JOINT PRICING AND ORDERING DECISIONS FOR PERISHABLE PRODUCTS

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Summary

Increasing adoption of dynamic pricing for perishable products is witnessed in retail and manufacturing industries. In these industries, the integration of pricing and ordering decisions significantly increases the total profit by better matching demand and supply. Hence, this study focuses on joint pricing and ordering decisions for perishable products.

A periodic review inventory problem with dynamic pricing for perishable products is first studied. In any given period, the inventory consists of products of different ages, purchased by different demand classes. Demands for products of different ages are assumed to be dependent on the price of itself and independent to each other. A discrete time dynamic programming model is developed to determine the optimal order quantity for a new product (product of age 1) and the optimal prices for products of different ages which maximize the total profit over a multiple period horizon. Furthermore, it is proven that the expected profit from dynamic pricing is never worse than the expected profit from static pricing.

The study is further extended to consider substitution among products of different ages and the corresponding demand transfers between demand classes. Demands for products of different ages are assumed to be dependent on not only the price of itself but also the prices of substitutable products, i.e., products of “neighboring ages”. The products of neighboring ages are defined by the products that are a period older or younger than the target products. For a product with a two period lifetime, the optimal order quantity and the optimal price for the new product (product of age 1) and the optimal discounted price

for the old product (product of age 2) are obtained. The computational results show that the total profit significantly increases when demand transfers between new and old products are considered. For a product with the lifetime longer than two periods, a heuristic based on the optimal solution for a single period problem is proposed for a multiple period problem.

Finally, this study considers a problem where the product of only one age is sold at each period and the price of the product will increase as the time at which it perishes approaches to. Such problems can be encountered in the airline industry. To maximize the expected revenue, a discrete time dynamic programming model is developed to obtain the optimal prices and the optimal inventory allocations for the product with a two period lifetime. Three heuristics are then proposed when the lifetime is longer than two periods. The computational results show that the expected revenues from the proposed heuristics are very close to the maximum expected revenue from the dynamic programming model. An upper bound for the maximum expected revenue is computed and the difference between the upper bound and the maximum expected revenue decreases as the initial inventory increases. Furthermore, the study is extended to consider two other cases where the price for the product first increases and later decreases and where the price for the product always decreases and obtains the pricing and inventory allocation decisions.

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M	Lifetime of a perishable product
i	Age of the perishable product, $i = 1,\ldots, M$
y	Order quantity for a new product
x_i	Inventory level for a product of age i, $i = 1,\ldots, M$
S_i	Inventory allocation for a product of age i, $i = 1,\ldots, M$
p_1	Retail price of a new product
p_i	Discounted price for a product of age i, $i = 2,\ldots, M$
π_i	Penalty cost for a product of age i, $i = 1,\ldots, M$
h	Holding cost per period (regardless of ages)
c	Purchasing cost for a new product
α	Discounted factor per period
N	The number of studying periods
k	Index for the period, $k = 1,\ldots, N$
ε	Random variable
$f(\varepsilon)$	Probability density function of the random variable
$E[.]$	Expectation operator

Chapter 1 Introduction

1.1 Background

Inventory is spread throughout the supply chain from raw materials to semi-finished and final products that suppliers, manufacturers, distributors and retailers hold (Chopra and Meindl, 2004). The scale of all these inventory related operations is immense: In 2004, the total value of inventories in the United States exceeds 1.4 trillion dollars (Wilson, 2004).

Implementation of a good inventory management policy is highly effective in reducing the inventory costs. For example, inventory carrying cost as a percentage of Gross Domestic Product (GDP) declined by 50 percent over the last twenty years, since the United States Business logistics system became proficient in inventory management (Wilson, 2004). In next section, a brief introduction to inventory management is presented, including its history and its new trend.

1.1.1 Inventory management

Inventory theory began with the derivation of the Economic Order Quantity (EOQ) formula by Harris (1913). However, it was probably that the works of Arrow et al. (1951)

and Dvoretsky et al. (1952a,b) laid the foundation for later development in the mathematical inventory models.

During the 1950s, a large number of researchers turned their attention to mathematical inventory models. Bellman et al. (1955) showed how the methods of dynamic programming could be used to obtain structural properties for a stochastic inventory problem. Wagner and Whitin (1958) solved the dynamic lot sizing problem under time varying demand. A collection of highly sophisticated mathematical inventory models was found in the book edited by Arrow et al. (1958).

Most of the researchers during the 1950s considered a single storable product. That is, a product once in stock remains unchanged and fully usable for satisfying future demand. However, certain products may perish in storage so that they may become partially or entirely unfit for consumption. For example, fresh produce, meats and other stuffs become unusable after a certain time has elapsed. These products are perishable products, which have a limited useful lifetime.

Since 1960s, several researchers considered the stochastic inventory problem for perishable products. When the lifetime of perishable product is exactly one period, the ordering decisions in successive periods are independent and the problem reduces to a sequence of newsvendor problems. The newsvendor model is a crucial building block of stochastic inventory theory, where the decision maker facing stochastic demand for the perishable product that expires at the end of a single period, must decide how many units of the product to order with the objective of maximizing the expected profit.

When the product lifetime exceeds one period, determining the optimal ordering policies is quite complex, due to the overwhelming number of states which include all the inventory levels of each possible age stocks. The first analysis of the optimal ordering policy for perishable products was due to Van Zyl (1964). He considered the case where the product lifetime is two periods. Independently, Nahmias (1975) and Fries (1975) studied stochastic inventory problems when the lifetime of a perishable product is longer than two periods. Since the optimal ordering policy cannot be expressed in a simple form, the bulk of efforts have been spent in the development of efficient heuristics. For example, the fixed critical number order policy was proposed by Chazan and Gal (1977), Cohen (1976) and Nahmias (1976) under different assumptions. More studies about inventory management for perishable products can be found in the literature reviews provided by Nahmias (1982) and Raafat (1991).

Nowadays, inventory management for perishable products has been significantly improved with the help of advances in information technology and e-commence. For example, programs such as CPFR (collaborative planning forecasting and replenishment), QR (quick response) and VMI (vendor managed inventory) enable information sharing and collaboration among supply chain partners, which leads to lower inventory costs and higher service levels. However, despite significant efforts made in reducing supply chain costs via improved inventory management, a large portion of retailers still lose millions of dollars annually due to lost sales and excess inventory (Elmaghraby and Keskinocak, 2003). Therefore, many are now willing to coordinate inventory management with dynamic pricing in order to maximize the overall profit.

1.1.2 Dynamic pricing

Dynamic pricing is that the companies change prices dynamically over the time period. Determining the “right” price to charge a customer for a product is a complex task, requiring that a company have a wealth of information about its customer base and be able to set and adjust prices at minimal cost. However, in the past, companies had limited ability to track information about their customers’ tastes, and faced high costs in changing prices. Hence, companies always fixed the price of a product over a relatively long time period, i.e., the prices are usually static.

Nowadays, the rapid development of information technologies and the corresponding growth of Internet have opened the door for the adoption of dynamic pricing in practice. New technologies and Internet allow retailers to collect information not only about the sales, but also about demographic data and customer preferences. Due to the ease of making price changes on the Internet, dynamic pricing strategies are now frequently used in e-commerce environments. Although price changes are still costly in traditional retail stores, this may soon change with the introduction of new technologies such as Electronic Shelf Labeling System (Southwell, 2002).

Early applications of dynamic pricing have been mainly in industries characterized by perishability of the product, fixed capacity and possibility to segment customers (Weatherford and Bodily, 1992). For example, in the airline industry, there is usually a fixed capacity (seats on a flight) and these seats will perish when the flight leaves the gate. Airlines charge different prices for identical seats on the same flight. In airline reservation systems, limits are placed on the number of seats available of each fare class. Effective application of fare class booking limits allows airlines to generate incremental revenues. The term yield management (YM), or more appropriately revenue management (RM), has typically been employed to refer to the airlines' practice of enhancing revenues through the efficient control of seat inventories. Both American and United airlines reported that YM adds several hundred million dollars to the bottom line each year. (Weatherford, 1991). Since YM has been used successfully in the airline industry, the application of YM has been extended to other industries such as hotels and telecommunication (Bitran and Mondschein, 1995 and Nair and Bapna, 2001).

In recent years, we have witnessed an increased adoption of dynamic pricing for perishable products in retail and manufacturing industries. For example, in food industry, perishable products such as bread or fresh produces (vegetables, dairy products) have very short shelf life times. When these products come in fresh, they are usually priced at the retail price. However, when the products left are close to their expiry dates, the retailer sells them at discounted prices, therefore attracting customers who are more price sensitive, with the aim of generating more revenue through higher sales. This practice is widely employed in the electronics industry as well. For instance, prices of CPUs drop several times throughout their short life times whenever new CPUs are introduced to the market. In these industries, the profits of the retailer may be significantly increased by dynamic pricing and/or coordinating inventory and pricing decisions.

1.2 Motivation of the study

Initially, many researchers focus on pricing alone as a tool to improve the total profit. However, the integration of pricing with inventory (ordering) decisions optimizes the system rather than individual elements and thus significantly improves the profit of the company. This integration is still in its early stages in many retail and manufacturing companies, but it has the potential to radically improve supply chain efficiencies in much the same way as RM has changed airline, hotel and car rental companies (Chan et al., 2004).

Most researchers such as Zabel (1972), Thowsen (1975) and Federgruen and Heching (1999) focus on the joint pricing and ordering decisions for a single storable product. However, due to rapid developments of new technologies, product value quickly diminishes and more products can be considered as perishable products. In contrast with a single storable product, a single perishable product can be differentiated with respect to its ages. Products of different ages may capture different market segments. By differentiating prices for products of different ages, additional revenue and profit can be obtained. Thus, there is a great need to investigate the coordination of pricing and ordering decisions for perishable products, which may add a lot of money to the bottom line.

When prices for products of different ages are differentiated, substitution among products of different ages is observed among customers. If the prices for new and old products are sufficiently close, the customers may decide which products to purchase based on the prices of new (target) and old (substitute) products, rather than the price of the target products only. For example, a customer intending to purchase a newer version product and finding it too expensive may purchase an attractively priced older version product, instead. Such demand transfers between new and old products make the pricing and ordering decision problem more complicated. The solution to this problem will be of great value to the company.

Traditional RM problems have assumed that prices are fixed and solved for the optimal inventory allocation for each fare class (Littlewood, 1972; Belobaba, 1987, 1989; etc). The revenue is protected by adjusting the inventory allocation for each fare class. However, among various techniques to maximize the revenue, both price and inventory allocation are major control tools. The prices charged for different fare customers would influence demand and should be considered as decision variables, not fixed quantities. The integration of price and inventory allocation decisions should receive more attention that it deserves (Mcgill and van Ryzin, 1999).

1.3 Scope and objectives of the study

In this study, we focus on the joint pricing and ordering decisions for perishable products. The aim of this research is shown as follows:

(1) To study the integration of dynamic pricing and ordering decisions for a perishable product with a limited period lifetime. In any given period, the inventory contains products of different ages. At the beginning of each period, two decisions are made: what are the optimal prices charged for products of different ages and how many quantities are ordered for a new product. The objective is to maximize the total profit over multiple periods.

(2) To compare the expected profit from dynamic pricing with that from static pricing and identify when dynamic pricing provides a significant increase in the total profit compared to static pricing.

(3) To consider the substitution among products of different ages. The optimal prices for products of different ages and the optimal order quantity for a new product at each period are determined with the objective of maximizing the total profit over the multiple periods. In addition, the effect of substitution on the expected profit increase is measured.

(4) To incorporate the pricing decision into a typical RM problem. At the beginning of each period, the price and the inventory allocation for the period are jointly determined.

The insights obtained from this thesis may help to make pricing and ordering (or production capacity) decisions for perishable products and mass customized products (products with short life cycles) effectively and efficiently, to significantly increase the total profit.

Some researchers consider the competition among different retailers and apply game theory to decide the equilibrium prices of each retailer. However, this thesis does not consider such competition, since this condition will make our problem intractable. Instead, we assume that a retailer operates in a market with imperfect competition. This assumption can be justified by assuming that the retailer may be a monopolist or the product he sells may be new and innovative.

1.4 Organization

This dissertation contains 6 chapters. In Chapter 2, literatures related to this study will be reviewed. The topics covered in the literature review include: joint pricing and inventory decisions, substitution and RM.

Chapter 3 focuses on the integration of dynamic pricing and ordering decisions for perishable products. The product with a two period lifetime is first considered and a periodic review policy is used. Hence, in any given period the inventory consists of products with two different ages. The new product (product of age 1) is sold at the retail price while the old product (product of age 2) is sold at a discounted price. Demands for products of two ages come from two independent demand classes. At the beginning of each period, the optimal order quantity for new products is determined, and the optimal discounted price for old products is determined given the remaining inventory level of old products. The results are then extended to a product with the lifetime of longer than two periods, and hence with more than two demand classes.

Chapter 4 extends the work in Chapter 3 by considering the substitution among products of different ages. Demands for products of different ages are assumed to be dependent on not only the price of itself but also the prices of substitutable products, i.e., products of "neighboring ages". The products of neighboring ages are defined by the products that are a period older or younger than the target products. A periodic review policy is used. The objective is to find the optimal prices for products of different ages and the optimal order quantity for a new product with the objective of maximizing the total profit over the multiple periods.

Chapter 5 jointly determines the price and the inventory allocation for a perishable product. The price of the product is assumed to increase as the time at which it perishes approaches to, as in the airline industry. Demand for the product is price sensitive. To maximize the expected revenue, a discrete time dynamic programming model is developed to obtain the optimal prices and the optimal inventory allocations for the product with a two period lifetime. Three heuristics are then proposed when the lifetime is longer than two periods. These results are extended to (i) the case in which the price for the product always decreases; and (ii) the case in which the price for the product first increases and later decreases.

Chapter 6 summarizes the studies covered in this dissertation and gives some directions for future works.

Chapter 2 Literature Review

Chapter 2 reviews the previous studies relevant to joint pricing and inventory decisions, substitution and RM. Section 2.1 presents a classification table with the objective of intelligibly describing the literatures. The studies on integration of pricing and inventory decisions for a single product will be reviewed in Section 2.2. Section 2.3 introduces the studies on multiple products substitution problems. Finally, the studies on RM will be elaborated in Section 2.4.

2.1 Classification

There are voluminous research works in the area of pricing and inventory control. Hence, it is useful to provide a classification table which is used to describe the papers that will be reviewed in the following sections.

Table 2.1 Legend for classification system

Elements	Descriptions
Length of horizon	Single period / Multiple periods / Infinite horizon
Pricing strategy	Static pricing / Dynamic pricing
Demand type	Deterministic demand / Stochastic demand
Demand input parameters	Price, Time, Inventory, Reserved price
Review policy	Periodic review / Continuous review
Sales	Backlogging / Lost sales
Replenishment	Yes / No
Capacity limits	Yes / No
Set-up cost	Yes / No
Products	Single product (a single storable product, a single perishable product) Multiple substitutable products

2.2 Joint pricing and inventory decisions for a single product

Pricing and inventory control strategies have traditionally been determined by entirely separate units of a company's organization, without proper mechanisms to coordinate these two planning areas (Federgruen and Heching, 1999). Such dichotomy has also been observed in the academic literature. More specifically, single product inventory models assume that the price is known, and hence the demand distribution at each period is exogenously specified. Since expected revenues are constant under this assumption, these models focus on minimizing the expected costs. (Lee and Nahmias,1993 and Porteus, 2003). On the other hand, the literature on dynamic pricing assumes that with the exception of an initial procurement at the beginning of the planning horizon, no subsequent orders are allowed. (Gallego and van Ryzin, 1994, Bitran and Mondschein, 1997, etc).

The need to integrate inventory control and pricing was first studied by Whitin (1955) who addressed a single period problem. More research works on a single period problem are reviewed in Section 2.2.1.

2.2.1 The newsvendor model with pricing

The original newsvendor problem assumes that pricing is an exogenous decision. In contrast, Whitin (1955) added the pricing decision to the newsvendor problem, where the selling price and the order quantity are determined simultaneously. Under the assumption of deterministic demand, the optimal price and the optimal order quantity are obtained with the objective for maximizing the expected profit.

Mills (1959) considered the similar problem under stochastic demand. The additive demand $D(p,\varepsilon) = y(p) + \varepsilon$ was used, where $y(p)$ is a decreasing function of price p and ε is a random variable defined within some range. The study showed that the optimal price under stochastic demand is always no greater than the optimal price under deterministic demand, the riskless price. Both Lau and Lau (1988) and Polatoglu (1991) studied linear additive demand where $y(p) = b - ap$ under different assumptions.

On the other hand, Karlin and Carr (1962) used the multiplicative demand $D(p,\varepsilon) = y(p)\varepsilon$. They showed that the optimal price under stochastic demand is always no smaller than the riskless price, which is the opposite of the corresponding relationship obtained by Mills (1959) for the additive demand case.

Petruzzi and Dada (1999) provided a unified framework to reconcile this apparent contradiction by introducing the notion of a base price and demonstrating that the optimal price can be interpreted as the base price plus a premium. In addition, they presented a comprehensive review that synthesized existing results for the single period problem.

The papers reviewed above focus on a single period problem. A natural extension of this problem is a problem involving multiple periods, where the remaining inventories from one period are carried forward to meet demand in subsequent periods. The relevant literature will be reviewed in next section.

2.2.2 Multiple period inventory models with pricing

2.2.2.1 Dynamic pricing

Deterministic demand

Rajan et al. (1997) focused on price changes that occurred within an order cycle when the seller sold a single perishable product. The seller ordered the new product every T periods, which was delivered instantaneously. Deterministic demand for the product was a decreasing function of the age of the product as well as price. Given the assumption of deterministic demand and zero lead times, the seller depleted her entire inventory within each order cycle (i.e., no lost sales and backlogging are incurred). The optimal price within an order cycle p_t^*, the optimal cycle length *T*, and the optimal order quantity *Q* were obtained which maximized the average profit over time.

This thesis determines the optimal price and the optimal order quantity under stochastic demand, which is significantly different from the previous studies under deterministic demand.

Stochastic demand

The following three papers consider a single storable product. Demand in consecutive periods is independent, but their distributions depend on the product's price following a specified stochastic demand function. A periodic review policy is used. At the beginning of each period, before demand is realized, the seller must decide how many inventories to order and the price charged for these inventories.

Zabel (1972) was one of the earliest researchers who studied this multiple period problem under stochastic demand. Under the assumption of lost sales, Zabel considered both multiplicative and additive demand with a stochastic component, and found that the latter had properties that made the problem easier to solve. For additive demand, the author showed that a unique solution was obtained under certain conditions.

Similarly, Thowsen (1975) considered the problem of determining the price and the order quantity under additive demand. He extended Zabel's analysis to the case where backlogging was allowed. A base stock list price (BSLP) policy is proved to be optimal under certain conditions.

A BSLP policy is defined as follows: (i) if the inventory at the beginning of period t, x_t, is less than some base stock level y_t^*, place an order and bring the inventory level up to y_t^*, and charge p_t^*; (ii) if $x_t > y_t^*$, order nothing and offer the product at a discounted price of $p_t^*(x_t)$, where $p_t^*(x_t)$ is decreasing in x_t.

Recently, Federgruen and Heching (1999) addressed both finite and infinite horizon models for a similar problem under a non-stationary demand function $D_t = \gamma_t(p_t) * \varepsilon_t + \delta_t(p_t)$. Excess demand was assumed to be fully backlogged. Federgruen and Heching showed that the expected profit was concave and the optimal price was a non-increasing function of the inventory level. The authors provided an efficient algorithm to compute the optimal price. Using a numerical study, they showed that dynamic pricing provided 2% increase in expected profit over static pricing.

While the papers above focus on a single storable product, this study considers a single perishable product which can be differentiated with respect to its ages. At any period, the inventory consists of products of different ages. The optimal prices for products of different ages and the optimal order quantity for the new product (product of age 1) are simultaneously determined at the beginning of each period.

2.2.2.2 Static pricing

Although most previous studies focused on dynamic pricing, some researchers have also considered the problem of choosing a static or constant price over the lifetime of a product.

The earliest known example of integrating a static price decision with inventory decisions was that of Kunreuter and Schrage (1973). They considered a problem with deterministic demand, a linear function of price, and varying over a season. Their model did not assume lost sales or backlogging, since demand was exactly predicted by the price and time. The objective was to determine price, production per period, and production quantities so as to maximize profit. A “hill-climbing” algorithm was provided to compute the upper and the lower bounds for the price decision.

Gillbert (1999) focused on a similar problem but assumed that demand was a multiplicative function of seasonality, i.e., $d_t(p) = \beta_t D(p)$. Gillbert also assumed that holding costs and production set-up costs was invariant over time and the total revenue was concave. He developed a solution approach that guaranteed the optimality for this problem, employing a Wagner-Whitin time approach for determining production periods.

Even though less attractive in e-commerce environments, static pricing is particularly easy to implement in the traditional businesses where price changes are still costly. It would be valuable to identify when dynamic pricing provides a significant increase in total profit compared to static pricing. This comparison will help the companies to decide whether it is worth the extra efforts to employ dynamic pricing.

2.3 Multiple products with substitution

The literatures reviewed in Section 2.2 focus on a single product. Affected by shorter product lifetimes and even quickening technological developments, more and more new products are frequently introduced to the markets. Hence, the problems which allow for substitution between new products and existing products have attracted the attention of the researchers. The studies considering multiple product substitution problems will be reviewed in this section.

2.3.1 Multiple product inventory models with substitution

The earliest work on obtaining the optimal inventory policies for multiple substitutable products was due to Veinott (1965). This study was generalized by Ignall and Veinott (1969) and extended to perishable inventories by Deuermeyer (1980).

Analysis of single period two product substitution problems appeared in Mcgillivray and Silver (1978), Parlar and Goyal (1984), Pasternack and Drezner (1991), and Gerchak et al. (1996). In particular, Gerchak et al. presented several different models of a two product substitution problem with random yield and focused on identifying structural properties of the optimal policy.

Bitran and Dasu (1992) considered planning problems with multiple products, stochastic yields, and substitutable demands. Drawing on insights from the two period problem, a class of heuristics was provided for solving the multiple period problem with no capacity constraint.

Bitran and Leong (1992) also examined multiple period, multiple product planning problems with stochastic yields and substitutable demands. They formulated the problem under service constraints and provided near optimal solution to an approximate problem with fixed planning horizon. They also proposed simple heuristics for the problem, solved with rolling horizons. Common to these two papers is the approach of approximating the stochastic problem with a deterministic one.

Recently, Bassok et al. (1999) studied a single period multiple product inventory problem with substitution. They considered N products and N demand classes with downward substitution, i.e., excess demand for class i can be satisfied using product j for $i > j$. The problem was modeled as a two-stage stochastic program. A greedy allocation policy was shown to be optimal. Additional properties of the profit function and several interesting properties for the optimal solutions were obtained.

Hsu and Bassok (1999) considered a similar substitution problem of Bassok et al. (1999). However, their model had one raw material as the production input and produced N different products as outputs. By efficiently solving a two-stage stochastic problem, the optimal production input and allocation of units to lower functionality demands were obtained.

While the literatures above consider the “pure” inventory policy for multiple products, this study determines not only the optimal order quantity for a new product but also the optimal prices for multiple existing products.

2.3.2 Pricing decisions for multiple products

Gallego and van Ryzin (1997) considered a multiple period pricing problem with multiple products sharing common resources. Demand for each product was a stochastic function of time and the product prices. An upper bound for the expected revenue was obtained by analyzing this problem under the assumption of deterministic demand. The solution for deterministic demand was employed for two heuristics for a stochastic problem that were shown to be asymptotically optimal as the expected sales volume goes to infinity.

Instead of approximating the stochastic problem with a deterministic one, the stochastic problem needs to be further optimized. In addition, the ordering quantities for multiple products should also be determined, rather than the prices alone.

2.3.3 Joint pricing and ordering decisions for two substitutable products

The first paper that combined the pricing and capacity decisions was Birge et al. (1998), who addressed a single period problem. By assuming demand to be uniformly distributed, they obtained the optimal pricing and capacity decisions for two substitutable products. In addition, they presented numerical results to show that pricing and capacity decisions were affected significantly by the experimental parameters.

Similarly, Karakul and Chan (2003) formulated a single period problem of two products which the new product can be a substitute in case the existing product runs out. The objective is to find the optimal price of the new product and inventory levels for both new and existing products in order to maximize the single period expected profit. The authors showed that the problem could be transformed to a finite number of single variable optimization problem. The single variable functions to be optimized have only two possible roots under certain demand distributions for the new product. They also showed that besides the expected profit, both the price and production quantity of new products were higher when it was offered as a substitute.

The papers reviewed above analyze a single period, two products problem. In contrast, this study first considers a multiple period, two products problem under general demand distributions. The study is further extended to consider a multiple period, multiple products problem with substitutable demands.

2.4 Revenue management

From a historical perspective, the interest in revenue management practices started with the pioneering research of Littlewood (1972) on airline. However, it was probably after the work of Belobaba (1987, 1989) and the American Airline success that the field really took off. The publication of a survey paper by Weatherford and Bodily (1992), where a taxonomy of the field and an agenda for future work were proposed, was another symptom of this revival. At this stage, however, much of the work was done on capacity management and overbooking with little discussion of dynamic pricing policy. Prices in these original models are assumed to be fixed and managers were in charge of opening and closing different fare classes as demand evolved. During the 90's, the increasing interest in RM became evident in the different applications that were considered. Models became industry specific (e.g. airlines, hotels, or retail stores) with a higher degree of complexity (e.g. multi-class and multi-period stochastic formulations). Furthermore, it was in the last decade that pricing policies really became an active component of the RM literature. Today, dynamic pricing in a RM context is an active field of research that has reached a certain level of maturity.

2.4.1 Single-leg seat inventory control

The problem of seat inventory control across multiple fare classes has been studied by many researchers since 1972. There has been significant progress from Littlewood's rule for two fare classes, to the expected marginal seat revenue (EMSR) rule for multiple fare classes, to optimal booking limits for single-leg flight.

Littlewood (1972) studied a stochastic two-price, single-leg airline RM model and proposed a marginal seat revenue principle. The principle suggested that booking requests for the lower fare class can be declined if the seat could be sold later to the higher fare class. Bhatia and Parekh (1973), and Richter (1982) used the marginal seat revenue principle to develop simple decision rules which were employed to determine optimal booking limits in a nested fare inventory system.

Belobaba (1989) extended Littlewood’s rule to multiple-fare classes and proposed an EMSR rule. The EMSR method did not produce optimal booking limits except in the two fare class, however, it was particularly easy to implement. Methods for obtaining optimal booking limits for single-leg seat inventory control were provided in Curry (1990), Wollmer (1992), Brumelle and McGill (1993), and Robinson (1994). These studies also showed that the Belobaba’s heuristics was sub-optimal.

A comprehensive overview for perishable assets RM was founded in Weatherford and Bodily (1992). Subramanian et al. (1999) formulated the airline seat allocation problem on a single-leg flight into a discrete-time Markov decision process. The model allowed cancellation, no-shows, and overbooking. They showed that an optimal booking policy was characterized by seat and time dependent booking limits for each fare class. Because of fare-dependent cancellation refunds, the optimal booking limits may not be nested. Independently, Liang (1999), and Feng and Xiao (2001) studied a continuous-time, dynamic seat inventory control problem. Both of them proved that a threshold control policy was optimal. Zhao and Zheng (2001) considered a more general airline seat allocation problem that allows diversion/upgrade and no-shows and showed that a similar threshold control policy was optimal. Other studies on airline RM problems can be found in Rothstein (1971), Hersh and Ladany (1978), Pfeifer (1989), Brumelle et al. (1990), Ladany and Arbel (1991), Smith et al. (1992), Lee and Hersh (1993), Bassok and Ernst (1995), Weatherford (1997), Talluri and van Ryzin (1999), and Chatwin (1999).

The papers reviewed in this section assume that prices are predetermined and never allowed to decrease. Under this assumption, the optimal booking limit for each fare class is obtained. In contrast, this study determines not only the optimal booking limit but also the optimal price for each fare class. Furthermore, the study considers two more cases, (i) the case where the price first increases and later decreases and (ii) the case where the price always decrease, and obtains the price and inventory decisions, simultaneously.

2.4.2 Dynamic pricing

The following papers focus on market environments where there is no opportunity for inventory replenishment over the selling horizon. These markets arise when the seller faces a shorter selling horizon, e.g., when the product itself is a short life cycle product, such as fashion apparel or holiday products, or is at the end of its life cycle (e.g. clearance items). In these markets, production/delivery lead times prevent replenishment of inventory and hence, the seller has a fixed inventory on hand and must determine how to price the product over the remaining selling horizon.

The first researchers to study dynamic product pricing were Kincaid and Darling (1963). They investigated two continuous time models, where demand followed a Poisson process with fixed intensity λ. An arriving customer at time t had a reservation price r_t for the product, i.e., the maximum price the customer was willing to pay. The reservation price r_t was a random variable with distribution $F(r_t)$. In the first model, the seller did not post prices but receives offers from potential buyers, which he/she either accepted or rejected. In the second model, the seller announced a price p_t and arriving customers purchased the product only if $r_t \geq p_t$. The demand process in this situation was Poisson with intensity $\lambda(1-F(p_t))$ .Optimality conditions for the maximum revenue and the optimal price were derived for both cases.

Gallego and van Ryzin (1994) modeled the demand as a homogenous Poisson process with intensity $\lambda(p)$, where $\lambda(p)$ was non-increasing function of p. For a "regular" demand function, they derived optimality conditions and showed that

(i) at a given point of time, the optimal price is a non-increasing function of the inventory level

(ii) for a given inventory level, the optimal price is a non-decreasing function of the duration of the selling horizon.

The optimal price path that Gallego and van Ryzin obtained was that the price jumped up after each sale, then decayed slowly until the next sale, and jumped up again.

Bitran and Mondschein (1997) and Zhao and Zheng (2000) generalized the model of Gallego and van Ryzin (1994) by assuming the demand as a non-homogenous Poisson process with intensity $\lambda(p,t)=\lambda_t(1-F_t(p))$. They showed that the property (i) held under this more general assumption; however, the property (ii) may not hold when $F_t(.)$ changed over time. Zhao and Zheng (2000) showed a necessary condition for the property (ii) to hold, namely that the probability that a customer was willing to pay a premium decreased over time.

Bitran and Mondschein (1997) considered a periodic pricing review policy where the prices were revised only at a finite set of times and were never allowed to rise. This policy can be applied for pricing seasonal products in the retailing industry. Demand distribution was assumed to be Poisson. The authors used empirical analysis to develop conjectures as to the structure of the optimal policy and the optimal revenue but no theoretical results were presented.

The optimal pricing policy in Gallego and van Ryzin (1994) required continuous updating of prices over time, which is not practical. Therefore, Gallego and van Ryzin presented the fixed price heuristics. These simple heuristics are proved to be asymptotically optimal as the volume of expected sales and the number of selling periods go to infinity.

Another focus on the continuous time problem is the case where prices have to be chosen from a discrete set of allowable prices $\{p_1, p_2, ..., p_k\}$. In addition to continuous price paths and fixed price heuristics, Gallego and van Ryzin (1994) discussed this issue and showed that the policies with at most one price change were asymptotically optimal as the initial capacity and/or the time to sell increased.

Inspired by the proposed pricing policy that allow at most one price change, Feng and Gallego (1995) focused on a very specific question: what was the optimal time to switch between two pre-determined prices in a fixed selling season. They considered both the typical retail situation of switching from an initial high price to a lower price as well as the case more common in the airlines of switching from an initial low price to a higher one later in the season. By assuming that demand was a Poisson process which was a function of price, Feng and Gallego showed that the optimal policy for this problem was a threshold policy, whereby a price was changed (decreased or increased) when the time left in the horizon passed a threshold (below or above) that depends on the unsold inventory. For the problem where the direction of price change was not specified, they showed a dual policy, with two sequences of monotone time thresholds. Although they did not explicitly consider the choice of the two starting prices for the problem, a company could use the policy they developed to determine the expected revenue for each pair of prices and chose the pair that maximizes the expected revenue. Feng and Gallego (2000) discussed Markovian demand and Feng and Xiao (1999) generalized the two price model to consider risk preference.

Feng and Xiao (2000a) further extended their previous model by considering multiple predetermined prices. Similar to Feng and Gallego (1995), they assumed that price changes were either decreasing or increasing, i.e., monotone and non-reversible. The initial inventory was fixed and demand was a Poisson process with constant intensity rate. Under these assumptions, the authors developed an exact solution for this continuous time model and showed that the objective function of maximizing the revenue was piecewise concave with respect to time and inventory.

Independently, Chatwin (2000) and Feng and Xiao (2000b) provided a systematic analysis of the pricing policy and the expected revenue for the problem within a finite set of prices. In these two papers, it is shown that the maximum expected revenue is concave on both the remaining inventory and duration of the selling horizon. For a given inventory level, the optimal price is a non-increasing function of the remaining time. At any given time, the optimal price is a non-increasing function of the remaining inventory. An upper bound on the maximum numbers of price changes is also reported. In addition, Feng and Xiao (2000b) showed that there was a maximum subset $P_0 \subseteq \{p_1,..., p_k\}$ such that the revenue rate was increasing and concave within P_0 and the optimal price at any time belonged necessarily to P_0. This observation was particularly useful since it narrowed down the set of potential optimal prices making the computation of the optimal prices much easier.

The papers reviewed in this section require continuous changing of prices over time, which may not be practical. In contrast, this study focuses on periodically updating the prices and determines the optimal prices and the optimal order quantity for perishable products simultaneously.

Chapter 3 Dynamic pricing and ordering decision for perishable products with multiple demand classes

Chapter 3 focuses on the integration of dynamic pricing and ordering decisions for perishable products. In Section 3.2, a dynamic programming model is developed for the product with a two period lifetime. The optimal order quantity for the newer product and the optimal price for the older product are obtained. Furthermore, we prove that the expected profit obtained from dynamic pricing is always higher than the expected profit from static pricing. Numerical results for the product with a two period lifetime are presented in Section 3.3. The study is further extended to consider a more general problem where the lifetime of the product is longer than two periods, as shown in Section 3.4.

3.1 Introduction

Advances in information technology and e-commerce have played an important role in improving the inventory management of perishable products. With advanced tools such as CPFR (collaborative planning forecasting and replenishment), QR (quick response) and VMI (vendor managed inventory), supply chain partners can share the information and collaborate with each other, which leads to lower inventory costs and increased service levels. However, despite significant efforts made in reducing supply chain costs, a large portion of retailers still lose millions of dollars annually due to lost sales and excess inventory (Elmaghraby and Keskinocak, 2003). Therefore, many are now willing to re-examine their pricing policies and explore dynamic pricing for the maximization of their profit.

This chapter focuses on the integration of dynamic pricing and ordering decisions for perishable products under stochastic demand. The product is first assumed to have a two period lifetime and a periodic review policy is used. Hence, in any given period the inventory consists of products with two different ages. The new product (product of age 1) is sold at the retail price while the old product (product of age 2) is sold at a discounted price. We assume demands for two different ages of products come from two independent demand classes. Moreover, demand for the old product is dependent on the discounted price. At the beginning of each period, the optimal order quantity for new products is determined, and the optimal discounted price for old products is determined given the remaining inventory level of old products. The approach of offering a promotional discount for old products helps the retailer to increase sales, reduce the inventory level, and thus obtain higher profits. We also extend the results to a product with the lifetime of longer than two periods, and hence with more than two demand classes.

The proposed model makes an assumption that could be controversial, which is independence of different demand classes. Examples of such independence can be easily found in practice, such as in electronics industry. Due to fast developments in technologies, new products are significantly improved compared with existing products in terms of performance, design, etc. Thus, the customers who are interested in the new product are more performance oriented and thus, are not affected by the price of old existing products. Similarly, the customers who are more price sensitive cannot afford to purchase the new product and focus on the availability and price of old products.

3.2 Pricing and ordering decisions for a product with a two period lifetime

In this section, we formulate and analyze a multiple period problem for a perishable product with a two period lifetime.

3.2.1 Assumptions and notations

We consider a perishable product with a two period lifetime, represented by $M = 2$. A periodic review policy is used. Hence, in any given period, the inventory consists of products with two different ages. Let index $i = 1, 2$ denotes the ages of the product, where $i = 1$ (2) represents that the product is new (old). In each period, two independent demand classes, denoted by Type i customers, purchase the product of age i. Type 1 customers purchase the new product regardless of the price and availability of old products and Type 2 customers purchase old products regardless of the price and availability of the new product.

Demands of Type 1 customers t_1 at each period are nonnegative, independent and identically distributed (*i.i.d.*) with a known probability distribution $g_1(t_1)$. Demands of

Type 2 customers are dependent on the discounted price p_2, represented by a given linear stochastic demand function:

$$t_2 = \mu_2(p_2) + \varepsilon_2 \tag{3.1}$$

$\mu_2(p_2)$ is mean demand of Type 2 customers and $\mu_2(p_2) = b_2 - a_2 p_2$, where $a_2 > 0$. ε_2 is an *i.i.d.* random variable with a known probability density function $f_2(\varepsilon_2)$ and is bounded in $[\varepsilon_2^{\min}, \varepsilon_2^{\max}]$. In addition, $E(\varepsilon_2) = 0$, where $b_2 > -\varepsilon_2^{\min}$.

p_2 is confined to a finite interval $[p_2^{\min}, p_2^{\max}]$, where $p_2^{\max} < \dfrac{b_2 + \varepsilon_2^{\min}}{a_2}$. The upper bound of $p_2^{\max}$ prevents negative demands of Type 2 customers. The salvage value of any unsold items after their lifetimes is zero. In case t_i exceeds the available inventory of age *i* (*i* = 1, 2), the excessive demand is lost.

The following notation is employed as follows:

y = order quantity for a new product

x_i = inventory level for a product of age i, i = 1,2

p_1 = retail price of a new product

p_i = discounted price for a product of age i, i = 2 where $p_2 < p_1$

π_i = penalty cost for a product of age i, i = 1, 2 where $\pi_2 < \pi_1$

h = holding cost per period (regardless of ages)

c = purchasing cost for a new product

α = discounted factor per period

The index k is defined to represent the period, for $k = 1,..., N$, where N is the number of studying periods. Denote y_k as the quantity ordered at Period k and x_{2k} as the remaining quantity carried forward from Period $k-1$ to Period k after the realization of demand of Type 1 customers during Period $k-1$.

Observation 3.1: The order quantity y_k is only dependent on demand of Type 1 customers at Period k and the price sensitive demand of Type 2 customers at Period $k+1$.

Proof: During Period k, the order quantity y_k is demanded only by Type 1 customers. The remaining quantity of y_k carried forward to Period $k+1$, $x_{2,k+1}$, can only be used to satisfy Type 2 customers since the quantity x_{2k} is disposed of at the end of Period k, and the new order that arrives at Period $k+1$, y_{k+1}, can only be used to satisfy Type 1 customers. Furthermore, any $x_{2,k+1}$ unsold at the end of the period $k+1$ must be disposed of. Therefore, the influence of the order quantity y_k lasts for two periods only and y_k determines $x_{2,k+1}$. □

From Observation 3.1, an N period problem can be reduced to a two period problem as follows: At the beginning of Period 1, the retailer determines the order quantity for the new product. After the realization of demand for Period 1, the remaining products which become old are carried to Period 2. Given the inventory level for old products, the price is determined at the beginning of Period 2. No replenishment is allowed during the planning horizon of two periods.

3.2.2 Dynamic programming model

The dynamic programming model is developed to compute the expected profit over two periods for a given y. The maximum expected profit is computed recursively backward in time, starting from Period 2 to Period 1.

$V_2(x_2) = \underset{p_2}{Max}[\varphi_2(x_2; p_2)]$ is the maximum expected profit during Period 2 for a given inventory level x_2.

$\varphi_2(x_2; p_2)$ is the expected profit from Type 2 customers incurred at Period 2, including the expected revenue, holding cost for excess inventory and penalty cost for unsatisfied demand $L_2(x_2; p_2)$.

$$\varphi_2(x_2; p_2) = p_2 E[\min(x_2, t_2)] - L_2(x_2, p_2)$$

where $L_2(x_2; p_2) = hE[x_2 - t_2]^+ + \pi_2 E[t_2 - x_2]^+$ and $t_2 = b_2 - a_2 p_2 + \varepsilon_2$.

The maximum expected total profit over two periods V_1 is computed as follows:

$$V_1 = \underset{y}{Max}[\varphi_1(y) + \alpha E(V_2(x_2))] \tag{3.2}$$

where $x_2 = [y - t_1]^+$ represents the recursive function for the inventory level.

$\varphi_1(y)$ is the expected profit from Type 1 customers incurred at Period 1, including the expected revenue, holding cost for excess inventory and penalty cost for unsatisfied demand $L_1(y)$ and the purchasing cost.

$$\varphi_1(y) = p_1 E[\min(y, t_1)] - L_1(y) - cy$$

where $L_1(y) = hE[y - t_1]^+ + \pi_1 E[t_1 - y]^+$.

We denote $J_1(y)$ as the expected profit over two periods when the initial inventory level at Period 1 is y.

$$J_1(y) = \varphi_1(y) + \alpha E[V_2(x_2)] \tag{3.3}$$

3.2.2.1 Optimal discounted price

The optimal discounted price is obtained by maximizing the expected profit of Type 2 customers, $\varphi_2(x_2; p_2)$, and satisfies the following properties:

Lemma 3.1: $\varphi_2(x_2; p_2)$ is concave with respect to p_2 for a given x_2.

Proof: $\varphi_2(x_2; p_2)$ is expanded as follows:

$$\begin{aligned}
\varphi_2(x_2; p_2) = {} & p_2 E[Min(x_2, b_2 - a_2 p_2 + \varepsilon_2)] - hE[x_2 - b_2 + a_2 p_2 - \varepsilon_2]^+ \\
& - \pi_2 E[b_2 - a_2 p_2 + \varepsilon_2 - x_2]^+ \\
= {} & \int_{\varepsilon_2^{\min}}^{x_2 - b_2 + a_2 p_2} [p_2(b_2 - a_2 p_2 + \varepsilon_2) - h(x_2 - b_2 + a_2 p_2 - \varepsilon_2)] f_2(\varepsilon_2) d\varepsilon_2 \\
& + \int_{x_2 - b_2 + a_2 p_2}^{\varepsilon_2^{\max}} [p_2 x_2 - \pi_2(b_2 - a_2 p_2 + \varepsilon_2 - x_2)] f_2(\varepsilon_2) d\varepsilon_2
\end{aligned} \tag{3.4}$$

Taking the first order and second order partial derivatives of (3.4) with respect to p_2, we have

$$\begin{aligned}\frac{\partial \varphi_2}{\partial p_2} &= \int_{\varepsilon_2^{\min}}^{x_2-b_2+a_2p_2} [b_2 - 2a_2p_2 + \varepsilon_2 - a_2h] f_2(\varepsilon_2)d\varepsilon_2 \\ &\quad + \int_{x_2-b_2+a_2p_2}^{\varepsilon_2^{\max}} (x_2 + a_2\pi_2) f_2(\varepsilon_2)d\varepsilon_2\end{aligned} \tag{3.5}$$

and

$$\frac{\partial^2 \varphi_2}{\partial p_2^2} = -\int_{\varepsilon_2^{\min}}^{x_2-b_2+a_2p_2} 2a_2 f_2(\varepsilon_2)d\varepsilon_2 - a_2^2(p_2 + h + \pi_2) f_2(x_2 - b_2 + a_2p_2)$$

respectively.

Since the second order derivative is always less than zero, $\varphi_2(x_2; p_2)$ is concave with respect to p_2 for a given x_2. □

Lemma 3.2: Let $\hat{p}_2$ denote the value of p_2 satisfying the stationary condition of (3.5). $\hat{p}_2$ is also bounded in $[p_2^{\min}, p_2^{\max}]$. Hence, the optimal discounted price, p_2^*, is determined as follows:

$$p_2^* = \begin{cases} p_2^{\max} & \hat{p}_2 \geq p_2^{\max} \\ \hat{p}_2 & p_2^{\min} < \hat{p}_2 < p_2^{\max} \\ p_2^{\min} & \hat{p}_2 \leq p_2^{\min} \end{cases}$$

Lemma 3.2 is directly obtained from Lemma 3.1. Some optimal properties of y^* are provided in the following section, when the noise variable ε_2 follows an IFR distribution, such as the Uniform, Exponential, Erlang, Normal or Truncated Normal distributions (Porteus, 2003). IFR distributions are widely used in modeling demand distributions because of their robustness.

3.2.2.2 Optimality properties of y^* when ε_2 follows IFR distributions

Under the assumption that the noise variable ε_2 follows an IFR distribution, where the hazard rate $\lambda_2(\varepsilon_2)$ is defined by $\dfrac{f_2(\varepsilon_2)}{1-F_2(\varepsilon_2)}$, the optimal discounted price p_2^* satisfies the following optimality properties:

Lemma 3.3: p_2^* is a non-increasing function of x_2 when the hazard rate $\lambda_2(\varepsilon_2) \geq \dfrac{1}{a_2(p_2^{\min}+h+\pi_2)}$.

Proof: From (3.5), $\hat{p}_2$ satisfies

$$\begin{aligned}\left.\frac{\partial \varphi_2}{\partial p_2}\right|_{p_2=\hat{p}_2} &= \int_{\varepsilon_2^{\min}}^{x_2-b_2+a_2\hat{p}_2}[b_2-2a_2\hat{p}_2+\varepsilon_2-a_2h]f_2(\varepsilon_2)d\varepsilon_2 \\ &+\int_{x_2-b_2+a_2\hat{p}_2}^{\varepsilon_2^{\max}}[x_2+a_2\pi_2]f_2(\varepsilon_2)d\varepsilon_2=0\end{aligned} \tag{3.6}$$

From (3.6), $\hat{p}_2$ is a function of x_2, denoted by $\hat{p}_2(x_2)$. By taking the first order derivative of (3.6) with respect to x_2, we obtain

$$1-F_2[x_2-b_2+a_2\hat{p}_2(x_2)]-2a_2(\frac{d\hat{p}_2(x_2)}{dx_2})F_2[x_2-b_2+a_2\hat{p}_2(x_2)]$$
$$-a_2(\pi_2+\hat{p}_2(x_2)+h)[1+a_2(\frac{d\hat{p}_2(x_2)}{dx_2})]f_2[x_2-b_2+a_2\hat{p}_2(x_2)]=0$$

Rearranging the terms,

$$a_2\frac{d\hat{p}_2(x_2)}{dx_2}=\frac{1-F_2[x_2-b_2+a_2\hat{p}_2(x_2)]-a_2(\pi_2+\hat{p}_2(x_2)+h)f_2[x_2-b_2+a_2\hat{p}_2(x_2)]}{2F_2[x_2-b_2+a_2\hat{p}_2(x_2)]+a_2(\pi_2+\hat{p}_2(x_2)+h)f_2[x_2-b_2+a_2\hat{p}_2(x_2)]} \tag{3.7}$$

Given that $\lambda_2(x_2-b_2+a_2\hat{p}_2(x_2))=\frac{f_2(x_2-b_2+a_2\hat{p}_2(x_2))}{1-F_2(x_2-b_2+a_2\hat{p}_2(x_2))}\geq\frac{1}{a_2(p_2^{\min}+h+\pi_2)}$,

$\hat{p}_2(x_2)\geq p_2^{\min}$ and the denominator of (3.7) is positive, hence $a_2\frac{d\hat{p}_2(x_2)}{dx}\leq 0$. Therefore, it follows that p_2^* is a non-increasing function of x_2. □

Lemma 3.3 provides the optimal pricing policy. Since p_2^* is a non-increasing function of x_2, there exist two threshold values, x_2^m and x_2^n, which satisfy $x_2^m\leq x_2^n$, $\hat{p}_2(x_2^m)=p_2^{\max}$ and $\hat{p}_2(x_2^n)=p_2^{\min}$. If $x_2\geq x_2^n$, the optimal discounted price p_2^* equals to $p_2^{\min}$. If $x_2\leq x_2^m$, the optimal discounted price p_2^* equals to $p_2^{\max}$.

Lemma 3.4: The maximum expected profit of Type 2 customers, $V_2(x_2)$, is concave with respect to x_2.

Proof: Define $V_2(x_2)$ as

$$V_2(x_2)=\begin{cases} V_{2,1}(x_2) \text{ obtained when } p_2^*=p_2^{\min} & x_2 \ge x_2^n \\ V_{2,2}(x_2) \text{ obtained when } p_2^*=\hat{p}_2 & x_2^m < x_2 < x_2^n \\ V_{2,3}(x_2) \text{ obtained when } p_2^*=p_2^{\max} & x_2 \le x_2^m \end{cases}$$

where the thresholds x_2^m, x_2^n are calculated by setting (3.5) to be zero under the condition that $p_2=p_2^{\max}$ and $p_2=p_2^{\min}$, i.e.,

$$\left.\frac{\partial \varphi_2}{\partial p_2}\right|_{p_2=p_2^{\max}} = \int_{\varepsilon_2^{\min}}^{x_2^m-b_2+a_2p_2^{\max}} [b_2-2a_2p_2^{\max}+\varepsilon_2-a_2h] f_2(\varepsilon_2)d\varepsilon_2 + \int_{x_2^m-b_2+a_2p_2^{\max}}^{\varepsilon_2^{\max}} [x_2^m+a_2\pi_2] f_2(\varepsilon_2)d\varepsilon_2 = 0$$

$$\left.\frac{\partial \varphi_2}{\partial p_2}\right|_{p_2=p_2^{\min}} = \int_{\varepsilon_2^{\min}}^{x_2^n-b_2+a_2p_2^{\min}} [b_2-2a_2p_2^{\min}+\varepsilon_2-a_2h] f_2(\varepsilon_2)d\varepsilon_2 + \int_{x_2^n-b_2+a_2p_2^{\min}}^{\varepsilon_2^{\max}} [x_2^n+a_2\pi_2] f_2(\varepsilon_2)d\varepsilon_2 = 0$$

Consider the following three cases:

Case (1) $x_2 \ge x_2^n$

$$\begin{aligned} V_{2,1}(x_2) &= \int_{\varepsilon_2^{\min}}^{x_2-b_2+a_2p_2^{\min}} [p_2^{\min}(b_2-a_2p_2^{\min}+\varepsilon_2)-h(x_2-b_2+a_2p_2^{\min}-\varepsilon_2)] f_2(\varepsilon_2)d\varepsilon_2 \\ &+ \int_{x_2-b_2+a_2p_2^{\min}}^{\varepsilon_2^{\max}} [p_2^{\min}x_2-\pi_2(b_2-a_2p_2^{\min}+\varepsilon_2-x_2)] f_2(\varepsilon_2)d\varepsilon_2 \end{aligned} \tag{3.8}$$

Since x_2 is independent of $p_2^{\min}$, the first and second order derivatives of (3.8) with respect to x_2 are

$$\frac{dV_{2,1}(x_2)}{dx_2} = -\int_{\varepsilon_2^{\min}}^{x_2-b_2+a_2p_2^{\min}} hf_2(\varepsilon_2)d\varepsilon_2 + \int_{x_2-b_2+a_2p_2^{\min}}^{\varepsilon_2^{\max}} [p_2^{\min} + \pi_2] f_2(\varepsilon_2)d\varepsilon_2$$

and

$$\frac{d^2V_{2,1}(x_2)}{dx_2^2} = -(p_2^{\min} + h + \pi_2) f_2(x_2 - b_2 + a_2 p_2^{\min}) \le 0$$

respectively. Thus, $V_{2,1}(x_2)$ is a concave function of x_2 when $x_2 \ge x_2^n$.

Case (2) $x_2^m < x_2 < x_2^n$

$$\begin{aligned} V_{2,2}(x_2) = & \int_{\varepsilon_2^{\min}}^{x_2-b_2+a_2\hat{p}_2(x_2)} [\hat{p}_2(x_2)(b_2 - a_2\hat{p}_2(x_2) + \varepsilon_2) - h(x_2 - b_2 + a_2\hat{p}_2(x_2) - \varepsilon_2)] f_2(\varepsilon_2)d\varepsilon_2 \\ & + \int_{x_2-b_2+a_2\hat{p}_2(x)}^{\varepsilon_2^{\max}} [\hat{p}_2(x_2)x_2 - \pi_2(b_2 - a_2\hat{p}_2(x_2) + \varepsilon_2 - x_2)] f_2(\varepsilon_2)d\varepsilon_2 \end{aligned} \tag{3.9}$$

The first order derivative of $V_{2,2}(x_2)$ with respect to x_2 is

$$\begin{aligned} \frac{dV_{2,2}(x_2)}{dx_2} = & \int_{x_2-b_2+a_2\hat{p}_2(x)}^{\varepsilon_2^{\max}} (\hat{p}_2(x_2) + \pi_2) f_2(\varepsilon_2)d\varepsilon_2 - \int_{\varepsilon_2^{\min}}^{x_2-b_2+a_2\hat{p}_2(x)} hf_2(\varepsilon_2)d\varepsilon_2 \\ & + \frac{d\hat{p}_2(x_2)}{dx_2} \int_{\varepsilon_2^{\min}}^{x_2-b_2+a_2\hat{p}_2(x_2)} (b_2 - 2a_2\hat{p}_2(x_2) + \varepsilon_2 - a_2h) f_2(\varepsilon_2)d\varepsilon_2 \\ & + \frac{d\hat{p}_2(x_2)}{dx_2} \int_{x_2-b_2+a_2\hat{p}_2(x)}^{\varepsilon_2^{\max}} (x_2 + a_2\pi_2) f_2(\varepsilon_2)d\varepsilon_2 \end{aligned} \tag{3.10}$$

Note that the sum of the 3rd and 4th terms in (3.10) is zero because of the optimality condition given in (3.5) when $p_2 = \hat{p}_2$. Then (3.10) is reduced to

$$\frac{dV_{2,2}(x_2)}{dx_2} = \int_{x_2-b_2+a_2\hat{p}_2(x)}^{\varepsilon_2^{\max}} [\hat{p}_2(x_2)+\pi_2] f_2(\varepsilon_2) d\varepsilon_2 - \int_{\varepsilon_2^{\min}}^{x_2-b_2+a_2\hat{p}_2(x_2)} hf_2(\varepsilon_2) d\varepsilon_2$$

The second order derivative of $V_{2,2}(x_2)$ is shown as follows.

$$\frac{d^2V_{2,2}(x_2)}{dx_2^2} = \frac{d\hat{p}_2(x_2)}{dx_2}[1-F_2(x_2-b_2+a_2\hat{p}_2(x_2))]$$
$$-[1+a_2\frac{d\hat{p}_2(x_2)}{dx_2}][\hat{p}_2(x_2)+h+\pi_2] f_2(x_2-b_2+a_2\hat{p}_2(x_2))$$

In order to prove $\frac{d^2V_{2,2}(x_2)}{dx_2^2} \le 0$, it suffices to prove that $-1 \le a_2 \frac{d\hat{p}_2(x_2)}{dx_2} \le 0$.

As stated in the assumption, $a_2 \ge 0$. From Lemma 3.3, $\frac{d\hat{p}_2(x_2)}{dx_2} \le 0$.

Also from (3.7), we can easily prove that

$$1+a_2\frac{d\hat{p}_2(x_2)}{dx_2} = \frac{1-F_2[x_2-b_2+a_2\hat{p}_2(x_2)]-2a_2\frac{d\hat{p}_2(x_2)}{dx_2}F_2[x_2-b_2+a_2\hat{p}_2(x_2)]}{a_2(\pi_2+\hat{p}_2(x_2)+h) f_2[x_2-b_2+a_2\hat{p}_2(x_2)]} \quad (3.11)$$

The right hand side of (3.11) is larger than zero since $\frac{d\hat{p}_2(x_2)}{dx_2} \le 0$. Therefore, it follows that $1+a_2(\frac{d\hat{p}_2(x_2)}{dx_2}) \ge 0$. In conclusion, $V_{2,2}(x_2)$ is concave with respect to x_2 when $x_2^m < x_2 < x_2^n$.

Case (3) $x_2 \le x_2^m$

$$V_{2,3}(x_2) = \int_{\varepsilon_2^{\min}}^{x_2 - b_2 + a_2 p_2^{\max}} [p_2^{\max}(b_2 - a_2 p_2^{\max} + \varepsilon_2) - h(x_2 - b_2 + a_2 p_2^{\max} - \varepsilon_2)] f_2(\varepsilon_2) d\varepsilon_2 \\ + \int_{x_2 - b_2 + a_2 p_2^{\max}}^{\varepsilon_2^{\max}} [p_2^{\max} x_2 - \pi_2 (b_2 - a_2 p_2^{\max} + \varepsilon_2 - x_2)] f_2(\varepsilon_2) d\varepsilon_2 \tag{3.12}$$

Since x_2 is independent of $p_2^{\max}$, the first and second order derivatives of (3.12) with respect to x_2 are given as follows:

$$\frac{dV_{2,3}(x_2)}{dx_2} = -\int_{\varepsilon_2^{\min}}^{x_2 - b_2 + a_2 p_2^{\max}} h f_2(\varepsilon_2) d\varepsilon_2 + \int_{x_2 - b_2 + a_2 p_2^{\max}}^{\varepsilon_2^{\max}} [p_2^{\max} + \pi_2] f_2(\varepsilon_2) d\varepsilon_2 \tag{3.13}$$

$$\frac{d^2 V_{2,3}(x_2)}{dx_2^2} = -(p_2^{\max} + h + \pi_2) f_2(x_2 - b + a p_2^{\max}) \le 0$$

Thus, the profit $V_{2,3}(x_2)$ is a concave function of x_2 when $x_2 \le x_2^m$.

Finally, we focus on the boundary conditions at the threshold values of x_2^m and x_2^n in order to show the overall concavity. At the thresholds x_2^m and x_2^n, $V_2(x_2)$ is continuous, which can be seen from (3.8), (3.9) and (3.12). Furthermore, it can easily be obtained that $\left.\frac{dV_{2,1}(x_2)}{dx_2}\right|_{x_2=(x_2^n)^+} = \left.\frac{dV_{2,2}(x_2)}{dx_2}\right|_{x_2=(x_2^n)^-}$ and $\left.\frac{dV_{2,2}(x_2)}{dx_2}\right|_{x_2=(x_2^m)^+} = \left.\frac{dV_{2,3}(x_2)}{dx_2}\right|_{x_2=(x_2^m)^-}$.

Therefore, we draw conclusion that the continuous profit function $V_2(x_2)$ is concave with respect to x_2. □

Let $V_2^{'}(x_2)$ and $V_2^{''}(x_2)$ denote the first and second order derivatives of $V_2(x_2)$ with respect to x_2. The following theorem computes the expected profit $J_1(y)$.

Theorem 3.1: The expected profit from Type 1 and Type 2 customers, $J_1(y)$, is concave with respect to *y*.

Proof: From Equation (3.3),

$$J_1(y) = \varphi_1(y) + \alpha E[V_2(x_2)] \tag{3.14}$$

where $x_2 = [y - t_1]^+$.

From the proof of Lemma 3.4, $V_2(x_2)$ is represented as follows.

$$V_2(x_2) = \begin{cases} V_{2,1}(y-t_1) \text{ obtained when } p_2^* = p_2^{\min} & y - t_1 \geq x_2^n \\ V_{2,2}(y-t_1) \text{ obtained when } p_2^* = \hat{p}_2 & x_2^m < y - t_1 < x_2^n \\ V_{2,3}(y-t_1) \text{ obtained when } p_2^* = p_2^{\max} & 0 \leq y - t_1 \leq x_2^m \\ V_{2,3}(0) \text{ obtained when } p_2^* = p_2^{\max} & 0 \leq y < t_1 \end{cases}$$

Expanding (3.14), we have

$$\begin{aligned} J_1(y) = {} & p_1 \int_0^y t_1 g_1(t_1) dt_1 + p_1 \int_y^\infty y g_1(t_1) dt_1 - h \int_0^y (y - t_1) g_1(t_1) dt_1 - \pi_1 \int_y^\infty (t_1 - y) g_1(t_1) dt_1 \\ & + \alpha \Big[\int_0^{y - x_2^n} V_{2,1}(y - t_1) g_1(t_1) dt_1 + \int_{y - x_2^n}^{y - x_2^m} V_{2,2}(y - t_1) g_1(t_1) dt_1 + \int_{y - x_2^m}^{y} V_{2,3}(y - t_1) g_1(t_1) dt_1 \\ & + \int_y^\infty V_{2,3}(0) g_1(t_1) dt_1 \Big] - cy \end{aligned}$$

By taking the derivatives of $J_1(y)$ with respect to *y*, we obtain

$$\begin{aligned} \frac{dJ_1}{dy} = {} & p_1 \int_y^\infty g_1(t_1) dt_1 - h G_1(y) + \pi_1 [1 - G_1(y)] + \alpha \Big[\int_0^{y - x_2^n} V_{2,1}'(y - t_1) g_1(t_1) dt_1 \\ & + \int_{y - x_2^n}^{y - x_2^m} V_{2,2}'(y - t_1) g_1(t_1) dt_1 + \int_{y - x_2^m}^{y} V_{2,3}'(y - t_1) g_1(t_1) dt_1 \Big] - c \end{aligned}$$

$$\frac{d^2 J_1}{dy^2} = -p_1 g_1(y) - (h+\pi_1) g_1(y) + \alpha[\int_0^{y-x_2^n} V_{2,1}^{"}(y-t_1) g_1(t_1) dt_1 + \int_{y-x_2^n}^{y-x_2^m} V_{2,2}^{"}(y-t_1) g_1(t_1) dt_1$$
$$+ \int_{y-x_2^m}^{y} V_{2,3}^{"}(y-t_1) g_1(t_1) dt_1 + V_{2,3}^{'}(0) g_1(y)]$$

Substituting $x_2 = 0$ into (3.13) and recalling $p_2^{\max} < \dfrac{b_2 + \varepsilon_2^{\min}}{a_2}$

$$V_{2,3}^{'}(0) = \left.\frac{dV_{2,3}(x_2)}{dx_2}\right|_{x_2=0} = -\int_{\varepsilon_2^{\min}}^{0-b_2+a_2 p_2^{\max}} hf_2(\varepsilon_2) d\varepsilon_2 + \int_{0-b_2+a_2 p_2}^{\varepsilon_2^{\max}} [p_2^{\max} + \pi_2] f_2(\varepsilon_2) d\varepsilon_2$$
$$= p_2^{\max} + \pi_2$$

Consequently,

$$\frac{d^2 J_1}{dy^2} = -(p_1 - \alpha p_2^{\max}) g_1(y) - (h + \pi_1 - \alpha\pi_2) g_1(y) + \alpha[\int_0^{y-x^n} V_{2,1}^{"}(y-t_1) g_1(t_1) dt_1$$
$$+ \int_{y-x^n}^{y-x^m} V_{2,2}^{"}(y-t_1) g_1(t_1) dt_1 + \int_{y-x^m}^{y} V_{2,3}^{"}(y-t_1) g_1(t_1) dt_1] \qquad (3.15)$$

The first two terms of (3.15) are always negative due to the assumptions that $p_1 > p_2^{\max}$ and $\pi_1 > \pi_2$. The last three terms are also always negative from Lemma 3.4. Therefore, the expected profit $J_1(y)$ is concave with respect to *y*. □

From Theorem 3.1, the unique order quantity y^* exists and the concavity of $J_1(y)$ with respect to *y* enables efficient algorithms such as gradient search to be employed to obtain y^*.

3.2.2.3 Bounds for y^* under general demand distributions

For other demand distributions that do not satisfy the conditions given in Lemma 3.3, $p_2^*(x_2)$ may not be a non-increasing function of x_2. Therefore, we provide bounds for the optimal order quantity y^*.

Denote y_1^* as the optimal order quantity obtained by solving the newsvendor problem only for Type 1 customers' demands with the associated revenue and cost parameters. Similarly, let $y_2^*(p_2)$ stand for the optimal order quantity computed by solving the newsvendor problem only for Type 2 customers' demands when the discounted price is p_2.

Proposition 3.1: An upper bound of y^* is equivalent to $\overline{y} = y_1^* + \underset{p_2}{Max}[y_2^*(p_2)]$**.**

The upper bound is obtained by dedicating different orders to fulfill Type 1 and Type 2 customers. This means orders placed for Type 2 customers cannot be consumed by Type 1 customers and vice versa. The overall problem is then reduced to two independent newsvendor problems, and it is obvious that the sum of these two optimal quantities will be an upper bound of y^*. Similarly, a lower bound of y^* is obtained as provided in Proposition 3.2.

Proposition 3.2: A lower bound of y^* is equivalent to $\underline{y} = y_1^*$**.**

3.2.2.4 Comparison of the expected profit from dynamic pricing and static pricing

In this section, we compute the expected profit from dynamic pricing with that from static pricing, where a constant discounted price is applied for the old product regardless of its inventory level.

Theorem 3.2: Given the same order quantity *y*, the expected profit from dynamic pricing, $J_1(y)$, is never worse than $J_1^{ST}(y)$, the expected profit from static pricing.

Proof: Given the same order quantity *y*, the expected profit for Type 1 customers in static pricing is equivalent to $\varphi_1(y)$ in dynamic pricing. Thus, the difference between $J_1(y)$ and $J_1^{ST}(y)$ is due to the difference in the maximum expected profit from Type 2 customers. Hence, it suffices to compare the maximum expected profit from Type 2 customers under two different pricing strategies.

Recall that $x_2 = [y - t_1]^+$ is the remaining stock available for Type 2 customers. Suppose that p_2^S is the optimal discounted price for static pricing and obtained at the beginning of Period 1 by solving a dynamic programming model developed for static pricing. Then the maximum expected profit from Type 2 customers, denoted by $V_2^{ST}(x_2)$, is computed as follows:

$$V_2^{ST}(x_2) = \varphi_2(x_2; p_2^S) - L_2(x_2; p_2^S)$$

For the same inventory level x_2, the following equation computes the maximum expected profit from Type 2 customers under dynamic pricing.

$$V_2(x_2) = \underset{p_2}{Max}\{\varphi_2(x_2;\, p_2) - L_2(x_2;\, p_2)\} \geq \varphi_2(x_2;\, p_2^S) - L_2(x_2;\, p_2^S) = V_2^{ST}(x_2) \qquad (3.16)$$

From (3.16), we prove that $V_2(x_2)$ is never worse than $V_2^{ST}(x_2)$. □

3.3 Numerical study for a product with a two period lifetime

In this section, we investigate how dynamic pricing performs under various parameters and this will lead us to identify when dynamic pricing provides a significant increase in the expected profit compared to static pricing. Moreover, the quality of the upper and the lower bounds for the optimal order quantity y^*, provided in Propositions 3.1 and 3.2, is examined.

3.3.1 Experimental design

In this numerical study, demand of Type 1 customers is assumed to follow a Normal distribution with mean μ_1 and variance σ_1^2. Demand of Type 2 customers is price-sensitive and has an additive stochastic demand function, i.e., $t_2 = \mu_2(p_2) + \varepsilon_2$, where $\mu_2(p_2) = b_2 - a_2 p_2$ is assumed to be a linear function of the discounted price p_2 and the noise variable ε_2 follows a truncated Normal distribution which is bounded by $\varepsilon_2^{\min} = -3\sigma_2$ and $\varepsilon_2^{\max} = 3\sigma_2$, where σ_2 is the standard deviation of the Normal distribution.

We are particularly interested in the effects of demand variability on the profit increase from dynamic pricing compared to static pricing. Thus, σ_1 and σ_2 are set to different levels, referring to different levels of demand variability. Different price sensitivities of Type 2 customers are also considered by changing the slope value a_2 of $\mu_2(p_2) = b_2 - a_2 p_2$. Two different values for the purchasing cost c are considered, as provided in Table 3.1.

The holding cost h is a constant value in each period as well as the retail price of products p_1. The values of the penalty costs π_1 and π_2 are provided in Table 3.2, while the feasibility condition $\pi_1 > \pi_2$ is satisfied. As stated in the assumptions, the lifetime of the product, M, is two periods.

Table 3.1 provides seven constants and their respective values. Table 3.2 summarizes the experimental variables and their respective values used in this study.

Table 3.1 Constants in the numerical study

Parameters	Values
p_1	25
μ_1	50
b_2	100
h	1
π_1	12
π_2	5
M	2

Table 3.2 Variables in the numerical study

Parameters	Values		
σ_1	10	20	30
σ_2	10	20	30
a_2	4	5	6
c	-	10	15

3.3.2 Profit increase from dynamic pricing

As shown in Theorem 3.2, dynamic pricing provides a higher profit than static pricing in all cases. The profit increase varies from 1% to 50%.

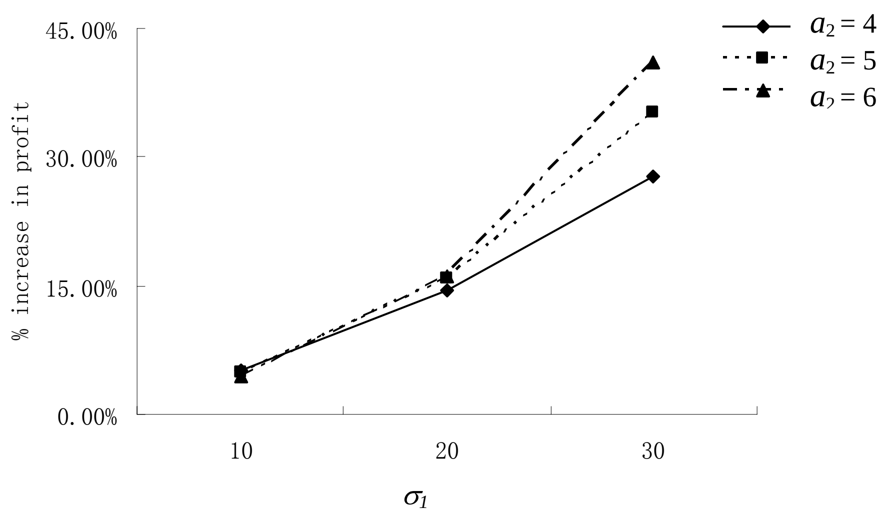

Figure 3.1 Profit increase from dynamic pricing under different σ_1 (when $\sigma_2 = 10$ and $c = 15$)

Given a fixed c and σ_2, we observe that the difference in total profit between dynamic pricing and static pricing increases as σ_1 increases. This trend is observed for all choices of σ_2 and c. The particular scenario satisfying $\sigma_2 = 10$ and $c = 15$ is shown in Figure 3.1. When σ_1 increases, the inventory level x_2 greatly fluctuates, and a more flexible pricing strategy is necessary to control demand of Type 2 customers so that excessive stockouts or stock expirations can be avoided. Dynamic pricing provides such flexibility.

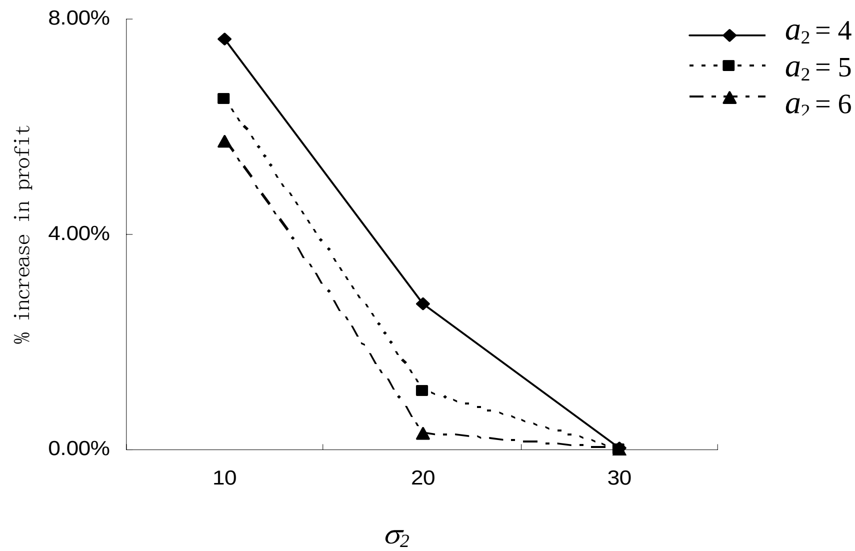

Figure 3.2 Profit increase from dynamic pricing under different σ_2 (when $\sigma_1 = 10$ and $c = 15$)

As σ_2 increases, we observe that the expected profit from both dynamic pricing and static pricing decrease, and the difference in total profit between the two pricing strategies diminishes. This is because the process of adjusting p_2 for demand of Type 2 customers becomes more difficult as σ_2 increases. Dynamic pricing, even though effectively controls demand of Type 2 customers, may still cause costly stockouts or stock expirations due to this high uncertainty. The particular scenario satisfying $\sigma_1 = 10$ and $c = 15$ is shown in Figure 3.2. The same trend is also observed for all choices of σ_1 and c.

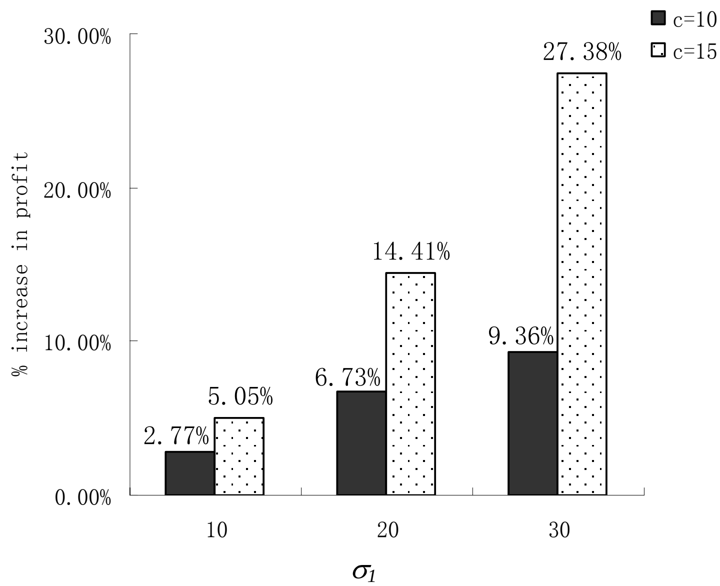

Figure 3.3 Profit increase from dynamic pricing under different σ_1 and c (when a_2 = 4 and σ_2 = 10)

Given a fixed a_2 and σ_2, we observe that the difference in total profit between dynamic pricing and static pricing increases as the purchasing cost c increases. This trend is observed for all choices of σ_2 and a_2. The particular scenario satisfying σ_2 = 10 and a_2 = 4 is shown in Figure 3.3. Implementing dynamic pricing, stock expirations as well as stockouts may be greatly reduced. This reduction in product wastage incurs higher cost savings when c is higher.

3.3.3 The upper and the lower bounds for y^*

From Proposition 3.1, an upper bound for y^* is equivalent to $\overline{y} = y_1^* + \underset{p_2}{Max}[y_2^*(p_2)]$, where p_2 is confined to the finite interval $[p_2^{\min}, p_2^{\max}]$. Hence, a closed form for this upper bound is $\overline{y} = y_1^* + y_2^*(p_2^{\min})$.

Given a fixed c and several cost constants shown in Table 3.2, the upper and the lower bounds for the optimal order quantity y^* are obtained under different values of price sensitivity a_2, as shown in Figure 3.4.

We observe that the upper bound decreases as the price sensitivity a_2 increases, which reduces the searching space for y^* and thus improves the computing speed.

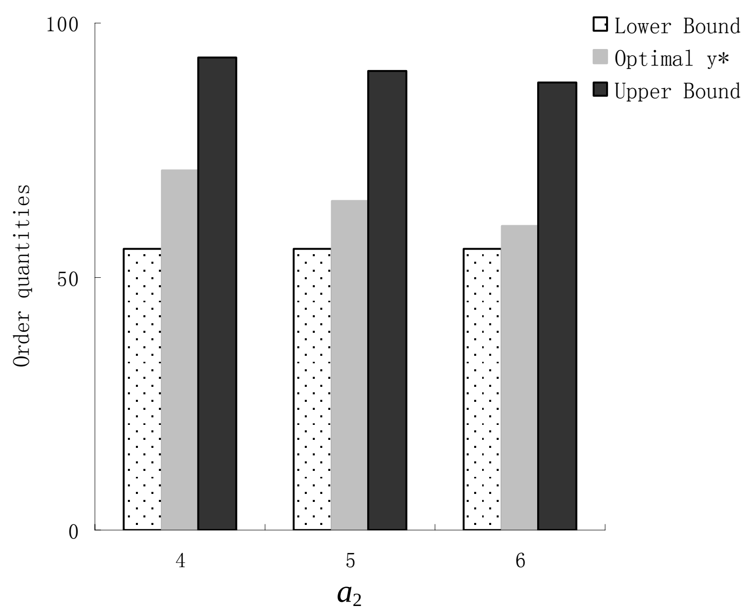

Figure 3.4 Comparisons of the bounds for y^* (when c = 10)

3.4 Pricing and ordering decisions for a product with an $M \geq 3$ period lifetime

In this section, we extend the previous results to a more general case where the lifetime of a perishable product is longer than two periods. This extension has strong practical implications. Under the proposed model in Section 3.2, there is only one chance where the decision maker could adjust the price for a markdown. However, in practice, retailers often employ successive markdowns to sell old products. The optimal prices for multiple markdowns are determined in this section. Furthermore, the optimal retail price and the optimal order quantity for the new product when it is first introduced to the market are determined as well.

3.4.1 Model assumptions

The lifetime of a perishable product is assumed to be longer than two periods, represented by $M \geq 3$. Hence for any given period, the inventory consists of stocks with M different ages, purchased by M independent demand classes. Type i customers purchase the products of age i in each period, for $i = 1,...,M$, while the age of new stock replenished is one.

The index k is defined to represent the period, for $k = 1,...,N$, where N is the number of studying periods. Let p_{ik} denote the discounted price for the product of age i at Period k. The corresponding demand is assumed to be dependent on p_{ik}, represented by a given linear stochastic demand function:

$$t_{ik} = b_i - a_i p_{ik} + \varepsilon_{ik} \qquad a_i > 0,\ b_i \geq -\varepsilon_{ik}^{\min} \quad i = 1,...,M \quad \text{and} \quad k = 1,...,N$$

where ε_{ik} is an *i.i.d.* random variable across different periods, which implies that demands of different types of customers at Period k are independent. The variable ε_{ik} is bounded in $[\varepsilon_{ik}^{\min}, \varepsilon_{ik}^{\max}]$.

Using the ideas of Observation 3.1, an N period problem can be reduced to an M period problem as follows: At Period i ($i = 1,..., M$), only the products of age i are sold and purchased by Type i customers. The retailer only replenishes at the beginning of Period 1, deciding the optimal order quantity x_1. At the beginning of Period i ($i = 1,..., M$), the optimal price for the remaining products of age i is determined.

Hence, demand of Type i customers and the discounted price for the product of age i can be simplified as t_i and p_i respectively. Similarly, ε_{ik} can be written as ε_i. The simplified linear stochastic demand function is shown as follows:

$$t_i = b_i - a_i p_i + \varepsilon_i$$

The discounted price p_i is confined to $[p_i^{\min}, p_i^{\max}]$, where $p_i^{\max} < \dfrac{b_i + \varepsilon_i^{\min}}{a_i}$ prevents negative demand t_i. We also assume that $p_{i-1}^{\min} \geq p_i^{\max}$, implying no overlap of the price intervals. The variable ε_i has a known probability density function $f_i(\varepsilon_i)$ and is bounded in $[\varepsilon_i^{\min}, \varepsilon_i^{\max}]$ satisfied with $E(\varepsilon_i) = 0$.

In case t_i exceeds the available inventory of age i, the excessive demand is lost.

Without loss of generality, we assume that $\pi_i > \pi_{i+1}$.

Apart from lost sales, we also assume that the stockouts can be satisfied by an "alternative" source (Lee et al., 2000 and Chew et al., 2006a). Under this assumption, if there is not enough stock to satisfy the demand, the retailer will meet the stockouts by obtaining some units from an “alternative” source with additional costs, representing the penalty cost to this stockouts.

This “alternative” source may be B2B marketplaces or third party manufacturers. With the advances of Electronic Data Interchange and rapid cargo transportation, “alternative” sources are easily identified and the leadtimes from the sources are often neglectable. Thus, compared with the long lead time for manufacturing (e.g., semi-conductor), we assume that the lead time from the “alternative” source is zero.

3.4.2 Pricing and ordering decisions under lost sales

3.4.2.1 Dynamic programming model

The dynamic programming model is developed to compute the expected profit given the inventory level for the product of age *i*, where $i = 1,...,M$.

$V_i^L(x_i)$, the maximum expected profit for the remaining periods when starting at Period *i* and with initial inventory x_i, is computed as follows:

$$V_i^L(x_i) = \underset{p_i}{Max}[\varphi_i^L(x_i;\, p_i) + \alpha E(V_{i+1}^L(x_{i+1}))] \quad \text{for } \; i = 1,...,M \tag{3.17}$$

$\varphi_i^L(x_i;\, p_i)$ represents the expected profit incurred at Period *i*, including the expected revenue, holding cost for excess inventory and penalty cost for unsatisfied demand $L_i(x_i;\, p_i)$.

$$\varphi_i^L(x_i;\, p_i) = p_i E[\min(x_i, t_i)] - L_i(x_i;\, p_i)$$

where $L_i(x_i; p_i) = hE[x_i - t_i]^+ + \pi_i E[t_i - x_i]^+$

The recursive function for the inventory level is $x_{i+1} = [x_i - t_i]^+$.

We denote $J_i^L(x_i; p_i)$ as the expected profit over the final *i* periods.

$$J_i^L(x_i; p_i) = \varphi_i^L(x_i; p_i) + \alpha E(V_{i+1}^L(x_{i+1})) \tag{3.18}$$

x_i and $V_i^L(x_i)$ are computed recursively backward in time, starting at Period *M* and ending at Period 1. The boundary condition $V_M^L(x_M) = \underset{p_M}{Max}[\varphi_M^L(x_M; p_M)]$ is the maximum expected profit during Period *M* given the initial inventory level x_M. Conversely, the value of $V_1^L(x_1) = \underset{p_1}{Max}[\varphi_1^L(x_1; p_1) + \alpha E(V_2^L(x_2))] - cx_1$ is the maximum expected profit over *M* periods when the initial inventory at Period 1 is x_1.

3.4.2.2 Optimal order quantity and optimal prices

In order to solve the dynamic programming model developed in Section 3.4.2.1 efficiently, $J_i^L(x_i; p_i)$ must be shown to be concave with respect to p_i for a given x_i. In addition, $V_i^L(x_i)$ must be concave with respect to x_i, for $i = 1,..., M$. We show the concavity starting from the last period and employ the backward recursive induction.

i) *i = M* (*last period*)

The optimal price p_M^* maximizes the expected profit for the last period, $J_M^L(x_M; p_M)$. The unsold products at the end of the last period have no salvage value.

Lemma 3.5:

(i) The expected profit $J_M^L(x_M; p_M)$ is concave with respect to p_M for a given x_M.

(ii) The optimal discounted price p_M^* is a non-increasing function of x_M when the hazard rate $\lambda_M(\varepsilon_M) \geq \dfrac{1}{a_M(p_M^{\min} + h + \pi_M)}$.

(iii) The maximum expected profit $V_M^L(x_M)$ is concave with respect to x_M.

Proof: (i) At Period *M*, $J_M^L(x_M; p_M)$ is shown as follows:

$$J_M^L(x_M; p_M) = \varphi_M^L(x_M; p_M) = p_M E[Min(x_M, t_M)] - L_M(x_M; t_M)$$

The first and second partial derivatives of $J_M^L(x_M; p_M)$ with respect to p_M are shown as follows.

$$\frac{\partial J_M^L}{\partial p_M} = \int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M p_M} (b_M - 2a_M p_M + \varepsilon_M - a_M h) f_M(x_M - b_M + a_M p_M) + (x_M + a_M \pi_M)[1 - F_M(x_M - b_M + a_M p_M)] \tag{3.19}$$

$$\frac{\partial^2 J_M^L}{\partial p_M^2} = -2a_M F_M(x_M - b_M + a_M p_M) - a_M^2 (p_M + h + \pi_M) f_M(x_M - b_M + a_M p_M)$$

Given an inventory level x_M, $J_M^L(x_M; p_M)$ is concave with respect to p_M since $\frac{\partial^2 J_M^L}{\partial p_M^2} \le 0$.

(ii) Let $\hat{p}_M$ denote the value of price p_M which satisfies $\frac{\partial J_M^L}{\partial p_M} = 0$ for a given x_M.

$$\left.\frac{\partial J_M^L}{\partial p_M}\right|_{p_M = \hat{p}_M} = \int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M \hat{p}_M} [b_M - 2a_M p_M + \varepsilon_M - a_M h] f_M(\varepsilon_M) d\varepsilon_M + \int_{x_M - b_M + a_M \hat{p}_M}^{\varepsilon_M^{\max}} [x_M + a_M \pi_M] f_M(\varepsilon_M) d\varepsilon_M = 0 \tag{3.20}$$

Note that (3.20) expresses the stationary point $\hat{p}_M$ as a function of x_M, denoted as $\hat{p}_M(x_M)$. Since $\hat{p}_M$ is bounded in $[p_M^{\min}, p_M^{\max}]$, the optimal discounted price p_M^* at Period M is determined as follows.

$$p_M^* = \begin{cases} p_M^{\min} & \hat{p}_M \le p_M^{\min} \\ \hat{p}_M & p_M^{\min} < \hat{p}_M < p_M^{\max} \\ p_M^{\max} & \hat{p}_M \ge p_M^{\max} \end{cases}$$

Taking the first order derivative of $\hat{p}_M(x_M)$ with respect to x_M based on (3.20) and rearranging the terms, we obtain

$$a_M \frac{d\hat{p}_M(x_M)}{dx_M} = \frac{1 - F_M[x_M - b_M + a_M \hat{p}_M(x_M)] - a_M(h + \hat{p}_M(x_M) + \pi_M) f_M[x_M - b_M + a_M \hat{p}_M(x_M)]}{2F_M[x_M - b_M + a_M \hat{p}_M(x_M)] + a_M(h + \hat{p}_M(x_M) + \pi_M) f_M[x_M - b_M + a_M \hat{p}_M(x_M)]} \tag{3.21}$$

Given that $\lambda_M(x_M - b_M + a_M \hat{p}_M) = \dfrac{f_M(x_M - b_M + a_M \hat{p}_M)}{1 - F_M(x_M - b_M + a_M \hat{p}_M)} \geq \dfrac{1}{a_M(p_M^{\min} + h + \pi_M)}$,

$\hat{p}_M \geq p_M^{\min}$ and the denominator of (3.21) is non-positive, hence $-1 \leq a_M \dfrac{d\hat{p}_M(x_M)}{dx_M} \leq 0$.

Therefore, it follows that p_M^* is a non-increasing function of the inventory level x_M.

(iii) Finally, we prove that $V_M^L(x_M)$ is concave with respect to x_M.

Let $V_M^L(x_M)$ be defined as follows.

$$V_M^L(x_M) = \begin{cases} V_{M,1}(x_M) \text{ obtained when } p_M^* = p_M^{\min} & x_M \geq x_M^n \\ V_{M,2}(x_M) \text{ obtained when } p_M^* = \hat{p}_M & x_M^m < x_M < x_M^n \\ V_{M,3}(x_M) \text{ obtained when } p_M^* = p_M^{\max} & x_M \leq x_M^m \end{cases}$$

where the thresholds x_M^n and x_M^m are calculated by setting (3.19) to be zero under the conditions $p_M = p_M^{\min}$ and $p_M = p_M^{\max}$.

Consider the following three cases:

Case (1) $x_M \geq x_M^n$

$$\begin{aligned} V_{M,1}(x_M) = & \int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M p_M^{\min}} [p_M^{\min}(b_M - a_M p_M^{\min} + \varepsilon_M) - h(x_M - b_M + a_M p_M^{\min} - \varepsilon_M)] f_M(\varepsilon_M) d\varepsilon_M \\ & + \int_{x_M - b_M + a_M p_M^{\min}}^{\varepsilon_M^{\max}} [p_M^{\min} x_M - \pi_M (b_M - a_M p_M^{\min} + \varepsilon_M - x_M)] f_M(\varepsilon_M) d\varepsilon_M \end{aligned} \tag{3.22}$$

The first and second order derivatives of (3.22) with respect to x_M are shown as follows:

$$\frac{dV_{M,1}}{dx_M} = -\int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M p_M^{\min}} hf_M(\varepsilon_M)d\varepsilon_M + \int_{x_M - b_M + a_M p_M^{\min}}^{\varepsilon_M^{\max}} (p_M^{\min} + \pi_M) f_M(\varepsilon_M)d\varepsilon_M \tag{3.23}$$

$$\frac{d^2V_{M,1}}{dx_M^2} = -(p_M^{\min} + h + \pi_M) f_M(x_M - b_M + a_M p_M^{\min}) \le 0$$

Thus, $V_{M,1}(x_M)$ is concave with respect to x_M when $x_M \ge x_M^n$.

Case (2) $x_M^n < x_M < x_M^m$

$$\begin{aligned} V_{M,2}(x_M) = & \int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M \hat{p}_M} [\hat{p}_M(b_M - a_M \hat{p}_M + \varepsilon_M) - h(x_M - b_M + a_M \hat{p}_M - \varepsilon_M)] f_M(\varepsilon_M)d\varepsilon_M \\ & + \int_{x_M - b_M + a_M \hat{p}_M}^{\varepsilon_M^{\max}} [\hat{p}_M x_M - \pi_M(b_M - a_M \hat{p}_M + \varepsilon_M - x_M)] f_M(\varepsilon_M)d\varepsilon_M \end{aligned} \tag{3.24}$$

The first and second order derivatives of (3.24) with respect to x_M are given as follows:

$$\frac{dV_{M,2}}{dx_M} = -\int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M \hat{p}_M} hf_M(\varepsilon_M)d\varepsilon_M + \int_{x_M - b_M + a_M \hat{p}_M}^{\varepsilon_M^{\max}} [\hat{p}_M + \pi_M] f_M(\varepsilon_M)d\varepsilon_M \tag{3.25}$$

$$\begin{aligned} \frac{d^2V_{M,2}}{dx_M^2} = & \frac{d\hat{p}_M(x_M)}{dx_M}[1 - F_M(x_M - b_M + a_M \hat{p}_M(x_M))] \\ & - [1 + a_M \frac{d\hat{p}_M(x_M)}{dx_M}](\hat{p}_M(x_M) + h + \pi_M) f_M(x_M - b_M + a_M \hat{p}_M(x_M)) \end{aligned} \tag{3.26}$$

Since $-1 \le a_M \dfrac{d\hat{p}_M(x_M)}{dx_M} \le 0$, (3.26) is negative. Therefore, $V_{M,2}(x_M)$ is concave with respect to x_M when $x_M^n < x_M < x_M^m$.

Case (3) $x_M \le x_M^m$

$$V_{M,3}(x_M) = \int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M p_M^{\max}} [p_M^{\max}(b_M - a_M p_M^{\max} + \varepsilon_M) - h(x_M - b_M + a_M p_M^{\max} - \varepsilon_M)] f_M(\varepsilon_M) d\varepsilon_M$$
$$+ \int_{x_M - b_M + a_M p_M^{\max}}^{\varepsilon_M^{\max}} [p_M^{\max} x_M - \pi_M(b_M - a_M p_M^{\max} + \varepsilon_M - x_M)] f_M(\varepsilon_M) d\varepsilon_M \tag{3.27}$$

Since x_M is independent of $p_M^{\max}$, the first and second order derivatives of (3.27) with respect to x_M are shown as follows:

$$\frac{dV_{M,3}}{dx_M} = -\int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M p_M^{\max}} h f_M(\varepsilon_M) d\varepsilon_M + \int_{x_M - b_M + a_M p_M^{\max}}^{\varepsilon_M^{\max}} (p_M^{\max} + \pi_M) f_M(\varepsilon_M) d\varepsilon_M \tag{3.28}$$

$$\frac{d^2 V_{M,3}}{dx_M^2} = -(p_M^{\max} + h + \pi_M) f_M(x_M - b_M + a_M p_M^{\max}) \le 0$$

Thus, $V_{M,3}(x_M)$ is concave with respect to x_M when $x_M \le x_M^m$.

Finally, we focus on the boundary conditions at the threshold values x_M^n and x_M^m in order to show overall concavity. At the thresholds x_M^n and x_M^m, $V_M^L(x_M)$ is continuous, which can be obtained from (3.22), (3.24) and (3.27). Furthermore, we can easily show that the gradients at x_M^n for cases (1) and (2) are the same. The same is true for the gradients at x_M^m for cases (2) and (3). Hence $V_M^L(x_M)$ is concave with respect to x_M. □

ii) $i = 1,\dots, M-1$

In order to complete the proof, Theorem 3.3 is shown in the followings:

Theorem 3.3: Assuming that $V_{i+1}^L(x_{i+1})$ is a continuous function and concave with respect to x_{i+1},

(i) The expected profit $J_i^L(x_i; p_i)$ is concave with respect to p_i for a given x_i.

(ii) The optimal discounted price p_i^* is a non-increasing function of x_i when the hazard rate $\lambda_i(\varepsilon_i) \geq \dfrac{1}{a_i(p_i^{\min} - \alpha p_{i+1}^{\max} + h + \pi_i - \alpha\pi_{i+1})}$.

(iii) The maximum expected profit $V_i^L(x_i)$ is concave with respect to x_i.

Proof: For the given assumption that $V_{i+1}^L(x_{i+1})$ is a continuous function and concave with respect to x_{i+1}, $V_{i+1}^L(x_{i+1})$ is represented as follows:

$$V_{i+1}^{L}(x_{i+1})=\begin{cases} V_{i+1,1}(x_i-t_i)\text{ obtained when } p_{i+1}^{*}=p_{i+1}^{\min} & x_i\geq t_i+x_{i+1}^{n} \\ V_{i+1,2}(x_i-t_i)\text{ obtained when } p_{i+1}^{*}=\hat{p}_{i+1} & t_i+x_{i+1}^{m}<x_i<t_i+x_{i+1}^{n} \\ V_{i+1,3}(x_i-t_i)\text{ obtained when } p_{i+1}^{*}=p_{i+1}^{\max} & t_i<x_i\leq t_i+x_{i+1}^{m} \\ V_{i+1,3}(0)\text{ obtained when } p_{i+1}^{*}=p_{i+1}^{\max} & x_i\leq t_i \end{cases}$$

where $x_{i+1}=[x_i-t_i]^{+}$ and $t_i=b_i-a_i p_i+\varepsilon_i$.

(i) It suffices to show that $\dfrac{\partial^2 J_i^L(x_i,p_i)}{\partial p_i^2}\leq 0$.

$$\begin{aligned} J_i^L(x_i,p_i)=\varphi_i^L(x_i,p_i)+\alpha[&\int_{\varepsilon_i^{\min}}^{x_i-x_{i+1}^{n}-b_i+a_i p_i}V_{i+1,1}(x_i-b_i+a_i p_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i \\ &+\int_{x_i-x_{i+1}^{n}-b_i+a_i p_i}^{x_i-x_{i+1}^{m}-b_i+a_i p_i}V_{i+1,2}(x_i-b_i+a_i p_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i \\ &+\int_{x_i-x_{i+1}^{m}-b_i+a_i p_i}^{x_i-b_i+a_i p_i}V_{i+1,3}(x_i-b_i+a_i p_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i+\int_{x_i-b_i+a_i p_i}^{\varepsilon_i^{\max}}V_{i+1,3}(0)f_i(\varepsilon_i)d\varepsilon_i] \end{aligned} \tag{3.29}$$

$$\begin{aligned} \frac{\partial^2 J_i^L}{\partial p_i^2}=&-2a_iF_i(x_i-b_i+a_i p_i)-a_i^2(p_i+h+\pi_i)f_i(x_i-b_i+a_i p_i) \\ &+\alpha a_i^2[\int_{\varepsilon_i^{\min}}^{x_i-x_{i+1}^{n}-b_i+a_i p_i}V_{i+1,1}^{''}(x_i-b_i+a_i p_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i \\ &+\int_{x_i-x_{i+1}^{n}-b_i+a_i p_i}^{x_i-x_{i+1}^{m}-b_i+a_i p_i}V_{i+1,2}^{''}(x_i-b_i+a_i p_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i \\ &+\int_{x_i-x_{i+1}^{m}-b_i+a_i p_i}^{x_i-b_i+a_i p_i}V_{i+1,3}^{''}(x_i-b_i+a_i p_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i+V_{i+1,3}^{'}(0)f_i(x_i-b_i+a_i p_i)] \end{aligned}$$

Note that $V_{i+1,3}^{'}(0)=\left.\dfrac{dV_{i+1,3}(x_i)}{dx_i}\right|_{x_i=0}=p_{i+1}^{\max}+\pi_{i+1}$ which is obtained by substituting $x_i=0$ in (3.28).

Since $\pi_i \geq \pi_{i+1}$ and $p_i \geq p_{i+1}^{\max}$, the sum of the 1st, 2nd and 6th terms is negative. Furthermore, the 3rd, 4th and 5th terms are less than zero, based on the assumption that $V_{i+1}^L(x_{i+1})$ is concave with respect to x_{i+1}. Therefore, $J_i^L(x_i;p_i)$ is concave with respect to p_i.

(ii) Let $\hat{p}_i$ denote the value of price p_i that satisfies the stationary condition $\dfrac{\partial J_i^L}{\partial p_i} = 0$.

$$\begin{aligned}
\left.\frac{\partial J_i^L}{\partial p_i}\right|_{p_i=\hat{p}_i} = & \int_{\varepsilon_i^{\min}}^{x_i-b_i+a_i\hat{p}_i} [b_i - 2a_i\hat{p}_i + \varepsilon_i - a_i h] f_i(\varepsilon_i) d\varepsilon_i + (x_i + a_i\pi_i)[1 - F_i(x_i - b_i + a_i\hat{p}_i)] \\
& + \alpha a_i [\int_{\varepsilon_i^{\min}}^{x_i - x_{i+1}^n - b_i + a_i\hat{p}_i} V_{i+1,1}^{'}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\
& + \int_{x_i - x_{i+1}^n - b_i + a_i\hat{p}_i}^{x_i - x_{i+1}^m - b_i + a_i\hat{p}_i} V_{i+1,2}^{'}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\
& + \int_{x_i - x_{i+1}^m - b_i + a_i\hat{p}_i}^{x_i - b_i + a_i\hat{p}_i} V_{i+1,3}^{'}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i] = 0
\end{aligned} \tag{3.30}$$

Note that (3.30) expresses the stationary point $\hat{p}_i$ as a function of x_i, denoted by $\hat{p}_i(x_i)$. Since $\hat{p}_i$ is bounded in $[p_i^{\min}, p_i^{\max}]$, we can determine the optimal discounted price at Period i, p_i^*, as follows.

$$p_i^* = \begin{cases} p_i^{\min} & \hat{p}_i \leq p_i^{\min} \\ \hat{p}_i & p_i^{\min} < \hat{p}_i < p_i^{\max} \\ p_i^{\max} & \hat{p}_i \geq p_i^{\max} \end{cases}$$

Taking the first order derivative of $\hat{p}_i(x_i)$ with respect to x_i based on (3.30) and rearranging the terms, we obtain

$$a_i \frac{d\hat{p}_i(x_i)}{dx_i} = \frac{N^L}{D^L}$$

where

$$N^L = 1 - F_i(x_i - b_i + a_i\hat{p}_i) - a_i(\hat{p}_i - \alpha p_{i+1}^{\max} + h + \pi_i - \alpha\pi_{i+1}) f_i(x_i - b_i + a_i\hat{p}_i) + \alpha \int_{\varepsilon_i^{\min}}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i + \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i$$

$$D^L = 2F_i(x_i - b_i + a_i\hat{p}_i) + a_i(\hat{p}_i - \alpha p_{i+1}^{\max} + h + \pi_i - \alpha\pi_{i+1}) f_i(x_i - b_i + a_i\hat{p}_i) - \alpha \int_{\varepsilon_i^{\min}}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i + \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i$$

and

$$\begin{aligned}\int_{\varepsilon_i^{\min}}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i = & \int_{\varepsilon_i^{\min}}^{x_i - x_{i+1}^n - b_i + a_i\hat{p}_i} V_{i+1,1}^{"}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\ & + \int_{x_i - x_{i+1}^n - b_i + a_i\hat{p}_i}^{x_i - x_{i+1}^m - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\ & + \int_{x_i - x_{i+1}^m - b_i + a_i\hat{p}_i}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i\end{aligned}$$

Given that the hazard rate

$$\lambda_i(x_i - b_i + a_i\hat{p}_i(x_i)) = \frac{f_i(x_i - b_i + a_i\hat{p}_i(x_i))}{1 - F_i(x_i - b_i + a_i\hat{p}_i(x_i))} \geq \frac{1}{a_i(p_i^{\min} - \alpha p_{i+1}^{\max} + h + \pi_i - \alpha\pi_{i+1})},$$

$\hat{p}_i(x_i) \geq p_i^{\min}$ and the denominator is non-positive, hence $-1 \leq a_i(\frac{d\hat{p}_i(x_i)}{dx_i}) \leq 0$.

Therefore, $\hat{p}_i(x_i)$ is a non-increasing function of the inventory level x_i. It follows that p_i^* is also a non-increasing function of the inventory level x_i.

(iii) Finally we prove that $V_i^L(x_i)$ is concave with respect to x_i.

$V_i^L(x_i)$ is shown as follows.

$$V_i^L(x_i) = \begin{cases} V_{i,1}(x_i) \text{ obtained when } p_i^* = p_i^{\min} & x_i \ge x_i^n \\ V_{i,2}(x_i) \text{ obtained when } p_i^* = \hat{p}_i & x_i^m < x_i < x_i^n \\ V_{i,3}(x_i) \text{ obtained when } p_i^* = p_i^{\max} & x_i \le x_i^m \end{cases}$$

where the thresholds x_i^m and x_i^n are calculated by satisfying $\frac{\partial J_i^L}{\partial p_i} = 0$ under the conditions that $p_i = p_i^{\min}$ and $p_i = p_i^{\max}$.

Finally, we focus on the boundary conditions at the threshold values x_i^m and x_i^n in order to show overall concavity. At the thresholds x_i^m and x_i^n, $V_i^L(x_i)$ is continuous, because $V_{i,1}(x_i^n) = V_{i,2}(x_i^n)$ and $V_{i,2}(x_i^m) = V_{i,3}(x_i^m)$.

Furthermore, it can easily be proved that $\left.\frac{dV_{i,1}(x_i)}{dx_i}\right|_{x_i=(x_i^n)^+} = \left.\frac{dV_{i,2}(x_i)}{dx_i}\right|_{x_i=(x_i^n)^-}$ and $\left.\frac{dV_{i,2}(x_i)}{dx_i}\right|_{x_i=(x_i^m)^+} = \left.\frac{dV_{i,3}(x_i)}{dx_i}\right|_{x_i=(x_i^m)^-}$. Therefore, we draw conclusion that the continuous profit function $V_i^L(x_i)$ is concave with respect to x_i. □

From (ii) in Theorem 3.3, we obtain the optimal pricing policy. The optimal discounted price at each period is determined based on the inventory level x_i. Since p_i^* is a non-increasing function of x_i, there must exist two thresholds x_i^m and x_i^n, satisfying the following conditions $x_i^m \le x_i^n$, $\hat{p}_i(x_i^m) = p_i^{\max}$ and $\hat{p}_i(x_i^n) = p_i^{\min}$. If $x_i \ge x_i^n$, the optimal price p_i^* equals to $p_i^{\min}$. If $x_i \le x_i^m$, the optimal price p_i^* is equivalent to $p_i^{\max}$.

From (iii) in Theorem 3.3, the unique optimal order quantity $y^* = x_1^*$ exists and the concavity of $V_i^L(x_i)$ with respect to x_i enables efficient searching algorithms to be employed.

3.4.3 Pricing and ordering decisions under "alternative" source

3.4.3.1 Dynamic programming model

The dynamic programming model is developed to compute the expected profit given the inventory level for the product of age *i*, where $i = 1,...,M$.

$V_i^A(x_i)$, the maximum expected profit for the remaining periods when starting at Period *i* and with initial inventory x_i, is computed as follows:

$$V_i^A(x_i) = \underset{p_i}{Max}[\varphi_i^A(x_i; p_i) + \alpha E(V_{i+1}^A(x_{i+1}))] \quad \text{for} \quad i = 1,...,M \tag{3.31}$$

$\varphi_i^A(x_i;p_i)$ represents the expected profit incurred at Period *i*, including the expected revenue, holding cost for excess inventory and penalty cost for unsatisfied demand $L_i(x_i;p_i)$.

$$\varphi_i^A(x_i;p_i) = p_i E(t_i) - L_i(x_i;p_i)$$

where $L_i(x_i;p_i) = hE[x_i - t_i]^+ + \pi_i E[t_i - x_i]^+$

The recursive function for the inventory level is $x_{i+1} = [x_i - t_i]^+$.

We denote $J_i^A(x_i;p_i)$ as the expected profit over the final *i* periods.

$$J_i^A(x_i;p_i) = \varphi_i^A(x_i;p_i) + \alpha E(V_{i+1}^A(x_{i+1})) \tag{3.32}$$

x_i and $V_i^A(x_i)$ are computed recursively backward in time, starting at Period *M* and ending at Period 1. The boundary condition $V_M^A(x_M) = \underset{p_M}{Max}[\varphi_M^A(x_M;p_M)]$ is the maximum expected profit during Period *M* given the initial inventory level x_M. Conversely, the value of $V_1^A(x_1) = \underset{p_1}{Max}[\varphi_1^A(x_1;p_1) + \alpha E(V_2^A(x_2))] - cx_1$ is the maximum expected total profit over *M* periods when the initial inventory at Period 1 is x_1.

3.4.3.2 Optimality properties

In order to solve the dynamic programming model developed in Section 3.4.3.1 efficiently, $J_i^A(x_i;p_i)$ must be shown to be concave with respect to p_i for a given x_i.

In addition, $V_i^A(x_i)$ must be concave with respect to x_i, for $i = 1,...,M$. We show the concavity starting from the last period and employ the backward recursive induction.

i) $i = M$ (*last period*)

The optimal price p_M^* maximizes the expected profit for the last period, $J_M^A(x_M; p_M)$. The unsold products at the end of the period are of no salvage value.

Lemma 3.6:

(i) The expected profit $J_M^A(x_M; p_M)$ is concave with respect to p_M for a given x_M.

(ii) The optimal discounted price p_M^* is a non-increasing function of x_M.

(iii) The maximum expected profit $V_M^A(x_M)$ is concave with respect to x_M.

Proof: (i) At Period M, the expected profit $J_M^A(x_M; p_M)$ is shown as follows:

$$J_M^A(x_M; p_M) = \varphi_M^A(x_M; p_M) = p_M E(t_M) - L_M(x_M; t_M)$$

The first and second partial derivatives of $J_M^A(x_M, p_M)$ with respect to p_M are shown as follows.

$$\frac{\partial J_M^A}{\partial p_M} = b_M - 2a_M p_M - a_M h F_M(x_M - b_M + a_M p_M) + a_M \pi_M [1 - F_M(x_M - b_M + a_M p_M)] \tag{3.33}$$

$$\frac{\partial^2 J_M^A}{\partial p_M^2} = -2a_M - a_M^2 (h + \pi_M) f_M(x_M - b_M + a_M p_M)$$

Given the inventory level x_M, $J_M^A(x_M, p_M)$ is concave with respect to p_M since $\frac{\partial^2 J_M^A}{\partial p_M^2} \le 0$.

(ii) Let $\hat{p}_M$ denote the value of price p_M which satisfies $\frac{\partial J_M^A}{\partial p_M} = 0$ for a given x_M.

$$\left.\frac{\partial J_M^A}{\partial p_M}\right|_{p_M = \hat{p}_M} = b_M - 2a_M \hat{p}_M - a_M h F_M(x_M - b_M + a_M \hat{p}_M) + a_M \pi_M [1 - F_M(x_M - b_M + a_M \hat{p}_M)] \tag{3.34}$$

Note that (3.34) expresses the stationary point $\hat{p}_M$ as a function of x_M, denoted as $\hat{p}_M(x_M)$. Since $\hat{p}_M$ is bounded in $[p_M^{\min}, p_M^{\max}]$, the optimal discounted price p_M^* at Period *M* is determined as follows.

$$p_M^* = \begin{cases} p_M^{\min} & \hat{p}_M \le p_M^{\min} \\ \hat{p}_M & p_M^{\min} < \hat{p}_M < p_M^{\max} \\ p_M^{\max} & \hat{p}_M \ge p_M^{\max} \end{cases}$$

Taking the first order derivative of $\hat{p}_M(x_M)$ with respect to x_M based on (3.34) and rearranging the terms, we obtain

$$a_M \frac{d\hat{p}_M(x_M)}{dx_M} = \frac{-a_M(h+\pi_M)f_M[x_M - b_M + a_M\hat{p}_M(x_M)]}{2 + a_M(h+\pi_M)f_M[x_M - b_M + a_M\hat{p}_M(x_M)]}$$

It is straightforward that $-1 \le \frac{a_M d\hat{p}_M(x_M)}{dx_M} \le 0$. Thus, $\hat{p}_M(x_M)$ is a non-increasing function of x_M. It follows that p_M^* is also a non-increasing function of x_M.

(iii) Next, we prove that $V_M^A(x_M)$ is concave with respect to x_M.

Let $V_M^A(x_M)$ be defined as follows.

$$V_M^A(x_M) = \begin{cases} V_{M,1}(x_M) \text{ obtained when } p_M^* = p_M^{\min} & x_M \ge x_M^n \\ V_{M,2}(x_M) \text{ obtained when } p_M^* = \hat{p}_M & x_M^m < x_M < x_M^n \\ V_{M,3}(x_M) \text{ obtained when } p_M^* = p_M^{\max} & x_M \le x_M^m \end{cases}$$

where the thresholds x_M^n and x_M^m are calculated by setting (3.33) to be zero under the conditions $p_M = p_M^{\min}$ and $p_M = p_M^{\max}$.

Consider the following three cases:

Case (1) $x_M \ge x_M^n$

$$\begin{aligned} V_{M,1}(x_M) = p_M^{\min}(b_M - a_M p_M^{\min}) - \int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M p_M^{\min}} h(x_M - b_M + a_M p_M^{\min} - \varepsilon_M) f_M(\varepsilon_M) d\varepsilon_M \\ - \int_{x_M - b_M + a_M p_M^{\min}}^{\varepsilon_M^{\max}} \pi_M(b_M - a_M p_M^{\min} + \varepsilon_M - x_M) f_M(\varepsilon_M) d\varepsilon_M \end{aligned} \quad (3.35)$$

The first and second order derivatives of (3.35) with respect to x_M are shown as follows:

$$\frac{dV_{M,1}}{dx_M} = -\int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M p_M^{\min}} hf_M(\varepsilon_M)d\varepsilon_M + \int_{x_M - b_M + a_M p_M^{\min}}^{\varepsilon_M^{\max}} \pi_M f_M(\varepsilon_M)d\varepsilon_M \tag{3.36}$$

$$\frac{d^2V_{M,1}}{dx_M^2} = -(h + \pi_M) f_M(x_M - b_M + a_M p_M^{\min}) \le 0$$

Thus, the profit $V_{M,1}(x_M)$ is concave with respect to x_M when $x_M \ge x_M^n$.

Case (2) $x_M^n < x_M < x_M^m$

$$\begin{aligned} V_{M,2}(x_M) = {} & \hat{p}_M(b_M - a_M \hat{p}_M) - \int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M \hat{p}_M} h(x_M - b_M + a_M \hat{p}_M - \varepsilon_M) f_M(\varepsilon_M)d\varepsilon_M \\ & - \int_{x_M - b_M + a_M \hat{p}_M}^{\varepsilon_M^{\max}} \pi_M(b_M - a_M \hat{p}_M + \varepsilon_M - x_M) f_M(\varepsilon_M)d\varepsilon_M \end{aligned} \tag{3.37}$$

The first and second order derivatives of (3.37) with respect to x_M are given as follows:

$$\frac{dV_{M,2}}{dx_M} = -\int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M \hat{p}_M} hf_M(\varepsilon_M)d\varepsilon_M + \int_{x_M - b_M + a_M \hat{p}_M}^{\varepsilon_M^{\max}} \pi_M f_M(\varepsilon_M)d\varepsilon_M \tag{3.38}$$

$$\frac{d^2V_{M,2}}{dx_M^2} = -[1 + a_M \frac{d\hat{p}_M(x_M)}{dx_M}](h + \pi_M) f_M(x_M - b_M + a_M \hat{p}_M) \tag{3.39}$$

Since $-1 \le a_M \dfrac{d\hat{p}_M(x_M)}{dx_M} \le 0$, (3.39) is negative. Therefore, the profit $V_{M,2}(x_M)$ is concave with respect to x_M when $x_M^n < x_M < x_M^m$.

Case (3) $x_M \le x_M^m$

$$V_{M,3}(x_M) = p_M^{\max}(b_M - a_M p_M^{\max}) - \int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M p_M^{\max}} h(x_M - b_M + a_M p_M^{\max} - \varepsilon_M) f_M(\varepsilon_M) d\varepsilon_M$$
$$- \int_{x_M - b_M + a_M p_M^{\max}}^{\varepsilon_M^{\max}} \pi_M (b_M - a_M p_M^{\max} + \varepsilon_M - x_M) f_M(\varepsilon_M) d\varepsilon_M \qquad (3.40)$$

Since x_M is independent of $p_M^{\max}$, the first and second order derivatives of (3.40) with respect to x_M are shown in (3.41) and (3.42).

$$\frac{dV_{M,3}}{dx_M} = -\int_{\varepsilon_M^{\min}}^{x_M - b_M + a_M p_M^{\max}} h f_M(\varepsilon_M) d\varepsilon_M + \int_{x_M - b_M + a_M p_M^{\max}}^{\varepsilon_M^{\max}} \pi_M f_M(\varepsilon_M) d\varepsilon_M \qquad (3.41)$$

$$\frac{d^2 V_{M,3}}{dx_M^2} = -(h + \pi_M) f_M(x_M - b_M + a_M p_M^{\max}) \le 0 \qquad (3.42)$$

Thus, the profit $V_{M,3}(x_M)$ is concave with respect to x_M when $x_M \le x_M^m$.

Finally, we focus on the boundary conditions at the threshold values x_M^n and x_M^m in order to show overall concavity. At the thresholds x_M^n and x_M^m, $V_M^A(x_M)$ is continuous, which can be obtained from (3.35), (3.37) and (3.40). Furthermore, we can easily show that the gradients at x_M^n for cases (1) and (2) are the same. The same is true for the gradients at x_M^m for cases (2) and (3). Hence $V_M^A(x_M)$ is concave with respect to x_M. □

ii) $i = 1,\ldots, M - 1$

In order to complete the proof, Theorem 3.4 is shown in the followings:

Theorem 3.4: Assuming that $V_{i+1}^A(x_{i+1})$ is a continuous function and concave with respect to x_{i+1},

(i) The expected profit $J_i^A(x_i; p_i)$ is concave with respect to p_i for a given x_i.

(ii) The optimal discounted price p_i^* is a non-increasing function of x_i.

(iii) The maximum expected profit $V_i^A(x_i)$ is concave with respect to x_i.

Proof: For the given assumption that $V_{i+1}^A(x_{i+1})$ is a continuous function and concave with respect to x_{i+1}, $V_{i+1}^A(x_{i+1})$ is represented as follows:

$$V_{i+1}^A(x_{i+1}) = \begin{cases} V_{i+1,1}(x_i - t_i) \text{ obtained when } p_{i+1}^* = p_{i+1}^{\min} & x_i \geq t_i + x_{i+1}^n \\ V_{i+1,2}(x_i - t_i) \text{ obtained when } p_{i+1}^* = \hat{p}_{i+1} & t_i + x_{i+1}^m < x_i < t_i + x_{i+1}^n \\ V_{i+1,3}(x_i - t_i) \text{ obtained when } p_{i+1}^* = p_{i+1}^{\max} & t_i < x_i \leq t_i + x_{i+1}^m \\ V_{i+1,3}(0) \text{ obtained when } p_{i+1}^* = p_{i+1}^{\max} & x_i \leq t_i \end{cases}$$

where $x_{i+1} = [x_i - t_i]^+$ and $t_i = b_i - a_i p_i + \varepsilon_i$.

(i) It suffices to show that $\frac{\partial^2 J_i^A(x_i, p_i)}{\partial p_i^2} \le 0$.

$$J_i^A(x_i, p_i) = \varphi_i^A(x_i, p_i) + \alpha[\int_{\varepsilon_i^{\min}}^{x_i - x_{i+1}^n - b_i + a_i p_i} V_{i+1,1}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i$$
$$+ \int_{x_i - x_{i+1}^n - b_i + a_i p_i}^{x_i - x_{i+1}^m - b_i + a_i p_i} V_{i+1,2}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \tag{3.43}$$
$$+ \int_{x_i - x_{i+1}^m - b_i + a_i p_i}^{x_i - b_i + a_i p_i} V_{i+1,3}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i + \int_{x_i - b_i + a_i p_i}^{\varepsilon_i^{\max}} V_{i+1,3}(0) f_i(\varepsilon_i) d\varepsilon_i]$$

$$\frac{\partial^2 J_i^A}{\partial p_i^2} = -2a_i - a_i^2 (h + \pi_i) f_i(x_i - b_i + a_i p_i)$$
$$+ \alpha a_i^2 [\int_{\varepsilon_i^{\min}}^{x_i - x_{i+1}^n - b_i + a_i p_i} V''_{i+1,1}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i$$
$$+ \int_{x_i - x_{i+1}^n - b_i + a_i p_i}^{x_i - x_{i+1}^m - b_i + a_i p_i} V''_{i+1,2}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i$$
$$+ \int_{x_i - x_{i+1}^m - b_i + a_i p_i}^{x_i - b_i + a_i p_i} V''_{i+1,3}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i + V'_{i+1,3}(0) f_i(x_i - b_i + a_i p_i)]$$

Note that $V'_{i+1,3}(0) = \left.\frac{dV_{i+1,3}(x_i)}{dx_i}\right|_{x_i = 0} = \pi_{i+1}$ which is obtained by substituting $x_i = 0$ in (3.41).

Since $\pi_i \ge \pi_{i+1}$ and $p_i \ge p_{i+1}^{\max}$, the sum of the 1st, 2nd and 6th terms is negative. Furthermore, the 3rd, 4th and 5th terms are less than zero, based on the assumption that $V_{i+1}^A(x_{i+1})$ is concave with respect to x_{i+1}. Therefore, $J_i^A(x_i; p_i)$ is concave with respect to p_i.

(ii) Let $\hat{p}_i$ denote the value of price p_i that satisfies the stationary condition $\frac{\partial J_i^A}{\partial p_i} = 0$.

$$\begin{aligned}
\left.\frac{\partial J_i^A}{\partial p_i}\right|_{p_i=\hat{p}_i} &= b_i - 2a_i\hat{p}_i - a_i hF_i(x_i - b_i + a_i\hat{p}_i) + a_i\pi_i[1 - F_i(x_i - b_i + a_i\hat{p}_i)] \\
&+ \alpha a_i[\int_{\varepsilon_i^{\min}}^{x_i - x_{i+1}^n - b_i + a_i\hat{p}_i} V'_{i+1,1}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\
&+ \int_{x_i - x_{i+1}^n - b_i + a_i\hat{p}_i}^{x_i - x_{i+1}^m - b_i + a_i\hat{p}_i} V'_{i+1,2}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\
&+ \int_{x_i - x_{i+1}^m - b_i + a_i\hat{p}_i}^{x_i - b_i + a_i\hat{p}_i} V'_{i+1,3}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i] = 0
\end{aligned} \tag{3.44}$$

Note that (3.44) expresses the stationary point $\hat{p}_i$ as a function of x_i, denoted by $\hat{p}_i(x_i)$. Since $\hat{p}_i$ is bounded in $[p_i^{\min}, p_i^{\max}]$, we can determine the optimal discounted price at Period i, p_i^*, as follows.

$$p_i^* = \begin{cases} p_i^{\min} & \hat{p}_i \le p_i^{\min} \\ \hat{p}_i & p_i^{\min} < \hat{p}_i < p_i^{\max} \\ p_i^{\max} & \hat{p}_i \ge p_i^{\max} \end{cases}$$

Taking the first order derivative of (3.44) with respect to x_i and rearranging the terms, we obtain

$$a_i \frac{d\hat{p}_i(x_i)}{dx_i} = \frac{N^A}{D^A}$$

where

$$N^A = -a_i[(h+\pi_i - \alpha\pi_{i+1}) f_i(x_i - b_i + a_i\hat{p}_i) - \alpha \int_{\varepsilon_i^{\min}}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i + \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i]$$

$$D^A = 2 + a_i[(h+\pi_i - \alpha\pi_{i+1}) f_i(x_i - b_i + a_i\hat{p}_i) - \alpha \int_{\varepsilon_i^{\min}}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i + \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i]$$

and

$$\begin{aligned}
\int_{\varepsilon_i^{\min}}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i = & \int_{\varepsilon_i^{\min}}^{x_i - x_{i+1}^n - b_i + a_i\hat{p}_i} V_{i+1,1}^{"}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\
& + \int_{x_i - x_{i+1}^n - b_i + a_i\hat{p}_i}^{x_i - x_{i+1}^m - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\
& + \int_{x_i - x_{i+1}^m - b_i + a_i\hat{p}_i}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}^{"}(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i
\end{aligned}$$

It is obvious that $-1 \le a_i \dfrac{d\hat{p}_i(x_i)}{dx_i} \le 0$. Therefore, $\hat{p}_i(x_i)$ is a non-increasing function of x_i. It follows that p_i^* is also a non-increasing function of x_i.

(iii) Next we prove that $V_i^A(x_i)$ is concave with respect to x_i.

$V_i^A(x_i)$ is shown as follows.

$$V_i^A(x_i) = \begin{cases} V_{i,1}(x_i) \text{ obtained when } p_i^* = p_i^{\min} & x_i \ge x_i^n \\ V_{i,2}(x_i) \text{ obtained when } p_i^* = \hat{p}_i & x_i^m < x_i < x_i^n \\ V_{i,3}(x_i) \text{ obtained when } p_i^* = p_i^{\max} & x_i \le x_i^m \end{cases}$$

where the thresholds x_i^m and x_i^n are calculated by satisfying $\dfrac{\partial J_i^A}{\partial p_i} = 0$ under the conditions that $p_i = p_i^{\min}$ and $p_i = p_i^{\max}$.

Finally, we focus on the boundary conditions at the threshold values x_i^m and x_i^n in order to show overall concavity. At the thresholds x_i^m and x_i^n, $V_i^A(x_i)$ is continuous, because $V_{i,1}(x_i^n) = V_{i,2}(x_i^n)$ and $V_{i,2}(x_i^m) = V_{i,3}(x_i^m)$.

Furthermore, it can easily be proved that $\left.\frac{dV_{i,1}(x_i)}{dx_i}\right|_{x_i=(x_i^n)^+} = \left.\frac{dV_{i,2}(x_i)}{dx_i}\right|_{x_i=(x_i^n)^-}$ and $\left.\frac{dV_{i,2}(x_i)}{dx_i}\right|_{x_i=(x_i^m)^+} = \left.\frac{dV_{i,3}(x_i)}{dx_i}\right|_{x_i=(x_i^m)^-}$. Therefore, we draw conclusion that the continuous profit function $V_i^A(x_i)$ is concave with respect to x_i. □

From (ii) in Theorem 3.4, we obtain the optimal pricing policy. The optimal discounted price at each period is determined based on the inventory level x_i. Since p_i^* is a non-increasing function of x_i, there must exist two thresholds x_i^m and x_i^n, satisfying the following conditions $x_i^m \le x_i^n$, $\hat{p}_i(x_i^m) = p_i^{\max}$ and $\hat{p}_i(x_i^n) = p_i^{\min}$. If $x_i \ge x_i^n$, the optimal price p_i^* equals $p_i^{\min}$. If $x_i \le x_i^m$, the optimal price p_i^* is equivalent to $p_i^{\max}$.

From (iii) in Theorem 3.4, the unique optimal order quantity $y^* = x_1^*$ exists and the concavity of $V_i^A(x_i)$ with respect to x_i enables efficient searching algorithms to be employed.

3.4.4 Comparison of the maximum expected profit under "alternative" source and lost sales

In this section, we compute the maximum expected profit from the dynamic programming model under the assumptions of "alternative" source and lost sales, where π_i^A (π_i^L) represents the penalty cost under the assumption of alternative source (lost sales).

Theorem 3.5: When $\pi_i^A \le p_i^{\min} + \pi_i^L$, the maximum expected profit from the dynamic programming model under "alternative" source, $V_i^A(x_i)$, is greater than or equals to $V_i^L(x_i)$, the maximum expected profit from the dynamic programming model under lost sales for a given inventory level x_i.

Proof: We prove that $V_i^A(x_i) \ge V_i^L(x_i)$ for $i = 1,\ldots,M$ by induction.

At Period M, given the inventory level x_M, the difference between $V_M^A(x_M)$ and $V_M^L(x_M)$ is computed as follows:

$$J_M^A(x_M, p_M) - J_M^L(x_M, p_M) = p_M E(t_M) - L_M(x_M, t_M) \\ - [p_M E(t_M) - L_M(x_M, t_M) - p_M E(t_M - x_M)^+] \ge 0 \tag{3.45}$$

Given that $\pi_M^A \le p_M^{\min} + \pi_M^L$, Equation (3.45) is larger or equivalent to zero.

From (3.45), $V_M^A(x_M) = \underset{p_M}{Max}[J_M^A(x_M, p_M)] \ge \underset{p_M}{Max}[J_M^L(x_M, p_M)] = V_M^L(x_M)$.

In order to complete the proof, we must show that $V_i^A(x_i) \geq V_i^L(x_i)$, assuming that $V_{i+1}^A(x_{i+1}) \geq V_{i+1}^L(x_{i+1})$.

For Period i ($i = M - 1,\ldots,1$), given the inventory level x_i, the difference between $V_i^A(x_i)$ and $V_i^L(x_i)$ is:

$$\begin{aligned} J_i^A(x_i, p_i) - J_i^L(x_i, p_i) = {} & p_i E(t_i) - L_i(x_i, t_i) + \alpha E(V_{i+1}^A(x_{i+1}) \\ & - [p_i E(t_i) - L_i(x_i, t_i) + \alpha E(V_{i+1}^L(x_{i+1}) - p_i E(t_i - x_i)^+] \geq 0 \end{aligned} \quad (3.46)$$

From (3.46), we obtain that $V_i^A(x_i) = \underset{p_i}{Max}[J_i^A(x_i, p_i)] \geq \underset{p_i}{Max}[J_i^L(x_i, p_i)] = V_i^L(x_i)$.

Thus, Theorem 3.5 is proven. □

Theorem 3.5 provides an upper bound for the maximum expected profit under the lost sales assumption. From (iii) in Theorem 3.4, this upper bound can be efficiently computed.

3.5 Numerical study for a product with an $M \geq 3$ period lifetime

In this section, we investigate how the upper bound obtained in Section 3.4.4 performs under different levels of demand variability. In addition, the optimal order quantity is computed under two different assumptions, “alternative” source and lost sales. Furthermore, we compute the maximized expected profit from both dynamic pricing and static pricing under the assumption of “alternative” source.

3.5.1 Experimental design

In this numerical study, we consider a product with lifetime of three periods. Demand of Type i (i = 1, 2 ,3) customers is price-sensitive and has an additive stochastic demand function, i.e., $t_i = \mu_i(p_i) + \varepsilon_i$, where $\mu_i(p_i) = b_i - a_i p_i$ is assumed to be a linear function of the price p_i and the noise variable ε_i follows a truncated Normal distribution which is bounded by $\varepsilon_i^{\min} = -3\sigma_i$ and $\varepsilon_i^{\max} = 3\sigma_i$, where σ_i is the standard deviation of the Normal distribution.

We are particularly interested in the effects of demand variability on the performance of this upper bound. Thus, σ_1, σ_2 and σ_3 are set to different values, referring to different levels of demand variability.

Table 3.3 summarizes the experimental variables and their respective values used in this numerical study. Several constants and their respective values are also provided in Table 3.4.

Table 3.3 Variables in the numerical study

Parameters	Values		
σ_1	0.1*b_1	0.2*b_1	0.3*b_1
σ_2	0.1*b_2	0.2*b_2	0.3*b_2
σ_3	0.1*b_3	0.2*b_3	0.3*b_3

Table 3.4 Constants in the numerical study

Parameters	Values	Parameters	Values
b_1	72	a_1	6
b_2	40	a_2	5
b_3	24	a_3	4
π_1	11.5	h	0.2
π_2	7.5	c	2
π_3	5.5	M	3

3.5.2 Comparison of the maximum profit under "alterative" source and lost sales

As shown in Theorem 3.5, the maximum expected profit obtained under "alternative" source is never worse than that obtained under lost sales. The ratio of the maximum expected profit under lost sales, to the maximum expected profit under "alternative" source is between 91% and 97% under different levels of demand variability.

The ratio decreases as the demand variabilities σ_1, σ_2 and σ_3 increase. For example, when σ_1 increases, the inventory level x_2 greatly fluctuates. Dynamic pricing under lost sales obtains less expected revenue compared to dynamic pricing under "alternative" source. Due to this uncertainty, the ratio decreases. The particular scenario satisfying $\sigma_2 = 0.1*b_2$ and $\sigma_3 = 0.1*b_3$ is shown in Figure 3.5.

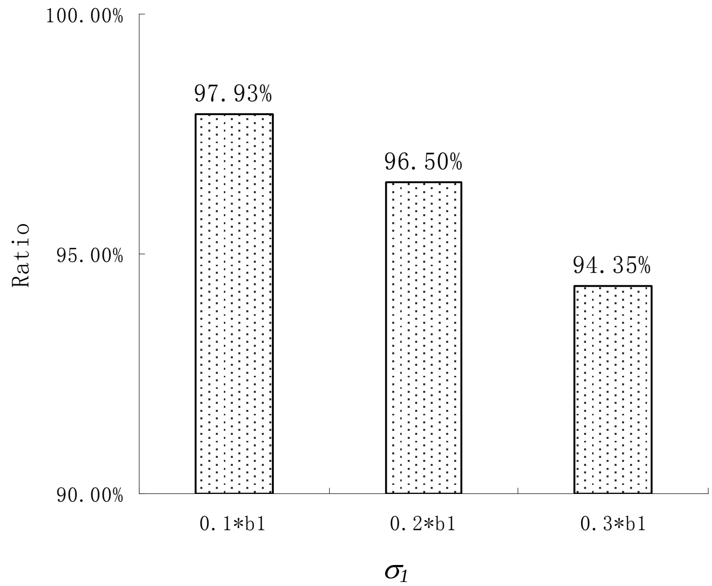

Figure 3.5 Ratio under different σ_1 (when $\sigma_2 = 0.1{*}b_2$ and $\sigma_3 = 0.1{*}b_3$)

The optimal order quantity obtained from the dynamic programming model under lost sales is greater than that from the dynamic programming model under “alternative” source. This phenomenon should be explained as follows. For the assumption of lost sales, more products are ordered to avoid the excessive demand. However, “alternative” source has a second chance to purchase the products, there is no need to build high inventory to buffer the unexpected demand. This trend is observed for all combinations of σ_1, σ_2 and σ_3. The particular scenario satisfying $\sigma_2 = 0.1{*}b_2$ and $\sigma_3 = 0.1{*}b_3$ is shown in Figure 3.6.

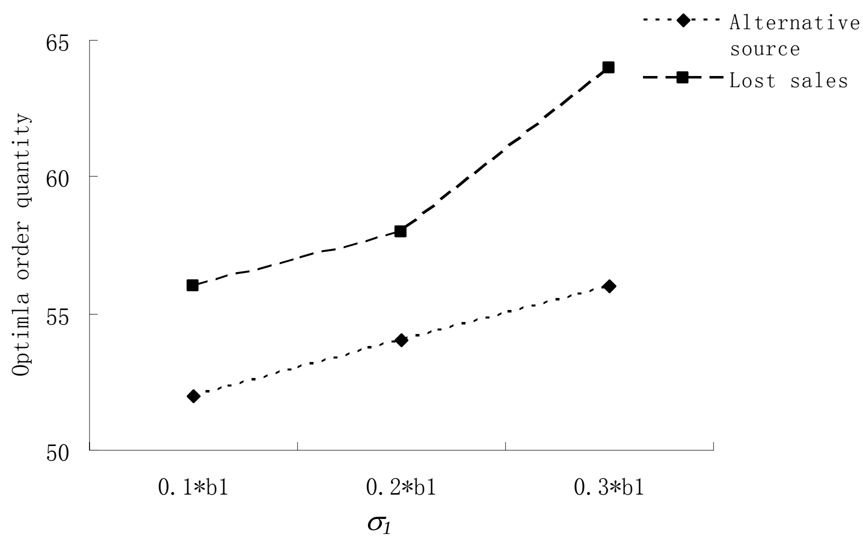

Figure 3.6 Optimal order quantity under different σ_1 (when $\sigma_2 = 0.1*b_2$ and $\sigma_3 = 0.1*b_3$)

3.5.3 Profit increase from dynamic pricing under "alternative" source

Our numerical results show that the expected profit from dynamic pricing is never worse than that from static pricing under the "alternative" source assumption. The profit increase from dynamic pricing becomes more significant as the demand variability σ_1 becomes higher. The particular scenario satisfying $\sigma_2 = 0.1*b_2$ and $\sigma_3 = 0.1*b_3$ is shown in Figure 3.7. The same trend is also observed for all choices of σ_2 and σ_3. When σ_1 increases, the inventory level x_2 greatly fluctuates, and a more flexible pricing strategy is necessary to control demand of Type 2 customers so that excessive stockouts or stock expirations can be avoided. Dynamic pricing provides such flexibility.

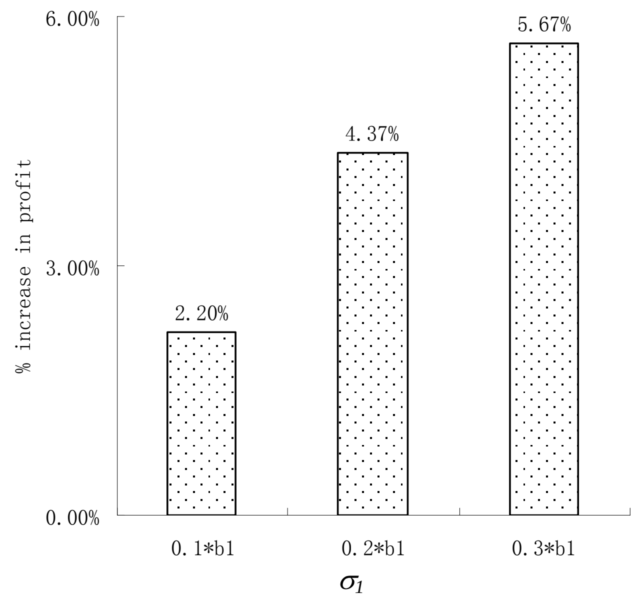

Figure 3.7 Profit increase from dynamic pricing under different σ_1
(when $\sigma_2 = 0.1*b_2$ and $\sigma_3 = 0.1*b_3$)

As σ_2 increases, the difference in the expected profit between dynamic pricing and static pricing may become greater or smaller. When σ_2 increases, the inventory level x_3 greatly fluctuates. Dynamic pricing could flexibly control demand of Type 3 customers so that excessive stockouts or stock expirations can be avoided. However, at the same time, the process of adjusting p_2 for demand of Type 2 customers becomes more difficult, which may cause more stockouts or stock expirations. Hence, it is hard to identify whether the difference in the expected profit under two different pricing strategies will increase or decrease, as σ_2 increases.

As σ_3 increases, we observe that the expected profit from both dynamic pricing and static pricing decrease, and the difference in the expected profit between these two pricing strategies diminishes. This is because the process of adjusting p_3 for demand of Type 3 customers becomes more difficult as σ_3 increases. Since σ_3 is large, dynamic pricing, even though controls demand of Type 3 customers, may still cause costly stockouts or stock expirations due to this high uncertainty. The particular scenario satisfying σ_1 = 0.1*b_1 and σ_2 = 0.1*b_2 is shown in Figure 3.8. The same trend is also observed for all choices of σ_1 and σ_2.

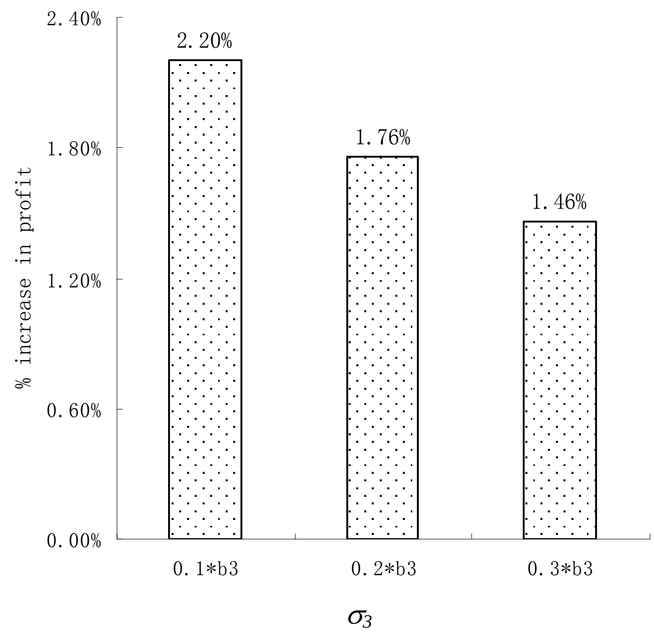

Figure 3.8 Profit increase from dynamic pricing under different σ_3
(when $\sigma_1 = 0.1*b_2$ and $\sigma_2 = 0.1*b_2$)

3.6 Summary

In this study, we first develop a discrete time dynamic programming model for a perishable product with a two period lifetime. Under certain conditions, the optimal discounted price for the old product is a non-increasing function of the inventory level. From this property, we prove that the expected profit is a concave function with respect to the order quantity for the new product. This concavity enables efficient algorithms to be employed to obtain the optimal order quantity for the new product. Even when this property does not hold, still an upper and a lower bound for the optimal order quantity are provided. We also prove that the expected profit from dynamic pricing is never worse than the expected profit from static pricing. The computational results show that the profit increase from dynamic pricing becomes more significant as the demand uncertainty of Type 1 customers and the purchasing cost become higher.

We further consider a more general problem, where the lifetime of the product is longer than two periods. The problem is analyzed under two different assumptions, lost sales and “alternative” source. For each case, a dynamic programming model is developed with the objective of maximizing the total profit over the finite number of periods. The optimal prices for products of different ages and the optimal order quantity for the new product are obtained. Moreover, we prove that the maximum expected profit under “alternative” source is never worse than the one under lost sales under certain conditions. Our numerical results show that the ratio of the maximum expected profit from lost sales, to the maximum expected profit from “alternative” source is between 91% and 97% under different levels of demand variability. In addition, the optimal order quantity obtained from the dynamic programming model under lost sales is greater than the one from the dynamic programming model under “alternative” source.

Chapter 4 Optimal dynamic pricing and ordering decisions for perishable products

Chapter 4 extends the work of Chapter 3 by considering substitution among products of different ages and the corresponding demand transfers between demand classes. In Section 4.2, the assumptions and notation are provided. A product with the lifetime of two or more periods is considered and the dynamic programming model for a multiple period profit maximization problem is developed. In Section 4.3, the model for the product with the lifetime of two periods is analyzed. The computational results for the product with the lifetime of two periods are presented in Section 4.4. For a product with the lifetime of longer than two periods, a heuristic based on the optimal solution for a single period problem is proposed in Section 4.5.

4.1 Introduction

Companies today are facing the increasingly volatile business environments, characterized by shorter product life cycles and ever quickening technological developments. In order to achieve competitive edges, new (versions of) products must frequently be introduced to the market. When new versions of products enter the market, old (versions of) products may be offered at discounted prices. This discount enables a quick reduction of the inventory and is easily found in practice, such as in electronics and automobile industries. The retail price of new products as well as the discounted prices of old products must carefully be determined. If the prices for new and old products are sufficiently close, the customers may decide which products to purchase based on the prices of both products, rather than the price of the target products only. For example, a customer intending to purchase a newer version product and finding it too expensive may purchase an attractively priced older version product, instead. Thus, in order to maximize the profit, the price of a new product and the discounted prices of old products must simultaneously be determined, considering such demand transfers between new and old products.

This chapter considers a finite horizon problem for a perishable product with a limited period lifetime, where substitution among products of different ages is allowed. Demands for products of different ages are assumed to be dependent on the prices of itself and substitutable products, i.e., products of "neighboring ages". The products of neighboring ages are defined by the products that are a period older or younger than the target products. A periodic review policy is used. The objective is to find the optimal prices for products of different ages and the optimal order quantity for a new product with the objective of maximizing the total profit over the multiple periods.

4.2 Problem formulation

In this section, we consider a perishable product with an M period lifetime. Let index $i = 1,\ldots,M$ denotes the ages of the products, where $i = 1$ represents that the product is new. Hence in any period, there exist products of M different ages. The following notation is employed in this chapter:

y = order quantity for a new product

x_i = inventory level for a product of age i , $i = 1,\ldots,M$

p_1 = retail price of a new product

p_i = discounted price for a product of age i, $i = 2,\ldots,M$

π_i = penalty cost for a product of age i, $i = 1,\ldots,M$

h = holding cost per period (regardless of ages)

c = purchasing cost for a new product

α = discounted factor per period

We assume that each aged product is purchased by a distinctive demand class. For products of age i, the price that the customers from a respective demand class, demand class i, are willing to pay is assumed to be confined in an interval $[p_i^{\min}, p_i^{\max}]$. Moreover, price intervals of demand classes are non-overlapping with $p_i > p_{i+1}$. Even though demand classes are categorized by these price intervals, we allow the customers of each class to move up or down to neighboring demand classes, depending on the differential pricing. In particular, demand for class i is dependent on p_{i-1}, p_i, p_{i+1} and is represented by a given linear stochastic demand function

$$\begin{aligned} t_i &= \mu_i(p_{i-1}, p_i, p_{i+1}) + \varepsilon_i \\ &= b_i - a_i p_i + l_{i+1,i} p_{i+1} + l_{i-1,i} p_{i-1} + \varepsilon_i \end{aligned} \tag{4.1}$$

where $l_{10} = l_{01} = l_{M+1,M} = l_{M,M+1} = 0$

$\mu_i(p_{i-1}, p_i, p_{i+1})$ is mean demand for class i, $b_i - a_i p_i + l_{i+1,i} p_{i+1} + l_{i-1,i} p_{i-1}$, and satisfied with $a_i, l_{i+1,i}, l_{l-1,i} \geq 0$. ε_i *is* an *i.i.d*. random variable with a known probability density function $f_i(\varepsilon_i)$ and is bounded in $[\varepsilon_i^{\min}, \varepsilon_i^{\max}]$. In addition, $E(\varepsilon_i) = 0$, where $b_i > -\varepsilon_i^{\min}$.

Note that $l_{i,i+1}$ is the transfer rate (demand transfer per unit price increase) of demand class i to demand class i+1 with respect to the price differences between the respective demand classes, and a_i represents the loss rate (demand loss per unit price increase) of demand class i with respect to p_i.

In our proposed demand function, we allow demand transfers, i.e., the demand class i customers may purchase products of ages i-1 and i+1 instead, which transfers demand of class i to demands of classes i-1 and i+1. Thus, the products of different ages considered in our model can be treated as different products. These products can be substituted by each other to a certain extent, depending on the attractiveness of the degree in the pricing differences. Without loss of generality, we assume that $\pi_i > \pi_{i+1}$.

From the demand function given in (4.1), we note that the substitutability among different products is caused by price differences, not by shortages in one product. The shortage in one product is assumed to be satisfied by an "alternative" source (Lee et al., 2000). Under this assumption, if there is not enough stock to satisfy the demand, the retailer will meet the stockouts by obtaining some units from an "alternative" source with additional costs, representing the penalty cost to these stockouts.

The dynamic programming model is developed to compute the expected profit given the inventory levels for products of M different ages. The index k is defined to represent the period, for k = 1 ,…, N, where N is the number of studying periods.

$V_k(x_{2k},...,x_{Mk})$, the maximum expected profit for the remaining periods, starting at Period k and with the inventory levels $(x_{2k},...,x_{Mk})$, is computed as follows:

$$V_k(x_{2k},...,x_{Mk}) = \underset{y_k, p_{1k},...,p_{Mk}}{Max} [\varphi_k(x_{2k},...,x_{Mk}; y_k, p_{1k},...,p_{Mk}) + \alpha E[V_{k+1}(x_{2k+1},...,x_{Mk+1})]] \quad (4.2)$$

where the recursive function for the inventory level is $x_{2k+1} = [y_k - t_{1k}]^+$ and $x_{i+1,k+1} = [x_{ik} - t_{ik}]^+$ for i = 2 ,…, M – 1.

$\varphi_k(x_{2k},...,x_{Mk}; y_k, p_{1k},...,p_{Mk})$ represents the expected profit for products of M different ages in Period k. The expected profit is obtained by computing the expected revenue $R_k(p_{1k},...,p_{Mk})$, the expected cost $C_k(x_{2k},...,x_{Mk}; y_k, p_{1k},...,p_{Mk})$ and the purchasing cost for the new product. $L_{ik}(x_{ik}, t_{ik})$ represents the expected cost for a product of age i at Period k.

$$\varphi_k(x_{2k},...,x_{Mk}; y_k, p_{1k},...,p_{Mk}) = R_k(p_{1k},...,p_{Mk}) - C_k(x_{2k},...,x_{Mk}; y_k, p_{1k},...,p_{Mk}) - cy_k$$

(4.3)

where $R_k(p_{1k},...,p_{Mk}) = \sum_{i=1}^{M} p_{ik} E(t_{ik})$

$$C_k(x_{2k},...,x_{Mk}; y_k, p_{1k},...,p_{Mk}) = L_{1k}(y_k, t_{1k}) + \sum_{i=2}^{M} L_{ik}(x_{ik}, t_{ik})$$

$$L_{ik}(x_{ik}, t_{ik}) = hE[x_{ik} - t_{ik}]^{+} + \pi_{ik} E[t_{ik} - x_{ik}]^{+} \quad \text{for} \quad i = 1,\ldots,M$$

We denote $J_k(x_{2k,}..., x_{Mk}; y_k, p_{1k},..., p_{Mk})$ as the expected profit over the last k periods.

$$J_k(x_{2k},..., x_{Mk}; y_k, p_{1k},..., p_{Mk}) = \varphi_k(x_{2k},..., x_{Mk}; y_k, p_{1k},..., p_{Mk}) + \alpha E[V(x_{2k+1},..., x_{Mk+1})] \tag{4.4}$$

These optimality functions are computed recursively backward in time, starting at Period N and ending at Period 1. The boundary condition $V_N(x_{2N},..., x_{MN}) = \underset{y_N, p_{1N},..., p_{MN}}{Max}[\varphi_N(x_{2N},..., x_{MN}; y_N, p_{1N},..., p_{MN})]$ is the maximum expected profit for Period N (the last period), given the inventory level $(x_{2N},..., x_{MN})$. Conversely, the value of $V_1(x_{21},..., x_{M1})$ is the maximum expected profit over N periods when the initial inventory at Period 1 is $(x_{21},..., x_{M1})$.

4.3 Pricing and ordering decisions for a product with a two period lifetime

In this section, a multiple period problem for a product with a two period lifetime (M = 2) is considered, where demands for both new and old products are dependent on the retail price of the new product (product of age 1) as well as the discounted price of the old product (product of age 2).

4.3.1 Additional assumption

For the product with lifetime of M = 2, we assume $a_1, a_2 \geq l_{1,2} \geq l_{2,1}$. The assumption of $a_1 \geq l_{1,2}$ ($a_2 \geq l_{2,1}$) ensures that the demand transfer from class 1 (2) to classes 2 (1) is less than or equal to demand loss of class 1 (2). $l_{1,2} \geq l_{2,1}$ holds because the customers may want to purchase an attractively priced old product, instead of a new product. However, the customers who intend to purchase an old product seldom purchase an expensive new product, instead.

4.3.2 Multiple period problem

In order to solve the dynamic programming model developed in Section 2 (when M = 2) efficiently, $J_k(x_{2k}; y_k, p_{1k}, p_{2k})$ must be shown to be jointly concave with respect to y_k, p_{1k} and p_{2k}, given an inventory level x_{2k}. In addition, $V_k(x_{2k})$ must be concave with respect to x_{2k} for k = 1 ,…, N. We show the concavity starting from the last period and by backward recursive induction.

i) *k = N* (*Last Period*)

The optimal solution y_N^*, p_{1N}^* and p_{2N}^* maximizes the expected profit for the last Period, $\varphi_N(x_{2N}; y_N, p_{1N}, p_{2N})$. The unsold products at the end of the period have no salvage value.

Theorem 4.1: $\varphi_N(x_{2N}; y_N, p_{1N}, p_{2N})$ is jointly concave with respect to y_N, p_{1N} and p_{2N}, given an inventory level x_{2N}.

Proof: H_N represents the Hessian Matrix of $\varphi_N(x_{2N}; y_N, p_{1N}, p_{2N})$ and its determinant is as follows:

$$\det(H_N) = \begin{vmatrix} \frac{\partial^2 \varphi_N}{\partial^2 p_{2N}^2} & \frac{\partial^2 \varphi_N}{\partial p_{2N} \partial y_N} & \frac{\partial^2 \varphi_N}{\partial p_{2N} \partial p_{1N}} \\ \frac{\partial^2 \varphi_N}{\partial y_N \partial p_{2N}} & \frac{\partial^2 \varphi_N}{\partial y_N^2} & \frac{\partial^2 \varphi_N}{\partial y_N \partial p_{1N}} \\ \frac{\partial^2 \varphi_N}{\partial p_{1N} \partial p_{2N}} & \frac{\partial^2 \varphi_N}{\partial p_{1N} \partial y_N} & \frac{\partial^2 \varphi_N}{\partial^2 p_{1N}^2} \end{vmatrix}$$

Let ***A*** represents $-(h+\pi_{1N}) f_{1N}(y_N - b_1 + l_{2,1} p_{2N} + a_1 p_{1N})$ and ***B*** refers to $-(h+\pi_{2N}) f_{2N}(x_{2N} - b_2 + a_2 p_{2N} - l_{1,2} p_{1N})$. It is obvious that both ***A*** and ***B*** are negative.

The 1st leading principal minor is proven to be negative.

$$\begin{aligned} \frac{\partial^2 \varphi_N}{\partial^2 p_{2N}^2} &= -2a_2 - l_{2,1}^2 (h+\pi_{2N}) f_{1N}(y_N - b_1 + a_1 p_{1N} - l_{2,1} p_{2N}) \\ &\quad - a_2^2 (h+\pi_{2N}) f_{2N}(x_{2N} - b_2 + a_2 p_{2N} - l_{1,2} p_{1N}) \\ &= -2a_2 + l_{2,1}^2 A + a_2^2 B \le 0 \end{aligned}$$

The 2nd leading principal minor is proven to be positive.

$$\frac{\partial^2 \varphi_N}{\partial y_N \partial p_{2N}} = \frac{\partial^2 \varphi_N}{\partial p_{2N} \partial y_N} = l_{2,1}(h+\pi_{1N}) f_{1N}(y_N - b_1 + l_{2,1} p_{2N} + a_1 p_{1N}) = -l_{2,1} A$$

$$\frac{\partial^2 \varphi_N}{\partial {y_N}^2} = -(h+\pi_{1N}) f_{1N}(y_N - b_1 + l_{2,1} p_{2N} + a_1 p_{1N}) = A$$

Hence

$$\begin{vmatrix} \frac{\partial^2 \varphi_N}{\partial p_{2N}^2} & \frac{\partial^2 \varphi_N}{\partial p_{2N} \partial y_N} \\ \frac{\partial^2 \varphi_N}{\partial y_N \partial p_{2N}} & \frac{\partial^2 \varphi_N}{\partial y_N^2} \end{vmatrix} = \begin{vmatrix} -2a_2 + l_{2,1}^2 A + a_2^2 B & -l_{2,1} A \\ -l_{2,1} A & A \end{vmatrix} = -2a_2 A + a_2^2 AB \geq 0$$

The 3rd leading principal minor is shown to be negative.

$$\frac{\partial^2 \varphi_N}{\partial p_{1N} \partial y_N} = \frac{\partial^2 \varphi_N}{\partial y_N \partial p_{1N}} = -a_1 (h+\pi_{1N}) f_{1N}(y_N - b_1 + l_{2,1} p_{2N} + a_1 p_{1N}) = a_1 A$$

$$\begin{aligned} \frac{\partial^2 \varphi_N}{\partial p_{1N} \partial p_{2N}} &= l_{2,1} + l_{1,2} + a_1 l_{2,1} (h+\pi_{1N}) f_{1N}(y_N - b_1 + l_{2,1} p_{2N} + a_1 p_{1N}) \\ &\quad + a_2 l_{1,2} (h+\pi_{2N}) f_{2N}(x_{2N} - b_2 + l_{1,2} p_{1N} + a_1 p_{1N}) \\ &= l_{2,1} + l_{1,2} - a_1 l_{2,1} A - a_2 l_{1,2} B \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \varphi_N}{\partial^2 p_{1N}^2} &= -2a_1 - a_1^2 (h+\pi_{2N}) f_{1N}(y_N - b_1 + a_1 p_{1N} - l_{2,1} p_{2N}) \\ &\quad - l_{1,2}^2 (h+\pi_{2N}) f_{2N}(x_{2N} - b_2 + a_2 p_{2N} - l_{1,2} p_{1N}) \\ &= -2a_1 + a_1^2 A + l_{1,2}^2 B \end{aligned}$$

$$\det(H_N)=\begin{vmatrix} -2a_2+l_{2,1}^2A+a_2^2B & -l_{2,1}A & (l_{2,1}+l_{1,2}-a_1l_{2,1}A-a_2l_{1,2}B) \\ -l_{2,1}A & \mathrm{A} & a_1A \\ (l_{2,1}+l_{1,2}-a_1l_{2,1}A-a_2l_{1,2}B) & a_1A & -2a_1+a_1^2A+l_{1,2}^2B \end{vmatrix}$$

$$=A*[4a_1a_2-(l_{2,1}+l_{1,2})^2-2a_2B(a_1a_2-l_{2,1}l_{1,2})]\le 0$$

Since the value of the j^{th} leading principal is either zero or has the sign of $(-1)^j$ for all j (j = 3), the symmetric matrix H_N is negative semi-definite.

Thus, $\varphi_N(x_{2N};y_N,p_{1N},p_{2N})$ is concave with respect to y_N, p_{1N} and p_{2N}, given an inventory level x_{2N} at the last period $k = N$. □

From Theorem 4.1, the unique optimal solution y_N^*, p_{1N}^* and p_{2N}^* exists and from the joint concavity, efficient algorithms such as the steepest ascent method can be employed to obtain the optimal solution.

Lemma 4.1: when $k = N$

(i) $p_{2N}^*(x_{2N})$ is a non-increasing function of x_{2N}.

(ii) $p_{1N}^*(x_{2N})$ is a non-decreasing function of x_{2N}.

(iii) $y_N^*(x_{2N})$ is a non-increasing function of x_{2N}.

(iv) $y_N^*(x_{2N}) = F^{-1}(\frac{\pi_{1N} - c_N}{h + \pi_{1N}}) + \mu_{1N}^*(p_{1N}^*, p_{2N}^*)$.

Proof: The optimal $y_N^*(x_{2N})$, $p_{1N}^*(x_{2N})$ and $p_{2N}^*(x_{2N})$ can be obtained by the following KKT conditions.

$$\begin{aligned}\frac{\partial \varphi_N}{\partial p_{1N}} &= b_1 - 2a_1 p_{1N}^* + p_{2N}^*(l_{2,1} + l_{1,2}) - a_1 h F_{1N}(y_N^* - b_1 + a_1 p_{1N}^* - l_{2,1} p_{2N}^*) \\ &+ \pi_{1N} a_1 [1 - F_{1N}(y_N^* - b_1 + a_1 p_{1N}^* - l_{2,1} p_{2N}^*)] + l_{1,2} h F_{2N}(x_{2N} - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*) \\ &- l_{1,2} \pi_{2N} [1 - F_{2N}(x_{2N} - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*)] - \lambda_{1N}^* + \lambda_{2N}^* = 0\end{aligned} \tag{4.5}$$

$$\begin{aligned}\frac{\partial \varphi_N}{\partial p_{2N}} &= b_2 - 2a_2 p_{2N}^* + p_{1N}^*(l_{2,1} + l_{1,2}) + l_{2,1} h F_{1N}(y_N^* - b_1 + a_1 p_{1N}^* - l_{2,1} p_{2N}^*) \\ &- \pi_{1N} l_{2,1} [1 - F_{1N}(y_N^* - b_1 + a_1 p_{1N}^* - l_{2,1} p_{2N}^*)] - a_2 h F_{2N}(x_N - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*) \\ &+ a_2 \pi_{2N} [1 - F_{2N}(x_N - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*)] - \lambda_{3N}^* + \lambda_{4N}^* = 0\end{aligned} \tag{4.6}$$

$$\begin{aligned}\frac{\partial \varphi_N}{\partial y_N} &= -h F_{1N}(y_N^* - b_1 + a_1 p_{1N}^* - l_{2,1} p_{2N}^*) \\ &+ \pi_{1N} [1 - F_{1N}(y_N^* - b_1 + a_1 p_{1N}^* - l_{2,1} p_{2N}^*)] - c = 0\end{aligned} \tag{4.7}$$

$$\lambda_{1N}^*(p_{1N}^* - p_{1N}^{\max}) = 0 \tag{4.8}$$

$$\lambda_{2N}^*(p_{1N}^{\min} - p_{1N}^*) = 0 \tag{4.9}$$

$$\lambda_{3N}^*(p_{2N}^* - p_{2N}^{\max}) = 0 \tag{4.10}$$

$$\lambda^*_{4N}(p^{\min}_{2N} - p^*_{2N}) = 0 \tag{4.11}$$

$$\lambda^*_{1N}, \lambda^*_{2N}, \lambda^*_{3N}, \lambda^*_{4N} \geq 0 \tag{4.12}$$

$$p^{\min}_{1N} \leq p_{1N} \leq p^{\max}_{1N} \tag{4.13}$$

$$p^{\min}_{2N} \leq p_{2N} \leq p^{\max}_{2N} \tag{4.14}$$

Incorporating (4.7) with (4.5) and (4.6), we obtain

$$\begin{aligned} &b_1 - 2a_1 p^*_{1N} + p^*_{2N}(l_{2,1} + l_{1,2}) + l_{1,2} hF_{2N}(x_{2N} - b_2 + a_2 p^*_{2N} - l_{1,2} p^*_{1N}) \\ &- l_{1,2}\pi_{2N}[1 - F_{2N}(x_{2N} - b_2 + a_2 p^*_{2N} - l_{1,2} p^*_{1N})] + a_1 c - \lambda^*_{1N} + \lambda^*_{2N} = 0 \end{aligned} \tag{4.15}$$

$$\begin{aligned} &b_2 - 2a_2 p^*_{2N} + p^*_{1N}(l_{2,1} + l_{1,2}) - a_2 hF_{2N}(x_{2N} - b_2 + a_2 p^*_{2N} - l_{1,2} p^*_{1N}) \\ &+ a_2\pi_{2N}[1 - F_{2N}(x_{2N} - b_2 + a_2 p^*_{2N} - l_{1,2} p^*_{1N})] - l_{2,1} c - \lambda^*_{3N} + \lambda^*_{4N} = 0 \end{aligned} \tag{4.16}$$

$a_2 *$ (4.15) + $l_{1,2} *$ (4.16)

$$\begin{aligned} &(2a_1 a_2 - l_{2,1} l_{1,2} - l^2_{1,2}) p^*_1 + a_2(l_{1,2} - l_{2,1}) p^*_2 - a_2 \lambda^*_{1N} + a_2 \lambda^*_{2N} - l_{1,2}\lambda^*_{3N} + l_{1,2}\lambda^*_{4N} \\ &- a_2 b_1 - l_{1,2} b_2 - a_1 a_2 c + l_{2,1} l_{1,2} c = 0 \end{aligned} \tag{4.17}$$

Taking the first derivative of (4.17) with respect to x_{2N}, we obtain

$$\begin{aligned} &(2a_1 a_2 - l_{2,1} l_{1,2} - l^2_{1,2}) \frac{dp^*_{1N}}{dx_{2N}} + a_2(l_{1,2} - l_{2,1}) \frac{dp^*_{2N}}{dx_{2N}} \\ &+ a_2 \frac{d\lambda^*_{1N}}{dx_{2N}} - a_2 \frac{d\lambda^*_{2N}}{dx_{2N}} + l_{1,2} \frac{d\lambda^*_{3N}}{dx_{2N}} - l_{1,2} \frac{d\lambda^*_{4N}}{dx_{2N}} = 0 \end{aligned} \tag{4.18}$$

$a_2 *$ (4.15) - $l_{1,2} *$ (4.16)

$$\begin{aligned}&l_{1,2}b_2 - a_2b_1 - (3a_2l_{1,2} + a_2l_{2,1})p_{2N}^* + (2a_1a_2 + l_{2,1}l_{1,2} + l_{1,2}^2)p_{1N}^* \\ &- 2a_2l_{1,2}hF_{2N}(x_{2N} - b_2 + a_2p_{2N}^* - l_{1,2}p_{1N}^*) + a_2\lambda_{1N}^* - a_2\lambda_{2N}^* - l_{1,2}\lambda_{3N}^* + l_{1,2}\lambda_{4N}^* \\ &+ 2a_2l_{1,2}\pi_{2N}[1 - F_{2N}(x_{2N} - b_2 + a_2p_{2N}^* - l_{1,2}p_{1N}^*)] - a_1a_2c - l_{2,1}l_{1,2}c = 0\end{aligned} \tag{4.19}$$

Taking the first derivative of (4.19) with respect to x_{2N},

$$\begin{aligned}&(2a_1a_2 + l_1l_2 + l_1^2)\frac{dp_{1N}^*}{dx_{2N}} - a_2(3l_{1,2} + l_{2,1})\frac{dp_{2N}^*}{dx_{2N}} + a_2\frac{d\lambda_{1N}^*}{dx_{2N}} - a_2\frac{d\lambda_{2N}^*}{dx_{2N}} - l_{1,2}\frac{d\lambda_{3N}^*}{dx_{2N}} + l_{1,2}\frac{d\lambda_{4N}^*}{dx_{2N}} \\ &= 2a_2l_2(h + \pi_{2N})f_{2N}(x_{2N} - b_2 + a_2p_{2N}^* - l_2p_{1N}^*)[1 + a_2\frac{dp_{2N}^*}{dx_{2N}} - l_{1,2}\frac{dp_{1N}^*}{dx_{2N}}]\end{aligned}$$

(4.20)

For (i) and (ii) in Lemma 4.1, it is necessary and sufficient to prove that $\frac{dp_{2N}^*}{dx_{2N}} \leq 0$ and $\frac{dp_{1N}^*}{dx_{2N}} \geq 0$. We consider nine cases.

<u>Case 1</u>: $\lambda_{1N}^* = \lambda_{2N}^* = \lambda_{3N}^* = \lambda_{4N}^* = 0$

Substituting $\lambda_{1N}^* = \lambda_{2N}^* = \lambda_{3N}^* = \lambda_{4N}^* = 0$ in (4.17) and (4.19), we obtain new (4.18) and (4.20). After solving them, $\frac{dp_{2N}^*}{dx_{2N}} \leq 0$ and $\frac{dp_{1N}^*}{dx_{2N}} \geq 0$ are obtained, respectively.

<u>Case 2</u>: $\lambda_{1N}^* > 0$ and $\lambda_{2N}^* = \lambda_{3N}^* = \lambda_{4N}^* = 0$

From (4.8), we obtain that $p_{1N}^{*} - p_{1N}^{\max} = 0$ when $\lambda_{1N}^{*} > 0$.

Taking the first derivative of (4.8) with respect to x_{2N},

$$\frac{d\lambda_{1N}^{*}}{dx_{2N}}(p_{1N}^{*} - p_{1N}^{\max}) + \lambda_{1N}^{*}\frac{dp_{1N}^{*}}{dx_{2N}} = \lambda_{1N}^{*}\frac{dp_{1N}^{*}}{dx_{2N}} = 0 \tag{4.21}$$

After substituting $\lambda_{1N}^{*} > 0$ and $\lambda_{2N}^{*} = \lambda_{3N}^{*} = \lambda_{4N}^{*} = 0$ in (4.17) and (4.19), we solve the new (4.18) and (4.20) and obtain $\frac{dp_{2N}^{*}}{dx_{2N}} \leq 0$. In addition, $\frac{dp_{1N}^{*}}{dx_{2N}} = 0$ is directly obtained from (4.21), when $\lambda_{1N}^{*} > 0$. Hence, we obtain $\frac{dp_{2N}^{*}}{dx_{2N}} \leq 0$ and $\frac{dp_{1N}^{*}}{dx_{2N}} = 0$, when $p_{1N}^{*} - p_{1N}^{\max} = 0$.

<u>Case 3</u>: $\lambda_{2N}^{*} > 0$ and $\lambda_{1N}^{*} = \lambda_{3N}^{*} = \lambda_{4N}^{*} = 0$

Similarly, we obtain that $\frac{dp_{2N}^{*}}{dx_{2N}} \leq 0$ and $\frac{dp_{1N}^{*}}{dx_{2N}} = 0$, when $p_{1N}^{*} - p_{1N}^{\min} = 0$.

<u>Case 4</u>: $\lambda_{3N}^{*} > 0$ and $\lambda_{1N}^{*} = \lambda_{2N}^{*} = \lambda_{4N}^{*} = 0$

Similarly, we obtain that $\frac{dp_{1N}^{*}}{dx_{2N}} \geq 0$ and $\frac{dp_{2N}^{*}}{dx_{2N}} = 0$, when $p_{2N}^{*} - p_{2N}^{\max} = 0$.

<u>Case 5</u>: $\lambda_{4N}^{*} > 0$ and $\lambda_{1N}^{*} = \lambda_{2N}^{*} = \lambda_{3N}^{*} = 0$

Similarly, we obtain that $\frac{dp_{1N}^*}{dx_{2N}} \geq 0$ and $\frac{dp_{2N}^*}{dx_{2N}} = 0$, when $p_{2N}^* - p_{2N}^{\min} = 0$.

Case 6: $\lambda_{1N}^* > 0$, $\lambda_{3N}^* > 0$ and $\lambda_{2N}^* = \lambda_{4N}^* = 0$

From $\lambda_{1N}^* > 0$ and $\lambda_{3N}^* > 0$, we obtain that $p_{1N}^* - p_{1N}^{\max} = 0$ from (4.8) and $p_{2N}^* - p_{2N}^{\max} = 0$ from (4.10).

Taking the first derivatives of (4.8) and (4.10) with respect to x_{2N}, we obtain

$$\frac{d\lambda_{1N}^*}{dx_{2N}}(p_{1N}^* - p_{1N}^{\max}) + \lambda_{1N}^* \frac{dp_{1N}^*}{dx_{2N}} = \lambda_{1N}^* \frac{dp_{1N}^*}{dx_{2N}} = 0 \tag{4.22}$$

and

$$\frac{d\lambda_{3N}^*}{dx_{2N}}(p_{2N}^* - p_{2N}^{\max}) + \lambda_{3N}^* \frac{dp_{2N}^*}{dx_{2N}} = \lambda_{3N}^* \frac{dp_{2N}^*}{dx_{2N}} = 0 \tag{4.23}$$

Hence, we obtain that $\frac{dp_{1N}^*}{dx_{2N}} = 0$ and $\frac{dp_{2N}^*}{dx_{2N}} = 0$ when $p_{1N}^* - p_{1N}^{\max} = 0$ and $p_{2N}^* - p_{2N}^{\max} = 0$.

Case 7: $\lambda_{1N}^* > 0$, $\lambda_{4N}^* > 0$ and $\lambda_{2N}^* = \lambda_{3N}^* = 0$

Case 8: $\lambda_{2N}^* > 0$, $\lambda_{3N}^* > 0$ and $\lambda_{1N}^* = \lambda_{4N}^* = 0$

<u>Case 9</u>: $\lambda_{2N}^* > 0$, $\lambda_{4N}^* > 0$ and $\lambda_{1N}^* = \lambda_{3N}^* = 0$

The condition that $\frac{dp_{1N}^*}{dx_{2N}} = 0$ and $\frac{dp_{2N}^*}{dx_{2N}} = 0$ also holds for Cases 7 to 9.

Thus, we prove that $p_{2N}^*(x_{2N})$ is a non-increasing function of x_{2N} and $p_{1N}^*(x_{2N})$ is a non-decreasing function of x_{2N}, respectively.

As for (iii) in Lemma 4.1, we take the first derivative of (4.7) with respect to x_{2N}, which is $(\frac{dy_N^*}{dx_{2N}} + a_1 \frac{dp_{1N}^*}{dx_{2N}} - l_{2,1} \frac{dp_{2N}^*}{dx_{2N}}) f_{1N}(y_N^* - b_1 + a_1 p_{1N}^* - l_{2,1} p_{2N}^*) = 0$, and then subsequently obtain $\frac{dy_N^*}{dx_{2N}} \leq 0$.

The property (iv) in Lemma 4.1 is directly obtained from (4.7). □

As the inventory level of the old product increases, the optimal price of the old product must be reduced to increase demand for the old product (as in (i) of Lemma 4.1). This in turn reduces demand for the new product. The optimal order quantity for the new product must be reduced accordingly (as in (iii) of Lemma 4.1). Reduction in the order quantity for the new product increases the optimal price for the new product due to our assumption of $l_{1,2} \geq l_{2,1}$ (as in (ii) of Lemma 4.1). A closed form for the optimal order quantity $y_N^*(x_{2N})$ is obtained from $\mu_{1N}^*(p_{1N}^*, p_{2N}^*)$ (as in (iv) of Lemma 4.1).

Lemma 4.2:

(i) $(\lambda_{1N}^* - \lambda_{2N}^*)\dfrac{dp_{1N}^*}{dx_{2N}} + (\lambda_{3N}^* - \lambda_{4N}^*)\dfrac{dp_{2N}^*}{dx_{2N}} = 0$

(ii) $[1 + a_2 \dfrac{dp_{2N}^*}{dx_{2N}} - l_{1,2} \dfrac{dp_{1N}^*}{dx_{2N}}] \geq 0$

Proof: The above two equations follow Lemma 4.1, which can be easily obtained by solving (4.18) and (4.20) under each case. □

Lemma 4.2 provides the necessary conditions for Theorem 4.2.

Theorem 4.2: The maximum expected profit at Period $N, V_N(x_{2N})$, is concave with respect to x_{2N}.

Proof: Taking the first derivative of $V_N(x_{2N}) = \varphi_N(x_{2N}; y_N^*, p_{1N}^*, p_{2N}^*)$ with respect to x_{2N}, we obtain

$$\begin{aligned}
\frac{dV_N}{\partial x_{2N}} &= (b_1 - 2a_1 p_{1N}^*)\frac{dp_{1N}^*}{dx_{2N}} + l_{2,1}(p_{1N}^* \frac{dp_{2N}^*}{dx_{2N}} + p_{2N}^* \frac{dp_{1N}^*}{dx_{2N}}) \\
&\quad + (b_2 - 2a_2 p_{2N}^*)\frac{dp_{2N}^*}{dx_{2N}} + l_{1,2}(p_{1N}^* \frac{dp_{2N}^*}{dx_{2N}} + p_{2N}^* \frac{dp_{1N}^*}{dx_{2N}}) \\
&\quad - h[\frac{dy_N^*}{dx_{2N}} + a_1 \frac{dp_{1N}^*}{dx_{2N}} - l_{2,1} \frac{dp_{2N}^*}{dx_{2N}}]F_{1N}(y_N^* - b_1 + a_1 p_{1N}^* - l_{2,1} p_{2N}^*) \\
&\quad + \pi_{1N}[\frac{dy_N^*}{dx_{2N}} + a_1 \frac{dp_{1N}^*}{dx_{2N}} - l_{2,1} \frac{dp_{2N}^*}{dx_{2N}}][1 - F_{2N}(y_N^* - b_1 + a_1 p_{1N}^* - l_{2,1} p_{2N}^*)] \\
&\quad - h[1 + a_2 \frac{dp_{2N}^*}{dx_{2N}} - l_{1,2} \frac{dp_{1N}^*}{dx_{2N}}]F_{2N}(x_{2N} - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*) + \\
&\quad + \pi_{2N}[1 + a_2 \frac{dp_{2N}^*}{dx_{2N}} - l_{1,2} \frac{dp_{1N}^*}{dx_{2N}}][1 - F_{2N}(x_{2N} - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*)] - c\frac{dy_N^*}{dx_{2N}}
\end{aligned} \tag{4.24}$$

By substituting (4.5)-(4.7) in the above equation, Equation (4.24) is reduced to

$$\frac{dV_N}{\partial x_{2N}} = -hF_{2N}(x_{2N} - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*) + \pi_{2N}[1 - F_{2N}(x_{2N} - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*)]$$
$$- (\lambda_{1N}^* - \lambda_{2N}^*)\frac{dp_{1N}^*}{dx_{2N}} - (\lambda_{3N}^* - \lambda_{4N}^*)\frac{dp_{2N}^*}{dx_{2N}} \tag{4.25}$$

Using (i) in Lemma 4.2, Equation (4.25) is further simplified as follows:

$$\frac{dV_N}{\partial x_{2N}} = -hF_{2N}(x_{2N} - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*) + \pi_{2N}[1 - F_{2N}(x_{2N} - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*)]$$

$$\left.\frac{dV_N}{\partial x_{2N}}\right|_{x_{2N}=0} = -hF_{2N}(0 - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*) + \pi_{2N}[1 - F_{2N}(0 - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*)]$$
$$\leq \pi_{2N}$$

Taking the second derivative of $V_N(x_{2N}) = \varphi_N(x_{2N}; y_N^*, p_{1N}^*, p_{2N}^*)$ with respect to x_{2N}, we obtain

$$\frac{d^2V_N}{dx_{2N}^2} = -(h + \pi_N)[1 + a_2\frac{dp_{2N}^*}{dx_{2N}} - l_{1,2}\frac{dp_{1N}^*}{dx_{2N}}] f_{2N}(x_{2N} - b_2 + a_2 p_{2N}^* - l_{1,2} p_{1N}^*)$$

The sign of $\frac{d^2V_N}{dx_{2N}^2}$ is negative, because $[1 + a_2\frac{dp_{2N}^*}{dx_{2N}} - l_{1,2}\frac{dp_{1N}^*}{dx_{2N}}]$ is positive from (ii) in Lemma 4.2. □

ii) k = 1, …, N - 1

In order to complete the proof, Theorems 4.3 and 4.4 are shown in the followings:

Theorem 4.3: Assuming that $V_{k+1}(x_{2,k+1})$ is concave with respect to $x_{2,k+1}$ and $V'_{k+1}(0) \le \pi_{2,k+1}$, $J_k(x_{2k}; y_k, p_{1k}, p_{2k})$ is jointly concave with respect to y_k, p_{1k} and p_{2k} given an inventory level x_{2k}.

Proof: For Period *k*, $J_k(x_{2k}; y_k, p_{1k}, p_{2k})$ is developed as follows:

$$J_k(x_{2k}; y_k, p_{1k}, p_{2k}) = \varphi_k(x_{2k}; y_k, p_{1k}, p_{2k}) + \alpha \int_{\varepsilon_{1k}^{\min}}^{y_k - b_1 + a_1 p_{1k} - l_{2,1} p_{2k}} V_{k+1}(y_k - t_{1k}) f_{1k}(\varepsilon_{1k}) d\varepsilon_{1k} + \alpha \int_{y_k - b_1 + a_1 p_{1k} - l_{2,1} p_{2k}}^{\varepsilon_{1k}^{\max}} V_{k+1}(0) f_{1k}(\varepsilon_{1k}) d\varepsilon_{1k}$$

where $t_{1k} = b_1 - a_1 p_{1k} + l_{2,1} p_{2k}$.

H_k represents the Hessian Matrix of $J_k(x_{2k}; y_k, p_{1k}, p_{2k})$ and its determinant is as follows:

$$\det(H_k) = \begin{vmatrix} \dfrac{\partial^2 J_k}{\partial^2 p_{2k}^2} & \dfrac{\partial^2 J_k}{\partial p_{2k} \partial y_k} & \dfrac{\partial^2 J_k}{\partial p_{2k} \partial p_{1k}} \\ \dfrac{\partial^2 J_k}{\partial y_k \partial p_{2k}} & \dfrac{\partial^2 J_k}{\partial y_k} & \dfrac{\partial^2 J_k}{\partial y_k \partial p_{1k}} \\ \dfrac{\partial^2 J_k}{\partial p_{1k} \partial p_{2k}} & \dfrac{\partial^2 J_k}{\partial p_{1k} \partial y_k} & \dfrac{\partial^2 J_k}{\partial^2 p_{1k}^2} \end{vmatrix}$$

Let $\boldsymbol{A_k}$ represents $-(h + \pi_{1k}) f_{1k}(y_k - b_1 + l_{2,1} p_{2k} + a_1 p_{1k})$ and $\boldsymbol{B_k}$ refers to

$-(h + \pi_{2k}) f_{2k}(x_{2k} - b_2 + a_2 p_{2k} - l_{1,2} p_{1k})$.

Let $\boldsymbol{C_k}$ represents

$$\alpha \int_{\varepsilon_{1k}^{\min}}^{y_k - b_1 + a_1 p_{1k} - l_{2,1} p_{2k}} V_{k+1}^{''}(y_k - t_{1k}) f_{1k}(\varepsilon_{1k}) d\varepsilon + \alpha V_{k+1}^{'}(0) f_{1k}(y_k - b_1 + a_1 p_{1k} - l_{2,1} p_{2k}),$$

where $t_{1k} = b_1 - a_1 p_{1k} + l_{2,1} p_{2k}$.

The determinant of H_k is simplified as follows:

$$\det(H_k) = \begin{vmatrix} -2a_2 + l_{2,1}^2 (A_k + C_k) + a_2^2 B_k & -l_{2,1}(A_k + C_k) & l_{2,1} + l_{1,2} - a_1 l_{2,1}(A_k + C_k) - a_2 l_{1,2} B_k \\ -l_{2,1}(A_k + C_k) & (A_k + C_k) & a_1 (A_k + C_k) \\ l_{2,1} + l_{1,2} - a_1 l_{2,1}(A_k + C_k) - a_2 l_{1,2} B_k & a_1 (A_k + C_k) & -2a_1 + a_1^2 (A_k + C_k) + l_{1,2}^2 B_k \end{vmatrix}$$

It is obvious that both $\boldsymbol{A_k}$ and $\boldsymbol{B_k}$ are negative. In addition, $(\boldsymbol{A_k} + \boldsymbol{C_k})$ is negative from the assumption that $V_{k+1}(x_{2,k+1})$ is concave with respect to $x_{2,k+1}$ and $V_{k+1}^{'}(0) \le \pi_{2,k+1}$.

Hence,

The 1st leading principle minor, $-2a_2 + l_{2,1}^2 (A_k + C_k) + a_2^2 B_k$, is negative.

The 2nd leading principle minor, $-2a_2 (A_k + C_k) + a_2^2 B_k (A_k + C_k)$, is positive.

The 3rd leading principle minor, $(A_k + C_k) * [4a_1 a_2 - (l_{2,1} + l_{1,2})^2 - 2a_2 B_k (a_1 a_2 - l_{2,1} l_{1,2})]$, is negative from the assumption $a_1, a_2 \ge l_{1,2} \ge l_{2,1} \ge 0$.

Since the value of the j^{th} leading principal is either zero or has the sign of $(-1)^j$ for all j (j = 3), the symmetric matrix H_k is negative semi-definite.

Thus, $J_k(x_{2k}; y_k, p_{1k}, p_{2k})$ is jointly concave with respect to y_k, p_{1k} and p_{2k} given an inventory level x_{2k}. □

Lemma 4.3: when $k = 1,\ldots,N-1$

(i) $p^*_{2k}(x_{2k})$ is a non-increasing function of x_{2k}.

(ii) $p^*_{1k}(x_{2k})$ is a non-decreasing function of x_{2k}.

(iii) $y^*_k(x_{2k})$ is a non-increasing function of x_{2k}.

(iv) $(\lambda^*_{1k} - \lambda^*_{2k})\dfrac{dp^*_{1k}}{dx_{2k}} + (\lambda^*_{3k} - \lambda^*_{4k})\dfrac{dp^*_{2k}}{dx_{2k}} = 0$

(v) $[1 + a_2\dfrac{dp^*_{2k}}{dx_{2k}} - l_{1,2}\dfrac{dp^*_{1k}}{dx_{2k}}] \geq 0$

Proof: The optimal $y^*_k(x_{2k})$, $p^*_{1k}(x_{2k})$ and $p^*_{2k}(x_{2k})$ can be obtained by the following KKT conditions.

$$\begin{aligned}\frac{\partial\varphi_k}{\partial p_{1k}} &= b_1 - 2a_1 p_{1k}^* + p_{2k}^*(l_{2,1}+l_{1,2}) - a_1 hF_{1k}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*) \\ &+ \pi_{1k} a_1 [1 - F_{1k}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*)] + l_{1,2} hF_{2k}(x_{2k} - b_2 + a_2 p_{2k}^* - l_{1,2} p_{1k}^*) \\ &- l_{1,2}\pi_{2k}[1 - F_{2k}(x_{2k} - b_2 + a_2 p_{2k}^* - l_{1,2} p_{1k}^*)] - \lambda_{1k}^* + \lambda_{2k}^* \\ &+ \alpha a_1 \int_{\varepsilon_k^{\min}}^{y_k^* - b_1 + a_k p_{1k}^* - l_{2,1} p_{2k}^*} V_{k+1}^{'}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^* - \varepsilon_{1k}) f(\varepsilon_{1k}) d\varepsilon = 0\end{aligned} \tag{4.26}$$

$$\begin{aligned}\frac{\partial\varphi_k}{\partial p_{2k}} &= b_2 - 2a_2 p_{2k}^* + p_{1k}^*(l_{2,1}+l_{1,2}) + l_{2,1} hF_{1k}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*) \\ &- \pi_{1k} l_{2,1}[1 - F_{1k}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*)] - a_2 hF_{2k}(x_{2k} - b_2 + a_2 p_{2k}^* - l_{1,2} p_{1k}^*) \\ &+ a_2 \pi_{2k}[1 - F_{2k}(x_{2k} - b_2 + a_2 p_{2k}^* - l_{1,2} p_{1k}^*)] - \lambda_{3k}^* + \lambda_{4k}^* \\ &- \alpha l_{2,1} \int_{\varepsilon_k^{\min}}^{y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*} V_{k+1}^{'}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^* - \varepsilon_{1k}) f(\varepsilon_{1k}) d\varepsilon = 0\end{aligned} \tag{4.27}$$

$$\begin{aligned}\frac{\partial\varphi_k}{\partial y_k} &= -hF_{1k}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*) \\ &+ \pi_{1k}[1 - F_{1k}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*)] - c \\ &+ \alpha \int_{\varepsilon_k^{\min}}^{y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*} V_{k+1}^{'}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^* - \varepsilon_{1k}) f(\varepsilon_{1k}) d\varepsilon = 0\end{aligned} \tag{4.28}$$

$$\lambda_{1k}^*(p_{1k}^* - p_{1k}^{\max}) = 0 \tag{4.29}$$

$$\lambda_{2k}^*(p_{1k}^{\min} - p_{1k}^*) = 0 \tag{4.30}$$

$$\lambda_{3k}^*(p_{2k}^* - p_{2k}^{\max}) = 0 \tag{4.31}$$

$$\lambda_{4k}^*(p_{2k}^{\min} - p_{2k}^*) = 0 \tag{4.32}$$

$$\lambda_{1k}^*, \lambda_{2k}^*, \lambda_{3k}^*, \lambda_{4k}^* \geq 0 \tag{4.33}$$

$$p_{1k}^{\min} \le p_{1k} \le p_{1k}^{\max} \tag{4.34}$$

$$p_{2k}^{\min} \le p_{2k} \le p_{2k}^{\max} \tag{4.35}$$

Incorporating (4.28) with (4.26) and (4.27), we obtain (4.15) and (4.16).

Using the proofs of Lemma 4.1, five properties of Lemma 4.3 can be proved. □

Theorem 4.4: The maximum expected profit, $V_k(x_{2k})$, is concave with respect to x_{2k}.

Proof: The equation (4.2) is developed as follows:

$$\begin{aligned} V_k(x_{2k}) &= \varphi_k(x_{2k}; y_k^*, p_{1k}^*, p_{2k}^*) - cy_k^* \\ &+ \alpha \int_{\varepsilon_{1k}^{\min}}^{y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*} V_{k+1}(y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^* - \varepsilon_{1k}) f_{1k}(\varepsilon_{1k}) d\varepsilon_{1k} \\ &+ \alpha \int_{y_k^* - b_1 + a_1 p_{1k}^* - l_{2,1} p_{2k}^*}^{\varepsilon_{1k}^{\max}} V_{k+1}(0) f_{1k}(\varepsilon_{1k}) d\varepsilon_{1k} \end{aligned}$$

Taking the first derivative of $V_k(x_{2k})$ with respect to x_{2k}, we obtain

$$\begin{aligned} \frac{dV_k(x_k)}{dx_{2k}} &= \frac{d\varphi_k^*}{dx_{2k}} + c\frac{dy_k^*}{dx_{2k}} \\ &+ \alpha\left[\frac{dy_k^*}{dx_{2k}} + a_1\frac{dp_{1k}^*}{dx_{2k}} - l_{2,1}\frac{dp_{2k}^*}{dx_{2k}}\right] \int_{\varepsilon_{1k}^{\min}}^{y_k^* - b_1 + a_1 p_1^* - l_{2,1} p_2^*} V_{k+1}'(y_k^* - b_1 + a_1 p_1^* - l_{2,1} p_2^* - \varepsilon_{1k}) f_{1k}(\varepsilon_{1k}) d\varepsilon_{1k} \end{aligned} \tag{4.36}$$

By substituting (4.15)-(4.17) in the equation, Equation (4.36) is simplified as follows:

$$\frac{dV_k}{\partial x_{2k}} = -hF_{2k}(x_{2k} - b_2 + a_2 p_{2k}^* - l_{1,2} p_{1k}^*) + \pi_{2k}[1 - F_{2k}(x_{2k} - b_2 + a_2 p_{2k}^* - l_{1,2} p_{1k}^*)]$$

$$\left.\frac{dV_k}{\partial x_{2k}}\right|_{x_N=0} = -hF_{2k}(0 - b_2 + a_2 p_{2k}^* - l_{1,2} p_{1k}^*) + \pi_{2k}[1 - F_{2k}(0 - b_2 + a_2 p_{2k}^* - l_{1,2} p_{1k}^*)]$$
$$\leq \pi_{2k}$$

Taking the second derivative of $V_k(x_{2k})$ with respect to x_{2k}, we obtain

$$\frac{d^2V_k}{dx_{2k}^2} = -(h + \pi_{2k})[1 + a_2 \frac{dp_{2k}^*}{dx_{2k}} - l_{1,2} \frac{dp_{1k}^*}{dx_{2k}}] f_{2k}(x_{2k} - b_2 + a_2 p_{2k}^* - l_{1,2} p_{1k}^*)$$

The sign of $\frac{d^2V_k}{dx_{2k}^2}$ is negative, because $[1 + a_2 \frac{dp_{2k}^*}{dx_{2k}} - l_{1,2} \frac{dp_{1k}^*}{dx_{2k}}]$ is positive from Lemma 4.3. □

Hence, the joint concavity of $J_k(x_{2k}; y_k, p_{1k}, p_{2k})$ with respect to y_k, p_{1k} and p_{2k} given an inventory level x_{2k} is shown. In addition, $V_k(x_{2k})$ is proved to be concave with respect to x_{2k}.

4.3.3 Special cases

In this section, we discuss several special cases, where the optimal solution can be easily obtained.

i) $l_{2,1} = l_{1,2} = 0$

The condition $l_{1,2} = 0$ implies no downward substitution. From the assumption $l_{1,2} \geq l_{2,1}$, $l_{2,1} = l_{1,2} = 0$. Under this condition, the optimal prices and the optimal order quantity can be obtained by the Chew et al. (2005a) algorithm.

Examples of $l_{2,1} = l_{1,2} = 0$ can be found in practice. Due to fast developments in technologies, new products are significantly improved, compared with existing products in terms of performance, design, etc. Thus, the customers who are interested in new products are more performance oriented and thus, are not affected by the pricing of existing products. Similarly, the customers who are more price sensitive cannot afford to purchase new products and focus on old products only.

ii) $l_{2,1} = 0$

The condition $l_{2,1} = 0$ implies no upward substitution: only the customers who initially plan to purchase the new product may purchase the old product, instead. The optimal prices and the optimal order quantity are obtained by the proposed solution procedure with $l_{2,1} = 0$.

iii) $l_{2,1} = l_{1,2}$

The condition $l_{2,1} = l_{1,2}$ shows that the rate of the upward substitution equals to that of the downward substitution. This is a special case in a full substitution problem, which includes both the upward and the downward substitution.

Under this condition, a closed form for p_{1k}^* is obtained from (4.17) under Case 1, $p_{1k}^* = \dfrac{a_2 b_1 + l_{2,1} b_2 + a_1 a_2 c - l_{2,1}^2 c}{(2a_1 a_2 - 2l_{2,1}^2)}$. Given the closed form for p_{1k}^*, the optimal solution can be obtained efficiently. Note that the optimal p_{1k}^* is independent of x_{2k}. This is due to the balance between the upward and the downward substitution.

4.4 Numerical study for a product with a two period lifetime

This study successfully considers demand transfers between new and old products. Previous research determines the prices and the order quantity by assuming $l_{1,2} = l_{2,1} = 0$ (Chew et al., 2005a). In this numerical study, we first compute the total expected profit increase by employing the proposed method, compared with the conventional method (Chew et al., 2005a). Furthermore, we investigate the effects of the parameter changes on the profit obtained from the substitution effect. Finally, we investigate whether the initial inventory (the initial state) significantly affects the average profit per period as the number of periods increases.

4.4.1 Experimental design

In this numerical study, demands for products of age *i* (*i* = 1, 2) at Period *k* are price sensitive and have an additive stochastic demand function, $t_{ik} = \mu_{ik}(p_{1k}, p_{2k}) + \varepsilon_{ik}$, where $\mu_{ik}(p_{1k}, p_{2k})$ is a linear function of the regular price p_{1k} and the discounted price p_{2k}. The noise variable ε_{ik} follows a truncated normal distribution which is bounded by

$\varepsilon_{ik}^{\min} = -3\sigma_{ik}$ and $\varepsilon_{ik}^{\max} = 3\sigma_{ik}$, where σ_{ik} is the standard deviation of the normal distribution.

We are particularly interested in the effects of transfer rates on the profit increase obtained from the proposed method. Thus, $l_{2,1}$ and $l_{1,2}$ are set to two different levels, as provided in Table 4.1, while the condition $l_{1,2} \geq l_{2,1}$ is satisfied.

Table 4.1 Variables in the numerical study

Parameters	Low Level (-)	High Level (+)
$l_{1,2}$	2	3
$l_{2,1}$	1	2

The holding cost h is a constant value in each period as well as the purchasing cost c. The values of the penalty cost π_1 and π_2 are also provided in Table 4.2, while the condition $\pi_1 \geq \pi_2$ is satisfied.

The price sensitivity parameters, a_1 and a_2, are provided in Table 4.2, where the condition $a_1, a_2 \geq l_{1,2} \geq l_{2,1} \geq 0$ is always satisfied.

Table 4.2 Constants in the numerical study

Parameters	Values	Parameters	Values
b_1	200	b_2	100
a_1	5	a_2	5
σ_1	20	σ_2	20
π_1	20	π_2	15
h	1	c	10

4.4.2 Profit increase from the substitution effect

The effects of demand transfers between new and old products are studied by comparing the expected profit obtained under $l_{1,2} \geq l_{2,1} > 0$ with $l_{1,2} = l_{2,1} = 0$.

As shown in Table 4.3, the expected profit obtained under $l_{1,2} \geq l_{2,1} > 0$ is always higher. As the transfer rates, $l_{1,2}$ and/or $l_{2,1}$, increase, the profit increase becomes greater. Hence, the effects of demand transfers between new and old products should be seriously considered.

Table 4.3 Percentage profit increase by the substitution effect with $x_{2k} = 80$

$(l_{1,2}, l_{2,1})$	$l_{1,2}$	$l_{2,1}$	% increase in profit*
(3,2)	+	+	51.37%
(3,1)	+	-	30.65%
(2,2)	-	+	26.45%
(2,1)	-	-	10.08%

$$\text{* \% increase in profit} = \frac{(\text{profit when } l_{1,2} \geq l_{2,1} > 0 - \text{profit when } l_{1,2} = l_{2,1} = 0)}{\text{Profit when } l_{1,2} = l_{2,1} = 0} * 100\%$$

4.4.3 Sensitivity analysis of the optimal prices

The changes in the parameters of a_1, a_2, $l_{1,2}$ and $l_{2,1}$ have different effects on the optimal prices p_1^* and p_2^*. As a_1 or a_2 increases by one unit given the other parameters unchanged, the optimal prices of both new and old products decrease, compared with the base scenario, as shown in Table 4.4. Increases in a_1 (a_2) imply that demand for the new (old) product becomes more sensitive to the price of the new (old) product and this will reduce the price of the new (old) product. Consequently, demand for the old (new) product decreases and in order to compensate for this reduction in demand for the old (new) product, the price of the old (new) product must also be reduced. As a result, both p_1^* and p_2^* decrease. In contrast, if $l_{1,2}$ or $l_{2,1}$ increases by one unit, given the other parameters unchanged, both p_1^* and p_2^* consistently increase.

Table 4.4 Optimal solutions under different price sensitivity parameters with x_{2k} = 80

Legend	a_1	a_2	$l_{1,2}$	$l_{2,1}$	p_1^*	p_2^*	Profit
Base	5	5	3	2	38	29	3668.5
a_1	6 ↑	unchanged	unchanged	unchanged	31 ↓	25 ↓	-23.36% ↓
a_2	unchanged	6 ↑	unchanged	unchanged	35 ↓	23 ↓	-15.85% ↓
$l_{1,2}$	unchanged	unchanged	4 ↑	unchanged	45 ↑	39 ↑	+31.84% ↑
$l_{2,1}$	unchanged	unchanged	unchanged	3 ↑	45 ↑	34 ↑	+27.75% ↑

It is important to obtain accurate estimates of the parameters a_1, a_2, $l_{1,2}$ and $l_{2,1}$, which represent the customer behaviors. Table 4.4 also shows that the total profit is sensitive to these parameter values, and thus, even a slight misestimation of the parameters will result in a highly erroneous profit estimate.

As shown in Figure 4.1, the difference in the prices (p_1^* - p_2^*) increases as the inventory level of the old product increases. As the inventory level of the old product increases, the optimal price for the old product decreases. Consequently, demand for the old product increases and demand for the new product decreases due to demand transfers. Thus, the optimal order quantity for the new product must be reduced and the optimal price for the new product increases due to our assumption of $l_{1,2} \geq l_{2,1}$. Since p_1^* increases and p_2^* decreases, the difference in the prices becomes greater.

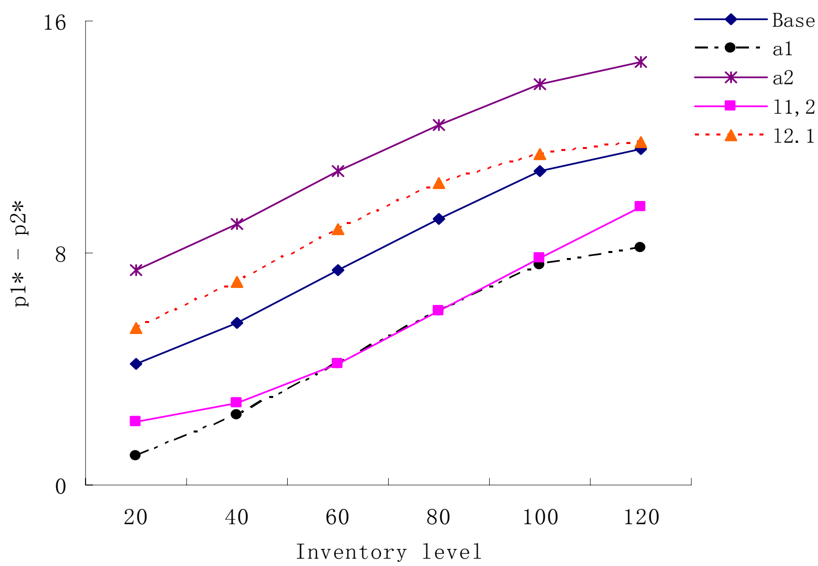

Figure 4.1 p_1^* - p_2^* under different inventory levels

4.4.4 Effect of initial inventory

Figure 4.2 shows that the effect of the initial inventory level on the average profit fast decreases as the number of periods increases. The average profit is not greatly affected by the initial inventory level as the number of periods exceeds 3.

Due to this fast convergence of the average profit, the profit for a finite horizon problem with a large number of periods can be approximated by the profit for an infinite horizon problem. For the infinite horizon problem, the average profit is computed by the value iteration method, regardless of the initial inventory level.

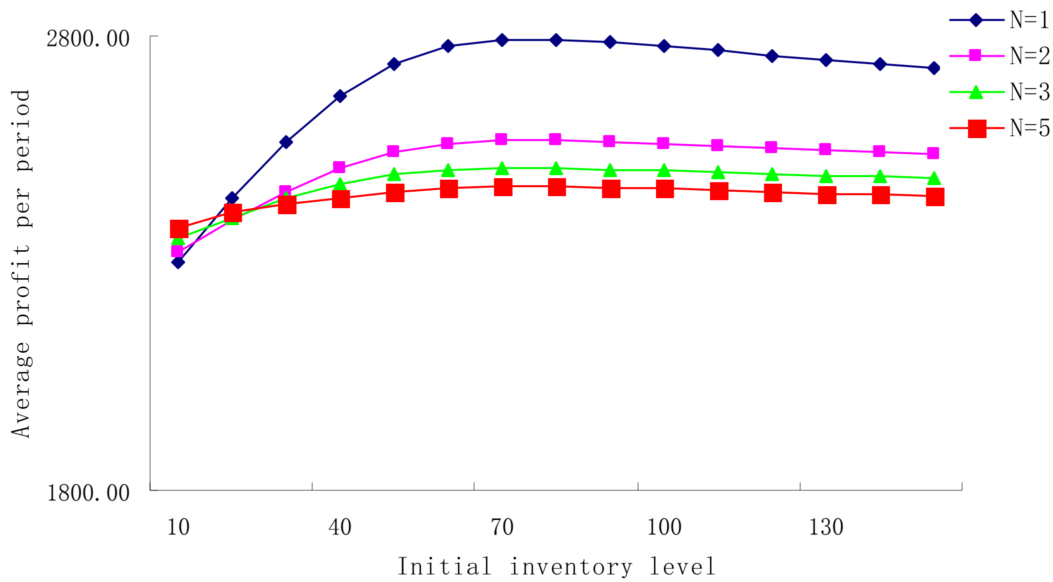

Figure 4.2 Average profit under different N, given $l_{1,2} = 2$ and $l_{2,1} = 2$

4.5 Pricing and ordering decisions for a product with an $M \geq 3$ period lifetime

In this section, we consider a general problem, where the lifetime of a perishable product is M Periods ($M \geq 3$).

A single period problem is first considered. The optimal order quantity y^* and the optimal prices for products of different ages p_i^* (i = 1,…,M) are determined for the objective of maximizing the expected profit, $\varphi(x_2,..., x_M; y, p_1,..., p_M)$.

Lemma 4.4: The expected revenue $R(p_1,..., p_M)$ is concave with respect to $p_1, p_2,..., p_M$ under the condition $a_i \geq l_{i+1,i} + l_{i-1,i} + l_{i,i+1} + l_{i,i-1}$.

Proof: Lemma 4.4 is proven by the definition of concavity:

Since $a_i \geq l_{i+1,i} + l_{i-1,i} + l_{i,i+1} + l_{i,i-1}$, we obtain

$$
\begin{aligned}
& R[\lambda p_{1x} + (1-\lambda) p_{1y}, ..., \lambda p_{Mx} + (1-\lambda) p_{My}] - \lambda R[p_{1x}, ..., p_{Mx}] - (1-\lambda) R[p_{1y}, ..., p_{My}] \\
&= \sum_{i=1}^{M} [\lambda p_{ix} + (1-\lambda) p_{iy}] * [\lambda \mu_{ix} + (1-\lambda) \mu_{iy}] - \lambda \sum_{i=1}^{M} p_{ix} \mu_{ix} - (1-\lambda) \lambda \sum_{i=1}^{M} p_{iy} \mu_{iy} \\
&= \lambda (1-\lambda) \sum_{i=1}^{M} [p_{ix} (\mu_{iy} - \mu_{ix}) + p_{iy} (\mu_{ix} - \mu_{iy})] \\
&= \lambda (1-\lambda) \sum_{i=1}^{M} [(p_{ix} - p_{iy}) * (\mu_{iy} - \mu_{ix})] \\
&= \lambda (1-\lambda) \sum_{i=1}^{M} (p_{ix} - p_{iy}) * [a_i (p_{ix} - p_{iy}) + l_{i+1,i} (p_{i+1,y} - p_{i+1,x}) + l_{i-1,i} (p_{i-1,iy} - p_{i-1,ix})] \\
&\geq \lambda (1-\lambda) \sum_{i=1}^{M} \frac{l_{i,i+1} + l_{i+1,i}}{2} [(p_{ix} - p_{iy}) - (p_{i+1,x} - p_{i+1,y})]^2 \\
&\geq 0
\end{aligned}
$$

Hence, the expected revenue $R(p_1, ..., p_M)$ is concave with respect to $p_1, p_2, ..., p_M$. □

The assumption $a_i \geq l_{i+1,i} + l_{i-1,i} + l_{i,i+1} + l_{i,i-1}$ implies the following: Equation (4.1) can be rewritten from the point of the price differences, which determine demand transfers from classes *i*-1 and *i*+1 to class *i*.

$$
t_i = b_i - s_i p_i + l_{i+1,i} (p_{i+1} - p_i) + l_{i-1,i} (p_{i-1} - p_i) + \varepsilon_i
$$

where $a_i = s_i + l_{i+1,i} + l_{l-1,i}$, for i = 1,…, M

Since $a_i = s_i + l_{i+1,i} + l_{l-1,i}$, the condition $s_i \geq l_{i,i-1} + l_{i,i+1}$ holds to ensure $a_i \geq l_{i+1,i} + l_{i-1,i} + l_{i,i+1} + l_{i,i-1}$. Even if such conditions are not satisfied, Lemma 4.4 still holds under the condition $l_{i,i+1} = l_{i+1,i}$, for i = 1,…, M.

Lemma 4.5: The expected cost $C(x_2,...,x_M;y,p_1,...,p_M)$ is jointly convex with respect to y and p_i for $i = 1,\ldots, M$.

Proof: The joint convexity $C(x_2,...,x_M;y,p_1,...,p_M)$ with respect to y and p_i is obtained by the convexity of $L_i(x_i;t_i)$ and the linearity of the demand functions, as follows:

$$C(x_2,...,x_M;y,p_1,...,p_M) = \sum_{i=1}^{M} L_i(x_i;t_i)$$

$$\begin{aligned}&C(x_2,...,x_M;\lambda y_{1x} + (1-\lambda)y_{1y}, \lambda p_{1x} + (1-\lambda)p_{1y},...,\lambda p_{Mx} + (1-\lambda)p_{My})\\ &- \lambda C(x_2,...,x_M;y_{1x},p_{1x},...,p_{Mx}) - (1-\lambda)C(x_2,...,x_M;y_{1y},p_{1y},...,p_{My}) \le 0\end{aligned}$$ □

Theorem 4.5: The expected profit $\varphi(x_2,...,x_M;y,p_1,...,p_M)$ is jointly concave with respect to $p_1,p_2,...,p_M$ and y.

Proof: The expected revenue $R(p_1,...,p_M)$ is concave with respect to $p_1,p_2,...,p_M$, while the expected cost $C(x_2,...,x_M;y,p_1,...,p_M)$ is jointly convex with respect to y and p_i by Lemma 4.5. The purchasing cost is concave with respect to y. Thus, the expected profit $\varphi(x_2,...,x_M;y,p_1,...,p_M)$ is jointly concave with respect to $p_1,p_2,...,p_M$ and y. □

Thus, the optimal solution $p_1^*,p_2^*,...,p_M^*$ and y^* for a singe period problem can be obtained by an efficient searching algorithm.

A multiple period problem for products of M different ages is formulated as a stochastic dynamic programming model. However, the optimal solution for this problem is hard to obtain due to the overwhelming number of states x_{ik}. Hence, a heuristic based on the optimal single period solution is applied to determine the prices for products of M different ages and the order quantity for the new product. One possible implementation of this heuristic is as follows: At the beginning of the period, given the inventory levels of the old products, the optimal order quantity for the new product and the optimal prices for both new and old products are computed for this period. After the realization of actual demand for this period, the remaining inventories are carried over to the next period where all of them will age by one period. We carry out this procedure repeatedly at the beginning of every period to compute the order quantity for the new product and the product prices.

4.6 Summary

In this study, we determine the optimal prices for products of different ages and the optimal order quantity for the new product, with the objective of maximizing the total profits over the finite number of periods. The problem for a product with lifetime of two periods is first analyzed. Given the inventory level of the old product, the expected profit is jointly concave with respect to the order quantity for the new product and the product prices (the price of the new product and the discounted price of the old product). This concavity enables an efficient algorithm to be employed to obtain the optimal solution. Furthermore, several optimality properties are obtained. The computational results show that the total profit significantly increases when demand transfers between products of different ages are considered. As the loss rates increase, the optimal prices for both new and old products decrease. In addition, the optimal prices increase with increase of the transfer rates. For the product with lifetime of longer than two periods, the optimal prices for products of different ages and the optimal order quantity for the new product are obtained for a single period problem. Based on the optimal single period solution, we propose a heuristic for a multiple period problem.

Chapter 5 Joint pricing and inventory allocation decisions for perishable products

Chapter 5 jointly determines the price and the inventory allocation for a perishable product with a limited useful lifetime. We assume that the price of the product will increase as the time at which it perishes approaches to, as in the airline industry. To maximize the expected revenue, a discrete time dynamic programming model is developed to obtain the optimal prices and the optimal inventory allocations for the product with a two period lifetime. Three heuristics are then proposed when the lifetime is longer than two periods. The computational results show that the expected revenues from the proposed heuristics are very close to that from the optimal solution. These results are extended to (i) the case in which the price for the product always decreases; and (ii) the case in which the price for the product first increases and later decreases.

5.1 Introduction

There has been very little published research on joint capacity allocation and pricing decisions in the RM literature. Traditional approaches have assumed that prices are fixed and solved for the optimal allocation quantities. For example, airlines charge different prices for identical seats on the same flight. Given the fixed prices, the booking limit for each fare class is determined and implemented in the airline reservation systems. Effective application of fare class booking limits allows airlines to generate incremental revenues (Belobaba, 1989). However, the prices charged for different fare classes would influence demand and should be considered as decision variables, not fixed quantities. The integration of price and inventory decisions should receive more attention than it deserves (Mcgill and van Ryzin, 1999).

In this chapter, we formulate a discrete time dynamic programming model to determine the price and the capacity allocation for a perishable product within a fixed capacity. A periodic review policy is used. The price for the product is assumed to increase as the time at which the product will perish approaches. Demand for the product is a linear function of the price. At the beginning of each period, given the inventory level of the product, the optimal price and the optimal inventory allocation are determined for the objective of maximizing the expected revenue.

The proposed model makes an assumption that the prices will increase as the time approaches, as in the airline industry. Similar assumptions apply to rooms at hotels, cabins on cruise liners and cars at rental agencies (Weatherford, 1992 and Mcgill and van Ryzin, 1999). In order to make our problem more general, this assumption will be relaxed in Section 5.5.

We place the problem formulation of this chapter in the context of the taxonomy of RM problems developed by Weatherford and Bodily (1992). Our problem formulation is described as an A1-B1-C3-D1-En-F3-G1-H1-I1-J1-K1-L1-M2-N3 PRAM problem. In other words, it has discrete resource, fixed capacity, prices that are set jointly with the allocation decision, buildup willingness to pay (relaxed in the extension), as many discount price classes as there are prices, random and independent reservation demand, certain show-up of discount and full-price reservations, lost turned-down reservation, no group reservation and no diversion or displacement, no bumping procedure (there is no overbooking), effectively nested asset control, and a dynamic decision rule.

The rest of this chapter is organized as follows: In Section 5.2, the assumptions and notation are provided. A discrete time dynamic programming model is developed for a perishable product with lifetime of two or more periods. In Section 5.3, a product with a two period lifetime is first considered. The optimal prices and the optimal inventory allocations are obtained. For the product with the lifetime longer than two periods, three heuristics are proposed to determine the prices and the inventory allocations. The computational results are presented in Section 5.4. Two different extensions are discussed in Section 5.5.

5.2 Problem formulation

We consider a perishable product with an M period lifetime, where $M \geq 2$. Let index i = 1,…, M denotes the ages of the products. A periodic review policy is assumed. The initial inventory level Q, (e.g. seat capacity in an airplane), is given at the beginning of Period 1. No replenishment is allowed throughout the lifetime. At Period i (i = 1,…,M), only the product of age i is sold. The price for the product at Period i is represented by p_i. Demand for the product at Period i is denoted by t_i, following a stochastic additive demand function $t_i = \mu(p_i) + \varepsilon_i$, for i = 1,…, M.

$\mu(p_i)$ is mean demand at Period i and $\mu(p_i) = b_i - a_i p_i$, where $a_i, b_i \geq 0$. ε_i is an *i.i.d*. random variable with a known probability density function $f_i(\varepsilon_i)$ and is bounded in $[\varepsilon_i^{\min}, \varepsilon_i^{\max}]$. In addition, $E(\varepsilon_i) = 0$, where $b_i > -\varepsilon_i^{\min}$.

The additional notation employed in this chapter is as follows:

S_i = inventory assigned at Period i

x_i = inventory level at the beginning of Period i, $x_1 = Q$

α = discounted factor per period

p_i is confined to the finite interval $[p_i^{\min}, p_i^{\max}]$ where $p_i^{\max} < \dfrac{b_i + \varepsilon_i^{\min}}{a_i}$. The upper bound $p_i^{\max}$ prevents negative demands. Moreover, price intervals at different periods are non overlapping with $p_1 < ... < p_M$.

If $t_i > S_i$, the excessive demand is lost.

The dynamic programming model is developed to compute the expected revenue over M periods.

$V_i(x_i)$, the maximum expected revenue for the remaining periods when starting at Period i and with the inventory x_i, is computed as follows:

$$V_i(x_i) = \underset{p_i, S_i}{Max}[\varphi_i(x_i; p_i, S_i) + \alpha E(V_{i+1}(x_{i+1})]$$

where $\varphi_i(x_i; p_i, S_i)$ represents the expected revenue at Period *i*, $p_i E[\min(t_i, S_i)]$.

$x_{i+1} = [x_i - S_i + (S_i - t_i)^+]$ is the recursive function for the inventory level.

x_i and $V_i(x_i)$ are computed recursively backward in time, starting from Period *M* to Period 1. The boundary condition $V_M(x_M) = \underset{p_M}{Max}[\varphi_M(x_M; p_M)]$ is the maximum expected revenue at Period *M* for a given x_M, where $S_M = x_M$. Conversely, $V_1(x_1) = \underset{p_1, S_1}{Max}[\varphi_1(x_1; p_1, S_1) + \alpha E V_2[(x_2)]$ is the maximum expected revenue over *M* periods when the initial inventory at Period 1 is *Q*, i.e., $x_1 = Q$.

5.3 Joint pricing and inventory allocation decisions

In this section, we first consider a product with a two period lifetime. The optimal prices and the optimal inventory allocations are obtained. After that, we consider a more general problem, where the lifetime of the product is longer than two periods.

5.3.1 When the lifetime of the product is two periods

The optimal prices and the optimal inventory allocation are obtained by solving the dynamic programming model developed in Section 5.2 when *M* = 2. We start from the last period and employ the backward recursive induction.

At Period 2, the noise variable ε_2 is assumed to follow an IFR distribution (has an increasing hazard rate), where the hazard rate $\lambda_2(\varepsilon_2)$ is defined by $\dfrac{f_2(\varepsilon_2)}{1 - F_2(\varepsilon_2)}$. The unsold products at the end of the last period have no salvage value. The optimal price p_2^* satisfies the following optimality properties:

Lemma 5.1: At Period 2,

(i) The expected revenue $J_2(x_2; p_2)$ is concave with respect to p_2 for a given x_2.

(ii) The optimal price p_2^* is a non-increasing function of x_2 under the condition that $\lambda_2(\varepsilon_2) \geq \dfrac{1}{a_2 p_2^{\min}}$.

(iii) The maximum expected revenue $V_2(x_2)$ is concave with respect to x_2.

(iv) The maximum expected revenue $V_2(x_2)$ monotone increases with respect to x_2.

Proof: See the Appendix. □

The concavity of $J_2(x_2; p_2)$ with respect to p_2 for a given x_2 enables efficient algorithms such as gradient search to be employed to obtain p_2^*.

Let $V_2^{'}(x_2)$ denote the first order derivative of $V_2(x_2)$ with respect to x_2 and $J_1(x_1; p_1, S_1)$ stands for the expected revenue over two periods. Once p_2^* and $V_2(x_2)$ are obtained, the following theorem computes the optimal inventory allocation S_1^* to maximize $J_1(x_1; p_1, S_1)$.

Theorem 5.1: For a given p_1, there exists a unique S_1^* that maximizes the expected revenue $J_1(x_1; p_1, S_1)$.

Proof: $J_1(x_1; p_1, S_1)$ is expanded as follows:

$$
\begin{aligned}
J_1(x_1; p_1, S_1) &= \varphi_1(x_1; p_1, S_1) + \alpha E[V_2(x_2)] \\
&= p_1 E(t_1) - p_1 \int_{S_1 - b_1 + a_1 p_1}^{\varepsilon_{\max}} (b_1 - a_1 p_1 + \varepsilon_1 - S_1) f_1(\varepsilon_1) d\varepsilon_1 \\
&\quad + \alpha\Big[\int_{\varepsilon_{\min}}^{S_1 - b_1 + a_1 p_1} V_2(Q - b_1 + a_1 p_1 - \varepsilon_1) f_1(\varepsilon_1) d\varepsilon_1 + \int_{S_1 - b_1 + a_1 p_1}^{\varepsilon_{\max}} V_2(Q - S_1) f_1(\varepsilon_1) d\varepsilon_1 \Big]
\end{aligned}
$$

The first order derivative of $J_1(x_1; p_1, S_1)$ with respect to S_1 is obtained as follows.

$$
\frac{\partial J_1}{\partial S_1} = \int_{S_1 - b_1 + a_1 p_1}^{\varepsilon^{\max}} [p_1 - V_2^1(Q - S_1)] f_1(\varepsilon_1) d\varepsilon_1
$$

From Lemma 5.1, $V_2(x_2)$ is monotone increasing with respect to x_2. In addition, $V_2(0) = 0$ can be obtained from (A12). Hence there exists a unique S_1^* that satisfies $p_1 - V_2^{'}(Q - S_1^{*}) = 0$ **,** given a particular p_1. □

From Theorem 5.1, the optimal allocation S_1^* is unique for a given p_1. We assume that only a finite set of prices is applicable (in practice, prices usually take discrete values in a bounded interval). A procedure to compute the optimal p_1^* and S_1^* is provided as follows:

(1) Compute S_1^* that maximizes $J_1(x_1;p_1,S_1)$ for every p_1 in $[p_1^{\min}, p_1^{\max}]$.

(2) Select p_1^* and S_1^* with the maximum $J_1(x_1;p_1^*,S_1^*)$.

5.3.2 Proposed heuristics for a product with the lifetime longer than two periods

When the lifetime of the product is longer than two periods, it is hard to efficiently obtain the optimal prices and the optimal inventory allocation from the dynamic programming model. As the concavity of $J_1(x_1;p_1,S_1)$ with respect to S_1 does not always hold, the optimal solutions have to be computed through extensive enumerations. Thus, the solution time may be too long to be of practical interest. To overcome this problem, three heuristics are proposed to compute the prices and the inventory allocations.

5.3.2.1 Heuristic 1 (H1)

In order to develop a simple heuristics, we first assume that the inventory allocated to a period is not used or carried forward to the next period even if there are excess of inventory. Under the assumption, the revenue at each period is solely determined by the amount of inventory allocation to the period and its prices. Thus, the problem is to determine how much inventory to be allocated to each period and how to price it. The optimal inventory allocation and the optimal price at each period can be obtained from

Lemma 5.1. Denote $R_k^*(x_k)$ as the optimal revenue for a given inventory level x_k at Period k. The solution approach is as follows:

(1) Compute $R_k^*(x_k)$ for each x_k = 1,…,Q and k = 1,…,M.

(2) Solve $\underset{x_1,\ldots,x_M}{Max}\sum_{k=1}^{M} R_k^*(x_k)$ subject to $\sum_{k=1}^{M} x_k = Q$ and the solution x_k^* (k = 1,…,M) is the inventory allocation S_k for the corresponding period.

(3) Compute the optimal price p_k^* (k = 1,…,M) for the given inventory allocation S_k from Lemma 5.1.

To get a better solution, we can implement this heuristics on a rolling horizon basis; after the end of each period when demand has been realized, given the known remaining inventory level, the above mathematical programming is solved again to obtain the updated decisions of the price and the inventory allocation for the remaining periods.

5.3.2.2 Heuristic 2 (H2)

Drawing on insights from a two period problem analyzed in Section 5.3.1, a simple heuristics is provided to determine the allocation and the price at each period. The algorithm starts from the last period, because the customers in the last period will pay higher price and these demands should be satisfied with higher priority. The inventory allocation and the price for Period M, S_M and p_M, are first computed by solving a two period problem for Periods M–1 and M. With the remaining capacity $Q - S_M$, the inventory allocation and the price for Period M–1 are then determined again by solving a two period problem for Periods M–2 and M–1. Similarly, S_k and p_k for Period k, k = M–2,…,3 are computed by solving a two period problem for Periods k–1 and k. Finally, S_1, S_2, p_1 and p_2 are simultaneously determined.

5.3.2.3 Heuristic 3 (H3)

We propose a heuristics to determine the inventory allocation for Period i (i = 1, …, M). Denote B_i^j as the optimal protection level for Period j from Period i and $G_i(.)$ as the cumulative density function for t_i, where $t_i = \mu_i(p_i) + \varepsilon_i$ and ε_i has a known probability density function $f_i(\varepsilon_i)$.

At the beginning of Period i (i = 1,…, M), we first obtain the prices for the remaining $M - i + 1$ periods from H2, where $p_i < \ldots < p_M$. For the given prices p_i ,…, p_M, the inventory allocation at Period i is computed as follows:

(1) Start from $j = M$

Compute B_i^j that satisfies $B_i^j = G_j^{-1}[1 - \frac{p_i}{p_j}]$ for all $i < j$.

Similarly, for $j = M-1,...,i+1$, compute B_i^j for all $i < j$.

(2) From B_i^j obtained in (1), the inventory allocation for Period i is obtained by $S_i = Max(0, Q - \sum_{j=i+1}^{M} B_i^j)$.

After the end of each period when demand has been realized, given the known remaining inventory, the above procedure is repeated to determine the price and the inventory allocation for the next period.

The proposed methodology is motivated by the EMSR method (Belobaba, 1989). However, in the EMSR method, the price at each period is assumed to be known, but in our heuristics, the price is dynamically obtained from H2 at the beginning of each period.

All of the three proposed heuristics take account actual demands and dynamically update the pricing and inventory decision over the lifetime of the product. This may improve the company's revenue significantly.

5.4 Performance analysis of proposed heuristics

In this section, we compute the expected revenue from the proposed heuristics and the maximum expected revenue from the dynamic programming model. The comparison on the expected revenue is provided to study the performance of the proposed heuristics. However, the computation time of the dynamic programming model increases significantly with an increase of the product's lifetime M, since enumerations are required for obtaining the maximum expected revenue when $M \geq 3$. In order to examine (measure) the performance of the proposed heuristics for a large M, an upper bound for $V_1(Q)$ is also computed in Section 5.4.3 and compared with the maximum expected revenue.

5.4.1 Experimental design

In this numerical study, demand at Period i is price-sensitive and has an additive stochastic demand function, i.e., $t_i = \mu(p_i) + \varepsilon_i$, where $\mu(p_i) = b_i - a_i p_i$ is assumed to be a linear function of the discounted price p_i and the noise variable ε_i follows a truncated Normal distribution which is bounded by $\varepsilon_i^{\min} = -3\sigma_i$ and $\varepsilon_i^{\max} = 3\sigma_i$, where σ_i is the standard deviation of the Normal distribution.

We are particularly interested in the effects of demand variability on the revenue increase. Thus, σ_i is set to different levels, referring to different levels of demand variability.

Table 5.1 summarizes the experimental variables and their respective values used in this study. Seven constants and their respective values are also provided in Table 5.2.

Table 5.1 Variables in the numerical study

Parameters	Low level (-)	High Level (+)
σ_1	0.1* b_1	0.2* b_1
σ_2	0.1* b_2	0.2* b_2
σ_3	0.1* b_3	0.2* b_3
Q	20	30

Table 5.2 Constants in the numerical study

Parameters	Values	Parameters	Values
b_1	20	a_1	4
b_2	20	a_2	2
b_3	20	a_3	1
M	3		

For each scenario, the numerical experiments are replicated 100 times and the expected revenues under the three proposed heuristics are computed. The Common Random Number technique is employed to synchronize the results.

5.4.2 Expected revenue from dynamic programming and proposed heuristics

The performance of the proposed heuristics is measured by comparing the expected revenue obtained from the proposed heuristics with that from the dynamic programming model.

As shown in Table 5.3, the expected revenues from the proposed heuristics are close to that from the dynamic programming model. The difference in the expected revenue between the heuristics and the dynamic programming model is within 4% except for the high level of σ_3. A statistical analysis is performed and no significant difference among the performance of the heuristics is observed.

Table 5.3 Expected revenue from dynamic programming and proposed heuristics when Q = 30

($\sigma_1, \sigma_2, \sigma_3$)	Maximum revenue from dynamic programming	Revenue from H1	Revenue from H2	Revenue from H3
(-, -, -)	171.36	170.14	170.62	170.8
(-, -, +)	167.34	157.37	158.37	158.33
(-,+, -)	167.95	162.5	164.23	164.25
(-,+,+)	162.85	152.27	151.85	149.74
(+,-,-)	168.22	164.34	163.62	165.94
(+,-,+)	162.83	157.3	157.96	157.41
(+,+,-)	163.49	159.13	160.44	160.44
(+,+,+)	157.34	151.77	150.44	149.71

5.4.3 Upper bound for the maximum expected revenue

An upper bound V^{UP} for the maximum expected revenue $V_1(Q)$ is computed as follows:

(1) Compute $R_i^*(x_i)$ for each $x_i = 1, \ldots, b_i - a_i p_i^{\min} + \varepsilon_i^{\max}$ and $i = 1, \ldots, M$.

(2) The optimal inventory level x_i^* for Period i ($i = 1, \ldots, M$) is obtained by $\underset{for\ \mathrm{all}\ x_i}{Max}[R_i^*(x_i)]$.

(3) $V^{UP} = \sum_{i=1}^{M} \{\underset{x_i}{Max}[R_i^*(x_i)]\}$.

In the above, an M period problem is reduced to M independent newsvendor problems and it is obvious that V^{UP}, the sum of the maximum expected revenues among M periods, is strictly greater than $V_1(Q)$ and $\sum_{i=1}^{M} x_i^*$ is never worse than Q.

As the demand variability increases, the difference between the maximum expected revenue from the dynamic programming model and the upper bound increases, as shown in Table 5.4. We also observe that this difference decreases as Q increases. In practice (as in the airline seat allocation), reasonably large values for Q are experienced. For such higher values of Q, the upper bound obtained in this study is reasonably close to the maximum expected revenue from the dynamic programming model. Hence, this upper bound can be effectively applied to analyze the performance of heuristics solutions.

Table 5.4 Comparisons between $V_1(Q)$ and V^{UP}

$(\sigma_1, \sigma_2, \sigma_3)$	Q = 30			Q = 20		
	$V_1(Q)$	V^{UP}	% difference in revenue*	$V_1(Q)$	V^{UP}	% difference in revenue*
(-, -, -)	171.36	174.95	2.1%	150.75	174.95	13.8%
(-, -, +)	167.34	175.02	4.4%	140.2	175.02	19.9%
(-,+, -)	167.95	174.93	4.0%	142.3	174.93	18.7%
(-,+,+)	162.85	175	6.9%	135.6	175	22.5%
(+,-,-)	168.22	174.97	3.9%	150.08	174.97	14.2%
(+,-,+)	162.83	175.03	7.0%	140.68	175.03	19.6%
(+,+,-)	163.49	174.95	6.6%	140.81	174.95	19.5%
(+,+,+)	157.34	175.01	10.1%	132.62	175.01	24.2%

$$* \text{difference in revenue} = \frac{\text{(the upper bound - the maximum revenue)}}{\text{the upper bound}} * 100\%$$

5.5 Extensions

The model developed in Section 5.3 can be applied to the airline industry, where prices for tickets typically rise as the flight time approaches to. However, this clearly does not apply in all circumstances of price changes, e.g., (i) monotone markdown prices for fashion apparel, which perishes when the appropriate season is passed; (ii) price for the product first increases and later decreases, which following an increase-decrease pattern. For example, a food product for a special holiday; a fraction of the customers will be willing to pay a higher price closer to the holiday. This behavior disappears immediately after the target day.

5.5.1 Markdown prices

In the fashion industry, consumers are unwilling to pay high prices toward the end of the season because they will enjoy the product for a short period of time. Hence, the companies often employ successive markdowns to sell fashion apparel which perishes when the appropriate season is passed. The similar examples can be found in the electronic industry.

Bitran and Mondschein (1997) considered a periodic pricing review policy where the prices were revised only at a finite set of times and were never allowed to rise. This policy can be applied for seasonal products in the retailing industry, which are successively discounted during the season. The demand distribution was assumed to be Poisson. The authors used empirical analysis to develop conjecture as to the structure of the optimal policy and the optimal revenue but no theoretical results are presented.

Recently, Chew et al. (2005a) developed a discrete time dynamic programming model for perishable products. Under the assumption of "alternative" source, the optimal expected profit is concave with respect to the inventory level. From this property, they compute the optimal expected profit efficiently and employ this value as an upper bound for the optimal expected profit under lost sales. The computational results of Chew et al. (2005a) show that the ratio of the optimal expected profit under lost sales to the one under "alternative" source, is between 91% and 97% under different levels of demand variability.

5.5.2 Price follows an increase-decrease pattern

In this section, we formulate a discrete programming model to determine the price and the inventory allocation for a perishable product. A periodic review policy is used. The price for the product is assumed to first increase and later decrease, following an increase-decrease pattern. Demand for the product is price sensitive. At the beginning of each period, given the inventory of the product, the optimal price and the optimal allocation are determined for the objective of maximizing the total revenue.

The proposed model stems from many real problems in industries. For example, prices of a product for a specified holiday will follow an increase-decrease pattern. Because customers are willing to pay a higher price closer to the holiday, continuing lower prices may hurt potential revenues. Thus, retailer will employ higher prices. After the holiday, the retailer employs the discounted prices to attract the customers and then reduce the inventory. Consequently, markup and markdown prices are mixed in the selling periods. This price pattern can also be seen in the airline industry, which implies a cheaper fare at the last period.

Customers will hardly be willing to buy a product whose price oscillates, from their point of view, randomly over the season (Bitran and Mondschein, 1997). Thus, we assume that only one switch is employed in the selling periods.

Having one switch among the prices makes the proposed model more practical. Compared with the previous models which allows to move the price in one direction only (either markup or markdown), the proposed model permits price to move in both directions. This gives management flexibility to obtain the higher revenues. If only markdown policy is allowed, once a higher price is dumped, it will no longer be offered even if the product is being sold successfully.

This pricing pattern has also been considered for new products introduction (Dolan and Jeuland 1981; Jeuland and Dolan 1982; and Kalish 1983). Kalish and Sen's (1986) intuitive explanation for such pricing pattern is that if early adopters have a strong positive effect on late adopters, a low introductory price should encourage them to adopt this product. Once a product is established, the rises in price are attributed to strong sales. Subsequently, when demand saturates and begins to decrease, the price is also decreased in order to increase the sales and reduce the remaining inventories. Hence, the prices for the new product first increase and later decrease over the lifetime of the product.

Under the above mentioned pattern, it is more profitable to reserve enough inventories of the new product for future customers (late adopters) who will pay higher price. Thus, the price and the capacity allocation for the products at each period must be simultaneously determined in order to maximize the total revenue over the selling periods.

5.5.2.1 Dynamic programming model

We consider a perishable product with an M period lifetime. Let $i = 1,\ldots,M$ denote the ages of the product. At Period i ($i = 1,\ldots,M$), only the product of age i is sold. The price for the product increases during the first R periods and then decreases in the remaining M - R periods. Hence, for the first R –1 periods, some capacities have to be reserved for the future customers who will pay higher price. During the remaining $M – R+$

1 periods, older products will be offered at discounted prices and the capacities will not be reserved, which implies that $S_i = x_i$ when $i = R,\ldots,M$.

p_i is confined to the finite interval $[p_i^{\min}, p_i^{\max}]$ where $p_i^{\max} < \dfrac{b_i + \varepsilon_i^{\min}}{a_i}$. The upper bound $p_i^{\max}$ prevents negative demands. We also assume that $p_j^{\max} < p_{j+1}^{\min}$ for $j = 1,\ldots,R-1$ and $p_j^{\max} < p_{j+1}^{\min}$ for $j = R,\ldots,M-1$.

If $t_i > S_i$, the excessive demand is lost.

The dynamic programming model is developed to compute the expected revenue over M periods.

$V_i(x_i)$, the maximum expected revenue for the remaining periods when starting at Period i and with the inventory x_i, is computed as follows:

$$V_i(x_i) = \underset{p_i, S_i}{Max}[\varphi_i(x_i; p_i, S_i) + \alpha E(V_{i+1}(x_{i+1})]$$

where $\varphi_i(x_i; p_i, S_i)$ represents the expected revenue at Period i.

$$\varphi_i(x_i; p_i, S_i) = \begin{cases} p_i E[\min(t_i, S_i)] & i \le M \\ p_i E[\min(t_i, x_i)] & i \ge M+1 \end{cases}$$

The recursive function for the inventory level x_i is shown as follows:

$$x_{i+1} = \begin{cases} [x_i - S_i + (S_i - t_i)^+] & i \le M \\ [x_i - t_i]^+ & i \ge M+1 \end{cases}$$

We denote $J_i(x_i; p_i, S_i)$ as the expected revenue over the last *i* periods.

$$J_i(x_i; p_i, S_i) = \varphi_i(x_i; p_i, S_i) + \alpha E V_{i+1}[(x_{i+1})]$$

Note that $S_i = x_i$ when *i* = *R*,…,*M*. Hence, $\varphi_i(x_i; p_i, S_i)$ and $J_i(x_i; p_i, S_i)$ can be simplified and written as $\varphi_i(x_i; p_i)$ and $J_i(x_i; p_i)$ respectively, for *i* = *R*,…,*M*.

x_i and $V_i(x_i)$ are computed recursively backward in time, starting from Period *M* to Period 1. The boundary condition $V_M(x_M) = \underset{p_M}{Max}[\varphi_M(x_M; p_M)]$ is the maximum expected revenue at Period *M* for a given x_M, where $S_M = x_M$. Conversely, $V_1(x_1) = \underset{p_1, S_1}{Max}[\varphi_1(x_1; p_1, S_1) + \alpha E(V_2(x_2))]$ is the maximum expected revenue over *M* periods when the initial inventory at Period 1 is *Q*, i.e., $x_1 = Q$.

5.5.2.2 Joint pricing and inventory allocation decisions

In this section, we determine the inventory allocation and the price for a perishable product with an *M* period lifetime where the price for the product increases during the first *R* periods and then decreases in the remaining *M* – *R* periods. Initially, a special case of *R* = 2 is considered, followed by more general cases of *R* ≥ 3.

i) *R* = 2

In order to solve the dynamic programming model developed in Section 5.5.2.1, $J_i(x_i; p_i)$ must be shown to be concave with respect to p_i for a given x_i. In addition, $V_i(x_i)$ must be concave with respect to x_i and monotone increases with respect to x_i, for i = 2,…, M.

We start from the last period (Period M) and employ the backward recursive function to show the properties hold. At Period M, $J_M(x_M, p_M)$ is concave with respect to p_M for a given x_M and $V_M(x_M)$ is concave with respect to x_M, as shown in Lemma 5.1. For Period i = M -1,…, 2, the optimality properties are proven by Lemma 5.2.

Lemma 5.2: when i = M -1,…, 2

(i) The expected revenue $J_i(x_i; p_i)$ is concave with respect to p_i for a given x_i.

(ii) The optimal price p_i^* is a non-increasing function of x_i under the condition that $\lambda_i(\varepsilon_i) \geq \dfrac{1}{a_i(p_i^{\min} - \alpha p_{i+1}^{\max})}$.

(iii) The maximum expected revenue $V_i(x_i)$ is concave with respect to x_i.

(iv) The maximum expected revenue $V_i(x_i)$ monotone increases with respect to x_i.

The optimal prices p_i^* at Period i (i = M,…, 2) exists and the concavity of $J_i(x_i; p_i)$ with respect to p_i for a given x_i enables efficient algorithms such as gradient search to be employed to obtain p_i^*. Furthermore, the optimal price p_1^* and the optimal inventory allocation S_1^* at Period 1 can be computed following the procedures provided in Section 5.3.1.

ii) $R \geq 3$

The computation time of dynamic programming significantly increases when $R \geq 3$, since enumerations are required to obtain the optimal prices and the optimal inventory allocations. Hence, a heuristics is applied to determine the prices and the inventory allocations. One possible implementation of this heuristics is as follows: For the first R periods, one of the three heuristics proposed in Section 5.3.2 is employed to determine the inventory allocation and the price at each period. For the remaining $M - R$ periods, the optimal discounted prices can be efficiently computed from Lemma 5.2.

5.6 Summary

In this study, we first develope a discrete time dynamic programming model to determine the optimal inventory allocations and the optimal prices for a perishable product with a two period lifetime. The price for the product is first assumed to increase as the time at which it perishes approaches to and this assumption is relaxed in the extension. Several optimality properties are obtained. Since such properties do not hold when the lifetime of the product is longer than two periods, three heuristics are proposed to obtain the inventory allocations and the prices. The computational results show that the expected revenues from the proposed heuristics are very close to the maximum expected revenue from the dynamic programming model. An upper bound for the maximum expected revenue is computed. Our numerical study shows that the difference between the upper bound and the maximum expected revenue decreases when the initial inventory level increases.

Finally, we consider two different extensions. In the first extension, the price for the product is assumed to decrease during the product's lifetime. The optimal markdown prices can be obtained from Chew et al. (2005a). In the second extension, we assume that the price for the product first increases and later decreases. The optimal inventory allocation and the optimal price at each period are obtained when the price increases during the first two periods and then decreases. For more general cases, a heuristics is proposed to determine the inventory allocations and the prices.

Chapter 6 Conclusions and future work

The main purpose of this thesis is to develop a mathematical model to determine the optimal prices for products of different ages and the optimal order quantity for the new product (product of age 1) so as to maximize the multiple periods profit. This chapter concludes the study by presenting a summary of research findings and discussing the implications and limitations of this research, as well as suggesting several directions for future research.

6.1 Main findings

In the first part of this thesis (Chapter 3), we first develop a dynamic programming model for a perishable product with a two period lifetime. Under certain conditions, the optimal discounted price for the old product is a non-increasing function of the inventory level. From this property, we obtain the optimal pricing policy and prove that the expected profit is a concave function with respect to the order quantity for the new product. This concavity enables efficient algorithms to be employed to obtain the optimal order quantity for the new product. Even when this property does not hold, still an upper and a lower bound for the optimal order quantity are provided. We also prove that the expected profit from dynamic pricing is never worse than the expected profit from static pricing. Our numerical study shows that the profit increase from dynamic pricing becomes more significant as the demand uncertainty of Type 1 customers and the purchasing cost become higher.

We further extend our results to a more general case, where the lifetime of the product is longer than two periods. This problem is analyzed under two different assumptions, lost sales and “alternative” source. For each case, a dynamic programming model is developed with the objective of maximizing the total profit over the finite number of periods. The optimal discounted prices for products of different ages and the optimal order quantity for the new product are obtained. Moreover, we prove that the maximum expected profit under “alternative” source is never worse than the one under lost sales under certain conditions. Our numerical study shows that the ratio of the optimal profit from lost sales, to the optimal profit from “alternative” source is between 91% and 97% under different levels of demand variability. In addition, the optimal order quantity obtained from the dynamic programming model under lost sales is greater than that under “alternative” source.

In the second part of this thesis (Chapter 4), we determine the optimal prices for products of different ages and the optimal order quantity for the new product, for the objective of maximizing the total profits over the finite number of periods. The problem for a product with lifetime of two periods is first analyzed. Given the inventory level of the old product, the expected profit is jointly concave with respect to the order quantity for the new product and the product prices (the price of the new product and the discounted price of the old product). This concavity enables an efficient algorithm to be employed to obtain the optimal solution. Furthermore, several optimality properties are obtained. For the product with the lifetime of longer than two periods, the optimal prices for products of different ages and the optimal order quantity for the new product are obtained for a single period problem. Based on the optimal single period solution, we propose a heuristic for a multiple period problem.

The computational results for a product with a two period lifetime show that the total profit significantly increases when demand transfers between products of different ages are considered. As the loss rates increase, the optimal prices for both new and old products decrease. In addition, the optimal prices increase with increase of the transfer rates. These findings show that demand transfers between products of different ages should be seriously considered in practice when the retailers make their pricing and ordering decisions.

In the third part of this thesis (Chapter 5), we first develop a discrete time dynamic programming model to determine the optimal inventory allocations and the optimal prices for a perishable product with a two period lifetime. The price for the product is first assumed to increase as the time at which it perishes approaches to. Several optimality properties are obtained. Since such properties do not always hold when the lifetime of the product is longer than two periods, three heuristics are proposed to obtain the inventory allocations and the prices. The computational results show that the expected revenues from the proposed heuristics are very close to the maximum expected revenue from the dynamic programming model. An upper bound for the maximum expected revenue is computed and the difference between the upper bound and the maximum expected revenue decreases when the initial inventory level increases.

Finally, we consider two different extensions. In the first extension, the price for the product is assumed to decrease during the product's lifetime. The optimal markdown prices can be obtained from Chew et al. (2005a). In the second extension, we assume that the price for the product first increases and later decreases. The optimal inventory allocation and the optimal price at each period are obtained when the price increases during the first two periods and then decreases. For more general cases, a heuristics is proposed to determine the inventory allocations and the prices.

In this study, we assume that the demand function follows an additive form. The additive demand function has its limitations because it assumes that the expected demand is a linear function of prices. However, this demand function is commonly employed in literature relating to pricing and inventory problems (Thowsen 1975, Lau and Lau 1988, Polatoglu 1991 and Abad 1996). Though most actual demand functions may not behave in this way, the model should still be able to provide useful insights on the general trend when the parameters change.

6.2 Suggestions for future work

General Demand Functions

Instead of an additive demand function used in this study, it would be interesting that a general demand function $D = \mu(p)\varepsilon + \beta(p)$ is considered. (The cases of $\mu(p) = 1$ and $\beta(p) = 0$ are often referred to as the additive and multiplicative function, respectively.) An updated dynamic programming model is obtained by substituting the general demand function in the model. The total profit can be computed by enumerations.

The comparison between the total profit obtained from a general demand function and that from an additive demand function is desirable. Smaller difference in the total profit suggests that the approach of approximating the problem under a general demand function with the problem under an additive demand function is possible. Hence, a heuristic based on the optimal solutions of this study can be proposed for this more complicated problem under the general demand function.

Demand Learning

Most of the existing works including this thesis assume that a firm has knowledge about the parameters of demand distribution. However, in real life, there are many situations where a firm does not have full knowledge of the parameters of the demand distribution, when new products are introduced for example, or the demand distribution may be changing in ways that are not predictable.

Although some research has been done in the area of demand learning, relatively little work is available on combining demand learning with pricing and ordering decisions. The key problem in demand learning is how to update the demand distribution including unknown parameters. In order to solving this problem, Bayesian approach is the best choice. The current demand distribution is updated by using probability and statistics knowledge. For example, let $g(d/w)$ represent the demand density function of an unknown parameter w. Let $f(w)$ be the known prior density function of w. Given sufficient statistic data S, the posterior density function of unknown parameter is obtained and denoted as $f(w/S)$. Then, the new demand distribution is updated using $g(d/w)$ and $f(w/S)$.

Incorporated the updated demand distribution into the dynamic programming model developed in this thesis, our problem is extended to a more general problem considering demand learning. This extension is valuable, since demand learning will help retailer effectively identify the changes of current demand and efficiently adjust their pricing and order decisions.

Strategic Customers

Most existing works on dynamic pricing assume myopic customers. A myopic customer is one who makes a purchase immediately if the price is below his valuation (reserved price), without considering future prices. By assuming myopic customers, the retailer can ignore the effects of future markdowns on current customer purchases, and only focuses on determining the current price. In contrast, dynamic pricing decisions for a retailer facing strategic customers are more complex, since a strategic customers will take into account the future path of prices when making purchasing decisions. In this case, the retailer has to consider the effects of future as well as current prices on customers' purchasing decisions. Hence, an interesting but challenging research direction would be to incorporate the customers' strategic purchasing behaviors into the pricing decisions.

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Appendix

Proof of Lemma 5.1:

(i) The expected profit at Period 2 is

$$J_2(x_2; p_2) = \varphi_2(x_2, p_2) = p_2 E[Min(x_2, t_2)] \tag{A1}$$

The first and second partial derivatives of $J_2(x_2; p_2)$ with respect to p_2 are shown as follows:

$$\frac{\partial J_2}{\partial p_2} = \int_{\varepsilon_2^{\min}}^{x_2 - b_2 + a_2 p_2} (b_2 - 2a_2 p_2 + \varepsilon_2) f_2(x_2 - b_2 + a_2 p_2) + x_2[1 - F_2(x_2 - b_2 + a_2 p_2)] \tag{A2}$$

$$\frac{\partial^2 J_2}{\partial p_2^2} = -2a_2 F_2(x_2 - b_2 + a_2 p_2) - a_2^2 p_2 f_2(x_2 - b_2 + a_2 p_2) \tag{A3}$$

Hence, $J_2(x_2; p_2)$ is concave with respect to p_2 for a given inventory level x_2.

(ii) Let $\hat{p}_2$ denote the value of price p_2 which satisfies $\frac{\partial J_2}{\partial p_2} = 0$ for a given x_2.

$$\left.\frac{\partial J_2}{\partial p_2}\right|_{p_2 = \hat{p}_2} = \int_{\varepsilon_2^{\min}}^{x_2 - b_2 + a_2 \hat{p}_2} [b_2 - 2a_2 \hat{p}_2 + \varepsilon_2] f_2(\varepsilon_2) d\varepsilon_2 + \int_{x_2 - b_2 + a_2 \hat{p}_2}^{\varepsilon_2^{\max}} x_2 f_2(\varepsilon_2) d\varepsilon_2 = 0 \tag{A4}$$

Note that (A4) expresses the stationary point $\hat{p}_2$ as a function of x_2, denoted as $\hat{p}_2(x_2)$. Since $\hat{p}_2$ is bounded in $[p_2^{\min}, p_2^{\max}]$, the optimal price p_2^* at Period 2 is determined as follows.

$$p_2^* = \begin{cases} p_2^{\min} & \hat{p}_2 \le p_2^{\min} \\ \hat{p}_2 & p_2^{\min} < \hat{p}_2 < p_2^{\max} \\ p_2^{\max} & \hat{p}_2 \ge p_2^{\max} \end{cases} \tag{A5}$$

Taking the first order derivative of $\hat{p}_2(x_2)$ with respect to x_2 based on (A4) and rearranging the terms, we obtain

$$a_2(\frac{d\hat{p}_2(x_2)}{dx_2}) = \frac{1 - F_2[x_2 - b_2 + a_2\hat{p}_2(x_2)] - a_2\hat{p}_2(x_2)f_2[x_2 - b_2 + a_2\hat{p}_2(x_2)]}{2F_2[x_2 - b_2 + a_2\hat{p}_2(x_2)] + a_2\hat{p}_2(x_2)f_2[x_2 - b_2 + a_2\hat{p}_2(x_2)]} \tag{A6}$$

Given that $\lambda_2(x_2 - b_2 + a_2\hat{p}_2(x_2) = \dfrac{f_2(x_2 - b_2 + a_2\hat{p}_2(x_2))}{1 - F_2(x_2 - b_2 + a_2\hat{p}_2(x_2))} \ge \dfrac{1}{a_2 p_2^{\min}}$,

$\hat{p}_2 \ge p_2^{\min}$ and the denominator of (A6) is non-positive, hence $-1 \le a_2 \dfrac{d\hat{p}_2(x_2)}{dx_2} \le 0$.

Therefore, it follows that p_2^* is a non-increasing function of the inventory level x_2.

(iii) Finally, we prove that $V_2(x_2)$ is concave with respect to x_2.

Let $V_2(x_2)$ be defined as follows.

$$V_2(x_2) = \begin{cases} V_{2,1}(x_2) \text{ obtained when } p_2^* = p_2^{\min} & x_2 \ge x_2^n \\ V_{2,2}(x_2) \text{ obtained when } p_2^* = \hat{p}_2 & x_2^m < x_2 < x_2^n \\ V_{2,3}(x_2) \text{ obtained when } p_2^* = p_2^{\max} & x_2 \le x_2^m \end{cases}$$

where the thresholds x_2^n and x_2^m are calculated by setting (A2) to be zero under the conditions $p_2 = p_2^{\min}$ and $p_2 = p_2^{\max}$.

Consider the following three cases:

(1) $x_2 \geq x_2^n$

$$V_{2,1}(x_2) = \int_{\varepsilon_2^{\min}}^{x_2 - b_2 - a_2 p_2^{\min}} p_2^{\min}(b_2 - a_2 p_2^{\min} + \varepsilon_2) f_2(\varepsilon_2) d\varepsilon_2 + \int_{x_2 - b_2 - a_2 p_2^{\min}}^{\varepsilon_2^{\max}} p_2^{\min} x_2 f_2(\varepsilon_2) d\varepsilon_2 \tag{A7}$$

The first and second order derivatives with respect to x_2 are shown as follows:

$$\frac{dV_{2,1}}{dx_2} = p_2^{\min}[1 - F_2(x_2 - b_2 - a_2 p_2^{\min})] \geq 0 \tag{A8}$$

$$\frac{d^2 V_{2,1}}{dx_2^2} = -p_2^{\min} f_2(x_2 - b_2 - a_2 p_2^{\min}) \leq 0$$

Thus, $V_{2,1}(x_2)$ is concave with respect to x_2 when $x_2 \geq x_2^n$.

(2) $x_2^n < x_2 < x_2^m$

$$V_{2,2}(x_2) = \int_{\varepsilon_2^{\min}}^{x_2 - b_2 - a_2 \hat{p}_2(x_2)} \hat{p}_2(x_2)(b_2 - a_2 \hat{p}_2(x_2) + \varepsilon_2) f_2(\varepsilon_2) d\varepsilon_2 + \int_{x_2 - b_2 - a_2 \hat{p}_2(x_2)}^{\varepsilon_2^{\max}} \hat{p}_2(x_2) x_2 f_2(\varepsilon_2) d\varepsilon_2 \tag{A9}$$

The first and second order derivatives with respect to x_2 are given as follows:

$$\frac{dV_{2,2}}{dx_2} = \int_{x_2 - b_2 - a_2 \hat{p}_2(x_2)}^{\varepsilon_2^{\max}} \hat{p}_2(x_2) f_2(\varepsilon_2) d\varepsilon_2 \geq 0 \tag{A10}$$

$$\begin{aligned} \frac{d^2 V_{2,2}}{dx_2^2} = & \frac{d\hat{p}_2(x_2)}{dx_2}[1 - F_2(x_2 - b_2 - a_2 \hat{p}_2(x_2))] \\ & -[1 + a_2 \frac{d\hat{p}_2(x_2)}{dx_2}] \hat{p}_2(x_2) f_2(x_2 - b_2 - a_2 \hat{p}_2(x_2)) \end{aligned} \tag{A11}$$

Since $-1 \le a_2 \dfrac{d\hat{p}_2(x_2)}{dx_2} \le 0$, (A11) is negative. Therefore, $V_{2,2}(x_2)$ is concave with respect to x_2 when $x_2^n < x_2 < x_2^m$.

(3) $x_2 \le x_2^m$

$$V_{2,3}(x_2) = \int_{\varepsilon_2^{\min}}^{x_2 - b_2 - a_2 p_2^{\max}} p_2^{\max}(b_2 - a_2 p_2^{\max} + \varepsilon_2) f_2(\varepsilon_2) d\varepsilon_2 + \int_{x_2 - b_2 - a_2 p_2^{\max}}^{\varepsilon_2^{\max}} p_2^{\max} x_2 f_2(\varepsilon_2) d\varepsilon_2 \quad \text{(A12)}$$

Since x_2 is independent of $p_2^{\max}$, the first and second order derivatives with respect to x_2 are given as follows:

$$\frac{dV_{2,3}}{dx_2} = p_2^{\max}[1 - F_2(x_2 - b_2 - a_2 p_2^{\max})] \ge 0 \quad \text{(A13)}$$

$$\frac{d^2V_{2,3}}{dx_2^2} = -p_2^{\max} f_2(x_2 - b_2 - a_2 p_2^{\max}) \le 0 \quad \text{(A14)}$$

Thus, $V_{2,3}(x_2)$ is concave with respect to x_2 when $x_2 \le x_2^m$.

Finally, we focus on the boundary conditions at the threshold values x_2^n and x_2^m in order to show overall concavity. At the thresholds x_2^n and x_2^m, $V_2(x_2)$ is continuous, which can be obtained from (A7), (A9) and (A12). Furthermore, we can easily show that the gradients at x_N^n for cases (1) and (2) are the same. The same is true for the gradients at x_N^m for cases (2) and (3). Hence $V_2(x_2)$ is concave with respect to x_2. Property (iv) is directly obtained from (A8), (A10) and (A13). □

Proof of Lemma 5.2:

We show by induction that $J_i(x_i; p_i)$ is concave with respect to p_i and then prove that $V_i(x_i)$ is concave with respect to x_i.

First we assume that $V_{i+1}(x_{i+1})$ is a continuous function and concave with respect to x_{i+1}. The first derivative of $V_{i+1}(x_{i+1})$ with respect to x_{i+1} is assumed to be positive.

$V_{i+1}(x_{i+1})$ is represented as follows.

$$V_{i+1}(x_{i+1}) = \begin{cases} V_{i+1,1}(x_i - t_i) \text{ obtained when } p_{i+1}^* = p_{i+1}^{\min} & x_i \geq t_i + x_{i+1}^n \\ V_{i+1,2}(x_i - t_i) \text{ obtained when } p_{i+1}^* = \hat{p}_{i+1} & t_i + x_{i+1}^m < x_i < t_i + x_{i+1}^n \\ V_{i+1,3}(x_i - t_i) \text{ obtained when } p_{i+1}^* = p_{i+1}^{\max} & t_i < x_i \leq t_i + x_{i+1}^m \\ V_{i+1,3}(0) \text{ obtained when } p_{i+1}^* = p_{i+1}^{\max} & x_i \leq t_i \end{cases}$$

where $x_{i+1} = [x_i - t_i]^+$ and $t_i = b_i - a_i p_i + \varepsilon_i$

(1) It suffices to show that $\dfrac{\partial^2 J(x_i; p_i)}{\partial p_i^2} \leq 0$.

$$\begin{aligned} J_i(x_i; p_i) = \varphi_i(x_i; p_i) + \alpha[& \int_{\varepsilon_i^{\min}}^{x_i - x_{i+1}^n - b_i + a_i p_i} V_{i+1,1}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\ & + \int_{x_i - x_{i+1}^n - b_i + a_i p_i}^{x_i - x_{i+1}^m - b_i + a_i p_i} V_{i+1,2}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \\ & + \int_{x_i - x_{i+1}^m - b_i + a_i p_i}^{x_i - b_i + a_i p_i} V_{i+1,3}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i + \int_{x_i - b_i + a_i p_i}^{\varepsilon_i^{\max}} V_{i+1,3}(0) f_i(\varepsilon_i) d\varepsilon_i] \end{aligned} \quad \text{(A15)}$$

$$\frac{\partial^2 J_i}{\partial p_i^2} = -2a_i F_i(x_i - b_i + a_i p_i) - a_i^2 p_i f_i(x_i - b_i + a_i p_i)$$
$$+ \alpha a_i^2 [\int_{\varepsilon_i^{\min}}^{x_i - x_{i+1}^n - b_i + a_i p_i} V''_{i+1,1}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i$$
$$+ \int_{x_i - x_{i+1}^n - b_i + a_i p_i}^{x_i - x_{i+1}^m - b_i + a_i p_i} V''_{i+1,2}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i$$
$$+ \int_{x_i - x_{i+1}^m - b_i + a_i p_i}^{x_i - b_i + a_i p_i} V''_{i+1,3}(x_i - b_i + a_i p_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i]$$
$$+ \alpha a_i^2 V'_{i+1,3}(0) f_i(x_i - b_i + a_i p_i)$$

Note that $V'_{i+1,3}(0) = \left. \frac{dV_{i+1,3}(x_i)}{dx_i} \right|_{x_i = 0} = p_{i+1}^{\max}$.

Since $p_i \geq p_{i+1}^{\max}$, the sum of the 1st, 2nd and 6th terms is negative. Furthermore, the 3rd, 4th and 5th terms are less than zero, based on the assumption that $V_{i+1}(x_{i+1})$ is concave with respect to x_{i+1}. Therefore, $J_i(x_i; p_i)$ is concave with respect to p_i.

(2) Let $\hat{p}_i$ denote the value of price p_i that satisfies the stationary condition $\frac{\partial J_i}{\partial p_i} = 0$.

$$\left. \frac{\partial J_i}{\partial p_i} \right|_{p_i = \hat{p}_i} = \int_{\varepsilon_i^{\min}}^{x_i - b_i + a_i \hat{p}_i} (b_i - 2a_i \hat{p}_i + \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i + x_i [1 - F_i(x_i - b_i + a_i \hat{p}_i)]$$
$$+ \alpha a_i [\int_{\varepsilon_i^{\min}}^{x_i - x_{i+1}^n - b_i + a_i \hat{p}_i} V'_{i+1,1}(x_i - b_i + a_i \hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \qquad \text{(A16)}$$
$$+ \int_{x_i - x_{i+1}^n - b_i + a_i \hat{p}_i}^{x_i - x_{i+1}^m - b_i + a_i \hat{p}_i} V'_{i+1,2}(x_i - b_i + a_i \hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i$$
$$+ \int_{x_i - x_{i+1}^m - b_i + a_i \hat{p}_i}^{x_i - b_i + a_i \hat{p}_i} V'_{i+1,3}(x_i - b_i + a_i \hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i] = 0$$

Note that (A16) express the stationary point $\hat{p}_i$ as a function of x_i, denoted by $\hat{p}_i(x_i)$. Since $\hat{p}_i$ is bounded in $[p_i^{\min}, p_i^{\max}]$, we can determine the optimal discounted price at Period *i*, p_i^*, as follows.

$$p_i^* = \begin{cases} p_i^{\min} & \hat{p}_i \le p_i^{\min} \\ \hat{p}_i & p_i^{\min} < \hat{p}_i < p_i^{\max} \\ p_i^{\max} & \hat{p}_i \ge p_i^{\max} \end{cases}$$

Taking the first order derivative of $\hat{p}_i(x_i)$ with respect to x_i based on (A16) and rearranging the terms, we obtain

$$a_i \frac{d\hat{p}_i(x_i)}{dx_i} = \frac{N}{D}$$

where

$$\begin{aligned} N = {} & 1 - F_i(x_i - b_i + a_i\hat{p}_i) - a_i(\hat{p}_i - \alpha p_{i+1}^{\max}) f_i(x_i - b_i + a_i\hat{p}_i) \\ & + \alpha \int_{\varepsilon_i^{\min}}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}''(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \end{aligned}$$

$$\begin{aligned} D = {} & 2F_i(x_i - b_i + a_i\hat{p}_i) + a_i(\hat{p}_i - \alpha p_{i+1}^{\max}) f_i(x_i - b_i + a_i\hat{p}_i) \\ & - \alpha \int_{\varepsilon_i^{\min}}^{x_i - b_i + a_i\hat{p}_i} V_{i+1}''(x_i - b_i + a_i\hat{p}_i - \varepsilon_i) f_i(\varepsilon_i) d\varepsilon_i \end{aligned}$$

and

$$\int_{\varepsilon_i^{\min}}^{x_i-b_i+a_i\hat{p}_i} V_{i+1}''(x_i-b_i+a_i\hat{p}_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i = \int_{\varepsilon_i^{\min}}^{x_i-x_{i+1}^n-b_i+a_i\hat{p}_i} V_{i+1,1}''(x_i-b_i+a_i\hat{p}_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i$$
$$+\int_{x_i-x_{i+1}^n-b_i+a_i\hat{p}_i}^{x_i-x_{i+1}^m-b_i+a_i\hat{p}_i} V_{i+1}''(x_i-b_i+a_i\hat{p}_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i\ .$$
$$+\int_{x_i-x_{i+1}^m-b_i+a_i\hat{p}_i}^{x_i-b_i+a_i\hat{p}_i} V_{i+1}''(x_i-b_i+a_i\hat{p}_i-\varepsilon_i)f_i(\varepsilon_i)d\varepsilon_i$$

Given that the hazard rate

$\lambda_i(x_i-b_i+a_i\hat{p}_i(x_i)) = \dfrac{f_i(x_i-b_i+a_i\hat{p}_i(x_i))}{1-F_i(x_i-b_i+a_i\hat{p}_i(x_i))} \geq \dfrac{1}{a_i(p_i^{\min}-\alpha p_{i+1}^{\max})}$, $\hat{p}_i(x_i)\geq p_i^{\min}$ and

the denominator is non-positive, hence $-1\leq a_i(\dfrac{d\hat{p}_i(x_i)}{dx_i})\leq 0$. Therefore, $\hat{p}_i(x_i)$ is a non-increasing function of the inventory level x_i . It follows that p_i^* is also a non-increasing function of the inventory level x_i .

(3) Next we prove that $V_i(x_i)$ monotone increases and is concave with respect to x_i .

$V_i(x_i)$ is shown as follows.

$$V_i(x_i) = \begin{cases} V_{i,1}(x_i)\text{ obtained when } p_i^* = p_i^{\min} & x_i \geq x_i^n \\ V_{i,2}(x_i)\text{ obtained when } p_i^* = \hat{p}_i & x_i^m < x_i < x_i^n \\ V_{i,3}(x_i)\text{ obtained when } p_i^* = p_i^{\max} & x_i \leq x_i^m \end{cases}$$

where the thresholds x_i^m and x_i^n are calculated by satisfying $\dfrac{\partial J_i}{\partial p_i}=0$ under the conditions that $p_i = p_i^{\min}$ and $p_i = p_i^{\max}$.

Finally, we focus on the boundary conditions at the threshold values x_i^m and x_i^n in order to show overall concavity. At the thresholds x_i^m and x_i^n, $V_i(x_i)$ is continuous, because $V_{i,1}(x_i^n) = V_{i,2}(x_i^n)$ and $V_{i,2}(x_i^m) = V_{i,3}(x_i^m)$.

Furthermore, it can easily be proved that $\left.\frac{dV_{i,1}(x_i)}{dx_i}\right|_{x_i=(x_i^n)^+} = \left.\frac{dV_{i,2}(x_i)}{dx_i}\right|_{x_i=(x_i^n)^-} \geq 0$ and $\left.\frac{dV_{i,2}(x_i)}{dx_i}\right|_{x_i=(x_i^m)^+} = \left.\frac{dV_{i,3}(x_i)}{dx_i}\right|_{x_i=(x_i^m)^-} \geq 0$. Therefore, we draw conclusion that the continuous profit function $V_i(x_i)$ not only monotonically increases with respect to x_i but also is concave with respect to x_i. □