# TWO FREE BOUNDARY PROBLEMS IN OPTIMAL INVESTMENT

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# Contents

# Abstract

This thesis is based on two of our recent working papers (see Dai and Zhong (2008a), Dai and Zhong (2008b)). We have considered two free boundary problems in optimal investment. Problem I is concerned with the optimal decision to sell or buy a stock in a given period with reference to the ultimate average of the stock price. Strictly speaking, we aim to determine an optimal selling (buying) time so as to maximize (minimize) the expectation of the ratio of the selling (buying) price to the ultimate average price over the period. This is an optimal stopping time problem which can be formulated as a variational inequality problem. We provide a partial differential equation (PDE) approach to study the optimal strategy. Problem II concerns numerical solutions for the continuous-time portfolio selection with proportional transaction costs which is described as a singular stochastic control problem. The associated value function is governed by a variational inequality with gradient constraints. We propose a penalty method to deal with the gradient constraints and then employ the finite difference discretization. Convergence analysis and numerical results are presented. In addition, we show that the standard penalty method can be applied in the case of single risky asset where the problem can be reduced to a standard variational inequality.

# Chapter 1

# Introduction

This chapter summarizes the work of this thesis. Firstly, we give the overviews of the two free boundary problems involved in our research. Problem I is on the decision of selling/buying a stock with reference to the ultimate average. Problem II is related to the optimal investment and consumption in the presence of transaction costs.

## 1.1 The overviews

### 1.1.1 Stock selling/buying with reference to the ultimate average

Assume that the discounted stock price evolves according to

$$dS_t = \alpha S_t dt + \sigma S_t dB_t,$$

where constants $\alpha \in (-\infty, +\infty)$ and $\sigma > 0$ are the discounted expected rate of return and volatility, respectively, and $\{B_t; t > 0\}$ is a standard Brownian motion on a filtered probability space $(\mathbb{S}, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with $B_0 = 0$ almost surely. We are interested in the following optimal decision to sell or buy a stock in a given period $[0, T]$ with reference to the ultimate average:

$$\text{Buy case: } \min_{0 \le \nu \le T} \mathbb{E}\left(\frac{S_\nu}{A_T}\right), \tag{1.1.1}$$

$$\text{Sell case: } \max_{0 \le \nu \le T} \mathbb{E}\left(\frac{S_\nu}{A_T}\right), \tag{1.1.2}$$

where $\mathbb{E}$ stands for the expectation, $\nu \in \mathfrak{T}$, the set of all $\mathscr{F}_t$ stopping time, and the benchmark value $A_T$ is taken as either geometric or arithmetic average price over the period [0,T], namely,

$$A_T = \begin{cases} \exp\left(\frac{1}{T}\int_0^T \log S_\nu d\nu\right), & \text{geometric average,} \\ \frac{1}{T}\int_0^T S_\nu d\nu, & \text{arithmetic average.} \end{cases} \tag{1.1.3}$$

Problem (1.1.1) and (1.1.2) are motivated by Shiryaev, Xu and Zhou (2008) that studied the optimal stock selling strategy with reference to the ultimate maximum, that is, the benchmark $A_T$ is taken as $\max\limits_{0 \le \nu \le T} S_\nu$. They derived a surprising optimal selling strategy: one either sells the stock immediately or holds it until expiry. More precisely, if $\alpha > \sigma^2/2$, it is optimal to hold the stock until expiry; if $\alpha \le \sigma^2/2$, it is optimal to sell the stock immediately at time zero.

To analyze the optimal strategy, Shiryaev, Xu and Zhou (2008) adopted a stochastic analysis approach which was also employed by Graversen, Peskir and Shiryaev (2001), Pedersen (2003) and Du Toit and Peskir (2007) where various models of predicting the maximum of a Brownian motion were studied. In our thesis, the arithmetic average involved makes the problems intractable. We will instead make use of a partial differential equation (PDE) approach.

The PDE formulation, described by a variational inequality, seemingly resembles the pricing model of American-style Asian options which has been extensively studied by numerous researchers (e.g., Geman and Yor (1993), Roger and Shi (1995), Wu, Kwok and Yu (1999), Vecer (2001), Ben-Ameur, Breton and Ecuyer

(2002), Wu and Fu (2003), Halluin, Forsyth and Labahn (2005), Dai and Kwok (2006), and reference therein). However, the present problem gets a different obstacle function involved and theoretical analysis of the resulting optimal strategy is distinct from the previous ones.

### 1.1.2 Penalty method for portfolio selection with proportional transaction costs

#### One risky asset

The study of portfolio optimization and consumption problems via stochastic control in continuous time was initiated by Merton (1969, 1971). In the absence of transaction costs, he showed that the optimal strategy of a CRRA investor is to allocate constant fraction ("Merton line") of total wealth in each asset and to consume at a constant rate. Such a strategy leads to incessant trading, which is impracticable in a real market with transaction costs. This motivated Magill and Constantinides (1976) to introduce proportional transaction costs into Merton's model. They provided a fundamental insight that there exists a no-trading region. Mathematically, the portfolio selection with proportional transaction costs is described as a singular stochastic control problem whose value function is governed by a variational inequality with gradient constraints. Since then, there have been extensive literatures studying the optimal transaction policies for an investor facing proportional transaction costs. Some examples are as follows. Davis and Norman (1990) showed that the optimal policy of the infinite horizon problem is determined by the solution of a free boundary problem and can be calculated by solving an ordinary differential equation (ODE). In 1994, Shreve and Soner provided rigorous mathematical analysis of the optimal policies, relying on the concept of viscosity solutions to Hamilton-Jacobi-Bellman (HJB) equations. Muthmuraman (2006) considered the infinite horizon problem and provided a computational scheme that transforms the resulting free boundary problem to a sequence of fixed boundary problems.

Many other authors studied the finite horizon problem. Gennotte and Jung (1994) employed dynamic programming method to numerically solve the finite horizon investment problem, while Liu and Loewenstein (2002) first examined the behaviors of the free boundaries of the finite horizon optimal investment problem by virtue of a sequence of approximate analytical solutions. Dai and Yi (2006) considered the same problem and obtained an equivalent standard variational inequality by which they completely characterize the behaviors of the free boundaries. It is worthwhile pointing out that Dai and Yi (2006) essentially established a connection between optimal stopping and singular control problems, which though well-known [cf. Karatzas and Shreve (1984) or Soner and Shreve (1991)], had never been revealed for the present problem. The idea of Dai and Yi (2006) was further extended by Dai et al. (2007) and Dai, Xu and Zhou (2007) to deal with the consumption case and the continuous-time mean-variance framework, respectively.

### Multiple risky assets

The multiple risky assets case with transaction costs is much more challenging. Relatively, the number of papers that concern this problem is much lower and most of existing literatures rely on the numerical methods. Alkian, Menaldi and Sulem (1996) considered the case that stock returns are uncorrelated. They showed the existence and uniqueness of the value function by restricting the risk aversion coefficient to lie in $(0, 1)$. They also numerically studied the optimal policies by use of policy iteration together with multigrid method. Liu (2004) considered the problem under two assumptions: uncorrelated asset returns and constant absolute risk aversion (CARA) utility, which result in the separability of optimal policies for transactions on each asset and the multidimensional problem collapses to a set of one dimensional problems which can be characterized by ODEs. Muthuraman and Kumar (2006) extended the approach of Muthmuraman (2006) to the case of multiple risky assets and solve it by the finite element method. All of the above papers on multiple risky assets are confined to infinite horizon problems.

### Penalty method

The standard penalty method has been extensively studied for the variational inequality arising from the American option pricing. For example, Forsyth and Vetzal (2002) first provided a convergence analysis and demonstrated its efficiency for pricing American options. An extension to the jump-diffusion model was made by D'Halluin, Forsyth and Labahn (2005). Dai, Kwok and You (2007) established a linkage between the intensity-based framework and penalty method of optimal stopping problems. However, it should be emphasized that the variational inequality of American option pricing model is different from that resulted from a singular stochastic problem because the latter gets the gradient constraints involved. Fortunately, the efficiency of the penalty method to the latter case has been verified by Dai, Kwok and Zong (2007) which dealt with another singular control problem arising from the pricing of guaranteed minimum withdrawal benefits.

## 1.2 The scope of this thesis

The scope of our work involves two aspects including theoretical analysis in Chapter 2 and numerical analysis in Chapter 3.

First of all, we are going to formulate problem I as a standard variational inequality and then make use of a PDE approach to fully characterize the optimal selling/buying strategy with reference to the ultimate average. It turns out that the optimal selling strategy is still bang-bang, while the optimal buying strategy can be a feedback one subject to the type of average and parameter values.

Secondly, we assume the investor to be of CRRA, and then focus on on log or power utility function when we consider problem II. We will propose a penalty method combined with the finite difference discretization to numerically solve the variational inequality with gradient constraints that the value function satisfies. We only confine to the finite horizon problem, and it is straightforward to extend to the infinite horizon case. Moreover, convergence analysis is provided in single risky asset case. When such a problem can be reduced to a double obstacle problem, we will make use of the standard penalty method (cf. Forsyth and Vetzal (2002) and Dai, Kwok and You (2007)) to achieve a better order of convergence. In addition, a comprehensive numerical analysis is provided.

## 1.3 The outline

The rest of this thesis is organized as follows. The formulation of problem I and theoretical analysis of the optimal selling/buying strategy are presented in Chapter 2. Chapter 3 is devoted to the numerical solutions of portfolio selection with transaction costs. The numerical examples confirm the theoretical analysis in Dai and Yi (2006) and Dai et al. (2007) in single risky asset case. We conclude the thesis in chapter 4.

# Chapter 2

# Optimal Stock Selling/Buying Strategy with reference to the Ultimate Average

In this chapter, we will make use of a PDE approach to study the optimal selling/buying strategy with reference to the ultimate average. The organization is as follows. In the first section, we formulate problem (1.1.1) and (1.1.2) as variational inequality problems. In section 2, we confine to the case of geometric average in which the problems allow analytical solutions and the optimal strategy can be readily figured out. Section 3 is devoted to the case of arithmetic average. Due to lack of analytical solutions, we provide a thorough theoretical analysis on the optimal strategy. We finish this chapter with an Appendix.

## 2.1 PDE formulation

In this section, we will provide a PDE formulation for the optimal stopping problems (1.1.1) and (1.1.2). Let us begin with the buy case.

### 2.1.1 Buy case

As in (1.1.3), we denote by $A_t$ the running average over $[0,t]$. Then, we can write the value function associated with problem (1.1.1) as

$$\varphi(S_t, A_t, t) \doteq \min_{t\le\nu\le T} \mathbb{E}_t\left(\frac{S_\nu}{A_T}\right) = \min_{t\le\nu\le T} \mathbb{E}_t\left[S_\nu\ \mathbb{E}_\nu\left(\frac{1}{A_T}\right)\right], \tag{2.1.1}$$

where $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot\,|\,\mathscr{F}_t)$.

Denote $\phi(S_t, A_t, t) \doteq \mathbb{E}_t\left(\frac{1}{A_T}\right)$. Since

$$\begin{aligned} dA_t &= \begin{cases} \frac{A_t}{t}\log\frac{S_t}{A_t}dt, & \text{geometric case} \\ \frac{S_t - A_t}{t}dt, & \text{arithmetic case} \end{cases} \\ &\doteq f(S_t, A_t, t)dt, \end{aligned}$$

it is easy to verify that $\phi$ satisfies

$$\begin{cases} \mathscr{L}_0\phi = 0, \quad 0 < S, A < \infty,\ t \in (0,T), \\ \phi(S, A, T) = \frac{1}{A}, \end{cases} \tag{2.1.2}$$

where $\mathscr{L}_0 = -\partial_t - \frac{1}{2}\sigma^2 S^2 \partial_{SS} - \alpha S\partial_S - f(S,A,t)\partial_A$.

It follows from (2.1.1) that

$$\varphi(S_t, A_t, t) = \min_{t\le\nu\le T} E_t\left[S_\nu \phi(S_\nu, A_\nu, \nu)\right], \tag{2.1.3}$$

which is the unique viscosity solution to the following HJB equation (or variational inequality equation)

$$\begin{cases} \max\{\mathscr{L}_0\varphi, \varphi - S\phi\} = 0, \quad 0 < S, A < \infty,\ t \in (0,T), \\ \varphi(S, A, T) = \frac{S}{A}, \end{cases} \tag{2.1.4}$$

**Remark 2.1.1.** *(2.1.2) resembles the well-known pricing model of a European-style Asian option, whereas (2.1.4) resembles that of an American-style Asian option. See, for example, Barles, Daher and Romano (1995), Wilmott, Dewynne and Howison (1995) or Jiang and Dai (2004).*

Next, we will show that problem (2.1.2) and (2.1.4) can be reduced to one-dimensional time dependent problems. Indeed, by the transformation

$$z = \frac{A}{S},\ \tau = T - t,\ V(z,\tau) = \varphi(S,A,t) \text{ and } \Phi(z,\tau) = S\phi(S,A,t), \tag{2.1.5}$$

(2.1.4) reduces to

$$\begin{cases} \max\{\mathscr{L}_1 V,\ V - \Phi\} = 0, \text{ in } D, \\ V(z,0) = \frac{1}{z}, \end{cases} \tag{2.1.6}$$

where $D = (0,\infty) \times (0,T)$,

$$\mathscr{L}_1 = \partial_\tau - \frac{1}{2}\sigma^2 z^2 \partial_{zz} - (\sigma^2 - \alpha) z \partial_z - \overline{f}(z,\tau)\partial_z, \tag{2.1.7}$$

$$\overline{f}(z,\tau) = \begin{cases} -\frac{z}{T-\tau}\log z, \text{ geometric average}, \\ \frac{1}{T-\tau}(1-z), \text{ arithmetic average}, \end{cases}$$

and $\Phi(z,\tau)$ satisfies

$$\begin{cases} \mathscr{L}_1 \Phi = \sigma^2 z \Phi_z + (\sigma^2 - \alpha)\Phi, \text{ in } D, \\ \Phi(z,0) = \frac{1}{z}. \end{cases} \tag{2.1.8}$$

In physics, problem (2.1.6) is known as an upper obstacle problem and $\Phi$ is the upper obstacle.

### 2.1.2 Sell case

In a similar way, we introduce the value function associated with problem (1.1.2) as follows:

$$\psi(S_t, A_t, t) \doteq \max_{t \le \nu \le T} \mathbb{E}_t\left(\frac{S_\nu}{A_T}\right) = \max_{t \le \nu \le T} \mathbb{E}_t\left[S_\nu \phi(S_\nu, A_\nu, \nu)\right]$$

satisfying

$$\begin{cases} \min\{\mathscr{L}_0\psi, \psi - S\phi\} = 0, \quad 0 < S, A < \infty,\ t \in (0,T), \\ \psi(S,A,T) = \frac{S}{A}, \end{cases} \tag{2.1.9}$$

where $\mathscr{L}_0$ and $\phi$ are the same as in the buy case. In terms of the same transformation as in (2.1.5), we can deduce that $U(z,\tau) \doteq \psi(S,A,t)$ satisfies the following lower obstacle problem:

$$\begin{cases} \min\{\mathscr{L}_1 U, U - \Phi\} = 0, \text{ in } D, \\ U(z,0) = \frac{1}{z}, \end{cases} \tag{2.1.10}$$

where $\mathscr{L}_1$ and $\Phi$ are as given in (2.1.5) and (2.1.7).

## 2.2 Geometric average case

In this section, we confine to the geometric average case in which analytical solutions to the variational inequality problem (2.1.6) and (2.1.10) are available. Then we can explicitly work out the optimal strategies.

Let us first present the analytical expression of the obstacle function $\Phi$.

**Lemma 2.2.1.** *Let $\Phi$ be the obstacle function in the geometric average case, i.e., the solution to problem (2.1.8) with $\overline{f}(z,\tau) = -\frac{z}{T-\tau}\log z$. Then*

$$\Phi(z,\tau) = z^{\frac{\tau - T}{T}} \exp(g(\tau)), \tag{2.2.1}$$

*where $g(\tau) = \frac{\sigma^2\tau^3}{6T^2} - (\alpha - \frac{\sigma^2}{2})\frac{\tau^2}{2T}$.*

**Proof**. Substituting into (2.1.8), it is easy to verify the result.

**Remark 2.2.2.** *The expression of $\Phi(z,\tau)$ can also be derived by computing expectation. We place the derivation in Appendix 2.4.1.*

### 2.2.1 Optimal buying strategy

**Proposition 2.2.3.** *Let $V$ be the solution to problem (2.1.6) in the geometric average case. Then*

$$V(z,\tau) = \begin{cases} \Phi(z,\tau)\exp(-\frac{\sigma^2}{2T}\tau^2 + \alpha\tau), & \text{if } \alpha \le 0, \\ \Phi(z,\tau), & \text{if } 0 < \alpha < \sigma^2, 0 \le \tau \le \frac{\alpha}{\sigma^2}T, \\ \Phi(z,\tau)\exp(-\frac{\sigma^2}{2T}(\tau - \frac{\alpha}{\sigma^2}T)^2), & \text{if } 0 < \alpha < \sigma^2, \frac{\alpha}{\sigma^2}T < \tau < T, \\ \Phi(z,\tau), & \text{if } \alpha \ge \sigma^2, \end{cases} \tag{2.2.2}$$

*for any $(z,\tau) \in D$.*

**Proof**. We postulate that $V(z,\tau)$ takes the form of $\Phi(z,\tau)\exp(b(\tau))$, namely,

$$V(z,\tau) = z^{\frac{\tau - T}{T}} \exp(g(\tau) + b(\tau)).$$

Substituting into (2.1.6), we get

$$\begin{cases} \max\left\{b'(\tau) + \frac{\sigma^2}{T}\tau - \alpha, b(\tau)\right\} = 0, \tau \in (0,T), \\ b(0) = 0, \end{cases}$$

which has a unique solution

$$b(\tau) = \begin{cases} -\frac{\sigma^2}{2T}\tau^2 + \alpha\tau, & \text{if } \alpha \le 0, \\ 0, & \text{if } 0 < \alpha < \sigma^2 \text{ and } 0 \le \tau \le \frac{\alpha}{\sigma^2}T, \\ -\frac{\sigma^2}{2T}(\tau - \frac{\alpha}{\sigma^2}T)^2, & \text{if } 0 < \alpha < \sigma^2 \text{ and } \frac{\alpha}{\sigma^2}T < \tau \le T, \\ 0, & \text{if } \alpha \ge \sigma^2. \end{cases}$$

This completes the proof.

Now let us define the buying region. It is worth pointing out that $\tau = T$ (i.e. $t = 0$) should be considered. Since $z = 1$ at $\tau = T$, we introduce $\widehat{D} = D \cup \{(z,\tau) = (1,T)\}$. Then, the buying region can be defined as follows:

$$BR = \{(z,\tau) \in \widehat{D} : V(z,\tau) = \Phi(z,\tau)\}. \tag{2.2.3}$$

Noting that (2.2.2) is also true for $(z,\tau) = (1,T)$, we immediately have the following corollary.

**Corollary 2.2.4.** *Let $BR$ be the buying region as defined in (2.2.3).*

*i) If $\alpha \leq 0$, then $BR = \emptyset$;*

*ii) If $0 < \alpha < \sigma^2$, then $BR = \{(z,\tau) \in \widehat{D} : 0 < \tau \leq \frac{\alpha}{\sigma^2}T\}$;*

*iii) If $\alpha \geq \sigma^2$, then $BR = \widehat{D}$.*

The corollary indicates that, given the geometric average as benchmark, one should never buy the stock before expiry if $\alpha \leq 0$, and one should buy the stock immediately if $\alpha \geq \sigma^2$. When $0 < \alpha < \sigma^2$, one should not buy the stock until $\tau = \frac{\alpha}{\sigma^2}T$.

### 2.2.2 Optimal selling strategy

In a similar way, we can find the analytical solution of variational inequality (2.1.10) in the geometric average case.

**Proposition 2.2.5.** *Let $U$ be the solution to problem (2.1.10) in the geometric average case. Then*

$$U(z,\tau) = \begin{cases} \Phi(z,\tau), & \text{if } \alpha \leq 0, \\ \Phi(z,\tau), & \text{if } 0 < \alpha \leq \frac{\sigma^2}{2} \text{ and } \frac{2\alpha}{\sigma^2}T \leq \tau \leq T, \\ \Phi(z,\tau)\exp(-\frac{\sigma^2}{2T}\tau^2 + \alpha\tau), & \text{if } 0 < \alpha \leq \frac{\sigma^2}{2} \text{ and } 0 < \tau < \frac{2\alpha}{\sigma^2}T, \\ \Phi(z,\tau)\exp(-\frac{\sigma^2}{2T}\tau^2 + \alpha\tau), & \text{if } \alpha > \frac{\sigma^2}{2}, \end{cases} \tag{2.2.4}$$

*for any $(z,\tau) \in \widehat{D}$.*

The proof resembles that of Proposition 2.2.3 and is omitted.

Similarly, we can define the corresponding selling region as follows:

$$SR = \{(z,\tau) \in \widehat{D} : U(z,\tau) = \Phi(z,\tau)\}. \tag{2.2.5}$$

By Proposition 2.2.5, we obtain the following corollary.

**Corollary 2.2.6.** *Let $SR$ be the selling region as defined in (2.2.5).*

*i) If $\alpha \leq 0$, then $SR = \widehat{D}$;*

*ii) If $0 < \alpha \leq \frac{\sigma^2}{2}$, then $SR = \{(z,\tau) \in \widehat{D} : \frac{2\alpha}{\sigma^2}T \leq \tau \leq T\}$;*

*iii) If $\alpha > \frac{\sigma^2}{2}$, then $SR = \emptyset$.*

We emphasize that in the scenario of $0 < \alpha \leq \frac{\sigma^2}{2}$, $\{(z,\tau) = (1,T)\}$ is always in $SR$. Therefore, combining with part i) and ii) in the corollary, we conclude that the optimal selling strategy is a bang-bang one. That is, one should never sell the stock before expiry if $\alpha > \frac{\sigma^2}{2}$, and one should immediately sell the stock at time 0 if $\alpha \leq \frac{\sigma^2}{2}$.

## 2.3 Arithmetic average case

Unlike the geometric average case, analytical solutions are no longer available in the arithmetic average case. We will make use of a PDE approach to investigate the optimal strategy.

First, let us make the following transformation which plays a critical role in the analysis[1]:

$$\begin{aligned} x &= \log((T-\tau)z), \ F(x,\tau) = \log\left(\frac{\Phi(z,\tau)}{T}\right), \\ \overline{V}(x,\tau) &= \log\left(\frac{V(z,\tau)}{\Phi(z,\tau)}\right), \text{ and } \overline{U}(x,\tau) = \log\left(\frac{U(z,\tau)}{\Phi(z,\tau)}\right). \end{aligned} \tag{2.3.1}$$

Then, (2.1.8), (2.1.6) and (2.1.10) reduce to

$$\begin{cases} F_\tau - \frac{\sigma^2}{2}(F_{xx} + F_x^2) + (\alpha - \frac{3\sigma^2}{2} - e^{-x})F_x + (\alpha - \sigma^2) = 0, \text{ in } \Omega, \\ F(x,0) = -x, \end{cases} \tag{2.3.2}$$

[1] Without the help of this transformation, it seems hard (at least for us) to prove the existence of the optimal selling boundary as a single function of time and its monotonicity in $\alpha$.

$$\begin{cases} \max\{\mathscr{L}\,\overline{V} + \sigma^2 F_x - (\alpha - \sigma^2),\ \overline{V}\} = 0, \ \text{ in } \Omega, \\ \overline{V}(x,0) = 0, \end{cases} \tag{2.3.3}$$

and

$$\begin{cases} \min\{\mathscr{L}\,\overline{U} + \sigma^2 F_x - (\alpha - \sigma^2),\ \overline{U}\} = 0, \ \text{in } \Omega, \\ \overline{U}(x,0) = 0, \end{cases} \tag{2.3.4}$$

respectively, where $\Omega = (-\infty, \infty) \times (0, T)$ and

$$\mathscr{L} = \partial_\tau - \frac{\sigma^2}{2}\left[\partial_{xx} + (\partial_x)^2 + 2F_x\partial_x\right] + (\alpha - \frac{\sigma^2}{2} - e^{-x})\partial_x.$$

Correspondingly, we can define the buying and selling regions. Note that $x = -\infty$ when $\tau = T$. It is convenient to introduce $\widehat{\Omega} = \Omega \cup \{x = -\infty\}$ and define

$$BR_x = \{(x,\tau) \in \widehat{\Omega} : \overline{V}(x,\tau) = 0\} \tag{2.3.5}$$

and

$$SR_x = \{(x,\tau) \in \widehat{\Omega} : \overline{U}(x,\tau) = 0\}. \tag{2.3.6}$$

### 2.3.1 Two lemmas

Let us introduce two lemmas which are useful for both buy and sell cases.

**Lemma 2.3.1.** *Let $\overline{V}(x,\tau)$ and $\overline{U}(x,\tau)$ be the solutions to problem (2.3.3) and (2.3.4), respectively. Then,*

$$\overline{V}(x,\tau) < \overline{U}(x,\tau) \text{ in } \Omega.$$

**Proof**. It is easy to see that $\mathscr{L}\overline{V} \le \mathscr{L}\overline{U}(x,\tau)$ in $\Omega$ and $\overline{V}(x,0) = \overline{U}(x,0)$. Applying the strong maximum principle gives the result.

**Lemma 2.3.2.** *Suppose $F(x,\tau)$ is the solution to (2.3.2), then $F(x,\tau)$ has the following properties.*

*i)* $-1 < F_x(x,\tau) < 0, \forall\ (x,\tau) \in \Omega$;

*ii)* $F_{xx}(x,\tau) < 0, \forall\ (x,\tau) \in \Omega$;

*iii)* $F_{x\tau}(x,\tau) \geq 0, \forall\ (x,\tau) \in \Omega$;

*iv)* $F_x(x,\tau;\alpha+\delta,\sigma) < F_x(x,\tau;\alpha,\sigma) + \frac{\delta}{\sigma^2}, \forall\ \delta > 0,\ (x,\tau) \in \Omega$;

*v)* $\lim_{x\to-\infty} F_x(x,\tau) = 0$ *and* $\lim_{x\to\infty} F_x(x,\tau) = -1, \forall\ \tau \in (0,T]$.

**Proof**. Denote $\widetilde{F}(x,\tau) \doteq F_x(x,\tau)$, $F^{xx}(x,\tau) \doteq F_{xx}(x,\tau)$ and $F^{x\tau}(x,\tau) \doteq F_{x\tau}(x,\tau)$. It is easy to verify that $\widetilde{F}, F^{xx}$ and $F^{x\tau}$ satisfy

$$\begin{cases} \widetilde{F}_\tau - \frac{\sigma^2}{2}(\widetilde{F}_{xx} + 2\widetilde{F}\widetilde{F}_x) + (\alpha - \frac{3\sigma^2}{2} - e^{-x})\widetilde{F}_x + e^{-x}\widetilde{F} = 0, \text{ in } \Omega, \\ \widetilde{F}(x,0) = -1, \end{cases}$$

$$\begin{cases} F^{xx}_\tau - \frac{\sigma^2}{2}(F^{xx}_{xx} + 2F_xF^{xx}_x + 2(F^{xx})^2) + (\alpha - \frac{3\sigma^2}{2} - e^{-x})F^{xx}_x + 2e^{-x}F^{xx} = e^{-x}F_x, \text{ in } \Omega, \\ F^{xx}(x,0) = 0, \end{cases}$$

and

$$\begin{cases} F^{x\tau}_\tau - \frac{\sigma^2}{2}(F^{x\tau}_{xx} + 2F_xF^{x\tau}_x + 2F_{xx}F^{x\tau}) + (\alpha - \frac{3\sigma^2}{2} - e^{-x})F^{x\tau}_x + e^{-x}F^{x\tau} = 0, \text{ in } \Omega, \\ F^{x\tau}(x,0) = e^{-x}, \end{cases}$$

respectively. By virtue of the (strong) maximum principle[2], we obtain part i), ii) and iii).

Next we prove part iv). Denote $\widetilde{F}^\delta(x,\tau) \doteq F_x(x,\tau;\alpha+\delta,\sigma)$ and $\widehat{F}(x,\tau) \doteq F_x(x,\tau;\alpha,\sigma) + \frac{\delta}{\sigma^2}$. Let $P = \widetilde{F}^\delta - \widehat{F}$. Then, it suffices to show $P < 0$ in $\Omega$. It is easy to check that $\widetilde{F}^\delta(x,\tau)$ and $\widehat{F}(x,\tau)$ satisfy

$$\begin{cases} \widetilde{F}^\delta_\tau - \frac{\sigma^2}{2}(\widetilde{F}^\delta_{xx} + 2\widetilde{F}^\delta\widetilde{F}^\delta_x) + (\alpha + \delta - \frac{3\sigma^2}{2} - e^{-x})\widetilde{F}^\delta_x + e^{-x}\widetilde{F}^\delta = 0, \text{ in } \Omega, \\ \widetilde{F}^\delta(x,0) = -1 \end{cases} \tag{2.3.7}$$

and

$$\begin{cases} \widehat{F}_\tau - \frac{\sigma^2}{2}(\widehat{F}_{xx} + 2\widehat{F}\widehat{F}_x) + (\alpha + \delta - \frac{3\sigma^2}{2} - e^{-x})\widehat{F}_x + e^{-x}\widehat{F} = \frac{\delta}{\sigma^2}e^{-x}, \text{ in } \Omega, \\ \widehat{F}(x,0) = -1 + \frac{\delta}{\sigma^2}, \end{cases} \tag{2.3.8}$$

[2]Apparently the solutions we consider do not grow too fast as state variables go to infinity. So, we can use the maximum principle of unbounded domain.

respectively. Subtracting (2.3.8) from (2.3.7), we obtain

$$\begin{cases} P_\tau - \frac{\sigma^2}{2}(P_{xx} + \widehat{F}P_x + \widetilde{F}_x^\delta P) + (\alpha + \delta - \frac{3\sigma^2}{2} - e^{-x})P_x + e^{-x}P = -\frac{\delta}{\sigma^2}e^{-x}, \text{ in } \Omega, \\ P(x,0) = -\frac{\delta}{\sigma^2}. \end{cases}$$

Applying the maximum principle gives the desired result.

The proof of part v) is placed in Appendix 2.4.2.

### 2.3.2 Optimal buying strategy

To begin with, we present the properties of $\overline{V}(x,\tau)$.

**Proposition 2.3.3.** *The variational inequality problem (2.3.3) has a unique solution $\overline{V}(x,\tau) \in W_p^{2,1}(\Omega_N)$, $1 < p < +\infty$, where $\Omega_N$ is any bounded set in $\Omega$. Moreover,*

$$0 \le \overline{V}_x \le 1 \text{ and } V_\tau \le 0 \text{ in } \Omega. \tag{2.3.9}$$

**Proof**. Using the penalty approximation method [cf. Friedman (1982)], it is not hard to show that (2.3.3) has a unique solution $\overline{V}(x,\tau) \in W_p^{2,1}(\Omega_N), 1 < p < +\infty$, where $\Omega_N$ is any bounded set in $\Omega$.

To show $0 \le \overline{V}_x \le 1$, we only need to confine to the noncoincidence set $\Lambda = \{(x,\tau) \in \Omega : \overline{V} > 0\}$. Denote $w = \overline{V}_x$ and $v = \overline{V}_x - 1$, then $w$ and $v$ satisfy

$$\begin{cases} w_\tau - \frac{\sigma^2}{2}(w_{xx} + 2ww_x + 2F_{xx}w + 2F_x w_x) + (\alpha - \frac{\sigma^2}{2} - e^{-x})w_x + e^{-x}w = -\sigma^2 F_{xx}, \\ \hfill \text{in } \Lambda \\ w|_{\partial\Lambda} = 0, \end{cases}$$

and

$$\begin{cases} v_\tau - \frac{\sigma^2}{2}(v_{xx} + 2vv_x + 2F_{xx}v + 2F_x v_x) + (\alpha - \frac{3\sigma^2}{2} - e^{-x})v_x + e^{-x}v = -e^{-x}, \text{in } \Lambda \\ v|_{\partial\Lambda} = -1, \end{cases}$$

respectively. Since $-\sigma^2 F_{xx} > 0$ and $-e^{-x} \le 0$, one can deduce $w \ge 0$ and $v \le 0$ in $\Lambda$ by the maximum principle, which is desired.

At last, let us prove $\overline{V}_\tau \le 0$. Denote $\widetilde{V}(x,\tau) = \overline{V}(x,\tau+\delta)$. It suffices to show $Q(x,\tau) \doteq \widetilde{V}(x,\tau) - \overline{V}(x,\tau) \le 0$ in $\Omega$, $\forall\, \delta > 0$. Suppose not, then

$$\Delta = \{(x,\tau) \in \Omega : Q(x,\tau) > 0\} \neq \emptyset.$$

It is easy to verify that $Q(x,\tau)$ satisfies

$$\begin{cases} Q_\tau - \frac{\sigma^2}{2}\Big(Q_{xx} + (\widetilde{V}_x + \overline{V}_x)Q_x + 2F_x Q_x\Big) + (\alpha - \frac{\sigma^2}{2} - e^{-x})Q_x \\ \qquad\qquad\qquad \le \delta\sigma^2 F_{x\tau}(\cdot,\cdot)(\widetilde{V}_x - 1) \le 0, \quad \text{in } \Delta, \\ Q|_{\partial\Delta} = 0, \end{cases}$$

where we have used part iii) of Lemma 2.3.2 and $\widetilde{V}_x \le 1$. Applying the maximum principle, we get $Q \le 0$ in $\Delta$, which conflicts with the definition of $\Delta$. The proof is complete.

In terms of Lemma 2.3.2 and Proposition 2.3.3, we will show the following theorem.

**Theorem 2.3.4.** *Let $BR_x$ be the buying region as defined in (2.3.5).*

*i) If $\alpha \le 0$, then $BR_x = \emptyset$;*

*ii) If $0 < \alpha < \sigma^2$, there is a monotonically increasing boundary $x_b^*(\tau) : (0,T) \to (-\infty,\infty)$ such that*

$$BR_x = \{(x,\tau) \in \Omega : x \ge x_b^*(\tau), 0 < \tau < T\}. \tag{2.3.10}$$

*Moreover,*

$$\lim_{\tau\to 0^+} x_b^*(\tau) = -\infty; \tag{2.3.11}$$

*iii) If $\alpha \ge \sigma^2$, then $BR_x = \widehat{\Omega}$;*

**Proof**. According to part i) in Lemma 2.3.2 and (2.3.3),

$$\mathscr{L}\overline{V} \le -\sigma^2(F_x + 1) + \alpha < 0, \text{ for } \alpha \le 0.$$

Applying the strong maximum principle, we infer $\overline{V} < 0$ in $\Omega$ for $\alpha \le 0$. Due to (2.3.9), we infer $\overline{V}(-\infty, T) < 0$. Part i) then follows.

If $\alpha \ge \sigma^2$, part i) in Lemma 2.3.2 leads to

$$\sigma^2 F_x - (\alpha - \sigma^2) \le 0.$$

So, $\overline{V} = 0$ is a solution to (2.3.3), which implies part iii).

It remains to show part ii). Since $\overline{V}_x \ge 0$, we can define a free boundary

$$x_b^*(\tau) = \inf\{x \in (-\infty, \infty) : \overline{V}(x, \tau) = 0\}, \text{ for any } \tau \in (0, T).$$

Due to $\overline{V}_\tau \le 0$, we infer that $x_b^*(\tau)$ is monotonically increasing with $\tau$. Let us prove $x_b^*(\tau) > -\infty$ for all $\tau$. Suppose not, then there exists a $\tau_0 > 0$, such that $x_b^*(\tau) = -\infty$ for all $\tau \in [0, \tau_0]$. This leads to $\overline{V}(x, \tau) = 0$, in $(-\infty, \infty) \times [0, \tau_0]$. By (2.3.3), we have $\mathscr{L}\,\overline{V} + \sigma^2 F_x - (\alpha - \sigma^2) \le 0$ in $(-\infty, \infty) \times (0, \tau_0]$, namely,

$$F_x \le \frac{\alpha}{\sigma^2} - 1 \text{ in } (-\infty, \infty) \times (0, \tau_0].$$

So,

$$\lim_{x \to -\infty} F_x(x, \tau) \le \frac{\alpha}{\sigma^2} - 1 < 0, \ \forall\ 0 < \tau \le \tau_0,$$

which is in contradiction with part v) in Lemma 2.3.2.

Further, the idea of the proof of $x_b^*(\tau) < \infty$ stems from Brezis and Friedman (1976) [cf. also the proof of Lemma 4.2 in Dai, Kwok and Wu (2004)]. Thanks to $\lim_{x \to \infty} F_x(x, \tau) = -1$, we see there exists an $x^* > 0$, such that $\sigma^2 F_x - (\alpha - \sigma^2) < -\frac{\alpha}{2}$ uniformly in $\tau$, if $x > x^*$, where $F_{x\tau}(x, \tau) \le 0$ is used. We now construct an auxiliary function $W(x) = \begin{cases} -\frac{\varepsilon(R-x)^2}{R} & \text{if } x^* < x < R, \\ 0 & \text{if } x \ge R, \end{cases}$ and aim to show $W(x) \le \overline{V}(x, \tau)$, $\forall x > x^*$, where $\varepsilon$ and $R$ will be determinate later.

Note that if $x^* < x < R$,

$$\begin{aligned} \mathscr{L}W - \frac{\alpha}{2} &\le -\frac{\sigma^2 \varepsilon}{R} - \frac{2\sigma^2 F_x \varepsilon (R - x)}{R} + (\alpha - \frac{\sigma^2}{2} - e^{-x}) \frac{2\varepsilon(R - x)}{R} - \frac{\alpha}{2} \\ &\le -\frac{\sigma^2 \varepsilon}{R} + 2\sigma^2 \|F_x\| \varepsilon + 2\|\alpha - \frac{\sigma^2}{2} - e^{-x}\| \varepsilon - \frac{\alpha}{2} < 0, \end{aligned}$$

by choosing $\varepsilon$ sufficiently small.

It is easy to see $\mathscr{L}W - \frac{\alpha}{2} = -\frac{\alpha}{2} < 0$, if $x \geq R$. So that

$$\mathscr{L}W + \sigma^2 F_x - (\alpha - \sigma^2) < 0, \ \forall x > x^*.$$

Since we can always choose $R$ sufficiently large such that

$$W(x^*) < \overline{V}(x^*, \tau), \forall \tau \in [0, T],$$

we obtain $W(x) \leq \overline{V}(x, \tau)$ in $[x^*, \infty) \times (0, T]$, by the comparison principle, which implies $x_b^*(\tau) \leq R < +\infty$.

In addition, due to the monotonicity of $x_b^*(\tau)$, we deduce that $\{x = -\infty\} \notin BR_x$. So, (2.3.10) follows.

At last, we need to prove (2.3.11). Assume contrary, i.e., $\lim_{\tau \to 0^+} x_b^*(\tau) = x_0 > -\infty$. Then, we have

$$\mathscr{L}\overline{V} + \sigma^2 F_x - (\alpha - \sigma^2) = 0, \ \forall \ x < x_0, 0 < \tau < T,$$

which, combined with $\overline{V}(x, 0) = 0$ for all $x$, gives

$$\overline{V}_\tau|_{\tau=0} = -\sigma^2 F_x|_{\tau=0} + (\alpha - \sigma^2) = \alpha > 0, \ \forall \ x < x_0.$$

This conflicts with $\overline{V}_\tau \leq 0$. The proof is complete.

Figure 1 presents a numerical example about the optimal buying boundary $x_b^*(\tau)$ in the arithmetic average case with $0 < \alpha < \sigma^2$.

### 2.3.3 Optimal selling strategy

Now let us look at the sell case. Similar to the buy case, we can deal with the scenario of $\alpha \leq 0$ and $\alpha \geq \sigma^2$. However, the scenario of $0 < \alpha < \sigma^2$ is more

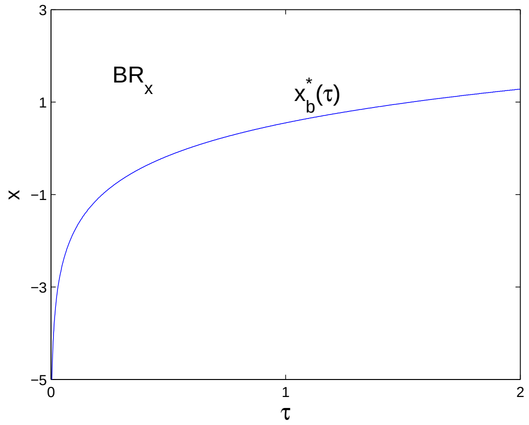


Figure 2.1: The optimal buying boundary $x_b^*(\tau)$ in the arithmetic average case. Parameter values used: $\alpha = 0.06,\ \sigma = 0.4,\ T = 2.$

challenging because we no longer have the monotonicity of $\overline{U}$ w.r.t. $\tau$. To overcome the difficulty, we introduce an auxiliary problem:

$$\begin{cases} \mathscr{L}\overline{U}^* + \sigma^2 F_x - (\alpha - \sigma^2) = 0, \text{ in } \Omega, \\ \overline{U}^*(x, 0) = 0. \end{cases} \tag{2.3.12}$$

**Lemma 2.3.5.** *Let $\overline{U}^*(x,\tau)$ be the solution to (2.3.12). Then for any $\tau \in (0, T]$,*

$$\begin{aligned} \lim_{x\to-\infty} \overline{U}^*(x,\tau) &> 0 \text{ if } \alpha > \frac{\sigma^2}{2}, \\ \lim_{x\to-\infty} \overline{U}^*(x,\tau) &= 0 \text{ if } \alpha = \frac{\sigma^2}{2}, \\ \lim_{x\to-\infty} \overline{U}^*(x,\tau) &< 0 \text{ if } \alpha < \frac{\sigma^2}{2}. \end{aligned}$$

We place the proof in Appendix 2.4.3.

**Proposition 2.3.6.** *The variational inequality problem (2.3.3) has a unique solution $\overline{U}(x,\tau) \in W_p^{2,1}(\Omega_N)$, $1 < p < +\infty$, where $\Omega_N$ is any bounded set in $\Omega$. Moreover, for any $(x,\tau) \in \Omega$,* *i)* $0 \leq \overline{U}_x \leq 1$;

*ii)* $\overline{U}(x,\tau;\alpha) \leq \overline{U}(x,\tau;\alpha+\delta)$ *for* $\delta > 0$;

*iii)* $\overline{U}(x,\tau) = \overline{U}^*(x,\tau) > 0$ *for* $\alpha \geq \frac{\sigma^2}{2}$. *And, for any* $\tau \in (0,T]$,

$$\lim_{x\to-\infty} \overline{U}(x,\tau) > 0, \textit{ if } \alpha > \frac{\sigma^2}{2}, \tag{2.3.13}$$

$$\lim_{x\to-\infty} \overline{U}(x,\tau) = 0, \textit{ if } \alpha = \frac{\sigma^2}{2}, \tag{2.3.14}$$

$$\lim_{x\to-\infty} \overline{U}(x,\tau) = 0, \textit{ if } \alpha < \frac{\sigma^2}{2}. \tag{2.3.15}$$

**Proof**. The proof of part i) is the same as that of Proposition 2.3.3. Now let us prove part ii). Suppose not, then

$$\mathscr{O} = \{(x,\tau) \in \Omega : H(x,\tau) < 0\} \neq \emptyset,$$

where $H(x,\tau) = \overline{U}(x,\tau;\alpha+\delta) - \overline{U}(x,\tau)$. Denote $F_x^\delta(x,\tau) = F_x(x,\tau;\alpha+\delta)$. It can be verified that

$$\begin{cases} H_\tau - \frac{\sigma^2}{2}(H_{xx} + H_x^2 + 2\overline{U}_x H_x + 2F_x^\delta H_x) + (\alpha + \delta - \frac{\sigma^2}{2} - e^{-x})H_x \\ \qquad\qquad\qquad \geq -(\sigma^2 F_x^\delta - \sigma^2 F_x - \delta)(1 - \overline{U}_x) \quad \text{in } \mathscr{O}, \\ H|_{\partial\mathscr{O}} = 0. \end{cases}$$

By part iv) in Lemma 2.3.2, $F_x^\delta < F_x + \frac{\delta}{\sigma^2}$, which, combines with $\overline{U}_x \leq 1$, gives

$$-(\sigma^2 F_x^\delta - \sigma^2 F_x - \delta)(1 - \overline{U}_x) \geq 0.$$

Again applying the maximum principle, we get $H \geq 0$, in $\mathscr{O}$, a contradiction with the definition of $\mathscr{O}$.

To show part iii), it is easy to see $\overline{U}_x^*(x,\tau) > 0$ in $\Omega$ by virtue of $F_{xx} < 0$ and the strong maximum principle. Combining with Lemma 2.3.5, we infer $\overline{U}^*(x,\tau) > 0$ in $\Omega$ when $\alpha \geq \frac{\sigma^2}{2}$. So, $\overline{U}^*(x,\tau)$ must be the solution to (2.3.4), which yields $\overline{U}(x,\tau) = \overline{U}^*(x,\tau)$ for $\alpha \geq \frac{\sigma^2}{2}$. Then (2.3.13) and (2.3.14) follow. To show (2.3.15), apparently we have $\lim_{x\to-\infty}\overline{U}(x,\tau)\geq 0$. Thanks to part ii) and (2.3.14), we infer $\lim_{x\to-\infty}\overline{U}(x,\tau)\leq 0$ for $\alpha<\frac{\sigma^2}{2}$, which leads to (2.3.15). This completes the proof.

We then study the optimal selling region.

**Theorem 2.3.7.** *Let $SR_x$ be the optimal selling region as defined in (2.3.6).*

*i) If $\alpha>\frac{\sigma^2}{2}$, then $SR_x=\emptyset$.*

*ii) If $\alpha=\frac{\sigma^2}{2}$, then $SR_x=\{x=-\infty\}$.*

*iii) If $0<\alpha<\frac{\sigma^2}{2}$, then $\{x=-\infty\}\subset SR_x$. Moreover, there is a free boundary $x_s^*(\tau):(0,T]\to(-\infty,+\infty)\cup-\infty$ such that*

$$SR_x=\left\{(x,\tau)\in\widehat{\Omega}:x\leq x_s^*(\tau)\right\}.$$

*iv) If $\alpha\leq 0$, then $SR_x=\widehat{\Omega}$.*

**Proof**. Part i) and ii) follow by part iii) of Proposition 2.3.6. The proof of part iv) is similar to that of part i) in Theorem 2.3.4. Now let us prove part iii). Thanks to (2.3.15), we immediately get $\{x=-\infty\}\subset SR_x$. Combined with $\overline{U}_x\geq 0$, we can define

$$x_s^*(\tau)=\sup\{x\in(-\infty,+\infty):\overline{U}(x,\tau)=0\},\text{ for any }\tau\in(0,T].$$

We only need to show that $x_s^*(\tau)<\infty$. Let $x_b^*(\tau)$ be the free boundary as given in part ii) of Theorem 2.3.4. Due to Lemma 2.3.1, we infer $x_s^*(\tau)\leq x_b^*(\tau)$, which, combined with $x_b^*(\tau)<\infty$, yields the desired the result. The proof is complete.

**Remark 2.3.8.** *Numerical results show that $x_s^*(\tau)$ is always monotonically increasing and $x_s^*(\tau)>-\infty$ when $0<\alpha<\frac{\sigma^2}{2}$. But currently we cannot prove this.*

As mentioned before, $x=-\infty$ at $\tau=T$ (i.e. $t=0$). By Theorem 2.3.7, $\{x=-\infty\}\subset SR_x$ if $\alpha\leq\frac{\sigma^2}{2}$, and $SR_x=\emptyset$ if $\alpha>\frac{\sigma^2}{2}$. We then obtain the bang-bang selling strategy as follows.

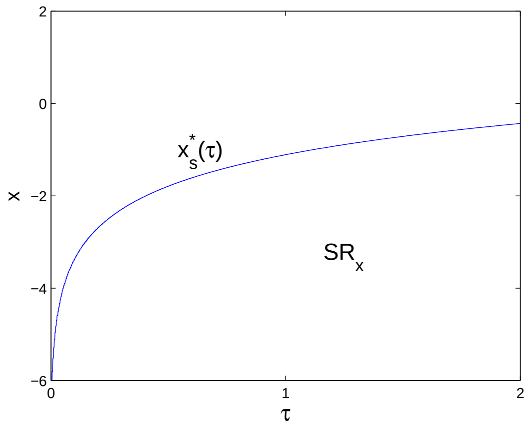


Figure 2.2: The optimal selling boundary $x_s^*(\tau)$ in the arithmetic average case. Parameter values used: $\alpha = 0.06,\ \sigma = 0.4,\ T = 2.$

**Corollary 2.3.9.** *It is optimal to sell the stock immediately at time* $0$ *if* $\alpha \le \frac{\sigma^2}{2}$, *and to hold the stock until expiry* $T$ *if* $\alpha > \frac{\sigma^2}{2}$.

**Remark 2.3.10.** *The main reason that we have the bang-bang selling strategy is that the average period is taken from time* $0$. *This leads the initial position being in the selling region for* $\alpha \le \frac{\sigma^2}{2}$. *If the average period is taken from some time horizon earlier than time* $0$, *then the initial position is likely to be beyond the selling region, which would result in a feedback strategy for* $0 < \alpha < \frac{\sigma^2}{2}$, *which will result in a feedback strategy. The same remark applies to the sell case with reference to the ultimate maximum or geometric average price.*

Figure 2 presents a numerical example about the optimal selling boundary $x_s^*(\tau)$ in the arithmetic average case with $0 < \alpha < \frac{\sigma^2}{2}$.

## 2.4 Appendix

### 2.4.1 The probability derivation of Lemma 2.2.1

**Proof**. Recall that $\Phi(z,\tau) = S_t\mathbb{E}_t\left(\frac{1}{A_T}\right)$, where the $z = \frac{A}{S}, \tau = T - t$. It suffices to calculate $S_t\mathbb{E}_t\left(\frac{1}{A_T}\right)$. Indeed,

$$
\begin{aligned}
& \mathbb{E}_t\left(\frac{S_t}{\exp\left\{\frac{1}{T}\int_0^T \log S_\nu d\nu\right\}}\right) \\
= \; & \frac{S_t}{\exp\left\{\frac{1}{T}\int_0^t \log S_\nu d\nu\right\}}\mathbb{E}_t\left(\exp\left\{-\frac{1}{T}\int_t^T \log S_\nu d\nu\right\}\right) \\
= \; & \left(\frac{S_t}{A_t}\right)^{\frac{t}{T}}\mathbb{E}_t\left(\exp\left\{-\frac{1}{T}\int_t^T (\log S_\nu - \log S_t)d\nu\right\}\right) \\
= \; & \left(\frac{S_t}{A_t}\right)^{\frac{t}{T}}\mathbb{E}\left(\exp\left\{-\frac{1}{T}\int_t^T \left((\alpha - \frac{\sigma^2}{2})(\nu - t) + \sigma B_{\nu-t}\right)d\nu\right\}\right) \\
= \; & \left(\frac{S_t}{A_t}\right)^{\frac{t}{T}}\exp\left(-(\alpha - \frac{\sigma^2}{2})\frac{(T-t)^2}{2T}\right)\mathbb{E}\left(\exp\left\{-\frac{1}{T}\int_0^{T-t}\sigma B_\nu d\nu\right\}\right) \\
= \; & \left(\frac{S_t}{A_t}\right)^{\frac{t}{T}}\exp\left(-(\alpha - \frac{\sigma^2}{2})\frac{(T-t)^2}{2T}\right)\mathbb{E}\left(\exp\left\{-\frac{\sigma}{T}\int_0^{T-t}(T-t-\nu)dB_\nu\right\}\right) \\
= \; & \left(\frac{S_t}{A_t}\right)^{\frac{t}{T}}\exp\left(-(\alpha - \frac{\sigma^2}{2})\frac{(T-t)^2}{2T}\right)\mathbb{E}\exp\left\{-\frac{\sigma}{T}B\left[\left(\frac{(T-t)^3}{3}\right)^{\frac{1}{2}}\right]\right\} \\
= \; & \left(\frac{S_t}{A_t}\right)^{\frac{t}{T}}\exp\left\{\frac{\sigma^2(T-t)^3}{6T^2} - (\alpha - \frac{\sigma^2}{2})\frac{(T-t)^2}{2T}\right\}.
\end{aligned}
$$

Thus, (2.2.1) follows.

### 2.4.2 The proof of part v) of Lemma 2.3.2

**Proof**. Note that

$$\begin{aligned} F(x,\tau) &= \log\left(\frac{\Phi(z,\tau)}{T}\right) = \log\left(\mathbb{E}_t\left(\frac{S_t}{TA_T}\right)\right) = \log \mathbb{E}_t\left[\left(\frac{tA_t}{S_t} + \int_t^T \frac{S_\nu}{S_t} d\nu\right)^{-1}\right] \\ &= \log \mathbb{E}\left[\left(e^x + \int_0^\tau e^{(\alpha - \frac{\sigma^2}{2})s + \sigma B_s} ds\right)^{-1}\right]. \end{aligned}$$

It follows

$$F_x(x,\tau) = \frac{-e^x \mathbb{E}\left[\left(e^x + \int_0^\tau e^{(\alpha-\frac{\sigma^2}{2})s+\sigma B_s} ds\right)^{-2}\right]}{\mathbb{E}\left[\left(e^x + \int_0^\tau e^{(\alpha-\frac{\sigma^2}{2})s+\sigma B_s} ds\right)^{-1}\right]}.$$

We then get the desired results by letting $x \to -\infty$ and $+\infty$.

### 2.4.3 The proof of Lemma 2.3.5

**Proof**. Let

$$\varphi^*(S_t, A_t, t) = \mathbb{E}_t\left(\frac{S_T}{A_T}\right),$$

which represents the value function associated with a simple strategy: holding the stock until expiry $T$. Similar to the transformation (2.3.1), we consider

$$U^*(z,\tau) = \varphi^*(S,A,t),\ \overline{U}^*(x,\tau) = \log\left(\frac{U^*(z,\tau)}{\Phi(z,\tau)}\right),$$

where the definitions of $x$, $z$ and $\tau$ are the same as earlier. It is easy to check that $\overline{U}^*(x,\tau)$ is the solution to (2.3.12). So, we only need to show

$$\begin{aligned} U^*(0,\tau) &> \Phi(0,\tau),\ \text{if } \alpha > \frac{\sigma^2}{2}, \\ U^*(0,\tau) &= \Phi(0,\tau),\ \text{if } \alpha = \frac{\sigma^2}{2}, \\ U^*(0,\tau) &< \Phi(0,\tau),\ \text{if } \alpha < \frac{\sigma^2}{2}. \end{aligned}$$

Let us only consider the case of $\alpha \geq \frac{\sigma^2}{2}$ since the case of $\alpha < \frac{\sigma^2}{2}$ is similar. Note that

$$\begin{aligned}
\Phi(0,\tau) &= \lim_{z\to 0^+} \Phi(z,\tau) = \lim_{z\to 0^+} T\mathbb{E}_t\left(\frac{S_t}{TA_T}\right) = T\lim_{z\to 0^+}\mathbb{E}_t\left(tz + \int_t^T \frac{S_\nu}{S_t}d\nu\right)^{-1} \\
&= T\mathbb{E}\left(\int_t^T \exp((\alpha-\frac{\sigma^2}{2})(\nu-t)+\sigma B_{\nu-t})d\nu\right)^{-1} \\
&\leq T\mathbb{E}\left(\int_0^\tau e^{\sigma B_\nu}d\nu\right)^{-1}, \text{ for } \tau\in(0,T).
\end{aligned}$$

In a similar way,

$$\begin{aligned}
U^*(0,\tau) &= \lim_{z\to 0^+} U^*(z,\tau) = \lim_{z\to 0^+} T\mathbb{E}_t\left(tz + \int_t^T \frac{S_\nu}{S_T}d\nu\right)^{-1} \\
&= T\mathbb{E}\left(\int_t^T \frac{\exp\left((\alpha-\frac{\sigma^2}{2})(\nu-t)+\sigma B_{\nu-t}\right)}{\exp\left((\alpha-\frac{\sigma^2}{2})(T-t)+\sigma B_{T-t}\right)}d\nu\right)^{-1} \\
&= T\mathbb{E}\left(\int_t^T \exp\left((\alpha-\frac{\sigma^2}{2})(\nu-T)+\sigma(B_{\nu-t}-B_{T-t})\right)d\nu\right)^{-1} \\
&\geq T\mathbb{E}\left(\int_0^\tau e^{\sigma B^*_\nu}d\nu\right)^{-1} \geq \Phi(0,\tau), \text{ for } \tau\in(0,T),
\end{aligned}$$

where $B^*_\nu = B_{(T-t)-\nu} - B_{T-t}$ is also a standard Brownian motion. In addition, we have the equality if and only if $\alpha = \frac{\sigma^2}{2}$. The proof is complete.

Chapter 3

# Penalty Methods for Continuous-Time Portfolio Selection with Proportional Transaction Costs

In this chapter, we will make use of the penalty method to numerically study the optimal trading strategies in the presence of transaction costs. The organization is as follow. In the first section, we present the problem formulation. Section 2 introduces a series of changes of variables that are also helpful for implementing penalty algorithm. In section 3, we describe the penalty method combined with the finite discretization. Section 4 is devoted to the convergence analysis. In section 5, we show that the standard penalty method can be employed in the single risky asset case. Numerical examples are given in section 6 before the chapter ends with an Appendix.

## 3.1 Model formulation

Suppose that there are $N+1$ assets available for investment: a risk free asset (bank account) and $N$ risky assets (stocks). Their prices, denoted by $S_0(t)$ and $S_i(t), i = 1, 2, \ldots, N$ respectively at time $t$, evolve according to the following equations:

$$\begin{aligned} dS_0 &= rS_0 dt, \\ dS_i &= S_i \left(\alpha_i dt + \sigma_i d\mathcal{B}_{it}\right), \end{aligned}$$

where $r > 0$ is the constant risk free rate, $\alpha_i > r$ and $\sigma_i > 0$ are constant expected rate of return and volatility, respectively, of the risky assets. The processes $\{\mathcal{B}_{it};\, t > 0\}$ are standard Brownian motions on a filtered probability space $\left(\mathcal{S}, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathcal{P}\right)$ with $\mathcal{B}_{i0} = 0$ almost surely and constant coefficients of correlation $\rho_{ij}$, namely, $\mathbb{E}(d\mathcal{B}_{it}d\mathcal{B}_{jt}) = \rho_{ij}dt$. We assume that the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is right-continuous and each $\mathscr{F}_t$ contains all $\mathcal{P}$-null sets of $\mathscr{F}$.

Assume that an investor holds a portfolio $X_t = (X_0(t), X_1(t), ..., X_N(t))$, where $X_0(t)$ and $X_i(t)$, $i = 1, 2, ..., N$, are dollar values in bank and the $i^{th}$ risky asset respectively at time $t$. In the presence of transaction costs, the equations describing their evolution are

$$dX_0 = (rX_0 - \kappa C(t))dt - \sum_{i=1}^{N}(1+\lambda_i)dL_i + \sum_{i=1}^{N}(1-\mu_i)dM_i, \qquad (3.1.1)$$

$$dX_i = \alpha_i X_i dt + \sigma_i X_i d\mathcal{B}_{it} + dL_i - dM_i. \qquad (3.1.2)$$

Here $C(t) \geq 0$ is the consumption rate and $\kappa$ is taken to be either 1 or 0 subject to whether consumption is considered or not. $L_i(t)$ and $M_i(t)$ are right-continuous (with left hand limits), nonnegative, and nondecreasing $\{\mathscr{F}_t\}_{t\geq 0}$-adapted processes with $L_i(0) = M_i(0) = 0$, representing cumulative dollar values for the purpose of buying and selling the $i^{th}$ stock respectively. The constants $\lambda_i \in [0, \infty)$ and $\mu_i \in [0, 1), i = 1, 2, \ldots, N$, appearing in these equations account for proportional transaction costs incurred on purchase and sale of the $i^{th}$ stock respectively. We will always assume $\lambda_i + \mu_i > 0$, $i = 1, 2, \ldots, N$.

Due to transaction costs, the investor's net wealth in monetary terms is $X_0 + \sum_{i=1}^{N}[(1-\mu_i)X_i^+ - (1+\lambda_i)X_i^-]$. With the requirement that net wealth at any time $t$ be positive, the solvency region $\mathscr{S}$ is defined as

$$\mathscr{S} = \left\{ x = (x_0, x_1, \ldots, x_N) \in \mathscr{R}^{N+1} : x_0 + \sum_{i=1}^{N}[(1-\mu_i)x_i^+ - (1+\lambda_i)x_i^-] > 0 \right\}.$$

Assume that the investor is given an initial position $x^0 \in \mathscr{S}$ at time 0. An investment and consumption strategy $(\{L_i\}, \{M_i\}, C)$ is admissible for $x$ starting from time $s \in [0, T)$ if $X_t$ given by (3.1.1)-(3.1.2) with $X_s = x$ is in $\mathscr{S}$. We let $\mathcal{A}_s(x)$ be the set of admissible investment strategies starting from time $s$. The investor aims to choose an admissible strategy so as to maximize the discounted expected utility of consumption and terminal wealth $W_T$, that is,

$$\sup_{(\{L_i\},\{M_i\},C)\in\mathcal{A}_0(x^0)} \mathbb{E}_0^{x^0}\left[\int_0^T \kappa e^{-\beta s}u(C(s))ds + e^{-\beta T}u(W_T)\right], \tag{3.1.3}$$

where $u(\cdot)$ is utility function and $\beta > 0$ is the discount rate. We will only confine to CRRA investors whose utility function takes the following form:

$$u(W) = \begin{cases} \frac{W^\gamma}{\gamma}, & \text{if } \gamma \neq 0, \gamma < 1, \\ \log W, & \text{if } \gamma = 0. \end{cases}$$

Define the value function by

$$V(x,t) = \sup_{(L_i,M_i,C)\in\mathcal{A}_t(x)} \mathbb{E}_t^{X_t = x}\left[\int_t^T \kappa e^{-\beta s}u(C(s))ds + e^{-\beta T}u(W_T)\right],$$

for $x \in \mathscr{S}$, $t \in [0, T)$. The problem is indeed a singular control problem for the displacement of the state variables $X_t$ due to control effort might be discontinuous. It turns out that the value function satisfies the following HJB equation [cf. Shreve and Soner (1994) or Alkian, Menaldi and Sulem (1996) or Fleming and Soner

(2006)]:

$$\max\left\{\frac{\partial V}{\partial t}+\mathscr{L}_0 V+\kappa u^*(\frac{\partial V}{\partial x_0}),\ \max_{1\le i\le N}\mathcal{L}_{0i}V,\ \max_{1\le i\le N}\mathcal{M}_{0i}V\right\}=0,\quad y\in\mathscr{S},\ t\in[0,T), \tag{3.1.4}$$

with the terminal condition

$$V(x,T)=u\left(x_0+\sum_{i=1}^{N}\left[(1-\mu_i)x_i^+-(1+\lambda_i)x_i^-\right]\right), \tag{3.1.5}$$

where

$$\begin{aligned}
\mathscr{L}_0 V &= \frac{1}{2}\sum_{i,j=1}^{N}\rho_{ij}\sigma_i\sigma_j x_i x_j\frac{\partial^2 V}{\partial x_i\partial x_j}+\sum_{i=1}^{N}\alpha_i x_i\frac{\partial V}{\partial x_i}+rx_0\frac{\partial V}{\partial x_0}-\beta V,\\
\mathcal{L}_{0i}V &= -(1+\lambda_i)\frac{\partial V}{\partial x_0}+\frac{\partial V}{\partial x_i},\ \mathcal{M}_{0i}V=(1-\mu_i)\frac{\partial V}{\partial x_0}-\frac{\partial V}{\partial x_i},\\
u^*(\nu) &= \max_{c\ge 0}(-c\nu+u(c))=\begin{cases}(\frac{1}{\gamma}-1)\nu^{\frac{\gamma}{\gamma-1}}, & \text{if } \gamma\neq 0,\gamma<1;\\ -\log\nu-1, & \text{if } \gamma=0.\end{cases}.
\end{aligned}$$

## 3.2 Change of variables

Due to the homotheticity of the utility function, it follows that for any positive constant $\rho$,

$$V(\rho x,t)=\begin{cases}\rho^\gamma V(x,t), & \text{if } \gamma\neq 0,\gamma<1;\\ g(t)\log\rho+V(x,t), & \text{if } \gamma=0,\end{cases}$$

where $g(t)=\frac{\kappa(1-e^{-\beta(T-t)})}{\beta}+e^{-\beta(T-t)}$. Take

$$\rho=\frac{1}{\sum_{i=0}^{N}x_i}\text{ and } y_i=\rho x_i,\ i=1,2,...,N^1.$$

Denote $y=(y_1,y_2,...,y_N)$ and $\varphi(y,t)=V(1-\sum_{i=1}^{N}y_i,y_1,y_2,\ldots,y_N,t)$, then

$$V(x_0,x_1,\ldots,x_N,t)=\begin{cases}\rho^\gamma\varphi(y,t), & \text{if } \gamma\neq 0,\gamma<1;\\ g(t)\log\rho+\varphi(y,t), & \text{if } \gamma=0.\end{cases} \tag{3.2.6}$$

[1] $y_i, i=1,2,\ldots,N$, can be consider as the fraction of the $i^{th}$ risky asset over the total wealth. In the following disscusion, we will adopt $y$ as state variable.

It is easy to verify that for $\gamma \neq 0$ and $\gamma < 1$, (3.1.4)-(3.1.5) are reduced to

$$\begin{cases} \max\Big\{\frac{\partial\varphi}{\partial t} + \mathscr{L}_1\varphi + \kappa(\frac{1}{\gamma}-1)\left(\gamma\varphi - \sum_{i=1}^N y_i\frac{\partial\varphi}{\partial y_i}\right)^{\frac{\gamma}{\gamma-1}}, \\ \qquad\qquad \max_{1\le i\le N}\mathcal{L}_{1i}\varphi,\ \max_{1\le i\le N}\mathcal{M}_{1i}\varphi\Big\} = 0, \\ \varphi(y,T) = \frac{\left(1-\sum_{i=1}^N(\mu_i y_i^+ + \lambda_i y_i^-)\right)^\gamma}{\gamma}, \quad y\in\Omega^N,\ t\in[0,T), \end{cases} \tag{3.2.7}$$

where $\Omega^N = \left\{y = (y_1, y_2, \ldots, y_N) \in \mathscr{R}^N : 1 - \sum_{i=1}^N(\mu_i y_i^+ + \lambda_i y_i^-) > 0\right\}$,

$$\begin{aligned} \mathscr{L}_1\varphi &= \sum_{k,l=1}^N a_{k,l}\frac{\partial^2\varphi}{\partial y_k\partial y_l} + \sum_{k=1}^N b_k\frac{\partial\varphi}{\partial y_k} - \theta\gamma\varphi, \\ \mathcal{L}_{1i}\varphi &= \sum_{k=1}^N(\delta_{ik}+\lambda_i y_k)\frac{\partial\varphi}{\partial y_k} - \lambda_i\gamma\varphi,\ \mathcal{M}_{1i}\varphi = \sum_{k=1}^N(-\delta_{ik}+\mu_i y_k)\frac{\partial\varphi}{\partial y_k} - \mu_i\gamma\varphi \end{aligned}$$

and

$$\begin{aligned} a_{k,l} &= y_k y_l\sum_{i,j=1}^N\frac{1}{2}\rho_{ij}\sigma_i\sigma_j(\delta_{il}-y_i)(\delta_{jk}-y_j), \\ b_k &= y_k\sum_{i=1}^N(\delta_{ik}-y_i)\Big[(\alpha_i - r) + \sum_{j=1}^N(\gamma-1)\rho_{ij}\sigma_i\sigma_j y_j\Big], \\ \theta &= \frac{\beta}{\gamma} - \left(r + \sum_{i=1}^N y_i\left(\alpha_i - r - \frac{1-\gamma}{2}\sum_{j=1}^N\rho_{ij}\sigma_i\sigma_j y_j\right)\right). \end{aligned}$$

Here $\delta_{ij}$ represents Kronecker index, i.e. $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise.

The above change of variables is well-known and has been widely adopted, see Davis and Norman (1990) for $N = 1$, and Alkian, Menaldi and Sulem (1996) and Muthuraman and Kuman (2006) for $N = 2$. These authors considered numerical implementation based on (3.2.7). However, numerical oscillation would be caused when applying penalty method to (3.2.7) because the associated penalty terms $K\lambda_i\gamma\varphi$ and $K\mu_i\gamma\varphi$, with $K$ large enough, may result in a singular matrix after discretization in the case of $\gamma < 0$. To overcome the difficulty, we further make the following transformation originally adopted by Dai and Yi (2006):

$$W(y,t) = \frac{\log(\gamma\varphi)}{\gamma}. \tag{3.2.8}$$

It follows

$$\begin{cases} \max\left\{\frac{\partial W}{\partial t} + \mathscr{L}W + \kappa f(W), \max_{1\le i\le N} \mathcal{L}_i W, \max_{1\le i\le N} \mathcal{M}_i W\right\} = 0, \\ W(y,T) = \log\left(1 - \sum_{i=1}^N (\mu_i y_i^+ + \lambda_i y_i^-)\right), \quad y \in \Omega^N,\ t \in [0,T), \end{cases} \tag{3.2.9}$$

where

$$\begin{aligned} \mathscr{L}W &= \sum_{k,l=1}^N a_{k,l}\left(\frac{\partial^2 W}{\partial y_k \partial y_l} + \gamma \frac{\partial W}{\partial y_k}\frac{\partial W}{\partial y_l}\right) + \sum_{k=1}^N b_k \frac{\partial W}{\partial y_k} - \theta, \\ \mathcal{L}_i W &= \sum_{k=1}^N (\delta_{ik} + \lambda_i y_k)\frac{\partial W}{\partial y_k} - \lambda_i,\ \mathcal{M}_i W = \sum_{k=1}^N (-\delta_{ik} + \mu_i y_k)\frac{\partial W}{\partial y_k} - \mu_i, \\ f(W) &= \left(\frac{1}{\gamma} - 1\right) e^{\frac{\gamma}{\gamma-1}W}\left(1 - \sum_{i=1}^N y_i \frac{\partial W}{\partial y_i}\right)^{\frac{-\gamma}{\gamma-1}}, \end{aligned}$$

and

$$\begin{aligned} a_{k,l} &= y_k y_l \sum_{i,j=1}^N \frac{1}{2}\rho_{ij}\sigma_i\sigma_j(\delta_{il} - y_i)(\delta_{jk} - y_j), \\ b_k &= y_k \sum_{i=1}^N (\delta_{ik} - y_i)\Big[(\alpha_i - r) + \sum_{j=1}^N (\gamma - 1)\rho_{ij}\sigma_i\sigma_j y_j\Big]. \end{aligned}$$

Another advantage of transformation (3.2.8) is that a slight modification of (3.2.9) applies to the case of logarithmic utility. Indeed, for $\gamma = 0$, let

$$W(y,t) = \frac{\varphi(y,t)}{g(t)}, \tag{3.2.10}$$

then it can be verified that $W(y,t)$ satisfies (3.2.9) with

$$\begin{aligned} f(W) &= -\frac{1 + \log g(t) + \log\left(1 - \sum_{i=1}^N y_i \frac{\partial W}{\partial y_i}\right) + W}{g(t)}, \\ \theta &= -r - \sum_{i=1}^N y_i\left(\alpha_i - r - \frac{1}{2}\sum_{j=1}^N \rho_{ij}\sigma_i\sigma_j y_j\right). \end{aligned}$$

Define

$$\begin{aligned} BR_i &= \{(y,t) \in \Omega^N \times [0,T) : \mathcal{L}_i W = 0\}, \\ SR_i &= \{(y,t) \in \Omega^N \times [0,T) : \mathcal{M}_i W = 0\}, \\ NTR_i &= \Omega^N \times [0,T) \backslash (BR_i \cup SR_i),\ \text{and } NTR = \cap_{i=1}^N NTR_{i.} \end{aligned}$$

Then, $NTR$ represents the no transaction region, $BR_i$, $SR_i$ and $NTR_i$ represent the buy region, sell region and no-trading region with regard to the $i^{th}$ stock, respectively.

## 3.3 The penalty method

Let us consider a penalty approximation to (3.2.9):

$$\begin{cases} -\frac{\partial W}{\partial t} - \mathscr{L}W - \kappa f(W) = K\sum_{i=1}^{N}\left[(\mathcal{L}_i W)^+ + (\mathcal{M}_i W)^+\right], \ y \in \Omega^N, \ t \in [0,T), \\ W(y,T) = \log\left(1 - \sum_{i=1}^{N}(\mu_i y_i^+ + \lambda_i y_i^-)\right), \end{cases} \tag{3.3.1}$$

where $K$ is a large positive constant. (3.3.1) is expected to converge to (3.2.9) as $K$ goes to infinity.

### 3.3.1 The control problem associated with (3.3.1).

The approximation (3.3.1) corresponds to the original problem (3.1.3) restricted to a class of admissible policies: $L_{it}$ and $M_{it}$ are absolutely continuous with bounded derivatives, i.e.

$$L_{it} = \int_0^t l_{is} ds, \ M_{it} = \int_0^t m_{is} ds, \ 0 \le l_{is} \le K, \ 0 \le m_{is} \le K, \ \text{for } i = 1, 2, \ldots, N.$$

Indeed, it is easy to see that the associated value function, denoted by $\overline{V}(x,t)$, satisfies (taking $\gamma \neq 0$ as an example)

$$\max_{(l_i, m_i, C)} \left\{ \frac{\partial \overline{V}}{\partial t} + \mathscr{L}_0 \overline{V} + \kappa \frac{C^\gamma}{\gamma} + \sum_{i=1}^{N} \left( l_i \mathcal{L}_{0i} \overline{V} + m_i \mathcal{M}_{0i} \overline{V} \right) \right\} = 0 \text{ in } \mathscr{S} \times [0,T).$$

The optimal strategies are

$$C = \left(\frac{\partial \overline{V}}{\partial x_0}\right)^{\frac{1}{\gamma-1}}, \; l_i = \begin{cases} K & \text{if } \frac{\partial \overline{V}}{\partial x_i} - (1+\lambda_i)\frac{\partial \overline{V}}{\partial x_0} \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and } m_i = \begin{cases} K & \text{if } (1-\mu_i)\frac{\partial \overline{V}}{\partial x_0} - \frac{\partial \overline{V}}{\partial x_i} \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

which yields

$$\frac{\partial \overline{V}}{\partial t} + \mathscr{L}_0 \overline{V} + \kappa u^* \left(\frac{\partial \overline{V}}{\partial x_0}\right) + K \sum_{i=1}^{N} \left[ \left(\mathcal{L}_{0i} \overline{V}\right)^+ + \left(\mathcal{M}_{0i} \overline{V}\right)^+ \right] = 0 \text{ in } \mathscr{S} \times [0,T). \quad (3.3.2)$$

Applying the transformations (3.2.6), (3.2.8) and (3.2.10), (3.3.2) with terminal condition is reduced to the penalty approximation (3.3.1).

### 3.3.2 Computation domain and boundary conditions

We are most interested in the $NTR$ that is much smaller than the solvency region. The intuition behind this is that it is not optimal for a risk averse investor to buy or short sell too much in one stock when the assets are not perfectly corrected. Then, we confine to

$$D^N = [y_{1\underline{m}}, y_{1\overline{m}}] \times \ldots \times [y_{N\underline{m}}, y_{N\overline{m}}] \subset \Omega^N$$

and impose the boundary conditions as follows:

$$\mathcal{L}_i W = 0 \text{ at } y_i = y_{i\underline{m}}, \; i = 1, 2, \ldots, N, \quad (3.3.3)$$

$$\mathcal{M}_i W = 0 \text{ at } y_i = y_{i\overline{m}}, \; i = 1, 2, \ldots, N, \quad (3.3.4)$$

which imply buying the $i^{th}$ risky asset at $y_{i\underline{m}}$, i.e. the fraction of the asset is too small and selling the $i^{th}$ risky asset at $y_{i\overline{m}}$, i.e. the fraction of the asset is too high. Figure 3.1 shows the computation domain (the rectangle) and boundary conditions when $N = 2$ and $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1\%$.

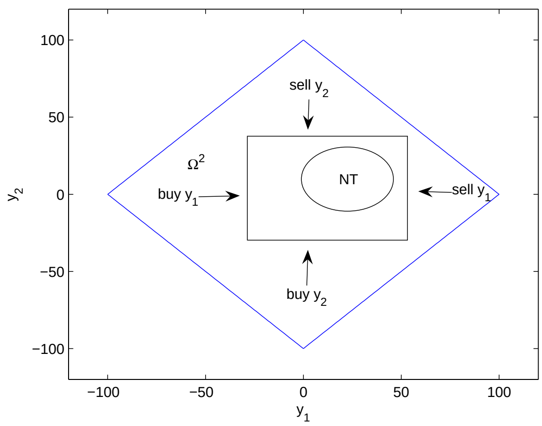


Figure 3.1: The computation domain and boundary conditions when $N = 2$

### 3.3.3 Finite difference discretization

Let $\Delta t$ be the time step, $\overline{n} = \frac{T}{\Delta t}$ and $t_n = n\Delta t$. Assume that we have a uniform grid, denoted by $D_h^N$. Let $h_i$ and $e_i$ be respectively the mesh size and the associated unit vector in $y_i$ direction. For illustration, let us perform discretization at $(y, t_n)$ with $y \in D_h^N$ and denote $W(y, t_n) = W^n(y)$.

The first order terms $\frac{\partial W}{\partial y_i}$ are discretized by the upwind scheme. For example,

$$b_i \frac{\partial W}{\partial y_i} \sim \begin{cases} b_i \frac{W^n(y+h_i e_i) - W^n(y)}{h_i} & \text{if } b_i > 0; \\ b_i \frac{W^n(y) - W^n(y-h_i e_i)}{h_i} & \text{if } b_i < 0. \end{cases}$$

Since the upwind scheme is only of the first order accuracy, we use the fully implicit approximation to the temporal term

$$\frac{\partial W}{\partial t} \sim \frac{W^{n+1}(y) - W^n(y)}{\Delta t}.$$

The term $\frac{\partial^2 W}{\partial y_i^2}$ is discretized as usual:

$$\frac{\partial^2 W}{\partial y_i^2} \sim \frac{W^n(y + h_i e_i) - 2W^n(y) + W^n(y - h_i e_i)}{h_i^2}.$$

As in Clift and Forsyth (2008), we discretize the cross terms $\frac{\partial^2 W}{\partial y_i \partial y_j}$, $i \neq j$, as follows:

$$
\begin{aligned}
a_{ij}\frac{\partial^2 W}{\partial y_i \partial y_j} \sim \frac{a_{ij}}{2h_i h_j}\Big[&W^n(y+h_ie_i+h_je_j)+W^n(y-h_ie_i-h_je_j)+2W^n(y)\\
&-W^n(y+h_ie_i)-W^n(y-h_ie_i)-W(y+h_je_j)-W(y-h_je_j)\Big] \text{ if } a_{ij}<0;
\end{aligned}
$$

$$
\begin{aligned}
a_{ij}\frac{\partial^2 W}{\partial y_i \partial y_j} \sim -\frac{a_{ij}}{2h_i h_j}\Big[&W^n(y+h_ie_i-h_je_j)+W^n(y-h_ie_i+h_je_j)+2W^n(y)\\
&-W^n(y+h_ie_i)-W^n(y-h_ie_i)-W^n(y+h_je_j)-W^n(y-h_je_j)\Big] \text{ if } a_{ij}>0.
\end{aligned}
$$

### 3.3.4 Newton iteration for nonlinear terms

(3.3.1) contains several nonlinear terms: the penalty terms, the nonlinear terms in $\mathscr{L}W$ due to transformation (3.2.8) if $\gamma \neq 0$, and the consumption term $f(W)$ if $\kappa = 1$. All these terms can be linearized by Newton iteration. Especially, owing to their non-smoothness, we linearize the penalty term using the following non-smooth Newton iteration as in Forsyth and Vetzal (2002). For illustration, let us take $K(\mathcal{L}_iW)^+$ for example. Assume that $W^l$ be the $l^{th}$ estimate for $W$. Then we linearize $K(\mathcal{L}_iW)^+$ as

$$
\begin{cases}
K\mathcal{L}_iW & \text{if } \mathcal{L}_iW^l \geq 0,\\
0 & \text{if } \mathcal{L}_iW^l < 0.
\end{cases}
$$

Here we emphasize that the upwind scheme should be applied for discretizing the first order terms in $\mathcal{L}_iW$.

## 3.4 Convergence analysis

In this section, we will focus on the convergence analysis, where we only confine to the case of single risky asset with log utility and without consumption, namely, $N = 1$, $\kappa = 0$ and $\gamma = 0$. To simplify the notation, we surrender the subscript due to the state variable, then $y = y_1, h = h_1, a = a_{11}, b = b_1$ and so on. In addition, the subscript $k$ below means $y = kh$.

The discrete scheme can be written as follows:

$$\begin{cases} (FW^n)_k =: -W_k^{n+1} + W_k^n + (AW^n)_k + \theta_k \Delta t \\ \qquad\qquad = P_{1k}^n \left( \frac{(E^+W^n)_k}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_k \right) + P_{2k}^n \left( \left(\frac{-\mu}{1-\mu y}\right)_k - \frac{(E^-W^n)_k}{h} \right), \\ W_k^{\overline{n}} = \log(1-(\mu y^+ + \lambda y^-))_k, \text{ for } k = \underline{m}+1, ..., \overline{m}-1, \; n = 0, ..., \overline{n}-1, \end{cases} \tag{3.4.1}$$

where

$$(AW^n)_k = -\frac{\Delta t}{h^2} a_k \left[ (E^+W^n)_k - (E^-W^n)_k \right] + \frac{\Delta t}{h} b_k I_{\{b_k<0\}} (E^+W^n)_k + \frac{\Delta t}{h} b_k I_{\{b_k>0\}} (E^-W^n)_k,$$

$$(E^+W^n)_k = W_{k+1}^n - W_k^n \quad \text{and} \quad (E^-W^n)_k = W_k^n - W_{k-1}^n,$$

$$P_{1k}^n = \begin{cases} Large, & \text{if } \frac{(E^+W^n)_k}{h} > \left(\frac{\lambda}{1+\lambda y}\right)_k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.2}$$

$$P_{2k}^n = \begin{cases} Large, & \text{if } \left(\frac{-\mu}{1-\mu y}\right)_k > \frac{(E^-W^n)_k}{h}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.3}$$

with

$$\begin{aligned} a_k &= \frac{1}{2}\sigma^2 \left(y^2(1-y)^2\right)_k, \quad b_k = \left((\alpha - r - \sigma^2 y) y (1-y)\right)_k, \\ \theta_k &= -\left(r + (\alpha - r) y - \frac{\sigma^2}{2} y^2\right)_k, \quad Large = K\Delta t. \end{aligned}$$

The first and last rows of $A$ will have to be modified to take into account the boundary conditions:

$$\frac{(E^+W^n)_k}{h} = \left(\frac{\lambda}{1+\lambda y}\right)_k, \text{ at } k = \underline{m}; \quad \frac{(E^-W^n)_k}{h} = \left(\frac{-\mu}{1-\mu y}\right)_k, \text{ at } k = \overline{m}.$$

The discrete form (3.2.9) can be written as

$$\begin{aligned} (FW^n)_k &\geq 0 \\ \left(\frac{\lambda}{1+\lambda y}\right)_k - \frac{(E^+W^n)_k}{h} &\geq 0 \\ \frac{(E^-W^n)_k}{h} - \left(\frac{-\mu}{1-\mu y}\right)_k &\geq 0 \end{aligned}$$

$$(FW_k^n = 0) \vee \left( \left( \frac{\lambda}{1+\lambda y} \right)_k = \frac{(E^+W^n)_k}{h} \right) \vee \left( \frac{(E^-W^n)_k}{h} = \left( \frac{-\mu}{1-\mu y} \right)_k \right) \quad (3.4.4)$$
$$W_k^{\overline{n}} = \log(1-(\mu y^+ + \lambda y^-))_k.$$

Here $(\cdot) \vee (\cdot) \vee (\cdot)$ denotes that at least one holds.

We aim to show that as $Large \to +\infty$, the discrete solution of (3.4.1) converges to (3.4.4). Thus, it suffices to prove the following theorem.

**Theorem 3.4.1.** *(Error in the penalty formulation) If $\frac{\Delta t}{h} < const.$, as $\Delta t, h \to 0$, then the penalty method for (3.4.1) solves*

$$FW_k^n \geq 0 \quad (3.4.5)$$
$$\left( \frac{\lambda}{1+\lambda y} \right)_k - \frac{(E^+W^n)_k}{h} \geq -\frac{C_0}{Large} \quad (3.4.6)$$
$$\frac{(E^-W^n)_k}{h} - \left( \frac{-\mu}{1-\mu y} \right)_k \geq -\frac{C_0}{Large} \quad (3.4.7)$$

$$(FW_k^n = 0) \vee \left( \left| \left( \frac{\lambda}{1+\lambda y} \right)_k - \frac{(E^+W^n)_k}{h} \right| \leq \frac{C_0}{Large} \right) \vee \left( \left| \frac{(E^-W^n)_k}{h} - \left( \frac{-\mu}{1-\mu y} \right)_k \right| \leq \frac{C_0}{Large} \right) \quad (3.4.8)$$

$$W_k^{\overline{n}} = \log(1-(\mu y^+ + \lambda y^-))_k,$$

*where constant $C_0 > 0$ is independent of $K, \Delta t, h, Large$.*

Let us proceed with one lemma.

**Lemma 3.4.2.** *(Bounds for discrete solution) Let $W_k^n$ be the solution to (3.4.1). Then*

$$-\|\theta\|_\infty T + \|W(T,y)\|_\infty \leq W_k^n \leq \|\theta\|_\infty T + \|W(T,y)\|_\infty, \text{ for all } n,\ k, \quad (3.4.9)$$

*where $\|\cdot\|_\infty$ refers to the $L_\infty$ norm.*

**Proof**. It is easy to see that $U_k^n = -\|\theta\|_\infty (T-t_n) - \|W(T,y)\|_\infty$ satisfies

$$-U_k^{n+1} + U_k^n + (AU^n)_k + \theta_k \Delta t \leq 0, \text{ and } U_k^{\overline{n}} \leq W_k^{\overline{n}}, \text{ for all } k,\ n. \quad (3.4.10)$$

Note that $A$ is an $M$-matrix. Due to the discrete maximum principle, we get the left hand side inequality.

Now let us prove the right hand side inequality. First, we prove

$$W_k^n \le U_k^n \text{ for all } k \text{ and } n \tag{3.4.11}$$

provided that $U$ satisfies

$$(FU^n)_k \ge 0, \tag{3.4.12}$$

$$\frac{(E^+U^n)_k}{h} \le \left(\frac{\lambda}{1+\lambda y}\right)_k \quad \text{and} \quad \frac{(E^-U^n)_k}{h} \ge \left(\frac{-\mu}{1-\mu y}\right)_k, \tag{3.4.13}$$

$$U_k^{\overline{n}} \ge W_k^{\overline{n}}, \tag{3.4.14}$$

for all $k$ and $n$. Suppose not, there exists a node $(k_0, n_0)$, such that $W_{k_0}^{n_0} - U_{k_0}^{n_0} > 0$. Without loss of generality, we assume

$$W_{k_0}^{n_0} - U_{k_0}^{n_0} = \max_{\{(k,n)\}} \{W_k^n - U_k^n\} > 0.$$

Moreover, we can assume $n_0$ is the maximum index of the nodes, if there are more than one maximum point.

Since $W_{k_0}^{n_0} - U_{k_0}^{n_0} \ge W_{k_0-1}^{n_0} - U_{k_0-1}^{n_0}$ and $W_{k_0}^{n_0} - U_{k_0}^{n_0} \ge W_{k_0+1}^{n_0} - U_{k_0+1}^{n_0}$, we deduce that

$$\frac{(E^+W^{n_0})_{k_0}}{h} = \frac{W_{k_0+1}^{n_0} - W_{k_0}^{n_0}}{h} \le \frac{U_{k_0+1}^{n_0} - U_{k_0}^{n_0}}{h} \le \left(\frac{\lambda}{1+\lambda y}\right)_{k_0}, \tag{3.4.15}$$

$$\frac{(E^-W^{n_0})_{k_0}}{h} = \frac{W_{k_0}^{n_0} - W_{k_0-1}^{n_0}}{h} \ge \frac{U_{k_0}^{n_0} - U_{k_0-1}^{n_0}}{h} \ge \left(\frac{-\mu}{1-\mu y}\right)_{k_0}. \tag{3.4.16}$$

Then, according to the terminal condition and boundary conditions, we are able to choose an interior node $(k_0, n_0)$, i.e. $\underline{m} < k_0 < \overline{m}$, and $0 \le n_0 < \overline{n}$.

Subtracting (3.4.12) from (3.4.1) at the node $(k_0, n_0)$, we have

$$\begin{aligned}(W^{n_0} - U^{n_0})_{k_0} + (A(W^{n_0} - U^{n_0}))_{k_0} &\le (W^{n_0+1} - U^{n_0+1})_{k_0} \\ &\quad + P_{1k_0}^{n_0}\left(\frac{(E^+W^{n_0})_{k_0}}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_{k_0}\right) \\ &\quad + P_{2k_0}^{n_0}\left(\left(\frac{-\mu}{1-\mu y}\right)_{k_0} - \frac{(E^-W^{n_0})_{k_0}}{h}\right) \\ &= (W^{n_0+1} - U^{n_0+1})_{k_0}, \qquad (3.4.17)\end{aligned}$$

where the equality is due to (3.4.15)-(3.4.16) and the definition of $P_{1k_0}^{n_0}$ and $P_{2k_0}^{n_0}$. Since $A$ is an $M$-matrix, $(A(W^{n_0} - U^{n_0}))_{k_0} \geq 0$. Thus,

$$(W^{n_0} - U^{n_0})_{k_0} \le (W^{n_0+1} - U^{n_0+1})_{k_0},$$

which is in contradiction with the selection of $n_0$. Then, (3.4.11) follows.

It is easy to verified that $U_k^n = \|\theta\|_\infty(T - t_n) + \|W(T, y)\|_\infty$, for all $k$, $n$, satisfies (3.4.12)-(3.4.14). This completes the proof.

**Proof of Theorem 3.4.1**. To establish the satisfaction of (3.4.5)-(3.4.8), it suffices to show that

$$P_{1k}^n\left(\frac{(E^+W^n)_k}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_k\right) \le C_0, \qquad (3.4.18)$$

$$P_{2k}^n\left(\left(\frac{-\mu}{1-\mu y}\right)_k - \frac{(E^-W^n)_k}{h}\right) \le C_0, \qquad (3.4.19)$$

where $C_0$ is independent of $K, \Delta t, h$. We will only show (3.4.18), and the proof of (3.4.19) is similar.

Let $(n, k_0)$ denote the node at which the penalty term

$$P_{1k}^n\left(\frac{(E^+W^n)_k}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_k\right)$$

achieves the maximum, then we can infer that

$$\frac{(E^+W^n)_{k_0}}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_{k_0} = \max_k\left\{\frac{(E^+W^n)_k}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_k\right\} > 0. \qquad (3.4.20)$$

It follows

$$\frac{(E^+W^n)_{k_0-1}}{h} - \frac{(E^+W^n)_{k_0}}{h} \le \left(\frac{\lambda}{1+\lambda y}\right)_{k_0-1} - \left(\frac{\lambda}{1+\lambda y}\right)_{k_0},$$

and thus

$$\begin{aligned}
(AW^n)_{k_0} &= \frac{\Delta t}{h} a_{k_0} \left(\frac{(E^+W^n)_{k_0-1}}{h} - \frac{(E^+W^n)_{k_0}}{h}\right) + \Delta t b_{k_0} \mathrm{I}_{\{b_{k_0}<0\}} \frac{(E^+W^n)_{k_0}}{h} \\
&\quad + \Delta t b_{k_0} \mathrm{I}_{\{b_{k_0}>0\}} \left[\left(\frac{\lambda}{1+\lambda y}\right)_{k_0-1} + \frac{(E^+W^n)_{k_0-1}}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_{k_0-1}\right] \\
&\le \frac{\Delta t}{h} a_{k_0} \left[\left(\frac{\lambda}{1+\lambda y}\right)_{k_0-1} - \left(\frac{\lambda}{1+\lambda y}\right)_{k_0}\right] + \Delta t b_{k_0} \mathrm{I}_{\{b_{k_0}<0\}} \left(\frac{\lambda}{1+\lambda y}\right)_{k_0} \\
&\quad + \Delta t b_{k_0} \mathrm{I}_{\{b_{k_0}>0\}} \left(\frac{\lambda}{1+\lambda y}\right)_{k_0-1} + \Delta t b_{k_0} \mathrm{I}_{\{b_{k_0}>0\}} \left(\frac{(E^+W^n)_{k_0}}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_{k_0}\right) \\
&\le 2\|a\|_\infty \|\frac{\lambda}{1+\lambda y}\|_\infty \frac{\Delta t}{h} + \|b\|_\infty \|\frac{\lambda}{1+\lambda y}\|_\infty \Delta t \\
&\quad + \|b\|_\infty \Delta t \left(\frac{(E^+W^n)_{k_0}}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_{k_0}\right).
\end{aligned}$$

Thanks to (3.4.1), we then obtain

$$\begin{aligned}
&(P^n_{1k_0} - \|b\|_\infty \Delta t) \left(\frac{(E^+W^n)_{k_0}}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_{k_0}\right) \\
\le\ & -W^{n+1}_{k_0} + W^n_{k_0} + (AW^n)_{k_0} + \theta_{k_0}\Delta t - \|b\|_\infty \Delta t \left(\frac{(E^+W^n)_{k_0}}{h} - \left(\frac{\lambda}{1+\lambda y}\right)_{k_0}\right) \\
\le\ & C_1 + C_2 + \|\theta\|_\infty \Delta t + 2\|a\|_\infty \|\frac{\lambda}{1+\lambda y}\|_\infty \frac{\Delta t}{h} + \|b\|_\infty \|\frac{\lambda}{1+\lambda y}\|_\infty \Delta t \\
\le\ & \frac{C_0}{2}
\end{aligned}$$

(3.4.18) then follows by choosing $K > 2\|b\|_\infty$. The proof is complete.

Now we move to the convergence of nonlinear iteration. Let $W^{n,l}$ be the $l^{th}$ estimate for $W^n$, and $W^{n,0} = W^{n+1}$. For notational convenience, we define

$$P_1^{n,l} = P_1^n(W^{n,l}) \quad \text{and} \quad P_2^{n,l} = P_2^n(W^{n,l}).$$

The iteration process for the nonlinear system (3.4.1)is as follows.

$$\text{For } l = 0, 1, \dots \text{ until convergence}$$

$$[(I+M) - \frac{1}{h}P_1^{n,l}E^+ + \frac{1}{h}P_2^{n,l}E^-]W^{n,l+1} = W^{n+1} - \theta\Delta t - P_1^{n,l}\frac{\lambda}{1+\lambda y} - P_2^{n,l}\frac{\mu}{1-\mu y}, \tag{3.4.21}$$

$$\text{If} \quad \frac{\|W^{n,l+1} - W^{n,l}\|_\infty}{\max(1, \|W^{n,l}\|_\infty)} < tol, \text{ quit.}$$

**Theorem 3.4.3.** *(Convergence of the nonlinear iteration) The algorithm for the nonlinear iteration scheme has the following properties.*

*i) The iterations converge monotonically, i.e.* $W^{n,l+1} \geq W^{n,l}$ *for* $l \geq 1$.

*ii) The nonlinear iteration (3.4.21) converges to the unique solution to equation (3.4.1), for any initial iterate value* $W^{n,0}$.

The proof is placed in Appendix 3.7.1, which is similar to Forsyth and Vetzal (2002). In contrast to Forsyth and Vetzal (2002), we are unable to prove the so-called "finite termination of iteration" due to the gradient constraints. However, our algorithm still converges for a given tolerance owing to the boundedness of discrete solution (Lemma 3.4.2) and monotone convergence.

## 3.5 The standard penalty method

At some occasions a singular stochastic control problem has a connection with an optimal stopping problem [cf. Karatzas and Shreve (1984)]. In other words, the variational inequality with gradient constraints arising from a singular stochastic control problem can be reduced to a standard variational inequality (i.e. complimentary problem or obstacle problem) in some cases, which enables us to make use of the standard penalty methods proposed by Forsyth and Vezval (2002) and Dai, Kwok and You (2007). In fact, Dai and Yi (2006) and Dai et al. (2007) have proved such a reduction for $N = 1$. Indeed, let

$$v = w_y, \tag{3.5.1}$$

which proves to satisfy the following double obstacle problem[2]:

$$\begin{cases} \min\left\{\max\left\{-v_t - \mathcal{T}v - \kappa\overline{f}(w), v - \frac{\lambda}{1+\lambda y}\right\}, v + \frac{\mu}{1-\mu y}\right\} = 0, \\ v(y,T) = -\frac{\mu}{1-\mu y}\ ,\ y \in [0, \frac{1}{\mu}),\ t \in [0,T). \end{cases} \tag{3.5.2}$$

Here

$$\begin{aligned} \mathcal{T}v &= \frac{1}{2}\sigma^2 y^2(1-y)^2 v_{yy} + \left[\alpha - r + (\gamma-1)\sigma^2 y + (1-2y)\sigma^2\right] y(1-y)v_y \\ &\quad + \left[(\alpha-r)(1-2y) + (\gamma-1)\sigma^2 y(2-3y)\right] v + \left[\alpha - r + (\gamma-1)\sigma^2 y\right] \\ &\quad + \gamma\sigma^2 y(1-y)v\left[(1-2y)v + y(1-y)v_y\right], \\ \overline{f}(w) &= \begin{cases} e^{\frac{\gamma}{\gamma-1}w}(1-yv)^{\frac{1}{\gamma-1}} y(v_y + v^2) & \text{if } \gamma \neq 0, \gamma < 1, \\ \frac{y}{g(t)(1-yv)}(v^2 + v_y) & \text{if } \gamma = 0. \end{cases} \end{aligned}$$

where $-\frac{\mu}{1-\mu y}$ and $\frac{\lambda}{1+\lambda y}$ are called the lower obstacle and the upper obstacle, respectively. Note that we only need to consider $y \geq 0$ because it can be shown that short selling is suboptimal in the case of $\alpha > r$ and $N = 1$ [see for example, Shreve and Soner (1994)].

Since the lower obstacle $-\frac{\mu}{1-\mu y}$ tends to infinity as $y \to \frac{1}{\mu}$, we impose a boundary condition

$$v(y,t) = -\frac{\mu}{1-\mu y} \text{ at } y = \frac{1}{\mu} - \epsilon \text{ with } 0 < \epsilon << 1. \tag{3.5.3}$$

The condition is deduced from the theoretical analysis in Dai and Yi (2006) and Dai et al. (2007) that the selling boundary never hits $y = \frac{1}{\mu}$.

It is seen that at $y = 0$, (3.5.2) reduces to

$$\begin{cases} \min\left\{\max\left\{-v_t - (\alpha-r)v - (\alpha-r), v - \lambda\right\}, v + \mu\right\} = 0, \\ v(0,T) = -\mu,\ \text{in } t \in [0,T), \end{cases}$$

[2] In Dai and Yi (2006) and Dai et al. (2007), they used a different state variable and the resulting double obstacle problem is slightly different.

solving which we obtain the boundary condition at $y = 0$:

$$v(0,t) = v(0,t) = \min\{(1-\mu)e^{(\alpha-r)(T-t)} - 1, \lambda\}, \forall t \in [0,T). \tag{3.5.4}$$

Hence, we will use the following penalty approximation:

$$\begin{cases} -v_t - \mathcal{T}v - \kappa\overline{f}(w) = -K(v - \frac{\lambda}{1+\lambda y})^+ + K(-\frac{\mu}{1-\mu y} - v)^+, \\ v(y,T) = -\frac{\mu}{1-\mu y}, \text{ in } y \in (0, \frac{1}{\mu} - \epsilon), \ t \in [0,T), \end{cases} \tag{3.5.5}$$

with the boundary conditions (3.5.3)-(3.5.4). The discretization is similar to that in Forsyth and Vetzal (2002) or Dai, Kwok and You (2007). We highlight that the Crank-Nicolson scheme will be used because the current penalty terms do not involve the first order terms.

In the following we will discuss the implementation of numerical methods respectively for $\kappa = 0$ and $\kappa = 1$. Without loss of generality, we only confine to $\gamma \neq 0$, $\gamma < 1$, and the case of $\gamma = 0$ is similar[3].

### 3.5.1 No-consumption case

In this case, at $y = 1$, (3.5.2) reduces to

$$\begin{cases} \min\left\{\max\left\{-v_t + (\alpha - r - (1-\gamma)\sigma^2)v - (\alpha - r - (1-\gamma)\sigma^2), v - \frac{\lambda}{1+\lambda}\right\}, v + \frac{\mu}{1-\mu}\right\} \\ \hfill = 0, \\ v(1,T) = -\frac{\mu}{1-\mu}, \text{ in } t \in [0,T), \end{cases}$$

which yields

$$v(1,t) = \max\left\{\min\left\{1 - \frac{1}{1-\mu}e^{-(\alpha-r-(1-\gamma)\sigma^2)(T-t)}, \frac{\lambda}{1+\lambda}\right\}, -\frac{\mu}{1-\mu}\right\}. \tag{3.5.6}$$

In terms of (3.5.6), we can solve (3.5.5) separately in $\{0 < y < 1\}$ and $\{1 < y < \frac{1}{\mu} - \epsilon\}$, which significantly reduces the size of computations.

[3]In the consumption case, the case of $\gamma = 0$ is simpler because the double obstacle problem (3.5.2) becomes a self-contained system.

### 3.5.2 Consumption case

In this case, at $y=1$, there is no explicit solution due to the presence of $\overline{f}(w)$. As a consequence, we have to solve the problem in $\left\{0<y<\frac{1}{\mu}-\epsilon\right\}$. Moreover, (3.5.2) is not a self-contained system for $w$ is involved. Fortunately, Dai et al. (2007) derived a relation between $w$ and $v$. It can be shown that there exists $y_s(t)$ such that

$$SR=\left\{(y,t)\in\Omega^1\times[0,T):y\geq y_s(t)\right\}.$$

From (3.5.1), we can write

$$w(y,t)=A(t)+\log(1-\mu y_s(t))-\int_y^{y_s(t)}v(\xi,t)d\xi, \tag{3.5.7}$$

where $A(t)$ is to be determined. Clearly $A(T)=0$. It is shown in Dai et al. (2007) that

$$A(t)=\frac{1-\gamma}{\gamma}\log\left(e^{\frac{\gamma}{1-\gamma}\int_t^T h(y_s(\zeta))d\zeta}\left(1+\int_t^T e^{-\frac{\gamma}{1-\gamma}\int_\tau^T h(y_s(\zeta))d\zeta}d\tau\right)\right), \tag{3.5.8}$$

where $h(y)=\frac{1}{(1-\mu y)^2}\left[\frac{\gamma-1}{2}\sigma^2y^2(1-\mu)^2+\Big(\alpha y(1-\mu)+r(1-y)\Big)(1-\mu y)\right]-\frac{\beta}{\gamma}$. A brief derivation of (3.5.8) is placed in Appendix 3.7.2.

Let $v^{n,l}(\cdot)$ and $w^{n,l}(\cdot)$ be the $l^{th}$ discrete solutions at time $t_n$. In terms of (3.5.7), we can have an iterative algorithm as follows.

Step 1: At time step $t=t_n$, start off with an initial guess of $w^n$, denoted by $w^{n,0}$.

Step 2: Find $v^{n,l+1}$ in virtue of the penalty method for (3.5.5) with $w=w^{n,l}$.

Step 3: Compute the corresponding boundary

$$y_s^{l+1}(t_n)=\min\left\{y\in(0,\frac{1}{\mu}-\epsilon):v^{n,l+1}(y,t_n)\leq-\frac{\mu}{1-\mu y}\right\}.$$

Step 4: Update $A(t_n)$ by (3.5.8), then compute $w^{n,l+1}$ by (3.5.7).

Step 5: Stop if $\frac{\left\|v^{n,l+1}-v^{n,l}\right\|}{\max\{1,\ \left\|v^{n,l}\right\|\}}<tol$. Otherwise, set $l=l+1$ and go back to Step 2.

## 3.6 Numerical results

In this section, we shall provide numerical analysis for both one stock case and two-stock case. First, we will study the effect of penalty parameter $K$ and the convergence rate. Then, we compare trading policies between consumption case and no consumption case. We further examine the effects of the parameter values such as expected return, transaction costs, correlation on the optimal strategies.

### 3.6.1 Penalty parameter $K$ and convergence rate

Let us first look at the convergence as the penalty parameter $K$ goes to infinity. Table 3.1 presents the values of $\varphi(y_M, 0)$ and $v(y_M, 0)$ against varying $K$, computed from the standard penalty method with $N = 1$. Here $y_M$ refers to the "Merton line" in the absence of transaction costs. It is apparent that the values converge as $K$ goes to infinity. Similar convergence for $N = 2$ can be observed from Table 3.2, where the penalty method for variational inequality with gradient constraints is adopted.

Next, we examine the order of convergence of the penalty methods. In Table 3.3, we list the numerical results for $N = 1$ obtained from the standard penalty method with the Crank-Nicolson scheme. When there is no consumption ($\kappa = 0$), the second order of convergence can be observed. When consumption is involved ($\kappa = 1$), the rate of convergence is however slower than the expected rate due to the upwind treatment of the consumption term. Table 3.4 lists the numerical results for $N = 2$ obtained from the penalty method with the fully implicit scheme. The apparent first order of convergence is revealed.

Table 3.1: Test of varying the penalty parameter $K$ on the double obstacle problem ($N = 1$).

| $K$ | $\varphi(y_M, 0)$ | $v(y_M, 0)$ |
|---|---|---|
| 10 | -7.123355 | -0.009888 |
| $10^3$ | -7.123726 | -0.009140 |
| $10^5$ | -7.123796 | -0.009136 |
| $10^6$ | -7.123798 | -0.009136 |
| $10^7$ | -7.123798 | -0.009136 |

Default parameter values: $\alpha = 0.15$, $\sigma = 0.4$, $r = 0.07$, $\beta = 0.1$, $\gamma = -1$, $\lambda = \mu = 0.01$, $T = 2$, $\kappa = 1$, $y_{\underline{m}} = 0$, $y_{\overline{m}} = 1$, $\Delta t = 5 \times 10^{-4}$, $h = 10^{-3}$. $\varphi(\cdot, \cdot)$ and $v(\cdot, \cdot)$ are the numerical solutions to (3.2.7) and (3.5.2). $y_M = \frac{\alpha - r}{(1-\gamma)\sigma^2}$ refers to the "Merton line" in the absence of transaction costs.

Table 3.2: Test of varying the penalty parameter $K$ on the gradient constraint problem ($N = 2$).

| $K$ | $\varphi(y_{1_M}, y_{2_M}, 0)$ |
|---|---|
| 0.1 | -7.104291 |
| 0.5 | -7.097533 |
| 1 | -7.097456 |
| $10^2$ | -7.097435 |
| $10^6$ | -7.097435 |

Default parameter values: $\alpha_1 = 0.15$, $\sigma_1 = 0.4$ $\alpha_2 = 0.12$, $\sigma_2 = 0.3$, $\rho = 0.2$, $r = 0.07$, $\beta = 0.1$, $\gamma = -1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 1$, $y_{1\underline{m}} = y_{2\underline{m}} = 0$, $y_{1\overline{m}} = y_{2\overline{m}} = 0.4$, $\Delta t = 5 \times 10^{-4}$, $h_1 = h_2 = 2 \times 10^{-3}$. $\varphi(\cdot, \cdot, \cdot)$ is the numerical solution to (3.2.7). $y_{i_M} = \frac{1}{1-\rho^2}\left(\frac{\alpha_i - r}{(1-\gamma)\sigma_i^2} - \rho\frac{\alpha_j - r}{(1-\gamma)\sigma_i\sigma_j}\right)$ refers to the "Merton line" of the $i^{th}$ risky asset in the absence of transaction costs, $i, j \in \{1, 2\}, i \neq j$.

Table 3.3: The convergence rate of the standard penalty method with Crank-Nicolson scheme ($N = 1$).

| $\bar{n}$ | $N_y$ | $\|\epsilon\|_\infty, \kappa = 0$ | Ratio | $\|\epsilon\|_\infty, \kappa = 1$ | Ratio |
|---|---|---|---|---|---|
| 600 | 100 | - | - | - | - |
| 1200 | 200 | 2.36e-5 | - | 3.06e-3 | - |
| 2400 | 400 | 9.39e-6 | 2.5 | 1.26e-3 | 2.4 |
| 4800 | 800 | 2.59e-6 | 3.6 | 4.13e-4 | 3.1 |
| 9600 | 1600 | 6.69e-7 | 3.9 | 1.53e-4 | 2.7 |
| 19200 | 3200 | 1.65e-7 | 4.1 | 5.16e-5 | 2.7 |

Default parameter values: $\alpha = 0.15$, $\sigma = 0.4$, $r = 0.07$, $\beta = 0.1$, $\gamma = -1$, $\lambda = \mu = 0.01$, $T = 0.5$, $y_{\underline{m}} = 0$, $y_{\overline{m}} = 1$, $K = \frac{10^3}{\Delta t}$, $\Delta t = \frac{T}{\bar{n}}$, $h = \frac{y_{\overline{m}} - y_{\underline{m}}}{N_y}$. Here, $\|\epsilon\|_\infty$ is the $L_\infty$-norm of the difference in the solution from the coarser gird, "Ratio" is the ratio of changes $\|\epsilon\|_\infty$ on the successive grids.

Table 3.4: The convergence rate of the penalty method with fully implicit scheme ($N = 2$).

| $\bar{n}$ | $N_{y_1}$ | $N_{y_2}$ | $\|\epsilon\|_\infty$ | Ratio |
|---|---|---|---|---|
| 100 | 20 | 20 | - | - |
| 200 | 40 | 40 | 7.96e-4 | - |
| 400 | 80 | 80 | 4.00e-4 | 2.0 |
| 800 | 160 | 160 | 2.00e-4 | 2.0 |
| 1600 | 320 | 320 | 1.00e-4 | 2.0 |

Default parameter values: $\alpha_1 = 0.15$, $\sigma_1 = 0.4$ $\alpha_2 = 0.12$, $\sigma_2 = 0.3$, $\rho = 0.2$, $r = 0.07$, $\beta = 0.1$, $\gamma = -1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 0.6$, $\kappa = 1$, $y_{i\underline{m}} = 0$, $y_{i\overline{m}} = 0.4$, $\Delta t = \frac{T}{\bar{n}}$, $h_i = \frac{y_{i\overline{m}} - y_{i\underline{m}}}{N_{y_i}}$, $K = \frac{10^3}{\Delta t}$, $i \in \{1, 2\}$.

### 3.6.2 Consumption vs no-consumption

We compare the optimal buying (selling) boundary between consumption case and no consumption case with single risky asset in Figure 3.2, which plots the shape of the $BR$, the $SR$ and the $NTR$ in $y$-$t$ plane for both the consumption case and the no-consumption case. It turns out that there are two time-dependent boundaries, one being the optimal buying boundary (the lower) and the other being the selling boundary (the upper), such that the $BR$ is below the buying boundary and the $SR$ is above the selling boundary and the $NTR$ is between them. This indicates that a risk averse investor prefers to buy low and sell high. Observe that the selling (buying) boundary in the consumption case is lower than the counterpart in the no consumption case. The intuition behind is that the investor has to keep a larger fraction of wealth in the bank account to maintain consumption. This is also consistent with the general observation that investors prefer present consumption.

In addition, Figure 3.2 also reveals that it is never optimal to buy the stock provided that the time is greater than a threshold value no matter whether consumption is involved. Apparently, no one would like to buy a stock if there is not enough time to recover the transaction costs incurred. Such a phenomenon, called "no-buying near maturity" feature, was first proved by Liu and Loewenstein (2002) for no-consumption case and Dai et al. (2007) for the consumption case, the threshold value $t_0 = T - \frac{1}{\alpha - r} \log\left(\frac{1+\lambda}{1-\mu}\right) = 2.818$ for the given example.

Dai and Yi (2006) proved that both the optimal buying and selling boundaries are monotonically decreasing with time $t$ in no-consumption case, which is consistent with the typical invest wisdom that the younger investor should allocate more wealth to stocks than the older investor [cf. Liu and Loeweinstein (2002)]. Our numerical results confirm this. Surprisingly, it may not be true in consumption case if the discount factor $\beta$ is big enough. Figure 3.3 presents an example with $\beta = 7$, where the optimal buying boundary in the consumption case is apparently not monotone. One possible reason is that the investor has to balance present consumption and terminal wealth according to the discount factor $\beta$.

Let us move to examine the case of $N = 2$. A time snapshot of the $SR_i, BR_i, NTR_i$, $i = 1, 2$, and $NTR$ is displayed in Figure 3.4. As in the case of single risky asset, the optimal trading strategy is to keep the ratio $(y_1, y_2)$ in the $NTR$ by selling high and buying low. In what follows, we only focus on the $NTR$. Figure 3.5 depicts the time snapshot of the $NTR$ at different time. It can be observed that as time approaches to maturity, the bottom and the left-hand sides of the $NTR$ match $\{y_1 = 0\}$ and $\{y_2 = 0\}$, respectively, which confirms the "no-buying near maturity". In Figure 3.6, we compare the no transaction region between the consumption case and no-consumption case. Same as the single risky asset case, it can be seen that larger fraction of wealth in each asset is allocated in the no-consumption case.

### 3.6.3 Parameter effects

#### The impact of the risky asset return

In Figure 3.7, we plot the optimal buying and selling boundaries with varying $\alpha$. It can be observed that the buying (selling) boundary is increasing with $\alpha$ in both the consumption case and the no-consumption case, which means that the bigger the return rate of the risky asset $\alpha$, the larger the fraction of wealth in the risky asset. If $\alpha = 0.18$, then $\alpha - r < (1 - \gamma)\sigma^2$ and the NTR is contained in the region $\{y < 1\}$ , which implies that leverage is always suboptimal. If $\alpha = 0.3$, then $\alpha - r > (1 - \gamma)\sigma^2$ and the NTR contains part of $\{y > 1\}$ , which indicates that leverage is likely needed. If $\alpha = 0.25$, then $\alpha - r = (1 - \gamma)\sigma^2$ and the selling boundary is exactly $y = 1$. All these results are consistent with the theoretical analysis in Liu and Loewenstein (2002), Dai and Yi (2006) and Dai et al. (2007).

### The impact of risk aversion

We now investigate the impacts of risk aversion and transaction costs on the optimal strategy. Let us only take the consumption case for illustration. By Figure 3.8, we can see that both the optimal buying and selling boundaries are increasing with $\gamma$, or equivalently, decreasing with the index of risk aversion $1-\gamma$. Moreover, the $NTR$ also expands as $\gamma$ increases. The intuition behind this is that a more risk averse investor would like to keep larger fraction of wealth in the bank account and trade more often to reduce the risk.

### The impact of transaction costs

Let us look at the dependence of the optimal trading strategies on the transaction costs. Keeping buying and selling costs equal, Figure 3.9 shows the $NTR$ expands quickly as the transaction costs increase, which means the investor trends to decrease the trading frequency to save transaction costs. In addition, similar to Liu and Loewenstein (2002), we find that the buying boundary is more sensitive to transaction costs than the optimal selling boundary.

### The impact of correlation

The impact of correlation between two stocks is displayed in Figure 3.10 and Figure 3.11. It can be observed that the $NTR$ elongates in the direction $(1,-1)$ and shrinks in the direction $(1,1)$ as positive correlation increases in Figure 3.10. On the contrary, Figure 3.11 shows that the $NTR$ elongates in the direction $(1,1)$ and shrinks in the direction $(1,-1)$ as negative correlation increases. These are the same as what Muthuraman and Kuman (2006) have observed for the infinite horizon problem. Further, we follow Muthuraman and Kuman (2006) to keep the Merton line fixed, then such an impact can be displayed more clear (see Figure 3.12).

# 3.7 Appendix

## 3.7.1 The proof of Theorem 3.4.3

To begin with, we prove the monotone property of (3.4.21). Writing (3.4.21) at the $(l-1)^{th}$ iteration, where $l \geq 1$, gives

$$\left[(I+M) - \frac{1}{h}P_1^{n,l-1}E^+ + \frac{1}{h}P_2^{n,l-1}E^-\right]W^{n,l} = W^{n+1} - \theta\Delta t - P_1^{n,l-1}\frac{\lambda}{1+\lambda y} - P_2^{n,l-1}\frac{\mu}{1-\mu y}. \tag{3.7.1}$$

Note that equation (3.7.1) always has a solution, since

$$\left[(I+M) - \frac{1}{h}P_1^{n,l-1}E^+ + \frac{1}{h}P_2^{n,l-1}E^-\right]$$

is an $M$-matrix.

Subtracting (3.7.1) from (3.4.21), we have

$$\begin{aligned}\left[(I+M) - \frac{1}{h}P_1^{n,l}E^+ + \frac{1}{h}P_2^{n,l}E^-\right](W^{n,l+1} - W^{n,l}) &= \left(P_1^{n,l} - P_1^{n,l-1}\right)\left(\frac{E^+W^{n,l}}{h} - \frac{\lambda}{1+\lambda y}\right) \\ &+ \left(P_2^{n,l} - P_2^{n,l-1}\right)\left(-\frac{\mu}{1-\mu y} - \frac{E^-W^{n,l}}{h}\right) \end{aligned} \tag{3.7.2}$$

Now we examine each of the components of the right hand side of (3.7.2). Observe

$$\left(P_1^{n,l} - P_1^{n,l-1}\right)\left(\frac{1}{h}E^+W^{n,l} - \frac{\lambda}{1+\lambda y}\right) \geq 0$$

and

$$\left(P_2^{n,l} - P_2^{n,l-1}\right)\left(-\frac{\mu}{1-\mu y} - \frac{1}{h}E^-W^{n,l}\right) \geq 0.$$

Therefore, we infer that

$$\left[(I+M) - \frac{1}{h}P_1^{n,l}E^+ + \frac{1}{h}P_2^{n,l}E^-\right](W^{n,l+1} - W^{n,l}) \geq 0.$$

Since $\left[(I+M) - \frac{1}{h}P_1^{n,l}E^+ + \frac{1}{h}P_2^{n,l}E^-\right]$ is an $M$-matrix, it follows that

$$W^{n,l+1} - W^{n,l} \geq 0.$$

Now we show that the solution obtained by the penalty iteration is unique. Suppose there are two solutions $W$ and $\overline{W}$ to the penalized equation( 3.4.1). Then

$$\left[(I+M)-\frac{1}{h}P_1E^+ +\frac{1}{h}P_2E^-\right]W = W^{n+1}-\theta\Delta t - P_1\frac{\lambda}{1+\lambda y}-P_2\frac{\mu}{1-\mu y}, \quad (3.7.3)$$

$$\left[(I+M)-\frac{1}{h}\overline{P}_1E^+ +\frac{1}{h}\overline{P}_2E^-\right]\overline{W} = W^{n+1}-\theta\Delta t - \overline{P}_1\frac{\lambda}{1+\lambda y}-\bar{P}_2\frac{\mu}{1-\mu y}. \quad (3.7.4)$$

Subtracting (3.7.4) from (3.7.3) gives

$$\begin{aligned}\left[(I+M)-\frac{1}{h}P_1E^+ +\frac{1}{h}P_2E^-\right](W-\overline{W}) &= (P_1-\overline{P}_1)\left(\frac{1}{h}E^+\overline{W}-\frac{\lambda}{1+\lambda y}\right)\\ &\quad +(P_2-\overline{P}_2)\left(-\frac{\mu}{1-\mu y}-\frac{1}{h}E^-\overline{W}\right).\end{aligned}$$

Using a similar argument as we used in proving monotone iteration, we obtain $W-\overline{W}\le 0$. Similarly we have $\overline{W}-W\ge 0$, and hence $W=\overline{W}$.

### 3.7.2 Derivation of (3.5.8)

As shown in Dai, et.al (2007), $v(.,t)\in C^1$ and $w(.,t)\in C^2$, and

$$w_t+\mathscr{L}w|_{y=y_s(t)}=0. \quad (3.7.1)$$

Thus,

$$w_y|_{y=y_s(t)} = -\frac{\mu}{1-\mu y_s(t)}, \quad w_{yy}|_{y=y_s(t)} = -\frac{\mu^2}{(1-\mu y_s(t))^2}.$$

Substituting into (3.7.1) gives

$$-A'(t) = -w_t(y_s(t),t) = \mathscr{L}w|_{y=y_s(t)} = \left(\frac{1}{\gamma}-1\right)e^{\frac{\gamma}{\gamma-1}A(t)}+h(y_s(t)). \quad (3.7.2)$$

Solving (3.7.2) with $A(T)=0$, we obtain (3.5.8).

### 3.7.3 Figures in 3.6

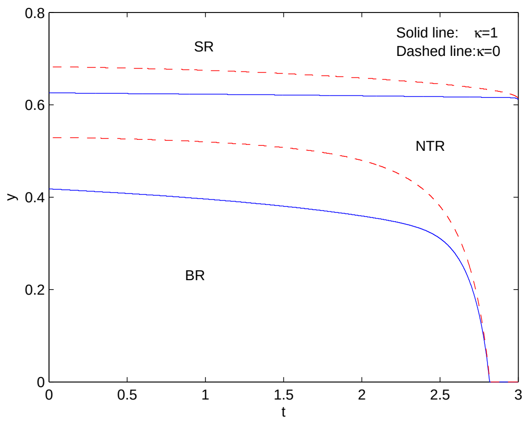


Figure 3.2: Shape of BR, SR and NTR, and comparison of the buying and selling boundaries between the consumption case and the no consumption case ($N = 1$). Default parameter values: $\alpha = 0.18$, $r = 0.07$, $\sigma = 0.3$, $\gamma = -1$, $\beta = 0.1$, $\lambda = \mu = 0.01$, $T = 3$.

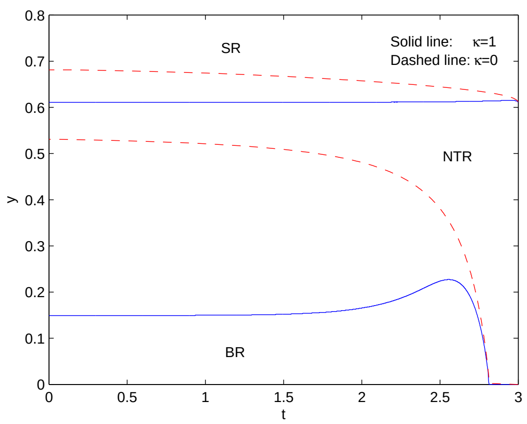


Figure 3.3: An example of non-monotone buying boundary in the consumption case ($N = 1$). Default parameter values: $\alpha = 0.18$, $r = 0.07$, $\sigma = 0.3$, $\gamma = -1$, $\beta = 7$, $\lambda = \mu = 0.01$, $T = 3$.

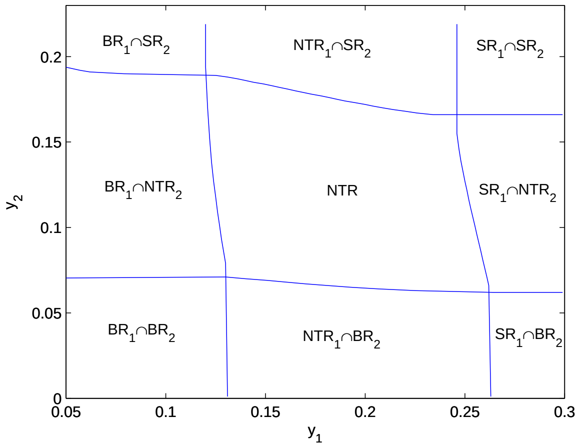


Figure 3.4: The time snapshot of NTR, $BR_i$, $SR_i$ and $NTR_i$, $i = 1, 2$, and $N = 2$. Default parameter values: $r = 0.07$, $\beta = 0.10$, $\alpha_1 = 0.15$, $\alpha_2 = 0.12$, $\sigma_1 = 0.4$, $\sigma_2 = 0.35$, $\rho = 0.20$, $\gamma = -1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 1$.

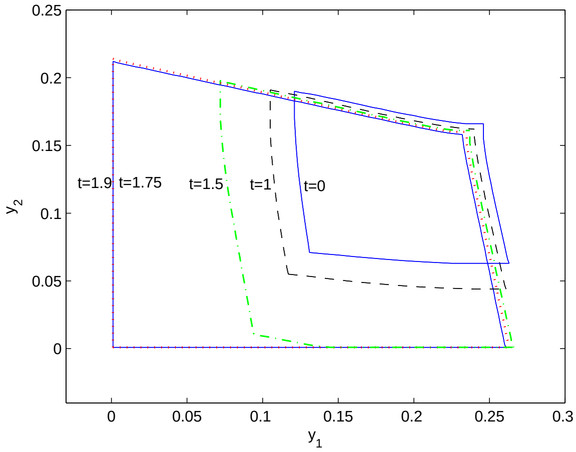


Figure 3.5: The different time snapshots of NTR $(N = 2)$. Default parameter values: $\alpha_1 = 0.15$, $\alpha_2 = 0.12$, $r = 0.07$, $\sigma_1 = 0.4$, $\sigma_2 = 0.35$, $\rho = 0.2$, $\gamma = -1$, $\beta = 0.1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 1$.

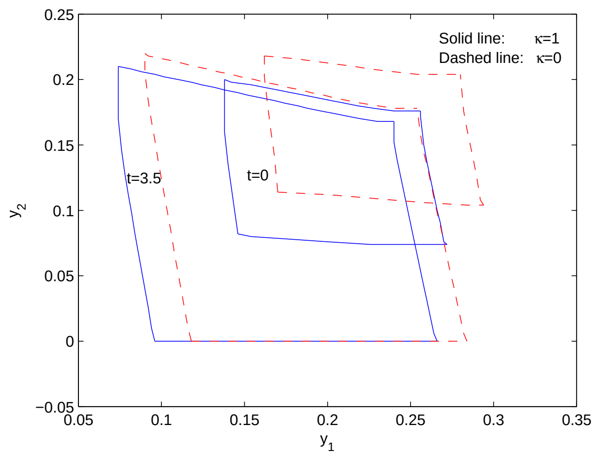


Figure 3.6: The comparison of the NTR between the consumption case and the no consumption case ($N = 2$). Default parameter values: $\alpha_1 = 0.15$, $\alpha_2 = 0.11$, $r = 0.07$, $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $\rho = 0.2$, $\gamma = -1$, $\beta = 0.1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 4$.

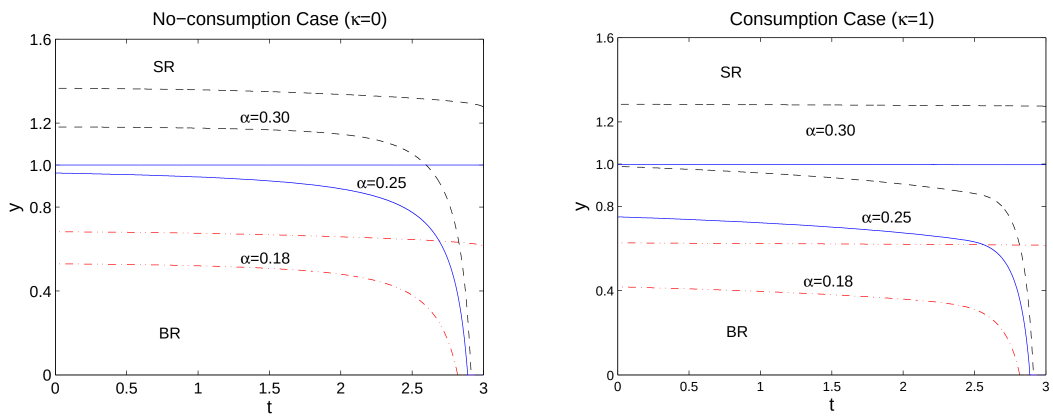


Figure 3.7: The impact of the risky asset return $\alpha$ on the optimal strategy ($N = 1$). Default parameter values: $r = 0.07$, $\sigma = 0.3$, $\gamma = -1$, $\beta = 0.1$, $\lambda = \mu = 0.01$, $T = 3$, $\kappa = 1$.

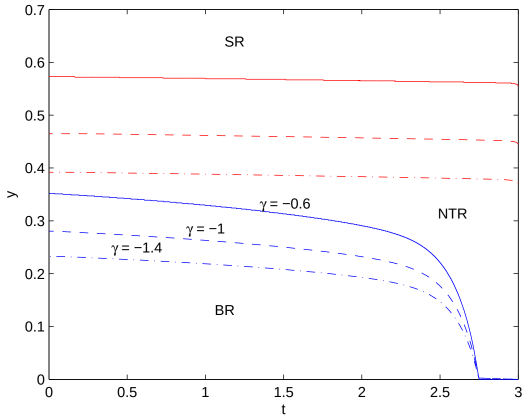


Figure 3.8: The impact of risk aversion $\gamma$ on optimal strategy ($N = 1$). Default parameter values: $\alpha = 0.15$, $r = 0.07$, $\sigma = 0.3$, $\beta = 0.1$, $\lambda = \mu = 0.01$, $T = 3$, $\kappa = 1$.

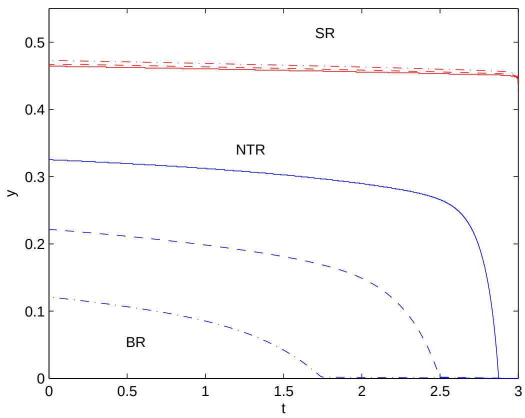


Figure 3.9: The impact of transaction costs on the optimal strategy ($N = 1$). Default parameter values: $\alpha = 0.15$, $r = 0.07$, $\sigma = 0.3$, $\gamma = -1$, $\beta = 0.1$, $\lambda = \mu$, $T = 3$, $\kappa = 1$.

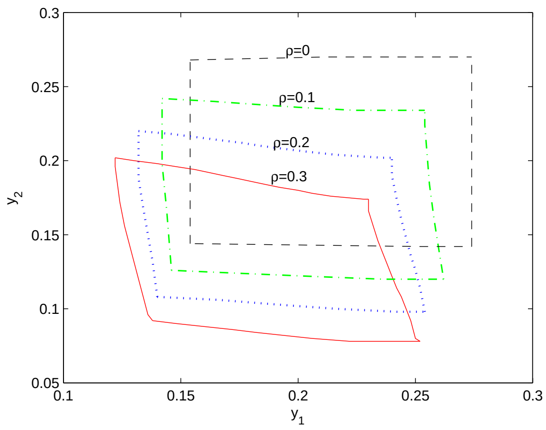


Figure 3.10: The impact of positive correlation on the NTR at $t = 0$ ($N = 2$). Parameter default values: $\alpha_1 = 0.14$, $\alpha_2 = 0.11$, $r = 0.07$, $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $\gamma = -1$, $\beta = 0.1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 0$.

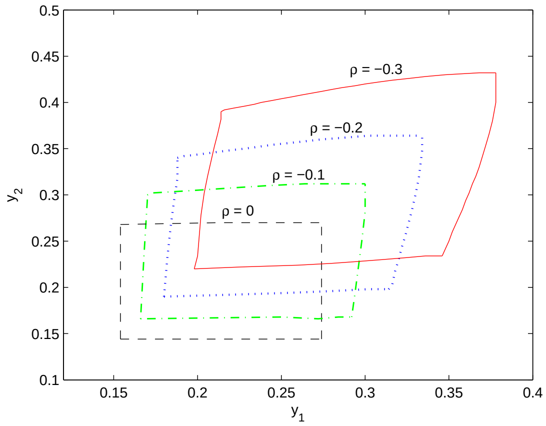


Figure 3.11: The impact of negative correlation on the NTR at $t = 0$ ($N = 2$). Default parameter values: $\alpha_1 = 0.14$, $\alpha_2 = 0.11$, $r = 0.07$, $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $\gamma = -1$, $\beta = 0.1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 0$.

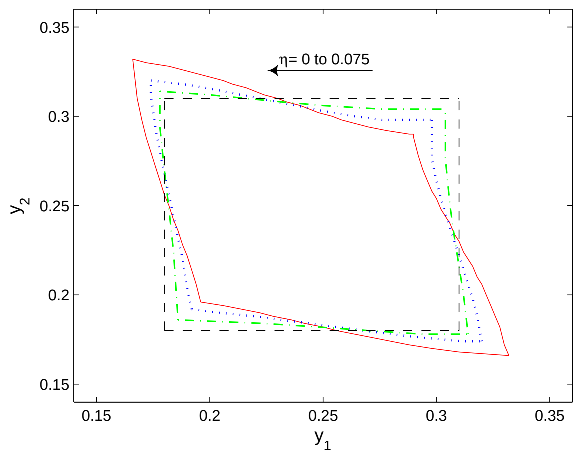


Figure 3.12: The impact of positive correlation on the NTR at $t = 0$ when the "Merton line" is fixed ($N = 2$). Default parameter values: $\alpha_1 = 0.15$, $\alpha_2 = 0.15$, $r = 0.07$, $\sigma_1^2 = \sigma_2^2 = (0.4 - \eta)^2 + \eta^2$, $\rho = \frac{2\eta(0.4-\eta)}{(0.4-\eta)^2+\eta^2}$, $\gamma = -1$, $\beta = 0.1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 0$. Given the parameter values, the "Merton line" as defined in Table 3.2 is constant. The positive correlation is measured by the parameter $\eta$ [cf. Muthuraman and Kuman (2006)].

Chapter 4

# Conclusion

Two free boundary problems in optimal investment are studied in this thesis. One is related to the optimal decision to sell/buy a stock in a given period with reference to the ultimate average of the stock price. The other is concerned with the numerical study of optimal investment and consumption problem with finite horizon in the presence of transaction costs.

By assuming the geometric Brownian motion of stock price, the first problem reduces to an optimal stopping problem which is formulated as a variational inequality problem. By virtual of the PDE approach, we have fully characterized the optimal buying (selling) strategy. It turns out that the optimal selling strategy is bang-bang, which is the same as that obtained by Shiryaev, Xu and Zhou (2008) taking the ultimate maximum of the stock price as the benchmark. However, the optimal buying strategy can be a feedback one subject to the type of average and parameter values. More precisely, for the sell case, if $\alpha > \frac{\sigma^2}{2}$, it is optimal to hold the stock until expiry; if $\alpha \leq \frac{\sigma^2}{2}$, it is optimal to sell the stock immediately at time 0. For the buy case, if $\alpha \geq \sigma^2$, one should buy the stock immediately; if $\alpha \leq 0$, one should never buy the stock before expiry; if $0 < \alpha < \sigma^2$, there is an optimal buying boundary and one should buy the stock once the boundary is reached. Moreover, we show that optimal strategy only depends on the time to expiry for the geometric average case, and on the ratio of stock price to the running average in addition to the time to expiry for the arithmetic average case.

It is worth pointing out that the bang-bang strategy for the sell case is the same as that obtained by Shiryaev, Xu and Zhou (2008) taking the ultimate maximum as benchmark. This, from another angle, justifies the definition of the "goodness index" presented in their paper. Nevertheless, we highlight that the bang-bang selling strategy heavily depends on the fact that the average period is taken from time 0. If we take the average period from some time horizon earlier than time 0, this can also lead to a feedback selling strategy, as we have seen in the buy case.

Mathematically, Problem II is equivalent to a variational inequality problem with gradient constraints. We have provided a general framework of penalty approximation method to numerically solve this variational inequality. Such a penalty approximation has a good interpretation that we restrict a class of policies being absolutely continuous and bounded. This is in contrast to the relation between the penalty approximation and the intensity framework for an optimal stopping problem [cf. Dai, Kwok and You (2007)]. In terms of a series of transformations, we obtain a unified variational inequality with gradient constraints that the value function satisfies both for power utility and log utility, and it is straightforward to apply the penalty method to the variational inequality. Convergence analysis is provided as well.

When there is only one risky asset, Dai and Yi (2006) and Dai et al. (2007) established a linkage with a standard variational inequality (obstacle problem). In this case, we can make use of the standard penalty method as in Forsyth and Vetzal (2002) and Dai, Kwok and You (2007), which allows us to adopt the Crank-Nicolson scheme. Then the better order of convergence can be achieved.

In addition, we carry out a comprehensive numerical analysis on the behaviors of the optimal buying and selling boundaries. The effects of parameter values on the optimal boundaries are investigated as well. In the case of single risky asset, numerical results demonstrate the theoretical analysis in Dai and Yi (2006) and Dai et al. (2007). Moreover, we offer an example that the optimal buying and selling boundaries may not be monotone when consumption is involved. In the case of multiple risky assets, we find that one should never buy any risky assets when time is close to maturity. Such a phenomenon has been proved by Liu and Loewenstein (2002), Dai and Yi (2006) and Dai et al. (2007) for the single risky asset case, but has never been revealed for the multiple risky assets case.

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