# SPARSE FLEXIBILITY STRUCTURES: DESIGN AND APPLICATION

**HUAN ZHENG**

**NATIONAL UNIVERSITY OF SINGAPORE**

**2007**

# SPARSE FLEXIBILITY STRUCTURES: DESIGN AND APPLICATION


**HUAN ZHENG**

(*Bachelor of Economics, Shanghai Jiao Tong University, China*)


A THESIS SUBMITTED

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY


DEPARTMENT OF DECISION SCIENCES

NATIONAL UNIVERSITY OF SINGAPORE


2007

# ACKNOWLEDGMENT

First of all, I would like to express my sincere gratefulness to my supervisors: Dr. Chung-Piaw Teo and Dr. Mabel Chou. I will not have finished my thesis and Ph.D. study without their continuous guidance and support. What I've learnt from them in the past four years, including passion and rigorous self-discipline for academic excellence and self-improvement, is a great benefit for my life.

I am grateful to my thesis committee members, Dr. Melvyn Sim and Dr. Hengqing Ye, for their encouragement and guidance. Their valuable suggestions and comments make my study more complete and hopefully more valuable. I also want to show thanks to my thesis examiner, Dr. Huang Huei Chuen, for her insightful comments.

I am very thankful to my parents who are always very understanding and supportive of the path I choose. I am also thankful to my husband, Kangning Wang. I will not select this wonderful career path without his support.

I would like to thank Mr. Lee Keng Leong for introducing me to the operations and issues in Food-From-The-Heart program and sharing with me his valuable thoughts on this subject.

Last but not least, I would like to thank my friends in NUS, Wenqing

Chen, Geoffrey Chua, Shanfei Feng, Hua Tao, Hua Wen and Na Xie, who have made my life in NUS truly exciting and memorable.

# CONTENTS

# ABSTRACT

Flexibility is a widely applicable concept in many business areas to help a company to deal with the demanding task of matching supply and demand in uncertain situations, without incurring much cost. Many companies in manufacturing, transportation and service industries have adopted flexibility as a key competitive tool. Flexibility practices, properly incorporated, could increase service levels, decrease response times without requiring additional capacity investment. The challenge is to effectively design a flexibility structure with a good performance, but with small implementation cost.

We first introduce the concept of "graph expander", which is widely used in graph theory, computer science and communication network design areas. We propose that a good flexibility structure possesses the properties of graph expander. Estimation on the performance of an expander flexibility structure is also proposed under the assumption of balanced and identical demands/supplies. We further examine the connections between the popular "chaining" structures and our expander structures, and propose that a "chain" is just the special case of an expander structure. The concept of "expander" can be further utilized to build an index to calibrate structures in terms of flexibility.

We then extend our analysis to a generalized unbalanced and non-identical demands/supplies case. Another approach called "constraint sampling" is applied to analyze the problem. The analysis also shows that a well designed sparse flexibility structure provides comparable performance to the full flexibility structure even when demands/supplies are unbalanced and non-identical.

We propose two heuristics to design good sparse flexibility structures based on the "graph expander" and "constraint sampling" concept. Both heuristics are simple and effective. These heuristics can be applied to a broad range of applications, such as process flexibility, transshipment, and cutting stock problems. We use real data from the Food-From-The-Heart (FFTH) program to support our conclusion. The theoretical results developed in our study are applied to fix the problem of their food-delivery operational system and enhance the operational performance. The result shows that by adding a little flexibility to the original dedicated system using our approach, the daily wastage of FFTH program can be reduced from more than 15 kilograms to only 2.808 kilograms. This result strongly supports the merits of our theoretical analysis.

## LIST OF FIGURES

# LIST OF TABLES

# 1. INTRODUCTION

With the evolvement of technology and the wave of globalization around the world, the operational environments for many manufacturing and service companies have become much more competitive and complicated. The complex environment and heightened customer expectation have brought vast uncertainties for all players in the supply chain. The ability to deal with the uncertainties effectively turns out to be the key issue of achieving a successful business in the fiercely competitive market. Uncertainties come from both internal situations in the company and external factors out in the market. Internal uncertainties are caused by incidents such as unexpected machine break-down, and could be tackled through well designed work schedule and frequent maintenance. The external uncertainties, on the other hand, come from the uncertainty of the demand and the supply sources: customers' order changes quickly and suppliers may fail to deliver raw materials on time. These external uncertainties are very hard to handle and usually out of the managers' control. Therefore, how to deal with the external uncertainties, especially the unexpected changes of demand and supply, is the greatest challenge faced by managers.

Some companies have adopted quick response strategies to enhance their competitive advantages. Zara, for instance, a ready-to-wear fashion garment maker and distributor, expanded quickly in the past several years. The main reason for Zara's success is its quick response strategy: they deliver new designs to their outlets twice-weekly and design customers' specific orders in just a few days. The quick response strategy is implemented by making its design and production process more flexible [57].

Besides Zara, more and more companies in a wide range of industries are beginning to treat flexibility as an important strategy to make their businesses successful. In the automobile industry, for example, companies are moving from focused factories to flexible factories. Ford Motor Company, for instance, invested $485 million in two Canadian engine plants to renovate and retool them with flexible system. It also has launched a plan for equipping most of its 30-odd engine and transmission plants all over the world with flexible systems.

> *"...'The initial investment is slightly higher, but long-term costs are lower in multiplies,'said Chris Bolen, manager of Ford's Windsor engine plant, which uses the flexible system to machine new three-valve-per-cylinder heads for Ford's 5.4-liter V8 engine...Ford says the system will help it meet changes in demand. 'If our business was hit by a significant downsizing from V8s to V6s or V6s to (four-cylinder engines) or diesels in North America, we'll be able to react to that without years of turnaround,' said Kevin Bennett, Ford director of power train manufacturing. 'It's essential we be able to react to the market more rapidly than in the past.'... "*

— Mark Phelan, "Ford Speeds Changeovers in Engine Production"

**Knight Ridder Tribune Business News.** Washington: Nov 6, 2002.

Similar initiatives to make plants more flexible have also been accepted, and are viewed as a strategic weapon in the automobile industry in the increasingly competitive global environment. A survey of North-American automobile industry conducted in 2004 shows that the plants of major automobile manufacturers, such as Ford and General Motor, are more flexible than those 20 years ago [55]. The survey showed that these flexible plants can produce much more types of cars to meet the rapidly changing customer demands while their capacities did not change very much. This kind of flexibility is called "process flexibility", one of the widely adopted flexibility strategies.

To enhance our understanding of the studies in flexibility, we briefly review the various classes of flexibility strategies in section 1.1.

## *1.1 Flexibility*

Flexibility, the ability of a system to respond or react to changes in external environments with little penalty in time, effort, or cost [54], is a general concept, which may have different interpretation in different settings. Sethi and Sethi [48] provided an extensive survey on the applications of flexibility in different areas. They categorized eleven types of flexibility, such as "machine flexility", "product flexibility", "routing flexibility", "resource flexibility", and etc. A recent survey conducted by Kara and Kayis [37] lists the factors causing the needs of flexibility, including both internal factors such as machine breakdown, workforce variations and etc, and external factors such as demand variations, customizations, short product life cycles and etc. This survey further describes 14 different types of flexibilities dealing with internal and external factors. Besides dealing with these traditional uncertainty factors, flexibility tools also begin to be implemented in e-business areas with the development of internet and IT technology. Shi and Daniels [49] reviewed the process flexibility literatures that deals with e-business issues and defined a new concept, "e-business flexibility", in their paper.

There are by now a vast literature on flexibility. One group of study focuses on how to measure and suitably implement these flexibility strategies. Das and Patel [17] suggested an "auditing" process to help a company identify its flexibility needs, and implementing the suitable flexibility strategies gradually. Anand and Ward [4] conduct an empirical study on the impacts of different types of flexibilities ("range" and "mobility") on the market shares and sales growths of companies under different environments (i.e. "unpredictable" and "volatility"), based on the data collected from 101 manufacturing firms. Their statistical results suggest that environment factors play important roles in determining suitable flexibility strategies. Jack and Raturi [34] identified the resources which might help companies to increase the volume flexibility based on case studies. Their study also showed positive correlations between volume flexibility and company's performances. They [35] used four metrics to measure the volume flexibility in the capital good industry. They tested 550 firms using 20 years worth of data, and indicated that higher volume flexibility may not lead to better financial performance since the implementation cost might be too high. These studies all focus on identifying the suitable flexibility tools for different companies at the strategic level, but do not consider how to design and implement flexibility at the operational level.

Another group of study focuses on identifying the guidelines to design effective flexibility structures which are cheap and easy to be implemented in daily operations. One widely adopted operational flexibility strategy is process flexibility. Process flexibility is an effective tool to enhance the flexibility of the operational process of a manufacturer or a service company. In section 1.2, we briefly describe the properties of process flexibility and thoroughly review the literature in this area.

## 1.2 Process Flexibility

Process flexibility can be defined as the ability of a system which enables a production facility to produce different types of products at the same time with little penalty in operational cost [36]. Process flexibility is an effective strategy that manufacturers can use to match fixed capacities with random demands for different products. Indeed it is common to find a plant employing the technique of process flexibility in automobile industries these days [55]. Process flexibility strategy is also widely used in services industries, where process flexibility is achieved by equipping a system with multi-skill agents [32].

To show why process flexibility can be used as an effective strategy to deal with uncertainties in different applications, we need to understand its

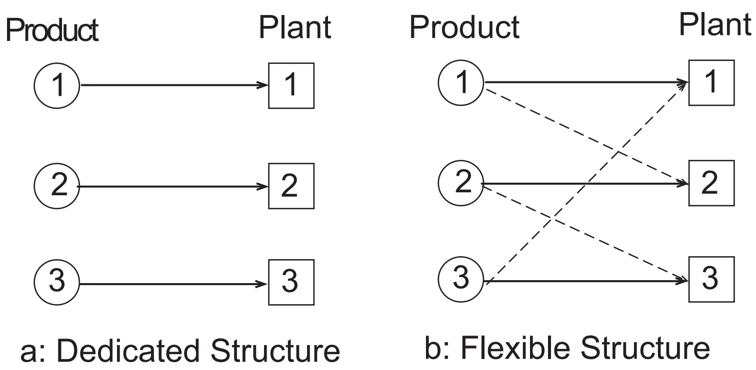



*Fig. 1.1:* The Mechanism of Process Flexibility

mechanism first. Figure 1.1 is a simple example illustrating the mechanism of process flexibility. There are two systems in Figure 1.1. Each system has three plants and three products. The demands of products are random and the capacities of plants are fixed. Figure 1.1-a is a traditional dedicated production system: product 1, 2 and 3 can only be produced in factory 1, 2 and 3 respectively. When demand of product 1 is more than the capacity of plant 1 and demand of product 2 is less than the capacity of plant 2 in the same time, this system fails to satisfy all the demand of product 1 while the capacity of plant 2 is not fully utilized. However, this situation is nicely handled in a flexible system (see Figure 1.1-b). In this flexible system, every product can be produced in 2 plants. The excessive demand of product 1 will be partially (or evenly fully) satisfied using the spare capacity of plant 2. This is the basic reason why the flexible system deals with demand uncertainties more effectively. In fact, the contribution of process flexibility partly stem from the fact that the capacities of different production facilities are partially pooled in the process flexibility structure.

Obviously, the flexibility structure is not unique. The number of possible flexibility structures for a $n$-product-$m$-plant system could be up to $(2^m - 1)^n$ [1]. Among these structures, the "full flexibility structure" (see Figure 1.2-A) attains the best performance, since each product can be produced in all plants. The full flexibility structure, on the other hand, has the highest implementation cost and management requirement. Other structures are known as "partial flexibility structure", in which each product is only connected to a few plants. A partial flexibility structure usually underperforms full flexibility structure, but the implementation cost could be significantly lower than full flexibility structure. Thus the trade-off between flexibility and implementation cost is an interesting and challenging problem in flexibility system design.

Among partial flexibility structures, "Chaining" is a widely accepted partial flexibility structure. A chain is a path connecting different products and products. Figure 1.2-B shows an example of a chaining structure with degree two (i.e. each node is connected with two links). Many studies (cf. [36], [29], [33]) have shown that chaining structures can achieve the performance close to full flexibility structure, and are much cheaper and easier to be implemented. To enhance our understanding of the advantage of chaining structure and the results and limitations of previous studies, a thorough

[1] We consider the choices for a single product first. The product could be produced in a single plant(choose 1 plant from the $m$ plants), or could be produced in two plants (choose 2 plants from the $m$ plants), and so on. The total number of possible options for the product is

$$\sum_{i=1}^{m} \binom{m}{i} = 2^m - 1.$$

For the system with $n$ products, the number of optional structures is $(2^m - 1)^n$.

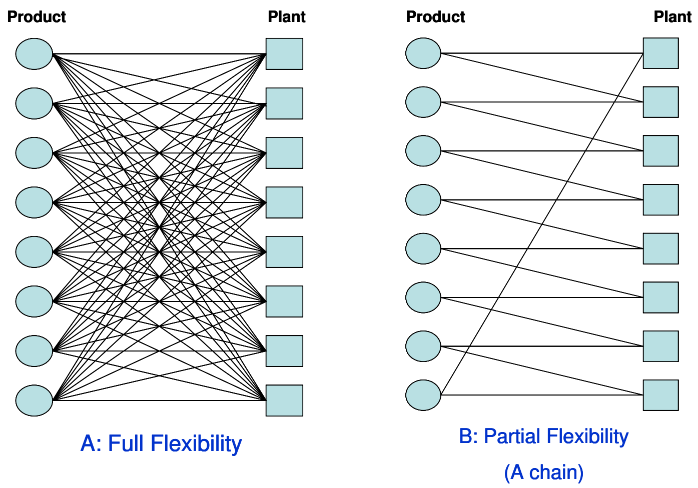



*Fig. 1.2:* An Example of Full Flexibility Structure and Partial Flexibility Structure

literature review on flexibility structure design is provided in section 1.2.1.

### *1.2.1 Literature Review on Process Flexibility*

Process Flexibility stems from a very hot topic "Flexible Manufacturing System" (cf. [50], [11]) in the 1980's. The focus of Flexible Manufacturing system (FMS) is the trade-off of investing on dedicated and flexible capacities (cf. [21], [56]). However, these early studies only consider full flexible resource, i.e. a plant can produce all types of products. The classical study about designing a partial flexibility structure was conducted by Jordan and Graves [36]. Their findings were based on the simulation study of a General Motor's production network. In this study, they calibrated the performance of sparse partial flexibility structures by comparing full flexibility structure and partial flexibility structures in an intensive simulation. The results showed that a partial flexibility structure, if well designed, could capture almost all the contribution of the full flexibility structure. They further proposed a chaining structure as the guideline for designing a good partial flexibility structure.

Jordan and Graves' study partially answers the question: how much flexibility is enough? This problem has puzzled many researchers and managers for a long time. Hence, the partial flexibility and chaining strategy has been applied and examined in various areas such as supply chain ([29], [10]), queuing ([8], [30]), revenue management ([23]), transshipment distribution network design ([40], [60]), manufacturing planning ([39]) and flexible work force scheduling ([18], [32], [59], [12]). For instance, Graves and Tomlin [29] extended the study to multi-stage supply chain problems and found out that "chaining" structures also work robustly well. Hopp et al. [32] observed similar results in their study of a work force scheduling problem in a ConWIP (constant work-in-process) queuing system. By comparing the performances of "cherry picking" and "skill-chaining" cross-training strategies, they observed that "skill-chaining", which is indeed a kind of the "chaining" strategy, outperforms others. They also showed that a chain with a low degree (the number of tasks a worker can handle) is able to capture the bulk of the contribution of a chain with high degree. That means the marginal contribution of the additional flexibility will decrease when the degree of flexibility increases.

Some studies address the side effects of implementing flexibility tools. Muriel et al. [42] showed that a surgery planning system (e.g. a hospital) with a limited flexibility structure could lead to a great increase in the variability of rescheduling and operation when the system need to meet an unexpected surge in emergency operation requirement. Bish et al. [9] also indicated that in the make-to-order environment, flexibility could introduce variability in the upstream of the supply chain, thus leading to higher inventory cost, greater production variability and more complicated management requirement.

All these studies show that partial flexibility is a cost-effective strategy: a well designed partial structures can capture most of the benefits of a full flexibility, but requires much less investment in fixed cost. However, there are very few analytical results on the performance of partial flexibility structures. One such study is conducted by Aksin and Karaesmen [3]. They applied network theories to the study of flexible structure. They argued that the flexibility of a structure is determined by the maximum network flow through products' demand to the plants. Unfortunately, this paper did not provide any guideline on the design of flexible structure. Instead, it focused on deriving the concavity of certain fixed process structure, as a function of the degree of each production nodes. Thus, it is still unclear how to exactly estimate the gap between a partial flexibility structure and a full flexibility structure. Furthermore, a theoretical justification of the existence of a well-performing sparse partial flexibility structure is also an open question.

Another problem arising from these studies is how to find the "well designed" partial flexibility structure. Chaining structure is widely accepted. A system with a long chain and a small degree is usually considered a good flexible system [36]. However this guideline is still not enough to identify and generate good partial flexibility structure. As shown in Figure 1.3, there are

6 different chains for a 3-product-3-plant. It is hard to determine which one

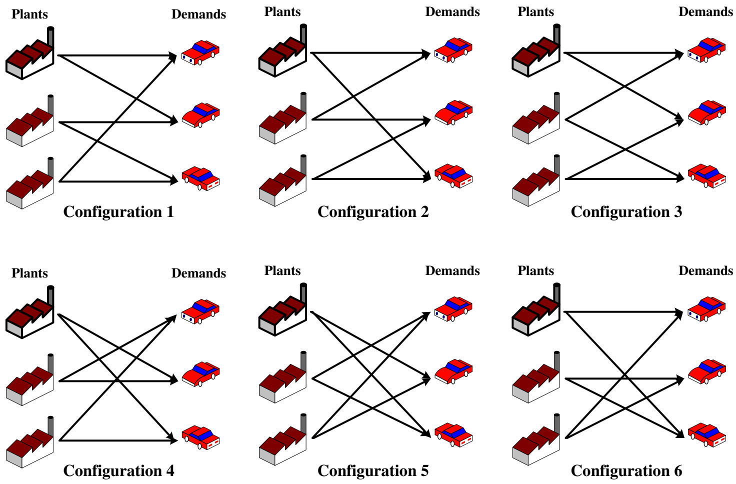



*Fig. 1.3:* Different Chaining Structures with the Same Degree and Length

is the best based on the current guideline. Actually, not all chains work well. For example, when the means of demands are nonidentical and the supplies are identical and fixed, a sparse structure with more arcs connected to the large demand node may outperform a chaining structure with same number of arcs. Furthermore, the benefit of chaining structures might be limited under certain conditions. Chou et al. [15], for instance, re-evaluated regular chaining structures with primary production and secondary production options, and the production cost for secondary production is expensive. Their results show that the profit increase by introducing a chaining structure to a dedicated system is no more than about 70% of the full flexibility structure when the secondary production cost is quite expensive and demand follows normal distributions. This observation is quite contrary to the belief that chaining structure could achieve almost all benefit of full flexibility in many simulation results (cf. [36], [32]). Therefore, further study is still needed to investigate the property of chaining structures, and to develop an effective flexibility structure design method.

Another important issue is to propose effective indices to measure the performance of flexibility structures (cf. [36], [29] and [33]). Jordan and Graves [36], for instance, used the performance of full flexibility structure as the benchmark, and developed a probabilistic index. The index focuses on the probability that the unsatisfied demand from a subset of product nodes of a flexibility structure would excess that of the full flexibility structure. The largest probability among all subsets is deemed as the index. A good flexibility structure thus should have a low index. The index comes directly from the function of flexibility: a more flexible structure should deal with demand uncertainty more effectively, and thus the unfilled demand of the structure should be same as the full flexibility structure most of the time. However, this index is usually very hard to compute if demands are not normally distributed or/and correlated.

The limitations of Jordan and Graves' index is partly overcome by another set of indices. These indices were proposed by Iravani et al [33] based on an extension to the study of the ConWIP flexibility system [32]. In this study, a suitably defined "structural flexibility matrix" (SF Matrix) $M$ was proposed to calibrate a system in terms of flexibility. An entry $(i, j)$ in $M$ represents the non-overlapping routes from demand node $i$ to supply node $j$, and $(i, i)$ is the degree of arcs connected to the demand node $i$. The largest eigenvalue and mean of the SF matrix $M$ are used as the indices to determine the flexibility of a structure. SF indices are much easier to compute and work very well in some simulation examples [33]. However, the SF indices were built based on the assumption of "fit", i.e. the demand of each product could be satisfied on average, which limits the application of SF indices in the situation that total capacity is greater than the average total demand. In addition, the SF matrix do not reflect the impacts of variance and co-variance. SF indices therefore may not work well when demands have large variances. Hence, a simple but effective index that works robustly well in a more general situation is needed.

## *1.3 Research Objectives*

The objectives of this thesis are:

- To examine the existence of a sparse partial flexibility structure, with a small number of links on average, and capturing almost all the benefit of full flexibility. We first show the intimate connection between flexibility structures and graph expander (a group of graphs with small number of arcs but well connected). Based on the graph expander theories, we propose a mathematically concise statement about the gap between the performances of the full flexibility structure and a "well defined" sparse partial flexibility structure when demands and supplies are balanced and identical. we further extend our study to the case when demands and supplies are non-identical and unbalanced, using constraint sampling approach.

- To present an efficient method to generate a good sparse flexibility structure whose performance is close to full flexibility structure. We build two different heuristics, based on the analysis using"graph expander" and "constraint sampling". Both methods are quite simple and effective, comparing to the traditional extensive simulation approach.

- To propose an effective but simple index to calibrate structures in respect of flexibility. Since flexibility is intimately related to graph expander, we introduce an index measuring the connectivity of graphs in graph theory. This index can be easily adjusted to measure flexibility of structures. This index is also easy to compute and can be widely used in various environments.

- To extend our structural design concept to a broader area. It is well known that a well designed sparse partial flexibility structure can capture the most benefit of full flexibility. We believe that this phenomenon may also exist in other areas, such as transshipment network design and cutting stock problems. We examine whether a good sparse structure exists in these two cases.

- To apply insights and results of our theoretical study to real business applications. The operational system of a non-profit organization in Singapore, "Food From The Heart", is studied, and the problems in the food-delivery operation is raised. We successfully reduce the wastage of current dedicated routing system by developing a flexible routing system via our expansion heuristic.

## 1.4 Research Contributions

In this study, we provide a theoretical justification to the existence of good sparse flexibility structures. A concept, “graph expander”, from graph theory is introduced to analyze the flexibility structures. The relationship between graph expander and flexibility structures is thoroughly investigated, providing a clearer understanding of partial flexibility structures. The expansion concept may also be adjusted to calibrate structures in terms of flexibility.

The sampling approach not only supports the existence of a good sparse structure in a general demand/supply settings, but also could be used to generate good structures for various applications.

Our theoretical results and observations also have important practical contributions. The expansion heuristic and sampling heuristic are quite easy to use and can be applied to different applications such as transshipment structure design and cutting stock problems. The expansion heuristic is quite robust and requires minimal information of demand/supply: only the mean of each demand/supply is needed. Our heuristics also have an impressive performance in real applications such as “Food From The Heart” problem.

## 1.5 Structure of Thesis

The remaining sections of the study are organized as follows. To provide a clear understanding of flexibility structures, the assumptions and issues in most flexibility studies are discussed in chapter 2. Two basic models, maximum flow model and minimum excess flow model, are also introduced in chapter 2. Chapter 3 will investigate the flexibility structure design problem using "graph expander" approach, and provide a theoretical justification to the existence of good sparse flexibility structures. A simple and effective heuristic to construct flexibility structures and a good index to measure structures in terms of flexibility will also be proposed in Chapter 3. Chapter 4 will study the problem using "constraint sampling" method in the situation when supplies/demands are non-identical and unbalanced. A sampling heuristic obtained from the insights of the analysis will be introduced as well. Chapter 5 will apply the theoretical results and heuristics to various applications, such as production planning problem, transshipment network design problem, and cutting stock problem. Chapter 6 is the case study of "Food From The Heart"(FFTH) program. Our heuristic will be applied in the real case to fix the operational problem in the program. Chapter 7 will conclude the study by summarizing the results and contributions, and listing some directions for future research.

# 2. MODELS AND ASSUMPTIONS

## *2.1 Flexibility Models*

Network flow models and queuing models are the most widely used models in the studies of flexibility structures. Network flow models consider the system as a bipartite graph and allocate the flow of commodity in the network structure to optimally match the demand and supply. These models are usually used in production planning problems with multiple plants and products in a single-stage supply chain (e.g. [36], [3] and [29]). Queuing models are more suitable for a single-product line production system with several sequential tasks, and workers in the production line have different service/production rates. The flexibility structures are used to define the cross-training scheme to balance the different service rates of the tasks (e.g. [32] and [33]). Though the two modeling approaches are quite different, the design strategies and flexibility structures obtained by both approaches are indeed the same. In this study, we focus on the network flow models.

### *2.1.1 Maximum Network Flow Model*

A flexibility structure can be represented by a bipartite graph $G = (A \bigcup B, F)$. The left-hand-side vertices $A$ denote the (random) demands of products. The right-hand-side vertices $B$ denote the (constant) capacities of plants. The arc $e \in F$ connects a node (say $a$) in $A$ to a node (say $b$) in $B$, and means that the product $a$ can be produced in plant $b$.

The purpose of this study is to design a good partial flexibility system with only a small number of arcs which can match the supply to the demand almost as well as a full flexibility system. To be more specific, we want to design a set $F$ in the bipartite graph $G$ with relatively small $|F|$ which can match supply to demand almost as well as $A \times B$, the full flexibility system. To evaluate how well a set $F$ can match the supply to the demand, we consider the following formulation: Consider any given set $F$, we let $D_1, \ldots, D_m$ denote any realized random demand of products in $A$, and $S_1, \ldots, S_n$ denote the fixed capacities of the plants in $B$. Let $x_{ij}$ denote the amount of product $i$ produced by plant $j$. Obviously, $x_{ij} = 0$ for all $(i, j) \notin F$. To measure how well $F$ can match the realized demand to the fixed capacity, we define $z_m(F)$, the maximum flow amount when $F$ is in place and all products are produced by one or more plants through the arcs in $F$, by solving the following optimization problem:

$$
\begin{aligned}
z_m(F) = \quad \max \quad & \sum_{j=1}^{n} \sum_{i:(i,j)\in F} x_{ij} \\
s.t. \quad & \sum_{j:(i,j)\in F} x_{ij} \le D_i \quad \forall\ i = 1, \ldots, m; \\
& \sum_{i:(i,j)\in F} x_{ij} \le S_j \quad \forall\ j = 1, \ldots, n; \\
& x_{ij} \ge 0 \quad \forall \quad (i, j) \in F
\end{aligned}
$$

Obviously, we seek $F$ with large $z_m(F)$. It is easy to see that for any realization of demands and supplies, $z_m(F)$ is the largest when $|F| = A \times B$, i.e. when $F$ is full flexibility structure.

### 2.1.2 Minimum Excess Flow Model

On the other hand, a flexibility model can also be measured by the unsatisfied demand $z_e(F)$, which is the minimum excess flow of $F$. Specifically, $Z_e(F)$ can be obtained by solving the following optimization problem.

$$
\begin{aligned}
z_e(F) = \quad & \min \quad \sum_{j=1}^{n} \Bigg( \sum_{i:(i,j)\in F} x_{ij} - S_j \Bigg)^{+} \\
& s.t. \quad \sum_{j:(i,j)\in F} x_{ij} = D_i \;\; \forall\; i = 1, \ldots, m; \\
& \qquad\quad x_{ij} \geq 0 \;\; \forall \;\; (i,j) \in F,
\end{aligned}
$$

where $(\cdot)^{+}$ stands for the positive part of $(\cdot)$.

We seek $F$ with small $z_e(F)$, which is just opposite to the direction of max-flow criterion.

### 2.1.3 Relationships

Since the total demand is $\sum_{i=1}^{m} D_i$, it is easy to see that $z_m(F)$ and $z_e(F)$ satisfy the following relationship:

$$z_m(F) + z_e(F) = \sum_{i=1}^{m} D_i.$$

The equation holds because of the network flow conservation axiom. This relationship shows that the excess flow model $(z_e(F))$ is essentially a re-statement of the classical maximum flow model ($z_m(F)$) - a structure minimizing the expected excess flow will simultaneously maximize the total flow through the network.

This relationship also helps to derive the following proposition which is useful in our subsequent analysis.

Proposition 1: Given any bipartite graph $G = (A \cup B, F)$,

$$z_e(F) = \max\left(0, \max_{S:S\subseteq A}\left\{\sum_{i\in S} D_i - \sum_{j\in N(S)} S_j\right\}\right),$$

where $N(S) = \{j \in B : (i,j) \in F,\ i \in S\}$ represents the neighbors of set $S$ in $G$.

*Proof:* The above proposition is an easy consequence of the max-flow-min-cut theorem: the arc between $A$ and $B$ has infinite capacity, and hence

$$z_m(F) = \min_{S\subseteq A}\left(\sum_{i\notin S} D_i + \sum_{j\in N(S)} S_j\right).$$

Since $z_e(F) = \sum_{i=1}^{m} D_i - z_m(F)$, the proposition follows. ∎

## 2.2 Chaining Strategy

One of the most well known concept in the area of flexibility structure design is the chaining strategy pioneered by Jordan and Graves (1995). Although chaining strategy arguably captures a key feature of good process flexibility structure, the way the capabilities are chained together also plays an important role in the performance, as demonstrated in the following example.

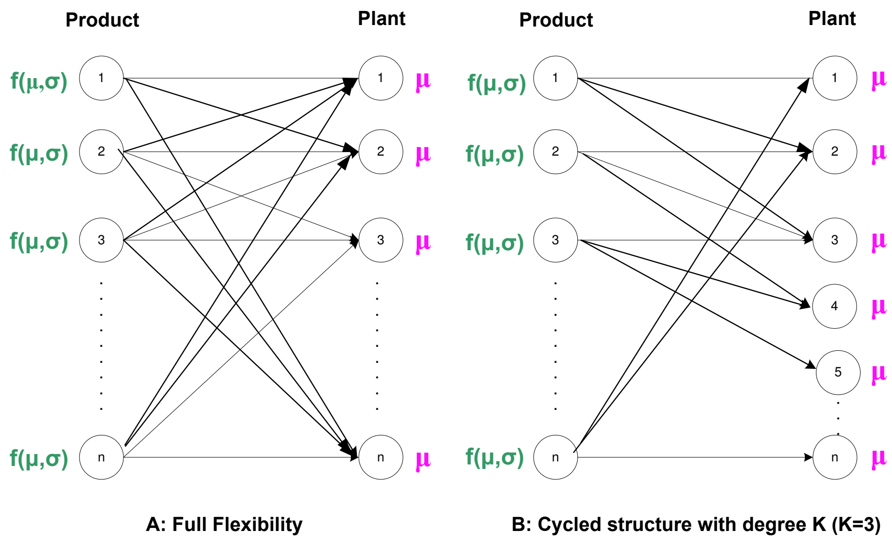



*Fig. 2.1:* Full Flexibility Structure and a Cycled Chain Partial Flexibility Structure.

Consider two flexibility structures as shown in Figure 2.1, where there are $n$ plant vertices and $n$ product vertices. We assume that each plant has a fixed capacity $\mu$ while the product's demand follows a distribution with a finite support, and with mean $\mu$ and standard deviation $\sigma$. It is clear that the expected total demand equals to the total supply. Full flexibility structure (2.1-A) and the cycling chain structure (2.1-B) are compared in terms of excess flow and maximum flow. We focus our comparisons on the difference between structure $A$ and $B$ as $n$ increases.

Consider the case when each demand follows a uniform distribution from 0 to 200 and each plant has a fixed capacity of 100. We conduct a simulation by sampling 200 scenarios for each demand node, and compare the expected excess flow of the regular chains with degree $k$ ($k = 2 \ldots 5$) and the full flexibility structure. As shown in Figure 2.2, the expected excess flow of full flexibility structure and chaining structures are quite close when $n$ is small, say 50. However, the gaps between regular graphs (i.e. chaining structures) and full flexibility increase quickly as $n$ increase, and regular graphs no longer can capture most benefit of full flexibility in this case.

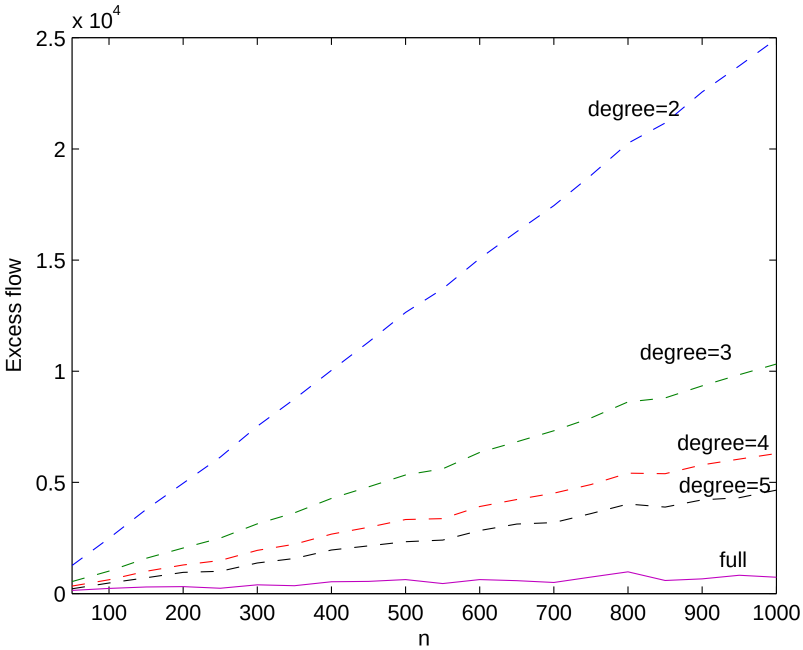



*Fig. 2.2:* The Performance Gaps Between Full Flexibility Structure and Regular chains.

This special case indicates that chaining strategy might greatly underperform in certain conditions. We can further provide a mathematical estimation to the expected excess flow of the chaining structures (Figure 2.1-B)

In the fully flexible system described in Figure 2.1-A, it is easy to see that the expected excess flow equals $\boldsymbol{E}[(\sum_{i=1}^{n} D_i - \sum_{j=1}^{n} S_j)^+]$. We can use the following lemma to obtain the upper bound for the excess flow of full flexibility ($A$).

Lemma 1: For a random variable $x$ following an arbitrary distribution with standard deviation $\sigma_x$, the mean of the positive part ($x^+$) has the following property:

$$\boldsymbol{E}(x^+) \leq \frac{1}{2}\left(\boldsymbol{E}(x) + \sqrt{(\boldsymbol{E}(x))^2 + \sigma_x^2}\right),$$

**Proof.** Since for any real number $x$, we know that

$$\begin{aligned} x &= x^+ - x^- \\ |x| &= x^+ + x^-, \end{aligned}$$

it is easy to get

$$\begin{aligned}
\boldsymbol{E}(x^+) &= \frac{\boldsymbol{E}(x) + \boldsymbol{E}(|x|)}{2} \\
&= \frac{\boldsymbol{E}(x) + \sqrt{(\boldsymbol{E}(|x|))^2}}{2} \\
&\leq \frac{\boldsymbol{E}(x) + \sqrt{\boldsymbol{E}(|x|^2)}}{2} \\
&= \frac{\boldsymbol{E}(x) + \sqrt{\boldsymbol{E}(x^2)}}{2} \\
&= \frac{\boldsymbol{E}(x) + \sqrt{(\boldsymbol{E}(x))^2 + \sigma_x^2}}{2}
\end{aligned}$$

■

In this example, we know that the expected demand is the same as the fixed capacity, and each demand is assumed to be independent, thus the mean of $\sum_{i=1}^n D_i - \sum_{j=1}^n S_j$ is 0 and variance is $\sum_{i=1}^n \sigma^2$. Therefore, the expected excess flow of structure $A$ is

$$\begin{aligned}
\boldsymbol{E}\Big(z_e(A)\Big) &= \boldsymbol{E}\Big(\sum_{i=1}^n D_i - \sum_{j=1}^n S_j\Big)^+ \\
&\leq \frac{1}{2}\left(0 + \sqrt{0 + \sum_{i=1}^n \sigma^2}\right) \\
&= \frac{\sqrt{n}\sigma}{2} \\
&\sim O(\sqrt{n})
\end{aligned}$$

To analyze the expected excess flow in the cycling chain structure ($B$) described in Figure 2.1-B, we first observe that every product node is connected to $k$ plant nodes in a regular way: product $i$ is connected to plant $i,\ i+1,\ \ldots,\ i+k-1$ if $i \leq n-k+1$; otherwise, node $i$ is linked to plant $i,\ i+1,\ \ldots,\ n,\ 1,\ 2,\ \ldots,\ i-(n-k+1)$. As such, each group of consecutive $(k-1)^2$ products is connected to exactly $(k-1)^2+(k-1)$ plants, which implies a total capacity of $(k^2-k)\mu$. WLOG, we assume that $n/(k-1)^2$ is an integer and divide the product nodes into $n/(k-1)^2$ groups of consecutive $(k-1)^2$ nodes. We observe that for each subgroup, the expected excess flow is at least $\boldsymbol{E}\left[\sum_{i=1}^{(k-1)^2} D_i-(k^2-k)\mu\right]^+$. We might assume $\sum_{i=1}^{(k-1)^2} D_i \sim N\Big((k-1)^2\mu,(k-1)\sigma\Big)$ for suitably large $k$. By the central limit theorem, the expected excess flow $\boldsymbol{E}(z_e(B))$ for the whole system satisfies

$$
\begin{aligned}
\boldsymbol{E}(Z_e(B)) &\geq \boldsymbol{E}\left[\frac{n}{(k-1)^2}\left(\sum_{i=1}^{(k-1)^2} D_i-(k^2-k)\mu\right)^+\right] \\
&= \frac{n}{(k-1)^2}\int_{(k^2-k)\mu}^{\infty}\left[x-(k^2-k)\mu\right] f_{N((k-1)^2\mu,(k-1)\sigma)}(x)dx \\
&= \frac{n}{(k-1)^2}(k-1)\sigma \\
&\quad \int_{(k^2-k)\mu}^{\infty}\left[\frac{x-(k-1)^2\mu}{(k-1)\sigma}-\frac{(k-1)\mu}{(k-1)\sigma}\right] f_{N((k-1)^2\mu,(k-1)\sigma)}(x)dx \\
&= \frac{n\sigma}{k-1}\int_{\mu/\sigma}^{\infty}(x-\frac{\mu}{\sigma})\phi(x)dx \\
&\sim \Omega(n),
\end{aligned}
$$

where $\phi(\cdot)$ is the density function of a standard normal variate. Therefore, for fixed $k$ and large $n$, the above chaining strategy may lead to a performance far inferior to that of full flexibility structure.

Therefore, as $n$ increases the excess flow of the cycling chain structure will be far greater than full flexibility structure. This observation indicates that the effectiveness of the above chaining strategy is limited in a system with large $n$ when the management focuses on the excess flow criterion(e.g. unsatisfied demand).

We will show in chapters 3 and 4 that it is possible to chain the process together in a suitable manner so that the above deficiency in the partial flexible structure will disappear. To be more specific, for fixed $k$ and suitably large $n$, there is a way to chain the process together which allows us to accrue benefits close to the fully flexible system!

## 2.3 Structural Flexibility Matrix

Another inspiring study on flexibility was recently done by Iravani, Van Oyen and Sims (2005). They proposed a "structural flexibility" method to explore flexibility systems such as cross-training workers, flexible machine planning, etc. They defined a "structural flexibility matrix" (SF matrix) $M$ to represent the flexibility of a system.

The SF matrix $(M_{ij})_{i,j=1}^{n}$ advances the chaining concept in several ways. On one hand, it makes the notion of chaining concrete by explicitly measuring the number of non-overlapping routes between node $i$ and node $j$, represented by $M_{ij}$ ($M_{ii}$ represents the number of arcs that are connected to node $i$). It also reduces the difficult problem of evaluating the expected value of $z_m(F)$ (or $z_e(F)$) to a simpler problem of computing the SF indices such as the mean of the entries in the SF matrix $M$ or the dominant eigenvalue of $M$. In their study, in order to have a fair analysis in capturing the characteristics of a good flexibility structure, they required the structure to be "fit". In our context, fitness of a process structure means that, *on average*, we can allocate all fluctuating demands to the dedicated capacities in the system in such a way that no excess flow would occur under such a structure.

Iravani, Oyen and Sims (2005) demonstrated through extensive simulation analysis that a flexibility structure with higher SF indices will attain better performance. More importantly, they also proved that a $D$-skill chaining structure (a cycled chaining structure with degree $D$ for each node) has the highest SF indices among all structures with $N$ demand nodes, $N$ supply nodes and $ND$ arcs, assuming that all supplies and demands are identical. This lent credible evidence to the usefulness of the SF approach, and the effectiveness of the $D$-skill chaining concept.

While the SF matrix and the chaining concept have so far been proven useful and effective in numerous situations when examining the process flexibility issues, we need to caution the readers that these approaches may not reveal the right insight all the time. We use the following example to illustrate the potential pitfall of such approaches.

> Consider two flexibility structures as shown in Figure 2.3. We assume that each plant has a capacity of 10 while the demands are uneven and random as specified in the following: the demand of product 1 is either 30 or 10, with equal probability; the demand of product 2 and 3 are either 10 or 0 with equal probability. Obviously, the mean of the total demand is 30, which is equal to the total capacity. Figure 2.3-A shows a simple chaining structure (i.e. a 2-skill chain) and the corresponding expected excess/maximum flow is 6.25/23.75.[1] Figure 2.3-B shows a flexible de-

[1] With prob. 0.5, the demand at node 1 is 30, leading to excess flow of at least 10.

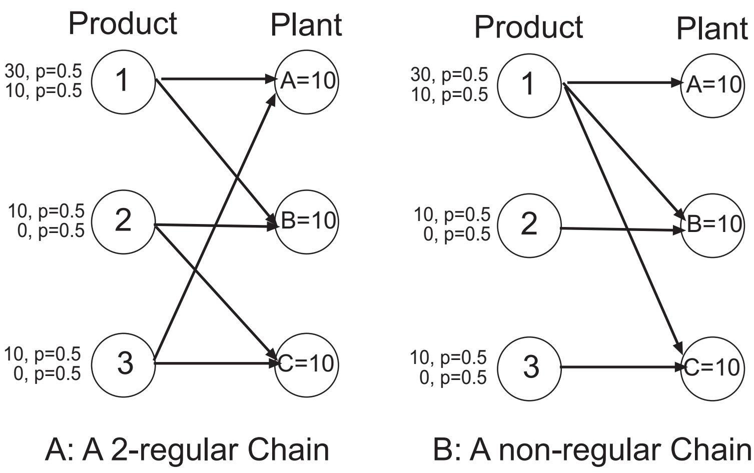



*Fig. 2.3:* Flexibility Structures in an Unbalanced System

sign without any regular chains. However, the corresponding expected excess/maximum flow is 5/25, [2] which indicates that the chaining strategy under-performs in this example.

In addition, it is easy to see that both process structures are fit and the SF indices of structure A are higher than those of structure B. As such, the fact that the expected excess/maximum flow of structure A is higher/lower than that of structure B also contradicts SF theory. ■

---

In this case, with prob. 0.25, node 2 and 3 will together contribute an excess of 10, if their demands are 10 each. This leads to excess flow of $0.5(10 + 0.25\times 10) = 6.25$. The expected maximum flow is 23.75, which is just the mean of total demand (30) minus the excess flow (6.25).

[2] There is excess flow only when the demand at node 1 is 30. This happens with probability 0.5. The expected excess flow is thus $0.5(0.5\times 10 + 0.25\times 20) = 5$, and the expected maximum flow is $30 - 5 = 25$.

## 2.4 Variance and Covariance

To the best of our knowledge, there are very few studies which take into account the impact of the variance and correlational structure of the uncertain parameters. If the variance can be arbitrarily large, then it is conceivable that a sparse process flexibility structure may be much less effective than a fully flexible structure, as demonstrated by the following example.

Consider a system with $n$ demand nodes and $n$ supply nodes, where $D_i = n$ with probability $1/n$; 0 otherwise for $i = 1, \ldots, n$, and $S_j = 1$ for $j = 1 \ldots, n$. Furthermore, the demands are correlated in such a way that $\sum_{i=1}^{n} D_i = n$ for all realizations, i.e., exactly one supply node has a value of $n$ and all other $n-1$ supply nodes are with values 0.

It is easy to see that there will not be any excess flow in the fully flexible system. On the other hand, in any partial flexible system with degree of flexibility bounded by some fixed $k$ (i.e., each supply node has at most $k$ neighbors), the excess flow is at least $n-k$, which is very large for a sparse process flexibility structure.

This result also holds in terms of the maximum flow. It is obvious that the flow for the full flexible system is $n$ and for a partial flexible system with its degree bounded by $k$ is at most $k$. As the number of demand node increases, the gap $n-k$ would be too large. ■

Note that the variance of the supply in the above example is $n-1$, a term which grows with the size of the network. To rule out such extreme cases, we make the following assumption throughout the rest of this study:

Assumption 1: For each $i = 1, \ldots, m$, $D_i < \lambda E(D_i)$ almost surely, for some constant $\lambda > 0$.

Note that we make no assumption on the relationship between different demands, except that the demands are within certain multiple of their means all the time. Moreover, this assumption can be easily adjusted to a system with random supplies and fixed demands, or random supplies and random demands.

We will show that partial flexibility is still a powerful tool to cope with uncertainty when system's variation is not very large in the following chapters. However, as shown in Example 2.3 in which the supplies/demands are not correlated, the chaining structure is not always the best choice. As a matter of fact, the chaining structure might still not be the best choice when the supplies/demands are correlated. We illustrate this observation by slightly modifying Example 2.3 in the following way. Suppose supplies are now correlated. Let us discuss the following 2 cases:

- *Demands are negatively correlated.*

  Suppose $D_2 = D_3 = 15 - \frac{D_1}{2}$ and $D_1$ could be 30 or 10 with equal probability. Obviously, $D_2$ and $D_3$ are negatively correlated to $D_1$ and there are only two possible scenarios of supplies with equal probability: $(30, 0, 0)$ and $(10, 10, 10)$. It is easy to see that the expected excess/maximum flow of structure A is 5/25.[3] and the expected ex-

[3] Expected excess flow $= 0.5 \times 0 + 0.5 \times 10 = 5$. Expected maximum flow $= 30 - 5 = 25$.

cess/maximum flow of structure B is 0/30. Therefore, structure B outperforms the chaining structure A in this case.

- *Demands are positively correlated.*

  Suppose $D_2 = D_3 = \frac{D_1}{2} - 5$ and $D_1$ could be 30 or 10 with equal probability. Obviously, $D_2$ and $D_3$ are positively correlated to $D_1$ and there are only two possible scenarios of supplies with equal probability: $(30, 10, 10)$ and $(10, 0, 0)$. It is clear that the expected excess/maximum flow under both structure A and B is equal to 10/20. The result is not surprising, because the system variance is larger when demands are positively correlated. In fact, the expected excess/maximum flow under fully flexible structure is also equal to 10/20. Therefore, structure B is not worse than A when supplies are positively correlated.

As illustrated in Example 2.3 and the above two cases, it is clear that the performance of structure B is quite robust no matter supplies are independent or correlated. This is no coincidence at all. Indeed, such a robust structure can be derived by applying graph expander theories, as we will explain in chapter 3. However, as demonstrated in Example 2.4, flexibility structure as a tool to cope with uncertainty still has its limit: it only works well when fluctuation is not very large, that is, when Assumption 1 holds. Therefore, when the variation of the system is medium, managers could count on good flexibility structure design to cope with uncertainty. However, when the variation is high, managers should also consider implementing other approaches such as increasing the system's capacity, or to actively engage in variability reduction program.

# 3. FLEXIBILITY STRUCTURES AND GRAPH EXPANDER

In this chapter, we introduce a concept "graph expander" to analyze partial flexibility structures. "Graph expander" is a well known concept and technique in areas of graph theory, communication network design, computer science and etc. We thoroughly explore the intimate relationship between graph expander and flexibility structures in a balanced and identical system, i.e. the system has $n$ products and $n$ plants, the mean of each product's demand is equal to the fixed capacity of a plant. The insights obtained from this study can be used to design good flexibility structures in a general unbalanced and non-identical system.

## *3.1 Graph Expander Review*

The study of expander graphs has been a rapidly developing subject in the area of mathematics and theoretical computer science. The concept of "expander" was first introduced by Bassalygo and Pinsker [6] in the study of communication networks. Over the past three decades, graph expander has been developed into a powerful tool with wide applications in many areas. Among their applications are "the design of explicit super efficient communication networks, constructions of error-correcting codes with very efficient encoding and decoding algorithms, derandomization of random algorithms, and analysis of algorithms in computational group theory." See Sarnak [46] for an entertaining account of this subject.

| | |
|---|---|
| Jordan and Graves [36]<br>Graphs with small number of arcs are almost as *flexible* as Complete Graphs. | Pinsker [43]<br>There exists a sparse graph with almost the same *expansion* as a complete graph. |
| Iravani et. al [33]<br>*Flexibility* can be captured by the largest eigenvalue of an associated matrix. | Tanner [53]<br>Graph *expansion* is bounded by the spectral gap of the adjacency matrix. |

*Tab. 3.1:* The Connections Between Graph Expander and Process Flexibility.

Many important results in the literature on graph expander are quite similar to the findings in the process flexibility area. Table 3.1 lists some examples. One example is about the similar performances of a good sparse structure and the complete graph. Pinsker [43] stated that there exists a sparse structure with $O(n)$ arcs that is almost as expansion as a complete graph in his study on concentration networks. Similarly, Jordan and Graves [36] suggested that a "well-designed" sparse structure with a small number of arcs is almost as flexible as the full flexibility structure, which is indeed a complete graph. Another example is about the similar measures of expansion and flexibility. Tanner [53] mentioned that the graph expansion of a graph is bounded by the spectral gap (the difference between the largest eigenvalue and the second largest eigenvalue of its adjacency matrix) in his study on applying graph expander theories to construct a long error-correcting codes from short error-correcting codes. Similarly, Iravani et. al [33] proposed that the flexibility of a structure can be measured by the largest eigenvalue of its SF matrix. These examples indicate that the concept of flexibility and graph expansion are intimately related. Motivated by these examples, we next use the notion of graph expander to analyze the process flexibility problem in this section.

Definition 1: A bipartite graph $G = (A \cup B, F)$,with partite sets $A$ and $B$, and edge set $F$, is a $(\alpha, \lambda, \Delta)$-expander if deg($v$)$\leq \Delta$ for every $v \in A$, and for all $S \subset A$, with $|S| \leq \alpha|A|$, then

$$|N(S)| \geq \lambda|S|,$$

where $N(S) = \{j \in B : (i, j) \in F, \quad \text{for some } i \in S\}$.

WLOG, we assume $|B| = |A| = n$. It is clear then that $\alpha\lambda \leq 1$. Intuitively, a (bipartite) graph expander is a graph with high connectivity - the neighborhood of small subsets in $A$ is expanded by a factor $\lambda > 1$. In order for these graphs to be interesting, we also impose that they have low degree, i.e., the degree of each node is constrained by $\Delta$. It is remarkable indeed that these graphs exist at all. However, a line of work initiated by Pinsker [43] culminating in recent work of Friedman [22] showed using probabilistic argument that in fact randomly selected graphs enjoy this property with high probability.

Theorem 1: [[5]] For any $n, \lambda \geq 1$, and $\alpha < 1$ with $\alpha\lambda < 1$, there exists a $(\alpha, \lambda, \Delta)$ expander, for any

$$\Delta \geq \frac{1 + \log_2 \lambda + (\lambda + 1) \log_2 e}{-\log_2(\alpha\lambda)} + \lambda + 1.$$

This result is by now folklore in the graph expander community. Note that the lower bound for $\Delta$ is independent of $n$. For completeness, we provide a proof of the above (adopted from Asratian [5]), using the probabilistic argument.

**Proof.** Consider the following probabilistic method to generate a flexibility structure: For each node in $A$, pick $\Delta$ neighbors in $B$ randomly. For each set $U$ with $|U| = z \leq \alpha n$, probability that all neighbors are contained in a set $V$ with $|V| = \lambda z$ is given by $(\lambda z/n)^{z\Delta}$. There are $\begin{pmatrix} n \\ z \end{pmatrix}$ and $\begin{pmatrix} n \\ \lambda z \end{pmatrix}$ ways to choose $U$ and $V$ respectively. Hence the probability that there exists such set $U$ and $V$ is at most

$$g_z = \begin{pmatrix} n \\ z \end{pmatrix} \begin{pmatrix} n \\ \lambda z \end{pmatrix} (\lambda z/n)^{z\Delta} \leq (\frac{ne}{z})^z (\frac{ne}{\lambda z})^{\lambda z} (\lambda z/n)^{z\Delta},$$

using the inequality $\begin{pmatrix} n \\ k \end{pmatrix} \leq (ne/k)^k$. Re-arranging the terms, and using the fact that $z \leq \alpha n$, we have

$$g_z \leq \left[ n^{1+\lambda-\Delta} e^{1+\lambda} \lambda^{\Delta-\lambda} z^{\Delta-\lambda-1} \right]^z \leq \left[ e^{1+\lambda} \lambda (\alpha\lambda)^{\Delta-\lambda-1} \right]^z.$$

By picking $\Delta$ at least as large as the lowerbound as shown in the theorem, we can ensure that $g_z \leq (1/2)^z$. Note that $\alpha\lambda < 1$ is crucial for this to hold. Hence the probability that there exists some set $U$ with $|U| \leq \alpha n$, with $|N(U)| \leq \lambda|U|$, is at most

$$\sum_{z=1}^{\alpha n} g_z < 1.$$

Hence $(\alpha, \lambda, \Delta)$-expander exists. ■

The existence of graph expanders can be established easily using probabilistic method. Explicit construction of graph expanders was proved to be much more difficult and requires a lot of sophisticated tools from number theory and graph theory. Initial attempts to construct good expanders exploited the connection between the expansion ratio and the second largest eigenvalue of the associated adjacency matrix of the underlying (regular) graph (see Lubotzky for the best possible graphs, aptly named Ramanujun graphs, that can be constructed this way using algebraic method). Reingold [44] used combinatorial graph product operation (zig-zag product) to produce large graph with near optimal expansion property. We refer the readers to the numerous surveys and articles for details on this subject.

## 3.2 An Expander is a Good Flexibility Structure

We can use theorem 1 to derive a few important insights for the process flexibility problem. Consider the case when there are $n$ products and $n$ plant nodes in the flexibility problem. All products have demand $D_i$ $(i = 1 \ldots n)$ with mean $\mu$ and all plants have capacity $S_j$ $(j = 1 \ldots n)$. By the assumption 1, there exists $\lambda > 1$ such that $P(D_i \geq \lambda\mu) = 0$. Under the full flexibility model, we expect the average excess to be

$$\boldsymbol{E}\left[\sum_{i=1}^{n} D_i - \sum_{j=1}^{n} S_j\right]^+ = \boldsymbol{E}\left[\sum_{i=1}^{n} D_i - n\mu\right]^+.$$

For a partial flexibility model, let $N(S)$ denote the set of plants $S$ $(S \subseteq A)$ is linked to. By proposition 1, the average excess is

$$\boldsymbol{E}\left[\max_{S\subseteq A}\left\{\sum_{i\in S} D_i - \sum_{j\in N(S)} S_j\right\}\right]^+.$$

Let

$$D_i^* = \begin{cases} D_i & \text{if } \sum_{i=1}^{n} D_i \leq n\mu \\ D_i\left(\frac{n\mu}{\sum_{i=1}^{n} D_i}\right) & \text{if } \sum_{i=1}^{n} D_i > n\mu \end{cases}$$

It is easy to see that $\sum_{i=1}^{n} D_i^* \leq n\mu$. Furthermore, if $\sum_{i=1}^{n} D_i \leq n\mu$, then

$$\sum_{i=1}^{n} D_i = \sum_{i=1}^{n} D_i^*.$$

For each subset $S$,

$$\begin{aligned}\sum_{i\in S} D_i - \sum_{j\in N(S)} S_j &= [\sum_{i\in S} D_i - \sum_{i\in S} D_i^*] + [\sum_{i\in S} D_i^* - \sum_{j\in N(S)} S_j] \\ &= [\sum_{i\in S} D_i - \sum_{i\in S} D_i^*]\chi(\sum_{i=1}^{n} D_i > n\mu) + [\sum_{i\in S} D_i^* - \sum_{j\in N(S)} S_j]\end{aligned}$$

where $\chi(\cdot)$ denote the indicator function.

If the process structure corresponds to an expander graph $(\alpha, \lambda, \Delta)$, then we have

$$\sum_{i\in S} D_i^* - \sum_{j\in N(S)} S_j \begin{cases} \leq \sum_{i\in S} D_i^* - \lambda|S|\mu, & \text{if } |S| \leq \alpha n; \\ \leq \sum_{i\in S} D_i^* - \lambda\alpha n\mu, & \text{if } |S| > \alpha n; \end{cases}$$

since $N(S)$ must contain at least $\lambda \min(|S|, \alpha n)$ plants, each with capacity $\mu$. By our assumption on the demand distribution, $\sum_{i\in S} D_i^* < \lambda|S|\mu$ almost surely. Furthermore, since $\sum_{i=1}^{n} D_i^* \leq n\mu$, we have

$$\sum_{i\in S} D_i^* - \lambda\alpha n\mu \leq (1-\alpha\lambda)n\mu.$$

Thus

$$\sum_{i\in S} D_i - \sum_{j\in N(S)} S_j \leq [\sum_{i\in S} D_i - \sum_{i\in S} D_i^*]\chi\big(\sum_{i=1}^{n} D_i > n\mu\big) + (1-\alpha\lambda)n\mu$$

Hence

$$
\begin{aligned}
& \boldsymbol{E}\left[\max_{S\subseteq A}\left\{\sum_{i\in S} D_i - \sum_{j\in N(S)} S_j\right\}\right]^+ \\
\leq\ & \boldsymbol{E}\left[\max_{S\subseteq A}\left\{(\sum_{i\in S} D_i - \sum_{i\in S} D_i^*)\chi\big(\sum_{i=1}^n D_i > n\mu\big) + (1-\alpha\lambda)n\mu\right\}\right]^+ \\
\leq\ & \boldsymbol{E}\left[\max_{S\subseteq A}(\sum_{i\in S} D_i - \sum_{i\in S} D_i^*)\chi\big(\sum_{i=1}^n D_i > n\mu\big)\right]^+ + (1-\alpha\lambda)n\mu \\
=\ & \boldsymbol{E}\left[\max_{S\subseteq A}\left\{(\sum_{i\in S} D_i - \sum_{i\in S} D_i \frac{n\mu}{\sum_{i=1}^n D_i})\chi\big(\sum_{i=1}^n D_i > n\mu\big)\right\}\right]^+ + (1-\alpha\lambda)n\mu \\
=\ & \boldsymbol{E}\left[\max_{S\subseteq A}\left\{\sum_{i\in S} D_i\left(1 - \frac{n\mu}{\sum_{i=1}^n D_i}\right)\chi\big(\sum_{i=1}^n D_i > n\mu\big)\right\}\right]^+ + (1-\alpha\lambda)n\mu \\
=\ & \Big(0 + (1-\alpha\lambda)n\mu\Big) P\left(\sum_{i=1}^n D_i \leq n\mu\right) \\
& + \boldsymbol{E}\left[\sum_{i=1}^n D_i - n\mu + (1-\alpha\lambda)n\mu \,\middle|\, \sum_{i=1}^n D_i > n\mu\right] P\left(\sum_{i=1}^n D_i > n\mu\right) \\
=\ & \boldsymbol{E}\left[\sum_{i=1}^n D_i - n\mu\right]^+ + (1-\alpha\lambda)n\mu
\end{aligned}
$$

Hence the expander graph obtains an average excess flow which is $(1-\alpha\lambda)n\mu$ above the full flexibility model. Note that the expander graph has $n\Delta$ links, whereas the full flexibility model has $n^2$ links. We have thus obtained the following theorem:

Theorem 2: In a flexibility system with $n$ products (with random demand $D_i$) and $n$ plants (each with capacity $S_j$), if (i) $P(D_i > \lambda\mu) = 0$, (ii) $E(D_i) = \mu$ for all $i$, and (iii) $S_j = \mu$ for all $j$, then there is a partial flexible structure $F$, with $|F| \leq n\Delta$, which attains nearly the benefits of the full flexibility model.

i.e.,

$$\boldsymbol{E}[z_e(F)] \leq \boldsymbol{E}[z_e(A \times B)] + \epsilon n\mu$$

for any $\epsilon > 0$, and for all suitably large $n$.

Note that the above result holds for arbitrarily correlated demand distribution. Using the relationship $z_m(F) = \sum_{i=1}^{m} D_i - z_e(F)$, we also have the following corollary:

Corollary 1: In a flexibility system with $n$ products (with random demand $D_i$) and $n$ plants (each with capacity $S_j$), if (i) $P(D_i > \lambda\mu) = 0$, (ii) $E(D_i) = \mu$ for all $i$, and (iii) $S_j = \mu$ for all $j$, then there is a partial flexible structure $F$, with $|F| \leq n\Delta$, which attains nearly the benefits of the full flexibility model. i.e.,

$$\boldsymbol{E}[z_m(F)] \geq \boldsymbol{E}[z_m(A \times B)] - \epsilon n\mu$$

for any $\epsilon > 0$, and for all suitably large $n$.

## 3.3 Numerical Test

We have shown that an expander structure could capture almost all the benefit of the full flexibility structure theoretically. On the other hand, the merit of cycled chaining strategies have been supported by many simulation studies. Though chaining structures may have good expansion properties, they are still different from expanders. To determine which structure is better, we conduct numerical studies to evaluate the performances of expanders and chaining structures (i.e. regular graph).

We evaluate a flexibility structure in two measures: the average performance and the worst case performance. The former is widely used in practice and theoretical analysis. The latter is also needed to examine the robustness of a structure. A good structure should still perform robustly in the worst case.

As shown in Figure 3.1, we will compare two different flexibility structures with 27 product nodes and 27 plant nodes. One is a regular graph with degree 3, which is a commonly used example of a chaining structures with degree 3 (see Figure 3.1-B). The other one is called as “Levi graph” (see Figure 3.1-A), which is an incidence graph of a generic configuration ([2]). A Levi graph is known to have a high graph expansion.

The Levi graph and regular graph in Figure 3.1 share many common properties: same number of arcs, same out-degree for each node, same number of demand and supply nodes, and similar configurations (they both contain a long chain visiting all supply and demand nodes). The only difference is that the Levi graph does not seem as regular as the regular graph and has better expansion property.

To compare the performances of these two structures, we assume, WLOG, that the supply is 2 for each supply node and the average total demand in the system is 54, which ensures balanced supply and demand. We consider independent demands and correlated demands.

**Independent demand.**

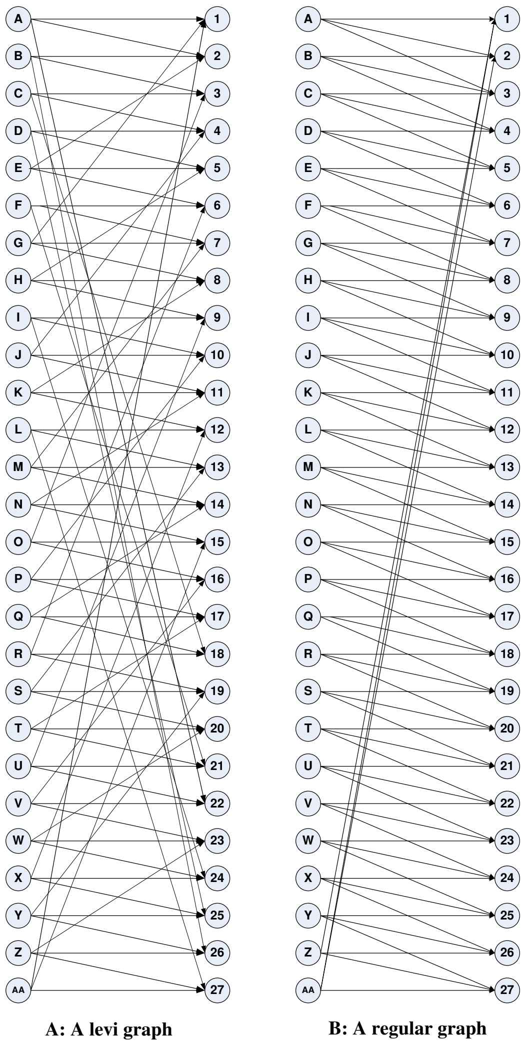
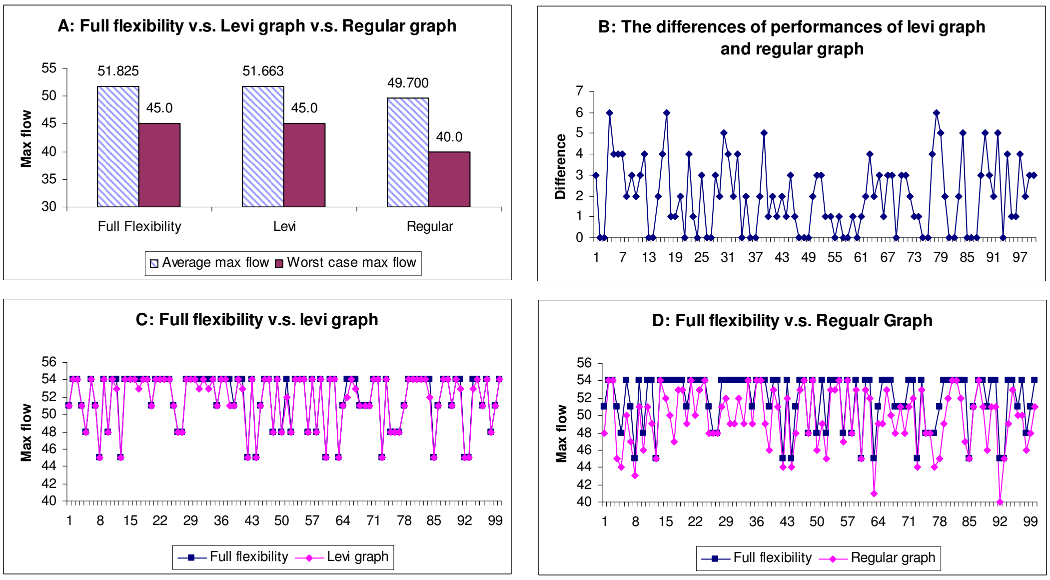



*Fig. 3.1:* A Levi Graph and a Regular Graph with Degree 3.

We assume the demand of a node is either 1 with probability 2/3 or 4 with probability 1/3. Thus, the average total demand is 54. This binomial distribution is used instead of common distributions such as normal distribution because binomial distribution easily sketches out the extreme points of the uncertainty set containing all distributions with a support in [1,4]. We then generate 100 scenarios of demands $\{(S^k_A, \ldots, S^k_{AA}) : k = 1, \ldots, 100\}$ and evaluate the performances of these two structures.

To effectively evaluate the impact of different structures, we eliminate the simulated scenarios in which the total demand is far less than the total supply. This is because when total demand is too small, whether the structure is able to handle uncertainties well has very little impact on the performance. As such, we only consider the simulated scenarios in which the total demand is no less than 45.




**Performance comparisons of different structures when supplies are independent**

*Fig. 3.2:* Comparisons between Levi Graph and Regular Graph when Demands are Independent.

As shown in Figure 3.2-A, the average max flow in Levi graph (51.67) is higher than the regular graph (49.7). Moreover, in the worst case, the maximum flow in Levi graph (45) is much higher than the regular graph (40). Figure 3.2-B shows the difference between the satisfied demand in Levi graph and the regular graph for each scenario. Interestingly, Levi graph outperforms the regular graph all the time. As a matter of fact, Figure 3.2-C shows that Levi graph has the same performance as the full flexibility structure in most scenarios, which implies that Levi graph is almost as flexible as a complete graph when demand and supply are identical and balanced.

**Correlated demand.**

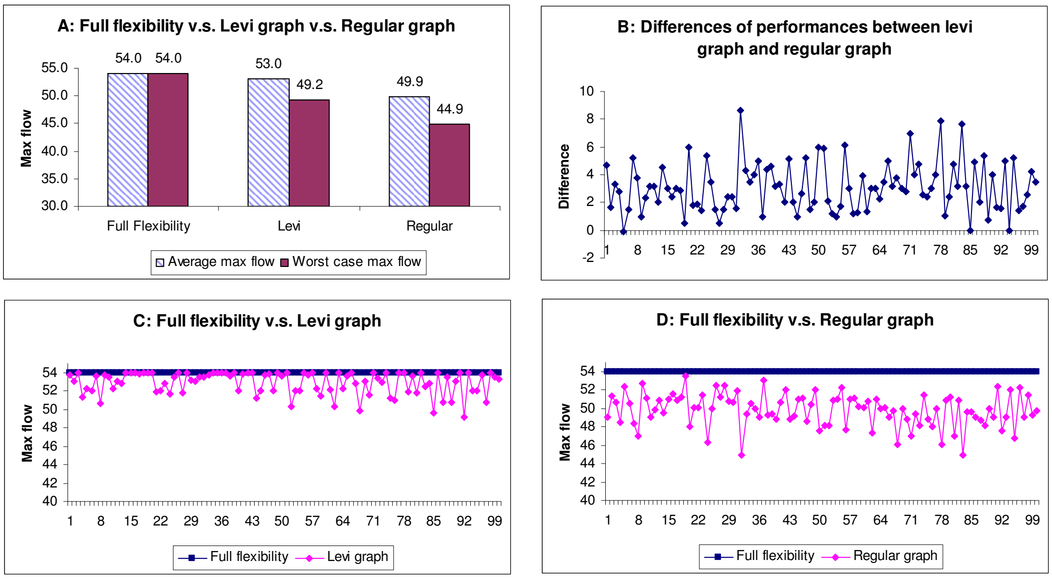
To study the case with correlated demand, we still fix each supply at 2, but generate the demand using the following equation:

$$D_i^k = 54 * \frac{u_i}{\sum_j u_j}, \ \forall\, i, \quad \text{and } k = 1, \ldots, 100,$$

where $u_i$ is a random variable with either value 1 or 4. This generator ensures that the mean of each demand is 2 and the total demand is 54. 100 scenarios of supplies are generated and the performances of different structures are shown in Figure 3.3.

The max flow in Levi graph (53) is still higher than the regular graph (49.9) on average. In the worst case, the max flow in Levi graph (49.2) is much higher than the regular graph (44.9). From Figure 3.3-B, we see that Levi graph outperforms the regular graph all the time. Moreover, as shown in Figure 3.3-C and D, the performance of Levi graph is very close to the full flexibility structure while the gap between the performances of the regular graph and the full flexibility structure is quite large.

In summary, no matter demands are independent or correlated, Levi graph has good and robust performances. Table 3.2 summarizes the perfor-




**Performance comparisons on different structures when supplies are correlated**

*Fig. 3.3:* Comparisons between Levi Graph and Regular Graph When Demands are Correlated.

mances of Levi graph against the regular graph. It is clear that Levi graph, which has a better graph connectivity, is more flexible than the regular graph. Further, its performance is very close to the full flexibility structure. Therefore, graph expansion provides important information in designing a robust flexibility structure.

| | | **Independent** | **Correlated** |
|---|---|---|---|
| **Average performance** | Levi $\geq$ Regular | $\surd$ | $\surd$ |
| | Levi $\approx$ Full | $\surd$ | $\surd$ |
| **Worst case performance** | Levi $\geq$ Regular | $\surd$ | $\surd$ |
| | Levi $\approx$ Full | $\surd$ | |

*Tab. 3.2:* Summary on the Performances of the Levi Graph and the Regular Graph.

## 3.4 Expander Heuristic

We have, for the case when the demand and supply nodes have the same mean and when total supply and total demand are balanced, established a connection between the subject of graph expander and process flexibility structure. For the case with non-identical supply and demand, We use the notion of "constraint sampling" method to prove in the next chapter that a similar theorem stating that a well designed sparse flexibility structure can capture almost all the benefit of a full flexibility structure. However, the analysis using constraint sampling establishes the existence of such a structure, but does not provide any guideline on the characteristics of good process flexibility structures. In this section, we build on the understanding of graph expansion concept to provide valuable insights that can be readily exploited for the design of good process flexibility structure.

Since our ultimate goal is to build a sparse process structure with high flexibility, we will use a greedy approach to build as much "flexibility" as possible into the system by adding one link at a time. Note that Ghosh and Boyd [25] have also recently proposed a heuristic to design graph with high connectivity, for the case of identical supply and demand. Our heuristic, on the other hand, works well in the case of non-identical supply and demand.

Definition 2: We define the node-expansion ratio for demand node $i$ in a flexibility structure $F$ to be

$$\delta_i = \frac{\sum_{j:(i,j)\in F} S_j}{\mu_i}.$$

Similarly, the node-expansion ratio for supply node $j$ in the flexibility structure $F$ is defined as

$$\delta_j = \frac{\sum_{i:(i,j)\in F} \mu_i}{S_j}.$$

Definition 3: The expansion vector of a flexibility structure $F$ is defined as:

$$\rho(F) = \left(\min_{i\in A} \delta_i, \min_{j\in B} \delta_j\right).$$

From Theorem 2 and Corollary 1, we know that an expander graph will provide a good flexibility structure. We also note that if $F$ is an expander, $\rho(F)$ will have large entries. Therefore, to obtain a sparse yet flexible structure, we aim to increase the expansion vector $\rho(F)$ while keeping the number of links low, which provides the basic logic of our heuristic.

Figure 3.4 briefly describes the steps of our heuristic. There are two main phases in this heuristic: getting a good base assignment (step 1) and incrementally adding arcs into the base assignment (step 2 to the end).

### Phase 1: Base Assignment Selection

To construct a flexible routing structure, we start with finding a good base (dedicated) assignment. In some cases (e.g. [36]), the currently used (dedicated) assignment is well designed and can be used as the base assignment. However, in practice, many currently adopted dedicated operational system is far from optimal. Therefore, we formulate the following stochastic integer programming problem to solve for the optimal dedicated system, in which

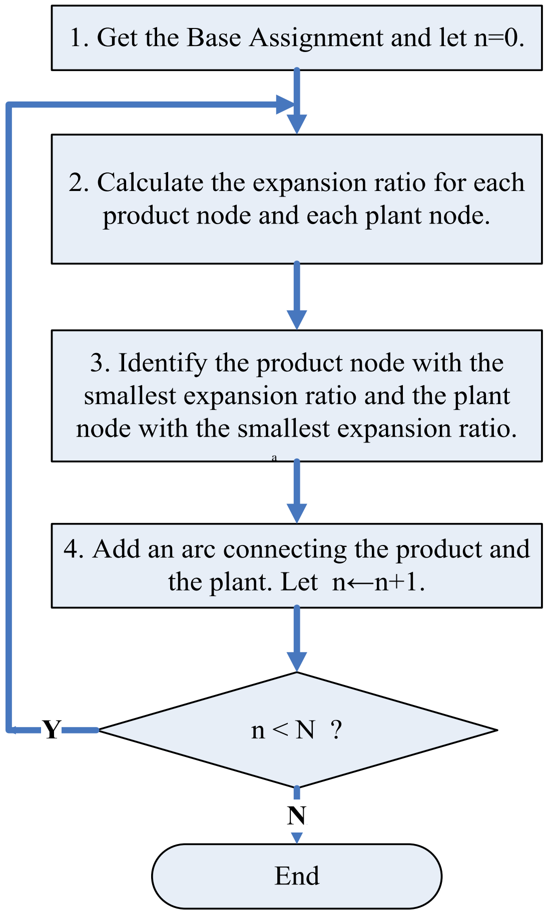



*Fig. 3.4:* Steps of Expander Heuristic.

each product will be produced in exactly one plant and each plant will produce at least one product:

$$
\begin{aligned}
IP_D = \min \quad & \boldsymbol{E}\Big[\sum_{j=1}^{n}\Big(\sum_{i=1}^{m} D_i x_{ij} - S_j\Big)^+\Big] \\
s.t. \quad & \sum_{i=1}^{m} x_{ij} \geq 1 \ \forall j \\
& \sum_{j=1}^{n} x_{ij} = 1 \ \forall\ i \\
& x_{ij} \in \{0,1\} \ \forall\ i,j
\end{aligned}
$$

We use the standard sampling average approach to solve $IP_D$ approximately - sample $M$ scenarios $\{(D_1^k, \ldots, D_m^k) : k = 1, \ldots, M\}$ (for some suitably large $M$), from the joint distribution of the supplies, and solve the following integer programming problem:

$$
\begin{aligned}
SIP_D = \min \quad & \frac{1}{M}\Big\{\sum_{k=1}^{M}\Big(\sum_{j=1}^{n}\Big(\sum_{i=1}^{m} D_i^k x_{ij} - S_j\Big)^+\Big)\Big\} \\
s.t. \quad & \sum_{i=1}^{m} x_{ij} \geq 1 \ \forall j \\
& \sum_{j=1}^{n} x_{ij} = 1 \ \forall\ i \\
& x_{ij} \in \{0,1\} \ \forall\ i,j
\end{aligned}
$$

The function $\left(\sum_{i=1}^{m} D_i^k x_{ij} - S_j\right)^+$ is piecewise linear convex and hence the above can be turned into a linear integer programming problem.

### Phase 2: Incremental Improvement

We then incrementally add arcs into the base assignment to improve its expansion vector, terminating when we reach the upper limit $N$ of the number of links allowed. We calculate the expansion ratios for all supply/demand nodes and identify the demand and supply node with the smallest expansion ratio. We then add an arc connecting these two nodes. In the case when these two nodes have already been connected, we select an arc which connects a demand node with the next smallest expansion ratio and the supply node with the smallest expansion ratio.

There are several advantages of this heuristic. Firstly, we can control the graph density by imposing the maximum number of arcs that can be added to the optimal dedicated network. It can be seen as the budget of investment on flexibility. Secondly, this heuristic is simple and effective. We do not need to go through intensive simulation for testing out the performances of various flexibility structures. Instead, our heuristic keeps improving the expansion vector with each added arc, which ensures a well-connected structure in the end.

## 3.5 Measure Flexibility via Expansion Index

Besides designing a good partial flexibility structure, another issue pertaining to the study of flexibility structures is how to measure flexibility. Good flexibility measures would help a manager to evaluate the flexibility of current operating system, or/and to select a suitable flexibility system from many candidate structures.

A good flexibility index should possess the following three properties: (i) easy to compute, (ii) minimal and accessible input requirement, and (iii) applicable to various situations. The prevailing flexibility indices, however, hardly have all the properties. For example, the Jordan and Graves' index [36], the probability that the unfilled demand for any subset of demand nodes is more than that of the full flexibility structure, hardly possesses property (i) and (ii) due to the demanding computational requirement (it is usually hard to compute the joint probability except for the situation that demands (or supplies) follow independent normal distributions) and the requirement for exact demand/supply distribution of each product/plant as inputs. Another widely-used indices, SF indices [33], is easy to compute and requires little information of demand/supply, but cannot be applied to a system with disconnected sub-graphes.

In this section, we developed a new index possessing all the three properties, based on the concept of graph expansion.

For a bipartite graph $G = (A \cup B, F)$, the incidence matrix $\mathcal{T}$ is defined in the following way: Each column of $\mathcal{T}$, say $\mathcal{T}_l$, represents an edge $l \in F$ with $\mathcal{T}_{li} = 1$, $\mathcal{T}_{lj} = -1$ and $\mathcal{T}_{lk} = 0, \forall k \neq i, j$, if the edge $l$ connects node $i \in A$ to $j \in B$. The Laplacian matrix $\mathcal{L} = \mathcal{T}\mathcal{T}'$ is a positive semidefinite matrix. The connectivity of the graph $G$ can be measured by the second smallest eigenvalue $\lambda_2(\mathcal{L})$ of $\mathcal{L}$, also known as the algebraic connectivity of $G$.

The relationship between $\lambda_2(\mathcal{L})$ and the connectivity property of $G$ is well-knwon (cf. [20, 25] and the literature therein). For instance, $\lambda_2(\mathcal{L}) > 0$ if and only if $G$ is connected. Furthermore, for any two graphs $G_1 = (A \cup B, F_1)$ and $G_2 = (A \cup B, F_2)$, if $F_1 \subseteq F_2$, then $\lambda_2(\mathcal{L}_1) \leq \lambda_2(\mathcal{L}_2)$. These properties suggest that $\lambda_2(\mathcal{L})$ can be used as an index to rank process flexibility structures - in general, a graph with more links are more flexible, and connected graphs are generally more flexible than disconnected graphs.

The above discussion is based on the assumption that supply and demand are balanced and identical. We can easily extend the notion of $\lambda_2(\mathcal{L})$ to a weighted graph by letting $\mathcal{T}_{li} = \sqrt{\mu_i S_j}$ and $\mathcal{T}_{li} = -\sqrt{\mu_i S_j}$ in the generalized unbalance and non-identical situation.

Definition 4: The expansion index for a structure $F$ is defined as $\lambda_2(F)$, which is the second smallest eigenvalue of $\mathcal{L}$, where $\mathcal{L} = \mathcal{T}\mathcal{T}'$.

The expansion index possesses all the three desired properties of a good index. It is easy to compute, only requires the demand/supply mean for each node as input, and can be applied to a general system with different number of supply nodes and demand nodes and unbalance supply and demand. Table 3.3 compares the Jordan and Graves index, SF indices and expansion index in respect of the three properties, and indicates that the expansion index is very practical.

We next use two numerical experiments to examine the power of the expansion index. The first experiment is a comparison between the Levi graph and regular graph discussed in chapter 3.3, and the second experiment evaluates the expansion index and SF indices via the examples used by Iravani et al [33].

We compare our expansion index with SF indices for evaluation. Remember the SF indices are simple tractable measures which can be used to "rank" the process structures. Consider the SF matrix $(M_{ij})_{i,j=1}^{n}$, where $M_{ij}$ $(i \neq j)$ represents the number of non-overlapping routes from supply node

| Index | Attributes | | |
|---|---|---|---|
| | Computational load | Input info. | Applicable areas |
| Jordan & Graves Index | Need to compute the joint probability of multiple variables. Only tractable when demands/supplies follow independent normal distribution. | The probability density function of each demand/supply node is needed. | The index's computation restrictions limits its application. The index may only be suitable to systems with a few number of supply/demand nodes. |
| SF indices | Can use a simple max-flow model to compute the total number of non-overlap routes between demand node $i$ and $j$. | The structure should "fits" the environment, i.e. the demand of each product should be satisfied on average. | Cannot be applied to a disconnected structure. |
| Expansion Index | Easiest to compute. | The mean of each demand/supply node is needed. | Suitable to a non-identical and unbalanced generalized system. |

*Tab. 3.3:* Comparisons among Difference Flexibility Indices

$i$ to $j$, and $M_{ii}$ represents the number of arcs connected to node $i$. The SF indices can be defined in terms of (i) the mean of the entries in $M$; or (ii) the dominated eigenvalue (i.e. largest eigenvalue) of $M$. A process structure with higher SF indices would be more flexible. Note the Jordan and Graves' index is not applicable here because the demand distribution is assumed unknown for both experiments.

- **Levi graph and regular graph.**

  This experiment tests the power of the expansion index using the example discussed in section 3.3. As shown in Table 3.4, $\lambda_2$(Levi) (0.55) is much higher than $\lambda_2$(Regular) (0.05), which is consistent with our simulation analysis. On the other hand, SF indices will give the highest rank to the regular graph among all structures with the same number of arcs.

- **Structural Flexibility examples.**

  To compare our index with SF indices, several numerical examples used by Iravani et al [33] are re-examined here. The purpose of the comparison is to check whether our index will get the same results as SF indices. We select two representative examples from the study of Iravani et al [33]. One example is a group of structures with random demand $\mu = (1.5, 1, 0.5, 0.5, 1, 1.5)$ and fixed capacity $S = (1, 1, 1, 1, 1, 1)$ as shown in Figure 3.5, which represents the non-identical demand/supply situation. The other one is a group of structures with random demand $\mu = (1, 1, 1, 1, 1, 1, 1, 1)$ and fixed capacity $S = (1, 1, 1, 1, 1, 1, 1, 1)$ as shown in Figure 3.6, which represents the identical demand situation.

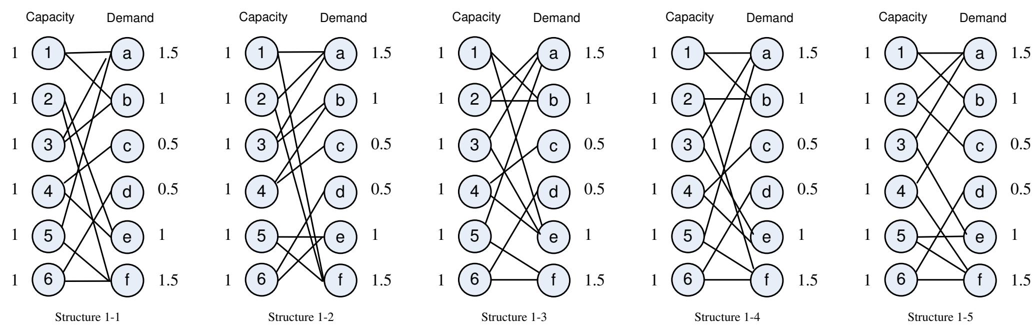



*Fig. 3.5:* SF Group 1: Structures with Demand $\mu = (1.5, 1, 0.5, 0.5, 1, 1.5)$

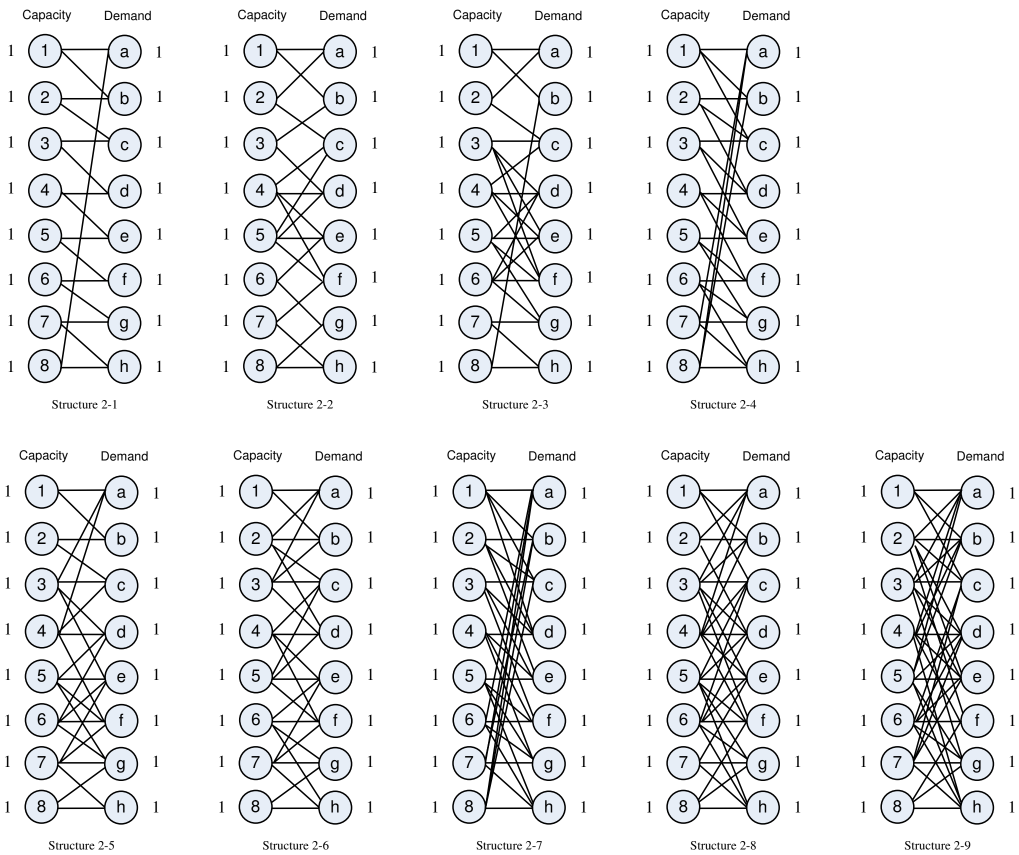



*Fig. 3.6:* SF Group 2: Structures with Demand $\mu = (1, 1, 1, 1, 1, 1, 1, 1)$

We obtain estimate for $E(z_e(F))$, the expected excess flow of structure $F$, to be used as the benchmark to evaluate the power of the expansion index and SF index. $E(z_e(F))$ is obtained from sampling 200 scenarios of demand, assuming uniformly distributed in (0, $2\mu_i$).

In both cases, the ranks obtained from the expansion index are consistent with the rank given by $E(z_e(F))$. The ranks given by SF index, on the other hand, are slightly different from the expansion index. Since the difference in performance, among different structures, are quite small due to the symmetric demand/supply settings provided in the example, it is hard to tell whether the expansion index outperforms the SF index or not. Nevertheless, the simulation result indeed suggest that our expansion index could be at least as good as SF indices.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Levi & regular | Structures | Levi | Regular | | | | | | | |
| | $\lambda_2(F)$ | 0.551 | 0.054 | | | | | | | |
| | # of arcs | 81 | 81 | | | | | | | |
| SF Group 1 | Structures | 1-1 | 1-2 | 1-3 | 1-4 | 1-5 | | | | |
| | $\lambda_2(F)$ | 0.130 | 0.111 | 0.141 | 0.156 | 0.194 | | | | |
| | SF $\mathcal{I}_{ei}$ | 7.53 | 7.61 | 8.34 | 8.46 | 9.58 | | | | |
| | # of arcs | 12 | 12 | 12 | 12 | 12 | | | | |
| | $E(z_e(F))$ | 0.74 | 0.77 | 0.72 | 0.71 | 0.66 | | | | |
| SF Group 2 | Structures | 2-1 | 2-2 | 2-3 | 2-4 | 2-5 | 2-6 | 2-7 | 2-8 | 2-9 |
| | $\lambda_2(F)$ | 0.152 | 0.198 | 0.218 | 0.586 | 0.338 | 0.298 | 1.387 | 0.966 | 1.806 |
| | SF $\mathcal{I}_{ei}$ | 16.00 | 18.25 | 20.93 | 24.00 | 23.68 | 26.17 | 32.00 | 34.12 | 40.52 |
| | # of arcs | 16 | 20 | 24 | 24 | 26 | 28 | 32 | 36 | 42 |
| | $E(z_e(F))$ | 0.79 | 0.66 | 0.64 | 0.57 | 0.61 | 0.62 | 0.5658 | 0.5660 | 0.5658 |

*Tab. 3.4:* Comparisons among Flexibility indices

# 4. FLEXIBILITY STRUCTURES AND CONSTRAINT SAMPLING

We have proven the existence of a good sparse flexibility structure in the balanced and identical situation using the graph expander concept. In this chapter, we extend our study to the general unbalanced and non-identical situation. To avoid the difficulty and complexity of graph theory techniques, we use a different tool, constraint sampling, to study the flexibility models.

We start from the max-flow model of full flexibility structure.

$$
\begin{aligned}
(MF): \quad z_m(A \times B) = \quad & \max && \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \\
& s.t. && \sum_{j=1}^{n} x_{ij} \le D_i \quad \forall\ i = 1,\ldots,m; \\
& && \sum_{i=1}^{m} x_{ij} \le S_j \quad \forall\ j = 1,\ldots,n; \\
& && x_{ij} \ge 0 \quad \forall \quad i = 1,\ldots m,\ j = 1,\ldots,n.
\end{aligned}
$$

This max-flow model has $m \times n$ decision variables and $m+n$ constraints [1]. If $D_i$ and $S_j$ are constant for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$, it is

[1] We ignore the nonnegative constraint here because the nonnegative parameters $S_j$ and $D_i$ in the max-flow model ensure the nonnegativity of decision variables.

well known that there exists an optimal solution $x^*$ with support in no more than $m + n$ decision variables. There is no loss of optimality if the other variables are discarded. Since $x_{ij}$ represents the network flow in arc $ij$ of the full flexibility structure, removing a non-supportive variable $x_{ij}$ is the same as removing arc $ij$ from the full flexibility structure. Hence, it is not hard to tell that the non-zero optimal solution $x^*$ represents a good sparse flexibility structure with at most $m + n$ arcs, and this structure has exactly the same performance as the full flexibility structure. In this chapter, we further investigate this phenomenon when $D_i$ and $S_j$ are random. Our goal is to identify a good sparse structure $F$ capturing almost all benefit of full flexibility structure. More specifically, $F$ satisifies

$$E_{D,S}\bigg(z_m(F)\bigg) \geq (1-\epsilon)E_{D,S}\bigg(z_m(A \times B)\bigg).$$

and

$$|F| \approx O(m+n) << mn.$$

We will use a simple and effective probabilistic constraint sampling method to analyze the problem. This method is based on a recent elegant result obtained by Calafiore and Campi [13] (see also de Farias and Van Roy [18]). To provide a good understanding of the constraint sampling approach, we will briefly review the result in the following section.

## 4.1 Constraint Sampling Review

Ever since the seminar paper by Dantzig [16] on linear programming under uncertainty, there have been a flurry of activities in the area of optimization under uncertainty(cf. [7] [41] and [14] for recent progress in this area ). This area of work focuses on finding a good solution to optimization problem with uncertainty in the data. The key idea is to represent the inherent uncertainty in the data using an "uncertainty set" to ensure tractability of the new optimization model, while maintaining a good approximation of the underlying uncertainty in the problem parameters. The robust optimization approach, for instance, can also be potentially used in a two stage stochastic integer programming framework to design a good support set for our problem, although it is not clear how the sparsity of the support set can be derived this way. We used instead a recent elegant constraint sampling approach by Calafiore and Campi [13] for our analysis.

The idea of constraint sampling is also independently developed by de Farias and Van Roy [18] in their study of the approximate dynamic programming method. In fact, they are more interested in estimating the error from constraint sampling, in the approximation for the cost-to-go function. Their approach and analysis, although couched in a different context, uses sampling distribution based on optimal policy to the original problem, and is very similar to the key idea used here. However, we take a small step further, and identify a condition on the behavior of the optimal solution to ensure that the error (loss of optimality) introduced in the constraint sampling approach will be small. More importantly, we show that this approach can be used to analyze various flexibility design problems in the operations management literature.

The problem addressed by Calafiore and Campi [13] can be formulated as follows:

$$UCP : \left\{ \min \ c^T x \ : f(x, \delta) \leq 0, \ \delta \in \mathbf{\Delta}, \ x \in \Psi \subseteq \mathbb{R}^n \right\},$$

where $x$ is the decision variable, $\Psi$ is a convex and closed region, $\delta$ is the random parameter in set $\mathbf{\Delta}$ ($\mathbf{\Delta} \subset \mathbb{R}^l$), and $f(x, \delta)$ is continuous and convex in $x$ for all $\delta$.

UCP problem is a difficult computational optimization problem since it involves infinitely many constraints, as the set $\mathbf{\Delta}$ may already be uncountable. Instead, Calafiore and Campi [13] proposed the "randomized constraint sampling" approach to construct "$\epsilon$-robust" feasible solution (i.e., the probability that the solution obtained will violate a random constraint is less than $\epsilon$). They generated $N$ constraints by sampling the uncertain parameter from $\mathbf{\Delta}$, and solved instead the following problem:

$$UCP_N : \left\{ \min \ c^T x \ : f(x, \delta^k) \leq 0, \ k = 1, \ldots, N, \ x \in \Psi \subseteq \mathbb{R}^n \right\},$$

where $\delta^k$'s are the parameters sampled.

They showed that the solution of the new problem will only violate a tiny portion of the original constraints if $N$ is large enough. Specifically, if $N \geq \frac{n}{\epsilon\beta} - 1$, the probability that optimal solution of $UCP_N$ (say $\hat{x}_N$) is $\epsilon$-robust feasible is more than $1 - \beta$. Here, $n$ is the dimension of $x$ and

$\epsilon, \beta \in (0, 1]$.

It is obvious that Calafiore and Campi's result is also valid for Uncertain Linear Programming (ULP) problems (cf. [18]), where the constraints $f(x, \delta) \leq 0$ are linear. We will briefly describe the intuition of the proof in this case, as it applies to our problem. Note that in our approach, the non-negative constraints $x \geq 0$ are always included in the subproblems, and we only sample from the other constraints in $\Delta$.

Their proof is based on the following idea: Let $z^{(1)}, \ldots, z^{(N+1)}$ represent $N+1$ independent random variable with same distribution in $\mathbf{\Delta}$. Construct the following N+1 problem:

$$ULP_N^k : \min \left\{ c^T x : \; x \geq 0, \; f(x, z^{(i)}) \leq 0, \; i = 1, \ldots, k-1, k+1, \ldots, N+1 \right\},$$

where $k = 1, 2, \ldots, N+1$.

Let $\hat{x}_N^k$ denote an optimal solution of $ULP_N^k$. In the case of multiple optimal solutions, we choose one which is lexicographically maximal. In addition, define a problem $ULP^{N+1}$ consisting of all $N+1$ constraints:

$$ULP^{N+1} : \min \left\{ c^T x : \; x \geq 0, \; f(x, z^{(i)}) \leq 0, \; i = 1, \ldots, N+1, \right\}$$

and denote $\hat{x}^{N+1}$ as the optimal solution of $ULP^{N+1}$.

Note that $ULP^{N+1}$ and $ULP_N^k$ differ in just one constraint - $f(x, z^{(k)}) \leq 0$. In the event that this constraint is not tight at the optimal solution for

$ULP^{N+1}$, we have $\hat{x}^{N+1} = \hat{x}_N^k$, and hence

$$f(\hat{x}_N^k, z^{(k)}) \leq 0.$$

As there are relatively fewer tight constraints than total number of constraints, if $N$ is sufficiently large, the event $f(\hat{x}_N^k, z^{(k)}) \leq 0$ holds with very high probability.

Proposition 2: [13] The probability that $\hat{x}_N^k$ will violate the $k$th (sampled) constraint $f(x, z^{(k)}) \leq 0$ is bounded above by $n/(N+1)$, i.e.,

$$\boldsymbol{P}\Big(f(\hat{x}_N^k, z^{(k)}) > 0\Big) \leq \frac{n}{N+1},$$

where $n$ is the dimension of decision variable $x$.

Note that for each $k$, the random variable $\hat{x}_N^k$ has the same distribution as $\hat{x}^N$, the solution obtained by solving $ULP^N$ with $N$ independently and randomly generated constraints. If the $j$th constraint in $\Delta$ is sampled with probability $q_j$ in the experiment, the proposition ensures that

$$\sum_{j \in \Delta} q_j P\Big(f(\hat{x}_N^k, z^{(k)}) > 0 | z^{(k)} = z^j\Big) = \sum_{j \in \Delta} q_j P\Big(f(\hat{x}^N, z^j) > 0\Big) \leq \frac{n}{N+1}. \tag{4.1}$$

Note that the above results hold as long as $z^{(k)}$'s are sampled in identical manner, using the probability distribution endowed on the space of random parameters $\Delta$.

## 4.2 Identifying Sparse Support Set

It is not difficult to fit the max-flow model (MF) into a general LP model as following.

$$(P): \ z_m(A \times B) = \max\left\{\sum_{i=1}^{\tilde{n}} c_i x_i : Ax \leq b, \ x_i \geq 0, \ i = 1, \ldots, \tilde{n}\right\},$$

where $A$ is a $\tilde{m} \times \tilde{n}$ matrix, and $b$ is random. $(MF)$ is just the special case of $(P)$ by letting $b = (D_1, \ldots, D_m, S_1, \ldots, S_n)^T$ and $A$ be the corresponding coefficient matrix with $\tilde{n} = mn$ and $\tilde{m} = m + n$. The following analysis will focus on the general model $(P)$.

The dual $(D)$ can be formulated as

$$z_m(A \times B) = \min\left\{\sum_{j=1}^{\tilde{m}} b_j y_j : A^T y \geq c, \ y_j \geq 0, \ j = 1, \ldots, \tilde{m}\right\}.$$

$(D)$ is a linear programming problem with $\tilde{m}$ variables and $\tilde{n}$ constraints. We can easily construct a structure $F$ with $N$ links by sampling $N$ constraints from $(D)$, and obtain the corresponding primal model

$$(D(F)): \quad z_m(F) = \min\left\{\sum_{j=1}^{\tilde{m}} b_j y_j : A_F^T y \geq c_F, \ y_j \geq 0, \ j = 1, \ldots, \tilde{m}\right\}.$$

and the dual model for $F$

$$(P(F)): \quad z_m(F) = \max\left\{\sum_{i \in F} c_i x_i : A_S x_F \leq b, \ x_i \geq 0, \ i \in F\right\}.$$

Note that unlike the uncertain convex programming problem, we do not have an endowed distribution for the set of constraints. The selection of the sampling distribution plays a key role in our analysis. We discuss next how the sampling distribution can be obtained.

Let $x^*$ denote an optimal solution in $z_m(A \times B)$. Note that since $b$ is random, $x^*$ is also a random vector. We assume that problem $(P)$ has an optimal solution $x^*$ with the following property:

(Property $\boldsymbol{A}$): $x_i^* \leq \lambda E(x_i^*)$ almost surely for some constant $\lambda > 0$ independent of $\tilde{n}$, and for all $i = 1, \ldots, \tilde{n}$.

The above property essentially states that the optimal primal solution $x^*$, as a function of the random $b$, should not be too far above its mean value. This property indeed holds for serval important classes of optimization problem. We will show in the next section that this property holds for the flexibility problem.

Theorem 1: Suppose Property $\boldsymbol{A}$ holds for $(P)$. Then there exists a set $F$ with cardinality $N = O(\frac{\lambda \tilde{m}}{\epsilon})$, such that

$$E_b(z_m(F)) \geq (1 - \epsilon) E_b(z_m(A \times B)).$$

We prove the above result via constraint sampling. We sample $N$ constraints in $(D)$ with replacement. The $i$th constraint in problem $(D)$ is selected with probability

$$\frac{c_i E(x_i^*)}{\sum_{i=1}^{\tilde{n}} c_i E(x_i^*)}.$$

We denote the set of constraints obtained by $F$.

**Proof.** For fixed $b$, let $x^*$ and $y^*$ denote the corresponding optimal primal and dual solution in $(P)$ and $(D)$ respectively. Similarly, let $x^*(F)$ and $y^*(F)$ denote the corresponding optimal primal and dual solution in $(P(F))$ and $(D(F))$ respectively.

$$\begin{aligned}
z_m(F) &= \sum_{j=1}^{\tilde{m}} b_j y_j^*(F) \\
&= \sum_{j=1}^{\tilde{m}} b_j y_j^*(F) + z_m(A \times B) - \sum_{i=1}^{\tilde{n}} c_i x_i^* \\
&\geq z_m(A \times B) + \sum_{j=1}^{\tilde{m}} y_j^*(F) \left( \sum_{i=1}^{n} A_{ji} x_i^* \right) - \sum_{i=1}^{\tilde{n}} c_i x_i^* \\
&= z_m(A \times B) + \sum_{1=1}^{\tilde{n}} x_i^* \left( \sum_{j=1}^{\tilde{m}} A_{ji} y_j^*(F) \right) - \sum_{i=1}^{\tilde{n}} c_i x_i^* \\
&= z_m(A \times B) + \sum_{i=1}^{\tilde{n}} \left( \sum_{j=1}^{\tilde{m}} A_{ji} y_j^*(F) - c_i \right) x_i^*.
\end{aligned}$$

Note that

$$\left( \sum_{j=1}^{\tilde{m}} A_{ji} y_j^*(F) - c_i \right) x_i^* \geq -c_i x_i^*$$

for all $y^*(F)$, but when the constraint $\sum_{j=1}^{\tilde{m}} A_{ji} y_j \geq c_i$ holds for $y^*(F)$, then

$$\left( \sum_{j=1}^{\tilde{m}} A_{ji} y_j^*(F) - c_i \right) x_i^* \geq 0.$$

Hence

$$
\begin{aligned}
E_F(z_m(F)) &\geq z_m(A\times B) + E_F\left[\sum_{i=1}^{\tilde{n}}\left(\sum_{j=1}^{\tilde{m}} A_{ji}y_j^*(F) - c_i\right)x_i^*\right] \\
&\geq z_m(A\times B) - E_F\left[\sum_{i=1}^{\tilde{n}} c_i x_i^* \middle| \sum_{j=1}^{\tilde{m}} A_{ji}y_j^*(F) < c_i\right] P\left(\sum_{j=1}^{\tilde{m}} A_{ji}y_j^*(F) < c_i\right) \\
&\geq z_m(A\times B) - \left[\sum_{i=1}^{\tilde{n}} c_i \lambda E_F(x_i^*) P\left(\sum_{j=1}^{\tilde{m}} A_{ji}y_j^*(F) < c_i\right)\right] \qquad \text{[by Property } A\text{]} \\
&= z_m(A\times B) - \lambda\left(\sum_{j=1}^{\tilde{n}} c_j E_F(x_j^*)\right)\left[\sum_{i=1}^{\tilde{n}} \frac{c_i E_F(x_i^*)}{\sum_{k=1}^{\tilde{n}} c_k E_F(x_k^*)} P\left(\sum_{j=1}^{\tilde{m}} A_{ji}y_j^*(F) < c_i\right)\right] \\
&\geq z_m(A\times B) - \frac{\lambda\tilde{m}}{N+1}\left(\sum_{j=1}^{\tilde{n}} c_j E_F(x_j^*)\right) \qquad \text{[from (4.1)]}
\end{aligned}
$$

Since $\sum_{i=1}^{\tilde{n}} c_i E_F(x_i^*) = E_F(z_m(A\times B))$,

$$E_{b,F}(z_m(F)) \geq (1-\epsilon)E_b(z_m(A\times B)).$$

Hence there exists a sparse support set $F$ with cardinality $N = O(\frac{\lambda\tilde{m}}{\epsilon})$ (independent of $\tilde{n}$) such that $E_b(z_m(F)) \geq (1-\epsilon)E_b(z_m(A\times B))$. ■

## 4.3 Sparse Flexibility Structure

From section 4.2, we know that a good sparse structure can be obtained by sampling the constraints of model (D) using certain sampling probability, assuming that $x^* \leq \lambda E(x^*)$. In this section, we will study how to find a proper $x^*$ for the max-flow model $(MF)$ and identify the conditions satisfying

$x^* \leq \lambda E(x^*)$.

The following simple structural result on the maximum flow will be useful for our analysis:

Lemma 2: For a realization $(D_1, \ldots, D_m)$ of demand and $(S_1, \ldots, S_n)$ of supply, an optimal solution of (P) is

$$x^*_{ij} = \frac{D_i S_j}{\max\left\{\sum_{i=1}^m D_i,\ \sum_{j=1}^n S_j\right\}}, \ \forall i = 1, \ldots, m, j = 1, \ldots, n,$$

and

$$z_m(A \times B) = \min\left\{\sum_{i=1}^m D_i,\ \sum_{j=1}^n S_j\right\}.$$

**Proof.** For full flexibility structure $A \times B$, it is easy to see that the optimal max flow through the structure is the minimum of the inflow (total demand) and the outflow (total capacity). Therefore $z_m(A \times B) = \min\left\{\sum_{i=1}^m D_i, \sum_{j=1}^n S_j\right\}$. We can also show that

$$\begin{aligned} x^*_{ij} &= \frac{D_i S_j}{\max\left\{\sum_{i=1}^m D_i, \sum_{j=1}^n S_j\right\}} \\ &= \left(\frac{D_i}{\sum_{i=1}^m D_i}\right)\left(\frac{S_j}{\sum_{j=1}^n S_j}\right)\min\left\{\sum_{i=1}^m D_i, \sum_{j=1}^n S_j\right\} \\ &= \left(\frac{D_i}{\sum_{i=1}^m D_i}\right)\left(\frac{S_j}{\sum_{j=1}^n S_j}\right)z_m(A \times B) \end{aligned}$$

$x^*$ is a feasible solution of the primal problem $(PF)$, because

$$\sum_{j=1}^{n} x_{ij}^* \leq D_i, \ \forall i = 1, \ldots, m;$$

and

$$\sum_{i=1}^{m} x_{ij}^* \leq S_j, \ \forall j = 1, \ldots, n.$$

Since

$$\sum_{i=1}^{m}\sum_{j=1}^{m} x_{ij}^* = \sum_{i=1}^{m}\left(\frac{D_i}{\sum_{i=1}^{m} D_i}\right)\sum_{j=1}^{n}\left(\frac{S_j}{\sum_{j=1}^{n} S_j}\right) z_m(A \times B) = z_m(A \times B),$$

$x^*$ is the optimal solution for (PF) of full flexibility structure. ■

This result has numerous implications for the process flexibility problem. For instance, suppose $m = n$, demand $D_i$ is bounded between $a(> 0)$ and $b(\geq a)$ for all $i = 1, \ldots, n$. Similarly, supply $S_j$ is also within the range $[a, b]$ for all $j = 1, \ldots, n$. Clearly

- $x_{ij}^* \leq \frac{b^2}{na}$;
- $E(x_{ij}^*(D)) \geq \frac{a^2}{nb}$.

In this case,

$$x_{ij}^* \leq \left(\frac{b}{a}\right)^3 E(x_{ij}^*)$$

Using $\lambda = b^3/a^3$ in Theorem 1, we obtain the existence of a $(1 - \epsilon)$ optimal solution with support at only $O(2b^3n/a^3\epsilon)$ variables. Note that each variable $x_{ij}$ corresponds to a link in the process flexibility structure. A sparse support set here translates to a sparse flexibility structure for the problem $(PF)$. Hence our approach provides an analytical explanation to the often observed empirical phenomenon: a small number of process capabilities is enough to reap the bulk of the benefits from process flexibility enhancement.

We note that in the manufacturing flexibility literature(c.f. [36]), demand is normally assumed to follow a truncated normal distribution at $[\mu - k\sigma, \mu + k\sigma]$, with $k < 3$, and $\mu > 3\sigma$. Here, $\mu$ and $\sigma$ denote the mean and standard deviation of the demand distribution. The distribution generated, properly scaled, can be seen to follow the above assumptions on the demands. Hence our approach provides an analytical explanation to the often observed empirical phenomenon: a small number of process capabilities is enough to reap the bulk of the benefits from process flexibility enhancement.

### *4.3.1 Supply Chain Flexibility*

Jordan and Graves [36] investigated how to design process flexibility structure in a single stage, multi-product and multi-plant supply chain. Graves and Tomlin [29] extended Jordan and Grave's work to multi-product, multi-stage supply chains, where each product need to be processed in every stage. They proposed a supply chain flexibility measure $g$, where higher $g$ indicates higher flexibility. Unfortunately, they did not show how to design a flexible supply chain network structure. In this section, we will show that sparse partial flexibility structures can also work very well in multi-stage supply chain system.

Consider the following supply chain design problem (see Figure 4.1):

There are $n_1$ products, $n_2$ plants, and $n_3$ suppliers. In the full flexibility scenario, each product can be produced at any plant, using material sources from any supplier. We assume that each unit of product consumes a unit of material from each supplier and occupies a unit capacity of a plant. We assume further that production capacity at the plants are $C_i$, $i = 1, \ldots, n_1$, and the suppliers have limited amount of materials, at capacity $B_i$, $i = 1, \ldots, n_3$. The demand for the products are random and denoted by the random variable $D_i$, $i = 1, \ldots, n_1$.

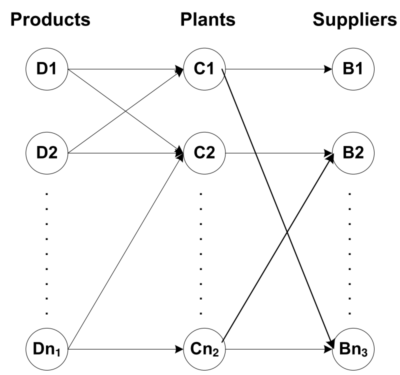



*Fig. 4.1:* A Supply Chain Flexibility Structure.

In the full flexibility scenario, it is easy to see that the expected amount of the total demand that can be satisfied in the supply chain is

$$E_D\left[\min\left(\sum_{i=1}^{n_1} D_i, \sum_{i=1}^{n_2} C_i, \sum_{i=1}^{n_3} B_i\right)\right].$$

The above problem can be formulated as the following set packing problem:

$$
\begin{aligned}
z_m(A \times B \times C) = \quad & \max \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} x_{ijk} \\
s.t. \quad & \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} x_{ijk} \leq D_i \ \forall i = 1, 2, \dots n_1; \\
& \sum_{i=1}^{n_1} \sum_{k=1}^{n_3} x_{ijk} \leq C_j \ \forall j = 1, 2, \dots n_2; \\
& \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} x_{ijk} \leq B_k \ \forall k = 1, 2, \dots n_3; \\
& x_{ijk} \geq 0 \ \forall \ i, j, k.
\end{aligned}
$$

For each realization of demand $D_i$, it is easy to see that there is an optimal solution given by

$$
x^*_{ijk} = \frac{D_i C_j B_k}{\max\left\{ \sum_{i=1}^{n_1} D_i \times \sum_{j=1}^{n_2} C_j, \sum_{i=1}^{n_1} D_i \times \sum_{k=1}^{n_3} B_k, \sum_{j=1}^{n_2} C_j \times \sum_{k=1}^{n_3} B_k \right\}}.
$$

With this result, we can show the existence of a sparse supply chain network in a very general settings for this problem. For instance, suppose $D_i, C_j, B_k$ follow any distribution with support $[a, b]$, where $0 < a \leq b$, and $n_1 = n_2 = n_3 = n$. The expected optimal solution

$$
E(x_{ijk}) \geq \frac{a^3}{(nb)^2} \ \forall i, j, k = 1, \dots n
$$

and the corresponding optimal solution

$$\begin{aligned} x_{ijk} &\leq \frac{b^3}{(na)^2} \\ &= \frac{b^5}{a^5}\frac{a^3}{n^2b^2} \\ &\leq \frac{b^5}{a^5}E(x_{ijk}). \end{aligned}$$

Let $\lambda = b^5/a^5$ so that Property $\boldsymbol{A}$ holds for this case. In fact, this result can be easily modified for a more general situation with bounded positive demand, capacity and supply.

Theorem 2: Suppose the optimal solution $x_{ijk}$ satisfies

$$\begin{aligned} x^*_{ijk} &= \frac{D_iC_jB_k}{\max\left\{\sum_{i=1}^{n_1} D_i \times \sum_{j=1}^{n_2} C_j, \sum_{i=1}^{n_1} D_i \times \sum_{k=1}^{n_3} B_k, \sum_{j=1}^{n_2} C_j \times \sum_{k=1}^{n_3} B_k\right\}} \\ &\leq \lambda E\left[\frac{D_iC_jB_k}{\max\left\{\sum_{i=1}^{n_1} D_i \times \sum_{j=1}^{n_2} C_j, \sum_{i=1}^{n_1} D_i \times \sum_{k=1}^{n_3} B_k, \sum_{j=1}^{n_2} C_j \times \sum_{k=1}^{n_3} B_k\right\}}\right], \end{aligned}$$

for some $\lambda > 0$, then there exists a sparse supply chain configuration $S$ with cardinality $|S| = O(\lambda(n_1 + n_2 + n_3)/\epsilon)$, such that the expected demand met by the sparse supply chain system is at least $(1 - \epsilon)E(Z^*(D))$.

## 4.4 Sampling Heuristic: Designing a Sparse Flexibility Structure

The study above has shown the existence of a good flexibility structure in most practical cases. We can also modify the constraint sampling method to build a sampling heuristic to design a good sparse flexibility structure. The basic idea, as shown in Figure 4.2, is to sample a group of different structures with $N$ links using the estimated sampling probability and select the best one. Note the number of links $N$ required is controlled explicitly by managers, depending on flexibility requirements and budget constraints. Adding more links in a structure would provide more flexibility but may increase the cost of implementation.

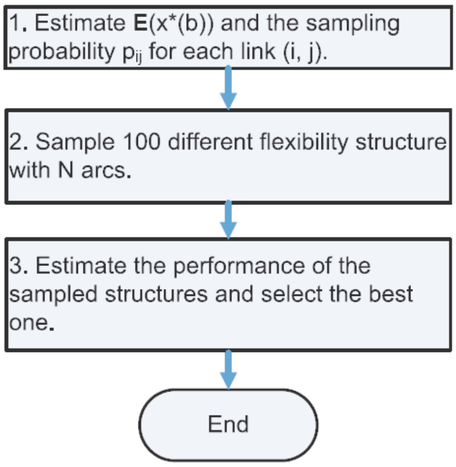



*Fig. 4.2:* Sampling Heuristic

The detailed steps to construct a flexibility structure with $m$ demand node and $n$ supply nodes are described as following. It can be easily modified and applied to many different applications, including flexibility structure design, transshipment network design, supply chain design, and etc. These applications will be discussed in Chapter 5.

[**Phase 1: Estimate Sampling Probabilities via a Monte-Carlo Simulation.**]

1. For each retailer, generate 100 realizations of demand and supply based on a specific demand distribution.
2. Estimate $E(x^*_{ij})$ using

$$E(x^*_{ij}) = \sum_{k=1}^{100} \left( \frac{D_i^k S_j^k}{\max\{\sum_{i=1}^m D_i^k,\ \sum_{j=1}^n S_j^k\}} \right) /100,$$

   where $D^k$ and $S^k$ is the demand and supply generated in $k$th instance.
3. Estimate $p_{ij}$ using

$$p_{ij} = \frac{E(x^*_{ij})}{\sum_{i=1}^m \sum_{j=1}^n E(x^*_{ij})}.$$

We next sample the arcs using the estimated probabilities $\{p_{ij}\}$. However, to ensure that no node will be disconnected from the rest of the network, we first generate an arc into each node. Furthermore, to avoid sampling an arc twice, we remove the arc from the sampling experiment once it has been selected. We do this by normalizing the corresponding sampling probability to zero. The procedure is described next.

[**Phase 2: Sampling Arcs.**]

1. For $k = 1, \ldots, m$, generate a random number $U_k \in (0, 1)$. If

$$\sum_{j=0}^{l-1} \frac{p_{kj}}{\sum_{j=1}^{n} p_{kj}} < U_k \leq \sum_{j=0}^{l} \frac{p_{kj}}{\sum_{j=1}^{n} p_{kj}}, \text{ for some } l = 1, \ldots, n,$$

   add arc $(k, l)$ to the structure and let $p_{kl} = 0$.

2. Order the arcs in lexicographical order, and let $\hat{p}_{n(i-1)+j} = p_{ij}$.

3. Generate a random number $U_k \in (0, \sum_{i=1}^{mn} \hat{p}_i)$. If

$$\sum_{i=1}^{l-1} \hat{p}_i < U \leq \sum_{i=1}^{l} \hat{p}_i,$$

   $l = 1, 2, \ldots, mn$, let $\hat{p}_l = 0$ and add arc $l$ to the network.

4. Repeat Step 3 until the number of arcs sampled reaches $N$.

We can generate several structures with $N$ links by repeating step 2. Here, 100 structures are sampled, and the best-performing one could be selected in the next evaluation step.

[**Phase 3: Evaluation Approaches**]

The sampled structures are evaluated in this phase and the most flexible one will be selected. Two different approaches can be adopted to measure the structures in terms of flexibility.

One approach is Monte-Carlo simulation. Another set of demand and supply instances are generated, and each structure are tested using the new set of data. The structure with highest expected max-flow (or lowest expected excess flow) will be selected. This simulation approach can precisely estimate the performance of structures, especially when a huge number of demand/supply scenarios are simulated. The disadvantage of the approach, on the other hand, is that the simulation could be very time-consuming.

Another approach is to use flexibility indices to measure the structures. We can use the expansion index discussed in chapter 3, or the SF indices [33]. The structure with highest rank indicated by the indices should be selected. The advantage of this approach is that the indices are usually easy to compute and effective. The disadvantage is that indices might be biased and some indices can only be used in limited structures.

# 5. APPLICATIONS

The results obtained in chapter 3 and 4 can be applied to many different problems. In this chapter, we will use the expansion heuristic and sampling heuristic to design sparse structures in production planning problem, transsshipment network design and cutting stock patterns selection problem. The purpose of the study is to test the effectiveness of graph expansion and sampling method in different business applications. Furthermore, the observation that a sparse structure could be almost as good as a complicated fully connected structure is also examined in the various situations.

## *5.1 Production Planning Problem*

Manufacturers are the pioneers in adopting and developing flexibility tools to help them make flexible production plans and effectively match random demand and supply. Many simulation studies have investigated the flexible production planning problem, and showed that a good flexible structure could greatly enhance the manufacturer's ability of matching (random) demand and supply. Most good structures, however, are generated via time-consuming intensive simulations, and might not be applicable in many situations.

In this section, we use the expansion heuristic discussed in chapter 3 to construct good flexible structures. We adopt the General Motor's production planning problem studied by Jordan and Graves [36] to examine the effectiveness of our heuristic. The basic settings of the example can be briefly described as follows. Consider a system containing 16 products and 8 plants. Products have normally distributed demands, Plants have constant capacities. The expected demand and capacity are shown in Figure 5.1. The standard deviation of each demand is 40% of its mean. The products are divided into three groups: product $A$ to $F$ are in group 1, $G$ to $M$ are in group 2, and $N$ to $P$ are in group 3. The demands of products in the same group are pairwise correlated with correlation coefficient $\rho = 0.3$, and demands of products in different groups are independent.

We use our heuristic to construct another flexibility structure (see Figure 5.1-D) by adding 6 more arcs into the base assignment. We then conduct a simulation study to compare the performances of our heuristic structure and the JG's (Jordan and Graves') structures. The JG's structures (Figure 5.1-B and C) are obtained from a simulation study ([36]) and have very good performances, close to full flexibility.

To test the robustness of the structures under consideration, we generate supplies from 8 different distributions: four for independent supplies and four for correlated normally-distributed supplies.

**Independent Supplies.**

As shown in Table 5.1, the four independent distributions are binomial, uniform, normal with a small standard deviation and normal with a big standard deviation. We simulate 100 scenarios of supplies from each distribution and

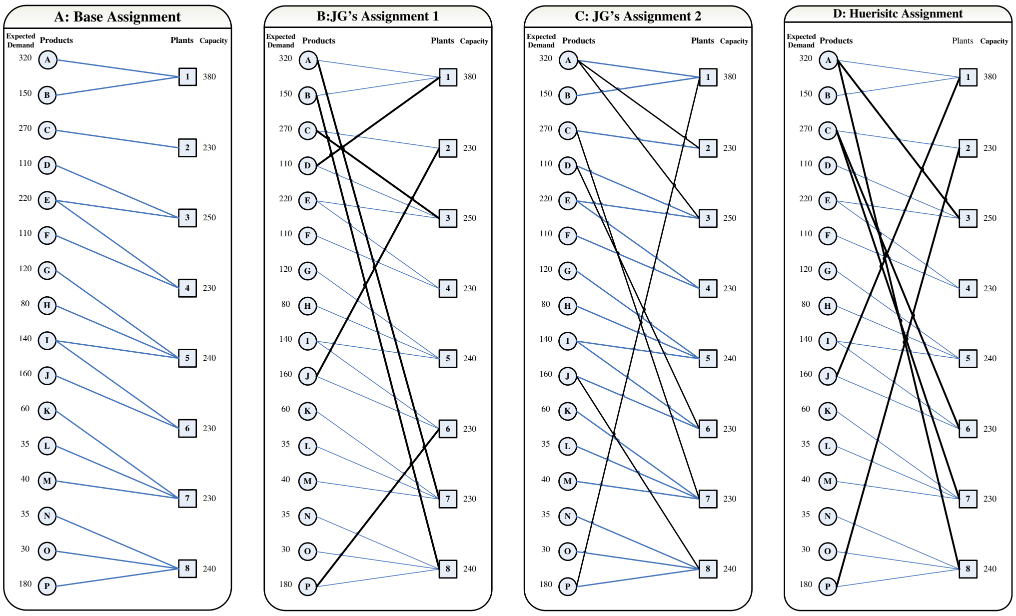



*Fig. 5.1:* Different Flexibility Structures for the GM Problem.

compare the average performance, the worst case performance, and the performance differences among the heuristic structure and JG's structures. In some cases, the max flow of a structure is quite low merely because the total supply in the system is low, not because the structure lacks flexibility. To address this issue, we measure the performance by the ratio of the max flow in a structure to the max flow in the full flexibility structure, which eliminates the impact of supply fluctuations and reflects how close the performance of a structure is to the full flexibility structure.

The results are shown in Figure 5.2, 5.3, 5.4 and 5.5. Except for comparison 3, our heuristic structure has better average performances than at

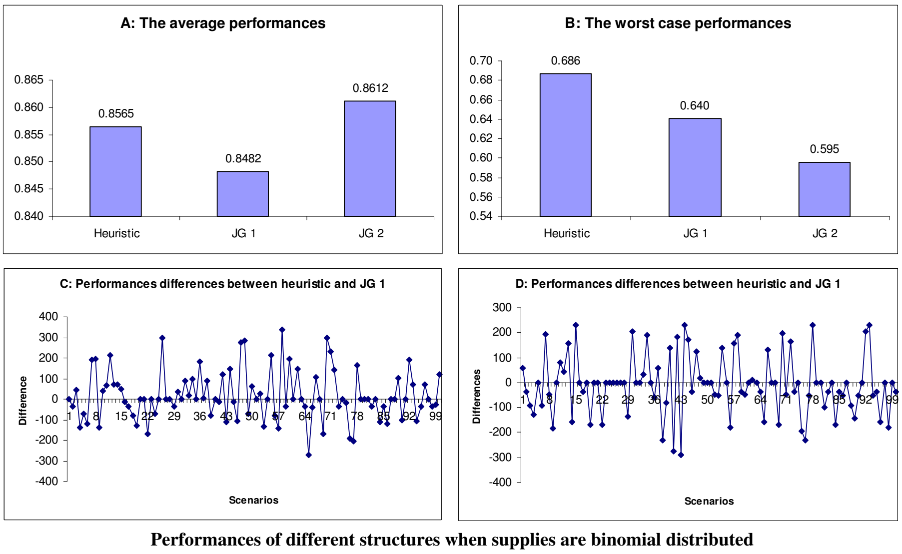
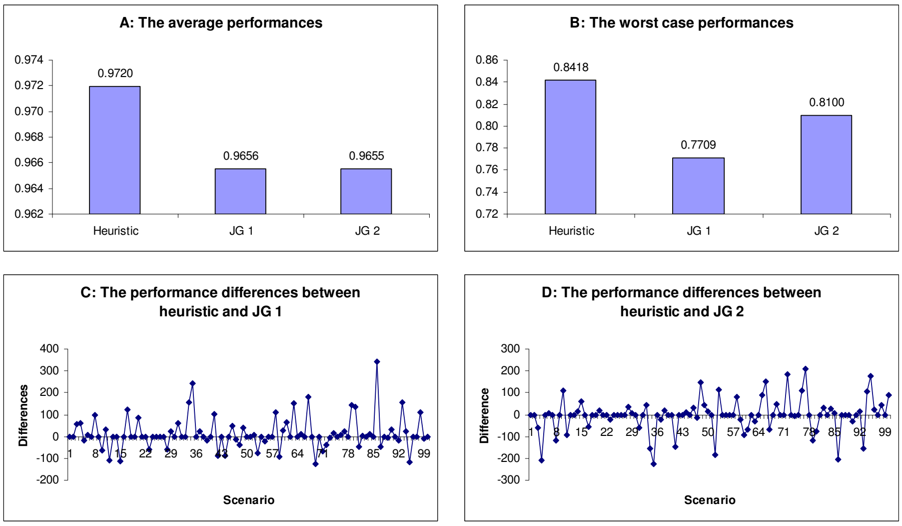



*Fig. 5.2:* GM Comparison 1: Demand Follows Independent Binomial Distribution.

least one of JG's structures, and better worst case performances than both JG's structures. The heuristic structure also outperforms JG 1 in most scenarios in comparison 1, 2 and 4, and outperforms JG 2 in most scenarios in comparison 4.

Note that JG's structures are designed based on simulation study which assumes the supplies are normally distributed with $\sigma_i = 0.4\mu_i$. Therefore, it is not surprising that, in comparison 3 where supplies are normally distributed with $\sigma_i = 0.4\mu_i$, our heuristic structure performance is not as good as JG's structures in some scenarios. As a matter of fact, when the standard deviation increases from $0.4\mu_i$ to $0.6\mu_i$, the performances of JG's structures are not as good as our heuristic structure, as shown in comparison 4. Note that JG's simulation also assumes correlations among supplies, which we will further analyze in the correlated supplies case.




**Performances of different structures when supplies are uniformly distributed**

*Fig. 5.3:* GM Comparison 2: Demand Follows Independent Uniform Distribution.

**Correlated Supplies.**

Following Jordan and Graves example [36], we divide the supply nodes into three groups: group one from node A to F, group two from node G to M, and group three from node N to P. Supplies in the same group are pair-wise correlated, but are independent with supplies in other groups. We consider four correlated normal distributions: $N(\mu_i, 0.4\mu_i)$ with correlation coefficient $\rho = 0.3$, $N(\mu_i, 0.4\mu_i)$ with $\rho = 0.5$, $N(\mu_i, 0.6\mu_i)$ with $\rho = 0.3$, and $N(\mu_i, 0.6\mu_i)$ with $\rho = 0.5$. Note that the distribution used in comparison 5 is exactly the distribution used in the simulation study on designing JG's structures ([36]).

The results of different structures' performances are shown in Figure 5.6, 5.7, 5.8 and 5.9. The heuristic structure has better average performances than JG 1 in all 4 comparisons, and better worst case performances than JG 1 in comparison 5, 7 and 8. The heuristic structure also beats JG 2 when the standard deviations of supplies do not follow $\sigma_i = 0.4\mu_i$, which was

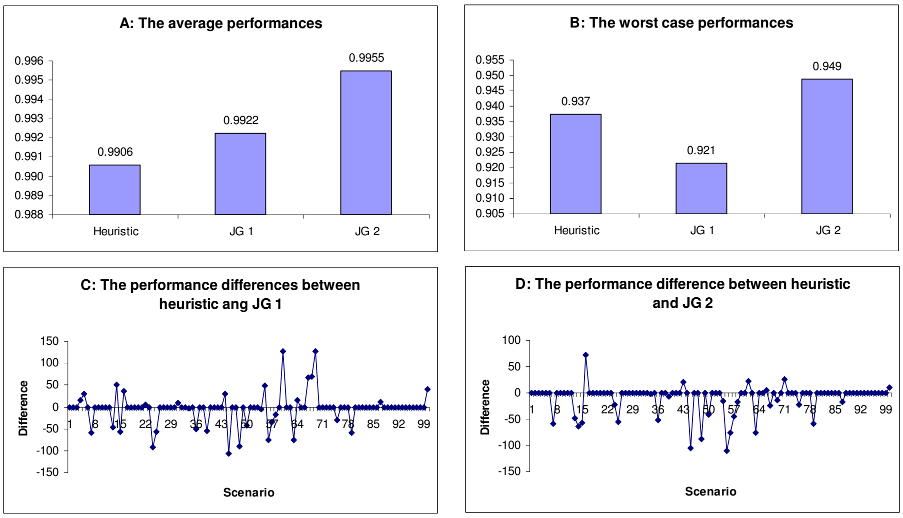
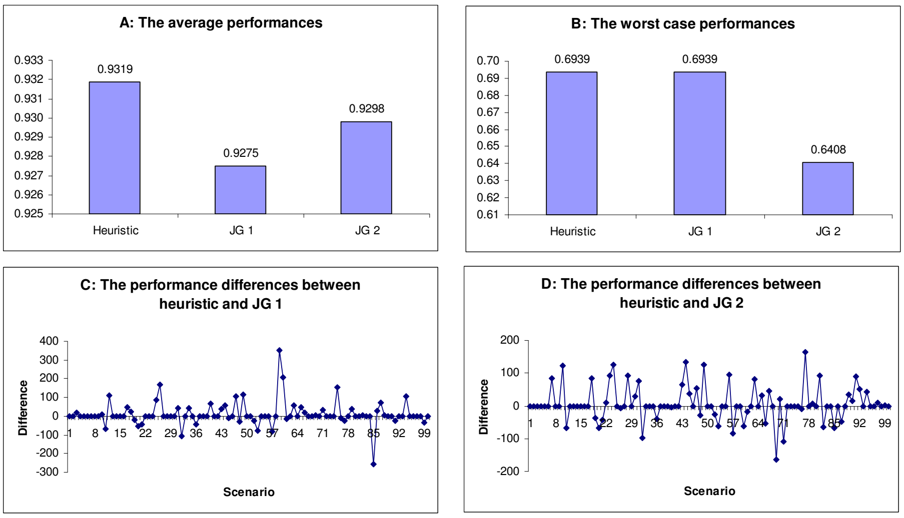



*Fig. 5.4:* GM Comparison 3: Demand Follows Independent Normal Distribution with $\sigma_i = 0.4\mu_i$.

assumed when JG 2 was designed. Indeed, it has better average and worst performances than JG 2 in comparison 7 and 8.

We note that when supply variance increases, the performances of all structures become worse. However, the performances of our heuristic structure seem to be more stable than JG's structures, and even more so when correlations are high. This suggests that our structure is robust in the worst situation (high variances and high correlations).

Table 5.1 summarizes the comparisons among different structures in all eight cases. Our heuristic structure outperforms JG's structures in most cases. It only slightly under-performs JG's structures when supplies follow independent/correlated normal distribution with $\sigma_i = 0.4\mu_i$, which is still acceptable because JG's structures are selected from an extensive simulation study assuming supplies follow normal distribution with $\sigma_i = 0.4\mu_i$ and JG's




**Performances of different structures when supplies are normally distributed with SD=0.6μ**

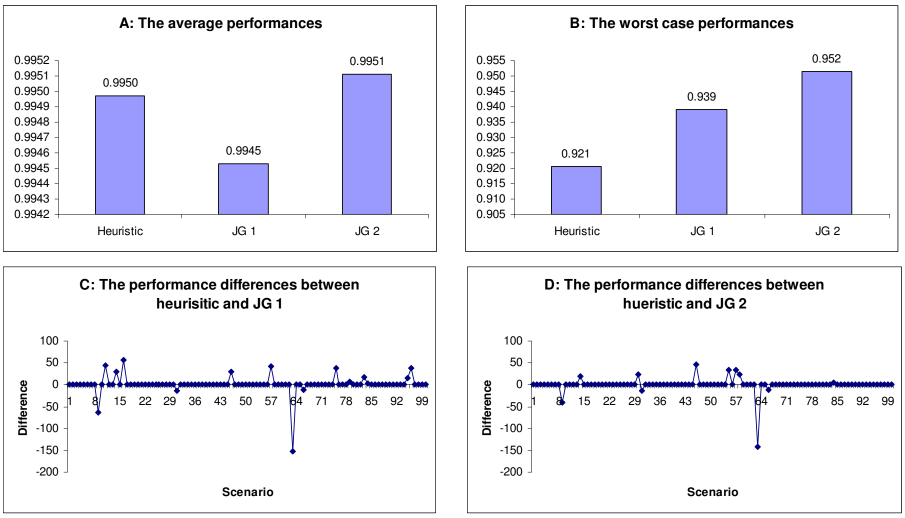
*Fig. 5.5:* GM Comparison 4: Demand Follows Independent Normal Distribution with $\sigma_i = 0.6\mu_i$.

structures perform almost as well as the full flexibility structure.

It is important to note that our structure is obtained from a simple heuristic, yet performs robustly well in most cases. It is also computationally efficient. This illustrates the importance of using expansion information in designing a flexible structure.


Performances of different structures when supplies are normally distributed with SD=0.4μ and ρ=0.3

*Fig. 5.6:* GM Comparison 5: Demand Follows Normal Distribution with $\sigma_i = 0.4\mu_i$ and $\rho = 0.3$.

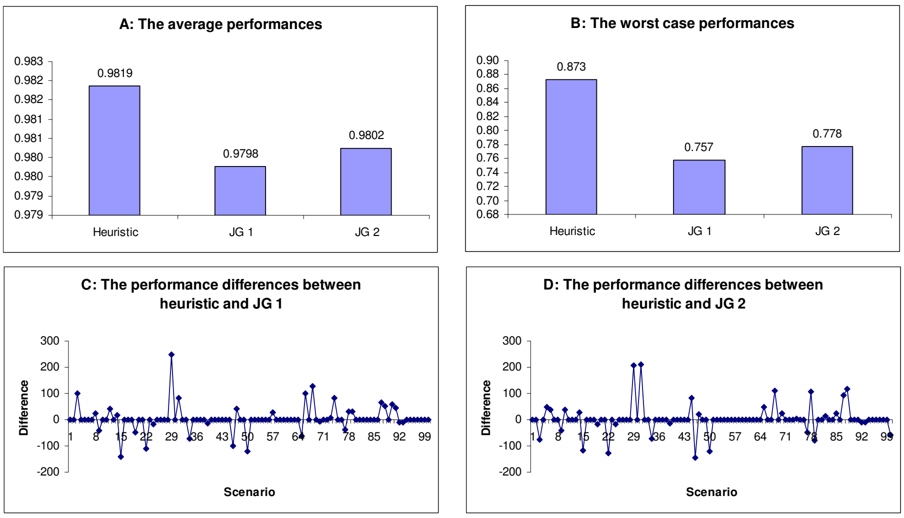



Performances of different structures when supplies are normally distributed with SD=0.4μ and ρ=0.5

*Fig. 5.7:* GM Comparison 6: Demand Follows Normal Distribution with $\sigma_i = 0.4\mu_i$ and $\rho = 0.5$.


**Performances of different structures when supplies are normally distributed with SD=0.6μ and ρ=0.3**

*Fig. 5.8:* GM Comparison 7: Demand Follows Normal Distribution with $\sigma_i = 0.6\mu_i$ and $\rho = 0.3$.




**Performances of different structures when supplies are normally distributed with SD=0.6μ and ρ=0.5**

*Fig. 5.9:* GM Comparison 8: Demand Follows Normal Distribution with $\sigma_i = 0.6\mu_i$ and $\rho = 0.5$.

| Supply distributions | | Average performance | | Worst Case Perforamnce | |
|---|---|---|---|---|---|
| | | Heuristic $\geq$ JG 1 | Heuristic $\geq$ JG 2 | Heuristic $\geq$ JG 1 | Heuristic $\geq$ JG 2 |
| Indep-dent | $S_i = 3\mu_i$ with prob. 1/4, $S_i = 1/3\mu_i$ with prob. 3/4. | $\surd$ | | $\surd$ | $\surd$ |
| | $S_i \sim U[0, 2\mu_i]$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| | $S_i \sim N(\mu_i, 0.4\mu_i)$ | | | $\surd$ | |
| | $S_i \sim N(\mu_i, 0.6\mu_i)$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| Corre-lated | $S_i \sim N(\mu_i, 0.4\mu_i)$, $\rho = 0.3$ | $\surd$ | | $\surd$ | |
| | $S_i \sim N(\mu_i, 0.4\mu_i)$, $\rho = 0.5$ | $\surd$ | | | |
| | $S_i \sim N(\mu_i, 0.6\mu_i)$, $\rho = 0.3$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ |
| | $S_i \sim N(\mu_i, 0.6\mu_i)$, $\rho = 0.5$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ |

*Tab. 5.1:* Summary of Performances Comparisons.

## 5.2 Transshipment Problem

Transshipment is a widely used inventory management strategy to cope with demand uncertainties and to effectively increase service levels. Consider $n$ retailers, each facing a random demand of $D_i$, $i = 1, \ldots, n$ for a common product. Retailer $i$ has $Q_i$ units of the product. In the event that $Q_i > D_i$, the retailers would wish to have options to transship the excess product to another retailer who may need it.

There is a huge literature on this problem. Studies on transshipment focus mainly on how to decide optimal inventory policy and optimal order quantity $Q_i^*$ for each retailer. They either deal with problems with two retailers (cf. [52]) or many identical retailers (cf. [38], [45] and [31]), with the assumption of complete pooling (i.e. a retailer could tranship its products to any other retailers).

Only a few papers discuss how to design a good transshipment networks. Lien et al [40] studied the impacts of the transshipment network structure. They compared the performances of different network configures: no transshipment, complete pooling, partial grouping, unidirectional chain and bidirectional chain (See Figure 5.10). They showed similar results to the findings of Jordan and Graves [36]: sparse transshipment network structures can capture almost all the benefit of complete pooling. They also indicated that the chaining structure, which is a kind of sparse structure, would outperform other sparse structures. In this section, we will show the existence of a good sparse transshipment network by sampling approach.

The transshipment structure design problem can be reduced to a variant

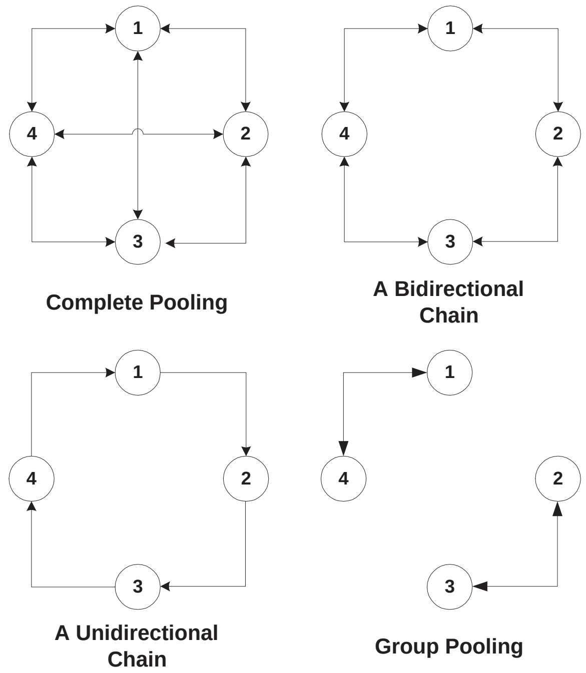



*Fig. 5.10:* Different Types of Transshipment Network Structures.

of the process flexibility problem, where there are $n$ plants and $n$ products. Each plant $i$ has capacity $(Q_i - D_i)^+$ (the left over at retailer $i$), which can be used to meet demand for other products. Each product has demand $(D_i - Q_i)^+$ (unfilled demand at retailer $i$). Note that in this case, both capacity and demand are random parameters in our problem, and $(Q_i - D_i)^+ \times (D_i - Q_i)^+ = 0$.

From the analysis of sampling approach in chapter 4, the existence of a sparse support structure for the transshipment problem is guaranteed by the following condition:

$$\begin{aligned} x_{ij}^* &= \frac{(D_i - Q_i)^+(Q_j - D_j)^+}{\max\left\{\sum_{i=1}^n (D_i - Q_i)^+, \sum_{j=1}^n (Q_j - D_j)^+\right\}} \\ &\leq \lambda E\left[\frac{(D_i - Q_i)^+(Q_j - D_j)^+}{\max\left\{\sum_{i=1}^n (D_i - Q_i)^+, \sum_{j=1}^n (Q_j - D_j)^+\right\}}\right] \end{aligned}$$

almost surely for some $\lambda > 1$, and for all $i, j$.

When $D_i$ are identical and independent random variables and take values in $\{0, 2\}$ with equal probability, $Q_i = 1$ for all $i = 1, \ldots, n$, then $(D_i - Q_i)^+$ and $(Q_i - D_i)^+$ are Bernoulli variable with equal probability. Note that

$$E((D_i - Q_i)^+) = E((Q_i - D_i)^+) = \frac{1}{2}.$$

Furthermore,

$$\sum_{i=1}^n (D_i - Q_i)^+ + \sum_{j=1}^n (Q_j - D_j)^+ = \sum_{i=1}^n |D_i - Q_i| = n.$$

Hence

$$
\begin{aligned}
x^*_{ij}(D) &= \frac{(D_i - Q_i)^+(Q_j - D_j)^+}{\max\left\{\sum_{i=1}^n (D_i - Q_i)^+, \sum_{j=1}^n (Q_j - D_j)^+\right\}} \qquad (5.1) \\
&\leq \frac{2}{n}(D_i - Q_i)^+(Q_j - D_j)^+ \\
&\leq \frac{8}{n}E\Big[(D_i - Q_i)^+(Q_j - D_j)^+\Big] \\
&\leq 8E\left[\frac{(D_i - Q_i)^+(Q_j - D_j)^+}{\max\left\{\sum_{i=1}^n (D_i - Q_i)^+, \sum_{j=1}^n (Q_j - D_j)^+\right\}}.\right]
\end{aligned}
$$

Property $\boldsymbol{A}$ holds for this example. ■

Remark 1: In the general case when $D_i$ ($\forall i = 1, \ldots, n$) has support in a continuous set, the event that both $\sum_{i=1}^n (D_i - Q_i)^+$ and $\sum_{j=1}^n (Q_j - D_j)^+$ are both zeros happen with measure zero. Therefore, we need not consider the case that the denominator in equation 5.1 is zero.

Remark 2: The sampling approach in our analysis uses the value

$$
E_D\left[\frac{(D_i - Q_i)^+(Q_j - D_j)^+}{\max\left\{\sum_{i=1}^n (D_i - Q_i)^+, \sum_{j=1}^n (Q_j - D_j)^+\right\}}\right]
$$

to obtain the variable sampling probability. This approach can gainfully employ the additional information on the covariance structure of $D_i$ and $D_j$, and the total excess and unfilled demand distribution $\sum_{i=1}^n (D_i - Q_i)^+$ and $\sum_{j=1}^n (Q_j - D_j)^+$ to obtain reliable sampling probabilities.

We state a combinatorial analogue of the above observation as a formal theorem. Suppose we distribute $n$ red and $n$ blue points randomly on a graph $G$ with $2n$ nodes, with each node covered by exactly one color. Let $c(i)$ denote the color assigned to node $i$. Let $e(G)$ denote the edge set in $G$. We say that $M \subset e(G)$ is a colored matching if it is a matching in $G$ with

$$M = \{(i,j) : c(i) \neq c(j), (i,j) \in e(G)\}.$$

Let $m(G)$ denote the cardinality of a maximum colored matching in $G$. Note that $m(G) \leq n$ for all realizations of the color distribution, and $E(m(G)) = n$ when $e(G) = K(2n)$, the complete graph on $2n$ nodes. Note that the graph $K(2n)$ has $O(n^2)$ edges. Theorem 1 and Property ***A*** shows that cardinality of the edge set $e(G)$ can be reduced much further, while sacrificing only a little in value of $E(m(G))$.

Theorem 3: For all $\epsilon > 0$, there exists $n(\epsilon) > 0$ such that for all $n \geq n(\epsilon)$, there exists a graph $G_n$ with $2n$ nodes and $O(n)$ edges, such that

$$n \geq E(m(G_n)) \geq (1-\epsilon)n.$$

■

### *5.2.1 Numerical Example*

We adopted the data provided by Jordan and Graves [36] to build our example for the transshipment network design problem. Consider a retail network containing 16 retailers. Demand for each retailer is uncertain and normally distributed. Expected demands are shown in Figure 5.11. The standard deviation is 40% of the expected demand. The retailers can be divided into 3 subgroups, retailer 1 to 6, 7 to 13 and 14 to 16. Demands of retailers in the same subgroup are correlated. The correlation coefficients are 0.3 pairwise for retailers within same subgroup. However, there are no correlations between demands of retailers in different subgroups.

We consider a single period decision. All unsold items will be abandoned and unmet demand will be lost. Note that in the case with transshipment, the optimal ordering quantity $Q^*$ will depend on the transshipment structure, the demand distribution, transportation, holding and shortage cost parameters. It is in general difficult to determine the optimal order-up-to quantity for this general problem. We assume instead a fixed ordering quantity, and focus instead on the design of the transshipment structure to facilitate the transshipment operations. For convenience, the order quantity ($Q^*$) for each retailer is set to be the mean of its demand.

We focus on how the transshipment network support structure can be suitably designed. Here we only consider the unidirectional transshipment structure. See Figure 5.11, where retailers are connected with directed arcs. An arc from retailer $i$ to $j$ means that retailer $i$ can send its unsold products to retailer $j$. However retailer $j$ cannot send its products to $i$ unless there is another arc $(j, i)$ connecting them.

It is obvious that the complete pooling network (where a retailer can tranship to any other retailer) will achieve the maximum savings. However, it would increase the complexity of the transshipment operations, as more transportation linkages between the retailers have to be pre-arranged. Therefore, a sparse network structure is preferred.

The proof to Theorem 4.2 indicates that in our sampling approach, we should set the probability of selecting arc $(i, j)$ to be

$$p_{ij} = \frac{E_D(x^*_{ij})}{\sum_{(i,j):i,j\in A} E_D(x^*_{ij})}, \tag{5.2}$$

where $A$ represents the set containing all retailers. Note that

$$E_D(x^*_{ij}) = E_D\left[\frac{(Q^*_i - D_i)^+(D_j - Q^*_j)^+}{\max\left\{\sum_{i=1}^{16}(Q^*_i - D_i), \sum_{j=1}^{16}(D_j - Q^*_j)\right\}}\right]. \tag{5.3}$$

We apply the sampling heuristic discussed in chapter 4 to design a good transshipment network.

We first generate 100 sets of demand scenarios to estimate $E_D(x^*_{ij})$ and calculate the sampling probability $p_{ij}$ of link $(i, j)$ for all $i, j = 1, \ldots 16$. For any given $N$, i.e. the number of links needed, We sample 100 different structures and select the best one among them using one of the following evaluation approaches.

Two evaluation approaches could be used to select a structures. One approach is simulation: another set of demand scenarios is sampled and the expected transshipment flow of each structure is calculated, the structure with highest flow will be selected. Another approach is to use the expansion index discussed in section 3.5. The structure with highest expansion index will be selected. This method has minimal computational requirement, while it may select a different structure from the simulation method due to the sensitivity and precision of the index.

**Simulation Evaluation.**

Another 100 sets of demand scenarios are generated to evaluate performance of transshipment networks. The network with highest expected transshipment flow will be selected. Figure 5.11, for instance, is a network we obtained from this sampling based approach, using only 32 transshipment links. The network obtained exhibits characteristics of good transshipment network structure: (i) The retailers with higher average demand should ideally be linked with more other nodes, since they would probably face oversupply or undersupply situation with a larger quantity. Moreover, (ii) retailers within the same group are positively correlated and hence ideally there should only be a small number of transshipment arcs within them, whereas retailers in different groups are more connected (i.e. more arcs linking them) because their demands are independent.

Figure 5.12 plots the performances of the networks obtained (in terms of average transshipped quantity) as the number of arcs increases. For each $N$, we plot the performance attained by the best network structure, the worst network structure, and the average performance of all 100 sampled structures. As shown in Figure 5.12, the performance gap among these three

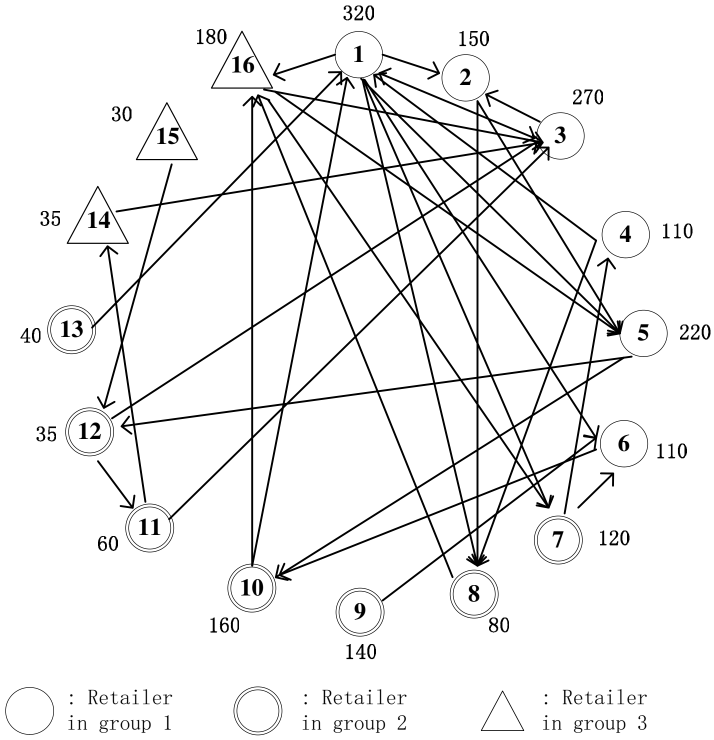



*Fig. 5.11:* Sampled Transshipment Network with 32 Arcs.

cases is very small, and quickly converges to zero as the number of links $N$ increases. This suggests that the sampling heuristic is quite stable and robust,and the performance of any sampled network is acceptable as long as $N$ is sufficiently large. The scheme of sampling 100 networks and selecting the best among them can be used to improve the transshipment network design when $N$ is small.

Another observation from Figure 5.12 is that as the number of links increases, the marginal contribution of additional links diminishes for all three cases - the best, the worst, and the average case. For the best case with 16 links, for instance, the expected transshipment quantity is only 46.7% of the complete pooling network. After increasing the number of arcs to 96, however, the transshipment quantity is already close to 98.5% the complete pooling network. Note that the complete pooling network has up to 240 arcs. Since the transshipment network obtained using the sampling approach gives a lower bound to the performance of the optimal transshipment network structure, it is expected that the optimal performance-flexibility curve should be even steeper. Nevertheless, our results further validate the fact that a sparse transshipment network structure can achieve the performance close to the complete pooling system.

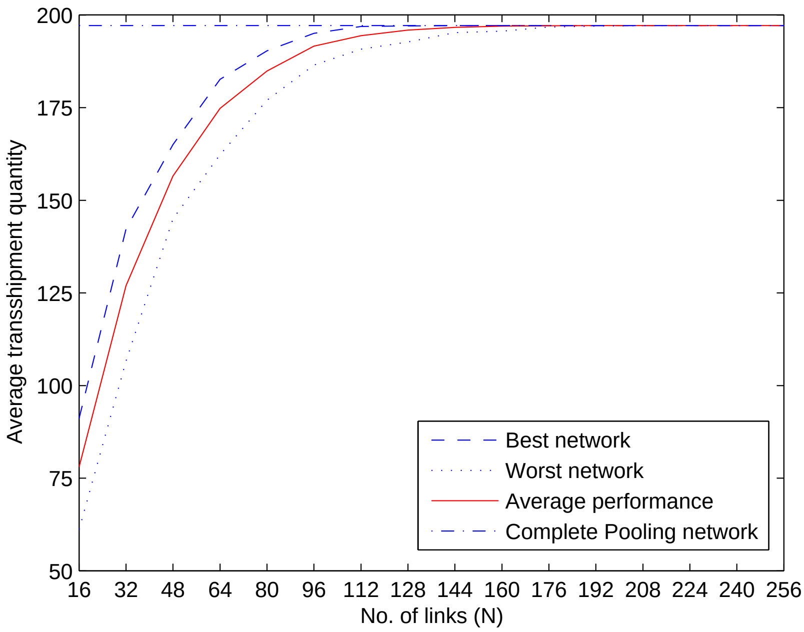



*Fig. 5.12:* Expected Transshipment Quantity as Flexibility Increase.

**Index Evaluation.**

The expansion index is used instead to measure the sampled structures in terms of flexibility, and the structure with highest index will be selected. To benchmark the effectiveness of the indexing method, the average transshipment quantity of the selected structure is also estimated using the set of demand scenarios generated in the previous simulation method part.

| $N$ | Index | Performance | $N$ | Index | Performance |
|---|---|---|---|---|---|
| 16 | 0.00 | 84.8617 | 144 | 703.55 | 196.8300 |
| 32 | 0.00 | 116.9658 | 160 | 796.59 | 197.1596 |
| 48 | 106.27 | 161.9175 | 176 | 1019.05 | 197.1596 |
| 64 | 184.68 | 174.3876 | 192 | 1086.75 | 197.1596 |
| 80 | 316.26 | 187.3700 | 208 | 1226.70 | 197.1596 |
| 96 | 415.34 | 194.8472 | 224 | 1326.29 | 197.1596 |
| 112 | 489.26 | 194.8267 | 240 | 1342.83 | 197.1596 |
| 128 | 632.63 | 196.4987 | 256 | 1367.8 | 197.1596 |

*Tab. 5.2:* Expansion Index and Average Transshipment Quantity for Different $N$.

Table 5.2 shows the expansion index and the expected performance of the selected structures for each $N$. A network with better performance and more links usually has higher expansion index, which suggests the effectiveness of the indexing method. It is also important to notice that expansion index is just a ranking measure: a structure with higher index indicates it would be more flexible, but the performance of a structure is not proportionally related to its expansion index.

Another observation from Table 5.2 is that the expansion index increases as the number of links in the network increases. It is consistent with the intuition: a structure with more arcs would be more flexible. This could also explain the interesting finding that the structures with links more than 160 have the same performances but the index still increases as $N$ becomes larger.

Figure 5.13 shows the comparison of the indexing method and simulation method. Certainly the simulation method will always select the best-

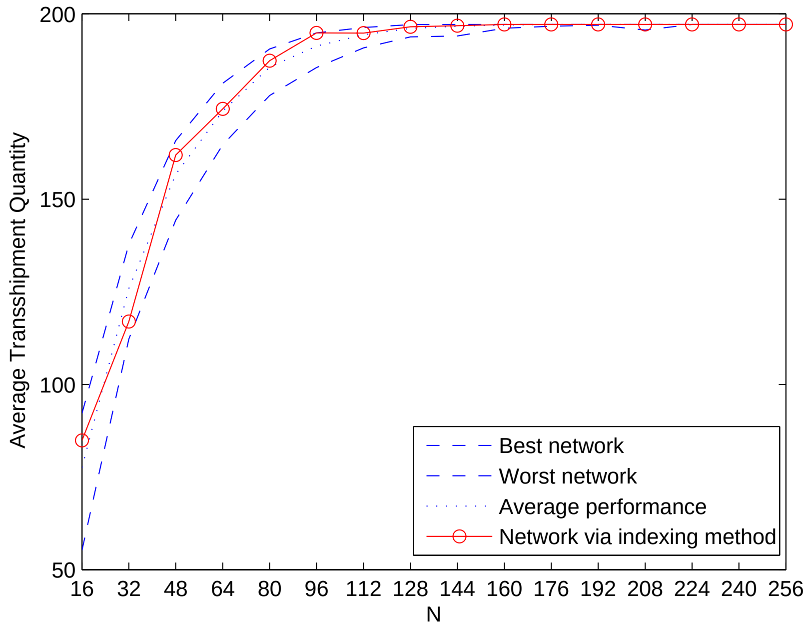



*Fig. 5.13:* Evaluation of the Networks Selected via Indexing Method

performing structure since the simulation approach is a more time consuming manner to evaluate structures. The indexing method can select a good network without conducting the time-consuming simulation. As shown in Figure 5.13, the performances of the structures selected via indexing method are above the average line almost all the time. This result suggests that the indexing method is both effective and efficient in practice.

## 5.3 *Cutting Stock Problems*

Cutting stock problems arise from many industrial applications. A typical cutting stock problem considers how to minimize the cost involved in the trim losses when cutting rolls of paper, metal slab and textiles, etc, while satisfying customer demand at the same time.

Cutting stock problem was first studied by Eisemann [19]. He discovered the trim losses phenomenon in cutting rolls of papers, textiles, etc and formulated a relaxed linear programming model which included all possible cutting patterns. However, he did not further his research to study how to effectively solve the integer programming problem. A mathematical analysis and a linear programming approach to solving the IP problem was proposed by Gilmore and Gomory (1961, 1963 and 1965). In these studies, Gilmore and Gomory proposed a column generation method, which generates knapsack sub-problems to add cutting patterns to the basis and can solve the large- scale cutting stock problems effectively. This method is, now, widely accepted as the standard approach to solving cutting stock problems.

The studies of Eisemann [19] and Gilmore and Gomory [26] have motivated numerous further studies extending on the fundamental ideas from the trim loss problem. These studies include multi-dimensional cutting stock problems ([28], [47]), online cutting decision problems ([51], [24]), and several business applications in the wood product industry, steel industry, paper industry and etc ([24], [58]). Vonderembse [58], for instance, studied an interesting cutting problem in the steel industry, in which the master steel rolls can be stretched or squeezed for a limited range, that is, the master roll width could be changed in a small range. Their results suggested that a longer master roll will obtain a lower total cost, because the number of optional patterns will increase. The increase in cutting patterns, however, would also increase the computational time required to gauge the optimal cutting patterns and exacerbate management complexity.

A typical cutting stock problem [19] can be described using the following example. As shown in Figure 5.14, a factory produces large master rolls with width $W$, and sells rolls with smaller width to customers. Customers may order final rolls with any width smaller than $W$. Obviously, a master roll could be cut by different combinations of final rolls. For instance, Figure 5.14 shows a cutting pattern (one roll 1, one roll 2 and one roll3). The factory thus need to decide how to cut the master rolls to satisfy the demand and minimize the number of master rolls cut.

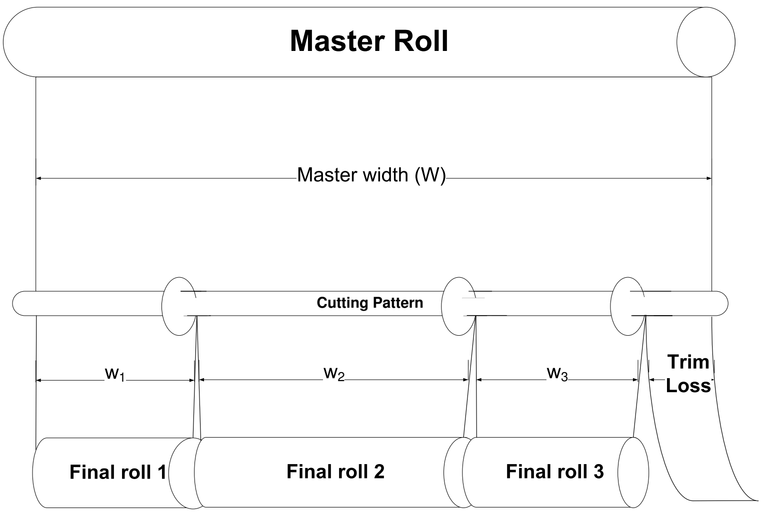



*Fig. 5.14:* A Cutting Stock Problem

This problem can be formulated as [[26]]:

$$\begin{aligned}
(P):\min \quad & Z(D)=\sum_{j=1}^{n} x_j \\
\text{s.t.} \quad & a_{ij}x_j \geq D_i \quad \forall i=1,\dots,m \\
& x_j \in \mathbb{Z}^+ \quad \forall j=1,\dots,n,
\end{aligned}$$

where $m$ is the number of different final rolls, $n$ is the number of patterns in the model, $x_j$ is the number of master rolls cut by pattern $j$ ($j = 1 \ldots n$), $D_i$ is the demand for final roll $i$ ($i = 1 \ldots m$), and $a_{ij}$ is the number of final roll $i$ produced using pattern $j$. Obviously, each column of the coefficient matrix represents a feasible cutting pattern. The feasible cutting patterns are in the convex hull

$$(CH): \left\{ y : \sum_{i=1}^{m} w_i y_i \leq W, y \geq 0, y \in \mathbb{Z}^m \right\}.$$

The optimal solution for a cutting stock problem can be obtained from $(P)$ containing all the possible cutting patterns of a master roll. This problem is not easy to solve because of the integer constraints and possibly exponential number of feasible cutting patterns, that is, $n >> m$ when $m$ is sufficient large.

For a relaxed $(P)$, however, there exists an optimal fractional solution $x^*$ with no more than $m$ supporting variables when $D$ is constant. There is no loss of optimality if the other decision variables are discarded. Based on this interesting observation, Gilmore and Gomory [26] proposed a column generation method to effectively solve $(P)$. The basic idea of the method is to first identify the cutting patterns for a relaxed $(P)$, and then solve the integer problem containing only these particular cutting patterns it has identified. In most applications, this method has proved to be very effective and accurate approximation to $(P)$. Therefore, for a cutting stock problem with a deterministic demand, a minimum cutting cost can be achieved by using a few selected cutting patterns. In this paper, we will consider the case when $D$ is random and examine whether this property still exists.

Let

$$
\begin{aligned}
(P(F)) : \min \quad & Z(F,D) = \sum_{j \in F} x_j \\
\text{s.t.} \quad & A_F x_F \geq D \\
& x_j \in \mathbb{Z}^+, \ \forall j \in F
\end{aligned}
$$

where $F$ denotes a subset of the set containing all cutting patterns, and $A_F$ denotes the columns (cutting patterns) indexed by the subset of variables in $F$. Our goal is to identify a subset $F$ ($|F| \approx O(m) << n$) so that

$$
\boldsymbol{E}_D\Big(Z(F,D)\Big) \leq (1+\epsilon)\boldsymbol{E}_D\Big(Z(D)\Big).
$$

## 5.4 Identify the Supporting Cutting Patterns

In this section, we will conduct a few numerical experiments to test whether there exists a small number of cutting patterns that are able to capture the maximum benefits from using all cutting patterns.

### 5.4.1 Study 1: identify the supporting patterns for a small-size example.

We first test on a small problem with five different final rolls, which is provided by ILOG CPLEX 9.1 Users Manual. The width of the master roll is 110. The final roll width and expected demand $\mu$ are shown in Table 5.3.

The corresponding IP problem $(P)$ for this small example is not difficult to solve as we can obtain all feasible cutting patterns by enumerating the

| Final Roll | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **Final Roll Width** ($w_i$) | 20 | 45 | 50 | 55 | 75 |
| **Expected Demand** ($\mu_i$) | 48 | 35 | 24 | 10 | 8 |

*Tab. 5.3:* Problem Settings of the Small-size example.

extreme point of the corresponding convex hull ($CH$). Table 5.4 lists all the 11 feasible cutting patterns.

| Pattern | Roll 1 | Roll 2 | Roll 3 | Roll 4 | Roll 5 |
|---|---|---|---|---|---|
| **1** | 0 | 0 | 0 | 2 | 0 |
| **2** | 0 | 0 | 1 | 1 | 0 |
| **3** | 0 | 0 | 2 | 0 | 0 |
| **4** | 0 | 1 | 0 | 1 | 0 |
| **5** | 0 | 1 | 1 | 0 | 0 |
| **6** | 1 | 0 | 0 | 0 | 1 |
| **7** | 1 | 2 | 0 | 0 | 0 |
| **8** | 2 | 0 | 0 | 1 | 0 |
| **9** | 3 | 0 | 1 | 0 | 0 |
| **10** | 3 | 1 | 0 | 0 | 0 |
| **11** | 5 | 0 | 0 | 0 | 0 |

*Tab. 5.4:* Cutting Patterns.

A simulation study can be conducted to assess whether all patterns play the same role in the cutting stock problem for different demand scenarios. To do so, we first consider two types of demand distributions: uniform distribution and normal distribution.

**Uniform distribution**

Suppose the demand for each final roll follows a uniform distribution from 0 to $2\mu_i$, for all $i = 1, \ldots 5$. We generate 500 scenarios of customer demands ($\{(d_1^k, \ldots, d_5^k) : \; k = 1, \ldots, 500\}$), and solve the IP problem $(P)$ with 11 different patterns using ILOG CPLEX 9.1. The frequency that a pattern $j$ is active in the simulation, that is, the corresponding optimal solution $x_j^*$ is nonzero, is recorded for all $j = 1, \dots, n$. As shown in Figure 5.15, only a few patterns are frequently active in the simulation, while many patterns appear only occasionally in the simulation. A pattern with a higher frequency is more likely to be used in a cutting process, and thus more likely to be selected as a supportive pattern. We select the patterns with a frequency greater than 200 to build the small support set $F$, and $F$ contains five patterns: pattern 1, 3, 6, 7, 9.

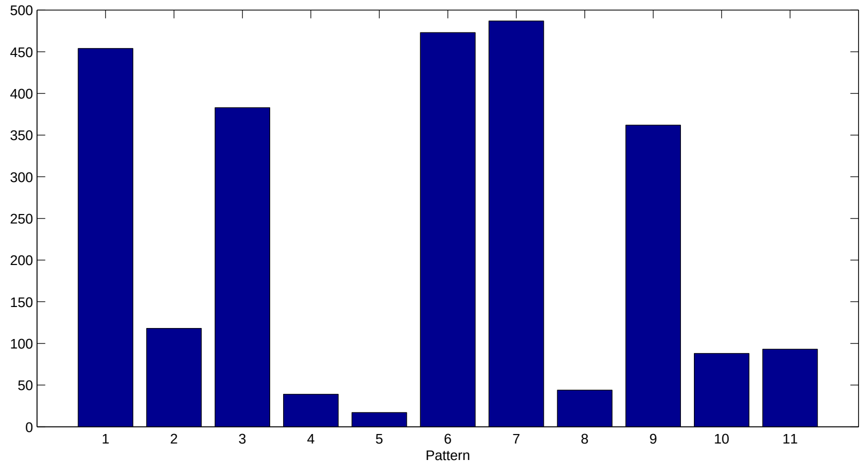



*Fig. 5.15:* The Frequencies of Patterns

Another 500 scenarios of demands are then generated to test the performance of the small pattern set $F$. As shown in Figure 5.16, the expected cutting cost of using the small cutting pattern set $F$, $\boldsymbol{E}[Z(F, D)]$ (41.082) is greater than the expected cutting cost for the optimal solution with all patterns, $\boldsymbol{E}[Z(D)]$ (40.234), by 2.1%. Figure 5.16 also shows the performance difference between $F$ and the complete pattern set in the first 100 simulation scenarios $(Z(F, D^k) - Z(D^k)/Z(D^k), k = 1, \dots, 100)$. The result shows $F$ has the same cutting cost as the complete set in 42 scenarios, the gap between $F$ and the complete set is less than 5% in 43 scenarios, while the gap is larger than 10% in only 7 scenarios. In summary, we can conclude that this small set $F$ captures nearly all the benefits of all the patterns considered. Through this simulation, we have been able to identify a sparse pattern set $F$ that satisfies

$$\boldsymbol{E}_D\Big(Z(F,D)\Big) \leq (1+\epsilon)\boldsymbol{E}_D\Big(Z(D)\Big).$$

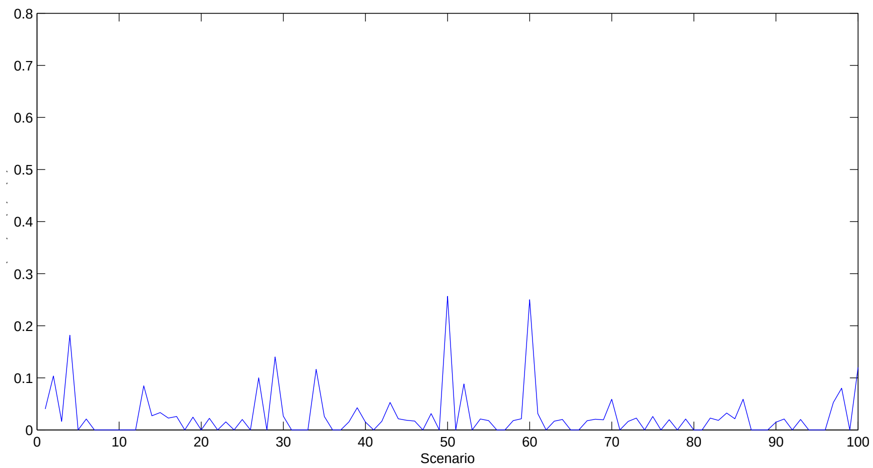



*Fig. 5.16:* Gaps Between $F$ and the Complete Set in the First 100 Scenarios

**Normal distribution.**

The above observation is not only valid when customer demands for final rolls are uniformly distributed, but also true when customer demands follow normal distributions. Figure 5.17 shows the simulation results when each demand follows an independent normal distribution with mean $\mu_i$ and standard deviation $0.4\mu_i$, $i = 1, \ldots, 5$.

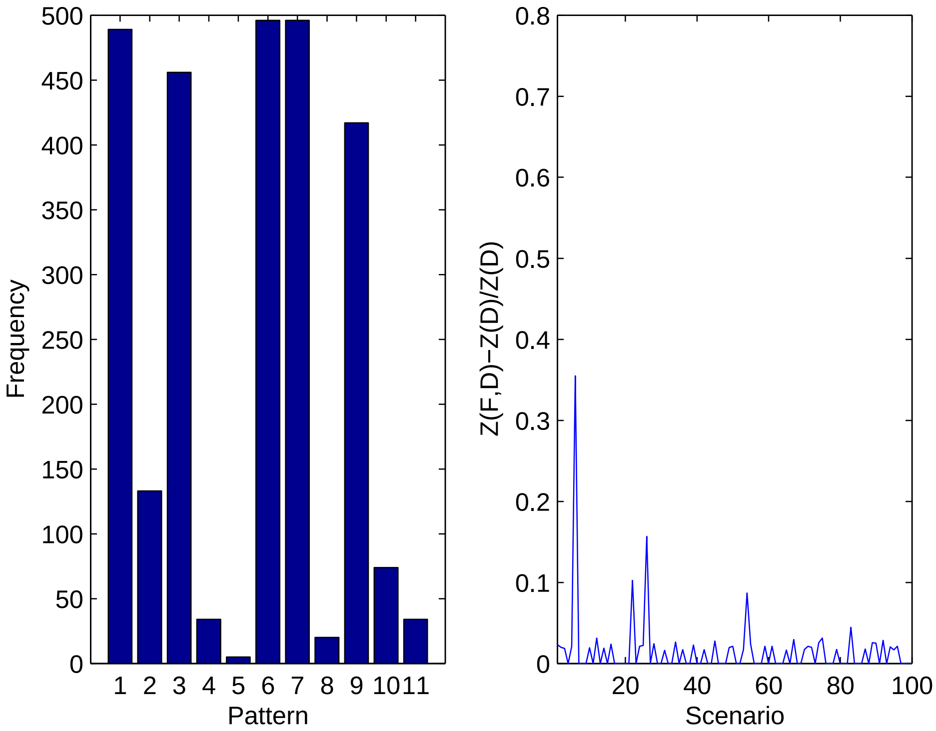



*Fig. 5.17:* Test on Normal Distributions.

Interestingly, the distribution of demand seems to have little impact on the IP activeness of patterns. As shown in Figure 5.17-a, the patterns' frequencies of being active are quite similar to the frequencies when demands are uniformly distributed. The patterns with frequencies greater than 200 (pattern 1, 3, 6, 7 and 9) are also the same as those identified in the uniform simulation. Therefore, we identify the same sparse set $F$ as the one tested when the customer demands are uniformly distributed.

Again, we generate another 500 demand scenarios to test the performance of $F$. As shown in Figure 5.17-b, the difference between the number of master rolls cut by $F$ and by the complete set in the first 100 scenarios are quite small. Most of these differences are no more than 5% of $Z(D^k), k = 1, \ldots 100$. On average, the number of master rolls used by $F$ (47.632) is only 1.3% higher than the optimal quantity (47.016) cut by the complete set. Hence, we can also observe that a small set of cut patterns can achieve almost the full benefits of the large complete pattern set in this experiment.

**Impact of Variances.**

The previous examples have shown us that the patterns' behavior is not dependant on the demand distribution types. Next, we will examine whether the variance of demand is a critical factor in selecting patterns and constructing the sparse set $F$.

We compare 6 different cases: (i)$D_i \sim U[0.8\mu_i, 1.2\mu_i]$, (ii) $D_i \sim U[0.4\mu_i, 1.6\mu_i]$, (iii)$D_i \sim U[0, 2\mu_i]$ , (iv) $D_i \sim N(\mu_i, 0.2\mu_i)$, (v) $D_i \sim N(\mu_i, 0.4\mu_i)$, (vi) $D_i \sim N(\mu_i, 0.6\mu_i)$, for all $i = 1, \ldots, 5$. Note that case (iii) and case (v) which we tested earlier are included for the purpose of comparison.

| Distribution | Pattern | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| $U[0.8\mu_i, 1.2\mu_i]$ | 500 | 86 | 500 | 16 | 0 | 500 | 500 | 19 | 500 | 72 | 0 |
| $U[0.4\mu_i, 1.6\mu_i]$ | 500 | 127 | 463 | 36 | 0 | 500 | 500 | 23 | 415 | 69 | 19 |
| $U[0, 2\mu_i]$ | 454 | 118 | 383 | 39 | 17 | 473 | 487 | 44 | 362 | 88 | 93 |
| $N(\mu_i, 0.2\mu_i)$ | 500 | 136 | 499 | 20 | 0 | 500 | 500 | 24 | 489 | 71 | 0 |
| $N(\mu_i, 0.4\mu_i)$ | 489 | 133 | 456 | 34 | 5 | 496 | 496 | 20 | 417 | 74 | 34 |
| $N(\mu_i, 0.6\mu_i)$ | 467 | 114 | 393 | 40 | 11 | 464 | 478 | 32 | 371 | 72 | 88 |

*Tab. 5.5:* Frequencies of Patterns under Different Distributions

Table 5.5 shows the patterns' frequencies when they are active in the 6 different distributions. Each pattern's frequencies for all distributions are quite close. Thus the sparse set ($F$) constructed is the same in each case.

This result suggests that the demand variance may not play a significant role in selecting active patterns.

The demand variance could, nevertheless, affect the performance of the sparse set $F$ in terms of the worse case performance. As shown in Table 5.6, the maximum gap between the number of master rolls cut by set $F$ $(Z(F,D))$ and by complete set $(Z(D))$ increases quickly as the demand's standard deviation increases. The average gap, however, increases only slightly as the demand variance increases.

| | Complete | | $F$ | | Gap (%) | |
|---|---|---|---|---|---|---|
| | mean | max | mean | max | mean | max |
| $U[0.8\mu_i, 1.2\mu_i]$ | 46.836 | 54 | 47.222 | 54 | 0.83% | 2.56% |
| $U[0.4\mu_i, 1.6\mu_i]$ | 47.892 | 68 | 48.368 | 68 | 1.04% | 6.67% |
| $U[0, 2\mu_i]$ | 46.28 | 81 | 47.39 | 82 | 2.88% | 38.89 % |
| $N(\mu_i, 0.2\mu_i)$ | 47.03 | 61 | 47.442 | 61 | 0.88% | 2.86% |
| $N(\mu_i, 0.4\mu_i)$ | 47.016 | 73 | 47.632 | 74 | 1.41% | 35.48% |
| $N(\mu_i, 0.6\mu_i)$ | 48.698 | 92 | 49.524 | 93 | 1.93% | 26.19% |

*Tab. 5.6:* $F$'s Performances Under Different Demand Distributions

### 5.4.2 Study 2: identify the supporting patterns for a large-scale problem

We adopt the example studied by Gilmore and Gomory [27]. The example considers a master roll with width 218 which results in 23 different final rolls with width and expected demand shown in Table 5.7.

$(P)$, under the large-scale problem settings, is no longer easy to solve, because it contains an exponentially large number of patterns. The integer property of the decision variable $x$ also makes the problem even harder to solve. Therefore, our study will focus on not only examining whether a good small supporting pattern set exists, but also how to find these patterns easily.

| No. | Final width | Mean | No. | Final width | Mean |
|---|---|---|---|---|---|
| 1 | 81 | 4415 | 13 | 49 | 133 |
| 2 | 70 | 291 | 14 | 46 | 529 |
| 3 | 68 | 4765 | 15 | 45 | 185 |
| 4 | 67 | 4827 | 16 | 44 | 94 |
| 5 | 66 | 781 | 17 | 41 | 393 |
| 6 | 64 | 263 | 18 | 38 | 142 |
| 7 | 63 | 274 | 19 | 35 | 411 |
| 8 | 60 | 390 | 20 | 33 | 309 |
| 9 | 56 | 802 | 21 | 32 | 56 |
| 10 | 52 | 824 | 22 | 31 | 171 |
| 11 | 51 | 2948 | 23 | 21 | 140 |
| 12 | 50 | 720 | | | |

*Tab. 5.7:* Problem Settings of the Large-scale Example.

To test the existence of a good small set of cutting patterns, we conduct another simulation by generating a new set of 500 scenarios of demands following the uniform distribution $U[0, 2\mu_i]$ ($i = 1 \dots 23$), and identify the small number of patterns that play an important role in efficiently fulfilling customer demands. We solve $(P)$ using the column generation method employed by Gilmore and Gomory [27] and record all generated and active patterns. 367 different active patterns are recorded during the simulation. It is evident that these patterns have significantly different frequencies. As shown in Figure 5.18, 60% of the patterns appear to be active less than 10 times, while a smaller portion of patterns (4%) are active more than 200 times. This strongly confirms the deduction that only a few patterns are pivotal in the cutting process as the majority of the patterns have very little effect.

**Pattern generation heuristic.**

For the large-scale problem, it is not suitable to count the frequency of pat-

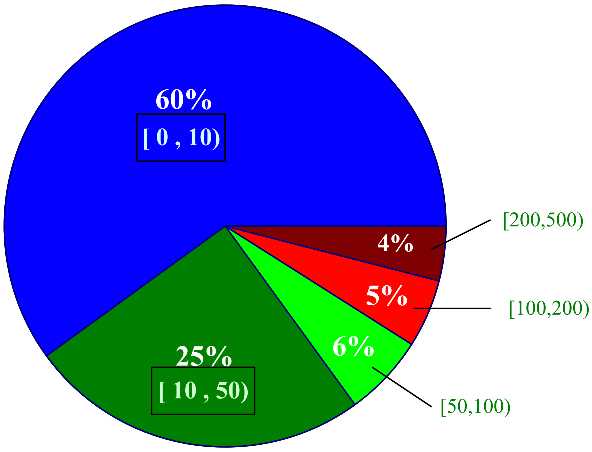



*Fig. 5.18:* The Percentage of Patterns with Different Frequencies

terns in a simulation to identify the supporting patterns, because the column generation method is very time consuming and the number of candidate patterns is very huge. Instead, we propose a simple heuristic to generate good patterns.

The idea of the heuristics is similar to searching over the extreme point in the simplex method. Initially, we use patterns which focus only on cutting final rolls of a specific size, so that if the demand of this final roll increases, the additional master rolls needed (i.e., the increase of the objective of $(P)$) is minimal. In addition, to reduce the trim loss, the left-over part after cutting the final roll, could be used to cutting other smaller rolls wherever possible. Figure 5.19 shows the steps of the pattern generation heuristic generating $N$ patterns for $m$ final rolls.

We then use the heuristic to generate 34 cutting patterns for the large-size problem. We first create a priority list by ordering the final rolls with

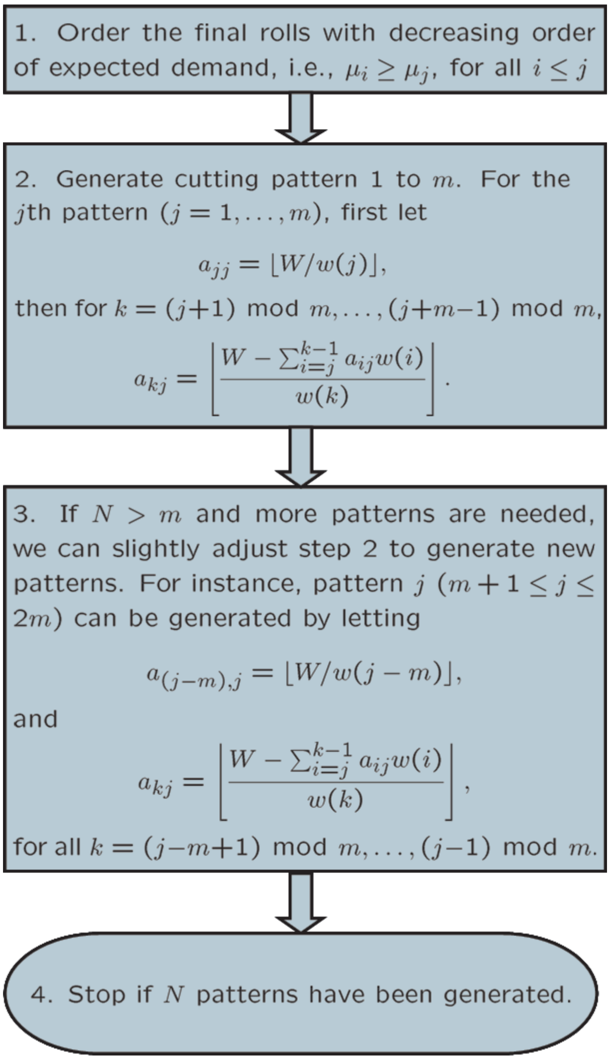



*Fig. 5.19:* Pattern Generation Heuristic.

decreasing order of expected demand. The first 23 cutting patterns can be generated iteratively using the following method. The $i$th pattern first cuts as many as possible pieces of the $i$th final roll in the priority list, that is, $\lfloor W/w(i) \rfloor$, and then uses the left-over part to cut as many as possible pieces of the $i + 1$th final roll, and goes on until the left-over cannot be cut into rolls any more.

If more patterns are needed, they can be generated by slightly adjusting the previous step. For example, patterns 24 to 46 can be generated iteratively as following. The $i$th pattern first cuts $\lfloor W/w(i-23) \rfloor - 1$ pieces of final roll $i - 23$ in the priority list, then uses the left-over part to cut as many as possible pieces of the next final roll in the priority list, and moves on until the left-over cannot be cut any further.

Table 5.8 shows a small set $F$ containing 34 patterns generated by the heuristic. This set of patterns is quite small compared to the 367 active patterns found in the column generation simulation. We next conduct simulations to examine the effectiveness of the heuristic.

**Simulation tests.**

We conduct two groups of simulations to examine the effectiveness of the heuristic. The first group is to evaluate the performance of $F$ generated for this large-scale example under different demand distributions. The second group is using our heuristic to generate cutting patterns for several variations of the original large-scale example and measure the performances of these patterns in simulations.

**Test 1. $F$'s performance in different demand distributions.**

We evaluate the performance of $F$ using six different distributions: (i)$D_i \sim U[0.8\mu_i, 1.2\mu_i]$, (ii) $D_i \sim U[0.4\mu_i, 1.6\mu_i]$, (iii)$D_i \sim U[0, 2\mu_i]$ , (iv) $D_i \sim N(\mu_i, 0.2\mu_i)$, (v) $D_i \sim N(\mu_i, 0.4\mu_i)$, (vi) $D_i \sim N(\mu_i, 0.6\mu_i)$, for all $i = 1, \ldots, 23$. The average and worst case performance of $(P)$ solved by column generation method will be used as the benchmark for evaluation.

As shown in Table 5.9, $F$ performs well in all six situations. The average performance gap between complete set and $F$ is no more than 4% in all cases. This result effectively insinuates that our pattern generation heuristic can be used in fact to find good cutting patterns. It also validates that a small set of patterns can provide almost all benefits of the complete cutting pattern set.

The impact of variance in the width of the final roll in the cutting process seems to be similar to the results shown in the small-size example earlier. As the variance in the roll width increases, the maximum gap between the performance of complete set and $F$ also increases quickly. Conversely, this increase of variance only causes a very slight rise in the average performance gap. Then, it is possibly true that variance plays a minor role when choosing and deciding on a cutting pattern.

**Test 2. $F$'s performance in different variations of the original problem.**

We consider 5 variations of the original problem: (i) only final roll 1, 3, 5, ..., 23 are needed; (ii) only final roll 1-6, 18-23 are needed; (iii) only final roll 1-12 are needed; (iv) only final roll 7-18 are needed; (v) only final roll 12-23 are needed. For each variation, a small set $F$ with 17 cutting patterns is generated using the heuristic.

A simulation is then conducted to evaluate the performance of $F$. 500 scenarios of demand data are generated following the uniform distribution $U[0, 2\mu_i]$. Table 5.10 lists the performance of our heuristic in different variations. The expected number of master rolls cut using the small set $F$ is very close to the result obtained by column generation method (within 7%).

The worst case performance gap between our heuristic and column generation method is around 15% in most cases, with is also consistent with the results in test 1. Hence, the simulation results would strongly support the effectiveness of the pattern generation heuristic.

| | Final roll | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 1 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 0 | 0 |
| 14 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 16 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 0 | 0 | 0 | 0 | 1 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 10 |
| 21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 22 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 23 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 1 |
| 24 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 25 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 26 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 27 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 28 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 29 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 30 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 31 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 5 | 0 | 0 | 0 | 0 |
| 34 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 1 | 0 | 0 | 1 |

*Tab. 5.8:* Generated Cutting Patterns

| Distribution | Column Generation mean | Column Generation max | $F$ mean | $F$ max | Gap (%) mean | Gap (%) max |
|---|---|---|---|---|---|---|
| $U[0.8\mu_i, 1.2\mu_i]$ | 6937.52 | 7760 | 7150.05 | 8002 | 3.07% | 7.51% |
| $U[0.4\mu_i, 1.6\mu_i]$ | 6982.21 | 9263 | 7202.90 | 9511 | 3.20% | 22.80% |
| $U[0, 2\mu_i]$ | 6999.35 | 11162 | 7260.98 | 11491 | 3.79% | 15.1 % |
| $N(\mu_i, 0.2\mu_i)$ | 7004.32 | 8347 | 7215.81 | 8573 | 3.03% | 3.72% |
| $N(\mu_i, 0.4\mu_i)$ | 6966.78 | 8614 | 7183.09 | 8840 | 3.08% | 11.42% |
| $N(\mu_i, 0.6\mu_i)$ | 7235.36 | 11156 | 7491.15 | 11406 | 3.65% | 13.7% |

*Tab. 5.9:* $F$'s Performances Under Different Demand Distributions

| Variation | Column Generation mean | Column Generation max | $F$ mean | $F$ max | Gap (%) mean | Gap (%) max |
|---|---|---|---|---|---|---|
| Roll 1,2,...23 | 4636.32 | 8104 | 4748.93 | 8187 | 2.43% | 17.15% |
| Roll 1-6, 18-23 | 5158.20 | 9017 | 5468.52 | 9720 | 6.02% | 18.31% |
| Roll 1-12 | 6619.48 | 10773 | 6851.58 | 11114 | 3.5% | 19.47 % |
| Roll 7-18 | 1820.47 | 2855 | 1888.02 | 2946 | 3.7% | 16.9% |
| Roll 12-23 | 628.15 | 1029 | 652 | 1081 | 3.8% | 8.8% |

*Tab. 5.10:* $F$'s Performances in Different Variations.

# 6. CASE STUDY: FOOD FROM THE HEART

## *6.1 Food From The Heart*

The "Food from the Heart" Program (FFTH) [1] was initiated by Henry and Christine Laimer, a couple from Vienna who have lived in Singapore since 1996. They were keen to start a charity project in Singapore after they were struck by a news report on bakeries throwing away unsold bread[1]. This provided the needed inspiration to launch FFTH, with the mission to ensure that unsold bread and pastries previously thrown away by bakeries go instead to the less fortunate.

Through their persistent lobbying, sound business plan, and support from the media, well known bakery chains in Singapore[2] have now pledged their support to this innovative program. The couple's experience in the logistics field provided adequate credibility for other corporate sponsors to be forthcoming for the program. For example, Fujitsu Asia provided the anchor for administering the delivery system while other firms like ACRS Automobile Centre and Omega Fusion assisted in cash or kind. The program also

[1] The Sunday Times in November 24, 2002 quoted that large quantities of unsold bread are dumped by bakeries daily.

[2] These bakeries include Prima Deli, Four Leaves, NTUC, Bakery's Corner, Sunshine, Delifrance, Hieotaud (Swiss Gourmet bakery), and Blossoms Cake House.

depended on the kind support from national organizations like the Central Singapore Community Development Council (CDC) and the National Volunteer Centre (NVC) for guidance[3].

To ensure that this program will not falter away like other attempts in the past, meticulous efforts have been put into the design of the delivery operations to address the concerns of the donors. Legal, security and hygiene concerns from the bakeries are painstakingly addressed by specially designed delivery operations and procedure. For instance, welfare homes have to sign a form of indemnity, absolving the bakeries from all legal duties should there be a case of food poisoning; Volunteers can only handle the breads in sealed bags or with gloves; The amount donated by each bakery is recorded and tallied every week for accountability. The volunteers will have to SMS the amount of bread donated to a central system before they deliver the breads. The homes will have to ensure that the breads are properly refrigerated, and were instructed not to consume stale breads.

With strong media support, Christine and Henry have recruited more than 1000 volunteers since 2002. These volunteers take turns to make the collection and delivery routines nightly, either using their own cars or public transport. The logistical concept used in the delivery operation is amazingly simple. The entire planning and monitoring system runs on a "Food-Trek" system, donated by Fujitsu Asia. An administrator for FFTH will select routes (bakery-home assignment) and assign a volunteer (based on the address of the volunteers, and mode of transport) to each route. The staff at

[3] For details on other organization aspects of this innovative program, we refer the readers to Lee (2004).

FFTH reacts to situations when certain volunteers are not able to fulfill their delivery responsibility. When such cases arise, FFTH has to instantly find volunteer replacements to cover the routes on the spot.

To reduce the burden on the volunteers, who normally volunteer one or two nights each week to deliver for FFTH, the administrator usually assigns to each volunteer a fixed route[4] and schedule, i.e., *to deliver from the same bakery to the same home at the same days of each week.* Furthermore, unless the supply from a bakery is exceptionally large, the administrator normally assigns only one volunteer to each bakery each night to avoid job overlap and complication on the ground.

## 6.2 Issues Arising from FFTH

While the assignment principles adopted by FFTH significantly reduce the complexity of the operations, the embedded rigidity inevitably introduces an un-intended problem in the operations - the mismatch between the supply, i.e., the amount of donated breads, and the demand, i.e., the amount required at each home. This problem is exacerbated by the fact that the supply from each bakery at each night is random since it depends on the amount of breads produced and sold at the bakery. Figure 6.1 shows the supply (amount of unsold breads) profile of a typical bakery. We can see that the supply fluctuates from day to day. This makes it even more difficult to match the supplies from the bakeries to the demands from the homes even though the demand at each home is fixed. It is ironical, as pointed out by one volunteer,

[4] So that each volunteer only needs to be familiar with one route, i.e., from a bakery to a home.

that a program on a mission to save food will end up with the donated breads being thrown away, especially when a large amount of leftover breads is brought to a home with only small demand while other homes could have used more breads.

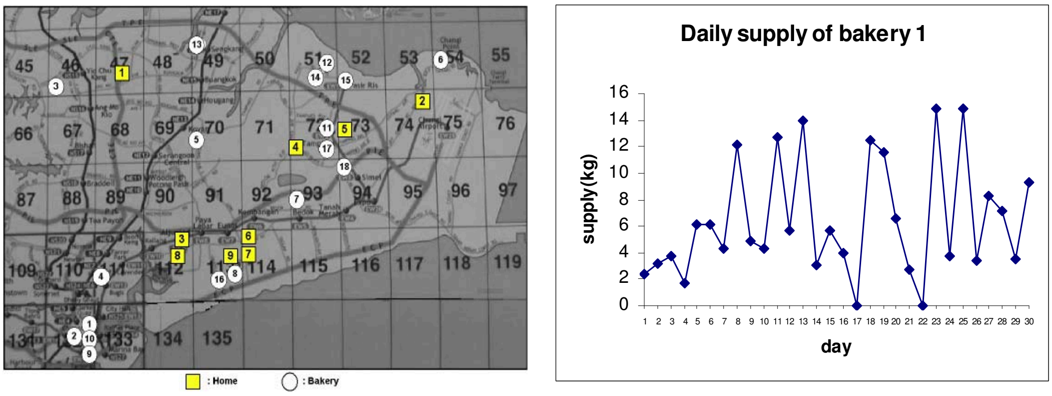



*Fig. 6.1:* Daily Supply of a Bakery (30 days)

Note that the rigidity of the current operations comes from the belief that each volunteer can only deliver from one bakery to one home each night. However, many volunteers are actually willing to visit reasonably more homes to reduce food wastage. This motivates us to look into how much better we can match the fluctuating supply from each bakery to the fixed demand at each home if we assign some volunteers to visit more than one home. It is obvious that a "full flexibility system" which requires each volunteer to visit all homes will best match the supply and the demand. But such a system is also expensive to operate. Therefore, we aim to design a good "partial flexibility system" which only requires each volunteer to visit a small number of homes but can match the supply to the demand almost as well as a full flexibility system.

## 6.3 Flexible Routing System

In this section, we apply our expansion heuristic to the "Food from the Heart" delivery problem. We singled out 9 homes with similar delivery characteristics such as delivery frequency and delivery time. For convenience, the homes are ordered in descending order of their demands. 18 bakeries have been assigned to send foods to these 9 homes by FFTH program. The 18 bakeries's daily supplies are recorded for 66 days from July to September 2003. The quantity of leftover breads collected during this time period showed wide fluctuation. The homes' demands are constant. The demands of homes, means and standard deviations of leftover foods in bakeries are shown in Figure 6.2. The units are in kilograms.

The current routes in use are not optimal, because they were designed by the staff of FFTH program in an ad hoc manner. We first replace the current routes by the optimal dedicated routes constructed from phase 1 of our heuristic. The advantage of this improvement is that the delivery operations will essentially remains intact, except that now the volunteers deliver breads to different locations. We obtained the optimal dedicated routes (Figure 6.2-A) by generating 100 different scenarios of daily supply profiles from the historical data and find the optimal dedicated routes using ILOG CPLEX 9.1. We use the performance of the optimal dedicated routing system as a more rational and stricter benchmark to assess the performance of flexible system designed by phase 2 of our heuristic.

Figure 6.2-B shows the new flexible routing system obtained by our heuristic. The newly added arcs help forming many long chains in this new

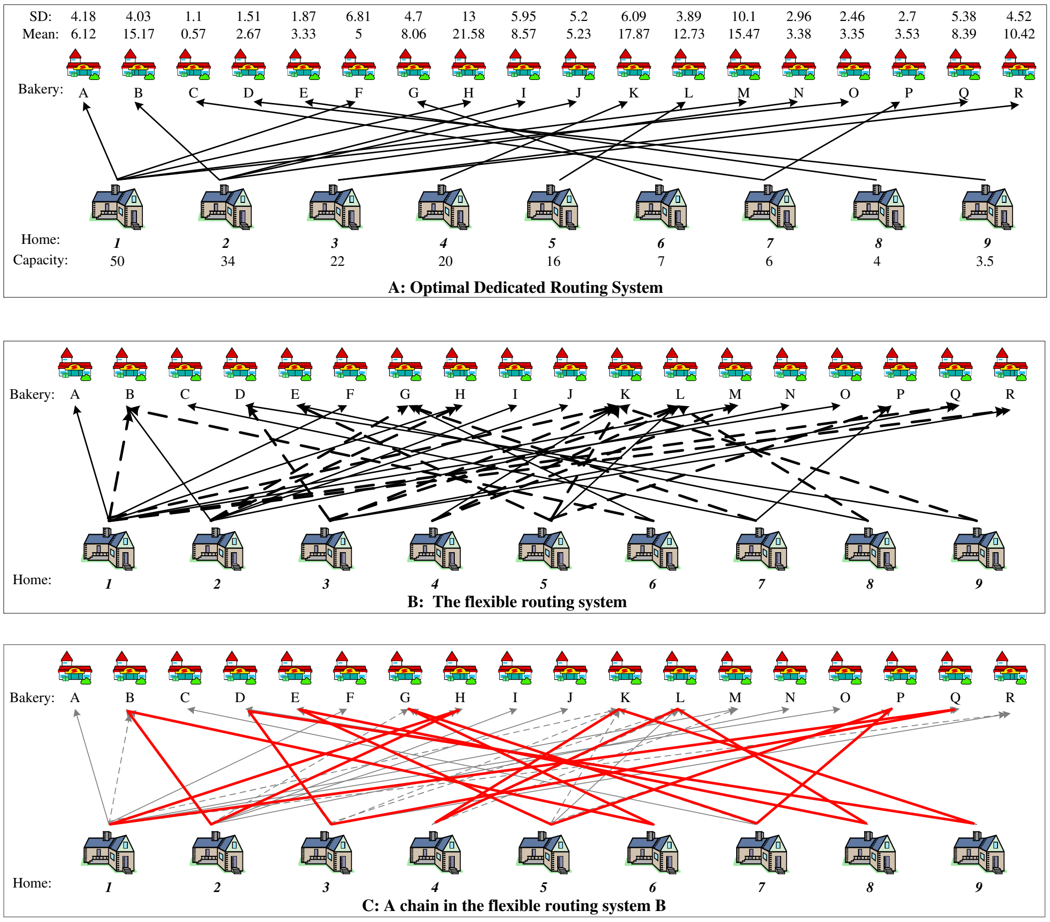



*Fig. 6.2:* The Different Routing Systems for FFTH Problem.

flexible system. A long chain that visits 9 homes and 9 bakeries is shown in Figure 6.2-C for illustration. Among the 18 arcs in the chain, 11 arcs are newly added by our heuristic. This result suggests that our heuristic is very effective to construct a flexibility structure which contains long chains.

We conduct simulation analysis to evaluate this flexible system. Note that it is reasonable to assume that the supply of each bakery is statistically independent. Hence, in our simulation analysis, a bakery's supply is generated by randomly selecting a number from its historical data. Daily supplies of 100 days are simulated. We use the expected daily excess as the measure to evaluate this system because the purpose of this case study is to test the effectiveness of our heuristic in real world problem. In this case, the effectiveness of our heuristic is to show how much we could help to decrease the food wastage in the FFTH program. Therefore, the expected daily oversupply, which is also widely accepted as a measure of a flexibility structure's performance in practice, is preferred to evaluate the flexible system in this case study.

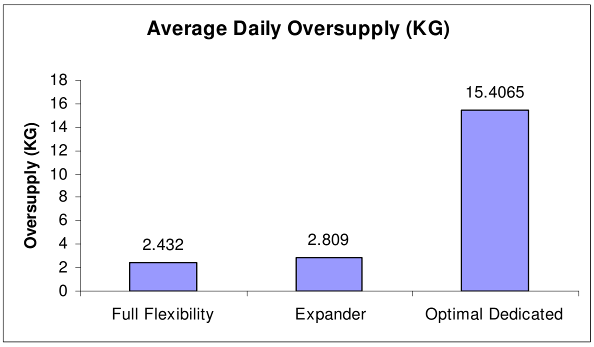



*Fig. 6.3:* Average Daily Excess.

Figure 6.3 shows the average daily excess in full flexibility system, the heuristic flexibility system and the optimal dedicated system. By adding 18 arcs to the optimal dedicated system, the average daily excess decreases significantly from 15.407 kilograms to 2.809 kilograms. It is only 20% of the optimal dedicated system's excess. Moreover, it is only 0.377 kilograms greater than the excess of the fully flexible system. On average, the food savings through the flexible routing system each day (148.64 kg[5]) is 99.7%

[5] The average daily food savings for the heuristic flexibility system=the average daily

of the foods sent by full flexibility system (149.02 kg [6]). This result not only suggests our heuristic works quite well in practice but also strongly supports that the exapnder flexible system, which has high expansion ratios and contains long chains, is the desired flexible structure.

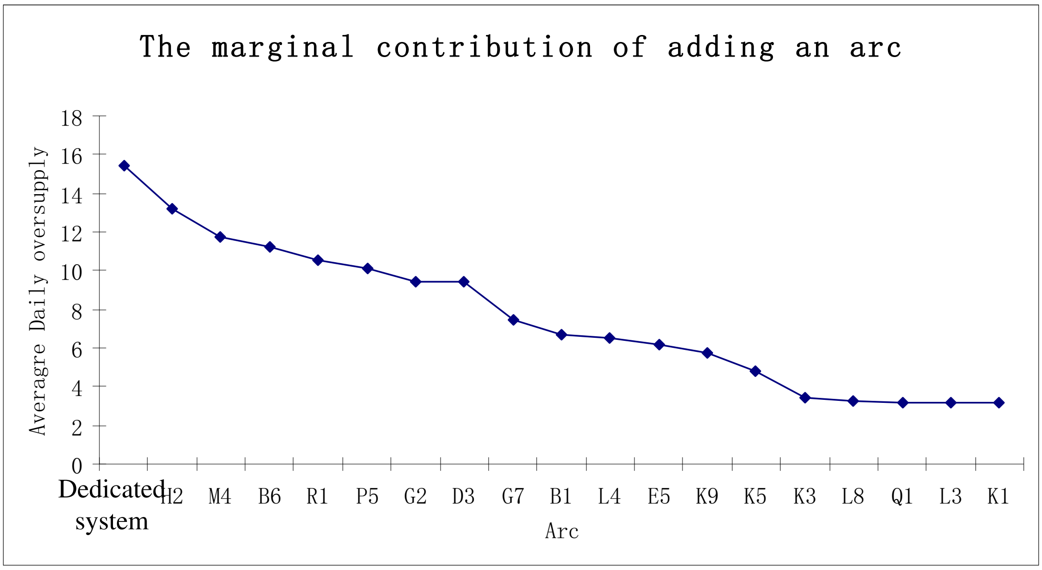



*Fig. 6.4:* Marginal Contribution of Each Arc.

Another interested finding in this case study is that the contributions of the added arcs decrease very quickly if they are added in a certain sequence. Figure 6.4 shows that the marginal contributions of arcs diminish very quickly if they are added in the sequence shown in $x$ axis. This result is consistent with the finding of Jordan and Graves[36]. It would also support that a sparse flexibility structure can capture the benefit of full flexibility structure. Therefore, the number of arcs we need to add to a base assignment is small.

leftover foods - the average daily oversupply of the heuristic= 151.45kg-2.809kg = 148.64 kg.

[6] The average daily food savings=151.45kg-2.432kg=149.02kg

In practice, people only need to design a structure with a small number of arcs to deal with uncertainties.

# 7. CONCLUSIONS

## *7.1 Summary of Results*

The purpose of this study is to provide a clear understanding of flexibility structures and find an effective way to design and analyze different flexibility structures in various applications.

In this study, the concept of graph expander is first introduced to investigate the performance of flexibility structures. We point out the connection between graph expansion and flexibility structure, and show that good expanders give rise to good process flexibility structures, in the case of identical mean supply and demand. We further analytically prove that there exists a sparse flexible network that has almost the same capability of a fully flexible system. This proof also provides an upper bound for the performance of any expander flexibility structure. This observation has numerous implications. First, The expander concept can be adjusted to construct a practical expansion heuristic to design a sparse flexibility structure under a generalized condition that demand and supply could be random. Secondly, a simple and effective expansion index can be developed to effectively calibrate the structures in terms of flexibility.

The "constraint sampling" method is then introduced to further support the existence of a good sparse flexibility structure under the generalized condition that demand and supply could be non-identical and unbalanced. Another design approach, the sampling method, is also proposed to construct a flexible sparse structure.

The observation that a well-designed sparse structure could be almost as flexible as full flexibility structure can be applied to various areas. In this study, we investigate manufacturing production planning problem, transshipment network design problem, and cutting stock patterns design problem. The expansion heuristic and sampling heuristic are also applied to these applications to design a good sparse structure. Furthermore, these approaches have been applied to solve a real bread delivery problem in a charity organization "Food From The Heart" [1] in Singapore.

## 7.2 Research Contributions

The theoretical contribution of our study is that we provide an analytical support to the observation that a well-designed sparse structure could be a good support to the completely connected flexibility structure, which is already indicated by numerous computational studies (c.f. [29] [32] [33] and [36]). More importantly, our study discover the relationship between a good sparse flexibility structure and an expander. Our results strongly suggest that a good flexibility structure is an expander with a large expansion ratio.

Our study has very important practical contributions. We propose two effective heuristics to design good sparse flexibility structures. These heuristics can be easily adjusted to build good sparse structures in different applications such as transshipment networks, supply chain flexibility structures, and etc. In practice, both heuristics are quite effective and robust. The expansion heuristic requires minimum demand and supply information: only mean of random demand/supply is needed. This heuristic would be quite helpful under the situation that the flexibility capacity investment should be decided before the exact demand/supply distribution is known, or the demand/supply is quite unstable and specific distribution cannot be used to model them. The sampling heuristic requires the full distribution of the random demand supply, and can be applied to a broad area such like cutting stock problem, transshipment network design and etc.

## 7.3 Future Studies

A basic assumption we made in this study is that the capacities allocated for products should be decided based on complete information, i.e. the demands and supplies are all known. This assumption is also widely used in many other studies (c.f. [36],[32],[29] and [33]). However, in the FFTH problem, bakeries' closing time are different, and the food should be delivered shortly after the closing time. Thus it is difficult to get all the information of bakeries' supplies in the system before a coordinator assigns a volunteer a delivery route. Though an expander structure still works well in this case, the online features of the FFTH problem limit the power of the expander. Therefore, the design of an online flexibility structure should be carefully studied in future research. One possible way is using the dynamic approach to remodel the FFTH problem and examine the performance of the new structure in the online situations.

Our results already showed that a good sparse flexibility structure should be an expander. However, it is not easy to find this expander. We propose a heuristic instead to solve the design problem. Hence, further research is still needed to find a better way to such structure. One possible way is to find a strong LP relaxation to the expander problem, which can significantly reduce the computational time. Another way is to introduce an efficient index to measure flexibility and combine this index with current sampling algorithm to build a more efficient sampling method.

# BIBLIOGRAPHY


[1] Food from the heart. *http://www.foodheart.org*.

[2] mathworld. *http://mathworld.wolfram.com/*.

[3] O. Z. Aksin and F. Karaesmen. Characterizing the performance of process flexibility structures. Working Paper, July 2004.

[4] G. Anand and P. T Ward. Fit, flexibility and performance in manufacturing: Coping with dynamic environments. *Production and Operations Management*, 13(4):369, 2004. TY - JOUR.

[5] A. Asratian, T. Denley, and R. Haggkvist. *Bipartite graphs and their applications*. Cambridge University Press, 1998.

[6] L. A. Bassalygo and M. S. Pinsker. Complexity of an optimum nonblocking switching network with reconnections. *Problemy Informatsii (English Translation in Problems of Information Transmission)*, 9:84–87, 1973.

[7] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Math. Oper. Research.*, 23(4):769–805, 1998.

[8] S. Benjafaar. Modeling and analysis of congestion in the design of facility layouts. *Management Science*, 48(5):679–704, 2002.

[9] E. Bish, A. Muriel, and S. Biller. Managing flexible capacity in a make-to-order environment. *Management Science*, 51:167–180, 2005.

[10] E. Bish and Q. Wang. Optimal investment strategies for flexible resources, considering pricing and correlated demands. *Operations Research*, 52(6):954–964, 2004.

[11] J. Browne, D. Dubois, K. Rathmill, S. P. Sethi, and K. E. Stecke. Classification of flexible manufacturing systems. *The FMS Magazine*, August:114–117, 1984.

[12] M. Brusco and T. Johns. Staffing a multiskilled workforce with varying levels of productivity: An analysis of cross-training policies. *Decision Science*, 29(2):499–515, 1998.

[13] G. Calafiore and M. C. Campi. Uncertainty convex programs: Randomized solutions and confidence levels. *Math. Programming*, 102(1):25–46, 2005.

[14] X. Chen, M. Sim, P. Sun, and J. Zhang. Stochastic programming: Convex approximation and modified linear decision rule. Working Paper, 2005.

[15] M. Chou, G. Chua, and C. P. Teo. Process flexibility: Sales versus profits. Working Paper July 2006.

[16] G. B. Dantzig. Linear programming under uncertainty. *Management Science*, 1:197–206, 1955.

[17] S. K Das and P. Patel. An audit tool for determining flexibility requirements in a manufacturing facility. *Integrated Manufacturing Systems*, 13(4):264, 2002. TY - JOUR.

[18] D. P. de Farias and B. Van Roy. On constraint sampling in the linear programming approach to approximate dynamic programming. *Math. Oper. Res.*, 29(3):462–478, 2004.

[19] K. Eismann. The trim problem. *Management Science*, 3:279–284, 1957.

[20] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematics Jornal*, 23:298–305, 1973.

[21] C. H. Fine and R. M. Freund. Optimal investment in product-flexible manufacturing capacity. *Management Science*, 36(4):449–466, 1990.

[22] J. Friedman. A proof of alon's second eigenvalue conjecture. *In STOC 03*, page To Appear, 2003.

[23] G. Gallego and R. Phillips. Revenue management of flexible products. *Manufacturing & Service Operations Management*, 6(4):321–337, 2004. TY - JOUR.

[24] R. Ghodsi and F. Sassani. Online cutting stock optimization with prioritized orders. *Assembly Automation*, 25(1):66, 2005. TY - JOUR.

[25] A. Ghosh and S. Boyd. Growing well-connected graphs. *Working Paper*, 2006.

[26] P. C. Gilmore and R. E. Gomory. A linear programming approach to the cutting-stock problem. *Operations Research*, 8:849–859, 1961.

[27] P. C. Gilmore and R. E. Gomory. A linear programming approach to the cutting stock problem–part ii. *Operations Research*, 11(6):863–888, 1963.

[28] P. C. Gilmore and R. E. Gomory. Multistage cutting stock problems of two and more dimensions. *Operations Research*, 13(1):94–120, 1965.

[29] S. C. Graves and B. T. Tomlin. Process flexibility in supply chain. *Management Science*, 49(7):907–919, 2003.

[30] S. Gurumurthi and S. Benjaafar. Modeling and analysis of flexible queueing systems. *Naval Research Logistics*, 51:755–782, 2004.

[31] Y. T. Herer, M. Tzur, and E. Yűcesan. The multi-location transshipment problem. *IIE Transactions*, To appear.

[32] W. J. Hopp, E. Tekin, and M. P. Van Oyen. Benefits of skill chaining in production lines with cross-trained workers. *Management Science*, 50(1):83–98, 2004.

[33] S. M. Iravani, M. P. Van Oyen, and K. T. Sims. Structural flexibility: A new perspective on the design of manufacturing and service operations. *Management Science*, 51(2):151–166, 2005.

[34] E. P. Jack and A. S. Raturi. Sources of volume flexibility and their impact on performance. *Journal Of Operations Management*, 20(5):519–548, 2002.

[35] E. P. Jack and A. S. Raturi. Measuring and comparing volume flexibility in the capital goods industry. *Production and Operations Management*, 12(4):480, 2003.

[36] W. C. Jordan and S. C. Graves. Principles on the benefits of manufacturing process fexibility. *Management Science*, 41(4):577–594, 1995.

[37] S. Kara and B. Kayis. Manufacturing flexibility and variability:

an overview. *Journal of Manufacturing Technology Management*, 15(6):466–478, 2004.

[38] K. S. Krishnan and V. R. K. Rao. Inventory control in n warehouse. *Journal of Industrial Engineering*, 16:212–215, 1965.

[39] M. Lahmar, H. Ergan, and S. Benjaafar. Resequencing and feature assignment on an automated assembly line. *IEEE Transactions on Robotics and Automation*, 19(1):89–102, 2003.

[40] R. Lien, S. M. Iravani, K. Smilowitz, and M. Tzur. Efficient and robust design for transshipment networks. Working Paper, 2005.

[41] A. E. B. Lim and J. G. Shanthikumar. Relative entropy, exponential utility and robust dynamic pricing. Working Paper, 2005.

[42] A. Muriel, A. Somasundaram, and Y. Zhang. Impact of partial manufacturing flexibility on production variability. *Working Paper*, 2001.

[43] M. Pinsker. On the complexity of a concentrator. In *In 7th International Teletraffic Conference*, pages 318/1–318/4. Stockholm, 1973.

[44] O. Reingold, S. Vadhan, and A. Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders and extractors. *Ann. of Math.*, 155:157–187, 2002.

[45] L. W. Robinson. Optimal and approximate policies in multiperiod, multilocation inventory models with transshipments. *Operations Research*, 38(2):278–295, 1990.

[46] P. Sarnak. What is an expander? *Notices of the AMS*, 51(7):762–763, 2004.

[47] S. S. Seiden and G. J. Woeginger. The two-dimensional cutting stock problem revisited. *Mathematical Programming*, 102(3):519, 2004. TY - JOUR.

[48] A. K. Sethi and S. P. Sethi. Flexibilityin manufacturing: a survey. *The International Journal of Flexible Manufacturing Systems*, 2:289–328, 1990.

[49] D. Shi and R. L. Daniels. A survey of manufacturing flexibility: Implications for e-business. *IBM Systems Journal*, 42(3):414–427, 2003.

[50] K. E. Stecke. Formulation and solution of nonlinear integer production planning problems for flexible manufacturing systems. *Management Science*, 29(3):273–288, 1983.

[51] P. E. Sweeney and R. W. Haessler. One-dimensional cutting stock decisions for rolls with multiple quality grades. *European Journal of Operations Research*, 44:224–255, 1990.

[52] G. Tagaras and M. Cohen. Pooling in two-location inventory systems with non-negligible replenishment lead times. *Management Science*, 38(8):1067–1083, 1992.

[53] R. M. Tanner. A recursive approach to low complexity codes. *IEEE Trans. Inform. Theory*, 27:533–547, 1981.

[54] D. M. Upton. The management ofmanufacturing flexibility. *California Management Review*, 36(2):72–89, 1994.

[55] J. Van Biesebroeck. Flexible manufacturing in the north-american automobile industry. Working Paper, 2004.

[56] J. A. Van Mieghem. Investment stragegies for flexible resources. *Management Science*, 44(8):1071–1078, 1998.

[57] C. Vitzthum. Spain's zara cut a dash with "fashion on demand". *The Wall Street Journal*, May 1998.

[58] M. A. Vonderembse. Exploring a design decision for a cutting stock problem in the steel industry: All design widths are not created equal. *IIE Transactions*, 27(3):358, 1995. TY - JOUR.

[59] R. B. Wallace and W. Whitt. A staffing algorithm for call centers with skill-based routing. 2004.

[60] D. Z. Yu, S. Y. Tang, H. Shen, and J. Niederhoff. On benefits of operational flexibility in a distribution network with transshipment. *Working Paper*, 2006.