=k_x^{n-1}}^{K} p_i \cdot [\Delta_x(n-1,x) + e_i(n-1)] \end{aligned} \tag{A1}$$

and when $x = 0$,

$$H(n,0) = \sum_{i=1}^{K} p_i \cdot e_i(n-1) + H(n-1,0), \tag{A2}$$

so the first difference of (A1) with respect to x when $x \ge 2$ is

$$\Delta_x(n,x) = \Delta t \cdot h + p_0 \cdot \Delta_x(n-1,x) + (1-p_0)\cdot \Delta_x(n-1,x-1)$$
$$+ \sum_{i=k_x^{n-1}}^{K} p_i \cdot [\Delta_x(n-1,x) + e_i(n-1)] - \sum_{i=k_{x-1}^{n-1}}^{K} p_i \cdot [\Delta_x(n-1,x-1) + e_i(n-1)]$$
$$= \Delta t \cdot h + (p_0 + \sum_{i=k_x^{n-1}}^{K} p_i)\cdot \Delta_x(n-1,x) + (1 - p_0 - \sum_{i=k_{x-1}^{n-1}}^{K} p_i)\cdot \Delta_x(n-1,x-1) \tag{A3}$$
$$- \sum_{i=k_{x-1}^{n-1}}^{k_x^{n-1}-1} p_i \cdot e_i(n-1)$$

and when $x = 1$,

$$\Delta_x(n,1) = \Delta t \cdot h + (p_0 + \sum_{i=k_1^{n-1}}^{K} p_i)\cdot \Delta_x(n-1,1) - \sum_{i=1}^{k_1^{n-1}-1} p_i \cdot e_i(n-1). \tag{A4}$$

We now look at the second difference of (A3) with respect to *x*. When $x \geq 3$,

$$\Delta_x(n,x) - \Delta_x(n,x-1)$$
$$= (p_0 + \sum_{i=k_x^{n-1}}^{K} p_i)\cdot \Delta_x(n-1,x) + (1 - p_0 - \sum_{i=k_{x-1}^{n-1}}^{K} p_i)\cdot \Delta_x(n-1,x-1) - \sum_{i=k_{x-1}^{n-1}}^{k_x^{n-1}-1} p_i \cdot e_i(n-1)$$
$$-\{(p_0 + \sum_{i=k_{x-1}^{n-1}}^{K} p_i)\cdot \Delta_x(n-1,x-1) + (1 - p_0 - \sum_{i=k_{x-2}^{n-1}}^{K} p_i)\cdot \Delta_x(n-1,x-2) - \sum_{i=k_{x-2}^{n-1}}^{k_{x-1}^{n-1}-1} p_i \cdot e_i(n-1)\}$$
$$= (p_0 + \sum_{i=k_x^{n-1}}^{K} p_i)\cdot[\Delta_x(n-1,x) - \Delta_x(n-1,x-1)] + (1 - p_0 - \sum_{i=k_{x-2}^{n-1}}^{K} p_i)\cdot[\Delta_x(n-1,x-1) - \Delta_x(n-1,x-2)]$$
$$+\{-\sum_{i=k_{x-1}^{n-1}}^{k_x^{n-1}-1} p_i \cdot [\Delta_x(n-1,x-1) + e_i(n-1)]\} + \sum_{i=k_{x-2}^{n-1}}^{k_{x-1}^{n-1}-1} p_i \cdot [\Delta_x(n-1,x-1) + e_i(n-1)].$$

According to that the first difference of $H(n-1,x)$ is nondecreasing in x and the definition of $k_x^{n-1}, k_{x-1}^{n-1}, k_{x-2}^{n-1}$, each of the four items in the above express is nonnegative, so when $x \geq 3$, $\Delta_x(n,x) - \Delta_x(n,x-1) \geq 0$.

Now consider the case when $1 \le x < 3$. From (A3) and (A4) we have:

$$
\begin{aligned}
&\Delta_x(n,2) - \Delta_x(n,1) \\
&= (p_0 + \sum_{i=k_2^{n-1}}^{K} p_i) \cdot \Delta_x(n-1,2) + (1 - p_0 - \sum_{i=k_1^{n-1}}^{K} p_i) \cdot \Delta_x(n-1,1) - \sum_{i=k_1^{n-1}}^{k_2^{n-1}-1} p_i \cdot e_i(n-1) \\
&-[(p_0 + \sum_{i=k_1^{n-1}}^{K} p_i) \cdot \Delta_x(n-1,1) - \sum_{i=1}^{k_1^{n-1}-1} p_i \cdot e_i(n-1)] \\
&= (p_0 + \sum_{i=k_2^{n-1}}^{K} p_i) \cdot [\Delta_x(n-1,2) - \Delta_x(n-1,1)] + \sum_{i=1}^{k_1^{n-1}-1} p_i \cdot [\Delta_x(n-1,1) + e_i(n-1)] \\
&+\{-\sum_{i=k_1^{n-1}}^{k_2^{n-1}-1} p_i \cdot [\Delta_x(n-1,1) + e_i(n-1)]\}.
\end{aligned}
$$

According to that the first difference of $H(n-1, x)$ is nondecreasing and the definition of $k_x^{n-1}, k_{x-1}^{n-1}, k_{x-2}^{n-1}$, each of the three items in the above express is also nonnegative, hence $\Delta_x(n, x) - \Delta_x(n, x-1) \ge 0$. So given that the first difference of $H(n-1, x), x \ge 0$ is nondecreasing in x, we have: the first difference of $H(n, x)$ is also nondecreasing in x. Thus by induction, the result follows. Ĭ

Proof of Theorem 3.1

(a) By Lemma 3.3 which states the first difference of $H(n, x)$ is nondecreasing in x for every n, we have part a.

(b) By Lemma 3.3 and given that $e_i(n) \ge e_j(n)$, $i < j$. we have part b.

(c) At the time point $n = 0$, it is obvious that we should satisfy the demand of class i if the on-hand inventory is above 0. So $x_i^*(0) = 0$. For a fixed on-hand inventory $x > 0$, when n increases from 0, we may still satisfy the demand of class i. When n arrives at a certain value, it may be better to reject the demand of class i. Define $n_i^x = \{y \mid \text{when } 0 \le n \le y, \Delta_x(n,x) + e_i(n) \ge 0, \text{ and } \Delta_x(y+1,x) + e_i(y+1) < 0\}$. For a fixed on-hand inventory $x > 0$, when $0 \le n \le n_i^x$, the demand of class i should be satisfied and when $n = n_i^x + 1$ the demand of class i should be rejected. According to that the first difference of $H(n,x)$ is nondecreasing in x and the definition of n_i^x, we have $n_i^x \le n_i^{x+1}$. If we can show: for any given i and $x > 0$, if $n \ge n_i^x + 1$, then $\Delta_x(n,x) + e_i(n) < 0$, then the proposition is proved. When $n = n_i^x + 1$, we have known that $\Delta_x(n,x) + e_i(n) < 0$. So if we can show the following statement is true, then the proposition is proved:

For any given i and $x > 0$, if $n \ge n_i^x + 1$, then

$$\Delta_x(n,x) + e_i(n) - \Delta_x(n-1,x) - e_i(n-1) = \Delta_x(n,x) - \Delta_x(n-1,x) + \pi\breve{E}_i \cdot \Delta t \le 0 .$$

Define k_x^n, k_{x-1}^n k_x^{n-1} and k_{x-1}^{n-1} as in proof of Lemma 3.3. We will attempt to show the above statement is true by induction and the outline of the procedure is given as follows:

Step 1:

First we show by induction that when $x = 1$, the following statement is true:

for $n \geq n_i^x + 1$, then $\Delta_x(n,x) - \Delta_x(n-1,x) + \pi\breve{E}_i \cdot \Delta t \leq 0$ and $k_x^n \leq k_x^{n-1}$.

Step 2:

Given that the relation $\Delta_x(n,x-1) - \Delta_x(n-1,x-1) + \pi\breve{E}_i \cdot \Delta t \leq 0$ and $k_{x-1}^n \leq k_{x-1}^{n-1}$.

Holds for $n \geq n_i^{x-1} + 1$, i.e., given the statement is true for inventory at $x-1$, $x \geq 2$, we show by induction that the statement is true for inventory x, i.e., for $n \geq n_i^x + 1$, it holds that $\Delta_x(n,x) - \Delta_x(n-1,x) + \pi\breve{E}_i \cdot \Delta t \leq 0$ and $k_x^n \leq k_x^{n-1}$.

Step 3: By using the induction from step 1 and step 2 the proposition is proved.

By definition of n_i^x, we know that when $n = n_i^x + 1$,

$\Delta_x(n,x) + e_i(n) - \Delta_x(n-1,x) - e_i(n-1) = \Delta_x(n,x) - \Delta_x(n-1,x) + \pi\breve{E}_i \cdot \Delta t < 0$. This is a strict inequality. The following two statements are true when $n = n_i^x + 1$ for any i and $x > 0$:

1) $\Delta_x(n,x) - \Delta_x(n-1,x) + \pi\breve{E}_i \cdot \Delta t \leq 0$.

2) $k_x^n \leq k_x^{n-1}$.

Step 1:

Now consider the case when $x = 1$. From the above we have known that when $n = n_i^1 + 1$, the following two statements are true.

1) $\Delta_x(n, x) - \Delta_x(n-1, x) + \pi \breve{E}_i \cdot \Delta t \le 0$

2) $k_x^n \le k_x^{n-1}$

Following is to show these two statements are true at time point $n+1$ if the two statements are true at time point n.

Now given the two statements are true at a certain time point $n \ge n_i^1 + 1$. From equation (A4), we have

$$\Delta_x(n+1,1) - \Delta_x(n,1) = (p_0 + \sum_{i=k_1^n}^{K} p_i) \cdot \Delta_x(n,1) - \sum_{i=1}^{k_1^n - 1} p_i \cdot e_i(n)$$
$$-[(p_0 + \sum_{i=k_1^{n-1}}^{K} p_i) \cdot \Delta_x(n-1,1) - \sum_{i=1}^{k_1^{n-1}-1} p_i \cdot e_i(n-1)]$$
$$= (p_0 + \sum_{i=k_1^{n-1}}^{K} p_i) \cdot [\Delta_x(n,1) - \Delta_x(n-1,1)] - \sum_{i=1}^{k_1^{n-1}-1} p_i \cdot [e_i(n) - e_i(n-1)]$$
$$+ \sum_{i=k_1^n}^{k_1^{n-1}-1} p_i \cdot [\Delta_x(n,1) + e_i(n)]$$
$$\le -\pi \breve{E}_i \cdot \Delta t + \sum_{i=k_1^n}^{k_1^{n-1}-1} p_i \cdot [\Delta_x(n,1) + e_i(n)].$$

According to the definition of k_x^n, we know that for $i \ge k_x^n$, then $\Delta_x(n, x) + e_i(n) < 0$, so $\Delta_x(n+1,1) - \Delta_x(n,1) \le -\pi \breve{E}_i \cdot \Delta t$, i.e., $\Delta_x(n+1,1) - \Delta_x(n,1) + \pi \breve{E}_i \cdot \Delta t \le 0$. Hence $\Delta_x(n+1, x) + e_i(n+1) \le \Delta_x(n, x) + e_i(n)$, so a demand of class i should be rejected at *n+1* if it should be rejected at time point n, so we have $k_x^{n+1} \le k_x^n$. Thus the two statements are true at time point $n+1$ if they are true at time point n. According to the induction, we have:

when $x = 1$, for $n \ge n_i^x + 1$, then $\Delta_x(n, x) - \Delta_x(n-1, x) + \pi\breve{E}_i \cdot \Delta t \le 0$ and $k_x^n \le k_x^{n-1}$.

Step 2:

We want to show if the inventory is $x-1,\ x \ge 2$, the following relation holds:

for $n \ge n_i^{x-1} + 1$, then $\Delta_x(n, x-1) - \Delta_x(n-1, x-1) + \pi\breve{E}_i \cdot \Delta t \le 0$ and $k_{x-1}^n \le k_{x-1}^{n-1}$,

then when the inventory is x, the following relation also holds true:

for $n \ge n_i^x + 1$, then $\Delta_x(n, x) - \Delta_x(n-1, x) + \pi\breve{E}_i \cdot \Delta t \le 0$ and $k_x^n \le k_x^{n-1}$.

Given the inventory is $x \ge 2$, from (A3), we have

$$
\begin{aligned}
&\Delta_x(n+1,x)-\Delta_x(n,x) \\
&=(p_0+\sum_{i=k_x^n}^{K}p_i)\cdot\Delta_x(n,x)-\sum_{i=1}^{k_x^n-1}p_i\cdot e_i(n)+\sum_{i=1}^{k_{x-1}^n-1}p_i[\Delta_x(n,x-1)+e_i(n)] \\
&-\{(p_0+\sum_{i=k_x^{n-1}}^{K}p_i)\cdot\Delta_x(n-1,x)-\sum_{i=1}^{k_x^{n-1}-1}p_i\cdot e_i(n-1)+\sum_{i=1}^{k_{x-1}^{n-1}-1}p_i[\Delta_x(n-1,x-1)+e_i(n-1)]\} \\
&=(p_0+\sum_{i=k_x^{n-1}}^{K}p_i)\cdot[\Delta_x(n,x)-\Delta_x(n-1,x)]-\sum_{i=1}^{k_x^{n-1}-1}p_i\cdot[e_i(n)-e_i(n-1)] \\
&+\sum_{i=k_x^n}^{k_x^{n-1}-1}p_i\cdot[\Delta_x(n,x)+e_i(n)]+\{-\sum_{i=k_{x-1}^n}^{k_{x-1}^{n-1}-1}p_i\cdot[\Delta_x(n-1,x-1)+e_i(n-1)]\} \\
&+\sum_{i=1}^{k_{x-1}^n-1}p_i[\Delta_x(n,x-1)+e_i(n)-\Delta_x(n-1,x-1)-e_i(n-1)].
\end{aligned}
\tag{A5}
$$

Since given at time point is n, $\Delta_x(n,x)-\Delta_x(n-1,x)+\pi\breve{E}_i\cdot\Delta t\le 0$, the first two terms in (A5) is bounded by

$$
(p_0+\sum_{i=k_x^{n-1}}^{K}p_i)\cdot[\Delta_x(n,x)-\Delta_x(n-1,x)]-\sum_{i=1}^{k_x^{n-1}-1}p_i\cdot[e_i(n)-e_i(n-1)]\le -\pi\breve{E}_i\cdot\Delta t\,.
$$

By definition of k_x^n, k_{x-1}^{n-1}, the third and fourth terms in (A5) are non-positive. Moreover, when inventory is $x-1$, if $n\ge n_i^{x-1}+1$, then $\Delta_x(n,x-1)-\Delta_x(n-1,x-1)+\pi\breve{E}_i\cdot\Delta t\le 0$, and the fact that $n_i^{x-1}\le n_i^x$ we have that the last term in (A5) is non-positive. Hence from (A5), we have $\Delta_x(n+1,x)-\Delta_x(n,x)\le -\pi\breve{E}_i\cdot\Delta t$. This implies that a demand of class i should be rejected at $n+1$ if it should be rejected at time point n, and we have $k_x^{n+1}\le k_x^n$. By induction, the following statement is true:

If $n\ge n_i^x+1$, then $\Delta_x(n,x)-\Delta_x(n-1,x)+\pi\breve{E}_i\cdot\Delta t\le 0$ and $k_x^n\le k_x^{n-1}$.

Step 3:

From Steps 1 and 2, the result follows. Ĭ

Proof of Lemma 3.4

(a) We prove it by induction. During the proof we need to show the first difference of $W_m(y_m)$ is nondecreasing in y_m. In particular, we will show: the first difference of $V_{m+1}(x_{m+1})$ is nondecreasing in $x_{m+1} \rightarrow$ the first difference of $W_m(y_m)$ is nondecreasing in $y_m \rightarrow$ the first difference of $V_m(x_m)$ is nondecreasing in x_m and so on.

Before proceed to the proof, we first copy two equations (3.5 and 3.6) to the following lines to bring some convenience:

$$W_m(y_m) = c \cdot y_m + H(N, y_m) + E[V_{m+1}(y_m - D_m)],\ m \in \{1,...,M\}, \qquad \text{(Equation 3.5, copied)}$$

$$V_m(x_m) = -cx_m + \min\{W_m(y_m \mid y_m \geq x_m)\}. \qquad \text{(Equation 3.6, copied)}$$

Step 1: We show: when *m*=*M*, the first difference of $V_m(x_m)$ is nondecreasing in x_m.

Consider Equation (3.5). When *m*=*M*,

$E[V_{m+1}(y_m - D_m)] = -E[c \cdot (y_m - D_m)] = -cy_m + c \cdot E[D_m]$. We can see that the first difference of $E[V_{m+1}(y_m - D_m)]$ (i.e., $E[V_{m+1}(y_m - D_m)] - E[V_{m+1}(y_m - 1 - D_m)]$) is nondecreasing in y_m. We have already known that the first difference of $H(N, y_m)$ is nondecreasing in y_m. So from Equation (3.5) we can see that the first difference of $W_m(y_m)$ is nondecreasing in y_m.

Now consider Equation (3.6). To simplify the notations, let $f_m(x_m) = \min\{W_m(y_m \mid y_m \geq x_m)\}$. So from Equation (3.6), we have: $V_m(x_m) = -cx_m + f_m(x_m)$. In the following we will show: the first difference of $f_m(x_m)$ is nondecreasing in x_m, i.e., $f_m(x_m+1) - f_m(x_m) \geq f_m(x_m) - f_m(x_m-1)$. Once it is done, then from Equation (3.6) we can see that when *m=M,* the first difference of $V_m(x_m)$ is nondecreasing in x_m, hence Step 1 of the proof is finished. In the following we show: $f_m(x_m+1) - f_m(x_m) \geq f_m(x_m) - f_m(x_m-1)$.

Let $\bar{y}_m = \arg\min_{y_m \geq 0} W_m(y_m)$. For $f_m(x_m) = \min\{W_m(y_m \mid y_m \geq x_m)\}$ and the property that the first difference of $W_m(y_m)$ is nondecreasing in y_m, which has been shown in the previous paragraphs, we have:

$$f_m(x_m) = \begin{cases} W_m(x_m), & x_m \geq \bar{y}_m \\ W_m(\bar{y}_m), & x_m \leq \bar{y}_m. \end{cases}$$

In the following we consider different cases to show $f_m(x_m+1) - f_m(x_m) \geq f_m(x_m) - f_m(x_m-1)$.

(i) when $(x_m+1) \leq \bar{y}_m$.

In this case, $f_m(x_m+1)-f_m(x_m)=f_m(x_m)-f_m(x_m-1)=0$. So the first difference of $f_m(x_m)$ is nondecreasing in x_m.

(ii) when $x_m=\bar{y}_m$.

In this case, $f_m(x_m)-f_m(x_m-1)=0$. For $\bar{y}_m=\arg\min_{y_m\geq 0} W_m(y_m)$, we have: $f_m(x_m+1)-f_m(x_m)=f_m(\bar{y}_m+1)-f_m(\bar{y}_m)\geq 0$. So the first difference of $f_m(x_m)$ is nondecreasing in x_m.

(iii) when $x_m-1\geq\bar{y}_m$.

In this case, $f_m(x_m)-f_m(x_m-1)=W_m(x_m)-W_m(x_m-1)$ and $f_m(x_m+1)-f_m(x_m)=W_m(x_m+1)-W_m(x_m)$. For the first difference of $W_m(x_m)$ is nondecreasing in x_m, we have: the first difference of $f_m(x_m)$ is nondecreasing in x_m.

So from the above we can see that first difference of $f_m(x_m)$ is nondecreasing in x_m. For $V_m(x_m)=-cx_m+f_m(x_m)$, we can see that: when *m*=*M*, the first difference of $V_m(x_m)$ is nondecreasing in x_m. Hence the first step is finished.

Step 2: Now assume that the first difference of $V_{m+1}(x_{m+1})$ is nondecreasing in x_{m+1}. In the following we prove: the first difference of $V_m(x_m)$ is nondecreasing in x_m.

Consider Equation (3.5): $W_m(y_m) = c \cdot y_m + H(N, y_m) + E[V_{m+1}(y_m - D_m)]$. For the first difference of $V_{m+1}(x_m)$ is nondecreasing in x_m, we can see that the first difference of the function $E[V_{m+1}(y_m - D_m)]$ is nondecreasing in y_m. We have known that the first difference of the function $H(N, y_m)$ (for the fixed *N*) is nondecreasing in y_m. The first difference of $c \cdot y_m$ is also nondecreasing in y_m. So from Equation (3.5), we can see the first difference of $W_m(y_m)$ is nondecreasing in y_m.

Now consider Equation 3.6: $V_m(x_m) = -cx_m + \min\{W_m(y_m \mid y_m \geq x_m)\}$, which is also written as $V_m(x_m) = -cx_m + f_m(x_m)$. For the first difference of $W_m(y_m)$ is nondecreasing in y_m, the first difference of $f_m(x_m)$ is nondecreasing in x_m. So from Equation 3.6, we have: the first difference of $V_m(x_m)$ is nondecreasing in x_m.

Step 3: From Steps 1 and 2, the result of Part (a) follows.

(b) Consider Equation 3.5: $W_m(y_m) = c \cdot y_m + H(N, y_m) + E[V_{m+1}(y_m - D_m)]$. In Part (a) we have shown: when the first difference of $V_{m+1}(x_{m+1})$ is nondecreasing in x_{m+1}, then from Equation 3.5 we can see that the first difference of $W_m(y_m)$ is nondecreasing in y_m. So from Part (a), Part (b) immediately follows. Ĭ

Proof of Theorem 3.2

Consider Equation 3.6:

$$V_m(x_m) = -cx_m + \min\{W_m(y_m \mid y_m \geq x_m)\}.$$

The optimal solution y_m^* to the minimization problem $\min\{W_m(y_m \mid y_m \geq x_m)\}$ depends on the relation between x_m and $\bar{y}_m$, where $\bar{y}_m = \arg\min_{y_m \geq 0} W_m(y_m)$. If $x_m \leq \bar{y}_m$, then the optimal solution $y_m^* = \bar{y}_m$, so the system should order inventory up to $\bar{y}_m$. If $x_m > \bar{y}_m$, then the optimal solution $y_m^* = x_m$, because $W_m(y_m)$ is nondecreasing in y_m over the range $y_m \geq \bar{y}_m$ (which comes from the fact: $W_m(y_m)$ arrives at its minimum at $\bar{y}_m$ and the first difference of $W_m(y_m)$ is nondecreasing in y_m). So when $x_m > \bar{y}_m$, then the system should not order to increase inventory, for increasing inventory just incurs more cost. Hence the optimal ordering policy at the beginning of period *m* is precisely a base stock policy with base stock level $\bar{y}_m$. So the result follows. Ĭ

Proof of Theorem 3.3

Let $V(x_1 \mid \bar{y})$ denote the expected total cost on the whole horizon under the myopic ordering policy with base stock $\bar{y}$, given initial inventory x_1. In the following we show that the optimal cost $V_1(x_1)$ has a lower bound which is equal to $V(x_1 \mid \bar{y})$, and then the theorem is proved.

From Equations (3.7) and (3.8) we have

$$V'_m(x_m) = \min_{y_m \ge x_m} \{G_m(y_m) + E[V'_{m+1}(y_m - D_m)]\} \ge \min_{y_m} G_m(y_m) + \min_{y_m} E[V'_{m+1}(y_m - D_m)]$$
$$= G_m(\bar{y}) + \min_{y_m} E[V'_{m+1}(x_{m+1})] \ge G_m(\bar{y}) + \min_{x_{m+1}} V'_{m+1}(x_{m+1})$$
$$\ge G_m(\bar{y}) + \min_{x_{m+1}}[G_{m+1}(\bar{y}) + \min_{x_{m+2}} V'_{m+2}(x_{m+2})] = G_m(\bar{y}) + G_{m+1}(\bar{y}) + \min_{x_{m+2}} V'_{m+2}(x_{m+2})$$
$$\ge \sum_{i=m}^{M} G_i(\bar{y}) + \min_{x_{M+1}} V'_{M+1}(x_{M+1}).$$

For $V'_{M+1}(x_{M+1}) = 0$, we obtain $V'_1(x_1) \ge \sum_{m=1}^{M} G_m(\bar{y})$, so

$$V_1(x_1) = -cx_1 + V_1'(x_1) \ge -cx_1 + \sum_{m=1}^{M} G_m(\bar{y}),$$

hence, a lower bound on the optimal cost $V_1(x_1)$ is obtained.

Now consider the cost $V(x_1 \mid \bar{y})$ under the myopic ordering policy. Under this policy, if the initial inventory x_1 is less than $\bar{y}$, then the system places an order to rise the inventory position up to $\bar{y}$ at the beginning of period 1, i.e., $y_1 = \bar{y}$. The net inventory x_2 at the beginning of period 2 before ordering should not be greater than $\bar{y}$ and the system should order up to $\bar{y}$, i.e., $y_2 = \bar{y}$. Based on the same reason, we have: the inventory position after ordering at any period should be equal to $\bar{y}$, i.e., $y_m = \bar{y}$, $m \in \{1,...,M\}$. According to the fact $x_{m+1} = \bar{y} - D_m$ and the definition of $G_m(y_m)$, the expected total cost $V(x_1 \mid \bar{y})$ is

$$V(x_1 \mid \bar{y}) = \sum_{m=1}^{M} \{E[c(\bar{y} - x_m)] + H(N, \bar{y})\} - E[c(\bar{y} - D_M)]$$
$$= -cx_1 + \sum_{m=1}^{M} G_m(\bar{y}),$$

which is exactly the lower bound on optimal cost $V_1(x_1)$. So the myopic ordering policy is optimal. Ĭ

Proof of Lemma 3.5

Under policy (S, dy), there exists a limiting distribution for $(y_m, \boldsymbol{B}_m)$. $(y_\infty, \boldsymbol{B}_\infty)$ is the random variable with such limiting distribution and Ω is the set of all possible values for $(y_m, \boldsymbol{B}_m)$. So

$$AC(S, dy) = \lim_{M\to\infty} \frac{1}{M} \sum_{m=0}^{M-1} \sum_{\Omega} C_m(y, \boldsymbol{B} \mid dy) P_m(y, \boldsymbol{B}) = E[C_\infty(y, \boldsymbol{B} \mid dy)]. \tag{A6}$$

Substitute (3.12) and (3.13) into the above equation and according to the fact that when $y_m < 0$, $C_m(y, \boldsymbol{B} \mid dy)$ is not less than the cost assuming all backorders in $\boldsymbol{B}$ come from the least important class K, we immediately have the lower bound $E[U_{LB}(S - D_L)]$.
Ĭ

Proof of Lemma 3.6

According to (A6) and (3.15), we have:

$$AC(S, dy) = E[C_\infty(y, \boldsymbol{B} \mid dy)] \le E[U_{UB}(y)] = E[U_{UB}(S - D_L)].$$ Ĭ

Proof of Lemma 3.7

Let Z denote the set of integers. We can see that the net inventory $y_m \in Z$ .The optimal cost $\mathrm{AC}(\omega_o, \nu_o)$ is

$$\mathrm{AC}(\omega_o,\nu_o) = \lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M-1}\sum_{\Omega} C_m(y,\boldsymbol{B}\mid\omega_o,\nu_o)P_m(y,\boldsymbol{B}\mid\omega_o,\nu_o)$$
$$\geq \lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M-1}\sum_{y\in Z} U_{LB}(y)P_m(y\mid\omega_o,\nu_o).$$

In the above expression, $y_m = S_m - D_L$, where S_m is the inventory position just after ordering at the beginning of ordering period *m*, and S_m is determined by the ordering policy. For we are considering any ordering polices in Ψ, S_m may be different from S_n, when $m \neq n$, and they may have some relation with each other. So the distribution of y_m is determined by ordering policy, and is independent on the rationing policy. For $S^* = \arg\min E[U_{LB}(S - D_L)]$, we have:

$$\mathrm{AC}(\omega_o,\nu_o) \geq \lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M-1}\sum_{y\in Z} U_{LB}(y)P_m(y\mid\omega_o,\nu_o) \geq E[U_{LB}(S^* - D_L)].$$ Ĭ

Appendix B

Complementary Results in Chapter 3

Part of the results of this numerical study in Chapter 3 is shown in Chapter 3. Other results in the numerical study are shown in the following tables.

Table B1 Comparison of policies when $\lambda_1 = \lambda_2 = 450, \pi\breve{E}_2 = 1.5$

$\pi\breve{E}_1 / \pi\breve{E}_2$	L/u	CR_{LB}	$CR_{cnM-dyM}$
3	0	0	0.46%
	1	3.19E-11	0.93%
	2	3.55E-07	1.19%
	3	1.07E-05	1.40%
	4	8.28E-05	1.49%
	0	0	5.94%

100	1	-3.92E-13	10.02%
	2	2.85E-08	11.43%
	3	1.88E-06	11.10%
	4	2.15E-05	10.38%
1000	0	0	8.60%
	1	-7.24E-12	14.38%
	2	9.96E-09	15.38%
	3	7.75E-07	12.04%
	4	7.10E-06	10.38%

Table B2 Comparison of policies when $\lambda_1 = \lambda_2 = 450, \tilde{\pi}_2 = 5$

$\tilde{\pi}_1 / \tilde{\pi}_2$	L/u	CR_{LB}	$CR_{cnM-dyM}$
	0	0	0.74%
	1	-4.35E-14	1.11%
3	2	8.38E-10	1.28%
	3	1.24E-07	1.47%
	4	1.83E-06	1.59%
	0	0	5.93%
	1	-1.97E-12	8.68%
100	2	3.46E-10	10.74%
	3	1.27E-07	10.71%
	4	2.28E-06	10.94%
	0	0	8.45%
	1	-1.87E-11	11.23%
1000	2	3.02E-10	15.05%
	3	1.14E-07	13.17%
	4	1.78E-06	12.09%

Table B3 Comparison of policies when $\lambda_1 = \lambda_2 = 450, \breve{\pi}_2 = 10$

$\breve{\pi}_1 / \breve{\pi}_2$	L/u	CR_{LB}	$CR_{cnM-dyM}$
3	0	0	0.71%
	1	-9.07E-14	1.04%
	2	3.48E-11	1.16%
	3	1.01E-08	1.35%
	4	2.72E-07	1.36%
100	0	0	5.49%
	1	-4.89E-12	7.81%
	2	1.69E-11	9.47%
	3	1.58E-08	9.69%
	4	6.21E-07	9.42%
1000	0	0	8.42%
	1	-3.90E-11	10.24%
	2	-6.21E-11	13.95%
	3	2.59E-08	14.09%
	4	5.89E-07	12.94%

Table B4 Comparison of policies when $\lambda_1 = \lambda_2 = 300, \pi\breve{E}_2 = 1.5$

$\pi\breve{E}_1 / \pi\breve{E}_2$	L/u	CR_{LB}	$CR_{cnM-dyM}$
3	0	0	0.62%
	1	2.29E-08	1.14%
	2	1.57E-05	1.34%
	3	0.000163	1.40%
	4	0.000712	1.42%
100	0	0	6.65%
	1	8.60E-10	9.90%
	2	1.78E-06	9.90%
	3	4.31E-05	8.54%
	4	0.000176	7.18%
1000	0	0	10.31%
	1	1.21E-10	12.03%
	2	7.13E-07	11.33%
	3	1.26E-05	8.47%
	4	6.69E-05	6.71%

Table B5 Comparison of policies when $\lambda_1 = \lambda_2 = 300, \pi\breve{E}_2 = 10$

$\pi\breve{E}_1 / \pi\breve{E}_2$	L/u	CR_{LB}	$CR_{cnM-dyM}$
3	0	0	0.70%
	1	7.88E-13	0.95%
	2	1.51E-08	1.25%
	3	1.00E-06	1.36%
	4	9.65E-06	1.36%
100	0	0	4.81%
	1	1.27E-12	6.71%
	2	1.45E-08	9.28%
	3	1.74E-06	9.82%
	4	1.66E-05	9.31%
1000	0	0	7.18%
	1	2.10E-11	9.84%
	2	1.83E-08	12.80%
	3	1.63E-06	12.11%
	4	1.19E-05	7.23%

Table B6 Comparison of policies when $\lambda_1 = \lambda_2 = 150, \breve{\pi}_2 = 1.5$

$\breve{\pi}_1 / \breve{\pi}_2$	L/u	CR_{LB}	$CR_{cnM-dyM}$
3	0	0	0.80%
	1	2.19E-05	1.31%
	2	0.000878	1.17%
	3	0.002968	1.15%
	4	0.005732	1.07%
100	0	0	6.73%
	1	3.10E-06	8.51%
	2	0.000149	5.96%
	3	0.000811	4.40%
	4	0.001858	3.23%
1000	0	0	8.73%
	1	5.33E-07	9.26%
	2	5.35E-05	6.09%
	3	0.000372	2.32%
	4	0.000921	2.03%

Table B7 Comparison of policies when $\lambda_1 = \lambda_2 = 150, \pi\breve{E}_2 = 5$

$\pi\breve{E}_1 / \pi\breve{E}_2$	L/u	CR_{LB}	$CR_{cnM-dyM}$
3	0	0	0.83%
	1	4.57E-07	1.08%
	2	4.60E-05	1.35%
	3	0.000391	1.29%
	4	0.001254	1.19%
100	0	0	4.98%
	1	7.43E-08	7.44%
	2	3.10E-05	7.32%
	3	0.000303	4.61%
	4	0.00096	4.06%
1000	0	0	5.20%
	1	5.21E-08	8.91%
	2	1.62E-05	6.53%
	3	0.000116	3.27%
	4	0.000405	3.17%

Table B8 Comparison of policies when $\lambda_1 = \lambda_2 = 150, \breve{\pi}_2 = 10$

$\breve{\pi}_1 / \breve{\pi}_2$	L/u	CR_{LB}	$CR_{cnM-dyM}$
3	0	0	0.65%
	1	3.79E-08	0.98%
	2	1.14E-05	1.15%
	3	0.000106	1.14%
	4	0.000476	1.13%
100	0	0	3.96%
	1	2.48E-08	6.19%
	2	1.07E-05	7.42%
	3	0.000153	4.95%
	4	0.000411	4.17%
1000	0	0	4.23%
	1	1.63E-08	7.66%
	2	9.21E-06	9.59%
	3	8.47E-05	6.12%
	4	0.000343	3.05%

Appendix C

Proofs in Chapter 4

Proof of Lemma 4.1

Prove it by showing the first difference $\Delta J_X^2(t_c, s)$ is non-decreasing in s .

From equation (4.3) we have

$$\begin{aligned}
&\Delta J_X^2(t_c, s+1) = J^2(t_c, s+1) - J^2(t_c, s) \\
&= h \cdot t_c - [t_c \cdot (\breve{\pi}_1 + h) + \pi_1] \cdot P(D_1^{t_c} \geq s+1) \qquad \text{(C.1)} \\
&+ (\breve{\pi}_1 + h) \cdot E[\tau_{s+1} \mid D_1^{t_c} \geq s+1] \cdot P(D_1^{t_c} \geq s+1)
\end{aligned}$$

and

$$\begin{aligned}
&\Delta J_X^2(t_c, s) = J^2(t_c, s) - J^2(t_c, s-1) \\
&= h \cdot t_c - [t_c \cdot (\breve{\pi}_1 + h) + \pi_1] \cdot P(D_1^{t_c} \geq s) \qquad \text{(C.2)} \\
&+ (\breve{\pi}_1 + h) \cdot E[\tau_s \mid D_1^{t_c} \geq s] \cdot P(D_1^{t_c} \geq s).
\end{aligned}$$

From (C.1) and (C.2) we have

$$
\begin{aligned}
&\Delta J_X^2(t_c, s+1) - \Delta J_X^2(t_c, s) \\
&= [t_c \cdot (\breve{\pi}_1 + h) + \pi_1] \cdot P(D_1^{t_c} = s) \\
&+ (\breve{\pi}_1 + h) \cdot \{E[\tau_{s+1} \mid D_1^{t_c} \geq s+1] \cdot P(D_1^{t_c} \geq s+1) - E[\tau_s \mid D_1^{t_c} \geq s] \cdot P(D_1^{t_c} \geq s)\}.
\end{aligned}
\tag{C.3}
$$

For

$$
\begin{aligned}
&E[\tau_s \mid D_1^{t_c} \geq s] \cdot P(D_1^{t_c} \geq s) \\
&= \sum_{i=s}^{\infty} E[\tau_s \mid D_1^{t_c} = i] \cdot P(D_1^{t_c} = i) \\
&= E[\tau_s \mid D_1^{t_c} \geq s+1] \cdot P(D_1^{t_c} \geq s+1) + E[\tau_s \mid D_1^{t_c} = s] \cdot P(D_1^{t_c} = s),
\end{aligned}
\tag{C.4}
$$

we obtain by substituting (C.4) into (C.3)

$$
\begin{aligned}
&\Delta J_X^2(t_c, s+1) - \Delta J_X^2(t_c, s) \\
&= [t_c \cdot (\breve{\pi}_1 + h) + \pi_1] \cdot P(D_1^{t_c} = s) + (\breve{\pi}_1 + h) \\
&\cdot \{E[\tau_{s+1} \mid D_1^{t_c} \geq s+1] \cdot P(D_1^{t_c} \geq s+1) - E[\tau_s \mid D_1^{t_c} \geq s+1] \cdot P(D_1^{t_c} \geq s+1)\} \\
&- (\breve{\pi}_1 + h) \cdot E[\tau_s \mid D_1^{t_c} = s] \cdot P(D_1^{t_c} = s) \\
&= \{[t_c \cdot (\breve{\pi}_1 + h) + \pi_1] - (\breve{\pi}_1 + h) \cdot E[\tau_s \mid D_1^{t_c} = s]\} \cdot P(D_1^{t_c} = s) + (\breve{\pi}_1 + h) \\
&\cdot \{E[\tau_{s+1} \mid D_1^{t_c} \geq s+1] \cdot P(D_1^{t_c} \geq s+1) - E[\tau_s \mid D_1^{t_c} \geq s+1] \cdot P(D_1^{t_c} \geq s+1)\}.
\end{aligned}
\tag{C.5}
$$

Given $D_1^{t_c} = s$, we can see that τ_s will be always less than or equal to t_c, so

$E[\tau_s \mid D_1^{t_c} = s] \leq t_c$, hence

$$
\begin{aligned}
&[t_c \cdot (\breve{\pi}_1 + h) + \pi_1] - (\breve{\pi}_1 + h) \cdot E[\tau_s \mid D_1^{t_c} = s] \\
&= (\breve{\pi}_1 + h) \cdot \{t_c - E[\tau_s \mid D_1^{t_c} = s]\} + \pi_1 \geq 0.
\end{aligned}
$$

We can also see that $\tau_{s+1} \geq \tau_s$, for τ_{s+1} is the time of demanding for $(s+1)$th unit of the product, while τ_{s+1} is the time of demanding for sth unit. So we have

$$E[\tau_{s+1} \mid D_1^{t_c} \geq s+1] \cdot P(D_1^{t_c} \geq s+1) > E[\tau_s \mid D_1^{t_c} \geq s+1] \cdot P(D_1^{t_c} \geq s+1).$$

So from (C.5) we have $\Delta J_X^2(t_c, s+1) - \Delta J_X^2(t_c, s) > 0$, thus the result follows. Ĭ

Proof of Theorem 4.1

According to Lemma 4.1, we immediately have it. Ĭ

Proof of Theorem 4.2

a) When $t_c = 0$, for any given $s > 0$, $\Delta J_X^2(t_c, s) = 0$, so $\Delta J_X^2(t_c, s) + e_2(t_c) = \pi_2 \geq 0$. According to equation (4.2) we have $s_2^a(t_c) = 0$.

b) For a given $s > 0$, $J^2(t_c, s)$ is continuous in t_c, so $\Delta J_X^2(t_c, s) + e_2(t_c)$ is also continuous in t_c. When $t_c = 0$, $\Delta J_X^2(t_c, s) = 0$. For $\pi_2 \geq 0$, we have $\Delta J_X^2(t_c, s) + e_2(t_c) = \pi_2 \geq 0$. For a given $s > 0$, when t_c increases from 0 and arrives at a certain value, we may had better reject all demands of class 2 from the time t_c to the end of period, i.e., $\Delta J_X^2(t_c, s) + e_2(t_c) > 0$ when t_c is a certain large value. According to the continuity of $\Delta J_X^2(t_c,s)+e_2(t_c)$, there exists at least one value of t_c where $\Delta J_X^2(t_c,s)+e_2(t_c)=0$. Let $t_2^s=\min\{t_c \mid \Delta J_X^2(t_c,s)+e_2(t_c)=0\}$. So when $t_2^s>t_c\geq 0$, $\Delta J_X^2(t_c,s)+e_2(t_c)>0$ and when $t_c=(t_2^s)^+$, $\Delta J_X^2(t_c,s)+e_2(t_c)<0$, where $(t_2^s)^+$ is the time when the remaining time is infinitesimally longer than t_2^s.

According to the definition of critical levels, if we can show: given inventory s, for any $t_c>t_c^s$, $\Delta J_X^2(t_c,s)+e_2(t_c)<0$, then the proposition is proved. According to the definition of t_2^s, we know that when $t_c=t_2^s$, $\Delta J_X^2(t_c,s)+e_2(t_c)=0$. So if we can show:

for any $t_c>t_2^s$, $\dfrac{\partial[\Delta J_X^2(t_c,s)+e_2(t_c)]}{\partial t_c}=\dfrac{\partial[\Delta J_X^2(t_c,s)]}{\partial t_c}+\breve{\pi}_2\leq 0$, then we have shown: for any $t_c>t_c^s$, $\Delta J_X^2(t_c,s)+e_2(t_c)<0$. Following is to show: for any $t_c>t_c^s$, then $\dfrac{\partial[\Delta J_X^2(t_c,s)]}{\partial t_c}+\breve{\pi}_2\leq 0$.

From equation (C2) we have

$$\begin{aligned}\frac{\partial[\Delta J_X^2(t_c,s)]}{\partial t_c}&=h-(\breve{\pi}_1+h)P(D_1^{t_c}\geq s)-t_c(\breve{\pi}_1+h)\frac{\partial P(D_1^{t_c}\geq s)}{\partial t_c}\\&-\pi_1\frac{\partial P(D_1^{t_c}\geq s)}{\partial t_c}+(\breve{\pi}_1+h)\frac{\partial\{E[\tau_s \mid D_1^{t_c}\geq s]\cdot P(D_1^{t_c}\geq s)\}}{\partial t_c}.\end{aligned}\tag{C6}$$

We have known τ_s is a continuous random variable. Let $p(\tau_s)$ be its probability density function. So

$$\frac{\partial P(D_1^{t_c}\geq s)}{\partial t_c}=p(\tau_s=t_c).$$

For $E[\tau_s \mid D_1^{t_c} \geq s] \cdot P(D_1^{t_c} \geq s) = \int_0^{t_c} \tau_s \cdot p(\tau_s) \cdot d\tau_s$, we have

$$\frac{\partial \{E[\tau_s \mid D_1^{t_c} \geq s] \cdot P(D_1^{t_c} \geq s)\}}{\partial t_c} = t_c \cdot p(\tau_s = t_c) \, .$$

Substitute the above equations to equation (C6) and we have

$$\frac{\partial [\Delta J_X^2(t_c, s)]}{\partial t_c} = h - (\tilde{\pi}_1 + h) P(D_1^{t_c} \geq s) - \pi_1 p(\tau_s = t_c). \tag{C7}$$

According to the condition in proposition we have $\pi_1 = \pi_2 = 0$, hence

$$\frac{\partial [\Delta J_X^2(t_c, s)]}{\partial t_c} + \tilde{\pi}_2 = \tilde{\pi}_2 + h - (\tilde{\pi}_1 + h) P(D_1^{t_c} \geq s) \, . \tag{C8}$$

According to the definition of t_2^s, we can know that when $t_c = t_2^s$,

$$\frac{\partial [\Delta J_X^2(t_c, s)]}{\partial t_c} + \tilde{\pi}_2 < 0 \, .$$

So, from equation (C8), we know when $t_c = t_2^s$, $\tilde{\pi}_2 + h - (\tilde{\pi}_1 + h) \cdot P(D_1^{t_c} \geq s) < 0$.

When t_c increases from t_2^s, $P(D_1^{t_c} \geq s)$ also increases, so for any $t_c > t_2^s$,

$$\frac{\partial [\Delta J_X^2(t_c, s)]}{\partial t_c} + \tilde{\pi}_2 < 0 \, ,$$

hence the result follows. ∎

Appendix D

Proofs in Chapter 5

Proof of Lemma 5.1

We prove it by induction. As k is fixed as 1, for the simplicity of symbols we in the following equations use $H_T(n,x)$ and $\Delta_x(n,x)$ and to replace $H_T(k,n,x)$ and $\Delta_x(k,n,x)$, respectively.

When $n=0$, $H_T(n,x) = -\alpha \cdot S_0(x)$. According to the assumption that the first difference of the salvage value function $S_0(x)$ is nonincreasing in x, we have: the first difference of $H_T(n,x)$ when n=0 is nondecreasing in x. Now assume that the first difference of $H(n-1,x), n \geq 1$, is nondecreasing in x, i.e. $\Delta_x(n-1,x)$ is nondecreasing in x. For a given on-hand inventory x at the beginning of interval n, i.e., at time point n, there exists a critical class k_x^{n-1} at time point $n-1$ such that when $i \geq k_x^{n-1}$, $\Delta_x(n-1,x) + \pi_i < 0$ (reject the demand from class i), and when $i < k_x^{n-1}$,

$\Delta_x(n-1,x)+\pi_i \geq 0$ (accept the demand from class i). Since $\Delta_x(n-1,x)$ is non-decreasing in x, and $\pi_i \geq \pi_j$, $i<j$, we have $k_x^{n-1} \geq k_{x-1}^{n-1} \geq k_{x-2}^{n-1}$.

From (5.6) in Section 5.2, we can know that when $x \geq 1$,

$$\begin{aligned}
&H_T(n,x) = x\cdot\Delta t\cdot h + p_0\cdot H_T(n-1,x)+(1-p_0)\cdot H_T(n-1,x-1)\\
&+\sum_{i=1}^{K} p_i\cdot\min[0,\Delta_x(n-1,x)+\pi_i]\\
&= x\cdot\Delta t\cdot h + p_0\cdot H_T(n-1,x)+(1-p_0)\cdot H_T(n-1,x-1)\\
&+\sum_{i=k_x^{n-1}}^{K} p_i\cdot[\Delta_x(n-1,x)+\pi_i].
\end{aligned} \tag{D1}$$

and when $x=0$,

$$H_T(n,0)=\sum_{i=1}^{K} p_i\cdot\pi_i + H_T(n-1,0). \tag{D2}$$

So the first difference of (D1) with respect to x when $x \geq 2$ is

$$\begin{aligned}
&\Delta_x(n,x) = \Delta t\cdot h + p_0\cdot\Delta_x(n-1,x)+(1-p_0)\cdot\Delta_x(n-1,x-1)\\
&+\sum_{i=k_x^{n-1}}^{K} p_i\cdot[\Delta_x(n-1,x)+\pi_i]-\sum_{i=k_{x-1}^{n-1}}^{K} p_i\cdot[\Delta_x(n-1,x-1)+\pi_i]\\
&= \Delta t\cdot h + (p_0+\sum_{i=k_x^{n-1}}^{K} p_i)\cdot\Delta_x(n-1,x)+(1-p_0-\sum_{i=k_{x-1}^{n-1}}^{K} p_i)\cdot\Delta_x(n-1,x-1)\\
&-\sum_{i=k_{x-1}^{n-1}}^{k_x^{n-1}-1} p_i\cdot\pi_i.
\end{aligned} \tag{D3}$$

and when $x=1$,

$$\Delta_x(n,1) = \Delta t \cdot h + (p_0 + \sum_{i=k_1^{n-1}}^{K} p_i) \cdot \Delta_x(n-1,1) - \sum_{i=1}^{k_1^{n-1}-1} p_i \cdot \pi_i \qquad \text{(A4)}$$

We now look at the second difference of (D3) with respect to *x*, when $x \geq 3$,

$$\begin{aligned}
&\Delta_x(n,x) - \Delta_x(n,x-1) \\
&= (p_0 + \sum_{i=k_x^{n-1}}^{K} p_i) \cdot \Delta_x(n-1,x) + (1 - p_0 - \sum_{i=k_{x-1}^{n-1}}^{K} p_i) \cdot \Delta_x(n-1,x-1) - \sum_{i=k_{x-1}^{n-1}}^{k_x^{n-1}-1} p_i \cdot \pi_i \\
&- \{(p_0 + \sum_{i=k_{x-1}^{n-1}}^{K} p_i) \cdot \Delta_x(n-1,x-1) + (1 - p_0 - \sum_{i=k_{x-2}^{n-1}}^{K} p_i) \cdot \Delta_x(n-1,x-2) - \sum_{i=k_{x-2}^{n-1}}^{k_{x-1}^{n-1}-1} p_i \cdot \pi_i\} \\
&= (p_0 + \sum_{i=k_x^{n-1}}^{K} p_i) \cdot [\Delta_x(n-1,x) - \Delta_x(n-1,x-1)] + (1 - p_0 - \sum_{i=k_{x-2}^{n-1}}^{K} p_i) \cdot [\Delta_x(n-1,x-1) - \Delta_x(n-1,x-2)] \\
&+ \{-\sum_{i=k_{x-1}^{n-1}}^{k_x^{n-1}-1} p_i \cdot [\Delta_x(n-1,x-1) + \pi_i]\} + \sum_{i=k_{x-2}^{n-1}}^{k_{x-1}^{n-1}-1} p_i \cdot [\Delta_x(n-1,x-1) + \pi_i].
\end{aligned}$$

According to that the first difference of $H_T(n-1,x)$ is nondecreasing in x and the definition of $k_x^{n-1}, k_{x-1}^{n-1}, k_{x-2}^{n-1}$, each of the four items in the above express is nonnegative, so we have: When $x \geq 3$, $\Delta_x(n,x) - \Delta_x(n,x-1) \geq 0$.

Now consider when $1 \leq x < 3$. From (D3) and (D4) we have

$$\Delta_x(n,2)-\Delta_x(n,1)$$
$$=(p_0+\sum_{i=k_2^{n-1}}^{K}p_i)\cdot\Delta_x(n-1,2)+(1-p_0-\sum_{i=k_1^{n-1}}^{K}p_i)\cdot\Delta_x(n-1,1)-\sum_{i=k_1^{n-1}}^{k_2^{n-1}-1}p_i\cdot\pi_i$$
$$-[(p_0+\sum_{i=k_1^{n-1}}^{K}p_i)\cdot\Delta_x(n-1,1)-\sum_{i=1}^{k_1^{n-1}-1}p_i\cdot\pi_i]$$
$$=(p_0+\sum_{i=k_2^{n-1}}^{K}p_i)\cdot[\Delta_x(n-1,2)-\Delta_x(n-1,1)]+\sum_{i=1}^{k_1^{n-1}-1}p_i\cdot[\Delta_x(n-1,1)+\pi_i]$$
$$+\{-\sum_{i=k_1^{n-1}}^{k_2^{n-1}-1}p_i\cdot[\Delta_x(n-1,1)+\pi_i]\}.$$

According to that the first difference of $H(n-1,x)$ is nondecreasing in x and the definition of $k_x^{n-1},k_{x-1}^{n-1},k_{x-2}^{n-1}$, each of the three items in the above expression is also nonnegative, hence $\Delta_x(n,x)-\Delta_x(n,x-1)\geq 0$.

So given that the first difference of $H(n-1,x),x\geq 0$, is nondecreasing in x, we have: the first difference of $H(n,x)$ is also nondecreasing in x. Thus by induction, the result follows. Ĭ

Proof of Lemma 5.2

We prove it by induction. In Lemma 5.1 we have shown: when k=1, for a given n, the first difference of $H_T(k,n,x)$ is nondecreasing in x. Now assume when $k=i$, $1\leq i<M$, for a given n, $N\geq n\geq 1$, the first difference of $H_T(k,n,x)$ is nondecreasing in x.

In the following we show: when $k = i + 1$, for a given *n*, the first difference of $H_T(k, n, x)$ is nondecreasing in *x*.

From equation (5.5) we can see that $H_T(i+1, n, x)$, which is the total cost from the beginning of interval *n* of the period $i+1$ to the end of the horizon, can be regarded as the cost from the beginning of interval *n* of period $i+1$ (i.e., time point $(i+1, n)$) to the end of period $i+1$ with terminal cost function $R_k(x)$. So $H_T(i+1, n, x)$ can be regarded as a single-period model. The difference between formula of $H_T(i+1, n, x)$ and that of $H_T(1, n, x)$ is the different terminal cost functions.

Based on the assumption, we know that the first difference of $H_T(i, N, x)$ is nondecreasing in *x*. Thus the first difference of the function $R_k(x) = \alpha \cdot H_T(i, N, x+Q)$ is also nondecreasing in *x*. From Lemma 5.1 we can see that when the first difference of the terminal cost function is nondecreasing in *x*, the first difference of the optimal cost function $H_T(1, n, x)$ of the single-period model is also nondecreasing in *x* for a given *n*. So we have: for a given *n*, the first difference of $H_T(i+1, n, x)$ is nondecreasing in *x*. Thus by induction the result follows. Ĭ

Proof of Theorem 5.1

According to Lemma 5.2, from the property that the first difference of $H_T(k, n, x)$ is nondecreasing in *x*, we have: there is a unique $x_i^*(k, n)$ such that when the on-hand inventory $x > x_i^*(k,n)$, then $\Delta_x(k,n,x)+\pi_i \geq 0$ and the system should satisfy the demand of class i at time point n, and when the on-hand inventory is equal to or below $x_i^*(k,n)$, then $\Delta_x H_T(k,n,x)+\pi_i < 0$ and the system should reject the demand of class i at time point n. Thus the result follows. Ĭ

Proof of Theorem 5.2

Proposition 1.6 of Chapter 3 in Bertsekas (1995) (page 146) has shown that: for dynamic programming problems with discount factor α, $0<\alpha<1$, infinite horizon and unbound cost per period, when the decision space for each state at any stage is finite, then the optimal cost function of infinite horizon can be obtained by limiting the cost of *m*-stage dynamic programming problem.

In the previous dynamic programming model where each period is divided into many small intervals, one interval is one stage. We may reformulate the dynamic inventory rationing problem in another way: one period is one stage. In this case, the state variable is the on-hand inventory at the beginning of a period, and the decision is to choose a rationing policy for the current period, i.e., how to ration stock at each time point in the current period. Let *C* denote the set of single-period rationing policies. In this new formulation, for each state x_k, the system needs to choose a policy from *C*. We can see that set *C* is infinite. In the following we show that it is enough to consider a finite subset of *C*, hence the theorem is proved.

It is obvious that when the on-hand inventory is very large, then the system does not need to reject demands of some classes to reserve stock for more important classes, i.e. should satisfy demands of all classes. So there exists such an extremely large value of on-hand inventory such that when the on-hand inventory is larger than it, then the system should satisfy demands of all classes during the period. Let $C^{'}$ denote the set of single-period rationing policies that satisfy this requirement. We can see that the optimal policies should be located in $C^{'}$. So it is enough to consider only policies in $C^{'}$, i.e., we consider only elements in $C^{'}$ as admissible single-period rationing policies. We can see that $C^{'}$ is finite. So, for each state, the decision space is finite. Thus, according to the Proposition 1.6 of Chapter 3 in Bertsekas (1995), the result follows. Ĭ

Proof of Theorem 5.3

Part (a)

From Lemma 5.2 and Theorem 5.2 (infer the property of the optimal cost function over infinite horizon from that of the *M*-period system), we immediately have Part (a).

Part (b)

From Theorem 5.1 and Theorem 5.2, we immediately have Part (b).

Part (c)

In the proof of Theorem 5.2, we have shown that the previous dynamic programming model with one interval as one stage can be reformulated as a new one with one period as one stage, and it is enough to consider the finite control space for each state. Hence, according to Proposition 1.3 of Chapter 3 in Bertsekas (1995) (page 143), we have: there exists an optimal stationary rationing policy. Ĭ