A MAINTENANCE MODEL FOR THE SUPPLY-BUFFER-DEMAND PRODUCTION SYSTEM

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Table of Contents

Summary

In this thesis we first make a brief literature review on the research area of "Maintenance". We classify the recent papers on maintenance into different categories and discuss them for each category; especially, we emphasize on the papers whose subjects are about the age-dependent maintenance, imperfect maintenance and the multi-unit systems maintenance, which are all involved in the system that we study.

Then we study a special kind of the multi-unit systems, the so-called Supply-Buffer-Demand production system, in which there is an inventory buffer between the supplying production unit and the demanding unit. We propose our maintenance model for this system, which is a more general model compared to the model presented by Chelbi and Rezg (2006) on a similar system. In the system we study, the supplying unit undergoes a maintenance action as soon as its age increasing by "T" or at its failure, whichever occurs first. Corrective maintenance is assumed to be perfect; while preventive maintenance is assumed to be imperfect in that it is perfect with probability "p" and minimal with probability "q". In every "N" maintenance actions, the system undergoes an enhanced preventive maintenance which is a perfect maintenance action, so that the system would definitely return to its initial state (age zero). There are stocks built up in the buffer whose capacity is "h", which are used to supply the demanding unit when the supplying unit undergoes maintenance.

We take the joint consideration of both the age-dependent maintenance planning

and the buffer inventory control in formulating the model. We minimize the expected total cost per unit of time for the system, under constraints of minimum required stationary availability level, minimum required reliability level, and maximum required inventory shortage rate level. We also propose numerical algorithms to obtain the optimal solutions for the decision variables of the model: the preventive maintenance age increment "T", the number of periods within a cycle "N", and the capacity of the buffer "h". The optimal maintenance and inventory policies for the system would then be determined.

List of Tables

List of Figures

List of Notation

$A_n(T)$ the expected maintenance costs (including the preventive and corrective maintenance) for the n^{th} period since the last perfect maintenance action (either corrective maintenance, or "enhanced" preventive maintenance, or preventive maintenance which is perfectly performed with probability p);

$AV_n(T)$ the expected available time of the unit M1 for the n^{th} period since the last perfect maintenance action;

$B_n(T)$ the expected time duration (including the operating time and maintenance time) for the n^{th} period since the last perfect maintenance action;

C_h holding cost for a unit of product during one unit of time;

C_s shortage cost for a unit of product during one unit of time;

d demand rate of the unit M2;

$EAV_n(T)$ the expected total available time duration of the unit M1 for the first periods within a cycle;

$EC_n(T)$ the expected total maintenance costs (corrective and preventive maintenance) for the first n periods within a cycle;

$ET_n(T)$ the expected total time duration (operating time and maintenance time) for the first n periods within a cycle;

$EC1_n(T, h)$ the expected total costs (including both the maintenance cost and inventory cost) for the first n periods within a cycle, under **Condition 1** of the inventory control policy;

$EC2_n\ (T,\ h)$ the expected total costs (including both the maintenance cost and inventory cost) for the first n periods within a cycle, under **Condition 2** of the inventory control policy;

$EShort1_n\,(T,\ h)$ the expected total number of shortage of the buffer for the first n periods within a cycle, under **Condition 1** of the inventory control policy;

$EShort2_n\,(T,\ h)$ the expected total number of shortage of the buffer for the first n periods within a cycle, under **Condition 2** of the inventory control policy;

$f\,(t)$ probability density function associated to the lifetime of the production unit M1;

$F(t)$ probability distribution function associated to the lifetime of the production unit M1;

Fa minimum stationary availability requirement;

Fr minimum reliability requirement for joint N periods;

Fs maximum stationary shortage rate requirement;

$G1_n\ (T, h)$ the expected total inventory costs (holding cost and shortage cost) for the n^{th} period since the last perfect maintenance action, under **Condition 1** of the inventory control policy;

$G2_n\ (T, h)$ the expected total inventory costs (holding cost and shortage cost) for the n^{th} period since the last perfect maintenance action, under **Condition 2** of the inventory control policy;

h buffer capacity;

M_p cost for a preventive maintenance action;

M_c cost for a corrective maintenance action ($M_c > M_p$);

M_e additional cost for an "enhanced" preventive maintenance action;

N number of periods within a cycle;

p the probability that a preventive maintenance action is perfect;

Pc precision criterion for the solution;

q the probability that a preventive maintenance action is imperfect;

$R(t)$ reliability function of the production unit M1;

$Ra_n(T)$ the probability that the system is reliable immediately after the maintenance action in the n^{th} period within a cycle, i.e. the probability that the system is reliable immediately after the beginning of Phase I of the $(n+1)^{th}$ period within a cycle;

$Rb_n(T)$ the probability that the system is reliable right before the maintenance action in the n^{th} period (i.e. it has survived a time T in the n^{th} period) within a cycle;

$S(N, T, h)$ total cost for the system per unit of time;

$SAV_N(T)$ the expected stationary availability of the production unit M1 within a cycle (N periods);

$Short1_n(T, h)$ the expected number of shortage of the buffer for the n^{th} period since the last perfect maintenance action, under **Condition 1** of the inventory control policy;

$Short2_n(T, h)$ the expected number of shortage of the buffer for the n^{th} period since the last perfect maintenance action, under **Condition 2** of the inventory control policy

$SShort1_N(T, h)$ the expected total number of shortage of the buffer per unit of time within a cycle (N periods), under **Condition 1** of the inventory control policy;

$SShort2_N(T, h)$ the expected total number of shortage of the buffer per unit of time within a cycle (N periods), under **Condition 2** of the inventory control policy;

T age increment by which a preventive maintenance action must be performed;

Umax maximum production rate of the unit M1 (Umax > d);

X virtual lifetime of the unit M1;

$Y_k(T)$ the probability that the system's virtual age restores to zero after the k^{th} period within a cycle;

$\Delta EAV_n(T)$ the expected available time of the unit M1 for the n^{th} period within a cycle;

$\Delta EC_n(T)$ the expected maintenance costs for the n^{th} period within a cycle;

$\Delta ET_n(T)$ the expected time duration for the n^{th} period within a cycle;

$\Delta EC1_n(T, h)$ the expected total costs (including both the maintenance cost and inventory cost) for the n^{th} period within a cycle, under **Condition 1** of the inventory control policy;

$\Delta EC2_n(T, h)$ the expected total costs (including both the maintenance cost and inventory cost) for the n^{th} period within a cycle, under **Condition 2** of the inventory control policy;

$\Delta EShort1_n(T, h)$ the expected number of shortage of the buffer for the n^{th} period within a cycle, under **Condition 1** of the inventory control policy;

$\Delta EShort2_n(T, h)$ the expected number of shortage of the buffer for the n^{th} period within a cycle, under **Condition 2** of the inventory control policy;

μ_p duration for a preventive maintenance action;

μ_c duration for a corrective maintenance action ($\mu_c > \mu_p$);

A MAINTENANCE MODEL FOR THE SUPPLY-BUFFER-DEMAND PRODUCTION SYSTEM

Chapter 1

Introduction to Maintenance

Maintenance, is repairing any kind of an engineering system (e.g. a mechanical or an electrical system) when it fails to perform normally, as well as taking actions to keep the system in good operating status and to prevent the deterioration. The European Federation of National Maintenance Societies defines maintenance as: "all actions which have as an objective to retain an item in or restore it to, a state in which it can perform the required function. The actions include the combination of all technical and corresponding administrative, managerial and supervision actions".

As deterioration process is prevalent in the engineering systems, maintenance measures are becoming necessary and crucial in ensuring the performances of the systems during their lives. More and more interest has been attracted into the area of maintenance during the past few years, and there are more papers published on this area.

In this thesis, we will study the problem of designing a maintenance scheme on the Supply-Buffer-Demand production system. The model we propose is an extensive study following previous works by Chelbi and Rezg (2006). The model extends the previous assumption on preventive maintenance from perfect maintenance to imperfect, and considers additional availability, reliability and inventory shortage requirements for the system. Numerical algorithms and examples to solve our model are also provided.

The organization of the thesis is as follows: In Chapter 1, we briefly introduce the research area of Maintenance and four mostly used methods to classify papers in this area; In Chapter 2 we review the existing literature on the classification method of maintenance topics, and all the topics we mention are closely related to or used in our model, so that the content of thesis can be self-contained; In Chapter 3, we define the problem and provide the assumptions assumed for the general system; In Chapter 4, we analyze the general system and derive the analytical results for the objective and constraint functions for our model; In Chapter 5 we define the mathematical optimization model and provide the algorithm to solve the model, and also examples for the algorithm are presented and analyzed.

We continue Chapter 1 with introducing the methods to classify papers in the research area of Maintenance. Papers in this area can be categorized into groups according to different classification standards, e.g. topics and areas, maintenance policies, complexity of the system, types of maintenance actions, source of publications etc. In the following there are some of the major classification standards.

1.1 Industrial standards classification

Generally speaking, a maintenance action can be technically classified into two major types: preventive maintenance (PM) and corrective maintenance (CM). According to Japanese Industrial Standards Z 8115-2000, preventive maintenance can be seen to consist of three subcategories: Hard Time Scheduled Maintenance (HTSM), On-Condition Maintenance (OCM) and Condition Monitoring Maintenance (CMM). On the other hand, corrective maintenance includes two subcategories: Emergency Maintenance and Normal Corrective Maintenance. This kind of classification is important for industrial concerns, as it involves the purchase and installation of hardware devices. For example, if CMM is chosen to prevent the potential fire hazards, usually detectors for smoke and temperature should be purchased to be installed at proper places to monitor the environmental conditions.

1.2 Optimization modeling classification

In the quantitative and modeling researches on the area of maintenance, the papers aim to compare the system performances under different circumstances to determine the optimal policy and its decision parameters. Wang (2002) summarized four objectives which an ordinary maintenance optimization problem would consider: minimizing maintenance cost rate of the system; maximizing the system reliability measures; minimizing maintenance cost rate while keeping the system reliability above a certain level; maximizing the system reliability measure while the cost for the maintenance is within some constraints. Beside these four optimization criteria which are used to formulate the objective functions for optimization modeling, Wang (2002) raised other factors which may characterize an optimal maintenance objective or serve as "constraints" in the optimization: maintenance policies, system configurations, shut-off rules, maintenance degree, maintenance cost, modeling tools, planning horizon, dependence, and system information are all factors describing certain aspects of the system that is studied.

1.3 Maintenance policies classification

Many different maintenance policies have been developed for different circumstances or requirements of the system which is studied. Generally, a system can be either a single-unit system or a multi-unit system. The study of single-unit systems is the foundation of studying the multi-unit systems. Therefore, most of the effort has been put into the studies of single-unit systems, and the corresponding maintenance policies have been discussed. Wang (2002) summarized six major policies for the single-unit systems: Age-dependent PM (preventive maintenance) policy, Periodic PM policy, Failure limit policy, Sequential PM policy, Repair limit policy, and Repair number counting and reference time policy.

Among all these policies, the most popular and common one is the Age-dependent PM policy, under which usually a unit is preventively maintained when its age reaches a predetermined value or it is repaired when it fails. Various circumstances have been investigated under this policy: many researchers have developed the extensive policies, such as age replacement policy, repair replacement policy, mixed age PM policy, or random age-dependent maintenance policy, etc; others would like to focus on discussing different maintenance properties, e.g. different types of PM (minimal, imperfect, perfect) or different cost structures; besides, other researchers have introduced additional decision variables and auxiliary parameters, including reference time, repair counting number, and probabilities for different failure types.

In addition to the Age-dependent PM policy, many other policies have been introduced by researchers, too. In the Periodic PM policy, a unit takes on preventive maintenance at fixed time kT (k=1, 2, …) or is repaired at failures, regardless of the age of the unit. Block replacement policy and "Periodic replacement with minimal repair at failures" policy are two basic policies in the category of periodic PM policy. In Failure limit policy, a unit is preventively maintained when its failure rate reaches a predetermined value or the unit is repaired when it fails. Under the Sequential PM policy, PM is conducted at unequal time intervals, and after each PM the next PM interval is specified to minimize the expected costs during the residual life. Repair limit policy consists of Repair cost limit policy and Repair time limit policy: in the former policy, PM is performed if the estimated cost is less than a threshold, otherwise a replacement action will be taken; while in the latter policy, researchers introduced a threshold called "repair time limit", which is used to decide whether to perform a repair or a replacement for the unit studied. The principle for Repair number counting policy is that the unit is minimally repaired at failures but replaced every fixed number of failures (e.g. every k failures, where k is a constant). The

Reference time policy, instead of using the number of failures (k) as a criterion, uses time (T) as a reference: before time T, the unit is minimally repaired upon failure; after time T, it will be replaced once it fails.

1.4 Maintenance topics or focuses classification

Besides the classifications stated above, there should be other classification standards: since the maintenance area spans over a wide range and has plenty of contents, the research papers on maintenance cannot be always covered by those purely mathematical models or model based policies. For example, some papers have investigated the qualitative aspects of maintenance field, such as papers focusing on maintenance management; other papers are discussing case studies of maintenance, illustrating how the knowledge of maintenance interacts with the practical situations. For these reasons, it will be a good classification to group the papers according to their maintenance-related topics or focuses. One way to group these papers is to classify them into the following topics: Preventive Maintenance; Condition-based Maintenance; Imperfect Maintenance; Maintenance Planning and Production Joint Models; Maintenance Management; Maintenance Application and practical Examples; and Techniques associated to Maintenance.

Chapter 2

Review on Maintenance Topics or Focuses

In this chapter, we will group the papers we have reviewed into different categories according to their Maintenance Topics or Focuses. Though there are many topics for this classification, here we only present the topics which are related to the system we are going to study later.

2.1 Preventive maintenance

Papers categorized into this section deals with normal or fundamental models and strategies on preventive maintenance. However, papers with specific focuses (e.g. imperfect maintenance) are categorized into other topics, although those papers may also be concern with preventive maintenance. Due to its prevalence and fundamental position in the maintenance research area, this topic has the most prolific papers and it has been investigated extensively almost since the very early period, at which time maintenance started to become an academic issue. Most of the maintenance policies stated in the subsection 1.3 constitute the majority part of this topic, and optimization methods discussed in the subsection 1.2 are greatly involved in the models on this topic. Up to now it is still a hot topic, as papers on further advancements for this topic still take a large percentage of recent papers on maintenance.

Examples of recent papers on this topic cover various aspects. Pascual et al (2008) proposed a model for a production system which takes into account stock piles, line and equipment redundancy, and the use of other production methods. Lu and Jiang (2007) compared the performance of corrective maintenance, preventive maintenance, and predictive maintenance for standby *k*-out-of-*n* systems; and found out that the corrective maintenance is more preferable when the system deteriorates slowly and the preventive maintenance does best when the failure rate is high. Coolen-Schrijner and Coolen (2007) used costs per unit of time over a single cycle to study adaptive strategies for age-replacement policy, when the system sends out some kind of feedback about its process information. Wang and Zhang (2006) determined an optimal bivariate replacement policy for the system, in which the successive operating times form a stochastically decreasing geometric process and the consecutive preventive repair times form a stochastically increasing geometric process. Chen (2008) minimized the make-span for a single-unit system which receives periodic maintenance, and he discussed the situation where a maintenance job cannot be completed within the given time for maintenance.

2.2 Imperfect maintenance

Imperfect Maintenance, which cannot bring the system to "as good as new" state, is in contrast with the simple perfect maintenance. It is necessary to clarify some terms which are frequently used in imperfect maintenance area: according to the literature review of Pham and Wang (1996), maintenance can be classified, based on the degree to which the operating condition of an item would be restored through maintenance actions, in the following way:

a. Perfect repair or perfect maintenance: a maintenance action which restores the system operating condition to be "as good as new". That is, upon perfect maintenance, a system has the same lifetime distribution and failure rate function as a brand new one.

b. Minimal repair or minimal maintenance: a maintenance action which restores the system to the failure rate it had when it failed. Minimal repair is first studied by Barlow and Proschan (1965). After the minimal repair, the system operating state is often called "as bad as old".

c. Imperfect repair or imperfect maintenance: a maintenance action does not make a system be like as good as new, but younger. Usually, it is assumed that imperfect maintenance restores the system operating state to somewhere between as good as new and as bad as old. Thus, imperfect maintenance (repair) is a general maintenance (repair) which can include two extreme cases: minimal maintenance (repair) and perfect maintenance (repair).

d. Worse repair or maintenance: a maintenance action which makes the system failure rate or actual age increases but the system does not break down. Thus, upon worse repair, the system's operating condition becomes worse than that just prior to its maintenance.

e. Worst repair or maintenance: a maintenance action which does not deliberately make the system failed or broken down.

We synthesize possible causes and circumstances, which Brown and Proschan (1983), Nakagawa and Yasui (1987) provided, for imperfect, worse or worst maintenance to happen:

a. Repair the wrong part;
b. Only partially repair the faulty part;
c. Repair (partially or completely) the faulty part but damage adjacent parts;
d. Incorrectly assess the condition of the unit inspected;
e. Perform the maintenance action not when called for but at his convenience (the timing for maintenance is off the schedule);
f. Hidden faults and failures which are not detected during maintenance;
g. Human errors such as wrong adjustments and further damage done during maintenance;
h. Replacement with faulty parts.

Imperfect maintenance has been studied ever since the early stage that the area of maintenance arose as an academic field, so the large number of accumulated papers on this topic could justify it to be an almost independent topic from the normal maintenance in subsection 2.1. Aven and Castro (2008) studied a system with two types of failures: the system is minimally maintained for type 1 failure; while for type 2 failure, the system is minimally maintained with probability p and perfectly maintained with probability 1-p. El-Ferik (2008), Sheu et al. (2004a), Ben-Daya

(2002), and Sheu et al. (2004b) dealt with "lot-sizing problem" with imperfect maintenance and production. Yun et al. (2004) tried to deal with parameter estimation by the method of maximum likelihood under the "proportional age reduction" models. Pascual and Ortega (2006) proposed a novel model to determine optimal life-cycle duration and intervals between overhauls by minimizing global maintenance costs, and also discussed the impact of a better warranty contract by offering an improved preventive maintenance program for the equipment.

2.3 Maintenance planning and production

The overall objective of maintenance planning is to study the interactions between normal maintenance actions and production/logistic processes, as well as make working schedules for the whole system so that various objectives could be satisfied. The driving force of this topic is that production/logistic processes scheduling and preventive maintenance planning decisions are interdependent in real-world situations, e.g. maintenance actions can affect available production time and conversely the elapsed production time affects the probability of system failure. However, this interdependency had been overlooked in early literature. Until recently some researchers just started to consider this interdependency in their works.

Diallo et al (2008) studied a system in which both preventive maintenance and spare parts inventory control policies are considered, and spare parts inventory control policy is a (s, Q) control policy. Cassady and Kutanoglu (2003) proposed an integrated model that simultaneously determines production scheduling and preventive maintenance planning decisions, so that the total weighted tardiness of jobs is minimized, which is something of filling the gap of research and worthy to be investigated further. Chelbi and Rezg (2006) considered a production and inventory joint model, in which there is a buffer stock "*h*" to make sure the continuous supply when the production system undergoes maintenance.

2.4 Maintenance for multi-unit systems

According to the complexity of the system that we study, we can classify a system into one of the two categories: a single-unit system or a multi-unit system. In the subsection "1.3 Maintenance Policies Classification", we have summarized the maintenance policies for single-unit systems. A multi-unit system, of course, can be seen as the combination of several single-unit systems.

Previous researchers have done literature reviews specifically on multi-unit systems: Cho and Parlar (1991) did a literature review specifically on the papers, which are related to optimal maintenance and replacement models for multi-unit systems, between the year 1976 and 1991. In this review, they classified the models in the surveyed articles into five categories: machine interference/repair models, group/block/cannibalization/opportunistic models, inventory/maintenance models, other maintenance/replacement models and inspection/maintenance models. When they introduced and discussed each category, they put much emphasis on the inventory/maintenance models, in which there are inventory spare stocks for repairable production units in the systems. Dekker et al (1997) did a literature review following Cho and Parlar (1991), which covers the articles between 1991 and 1997 on the same subject. This review distinguishes between stationary models, where a long-term stable situation is assumed, and dynamic models, which take into account the information that becomes available only on the short term. The stationary models are discussed in details according to the different categories: grouping corrective maintenance, grouping preventive maintenance, and opportunistic maintenance.

In the recent papers on multi-unit systems, Wang and Pham (2006) studied availability, maintenance cost, and optimal maintenance policies of the series system. The system has n constituting components and each component is assumed to be subject to correlated failure and repair, imperfect repair, shut-off rule, and arbitrary distributions of times to failure and repair. They modeled the system with quasi renewal processes, using system maintenance cost rates and system availability as the criteria. De Smidt-Destombes et al (2006) considered a k-out-of-N system with identical, repairable components. They studied relationship between the system availability and its controlling variables: maintenance policy, the spare part inventory level, the repair capacity, and repair job priority setting. Vaughan (2005) studied the inventory policy of spare parts for a system, which contains n identical components. He developed a stochastic dynamic programming model to solve the problem, and obtained the optimal policy $(s(k),S(k))$, in which k is the number of periods until the next scheduled preventive maintenance.

2.5 Maintenance on the Supply–Buffer-Demand system

For the research done on multi-unit systems maintenance, there has been an increasing interest on the joint production systems, and many papers have been published on this subject. Usually, a production system with consideration of maintenance has single or multiple production units which need to be maintained. A "joint" production system, however, not only consists of one production unit which needs to be maintained sometimes due to failure, but also it has one inventory buffer. In this way, the demand for products could be satisfied from the stocks in the inventory buffer when the production unit undergoes preventive maintenance or corrective maintenance. This joint production system combines the maintenance and the inventory problem into one system, so the maintenance optimization policies for such systems would turn into a joint consideration of both maintenance and inventory influences.

This joint production system can be roughly depicted as the following graph:

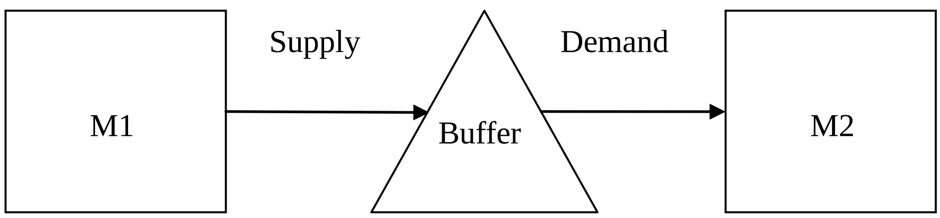

Figure 2.1 A two-machine serial production system with a buffer

This Supply-Buffer-Demand production system consists of two production units M1 and M2, and M1 supplies raw materials to M2 so that the need of M2 is satisfied.

M1 is unreliable and so it encounters random failures. Therefore, a buffer B between M1 and M2 is needed, to supply the need of M2 when M1 breaks down or undergoes maintenance.

As stated in the papers done by Meller and Kim (1996) and Kyriakidis and Dimitrakos (2006), a typical application area of such a system lies in automobile general assembly where M2 represents the assembly line and M1 represents one of the many parallel operations that directly supply the line. Another application is in seat assembly, where M2 represents the seat assembly line and M1 is the machine that produces seat covers and sends them to a large buffer that feeds the seat assembly line. Besides, an example of this production system could be an offshore oil exploration platform, which provides the crude oil to onshore refineries. The crude oil is transported by pipelines from the platform to storage tanks, from which it is further transported to the refinery. In this case the crude oil, the exploration platform, the refineries, and the storage tanks are the raw materials, M1, M2 and the buffer, respectively.

In recent years, many researchers have studied this Supply-Buffer-Demand production system and obtained their results for maintenance policies (and buffer size).

Van der Duyn Schouten and Vanneste (1995) studied such a system with capacitated buffer size, and they derived a preventive maintenance policy (n, N, k), which used both the age of the unit M1 and the content of the buffer as parameters of their policy. In their research, preventive maintenance is assumed to be perfect. In addition, both the lifetime distribution of M1 and buffer content distribution are assumed to be discrete, so that they could use embedding technique from Markov decision theory.

Meller and Kim (1996) derived a cost model for this system which includes the costs for preventive maintenance, unscheduled repairs, starving the second production unit M2, and the inventory. In this paper, the frequency for carrying out preventive maintenance is determined by the buffer content level, i.e. when the buffer level reaches the optimal buffer level b^* the unit M1 would be preventively maintained. They assumed the buffer level states as discrete states, and they used embedded stochastic process for Markov chain to compute this optimal buffer level b^*.

Cheung and Hausman (1997) based their work on three assumptions for the system: the constant time requirement for a preventive maintenance operation is short when compared with the mean time between failures (MTBF); sufficient capacity is present to allow rapid accumulation of safety stocks in the beginning of each machine life cycle; the time to accomplish buildup and depletion of safety stocks is small relative to the MTBF. Under these assumptions, they used perfect periodic preventive maintenance to formulate an analytical model for the cost rate of the system, then they minimized this cost rate to find out the optimal preventive maintenance scheduling and safety stocks level simultaneously.

Ben-Daya (2002) studied a preventive maintenance policy on this system: after preventive maintenance, the age of the system is reduced proportional to the preventive maintenance level; the system would return to be "as good as new" after replacement or *m* preventive maintenance actions, whichever occurs first. He also assumed that there are no shortages in this model, and the time for preventive maintenance and inspection is negligible. He finally derived the model for the expected total cost per unit of time, involving the setup cost, inspection cost, inventory holding cost, quality related costs, and preventive maintenance cost.

Sheu and Chen (2004) further developed the model by Ben-Daya (2002). They just extended the original model to classify the out-of-control state of the system into two categories: type I and type II. Minimal repair would be undertaken if it is in the type I state; otherwise the production is stopped and the system is restored for the type II state.

Kenne et al (2007) developed the analytical model for the total costs for maintenance, inventory and lost sale of the system. They used the age-dependent policy for preventive maintenance, and used a new inventory policy in consideration of reducing the holding cost, which is called "multiple threshold levels hedging point policy". Both preventive maintenance and corrective maintenance are assumed to be perfect in their model, and also the preventive maintenance duration is assumed to be shorter than MTTF.

Chapter 3

Problem Definition

In this chapter, we propose our study on the Supply-Buffer-Demand system. First, a previous model on the Supply-Buffer-Demand system by Chelbi and Rezg (2006) is introduced and its extensions are discussed. In the next, a more general maintenance model (compared to that of Chelbi and Rezg (2006)) on the Supply-Buffer-Demand system is raised, which involves "preventive maintenance", "imperfect maintenance", "maintenance planning and production", and "maintenance on multi-unit systems".

3.1 An existing model on the Supply–Buffer-Demand system and its extension

Chelbi and Rezg (2006) developed a model for the Supply-Buffer-Demand system based on the age-dependent maintenance policy on M1. They derived the expected total costs per unit of time for the system, which include maintenance cost, holding cost and shortage cost. In order to obtain the optimal maintenance time T and the buffer level h simultaneously in their model, the total cost rate is minimized while a minimum required stationary availability is satisfied as a constraint. Therefore, the optimal policy not only considers the cost rate but also takes into account the availability of the system. Maintenance is assumed to be perfect in their model, and the failures are assumed to be excluded during the buildup of the stocks in the buffer.

The model we develop is an extension for the model presented by Chelbi and Rezg (2006). In our model, the age-dependent preventive maintenance is not as perfect as assumed by Chelbi and Rezg (2006) or in other papers on Supply-Buffer-Demand systems. Instead, preventive maintenance is assumed to be imperfect which follows the (p, q) rule in our model, i.e. each preventive maintenance action is perfect with probability p and is imperfect with probability q. Therefore, in our model, an "enhanced" preventive maintenance is carried out every N maintenance actions (preventive or corrective), so that the state of the system can be totally restored to the perfect state ("as good as new"). Such an enhanced preventive maintenance action is assumed to be perfect, but it would cost more money than a normal preventive maintenance. The expected total cost rate would be formulated as the objective function, and the minimum required stationary availability should be satisfied as a constraint. In addition, as normal preventive maintenance is imperfect, the minimum reliability requirement should also be considered to be a constraint, to prevent the system from falling into a terribly unreliable state. Finally, we also consider the circumstances where the average quantity of shortage of the buffer should not exceed a certain limit. Beyond that limit some customers may be lost forever, as it may be impossible for M2 to backlog from external resources any more (instead from the buffer, which is the internal resource). The aim of the model is to determine the optimal decision variables T (age-dependent preventive maintenance increment age), *N* (enhanced preventive maintenance decision variable), and *h* (buffer capacity) simultaneously, since they interactively decide the cost rate and other constraints of the system. Our model, compared to the model given by Chelbi and Rezg (2006), considers more conditions for the system, and so it is a more general model to describe the Supply-Buffer-Demand system.

3.2 A general model for the Supply-Buffer-Demand system

The manufacturing system that we consider, as depicted in Figure 2.1, consists of a production unit M1 which produces raw materials and supplies them to the subsequent production unit M2. The system has the following characteristics:

1. M1 is an unreliable unit and it is subject to random failures. Maintenance actions are taken on M1 as soon as its age increases by *T* or at failure, whichever occurs first. Corrective maintenance is perfectly performed at M1's failure and restores M1's virtual age to zero. Preventive maintenance is imperfectly performed: the virtual age of M1 may return to zero after preventive maintenance with certain probability *p*, or the age does not change with probability 1-*p*. There is an "enhanced" preventive maintenance action for every *N* (*N* is a fixed number) maintenance actions (either corrective or preventive), which could restore M1 to be as good as new.
2. M2 is a reliable unit with no random failures, and its demand for raw materials is fixed to a constant rate.
3. A buffer stock, which is between the sequential units M1 and M2, is built up to

supply M2 with raw materials when M1 undergoes corrective maintenance or planned preventive maintenance. The buffer has a finite capacity *h*. As long as the buffer capacity is not reached, M1 operates at its maximum production rate *U*max. *U*max is bigger than the demand rate of M2, so the excess output is stored in the buffer. When the buffer is full, the production rate of M1 is lowered down to the demand rate of M2.

4. A period is defined as the time interval, which starts right after the completion of a maintenance action (or time zero) and ends until the completion of the next maintenance action. From this definition, we know that right after the end of each period, the production unit M1 may return to the state "as good as new" and its virtual age returns to zero (if it undergoes corrective maintenance or undergoes perfect preventive maintenance with probability *p*); otherwise it remains "as bad as old" and its virtual age does not change (if it undergoes minimal preventive maintenance with probability *q*). In a word, M1's virtual age increases by *T* or returns to zero for each period.
5. A cycle consists of *N* periods, which is defined as the time interval starting right after the completion of an "enhanced" preventive maintenance action (or time zero) and ending just until the completion of the next enhanced preventive maintenance action. According to this definition, the unit M1's virtual age returns to zero right after the end of each cycle. Therefore, there is a renewal process associated with the cycles of the system.
6. An enhanced preventive maintenance is a normal preventive maintenance getting

enhanced: either the cost of a preventive maintenance action, or the duration of an action, or both the cost and the duration are increased (such as costing more money to assign more personal to the maintenance action or taking more time to examine and maintain), in order to make sure that the enhanced preventive maintenance would become a perfect action. This contrasts with the normal preventive maintenance for the system, which is an imperfect action.

We formulate our mathematical model on this system. In our model, a minimum stationary availability level for M1 is required. A minimum reliability requirement for M1 should also be satisfied. For certain circumstances, the expected quantity for the average shortage of raw materials supplied to M2 is considered (i.e. neither M1 nor the buffer could supply M2), which should not exceed a maximum level. Our objective is to determine the age increment T, the size of the buffer capacity h, and the number of periods in a cycle N, so that the total cost per unit of time is minimized while requirements are simultaneously met.

The following assumptions are considered:

1. Lifetime probability distribution of M1 is known.
2. Maintenance duration is known and constant.
3. All costs, which are related to maintenance and inventory, are assumed to be known and constant.
4. Failures are detected instantaneously.
5. All the resources needed to perform the maintenance actions are available at the

right time.

6. Corrective maintenance and enhanced preventive maintenance actions are perfectly performed. Each action restores the supplying production unit M1 to be as good as new.
7. Preventive maintenance action is imperfect following the (p, q) rule: each action may be a perfect action with probability p, which restores the system to be as good as new; or it may be a minimal maintenance action with probability q=1-p, which does not change the age of the system, so that the system remains "as bad as old" state.
8. An "enhanced" preventive maintenance action only costs more money than a normal preventive maintenance action, while the time for its maintenance action is the same as a normal preventive maintenance action.
9. A corrective maintenance action costs more time and money than a preventive maintenance action.
10. The stocks in the buffer are imperishable with time.
11. The failure rate of M1 is an increasing failure rate.
12. The system initial state is time zero.

Relationship between "period" and "cycle" can be depicted in a figure:

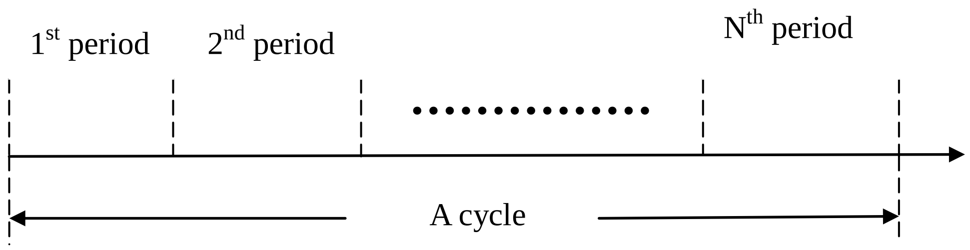

Figure 3.1 Relationship between "period" and "cycle"

According to the definitions of "period" and "cycle", we get to know that each period incurs the maintenance cost and causes the corresponding inventory costs due to the maintenance, so it is obvious that the total costs within a "cycle" would be computed on a period to period basis. As for each cycle, the system's state would return to the initial state (time zero) at the end of the cycle after the enhanced preventive maintenance, so the system is actually renewed after each cycle. The renewal theory could be used here to compute the total average cost per unit of time.

Similar to what has been raised in the works by Wang and Pham (1999) and by Chelbi and Rezg (2006), from the classical renewal reward theory we have the following conclusion: the total average cost per unit of time on an infinite horizon $S(N, T, h)$ is equivalent to the expected total average cost per unit of time within a renewal cycle. In this thesis, the times between consecutive enhanced preventive maintenance actions constitute renewal cycles. Therefore, in order to formulate the objective function $S(N, T, h)$, we only need to obtain both the expected total costs within a cycle and the expected total time within a cycle, and derive the quotient.

Chapter 4

Analysis and Theoretical Development

In this chapter, we analyze our general system and formulate the model. We derive the analytical results for the objective function and constraint functions for our model.

In Section 4.1 we develop the objective function for our model. First we analyze the cost and time for the maintenance policy and derive the corresponding analytical results in Section 4.1.1. Then we study and derive the analytical form of the cost for the inventory policy in Section 4.1.2. Finally based on the results in Sections 4.1.1 and 4.1.2, we obtain the total cost rate (including both maintenance cost and inventory cost) of the system in Section 4.1.3, and this cost rate is what we are going to minimize.

In Section 4.2 we develop the constraint functions for our model of the system. We analyze the requirements of our system, so that we derive the corresponding analytical forms of constraints for our model: availability constraint, reliability constraint, and shortage rate constraint. We first develop the stationary availability of the system in Section 4.2.1, and the constraint function for satisfying minimum availability is derived. We then develop the constraint function for satisfying minimum reliability requirement of the system in Section 4.2.2. Finally we study and develop the shortage rate of the system in Section 4.2.3, and the constraint function for satisfying maximum shortage rate is also derived.

4.1 Derivation of the total cost rate of the system

4.1.1 Derivation of cost and time for the age dependent maintenance policy

First we consider the age dependent preventive maintenance policy, as well as its related costs and time within a cycle. We define two symbols related to periods:

Definition 4.1

$A_n(T)$: the expected maintenance costs (including the preventive and corrective maintenance) for the n^{th} period since the last perfect maintenance action (either corrective maintenance, enhanced preventive maintenance, or preventive maintenance which is perfectly performed with probability p);

$B_n(T)$: the expected time duration (including the operating time and maintenance time) for the n^{th} period since the last perfect maintenance action.

Proposition 4.1 The expected maintenance costs and the expected time for the n^{th} period since the last perfect maintenance action are

$$A_n(T) = M_p \frac{R(nT)}{R[(n-1)T]} + M_c \frac{F(nT) - F[(n-1)T]}{R[(n-1)T]}; \tag{4.1}$$

$$B_n(T) = \frac{\int_{(n-1)T}^{nT} R(u)du}{R[(n-1)T]} + \mu_p \frac{R(nT)}{R[(n-1)T]} + \mu_c \frac{F(nT) - F[(n-1)T]}{R[(n-1)T]}. \tag{4.2}$$

Proof. For the n^{th} period since the last perfect maintenance action, the production unit M1 would have undergone (n-1) minimal preventive maintenance actions during the last (n-1) periods (as there has been no failure or perfect preventive maintenance on M1), so the virtual age of M1 is (n-1)T at the beginning of the n^{th} period. Therefore, probability distribution function for M1's lifetime X in the n^{th} period is the conditional probability given that M1 has survived for time (n-1)T, i.e. the conditional probability distribution function is

$$P[X-(n-1)T \le t \mid X > (n-1)T] = \{F[(n-1)T+t]-F[(n-1)T]\}/R[(n-1)T]. \qquad (4.3)$$

Therefore the conditional reliability function for M1 in the n^{th} period is

$$1-\{F[(n-1)T+t]-F[(n-1)T]\}/R[(n-1)T] = R[(n-1)T+t]/R[(n-1)T], \qquad (4.4)$$

and the expected lifetime in the n^{th} period is

$$\int_0^T R[(n-1)T+x]/R[(n-1)T]dx = [\int_{(n-1)T}^{nT} R(x)dx]/R[(n-1)T]. \qquad (4.5)$$

Thus, it is obvious that formulas (4.1) and (4.2) are correct from the results of formulas (4.3), (4.4), and (4.5). □

Continuing to define the other symbols:

Definition 4.2

EC_n (T): the expected total maintenance costs (corrective and preventive maintenance) for the first n periods within a cycle;

ET_n (T): the expected total time duration (operating time and maintenance time) for the first n periods within a cycle;

ΔEC_n (T): the expected maintenance costs for the n^{th} period within a cycle;

ΔET_n (T): the expected time duration for the n^{th} period within a cycle.

According to the definition, we have

$$\Delta EC_n(T) = EC_n(T) - EC_{n-1}(T)\,;\quad \Delta ET_n(T) = ET_n(T) - ET_{n-1}(T)\,; \tag{4.6}$$

$$EC_0(T) = 0,\ EC_n(T) = \sum_{i=1}^{n}[EC_i(T) - EC_{i-1}(T)] = \sum_{i=1}^{n}\Delta EC_i(T)\text{, n}\geq 1; \tag{4.7}$$

$$ET_0(T) = 0,\ ET_n(T) = \sum_{i=1}^{n}[ET_i(T) - ET_{i-1}(T)] = \sum_{i=1}^{n}\Delta ET_i(T)\text{, n}\geq 1; \tag{4.8}$$

Therefore, to obtain the formulas EC_n (T) and ET_n (T), we only need to get the expressions for ΔEC_n (T) and ΔET_n (T).

Proposition 4.2 ΔEC_n (T) and ΔET_n (T) can be obtained through

$$\Delta EC_i(T) = q^{i-1}R[(i-1)T]A_i(T) + \sum_{j=1}^{i-1}q^{j-1}R[(j-1)T]\{1 - q\frac{R(jT)}{R[(j-1)T]}\}\Delta EC_{i-j}(T) \tag{4.9}$$

$$\Delta ET_i(T) = q^{i-1}R[(i-1)T]B_i(T) + \sum_{j=1}^{i-1}q^{j-1}R[(j-1)T]\{1 - q\frac{R(jT)}{R[(j-1)T]}\}\Delta ET_{i-j}(T) \tag{4.10}$$

Proof. If there is no perfect maintenance action (either corrective or preventive) in the first (i-1) periods within a cycle, the production unit M1 would have undergone

(i-1) minimal preventive maintenance actions since the beginning of a cycle. Thus, the probability for such circumstance is $q^{i\text{-}1}R[(i\text{-}1)T]$ and the maintenance costs for the i^{th} period within a cycle is $A_i\,(T)$.

Otherwise, there is at least one perfect maintenance action in the first (i-1) periods within a cycle. If the first of these perfect maintenance actions happens at the j^{th} period, M1 would have undergone (j-1) minimal preventive maintenance actions in the first (j-1) periods and a perfect maintenance action (corrective or preventive) in the j^{th} period, so the probability for this circumstance is $q^{j-1}R[(j-1)T]\{1-q\dfrac{R(jT)}{R[(j-1)T]}\}$. On the other hand, since the prefect maintenance action reduces the virtual age of M1 to zero at the j^{th} period, the state of the system at the beginning of $(j+1)^{\text{th}}$ period of a cycle would be as if what it were at the beginning of a cycle. Thus, the state of the system at the beginning of i^{th} period of a cycle would be as if what it were at the beginning of the $(i\text{-}j)^{\text{th}}$ period of a cycle. Therefore, the expected maintenance costs of the i^{th} period within a cycle is mathematically equivalent to the expected maintenance costs for the $(i\text{-}j)^{\text{th}}$ period within a cycle, which is $\Delta EC_{i\text{-}j}\,(T)$ according to the definition.

Thus, the formula (4.9) has been proved. Similarly we can derive the formula (4.10). □

Since we have **proposition 4.2** and we know that $\Delta EC_1(T) = EC_1(T) = A_1(T)$ and $\Delta ET_1(T) = ET_1(T) = B_1(T)$, we could compute $\Delta EC_n\,(T)$ and $\Delta ET_n\,(T)$ with recursion method. Thus, we can obtain $EC_n\,(T)$ and $ET_n\,(T)$ through formulas (4.7) and (4.8).

4.1.2 Derivation of cost for the inventory control policy

Inventory control policy that we consider helps to smooth the supply to the unit M2. When the production unit M1 is undergoing maintenance, the supply from M1 will be ceased. To ensure that the demand of M2 is satisfied, the stocks from the buffer would be used as a temporary supply. However, depending upon the content in the buffer, the demand of M2 during M1's maintenance may or may not be fully covered by the stocks in the buffer. In some cases, the content in the buffer is rich enough and it can supply M2 during the whole process of maintenance; in other cases, however, the content in the buffer can only supply M2 for part of the process of maintenance, so in the remaining process of maintenance there will be a shortage cost C_s for each product which the buffer is unable to supply to M2. This represents the cost for additional efforts to supply M2 from external resources in the short run.

The evolution of buffer stock level in a period consists of three phases: in Phase I, the production unit M1 produces at its maximum production rate Umax, in order to build up the buffer stock h; and then in Phase II, M1 produces at the production rate of d, which is the demand rate of M2;[1] in Phase III, due to M1's failure or its virtual age increasing by T, the production of M1 stops for maintenance and the buffer stock supplies the demand of M2 instead. In Phase III, the maximum depletion time of the

[1] For example, in a production process including stamping and press punching in the automotive industry and die casting, when all three shifts of operation are utilized, the production unit can run at its maximum output rate Umax; when only one shift is utilized, it runs at a normal rate d.

buffer stock is h/d. This evolution process can be depicted as in Figure 4.1 and 4.2. Figure 4.1 characterizes the case where maintenance time exceeds h/d, so that the shortage and its related costs are incurred; while in Figure 4.2 a period without shortage is depicted.

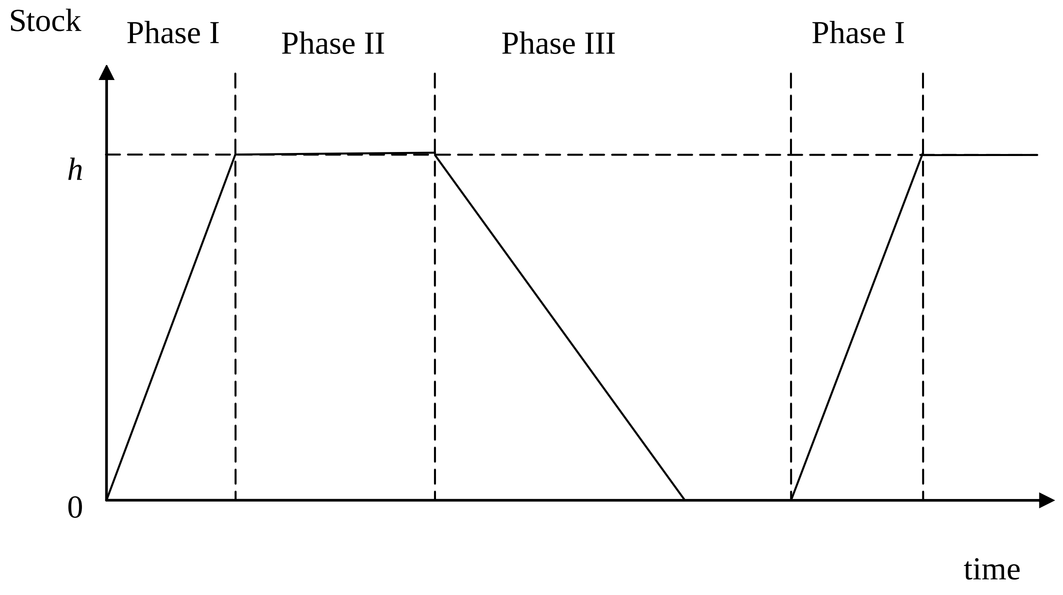

Figure 4.1 The buffer stock level in a period with shortage

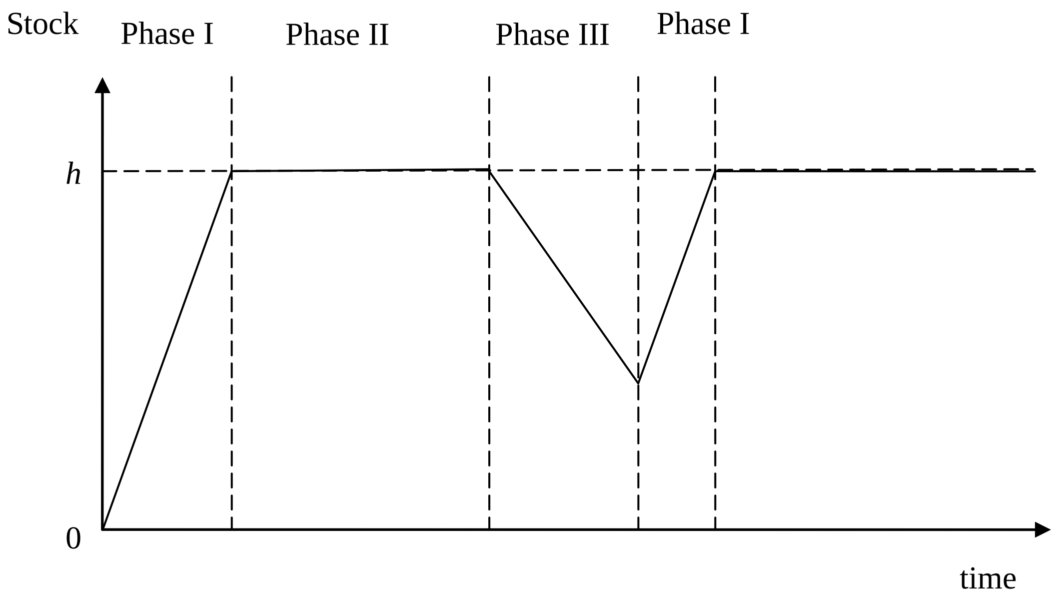

Figure 4.2 The buffer stock level in a period without shortage

The mathematical model for this inventory control policy is based on two assumptions:

1. The shortage cost rate C_s is much bigger than the holding cost rate C_h ($C_s >> C_h$).
2. Failures are excluded in Phase I of a period, i.e. there is no failure in the buffer stock buildup stage.

Previous papers have introduced the second assumption for modeling the inventory buffer of the system: Cheung and Hausman (1997) assumed that the time to accomplish the buildup of a safety stock is small, compared to the mean time to failures; Chelbi and Alt-Kadi (2004) made the same assumption as Cheung and Hausman (1997); Chelbi and Rezg (2006) assumed that failures are excluded in the phase of reconstitution of the buffer stock.

Obviously, to make the idealistic assumption "failures are excluded in Phase I" (the second assumption) be valid in some extent, it needs certain minimum requirements for the reliability of the unit M1, so that the risk of failure in Phase I would be relatively small enough. We will discuss the reliability requirements later in the subsection "4.2.2 Derivation of reliability and its minimum requirement".

Next we will formulate the expression for the expected total holding and shortage costs within a period. According to different range for h/d, which is the depletion time of full buffer stock, two forms of expressions under two different conditions are formulated respectively. It is noted that h/d should not be bigger than μ_c, i.e. there should be $h \le \mu_c d$. This is because in our model the longest time that the unit M1 ceases to work (which is also the longest time that the buffer works continuously) is μ_c (under corrective maintenance due to failure), after that M1 resumes working at least until the buffer stock is restored to h (according to the second assumption). Therefore, any product stored beyond $\mu_c d$ would just incur additional holding cost and it is not beneficial to the system at all.

We will then split the possible area $h/d \le \mu_c$ into two ranges, and each range corresponds to a condition for the inventory control policy:

Condition 1: $\mu_p \le \frac{h}{d} \le \mu_c$; i.e. full stocks in the buffer will be depleted during corrective maintenance, but will not be depleted after preventive maintenance;

Condition 2: $\frac{h}{d} < \mu_p$; i.e. full stocks in the buffer will be depleted during preventive maintenance.

Definition 4.3

$G1_n\ (T, h)$: the expected total inventory costs (holding cost and shortage cost) for the

n^{th} period since the last perfect maintenance action, under **Condition 1** of the inventory control policy;

$G2_n\ (T, h)$: the expected total inventory costs (holding cost and shortage cost) for the n^{th} period since the last perfect maintenance action, under **Condition 2** of the inventory control policy.

From the proof of **proposition 4.1**, we know that: for the n^{th} period since the last perfect maintenance action, the probability that unit M1 undergoes corrective maintenance is $\frac{F(nT)-F[(n-1)T]}{R[(n-1)T]}$, and the probability that M1 undergoes preventive maintenance is $\frac{R(nT)}{R[(n-1)T]}$.

For **Condition 1** $\mu_p \le \frac{h}{d} \le \mu_c$:

a. If the unit M1 undergoes corrective maintenance, there will be a shortage incurred, so we should compute the inventory costs according to Figure 4.3.

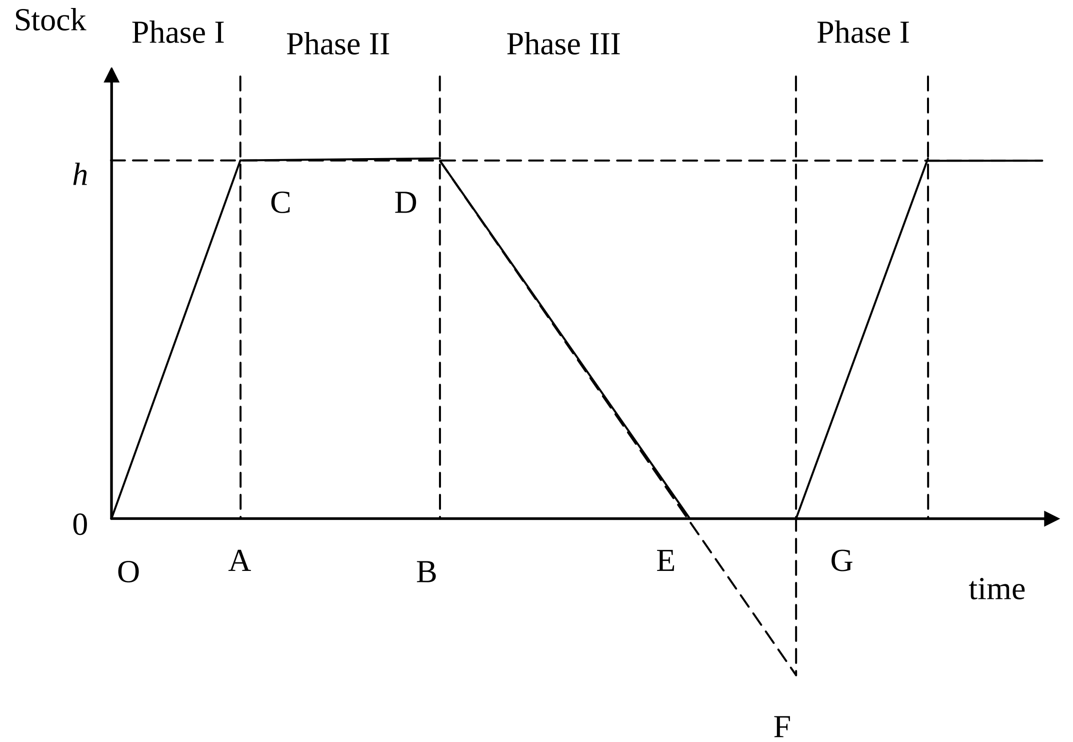

Figure 4.3 The buffer stock level in a period with shortage

The total holding cost for the n^{th} period since the last perfect maintenance action is the product of holding cost rate C_h multiplying the surface delimited by OCDEBA in Figure 4.3, which is

$$C_h[\frac{1}{2}\frac{h^2}{U_{\max}-d}+h(\frac{1}{R[(n-1)T]}\int_{(n-1)T}^{nT}R(t)dt-\frac{h}{U_{\max}-d})+\frac{1}{2}\frac{h^2}{d}]; \qquad (4.11)$$

while the total shortage cost for the n^{th} period since the last perfect maintenance action is the product of shortage cost rate C_s multiplying the surface delimited by EFG in Figure 4.3, which is

$$C_s[(\mu_c-\frac{h}{d})^2\frac{1}{2}d]. \qquad (4.12)$$

Therefore, the total inventory costs for the n^{th} period since the last perfect maintenance action is (when M1 undergoes corrective maintenance)

$$C_h[\frac{1}{2}\frac{h^2}{U_{\max}-d}+h(\frac{1}{R[(n-1)T]}\int_{(n-1)T}^{nT}R(t)dt-\frac{h}{U_{\max}-d})+\frac{1}{2}\frac{h^2}{d}]+C_s(\mu_c-\frac{h}{d})^2\frac{1}{2}d\text{ ; (4.13)}$$

b. If the unit M1 undergoes preventive maintenance, there will be no shortage, so we should compute the inventory costs according to Figure 4.4.

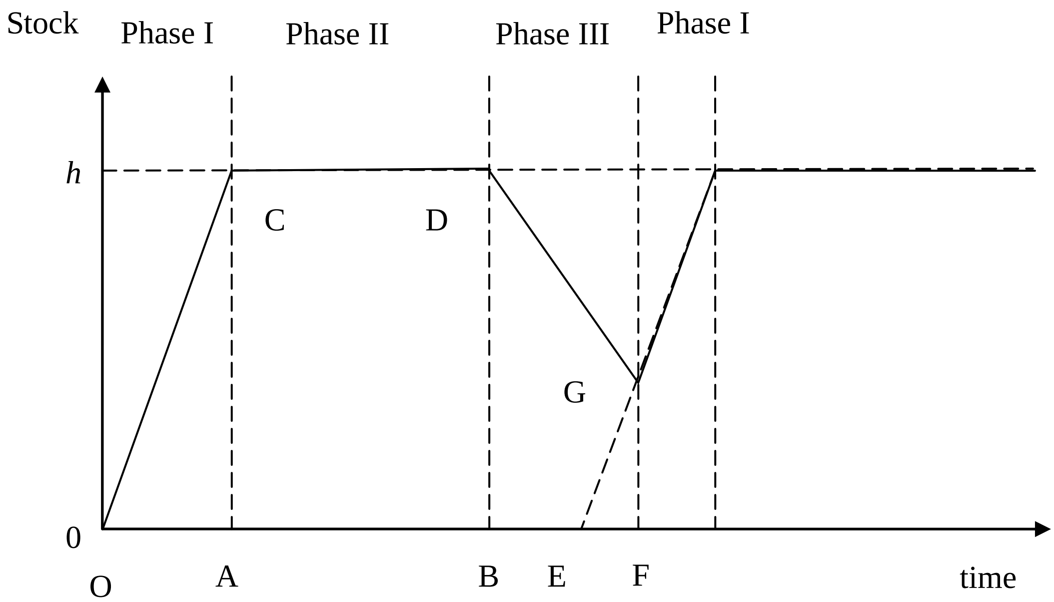

Figure 4.4 The buffer stock level in a period without shortage

The total holding cost for the n^{th} period since the last perfect maintenance action is the product of holding cost rate C_h multiplying the surface delimited by OCDGEBA in Figure 4.4,[2] which is

$$C_h[\frac{1}{2}\frac{h^2}{U_{\max}-d}+h(T-\frac{h}{U_{\max}-d})+h\mu_p-\frac{1}{2}\mu_p^2d-\frac{1}{2}\frac{(h-\mu_pd)^2}{d}]; \qquad (4.14)$$

[2] The surface delimited by EFG is reserved to be calculated in the next period, i.e. it is calculated in the $(n+1)^{th}$ period since the last perfect maintenance action.

Therefore, the total inventory costs for the n^{th} period since the last perfect maintenance is (when M1 undergoes preventive maintenance)

$$C_h[\frac{1}{2}\frac{h^2}{U_{max}-d}+h(T-\frac{h}{U_{max}-d})+h\mu_p-\frac{1}{2}\mu_p^2 d-\frac{1}{2}\frac{(h-\mu_p d)^2}{d}]. \tag{4.15}$$

Hence, for **Condition 1**, formulas (4.13) and (4.15) imply that: for the n^{th} period since the last perfect maintenance action, the expected total inventory costs are

$$G1_n(T,h)=\frac{R(nT)}{R[(n-1)T]}\{C_h[\frac{1}{2}\frac{h^2}{U_{max}-d}+h(T-\frac{h}{U_{max}-d})+h\mu_p-\frac{1}{2}\mu_p^2 d-\frac{1}{2}\frac{(h-\mu_p d)^2}{d}]\}$$
$$+\frac{F(nT)-F[(n-1)T]}{R[(n-1)T]}\{C_h[\frac{1}{2}\frac{h^2}{U_{max}-d}+h(\frac{1}{R[(n-1)T]}\int_{(n-1)T}^{nT}R(t)dt-\frac{h}{U_{max}-d})+\frac{1}{2}\frac{h^2}{d}]+C_s(\mu_c-\frac{h}{d})^2\frac{1}{2}d\} \tag{4.16}$$

For **Condition 2** $\frac{h}{d}<\mu_p$:

a. If the unit M1 undergoes corrective maintenance, there will be a shortage incurred. Similar to the derivation for **Condition 1**, the total holding cost for the n^{th} period since the last perfect maintenance action is the product of holding cost rate C_h multiplying the surface delimited by OCDEBA in Figure 4.3, which is

$$C_h[\frac{1}{2}\frac{h^2}{U_{max}-d}+h(\frac{1}{R[(n-1)T]}\int_{(n-1)T}^{nT}R(t)dt-\frac{h}{U_{max}-d})+\frac{1}{2}\frac{h^2}{d}]; \tag{4.17}$$

while the total shortage cost for the n^{th} period since the last perfect maintenance action is the product of shortage cost rate C_s multiplying the surface delimited by EFG in Figure 4.3, which is

$$C_s[(\mu_c - \frac{h}{d})^2 \frac{1}{2} d]. \quad (4.18)$$

Therefore, the total inventory costs for the n^{th} period since the last perfect maintenance action is (when M1 undergoes corrective maintenance)

$$C_h[\frac{1}{2}\frac{h^2}{U_{max}-d} + h(\frac{1}{R[(n-1)T]}\int_{(n-1)T}^{nT} R(t)dt - \frac{h}{U_{max}-d}) + \frac{1}{2}\frac{h^2}{d}] + C_s(\mu_c - \frac{h}{d})^2 \frac{1}{2} d\,; \quad (4.19)$$

b. If the unit M1 undergoes preventive maintenance, there will be a shortage incurred. Similar to the derivation for **Condition 1**, the total holding cost for the n^{th} period since the last perfect maintenance action is the product of holding cost rate C_h multiplying the surface delimited by OCDEBA in Figure 4.3, which is

$$C_h[\frac{1}{2}\frac{h^2}{U_{max}-d} + h(T - \frac{h}{U_{max}-d}) + \frac{1}{2}\frac{h^2}{d}]\,; \quad (4.20)$$

while the total shortage cost for the n^{th} period since the last perfect maintenance action is the product of shortage cost rate C_s multiplying the surface delimited by EFG in Figure 4.3, which is

$$C_s[(\mu_p - \frac{h}{d})^2 \frac{1}{2} d]\,; \quad (4.21)$$

Therefore, the total inventory costs for the n^{th} period since the last perfect maintenance action is (when M1 undergoes preventive maintenance)

$$C_h[\frac{1}{2}\frac{h^2}{U_{max}-d} + h(T - \frac{h}{U_{max}-d}) + \frac{1}{2}\frac{h^2}{d}] + C_s(\mu_p - \frac{h}{d})^2 \frac{1}{2} d\,; \quad (4.22)$$

Hence, for **Condition 2**, formulas (4.19) and (4.22) imply that: for the n^{th} period since the last perfect maintenance, the expected total inventory costs are

$$G2_n(T,h)=\frac{R(nT)}{R[(n-1)T]}\{C_h[\frac{1}{2}\frac{h^2}{U_{\max}-d}+h(T-\frac{h}{U_{\max}-d})+\frac{1}{2}\frac{h^2}{d}]+C_s(\mu_p-\frac{h}{d})^2\frac{1}{2}d\}$$
$$+\frac{F(nT)-F[(n-1)T]}{R[(n-1)T]}\{C_h[\frac{1}{2}\frac{h^2}{U_{\max}-d}+h(\frac{1}{R[(n-1)T]}\int_{(n-1)T}^{nT}R(t)dt-\frac{h}{U_{\max}-d})+\frac{1}{2}\frac{h^2}{d}]+C_s(\mu_c-\frac{h}{d})^2\frac{1}{2}d\}$$

(4.23)

In previous formulas, we have seen the form "$T-\frac{h}{U\max-d}$". Since we have made the assumption that "failures are excluded in Phase I", the interval for consecutive preventive maintenance actions should not be smaller than the reconstitution time of the buffer, i.e. there should be

$$T\geq\frac{h}{U\max-d}. \tag{4.24}$$

4.1.3 Derivation of total cost and time for the system

The Supply-Buffer-Demand system that we consider is a serial system, i.e. the unit M1 is in the upper stream of the system while the buffer and the unit M2 are in the lower stream of system, so the activities in the buffer could not influence the maintenance actions of M1. Therefore, the inventory control policy that the buffer adopts has no direct impact on the age dependent preventive maintenance policy of M1. This means that once the maintenance policy of M1 is fixed, the expected maintenance costs and time duration for a period would be fixed, and they could not be changed by different inventory policies. Thus, we get to know the following two points in the combination of two policies:

1. The inventory control policy has exactly the same period and the cycle as the preventive maintenance policy, and both the period and the cycle are solely determined by the preventive maintenance policy;
2. The expected total costs of both maintenance and inventory control in a period would be just the summation of maintenance costs and inventory costs.

From the results in previous two subsections 4.1.1 and 4.1.2, we have already obtained the formulas for the expected maintenance costs and the expected inventory costs for the n^{th} period since the last perfect maintenance, so we can derive the expected total costs (including both maintenance and inventory costs) for the first n periods within a cycle with the similar formulas as those in **proposition 4.2**. We also know that the expected total time duration for the first n periods within a cycle is the same for either preventive maintenance policy or inventory policy, and this duration is solely determined by preventive maintenance policy.

To derive the total cost rate (including maintenance and inventory costs) of the system, we first define the following symbols ("maintenance cost" in these definitions do not include the enhanced preventive maintenance cost):

Definition 4.4

$\Delta EC1_n\ (T,\ h)$: the expected total costs (including both the maintenance cost and inventory cost) for the n^{th} period within a cycle, under **Condition 1** of the inventory control policy;

$\Delta EC2_n$ (T, h): the expected total costs (including both the maintenance cost and inventory cost) for the n^{th} period within a cycle, under **Condition 2** of the inventory control policy;

$EC1_n$ (T, h): the expected total costs (including both the maintenance cost and inventory cost) for the first n periods within a cycle, under **Condition 1** of the inventory control policy;

$EC2_n$ (T, h): the expected total costs (including both the maintenance cost and inventory cost) for the first n periods within a cycle, under **Condition 2** of the inventory control policy.

Similar to **proposition 4.2**, we have

Proposition 4.3 $\Delta EC1_n$ (T, h) and $\Delta EC2_n$ (T, h) can be obtained from

$$\Delta EC1_i(T,h)$$
$$= q^{i-1}R[(i-1)T][A_i(T)+G1_i(T,h)]+\sum_{j=1}^{i-1} q^{j-1}R[(j-1)T]\{1-q\frac{R(jT)}{R[(j-1)T]}\}\Delta EC1_{i-j}(T,h), \tag{4.25}$$

$$\Delta EC2_i(T,h)$$
$$= q^{i-1}R[(i-1)T][A_i(T)+G2_i(T,h)]+\sum_{j=1}^{i-1} q^{j-1}R[(j-1)T]\{1-q\frac{R(jT)}{R[(j-1)T]}\}\Delta EC2_{i-j}(T,h), \tag{4.26}$$

Proof. Since we know that "the inventory control policy has exactly the same period and the cycle as the preventive maintenance policy", the expected total costs (including both the maintenance cost and inventory cost) for the i^{th} period since the last perfect maintenance action is $A_i(T)+G1_i(T, h)$ under **Condition 1** of inventory control policy. Therefore, using the same deduction as that in **proposition 4.2**, we can derive the formula (4.25). Similarly, under **Condition 2** of inventory control policy, we can obtain the formula (4.26). □

Under the help of **proposition 4.3**, we can obtain $EC1_n(T, h)$ and $EC2_n(T, h)$ from:

$$EC1_n(T,h) = \sum_{i=1}^{n}[EC1_n(T,h) - EC1_{n-1}(T,h)] = \sum_{i=1}^{n}\Delta EC1_i(T,h), \tag{4.27}$$

$$EC2_n(T,h) = \sum_{i=1}^{n}[EC2_n(T,h) - EC2_{n-1}(T,h)] = \sum_{i=1}^{n}\Delta EC2_i(T,h), \tag{4.28}$$

$$\Delta EC1_1(T,h) = EC1_n(T,h) = A_1(T) + G1_1(T,h), \tag{4.29}$$

$$\Delta EC2_1(T,h) = EC2_n(T,h) = A_1(T) + G2_1(T,h). \tag{4.30}$$

In summary, when temporarily not considering the enhanced preventive maintenance cost, we have obtained the expected total cost per unit of time within a cycle:

$$\frac{EC1_N(T,h)}{ET_N(T)}, \text{ if } \mu_p \le \frac{h}{d} \le \mu_c; \quad \frac{EC2_N(T,h)}{ET_N(T)}, \text{ if } \frac{h}{d} < \mu_p. \tag{4.31}$$

4.2 Optimal strategy meeting system requirements

4.2.1 Derivation of availability and its minimum requirement

In this subsection we will formulate the expected availability of the production unit M1 within a cycle. The joint consideration of total cost rate and stationary availability has significant engineering meanings in the applications. In fact, for an extreme example, where preventive maintenance cost is too small compared to corrective maintenance cost, the optimal maintenance strategy without consideration of availability could be: preventive maintenance is carried out constantly, without any operation of the production system, in order that the cost for corrective maintenance would never be incurred. Obviously, the system in this extreme situation would be nothing but a futile system. Thus, we know that the availability requirement is a must in formulating the models.

Previous papers have implemented this joint consideration into their models. Wang and Pham (1999) stated the importance of joint consideration of cost measures and availability measures, raising examples which may be needed in practice: "policies which minimize the maintenance cost rate while some availability requirements are satisfied, or policies that maximize the system availability while maintenance cost rate is less than some predetermined value". They also applied the former strategy into their maintenance model for a single production unit. Chelbi and Rezg (2006) required a minimum stationary availability level for their optimization model of a Supply-Buffer-Demand system, too.

Definition 4.5

$AV_n\,(T)$: the expected available time of the unit M1 for the n^{th} period since the last perfect maintenance;

$\Delta EAV_n\,(T)$: the expected available time of the unit M1 for the n^{th} period within a cycle;

$EAV_n\,(T)$: the expected total available time duration of the unit M1 for the first n periods within a cycle;

$SAV_N\,(T)$: the expected stationary availability of the production unit M1 within a cycle (N periods).

From the formula (4.2), we have

$$AV_n(T) = \frac{\int_{(n-1)T}^{nT} R(u)du}{R[(n-1)T]}\,; \tag{4.32}$$

Similar as **proposition 4.2**, we can prove that $\Delta EAV_n\,(T)$ can be obtained from:

Proposition 4.4

$$\Delta EAV_i(T) = q^{i-1}R[(i-1)T]AV_i(T) + \sum_{j=1}^{i-1} q^{j-1}R[(j-1)T]\{1 - q\frac{R(jT)}{R[(j-1)T]}\}\Delta EAV_{i-j}(T)\,. \tag{4.33}$$

With the conclusion of **proposition 4.4**, we can get $EAV_n\,(T)$ through

$$EAV_n(T) = \sum_{i=1}^{n}[EAV_n(T) - EAV_{n-1}(T)] = \sum_{i=1}^{n}\Delta EAV_i(T)\,. \tag{4.34}$$

Therefore, we obtain the expected stationary availability

$$SAV_N(T) = \frac{EAV_N(T)}{ET_N(T)}. \tag{4.35}$$

We know that our system starts at the age of zero at the beginning of each cycle. Thus, according to the definition of $SAV_N(T)$, it is obvious that for any $N>1$, there is

$$SAV_1(T) \geq SAV_N(T), \tag{4.36}$$

i.e. if all the other parameters and the decision variable T are fixed, the expected stationary availability of a cycle which consists of two or more periods is smaller than the expected stationary availability of a cycle which consists of only one period.

In our optimization model of the system, there is a minimum requirement level for this expected stationary availability, i.e. there is a form of inequality

$$SAV_N(T) \geq Fa \tag{4.37}$$

as one of the constraints (Fa is a constant), when we minimize the expected total cost rate of the system.

As for the property of this availability function, Chelbi and Rezg (2006) had proved that for N=1 (i.e. when preventive maintenance is always perfect) the stationary availability function is concave in T for any system with an increasing failure rate. But for N>1, whether our availability function is also concave in T is uncertain. The following figure (Figure 4.5) describes a numerical example for the situation when N=5 (assuming failure distribution to be Rayleigh distribution):

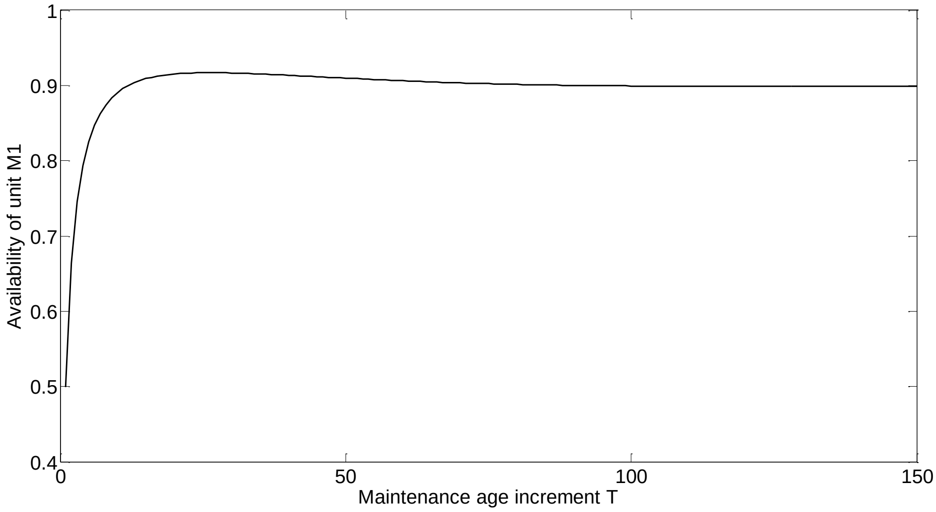

Figure 4.5 Availability vs. *T* when *N*=5

The availability function in the example of Figure 4.5 seems to be concave in *T*: the function increases sharply when *T* is increasing from very small values, and after the function reaches its maximum value it starts to decrease with a relatively small slope.

The trend in Figure 4.5 conforms our understanding about the relationship between availability and preventive maintenance age increment *T*: to maximize the availability, the age increment *T* must not be too small or too big. This is because if *T* is too small, the time interval between maintenance actions (either corrective or preventive) would be too small, so that most of the time within a cycle would be occupied by maintenance actions; on the other hand, if *T* is too big, the probability of system failure would become very high and the system would undergo corrective maintenance, which costs much more time than a preventive maintenance action.

4.2.2 Derivation of reliability and its minimum requirement

In this subsection, we will discuss the reliability and its minimum requirements for the unit M1 of our system. As it is known, we have introduced the imperfect preventive maintenance into the original model given by Chelbi and Rezg (2006). In other words we have brought in an uncertainty for the reliability status into our model: after each preventive maintenance action we are not sure of the exact state of the system any more, as we do not know whether the virtual age has been restored to zero (with probability p) or the virtual age has remained the same (with probability q). Thus, the reliability of the system decreases and hazards accumulate with time advancing, raising the probability of system failure and the corresponding corrective maintenance. Therefore, the enhanced preventive maintenance is needed to ensure that: for every N periods the system could return to its initial state, i.e. the virtual age of the system becomes zero again. In this way, we will also have a fixed length of a renewal cycle (N periods).

Maintaining a minimum level for the reliability is also very important to make sure that our assumption is valid. In the subsection “4.1.2 Derivation of cost for the inventory control policy”, we made the similar assumption as that in Cheung and Hausman (1997), Chelbi and Alt-Kadi (2004), and Chelbi and Rezg (2006): failures of the unit M1 are excluded in Phase I of a period, i.e. there is no failure in the buffer stock buildup stage. Moreover, in their models, preventive and corrective maintenance have been assumed to be perfect, i.e. immediately after the end of each period (in other words, at the beginning of the Phase I of each period) the unit M1 would start at virtual age of zero and its reliability is $R(0)=1$. Thus, for N consecutive periods in their models, the probability that the system is reliable at the beginning of the Phase I of all N consecutive periods, would be $R^N(0)=1$. Therefore, the probability that there is a failure in the Phase I of a period, would be small enough, so that the probability could be assumed to be neglected as in their models.

On the other hand, preventive maintenance has been assumed to be imperfect in our model, so immediately after each maintenance action the unit M1 would start at the virtual age of kT, where $k=0, 1, 2, \ldots$; while according to different virtual ages of M1, the probability that the unit M1 is reliable would be $R(kT)\leq 1$, $k=0, 1, 2, \ldots$. Therefore, at the beginning of the Phase I of each period, the reliability would be

$$R = \sum_{k=1}^{\infty} R(kT)P(\text{the virtual age of M1 is } kT) < 1. \tag{4.38}$$

Thus, the probability, that the system is reliable at the beginning of the Phase I of all N consecutive periods (i.e. for every period of a cycle), would be the product of N real positives whose values are all smaller than one, e.g. 0.9^N. Since such numbers as $0.9^{10}=0.3487$, $0.9^{20}=0.1215$, $0.9^{30}=0.0424$ are all smaller than one and are decreasing as N increases, there should be an upper limit for N. This upper limit could ensure that the probability such as 0.9^N is not small (e.g. at least bigger than 0.8000). We implement the reliability constraint into the model, just to make sure the probability, that there is a failure in Phase I of a period within a cycle, to be small enough, so that this probability could be neglected and our assumption about "no failure in Phase I" could be valid. To obtain the related reliabilities, first we define:

Definition 4.6

Y_k (T): the probability that the system's virtual age restores to zero after the k^{th} period within a cycle;

Rb_n (T): the probability that the system is reliable right before the maintenance action in the n^{th} period (i.e. it has survived a time T in the n^{th} period) within a cycle;

Ra_n (T): the probability that the system is reliable immediately after the maintenance action in the n^{th} period within a cycle, i.e. the probability that the system is reliable immediately after the beginning of Phase I of the $(n+1)^{\text{th}}$ period within a cycle.

Since the system that we study initiates at the age of zero and the preventive maintenance is imperfect, we could easily derive from the definitions:

$$Y_0(T)=1,\ Ra_0(T)=1,\ Rb_0(T)=1; \tag{4.39}$$

$$Ra_1(T)>Ra_n(T),\ Rb_1(T)>Rb_n(T),\ \forall\ n>1. \tag{4.40}$$

To derive a general form for Y_k (T), we have the following results:

Proposition 4.5 Y_k (T) can be obtained through the following recursion

$$Y_k(T)=1-\sum_{m=0}^{k-1} q^{k-m} R[(k-m)T]Y_m(T)\,. \tag{4.41}$$

Proof. According to the definition of Y_k (T), we know that Y_k (T) is equivalent to "1-P (the system's virtual age does not restore to zero after the k^{th} period)". Then it is left for us to compute the probability that the system does not return to perfect state after the k^{th} period within a cycle. As we have analyzed before, if the system does not return to perfect state after the k^{th} period within a cycle, the possible virtual age of the system would be T, $2T$, $3T$, …, kT after the k^{th} period within a cycle.

On the other hand, if the virtual age of the system is $(k\text{-}m)T$ after the k^{th} period within a cycle, two requirements must be satisfied: 1. the system's virtual age must return to zero after the m^{th} period; 2. after the m^{th} period, the system must undergo $(k\text{-}m)$ consecutive minimal preventive maintenance actions in the subsequent $(k\text{-}m)$ periods. According to the definition, the probability that corresponds to the first requirement is $Y_m(T)$, while the probability for the second requirement to happen is

$$\prod_{i=1}^{k-m}\{q\frac{R(iT)}{R[(i-1)T]}\}=q^{k-m}R[(k-m)T]\,. \tag{4.42}$$

Therefore, we will obtain

$$P(\text{the system's virtual age doesn't restore to zero after the } k^{\text{th}} \text{ periods})=\sum_{m=0}^{k-1}q^{k-m}R[(k-m)T] \tag{4.43}$$

□

With the results of **proposition 4.5**, we can obtain $Rb_n(T)$ and $Ra_n(T)$ from

Proposition 4.6

$$Rb_n(T)=\sum_{k=0}^{n-1}\{Y_k(T)q^{n-k-1}R[(n-k)T]\}\,; \tag{4.44}$$

$$Ra_n(T)=Y_n(T)+(1-Y_n(T))Rb_n(T)\,. \tag{4.45}$$

Proof. We prove the formula (4.45) first. When $Rb_n(T)$ is known, it is easy to derive $Ra_n(T)$: according to the definition, the n^{th} maintenance action within a cycle either restores the system virtual age to zero with probability $Y_n(T)$ or does not change the system age with probability 1-$Y_n(T)$. The former occasion would turn the reliability of the unit M1 to $R(0)=1$; while the latter occasion would keep the reliability of the unit M1 just as it was before the maintenance, i.e. the reliability remains to be $Rb_n(T)$. Thus, we have proved the formula (4.45).

If the system is reliable right before the maintenance action in the n^{th} period within a cycle, the virtual age of the unit M1 at that time would possibly be T, $2T$, $3T$, …, nT. If the virtual age of the system is $(n-k)T$ right before the maintenance action in the n^{th} period, the system must satisfy two requirements: 1. the system's virtual age must return to zero immediately after the k^{th} period; 2. after the k^{th} period, the system must undergo (n-k-1) consecutive minimal preventive maintenance actions in the subsequent (n-k-1) periods, and then survived for time T in the next period. According to the definition, the probability that the system satisfies the first requirement is $Y_k(T)$; while the probability that the system satisfies the second requirement is

$$\prod_{i=1}^{n-k-1}\{q\frac{R(iT)}{R[(i-1)T]}\}\frac{R[(n-k)T]}{R[(n-k-1)T]}=q^{n-k-1}R[(n-k)T]. \tag{4.46}$$

Therefore, the formula (4.44) is proved. □

With the help of **proposition 4.6**, we can have some numerical examples for $Rb_n(T)$ and $Ra_n(T)$. The definition for the reliability $Rb_n(T)$ implies that: with time advancing, the reliability of the unit M1 is decreasing if without any preventive maintenance or corrective maintenance. Therefore, although $Rb_n(T)$ may not be rigorously decreasing in T, at least it seems to have a tendency of decreasing with T. The following figure shows an example that Rb_n (T) is decreasing when T increases, in which n=10:

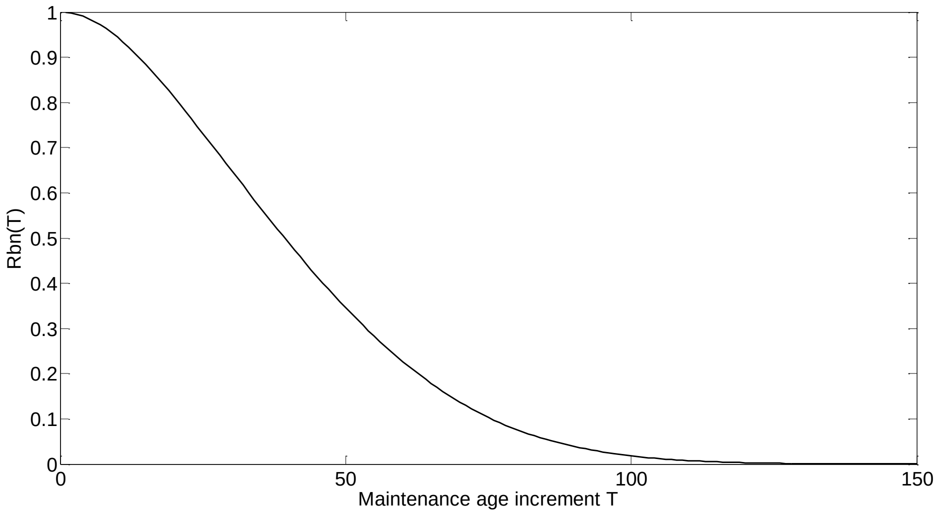

Figure 4.6 Reliability Rb_n (T) vs. T when n=10

On the other hand, we have numerical results for $Ra_n(T)$: the following figure depicts a case where n=10 (although we are not sure of the convexity of $Ra_n(T)$, it seems to be convex with T in this example):

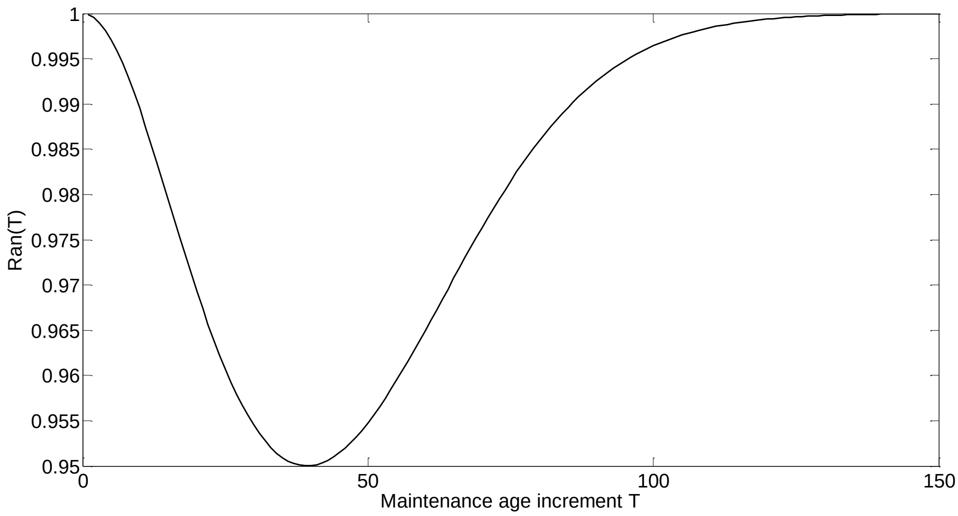

Figure 4.7 Reliability Ra_n (T) vs. T when n=10

An illustration to this phenomenon is: when T is relatively small, the reliability is decreasing when T increases (time advancing); however, when T is relatively big, the reliability becomes too small that there is a high probability of failure, so that the reliability increases as a result of undergoing corrective maintenance due to the failure of the unit M1.

Moreover, with the help of results from **proposition 4.6**, we can derive the reliabilities we have discussed previously in this subsection. The probability that the system is reliable at the beginning of the Phase I of every period of a cycle (i.e. for N consecutive periods within a cycle), is

$$\prod_{j=0}^{N-1} Ra_j(T) = Ra_0(T)\prod_{j=1}^{N-1} Ra_j(T) = \prod_{j=1}^{N-1} Ra_j(T), \tag{4.47}$$

in which $Ra_0(T)$=1 according to the formula (4.37). Apparently given the fixed value of T, this N-period joint reliability $\prod_{j=1}^{N-1} Ra_j(T)$ is a decreasing function of N.

In addition, from the formula (4.38) we have

$$\prod_{j=1}^{N-1} Ra_j(T) \le [Ra_1(T)]^{N-1}, \tag{4.48}$$

in which only when N=1 or 2 does the inequality sign become an equal sign.

As we have discussed before, this probability should have a lower bound as a constraint of our model, so that the reliability of the unit M1 remains high and our assumption that “there is no failure in the buffer stock buildup stage” could be close to the reality. In summary, we should have the constraint

$$\prod_{j=1}^{N-1} Ra_j(T) \ge Fr, \tag{4.49}$$

where Fr is a constant, representing the predetermined minimum reliability requirement for N consecutive periods (a cycle).

A numerical example of $\prod_{j=1}^{N-1} Ra_j(T)$ is depicted in the figure below (although the convexity of $\prod_{j=1}^{N-1} Ra_j(T)$ is uncertain, it seems to be convex in T in this case):

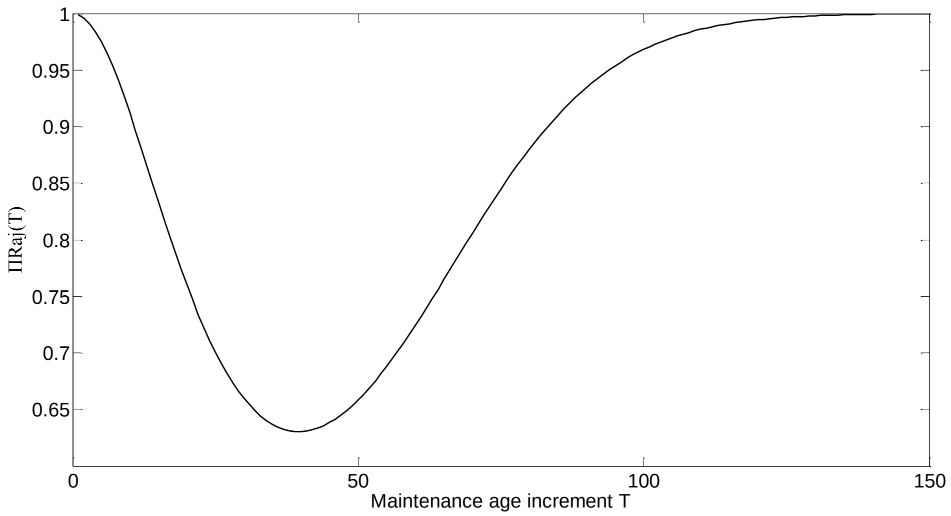

Figure 4.8 N-period joint reliability $\prod_{j=1}^{N-1} Ra_j(T)$ vs. T when n=10

Finally, we should know that $Rb_n(T)$ is very important in determining whether an additional cost M_e is incurred for an enhanced preventive maintenance action in every N periods: if the maintenance action is a preventive maintenance action for the N^{th} period, an additional cost M_e is needed to ensure this preventive maintenance to be perfect, so that the system would be definitely renewed in every N periods; on the contrary, if the maintenance action is a corrective maintenance action for the N^{th} period, there is no need to pay this additional cost M_e at all because the corrective maintenance is already perfect. Therefore, according to the definition of $Rb_n(T)$ and our analysis, the expected cost for the enhanced preventive maintenance action in a cycle would be

$$Rb_N(T)M_e + [1 - Rb_N(T)] * 0 = Rb_N(T)M_e. \tag{4.50}$$

4.2.3 Derivation of shortage rate and its maximum requirement

In our Supply-Buffer-Demand system, the inventories in the buffer are used to supply the demanding unit M2 when the supplying unit M1 is undergoing corrective or preventive maintenance. Since the capacity of the buffer is finite, there will be two possibilities when the buffer supplies the unit M2: 1. there are enough inventories in the buffer, so that all the demand of M2 during M1's maintenance is met by the inventories; 2. there are not enough inventories in the buffer, so that there will be shortage incurred (i.e. some demand of M2 during M1's maintenance can't be met by the buffer). For every unit of product that the buffer fails to supply to M2, there will be a shortage cost C_s incurred, which is associated with the additional effort to provide M2 with the unavailable product in the short run. In other words, if the inventory in the buffer is not enough and so a shortage is caused, the system would turn to help from an external resource temporarily, so that the demand of the unit M2 is satisfied. This could be illustrated in the following graph:

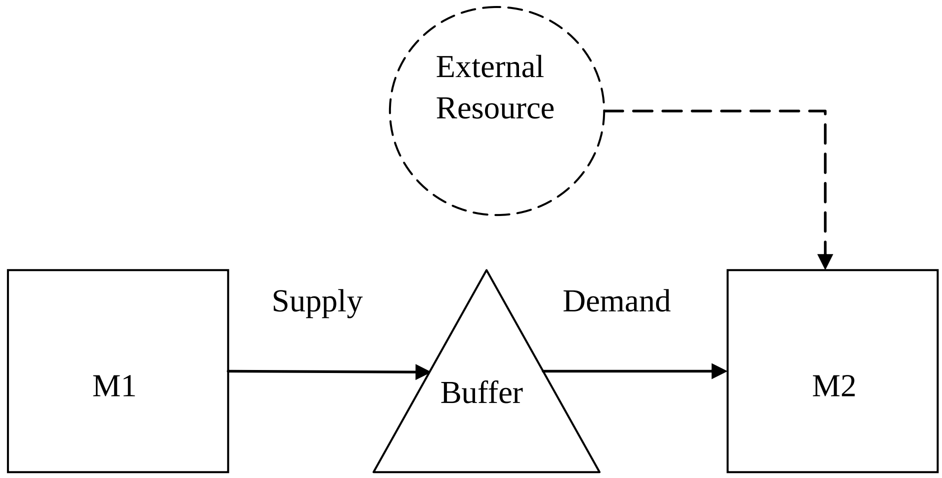

Figure 4.9 A Supply-Buffer-Demand system with shortage

Therefore, another problem may arise for the external resource: whether the external resource is sufficient to supply the demand of M2 when there is a shortage in the buffer. Although C_s represents the additional effort to provide M2 with the unavailable product in the short run, it is not guaranteed that all the shortage could be covered by the external resource. Sometimes the external resource is so plentiful that it can be seen as an infinite resource: in this case, there will be no further consideration for the external resource at all, because every unit of demand of M2 that is shortage could be supplied from the outside. In the opposite case, however, the external resource is finite: in such a case, there will be an upper limit that the system could "import" from the external resource when the system's buffer is in shortage. In this case, we should require that the shortage in the buffer should be satisfied by the external resource; otherwise other serious consequences maybe arise as a result of the unsatisfied shortage of the buffer, e.g. some customers may be lost forever if they could not get the products from the system.

In summary, we have analyzed that there are two cases that we should consider when we formulate the optimization model: 1. Infinite external resource; 2. Finite external resource. For the case of the finite external resource, there should be an upper limit for the shortage of the buffer, so that all the shortage could be covered by the external resource.

In the system that we study, we assume that the upper limit of the shortage is measured by the expected total number of shortage per unit of time within a cycle, i.e. the stationary shortage quantity rate within a cycle, which is equivalent to the stationary shortage rate. In this subsection, we will develop the formulas for the stationary shortage quantity rate.

Definition 4.7

$Short1_n(T, h)$: the expected number of shortage of the buffer for the n^{th} period since the last perfect maintenance action, under **Condition 1** of the inventory control policy;

$Short2_n(T, h)$: the expected number of shortage of the buffer for the n^{th} period since the last perfect maintenance action, under **Condition 2** of the inventory control policy;

$\Delta EShort1_n(T, h)$: the expected number of shortage of the buffer for the n^{th} period within a cycle, under **Condition 1** of the inventory control policy;

$\Delta EShort2_n(T, h)$: the expected number of shortage of the buffer for the n^{th} period within a cycle, under **Condition 2** of the inventory control policy;

$EShort1_n(T, h)$: the expected total number of shortage of the buffer for the first n periods within a cycle, under **Condition 1** of the inventory control policy;

$EShort2_n(T, h)$: the expected total number of shortage of the buffer for the first n periods within a cycle, under **Condition 2** of the inventory control policy;

$SShort1_N(T, h)$: the expected total number of shortage of the buffer per unit of time within a cycle (N periods), under **Condition 1** of the inventory control policy;

$SShort2_N(T, h)$: the expected total number of shortage of the buffer per unit of time within a cycle (N periods), under **Condition 2** of the inventory control policy.

From the proof of **proposition 4.1**, we already know that: for the n^{th} period since the last perfect maintenance action, the probability that the unit M1 undergoes corrective maintenance is $\frac{F(nT)-F[(n-1)T]}{R[(n-1)T]}$, and the probability that M1 undergoes preventive maintenance is $\frac{R(nT)}{R[(n-1)T]}$. With the same analysis as that in the subsection 4.1.2 for **Condition 1** and **Condition 2**, we can develop the formulas for $Short1_n(T, h)$ and $Short2_n(T, h)$.

For **Condition 1** $\mu_p \le \frac{h}{d} \le \mu_c$:

a. If the unit M1 undergoes corrective maintenance, there will be shortage incurred and it is $(\mu_c - \frac{h}{d})^2 \frac{1}{2} d$, which is according to the formula (4.12).
b. If the unit M1 undergoes preventive maintenance, there will be no shortage.

In summary, $Short1_n(T, h)$ can be obtained:

$$Short1_n(T,h) = \frac{F(nT) - F[(n-1)T]}{R[(n-1)T]}(\mu_c - \frac{h}{d})^2 \frac{1}{2} d\ . \quad (4.51)$$

For **Condition 2** $\frac{h}{d} < \mu_p$:

a. If the unit M1 undergoes corrective maintenance, there will be shortage incurred and it is $(\mu_c - \frac{h}{d})^2 \frac{1}{2} d$, which is according to the formula (4.18).

b. If the unit M1 undergoes preventive maintenance, there will be shortage incurred and it is $(\mu_p - \frac{h}{d})^2 \frac{1}{2} d$, which is according to the formula (4.21).

In summary, $Short2_n(T, h)$ can be obtained:

$$Short2_n(T,h) = \frac{F(nT) - F[(n-1)T]}{R[(n-1)T]}(\mu_c - \frac{h}{d})^2 \frac{1}{2} d + \frac{R(nT)}{R[(n-1)T]}(\mu_p - \frac{h}{d})^2 \frac{1}{2} d\ . \quad (4.52)$$

Similar as **proposition 4.2**, we can prove the following proposition:

Proposition 4.7 $\Delta EShort1_n(T, h)$ and $\Delta EShort2_n(T, h)$ can be obtained through

$$\Delta EShort1_i(T,h) = q^{i-1}R[(i-1)T]Short1_i(T,h) + \sum_{j=1}^{i-1} q^{j-1}R[(j-1)T]\{1 - q\frac{R(jT)}{R[(j-1)T]}\}\Delta EShort1_{i-j}(T,h)\ ; \quad (4.53)$$

$$\Delta EShort2_i(T,h) = q^{i-1}R[(i-1)T]Short2_i(T,h) + \sum_{j=1}^{i-1} q^{j-1}R[(j-1)T]\{1 - q\frac{R(jT)}{R[(j-1)T]}\}\Delta EShort2_{i-j}(T,h)\ . \quad (4.54)$$

With the conclusion from **proposition 4.7**, we can derive the formulas for $EShort1_n(T, h)$ and $EShort2_n(T, h)$ according to their definitions:

$$EShort1_n(T,h) = \sum_{i=1}^{n}[EShort1_n(T,h) - EShort1_{n-1}(T,h)] = \sum_{i=1}^{n}\Delta EShort1_i(T,h); \quad (4.55)$$

$$EShort2_n(T,h) = \sum_{i=1}^{n}[EShort2_n(T,h) - EShort2_{n-1}(T,h)] = \sum_{i=1}^{n}\Delta EShort2_i(T,h). \quad (4.56)$$

Furthermore, according to the definitions, $SShort1_N(T, h)$ and $SShort2_N(T, h)$ are

$$SShort1_N(T,h) = \frac{EShort1_N(T,h)}{ET_N(T)}; \quad (4.57)$$

$$SShort2_N(T,h) = \frac{EShort2_N(T,h)}{ET_N(T)}. \quad (4.58)$$

For the case of the "finite external resource", as we have discussed before, there should be an upper limit for the expected total number of shortage per unit of time within a cycle, so that all the shortage of the buffer could be satisfied by the external resource. In other words, we should have a constraint for our model, which is

$$SShort1_N(T,h) \le Fs, \text{ if } \mu_p \le \frac{h}{d} \le \mu_c; \quad (4.59)$$

$$SShort2_N(T,h) \le Fs, \text{ if } \frac{h}{d} < \mu_p; \quad (4.60)$$

where Fs is a constant, representing the predetermined maximum stationary shortage quantity rate requirement level for a cycle.

With the conclusion from formulas (4.57) and (4.58), we could have some numerical discussions for $SShort1_N(T, h)$ and $SShort2_N(T, h)$. Our numerical results show that for fixed T and N, $SShort1_N(T, h)$ and $SShort2_N(T, h)$ are decreasing functions of h (note that the domains for the variable h of these two functions are $[d\mu_p, d\mu_c]$ and $[0, d\mu_p]$ respectively). The following figure shows a numerical example about this relationship:

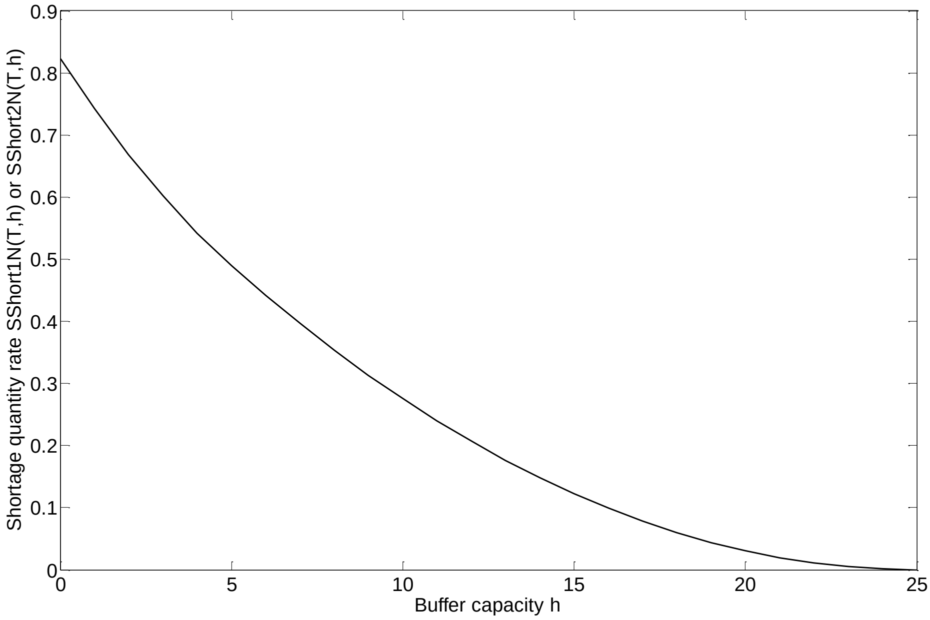

Figure 4.10 $SShort1_N(T, h)$ and $SShort2_N(T, h)$ vs. h when N=10

In this numerical example the variable N=10, and the domains for the variable h of $SShort1_N(T, h)$ and $SShort2_N(T, h)$ are [5, 25], [0, 5] respectively. In the Figure 4.10, the graph between h=0 and h=5 is the figure for $SShort2_N(T, h)$; the graph between h=5 and h=25 is the figure for $SShort1_N(T, h)$; when h=5, $SShort1_N(T, h)$ and $SShort2_N(T, h)$ are equivalent. The reason that the shortage quantity rate is the decreasing function of h is quite obvious: if h is bigger, there will be more inventories in the buffer to hedge against the risk of shortage, so there will be less shortage in a given time.

Our numerical results also show that: given fixed h and N, $SShort1_N(T, h)$ is the increasing function of T; when h is close to $d\mu_p$ in the domain [0, $d\mu_p$], $SShort2_N(T, h)$ is the increasing function of T; when h is close to 0 in the domain [0, $d\mu_p$], $SShort2_N(T, h)$ is decreasing with T when T is relatively small, and it is increasing with T when T is relatively big. Here are three figures showing the relationships between the shortage quantity rate and T under different circumstances (In the following three numerical examples the variable N=10, and the definition domains of the variable h of $SShort1_N(T, h)$ and $SShort2_N(T, h)$ are [5, 25], [0, 5] separately, i.e. $d\mu_c$=25, $d\mu_p$=5):

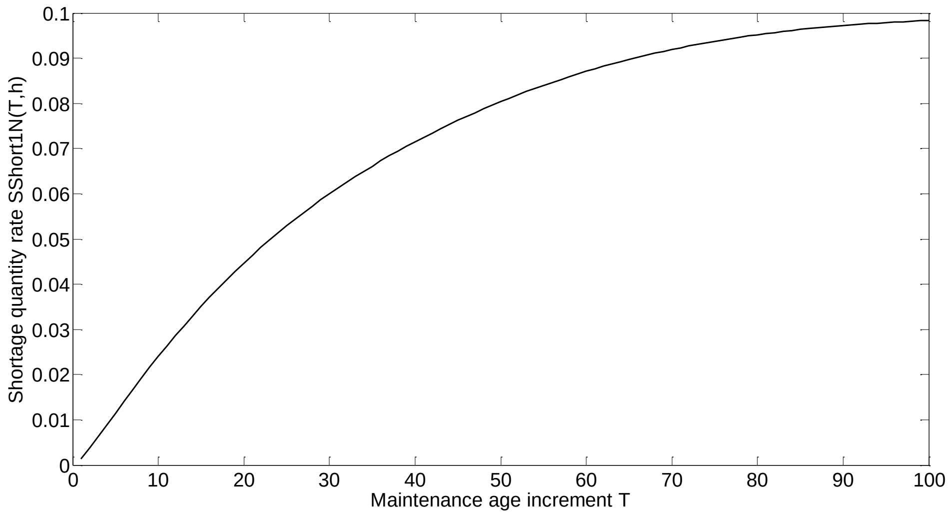

Figure 4.11 $SShort1_N(T, h)$ is an increasing function of T when h=18

According to the definitions for μ_p, μ_c and d, we know that $d\mu_p$ is the amount of products that are demanded by the unit M2 during preventive maintenance, while $d\mu_c$ is the amount of products demanded by the unit M2 during corrective maintenance. If h belongs to [$d\mu_p$, $d\mu_c$], the failure of the unit M1 and the subsequent corrective maintenance is the only source of shortage. Therefore, a bigger T would increase the probability of failure and thus increase the shortage quantity rate.

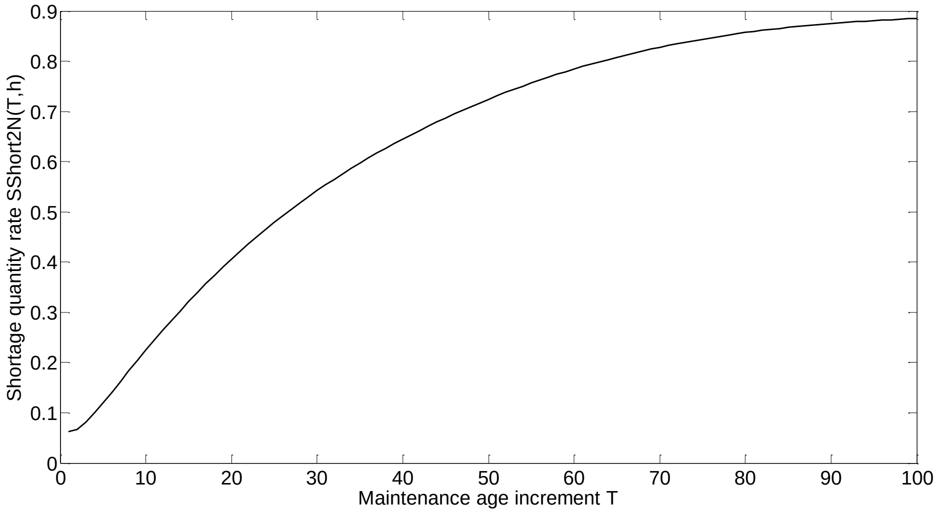

Figure 4.12 $SShort2_N\,(T, h)$ is an increasing function of T when h=4

If h belongs to [0, $d\mu_p$] but is close to $d\mu_p$, the failure and the subsequent corrective maintenance of the system is the primary source of shortage, while the shortage caused by the preventive maintenance would be just a minor.

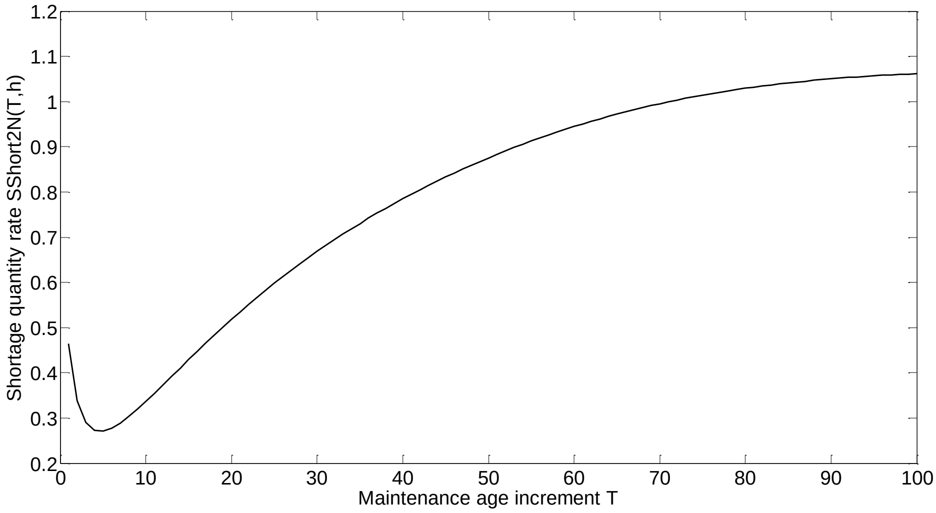

Figure 4.13 $SShort2_N\,(T, h)$ vs. T when h=2

The preventive maintenance would cause more shortage if h belongs to [0, $d\mu_p$]

but is close to 0. We know that T represents the frequency of the preventive maintenance; therefore, if T is getting too small it would increase the shortage rate.

These three figures have demonstrated that the shortage quantity rate $SShort1_N(T, h)$ and $SShort2_N(T, h)$ are increasing functions of T in general. Even for the case as in Figure 4.13, T should be small enough to impose an effect onto this "increasing" trend. On the other hand, however, as we have discussed in the subsection "4.2.1 Derivation of availability and its minimum requirement" before, T should not be "too small" since there is a minimum availability requirement for the system, which requires that T should be "big enough" to maintain the availability of the system. Therefore, when we develop our numerical algorithms later, we can treat $SShort1_N(T, h)$ and $SShort2_N(T, h)$ as increasing functions of T, without considering the special cases where T is too small.

Chapter 5

Methods and Results

In this chapter, we present the methods and processes to obtain the optimal policies (maintenance and inventory policies) for the system. First, we formulate the optimization models based on the theoretical results in Chapter 4. Next, we propose the numerical algorithms for solving the models. Finally, numerical examples for the algorithms are raised and discussed.

5.1 Optimization models

According to the two different assumptions for the External Resource which has been discussed in the subsection 4.2.3, we can formulate two optimization models: 1. Infinite external resource; 2. Finite external resource.

If the external resource is assumed to be infinite, we should formulate the comprehensive optimization model according to the analysis in the previous subsections and the formulas (4.24), (4.31), (4.37), (4.49), (4.50).

Optimization Model I

$$\text{Min } S(N,T,h) = \begin{cases} \dfrac{EC1_N(T,h) + Rb_N(T)M_e}{ET_N(T)}, \text{ if } \mu_p \le \dfrac{h}{d} \le \mu_c \\ \dfrac{EC2_N(T,h) + Rb_N(T)M_e}{ET_N(T)}, \text{ if } \quad \dfrac{h}{d} < \mu_p \end{cases}$$

$$
\text{subject to}\begin{cases} SAV_N(T) \geq Fa \\ \prod_{j=1}^{N-1} Ra_j(T) \geq Fr \\ T \geq \dfrac{h}{U\max - d} \\ N, T, h > 0 \\ N \text{ and } h \text{ are integers} \end{cases},
$$

We have formulated this Optimization Model I which minimizes the expected total cost (including preventive maintenance cost, corrective maintenance cost, inventory cost and enhanced preventive maintenance cost) per unit of time of the system, as well as take into consideration of the supplying unit M1's minimum stationary availability, and the minimum N-period joint reliability.

When *N*=1, the enhanced preventive maintenance will be carried out every period, i.e. there will be no imperfect preventive maintenance because they all become perfect preventive maintenance after being "enhanced". This will be the situation that is presented and discussed by Chelbi and Rezg (2006). Therefore, when *N*=1, the Optimization Model I would be reduced to the model presented by Chelbi and Rezg (2006).

If the external resource is assumed to be finite, we should formulate the comprehensive optimization model through formulas (4.24), (4.31), (4.37), (4.49), (4.50), (4.59), (4.60).

Optimization Model II

$$\text{Min } S(N,T,h)=\begin{cases}\dfrac{EC1_N(T,h)+Rb_N(T)M_e}{ET_N(T)}, \text{ if } \mu_p \le \dfrac{h}{d} \le \mu_c \\ \dfrac{EC2_N(T,h)+Rb_N(T)M_e}{ET_N(T)}, \text{ if } \quad \dfrac{h}{d} < \mu_p \end{cases}$$

$$\text{subject to}\begin{cases} SAV_N(T) \ge Fa \\ \prod_{j=1}^{N-1} Ra_j(T) \ge Fr \\ SShort1_N(T,h) \le Fs, \text{ if } \mu_p \le \dfrac{h}{d} \le \mu_c \\ SShort2_N(T,h) \le Fs, \text{ if } \dfrac{h}{d} < \mu_p \\ T \ge \dfrac{h}{U\max - d} \\ N,T,h > 0 \\ N \text{ and } h \text{ are integers} \end{cases},$$

We have formulated this Optimization Model II which minimizes the expected total cost (including preventive maintenance cost, corrective maintenance cost, inventory cost and enhanced preventive maintenance cost) per unit of time of the system, as well as take into consideration of the supplying unit M1's minimum stationary availability, the minimum N-period joint reliability, and the maximum expected number of total shortage per unit of time.

The objective functions of the Optimization Model I and II are the same:

$$S(N,T,h)=\begin{cases}\dfrac{EC1_N(T,h)+Rb_N(T)M_e}{ET_N(T)}, \text{ if } \mu_p \le \dfrac{h}{d} \le \mu_c \\ \dfrac{EC2_N(T,h)+Rb_N(T)M_e}{ET_N(T)}, \text{ if } \quad \dfrac{h}{d} < \mu_p \end{cases}. \tag{5.1}$$

We will analyze the function $S(N, T, h)$ and its variables. When T and N are both fixed, h will be a tradeoff between inventory holding cost and shortage cost: a bigger h is increasing the holding cost but reducing the shortage cost; while a smaller h is increasing the shortage cost but reducing the holding cost. When h and N are both fixed, T will be a tradeoff between preventive maintenance cost and the sum of corrective maintenance cost and shortage cost: a bigger T is increasing the probability of failure, so that the corrective maintenance cost and shortage cost are both increasing; while a smaller T is increasing the frequency of preventive maintenance so that the preventive maintenance cost is rising. Although generally the enhanced preventive maintenance cost only plays a minor role in the total cost, especially when N is large, it is still needed to note that: when T is increasing, the reliability $Rb_N(T)$ will decrease and $ET_N(T)$ will increase, so a bigger T can reduce the enhanced preventive maintenance cost rate.

Here are two examples of figures for the objective function (5.1). Although it is uncertain whether the total cost rate is convex in T or h, it seems to be convex with T and h in these two figures respectively.

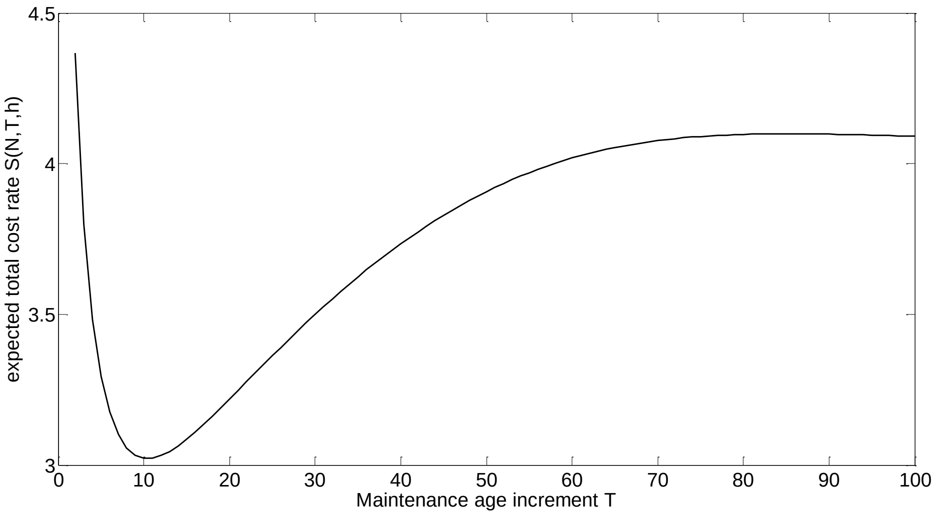

Figure 5.1 Total cost rate $S(N, T, h)$ vs. T when N=10 and h=18

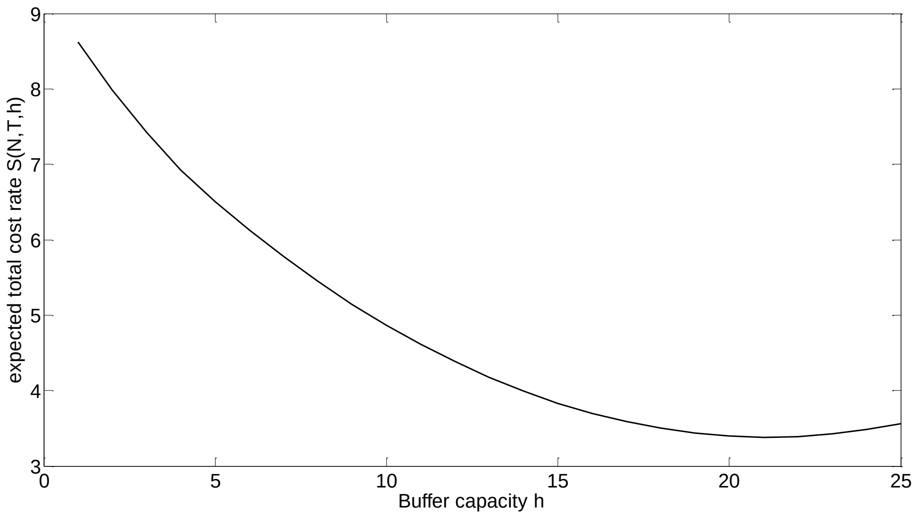

Figure 5.2 Total cost rate *S*(*N*, *T*, *h*) vs. *h* when *N*=10 and *T*=30

Figure 5.1 depicts that the expected total cost rate function *S*(*N*, *T*, *h*) is generally convex with *T* on the whole and gets its minimal value around *T*=10. Specifically, *S*(*N*, *T*, *h*) is convex with *T* when *T* is not too big; when *T* is getting too big, around more than *T*=80, the function comes into a generally steady state in which it is decreasing very slowly, i.e. the absolute value of its slope is very small.

Figure 5.2 depicts that the expected total cost rate function *S*(*N*, *T*, *h*) is convex in *h* and it gets its minimal value at around *h*=20.

5.2 Numerical algorithms to solve the models

To solve the optimization models, the first step is determining the possible values or domains for the decision variables from the constraints of the model. Then we enumerate all the possible values of the variables to calculate the total cost rate. Finally we compare all the results of the total cost rate, in order to find out the optimal values for the decision variables which minimize the total cost rate.

First, we know that the Optimization Model I and II both have the same availability requirement which is given by formula (4.37). From the formulas (4.36) and (4.37), we know that to meet the needs of the availability constraint of the model, T should satisfy

$$SAV_1(T) \geq SAV_N(T) \geq Fa \,. \tag{5.2}$$

Because Chelbi and Rezg (2006) had proved that for N=1 the stationary availability function is concave in T for any system with an increasing failure rate (i.e. $SAV_1(T)$ is concave in T), there should be just two values T_a, T_b ($T_a<T_b$) for T which are the solutions for equation

$$SAV_1(T) = Fa \,, \tag{5.3}$$

under appropriate assignment of the availability requirement Fa. (If there is only one solution for this equation (5.3), then this only solution is the only value for T which satisfies the availability requirement of the model, so it will be the only possible solution for T of the optimization model; if there is no solution for the equation (5.3), there will be no solution for the optimization model.)

Due to the concavity of the function $SAV_1(T)$, there should be

$$SAV_1(T) > Fa, \forall T \in [T_a, T_b]; \tag{5.4}$$

$$SAV_1(T) < Fa, \forall T < T_a \; or \; \forall T > T_b; \tag{5.5}$$

The results of the formulas (5.2) and (5.5) imply that for any N, there is

$$SAV_N(T) \le SAV_1(T) < Fa, \forall T < T_a \text{ or } \forall T > T_b . \tag{5.6}$$

Therefore, we know that the possible field for T which satisfies the availability requirement in formula (5.2) is $[T_a, T_b]$.

Secondly, we can determine the possible integers for N according to the N-period reliability requirement which is given by formula (4.49). From the formulas (4.48) and (4.49), we know that to meet the needs of the reliability constraint of the model, T and N should satisfy

$$[Ra_1(T)]^{N-1} \ge \prod_{j=1}^{N-1} Ra_j(T) \ge Fr . \tag{5.7}$$

If N=1, formula (5.7) is valid for any T. If N>1, from formula (5.7) we derive

$$\log[Ra_1(T)]^{N-1} \ge \log Fr . \tag{5.8}$$

Before we proceed, we derive the exact form and property of $Ra_1(T)$ first. We have the following proposition:

Proposition 5.1 $Ra_1(T)$ is convex with respect to $R(T)$.

Proof. According to the formula (4.39), we have

$$Y_0(T) = Ra_0(T) = Rb_0(T) = 1 . \tag{5.9}$$

Then according to the formulas (4.41), (4.44) and (5.9), we have

$$Y_1(T) = 1 - \sum_{m=0}^{0} q^1 R(T) Y_0(T) = 1 - qR(T) . \tag{5.10}$$

$$Rb_1(T) = \sum_{k=0}^{0}[Y_0(T)q^0R(T)] = R(T). \quad (5.11)$$

Finally, according to the formulas (4.45), (5.10) and (5.11), we have

$$Ra_1(T) = Y_1(T) + (1 - Y_1(T))Rb_1(T) = qR^2(T) - qR(T) + 1. \quad (5.12)$$

Then it is obvious that $Ra_1(T)$ is convex with respect to $R(T)$. □

Previously in the first step we have known that: to satisfy the availability requirement, T should be in the interval $[T_a, T_b]$. Since $R(T)$ is a decreasing function in T according to its definition, the argument that "T is in the interval $[T_a, T_b]$" is equivalent to the argument that "$R(T)$ is in the interval $[R(T_a), R(T_b)]$". Then according to **Proposition 5.1** that $Ra_1(T)$ is convex in $R(T)$, we could get the range for $Ra_1(T)$ on the interval $[T_a, T_b]$, and its maximal value on the interval $[T_a, T_b]$ is

$$\max_{T\in[T_a,T_b]} Ra_1(T) = \max(Ra_1(T_a), Ra_1(T_b)) < 1. \quad (5.13)$$

Since $Ra_1(T)$ and Fr are both between 0 and 1, from the formula (5.8) we have

$$N \le \frac{\log Fr}{\log[Ra_1(T)]} + 1 \le \frac{\log Fr}{\log[\max_{T\in[T_a,T_b]} Ra_1(T)]} + 1. \quad (5.14)$$

Therefore, the possible integer values of N are 1, 2, 3, ..., $[\frac{\log Fr}{\log[\max_{T\in[T_a,T_b]} Ra_1(T)]} + 1]$.

Next, we will obtain the constraints of the model for every possible integer value of N. For a given possible value of N, we can derive the exact forms of the availability and reliability constraints, $SAV_N(T) \geq Fa$ and $\prod_{j=1}^{N-1} Ra_j(T) \geq Fr$, only with T as the variable. On the other hand, according to the formula (5.8), we know that given N there should be

$$[Ra_1(T)]^{N-1} \geq Fr\,. \tag{5.15}$$

Since we know that $Ra_1(T)$ is convex in $R(T)$, there must exist $R(T_c)<R(T_d)$ which are the two solutions for the equation

$$[Ra_1(T)]^{N-1} = Fr\,. \tag{5.16}$$

in which $Ra_1(T)$ is seen as the function of $R(T)$, and only if $R(T)\leq R(T_c)$ or $R(T)\geq R(T_d)$ that the formula (5.15) satisfies. Since we also know that $R(T)$ is a decreasing function of T, the previous conclusion is equivalent to: there exist $T_c > T_d$ which are two solutions for the formula (5.16) (in which $Ra_1(T)$ is seen as the function of T) and only if $T\leq T_d$ or $T\geq T_c$ that the formula (5.15) satisfies. Let W denote the field that satisfies both the formulas (5.4) and (5.15), i.e.

$$W=[T_a, T_b]\cap([0, T_d]\cup[T_c,+\infty)). \tag{5.17}$$

Finally, we will consider the possible values for the integer h. The original possible interval for h is $[0, d\mu_c]$, so the integer h may get the values of 1, 2, 3, … , $[d\mu_c]$, where $[d\mu_c]$ represents the gauss function of $d\mu_c$ (the maximum integer that is no bigger than $d\mu_c$). For the Optimization Model II, we should consider an additional constraint: the shortage quantity rate constraint. Given a value for h, we can solve the possible field for T which satisfies the shortage rate constraints formulas (4.59) and (4.60) ($SShort1_N(T,h) \leq Fs$, $SShort2_N(T,h) \leq Fs$), as these two formulas are either increasing or convex functions of T. Then we combine the result field with the field W which we obtained previously. There is one thing to be noted: we have known that the shortage quantity rate $SShort1_N$ (T, h) (or $SShort2_N$ (T, h)) is a decreasing function of h. Therefore, if there is no solution of T for formulas (4.59) and (4.60) when $h=j$ ($1 \leq j \leq [d\mu_c]$), there will be no solution of T for formulas (4.59) and (4.60) for any $h<j$.

Following the previous steps, we could obtain the possible field for T given the fixed values of N and h. Then the optimal value of T which minimizes the expected total cost rate could be computed. Varying the fixed values of N and h within their possible fields, we can get the corresponding optimal values T^* and variable sets (N, T^*, h). After comparing the different variable sets, the optimal set (N^*, T^*, h^*) which minimizes the expected total cost rate is finally determined.

Figure 5.3 describes the proposed procedure:

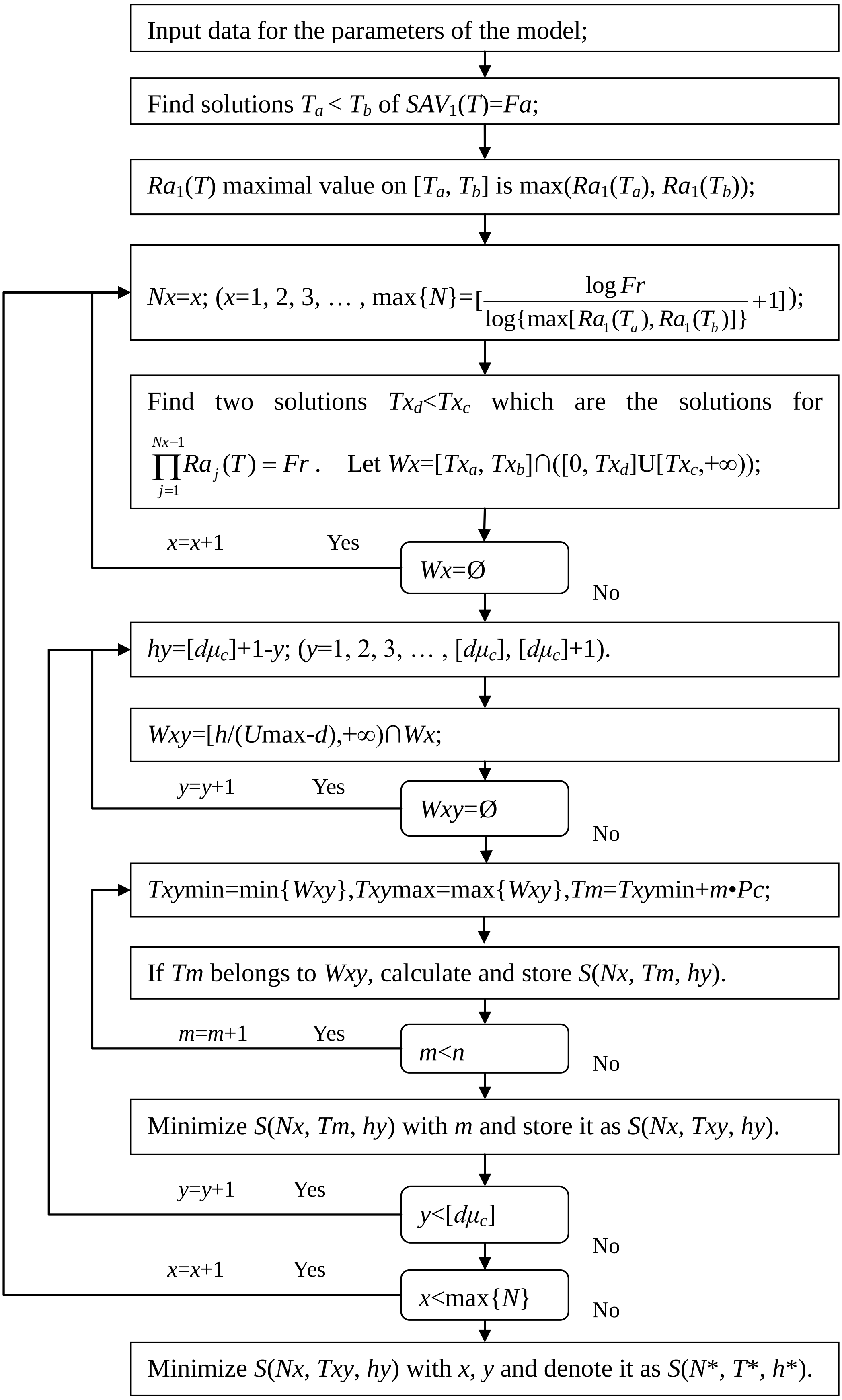

Figure 5.3 Numerical algorithms to find the optimal solution for Model I

There are three points to be noted for Figure 5.3:

1. When Nx=1, there will be Tx_e=0, Tx_f=+∞;
2. Pc represents the precision criterion for the solution;
3. The range for m here is m=0, 1, 2, 3, … , n=[(Txymax-Txymin)/Pc], in which [(Txymax-Txymin)/Pc] is the gauss function of (Txymax - Txymin)/Pc).

Optimization Model II has almost the same numerical algorithms as Optimization Model I, except that:

After determining the value of h=hy, we should find solutions for $SShort1_N(T,h)=Fs$ (if $d\mu_p \leq hy$) or $SShort2_N(T,h)=Fs$ (if $hy < d\mu_p$). If there is only one solution Typ, then Wy=[0, Typ]; if there are two solutions $Typ < Tyq$, then Wy=[0,Typ]U[Tyq,+∞). Wxy should change to be Wxy=[h/(Umax-d),+∞)∩Wx∩Wy.

5.3 Numerical examples for solving models and discussions

To illustrate our approach to find the optimal solution to our models, the following input data were used:

- Costs (in monetary units): M_p=10, M_c=70, M_e=5, C_h=0.1, C_s =10.
- Demand: d=5 unit/time unit.
- Supply: Umax=20 unit/time unit
- Supplying unit time to failure distribution $F(\bullet)$: Weibull distribution with shape parameter 2 and the scale parameter 50, leading to an average lifetime MTTF=44.3

time units. We have an increasing failure rate in this case.

● Maintenance time: μ_p=1 time unit, μ_c=5 time units.

● Preventive maintenance imperfect probability: q=0.2, p=1-q=0.8.

● Minimum required availability level: Fa=91%.

● Minimum required reliability level: Fr=90%.

● Maximum required shortage rate level: Fs=0.03.

● Precision level for the solution: Pc=0.01.

Using the procedure described in Figure 5.3 (and the additional procedure for Optimization Model II), we obtain the following results (Table 5.1): according to these results, the optimal solution for Optimization Model I which minimizes the expected total cost per unit of time of the system, while satisfying the constraints of a 91% minimum stationary availability and a 90% minimum N-period joint reliability, consists in performing preventive maintenance after 14.91 time units of operation without failure, performing an enhanced preventive maintenance action (perfect) every 3 periods, and forming a buffer stock of 19 units after the completion of each maintenance action. By doing so, it would cost 3.083387 monetary units per unit of time to operate the system according to the proposed policies. The optimal solution in Table 5.1 for Optimization Model II which satisfies an additional constraint (compared to Optimization Model I) of a 0.03 maximum expected number of total shortage per unit of time, consists of the same performing actions as those of Optimization Model I and it costs the same too.

Tables 5.2, 5.3 and 5.4 are the sensitivity analysis for specific parameters, when all the other parameters stay constant as the values assigned in the basic case.

Table 5.2 compares the results for different values of the maximum allowed expected shortage rate *Fs*. These results indicate that: if the optimal solution for Optimization Model I generates an expected shortage rate smaller than the maximum required shortage rate level, the optimal solution of Optimization Model II would be the same as that of Optimization Model I; and any increase in the maximum required shortage rate level does not change the optimal solution for Optimization Model II. However, if we decrease the maximum required shortage rate level, i.e. with tighter shortage rate requirement, the optimal solution for Model II may change and h^* may increase, so that more stocks could be stored in the buffer to reduce the expected total shortage per unit of time.

Tables 5.3 and 5.4 compare the results for different values of the additional cost M_e for the enhanced preventive maintenance. The results for Model I and Model II show that: the optimal number of periods in a cycle N^* will increase if the additional cost M_e is increased; while if the additional cost is decreased, N^* will decrease. The reason for this is that the enhanced preventive maintenance is carried out every N periods, so a bigger additional cost M_e would require a bigger N to reduce the additional cost per unit of time. From Table 5.1, we know that the maximum possible value for N, which satisfies all the constraints of the Model I and II, is 5. Tables 5.3 and 5.4 show that if the additional cost M_e is as high as 10, N should take its maximum possible value 5, so that the additional cost per unit of time M_e can be reduced. While if M_e is as low as 1.5, N* should take the value of 1, as in this case M_e is so low that it is economy to turn every preventive maintenance into perfect maintenance.

Table 5.1 The optimal solution for the Optimization Model I and II

	Max{N}	N*	T*	h*	$S(N^*,T^*,h^*)$	ΠRa_j	SAV_N	$SShort1_N$
Model I	5	3	14.91	19	3.083387	0.9648	0.9100	
Model II	5	3	14.91	19	3.083387	0.9648	0.9100	0.023820

Table 5.2 Comparative analysis for different required shortage rate level of Model II

	Fs	N*	T*	h*	$S(N^*,T^*,h^*)$	ΠRa_j	SAV_N	$SShort1_N$
	0.04	3	14.91	19	3.083387	0.9648	0.9100	0.023820
Basic case	0.03	3	14.91	19	3.083387	0.9648	0.9100	0.023820
	0.02	4	15.28	20	3.086509	0.9430	0.9100	0.017373
	0.01	2	14.27	21	3.119921	0.9856	0.9100	0.009605

Table 5.3 Comparative analysis for different enhanced maintenance costs of Model I

	M_e	N*	T*	h*	$S(N^*,T^*,h^*)$	ΠRa_j	SAV_N
	10.0	5	15.53	19	3.140217	0.9212	0.9100
Basic case	5.0	3	14.91	19	3.083387	0.9648	0.9100
	1.5	1	13.08	17	2.920753	1.0000	0.9100

Table 5.4 Comparative analysis for different enhanced maintenance costs of Model II

	M_e	N^*	T^*	h^*	$S(N^*,T^*,h^*)$	ΠRa_j	SAV_N	$SShort1_N$
	10.0	5	15.53	19	3.140217	0.9212	0.9100	0.025778
Basic case	5.0	3	14.91	19	3.083387	0.9648	0.9100	0.023820
	1.5	1	13.08	18	2.926467	1.0000	0.9100	0.023065

Chapter 6

Conclusions

In this thesis we first did a literature review on the research area of Maintenance, according to the different categories of previous papers on this subject. Though there are many topics or focuses for the area of maintenance, we only reviewed papers on four major topics: “Preventive Maintenance”, “Imperfect Maintenance”, “Maintenance Planning and Production”, and “Maintenance for Multi-unit Systems”. The system we study is associated with all these four topics.

Then we proposed our model which is a more general model based on the work of Chelbi and Rezg (2006), in order to study a special kind of the multi-unit systems. The system we study is a so-called Supply-Buffer-Demand production system, in which there is an inventory buffer between the supplying production unit and the demanding unit. In such a system the supplying unit receives preventive or corrective maintenance due to its random failures, so there are stocks stored in the buffer which are used to supply the demanding unit when the supplying unit undergoes maintenance.

We took into account the joint consideration of both the age dependent maintenance planning and the buffer inventory control in formulating the model. We developed the expressions for the expected total cost per unit of time, minimum required stationary availability, minimum required reliability, and maximum required shortage rate. Our strategy is to minimize the expected total cost per unit of time, while satisfying the constraints of minimum required stationary availability level, minimum required reliability level, and maximum required shortage rate level.

According to two different assumptions of the system, we formulated two analytical optimization models, in order to solve the optimal solutions for "preventive maintenance age increment", "number of periods in a cycle", and "the capacity of the buffer". These optimal solutions of the models determine the optimal maintenance and inventory policies for the system. We developed corresponding numerical algorithms to solve the optimal solutions. Numerical examples were raised to test the algorithms that we provided, and comparative analyses of the numerical examples were made to show how the optimal solutions are influenced by the variations of the input parameters.

Further research may challenge the idealistic assumptions that we made when formulating our model, so that a more general model could be obtained. Also, since our model applied the age dependent preventive maintenance into the system, further investigations could be applying other preventive maintenance strategies to the system.

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