SOCIAL SECURITY, WELFARE AND ECONOMIC GROWTH

YEW SIEW LING

(BA (Education) with Honours, USM, Malaysia;
Master in Economics, UPM, Malaysia)

A THESIS SUBMITTED

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ECONOMICS

NATIONAL UNIVERSITY OF SINGAPORE

2009

ACKNOWLEDGEMENTS

Firstly, I would like to express my deepest gratitude to my supervisor and mentor, Professor Zhang Jie, for his patience, motivation, enthusiasm, and immense knowledge which have stimulated my interest in doing research. His constant guidance and encouragement have made possible the completion of this thesis. All the three papers in this thesis are coauthored with my supervisor.

I would also like to express my sincere gratitude to my co-supervisor Associate Professor Zeng Jinli for his kind advice and continuous support of my PhD study and research. I have also benefited a lot from his research-informed teaching of the module "Advanced Economic Growth".

Very special thanks go to Associate Professor Aditya Goenka, Dr. Emily Cremers, Dr. Lu Jingfeng, Associate Professor Liu Haoming, Associate Professor Tsui Ka Cheng, Dr. Aamir Rafique Hashmi, Professor Basant K. Kapur, the graduate committee members, and all the teachers that have taught me before for their kind guidance, useful suggestions and insightful comments.

I would also like to thank my friends at NUS for the stimulating discussions and comments.

Last but not the least; I thank my family for their support during my PhD study.

TABLE OF CONTENTS

SUMMARY

This thesis examines the implications of social security in a dynastic family model with altruistic bequest and endogenous fertility.

The first chapter focuses on the optimal scale of pay-as-you-go (PAYG) social security in a dynastic family model with human capital externalities, fertility, bequest and endogenous growth. If the taste for the number of children is sufficiently weak relative to the taste for the welfare of children, social security can be welfare enhancing by reducing fertility and raising human capital investment per child.

The second chapter explores the optimal PAYG social security and education subsidization in a dynastic family model with two types of capital, endogenous fertility and positive spillovers from average human capital. Such spillovers reduce the private return on human capital investment relative to the return on having an additional child, thereby leading to under-investment in human capital and over-reproduction of population. This chapter shows that social security and education subsidization together can fully eliminate such efficiency losses and achieve the socially optimal allocation under plausible conditions. But none of them can do so alone.

Since rising life expectancy has created financial pressure on maintaining a balanced budget for PAYG social security programs in many countries, the last chapter considers life expectancy as an endogenous variable. This chapter investigates long-run optimal tax rates of PAYG social security and public health and explores how they affect fertility, life expectancy, capital intensity, output per worker and welfare in a dynastic model with altruistic bequests and endogenous fertility. If the taste for the number of children is weaker but sufficiently close or equal to that for the welfare of children, social security and public health can reduce fertility and raise life expectancy, capital intensity and output per worker. The simulation results show that social security and public health can be welfare enhancing by reducing fertility and raising capital intensity.

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

Optimal social security in a dynastic model with human capital externalities, fertility and endogenous growth

1.1. Introduction

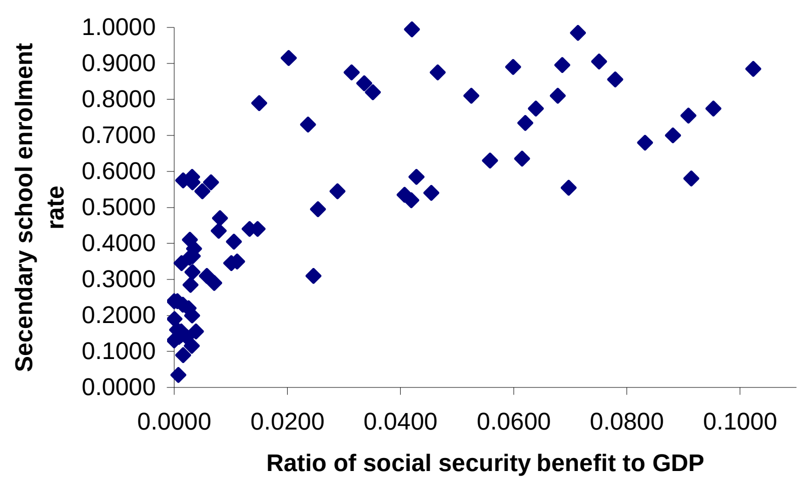
In this paper we investigate the implication of human capital externalities for optimal pay-as-you-go (PAYG) social security in a dynastic family model with two types of capital and with endogenous fertility. Human capital accumulation has been recognized as a key factor for earnings; see, e.g., some related studies in the survey article of Lemieux (2006). Yet, the outcome of human capital accumulation for children is under the influence of parental factors as well as social factors outside their families (i.e. external to families). According to empirical evidence by Solon (1999), about half of children's earnings are correlated with their parental earnings. This evidence suggests that non-parental factors or human capital externalities may be quantitatively substantial in the formation of one's human capital. Indeed, some empirical studies find evidence on human capital externalities in the determination of individuals' earnings through channels such as ethnic groups, neighborhoods, work places, or state funding of schools; see, e.g., Borjas (1992, 1994, 1995), Rauch (1993), Davies (2002) and Moretti (2004a, 2004b). For example, according to the studies of Borjas, the earnings of children are affected significantly not only by the earnings of their parents, but also by the mean earnings of the ethnic group in the parents' generation through ethnic neighborhoods in the United States. Also, Moretti (2004b) finds evidence on the effects of human capital externalities on individuals' earnings in manufacturing establishments across cities in the Unites States with different levels of human capital. The existence of human capital externalities found in the literature implies that the private rate of return to human capital investment should be lower than the social rate of return. This tends to engender underinvestment in human capital and thus may have strong policy implications for optimal social security.

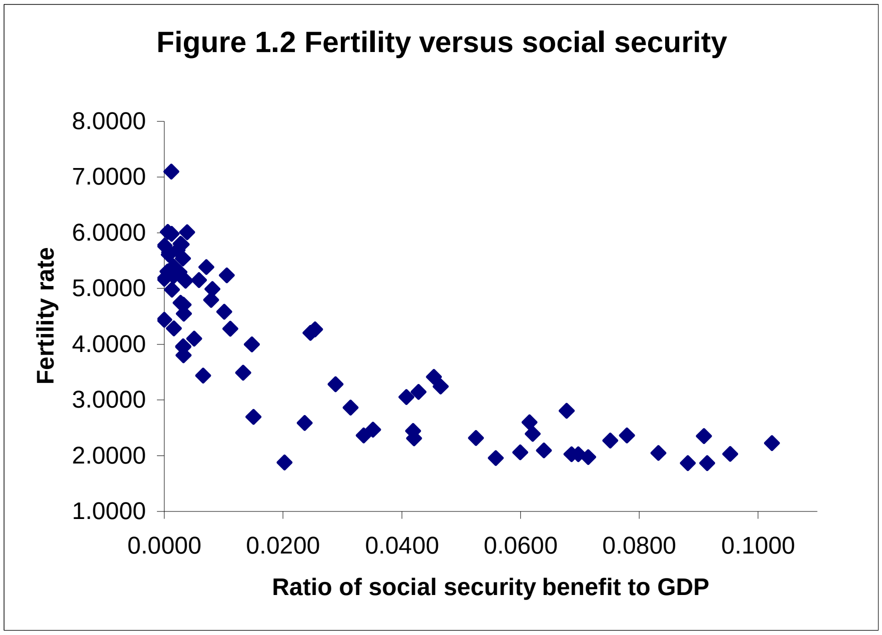
As important family decisions according to the well known trade-off between the quality and quantity of children in Becker and Lewis (1973), human capital investment and fertility have been found to be responsive to social security and thus serve as channels through which social security affects economic growth and population growth in Zhang (1995). Using cross-country data for the period 1960-2000, Zhang and Zhang (2004) investigate the effect of social security on growth and growth determinants (savings, human capital investment, and fertility).[1] Their empirical analysis allows for feedback from growth to social security and treats growth, fertility, human capital investment and savings as endogenous variables using the IV estimation method. They also allow for country-specific fixed effects in a panel regression. They show that the ratio of social security benefits to GDP has a positive effect on human capital investment and a negative effect on fertility, as suggested in Figures 1.1 and 1.2 that plot secondary school enrolment and fertility respectively against the ratio of social security benefits to GDP in 70 countries of market economies. It is thus interesting to extend this line of research to explore the

[1] Their data for social security benefits under statutory schemes are from the International Labor Office (ILO, various years); secondary school enrollment ratios and adult populations' education attainment, used as proxies for human capital investment and human capital stock respectively, are from UNESCO; GDP, consumption and saving are based on the Penn World Table by Summers and Heston (1988) and Heston, Summers and Aten (2002); government education, government consumption, government transfers, population, fertility net of child mortality, revolutions, coups and assassinations are from Barro and Lee (1994) and the United Nations' Demographic Yearbook (various years).

welfare implication and the optimal scale of social security in a dynastic family model with both human capital and fertility. This task is highly relevant today when many countries have been debating on whether PAYG social security should be reformed.

Figure 1.1 Secondary school enrolment versus social security across 70 countries





While most studies on social security focus on its implication for capital accumulation, few have paid close attention to its welfare implication. Among them, Cooley and Soares (1999) have used a majority voting mechanism to justify why social security receives a majority support once it is already in place, although their model does not explain why it was instituted in the first place. Also, Zhang and Zhang (2007) have considered optimal social security with investment externalities in the final production sector in an extended neoclassical growth model without sustainable growth. However, having ignored human capital accumulation, these models do not capture the interaction between social security on the one hand and the trade-off between the quality and quantity of children on the other.

The inclusion of human capital investment can be highly relevant in the analysis of optimal social security. On the one hand, the payroll tax for social security reduces the after-tax wage rate or the after-tax rate of return on human capital investment, thereby tending to reduce human capital investment. Thus, considering human capital investment in the analysis may make it more likely for social security to reduce welfare. On the other hand, when social security reduces fertility, human capital investment per child may rise via the trade-off between the quantity and quality of children. Because of these opposing forces, social security may engender a welfare gain only when the human capital externality causes fertility to be above its first-best level and causes human capital investment per child to be below its first-best level. If social security does improve welfare, it is also interesting in theory and relevant in practice to gauge the size of the optimal social security tax rate numerically for plausible parameterizations and compare it to the observed social security payroll tax rates in the real world.

The rest of the paper proceeds as follows. The next section introduces the model. Sections 1.3 and 1.4 determine the equilibrium solution and derive the results. Section 1.5 concludes.

1.2. The model

The model is an extension of Zhang and Zhang (2007) to incorporate human capital accumulation and to explore the welfare implication of social security with an externality in the form of spillovers of average human capital to all children's learning. This extension departs from the neoclassical growth model toward an endogenous growth model. The model economy is inhabited by overlapping generations of a large number of identical agents who live for three periods. In their first period of life, they embody human capital and do not make any decision. In their second period of life, they work and make decisions on life-cycle savings and on the number and education of identical children. In their third period of life, they retire and decide only on the allocation between the amount of bequests to children and their own old-age consumption. The mass of the working generation in period t is denoted by L_t.

The preferences of the coexisting old parent and young working members in a family are assumed to be identical, defined over the consumption levels of the old and young members, $C_{o,t}$ and $C_{y,t}$ respectively, and the number of children N_t of family members in all generations:

$$U_0 = \sum_{t=0}^{\infty} \alpha^t V(C_{o,t}, C_{y,t}, N_t), \quad \alpha \in (0,1),$$

where α is the discounting factor. [2] The period-utility function $V(C_{o,t,} C_{y,t}, N_t)$ captures what contributes to family members' welfare within a period: the consumption of coexisting old and young members as well as the number of children. The old-age consumption of a period-t young member will be reflected in $V(C_{o,t+1,} C_{y,t+1}, N_{t+1})$ next period. In this way, we can incorporate the life-cycle consumption-saving consideration into a dynastic family model along with the trade-off between the welfare and the number of children in a recursive manner. We assume that the period-utility function $V(\cdot,\cdot,\cdot)$ is increasing and concave and meets the

[2] There are various assumptions on preferences in the overlapping-generations models dealing with the demographic changes in the economy. Becker and Barro (1988) assume dynastic preferences where the discount factor is a function of the number of children. However, that assumption may not lead to analytical solutions with two types of capital. To obtain analytical solutions, we assume that α is independent of the number of children as in Lapan and Enders (1990) and Zhang (1995).

Inada conditions to ensure an interior optimal solution: $\partial V / \partial x \to 0$ as $x \to \infty$ for $x = C_o, C_y$; and $\partial V / \partial x \to \infty$ as $x \to 0$ for $x = C_o, C_y, N$.

The utility function in our model allows coexisting old and young members in the same family to value each other's consumption, in addition to their appreciation of the number of children and future generations' welfare in their family. In the conventional dynastic family model, by contrast, there is just one period in adulthood in which parents value the future consumption of children but not vise versa (downward altruism), since parental consumption would have become *sunk* when children grow up and make their own decisions. When young and old adults coexist and choose consumption in the same period in a family, however, the conventional assumption would rule out possible altruism from working family members toward their parents' old-age consumption and hence would create generational conflicts. In this sense, our approach here complements approaches featuring generational conflicts between coexisting old and working agents in conventional dynastic family models with only downward altruism as well as in conventional life-cycle models without any form of altruism. Further, our use of a dynastic family model, rather than a simple non-altruistic life-cycle model, is partly based on evidence in Zhang and Zhang (2004) that social security has an insignificant effect on private savings.[3]

However, the literature on the existence and on the form of altruism is divided in theory as well as in empirical evidence. On the one hand, empirical studies supporting the altruistic model include Tomes (1981), Laitner and Juster (1996), and

[3] A dynastic family model and a life cycle model have very different implications concerning how PAYG social security affects private savings rates. As is well known in the literature, the effect of social security on savings is neutral in the former model (e.g., Barro, 1974) but negative in the latter (e.g., Feldstein, 1974).

Laitner and Ohlsson (2001), among others. For example, the empirical studies of Laitner and Ohlsson (2001) show that the bequest behavior in Sweden and the U.S. offers support for the altruistic model. On the other hand, Altonji, Hayashi and Kotlikoff (1997) and Horioka (2002), among others, cast doubt on the hypothesis that altruism motivates intergenerational transfers. According to Horioka, the selfish life-cycle model is dominant both in the United States and Japan. These empirical studies use data in developed countries whereby the presence of social security and welfare systems might have weakened interactions among generations within families that are needed for detecting altruism. In particular, the traditional role of children in supporting old parents may no longer be necessary in these countries. By contrast, Raut and Tran (2005) use a sample of 7128 households from the Indonesian Family Life Survey (IFLS) data set in a developing country and find supporting evidence for the two-sided altruism model. Their estimated difference in the transfer-income derivatives between parents and children in the Indonesian data set is as high as 0.956, which is close to 1 as implied by altruistic models of intergenerational transfers and is much higher than an estimated counterpart 0.13 in Altonji et al. (1997) based on the US data set. Overall, our use of a dynastic model with two-sided altruism is consistent with some of the existing empirical evidence in a divided body of the related literature.

For tractability, we assume $V(C_{o,t}, C_{y,t}, N_t) = \beta \ln C_{o,t} + \alpha(\ln C_{y,t} + \rho \ln N_t)$. Here, $\beta \in (0,1)$ is the taste for utility derived from the consumption of the old parent, $\alpha \in (0,1)$ is the taste for utility from the young-age consumption and the number of children of each working member, and $\rho > 0$ is the taste for utility from the number of children relative to that from young-age consumption. If we equally value consumption undertaken by each of coexisting old and working members in a family in their identical utility function, then the values of α and β may depend on the relative length of working-age versus old-age lifetime. Since in reality the working period is longer than the retirement period, α may be greater than β. We rewrite the utility function as

$$U_0 = \sum_{t=0}^{\infty} \alpha^t [\beta \ln C_{o,t} + \alpha(\ln C_{y,t} + \rho \ln N_t)], \;\; 0 < \alpha, \beta < 1, \;\; \rho > 0. \tag{1.1}$$

For an initial old agent in period 0 who had chosen N_{-1} children, the only remaining decision is the trade-off between his or her own old-age consumption $C_{o,0}$ and the amount of bequests to children B_0.

Some observers may regard "upward" altruism in equation (1.1) as a more indirect phenomenon and "downward" altruism as a more direct one. However, assuming a preference with downward altruism and without upward altruism for the welfare assessment of social security would ignore the rise in utility for the initial old generation who receives social security benefits. In fact, the preference in (1.1) can be interpreted as a government objective assigning different weights (α, β), such that $\beta = 1 - \alpha$, to utilities derived from old-age consumption of an elderly $\beta \ln C_{o,0}$ and from a worker who has downward altruism $W_0 \equiv \sum_{t=0}^{\infty} \alpha^t \left(\ln C_{y,t} + \beta \ln C_{o,t+1} + \rho \ln N_t \right)$. According to this alternative interpretation, we can rewrite (1.1) as $U_0 = \beta \ln C_{o,0} + \alpha W_0$ that captures the consequences of social security on the welfare of coexisting elderly and working members with downward altruism in individuals' preferences. We will elaborate more on this alternative interpretation later.

Each young adult devotes one unit of time endowment to rearing children and working. Rearing a child requires v units of time, implying an upper bound $1/v$ on N; otherwise N may approach infinity. The amount of working time per worker is equal to 1-vN that earns $(1-\tau_t)(1-vN_t)W_tH_t$ where W is the wage rate per unit of effective labor, H_t is his or her human capital, and τ is the (payroll) tax rate for social security contributions. A young adult in period t also receives a bequest B_t from his or her old parent.[4] He or she spends the earnings and the received bequest on young-age consumption $C_{y,t}$, retirement savings S_t, and education for each child E_t. An old agent spends part of his or her savings plus interest income and social security benefits on own consumption and leaves the rest as bequests to children. The budget constraints can be written as:

$$C_{y,t} = B_t + (1-\tau_t)(1-vN_t)W_tH_t - S_t - E_tN_t, \tag{1.2}$$

$$C_{o,t+1} = S_tR_{t+1} + T_{t+1} - B_{t+1}N_t, \tag{1.3}$$

where R is the interest factor and T the amount of social security benefits per retiree.

As practiced in many countries such as France and Germany, the amount of social security benefits received by a retiree depends on his or her own earnings in

[4] Intentional bequests made by parents can be in the forms of inter vivos gifts and post-mortem bequests. Bequests in this model are of the inter vivos form which is consistent with the empirical evidence (i.e., Gale and Scholz, 1994) that suggests inter vivos are substantial. However, we expect both forms of inter vivos and post-mortem bequests to yield similar qualitative result concerning the effect of social security on the bequests cost of a child. This is because when parents value their children's welfare, a rise in the social security tax rate would increase the amount of intentional bequests to offset the increased tax burden on their children, regardless of whether the bequests are made in the form of inter vivos or post-mortem bequests.

working age according to a replacement rate ϕ_t, that is, $T_t = \phi_t(1 - vN_{t-1})W_{t-1}H_{t-1}$.[5] With this formula linking the amount of one's social security benefits to his or her own past earnings, a worker who has more children (hence less labor time) will not only earn less wage income today, but also receive less social security benefits in old age. The social security program is assumed to be always balanced in a typical PAYG fashion: $T_t = \bar{N}_{t-1}\tau_t(1 - v\bar{N}_t)W_t\bar{H}_t$, whereby the bar above a variable indicates its average level in the economy. With identical agents in the same generation, in equilibrium we have $\bar{N} = N$ and $\bar{H} = H$ by symmetry.

The production of the single final good is

$$Y_t = DK_t^{\theta}[L_t(1 - vN_t)H_t]^{1-\theta}, \; D > 0, \; 0 < \theta < 1, \tag{1.4}$$

where K_t is the aggregate stock of physical capital and L_t is the total number of workers. Since one period in this model corresponds to about 30 years, it is reasonable to assume that both physical capital and human capital depreciate fully within one period. This assumption will greatly help us obtain reduced form solutions.

The education of a child, H_{t+1}, depends on the investment of the final good per child, E_t, the human capital of his or her parent, H_t, and the average human capital in the economy, $\bar{H}_t$:

$$H_{t+1} = AE_t^{\delta}(H_t^{\varepsilon}\bar{H}_t^{1-\varepsilon})^{1-\delta}, \; A > 0, \; 0 < \delta < 1, \; 0 < \varepsilon \le 1. \tag{1.5}$$

[5] The essence of the results will remain valid if the amount of social security benefits is less than proportional to individuals' own earnings (as in the United States) or is independent of individuals' own earnings, though quantitatively different. As shown in Zhang and Zhang (2003), the more heavily the social security benefits depend on one's own past earnings, the more likely the increase in the social security payroll tax rate will have a negative effect on fertility and a positive effect on the growth rate of per capita output.

When $\varepsilon = 1$, there is no externality from average human capital in this model. However, when $\varepsilon < 1$, the externality takes the form of positive spillovers from average human capital to the formation of human capital of every child. The assumption concerning the existence of positive externalities in the production of human capital is consistent with the empirical evidence on human capital externalities in the literature that we mentioned earlier.[6]

Factors are paid by their marginal products; and the price of the sole final good is normalized to unity. The wage rate per unit of effective labor and the real interest factor are then given by

$$W_t = (1-\theta) D \mu_t^{\theta}, \tag{1.6}$$

$$R_t = \theta D \mu_t^{\theta - 1}, \tag{1.7}$$

where $\mu_t = K_t / [L_t (1 - v N_t) H_t]$ is the physical capital-effective labor ratio. The physical capital market clears when

$$K_{t+1} = L_t S_t . \tag{1.8}$$

The working population evolves according to $L_{t+1} = L_t N_t$.

1.3. The equilibrium and results

[6] Human capital externalities on individuals' earnings may arise in the production of human capital as in Tamura (1991) and in the production of goods as in Lucas (1988). In fact, many related empirical studies such as Moretti (2004a, 2004b) focus on human capital externalities in the production of goods by following the formulation in Lucas (1988); some empirical studies such as Borjas (1992, 1995) focus on human capital externalities from the parents' generation to the formation of children's skills as in our model. However, both forms of human capital externalities share the same essence that the average or aggregate level of human capital has a positive spillover on each individual's earnings. As a result, they should lead to the same problem of underinvestment in human capital. Therefore, assuming human capital externalities either in the production of goods or in the production of human capital is expected to yield similar results concerning optimal social security. For ease of exposition, we only focus on the latter in this paper.

We now solve the dynastic family's problem, track down the equilibrium allocation, and derive the solution for the welfare level for our welfare analysis of social security in Section 1.4.

1.3.1. Equilibrium solution for the dynastic family problem

The problem of a dynastic family is to maximize utility in (1.1) subject to budget constraints (1.2) and (1.3), the education technology (1.5) and the earning dependent benefit formula, taking the social security tax and replacement rates as given. This problem can be rewritten as the following:

$$\max_{B_t, N_t, S_t, H_{t+1}} \sum_{t=0}^{\infty} \alpha^t \{\beta \ln[S_{t-1}R_t + \phi_t(1 - vN_{t-1})W_{t-1}H_{t-1} - B_t N_{t-1}] + \alpha \ln[B_t + (1-\tau_t)(1 - vN_t)W_t H_t - S_t - N_t H_{t+1}^{1/\delta} A^{-1/\delta} H_t^{-\varepsilon(1-\delta)/\delta} \overline{H}_t^{-(1-\varepsilon)(1-\delta)/\delta}] + \alpha\rho \ln N_t\}$$

where we have used the budget constraints, the earning dependent benefit formula and the education technology for substitution. For $t \geq 0$, the first-order conditions are given as follows:[7]

$$B_t: \ \frac{\alpha}{C_{y,t}} = \frac{\beta N_{t-1}}{C_{o,t}}, \tag{1.9}$$

$$S_t: \ \frac{1}{C_{y,t}} = \frac{\beta R_{t+1}}{C_{o,t+1}}, \tag{1.10}$$

$$N_t: \ \frac{vW_t H_t (1-\tau_t) + E_t}{C_{y,t}} + \frac{vW_t H_t \phi_{t+1} \beta}{C_{o,t+1}} + \frac{\beta B_{t+1}}{C_{o,t+1}} = \frac{\rho}{N_t}, \tag{1.11}$$

[7] Note that the transversality conditions are satisfied in this model because the Bellman equation of this maximization problem meets Blackwell's sufficient conditions to be a contraction with $\alpha < 1$.

$$H_{t+1}: \quad \frac{\alpha(1-vN_{t+1})W_{t+1}(1-\tau_{t+1})}{C_{y,t+1}} + \frac{\alpha\beta\phi_{t+2}(1-vN_{t+1})W_{t+1}}{C_{o,t+2}} + \frac{\alpha N_{t+1}E_{t+1}(1-\delta)\varepsilon}{C_{y,t+1}\delta H_{t+1}}$$

$$= \frac{N_t E_t}{C_{y,t} H_{t+1}\delta}. \tag{1.12}$$

It is easy to verify that the preference with downward altruism and without upward altruism should lead to the same first-order conditions as those listed above. The only difference is that (1.9) is derived by an elderly at the beginning of period t while the other first-order conditions are derived by a worker in period t when the altruism is downward only in the form $W_0 \equiv \sum_{t=0}^{\infty} \alpha^t \left(\ln C_{y,t} + \beta \ln C_{o,t+1} + \rho \ln N_t \right)$. Thus, the equilibrium solution for allocations of time and output and for fertility must be the same regardless of whether the preference has only downward altruism or has both upward and downward altruism as in (1.1).

In (1.9), the marginal loss in the old parent's utility from giving a bequest to each child is equal to the marginal gain in children's utility. In (1.10), the marginal loss in utility from saving is equal to the marginal gain in utility in old age through receiving the return to saving. In (1.11), the marginal loss in utility from having an additional child, through giving up a fraction of wage income and earnings-dependent social security benefits, leaving a bequest to this child and spending on the education of this child, is equal to the marginal gain in utility from enjoying the child. In (1.12), the marginal loss in the parent's utility from investing an additional unit of income in children's education is equal to the marginal gain in children's utility through increasing their wage income and earnings-dependent social security benefits and making them more effective in teaching their own children. These first-order conditions hold for all $t \geq 0$.

Definition. Given an initial state (N_{-1}, K_0, H_0), a competitive equilibrium in the economy with PAYG social security is a sequence of allocations $\left\{B_t, C_{y,t}, C_{o,t}, K_{t+1}, H_{t+1}, N_t, S_t, \phi_t, \tau_t, T_t, Y_t\right\}_{t=0}^{\infty}$ and prices $\left\{R_t, W_t\right\}_{t=0}^{\infty}$ such that (i) taking prices and the tax and replacement rates $\left\{\phi_t, \tau_t\right\}_{t=0}^{\infty}$ as given, firms and households optimize and their solutions are feasible, (ii) the social security budget is balanced, and (iii) all markets clear with $K_{t+1} = L_t S_t$ and per worker labor being equal to $1 - vN_t$.

Specifically, these equilibrium conditions correspond to the first-order conditions of firms and households, the budget constraints of households and the government, the technologies, the capital market clearing condition, and the amount of labor supply per worker equal to $1 - vN_t$, for $t \geq 0$. In addition, as mentioned earlier, we have $X = \bar{X}$ for $X = K, N, H$ in equilibrium by symmetry. Moreover, with the log utility, the Cobb-Douglas functions for both the education and the production technologies and the full depreciation of capital within one period, we expect the proportional allocations of time and output and the tax/replacement rates of social security to be constant over time, given any initial state.

Letting the fraction of output per worker spent on item X_t be a time-invariant lower-case variable $x = X_t / y_t$ where $y_t = Y_t / L_t$, we transform the variables in the budget constraints and first-order conditions into their relative ratios to output per worker. The transformed budget constraints take the form:

$c_y = b + (1-\tau)(1-\theta) - s - eN$ and $c_o = N(\theta + \tau(1-\theta) - b)$ for $t > 0$ and $c_{o,0} = N_{-1}(\theta + \tau(1-\theta) - b)$ for a predetermined N_{-1}. Similarly, the transformed first-order conditions are:

$$\frac{\beta N}{c_o} = \frac{\alpha}{c_y} \qquad \text{(for } t > 0\text{)}, \tag{1.13}$$

$$\frac{\beta N_{-1}}{c_{o,0}} = \frac{\alpha}{c_y} \qquad \text{(for } t = 0\text{)}, \tag{1.14}$$

$$\frac{1}{c_y} = \frac{\beta \theta N}{c_o s} \ , \tag{1.15}$$

$$\frac{v(1-\tau)(1-\theta)}{(1-vN)c_y} + \frac{e}{c_y} + \frac{v\alpha\tau(1-\theta)}{(1-vN)c_y} + \frac{\alpha b}{Nc_y} = \frac{\rho}{N}, \tag{1.16}$$

$$\frac{\alpha(1-\tau)(1-\theta)}{c_y} + \frac{\alpha^2\tau(1-\theta)}{c_y} + \frac{\alpha Ne(1-\delta)\varepsilon}{c_y\delta} = \frac{Ne}{c_y\delta}. \tag{1.17}$$

It is worth mentioning that (1.17) can be derived by using $E_t = e_t y_t$, $T_t = \phi_t(1-vN_{t-1})W_{t-1}H_{t-1}$ (through updating), $W_t = (1-\theta)y_t/(1-vN_t)H_t$ and $R_t = \theta y_t / k_t$. The left-hand side of (1.16) contains four cost components of a child. The first cost component is the forgone wage income of spending time rearing a child, which falls with the social security tax rate, other things being equal. The second cost component is human capital investment per child, which may rise or fall with the social security tax rate. The third cost component is the forgone social security benefit of spending time rearing a child, which rises with the tax rate through the linkage between the replacement rate and the tax rate under a balanced social security budget. The fourth cost component is the bequest cost of a child, which should rise with the social security tax rate since altruistic parents are tempted to reduce the tax burdens of social security on their children.

When the tax rate rises, the subsequent rise in the third cost component partially offsets the fall in the first cost component, and the overall time cost of having a child is likely to fall. Thus, there are opposing effects of a rise in the tax rate on fertility: the fall in the time cost of having a child tends to raise fertility, while the possible rises in the costs of both human capital investment per child and bequests tend to reduce fertility. The net effect on fertility will depend on the taste for the number, relative to the welfare, of children. When the taste for the welfare of every child, α , becomes stronger, the third and fourth cost components of a child in (1.16) become larger and hence it is more likely that social security reduces fertility. By contrast, when the taste for the number of children, ρ , becomes larger, the marginal benefit of a child becomes larger and hence it is more likely for a rise in social security taxes or benefits to raise fertility.

From these equilibrium conditions, we obtain the following constant allocation rules:

$$b = \frac{\alpha[\theta + \tau(1-\theta)] - (1-\tau)(1-\theta)\beta + \alpha\theta\beta}{\alpha + \beta} + \frac{\alpha\beta\delta(1-\theta)[1-\tau(1-\alpha)]}{(\alpha+\beta)[1-\alpha\varepsilon(1-\delta)]}, \tag{1.18}$$

$$c_y = \frac{\alpha[\theta + \tau(1-\theta) - b]}{\beta}, \tag{1.19}$$

$$s = \alpha\theta, \tag{1.20}$$

$$N = \frac{N_n}{v\{N_n + (1-\theta)(\alpha+\beta)[1-\tau(1-\alpha)][1-\alpha\varepsilon(1-\delta)]\}}, \tag{1.21}$$

where the numerator of N is

$$N_n = \alpha\left\{\left[1-\alpha\varepsilon(1-\delta)\right]\left[\rho(1-\alpha\theta)-\alpha\theta(1+\beta)-\alpha(1-\theta)\tau+\beta(1-\theta)(1-\tau)\right]-\right.$$
$$\left.\delta(1-\theta)\left[1-\tau(1-\alpha)\right]\left[\alpha(\rho+\beta)+\alpha+\beta\right]\right\},$$

$$e = \frac{\alpha(1-\theta)\delta[1-\tau(1-\alpha)]}{N[1-\alpha\varepsilon(1-\delta)]}, \tag{1.22}$$

$$c_o = \frac{c_y\beta N}{\alpha} \quad (t>0), \tag{1.23}$$

$$c_{o,0} = \frac{c_y\beta N_{-1}}{\alpha}. \tag{1.24}$$

Note that the above solutions for the proportional allocation $(c_y, s, b, e, c_o, c_{o,0})$ and for fertility N are indeed constant over time as expected, for any constant tax rate. That is, the time-invariant proportional allocation solutions given here satisfy all the equilibrium conditions including the budget constraints, the first-order conditions and the market clearing conditions in all periods. Thus, they are valid solutions in all times on the entire equilibrium path. Also, we can easily observe that if N_n >0 then fertility N is positive in (1.21). However, since the log utility function excludes corner solutions for fertility, the presence of non-convexity in the form of $B_{t+1}N_t$ or E_tN_t in the budget constraints (1.2) and (1.3) may lead to a situation in which there is no solution for fertility for some parameter values. As shown in Zhang et al. (2001) and Zhang (1995), the sufficient condition for the solution to be optimal is a sufficiently large taste parameter for the number of children (ρ) such that an interior solution for fertility exists. Under these restrictions on ρ and α, there is a unique optimal interior solution in this model. This is explicitly given below:

Lemma 1.1. *There exists a unique equilibrium interior solution* $(c_y, c_o, c_{o,0}, e, N, s)$ *if the taste for the number of children is strong enough such that*

$$\rho > \underline{\rho} \equiv \frac{[1-\alpha\varepsilon(1-\delta)][\alpha\theta(1+\beta)-(1-\theta)\beta]+\alpha\delta\beta(1-\theta)+\delta(1-\theta)(\alpha+\beta)}{[1-\alpha\varepsilon(1-\delta)](1-\alpha\theta)-\alpha\delta(1-\theta)}.$$

Also, $b > 0$ *if the discount factor* α *is large enough and the externality* $1-\varepsilon$ *is weak enough.*

Proof. The proof is relegated to Appendix A.

Some features of the solution merit attention. First, these constant proportional allocation rules satisfy the equilibrium conditions for $t \geq 0$, given any initial state and any constant tax rate. The government budget constraint implies that for any given tax rate, there is a corresponding replacement rate. Second, these proportional allocation rules are consistent across generations, in the sense that agents in any generation will choose these optimal proportional allocation rules when expecting other generations do so, because they have the same first-order conditions and budget constraints in this recursive structure. As a result, these proportional allocation rules are the equilibrium solution on the entire equilibrium path of the economy, satisfying the equilibrium conditions (the first-order conditions, budget constraints and market clearing conditions) in all periods. These features allow us to obtain an analytical solution for the levels of the variables of interest in every period, starting from the initial period. Thus, we can analyze how social security affects the economy and what is its optimal scale to maximize social welfare.

We now ask how the solution responds to a rise in the social security tax rate:

Lemma 1.2. *A rise in the social security tax rate has a positive effect on the ratio of bequests per child to output per worker and on the fraction of output spent on young-age consumption, a negative effect on the ratio of total education spending to output per worker, but no effect on the saving rate. Also, defining* $\overline{\rho} \equiv \alpha(1+\beta)/(1-\alpha)$*, a rise in the tax rate reduces fertility if* $\underline{\rho} < \rho < \overline{\rho}$*, increases fertility if* $\rho > \overline{\rho}$*, and has no effect on fertility if* $\rho = \overline{\rho}$*.*

Proof. The first part of the lemma emerges from differentiating (1.18)-(1.20) and (1.22), respectively, with respect to τ. For the second part, we differentiate (1.21) with respect to τ:

$$\text{sign } \frac{\partial N}{\partial \tau} = \rho(1-\alpha) - \alpha(1+\beta)$$

which leads to the claim on how fertility responds to a tax rate change. Finally, as noted above, ρ needs to be large enough for the *existence* of an interior solution for fertility (i.e. the taste for the number of children is strong enough). Specifically, fertility is positive if

$$\rho > \underline{\rho} \equiv \frac{[1-\alpha\varepsilon(1-\delta)][\alpha\theta(1+\beta)-(1-\theta)\beta] + \alpha\delta\beta(1-\theta) + \delta(1-\theta)(\alpha+\beta)}{[1-\alpha\varepsilon(1-\delta)](1-\alpha\theta) - \alpha\delta(1-\theta)}$$

at $\tau = 0$, from equation (1.21). It is easy to verify that sign $(\overline{\rho} - \underline{\rho}) = (1-\delta)(1-\alpha\varepsilon) > 0$, i.e. $\overline{\rho} > \underline{\rho}$. As a result, there is a nonempty range of ρ for our analysis. □

The unresponsiveness of the saving rate to social security reflects the Ricardian equivalence hypothesis in a dynastic model. Also, in such a model altruistic parents respond to a rise in the social security tax and benefit by giving more bequests to each child so as to offset the increased tax burden on future generations. These results are well recognized in the literature; see Barro (1974) and Zhang (1995). Further, when a higher social security tax rate reduces the after-tax return to human capital investment, young parents reduce their total education spending for their children as a fraction of their output. As a result, a rise in the social security tax rate raises the fraction of income spent on consumption.

However, a rise in the tax rate may reduce, increase, or have no effect on fertility, depending on the relative strength of the taste for the number versus the welfare of children (ρ versus α), as mentioned earlier. By Lemma 1.2, if ρ is small enough relative to α, then a rise in the tax rate will reduce fertility. This negative net effect of social security on fertility is consistent with empirical evidence in the literature (e.g. Cigno and Rosati, 1992; Zhang and Zhang, 2004).

1.3.2. Dynamic equilibrium path

Most existing studies of social security focus on steady-state solutions (e.g. Zhang, 1995; Zhang and Zhang, 2003). To fully capture the welfare impact of social security, we also need to track down the entire dynamic equilibrium path starting from any initial level of capital K_0 and any predetermined fertility rate N_{-1}. In doing so, substituting $y_t = D\mu_t^{\theta}\left(1 - vN_t\right)H_t$ into the solutions for H_{t+1}, k_{t+1} and using (1.5), (1.8) and the solutions from (1.18) to (1.24) yields

$$\mu_{t+1} = \frac{sD^{1-\delta}\mu_t^{\theta(1-\delta)}}{A\left[e(1-vN)\right]^{\delta} N}, \tag{1.25}$$

$$\mu_{\infty} = \left[\frac{sD^{1-\delta}}{A\left[e(1-vN)\right]^{\delta} N}\right]^{\frac{1}{1-\Gamma}}, \quad \Gamma \equiv \theta(1-\delta), \tag{1.26}$$

where μ_t is globally convergent to μ_{∞} because $0 < \Gamma \equiv \theta(1-\delta) < 1$.

Using (1.25) and (1.26) and taking log, we have

$$\ln \mu_{t+1} = (1-\Gamma)\ln \mu_{\infty} + \Gamma \ln \mu_t . \tag{1.27}$$

By solving the log-linear first-order difference equation (1.27), we have

$$\ln \mu_t = \Gamma^t \ln \mu_0 + (1-\Gamma^t)\ln \mu_{\infty} . \tag{1.28}$$

With the solution in (1.28), we can now solve for the log-linear first-order difference equation for human capital per worker:

$$\ln H_t = \ln H_0 + \left(\frac{\theta\delta}{1-\Gamma}\right)(1-\Gamma^t)\ln \mu_0 + (\theta\delta)\left(\frac{-1}{1-\Gamma} + t + \frac{\Gamma^t}{1-\Gamma}\right)\ln \mu_{\infty}$$

$$+ t \ln A\left[eD(1-vN)\right]^{\delta} . \tag{1.29}$$

From the solutions in (1.28) and (1.29), we can also solve for $\ln y_t$:

$$\ln y_t = \ln D(1-vN) + \theta \ln \mu_t + \ln H_t . \tag{1.30}$$

Clearly, the economy converges globally toward its balanced growth path.

1.3.3. Solution for the welfare level $U_0(\tau)$

With the full characterization of the equilibrium path of the model, we can now solve for the welfare level. Using the solutions for (s, c_y, c_0, e, b, N) and for the sequence $\left\{\ln y_t\right\}_0^{\infty}$ given an initial state (N_{-1}, k_0, H_0), we can obtain

$$U_0(\tau) = \beta \ln\left(\frac{\beta N_{-1} c_y y_o}{\alpha}\right) + \alpha \sum_{t=0}^{\infty} \alpha^t \left[\ln c_{y,t} + \ln y_t + \rho \ln N_t + \beta \ln c_{0,t+1} + \beta \ln y_{t+1}\right] \quad (1.31)$$
$$= B_0 + B_1 + F(\tau)$$

where B_0 and B_1 are constants (unresponsive to time or to the social security tax), and $F(\tau)$ is a function of the tax rate via $c_y, e,$ and N . Also, the constant B_0 does not vary with the degree of the externality, whereas B_1 does:

$$B_1 = \Psi_c \ln\left\{\frac{\alpha}{(\alpha+\beta)[1-\alpha\varepsilon(1-\delta)]}\right\} + \Psi_{en} \ln\left[\frac{\alpha\delta(1-\theta)}{1-\alpha\varepsilon(1-\delta)}\right]$$

Where

$$\Psi_c \equiv \frac{\alpha+\beta}{1-\alpha} > 0,$$

$$\Psi_{en} \equiv \frac{\alpha\delta(1-\theta)(\alpha+\beta)}{(1-\alpha)^2[1-\alpha\theta(1-\delta)]} > 0.$$

The expression of the function $F(\tau)$ is

$$
\begin{aligned}
F(\tau) = {} & \Psi_c \ln\{(1-\alpha\theta)[1-\alpha\varepsilon(1-\delta)] - \alpha\delta(1-\theta)[1-\tau(1-\alpha)]\} + \\
& \Psi_{en} \ln[1-\tau(1-\alpha)] + \Psi_l \ln(1-vN) + \Psi_n \ln N \\
& = \Psi_c \ln\left\{(1-\alpha\theta)\left[1-\alpha\varepsilon(1-\delta)\right] - \alpha\delta(1-\theta)\left[1-\tau(1-\alpha)\right]\right\} + \\
& (\Psi_{en}+\Psi_l)\ln\left[1-\tau(1-\alpha)\right] - \\
& (\Psi_l+\Psi_n)\ln\left\{\alpha\left[1-\alpha\varepsilon(1-\delta)\right]\left[\rho(1-\alpha\theta) - \alpha\theta(1+\beta) + \beta(1-\theta)\right] - \right. \\
& \alpha\delta(1-\theta)\left[1-\tau(1-\alpha)\right]\left[\alpha(\rho+\beta)+\alpha+\beta\right] + \\
& \left.\left[1-\alpha\varepsilon(1-\delta)\right](1-\theta)(\alpha+\beta)(1-\tau)\right\} + \\
& \Psi_n \ln(\alpha/v)\left\{\left[1-\alpha\varepsilon(1-\delta)\right]\left[\rho(1-\alpha\theta) - \alpha\theta(1+\beta) - (1-\theta)(\tau(\alpha+\beta)-\beta)\right] - \right. \\
& \left.\delta(1-\theta)\left[1-\tau(1-\alpha)\right]\left[\alpha(\rho+\beta)+\alpha+\beta\right]\right\} + \\
& \Psi_l \ln\left\{\left[1-\alpha\varepsilon(1-\delta)\right](1-\theta)(\alpha+\beta)\right\}
\end{aligned}
$$

where

$$
\Psi_l \equiv \frac{(1-\theta)(\alpha+\beta)[1-\alpha(1-\delta)]}{[1-\alpha\theta(1-\delta)](1-\alpha)^2} > 0,
$$

$$
\Psi_n \equiv \frac{\alpha[1-\alpha\theta(1-\delta)][\rho(1-\alpha)-\alpha(1+\beta)] + \alpha(\alpha+\beta)(1-\theta)(1-\delta)}{[1-\alpha\theta(1-\delta)](1-\alpha)^2}.
$$

The sign of Ψ_n should be assumed to be positive for the following reasons. Differentiating $F(\tau)$ at $\tau = 0$ with respect to N gives rise to a solution for fertility $N = (1/v)\Psi_n/(\Psi_l+\Psi_n)$. This corresponds to the social planner's solution for fertility which is independent of the degree of the externality $1-\varepsilon$. In order to have a well defined social planner solution, we must assume $\Psi_n > 0$, that is

$$
\rho > \rho^* \equiv \frac{\alpha(1+\beta)[1-\alpha\theta(1-\delta)] - (\alpha+\beta)(1-\theta)(1-\delta)}{[1-\alpha\theta(1-\delta)](1-\alpha)}.
$$

It means that the taste for the number of children should be strong enough relative to the taste for the welfare of children to ensure positive fertility in the social planner's solution. It is easy to verify that, when $\varepsilon \to 1$, $\underline{\rho} \to \rho^*$. This is because in the absence of externalities in this dynastic family model the competitive equilibrium solution without social security would become the same as the social planner solution. This is a feature of dynastic models, as opposed to conventional life-cycle models whose competitive equilibrium solutions are typically not Pareto optimal even in the absence of externalities or other frictions.

Interestingly, even if fertility were treated as exogenous as in most studies of social security, a change in the tax rate would still have an impact on welfare in (1.31) through affecting both consumption and education spending in this dynastic family model, contrary to the result obtained in Barro (1974). The key reason is that a rise in the contribution rate for social security reduces the after-tax wage rate and hence reduces the fraction of income spent on human capital investment for all children. Thus, when fertility were treated as exogenous in the model, the welfare effect of social security would be negative. The main task next is to investigate how social security affects welfare with endogenous fertility and with human capital investment, and what the optimal social security tax rate should be.

1.4. Welfare implications

For comparison purposes, we begin with the case without human capital externality $\varepsilon = 1$ and then look at the case with the human capital externality $0 < \varepsilon < 1$.

1.4.1. Without externality from average human capital

Absent externalities with $\varepsilon = 1$, the welfare implication of social security is given below:

Proposition 1.1. *For* $\varepsilon = 1$ *and* $\rho > \underline{\rho}$ *, the competitive equilibrium without social security,* $\tau^* = 0$ *, is Pareto optimal.*

Proof. The proof is relegated to Appendix A.

In the absence of the externality, Proposition 1.1 provides the condition for an interior solution and describes the first-best nature of the competitive solution without social security. The intuition is as follows. First, with or without endogenous fertility in this model, social security is not neutral in general because the social security payroll tax distorts human capital investment at the margin, as opposed to the Ricardian equivalence hypothesis in Barro (1974). In the absence of the externality, this distortion creates a departure from the first-best solution. Second, with endogenous fertility social security further reduces welfare by changing fertility, consumption and education spending from their first-best levels in the absence of the externality. This conclusion is in line with the traditional view against PAYG social security in the literature. In the rest of this section, we will see how the externality from average human capital can justify social security.

1.4.2. With the externality from average human capital

With the human capital externality, the competitive equilibrium solution departs from the social planner solution in the following ways:

Lemma 1.3. *A stronger human capital externality (a smaller ε) leads to a higher fertility rate and a lower fraction of income spent on human capital investment for all children.*

Proof. Differentiate fertility with respect to ε in (1.21):

$$\left.\frac{\partial N}{\partial \varepsilon}\right|_{\tau=0} = \frac{-\alpha^2\delta(1-\delta)(1-\theta)^2(\alpha+\beta)[\alpha(\rho+\beta)+\alpha+\beta]}{vN_d{}^2} < 0$$

where $N_d = N_n + (1-\theta)(\alpha+\beta)[1-\alpha\varepsilon(1-\delta)]$ stands for the denominator of fertility for $\tau = 0$. In addition, from (1.22), it is clear that $\partial eN / \partial\varepsilon > 0$ for $\tau = 0$. □

Lemma 1.3 points out the efficiency loss due to the human capital externality that reduces the private rate of return to human capital investment from the social rate and hence causes under-investment in human capital. Through the trade-off between the quality and quantity of children, it also causes over-reproduction of the population. Therefore, a welfare maximizing scale of social security is to reduce fertility and raise human capital investment to some ideal extent in this model of endogenous growth driven by human capital investment.

With the externality such that $0 < \varepsilon < 1$, the welfare implication of social security is given below:

Proposition 1.2. *For $0 < \varepsilon < 1$ and $\underline{\rho} < \rho < \overline{\rho}$, the optimal level of the social security tax rate τ^* exists and is unique and positive if the taste for the number of children is* *sufficiently weak relative to the taste for the welfare of children, that is,* if ρ is close enough to $\underline{\rho}$ for a given α.

Proof. The proof is relegated to Appendix A.

In the following section, we perform a quantitative assessment of the value of the optimal social security tax rate for plausible parameterizations to find out whether it can approximate the observed rates in the real world. For comparison purposes, we begin with a case without externality ($\varepsilon = 1$) and then look at cases with the externality ($0 < \varepsilon < 1$).

1.4.3. Numerical examples

As mentioned earlier, it is sufficient to focus on $F(\tau)$ in dealing with the relationship between the welfare level U_0 and the tax rate. However, in order to fully capture how welfare varies with the degree of the human capital externality as well, we use $B_1 + F(\tau)$ in equation (1.31) as the measure of welfare in our numerical results. Concerning the parameterization, the values of parameters are either in line with those in the literature if any (e.g., $\alpha = 0.6$, $\theta = 0.33$), or they are chosen to yield plausible values for fertility and the fractions of income invested in both types of capital (e.g. $v = 0.1$, $\beta = 0.3$, $\delta = 0.27$ and $\rho = 0.93$). Taking one period as 30 years, the value of the discounting factor at $\alpha = 0.6$ corresponds to an annual discounting factor of 0.9855 as in Gomme, Kydland and Rupert (2001). Here, a smaller share parameter associated with physical inputs in education ($\delta = 0.27$) than in production reflects the fact that education is less physical (more human) capital intensive than production. Moreover, we set D =20, μ_0 =2, A =10 and H_0 =5, which are non-essential for the result.

A key parameter for the human capital externality is ε. In a log linear version of the determination of children's human capital or skills in equation (1.5), the coefficient on log average human capital in the parental generation is equal to $(1-\varepsilon)(1-\delta)$. In a similar equation, Borjas (1995) runs regressions of children's skills on two variables: parental skills and the mean skills of the ethnic group of the parents' generation. In doing so, he uses data sets in the United States and uses either education attainment or the log real wage as the proxy for skills. The estimated coefficient on the mean human capital or mean skills of the ethnic group in the parents' generation (defined as ethnic capital therein) is 0.18 when education attainment is used as the proxy, and is 0.30 when the log wage is used. Applying his estimates to the coefficient $(1-\varepsilon)(1-\delta)$ in our model, we have either $\varepsilon = 0.75$ or $\varepsilon = 0.6$. Note that both education attainment and real wage are only approximate indicators of human capital or skills. The former does not capture the quality of education, whereas the latter may include possible factors that are not determined in the production of human capital in the real world such as human capital externalities in the production of goods. To be more conservative on the strength of the human capital externality in the production of human capital, we thus regard 0.7 as the lower bound for ε (or 0.3 as the upper bound on $1-\varepsilon$) and $0.7 \le \varepsilon \le 0.85$ as a plausible range. We will vary it gradually toward the case without any externality ($\varepsilon = 1$) for better comparisons.

In Table 1.1, we report the numerical results on the optimal social security tax rate, fertility and human capital investment per child relative to output per worker, corresponding to the values of ε from 1 to 0.7 in five cases. Case 1 has no externality ($\varepsilon = 1$) and gives the Pareto optimal solution without social security ($\tau = 0$). In Case 2 through to 5, the externality is present ($0.7 \le \varepsilon < 1$).

It is worth noticing the following results. Given any social security tax rate, when the externality becomes stronger from case to case in Table 1.1, fertility rises but human capital investment per child relative to income per worker falls because the externality leads to over-reproduction of the population and under-investment in human capital. Also, given any degree of the externality, when the social security tax rate rises in each case, fertility falls but human capital investment per child relative to income per worker rises. These observations reflect the results in Lemmas 1.2 and 1.3.

According to Propositions 1.1 and 1.2, social security can be welfare enhancing or reducing, depending on whether the externality is present. Table 1.1 illustrates that the optimal tax rate is zero when there is no externality ($\varepsilon = 1$), and is positive when there are positive externalities ($\varepsilon < 1$). Also, it shows that when the externality becomes stronger (smaller ε), the optimal tax rate becomes higher accordingly. In particular, for an externality at $\varepsilon = 0.9$, the corresponding optimal social security contribution rate is about 9%, while for the value of ε in the plausible range from $\varepsilon = 0.85$ to $\varepsilon = 0.7$ the optimal social security rate is in the range from 12% to 22%. These high contribution rates are in line with the observed contribution rates for social security in many industrial nations.

Table 1.1 Simulation results for various levels of the externality

Parameterization: $\alpha = 0.6, \beta = 0.3, \delta = 0.27, \rho = 0.93, \theta = 0.33, v = 0.1$

We denote the optimal tax rate as τ^* and highlight the highest welfare by bold fonts.

Case 1.

$\varepsilon = 1$	$\tau^* = 0$	$\tau = 0.05$	$\tau = 0.10$	$\tau = 0.20$	$\tau = 0.30$	$\tau = 0.40$
$B_1 + F(\tau)$	**-3.4412**	-3.4433	-3.4510	-3.4942	-3.6246	-4.3344
N	1.3897	1.2648	1.1307	0.8309	0.4799	0.0632
e	0.1390	0.1496	0.1640	0.2138	0.3542	2.5664

Case 2.

$\varepsilon = 0.90$	$\tau = 0$	$\tau = 0.05$	$\tau^* = 0.086$	$\tau = 0.10$	$\tau = 0.20$	$\tau = 0.25$
$B_1 + F(\tau)$	-3.4518	-3.4480	**-3.4471**	-3.4472	-3.4605	-3.4793
N	1.6620	1.5449	1.4452	1.4194	1.1391	0.9820
e	0.1078	0.1137	0.1195	0.1212	0.1447	0.1642

Case 3.

$\varepsilon = 0.85$	$\tau = 0$	$\tau = 0.05$	$\tau = 0.10$	$\tau^* = 0.123$	$\tau = 0.15$	$\tau = 0.20$
$B_1 + F(\tau)$	-3.4624	-3.4569	-3.4538	**-3.4534**	-3.4540	-3.4590
N	1.7784	1.6646	1.5426	1.4913	1.4116	1.2705
e	0.0972	0.1018	0.1076	0.1104	0.1151	0.1252

Case 4.

$\varepsilon = 0.80$	$\tau = 0$	$\tau = 0.05$	$\tau = 0.10$	$\tau^* = 0.157$	$\tau = 0.20$	$\tau = 0.25$
$B_1 + F(\tau)$	-3.4751	-3.4683	-3.4635	**-3.4613**	-3.4630	-3.4699
N	1.8841	1.7732	1.6544	1.5002	1.3895	1.2412
e	0.0887	0.0923	0.0970	0.1043	0.1106	0.1212

Case 5.

$\varepsilon = 0.70$	$\tau = 0$	$\tau = 0.05$	$\tau = 0.10$	$\tau = 0.20$	$\tau^* = 0.217$	$\tau = 0.25$
$B_1 + F(\tau)$	3.5042	-3.4959	-3.4889	-3.4808	**-3.4806**	-3.4816
N	2.0687	1.9628	1.8494	1.5969	1.5418	1.4558
e	0.0757	0.0782	0.0813	0.0902	0.0926	0.0968

Note: First-best solution is the case when $\varepsilon = 1$ and $\tau = 0$.

In Table 1.2 we examine whether the results concerning the optimal tax rate of social security are sensitive to variations in the parameters $(\alpha, \beta, \delta, \rho, \theta, v)$. In doing so, we consider variations in one parameter at a time, starting from the parameterization in Table 1.1. First, a higher value of the taste for the welfare of children (α) yields a higher optimal tax rate of social security and the magnitudes of the changes in the optimal tax rate are large. This is because the efficiency loss of the human capital externality is a dynamic loss through underinvestment in human capital. Thus, the more individuals value their children's welfare, the greater the efficiency loss of the human capital externality and therefore the higher the optimal tax rate of social security. The variations in the taste for the welfare of children may reflect cultural changes over time or increases in women's education attainment and labor participation rates. Second, a larger share parameter for the physical input in the production of human capital (δ) leads to a higher optimal tax rate of social security and the magnitudes of the changes in the optimal tax rate are large as well. The reason for this result is that this share parameter measures the role of human capital investment in the accumulation of human capital. That is, with a larger share parameter δ, parental human capital investment becomes more important in the formation of children's human capital and therefore the efficiency loss of the human capital externality is larger for a given ε. Holding $\varepsilon = 0.7$ and $\tau = 0$, for example, if $\delta = 0.2$ then the fraction of income invested in human capital per child is equal to 4.6% and the welfare level is equal to -3.0232, whereas if $\delta = 0.27$ then the fraction of income invested in human capital per child is equal to 7.6% and the welfare level is equal to -3.5042 as given in Table 1.1. Thus, the optimal tax rate should be much higher in the latter case than in the former (21.7% versus 16.9% according to Tables 1.1 and 1.2).

Table 1.2 Simulated optimal tax rates: sensitivity analysis

Starting from the baseline parameterization					
Parameter	ε=0.95	ε=0.9	ε=0.85	ε=0.8	ε=0.7
Varying α					
α=0.55	0.0252	0.0481	0.0689	0.0879	0.1213
α=0.65	0.0590	0.1118	0.1586	0.2005	0.2726
Varying δ					
δ =0.2	0.0354	0.0673	0.0962	0.1226	0.1689
δ =0.32	0.0501	0.0956	0.1371	0.1751	0.2253
Varying ρ					
ρ =0.85	0.0451	0.0859	0.1229	0.1567	0.2160
ρ =1	0.0437	0.0834	0.1196	0.1528	0.2116
Varying θ					
θ =0.28	0.0429	0.0819	0.1175	0.1501	0.2080
θ =0.38	0.0467	0.0889	0.1272	0.1622	0.2236
Varying β					
β =0.05	0.0417	0.0796	0.1142	0.1459	0.2021
β =0.4	0.0451	0.0860	0.1232	0.1572	0.2170
Varying v					
v =0.05	0.0449	0.0856	0.1227	0.1567	0.2166
v =0.2	0.0449	0.0856	0.1227	0.1567	0.2166

By contrast, variations in the other parameters produce relatively little changes in the optimal tax rate of social security in Table 1.2. This is because these parameters are less relevant for human capital investment, which channels the efficiency loss of the human capital externality, than (α, δ). Among these cases, the variations in the taste for the number of children may reflect cultural changes over time or government policy associated with children (e.g. child benefits). In particular, one may want to know whether the results for optimal PAYG social security are robust when the change in this taste parameter creates significant changes in fertility (e.g. from the "baby boom" to the "baby bust"). To see this, we first raise the level of the taste for the number of children ρ to 1.0. Such a rise in ρ may reflect monetary incentives for having children provided in several countries (e.g. child benefits). Holding $\varepsilon = 0.8$, this rise in ρ raises fertility to 2.165 and reduces the fraction of investment in human capital per child to 7.7% without social security; these changes are significant in magnitude, compared to Case 4 of Table 1.1 at $\tau = 0$. Despite the significant rise in fertility, the optimal tax rate for social security only declines slightly from 15.7% in Case 4 of Table 1.1 to 15.3% in Table 1.2. Now, we lower the value of the taste for the number of children to 0.85 to capture a possible reason such as cultural change for sharp declines in fertility since the 1970s. The result of this decline in ρ is a substantial decline in fertility to 1.537 and a rise in the fraction of income for human capital investment per child to 10.9% without social security. Again, there is only a slight rise in the optimal tax rate for social security to 15.7%. These substantial changes in fertility resemble what is usually called as the "baby boom" and the "baby bust". However, the optimal tax rate for social security remains in a narrow range from 15% to 16%. It is also worth mentioning that for the value of ε in the plausible range from $\varepsilon = 0.85$ to $\varepsilon = 0.7$, the respective elasticity of fertility to tax rate at 5% is in the range from -0.06 to -0.07. This range of elasticity of fertility is close to the calibrated elasticity of fertility to tax rate at 4%, -0.09, in Ehrlich and Kim (2007) that use actual U.S. data from 1960-1991.

1.5. Conclusion

In this paper we have examined the welfare implication of social security by incorporating life-cycle savings, bequests, human capital investment and fertility in a dynastic family model. In achieving this, we have overcome difficulties in tracking down the entire equilibrium path of capital accumulation and deriving an explicit solution for the welfare level with both human and physical capital. We have shown analytically that scaling up PAYG social security improves welfare when there are externalities under the same condition it reduces fertility and raises capital intensity, until reaching an optimal tax rate. Quantitatively, for an externality in the range of ε from $\varepsilon = 0.85$ to $\varepsilon = 0.7$, our model can generate optimal social security contribution rates in a range of 12%-22%. This is very much in line with the actual range of the contribution rates in many industrial countries.

In terms of the underlying driving forces, our results hinge on assumptions of altruistic intergenerational transfers, human capital spillovers and endogenous fertility. Whether the results in this paper are useful contributions depends on whether these assumptions are plausible. Among them, the assumption of endogenous fertility follows the Beckerian approach. As a necessary condition for PAYG social security programs to mitigate the efficiency loss of the human capital externality, the negative response of fertility to social security is consistent with empirical evidence in some existing studies such as Cigno and Rosati (1992) and Zhang and Zhang (2004). The human capital externality works through the trade-off between the number and the quality of children, implying a below optimal level of human capital investment per child and an above optimal level of fertility. This differs from the implication of the investment externality explored in Zhang and Zhang (2007) that leads to suboptimal investment in physical capital and suboptimal fertility. Both types of externalities have received some supporting empirical evidence in the literature; their relative significance is an empirical task and awaits future research. If both externalities are present at the same time, we expect the results to remain similar qualitatively in the sense that they render a welfare improving role for unfunded social security. However, the results may differ quantitatively because of their different impacts on human capital accumulation, physical capital accumulation and fertility. Among these driving forces, the assumption of altruistic intergenerational transfers in dynastic families is more controversial since there are different views and evidence with regard to the existence or the extent of altruism among family members in different generations. Though the related literature is inconclusive, some of the existing empirical studies have found supporting evidence for intergenerational altruism.

Such a combination of these factors has not been used in the welfare analysis of social security, to the best of our knowledge. Thus, our results are complementary to Cooley and Soares (1999) that justifies why PAYG social security receives a majority support once it has already been put in place in an overlapping-generations model with selfish agents. Unlike their results, our model can also explain why social security has been instituted in many countries in the first place. While we focus on social security in this paper, there are other fiscal instruments that can also mitigate the efficiency loss of the human capital externality. These additional instruments include subsidies on education investment and taxes on the number of children. Since the competitive solution differs from the social planner's solution in two dimensions in our paper (fertility and human capital investment), using the conventional education subsidy alone cannot eliminate the efficiency loss of the human capital externality to reach the social planner allocation. Moreover, taxes on the number of children have hardly been practiced in the developed countries. In fact, poor families with many children have often been provided with financial assistance from social programs, which can be traced back to the early 19th century in England (see, e.g., Boyer, 1989).

Population aging in the last several decades has created financial pressure on maintaining a balanced budget for PAYG social security programs in many countries. Different proposals for social security reform have emerged. Some of them aim at replacing pay-as-you-go social security with compulsory retirement savings in individual accounts. For instance, there was a failed referendum in New Zealand in 1997 calling for establishing a compulsory individually-based retirement savings scheme, which was regarded as a substitute for its public pension. The policy implication of our analysis in this paper is a call for caution against reform plans that abandon pay-as-you-go social security or reduce its scale significantly. In developed countries, population aging has been driven by two factors: below-replacement fertility rates and falling mortality rates. Our present paper abstracts from the second factor. For population aging driven by a permanent decline in fertility (caused by factors other than social security), our numerical results suggest that little change in the optimal contribution rate for PAYG social security may be made in the presence of the human capital externality.

Indeed, many reform proposals try to keep the contribution rates at today's level and only change the "design" of the pension system such as strengthening the funded component or weakening the intragenerational redistribution. According to Zhang (1995), however, an inframarginal funded pension component is neutral if the benefit is linked to one's own contribution, whereas it has a positive effect on fertility and negative effects on human capital investment and growth if the benefit is independent of one's own contribution. In either case of the relationship between pension benefits and contributions for an individual, the funded component is not useful to mitigate the efficiency loss caused by human capital externalities in this model. Finally, reducing intragenerational redistribution makes social security benefits more dependent on one's own contribution. According to Zhang and Zhang (2003), this stronger linkage of benefits to contributions at an individual level makes it more likely for PAYG social security to raise human capital investment and the growth rate and reduce fertility. Therefore, reducing intragenerational redistribution is likely to make PAYG social security more effective in mitigating the efficiency loss of the human capital externality.

Appendix A

Proof of Lemma 1.1. First, it is easy to verify that if $\rho > \underline{\rho}$ (as defined in the lemma) then $N_n > 0$ according to the solution for fertility in (1.21). Consequently, from (1.21) with $N_n > 0$, we must have $0 < N < 1/v$. Here, it is obvious that N>0 under $\rho > \underline{\rho}$ because then the numerator of N, i.e. N_n, is positive and the denominator is signed by the sum of two positive terms $(1-\theta)(\alpha+\beta)[1-\tau(1-\alpha)][1-\alpha\varepsilon(1-\delta)]$ and N_n >0. These facts under $\rho > \underline{\rho}$ also imply

$$Nv = \frac{N_n}{\{N_n + (1-\theta)(\alpha+\beta)[1-\tau(1-\alpha)][1-\alpha\varepsilon(1-\delta)]\}} < 1,$$

leading to $N < 1/v$. Note that $c_y > 0$ in (1.19) by substituting (1.18) into it. Then, $c_o > 0$ in (1.23), $c_{o,0} > 0$ in (1.24), and e >0 in (1.22). Clearly, $s > 0$ in (1.20).

To see the conditions for $b > 0$ in (1.18), we define its signing part as

$$f(\alpha) = \alpha[\theta + \tau(1-\theta)] - (1-\tau)(1-\theta)\beta + \alpha\beta\theta + \frac{\alpha\beta\delta(1-\theta)[1-\tau(1-\alpha)]}{1-\alpha\varepsilon(1-\delta)}$$

Obviously, $f'(\alpha) > 0$ and $f(0) = -(1-\tau)(1-\theta)\beta < 0$. Also, when $\alpha \to 1$ and when $\varepsilon \to 1$, $f \to [\theta + \tau(1-\theta)] - (1-\tau)(1-\theta)\beta + \theta\beta + \beta(1-\theta) > 0$. As a result, if α is large enough and $1-\varepsilon$ small enough, bequests must be positive. Note that $b < 0$ means intergenerational transfers from grown up children to old parents in this model. The results of our paper remain qualitatively the same regardless of the direction of intergenerational transfers. Note also that the solution for each of these household variables is unique under the stated conditions.

The remaining task is to argue for the optimality for any given time-invariant social security tax rate τ, leaving the optimal design of social security to a later stage. The optimality builds on the following facts. (i) Since the log utility excludes corner solutions, any solution for fertility or for consumption must be strictly positive. (ii) All choice variables lie in closed and bounded sets: $c_y, c_o, c_{o,0}, e$ and s are in $\left[0,1\right]$, b in [-1,1] and N in $\left[0,1/v\right]$. (iii) The utility function U_t is continuous in the interior values of the choice variables (c_y, c_o, N). (iv) The utility level U_t is bounded above under $\alpha < 1$ as will be clearly seen later in (1.31). By (i)-(iv), there is at least one optimum. From both (i) and the uniqueness of the interior solution, the optimum must correspond to this unique solution for any given time-invariant social security tax rate. □

Proof of Proposition 1.1. It is sufficient to focus on $F(\tau)$ in dealing with the equilibrium relationship between the welfare level $U_0(\tau)$ and the tax rate. According to (1.31), we have

$$F'(\tau) = \mathrm{P}(\tau)/G(\tau) \tag{1.32}$$

where the denominator $G(\tau)$ is positive

$$\begin{aligned}
G(\tau) \equiv & \left\{(1-\alpha\theta)\left[1-\alpha\varepsilon(1-\delta)\right]-\alpha\delta(1-\theta)\left[1-\tau(1-\alpha)\right]\right\} \\
& \left\{\alpha\left[1-\alpha\varepsilon(1-\delta)\right]\left[\rho(1-\alpha\theta)-\alpha\theta(1+\beta)-(1-\theta)(\tau(\alpha+\beta)-\beta)\right]- \right. \\
& \left. \alpha\delta(1-\theta)\left[1-\tau(1-\alpha)\right]\left[\alpha(\rho+\beta)+\alpha+\beta\right]\right\} \\
& [1-\alpha\theta(1-\delta)](1-\alpha)^2\left[1-\tau(1-\alpha)\right] \\
& \left\{\alpha\left[1-\alpha\varepsilon(1-\delta)\right]\left[\rho(1-\alpha\theta)-\alpha\theta(1+\beta)+\beta(1-\theta)\right]-\alpha\delta(1-\theta)\right. \\
& \left.\left[1-\tau(1-\alpha)\right]\left[\alpha(\rho+\beta)+\alpha+\beta\right]+\left[1-\alpha\varepsilon(1-\delta)\right](1-\theta)(\alpha+\beta)(1-\tau)\right\},
\end{aligned}$$

and the numerator $\mathrm{P}(\tau)$ is a cubic function:

$$\mathrm{P}(\tau) = a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0$$

with

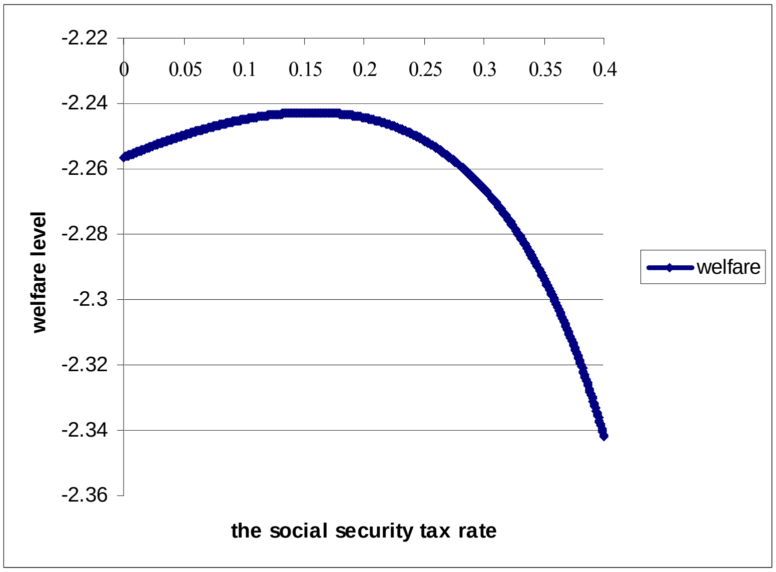
$$a_0 = -\alpha^3\delta(\alpha+\beta)(1-\theta)(1-\alpha\theta)(1-\delta)(1-\varepsilon)P_0$$

$$\begin{aligned} P_0 = &\{[1-\alpha\varepsilon(1-\delta)][\rho(1-\alpha\theta)-\alpha\theta(1+\beta)+\beta(1-\theta)]- \\ &\delta(1-\theta)[\alpha(\rho+\beta)+\alpha+\beta]\} \\ &\{\alpha[1-\alpha\varepsilon(1-\delta)][\rho(1-\alpha\theta)-\alpha\theta(1+\beta)+\beta(1-\theta)]- \\ &\alpha\delta(1-\theta)[\alpha(\rho+\beta)+\alpha+\beta]+ \\ &[1-\alpha\varepsilon(1-\delta)](\alpha+\beta)(1-\theta)\}(1-\alpha)+ \\ &\{(1-\alpha\theta)[1-\alpha\varepsilon(1-\delta)]-\alpha\delta(1-\theta)\}[1-\alpha\varepsilon(1-\delta)] \\ &[\rho(1-\alpha)-\alpha(1+\beta)][\alpha(\rho+\beta)+\alpha+\beta](1-\theta). \end{aligned}$$

Observe that if $\varepsilon = 1$ in the expression for a_0, then a_0=0 and hence $F'(\tau) = 0$ in equation (1.32) at $\tau = 0$ and $\varepsilon = 1$. In other words, for $\varepsilon = 1$, $\tau^* = 0$ maximizes $F(\tau)$, namely that, if there were no externality, the competitive solution without social security would be Pareto optimal. □

Proof of Proposition 1.2. For $0 < \varepsilon < 1$, a unique optimal level of positive social security τ^* exists if there are conditions leading to (i) $F'(\tau) > 0$ at $\tau = 0$, (ii) $F''(\tau) < 0$ at τ^*, and (iii) $F'(\tau) < 0$ for τ exceeding an upper limit $\bar{\tau}$. See Figure 1.3 for a numerical illustration.

Figure 1.3 Welfare with social security and externalities at $\varepsilon = 0.8$



Note: The welfare level refers to $F(\tau)$. The parameterization is $\varepsilon = 0.8, \alpha = 0.6, \beta = 0.3, \delta = 0.27, \rho = 0.93, \theta = 0.33, v = 0.1$

From equation (1.32), the condition for (i) $F'(\tau) > 0$ at $\tau = 0$ corresponds to a_0 > 0. We show that $a_0 > 0$ can emerge from a sufficiently small ρ for a given α under $0 < \varepsilon < 1$. That is, with the externality, the optimal social security tax rate is positive if the taste for the number of children is sufficiently weak relative to the taste for the welfare of children. According to equation (1.32), the task of showing $a_0 > 0$ is reduced to the task of showing $P_0 < 0$ under $0 < \varepsilon < 1$.

From the expression for P_0 in equation (1.32), it can be observed that for a sufficiently small ρ relative to α, P_0 < 0 and thus a_0 > 0, leading to $F'(\tau) > 0$ at $\tau = 0$. This is because the sign for the first term of P_0 is positive under $\rho > \underline{\rho}$ and the sign for the second term depends on the value of ρ relative to that of $\bar{\rho} \equiv \alpha(1+\beta)/(1-\alpha)$. Specifically, the second term is positive (zero, or negative) when ρ is greater than (equal to, or smaller than) $\bar{\rho}$. Also, the first term of P_0 approaches zero if $\rho \to \underline{\rho}$. Thus, when ρ is sufficiently small such that

$$\rho \to \underline{\rho} \equiv \frac{[1-\alpha\varepsilon(1-\delta)][\alpha\theta(1+\beta)-(1-\theta)\beta]+\alpha(1-\theta)\delta\beta+(1-\theta)\delta(\alpha+\beta)}{[1-\alpha\varepsilon(1-\delta)](1-\alpha\theta)-\alpha(1-\theta)\delta} < \bar{\rho}$$

,

then P_0 is negative and thus $a_0 > 0$, leading to $F'(\tau) > 0$ at $\tau = 0$ under these conditions. Conversely, if ρ is sufficiently large such that $\rho \geq \bar{\rho} \equiv \alpha(1+\beta)/(1-\alpha)$, then $a_0 < 0$ and thus $F'(\tau) < 0$ at $\tau = 0$. Since $F'(\tau)$ is a continuous function in the interval $(\underline{\rho}, \infty)$, there exists a range of sufficiently small $\rho \in (\underline{\rho}, \bar{\rho})$ relative to α, such that $a_0 > 0$ for $0 < \varepsilon < 1$ and thus $F'(\tau) > 0$ at $\tau = 0$.

For a small enough $\rho \in (\underline{\rho}, \bar{\rho})$, the second-order condition is also satisfied. This can be seen from

$$F''(\tau) = -\Psi_c \frac{\alpha^2\delta^2(1-\theta)^2(1-\alpha)^2}{\{(1-\alpha\theta)[1-\alpha\varepsilon(1-\delta)]-\alpha\delta(1-\theta)[1-\tau(1-\alpha)]\}^2} - \Psi_{en}\frac{(1-\alpha)^2}{[1-\tau(1-\alpha)]^2} - \Psi_l \frac{(vN')^2}{(1-vN)^2} - \Psi_l \frac{vN''}{1-vN} + \Psi_n \frac{N''}{N} - \Psi_n \frac{(N')^2}{N^2}$$

where

$$N''(\tau) = \frac{2(1-\theta)\{[1-\alpha\varepsilon(1-\delta)-\alpha\delta(1-\alpha)](\alpha+\beta)-\alpha^2\delta(1-\alpha)(\rho+\beta)\}}{N_d} N'(\tau)$$
$$= \frac{2(1-\theta)\{[1-\alpha\varepsilon(1-\delta)-\alpha\delta](\alpha+\beta)-\alpha^2\delta[\rho(1-\alpha)-\alpha(1+\beta)]\}}{N_d} N'.$$

Since the coefficient on N' on the right-hand side is positive for $\rho \in (\underline{\rho}, \overline{\rho})$, $N'' < 0$ for the same range of ρ. Also, since $\rho \to \underline{\rho}$ reduces N toward zero (hence raises $1/N$ to infinity), a sufficiently small $\rho \in (\underline{\rho}, \overline{\rho})$ leads to $F''(\tau) < 0$.

Finally, when the tax rate is already very high, a further rise in the tax rate will drive human capital investment down to suboptimal levels, causing $F'(\tau) < 0$.

Combining all the arguments above together, there exists a unique optimal $\tau^* > 0$ such that $F'(\tau^*) = 0$ and $F''(\tau^*) < 0$ under these stated conditions. □

1.6. Reference

Altonji, J.G., Hayashi, F., Kotlikoff, L.J., 1997. Parental altruism and *inter vivos* transfers: theory and evidence. Journal of Political Economy 105, 1121–1166.

Barro, R.J., 1974. Are government bonds net wealth? Journal of Political Economy 82, 1095–1117.

Barro R.J., Lee, J-W., 1994. Data set for a panel of 138 countries. http://www.nber.org/pub/barro.lee

Becker, G.S., Barro, R.J., 1988. A reformulation of the economic theory of fertility. Quarterly Journal of Economics 103, 1-26.

Becker, G.S., Lewis, H.G., 1973. On the interaction between the quantity and quality of children. Journal of Political Economy 81, S279-S288.

Borjas, G.J., 1992. Ethnic capital and intergenerational mobility. Quarterly Journal of Economics 107, 123-150.

Borjas, G.J., 1994. Long-run convergence of ethnic skill differentials: the children and grandchildren of the great migration. Industrial and Labor Relations Review 47, 553-573.

Borjas, G.J., 1995. Ethnicity, neighborhoods, and human-capital externalities. American Economic Review 85, 365-390.

Boyer, G.R., 1989. Malthus was right after all: poor relief and birth rates in southeastern England. Journal of Political Economy 97, 93-114.

Cigno, A., Rosati, E., 1992. The effects of financial markets and social security on saving and fertility behavior in Italy. Journal of Population Economics 5, 319–341.

Cooley, T. F., Soares, J., 1999. A positive theory of social security based on reputation. Journal of Political Economy 107, 135-160.

Davies, J.B., 2002. Empirical evidence on human capital externalities. University of Western Ontario, mimeo.

Ehrlich I., Kim J., 2007. Social security and demographic trends: theory and evidence from the international experience. Review of Economic Dynamics 10, 55-77.

Feldstein, M.S., 1974. Social security, induced retirement, and aggregate capital formation. Journal of Political Economy 23, 905-926.

Gale, W.G., Scholz, J.K., 1994. Intergenerational transfers and the accumulation of wealth. Journal of Economic Perspectives 8, 145–160.

Gomme, P., Kydland, F. E., Rupert, P., 2001. Home production meets time to build. Journal of Political Economy 109, 1115-1131.

Heston, A., Summers, R., Aten, B., 2002. Penn World Table Version 6.1. Center for International Comparisons at the University of Pennsylvania (CICUP).

Horioka, C.Y., 2002. Are the Japanese selfish, altruistic or dynastic? Japanese Economic Review 53, 26-54.

ILO, various years. The cost of social security. United Nations.

Laitner, J., Juster, F.T., 1996. New evidence on altruism: a study of TIAA-CREF retirees. American Economic Review 86, 893-908.

Laitner, J., Ohlsson, H., 2001. Bequest motives: a comparison of Sweden and the United States. Journal of Public Economics 79, 205–236.

Lapan, H.E., Enders, W., 1990. Endogenous fertility, Ricardian equivalence, and the debt management policy. Journal of Public Economics 41, 227-248.

Lemieux, T. 2006. The “Mincer equation” thirty years after schooling, experience, and earnings. In: Grossbard, S. (ed.), Jacob Mincer: A Pioneer of Modern Labor Economics, New York: Springer, 127-145.

Lucas, R., 1988. On the mechanics of economic development. Journal of Monetary Economics 22, 3–42.

Moretti, E., 2004a. Estimating the social returns to higher education: evidence from longitudinal and repeated cross-section data. Journal of Econometrics 121, 175–212.

Moretti, E., 2004b. Workers’ education, spillovers, and productivity: evidence from plant-level production functions. American Economic Review 94, 656-690.

Rauch, J.E., 1993. Productivity gains from geographic concentration of human capital: evidence from the cities. Journal of Urban Economics 34, 380–400.

Raut, L.K., Tran, L.H., 2005. Parental human capital investment and old-age transfers from children: is it a loan contract or reciprocity for Indonesian families? Journal of Development Economics 77, 389-414.

Solon, Gary, 1999. Intergenerational mobility in the labor market. In: Ashenfelter, O., Card, D. (eds.), Handbook of Labor Economics, Volume 3A, Amsterdam: Elsevier, 1761 - 1800.

Summers, R., Heston, A., 1988. A new set of international comparison of real product and price levels estimates for 130 countries. Review of Income and Wealth 34, 1–25.

Tamura, R., 1991. Income convergence in an endogenous growth model. Journal of Political Economy 99, 522-540.

Tomes, Nigel., 1981. The family, inheritance, and the intergenerational transmission of inequality. Journal of Political Economy 89, 928-958.

United Nations, various years. Demographic Yearbook.

UNESCO, various years. Statistical Yearbook.

Zhang, J., 1995. Social security and endogenous growth. Journal of Public Economics 58, 185–213.

Zhang, J., Zhang, J., Lee, R., 2001. Mortality decline and long-run economic growth. Journal of Public Economics 80, 485–507.

Zhang, J., Zhang, J., 2003. Long-run effects of unfunded social security with earnings-dependent benefits. Journal of Economic Dynamics & Control 28, 617-641.

Zhang, J., Zhang, J., 2004. How does social security affect economic growth: evidence from cross-country data. Journal of population Economics 17, 473-500.

Zhang, J., Zhang, J., 2007. Optimal social security in a dynastic model with investment externalities and endogenous fertility. Journal of Economic Dynamics & Control 31, 3345-3567.

CHAPTER 2

Pareto optimal social security and education subsidization in a dynastic model with human capital externalities, fertility and endogenous growth

2.1. Introduction

Social security and education subsidization have long been key elements of public policy in most countries and have received a great deal of attention in economic studies and public policy debates. Concerning education subsidization in practice, according to OECD (2008), governments of OECD countries on average spent about 5% of GDP on education. In fact, the recognition of the role of education in development started at the birth of modern economics as a profession: Adam Smith in *The Wealth of Nations* discusses the importance of education in length. The recent endogenous growth theory also recognizes education for human capital accumulation as an engine of sustainable growth as shown in Lucas (1988). For instance, according to Azariadis and Drazen (1990), rapid growth cannot occur without a sufficiently high level of human capital investment relative to income; and according to Laitner (1993), human capital accumulation through general education adds 30% to 50% to long-run growth in per capita output. Moreover, Zhang (1996) finds that education subsidization financed by labor income taxation alleviates under-investment caused by human capital externalities, while Zhang and Casagrande (1998) find empirical evidence that education subsidies promote economic growth but have little effect on fertility in a cross-country data set.

Similarly, social security has been established in most developed countries, mainly in the form of a pay-as-you-go (PAYG) system with payroll tax rates ranging from 10% to 20% or higher (see Social Security Administration and International Social Security Association, 2008). Its impacts on savings and economic growth have been examined extensively without reaching consensus in Barro (1974), Feldstein (1974), Hubbard and Judd (1987), Zhang (1995), Rosati (1996), Corneo and Marquardt (2000), Sanchez-Losada (2000), and Kemnitz and Wigger (2000), among others. Among them, Zhang (1995) shows that social security can promote growth by raising human capital investment and reducing fertility without changing the saving rate. Using a cross-country panel data set, Zhang and Zhang (2004) indeed find evidence that social security has a negative effect on fertility, positive effects on secondary school enrolment and economic growth, but no statistically significant effect on the saving rate. Also, Cooley and Soares (1999) argue that once social security is instituted it will be supported by a majority due to generational conflicts. Moreover, Zhang and Zhang (2007) and Yew and Zhang (2009) find that social security can improve social welfare because of spillovers from aggregate physical capital in a neoclassical growth model or from average human capital in an endogenous growth model. However, in these models social security cannot lead to the socially optimal allocation.

With few exceptions, the existing literature has studied the implications of social security and education subsidization separately. The exceptions include Kaganovich and Zilcha (1999), Pecchenino and Pollard (2002) and Rojas (2004). All these studies show that higher social security leads to lower welfare or slower growth in the presence of education subsidization. In Rojas (2004), an increase in education subsidies is also welfare reducing.

Differing from the previous work, in our paper social security and education subsidization together, rather than alone, can eliminate the efficiency losses of human capital externalities and achieve the socially optimal allocation when social security reduces fertility. Indeed, some empirical studies find evidence of human capital externalities in the determination of individuals' earnings through channels such as ethnic groups, neighborhoods, work places, or state funding of schools; see, e.g., Borjas (1992, 1994, 1995), Rauch (1993), Davies (2002) and Moretti (2004a,b). Such externalities lower the private rate of return on human capital investment from its social rate, and thus lead to under-investment in human capital. At the same time, via the well-known trade-off between the quality and quantity of children in the spirit of Becker and Lewis (1973), the externality also leads to over-reproduction of the population. The combination of too many children and too little education is indeed a typical phenomenon in early development, and therefore the analysis of government policies dealing with such efficiency losses of the human capital externality may be highly relevant in the real world.

As a conventional policy instrument, education subsidization reduces the cost of human capital investment and hence appears to be an ideal means to tackle the under-investment problem. However, once fertility is optimally chosen by individuals as in this model, education subsidization also reduces the education cost of a child. It may therefore tend to increase fertility, although it may reduce fertility indirectly via the trade-off between the quantity and quality of children. In addition, the accompanying tax on wage income weakens the positive effect of education subsidization on human capital investment but strengthens the positive effect of education subsidization on fertility by reducing the after-tax wage rate. Thus, education subsidization alone cannot fully eliminate the efficiency losses from human capital externalities once fertility is chosen optimally by individuals.

Similarly, social security financed by payroll taxation alone cannot eliminate the efficiency losses from the human capital externality either, because of its conflicting effects on fertility and human capital investment as well. On the one hand, social security raises the cost of a child by raising the foregone earnings-dependent social security benefits for time-intensive childrearing, and by raising bequests to children for easing their increased tax burden. It also raises the benefit of human capital investment when social security benefits are earnings-dependent. However, the payroll tax for social security exerts opposite effects on the cost of a child and on the benefit of human capital investment. The net effects on fertility and on human capital investment are unclear and dependent on the discounting factor in preferences. If the net effect of social security on fertility is negative as in the empirical evidence in Zhang and Zhang (2004), social security and education subsidization financed by payroll taxation together can eliminate the under-investment and over-reproduction problems and achieve the socially optimal allocation. In deriving these results, we start with a general model for general Pareto optimal government policy rules and then provide an example with log utility and Cobb-Douglas technologies.

The remainder of the paper is organized as follows. The next section introduces the model. Section 2.3 deals with the social planner problem. Section 2.4

determines the competitive equilibrium and derives the results for Pareto optimal social security and education subsidization in a general case. Section 2.5 illustrates our results in a special case with logarithmic utility and Cobb-Douglas technologies. Section 2.6 concludes.

2.2. The model

The model has an infinite number of discrete periods and overlapping-generations of identical agents who live for three periods. In their first period of life, they embody human capital and do not make any decision. In their second period of life, they work and make decisions on lifecycle savings and on the number and education of identical children. In their third period of life, they retire and decide only on the allocation between the amount of bequests to children and their own old-age consumption.[8] The mass of the working generation in period t is denoted by L_t.

The preferences of the coexisting old parent and young working members in a family are assumed to be identical and are defined over the consumption levels of the old and young members, $C_{o,t}$ and $C_{y,t}$, respectively, and the number of children N_t of family members in all generations as in Zhang and Zhang (2007):

$$\sum_{t=0}^{\infty} \alpha^t U(C_{o,t}, C_{y,t}, N_t), \quad \alpha \in (0,1), \tag{2.1}$$

[8] Intentional bequests made by parents can be in the forms of inter vivos gifts and post-mortem bequests. Bequests in this model are of the inter vivos form which is consistent with the empirical evidence (i.e., Gale and Scholz, 1994) that suggests inter vivos are substantial. However, we expect both forms of inter vivos and post-mortem bequests to yield similar qualitative result concerning the effect of social security on the bequests cost of a child. This is because when parents value their children's welfare, a rise in the social security tax rate would increase the amount of intentional bequests to offset the increased tax burden on their children, regardless of whether the bequests are made in the form of inter vivos or post-mortem bequests.

where α is the discounting factor.[9] The period-utility function $U(C_{o,t}, C_{y,t}, N_t)$ is defined over what can contribute to living family members' welfare within a period: the consumption of coexisting old and young members as well as the number of children.[10] The old-age consumption of a period-*t* young member will be reflected in $U(C_{o,t+1,} C_{y,t+1}, N_{t+1})$ next period. In this way, we can incorporate the lifecycle consumption-saving consideration into a dynastic family model along with the trade-off between the welfare and the number of children in a recursive manner. We assume that the period-utility function $U(\cdot,\cdot,\cdot)$ is increasing and concave and meets the Inada conditions to ensure an interior optimal solution: $\partial U / \partial x \to 0$ as $x \to \infty$ for $x = C_o, C_y$; and $\partial U / \partial x \to \infty$ as $x \to 0$ for $x = C_o, C_y, N$.[11]

Each young adult divides one unit of time endowment between rearing children and working. Rearing a child requires *v* fixed units of time, implying an upper bound 1/*v* on *N*; otherwise *N* may approach infinity. The amount of working time per worker is equal to 1-*vN* that earns $(1-\tau_t)(1-vN_t)W_t H_t$ where *W* is the wage

[9] There are various assumptions on preferences in the overlapping-generations models dealing with the demographic changes in the economy. Becker and Barro (1988) assume dynastic preferences where the discount factor is a function of the number of children. However, that assumption may not lead to analytical solutions with two types of capital. To obtain analytical solutions, we assume that α is independent of the number of children as in Lapan and Enders (1990) and Zhang (1995).

[10] When young and old adults coexist and choose consumption in the same period in a family, the conventional assumption would rule out possible altruism from working family members toward their parents' old-age consumption and hence would create generational conflicts. In this sense, our approach here complements approaches featuring generational conflicts between coexisting old and working agents in conventional dynastic family models with only downward altruism as well as in conventional lifecycle models without any form of altruism. Further, our use of a dynastic family model, rather than a non-altruistic lifecycle model, is partly based on empirical evidence in Zhang and Zhang (2004) that social security has a statistically insignificant effect on private savings. As is well known in the literature, the effect of social security on savings should be neutral in the dynastic family model (e.g., Barro, 1974) but negative in the lifecycle model (e.g., Feldstein, 1974).

[11] The literature on the existence and on the form of altruism is divided in theory as well as in empirical evidence. Some empirical studies find supporting evidence for the altruistic model, e.g., Tomes (1981), Laitner and Juster (1996), and Laitner and Ohlsson (2001). In particular, the empirical studies of Laitner and Ohlsson (2001) show that the bequest behavior in Sweden and the U.S. supports the altruistic model.

rate per unit of effective labor, H_t is human capital for each worker, and τ is the labor income tax rate. A young adult in period t also receives a bequest B_t from his or her old parent. He or she spends the after-tax earnings and the received bequest on young-age consumption $C_{y,t}$, retirement savings S_t, and education for each child $(1-\gamma_t)E_t$, where γ_t is the rate of education subsidies. An old agent spends part of his or her savings plus interest income and social security benefits on own consumption and leaves the rest as bequests to children. The household budget constraints can be written as:

$$C_{y,t} = B_t + (1-\tau_t)(1-vN_t)W_t H_t - S_t - (1-\gamma_t)E_t N_t, \tag{2.2}$$

$$C_{o,t} = S_{t-1}R_t + T_t - B_t N_{t-1}, \tag{2.3}$$

where R is the interest factor and T the amount of social security benefits per retiree.

As practiced in many countries such as France and Germany, the amount of social security benefits received by a retiree depends on his or her own earnings in working age according to a replacement rate ϕ_t, that is, $T_t = \phi_t(1-vN_{t-1})W_{t-1}H_{t-1}$.[12] With this formula linking the amount of one's social security benefits to one's own past earnings, a worker who has more children (hence less labor time) will not only earn less wage income today, but also receive less social security benefits in old age. The government is assumed to run a balanced budget in every period: $T_t = \bar{N}_{t-1}\left[\tau_t(1-v\bar{N}_t)W_t\bar{H}_t - \gamma_t E_t \bar{N}_t\right]$, whereby the bar above a variable indicates its

[12] The essence of the results will remain valid if the amount of social security benefits is less than proportional to individuals' own earnings (as in the United States) or is independent of individuals' own earnings, though quantitatively different. As shown in Zhang and Zhang (2003), the more heavily the social security benefits depend on one's own past earnings, the more likely the increase in the tax rate for social security will have a negative effect on fertility and a positive effect on the growth rate of per capita output.

average level in the economy. With identical agents in the same generation, in equilibrium we have $\bar{N} = N$ and $\bar{H} = H$ by symmetry.

The production of the single final good uses a constant-return-to-scale technology:

$$Y_t = F(K_t, L_t(1 - vN_t)H_t) \tag{2.4}$$

where K_t is aggregate physical capital and L_t is the total number of workers. Since one period in this model corresponds to about 30 years, it is reasonable to assume that both physical capital and human capital depreciate fully within one period. The function $F(\cdot,\cdot)$ is assumed to be increasing and concave. In per worker terms y_t and k_t, we have $y_t = Y_t / L_t = f(k_t, (1 - vN_t)H_t)$ and $k_{t+1} = K_{t+1} / L_{t+1}$.

A constant-return-to-scale education technology is available for a child to embody human capital, H_{t+1}, depending on the parental investment of the final good per child, E_t, parental human capital, H_t, and the average human capital in the economy, $\bar{H}_t$, as in Tamura (1991):

$$H_{t+1} = H(E_t, H_t, \bar{H}_t) . \tag{2.5}$$

The function $H(\cdot,\cdot,\cdot)$ is also assumed to be increasing and concave. Under this assumption, there is an externality in the form of positive spillovers from average human capital to the formation of human capital of every child, in line with the empirical evidence on human capital externalities in the literature that we mentioned earlier.[13]

[13] Human capital externalities on individuals' earnings may arise in the production of human capital as in Tamura (1991) and in the production of goods as in Lucas (1988). In fact, many related empirical studies such as Moretti (2004a, 2004b) focus on human capital externalities in the production of goods

Assuming perfect competition, the interest factor and the before-tax wage rate per unit of effective labor are equal to their marginal products:

$$R_t = \frac{\partial Y_t}{\partial K_t} = \frac{\partial f(k_t, (1-vN_t)H_t)}{\partial k_t}, \tag{2.6}$$

$$W_t = \frac{\partial Y_t}{\partial [L_t(1-vN_t)H_t]} = \frac{\partial f(k_t, (1-vN_t)H_t)}{\partial [(1-vN_t)H_t]}. \tag{2.7}$$

The price of the sole final good is normalized to unity.

The physical capital market clears when

$$K_{t+1} = L_t S_t. \tag{2.8}$$

The working population evolves according to $L_{t+1} = L_t N_t$.

Feasibility in the economy is given below:

$$C_{o,t} = y_t N_{t-1} - N_{t-1} C_{y,t} - N_{t-1} N_t k_{t+1} - E_t N_t N_{t-1}. \tag{2.9}$$

2.3. The social planner problem

Starting from an initial state (N_{-1} , K_0 , H_0), the social planner chooses a sequence $\left(C_{y,t}, N_t, k_{t+1}, E_t, H_{t+1}, \overline{H}_{t+1}\right)$ to maximize utility in (2.1) subject to feasibility and technologies in the economy as the following:

---

by following the formulation in Lucas (1988); some empirical studies such as Borjas (1992, 1995) focus on human capital externalities from the parents' generation to the formation of children's skills as in our model. However, both forms of human capital externalities share the same essence that the average or aggregate level of human capital has a positive spillover on each individual's earnings. As a result, they should lead to the same problem of underinvestment in human capital. Therefore, assuming human capital externalities either in the production of goods or in the production of human capital is expected to yield similar results concerning optimal government policy. For ease of exposition, we only focus on the latter in this paper.

$$\max \sum_{t=0}^{\infty} \Big\{ \alpha^t U\Big(f(k_t, (1-vN_t)H_t)N_{t-1} - N_{t-1}C_{y,t} - N_{t-1}N_t k_{t+1} - E_t N_t N_{t-1},\ C_{y,t},\ N_t \Big)$$
$$+\lambda_t [H(E_t, H_t, \bar{H}_t) - H_{t+1}] \Big\}.$$

Here, we have used the feasibility in the economy and the production technology for substitution and introduced a multiplier λ_t for the education technology. It is worth noting that the social planner, unlike an individual in a decentralized economy, can choose average human capital $\bar{H}_{t+1}$. Due to the presence of the products $N_t k_{t+1}$ and $N_t E_t$, the feasible set of the choice variables in feasibility (2.9) may not be a convex set; that is, $C_{o,t}$ and hence $U(\cdot)$ may not be a concave function of variables (E_t, k_{t+1}, N_t). We thus need to assume the following:

Assumption 2.1. $U\Big(f(k_t, (1-vN_t)H_t)N_{t-1} - N_{t-1}C_{y,t} - N_{t-1}N_t k_{t+1} - E_t N_t N_{t-1},\ C_{y,t},\ N_t \Big)$ *is concave in* $(C_{y,t}, E_t, k_{t+1}, N_t)$.

Denote $U_t \equiv U(C_{o,t}, C_{y,t}, N_t)$ and $f_t \equiv f\big(k_t, (1-vN_t)H_t\big)$ for notational ease. The first-order conditions are given below for $t \geq 0$:

$$C_{y,t}:\ \frac{\partial U_t}{\partial C_{o,t}} N_{t-1} = \frac{\partial U_t}{\partial C_{y,t}}, \tag{2.10}$$

$$k_{t+1}:\ \frac{\partial U_t}{\partial C_{o,t}} N_{t-1} = \alpha \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \frac{\partial f_{t+1}}{\partial k_{t+1}}, \tag{2.11}$$

$$N_t:\ \frac{\partial U_t}{\partial C_{o,t}} N_{t-1} \left[v \frac{\partial f_t}{\partial\big((1-vN_t)H_t\big)} H_t + k_{t+1} + E_t \right] = \frac{\partial U_t}{\partial N_t} + \alpha \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \frac{C_{o,t+1}}{N_t}, \tag{2.12}$$

$$E_t: \quad \alpha^t \frac{\partial U_t}{\partial C_{o,t}} N_t N_{t-1} = \lambda_t \frac{\partial H_{t+1}}{\partial E_t},$$

$$H_{t+1}, \bar{H}_{t+1}: \; \lambda_t = \lambda_{t+1}\left(\frac{\partial H_{t+2}}{\partial H_{t+1}} + \frac{\partial H_{t+2}}{\partial \bar{H}_{t+1}}\right) + \alpha^{t+1} \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \frac{\partial f_{t+1}}{\partial[(1-vN_{t+1})H_{t+1}]}(1-vN_{t+1})N_t \,.$$

The first-order conditions with respect to E_t and H_{t+1} or $\bar{H}_{t+1}$ lead to the optimal condition concerning human capital investment:

$$\frac{\partial U_t}{\partial C_{o,t}} N_{t-1} = \alpha \frac{\partial H_{t+1}}{\partial E_t} \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \times \left[\frac{\partial f_{t+1}}{\partial\left((1-vN_{t+1})H_{t+1}\right)}(1-vN_{t+1}) + N_{t+1}\left(\frac{\partial H_{t+2}}{\partial H_{t+1}} + \frac{\partial H_{t+2}}{\partial \bar{H}_{t+1}}\right) \Big/ \frac{\partial H_{t+2}}{\partial E_{t+1}}\right]. \tag{2.13}$$

The system of equations (2.9) - (2.13) and our assumptions about the functions for preferences and technologies provide necessary and sufficient conditions for the existence and uniqueness of the socially optimal allocation. [14]

Definition 2.1. *For* $t \geq 0$ *and a given state* (k_0, H_0, L_0, N_{-1}), *a socially optimal allocation is a sequence* $\left\{C_{y,t}, C_{o,t}, E_t, N_t, k_{t+1}, H_{t+1} = \bar{H}_{t+1}\right\}_{t=0}^{\infty}$ *satisfying the technologies in (2.4) and (2.5), the feasibility in (2.9) and the optimal conditions in (2.10)-(2.13).*

The main purpose of our paper is to find optimal social security and education subsidization that can decentralize the socially optimal allocation into a competitive equilibrium allocation.

[14] Note that the transversality conditions are satisfied in this model because the Bellman equation of this maximization problem meets Blackwell's sufficient conditions to be a contraction with $\alpha < 1$.

2.4. The competitive equilibrium and results

In the decentralized economy, each consumer maximizes utility in (2.1) subject to budget constraints (2.2) and (2.3), the education technology (2.5) and the earnings-dependent benefit formula, taking the rates of the income tax, the education subsidy and the replacement rate as given. This problem can be written as the following:

$$\max \sum_{t=0}^{\infty}\Big\{ \alpha^t U\big(S_{t-1}R_t+\phi_t(1-vN_{t-1})W_{t-1}H_{t-1}-B_tN_{t-1},$$
$$B_t+(1-\tau_t)(1-vN_t)W_tH_t-S_t-(1-\gamma_t)E_tN_t,\ N_t\big)+$$
$$\psi_t[H(E_t,H_t,\bar{H}_t)-H_{t+1}]\Big\}$$

by choice of $(B_t, E_t, N_t, H_{t+1}, S_t)$ where we have used the budget constraints and the earnings-dependent benefit formula for substitution. For $t \geq 0$, the first-order conditions are given as follows:

$$B_t:\ \frac{\partial U_t}{\partial C_{o,t}}N_{t-1}=\frac{\partial U_t}{\partial C_{y,t}}, \tag{2.14}$$

$$S_t:\ \frac{\partial U_t}{\partial C_{y,t}}=\alpha\frac{\partial U_{t+1}}{\partial C_{o,t+1}}R_{t+1}, \tag{2.15}$$

$$N_t:\ \frac{\partial U_t}{\partial N_t}=\frac{\partial U_t}{\partial C_{y,t}}\big[v(1-\tau_t)W_tH_t+(1-\gamma_t)E_t\big]+\alpha\frac{\partial U_{t+1}}{\partial C_{o,t+1}}\big(vW_tH_t\phi_{t+1}+B_{t+1}\big)\ , \tag{2.16}$$

$$E_t:\ \alpha^t\frac{\partial U_t}{\partial C_{y,t}}(1-\gamma_t)N_t=\psi_t\frac{\partial H_{t+1}}{\partial E_t},$$

$$H_{t+1}:\quad \begin{aligned}\psi_t &= \psi_{t+1}\frac{\partial H_{t+2}}{\partial H_{t+1}}+\alpha^{t+2}\frac{\partial U_{t+2}}{\partial C_{o,t+2}}\phi_{t+2}(1-vN_{t+1})W_{t+1}+\\ &\alpha^{t+1}\frac{\partial U_{t+1}}{\partial C_{y,t+1}}(1-\tau_{t+1})(1-vN_{t+1})W_{t+1}.\end{aligned}$$

The first-order conditions with respect to E_t and H_{t+1} lead to the following optimal condition concerning human capital investment:

$$\frac{\partial U_t}{\partial C_{y,t}}(1-\gamma_t)N_t = \alpha\frac{\partial H_{t+1}}{\partial E_t}\left[\frac{\partial U_{t+1}}{\partial C_{y,t+1}}(1-vN_{t+1})W_{t+1}(1-\tau_{t+1}) + \alpha\frac{\partial U_{t+2}}{\partial C_{o,t+2}}\phi_{t+2}(1-vN_{t+1})W_{t+1} + \frac{\partial U_{t+1}}{\partial C_{y,t+1}}(1-\gamma_{t+1})N_{t+1}\frac{\partial H_{t+2}}{\partial H_{t+1}}\bigg/\frac{\partial H_{t+2}}{\partial E_{t+1}}\right]. \tag{2.17}$$

In (2.14), the marginal loss in the old parent's utility from giving a bequest to each child is equal to the marginal gain in children's utility. In (2.15), the marginal loss in utility in young working age from saving is equal to the marginal gain in utility in old age through receiving the return to saving. In (2.16), the marginal loss in utility from having an additional child, through giving up a fraction of wage income and earnings-dependent social security benefits, leaving a bequest to this child and spending on the education of this child, is equal to the marginal gain in utility from enjoying the child. In (2.17), the marginal loss in the parent's utility from investing an additional unit of income in children's education is equal to the marginal gain in children's utility through increasing their wage income and earnings-dependent social security benefits and making them more effective in teaching their own children. A key difference between the individual choice of human capital investment and the social planner's is that individuals cannot choose average human capital, unlike the social planner.

In these optimal conditions, it is also worth noting that none of social security, education subsidization and payroll taxation creates any wedge in the consumption-saving-bequest trade-off in (2.14) and (2.15). This is because private intergenerational transfers can counteract public intergenerational transfers in a Ricardian world inhabited by dynastic families as in the literature (e.g. Barro, 1974; Zhang, 1995). When social security transfers income from workers to retirees, $C_{y,t}$ falls and $C_{o,t}$ rises. In response to this change, an old parent can restore the balance between marginal utilities of consumption across generations by leaving more bequests to children B_t in a dynastic model. When education subsidization financed by labor income taxation reduces the cost of education, there are conflicting impacts on $C_{y,t}$, the net impact can also motivate an old parent to change the amount of bequests B_t to regain the balance between the marginal utilities of consumption across generations.

However, all of education subsidization, social security and payroll taxation create wedges in the quantity-quality trade-off concerning children. Clearly, education subsidization reduces not only the cost of human capital investment relative to the benefit in (2.17) and but also the education cost of a child in (2.16). In addition, by increasing the earnings-dependent benefit, social security increases both the cost of a child in (2.16) and the benefit of human capital investment in (2.17). Moreover, social security also increases the bequest cost of a child in (2.16). Conversely, the payroll tax reduces both the cost of a child in (2.16) and the benefit of human capital investment in (2.17) by reducing the after-tax wage rate.

We define the competitive equilibrium below:

Definition 2.2. *For $t \geq 0$ and a given initial state (N_{-1}, K_0, H_0, L_0), a competitive equilibrium with education subsidization and PAYG social security financed by a* *labor income tax is a sequence of allocations* $\left\{B_t, C_{y,t}, C_{o,t}, K_{t+1}, H_{t+1}, N_t, S_t, Y_t\right\}_{t=0}^{\infty}$, *prices* $\left\{R_t, W_t\right\}_{t=0}^{\infty}$ *and government policies* $\left\{\tau_t, \gamma_t, \phi_t\right\}_{t=0}^{\infty}$ *such that: (i) taking the prices and the government policies as given, firms and households optimize and their solutions satisfy the budget constraints (2.2) and (2.3), the technologies (2.4) and (2.5), the optimal conditions (2.6), (2.7), and (2.14)-(2.17); (ii) the government budget is balanced, and (iii) all markets clear with* $K_{t+1} = L_t S_t$ *and per worker labor being equal to* $1 - vN_t$ *; (iv)* $X = \bar{X}$ *for* $X = H, K, N, H$ *by symmetry.*

We now derive Pareto optima government policies to decentralize the socially optimal allocation into a competitive allocation.

Proposition 2.1. *For* $t \geq 0,$ *Pareto optimal* $\left\{\tau_t, \gamma_t, \phi_t\right\}_{t=0}^{\infty}$ *are characterized implicitly by the following equations:*

$$\phi_{t+1} \frac{W_t H_t}{N_t R_{t+1}} = v\tau_t W_t H_t + \gamma_t E_t, \tag{2.18}$$

$$\begin{aligned} N_{t+1}\left(\frac{\partial H_{t+1}}{\partial E_t} \Big/ \frac{\partial H_{t+2}}{\partial E_{t+1}}\right)\frac{\partial H_{t+2}}{\partial \bar{H}_{t+1}} &= \gamma_t R_{t+1} - \frac{\partial H_{t+1}}{\partial E_t}\tau_{t+1}(1 - vN_{t+1})W_{t+1} \\ &+ \frac{\partial H_{t+1}}{\partial E_t}\frac{(1 - vN_{t+1})W_{t+1}}{R_{t+2}}\phi_{t+2} - \frac{\partial H_{t+1}}{\partial E_t} N_{t+1}\left(\frac{\partial H_{t+2}}{\partial H_{t+1}} \Big/ \frac{\partial H_{t+2}}{\partial E_{t+1}}\right)\gamma_{t+1}, \end{aligned} \tag{2.19}$$

$$\phi_t (1 - vN_{t-1})W_{t-1}H_{t-1} = N_{t-1}[\tau_t (1 - vN_t)W_t H_t - \gamma_t E_t N_t]. \tag{2.20}$$

For $\partial H_{t+1} / \partial \bar{H}_t = 0$, *no government intervention* $\tau_t = \gamma_t = \phi_{t+1} = 0$ *is Pareto optimal. For* $\partial H_{t+1} / \partial \bar{H}_t > 0$, *no government intervention* $\tau_t = \gamma_t = \phi_{t+1} = 0$ *leads to a situation whereby the net marginal benefit of human capital investment is lower in the* *competitive equilibrium than in the socially optimal allocation, other things being equal. Neither social security nor education subsidization alone can be Pareto optimal.*

Proof. A sequence of government policy $\{\tau_t, \gamma_t, \phi_t\}_{t=0}^{\infty}$ is Pareto optimal if and only if it transforms the system of equations characterizing the competitive equilibrium in Definition 2.2 to the same system of equations characterizing the socially optimal allocation in Definition 2.1. To begin with, note that the technologies in (2.4) and (2.5) are the same in both the competitive equilibrium and the socially optimal allocation.

The household budget constraints (2.2) and (2.3) and the market clearing condition $S_{t-1} = N_{t-1}k_t$ lead to

$$\begin{aligned} C_{o,t} = {} & N_{t-1}k_t R_t + N_{t-1}(1 - vN_t)W_t H_t - N_{t-1}N_t k_{t+1} - N_{t-1}N_t E_t - N_{t-1}C_{y,t} + T_t \\ & - N_{t-1}\tau_t (1 - vN_t)W_t H_t + N_{t-1}\gamma_t E_t N_t . \end{aligned}$$

Substituting the government budget constraint and $y_t = k_t R_t + (1 - vN_t)W_t H_t$ (due to the constant-return-to-scale technology), we get

$$C_{o,t} = N_{t-1}y_t - N_{t-1}C_{y,t} - N_{t-1}N_t k_{t+1} - N_{t-1}N_t E_t$$

which is the feasibility condition in (2.9) for the economy.

The optimal conditions with respect to intergenerational transfers within a family in (2.10) and (2.14) are the same between the socially optimal allocation and the competitive equilibrium. Substituting (2.6) and (2.14) into the optimal condition (2.15) concerning lifecycle savings in the competitive equilibrium, we obtain the same optimal condition (2.11) in the socially optimal allocation.

To compare optimal conditions (2.12) and (2.16) concerning fertility, we rewrite the latter in a few steps. From (2.3) we can express intergenerational transfers as $B_{t+1} = k_{t+1}\partial f_{t+1} / \partial k_{t+1} - C_{o,t+1} / N_t + \phi_{t+1}(1 - vN_t)W_t H_t / N_t$. Combining this and (2.7) into (2.16), we can rewrite (2.16) as

$$\frac{\partial U_t}{\partial N_t} = \frac{\partial U_t}{\partial C_{o,t}} N_{t-1}\left[v(1-\tau_t)\frac{\partial f_t}{\partial(1-vN_t)H_t} H_t + (1-\gamma_t)E_t \right] + \alpha \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \times$$
$$\left[vW_t H_t \phi_{t+1} - \frac{C_{o,t+1}}{N_t} + k_{t+1}\frac{\partial f_{t+1}}{\partial k_{t+1}} + \phi_{t+1}\frac{(1-vN_t)W_t H_t}{N_t} \right]$$
$$= \frac{\partial U_t}{\partial C_{o,t}} N_{t-1}\left[v(1-\tau_t)\frac{\partial f_t}{\partial(1-vN_t)H_t} H_t + k_{t+1} + (1-\gamma_t)E_t + \phi_{t+1}\frac{W_t H_t}{N_t R_{t+1}} \right] - \alpha \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \times$$
$$\left(\frac{C_{o,t+1}}{N_t} \right)$$

where $\partial U_t / \partial C_{o,t} = \alpha (\partial U_{t+1} / \partial C_{o,t+1}) R_{t+1}$ and $R_{t+1} = \partial f_{t+1} / \partial k_{t+1}$ are also used. Thus, the net benefit of having an additional child in the competitive equilibrium (i.e. the left-hand side less the right-hand side) minus the counterpart in the socially optimal allocation is:

$$\frac{\partial U_t}{\partial C_{o,t}} N_{t-1}\left(v\tau_t W_t H_t + \gamma_t E_t - \phi_{t+1}\frac{W_t H_t}{N_t R_{t+1}} \right)$$

which should equal zero so as to transform (2.16) to the same as (2.12). That is, the wedges created by τ_t, γ_t and ϕ_{t+1} should cancel out one another entirely in the optimal condition with respect to the number of children in order to obtain the socially optimal allocation. In particular, when $\tau_t = \gamma_t = \phi_{t+1} = 0$, the optimal condition in (2.16) is indeed the same as that in (2.12). In general, when this overall wedge signed by the terms in the parentheses of the above expression is equal to zero without restricting τ, γ and ϕ to zero, i.e. $v\tau_t W_t H_t + \gamma_t E_t - \phi_{t+1} W_t H_t / \left(N_t R_{t+1} \right) = 0$, the optimal condition (2.16) concerning fertility becomes the same as (2.12) in the socially optimal allocation. This justifies condition (2.18). However, when there is no social security with $\phi_{t+1} \le 0$, the net marginal benefit of having an additional child with education subsidization financed by payroll taxation, $\tau_t > 0$ and $\gamma_t > 0$, is always greater in the competitive equilibrium than in the socially optimal allocation, as can be seen in the above expression. This fact calls for a negative net effect of social security, $\phi_{t+1} > 0$, on the net benefit of having a child for the full cancelation of wedges caused by education subsidization and payroll taxation.

To compare the optimal condition (2.17) with (2.13) concerning human capital investment, we rewrite (2.17) in the following steps. Using (2.14) and (2.15) for substitution, (2.17) can be written as:

$$\frac{\partial U_t}{\partial C_{o,t}}(1-\gamma_t)N_{t-1} = \alpha \frac{\partial H_{t+1}}{\partial E_t} \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \times$$
$$\left[(1-vN_{t+1})W_{t+1}(1-\tau_{t+1}) + \phi_{t+2}\frac{(1-vN_{t+1})W_{t+1}}{R_{t+2}} + N_{t+1}(1-\gamma_{t+1})\frac{\partial H_{t+2}}{\partial H_{t+1}} \Big/ \frac{\partial H_{t+2}}{\partial E_{t+1}}\right].$$

The net marginal benefit of human capital investment in the competitive equilibrium (i.e. the right-hand side less the left-hand side) minus the counterpart in the socially optimal allocation is equal to

$$\frac{\partial U_t}{\partial C_{o,t}}\gamma_t N_{t-1} - \alpha \frac{\partial H_{t+1}}{\partial E_t}\frac{\partial U_{t+1}}{\partial C_{o,t+1}}\left[\tau_{t+1}(1-vN_{t+1})W_{t+1} - \phi_{t+2}\frac{(1-vN_{t+1})W_{t+1}}{R_{t+2}} + N_{t+1}\gamma_{t+1}\right.$$

$$\left.\frac{\partial H_{t+2}}{\partial H_{t+1}}\bigg/\frac{\partial H_{t+2}}{\partial E_{t+1}} + N_{t+1}\frac{\partial H_{t+2}}{\partial \bar{H}_{t+1}}\bigg/\frac{\partial H_{t+2}}{\partial E_{t+1}}\right]$$

$$= \alpha \frac{\partial U_{t+1}}{\partial C_{o,t+1}}\left[\gamma_t R_{t+1} - \tau_{t+1}(1-vN_{t+1})W_{t+1}\frac{\partial H_{t+1}}{\partial E_t} + \phi_{t+2}\frac{(1-vN_{t+1})W_{t+1}}{R_{t+2}}\frac{\partial H_{t+1}}{\partial E_t} - N_{t+1}\gamma_{t+1}\right.$$

$$\left.\frac{\partial H_{t+1}}{\partial E_t}\frac{\partial H_{t+2}}{\partial H_{t+1}}\bigg/\frac{\partial H_{t+2}}{\partial E_{t+1}} - N_{t+1}\frac{\partial H_{t+1}}{\partial E_t}\frac{\partial H_{t+2}}{\partial \bar{H}_{t+1}}\bigg/\frac{\partial H_{t+2}}{\partial E_{t+1}}\right].$$

Setting the terms in the brackets of the above expression at zero equalizes the net marginal benefits of human capital investment in the competitive equilibrium and in the socially optimal allocation and therefore justifies condition (2.19). When there is an externality in the form $\partial H_{t+2} / \partial \bar{H}_{t+1} > 0$, it is clear that in the above expression the overall wedge of τ_t, γ_t and ϕ_{t+1} should be non-zero to counteract the efficiency loss of the externality so as to make (2.17) be the same as (2.13). When there is no externality $\partial H_{t+2} / \partial \bar{H}_{t+1} = 0$, it is also clear that $\tau_t = \gamma_t = \phi_{t+1} = 0$ makes the above expression be equal to zero.

Also, when there is no education subsidization, condition (2.18) becomes $\phi_{t+2} = v\tau_{t+1}N_{t+1}R_{t+2}$. Making use of this for substitution, the net benefit of human capital investment will be lower in the competitive equilibrium than in the socially optimal allocation if $\phi > 0, \tau > 0,$ and $\gamma = 0$:

$$\alpha \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \frac{\partial H_{t+1}}{\partial E_t} \left[-\tau_{t+1}(1-vN_{t+1})W_{t+1} + \phi_{t+2} \frac{(1-vN_{t+1})W_{t+1}}{R_{t+2}} - N_{t+1} \frac{\partial H_{t+2}}{\partial \bar{H}_{t+1}} \Bigg/ \frac{\partial H_{t+2}}{\partial E_{t+1}} \right]$$

$$= \alpha \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \frac{\partial H_{t+1}}{\partial E_t} \left[-\phi_{t+2} \frac{(1-vN_{t+1})W_{t+1}}{vN_{t+1}R_{t+2}} + \phi_{t+2} \frac{(1-vN_{t+1})W_{t+1}}{R_{t+2}} - N_{t+1} \frac{\partial H_{t+2}}{\partial \bar{H}_{t+1}} \Bigg/ \frac{\partial H_{t+2}}{\partial E_{t+1}} \right]$$

$$= \alpha \frac{\partial U_{t+1}}{\partial C_{o,t+1}} \frac{\partial H_{t+1}}{\partial E_t} \left[\phi_{t+2} \frac{(1-vN_{t+1})W_{t+1}}{R_{t+2}} \left(1 - \frac{1}{vN_{t+1}} \right) - N_{t+1} \frac{\partial H_{t+2}}{\partial \bar{H}_{t+1}} \Bigg/ \frac{\partial H_{t+2}}{\partial E_{t+1}} \right] < 0$$

because $1-1/(vN) = -(1-vN)/vN < 0$ for an interior solution for labor.

Condition (2.20) is merely the government budget constraint. Clearly, conditions (2.18)-(2.20) characterize Pareto optimal $\{\tau_t, \gamma_t, \phi_t\}_{t=0}^{\infty}$ because it makes the system of equations characterizing the competitive equilibrium be the same as the system of equations characterizing the socially optimal allocation in all periods. □

Proposition 2.1 states how social security and education subsidization financed by payroll taxation together can fully eliminate the efficiency losses of the human capital externality to achieve the socially optimal allocation. It also states that neither education subsidization nor social security alone financed by payroll taxation can achieve the socially optimal allocation. On the one hand, education subsidization financed by payroll taxation reduces the marginal cost of having an additional child, thereby making the use of social security necessary to drive up the marginal cost of having a child to the socially optimal level. On the other hand, social security cannot fully cancel out the negative effect of the accompanying payroll tax on the net marginal benefit of human capital investment when their effects on the net benefit of having a child are fully canceled out.

It is important to observe that the Pareto optimal government policy rules in (2.18)-(2.20) are very general in nature and may include opposite policies from social security or education subsidization. For instance, Pareto optimal $\gamma, \phi,$ or τ may be negative. The exact nature or interpretation of the Pareto optimal government policy hinges on how social security affects fertility. Given that the human capital externality reduces the private return on education spending relative to the return on having an additional child, we expect fertility to be too high and education spending to be too low in the competitive equilibrium without government intervention. This situation is indeed similar to what we observe in countries in the early development stage. Starting from this situation, if social security financed by payroll taxation reduces fertility then it can be helpful to change fertility toward its first-best level along with education subsidization.

However, there is no guarantee that social security financed by payroll taxation can reduce fertility because social security and payroll taxation have opposing effects on the cost of having an additional child in (2.16): a positive one via the replacement rate and bequests and a negative one via the payroll tax rate. The former effect is stronger if the discounting factor α is larger. Thus, other things being equal, it is more likely for social security to raise the cost of a child and hence to reduce fertility if the discounting factor is greater. Intuitively, a greater discounting factor in the preference means a stronger motive for investing in human capital and for leaving bequests relative to the motive for having more children. In this regard, Cigno and Rosati (1992) and Zhang and Zhang (2004) find empirical evidence that social security reduces fertility, supporting our focusing on social security as part of the Pareto optimal government policy rather than on an opposite policy.

In the next section, we consider an example with log utility and Cobb-Douglas technologies in order to derive the reduced form of the Pareto optimal government policy. The example will help convince that the Pareto optimal combination of social security and education subsidization financed by labor income taxation is non-empty for plausible parameterizations.

2.5. Example: logarithmic utility and Cobb-Douglas technologies

Let the utility function be $U(C_{o,t}, C_{y,t}, N_t) = \beta \ln C_{o,t} + \alpha(\ln C_{y,t} + \rho \ln N_t)$, where $\beta \in (0,1)$ is the taste for utility derived from the consumption of the old parent, $\alpha \in (0,1)$ is the taste for utility from the young-age consumption and the number of children of each working member, and $\rho > 0$ is the taste for utility from the number of children relative to that from young-age consumption. If we equally value consumption undertaken by each of coexisting old and working members in a family in their identical utility function, then the values of α and β may depend on the relative length of lifetime in working age to old age. Since in reality the working period is longer than the retirement period, α may be greater than β. We rewrite the utility function as

$$\sum_{t=0}^{\infty} \alpha^t [\beta \ln C_{o,t} + \alpha(\ln C_{y,t} + \rho \ln N_t)], \;\; 0 < \alpha, \beta < 1, \;\; \rho > 0. \tag{2.21}$$

The production and education functions now take the following respective forms:

$$Y_t = DK_t^{\theta}[L_t(1-vN_t)H_t]^{1-\theta},\ D>0,\ 0<\theta<1, \tag{2.22}$$

$$H_{t+1} = AE_t^{\delta}(H_t^{\varepsilon}\bar{H}_t^{1-\varepsilon})^{1-\delta},\ A>0,\ 0<\delta<1,\ 0<\varepsilon\le 1. \tag{2.23}$$

When $\varepsilon = 1$, there is no externality from average human capital in this model. However, when $\varepsilon < 1$, the externality takes the form of positive spillovers from average human capital to the formation of human capital of every child. The wage rate per unit of effective labor and the real interest factor are then given by

$$W_t = (1-\theta)D\mu_t^{\theta}, \tag{2.24}$$

$$R_t = \theta D\mu_t^{\theta-1}, \tag{2.25}$$

where $\mu_t = K_t/[L_t(1-vN_t)H_t]$ is the physical capital-effective labor ratio.

The first-order conditions of the social planner problem in (2.10)-(2.13) become the following:

$$\frac{1}{c_{o,t}} = \frac{\alpha}{\beta c_{y,t}N_{t-1}}, \tag{2.26}$$

$$\begin{aligned}&\frac{\beta N_{t-1}\left[\left(v(1-\theta)y_t/(1-vN_t)\right)+k_{t+1}+E_t\right]}{C_{o,t}}\\ &= \frac{\alpha\rho}{N_t} + \frac{\alpha\beta(y_{t+1}-C_{y,t+1}-N_{t+1}k_{t+2}-E_{t+1}N_{t+1})}{C_{o,t+1}},\end{aligned} \tag{2.27}$$

$$\frac{N_{t-1}}{C_{o,t}} = \frac{\alpha\theta y_{t+1}}{C_{o,t+1}k_{t+1}}, \tag{2.28}$$

$$\frac{N_{t-1}E_t}{\delta C_{o,t}} = \frac{\alpha}{C_{o,t+1}}\left[(1-\theta)y_{t+1} + \frac{N_{t+1}E_{t+1}(1-\delta)}{\delta}\right]. \tag{2.29}$$

The first-order conditions in the individual utility maximization in (2.14)-(2.17) become the following:

$$\frac{\beta}{C_{o,t}} N_{t-1} = \frac{\alpha}{C_{y,t}}, \tag{2.30}$$

$$\frac{1}{C_{y,t}} = \frac{\beta R_{t+1}}{C_{o,t+1}}, \tag{2.31}$$

$$\frac{\rho}{N_t} = \frac{1}{C_{y,t}} \left[v(1-\tau_t) W_t H_t + (1-\gamma_t) E_t \right] + \frac{\beta}{C_{o,t+1}} \left(v W_t H_t \phi_{t+1} + B_{t+1} \right), \tag{2.32}$$

$$\begin{aligned} \frac{1}{C_{y,t}} (1-\gamma_t) N_t = \frac{\alpha \delta H_{t+1}}{E_t} \Bigg[& \frac{(1-vN_{t+1}) W_{t+1} (1-\tau_{t+1})}{C_{y,t+1}} + \\ & \frac{\beta}{C_{o,t+2}} \phi_{t+2} (1-vN_{t+1}) W_{t+1} + \frac{\varepsilon(1-\delta)(1-\gamma_{t+1}) E_{t+1} N_{t+1}}{C_{y,t+1} H_{t+1}} \Bigg]. \end{aligned} \tag{2.33}$$

With the log utility and Cobb-Douglas technologies and with the full depreciation of capital within one period, we expect the proportional allocations of time and output and the rates of the tax, the subsidy and the replacement to be constant over time, given any initial state. Thus, we can transform the variables in the overall feasibility, the budget constraints and the first-order conditions into their relative ratios to output per worker. For notational ease, we denote the fraction of output per worker spent on item X_t by a lower-case variable $x_t = X_t / y_t$.

In the social planner problem, the transformed feasibility in the economy and the transformed first-order conditions are gives as follows:

$$c_{o,t} = N_{t-1}(1 - c_{y,t} - s_t - e_t N_t), \tag{2.34}$$

$$\frac{\beta N}{c_o} = \frac{\alpha}{c_y} \qquad \text{(for } t > 0 \text{)}, \tag{2.35}$$

$$\frac{\beta N_{-1}}{c_{o,0}} = \frac{\alpha}{c_y} \qquad \text{(for } t = 0 \text{)}, \tag{2.36}$$

$$\frac{v(1-\theta)}{c_y(1-vN)} - \frac{\alpha\left[1-\theta(1+\alpha)\right]}{c_y N} + \frac{e(1+\alpha)}{c_y} + \frac{\alpha}{N} = \frac{\rho}{N}, \tag{2.37}$$

$$s = \alpha\theta, \tag{2.38}$$

$$eN = \frac{\alpha(1-\theta)\delta}{1-\alpha(1-\delta)}. \tag{2.39}$$

From these conditions, we obtain the following constant allocation rules in the social planner allocation, denoted by a superscript SP :

$$c_y^{SP} = \frac{\alpha(1-\alpha)[1-\alpha\theta(1-\delta)]}{(\alpha+\beta)[1-\alpha(1-\delta)]}, \tag{2.40}$$

$$s^{SP} = \alpha\theta, \tag{2.41}$$

$$N^{SP} = \frac{N_n^{SP}}{v\left\{N_n^{SP} + (1-\theta)(\alpha+\beta)[1-\alpha(1-\delta)]\right\}}, \tag{2.42}$$

where

$$N_n^{SP} = \alpha\left\{(1-\alpha)\rho[1-\alpha\theta(1-\delta)] + (1\text{-}\delta)\left[\beta(1-\theta+\alpha^2\theta) - \alpha\theta(1-\alpha)\right] - \alpha(\delta+\beta)\right\},$$

$$e^{SP} = \frac{\alpha(1-\theta)\delta}{N^{SP}[1-\alpha(1-\delta)]}. \tag{2.43}$$

The transformed budget constraints and first-order conditions in the competitive economy are:

$$c_{y,t} = b_t + (1-\tau_t)(1-\theta) - s_t - (1-\gamma_t)e_t N_t, \tag{2.44}$$

$$c_{o,t} = N_{t-1}(\theta + \tau_t(1-\theta) - b_t - \gamma_t e_t N_t), \tag{2.45}$$

$$\frac{\beta N}{c_o} = \frac{\alpha}{c_y} \qquad \text{(for } t>0\text{)}, \tag{2.46}$$

$$\frac{\beta N_{-1}}{c_{o,0}} = \frac{\alpha}{c_y} \qquad \text{(for } t=0\text{)}, \tag{2.47}$$

$$\frac{1}{c_y} = \frac{\beta\theta N}{c_o s} \quad , \tag{2.48}$$

$$\frac{v(1-\tau)(1-\theta)}{(1-vN)c_y} + \frac{(1-\gamma)e}{c_y} + \frac{v\alpha\left[\tau(1-\theta)-\gamma eN\right]}{(1-vN)c_y} + \frac{\alpha b}{Nc_y} = \frac{\rho}{N}, \tag{2.49}$$

$$\alpha\delta\left\{(1-\tau)(1-\theta)+\alpha\left[\tau(1-\theta)-\gamma eN\right]\right\} = (1-\gamma)Ne\left[1-\alpha(1-\delta)\varepsilon\right]. \tag{2.50}$$

From these equilibrium conditions, we obtain the following constant allocation rules in the competitive equilibrium, denoted by a superscript CE:

$$\begin{aligned} b^{CE} &= \theta + \tau(1-\theta) - \gamma\left[\frac{\alpha(1-\theta)[1-\tau(1-\alpha)]\delta}{\Pi}\right] - \\ &\quad \beta\left\{\frac{(1-\alpha\theta)\Pi - \alpha(1-\theta)[1-\tau(1-\alpha)]\delta}{(\alpha+\beta)\Pi}\right\} \end{aligned}, \tag{2.51}$$

$$c_y^{CE} = \frac{\alpha\left\{(1-\alpha\theta)\Pi - \alpha(1-\theta)[1-\tau(1-\alpha)]\delta\right\}}{(\alpha+\beta)\Pi}, \tag{2.52}$$

$$s^{CE} = \alpha\theta, \tag{2.53}$$

$$N^{CE} = \frac{N_n^{CE}}{v\left\{N_n^{CE} + (1-\theta)(\alpha+\beta)[1-\tau(1-\alpha)][1-\alpha\varepsilon(1-\delta)](1-\gamma)\Pi\right\}}, \tag{2.54}$$

where the numerator of N^{CE} is

$$\begin{aligned} N_n^{CE} &= \alpha\Pi\Big\{\rho\left[(1-\alpha\theta)\Pi - \alpha(1-\theta)\delta\left(1-\tau(1-\alpha)\right)\right] - \\ &\quad \Pi\left[\left(\tau(1-\theta)+\theta\right)(\alpha+\beta) - \beta(1-\alpha\theta)\right] - \\ &\quad \delta(1-\theta)\left[1-\tau(1-\alpha)\right]\left[\alpha\beta + (\alpha+\beta)(1-\gamma-\gamma\alpha)\right]\Big\}, \end{aligned}$$

$$e^{CE} = \frac{\alpha(1-\theta)\delta[1-\tau(1-\alpha)]}{N^{CE}\Pi}, \tag{2.55}$$

where $\Pi \equiv (1-\gamma)[1-\alpha\varepsilon(1-\delta)] + \alpha^2\delta\gamma$. We summarize the competitive solution and the socially optimal solution in Table 2.1.

Table 2.1 Comparison between the competitive solution and the socially optimal solution

Competitive solution	Socially optimal solution
$c_y^{CE} = \dfrac{[1-s^{CE}-(N^{CE}e^{CE})]\alpha}{(\alpha+\beta)}$	$c_y^{SP} = \dfrac{[1-s^{SP}-(N^{SP}e^{SP})]\alpha}{(\alpha+\beta)}$
$s^{CE} = \alpha\theta$	$s^{SP} = \alpha\theta$
$N^{CE} = \dfrac{N_n^{CE}}{v\left[N_n^{CE} + (1-\theta)\left(1-\tau(1-\alpha)\right) - \gamma\alpha(N^{CE}e^{CE})\right]}$ $N_n^{CE} \equiv (\rho-\alpha)c_y^{CE} - \alpha\left[s^{CE} - (1-\tau)(1-\theta)\right] - (1-\gamma)(N^{CE}e^{CE})(1+\alpha)$	$N^{SP} = \dfrac{N_n^{SP}}{v\left\{N_n^{SP} + (1-\theta)(\alpha+\beta)[1-\alpha(1-\delta)]\right\}}$ $N_n^{SP} \equiv \left\{(\rho-\alpha)c_y^{SP} + \alpha - (1+\alpha)\left[s^{SP} + (N^{SP}e^{SP})\right]\right\} \times \left[1-\alpha(1-\delta)\right](\alpha+\beta)$
$e^{CE} = \dfrac{\alpha(1-\theta)\left(1-\tau(1-\alpha)\right)\delta}{N^{CE}\left[(1-\gamma)\left(1-\alpha\varepsilon(1-\delta)\right) + \alpha^2\delta\gamma\right]}$	$e^{SP} = \dfrac{\alpha(1-\theta)\delta}{N^{SP}\left(1-\alpha(1-\delta)\right)}$

Note: the superscripts 'CE' and 'SP' refer to the competitive solution and the socially optimal solution, respectively.

Because log utility excludes corner solutions for fertility, the presence of non-convexity in the form of $N_t k_{t+1}$ or $E_t N_t$ in the feasibility in the economy (2.9) or in the form of $N_t B_{t+1}$ or $E_t N_t$ in the budget constraints (2.2) or (2.3) may lead to a situation in which there is no solution for fertility for some parameter values. This situation is ruled out by Assumption 2.1 with general functional forms. In our example with the specific functional forms, the restriction for a unique optimal interior solution is explicitly given below:

Lemma 2.1. *There exists a unique interior solution* $(c_y, c_o, c_{o,0}, e, N, s)$ *in the social planner problem if the taste for the number of children is strong enough such that*

$$\rho > \underline{\rho}^{SP} \equiv \frac{\alpha(1+\beta)[1-\alpha\theta(1-\delta)]-(1-\theta)(1-\delta)(\alpha+\beta)}{[1-\alpha\theta(1-\delta)](1-\alpha)}.$$

Proof. The proof is relegated to Appendix B.

The remaining task is to find out the Pareto optimal rates of the income tax and the education subsidy, and the Pareto optimal ratio of social security benefits to wage income $T_t/[N_{t-1}(1-vN_t)W_tH_t]$.

2.5.1. Pareto optimal social security and education subsidization

For notational ease, let us define an upper bound on the taste parameter:

$$\bar{\rho} \equiv \frac{\delta(1-\theta)(\alpha+\beta)+\alpha(1-\alpha)[1-\alpha\theta(1-\delta)](1+\beta)}{[1-\alpha\theta(1-\delta)](1-\alpha)^2}.$$

It is easy to verify that under $\rho < \bar{\rho}$, social security financed by payroll taxation reduces fertility as in Yew and Zhang (2009). The optimal government policy in our example is given below.

Proposition 2.2. *For* $t \geq 0$ and $0 < \varepsilon < 1$, *if* $\rho < \bar{\rho}$, *then the Pareto optimal social security and education subsidization financed by payroll taxation are characterized by the following equations:*

$$\gamma^* = \frac{\alpha(1-\delta)(1-\varepsilon)+\big(1-\alpha(1-\delta)\big)(1-\alpha)\tau^*}{1-\alpha\varepsilon(1-\delta)-\alpha^2\delta} > 0,$$

$$\tau^* = \frac{\gamma^*(e^{SP}N^{SP})\left[(1-\theta)(1+\alpha)\left(1-\alpha(1-\delta)\right)(\alpha+\beta)+\alpha N_n^{SP}\right]}{(1-\theta)\left[(1-\theta)\alpha\left(1-\alpha(1-\delta)\right)(\alpha+\beta)-(1-\alpha)N_n^{SP}\right]}$$

$$= \alpha(1-\delta)(1-\varepsilon)\delta\Big\{(\alpha+\beta)\left[(1-\theta)(1+\alpha\delta)-\alpha^3[1-\alpha\theta(1-\delta)]\right]+$$

$$\alpha^2(\rho-\alpha)[1-\alpha\theta(1-\delta)](1-\alpha)\Big\}/$$

$$\Big\{[1-\alpha(1-\delta)]\Big[\alpha\ \ (\alpha+\beta)\Big(\left(1-\alpha\varepsilon(1-\delta)\right)(1-\alpha)\left(1-\alpha\theta(1-\delta)\right)+$$

$$\delta(1-\theta)(1-\delta)(1-\varepsilon)\Big)-(1-\alpha)^2(\rho-\alpha)\left(1-\alpha\theta(1-\delta)\right)\left(1-\alpha\varepsilon(1-\delta)\right)\Big]\Big\}$$

$$>0,$$

$$\frac{T_t}{N_{t-1}(1-vN_t)W_tH_t} = \frac{\tau^*(1-\theta)-\gamma^*(e^{SP}N^{SP})}{1-\theta} > 0\,.$$

When $\varepsilon = 1$, $\gamma^* = \tau^* = T_t = 0$.

Proof. We obtain the optimal tax rate and the subsidy rate by equalizing the competitive solution with the social planner's given in Table 2.1. It is easy to verify that under $0<\varepsilon<1$ and $\rho<\bar{\rho}$, $\gamma^*>0$ and $\tau^*>0$. Obviously, $s^{CE}=s^{SP}$. Substituting (τ^*, γ^*) into the competitive solution leads to $c_y^{CE}=c_y^{SP}$, $N^{CE}=N^{SP}$, and $e^{CE}=e^{SP}$. Thus, the competitive equilibrium under (τ^*, γ^*) is first-best. The social security benefit relative to the payroll tax revenue is obtained by using the government budget constraint, which is positive because

$$\frac{\tau^*(1-\theta)}{\gamma^*(e^{SP}N^{SP})} = \frac{(1-\theta)(1+\alpha)\left(1-\alpha(1-\delta)\right)(\alpha+\beta)+\alpha N_n^{SP}}{(1-\theta)\alpha\left(1-\alpha(1-\delta)\right)(\alpha+\beta)-(1-\alpha)N_n^{SP}} > 1\,.$$

It follows that $\tau^*(1-\theta)-\gamma^*(e^{SP}N^{SP})>0$. It is also obvious that, when $\varepsilon=1$, $\gamma^*=\tau^*=T_t=0$.

An alternative proof of Proposition 2.2 can also be made by applying the specific functional forms to the general rules in (2.18)-(2.20) of Proposition 2.1. □

Proposition 2.2 illustrates that neither education subsidization nor social security can achieve the first-best alone, albeit each of them tips children's quantity-quality trade-off in the right direction when the taste for the number of children is not too strong. When education subsidization and social security are implemented together, they can reinforce each other to fully eliminate the efficiency loss of human capital externalities under the same condition under which social security reduces fertility.

2.5.2. Numerical examples

Now, we perform a quantitative assessment of the optimal tax and subsidy rates and the optimal ratio of social security benefits to wage income for plausible parameterizations. The purpose is to find out whether the simulated optimal values can approximate the observed counterpart in the real world. For a better comparison, we begin with a case without externality ($\varepsilon = 1$) and then look at cases with the externality ($0 < \varepsilon < 1$).

The values of parameters are either chosen in line with those in the literature if available (e.g., $\alpha = 0.6$, $\theta = 0.33$), or chosen to yield plausible values for fertility and for the fractions of income invested in both types of capital (e.g. $v = 0.1$, $\beta = 0.3$, $\delta = 0.27$ and $\rho = 0.93$). Taking one period as 30 years, the value of the discounting factor at $\alpha = 0.6$ corresponds to an annual discounting factor of 0.9855 as in Gomme, Kydland and Rupert (2001). Here, a smaller share parameter associated with physical inputs in education ($\delta = 0.27$) than in production reflects the fact that education is less physical (more human) capital intensive than production.

A key parameter for the human capital externality is ε. In a log linear version of the determination of children's human capital or skills in equation (2.23), the coefficient on log average human capital in the parental generation is equal to $(1-\varepsilon)(1-\delta)$. In a similar equation, Borjas (1995) runs regressions of children's skills on two variables: parental skills and the mean skills of the ethnic group of the parents' generation. In doing so, he uses data sets in the United States and uses either education attainment or the log real wage as the proxy for skills. The estimated coefficient on the mean human capital or mean skills of the ethnic group in the parents' generation (defined as ethnic capital therein) is 0.18 when education attainment is used as the proxy, and is 0.30 when the log wage is used. Applying his estimates to the coefficient $(1-\varepsilon)(1-\delta)$ in our model, we have either $\varepsilon = 0.75$ or $\varepsilon = 0.6$. Note that both education attainment and real wage are only approximate indicators of human capital or skills. The former does not capture the quality of education, whereas the latter may include possible factors that are not determined in the production of human capital in the real world such as human capital externalities in the production of goods. To be more conservative on the strength of the human capital externality in the production of human capital, we thus regard 0.7 as the lower bound for ε (or 0.3 as the upper bound on $1-\varepsilon$) and $0.7 \leq \varepsilon \leq 0.85$ as a plausible range. We will vary it gradually toward the case without any externality ($\varepsilon = 1$) for better comparisons.

In Table 2.2, we report the numerical results of the optimal rates of the income tax and the education subsidy and the optimal ratio of social security benefits to wage income, corresponding to the values of ε from 1 to 0.7 in six cases. Case 1 has no externality ($\varepsilon = 1$) and gives the Pareto optimal solution without any government intervention ($\tau = \gamma = 0$). In Case 2 through to 6, the externality is present ($0.7 \leq \varepsilon < 1$). When the externality becomes stronger (smaller ε), the optimal income tax rate becomes higher accordingly and hence leads to a higher optimal rate of education subsidies and a higher optimal ratio of social security benefits to wage income. In particular, for an externality at $\varepsilon = 0.95$, the corresponding optimal rates for the income tax, the education subsidy, and social security benefits are about 7%, 8%, and 5%, respectively. For the value of ε in the plausible range from $\varepsilon = 0.85$ to $\varepsilon = 0.7$ the corresponding optimal rates for the income tax, the education subsidy, and social security benefits are in the range from 18% to 31%, 20% to 34%, and 13% to 21%, respectively. These high contribution rates are in line with the observed counterparts in many industrial nations. Note that the optimal rates for the social security benefits at any given level of externalities are close to those obtained in Chapter 1. These simulation results therefore imply that social security alone cannot fully eliminate the efficiency loss of human capital externalities under the same condition that human capital investment is still below the socially optimal level though fertility is at the socially optimal level.

Table 2.2 Simulations with first-best tax rates and the share of social security benefits

Parameters: $\alpha = 0.6, \beta = 0.3, \delta = 0.27, \rho = 0.93, \theta = 0.33, v = 0.1$						
Variables	ε =1	ε =0.95	ε =0.9	ε =0.85	ε =0.8	ε =0.7
τ^*	0	0.0711	0.1319	0.1846	0.2306	0.3071
γ^*	0	0.0778	0.1444	0.2021	0.2524	0.3362
$\dfrac{\tau^*(1-\theta) - \gamma^*(N^{SP}e^{SP})}{1-\theta}$	0	0.0487	0.0903	0.1263	0.1578	0.2102

2.6. Conclusion

In this paper we have derived Pareto social security and education subsidization financed by payroll taxation that can fully eliminate the efficiency losses of human capital externalities in a dynastic family model with two types of capital and endogenous fertility. We have also shown that neither conventional education subsidization nor social security financed by payroll taxation alone can bring fertility and education spending to their first-best levels at the same time. When social security reduces fertility, it reinforces education subsidization to tip children's quality-quantity trade-off in the right direction until education spending and fertility reach their first-best levels. The results differ from those in existing studies that typically obtain second best allocations when fertility is endogenous.

Our result may be useful contributions with model assumptions based on existing empirical evidence of the prevalence of human capital externalities, the insignificant effect of social security on savings, and the negative effect of social security on fertility. The policy implication of our analysis in this paper is a warning against reform plans calling for transforming PAYG social security to compulsory individual saving schemes. According to Zhang (1995), an inframarginal funded component is neutral if the benefit is linked to one's own contribution and hence, compulsory individual saving schemes are not useful to mitigate the efficiency loss of human capital externalities in this model. Since education subsidization and PAYG social security can reinforce each other to fully eliminate the efficiency loss caused by human capital externalities in this model, our analysis also calls for a caution against reform plans that cut public funding for education in the last two decades in some industrial nations.

Appendix B

Proof of Lemma 2.1. First, it is easy to verify that for the social planner's solution, if $\rho > \underline{\rho}^{SP}$ (as defined in the lemma) then $N_n^{SP} > 0$ according to the solution for fertility in (2.42). Consequently, with $N_n^{SP} > 0$, we must have $0 < N^{SP} < 1/v$. Here, it is obvious that $N^{SP} > 0$ under $\rho > \underline{\rho}^{SP}$ because then the numerator of N^{SP}, i.e. N_n^{SP}, is positive and the denominator, $N_n^{SP} + (1-\theta)(\alpha+\beta)[1-\alpha(1-\delta)]$, is also positive. These facts under $\rho > \underline{\rho}^{SP}$ also imply

$$N^{SP}v = \frac{N_n^{SP}}{\left\{N_n^{SP} + (1-\theta)(\alpha+\beta)[1-\alpha(1-\delta)]\right\}} < 1,$$

leading to $N^{SP} < 1/v$. Note that e^{SP} >0 in (2.43) and $c_y^{SP} > 0$ in (2.40). Then, $c_o^{SP} > 0$ in (2.35), and $c_{o,0}^{SP} > 0$ in (2.36). Clearly, $s^{SP} > 0$ in (2.38). Note also that the solution for each of these variables is unique under the stated conditions.

The remaining task is to argue for the optimality. The optimality builds on the following facts. (i) Since the log utility excludes corner solutions, any solution for fertility or for consumption must be strictly positive. (ii) All choice variables lie in closed and bounded sets: $c_y, c_o, c_{o,0}, e$ and s are in $\left[0,1\right]$, and N in $\left[0,1/v\right]$. (iii) The utility function U_t is continuous in the interior values of the choice variables (c_y, c_o, N). (iv) The utility level U_t is bounded above under $\alpha < 1$. By (i)-(iv), there is at least one optimum. From both (i) and the uniqueness of the interior solution, the optimum must correspond to this unique solution. □

2.7. Reference

Azariadis, C., Drazen, A., 1990. Threshold externalities in economic development. Quarterly Journal of Economics 105, 501-26.

Barro, R.J., 1974. Are government bonds net wealth? Journal of Political Economy 82, 1095–1117.

Becker, G.S., Barro, R.J., 1988. A reformulation of the economic theory of fertility. Quarterly Journal of Economics 103, 1-26.

Becker, G.S., Lewis, H.G., 1973. On the interaction between the quantity and quality of Children. Journal of Political Economy 81, S279-S288.

Borjas, G.J., 1992. Ethnic capital and intergenerational mobility. Quarterly Journal of Economics 107, 123-150.

Borjas, G.J., 1994. Long-run convergence of ethnic skill differentials: the children and grandchildren of the great migration. Industrial and Labor Relations Review 47, 553-573.

Borjas, G.J., 1995. Ethnicity, neighborhoods, and human-capital externalities. American Economic Review 85, 365-390.

Cigno, A., Rosati, E., 1992. The effects of financial markets and social security on saving and fertility behavior in Italy. Journal of Population Economics 5, 319–341.

Cooley, T. F., Soares, J., 1999. A positive theory of social security based on reputation. Journal of Political Economy 107, 135-160.

Corneo, G., Marquardt, M., 2000. Public pension, unemployment insurance, and growth. Journal of Public Economics 75, 293–311.

Davies, J.B., 2002. Empirical evidence on human capital externalities. University of Western Ontario, mimeo.

Feldstein, M.S., 1974. Social security, induced retirement, and aggregate capital formation. Journal of Political Economy 23, 905-926.

Gale, W.G., Scholz, J.K., 1994. Intergenerational transfers and the accumulation of wealth. Journal of Economic Perspectives 8, 145–160.

Gomme, P., Kydland, F. E., Rupert, P., 2001. Home production meets time to build. Journal of Political Economy 109, 1115-1131.

Hubbard, R.G., Judd, K.L., 1987. Social security in individual welfare: precautionary saving, borrowing constraints, and the payroll tax. American Economic Review 77, 630–646.

ILO, various years. *The Cost of Social Security*. United Nations.

Kaganovich, M. and Zilcha, I., 1999. Education, social security, and growth. Journal of Public Economics 71, 289-309.

Kemnitz, A., Wigger, B. U., 2000. Growth and social security. European Journal of Political Economy 16, 673-683.

Laitner, J., 1993. Long-run growth and human capital. Canadian Journal of Economics 26, 796-814.

Laitner, J., Juster, F.T., 1996. New evidence on altruism: a study of TIAA-CREF retirees. American Economic Review 86, 893-908.

Laitner, J., Ohlsson, H., 2001. Bequest motives: a comparison of Sweden and the United States. Journal of Public Economics 79, 205–236.

Lapan, H.E., Enders, W., 1990. Endogenous fertility, Ricardian equivalence, and the debt management policy. Journal of Public Economics 41, 227-248.

Lucas, R., 1988. On the mechanics of economic development. Journal of Monetary Economics 22, 3–42.

Moretti, E., 2004a. Estimating the social returns to higher education: evidence from longitudinal and repeated cross-section data. Journal of Econometrics 121, 175–212.

Moretti, E., 2004b. Workers' education, spillovers, and productivity: evidence from plant-level production functions. American Economic Review 94, 656-690.

OECD. Education at a Glance 2008. www.oecd.org

Pecchenino, R.A., Pollard, P.S., 2002. Dependent children and aged parents: funding education and social security in an aging economy. Journal of Macroeconomics 24, 145-169.

Rauch, J.E., 1993. Productivity gains from geographic concentration of human capital: evidence from the cities. Journal of Urban Economics 34, 380–400.

Rojas, J.A., 2004. On the interaction between education and social security. Review of Economic Dynamics 7, 932-957.

Rosati, F.C., 1996. Social security in a non-altruistic model with uncertainty and endogenous fertility. Journal of Public Economics 60, 283-294.

Sánchez-Losada, F., 2000. Growth effects of an unfunded social security system when there is altruism and human capital. Economics Letters 69, 95-99.

Smith, Adam, 1776. *An Inquiry into the Nature and Causes of the Wealth of Nations*, London: W. Strahan and T. Cadell.

Social Security Administration and International Social Security Association, 2008.

Social Security Programs throughout the World. www.ssa.gov

Tamura, R., 1991. Income convergence in an endogenous growth model. Journal of Political Economy 99, 522-540.

Tomes, Nigel., 1981. The family, inheritance, and the intergenerational transmission of inequality. Journal of Political Economy 89, 928-958.

Yew, S.L., Zhang, J., 2009. Optimal social security in a dynastic model with human capital externalities, fertility and endogenous growth. Journal of Public Economics 93, 605-619.

Zhang, J., 1995. Social security and endogenous growth. Journal of Public Economics 58,185–213.

Zhang, J., 1996. Optimal public investment in education and endogenous growth. Scandinavian Journal of Economics 98, 387-404.

Zhang, J., Casagrande, R., 1998. Fertility, growth, and flat-rate taxation for education subsidies. Economics Letters 60, 209-216.

Zhang, J., Zhang, J., 2003. Long-run effects of unfunded social security with earnings-dependent benefits. Journal of Economic Dynamics & Control 28, 617-641.

Zhang, J., Zhang, J., 2004. How does social security affect economic growth: evidence from cross-country data. Journal of population Economics 17, 473-500.

Zhang, J., Zhang, J., 2007. Optimal social security in a dynastic model with investment externalities and endogenous fertility. Journal of Economic Dynamics & Control 31, 3345-3567.

CHAPTER 3

Golden-rule social security and public health in a dynastic model with endogenous life expectancy and fertility

3.1. Introduction

Most of the developed nations have instituted pay-as-you-go (PAYG) social security programs and public health programs (see, e.g., Aaron, 1985; Lee and Tuljapurkar 1997) for several decades. At the same time, they have observed dramatic increases in life expectancy and declines in fertility, decelerating population growth and leading to population aging. According to OECD (2007), population growth rates for all OECD countries between 1990 and 2005 averaged a little over 0.6% per year, half the rate observed in the 1960s and 1970s. During the same period, the percentage of the population aged 65 or older has risen in all these countries and is expected to rise further in the coming decades. As pointed out by Tang and Zhang (2007), there were upward trends in the ratio of public to private health expenditure and in life expectancy in the time series data of the United States for the period 1870-2000. The steady population aging has caused serious concerns about future economic growth, the pressure on funding social security and public health care, and the wellbeing of a greyer population.

Therefore, it is interesting to explore the implications of PAYG social security and public health for fertility, life expectancy, capital accumulation, economic growth and welfare. We will carry out this task in a dynastic model of neoclassical growth with altruistic bequests, endogenous fertility and actuarially fair annuity markets.

In our model, a rise in the tax rate for social security has opposing effects on fertility and capital accumulation. On the one hand, by increasing the bequest cost of having a child, the tax rise tends to reduce fertility and raise capital intensity. On the other hand, by reducing the after tax wage rate, the opportunity cost of spending time rearing a child falls and hence, the tax rise tends to increase fertility and reduce capital intensity. Moreover, the forgone social security benefits of spending time rearing a child rises with the tax rate under a PAYG system, thereby adding to the cost of a child to channel a negative effect of PAYG social security on fertility and a positive effect on capital intensity. A rise in the tax rate for public health care also exerts conflicting effects on fertility and capital accumulation. On the one hand, when the tax rate for public health increases, the time cost of spending time rearing a child falls and thus fertility may rise and capital intensity may fall. When higher public health spending drives up life expectancy, agents may shift focus from the number of children toward old-age consumption, thereby tending to reduce fertility and raise saving and capital intensity. Our main finding is that the net effect of a tax rise in social security or public health on fertility will depend on the taste for the number, relative to the welfare, of children. When the taste for the number of children is weaker but sufficiently close or equal to that for the welfare of children, social security and public health can reduce fertility and thus can raise both capital and output per worker.

The opposite movement of fertility and capital intensity affects welfare. On the one hand, a reduction in fertility reduces welfare as households obtain utility from the number of children. On the other hand, however, an increase in capital intensity increases labor productivity and increases welfare. The net welfare effect will depend on the relative strength of the tastes for the welfare and number of children. We illustrate numerically that when the taste for the number of children is weaker but sufficiently close or equal to that for the welfare of children, social security and public health can be welfare enhancing by reducing fertility and raising capital intensity. Due to the complexity of the model, our analysis and results are limited to the steady state. Therefore, our optimal social security and public health should be associated with the notion of the "golden rule" in the literature on neoclassical growth.

Our analysis differs from the large body of related literature. Samuelson (1958), Diamond (1965), Barro (1974), Feldstein (1974), Hubbard and Judd (1987), Zhang (1995), Rosati (1996), and Corneo and Marquardt (2000) investigate the impact of social security on savings; Kaganovich and Zilcha (1999), Cooley and Soares (1999), Zhang and Zhang (2007), and Yew and Zhang (2009), among others, study the welfare implications of social security; Zhang (1995), Sanchez-Losada (2000), and Kemnitz and Wigger (2000) show that with human capital, social security can promote growth, which is consistent with the empirical evidence in Zhang and Zhang (2004) that social security has positive effects on human capital investment and on the growth rate of per capita income.

However, most of these studies dealing with social security usually do not consider public health and life expectancy at the same time, as they usually treat life expectancy as an exogenous parameter. For instance, Ehrlich and Lui (1991), Hu (1999), de la Croix and Licandro (1999), and Zhang and Zhang (2001), conclude that higher life expectancy increases the rate of return to human capital investment and leads to higher human capital investment and faster per capita growth; Barro (1997) and Barro and Sala-i-Martin (1995) find empirical evidence that life expectancy has a positive effect on economic growth when income is low, and that the growth effect fades away when income is high; and Zhang et al. (2001) show that a rise in longevity has direct as well as indirect effects on fertility, human capital investment, and growth in a dynastic family model with social security. However, there are some empirical studies that provide evidence that life expectancy can be affected by factors such as average income. For instance, Preston (1975) empirically shows that in aggregate data, income contributes positively to life expectancy. Hence, the inclusion of life expectancy as an endogenous variable in the analysis of social security is highly relevant.

Though there are studies that consider endogenous life expectancy, these studies usually do not consider social security at the same time. For instance, Ehrlich and Chuma (1990) concern the role of endowed wealth, health, and other initial conditions in determining the demand for health and longevity, among others; Leung et al. (2004) consider gender-specific factors in the determination of life expectancy; Chakraborty and Das (2005) show that in the absence of perfect annuities markets, the interplay between income and mortality can generate poverty traps by assuming a positive relationship between probability of survival and private health investment; and Tang and Zhang (2007) investigate health investment, human capital investment, and life cycle savings and show that subsidies on health and human capital investment can improve welfare.

There are a few exceptions that model endogenous life expectancy in the studies on social security and health. Davies and Kuhn (1992) consider the intake of health related goods that endogenously affect life expectancy and show that a social security system would encourage suboptimal health investment, leading to excessive longevity, in the presence of a moral hazard problem. Philipson and Becker (1998) consider life expectancy under the influence of public programs, such as health care and social insurance and pointed out that all forms of old-age income annuity, such as private life insurance or social security programs, would have a similar effect on life prolongation. Zhang et al. (2006) analyze the relationships between life-cycle saving and health investments in different stages of life, and examine the effects of public pensions and health subsidies on health investments, longevity, capital accumulation, and welfare. However, these studies have ignored the combination of such important factors as altruistic intergenerational transfers and endogenous fertility that may lead to very different results.

The rest of the paper is organized as follows. Section 3.2 introduces the model. Section 3.3 characterizes the equilibrium and examines the long-run effects of social security and public health on fertility, life expectancy, capital intensity, and output per worker. Section 3.4 discusses the welfare implication of social security and public health and optimal rates in the long run numerically. Section 3.5 concludes.

3.2. The model

The model economy is inhabited by overlapping generations of a large number of identical agents who live for three periods. In the first period of life, agents do not make any decision. In their second period of life, they work and make decisions on life-cycle savings, the number of children, the amount of bequests to children and their own consumption; they retire when old.[15] Survival is certain from childhood through middle-age, but each middle-aged agent faces a probability $\overline{p} \in (0,1)$ to survive to old age. We assume that children and middle-aged agents (hence old-age survivors) in the same generation are identical.

The utility function of a middle-aged agent, V_t, is defined over own middle-age consumption, c_t, own old-age consumption, d_{t+1}, the number of children, n_t, and the utility of each identical child, V_{t+1}:[16]

$$V_t = \ln c_t + \overline{p}_t \beta \ln d_{t+1} + \eta \ln n_t + \alpha V_{t+1}, \quad \alpha, \beta, \in (0,1); \quad \eta > 0 \tag{3.1}$$

where α is the discounting factor.[17] The assumption of a logarithmic utility function helps to ensure tractability. Here, β is the taste for utility derived from own old-age consumption, η is the taste for utility derived from the number of children. We

[15] Intentional bequests made by parents can be in the forms of inter vivos gifts and post-mortem bequests. Bequests in this model are of the inter vivos form which is consistent with the empirical evidence (i.e., Gale and Scholz, 1994) that suggests inter vivos are substantial. However, we expect both forms of inter vivos and post-mortem bequests to yield similar qualitative result concerning the effect of social security on the bequests cost of a child. This is because when parents value their children's welfare, a rise in the social security tax rate would increase the amount of intentional bequests to offset the increased tax burden on their children, regardless of whether the bequests are made in the form of inter vivos or post-mortem bequests.

[16] Our use of an altruistic model is consistent with some of the existing empirical evidence. See Tomes (1981), Laitner and Juster (1996), and Laitner and Ohlsson (2001), for instance. In particular, the empirical studies of Laitner and Ohlsson (2001) show that the bequest behavior in Sweden and the U.S. offers support for the altruistic model.

[17] There are various assumptions on preferences in the overlapping-generations models dealing with the demographic changes in the economy. Becker and Barro (1988) assume dynastic preferences where the discount factor is a function of the number of children. However, that assumption may not lead to analytical solutions with endogenous life expectancy. To obtain analytical solutions, we assume that α is independent of the number of children as in Lapan and Enders (1990) and Zhang (1995).

assume that the survival rate is increasing in public health, $\bar{M}_t$, at a decreasing rate: $\bar{p}_t = a_0 - a_1 / e^{a_2 \bar{M}_t}$, where $a_{0,} a_{1,} a_2 > 0; a_0 > a_1$.

In period t, a middle-aged agent devotes vn_t units of time endowment to rearing children where $0 < v < 1$ is fixed. The remaining $(1 - vn_t)$ units of time is devoted to working that earns $(1 - \tau_t^T - \tau_t^M)(1 - vn_t)w_t$ where w is the wage rate per unit of labor, τ^T is the contribution rate for social security, and τ^M is the tax rate for public health. This agent receives a bequest with earned interest, $b_t(1 + r_t)$, from his or her old parent at the beginning of period *t*, and leaves a bequest, b_{t+1}, to each child at the end of period *t* so that children receive bequests regardless of their parents' survival status at old age. He or she spends the earnings and the received bequest with earned interest on own middle-age consumption, c_t, retirement savings via actuarially fair annuity markets $s_t(1 - vn_t)w_t$, and bequests to children $b_{t+1} n_t$ where s_t is the saving rate. An old agent spends his or her savings plus interest income and social security benefits on own consumption, d_{t+1}. The budget constraints can be written as:

$$c_t = b_t(1 + r_t) + (1 - \tau_t^T - \tau_t^M - s_t)(1 - vn_t)w_t - b_{t+1} n_t, \tag{3.2}$$

$$d_{t+1} = (1 + r_{t+1}) s_t (1 - vn_t) w_t / \bar{p}_t + T_{t+1}, \tag{3.3}$$

where *T* is the amount of social security benefits per retiree.

As practiced in many countries such as the U.S., France and Germany, the amount of social security benefits received by a retiree depends on his or her own earnings in working age according to a replacement rate ϕ.

The government budget constraints are given by

$$T_t = \phi_t(1 - vn_{t-1})w_{t-1} = \bar{n}_{t-1}\tau_t^T(1 - v\bar{n}_t)_t w_t / \bar{p}_{t-1},$$

$$\bar{M}_t = \tau_t^M(1 - v\bar{n}_t)w_t$$

where the bar above a variable indicates its average level in the economy.[18] With identical agents in the same generation, in equilibrium we have $\bar{n} = n; \bar{p} = p; \bar{M} = M$ by symmetry. In this model, we focus on public healthcare systems that are available in many industrial nations.

The production of the single final good is

$$Y_t = AK_t^{\theta}(1 - vn_t)^{1-\theta}\bar{K}_t^{\delta}, \quad A > 0, \quad \theta \in (0,1), \quad \delta \in [0, 1 - \theta) \tag{3.4}$$

where Y_t and K_t are output per worker and physical capital per worker, respectively; A is the total factor productivity parameter, θ is the share parameter of capital, and δ measures the strength of spillovers from average capital per worker $\bar{K}_t$. Since one period in this model corresponds to about 30 years, it is reasonable to assume that physical capital depreciates fully within one period. When $\delta = 0$, there is no externality from average physical capital in this model. However, when $\delta > 0$, the externality takes the form of positive spillovers from average physical capital to the production of the final good.[19] However, the exact degree of this externality is unclear. When $\delta = 1 - \theta$, the externality is strong enough to generate endogenous growth in an AK-style model. However, Jones (1995), using time series data in

[18] With this formula linking the amount of one's social security benefits to his or her own past earnings, a worker who has more children (hence more time to rearing children and less time to working) will not only earn less wage income today but also receive less social security benefits in old age.

[19] The investment externality has been emphasized in the literature on economic growth (e.g. Arrow, 1962; Romer, 1986; Lucas, 1993). Based on an international cross-section of country data, DeLong and Summers (1991) argued that the spillovers from equipment investment are very substantial. See also Bernstein and Nadiri (1988, 1989), and Nakanishi (2002)) for examples of externalities found in studies of research and development (R&D) stock.

OECD countries, finds empirical evidence against this type of model. We therefore limit our attention to $1-\theta > \delta \geq 0$.

Factors are paid by their marginal products; and the price of the sole final good is normalized to unity. The wage rate per unit of labor and the real interest factor are then given by

$$w_t = (1-\theta)Y_t/(1-vn_t), \tag{3.5}$$

$$1+r_t = \theta Y_t / K_t, \tag{3.6}$$

The physical capital market clears when

$$K_{t+1} = \left[s_t(1-vn_t)w_t + b_{t+1}n_t\right]/n_t. \tag{3.7}$$

3.3. The equilibrium and results

We now solve the dynastic family's problem and track down the equilibrium allocation.

3.3.1. Equilibrium solution for the dynastic family problem

The problem of a dynastic family is to maximize utility in (3.1) subject to budget constraints (3.2) and (3.3), the earnings dependent benefit formula, taking the probability to survive to old age, taxes and replacement rates as given. This problem can be rewritten as the following:

$$\max_{b_{t+1},n_t,s_t} \sum_{t=0}^{\infty} \alpha^t \{\ln[b_t(1+r_t) + (1-\tau_t^T - \tau_t^M - s_t)(1-vn_t)w_t - b_{t+1}n_t] +$$
$$\overline{p}_t \beta \ln[(1+r_{t+1})s_t(1-vn_t)w_t / \overline{p}_t + \phi(1-vn_t)w_t] + \eta \ln n_t\}$$

where we have used the budget constraints and the earnings dependent benefit formula for substitution. For $t \geq 0$, the first-order conditions are given as follows:[20]

$$b_{t+1}: \ \frac{n_t}{c_t} = \frac{\alpha(1+r_{t+1})}{c_{t+1}}, \tag{3.8}$$

$$s_t: \ \frac{1}{c_t} = \frac{\beta(1+r_{t+1})}{d_{t+1}}, \tag{3.9}$$

$$n_t: \ \frac{vw_t(1-\tau_t^T-\tau_t^M-s_t)+b_{t+1}}{c_t} + \frac{\beta\bar{p}_t}{d_{t+1}}\left[\frac{(1+r_{t+1})vw_t s_t}{\bar{p}_t}+\phi v w_t\right] = \frac{\eta}{n_t} \tag{3.10}$$

In (3.8), the marginal loss in utility from giving a bequest to each child is equal to the marginal gain in children's utility. In (3.9), the marginal loss in utility from saving is equal to the marginal gain in utility in old age through receiving the return to saving. In (3.10), the marginal loss in utility from having an additional child, through giving up a fraction of wage income, saving plus interest income and earnings-dependent social security benefits, and leaving a bequest to this child, is equal to the marginal gain in utility from enjoying the child. These first-order conditions hold for all $t \geq 0$.

The equilibrium of the economy is described below.

Definition. *Given an initial state (b_0, K_0), a competitive equilibrium in the economy with PAYG social security and public health is a sequence of allocations* $\left\{b_{t+1}, c_t, d_{t+1}, K_{t+1}, n_t, s_t, \phi_t, \tau_t^T, \tau_t^M, T_{t+1}, M_t, p_t, Y_t\right\}_{t=0}^{\infty}$ *and prices* $\left\{1+r_t, w_t\right\}_{t=0}^{\infty}$ *such that (i)*

[20] Note that the transversality conditions are satisfied in this model because the Bellman equation of this maximization problem meets Blackwell's sufficient conditions to be a contraction with $\alpha < 1$.

taking prices and government policies $\left\{\tau_t^T, \tau_t^M, \phi_t, M_t, T_{t+1}\right\}_{t=0}^{\infty}$ *as given, firms and households optimize and their solutions are feasible, (ii) the government budgets are balanced, (iii) all markets clear with* $K_{t+1} = \left[s_t(1-vn_t)w_t + b_{t+1}n_t\right]/n_t$ *and per worker labor being equal to* $(1-vn_t)$ *, and (iv)* $\bar{n} = n; \bar{p} = p; \bar{M} = M$ *by symmetry.*

Specifically, these equilibrium conditions correspond to the first-order conditions of firms and households, the budget constraints of households and the government, the production technology, the capital market clearing condition, and the amount of labor supply per worker equal to $(1-vn_t)$, for $t \geq 0$. In addition, as mentioned earlier, we have $\bar{n} = n; \bar{p} = p; \bar{M} = M$ in equilibrium by symmetry. Because the model is too complex to be tractable for its full dynamic path, we will only focus on the analysis of the steady state equilibrium.

Since labor income is a constant fraction, $(1-\theta)$, of output per worker in this model, letting $\gamma_c = c_t/(1-vn_t)w_t$, $\gamma_d = d_{t+1}/(1+r_{t+1})(1-vn_t)w_t$, $\gamma_b = b_{t+1}n_t/(1-vn_t)w_t$, we transform variables in the budget constraints and first-order conditions into their relative ratios to labor income in order to achieve the steady state solution. The transformed budget constraints take the form:

$$\gamma_c = \frac{\gamma_b\theta}{(1-\theta)(s+\gamma_b)} + (1-\tau^T-\tau^M-s-\gamma_b), \tag{3.11}$$

$$\gamma_d = \frac{s\theta+\tau^T(1-\theta)(s+\gamma_b)}{\bar{p}\theta}. \tag{3.12}$$

Similarly, the transformed first-order conditions are:

$$s+\gamma_b=\frac{\alpha\theta}{1-\theta}, \tag{3.13}$$

$$\beta\gamma_c=\gamma_d, \tag{3.14}$$

$$\frac{v(1-\tau^T-\tau^M)}{(1-vn)\gamma_c}+\frac{\gamma_b}{n\gamma_c}+\frac{\tau^T v\alpha}{(1-vn)\gamma_c}=\frac{\eta}{n}, \tag{3.15}$$

The expression (3.15) can be derived by using $1+r_{t+1}=\theta Y_{t+1}/K_{t+1}$, where $Y_{t+1}=(1-vn_{t+1})w_{t+1}/(1-\theta)$, and $K_{t+1}=s_t(1-vn_t)w_t+b_{t+1}n_t/n_t$. The left-hand side of (3.15) contains three cost components of a child. The first cost component is the forgone wage income of spending time rearing a child, which falls with the tax rates for social security or public health, other things being equal. The second cost component is the bequest cost of a child, which should rise with the tax rates for social security but may rise or fall with the tax rates for public health. On the one hand, altruistic parents are tempted to reduce the tax burdens of social security on their children and thus higher tax rates for social security increase the bequest cost of a child and tend to reduce fertility. On the other hand, with higher tax rates for public health, life expectancy rises and thus agents increase their life-cycle savings and may reduce the amount of bequests. The third cost component is the forgone social security benefit of spending time rearing a child, which rises with the tax rate for social security through the linkage between the replacement rate and the tax rate for social security under a balanced social security budget.

On the one hand, when the tax rate for social security rises, the subsequent rise in the third cost component partially offsets the fall in the first cost component, and the overall time cost of having a child is likely to fall. However, the possible rise in the bequest cost of a child due to higher tax rates for social security may reduce fertility. On the other hand, when the tax rate for public health rises, a fall in the time cost of having a child tends to increase fertility but the possible rise in the bequest cost of a child tends to reduce fertility. The net effect of social security or public health on fertility will depend on the taste for the number, relative to the welfare, of children. When the taste for the welfare of every child, α, becomes stronger, the third cost components of a child in (3.15) become larger and hence it is more likely that social security or public health reduces fertility. By contrast, when the taste for the number of children, η, becomes stronger, the marginal benefit of a child becomes larger and hence it is more likely for a rise in the tax rate for social security or for public health to raise fertility.

From these equilibrium conditions, we obtain the following steady-state allocation rules:

$$\gamma_b = \frac{\alpha\left\{\theta(\alpha+\beta p(n,\tau^M)) - \beta p(\tau^T,\tau^M)\left[1-(1-\theta)(\tau^T+\tau^M)-\alpha\theta\right]+(1-\theta)\alpha\tau^T\right\}}{(1-\theta)(\alpha+\beta p(\tau^T,\tau^M))} \tag{3.16}$$

$$\gamma_c = \frac{\alpha[\theta(1-\alpha)+(1-\tau^M)(1-\theta)]}{(1-\theta)(\alpha+\beta p(\tau^T,\tau^M))}, \tag{3.17}$$

$$s = \frac{\alpha\left\{\beta p(\tau^T,\tau^M)\left[1-(1-\theta)(\tau^T+\tau^M)-\alpha\theta\right]-(1-\theta)\alpha\tau^T\right\}}{(1-\theta)(\alpha+\beta p(\tau^T,\tau^M))}, \tag{3.18}$$

$$n = \frac{n_n}{v\left\{n_n+(1-\theta)(\alpha+\beta p(\tau^T,\tau^M))[1-\tau^T(1-\alpha)-\tau^M]\right\}}, \tag{3.19}$$

where the numerator of n is

$$n_n = \alpha\left\{\eta\left[\theta(1-\alpha)+(1-\theta)(1-\tau^M)\right]-\theta(\alpha+\beta p(\tau^T,\tau^M))+\right.$$
$$\left.\beta p(\tau^T,\tau^M)\left[1-(1-\theta)(\tau^T+\tau^M)-\alpha\theta\right]-(1-\theta)\alpha\tau^T\right\},$$

$$\gamma_d = \beta\frac{\alpha[\theta(1-\alpha)+(1-\tau^M)(1-\theta)]}{(1-\theta)(\alpha+\beta p(\tau^T,\tau^M))}, \tag{3.20}$$

Note that life expectancy $p(\tau^T,\tau^M)$ is a constant function in equilibrium: $p(\tau^T,\tau^M) = a_0 - a_1 / e^{a_2 M(\tau^T,\tau^M)}$ where M is a function of τ^T and τ^M via n in (3.19):

$$M(\tau^T,\tau^M) = \tau^M(1-\theta)\left(\frac{\alpha\theta}{n}\right)^{\frac{\theta+\delta}{1-(\theta+\delta)}}\left[A(1-vn)^{1-\theta}\right]^{\frac{1}{1-(\theta+\delta)}}.$$

We can easily observe that if n_n >0 then fertility n is positive in (3.19). However, since the log utility function excludes corner solutions for fertility, the presence of non-convexity in the form of $b_{t+1}n_t$ in the budget constraint (3.2) may lead to a situation in which there is no solution for fertility for some parameter values. As shown in Zhang et al. (2001) and Zhang (1995), the sufficient condition for the solution to be optimal is a sufficiently large taste parameter for the number of children (η) such that an interior solution for fertility exists. In order to obtain positive fertility in (3.19), we assume $\eta > \underline{\eta} \equiv [\alpha\theta - \beta(a_0-a_1)(1-\theta-\alpha\theta)]/(1-\alpha\theta)$. Further, we assume a strong enough taste for the welfare of children (α) such that bequests are positive: $\alpha > [\beta(a_0-a_1)(1-\theta)]/\left\{\theta\left[1+\beta(a_0-a_1)\right]\right\}$.[21] We now investigate how fertility, capital per worker and output per worker respond to rises in tax rates for unfunded social security and public health:[22]

[21] Kotlikoff and Summers (1981) indicate that bequests are important elements in accounting for capital accumulation.

[22] $\eta \to \alpha^-$ denotes η approaches α from below.

Proposition 3.1. *If $\alpha \geq \eta$, then a rise in the tax rate for unfunded social security, τ^T, reduces fertility, raises capital per worker, and raises output per worker. As $\eta \to \alpha^-$, then a rise in the tax rate for public health, τ^M, reduces fertility, raises capital per worker, and raises output per worker. All those effects of a rise in τ^M are also true when $\alpha = \eta$.*

Proof. The proof is relegated to Appendix C.

A rise in the tax rate for social security has opposing effects on fertility. On the one hand, by increasing the bequest cost of having a child, the tax rise tends to reduce fertility. On the other hand, by reducing the after tax wage rate, the opportunity cost of spending time rearing a child falls and therefore the tax rise tends to increase fertility. Moreover, the forgone social security benefit of spending time rearing a child rises with the tax rate for social security via the linkage between the replacement rate and the tax rate for social security under a balanced social security budget. In this way, it channels a negative effect of a rise in the tax rate on fertility. Thus, there are opposing effects of a rise in the tax rate for social security on fertility. The net effect of social security on fertility will depend on the taste for the number, relative to the welfare, of children. When the taste for the welfare of children, α, is not weaker than the taste for the number of children, η, i.e., when $\alpha \geq \eta$, the cost component of a child in terms of forgone social security benefits in equation (3.15) become larger and hence it is more likely that a rise in the tax rate for social security reduces fertility and leads to a rise in both capital and output per worker.

When the tax rate for public health increases, the time cost of spending time rearing a child falls and leads to higher fertility. However, when the tax rate for public health increases, the provision of public health per worker increases and hence life expectancy increases, given the fertility level. With higher life expectancy, agents receive lower returns on retirement savings and social security benefits and as a consequence, workers that expect to live longer in their old age would save more and leave less as bequests to children as a fraction of income. On the one hand, without any change in fertility, higher life expectancy reduces per child bequests as a fraction of income, and therefore, tends to increase fertility. However, when bequests per child fall, children's middle-age consumption falls as a consequence and hence the marginal cost of a child rises and fertility may fall to offset higher marginal costs of a child in equation (3.15). The net effect of a higher tax rate for public health on fertility therefore depends on the relative strength of the taste for the welfare and number of children. When the taste for the number of children, η, is weaker but sufficiently close or equal to that for the welfare of children, α, i.e., when $\eta \to \alpha^{-}$ or $\eta = \alpha$, a rise in the tax rate for public health reduces fertility and hence raises both the capital and output per worker.

Let us now investigate the effects of tax rates for social security and public health on the provision of public health per worker, life expectancy, the ratio of middle-age consumption to income and the ratio of old-age consumption to income.

Proposition 3.2. *If $\alpha \geq \eta$, then a rise in the tax rate for unfunded social security, τ^T, raises public health spending per worker, raises life expectancy, reduces the ratio of* *middle-age consumption to income, and reduces the ratio of old-age consumption to income. If $\eta \to \alpha^-$, a rise in the tax rate for public health, τ^M, raises public health spending per worker, raises life expectancy, reduces the ratio of middle-age consumption to income, and reduces the ratio of old-age consumption to income. All those effects of a rise in τ^M are also true when $\alpha = \eta$.*

Proof. The proof is relegated to Appendix C.

In the conventional dynastic model without health spending, social security is neutral with regard to consumption pattern over life stages via saving, which is well known as the Ricardian equivalence hypothesis (Barro, 1974; Zhang, 1995). When public health is present in our model, however, social security increases public health spending per worker (and hence life expectancy as well), if the taste for the welfare of children is not weaker than the taste for the number of children. The effect of social security on public health spending per worker, and hence life expectancy, works through the effect of social security on fertility. As shown in Proposition 3.1, when the taste for the welfare of children, α, is not weaker than the taste for the number of children, η, a rise in the tax rate for social security reduces fertility, and hence increases public health spending per worker. With higher public health spending per worker, life expectancy increases, which leads to lower ratios of middle-age and old-age consumption to income according to equations (3.17) and (3.20).

There are both direct and indirect effects of a rise in the tax rate for public health on public health spending per worker. The direct effect is that when the tax rate for public health increases, so does public health spending per worker increases, given any fertility level. The indirect effect of a rise in the tax rate for public health on public health spending per worker works through its effect on fertility. As shown in Proposition 3.1, when the taste for the number of children is weaker but sufficiently close or equal to that for the welfare of children, i.e., when $\eta \to \alpha^{-}$ or $\eta = \alpha$, a rise in the tax rate for public health reduces fertility, and hence, increases public health spending per worker. Since both the direct and indirect effects of a rise in the tax rate for public health increase public health spending per worker, life expectancy increases. As a consequence, a rise in the tax rate for public health leads to lower ratio of both middle-age consumption to income and old-age consumption to income according to equations (3.17) and (3.20).

Next, we turn to the impact of rises in tax rates for social security and public health on the fractions of middle-age earnings spent on savings and bequests.

Proposition 3.3. *A rise in the tax rate for unfunded social security, τ^{T}, has no effect on the fraction of middle-age earnings spent on the sum of savings and bequests $(s+\gamma_b)$. Similarly, a rise in the tax rate for public health, τ^{M}, also has no effect on the fraction of middle-age earnings spent on the sum of savings and bequests $(s+\gamma_b)$.*

Proof. The proof is relegated to Appendix C.

A rise in the tax rate for social security has the following effects on bequests and savings: a higher tax rate for social security increases the burden of children in paying higher social security contributions and hence, altruistic parents leave more bequests to children as in Barro (1974) and Zhang (1995). At the same time, parents expect to receive higher social security benefits and therefore, they tend to save less such that the fraction of middle-age earnings spent on the sum of savings and bequests $(s+\gamma_b)$ is unaffected by social security. On the other hand, when the tax rate for public health increases, life expectancy increases, and hence, agents tend to save more for longer life in old age and leave less bequests to children. By doing so, the fraction of middle-age earnings spent on the sum of savings and bequests $(s+\gamma_b)$ is also unaffected by the tax rate for public health.

Since social security and public health can increase capital per worker and output per worker as indicated in Proposition 3.1, it is intuitive to state the following proposition:

Proposition 3.4. *If $\eta \to \alpha^-$, the total increase in capital per worker (output per worker) due to increases in tax rates for unfunded social security, τ^T, and public health, τ^M, is higher than the total increase in capital per worker (output per worker) due to an increase in only one of these two tax rates. The above results are also true when $\alpha = \eta$.*

Proof. The proof is relegated to Appendix C.

Proposition 3.1 implies that a tax rise for social security or public health increases capital per worker and output per worker if the taste for the number of children, η, is weaker but sufficiently close or equal to that for the welfare of children, α, i.e., if $\eta \to \alpha^-$ or $\alpha = \eta$. Then it is obvious that increases in tax rates for both social security and public health generate a larger increase in capital per worker and output per worker than an increase in only one tax rate.

It is also interesting to compare the effects of a tax rise for social security on fertility, capital per worker and output per worker with those of a tax rise for public health on fertility, capital per worker and output per worker to explore which of the two tax policies, if used separately, is more effective in reducing fertility and raising both capital and output per worker. Proposition 3.5 summarizes the results:

Proposition 3.5. *If $\eta \to \alpha^-$, then the rate of decrease in fertility due to an increase in a tax rate for social security, τ^T, is larger than that due to an increase in a tax rate for public health, τ^M. At the same time, the rate of increase in capital per worker and output per worker due to an increase in a tax rate for unfunded social security, τ^T, is higher than the rate of increase in capital per worker and output per worker due to an increase in a tax rate for public health, τ^M. The above results are also true when $\alpha = \eta$.*

Proof. The proof is relegated to Appendix C.

According to Proposition 3.5, if the taste for the number of children, η, is weaker but sufficiently close or equal to that for the welfare of children, α, i.e., if $\eta \to \alpha^-$ or $\alpha = \eta$, then the tax rate for social security has stronger negative effects on fertility, and hence, it has stronger positive effects on both capital and output per worker than the tax rate for public health. This implies that a rise in the tax rate for social security may be more effective in reducing fertility and increasing both capital and output per worker than that for public health. The intuition is that a tax rise for social security imposes an additional cost component of a child in terms of forgone social security benefits of spending time rearing a child in equation (3.15), compared to a tax rise for public health. Therefore, social security exerts larger effects on fertility, capital per worker and output per worker than public health. The task next is to investigate how social security and public health affect welfare numerically with endogenous life expectancy and fertility.

3.4. Welfare implications through simulations

Due to the complexity of tracking down the full dynamic path for a complete welfare analysis in this complicated model, we only focus on the steady state for the welfare analysis. Such a steady-state welfare analysis yields results corresponding to what is coined as the "modified golden rule of capital accumulation" in the conventional neoclassical growth model. At the steady state, the welfare level V in (3.1) is given as follows:

$$\begin{aligned} V_{SS} = \{&(1+p\beta)\ln\gamma_c + (1+p\beta)\ln Y + \\ &(\eta + p\beta)\ln n + \ln(1-\theta) + p\beta\ln[\beta(1-\theta)/\alpha]\}/(1-\alpha) \end{aligned} \tag{3.21}$$

where γ_c, Y, p, n are at their respective steady state levels and are functions of τ^T and τ^M:

$$\gamma_c = \frac{\alpha[\theta(1-\alpha)+(1-\tau^M)(1-\theta)]}{(1-\theta)(\alpha+\beta p)}$$

$$p = a_0 - a_1 / e^{a_2 M}$$

$$M=\tau^{M}(1-\theta)\left(\frac{\alpha\theta}{n}\right)^{\frac{\theta+\delta}{1-(\theta+\delta)}}\left[A(1-vn)^{1-\theta}\right]^{\frac{1}{1-(\theta+\delta)}}$$

$$Y=A^{\frac{1}{1-(\theta+\delta)}}(\alpha\theta)^{\frac{\theta+\delta}{1-(\theta+\delta)}}n^{\frac{-(\theta+\delta)}{1-(\theta+\delta)}}(1-vn)^{\frac{1-\theta}{1-(\theta+\delta)}} \tag{3.22}$$

$$n=\frac{\tilde{n}_n}{\tilde{n}_d} \qquad (3.23)^{23}$$

where

$$\begin{aligned}\tilde{n}_n &= \mathrm{e}^{a_2M}\,\alpha\left\{\eta\left[\theta(1-\alpha)+(1-\tau^M)(1-\theta)\right]-\alpha\left[\theta+(1-\theta)\tau^T\right]\right\}+\\ &\quad \alpha\beta\left[(1-\theta)(1-\tau^T-\tau^M)-\alpha\theta\right]\left[\mathrm{e}^{a_2M}\,a_0-a_1\right],\\ \tilde{n}_d &= v\left\{\mathrm{e}^{a_2M}\,\alpha\left[(1-\theta)(1-\tau^T-\tau^M)-\alpha\theta+\eta\left(\theta(1-\alpha)+(1-\tau^M)(1-\theta)\right)\right]+\right.\\ &\quad \left.\beta\left[(1-\theta)(1-\tau^T-\tau^M)+\alpha\left(-\alpha\theta+(1-\tau^M)(1-\theta)\right)\right]\left[\mathrm{e}^{a_2M}\,a_0-a_1\right]\right\}.\end{aligned}$$

We now investigate the optimal tax rates of social security and public health. We first differentiate the welfare function in (3.21) with respect to the tax rate for social security or public health and obtain the following first-order conditions:

$$\begin{aligned}\frac{\partial V_{ss}}{\partial\tau^T}=0 \quad &\Leftrightarrow \quad (1+p\beta)\left[\frac{1}{\gamma_c}\frac{\partial\gamma_c}{\partial n}+\frac{1}{Y}\frac{\partial Y}{\partial n}\right]+\\ &\frac{\partial p}{\partial n}\beta\left[\ln\left(\frac{\beta(1-\theta)}{\alpha}\right)+\ln\gamma_c+\ln n+\ln Y\right]+\frac{1}{n}\left(\eta+p\beta\right)=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial V_{ss}}{\partial\tau^M}=0 \quad &\Leftrightarrow \quad \left[\frac{1+p\beta}{\theta(1-\alpha)+(1-\tau^M)(1-\theta)}\right]+\\ &\frac{\partial p}{\partial M}\beta Y\left[\left(\frac{1+p\beta}{\alpha+p\beta}\right)-\ln\left(\frac{\beta(1-\theta)}{\alpha}\right)-\ln\gamma_c-\ln n-\ln Y\right]=0.\end{aligned}$$

---

[23] By substituting $p(\tau^T,\tau^M)=a_0-a_1/e^{a_2M(\tau^T,\tau^M)}$ into the equation for fertility in (3.19), we obtain $n=\tilde{n}_n/\tilde{n}_d$.

The above first-order conditions implicitly determine the optimal tax rates of social security and public health in this complicated model. We next explore the implications of welfare in equation (3.21) using a numerical approach.

The values of parameters are either in line with those in the literature if any (e.g., $\alpha = 0.65$, $\theta = 0.25$), or they are chosen to yield plausible values for fertility and the survival probability to old-age (e.g. $v = 0.1$, $\beta = 0.5$, $\eta = 0.5$, $a_0 = 0.95$, $a_1 = 0.45$, $a_2 = 0.9$, and $A = 25$). Also, we set a low value for δ at 0.01 that can generate realistic values for the tax rates. We later will examine whether the existence of positive investment externalities is essential for social security or public health to improve welfare by setting δ at a zero level.

The numerical results show that the optimal tax rates for social security and public health are $(\tau^T, \tau^M) = (0.21, 0.09)$ as shown in Case 1 in Table 3.1 under the condition $\eta \to \alpha^-$. Given the parameterization, we can compare the implications of a tax rise in social security or public health on steady-state fertility, the ratio of middle-age consumption to income, the ratio of old-age consumption to income, the provision of public health per worker, life expectancy, capital per worker, output per worker and the welfare level for cases with or without social security or public health in Table 3.1. Case 2 shows the numerical results when both social security and public health are absent. Case 3 investigates the effect of social security when public health is absent and Case 4 investigates the effect of public health when social security is absent. Finally, we investigate Case 5 in which both social security and public health are present.

Table 3.1 Simulation results with the condition $\eta \to \alpha^-$

Parameterization:

$a_0 = 0.95, a_1 = 0.45, a_2 = 0.9, \alpha = 0.65, \eta = 0.5, \theta = 0.25, \beta = 0.5, \delta = 0.01, A = 25, v = 0.1$

	n	γ_c	γ_d	M	p	K	V
1. Optimal rates							
$\tau^T = 0.21$,							
$\tau^M = 0.09$	**1.964**	**0.623**	**0.312**	**1.668**	**0.85**	**2.139**	**11.611**
2. $\tau^T = \tau^M = 0$	2.796	0.806	0.403	0	0.5	1.188	10.998
3. $\tau^T > 0, \tau^M = 0$							
$\tau^T = 0.1$	2.509	0.806	0.403	0	0.5	1.431	11.043
$\tau^T = 0.3$	1.775	0.806	0.403	0	0.5	2.511	11.074
$\tau^T = 0.4$	1.297	0.806	0.403	0	0.5	4.063	11
4. $\tau^M > 0, \tau^T = 0$							
$\tau^M = 0.05$	2.754	0.684	0.342	0.775	0.726	1.22	11.425
$\tau^M = 0.1$	2.707	0.617	0.309	1.57	0.841	1.257	11.522
$\tau^M = 0.15$	2.653	0.572	0.286	2.39	0.898	1.301	11.455
5. $\left(\tau^T = \tau^M\right) > 0$							
$\tau^T = \tau^M = 0.05$	2.606	0.682	0.341	0.807	0.732	1.342	11.46
$\tau^T = \tau^M = 0.1$	2.372	0.613	0.307	1.721	0.854	1.572	11.58
$\tau^T = \tau^M = 0.2$	1.703	0.532	0.266	4.212	0.94	2.679	11.271

According to Proposition 3.1, the effects of a rise in the tax rate for social security or public health on fertility, capital per worker and output per worker depend on the relative strength between the taste for the welfare of children and that for the number of children. Table 3.1 illustrates that the rise in tax rates for social security or public health reduces fertility and raises capital per worker when the taste for the number of children is weaker but sufficiently close to the taste for the welfare of children i.e., when $\eta \to \alpha^-$.

Case 3 in Table 3.1 shows that when public health is absent, a rise in the tax rate for social security has no effect on the provision of public health per worker, life expectancy, the ratio of middle-age consumption to income and the ratio of old-age consumption to income. However, by comparing Case 3, Case 4 and Case 5 in Table 3.1, a rise in the tax rate for social security raises the provision of public health per worker and life expectancy but reduces the ratio of middle-age consumption to income and the ratio of old-age consumption to income when public health is present. Case 4 in Table 3.1 also shows that a rise in the tax rate for public health raises the provision of public health per worker and life expectancy but reduces the ratio of middle-age consumption to income and the ratio of old-age consumption to income. These results are consistent with Proposition 3.2.

Comparisons between Case 2, Case 3, Case 4 and Case 5 in Table 3.1 reflect Proposition 3.4 and Proposition 3.5. As shown in Table 3.1 for instance, when $\tau^T = \tau^M = 0$, capital per worker is 1.188. When $\tau^T = \tau^M = 0.1$, capital per worker increases to 1.572. Capital per worker at 1.572 is obviously higher than that at 1.431 when $\tau^T = 0.1$ and $\tau^M = 0$ or at 1.257 when $\tau^M = 0.1$ and $\tau^T = 0$. These results show that the increases in capital per worker and output per worker due to increases in both tax rates, τ^T and τ^M, are higher than the increases in capital per worker and output per worker due to an increase in only one of these two tax rates, and hence these results are consistent with Proposition 3.4.

By comparing Case 2, Case 3 and Case 4 in Table 3.1 for instance, it is also obvious that the rate of decrease in fertility by 2.87 when the tax rate for social security increases from $\tau^T = 0$ to $\tau^T = 0.1$ is larger than the rate of decrease in fertility by 0.89 when the tax rate for public health increases from $\tau^M = 0$ to $\tau^M = 0.1$, and as a consequence, the rate of increase in capital per worker or output per worker due to an increase in the tax rate for social security is higher than that due to the same amount of increase in the tax rate for public health. These results therefore reflect Proposition 3.5.

The simulation results also indicate that social security or public health can increase welfare by reducing fertility and raising capital per worker when the taste for the number of children is weaker but sufficiently close or equal to the taste for the welfare of children. When both social security and public health are absent as in Case 2, the welfare level is 10.998 in Table 3.1. By scaling up social security or public health, welfare increases and reaches the optimal level at 11.611 when $\tau^T = 0.21$ and $\tau^M = 0.09$ in Table 3.1. This implies that it is more efficient when both social security and public health are implemented together rather than separately. The optimal per worker public expenditure on health, M, at 1.668 in Table 3.1 is about 6% of the corresponding optimal output per worker at 25.856. The optimal per worker public expenditure on health at 6% of output per worker and the optimal tax rate for social security at 21% are close to the observed rates in industrial nations in which per capita public expenditure on health as a percentage of income per capita attains as high as around 8% (see World Health Statistics (2009)) and payroll tax rates for social security ranging from 10% to 20% or higher (see Social Security Administration and International Social Security Association (2006, 2008)).

In Table 3.2 we examine whether the simulation results concerning the optimal tax rates for social security and public health are sensitive to variations in the parameters ($a_0, a_1, a_2, \alpha, \eta, \theta, \beta, \delta, A, v$) and to the existence of investment externalities by varying δ from positive values to zero.[24] In doing so, we consider variations in one parameter at a time, starting from the parameterization in Table 3.1. First, a higher value of the taste for the welfare of children (α) yields a lower optimal tax rate of social security and the magnitudes of the changes in the optimal tax rate are large. This is because the more parents value their children's welfare than the number of children, the smaller the efficiency loss of the investment externalities and therefore the lower the optimal social security. Second, a larger share parameter of capital (θ) leads to a higher optimal tax rate of social security and the magnitudes of the changes in the optimal tax rate are large as well. The reason for this result is that this share parameter measures the role of physical capital investment in the accumulation of physical capital. That is, with a larger share parameter θ, physical capital investment becomes more important in the production of output and therefore the efficiency loss of the physical capital externality is larger for a given degree of investment externality (δ). Third, a larger degree of investment externality also requires a higher optimal tax rate of social security due to a larger efficiency loss of the externality. Notice that the optimal tax rate of public health is insensitive to the variations in α, θ or δ. This is because the optimal tax rate of public health depends mainly on how it affects the provision of public health per worker and life expectancy which are less relevant for physical capital investment, which channels the efficiency loss of the investment externality.

[24] The taste for the number of children, η, and the taste for the welfare of children, α, may change overtime due to cultural changes, government policies associated with children, increases in women's education attainment and labor participation rates.

By contrast, variations in the other parameters produce relatively little changes in the optimal tax rate of social security and public health in Table 3.2. This is because these parameters are either less relevant for physical capital investment, which channels the efficiency loss of the human capital externality, than (α, θ, δ), or less relevant for the provision of public health per worker and life expectancy.

It is worth mentioning that when the investment externality is absent (δ=0), the optimal tax rates for social security and public health are still positive. This is because when individuals value old-age consumptions and social security benefits received in old-age, there are still potential roles taken up by tax rates of social security and public health in improving the provision of public health per worker and life expectancy.

Table 3.2 Simulated optimal tax rates: sensitivity analysis

Parameter	τ^T	τ^M
Varying α		
α =0.6	0.26	0.09
α =0.7	0.17	0.09
Varying δ		
δ =0	0.18	0.09
δ =0.02	0.24	0.08
Varying η		
η =0.45	0.19	0.08
η =0.55	0.23	0.09
Varying θ		
θ =0.2	0.16	0.09
θ =0.3	0.26	0.08
Varying β		
β =0.45	0.22	0.08
β =0.55	0.2	0.09
Varying a_0		
a_0=0.9	0.21	0.09
a_0=1	0.21	0.09
Varying a_1		
a_1=0.4	0.21	0.08
a_1=0.5	0.21	0.09
Varying a_2		
a_2=0.85	0.21	0.09
a_2=0.95	0.2	0.09
Varying A		
A=20	0.21	0.09
A=30	0.2	0.09
Varying v		
v =0.05	0.21	0.11
v =0.15	0.2	0.08

3.5. Conclusion

In this paper we have examined the implications of PAYG social security and public health for fertility, life expectancy, capital per worker, output per worker and welfare in a dynastic model with altruistic bequest and endogenous fertility. We have shown analytically that if the taste for the welfare of children is not weaker than that for the number of children, scaling up social security reduces fertility, but raises capital per worker, output per worker, public health spending per worker and life expectancy. We have also shown analytically that if the taste for the number of children is weaker but sufficiently close or equal to that for the welfare of children, scaling up public health reduces fertility, but raises capital per worker, output per worker, public health spending per worker and life expectancy. A comparison of tax policies between social security and public health shows that social security may be more effective than public health in reducing fertility and raising both capital and output per worker when a tax rise for social security imposes an additional cost component of a child in terms of forgone social security benefits of spending time rearing a child compared to a tax rise for public health. Our simulation results reported in Tables 3.1 illustrate that scaling up social security or public health improves welfare by reducing fertility and raising capital intensity. Though social security and public health can be used separately to increase welfare, our simulation results show that the optimal welfare is reached when both social security and public health are implemented together. Our model can generate the optimal tax rate of social security at 21% and per worker public expenditure on health at 6% of output per worker at the same time. These optimal rates obtained jointly in this model are close to the observed rates for social security and per capita public expenditure on health as a percentage of per capita output in industrial nations.

The combination of such important factors as altruistic intergenerational transfers, and endogenous life expectancy and fertility has not been used together in exploring the implications of PAYG social security and public health for fertility, life expectancy, capital per worker, output per worker and welfare, to the best of our knowledge. Our results may have useful policy implications. Adopting both PAYG social security and public health may be appropriate for economies with high fertility, low life expectancy and low levels of capital per worker, output per worker and welfare. Our results also help to explain the popularity of PAYG social security and public health in developed economies. However, we recognize that private investment in both human capital and health may be relevant in exploring the welfare implication of social security when life expectancy and fertility are endogenous. This invites further research in this area.

Appendix C

Proof of Proposition 3.1. First, we substitute $p(\tau^T,\tau^M)=a_0-a_1/e^{a_2M(\tau^T,\tau^M)}$ into the equation for fertility in (3.19) to obtain $n=\tilde{n}_n/\tilde{n}_d$ as given in equation (3.23). We then differentiate $n=\tilde{n}_n/\tilde{n}_d$ in (3.23) with respect to τ^T and obtain

$$\frac{\partial n}{\partial \tau^T}=\frac{v\alpha(1-\theta)\left\{-\Lambda_1\left[\Gamma_1+\mathrm{e}^{a_2M}\left(\alpha-\eta\right)\right]+\Gamma_2a_2\left(\partial M/\partial\tau^T\right)\left(\alpha-\eta\right)\right\}}{(\tilde{n}_d)^2} \tag{3.24}$$

where

$$\Lambda_1=\left[1-\tau^M(1-\theta)-\alpha\theta\right]\left[\mathrm{e}^{a_2M}\,\alpha+\beta\left(\mathrm{e}^{a_2M}\,a_0-a_1\right)\right]>0\,,$$

$$\Gamma_1=\alpha\left[\mathrm{e}^{a_2M}\,\eta+\beta\left(\mathrm{e}^{a_2M}\,a_0-a_1\right)\right]>0\,,$$

$$\Gamma_2=a_1\beta\,\mathrm{e}^{a_2M}\left[1-\tau^T(1-\alpha)-\tau^M\right]\left[1-\tau^M(1-\theta)-\alpha\theta\right]>0 \text{ if } \alpha\geq\eta\text{, and}$$

$$\frac{\partial M}{\partial\tau^T}=-\frac{\tau^M\Pi_1\Pi_2}{a_2}\frac{\partial n}{\partial\tau^T}\,.$$

Note that $\Gamma_2>0$ if $\left[1-\tau^T(1-\alpha)-\tau^M\right]>0$ which is true if $\alpha\geq\eta$. Using the transformed budget constraint in equation (3.11), $\gamma_c=\gamma_b\theta/\left[(1-\theta)(s+\gamma_b)\right]+(1-\tau^T-\tau^M-s-\gamma_b)$, and equation (3.13), $(s+\gamma_b)=\alpha\theta/(1-\theta)$, we obtain $(\alpha\gamma_c-\gamma_b)/\alpha=1-\tau^T-\tau^M-\left[\alpha\theta/(1-\theta)\right]$. In addition, with positive fertility, the fertility equation in (3.15) implies $\eta\gamma_c>\gamma_b$. Thus, if $\alpha\geq\eta$, $(\alpha\gamma_c-\gamma_b)/\alpha=1-\tau^T-\tau^M-\left[\alpha\theta/(1-\theta)\right]>0$ and $\left[\alpha\theta-(1-\theta)(1-\tau^T-\tau^M)\right]<0$. The condition $\left[\alpha\theta-(1-\theta)(1-\tau^T-\tau^M)\right]<0$ implies $(1-\tau^T-\tau^M)>0$ which leads to $\left[1-\tau^T(1-\alpha)-\tau^M\right]>0$ and thus, $\Gamma_2>0$.

By substituting $\partial M / \partial \tau^{T}$ into equation (3.24) and after rearranging equation (3.24), we obtain

$$\frac{\partial n}{\partial \tau^{T}} = \frac{-v\alpha(1-\theta)\Lambda_1\left[\Gamma_1 + \mathrm{e}^{a_2 M}\left(\alpha-\eta\right)\right]}{(\tilde{n}_d)^2 + v\alpha(1-\theta)\tau^{M}\Pi_1\Pi_2\Gamma_2(\alpha-\eta)} \tag{3.25}$$

where

$$\Pi_1 = a_2(1-\theta)(\alpha\theta)^{\frac{(\theta+\delta)}{1-(\theta+\delta)}} A^{\frac{1}{1-(\theta+\delta)}} > 0\,,$$

$$\Pi_2 = \left[\left(\frac{\theta+\delta}{1-(\theta+\delta)}\right) n^{\frac{-1}{1-(\theta+\delta)}}(1-vn)^{\frac{1-\theta}{1-(\theta+\delta)}} + \left(\frac{1-\theta}{1-(\theta+\delta)}\right) n^{\frac{-(\theta+\delta)}{1-(\theta+\delta)}}(1-vn)^{\frac{\delta}{1-(\theta+\delta)}} v\right] > 0\,.$$

Therefore, if $\alpha \geq \eta$, then $\partial n / \partial \tau^{T}$ <0 in equation (3.25). By equations (3.5) and (3.7), $K = [A(1-vn)^{1-\theta}\alpha\theta / n]^{1/[1-(\theta+\delta)]}$ in the steady state, and hence if $\alpha \geq \eta$, then $\partial K / \partial \tau^{T} = (\partial K / \partial n)(\partial n / \partial \tau^{T}) > 0$. By equation (3.4), $Y = AK^{\theta+\delta}(1-vn)^{1-\theta}$ in the steady state, and with $\partial K / \partial \tau^{T} > 0$ and $\partial n / \partial \tau^{T} < 0$ if $\alpha \geq \eta$, we obtain $\partial Y / \partial \tau^{T} = (\partial Y / \partial K)(\partial K / \partial \tau^{T}) + (\partial Y / \partial n)(\partial n / \partial \tau^{T}) > 0$.

Similarly, by differentiating $n = \tilde{n}_n / \tilde{n}_d$ in (3.23) with respect to τ^{M}, we obtain

$$\frac{\partial n}{\partial \tau^{M}} = \frac{v\alpha(1-\theta)\left\{-\Lambda_2\left[\Gamma_1 + e^{a_2 M}(\alpha-\eta)\right] + \Gamma_2\, a_2\left(\partial M / \partial \tau^{M}\right)\left(\alpha-\eta\right)\right\}}{(\tilde{n}_d)^2} \tag{3.26}$$

where

$$\Lambda_2 = \left[\theta + (1-\theta)\tau^{T}\right]\left[\mathrm{e}^{a_2 M}\,\alpha + \beta\left(\mathrm{e}^{a_2 M}\, a_0 - a_1\right)\right] > 0$$

$$\frac{\partial M}{\partial \tau^M} = \frac{\Pi_1}{a_2}\left[n^{\frac{-(\theta+\delta)}{1-(\theta+\delta)}}(1-vn)^{\frac{1-\theta}{1-(\theta+\delta)}} - \tau^M \frac{\partial n}{\partial \tau^M}\Pi_2 \right].$$

By substituting $\partial M / \partial \tau^M$ into equation (3.26) and after rearranging equation (3.26), we obtain

$$\frac{\partial n}{\partial \tau^M} = \frac{v\alpha(1-\theta)\left\{ -\Lambda_2\left[\Gamma_1 + e^{a_2 M}(\alpha-\eta)\right] + \Gamma_2\Pi_1 n^{\frac{-(\theta+\delta)}{1-(\theta+\delta)}}(1-vn)^{\frac{1-\theta}{1-(\theta+\delta)}}(\alpha-\eta) \right\}}{(\tilde{n}_d)^2 + v\alpha(1-\theta)\tau^M \Pi_1\Pi_2\Gamma_2(\alpha-\eta)} \quad (3.27)$$

Note that if $\eta \to \alpha^-$, then the numerator of $\partial n / \partial \tau^M$ is negative but the denominator of $\partial n / \partial \tau^M$ is positive in equation (3.27). Therefore, if $\eta \to \alpha^-$, then $\partial n / \partial \tau^M$ <0 in equation (3.27). As stated earlier, $K = [A(1-vn)^{1-\theta}\alpha\theta / n]^{1/[1-(\theta+\delta)]}$ and $Y = AK^{\theta+\delta}(1-vn)^{1-\theta}$ in the steady state. If $\eta \to \alpha^-$, we therefore obtain $\partial K / \partial \tau^M = (\partial K / \partial n)(\partial n / \partial \tau^M) > 0$, and

$\partial Y / \partial \tau^M = (\partial Y / \partial K)(\partial K / \partial \tau^M) + (\partial Y / \partial n)(\partial n / \partial \tau^M) > 0$. It is obvious that the all the above results are also true for $\alpha = \eta$. □

Proof of Proposition 3.2. Recall that

$$M = \tau^M (1-\theta)\left(\frac{\alpha\theta}{n}\right)^{\frac{\theta+\delta}{1-(\theta+\delta)}}\left[A(1-vn)^{1-\theta}\right]^{\frac{1}{1-(\theta+\delta)}},$$

and $p(\tau^T, \tau^M) = a_0 - a_1 / e^{a_2 M(\tau^T, \tau^M)}$. By differentiating $p(\tau^T, \tau^M)$ with respect to τ^T, we then obtain

$$\frac{\partial p(\tau^T,\tau^M)}{\partial \tau^T} = a_1 a_2 e^{-a_2 M(\tau^T,\tau^M)} \frac{\partial M(\tau^T,\tau^M)}{\partial \tau^T}$$

where

$$\frac{\partial M(\tau^T,\tau^M)}{\partial \tau^T} = \frac{-\tau^M \Pi_1 \Pi_2}{a_2} \frac{\partial n}{\partial \tau^T}.$$

By Proposition 3.1, if $\alpha \geq \eta$, then $\partial n / \partial \tau^T$ <0, and hence, $\partial M / \partial \tau^T > 0$ and $\partial p / \partial \tau^T > 0$. Consequently, by equations (3.17) and (3.20), $\partial \gamma_c / \partial \tau^T < 0$ and $\partial \gamma_d / \partial \tau^T < 0$ when $\alpha \geq \eta$.

Similarly, by differentiating $p(\tau^T,\tau^M) = a_0 - a_1 / e^{a_2 M(\tau^T,\tau^M)}$ with respect to τ^M, we obtain

$$\frac{\partial p(\tau^T,\tau^M)}{\partial \tau^M} = a_1 a_2 e^{-a_2 M(\tau^T,\tau^M)} \frac{\partial M(\tau^T,\tau^M)}{\partial \tau^M}$$

where

$$\frac{\partial M(\tau^T,\tau^M)}{\partial \tau^M} = \frac{\Pi_1}{a_2}\left[n^{\frac{-(\theta+\delta)}{1-(\theta+\delta)}} (1-vn)^{\frac{1-\theta}{1-(\theta+\delta)}} - \tau^M \frac{\partial n}{\partial \tau^M} \Pi_2 \right].$$

By Proposition 3.1, if $\eta \to \alpha^-$ or $\alpha = \eta$, then $\partial n / \partial \tau^M < 0$, and hence, $\partial M / \partial \tau^M > 0$ and $\partial p / \partial \tau^M > 0$. By equations (3.17) and (3.20), we therefore obtain $\partial \gamma_c / \partial \tau^M < 0$ and $\partial \gamma_d / \partial \tau^M < 0$ if $\eta \to \alpha^-$ or $\alpha = \eta$. □

Proof of Proposition 3.3. Using equations (3.5) - (3.8), we can easily obtain

$$(s+\gamma_b) = \frac{\alpha\theta}{1-\theta}.$$

The claims follow through. □

Proof of Proposition 3.4. Proposition 3.1 implies that if $\eta \to \alpha^-$ or $\alpha = \eta$, then $\partial K / \partial \tau^T > 0$ $\partial K / \partial \tau^M > 0$, $\partial Y / \partial \tau^T > 0$ and $\partial Y / \partial \tau^M > 0$. Since $K = [A(1-vn)^{1-\theta} \alpha\theta / n]^{1/[1-(\theta+\delta)]}$ in the steady state, by totally differentiating capital per worker, we obtain

$$dK = \frac{\partial K}{\partial \tau^T} d\tau^T + \frac{\partial K}{\partial \tau^M} d\tau^M$$

which is obviously greater than $dK = (\partial K / \partial \tau^i) d\tau^i, i = T, M$, for $d\tau^i > 0, i = T, M$.

By substituting $K = [A(1-vn)^{1-\theta} \alpha\theta / n]^{1/[1-(\theta+\delta)]}$ into $Y = AK^{\theta+\delta}(1-vn)^{1-\theta}$, we can rewrite output per worker as a function of tax rates for social security and public health: $Y = Y(n(\tau^T, \tau^M))$. Hence, by totally differentiating output per worker, we obtain

$$dY = \frac{\partial Y}{\partial \tau^T} d\tau^T + \frac{\partial Y}{\partial \tau^M} d\tau^M$$

which is obviously greater than $dY = (\partial Y / \partial \tau^i) d\tau^i, i = T, M$, for $d\tau^i > 0, i = T, M$. □

Proof of Proposition 3.5. Proposition 3.1 implies that if $\eta \to \alpha^-$ or $\alpha = \eta$, then $\partial n / \partial \tau^T < 0$ and $\partial n / \partial \tau^M < 0$. If $\eta \to \alpha^-$ or $\alpha = \eta$, the sign for $\left(\partial n / \partial \tau^T - \partial n / \partial \tau^M\right)$ is given as follows:

$$sign\left(\frac{\partial n}{\partial \tau^T} - \frac{\partial n}{\partial \tau^M}\right) =$$
$$\left\{\Gamma_4\left[\Gamma_1 + e^{a_2 M}(\alpha - \eta)\right] - \Gamma_2 \Pi_1 n^{\frac{-(\theta+\delta)}{1-(\theta+\delta)}} (1-vn)^{\frac{1-\theta}{1-(\theta+\delta)}} (\alpha - \eta)\right\} < 0$$

where

$$\Gamma_4 = \left[\mathrm{e}^{a_2 M} \alpha + \beta\left(\mathrm{e}^{a_2 M} a_0 - a_1\right)\right]\left[\alpha\theta - (1-\theta)(1-\tau^T - \tau^M)\right] < 0$$

Note that from the proof of proposition 3.1, $\left[\alpha\theta - (1-\theta)(1-\tau^T - \tau^M)\right] < 0$ if $\alpha \geq \eta$. With $sign\left(\partial n / \partial \tau^T - \partial n / \partial \tau^M\right) < 0$, we have $\partial n / \partial \tau^T < \partial n / \partial \tau^M < 0$. Since $K = [A(1-vn)^{1-\theta} \alpha\theta / n]^{1/[1-(\theta+\delta)]}$ in the steady state, by combining $\partial n / \partial \tau^T < \partial n / \partial \tau^M < 0$ with $\partial K / \partial n < 0$, we thus obtain

$$\frac{\partial K}{\partial \tau^T} = \frac{\partial K}{\partial n}\frac{\partial n}{\partial \tau^T} > \frac{\partial K}{\partial \tau^M} = \frac{\partial K}{\partial n}\frac{\partial n}{\partial \tau^M}.$$

By substituting $K = [A(1-vn)^{1-\theta} \alpha\theta / n]^{1/[1-(\theta+\delta)]}$ into $Y = AK^{\theta+\delta}(1-vn)^{1-\theta}$, we obtain

$$Y = A^{\frac{1}{1-(\theta+\delta)}} (\alpha\theta)^{\frac{\theta+\delta}{1-(\theta+\delta)}} n^{\frac{-(\theta+\delta)}{1-(\theta+\delta)}} (1-vn)^{\frac{1-\theta}{1-(\theta+\delta)}}$$

and obviously, $\partial Y / \partial n < 0$. Therefore, we have

$$\frac{\partial Y}{\partial \tau^T} = \frac{\partial Y}{\partial n}\frac{\partial n}{\partial \tau^T} > \frac{\partial Y}{\partial \tau^M} = \frac{\partial Y}{\partial n}\frac{\partial n}{\partial \tau^M}.\ \square$$

3.6. Reference

Aaron, H., 1985. Social security: problems of maturity. Paper presented at International Institute of Public Finance Congress, Madrid.

Arrow, K.J., 1962. The economic implications of learning by doing. Review of Economic Studies 29, 155–173.

Barro, R.J., 1974. Are government bonds net wealth? Journal of Political Economy 82, 1095-1117.

Barro R.J., 1997. Determinants of economic growth: a cross-country empirical study. Cambridge MA: MIT Press.

Barro, R. J., Sala-i-Martin, X., 1995. Economic Growth, New York: McGraw-Hill.

Becker, G.S., Barro, R.J., 1988. A reformulation of the economic theory of fertility. Quarterly Journal of Economics 103, 1-26.

Bernstein, J., Nadiri, I., 1988. Interindustry R&D spillovers, rate of return, and production in high-tech industries. American Economic Review 78, 429-434.

Bernstein, J., Nadiri, I., 1989. Research and development and intraindustry spillovers: an empirical application of dynamic duality. Review of Economic Studies 56, 249-267.

Chakraborty, S., Das, M., 2005. Mortality, human capital and persistent inequality. Journal of Economic Growth 10, 159-192.

Cooley, T. F., Soares, J., 1999. A positive theory of social security based on reputation. Journal of Political Economy 107, 135-160.

Corneo, G., Marquardt, M., 2000. Public pension, unemployment insurance, and growth. Journal of Public Economics 75, 293-311.

Davies, J., Kuhn, P., 1992. Social security, longevity, and moral hazard. Journal of Public Economics 49, 91–106.

de la Croix, D., Licandro, O., 1999. Life expectancy and endogenous growth. Economics Letters 65, 255-263.

De Long J.B., Summers L.H., 1991. Equipment investment and economic growth. The Quarterly Journal of Economics 106, 445-502.

Diamond, P., 1965. National debt in a neoclassical growth model. American Economic Review 55, 1126–1150.

Diamond, P., Mirrlees, J., 1978. A model of social insurance with variable retirement. Journal of Public Economics 10, 295-336.

Eckstein, Z., Eichenbaum, M., Peled, D., 1985. Uncertain lifetimes and welfare enhancing properties of annuities' markets and social security. Journal of Public Economics 26, 303-326.

Ehrlich, I., and Chuma, H., 1990. A model of the demand for longevity and the value of life extension. Journal of Political Economy 98, 761-782.

Ehrlich, I., Lui, F.T., 1991. Intergenerational trade, longevity, and economic growth. Journal of Political Economy 99, 1029-1060.

Feldstein, M., 1974. Social security, induced retirement and aggregate capital accumulation. Journal of Political Economy 82, 905-926.

Gale, W.G., Scholz, J.K., 1994. Intergenerational transfers and the accumulation of wealth. Journal of Economic Perspectives 8, 145–160.

Gomme, P., Kydland, F. E., Rupert, P., 2001. Home production meets time to build. Journal of Political Economy 109, 1115-1131.

Hu, S., 1978. On the dynamic behavior of the consumer and optimal provision of social security. Review of Economic Studies 45, 437-445.

Hu, S., 1999. Economic growth in the perpetual-youth model: implications of the annuity market and demographics. Journal of Macroeconomics 21, 107-124.

Hubbard, R.G., Judd, K.L., 1987. Social security in individual welfare: precautionary saving, borrowing constraints, and the payroll tax. American Economic Review 77, 630-646.

Jones, C.I., 1995. Time series tests of endogenous growth models. Quarterly Journal of Economics 110, 495–525.

Kaganovich, M. and Zilcha, I., 1999. Education, social security, and growth. Journal of Public Economics 71, 289-309.

Karni, E., Zilcha, I., 1989. Aggregate and distributional effects of fair social security. Journal of Public Economics 40, 37-56.

Kemnitz, A., Wigger, B. U., 2000. Growth and social security. European Journal of Political Economy 16, 673-683.

Kotlikoff, L.J., Summers, L.H., 1981. The role of intergenerational transfers in aggregate capital accumulation. Journal of Political Economy 89, 706-732.

Laitner, J., 1988. Bequests, gifts and social security. Review of Economic Studies 55, 275-299.

Laitner, J., Juster, F.T., 1996. New evidence on altruism: a study of TIAA-CREF retirees. American Economic Review 86, 893-908.

Laitner, J., Ohlsson, H., 2001. Bequest motives: a comparison of Sweden and the United States. Journal of Public Economics 79, 205–236.

Lapan, H.E., Enders, W., 1990. Endogenous fertility, Ricardian equivalence, and the debt management policy. Journal of Public Economics 41, 227-248.

Lee, Ronald, Tuljapurkar, Shripad, 1997. Death and taxes: longer life, consumption, and social security. Demography 34, 67-81.

Leung, M. C., Zhang, J., Zhang, J., 2004. An economic analysis of life expectancy by gender with application to the United States. Journal of Health Economics 23, 737-759.

Lichtenberg, F. R., 2004. Sources of U.S. longevity increase, 1960-2001. The Quarterly Review of Economics and Finance 44, 369-389.

Lucas Jr., R.E., 1993. Making a miracle. Econometrica 61, 251–272.

Nakanishi, Y., 2002. Empirical evidence of inter-industry R&D spillover in Japan. Journal of Economic Research 7, 91–104.

OECD, 2007. Health at a glance 2007: OECD indicators.

Philipson, T.J., Becker, G.S., 1998. Old age longevity and mortality contingent claims. Journal of Political Economy 106, 551-573.

Preston, H. Samuel, 1975. The changing relation between mortality and level of economic development. Population Studies 29, 231-248.

Romer, P.M., 1986. Increasing returns and long-run growth. Journal of Political Economy 94, 1002–1037.

Rosati, F.C., 1996. Social security in a non-altruistic model with uncertainty and endogenous fertility. Journal of Public Economics 60, 283-294

Sala-I-Martin, X., 1996. A positive theory of social security. Journal of Economic Growth 1, 277-304.

Samuelson, P., 1958. An exact consumption–loan model of interest with or without the social contrivance of money. Journal of Political Economy 66, 467–482.

Samuelson, P., 1975. Optimum social security in a life-cycle model. International Economic Review 16, 539-544.

Sánchez-Losada, F., 2000. Growth effects of an unfunded social security system when there is altruism and human capital. Economics Letters 69, 95-99.

Sen, A. (1998). Mortality as an indicator of economic success and failure. The Economic Journal 108, 1-25.

Sheshinski, E., Weiss, Y., 1981. Uncertainty and optimal social security systems. Quarterly Journal of Economics 96, 189-206.

Social Security Administration and International Social Security Association, 2006. *Social Security Programs throughout the World: Europe, 2006*. www.ssa.gov

Social Security Administration and International Social Security Association, 2008. *Social Security Programs throughout the World: The Americas, 2007*. www.ssa.gov

Tang, K.K., Zhang, J., 2007. Health, education, and life cycle savings in the development process. Economic Inquiry 45, 615-630.

Tomes, Nigel., 1981. The family, inheritance, and the intergenerational transmission of inequality. Journal of Political Economy 89, 928-958.

Ulph, D., Hemming, R., 1980. Uncertain lifetime, imperfect insurance markets and the valuation of pension wealth. Review of Economic Studies 47, 587-593.

U.S. Census Bureau. Historical Statistics of the United States: From the Colonial Times to 1970, Washington, DC, 1975.

———. Statistical Abstract of the United States, Washington, DC, 2004 and other years.

World Health Statistics, 2009. World Health Organization.

Yew, S.L., Zhang, J., 2009. Optimal social security in a dynastic model with human capital externalities, fertility and endogenous growth. Journal of Public Economics 93, 605-619.

Zhang, J., 1995. Social security and endogenous growth. Journal of Public Economics 58, 185–213.

Zhang, J., Zhang, J., 2001. Longevity and economic growth in a dynastic family model with an annuity market. Economics Letters 72, 269-277.

Zhang, J., Zhang, J., 2004. How does social security affect economic growth: evidence from cross-country data. Journal of population Economics 17, 473-500.

Zhang, J., Zhang, J., 2007. Optimal social security in a dynastic model with investment externalities and endogenous fertility. Journal of Economic Dynamics & Control 31, 3345-3567.

Zhang, J., Zhang, J., Lee, R., 2001. Mortality decline and long-run economic growth. Journal of Public Economics 80, 485-507.

Zhang, J., Zhang, J., Leung, M.C.M., 2006. Health investment, saving, and public policy. Canadian Journal of Economics 39, 68-93.