ENTANGLEMENT AND GENERALIZED BELL INEQUALITIES

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Contents

Summary

Quantum entanglement, which has become a useful resource for quantum communication and computation, is a remarkable feature of quantum mechanics. Violation of Bell inequalities is one way to identify entanglement. The overall objective of this thesis is to develop Bell inequalities for multipartite systems of higher dimensions and explore their applications. To achieve this overall objective, different types of systems: N qubits, 2 QuNits, and 3 quNits are investigated. This thesis consists of three parts which are mainly based on the following papers.

1. J. L. Chen, C. F. Wu, L. C. Kwek and C. H. Oh, "Gisin's theorem for three qubits", Phys. Rev. Lett. **93**, 140407 (2004).

2. C. F. Wu, J. L. Chen, D. M. Tong, L. C. Kwek and C. H. Oh, "Quantum nonlocality of Heisenberg XX model with site-dependent coupling strength", J. Phys. A: Gen. Math. **37**, 11475 (2004).

3. C. F. Wu, J. L. Chen, L. C. Kwek, C. H. Oh and K. Xue, "Continuous multipartite entangled state in the Wigner representation and violation of Żukowski-Brukner inequality", Phys. Rev. A **71**, 022110 (2005).

4. J. L. Chen, C. F. Wu, L. C. Kwek, D. Kaszlikowski, M. Żukowski and C. H. Oh, "Multi-component Bell inequality and its violation for continuous-variable systems", Phys. Rev. A **71**, 032107 (2005).

5. C. F. Wu, J. L. Chen, L. C. Kwek and C. H. Oh, "A Bell inequality based on correlation functions for three qubits", Int. J. Quantum Inf. 3: 53-59 Suppl. S, NOV (2005).

6. C. F. Wu, J. L. Chen, L. C. Kwek and C. H. Oh, "Quantum nonlocality of N-qubit W state", Phys. Rev. A **73**, 012310 (2006).

7. J. L. Chen, C. F. Wu, L. C. Kwek and C. H. Oh, "Bell inequalities for three particles", e-print: quant-ph/0506230.

8. J. L. Chen, C. F. Wu, L. C. Kwek and C. H. Oh, "Violating Bell Inequalities Maximally for Two d-Dimensional Systems", e-print: quant-ph/0507227.

For N-qubit systems, quantum entanglement of quantum XX model is examined through its violation of the Żukowski-Brukner (ŻB) inequalities. There are two types of XX models considered in this thesis. It is shown that the quantum entanglement of these two models can be controlled by adjusting temperature and magnetic field strength. In addition to these discrete-variable systems, multipartite systems with continuous variables are found to possess quantum entanglement because the systems violate the Żukowski-Brukner inequalities under the Wigner representation. The degree of the violation increases with the increasing number of particles N.

For 2-quNit systems, a new type of Bell inequalities in terms of correlation functions are constructed based on multi-component measurements. Quantum entanglement of bipartite quantum systems of arbitrary dimensions with continuous variables is examined by violating the inequalities. In addition, maximal violation of the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequalities is investigated for non-maximally entangled states of 2 quNits. By extending the calculations to N=8000, it is shown that the degree of violation grows slowly with increasing dimensions. The results confirm that the violation is asymptotically constant when N goes to infinity. An approximate value of the constant is found numerically to be 3.9132. We construct a set of entangled states $|\Phi\rangle_{\text{app}}$ whose corresponding Bell quantities are closed to the actual maximal violations.

New Bell inequalities involving both probabilities and correlation functions for 3 qubits are found so that Gisin's theorem can indeed be generalized to 3 qubits. A new Bell inequality for tripartite systems of four dimensions is also formulated. Starting from this inequality, it is shown that another new Bell inequality with improved threshold visibility for three qubits can be constructed. The inequality is violated for any pure entangled state. It is also worth noting that the inequality is more resistant to noise than the known ones. In addition, an experimental setup for testing violation of local realism by using Bell inequalities for three qubits is proposed.

The results of the first two parts in this thesis play an important role in understanding quantum entanglement of quantum XX models and continuous-variable systems in general cases. Those entangled states presented in part II can potentially be useful for quantum cryptography as well as many other important fields of quantum information. The new Bell inequalities given in the third part (part III) provide a basis for testing quantum entanglement of composite quantum systems. By using the method given in part III, new Bell inequalities for 3 qubits are constructed. Given these new Bell inequalities, experimental observation of quantum nonlocality may be realized.

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Chapter 1

Introduction

Quantum information processing performs communication and computation beyond the limits achievable by the equivalent classical version. In this relatively young field, quantum entanglement, in which measurements on spatially separated quantum systems can influence instantaneously each other, plays a crucial role as a valuable resource. The importance of entanglement has been demonstrated in processes like quantum teleportation, quantum computation and quantum cryptography. Entanglement lies at the heart of the so called quantum nonlocality. Quantum nonlocality refers to the notion that a local realistic explanation of quantum mechanics is not possible and is quantitatively expressed by violation of Bell inequalities. Violation of a Bell inequality provides an important means to show that quantum entanglement cannot in any way be simulated by classical correlation. Violation of Bell inequalities which eliminates local realistic description is of importance in identifying the nonclassical resource for quantum information processing.

In the following section, we provide a brief review of quantum nonlocality, entanglement and Bell inequalities.

1.1 Historical Background

Quantum nonlocality is embodied in the EPR paradox, although it was not quite explicitly stated in the work. The EPR paradox also provides one of the most famous arguments concerning the foundation of quantum mechanics.

1.1.1 EPR Paradox

Einstein, Podolsky and Rosen (EPR) challenged the completeness of quantum mechanics in a 1935 paper [1]. In this paper, they did not doubt that quantum mechanics is correct, but they were dissatisfied with the description of a system by wave function in quantum mechanics. The EPR reasoning required completeness and locality: *"there is an element corresponding to each element of reality in a complete theory, and the real factual situation of the system A is independent of what is done with the system B, which is spatially separated from the former."* Element of physical reality was defined as *"If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity"*. As an example, they used the following wave function of a two-particle system to show that the description by wave function was incompatible with the completeness postulate,

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} e^{(i/\hbar)(x_1 - x_2 + x_0)p} dp. \tag{1.1}$$

According to EPR, a theory should meet the so called locality requirement in order to be considered a complete description. Suppose that a system consists of two particles 1 and 2 that are spatially separated. Then a measurement performed on particle 1 must not modify the description of particle 2. However, for the system described by the above wave function, after a measurement is performed on particle 1 and a corresponding outcome is obtained, the description of particle 2 does change. Hence, by EPR's argument, the description of a quantum system by wave function could not be considered complete.

In the following, we follow Ref. [1] to discuss the EPR argument in a little bit more detail . Consider the two-particle state (1.1). If one measures the momentum of the first particle and obtains a result p, the first particle will be in the corresponding eigenstate of momentum operator

$$u_p(x_1) = e^{(i/\hbar)px_1}. \tag{1.2}$$

Then Eq. (1.1) reads

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} \psi_p(x_2) u_p(x_1) dp, \tag{1.3}$$

with

$$\psi_p(x_2) = e^{-(i/\hbar)(x_2 - x_0)p}. \tag{1.4}$$

This tells us that the second particle will be in the eigenstate $\psi_p(x_2)$ of the momentum operator $\hat{P} = -i\hbar\frac{\partial}{\partial x_2}$ corresponding to the eigenvalue $-p$ after the momentum measurement performed on the first particle. If one measures the position of the first particle and obtains a result x, the first particle will be in the corresponding eigenstate of position operator

$$v_x(x_1) = \delta(x_1 - x). \tag{1.5}$$

Then Eq. (1.1) reads

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} \varphi_x(x_2) v_x(x_1) dx, \tag{1.6}$$

with

$$\varphi_x(x_2) = \int_{-\infty}^{\infty} e^{(i/\hbar)(x - x_2 + x_0)p} dp. \tag{1.7}$$

This tells us that the second particle will be in the eigenstate $\varphi_x(x_2)$ of the position operator $\hat{Q} = x_2$ corresponding to the eigenvalue $x + x_0$ after the position measurement performed on the first particle. Thus, the momentum and the position of the second particle are elements of reality. However, it is known that $\hat{P}$ and $\hat{Q}$ do not commute, or $\hat{Q}\hat{P} - \hat{P}\hat{Q} = i\hbar$. Therefore, it has been shown that it is possible for ψ_p and φ_x to be eigenfunctions of two noncommuting operators. So EPR questioned the completeness of quantum mechanics: "*We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete.*"

Einstein believed that there must be elements of reality that quantum mechanics ignores. Moreover, they argued that the incomplete description could be avoided by postulating the presence of hidden variables (HV). The variables are called hidden ones because no one knows how to incorporate them into the theory.

1.1.2 Entanglement

The EPR paradox drew attention to a phenomenon predicted by quantum mechanics known as quantum entanglement. One of the main differences between quantum physics and classical one is entanglement. The term entanglement was first introduced by Schrödinger [2], the work of Schrödinger was partially motivated by the EPR paper, who called it "Verschränkung". Central to the original seminal paper by Einstein, Podolsky and Rosen is an entangled state. There are two basic features that a situation typical for entanglement includes [3]. One is that individuals do not carry any information by themselves. The other feature is that the information in a total system is encoded in the joint properties of individuals.

To discuss basic features of entanglement, maximally entangled states of two-level systems (called qubits) are usually considered. One of the states can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 \otimes |1\rangle_2 - |1\rangle_1 \otimes |0\rangle_2), \tag{1.8}$$

where $|0\rangle_i, |1\rangle_i$ describe two orthogonal states of the i-th particle. This pure state is a coherent superposition of two product states with equal probability. Note that it cannot be written as the product of two terms, one describing the state of particle 1 and the other one describing the state of particle 2,

$$|\psi\rangle \neq |\psi\rangle_1 |\psi\rangle_2. \tag{1.9}$$

Unentangled states are those that can be described by two independent terms. The system is a simple composite of these two wave functions such that particle 1 is fully described by $|\psi\rangle_1$ and particle 2 by $|\psi\rangle_2$. But an entangled state cannot be factored into the product of two wave functions, and consequently cannot be thought of as a composite system in any classical sense. According to quantum mechanics, $|\psi\rangle$ contains all available information about the state of the qubits. If we write the state in the form of a density matrix

$$\begin{aligned}
\rho_{12} &= |\psi\rangle\langle\psi| \\
&= \frac{1}{2}(|0\rangle_{11}\langle 0| \otimes |1\rangle_{22}\langle 1| - |0\rangle_{11}\langle 1| \otimes |1\rangle_{22}\langle 0| \\
&- |1\rangle_{11}\langle 0| \otimes |0\rangle_{22}\langle 1| + |1\rangle_{11}\langle 1| \otimes |0\rangle_{22}\langle 0|),
\end{aligned} \tag{1.10}$$

and trace out one particle we obtain a density matrix describing the other particle,

$$\begin{aligned}
\rho_1 &= \mathrm{Tr}_2(|\psi\rangle\langle\psi|) = \frac{1}{2}(|0\rangle_{11}\langle 0| + |1\rangle_{11}\langle 1| = \frac{1}{2}\mathbf{1}_1 \\
\rho_2 &= \mathrm{Tr}_1(|\psi\rangle\langle\psi|) = \frac{1}{2}(|0\rangle_{22}\langle 0| + |1\rangle_{22}\langle 1| = \frac{1}{2}\mathbf{1}_2.
\end{aligned} \tag{1.11}$$

"Maximally entangled" means that when one traces over one of two particles to find the density operator ρ_i of the other particle, a density matrix $\rho_i = \frac{1}{2}\mathbf{1}_i$ describing a totally random state is obtained. This means that if particle 1 or 2 is measured, the result is completely random. Therefore, if one performs any local measurement on particle 1 or 2, no information about the state is generated.

Given the maximally entangled state in Eq. (1.8), it can be seen that information lives in the joint properties of individuals. Suppose there are two well separated observers (Alice and Bob) and a source which generates entangled pair of spin-1/2 particles described by the state $|\psi\rangle$ (1.8). One particle is sent to Alice and the other particle is sent to Bob. That there is no interaction possible between the particles is guaranteed by spatially separating Alice and Bob. Alice measures spin along $\hat{z}$ direction, or observable $\hat{s}_z$, on her particle in the computational basis $|0\rangle$, $|1\rangle$ of $\hat{s}_z$ and Bob does the same on his particle. With probability 1/2, Alice's measurement result is 0 and the state $|\psi\rangle$ reduces to state $|01\rangle$ after Alice's measurement and hence Bob's measurement result is 1. With probability 1/2, Alice's measurement result is 1 and the state $|\psi\rangle$ reduces to state $|10\rangle$ after Alice's measurement and hence Bob's measurement result is 0. Thus, Alice's result is 0 and 1 with probability 1/2 and so is Bob's result but their results are always opposite. Perfect correlations of the measurement outcomes occur. This is a situation typical for the entanglement. The consequence turns out to be so remarkable that Schrödinger was led to say that entanglement was not simply one of the many traits "but rather *the* characteristic trait of quantum mechanics".

The emergence of quantum computation and quantum information has revived the importance of quantum entanglement. It was used to produce unusual effects such as quantum teleportation [4], superdense coding [5] and quantum cryptography [6], etc. It should be noted that communication using entanglement is not superluminal. Superluminal communication [7] utilized the notion that the wavefunction reduced instantaneously over arbitrarily large distance after a measurement is performed. This would seem to suggest that signals can be sent from one end to the other end faster than the speed of light. According to the special theory of relativity, however, nothing can move with a velocity that exceeds the speed of light. It is therefore important to understand the instantaneously wavefunction reduction and the impossibility of superluminal communication.

Superluminal communication is not possible since one cannot actually use this instantaneous wavefunction reduction to transmit real messages from one side to another space separated side. For example, suppose that Alice and Bob need to decide at which time to meet – either in the morning or in the afternoon. In order to correlate their decisions, they might agree to measure spins of entangled particles emitted from a EPR source in a prearranged direction (say $\hat{z}$ direction). Alice will decide the time and then try to communicate her decision to Bob. A prior agreement is that the measurement of the spin of A along $\hat{z}$ direction would mean that the meeting is to be in the morning, whereas no such measurement along $\hat{z}$ would mean that afternoon is to be the meeting time. Thus, Alice either makes the measurement or she does not to transmit the decision. Corresponding to Alice's action, B will have definite spin along $\hat{z}$ direction and the time is morning, or otherwise the time is afternoon. However, Bob cannot 'read' this message although he can observe the particle B because there is no way Bob could know whether or not the spin of B was definite. A crucial point is that Bob cannot tell whether the spin of his particle was definite or not prior to his measurement. That is to say that Bob cannot discover whether his particle had already definite spin along the chosen direction due to Alice's measurement, or whether it was still in the original entangled state.

Entanglement lies at the heart of the so called quantum nonlocality. Quantum nonlocality is the fact that a local realistic explanation of quantum mechanics is not possible. The impossibility can be quantitatively expressed by violation of Bell inequalities. In next section, the Bell theorem is reviewed.

1.1.3 The Bell Theorem

For a long time, EPR paradox remained an inexplicable argument on the foundation of quantum mechanics. In 1964, J. Bell [8] put forward a proposal, the so called Bell theorem to solve the EPR paradox. He showed that the assumption of local hidden variables was not just a thought view, it resulted in constraints on measurement outcomes. In a local realistic model, the measurement results are determined by hidden variables, and the results obtained at one side are independent of any measurements carried out at the other space-separated side. Local realism imposes variable constraints on the statistical measurements on two or more physically separated systems. These constraints, called Bell inequalities, can be violated by the predictions of quantum mechanics and this is basis of the "Bell theorem". Thus, the Bell inequalities made it possible for the first time to eliminate local realistic description of quantum mechanics.

We next review the original Bell inequality given by J. Bell. In Ref. [8], the hidden variables are denoted by a parameter λ, and the system studied consists of two separated particles 1 and 2. Assume that the separated particles 1 and 2 have spin-1/2 and are in the quantum mechanical singlet state. The outcome of a measurement of spin components along direction $\hat{a}$ on particle 1 is

$$A(\hat{a}, \lambda) = \pm 1, \tag{1.12}$$

and the outcome of a measurement of spin components along direction $\hat{b}$ on particle 2 is

$$B(\hat{b}, \lambda) = \pm 1, \tag{1.13}$$

where $+1$ and -1 represent spin up and down, respectively. According to the locality requirement, the result of a measurement on A does not depend on what is measured on particle 2, and the result of a measurement on B does not depend on what is measured on particle 1. For particles in the singlet state, the spins of the two particles are anti-correlated. That is to say that if directions $\hat{a}$ and $\hat{b}$ are chosen to be the same then the outcomes of the measurements of the corresponding spin components of particles 1 and 2 are different, or the product of their measurement outcomes is -1. Under the hidden-variable theory, in order to agree with this quantum mechanical correlation, there must be a probability distribution ρ over the hidden variable λ such that

$$\int d\lambda \rho(\lambda) A(\hat{a}, \lambda) B(\hat{a}, \lambda) = -1, \tag{1.14}$$

and the distribution function ρ satisfies the normalization condition $\int d\lambda \rho(\lambda) = 1$. But Eq. (1.14) is guaranteed only if

$$A(\hat{a}, \lambda) = -B(\hat{a}, \lambda). \tag{1.15}$$

The correlation function of measurement of the spin components along directions $\hat{a}$ and $\hat{b}$ can be written as

$$E(\hat{a},\hat{b}) = \int d\lambda \rho(\lambda) A(\hat{a},\lambda) B(\hat{b},\lambda). \tag{1.16}$$

Introducing a third direction $\hat{c}$, and taking the difference

$$\begin{aligned} E(\hat{a},\hat{b}) - E(\hat{a},\hat{c}) &= \int d\lambda \rho(\lambda) [A(\hat{a},\lambda) B(\hat{b},\lambda) - A(\hat{a},\lambda) B(\hat{c},\lambda)] \\ &= -\int d\lambda \rho(\lambda) [A(\hat{a},\lambda) A(\hat{b},\lambda) - A(\hat{a},\lambda) A(\hat{c},\lambda)] \end{aligned} \tag{1.17}$$

by using Eq. (1.15). This can be factored to read

$$E(\hat{a},\hat{b}) - E(\hat{a},\hat{c}) = -\int d\lambda \rho(\lambda) A(\hat{a},\lambda) A(\hat{b},\lambda) [1 - A(\hat{b},\lambda) A(\hat{c},\lambda)] \tag{1.18}$$

because

$$[A(\hat{b},\lambda)]^2 = 1 \tag{1.19}$$

since $A(\hat{b},\lambda)$ is either plus or minus one. Similary, the factor $A(\hat{a},\lambda)A(\hat{b},\lambda)$ is either plus or minus one, and so is certainly less than or equal to plus one. Thus if one drops it from the the right hand side of Eq. (1.18) and takes absolute values, one obtains

$$|E(\hat{a},\hat{b}) - E(\hat{a},\hat{c})| \leq |\int d\lambda \rho(\lambda) [1 - A(\hat{b},\lambda) A(\hat{c},\lambda)]|. \tag{1.20}$$

By Eqs. (1.15, 1.16), the above equation gives

$$\begin{aligned} |E(\hat{a},\hat{b}) - E(\hat{a},\hat{c})| \quad &\leq 1 + \int d\lambda \rho(\lambda) A(\hat{b},\lambda) B(\hat{c},\lambda) \\ &\leq 1 + E(\hat{b},\hat{c}) \end{aligned} \tag{1.21}$$

which is the Bell theorem.

Quantum mechanically, the expectation value of the product of an arbitrary spin component of particle 1 and an arbitrary spin component of particle 2 measured on the singlet state is

$$\langle \vec{\sigma} \cdot \hat{a} \otimes \vec{\sigma} \cdot \hat{b} \rangle = -\hat{a} \cdot \hat{b} = -\cos\theta_{ab}. \tag{1.22}$$

The conflict between the local hidden-variable inequality (1.21) and the quantum mechanical expectation value (1.22) is obvious. For example, if choosing settings in which $\hat{b}$ makes an angle of $\pi/3$ to $\hat{a}$, $\hat{c}$ makes an angle of $\pi/3$ to $\hat{b}$, and $\hat{c}$ makes an angle of $2\pi/3$ to $\hat{a}$, then $\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \frac{1}{2}$ and $\hat{a}\cdot\hat{c} = -\frac{1}{2}$, the left hand side of inequality (1.21) is

$$|E(\hat{a},\hat{b}) - E(\hat{a},\hat{c})| = |(-\frac{1}{2}) - (\frac{1}{2})| = 1, \tag{1.23}$$

and the right hand side of inequality (1.21) is

$$1 + E(\hat{b},\hat{c}) = 1 + (-\frac{1}{2}) = \frac{1}{2}, \tag{1.24}$$

which do not obey the inequality (1.21).

The original Bell inequality is not exactly suitable for realistic experimental verification. However, there have been many attempts to formulate other versions of Bell inequalities that are more amenable to experimental tests. One of the most common form of Bell inequalities involves the study of correlation between two maximally entangled spin-1/2 particles. This inequality is known as the Clauser-Horne-Shimony-Holt (CHSH) inequality [9]. Here, we follow Bell's proof [10] to show the new version of the Bell Theorem. For the system discussed above, the measurement result $A(\hat{a},\lambda)$ or $B(\hat{b},\lambda)$ of spin components along direction $\hat{a}$ or $\hat{b}$ on particle 1 or 2 can take values ± 1 where $+1$ and -1 represent spin up and down, respectively. Thus the average values of these quantities satisfy

$$\begin{aligned} &|A(\hat{a},\lambda)| \leq 1, \\ &|B(\hat{b},\lambda)| \leq 1. \end{aligned} \tag{1.25}$$

The correlation function in the hidden variable theory can also be written as shown in Eq. (1.16). Then the following quantity can be written

$$\begin{aligned} E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') &= \int \{A(\hat{a},\lambda)B(\hat{b},\lambda) - A(\hat{a},\lambda)B(\hat{b}',\lambda)\}\rho(\lambda)d\lambda \\ &= \int A(\hat{a},\lambda)B(\hat{b},\lambda)\{1 \pm A(\hat{a}',\lambda)B(\hat{b}',\lambda)\}\rho(\lambda)d\lambda \\ &- \int A(\hat{a},\lambda)B(\hat{b}',\lambda)\{1 \pm A(\hat{a}',\lambda)B(\hat{b},\lambda)\}\rho(\lambda)d\lambda, \end{aligned} \tag{1.26}$$

where $\hat{a}'$ and $\hat{b}'$ are the alternative settings for observers.

By using Eq. (1.25), Eq. (1.26) reads

$$\begin{aligned}|E(\hat{a},\hat{b})-E(\hat{a},\hat{b}')| &\leq \int\{1\pm A(\hat{a}',\lambda)B(\hat{b}',\lambda)\}\rho(\lambda)d\lambda \\ &+ \int\{1\pm A(\hat{a}',\lambda)B(\hat{b},\lambda)\}\rho(\lambda)d\lambda,\end{aligned} \tag{1.27}$$

which can be written as

$$|E(\hat{a},\hat{b})-E(\hat{a},\hat{b}')| \leq \pm\{E(\hat{a}',\hat{b}')+E(\hat{a}',\hat{b})\}+2. \tag{1.28}$$

The CHSH inequality is then obtained, which must be satisfied by any local hidden variable theory,

$$-2 \leq E(\hat{a},\hat{b})-E(\hat{a},\hat{b}')+E(\hat{a}',\hat{b}')+E(\hat{a}',\hat{b}) \leq 2. \tag{1.29}$$

Quantum mechanically, $E(\hat{a},\hat{b})=-\hat{a}\cdot\hat{b}$. For appropriate angle settings in which $\hat{a}$ makes an angle of $\frac{\pi}{4}$ to $\hat{b}$ and an angle of $-\frac{\pi}{4}$ to $\hat{b}'$, $\hat{a}'$ makes an angle of $\frac{\pi}{4}$ to $\hat{b}$ and an angle of $\frac{3\pi}{4}$ to $\hat{b}'$, the above inequality (1.29) is violated. Cirel'son [11] first proved that the absolute value of the combination of correlation functions in Eq. (1.29) is bounded by $2\sqrt{2}$ for any quantum mechanical prediction, instead of the value 2 imposed by the local realism:

$$-2\sqrt{2} \leq E_{11}+E_{12}+E_{21}-E_{22} \leq 2\sqrt{2}, \tag{1.30}$$

which implies that quantum mechanics violates the CHSH inequality. If such a violation is observed experimentally, one can draw a conclusion that local realistic description does not exist since no local hidden-variable theory can reproduce the observed violation of the Bell inequality. This means that one has to abandon the concept of local realism.

However, such a perfect correlation cannot be achieved experimentally. If one considers the inefficiency of detector, one has to modify the correlation slightly to account for the imperfections, which are characterized by the quantum efficiency of detectors η $(0 \leq \eta \leq 1)$. For ideal detectors, $\eta = 1$ and the correlation is perfect. For non-ideal detectors, take two qubits for example, the correlation is modified by $E'_{ij} = \frac{\eta}{2-\eta}E_{ij}$ and the CHSH inequality is violated if $\eta > 2\sqrt{2}-2$ [12].

In 1951, Bohm [13] recrafted the EPR paradox into a clear form. In recent years his version of the EPR experiment has been realized in the laboratory. The EPR experiment is no longer a thought experiment but can be realistically carried out in the laboratory. The first experimental test of Bell inequality was performed by Freedman and Clauser [14] with photons from atomic cascade decays. But no precaution was taken to guarantee that the interaction between the two polarizing settings detection stations is not possible [15, 16] in the experiment. It is worth noting that Aspect and co-workers [17] published three landmark papers on tests of nonexistence of local hidden-variable theories via violation of the Bell theorem. In their experiments, possible interaction between the source and the analyzer was prevented by using fast switchings of the analyzer position. To this date, many experiments have been performed by using entangled photon pairs produced from parametric down conversion and other means and the violation of Bell inequalities is now fairly well established. Nature has shown us that the local realistic description is fundamentally wrong. For this reason, the Bell theorem, which eliminates local realism, is really a remarkable discovery of science.

The reason for the quantum violation of Bell inequalities is due to entanglement, the characteristic feature of quantum mechanics. It was shown [18] that any entangled pure state of two qubits violates some Bell inequality. For two-qubit pure states, "entangled" is equivalent to "violation of Bell inequality". For bipartite mixed states or multi-partite states, however, the situations are more complicated. After the original Bell inequality and the famous CHSH inequality, different types of Bell inequalities have been developed. In the next section, these Bell inequalities will be reviewed to show there are some problems that deserve further discussion.

1.2 Motivation and Goals

The theoretical foundation that no local and realistic theory can be compatible with all predictions of quantum mechanics was first laid down by Bell [8] through the violation of Bell inequalities. Since then, violation of Bell inequalities has also become an effective method to detect entanglement, a useful resource in quantum information. In this section, some formulation of Bell inequalities for different systems will be reviewed briefly. These previous investigations have directly influenced the studies to be presented in this thesis.

1.2.1 Motivation

Identifying and characterizing entanglement involving Bell inequalities are important issues in quantum information theory. The original Bell inequality [8] and the subsequent famous CHSH inequality [9] were formulated for the simplest composite quantum system, i.e. a system of two particles each in a two-dimensional Hilbert space (called qubits). For the purpose of experimental verification, the more suitable candidate is the CHSH inequality. The CHSH inequality was derived under such a condition: two local settings are provided for each observer. Take for example Alice who has local settings $\hat{a}$ and $\hat{a}'$.

Up to now, there are two different strategies to develop Bell inequalities for complex composite quantum systems, or multi-particles in high-dimensional Hilbert space. The first strategy is to limit the search to two local settings for each observer [8, 9]. The second strategy is to extend the search to three or more local settings per site [19]. Based on the former method, two kinds of Bell inequalities have been found that generalize the CHSH inequality to systems of multi-particles or higher dimensions.

One generalization of the CHSH inequality is Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequalities given by Collins et al [20] for two particles of arbitrarily high dimensions (2 quNits). The CGLMP inequalities were derived in terms of joint probabilities. The CGLMP inequalities offer a possibility to test entanglement of 2 quNits for both discrete-variable systems and continuous-variable systems. Although most protocols in quantum information processing were developed for a discrete-variable quantum system, similar protocols have also been carried out for a quantum system with continuous variables [21] recently. From the experimental perspective, continuous-variable states {for example, nondegenerate optical parametric amplification (NOPA) state [22]} are easier to generate than discrete entangled states. Therefore, identifying entanglement for a quantum system with continuous variables is necessary for realization of quantum information protocols. In 2002, Chen et al [23] showed that a 2-quNit continuous-variable system is quantum entangled by its violation of the CHSH inequality. So far there is no effort that contributes to the identification of entanglement for continuous-variable systems based on Bell inequalities for a general case, or a 2-quNit system by using nondichotomic observables.

The other generalization of the CHSH inequality is for N particles of two dimensions (N qubits). The work was done by two independent teams [24]. One is led by Żukowski and Brukner, and the other by Werner. It was shown that there exist general Bell inequalities [called Żukowski-Brukner (ŻB) inequalities in this thesis] which are sufficient and necessary conditions for N-qubit systems to be describable in a local and realistic theory. The most common model of N-qubit quantum systems is the Heisenberg model. The Heisenberg model has been shown to be a potential model for spin-spin interaction in a solid state quantum computer [25, 26, 27, 28]. Although an interesting type of entanglement, thermal entanglement, has been studied in the context of the Heisenberg models by using entanglement measure [29, 30, 31], no study has been given to identify entanglement of thermal states in a system of interaction spins by violation of the ŻB inequalities.

The ŻB inequalities also provide a useful tool to identify entanglement for multipartite systems by Wigner function. Bell [10] argued that the original EPR wave function does not violate local realism becasue its joint Wigner distribution function is positive. In a recent work, however, Banaszek and Wódkiewicz [32] considered parity measurement and interpreted the Wigner function as a correlation function. They then showed that the original EPR state and the two-mode squeezed vacuum state violate the CHSH inequality. Thus, it was shown that the positive definite Wigner function of the EPR state provides direct evidence of the nonlocality exhibited by this state. But for a multi-partite quantum system in 2-level Hilbert space, no work has been done to identify its entanglement. Given the general Bell inequalities for N-qubit states, it is possible to extend the test in the Wigner presentation to a multi-partite quantum system.

For N-qubit Bell inequalities, there remains another problem. That is "do all pure entangled states violate Bell inequalities for correlation function"? If Bell inequalities are violated by all pure entangled states, these Bell inequalities can be used to characterize entanglement. There are several important recent developments in characterizing entanglement based on Bell inequalities. In 1991, Gisin [18] demonstrated that every pure bipartite entangled state in two dimensions violates the CHSH inequality. This is known as Gisin's theorem. One year later, Bell inequalities for N qubits were developed by Mermin-Ardehali-Belinskii-Klyshko (MABK) [33, 34, 35]. However, Scarani and Gisin [36] noticed that there exist pure states of N-qubit that do not violate any of the inequalities. These states are the generalized Greenberger-Horne-Zeilinger (GHZ) [37] states given by

$$|\psi\rangle_{\rm GHZ} = \cos\xi|0\cdots0\rangle + \sin\xi|1\cdots1\rangle \tag{1.31}$$

with $0 \le \xi \le \pi/4$. The GHZ states [37] are for $\xi = \pi/4$. In Ref. [36], Scarani and Gisin noticed that for $\sin 2\xi \le 1/\sqrt{2^{N-1}}$ the states (1.31) do not violate the MABK inequalities. Based on the results, Scarani and Gisin were prompted to write that "this analysis suggests that MK (MABK [38]) inequalities, and more generally the family of Bell's inequalities with two observables per qubit, may not be the 'natural' generalizations of the CHSH inequality to more than two qubits" [36]. Soon after, Żukowski-Brukner and Werner [24] independently found general correlation-Bell inequalities (the ŻB inequalities) for N qubits. Using the ŻB inequalities, Żukowski *et al* in Ref. [38] showed that (a) For N =even, although the generalized GHZ states do not violate the MABK inequalities, they violate the ŻB inequalities and (b) For N =odd and $\sin 2\xi \le 1/\sqrt{2^{N-1}}$, the generalized GHZ states satisfy all known Bell inequalities involving correlation functions. Thus it seems that Gisin's theorem is invalid for N (odd numbers) qubits.

For M ($M > 2$) entangled N ($N > 2$)-dimensional quantum systems, the formulation of the corresponding Bell inequalities is still an open question. Only recently, the problem has been solved partly in the case of three three-dimensional particles (3 qutrits) in [39]. The authors developed a coincidence Bell inequality in terms of probabilities, which imposes a necessary condition on the existence of a local and realistic description. But for M ($M > 2$) entangled N ($N > 3$)-dimensional quantum systems, no Bell inequality in terms of either probabilities or correlation functions has so far been constructed.

Violation of Bell inequalities is a powerful tool to identify and characterize entanglement. It will be interesting to generalize the criteria for violation of local realism to the case of multi-partite high dimensional systems or to the case of more measurement choices for each observer than two. One can expect interesting practical applications of such generalizations.

1.2.2 Objectives and Significance

The primary purpose of this thesis is to develop Bell inequalities for multi-particles in high-dimensional Hilbert space based on the assumption of two local settings per site, and explore applications of these inequalities. More specifically, the aims of this thesis are

1. To determine entanglement in the context of quantum XX model, which is a N-qubit system, by using the Żukowski-Brukner inequalities. The identification of entanglement for a multi-partite system in the Wigner representation will be also discussed.

2. To investigate maximal violation of the Collins-Gisin-Linden-Massar-Popescu inequalities. An approximate value for the limit of the maximal violation when dimension goes to infinity is found. A set of entangled states $|\Phi\rangle_{\rm app}$ whose corresponding Bell quantities are closed to the actual maximal violations are constructed.

3. To derive a new set of Bell inequalities involving correlation functions for a quantum system of two particles in arbitrary high dimensional Hilbert space, namely, two quNits. Then the inequalities are used to determine entanglement of a continuous-variable system for a general case.

4. To develop new Bell inequalities for a 3-qubit system so as to determine whether Gisin's theorem can be generalized to 3 qubits.

5. To develop a new Bell inequality for systems of 3 particles in four dimensional Hilbert space. Start from the Bell inequality, a 3-qubit Bell inequality with improved visibility can be constructed. Based on the new 3-qubit inequality, a proposed experiment to test quantum nonlocality will be presented.

The present research may provide some insight in understanding entanglement of continuous-variable systems for more general cases: both 2 quNits and N qubits.

In addition, the study on quantum XX model may provide a method for identifying quantum entanglement of Heisenberg models. Those entangled states of 2 quNits presented in part II can potentially be useful for quantum information processing because of their high resistance to noise. The work in this thesis could also provide an exciting possibility to test violation of local realism for 3 four-level entangled states. It is also worth mentioning that the results in this thesis may be used to develop a simple procedure for generating 3-qubit Bell inequalities in further research. In other words, new Bell inequalities with improved visibility for a system of 3 particles in two-level space could be constructed using the method given in this thesis. Moreover, testing quantum nonlocality of three qubits by using Bell inequalities may be possible in our proposed experiment.

While methods do not need to be restricted the number of measurement choices for each observer to two, the method used in our study is limited to two local settings per site in the generalizations of the CHSH inequality to multi-partite systems in high dimensional Hilbert space. This thesis concentrates on the method of constructing Bell inequalities and explores their interesting applications. Moreover, the investigations in this thesis are restricted to quantum systems of pure states; research on mixed states, even though it is important, will not be considered here.

1.3 Organization of the Thesis

This thesis is devoted to the studies of entanglement and Bell inequalities in multi-partite systems. We study different systems from simple ones to complex ones. For N qubits, the Żukowski-Brukner inequalities are used as a tool to determine entanglement of Heisenberg model in Chapter 2. The last section in Chapter 2 relates the Żukowski-Brukner inequalities to entanglement identification of a multi-partite system in the Wigner representation. Bell inequalities for 2 particles in N-dimensional space will be discussed in Chapter 3. Maximal violation of local realism for 2 quNits is studied by using the Collins-Gisin-Linden-Massar-Popescu inequalities. In order to construct a new set of Bell inequalities involving correlation functions for 2 quNits, multi-component correlation functions have been introduced in this chapter. Then entanglement of a bipartite quantum system in high level with continuous variables is examined based on the new Bell inequalities. Chapter 4 is concerned with more complex quantum systems, or systems of 3 particles in high dimensional Hilbert space. New Bell inequalities for 3 particles in two and four-level systems are constructed. Especially for 3 qubits, Bell inequalities involving both probabilities and correlation functions are developed to show that Gisin's theorem can be generalized to 3 qubits. In addition, an observation on inequalities for 2 particles and 3 particles offers one method to construct new Bell inequalities with improved visibility for 3 qubits. In the last section of this chapter, an experimental setup for testing quantum nonlocality is proposed. All the problems investigated in this thesis are summarized in Chapter 5. Suggestions for further work are also presented in this chapter.

Chapter 2

N-Qubit Bell Inequalities and Applications

2.1 Bell Inequalities for N-qubit Systems

Entanglement and Bell inequalities concerning the non-existence of a local realism in quantum mechanics underly a fundamental role in quantum mechanics. Bell [8] proposed an inequality that could rule out the hidden variable description of quantum mechanics in 1964. Several variants of Bell inequalities have since been derived for two-body correlation functions to study the existence of local realism [24, 33, 34, 35]. A set of Bell inequalities for a state of N spin-1/2 particles were first derived by Mermin [33]. Following Mermin's work, Ardehali [34], Belinskii and Klyshko [35] have also developed a series of Bell inequalities for N qubits. Such inequalities are now known as Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities in Ref. [38]. N qubits refers to a quantum system of N particles each in two dimensional Hilbert space. Here the N-dimensional Hilbert space is simply a N-dimensional linear vector space. Recently more general Bell inequalities, called Żukowski-Brukner (ŻB) inequalities, for N qubits were proposed. The work is done by two independent teams [24]. The first one is led by Żukowski; the other is by Werner. One obtains, as corollaries from these generalized inequalities, the CHSH inequality [9] for two particles and the MABK inequalities for N particles [33, 34, 35].

The ŻB inequalities [24] were derived as follows. Consider N observers and suppose that they can each measure two dichotomic observables, parameterized by $\vec{n}_1$ and $\vec{n}_2$. The outcomes of j-th observer's measurement on the observable defined by $\vec{n}_1$ and $\vec{n}_2$ are represented by $A_j(\hat{n}_1)$ and $A_j(\hat{n}_2)$. Each outcome can take values

+1 or -1 under the assumption of local realism. The correlation function of the measurements performed by the N observers is the average over many runs of the experiment [24]

$$E(k_1, ..., k_N) = \langle \prod_{j=1}^{N} A_j(\vec{n}_{k_j}) \rangle. \tag{2.1}$$

For the predetermined results, one can construct the following identity [24]:

$$\sum_{s_1,...s_N=\pm1} S(s_1, ..., s_N) \prod_{j=1}^{N} [A_j(\vec{n}_1) + s_j A_j(\vec{n}_2)] = \pm 2^N, \tag{2.2}$$

where $S(s_1, ..., s_N)$ is an arbitrary function of $s_1, ..., s_N \in \{-1, 1\}$, and it can only take values ±1. There are totally 2^{2^N} different functions $S(s_1, ..., s_N)$. This identity can be proved in the following argument. For each observer j one has $|A_j(\vec{n}_1) + A_j(\vec{n}_2)| = 0/2$ and $|A_j(\vec{n}_1) - A_j(\vec{n}_2)| = 2/0$ because $A_j(\vec{n}_i) = \pm1$. So the product $\prod_{j=1}^{N}[A_j(\vec{n}_1) + s_j A_j(\vec{n}_2)]$ is nonzero only for one sign sequence, and the nonzero value of the product is $\pm 2^N$. One can use the correlation function defined in Eq. (2.1) to express the left hand side of the identity and obtain the following set of Bell inequalities [24]:

$$| \sum_{s_1,...s_N=\pm1} S(s_1, ..., s_N) \sum_{k_1,...k_N=1,2} s_1^{k_1-1} ... s_n^{k_N-1} E(k_1, ...k_N)| \leq 2^N. \tag{2.3}$$

There are a set of 2^{2^N} Bell inequalities for the correlation functions. Some of them are trivial and they are not violated by quantum mechanics. The function $S(s_1, ..., s_N)$ can be chosen properly to give nontrivial inequalities.

The ŻB inequalities for N qubits offer a possibility to test entanglement for both discrete-variable systems and continuous-variable systems. Two interesting applications of the ŻB inequalities will be described in the following sections.

2.2 Quantum Nonlocality of Quantum XX Model with Site Dependent Coupling Strength

The first application of the Żukowski-Brukner inequalities to be discussed is on quantum XX model[1], which is the most common model of a N-qubit system. The quantum XX model provides a simple model for the study of magnetic phenomena

[1]This work was published, see [2] in the publication list in Appendix A.

with nearest neighbor interactions. Recently there have been some studies of the model on the implementation of quantum processing on solid state devices. An interesting type of entanglement, thermal entanglement, was studied in the context of the Heisenberg XXX [29], XX [30], and XXZ [31] models. The simple Heisenberg model has been shown to have a potential as a model for spin-spin interaction in a solid state quantum computer [25]. It has been partially realized in quantum dots [25], nuclear spins [26], and optical lattices [27]. In a recent work, Imamoglu *et al* [28] realized quantum information processing using quantum dot spins and cavity QED, and obtained an effective interaction Hamiltonian based on the XY spin chain between two quantum dots. The effective Hamiltonian was shown to be capable of constructing the Controlled-Not gate [28]. The XY Hamiltonian is given by

$$H = \sum_{n=1}^{N} (J_1 S_n^x S_{n+1}^x + J_2 S_n^y S_{n+1}^y), \tag{2.4}$$

where $S^i = \sigma^i/2(i = x, y, z)$ and σ^i are Pauli operators. When $J_1 = J_2$, the XY model is known as the XX model. In the XY model, the interaction strength between neighboring sites is usually assumed to be independent of the sites. In most solid state models, however, the inter-site coupling strength is site dependent. Here we consider a special quantum XX model in which the interaction strength is assumed in a particular site dependent form. Interestingly, such a Hamiltonian has been shown to be useful for perfect state transfer in quantum spin networks [40].

We first examine eigenstates of the special XX model. Correlation functions needed to test entanglement can be constructed by using these eigenstate solutions.

2.2.1 Solution of the Special XX Model

The special Heisenberg XX model is described by the Hamiltonian

$$\begin{aligned} H &= 2\sum_{n=1}^{N-1} J_{n,n+1}(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) \\ &= \sum_{n=1}^{N-1} J_{n,n+1}(\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+), \end{aligned} \tag{2.5}$$

where $J_{n,n+1} = \sqrt{n(N-n)}$ is the coupling strength between lattices n and $n+1$. Obviously, the Hamiltonian H describes a nearest neighbor interaction open spin chain. For such a coupling, the Hamiltonian is linked to angular momentum and perfect state transfer can be realized in the quantum spin networks [40]. The Hamiltonian H possesses 2^N complete and orthonormal eigenstates, which span the Hilbert space of H. The Hilbert space of H can be divided into $N+1$ subspaces. In j-th subspace, there are $C_N^{(j-1)} = \frac{N!}{(j-1)!(N-j+1)!}$ states. This can be seen that the Hamiltonian H commuts with the operator of the total z-component of the spin, $\sigma_{tot}^z = \sum_{j=1}^N \sigma_j^z$, the total z-component of spin is conserved in each subspace. Thus, the j-th subspace consists of eigenstates with $(j-1)$ spins up and the others down and hence, there are C_N^{j-1} states.

The first subspace has only one (because $C_N^0 = 1$) eigenvector with zero value of eigenvalue, i.e.,

$$|\psi_0\rangle = |00\cdots 0\rangle, \qquad H|\psi_0\rangle = E_0|\psi_0\rangle, \quad E_0 = 0, \tag{2.6}$$

where we have denoted $|0\rangle$ as the state of spin-down $|\downarrow\rangle$, and $|1\rangle$ as the state of spin-up $|\uparrow\rangle$. The (vacuum) state $|\psi_0\rangle$ is a state with all spins down.

The second subspace contains N (because $C_N^1 = N$) first excitation states, which have the following forms

$$|\psi_1\rangle^{(k)} = \sum_{m=1}^{N} a_k(m)\phi(m), \quad H|\psi_1\rangle^{(k)} = E_1^{(k)}|\psi_1\rangle^{(k)}, \quad k = 1, 2, ..., N \tag{2.7}$$

where

$$\phi(m) = |00\cdots 1_m \cdots 0\rangle, \tag{2.8}$$

represents a state in which only the spin on the m-th lattice site is up.

The third subspace contains $C_N^2 = N(N-1)/2$ second excitation states, which have the following forms

$$\begin{aligned} |\psi_2\rangle^{(k)} &= \sum_{m1<m2}^{N} a_k(m1, m2)\phi(m1, m2), \\ H|\psi_2\rangle^{(k)} &= E_2^{(k)}|\psi_2\rangle^{(k)}, \quad k = 1, 2, ..., N(N-1)/2 \end{aligned} \tag{2.9}$$

where

$$\phi(m1, m2) = |\cdots 1_{m1} \cdots 1_{m2} \cdots\rangle \tag{2.10}$$

represents a state in which only the spins on the $m1$-th and $m2$-th lattice sites are up.

The fourth subspace contains $C_N^3 = N(N-1)(N-2)/6$ third excitation states, which have the following forms

$$|\psi_3\rangle^{(k)} = \sum_{m1<m2<m3}^{N} a_k(m1,m2,m3)\phi(m1,m2,m3),$$
$$H|\psi_3\rangle^{(k)} = E_3^{(k)}|\psi_3\rangle^{(k)}, \quad k = 1,2,...,N(N-1)(N-2)/6 \tag{2.11}$$

where

$$\phi(m1,m2,m3) = |\cdots 1_{m1}\cdots 1_{m2}\cdots 1_{m3}\cdots\rangle \tag{2.12}$$

represents a state in which only the spins on the $m1$-th, $m2$-th and $m3$-th lattice sites are up. Similarly the $(j+1)$-th subspace contains $C_N^j = \frac{N!}{j!(N-j)!}$ states, and the last subspace, $j = N$, contains only one state with all spins up.

For the special Heisenberg XX model, the site dependent coupling strength is of importance in perfect state transfer. In Ref. [40], the authors restricted their discussion to the first excitation states, or the second subspace. When restricting the Hamiltonian H to the second subspace, the Hamiltonian H in the matrix representation corresponds to the following $N \times N$ (because of $C_N^1 = N$) matrix

$$H = \frac{1}{2}\begin{pmatrix} 0 & J_{12} & 0 & 0 & \cdots & 0 \\ J_{12} & 0 & J_{23} & 0 & \cdots & 0 \\ 0 & J_{23} & 0 & J_{34} & \cdots & 0 \\ 0 & 0 & J_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & J_{N-1,N} \\ 0 & 0 & 0 & 0 & J_{N-1,N} & 0 \end{pmatrix}. \tag{2.13}$$

Equation (2.13) comes from the following consideration. As shown in Eq. (2.7), the first excitation states have the following form

$$|\psi_1\rangle^{(k)} = \sum_{m=1}^{N} a_k(m)|00\cdots 1_m\cdots 0\rangle, \quad k = 1,2,...,N.$$

One can associate N column vectors corresponding to N states $|100\cdots 0\rangle$, $|010\cdots 0\rangle$, $\cdots$, $|0\cdots 01\rangle$ respectively. Namely

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow |100\cdots 0\rangle, \quad \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow |01\cdots 0\rangle, \quad \cdots, \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \longrightarrow |00\cdots 01\rangle. \tag{2.14}$$

Therefore, the first excitation state becomes

$$|\psi_1\rangle^{(k)} = \begin{pmatrix} a_k(1) \\ a_k(2) \\ \vdots \\ a_k(N) \end{pmatrix} \tag{2.15}$$

with corresponding eigenvalues $E_1^{(k)}$. Accordingly, in the second subspace, the effective Hamiltonian H corresponds to Eq. (2.13).

The procedure of the perfect state transfer can be explained as follows [40]. Under the evolution $e^{-i\lambda t H}$, the state $|\psi_0\rangle$ becomes $e^{-i\lambda E_0 t}|\psi_0\rangle = |\psi_0\rangle$, namely the state $|\psi_0\rangle$ is unchanged during the evolution. Suppose initially one prepares the input qubit A in an unknown state $\alpha|0\rangle + \beta|1\rangle$, the state of the network becomes

$$\alpha|0_A 00\cdots 00_B\rangle + \beta|1_A 00\cdots 00_B\rangle = \alpha|\underline{0}\rangle + \beta|\underline{1}\rangle. \tag{2.16}$$

The coefficient α does not change in time as $|\underline{0}\rangle$ is the zero energy eigenstate of H. Since the operator of the total z-component of the spin, $\sigma^z_{tot} = \sum_{j=1}^{N} \sigma^z_j$, commutes with H, the total z-component of spin is conserved. Therefore the state $|\underline{1}\rangle = |1_A 00\cdots 00_B\rangle$ will evolve into a superposition of states with exactly one spin 'up' and all other spins 'down'. Thus the initial state of the network evolves in time t as

$$\alpha|\underline{0}\rangle + \beta|\underline{1}\rangle \rightarrow \alpha|\underline{0}\rangle + \sum_{n=1}^{N} \beta_n(t)|\underline{n}\rangle. \tag{2.17}$$

[Note that here the notation $|\underline{m}\rangle$ means $\phi(m) = |00\cdots 1_m \cdots 0\rangle$ as defined in Eq.(2.8)]. The dynamics is effectively confined to the second subspace. If we identify qubit A with vertex 1 and qubit B with vertex N then all we want to know is the probability amplitude that the network initially in state $|\underline{1}\rangle$, corresponding to $|1_A 00\cdots 00_B\rangle$, evolves after time t to state $|\underline{N}\rangle$, corresponding to $|0_A 00\cdots 01_B\rangle$, i.e.,

$$F(t) = \langle \underline{N}|e^{-i\lambda t H}|\underline{1}\rangle = \sum_{k=1}^{N} a_k(1) a_k^*(N) e^{-i\lambda t E_1^{(k)}}. \tag{2.18}$$

The faithful state transfer is obtained for times t such that $|F(t)| = 1$.

The coupling strength $J_{n,n+1} = \sqrt{n(N-n)}$ has definite significance. For such a coupling, the Hamiltonian H becomes the angular momentum operator S_x for the spin-$j = \frac{1}{2}(N-1)$ particle [40]. Therefore, it is easy to get the eigenvalues of H via the eigenvalues of S_z (since the eigenvalues of S_x and S_z are the same). A simple proof for the perfect state transfer can be proceeded as follows. For angular momentum, there is a disentangled formula:

$$\exp(re^{i\varphi}S_+ - re^{-i\varphi}S_-) = \exp(e^{i\varphi}\tan rS_+)(1+\tan^2 r)^{S_z}\exp(-e^{-i\varphi}\tan rS_-). \tag{2.19}$$

Since $H = S_x = (S_+ + S_-)/2$, for $r = \lambda t/2$, $\varphi = -\pi/2$ we then have

$$\begin{aligned} e^{-i\lambda tH} &= e^{-i\lambda tS_x} = \exp(re^{-i\pi/2}S_+ - re^{i\pi/2}S_-) \\ &= \exp(-i\tan rS_+)(1+\tan^2 r)^{S_z}\exp(-i\tan rS_-). \end{aligned} \tag{2.20}$$

It is easy to know that

$$\begin{aligned} &\exp(-i\tan rS_+) \\ &= 1 + \frac{-i\tan r}{1!}S_+ + \frac{(-i\tan r)^2}{2!}S_+^2 + \cdots + \frac{(-i\tan r)^{N-1}}{(N-1)!}S_+^{N-1}, \end{aligned} \tag{2.21}$$

since $S_+^N = 0$. It is clear that $\exp(-i\tan rS_+)$ is a upper-triangle matrix, and the matrix elements are given as

$$[\exp(-i\tan rS_+)]_{N,N} = 1, \tag{2.22}$$

and

$$\begin{aligned} &[\exp(-i\tan rS_+)]_{1,N} = (J_{12}J_{23}J_{34}\cdots J_{N-1,N})\frac{(-i\tan r)^{N-1}}{(N-1)!} = (-i\tan r)^{N-1}, \\ &[\exp(-i\tan rS_-)]_{N,1} = (-i\tan r)^{N-1}. \end{aligned} \tag{2.23}$$

And $(1+\tan^2 r)^{S_z}$ is a diagonal matrix with the matrix elements

$$[(1+\tan^2 r)^{S_z}]_{N,N} = (\cos^{-2} r)^{-(N-1)/2} = [\cos r]^{N-1}. \tag{2.24}$$

From these results, one has

$$\begin{aligned} F(t) &= (e^{-i\lambda tH})_{N,1} \\ &= [\exp(-i\tan rS_+)]_{N,N}\ [(1+\tan^2 r)^{S_z}]_{N,N}\ [\exp(-i\tan rS_-)]_{N,1} \\ &= 1\cdot[\cos r]^{N-1}\cdot(-i\tan r)^{N-1} = [-i\ \sin r]^{N-1} \\ &= \left[-i\ \sin\left(\frac{\lambda t}{2}\right)\right]^{N-1}. \end{aligned} \tag{2.25}$$

Therefore for some time t satisfying $\sin(\lambda t/2) = 1$, one obtains the faithful state transfer $|F(t)| = 1$.

It is well-known that, for spin-j particle, the operator S_z has $N = 2j + 1$ eigenvalues $\{-j, -j+1, ..., j-1, j\}$. From previous results, we know that there are N eigenvalues $E_1^{(k)}$ ($k = 1, 2, ..., N$) in the second subspace, which are the same as the eigenvalues of S_z, i.e., $E_1^{(k)} \in \{-\frac{N-1}{2}, -\frac{N-3}{2}, \cdots, \frac{N-1}{2}\}$. For convenience, denoting the set $\{-j, -j+1, ..., j-1, j\}$ by $\{v_1, v_2, ..., v_{N-1}, v_N\}$, then the energy spectrum has interesting structures: 1) For the first subspace, eigenvalue is given as

$$E_0 = v_1 + v_2 + \cdots + v_N = 0. \tag{2.26}$$

2) For the second subspace, the N eigenvalues are

$$E_1^k = v_1 + \cdots + v_{k-1} - v_k + v_{k+1} + \cdots + v_N. \tag{2.27}$$

Namely, a minus sign is for v_k, positive signs are for other $v_i (i \neq k)$. 3) For the third subspace, the C_N^2 eigenvalues are

$$\begin{aligned} &E_2^{k1,k2} = \\ &v_1 + \cdots + v_{k1-1} - v_{k1} + v_{k1+1} + \cdots + v_{k2-1} - v_{k2} + v_{k2+1} + \cdots + v_N. \end{aligned} \tag{2.28}$$

That is to say that v_{k1} and v_{k2} have minus signs, others have positive signs. The similar procedure can be done for the general n-th subspace. In the following, we give explicit solutions for the cases of $N = 2, 3, 4$.

a: For $N = 2$, the four eigenvalues and their corresponding eigenstates are

$$\begin{aligned} &E_0 = -1, \quad E_1 = 1, \quad E_2 = E_3 = 0. \\ &|\phi_0\rangle = \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle), \quad |\phi_1\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle), \\ &|\phi_2\rangle = |11\rangle, \qquad\qquad\quad |\phi_3\rangle = |00\rangle. \end{aligned} \tag{2.29}$$

b: For $N = 3$, there are eight eigenvalues and eigenstates.

$$
\begin{aligned}
&E_0 = E_1 = -2, \;\; E_2 = E_3 = 2, \;\; E_4 = E_5 = E_6 = E_7 = 0 \\
&|\phi_0\rangle = \frac{1}{2}(|011\rangle - \sqrt{2}|101\rangle + |110\rangle), \;\; |\phi_1\rangle = \frac{1}{2}(|001\rangle - \sqrt{2}|010\rangle + |100\rangle), \\
&|\phi_2\rangle = \frac{1}{2}(|011\rangle + \sqrt{2}|101\rangle + |110\rangle), \;\; |\phi_3\rangle = \frac{1}{2}(|001\rangle + \sqrt{2}|010\rangle + |100\rangle), \\
&|\phi_4\rangle = |111\rangle, \;\; |\phi_5\rangle = \frac{1}{\sqrt{2}}(-|011\rangle + |110\rangle), \\
&|\phi_6\rangle = \frac{1}{\sqrt{2}}(-|001\rangle + |100\rangle), \;\; |\phi_7\rangle = |000\rangle.
\end{aligned} \tag{2.30}
$$

c: For $N = 4$, there are sixteen eigenvalues and eigenstates.

$$
E_0 = E_7 = E_8 = E_{15} = 0, \;\; E_3 = E_{13} = -1, \;\; E_4 = E_{14} = 1, \;\; E_6 = -2,
$$
$$
E_9 = 2, \;\; E_1 = E_{11} = -3, \;\; E_2 = E_{12} = 3, \;\; E_5 = -4, \;\; E_{10} = 4.
$$

$$
\begin{aligned}
|\phi_0\rangle &= |0000\rangle, \\
|\phi_1\rangle &= \frac{1}{2\sqrt{2}}(-|1000\rangle + \sqrt{3}|0100\rangle - \sqrt{3}|0010\rangle + |0001\rangle), \\
|\phi_2\rangle &= \frac{1}{2\sqrt{2}}(|1000\rangle + \sqrt{3}|0100\rangle + \sqrt{3}|0010\rangle + |0001\rangle), \\
|\phi_3\rangle &= \frac{\sqrt{3}}{2\sqrt{2}}(|1000\rangle - \frac{1}{\sqrt{3}}|0100\rangle - \frac{1}{\sqrt{3}}|0010\rangle + |0001\rangle), \\
|\phi_4\rangle &= \frac{\sqrt{3}}{2\sqrt{2}}(-|1000\rangle - \frac{1}{\sqrt{3}}|0100\rangle + \frac{1}{\sqrt{3}}|0010\rangle + |0001\rangle), \\
|\phi_5\rangle &= \frac{1}{4}(|1100\rangle - 2|1010\rangle + \sqrt{3}|1001\rangle + \sqrt{3}|0110\rangle - 2|0101\rangle + |0011\rangle), \\
|\phi_6\rangle &= \frac{1}{2}(-|1100\rangle + |1010\rangle - |0101\rangle + |0011\rangle), \\
|\phi_7\rangle &= \frac{\sqrt{3}}{\sqrt{10}}(|1100\rangle - \frac{2}{\sqrt{3}}|1001\rangle + |0011\rangle), \\
|\phi_8\rangle &= \frac{5}{2\sqrt{10}}(-\frac{\sqrt{3}}{5}|1100\rangle - \frac{3}{5}|1001\rangle + |0110\rangle - \frac{\sqrt{3}}{5}|0011\rangle), \\
|\phi_9\rangle &= \frac{1}{2}(-|1100\rangle - |1010\rangle + |0101\rangle + |0011\rangle), \\
|\phi_{10}\rangle &= \frac{1}{4}(|1100\rangle + 2|1010\rangle + \sqrt{3}|1001\rangle + \sqrt{3}|0110\rangle + 2|0101\rangle + |0011\rangle), \\
|\phi_{11}\rangle &= \frac{1}{2\sqrt{2}}(-|1110\rangle + \sqrt{3}|1101\rangle - \sqrt{3}|1011\rangle + |0111\rangle),
\end{aligned}
$$

$$
\begin{aligned}
|\phi_{12}\rangle &= \frac{1}{2\sqrt{2}}(|1110\rangle + \sqrt{3}|1101\rangle + \sqrt{3}|1011\rangle + |0111\rangle),\\
|\phi_{13}\rangle &= \frac{\sqrt{3}}{2\sqrt{2}}(|1110\rangle - \frac{1}{\sqrt{3}}|1101\rangle - \frac{1}{\sqrt{3}}|1011\rangle + |0111\rangle),\\
|\phi_{14}\rangle &= \frac{\sqrt{3}}{2\sqrt{2}}(-|1110\rangle - \frac{1}{\sqrt{3}}|1101\rangle + \frac{1}{\sqrt{3}}|1011\rangle + |0111\rangle),\\
|\phi_{15}\rangle &= |1111\rangle. \qquad (2.31)
\end{aligned}
$$

It is also worth noting that we are able to find the analytical forms of the lowest and highest eigenvalues for H with arbitrary N. Obviously, the highest eigenvalue should be given by adding v_ms with all components with minus signs flipped. In the same way, the lowest eigenvalue is given by all components with plus signs flipped. For N is an odd number or even number, different results are given correspondingly,

$$
\begin{aligned}
&E_{\min} = \mathrm{n} - \mathrm{n}^2, \qquad \mathrm{E}_{\max} = \mathrm{n}^2 - \mathrm{n}, \qquad \text{for } \ \mathrm{N} = 2\mathrm{n} - 1\\
&E_{\min} = -\mathrm{n}^2, \qquad \mathrm{E}_{\max} = \mathrm{n}^2 \qquad \text{for } \ \mathrm{N} = 2\mathrm{n}. \qquad (2.32)
\end{aligned}
$$

where n is any arbitrary number larger than 1.

2.2.2 Violation of the Żukowski-Brukner Inequalities and the Threshold Temperature

When spin chains are subjected to environmental disturbance, they inevitably become thermal equilibrium states. The state of a system at finite temperature T is given by the Gibb's density operator $\rho(T) = \exp(-H/kT)/Z$, where $Z = \mathrm{Tr}[\exp(-H/kT)]$ is the partition function, H is the system Hamiltonian and k is the Boltzmann constant, which is set to unity for convenience in this thesis. At high temperature, the thermal state becomes maximally mixed and does not violate Bell inequalities of any kind. It is therefore interesting to consider the critical temperature at which a Bell inequality will be violated. For a two-qubit system, we have the CHSH inequality. For arbitrary number of qubits, we have the Żukowski-Brukner inequalities [24].

In this section, several simple examples of testing quantum nonlocality of the special Heisenberg XX model will be discussed by using the Żukowski-Brukner inequalities. In the simplest case of a two-qubit system, there are four eigenvalues of the Hamiltonian. So the thermal state is characterized by

$$\rho(T) = \frac{1}{Z}\sum_{\mu=0}^{3} e^{-\beta E_\mu}|\phi_\mu\rangle\langle\phi_\mu|, \tag{2.33}$$

where $\beta = 1/T$ and the partition function is calculated as

$$\begin{aligned} Z &= \mathrm{Tr}(e^{-\beta H}) = \sum_{\mu=0}^{3} e^{-\beta E_\mu} \\ &= 1 + 1 + e^{-\beta} + e^{\beta} \\ &= 2 + 2\cosh(\beta). \end{aligned} \tag{2.34}$$

To test quantum nonlocality for the state $\rho(T)$, correlation function Q_{ij} should be computed. Quantum mechanics predicts that

$$\begin{aligned} Q_{ij} &= \mathrm{Tr}[\rho(\hat{n}_1^i \cdot \vec{\sigma}) \otimes (\hat{n}_2^j \cdot \vec{\sigma})] \\ &= \frac{1}{Z}\sum_{\mu=0}^{3} e^{-\beta E_\mu}\mathrm{Tr}[|\phi_\mu\rangle\langle\phi_\mu|(\hat{n}_1^i \cdot \vec{\sigma}) \otimes (\hat{n}_2^j \cdot \vec{\sigma})] \\ &= \frac{1}{Z}\sum_{\mu=0}^{3} e^{-\beta E_\mu} Q_{ij}^{\mu}, \end{aligned} \tag{2.35}$$

where $\hat{n}_\alpha = (\sin\theta_\alpha, 0, \cos\theta_\alpha),\ \alpha = 1, 2$ and $i, j = 1, 2$. Q_{ij}^{μ} is the correlation function for the eigenstate $|\phi_\mu\rangle$,

$$Q_{ij}^{\mu} = \mathrm{Tr}[|\phi_\mu\rangle\langle\phi_\mu|(\hat{n}_i \cdot \vec{\sigma}) \otimes (\hat{n}_j \cdot \vec{\sigma})]. \tag{2.36}$$

It is easy to calculate that

$$\begin{aligned} Q_{ij}^{0} &= -\cos\theta_1^i\cos\theta_2^j - \sin\theta_1^i\sin\theta_2^j, \\ Q_{ij}^{1} &= -\cos\theta_1^i\cos\theta_2^j + \sin\theta_1^i\sin\theta_2^j, \\ Q_{ij}^{2} &= \cos\theta_1^i\cos\theta_2^j, \\ Q_{ij}^{3} &= \cos\theta_1^i\cos\theta_2^j. \end{aligned} \tag{2.37}$$

Thus the correlation function for the thermal state is written as

$$Q_{ij} = \frac{1}{1+\cosh\beta}(\cos\theta_1^i\cos\theta_2^j - \cos\theta_1^i\cos\theta_2^j\cosh\beta - \sin\theta_1^i\sin\theta_2^j\sinh\beta). \tag{2.38}$$

For a local and realistic description, the CHSH inequality is $-2 \leq Q_{11} + Q_{12} + Q_{21} - Q_{22} \leq 2$. By taking appropriate values $\beta = 15.2, \theta_1^1 = 0, \theta_1^2 = \pi/2, \theta_2^1 = -3\pi/4$, and

$\theta_2^2 = 3\pi/4$, the maximum value of $Q_{11}+Q_{12}+Q_{21}-Q_{22}$ under quantum mechanical prediction is $2\sqrt{2}$. The critical temperature $T_0 = 0.273$, above which the model is describable with a local and realistic description.

Unfortunately it is not possible to test quantum nonlocality of three qubits in this case since the correlation functions defined in a similar way are zero. Therefore, we focus on the next non-trivial case of a 4-qubit system and test the violation of local realistic description using the ŻB inequalities. The extension to arbitrary number of sites, albeit complicating, can also be done in the same manner. The Hamiltonian has sixteen eigenvalues, see Eq. (2.31). These eigenvalues and eigenstates completely determine the thermal state. The density operator $\rho(T)$ at the temperature T can be written as

$$\rho(T) = \frac{1}{Z}\sum_{\mu=0}^{15} e^{-\beta E_\mu}|\phi_\mu\rangle\langle\phi_\mu|, \tag{2.39}$$

with the partition function

$$\begin{aligned} Z &= \mathrm{Tr}(e^{-\beta H}) = \sum_{\mu=0}^{15} e^{-\beta E_\mu} \\ &= 4 + 4\cosh(3\beta) + 4\cosh\beta + 2\cosh(4\beta) + 2\cosh(2\beta). \end{aligned} \tag{2.40}$$

Similarly, correlation function Q_{ijkl} can be computed as follows,

$$\begin{aligned} Q_{ijkl} &= \mathrm{Tr}[\rho(\hat{n}_1^i\cdot\vec{\sigma})\otimes(\hat{n}_2^j\cdot\vec{\sigma})\otimes(\hat{n}_3^k\cdot\vec{\sigma})\otimes(\hat{n}_4^l\cdot\vec{\sigma})] \\ &= \frac{1}{Z}\sum_{\mu=0}^{15} e^{-\beta E_\mu}\mathrm{Tr}[|\phi_\mu\rangle\langle\phi_\mu|(\hat{n}_1^i\cdot\vec{\sigma})\otimes(\hat{n}_2^j\cdot\vec{\sigma})\otimes(\hat{n}_3^k\cdot\vec{\sigma})\otimes(\hat{n}_4^l\cdot\vec{\sigma})] \\ &= \frac{1}{Z}\sum_{\mu=0}^{15} e^{-\beta E_\mu} Q_{ijkl}^\mu, \end{aligned} \tag{2.41}$$

where $\hat{n}_\alpha = (\sin\theta_\alpha, 0, \cos\theta_\alpha)$, $\alpha = 1,2,3,4$ and $i,j,k,l = 1,2$. Q_{ijkl}^μ is the correlation function for the eigenstate $|\phi_\mu\rangle$,

$$Q_{ijkl}^\mu = \mathrm{Tr}[|\phi_\mu\rangle\langle\phi_\mu|(\hat{n}_1^i\cdot\vec{\sigma})\otimes(\hat{n}_2^j\cdot\vec{\sigma})\otimes(\hat{n}_3^k\cdot\vec{\sigma})\otimes(\hat{n}_4^l\cdot\vec{\sigma})]. \tag{2.42}$$

For instance, the quantum correlation for the ground state $|\phi_5\rangle$ is given by

$$\begin{aligned} Q^5_{ijkl} = \quad & \cos\theta^i_1 \cos\theta^j_2 \cos\theta^k_3 \cos\theta^l_4 + \frac{\sqrt{3}}{2} \cos\theta^k_3 \cos\theta^l_4 \sin\theta^i_1 \sin\theta^j_2 \\ & -\frac{\sqrt{3}}{4} \cos\theta^j_2 \cos\theta^l_4 \sin\theta^i_1 \sin\theta^k_3 + \frac{1}{2} \cos\theta^i_1 \cos\theta^l_4 \sin\theta^j_2 \sin\theta^k_3 \\ & +\frac{1}{2} \cos\theta^j_2 \cos\theta^k_3 \sin\theta^i_1 \sin\theta^l_4 - \frac{\sqrt{3}}{4} \cos\theta^i_1 \cos\theta^k_3 \sin\theta^j_2 \sin\theta^l_4 \\ & +\frac{\sqrt{3}}{2} \cos\theta^i_1 \cos\theta^j_2 \sin\theta^k_3 \sin\theta^l_4 + \sin\theta^i_1 \sin\theta^j_2 \sin\theta^k_3 \sin\theta^l_4. \end{aligned} \tag{2.43}$$

Other quantum correlation functions can also be calculated in a similar way. We list the correlation functions for each eigenstate of the 4-qubit Hamiltonian for easy reference, see Table 2.1. The correlation functions for the thermal state $\rho(T)$ are computed using Eq. (2.41).

Given a Bell inequality, a quantity can be associated to the inequality and the quantity is called Bell quantity. In the case of the CHSH inequality, the Bell quantity is $\mathcal{B}_{CHSH} = Q_{11} + Q_{12} - Q_{21} + Q_{22}$ and the CHSH inequality can be written as $-2 \leq \mathcal{B}_{CHSH} \leq 2$. When restricted to 4-qubit systems, we can write the corresponding Bell quantity for the 4-qubit ŻB inequality based on the calculated values of Q_{ijkl}:

$$\begin{aligned} \mathcal{B} = \quad & Q_{1111} - Q_{1112} - Q_{1121} - Q_{1122} - Q_{1211} - Q_{1212} \\ & -Q_{1221} + Q_{1222} - Q_{2111} - Q_{2112} - Q_{2121} + Q_{2122} \\ & -Q_{2211} + Q_{2212} + Q_{2221} + Q_{2222}. \end{aligned} \tag{2.44}$$

For a local realistic description, it is required that $-4 \leq \mathcal{B} \leq 4$ from Eq. (2.3). In Figure 2.1, we numerically compute the Bell quantity as a function of temperature. The results show that violation of the Bell inequality occurs at $T \leq T_0 = 0.626$. We call this critical value T_0 the threshold temperature. The maximum value of $\mathcal{B}$ for the state $\rho(T)$ approaches 7.917 when temperature is close to zero.

The Bell quantities $\mathcal{B}(|\phi_\mu\rangle)$ constructed from the correlation functions of each

correlation function	explicit expression
$Q^0_{ijkl} = Q^{15}_{ijkl}$	$\cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l$
$Q^1_{ijkl} = Q^{11}_{ijkl}$	$-\cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l - \frac{\sqrt{3}}{4}\cos\theta_3^k \cos\theta_4^l \sin\theta_1^i \sin\theta_2^j$ $+\frac{\sqrt{3}}{4}\cos\theta_2^j \cos\theta_4^l \sin\theta_1^i \sin\theta_3^k - \frac{3}{4}\cos\theta_1^i \cos\theta_4^l \sin\theta_2^j \sin\theta_3^k$ $-\frac{1}{4}\cos\theta_2^j \cos\theta_3^k \sin\theta_1^i \sin\theta_4^l + \frac{\sqrt{3}}{4}\cos\theta_1^i \cos\theta_3^k \sin\theta_2^j \sin\theta_4^l$ $-\frac{\sqrt{3}}{4}\cos\theta_1^i \cos\theta_2^j \sin\theta_3^k \sin\theta_4^l$
$Q^2_{ijkl} = Q^{12}_{ijkl}$	$-cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l + \frac{\sqrt{3}}{4}\cos\theta_3^k \cos\theta_4^l \sin\theta_1^i \sin\theta_2^j$ $+\frac{\sqrt{3}}{4}\cos\theta_2^j \cos\theta_4^l \sin\theta_1^i \sin\theta_3^k + \frac{3}{4}\cos\theta_1^i \cos\theta_4^l \sin\theta_2^j \sin\theta_3^k$ $+\frac{1}{4}\cos\theta_2^j \cos\theta_3^k \sin\theta_1^i \sin\theta_4^l + \frac{\sqrt{3}}{4}\cos\theta_1^i \cos\theta_3^k \sin\theta_2^j \sin\theta_4^l$ $+\frac{\sqrt{3}}{4}\cos\theta_1^i \cos\theta_2^j \sin\theta_3^k \sin\theta_4^l$
$Q^3_{ijkl} = Q^{13}_{ijkl}$	$-\cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l - \frac{\sqrt{3}}{4}\cos\theta_3^k \cos\theta_4^l \sin\theta_1^i \sin\theta_2^j$ $-\frac{\sqrt{3}}{4}\cos\theta_2^j \cos\theta_4^l \sin\theta_1^i \sin\theta_3^k + \frac{1}{4}\cos\theta_1^i \cos\theta_4^l \sin\theta_2^j \sin\theta_3^k$ $+\frac{3}{4}\cos\theta_2^j \cos\theta_3^k \sin\theta_1^i \sin\theta_4^l - \frac{\sqrt{3}}{4}\cos\theta_1^i \cos\theta_3^k \sin\theta_2^j \sin\theta_4^l$ $-\frac{\sqrt{3}}{4}\cos\theta_1^i \cos\theta_2^j \sin\theta_3^k \sin\theta_4^l$
$Q^4_{ijkl} = Q^{14}_{ijkl}$	$-\cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l + \frac{\sqrt{3}}{4}\cos\theta_3^k \cos\theta_4^l \sin\theta_1^i \sin\theta_2^j$ $-\frac{\sqrt{3}}{4}\cos\theta_2^j \cos\theta_4^l \sin\theta_1^i \sin\theta_3^k - \frac{1}{4}\cos\theta_1^i \cos\theta_4^l \sin\theta_2^j \sin\theta_3^k$ $-\frac{3}{4}\cos\theta_2^j \cos\theta_3^k \sin\theta_1^i \sin\theta_4^l - \frac{\sqrt{3}}{4}\cos\theta_1^i \cos\theta_3^k \sin\theta_2^j \sin\theta_4^l$ $+\frac{\sqrt{3}}{4}\cos\theta_1^i \cos\theta_2^j \sin\theta_3^k \sin\theta_4^l$
Q^6_{ijkl}	$\cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l + \cos\theta_1^i \cos\theta_4^l \sin\theta_2^j \sin\theta_3^k$ $-\cos\theta_2^j \cos\theta_3^k \sin\theta_1^i \sin\theta_4^l - \sin\theta_1^i \sin\theta_2^j \sin\theta_3^k \sin\theta_4^l$
Q^7_{ijkl}	$\cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l + \frac{2\sqrt{3}}{5}\cos\theta_2^j \cos\theta_4^l \sin\theta_1^i \sin\theta_3^k$ $\frac{2\sqrt{3}}{5}\cos\theta_1^i \cos\theta_3^k \sin\theta_2^j \sin\theta_4^l + \frac{3}{5}\sin\theta_1^i \sin\theta_2^j \sin\theta_3^k \sin\theta_4^l$
Q^8_{ijkl}	$\cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l + \frac{\sqrt{3}}{10}\cos\theta_2^j \cos\theta_4^l \sin\theta_1^i \sin\theta_3^k$ $\frac{\sqrt{3}}{10}\cos\theta_1^i \cos\theta_3^k \sin\theta_2^j \sin\theta_4^l - \frac{3}{5}\sin\theta_1^i \sin\theta_2^j \sin\theta_3^k \sin\theta_4^l$
Q^9_{ijkl}	$\cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l - \cos\theta_1^i \cos\theta_4^l \sin\theta_2^j \sin\theta_3^k$ $+\cos\theta_2^j \cos\theta_3^k \sin\theta_1^i \sin\theta_4^l - \sin\theta_1^i \sin\theta_2^j \sin\theta_3^k \sin\theta_4^l$
Q^{10}_{ijkl}	$\cos\theta_1^i \cos\theta_2^j \cos\theta_3^k \cos\theta_4^l - \frac{\sqrt{3}}{2}\cos\theta_3^k \cos\theta_4^l \sin\theta_1^i \sin\theta_2^j$ $-\frac{\sqrt{3}}{4}\cos\theta_2^j \cos\theta_4^l \sin\theta_1^i \sin\theta_3^k - \frac{1}{2}\cos\theta_1^i \cos\theta_4^l \sin\theta_2^j \sin\theta_3^k$ $-\frac{1}{2}\cos\theta_2^j \cos\theta_3^k \sin\theta_1^i \sin\theta_4^l - \frac{\sqrt{3}}{4}\cos\theta_1^i \cos\theta_3^k \sin\theta_2^j \sin\theta_4^l$ $-\frac{\sqrt{3}}{2}\cos\theta_1^i \cos\theta_2^j \sin\theta_3^k \sin\theta_4^l + \sin\theta_1^i \sin\theta_2^j \sin\theta_3^k \sin\theta_4^l$

Table 2.1: Quantum correlation functions for each pure state.

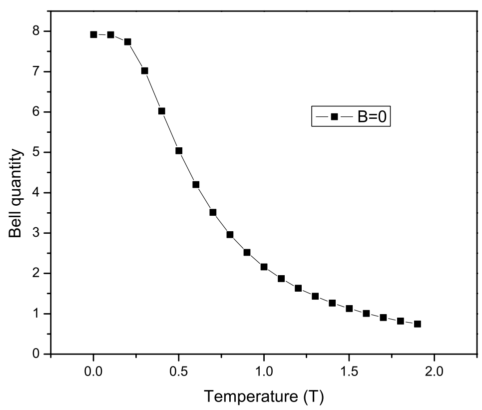

Figure 2.1: For a local realistic description of quantum mechanics, the Bell quantity $\mathcal{B}$ constructed from 4-qubit Żukowski-Brukner inequality must necessarily be less than 4. However, the Bell quantity for the XX model with site dependent coupling strength as a function of temperature T shows that there is a significant violation of Bell inequality at $T < 0.626$.

pure state $|\phi_\mu\rangle$ are also evaluated. The maximum values of $\mathcal{B}(|\phi_\mu\rangle)$ are respectively

$$\begin{aligned} \mathcal{B}_{max}(|\phi_\mu\rangle) = \quad & 4 && \text{for } |\phi_{0,15}\rangle \\ & 6.112 && \text{for } |\phi_{1,2,3,4,11,12,13,14}\rangle \\ & 7.917 && \text{for } |\phi_{5,10}\rangle \\ & 5.657 && \text{for } |\phi_{6,9}\rangle \\ & 4.866 && \text{for } |\phi_{7}\rangle \\ & 4.060 && \text{for } |\phi_{8}\rangle. \end{aligned} \tag{2.45}$$

That the maximum value of $\mathcal{B}$ for the thermal state is 7.917 can be qualitatively explained with the following argument. The thermal state $\rho(T)$ is the linear combination of $|\phi_\mu\rangle\langle\phi_\mu|$ weighted with the factors $e^{-\beta E_\mu} = e^{-E_\mu/T}$. For eigenvalue $E_5 = -4$, $\mathcal{B}_{max}(|\phi_5\rangle) = 7.917$, the power is $e^{4/T}$ and when T is small enough, the Bell quantity $\mathcal{B}$ is totally determined by the contribution of state $|\phi_5\rangle$. Another thing worth noting is that the eigenstates of special XX model do not lead to highest value of $\mathcal{B}_{max}$. We check the maximum values of the Bell quantities consist of correlation functions for the following three general states

$$
\begin{aligned}
|\phi'\rangle = \quad & \cos\alpha_1|1000\rangle + \sin\alpha_1\cos\alpha_2|0100\rangle \\
& + \sin\alpha_1\sin\alpha_2\cos\alpha_3|0010\rangle + \sin\alpha_1\sin\alpha_2\sin\alpha_3|0001\rangle
\end{aligned} \tag{2.46}
$$

$$
\begin{aligned}
|\phi''\rangle = \quad & \cos\alpha_1|1110\rangle + \sin\alpha_1\cos\alpha_2|1101\rangle \\
& + \sin\alpha_1\sin\alpha_2\cos\alpha_3|1011\rangle + \sin\alpha_1\sin\alpha_2\sin\alpha_3|0111\rangle
\end{aligned} \tag{2.47}
$$

$$
\begin{aligned}
|\phi'''\rangle = \quad & \cos\alpha_1|1100\rangle + \sin\alpha_1\cos\alpha_2|1010\rangle + \sin\alpha_1\sin\alpha_2\cos\alpha_3|1001\rangle \\
& + \sin\alpha_1\sin\alpha_2\sin\alpha_3\cos\alpha_4|0110\rangle + \sin\alpha_1\sin\alpha_2\sin\alpha_3\sin\alpha_4\cos\alpha_5|0101\rangle \\
& + \sin\alpha_1\sin\alpha_2\sin\alpha_3\sin\alpha_4\sin\alpha_5|0011\rangle
\end{aligned} \tag{2.48}
$$

and find that

$$
\begin{aligned}
\mathcal{B}_{max}(|\phi_0'\rangle) &= 6.217 \\
\mathcal{B}_{max}(|\phi_0''\rangle) &= 6.217 \\
\mathcal{B}_{max}(|\phi_0'''\rangle) &= 8.485
\end{aligned} \tag{2.49}
$$

for $|\phi_0'\rangle = 1/2(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$, $|\phi_0''\rangle = 1/2(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle)$ and $|\phi_0'''\rangle = 1/\sqrt{6}(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle)$ respectively. It is easy to see that violation degree of the 4-qubit ŻB inequality for state $|\phi_0'\rangle$ is higher than that for the eigenstates $|\phi_\mu\rangle, (\mu = 1, 2, 3, 4)$ listed in Eq. (2.31). The same results also apply for the eigenstates $|\phi_\mu\rangle, (\mu = 11, 12, 13, 14)$ and $|\phi_\mu\rangle, (\mu = 5, 6, 7, 8, 9, 10)$ respectively. We see that among all possible $\mathcal{B}_{max}$, the state $|\phi_0'''\rangle$ yields the largest violation.

Here, we consider the special Heisenberg XX model, modeling the nearest-neighbor interaction spin chain. For the 4-qubit special XX model, it is shown that since the correlation functions depend on the temperature, the violation of the 4-qubit ŻB inequality for the thermal state depends critically on the parameter. The effect of temperature for a local realistic description of quantum theory is determined by the threshold value of T below which the thermal state violates the 4-qubit ŻB inequality. Effect of external magnetic field will be discussed in the following section.

2.2.3 The Effect of External Magnetic Field

In this section, we would like to study the effect of magnetic field on the nonlocality of thermal state in a general way, for which the Hamiltonian we wish to consider is

$$H' = \sum_{n=1}^{N-1} J_{n,n+1}(\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+) + B\sum_{n=1}^{N}\sigma_z, \tag{2.50}$$

where B is the strength of the magnetic field. We will still consider the non-trivial case of a 4-qubit system. It is easy to verify that the eigenstates of H' are identical with the ones listed in Eq. (2.31) of H, but with different eigenvalues.

$$\begin{aligned}
&E'_0 = 4B, \quad E'_1 = -3+2B, \quad E'_2 = 3+2B, \quad E'_3 = -1+2B,\\
&E'_4 = 1+2B, \quad E'_5 = -4, \quad E'_6 = -2, \quad E'_7 = 0,\\
&E'_8 = 0, \quad E'_9 = 2, \quad E'_{10} = 4, \quad E'_{11} = -3-2B,\\
&E'_{12} = 3-2B, \quad E'_{13} = -1-2B, \quad E'_{14} = 1-2B, \quad E'_{15} = -4B.
\end{aligned} \tag{2.51}$$

The correlation function and Bell quantity $\mathcal{B}'$ are given by

$$Q'_{ijkl} = \frac{1}{Z'}\sum_{\mu=0}^{15} e^{-\beta E'_\mu} Q^\mu_{ijkl}, \tag{2.52}$$

and

$$\begin{aligned}
\mathcal{B}' = \quad & Q'_{1111} - Q'_{1112} - Q'_{1121} - Q'_{1122} - Q'_{1211} - Q'_{1212}\\
& -Q'_{1221} + Q'_{1222} - Q'_{2111} - Q'_{2112} - Q'_{2121} + Q'_{2122}\\
& -Q'_{2211} + Q'_{2212} + Q'_{2221} + Q'_{2222},
\end{aligned} \tag{2.53}$$

respectively, where $Z' = \mathrm{Tr}(e^{-\beta H'})$. Clearly the violation of the 4-qubit ŻB inequality depends not only on the temperature, but also on external magnetic field. Our numerical calculations on the effects of T and B are exhibited in Figure 2.2.

There are five curves corresponding to $B = 0.1, 0.5, 1.0, 1.5,$ and 2 respectively. When $B = 0.1$, the Bell quantity shows a similar variation of the violation of the Bell inequality as a function of T to that in the absence of magnetic field. With the increasing strength of external magnetic field, there is a decrease in the value of Bell quantity until $B = 0.5$. We see also that there is an increase in Bell quantity with B

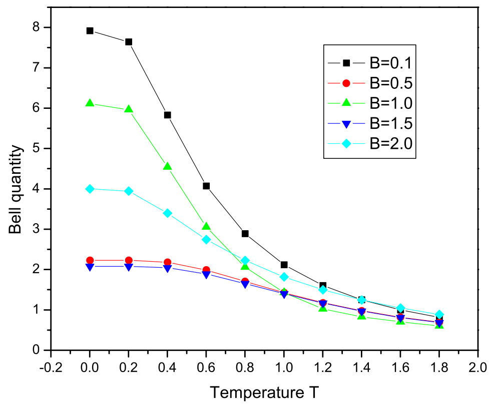

Figure 2.2: Bell quantity constructed from 4-qubit XX model with site dependent coupling strength for the cases with magnetic field $B = 0.1, 0.5, 1.0, 1.5,$ and 2.

until $B = 1.0$, and after that there is another decrease until $B = 1.5$ after which the system does not violate the Bell inequality. It is clear that the effect of magnetic field B on quantum nonlocality of the model is different from that of temperature T. The variation of the Bell quantity as a function of magnetic field can be explained qualitatively as follows. The $\rho'(T)$ is a different combination of $|\phi_\mu\rangle\langle\phi_\mu|$ from $\rho(T)$. The largest contribution of all the states $|\phi_\mu\rangle$ is determined by the value of B. When $B < 0.5$, it is the eigenstate, $|\phi_5\rangle$, which ultimately determines the maximal value of the Bell quantity ($\mathcal{B}_{max} = 7.917$) since $e^{-\beta E_5'} = e^{4/T}$ is the largest power among all the factors. When $0.5 < B < 1.5$, $|\phi_{11}\rangle$ takes the place of $|\phi_5\rangle$ with power $e^{(3+2B)/T}$ and $\mathcal{B}_{max} = 6.112$ at $B = 1.0$, for example. When $B > 1.5$, $e^{-\beta E_{15}'} = e^{4B/T}$ is the one with largest contribution and $\mathcal{B}_{max} = 4$. But there are two singular values of $B = 0.5$ and 1.5. In these two cases, $\mathcal{B}_{max} < 4$. The reason for the existence of singular values is that the largest factors of $e^{-\beta E_\mu'}$ are $e^{-\beta E_5'} = e^{-\beta E_{11}'} = e^{4\beta}$ for $B = 0.5$, $e^{-\beta E_{15}'} = e^{-\beta E_{11}'} = e^{6\beta}$ for $B = 1.5$, respectively. Thus the Bell quantity is determined principally using a combinations of these two elements of Q^{μ}_{ijkl}, namely, $e^{4\beta}(Q^5_{ijkl} + Q^{11}_{ijkl})$ and $e^{6\beta}(Q^{15}_{ijkl} + Q^{11}_{ijkl})$. Note that the maximum values of the Bell quantity for the two correlation functions $(Q^5_{ijkl} + Q^{11}_{ijkl})$ and $(Q^{15}_{ijkl} + Q^{11}_{ijkl})$ are 2.228 and 2.081 respectively. These two values are both less than 4. Which means that the system can be described in a local realistic description in the two cases since no violation of the ŻB inequality occurs. As a result, there does not exist a threshold temperature for the cases.

B	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
T_0	0.626	0.611	0.556	0.447	0.248	*None*	0.122	0.243
B	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5 *and above*
T_0	0.351	0.427	0.467	0.472	0.436	0.343	0.18	*None*

Table 2.2: Threshold temperatures of XX model with site dependent coupling strength for different strengths of the external magnetic field B. When $B = 0.5$ and $B = 1.5$ and above, the values of Bell quantity are no greater than 4 at all times. Therefore, no threshold temperatures exist for these cases.

The critical temperatures under different magnetic fields have been found (Table 2.2). The variation of T_0 with increasing strengths of B is complicated. The complexity arises mainly because the eigenstates contributing to the optimization of critical temperatures are different from those needed for the optimization of the Bell quantity in the absence of magnetic field. In the latter case, $\mathcal{B}_{max}$ is totally determined by the contribution of state with the largest weight or factor for sufficiently large β. In the former case, depending on the value of the external magnetic field, the eigenstates contributing to the optimization change and so the optimization is determined using a combination of the correlation functions from different states. In other words, the variation of T_0 with B is similar to that of $\mathcal{B}_{max}$ with B.

In summary, we examine the effects of temperature at different strengths of magnetic field in this section. For a fixed temperature, we can find the optimal value of the external magnetic field that violates the ŻB inequalities. Our results imply that quantum nonlocality could be controlled effectively by magnetic field and temperature. We have confined our argument to the 2,3,4-qubit cases. The violation of the ŻB inequalities for arbitrary number of qubit can also be done in the same manner.

In addition, quantum nonlocality of other kinds of spin chains and integrable models can also be tested by using similar method. In the next section, we study another form of XX model, in which coupling strength is taken to be constant.

2.3 Quantum Nonlocality of Quantum XX Model with Constant Coupling Strength

In this section, we look at the case in which the coupling strength is set to unity $J = 1$. The Hamiltonian is given by

$$H^c = \sum_{n=1}^{N-1}(\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+) + B\sum_{n=1}^{N}\sigma_z. \tag{2.54}$$

For this model, we continue to focus on the non-trivial case of a 4-qubit system and test the violation of local realistic description using the 4-qubit ŻB inequality. The eigenvalues and eigenstates are

$$\begin{aligned}
&E_0^c = E_1^c = 0, \quad E_2^c = -4B, \quad E_3^c = 4B,\\
&E_4^c = -1, \quad E_5^c = 1, \quad E_6^c = -\sqrt{5} \quad E_7^c = \sqrt{5},\\
&E_8^c = \frac{1}{2}(-1-\sqrt{5}-4B), \quad E_9^c = \frac{1}{2}(-1-\sqrt{5}+4B),\\
&E_{10}^c = \frac{1}{2}(1-\sqrt{5}-4B), \quad E_{11}^c = \frac{1}{2}(1-\sqrt{5}+4B),\\
&E_{12}^c = \frac{1}{2}(-1+\sqrt{5}-4B), \quad E_{13}^c = \frac{1}{2}(-1+\sqrt{5}+4B),\\
&E_{14}^c = \frac{1}{2}(1+\sqrt{5}-4B), \quad E_{15}^c = \frac{1}{2}(1+\sqrt{5}+4B),
\end{aligned} \tag{2.55}$$

and

$$\begin{aligned}
|\phi_0^c\rangle &= \frac{1}{\sqrt{3}}(|1100\rangle - |0110\rangle + |0011\rangle),\\
|\phi_1^c\rangle &= \frac{1}{\sqrt{2}}(-|0110\rangle + |1001\rangle),\\
|\phi_2^c\rangle &= |1111\rangle,\\
|\phi_3^c\rangle &= |0000\rangle,\\
|\phi_4^c\rangle &= \frac{1}{2}(-|1100\rangle + |1010\rangle - |0101\rangle + |0011\rangle),\\
|\phi_5^c\rangle &= \frac{1}{2}(-|1100\rangle - |1010\rangle + |0101\rangle + |0011\rangle),\\
|\phi_6^c\rangle &= \frac{1}{2\sqrt{5}}(|1100\rangle - \sqrt{5}|1010\rangle + 2|0110\rangle + 2|1001\rangle - \sqrt{5}|0101\rangle + |0011\rangle),\\
|\phi_7^c\rangle &= \frac{1}{2\sqrt{5}}(|1100\rangle + \sqrt{5}|1010\rangle + 2|0110\rangle + 2|1001\rangle + \sqrt{5}|0101\rangle + |0011\rangle),
\end{aligned}$$

$$
\begin{aligned}
|\phi_8^c\rangle &= \frac{1}{\sqrt{5+\sqrt{5}}}(-|1110\rangle + \frac{1}{2}(1+\sqrt{5})|1101\rangle - \frac{1}{2}(1+\sqrt{5})|1011\rangle + |0111\rangle),\\
|\phi_9^c\rangle &= \frac{1}{\sqrt{5+\sqrt{5}}}(-|1000\rangle + \frac{1}{2}(1+\sqrt{5})|0100\rangle - \frac{1}{2}(1+\sqrt{5})|0010\rangle + |0001\rangle),\\
|\phi_{10}^c\rangle &= \frac{1}{\sqrt{5-\sqrt{5}}}(|1110\rangle + \frac{1}{2}(1-\sqrt{5})|1101\rangle + \frac{1}{2}(1-\sqrt{5})|1011\rangle + |0111\rangle),\\
|\phi_{11}^c\rangle &= \frac{1}{\sqrt{5-\sqrt{5}}}(|1000\rangle + \frac{1}{2}(1-\sqrt{5})|0100\rangle + \frac{1}{2}(1-\sqrt{5})|0010\rangle + |0001\rangle),\\
|\phi_{12}^c\rangle &= \frac{1}{\sqrt{5-\sqrt{5}}}(-|1110\rangle + \frac{1}{2}(1-\sqrt{5})|1101\rangle - \frac{1}{2}(1-\sqrt{5})|1011\rangle + |0111\rangle),\\
|\phi_{13}^c\rangle &= \frac{1}{\sqrt{5-\sqrt{5}}}(-|1000\rangle + \frac{1}{2}(1-\sqrt{5})|0100\rangle - \frac{1}{2}(1-\sqrt{5})|0010\rangle + |0001\rangle),\\
|\phi_{14}^c\rangle &= \frac{1}{\sqrt{5+\sqrt{5}}}(|1110\rangle + \frac{1}{2}(1+\sqrt{5})|1101\rangle + \frac{1}{2}(1+\sqrt{5})|1011\rangle + |0111\rangle),\\
|\phi_{15}^c\rangle &= \frac{1}{\sqrt{5+\sqrt{5}}}(|1000\rangle + \frac{1}{2}(1+\sqrt{5})|0100\rangle + \frac{1}{2}(1+\sqrt{5})|0010\rangle + |0001\rangle).
\end{aligned}
\tag{2.56}
$$

It is easy to construct temperature dependent correlation functions based on these eigenvalues/states. These functions are

$$
\begin{aligned}
Q_{ijkl}^c &= \mathrm{Tr}[\rho^c(\hat{n}_1^i \cdot \vec{\sigma}) \otimes (\hat{n}_2^j \cdot \vec{\sigma}) \otimes (\hat{n}_3^k \cdot \vec{\sigma}) \otimes (\hat{n}_4^l \cdot \vec{\sigma})]\\
&= \frac{1}{Z^c}\sum_{\mu=0}^{15} e^{-\beta E_\mu^c} Q_{ijkl}^{c,\mu},
\end{aligned}
\tag{2.57}
$$

where $Z^c = \mathrm{Tr}(e^{-\beta H^c})$ and Bell quantity $\mathcal{B}^c$ is given from 4-qubit ŻB inequality,

$$
\begin{aligned}
\mathcal{B}^c = \quad & Q_{1111}^c - Q_{1112}^c - Q_{1121}^c - Q_{1122}^c - Q_{1211}^c - Q_{1212}^c\\
& - Q_{1221}^c + Q_{1222}^c - Q_{2111}^c - Q_{2112}^c - Q_{2121}^c + Q_{2122}^c\\
& - Q_{2211}^c + Q_{2212}^c + Q_{2221}^c + Q_{2222}^c.
\end{aligned}
\tag{2.58}
$$

By expressing correlation functions Q_{ijkl}^c for neighboring spins in terms of eigenstates of quantum XX model, quantum nonlocality of the model can be tested by its violation of the ŻB inequality. In Figure 2.3, the Bell quantities $\mathcal{B}^c$ for thermal state $\rho^c(T)$ plotted as a function of temperature and magnetic field are exhibited. There are four curves corresponding to $B = 0, 0.1, 0.5$, and 1.0. With the increasing value of temperature, Bell quantities decrease slowly in all the curves. When $B = 0$, it is calculated that the maximum value of the Bell quantity $\mathcal{B}^c$ approaches 7.754 when temperature is close to zero . To explain the result, Bell quantities $\mathcal{B}^c(|\phi_\mu^c\rangle)$

in terms of correlation functions for each pure state $|\phi_\mu^c\rangle$ are evaluated,

$$\begin{aligned}
\mathcal{B}^c_{max}(|\phi^c_\mu\rangle) = \quad & 4 & \text{for } |\phi^c_{1,2,3}\rangle \\
& 4.807 & \text{for } |\phi^c_{0}\rangle \\
& 5.657 & \text{for } |\phi^c_{4,5}\rangle \\
& 6.136 & \text{for } |\phi^c_{8,9,10,11,12,13,14,15}\rangle \\
& 7.754 & \text{for } |\phi^c_{6,7}\rangle.
\end{aligned} \tag{2.59}$$

It is worth noting that the Bell quantity of $\rho^c(T)$ is completely determined by the state $|\phi_\mu^c\rangle$ with the largest factor $e^{-E_\mu^c/T}$ when T is sufficiently small. After comparing the eigenvalues E_μ^c, it is found that state $|\phi_6^c\rangle$ has the largest factor $e^{-E_6^c/T} = e^{\sqrt{5}/T}$. Hence the Bell quantity $\mathcal{B}^c$ is totally determined by the contribution of state $\mathcal{B}^c_{max}(|\phi_6^c\rangle)$ which is equal to 7.754. For a local realistic description of quantum mechanics, the Bell quantity $\mathcal{B}^c$ must necessarily be less than 4. However, the Bell quantity as a function of temperature T shows that there is a significant violation of Bell inequality at $T < T_0^c = 0.374$.

When $B = 0.1$, the variation of the Bell quantity as a function of T is similar to the case in the absence of magnetic field. With the increasing strength of external magnetic field, the maximum value of the Bell quantity decreases and approaches the value 4 when B is about 1.0. Which of the states $|\phi_\mu^c\rangle$ will dominate the contribution to the Bell quantity $\mathcal{B}^c$ depends on the magnetic field B. When $B = 0.1$, it is the eigenstate $|\phi_6^c\rangle$ which ultimately determines the maximum value of the Bell quantity ($\mathcal{B}^c_{max} = 7.754$) since $e^{-E_6^c/T} = e^{\sqrt{5}/T}$ is the largest one among all the factors. When $B = 0.5$, $|\phi_8^c\rangle$ takes the place of $|\phi_6^c\rangle$ with factor $e^{\frac{1}{2}(1+\sqrt{5}+4B)/T} = e^{\frac{1}{2}(3+\sqrt{5})/T}$ and as a result, $\mathcal{B}^c_{max} = 6.136$. For the case of $B = 1$, $e^{-E_2^c/T} = e^{4B/T} = e^{4/T}$ is the one with largest contribution and $\mathcal{B}^c_{max} = 4$. In this case, the Bell inequality is not violated by the XX model, which means the model is describable in a local realistic theory when B is larger than 1. The data from Figure 2.3 suggests that the nonlocality of quantum XX model is determined by both temperature and strength of external magnetic field. These findings could serve as plausible evidence that the nonlocality of the XX model can be controlled by choosing appropriate strength of external magnetic field. These results are consistent with those given for that XX model with site dependent coupling strength in the previous section.

B	0	0.1	0.2	0.3	$\frac{\sqrt{5}-1}{4}$	0.4
T_0	0.374	0.345	0.24	0.021	$None$	0.112
B	0.5	0.6	0.7	0.8	$\frac{\sqrt{5}+1}{4}$	0.9 *and above*
T_0	0.21	0.239	0.184	0.016	$None$	$None$

Table 2.3: Threshold temperatures of XX model with constant coupling strength for different strengths of the external magnetic field B. When $B = \frac{\sqrt{5}-1}{4}$, $B = \frac{\sqrt{5}+1}{4}$ and $B \geq 0.9$, the values of Bell quantity are no greater than 4 at all times. Therefore, no threshold temperatures exist for these cases.

Table 2.3 summarizes the threshold temperatures under different magnetic fields. There is a decrease in the value of T_0 with increasing B until $B = \frac{\sqrt{5}-1}{4}$. We see also that there is an increase in T_0 with B until $B = 0.6$, and after that there is another decrease in T_0 until $B = \frac{\sqrt{5}+1}{4}$ after which there does not exist a threshold temperature. It is instructive to note that there exist two singular values of T_0, one at $B = \frac{\sqrt{5}-1}{4}$ and the other at $B = \frac{\sqrt{5}+1}{4}$. In these two cases, $\mathcal{B}^c_{max} < 4$ and hence there are no threshold temperatures. To explain these two singular values, the states contributing to the optimization of the threshold temperature are checked. For the case of $B = \frac{\sqrt{5}-1}{4}$, states $|\phi^c_6\rangle$ and $|\phi^c_8\rangle$ have the largest factors $e^{-E^c_6/T} = e^{-E^c_8/T} = e^{\sqrt{5}/T}$, and for the case of $B = \frac{\sqrt{5}+1}{4}$, states $|\phi^c_2\rangle$ and $|\phi^c_8\rangle$ have the largest factors $e^{-E^c_2/T} = e^{-E^c_8/T} = e^{(1+\sqrt{5})/T}$. In short, the Bell quantity is determined principally by using a combinations of two elements of $Q^{c,\mu}_{ijkl}$, $e^{\sqrt{5}/T}(Q^{c,6}_{ijkl} + Q^{c,8}_{ijkl})$ for $B = \frac{\sqrt{5}-1}{4}$ and $e^{(1+\sqrt{5})/T}(Q^{c,2}_{ijkl} + Q^{c,8}_{ijkl})$ for $B = \frac{\sqrt{5}+1}{4}$. It is also found that the maximum values of the Bell quantity for the two correlation functions $(Q^{c,6}_{ijkl}+Q^{c,8}_{ijkl})$ and $(Q^{c,2}_{ijkl}+Q^{c,8}_{ijkl})$ are 2.264 and 2.088 respectively. These values are less than 4 which means that the model does not violate the ŻB inequality in these two cases, and as a result, there are no threshold temperatures for these two cases. The results indicate that high threshold temperature can be achieved by adjusting the strength of magnetic field. High threshold temperature is needed for realization of quantum protocols in spin chains. The results provide important information on experimental realization of quantum computation and communication in the XX model.

Until now, quantum nonlocality of two types of quantum XX models has been investigated. These studies focus on discrete-variable quantum systems. By using violation of the ŻB inequalities, quantum nonlocality of continuous-variable systems

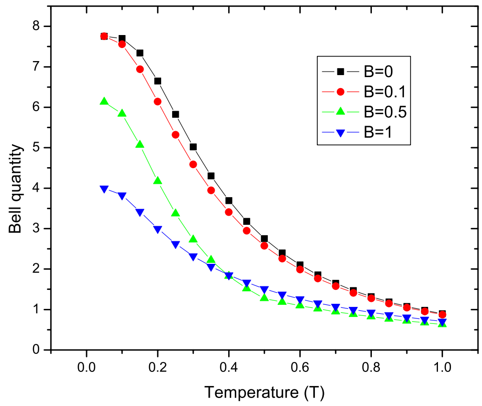

Figure 2.3: Variation of Bell quantity of XX model with constant coupling strength with T for the cases of $B = 0, 0.1, 0.5$, and 1.0.

can also be explored. In the next section, we look at the case of continuous model.

2.4 Violation of the Żukowski-Brukner Inequalities for Continuous-Variable Systems

2.4.1 Nonlocality and Wigner Function

Although most of the concepts and their applications in quantum information theory were initially developed for quantum systems with discrete variables, many quantum information protocols for continuous-variable systems have also been proposed [21]. Therefore, it is necessary to understand the nonlocal character of a quantum system with continuous variables. In recent years, quantum nonlocality for position-momentum variables associated with the original EPR states has been the object of interest. It is well known that quantum correlations for position-momentum variables can be analyzed in phase space by using the Wigner distribution function [41]. The Wigner function allows one to define a probability distribution in position-momentum phase space for a quantum state [42]. In Ref. [10], Bell used the phase space approach to investigate the nonlocality of the original EPR state. Recall that the original EPR state is in the form given in Eq. (1.1) which can be expressed as a δ-function:

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} e^{(2\pi i/\hbar)(x_1 - x_2 + x_0)p} dp.$$

Indeed, the state can be represented by a density matrix ρ which can be obtained from the Wigner function. The corresponding Wigner function for the EPR state is [10]

$$W(x_1, p_1; x_2, p_2) = 2\pi\delta(x_1 - x_2 + x_0)\delta(p_1 + p_2), \tag{2.60}$$

which is positive everywhere. In a local realistic theory, the correlation of measurement is given by Eq. (1.16). In this case, the distribution function is the Wigner function and the parameters describing hidden variables are x and p. Since the Wigner function is positive everywhere, it can be used to describe a local hidden variable correlation and hence the Bell inequality is not violated. According to Bell [10], the Wigner function would admit a local hidden variable description.

However, it should be noted that the choice of appropriate observables is important for testing the nonexistence of local realism for a given state. In other words, for an entangled state, the correlations in some type of measurements performed on the state cannot reveal nonlocal character of the state by violation of local realism. The Wigner function can be associated directly with the parity operator $(-1)^{\hat{n}}$ (where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the number operator) [43]. The Wigner representation of the parity operator is not a bounded reality corresponding to the dichotomic result of the measurement. This enables violation of Bell inequality for quantum states described by positive definite Wigner function. It is Banaszek and Wódkiewicz [32] who first demonstrated that positive definite Wigner function of the EPR state provides direct evidence of the nonlocality of the state. The proof was based on the fact that the correlation in the measurement of joint displaced parity operator on an entangled state is described by the Wigner function of the sate. The original EPR state is an unnormalized δ function. To avoid problems related to the singularity of the original EPR state, two-mode squeezed vacuum state produced through nondegenerate optical parametric amplification (NOPA)[22] was considered. The two-mode squeezed vacuum state generated in a nondegenerate optical parametric amplifier (NOPA)[22] is given by

$$|\text{NOPA}\rangle = e^{r(\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2)}|00\rangle = \sum_{n=0}^{\infty}\frac{(\tanh r)^n}{\cosh r}|nn\rangle, \tag{2.61}$$

where r is known as the squeezing parameter and $|nn\rangle \equiv |n\rangle_1 \otimes |n\rangle_2 = \frac{1}{n!}(\hat{a}_1^\dagger)^n(\hat{a}_2^\dagger)^n\,|00\rangle$.

The NOPA state |NOPA⟩ can also be written as [23]:

$$|\text{NOPA}\rangle = \sqrt{1-\tanh^2 r}\int dq \int dq' g(q,q';\tanh r)|qq'\rangle, \tag{2.62}$$

where $g\left(q,q';x\right) \equiv \frac{1}{\sqrt{\pi(1-x^2)}}\exp\left[-\frac{q^2+q'^2-2qq'x}{2(1-x^2)}\right]$ and $|qq'\rangle \equiv |q\rangle_1 \otimes |q'\rangle_2$, with $|q\rangle$ being the eigenstates of the position operator. Since $\lim_{x\to 1} g\left(q,q';x\right) = \delta(q-q')$, one has $\lim_{r\to\infty}\int dq\int dq' g(q,q';\tanh r)|qq'\rangle = \int dq|qq\rangle$, which is just the original EPR state by setting spatial separation between particles to be 0. Thus, in the infinite squeezing limit, $|\text{NOPA}\rangle\,|_{r\to\infty}$ becomes the original EPR state. Banaszek and Wódkiewicz then showed that the original EPR state and the two-mode squeezed vacuum state violate local realism since they violate generalized Bell inequalities such as the Clauser-Horne inequality [15] and the Clauser-Horne-Shimony-Holt (CHSH) [9] inequality. Thus, in Ref. [32], it was shown that despite its positive definiteness, the Wigner function of the EPR state could provide direct evidence of the nonlocality.

The observable measured in the experiment is displaced parity operator and the joint observables are $\hat{D}_1(\alpha)(-1)^{\hat{n}_1}\hat{D}_1^\dagger(\alpha)\otimes\hat{D}_2(\beta)(-1)^{\hat{n}_2}\hat{D}_2^\dagger(\beta)$. At the same time, the two-mode Wigner operator can be expressed as

$$\hat{W}(\alpha,\beta) = \frac{4}{\pi^2}\hat{D}_1(\alpha)(-1)^{\hat{n}_1}\hat{D}_1^\dagger(\alpha)\otimes\hat{D}_2(\beta)(-1)^{\hat{n}_2}\hat{D}_2^\dagger(\beta). \tag{2.63}$$

The above equation tells us that the correlation function measured in the experiment can be described by the Wigner function of the system. The correlation function of NOPA state is given as [44]

$$E(\alpha,\beta) = \exp\{-2\cosh 2r(|\alpha|^2+|\beta|^2) + 2\sinh 2r(\alpha\beta+\alpha^*\beta^*)\}. \tag{2.64}$$

In Ref. [32], the experimental setting for the displaced parity measurement is chosen as $\alpha = 0, \sqrt{\mathcal{J}}$ and $\beta = 0, -\sqrt{\mathcal{J}}$ where $\mathcal{J}$ is a positive constant describing the magnitude of the displacement. Properly choosing the constant, the Bell quantity constructed from the CHSH inequality by using the Wigner function approaches the value 2.19. In a local realistic theory, the CHSH-Bell quantity is required to be less than 2. Thus, a significant violation of the CHSH inequality takes place by using positive definite Wigner function of the original EPR state.

2.4.2 Wigner Function of N-Mode Squeezed States

No work has been aimed to address the relation between the nonlocality of arbitrary multipartite entangled states and the Wigner function until now. Recently, tripartite entangled state representation of the Wigner operator and the corresponding Wigner function have been found by Fan and Jiang [45]. They focused principally on a generalization of the Wigner function and its marginal distributions, without invoking the nonlocality issue. The general Bell inequalities involving correlation functions for N particles have been described in Ref. [24]. In this work, measurements on each particle are chosen from two arbitrary dichotomic observables. This general Bell theorem for general N-qubit states provides a useful tool to test the violation of local realism of multipartite quantum states described by the Wigner function. With this motivation, we derive an expression for the Wigner function of N-mode squeezed state in this section. By expressing the correlation function using the Wigner function, we show that the multipartite entangled state violates local realism, and this violation is enhanced with increasing number of particles[2].

To this end, we first choose parity operators as the observables for testing violation of local realism for a squeezed state. The Wigner function can be expressed as the expectation value of a product of displaced parity operators as follows

$$W(\alpha_1, \alpha_2, ..., \alpha_N) \propto \Pi(\alpha_1, \alpha_2, ..., \alpha_N), \tag{2.65}$$

with the joint displaced parity operator given by

$$\hat{\Pi}(\alpha_1, \alpha_2, ..., \alpha_N) \quad = \quad \hat{D}_1(\alpha_1)...\hat{D}_N(\alpha_N)(-1)^{\hat{n}_1+...+\hat{n}_N}\hat{D}_N^{-1}(\alpha_N)...\hat{D}_1^{-1}(\alpha_1). \tag{2.66}$$

In the above expression, $\hat{D}_i(\alpha_i) = \exp(\alpha_i\hat{a}^\dagger - \alpha_i^*\hat{a})$ denotes the displacement operator for the subsystem i, where $\hat{a}(\hat{a}^\dagger)$ is annihilation (creation) operator. The correlation function $E(\alpha_1, \alpha_2, ..., \alpha_N)$ given by the displaced parity operator $(-1)^{\hat{n}_1+...+\hat{n}_N}$ is proportional to the equivalent Wigner function [46]. In this way, we see that the nonlocal realistic description is embedded in the dichotomic correlation measurements given by the phase-space Wigner function for the multipartite entangled

[2]This work was published, see [3] in the publication list in Appendix A.

state,

$$W(\alpha_1, \alpha_2, ..., \alpha_N) \propto E(\alpha_1, \alpha_2, ..., \alpha_N) \equiv \Pi(\alpha_1, \alpha_2, ..., \alpha_N). \tag{2.67}$$

The correlation function, or equivalently the Wigner function for multipartite system, can be calculated by using the expectation value of the displaced parity operator on a N-mode squeezed state. This new squeezed state, a $SU(1,1)$ coherent state, is given as

$$|r\rangle = V|\mathbf{0}\rangle = \exp[r(W_+ - W_-)]|\mathbf{0}\rangle, \tag{2.68}$$

where $|\mathbf{0}\rangle = |00...0\rangle$ is a N-mode vacuum state, and

$$\begin{aligned} W_+ &= x\sum_{i=1}^{N} \hat{a}_i^{\dagger 2} + y\sum_{i<j=1}^{N} \hat{a}_i^{\dagger}\hat{a}_j^{\dagger}, \\ W_- &= x\sum_{i=1}^{N} \hat{a}_i^{2} + y\sum_{i<j=1}^{N} \hat{a}_i\hat{a}_j, \\ B &= \frac{1}{2}\sum_{i=1}^{N} \hat{a}_i^{\dagger}\hat{a}_i + \frac{N}{4}. \end{aligned} \tag{2.69}$$

W_+ is a N-mode squeezing operator and x and y are the coefficients which can be determined by the fact that the above formula satisfies the closed $SU(1,1)$ Lie algebra: $[W_+, W_-] = -2B, [W_+, B] = -W_+, [W_-, B] = W_-$. The final result is

$$W_+ = \frac{2-N}{2N}\sum_{i=1}^{N} \hat{a}_i^{\dagger 2} + \frac{2}{N}\sum_{i<j=1}^{N} \hat{a}_i^{\dagger}\hat{a}_j^{\dagger}, \tag{2.70}$$

$$W_- = \frac{2-N}{2N}\sum_{i=1}^{N} \hat{a}_i^{2} + \frac{2}{N}\sum_{i<j=1}^{N} \hat{a}_i\hat{a}_j. \tag{2.71}$$

The N-mode squeezed state is characterized by the squeezing parameter r.

The correlation function of the squeezed state is calculated in the following way. When r is zero, namely when no squeezing occurs, the correlation function is given by

$$\begin{aligned} E(\alpha_1, \alpha_2, ..., \alpha_N) &= \langle\mathbf{0}|\hat{\Pi}(\alpha_1, \alpha_2, ..., \alpha_N)|\mathbf{0}\rangle \\ &= \exp[-2\sum_{i=1}^{N} |\alpha_i|^2]. \end{aligned} \tag{2.72}$$

When $r \neq 0$, the new correlation function can be constructed from

$$E'(\alpha_1, \alpha_2, ..., \alpha_N) = \langle r|\hat{\Pi}(\alpha_1, \alpha_2, ..., \alpha_N)|r\rangle. \tag{2.73}$$

To calculate $E'(\alpha_1, \alpha_2, ..., \alpha_N)$, we first write the correlation function in the following form,

$$E(\alpha_1, \alpha_2, ..., \alpha_N) = \langle \mathbf{0}|\hat{D}_1(\alpha_1)...\hat{D}_N(\alpha_N)S\hat{D}_N^{-1}(\alpha_N)...\hat{D}_1^{-1}(\alpha_1)|\mathbf{0}\rangle, \tag{2.74}$$

with $S = (-1)^{\hat{n}_1+...+\hat{n}_N}$, so

$$\begin{aligned} E'(\alpha_1, \alpha_2, ..., \alpha_N) &= \langle r|\hat{D}_1(\alpha_1)...\hat{D}_N(\alpha_N)S\hat{D}_N^{-1}(\alpha_N)...\hat{D}_1^{-1}(\alpha_1)|r\rangle \\ &= \langle \mathbf{0}|V^{-1}\hat{D}_1(\alpha_1)...\hat{D}_N(\alpha_N)S\hat{D}_N^{-1}(\alpha_N)...\hat{D}_1^{-1}(\alpha_1)V|\mathbf{0}\rangle \\ &= \langle \mathbf{0}|[V^{-1}\hat{D}_1(\alpha_1)V]V^{-1}...V[V^{-1}\hat{D}_N(\alpha_N)V] \times \\ &\quad [V^{-1}SV][V^{-1}\hat{D}_N^{-1}(\alpha_N)V]V^{-1}...V[V^{-1}\hat{D}_1^{-1}(\alpha_1)V]|\mathbf{0}\rangle. \end{aligned} \tag{2.75}$$

Now the unitary transformation by parity operator $(-1)^{\hat{a}_i^\dagger \hat{a}_i}$ on $\hat{a}_i^\dagger$ and $\hat{a}_i$ is given by

$$\begin{aligned} (-1)^{\hat{a}_i^\dagger \hat{a}_i}\hat{a}_i^\dagger(-1)^{\hat{a}_i^\dagger \hat{a}_i} &= -\hat{a}_i^\dagger, \\ (-1)^{\hat{a}_i^\dagger \hat{a}_i}\hat{a}_i(-1)^{\hat{a}_i^\dagger \hat{a}_i} &= -\hat{a}_i. \end{aligned} \tag{2.76}$$

Since V is the exponential of a linear combination of $\hat{a}_i^{\dagger 2}$, $\hat{a}_i^\dagger\hat{a}_j^\dagger$, $\hat{a}_i\hat{a}_j$, and $\hat{a}_i^2$, clearly,

$$S^{-1}VS = V. \tag{2.77}$$

So the crucial observation is that the parity operator is invariant under the transformation V [43], $V^{-1}(-1)^{\hat{n}_1+...+\hat{n}_N}V = (-1)^{\hat{n}_1+...+\hat{n}_N}$. After writing $\hat{D}_i(\alpha_i') = V^{-1}\hat{D}_i(\alpha_i)V$, we have

$$\begin{aligned} E'(\alpha_1, \alpha_2, ..., \alpha_N) &= \langle \mathbf{0}|\hat{\Pi}(\alpha_1', \alpha_2', ..., \alpha_N')|\mathbf{0}\rangle \\ &= \exp[-2\sum_{i=1}^{N}|\alpha_i'|^2], \end{aligned} \tag{2.78}$$

where $\hat{\Pi}(\alpha_1', \alpha_2', ..., \alpha_N')$ is the squeezed displaced parity operator

$$\hat{\Pi}(\alpha_1', \alpha_2', ..., \alpha_N') = \hat{D}_1(\alpha_1')...\hat{D}_N(\alpha_N')(-1)^{\hat{n}_1+...+\hat{n}_N}\hat{D}_N^{-1}(\alpha_N')...\hat{D}_1^{-1}(\alpha_1'). \tag{2.79}$$

After some lengthy calculations, we arrive at the following relations,

$$V^{-1}\hat{a}_iV = \cosh r\hat{a}_i + \sinh r(\frac{2-N}{N}\hat{a}_i^\dagger + \frac{2}{N}\sum_{j\neq i}^{N}\hat{a}_j^\dagger), \tag{2.80}$$

$$V^{-1}\hat{a}_i^\dagger V = \cosh r\hat{a}_i^\dagger + \sinh r(\frac{2-N}{N}\hat{a}_i + \frac{2}{N}\sum_{j\neq i}^{N}\hat{a}_j). \tag{2.81}$$

The relations can be employed to yield squeezed displacement operator $\hat{D}_i(\alpha_i')$,

$$\begin{aligned}\hat{D}_i(\alpha_i') &= \exp(\alpha_i'\hat{a}^\dagger - \alpha_i'^*\hat{a}) \\ &= \exp\{[(\alpha_i \cosh r - \frac{2-N}{N}\alpha_i^* \sinh r - \frac{2}{N}\sum_{l\neq i}\alpha_l^* \sinh r)\hat{a}^\dagger \\ &\quad - (\alpha_i^* \cosh r - \frac{2-N}{N}\alpha_i \sinh r - \frac{2}{N}\sum_{l\neq i}\alpha_l \sinh r)\hat{a}]\}.\end{aligned} \tag{2.82}$$

Thus $\alpha_i' = \alpha_i \cosh r - \frac{2-N}{N}\alpha_i^* \sinh r - \frac{2}{N}\sum_{l\neq i}\alpha_l^* \sinh r$. Then the correlation function of N-mode squeezed state is given as

$$\begin{aligned}E'(\alpha_1, \alpha_2, ..., \alpha_N) &= \exp\{-2\cosh 2r \sum_{i=1}^{N} |\alpha_i|^2 \\ &\quad + \frac{4}{N}\sinh 2r \sum_{i<j}^{N}(\alpha_i\alpha_j + \alpha_i^*\alpha_j^*) \\ &\quad - \frac{N-2}{N}\sinh 2r \sum_{i=1}^{N}(\alpha_i^2 + \alpha_i^{*2})\}.\end{aligned} \tag{2.83}$$

The correlation function of the original EPR state is recovered in the limit of $r \to \infty$ for $N = 2$.

2.4.3 Violation of the Żukowski-Brukner Inequalities by the N-Mode Wigner Function

The N-mode NOPA (nondegenerate optical parametric amplification) field is equivalent to an entangled state of N oscillators. When $N = 2$, the correlation function in Ref. [32] is given,

$$\begin{aligned}E'(\alpha_1, \alpha_2) &= \exp\{-2\cosh 2r(|\alpha_1|^2 + |\alpha_2|^2) \\ &\quad + 2\sinh 2r(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)\}.\end{aligned} \tag{2.84}$$

When $N = 3$, the correlation function is

$$\begin{aligned}E'(\alpha_1, \alpha_2, \alpha_3) &= \exp\{-2\cosh 2r \sum_{i=1}^{3} |\alpha_i|^2 \\ &\quad + \frac{4}{3}\sinh 2r \sum_{i<j}^{3}(\alpha_i\alpha_j + \alpha_i^*\alpha_j^*) \\ &\quad - \frac{1}{3}\sinh 2r \sum_{i=1}^{3}(\alpha_i^2 + \alpha_i^{*2})\},\end{aligned} \tag{2.85}$$

and this is the same as the result given in Ref. [45]. The correlation function is determined by considering measurements corresponding to the settings $\alpha_1 = \{0, a\}$, $\alpha_2 = \{0, a\}$ and $\alpha_3 = \{-a, 0\}$, where a is a positive constant associated with the displacement magnitude. From these combinations, the following Bell quantity can be constructed from the 3-qubit Żukowski-Brukner inequality,

$$\begin{aligned}\mathcal{B}(3) \quad &= E'(0,0,0) + E'(0,a,-a) + E'(a,0,-a) - E'(a,a,0) \\ &= 1 + 2\exp\{(-4\cosh 2r - \frac{8}{3}\sinh 2r - \frac{4}{3}\sinh 2r)a^2\} \\ &- \exp\{(-4\cosh 2r + \frac{8}{3}\sinh 2r - \frac{4}{3}\sinh 2r)a^2\}. \qquad (2.86)\end{aligned}$$

For local hidden variables theories, we have the inequality [24] $-2 \leq \mathcal{B}(3) \leq 2$. If we perform an asymptotic analysis for large $|r|$ with $r < 0$, $\cosh 2r$ and $\sinh 2r$ can be replaced by $e^{-2r}/2$ and $-e^{-2r}/2$ respectively, and Eq. (2.86) becomes $\mathcal{B}(3) = 3 - \exp\{-\frac{8}{3}e^{-2r}a^2\}$ We see that when a^2/e^{2r} is large enough, the Bell inequality for three qubits is violated when $\mathcal{B}(3)$ approaches the value $\mathcal{B}_{\text{opt}} = 3$.

For $N = 4$, and choosing all α_i to be real, the correlation function can be written as

$$\begin{aligned}E'(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \quad &= \quad \exp\{(-2\cosh 2r - \sinh 2r) \\ &\times \sum_{i=1}^{4} \alpha_i^2 + 2\sinh 2r \sum_{i<j}^{4} \alpha_i \alpha_j\}. \qquad (2.87)\end{aligned}$$

Evaluating the quantity $\mathcal{B}(4)$ from the 4-qubit Żukowski-Brukner inequality, we have

$$\begin{aligned}\mathcal{B}(4) = \quad & -E'(\alpha_1^1, \alpha_2^1, \alpha_3^1, \alpha_4^1) + E'(\alpha_1^1, \alpha_2^1, \alpha_3^1, \alpha_4^2) + E'(\alpha_1^1, \alpha_2^1, \alpha_3^2, \alpha_4^1) \\ & + E'(\alpha_1^1, \alpha_2^1, \alpha_3^2, \alpha_4^2) + E'(\alpha_1^1, \alpha_2^2, \alpha_3^1, \alpha_4^1) + E'(\alpha_1^1, \alpha_2^2, \alpha_3^1, \alpha_4^2) \\ & + E'(\alpha_1^1, \alpha_2^2, \alpha_3^2, \alpha_4^1) - E'(\alpha_1^1, \alpha_2^2, \alpha_3^2, \alpha_4^2) + E'(\alpha_1^2, \alpha_2^1, \alpha_3^1, \alpha_4^1) \\ & + E'(\alpha_1^2, \alpha_2^1, \alpha_3^1, \alpha_4^2) + E'(\alpha_1^2, \alpha_2^1, \alpha_3^2, \alpha_4^1) - E'(\alpha_1^2, \alpha_2^1, \alpha_3^2, \alpha_4^2) \\ & + E'(\alpha_1^2, \alpha_2^2, \alpha_3^1, \alpha_4^1) - E'(\alpha_1^2, \alpha_2^2, \alpha_3^1, \alpha_4^2) - E'(\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^1) \\ & - E'(\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2). \qquad (2.88)\end{aligned}$$

Under a local realistic description, $\mathcal{B}(4) \leq 4$. By choosing appropriate measurements, we have $\mathcal{B}_{\text{opt}}(4) = 7.357$. That is to say that the 4-mode NOPA state shows strong nonlocality compared with 3-mode or 2-mode NOPA states.

$V = 2/\mathcal{B}_{\text{opt}}(N)$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$
ME states	0.707	0.5	0.354	0.25	0.177	0.125
Oscillator	0.913	0.667	0.544	0.4	0.318	0.229

Table 2.4: Threshold visibilities of maximally entangled states and entangled states of oscillator for $2 \leq N \leq 7$.

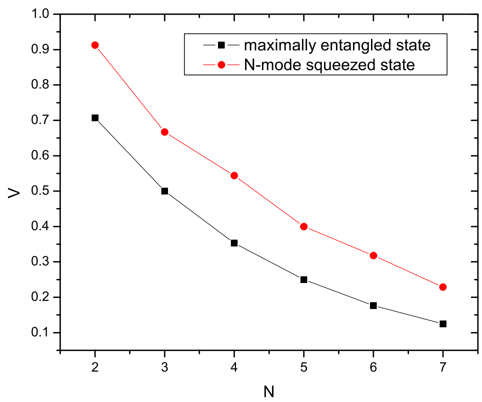

Figure 2.4: Critical visibilities of N-qubit Żukowski-Brukner inequalities (N = 2, 3, 4, 5, 6, 7) for both maximally entangled states and the entangled states of oscillator.

We also consider the strength of violation or visibility (V) [47] as the minimal amount V of the given entangled state $|\psi\rangle$ that one has to add to pure noise, ρ_{noise}, so that the resulting state violates local realism. The quantity V is thus the threshold visibility above which the state cannot be described by local realism, and it is sometimes called the critical visibility. More specifically, we consider Werner state of the form $\rho_w = V|\psi\rangle\langle\psi| + (1-V)\rho_{\text{noise}}$ where $\rho_{\text{noise}} = I/2^N$ is the completely mixed state. As shown in [24], for the maximally entangled state $|\psi\rangle_{\text{GHZ}} = 1/\sqrt{2}(|00\cdots0\rangle + |11\cdots1\rangle)$, the Werner state cannot be described by local realism if and only if $V > 1/\sqrt{2^{N-1}}$.

We repeat the calculation for entangled states of N oscillators (N=2,3,4,5,6,7) and their results are succinctly summarized in Table 2.4 and compared to the values for maximally entangled states. To see the variation of V with N, we also plot V versus the number of particles N both for maximally entangled (ME) states and N-mode squeezed states, see Figure 2.4. Naturally it is not surprising to see that the two systems show similar variations of V with increasing dimension N.

Alternatively, if one considers the optimal value of the violation for the Żukowski-Brukner inequalities, the optimal value for this violation grows with N. Increasing the number of qubits, in this case, will not bring us any closer to the classical regime, but rather it appears to discriminate better the quantum and the classical boundary. We also see that the entangled states of the oscillator do not violate the N-qubit Bell inequalities as much as the maximally entangled states do. However from the experimental perspective, NOPA state is easier to be generated than $|\psi\rangle_{\rm GHZ}$.

Our study shows that the multipartite entangled state in the Wigner representation exhibits nonlocal realism and this violation of local realism can be observed by using N-mode NOPA state. The violation of local realism for NOPA state can be manifested through the violation of N-particle Bell inequalities for correlation described by the Wigner function. This provides an exciting possibility to test the violation of local realism for the N-mode entangled state experimentally for the general case using the quantum N-mode squeezed state.

Chapter 3

2-QuNit Bell Inequalities and Applications

3.1 Bell Inequalities Involving Probabilities

One of the generalizations of the CHSH inequality is for 2 quNits. 2 quNits refers to a quantum system of two particles each in N-dimensional Hilbert space. In Ref. [20], a set of Bell inequalities, Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequalities, were achieved for two quNits. In this section, we will briefly review the CGLMP inequalities. The authors in Ref. [20] developed a powerful approach to the formulation of Bell inequalities. The method then was used to construct several families of Bell inequalities for bipartite higher-dimensional systems. Suppose that there are two observers Alice and Bob each can perform two possible N-outcome measurements (A_1 or A_2 for Alice, B_1 or B_2 for Bob). A local variable theory can be described by $4N^2$ probabilities $P(A_i = k, B_j = l)$, with $i, j = 1, 2$ and $k, l = 0, ..., N-1$. The probability $P(A_i = k, B_j = l)$ specifies that measurement A_i gives outcome k and measurement B_j gives outcome l.

They introduced the probability $P(A_i = B_j + k)$ that the measurement outcomes of A_i and B_j differ by k modulo N,

$$P(A_i = B_j + k) \equiv \sum_{m=0}^{N-1} P(A_i = m, B_j = m + k \mod N), \tag{3.1}$$

where mod is short for modulo. Take a system, for example, that consists of two particles with each particle in four-dimensional Hilbert space. If Alice chooses to measure observable A_1 and Bob chooses to measure observable B_1, the probability with their results that differ by 3 modulo 4 is

$$\begin{aligned} P(A_1 = B_1 + 3) &= P(A_1 = 3, B_1 = 0) + P(A_1 = 2, B_1 = 3) \\ &\quad + P(A_1 = 1, B_1 = 2) + P(A_1 = 0, B_1 = 1). \end{aligned} \tag{3.2}$$

The Bell inequalities given in Ref. [20] have the form

$$\begin{aligned} I_N &= \sum_{k=0}^{[N/2]-1} (1 - \frac{2k}{N-1}) \times \\ \{ &+ [P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) \\ &+ P(A_2 = B_2 + k) + P(B_2 = A_1 + k)] \\ &- [P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) \\ &+ P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1)]\} \leq 2. \end{aligned} \tag{3.3}$$

where I_N is the Bell quantity of the CGLMP inequalities. When $N = 2$, the CGLMP inequalities reduce to the CHSH inequality involving probabilities [20],

$$I_2 = P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1) \leq 2. \tag{3.4}$$

The quantum state considered in Ref. [20] was the maximally entangled state of two N-dimensional systems

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |jj\rangle. \tag{3.5}$$

Maximally entangled states are those that all its partial traces are maximally mixed. The observables considered in Ref. [20] were A_i for Alice and B_j for Bob. It was assumed that the observables have the following nondegenerate eigenstates [20]

$$\begin{aligned} |k\rangle_{A,i} &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \exp[i\frac{2\pi}{N} m(k + \alpha_i)]|m\rangle_A, \\ |l\rangle_{B,j} &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \exp[i\frac{2\pi}{N} m(-l + \beta_j)]|m\rangle_B, \end{aligned} \tag{3.6}$$

with $\alpha_1 = 0, \alpha_2 = 1/2, \beta_1 = 1/4$, and $\beta_2 = -1/4$. Thus the joint probabilities are [20]

$$\begin{aligned} P^{QM}(A_i = k, B_j = l) &= \langle\psi|kl\rangle\langle kl|\psi\rangle \\ &= \frac{1}{2N^3 \sin^2[\pi(k - l + \alpha_i + \beta_j)/N]}. \end{aligned} \tag{3.7}$$

It is clear that these probabilities depend on the difference between k and l and thus,

$$\begin{aligned} P^{QM}(A_i = B_j + c) &= \sum_{m=0}^{N-1} P^{QM}(A_i = m + c(\mod N), B_j = m) \\ &= NP^{QM}(A_i = c, B_j = 0), \end{aligned} \tag{3.8}$$

where mod is short for modulo. From these joint probabilities, quantum prediction of the Bell quantity can be written as [20]

$$\begin{aligned} I_N^{QM} &= 4N \sum_{k=0}^{[N/2]-1} (1 - \frac{2k}{N-1}) \times \\ & (\frac{1}{2N^3 \sin^2[\pi(k+1/4)/N]} - \frac{1}{2N^3 \sin^2[\pi(-k-3/4)/N]}). \end{aligned} \tag{3.9}$$

The authors calculated the maximum value that can be attained for the Bell quantity for quantum measurement on the entangled state. Specially, they found that [20]

$$\begin{aligned} I_3^{QM} &= 4/(-9 + 6\sqrt{3}) \simeq 2.87293, \\ I_4^{QM} &= \frac{2}{3}(\sqrt{2} + \sqrt{10 - \sqrt{2}}) \simeq 2.89624, \\ \lim_{N\to\infty} I_N^{QM} &= \frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(k+1/4)^2} - \frac{1}{(k+3/4)^2} \\ &\simeq 2.96981, \end{aligned} \tag{3.10}$$

The maximum value of the Bell quantity exceeds $2\sqrt{2}$ when dimension goes to infinity. They also showed numerically that these inequalities are optimal in the same sense as that the CHSH inequality is optimal for two-dimensional systems.

3.2 Bell Inequalities Involving Correlation Functions

The constraints on the correlations that local variable theories impose can also be written as Bell inequalities in terms of correlation functions. For 2 quNits, one type of correlation-Bell inequality was given by Fu [48]. In this paper, the author generalized the CHSH inequality to arbitrary high-dimensional systems based on correlation functions.

Recall that the CHSH inequality for two qubits reads

$$Q_{11} + Q_{12} - Q_{21} + Q_{22} \leq 2,$$

where Q_{ij} is known as correlation function of measurement on two qubits. The correlation functions can be expressed involving probabilities by ascribing $+1/-1$ to each probability,

$$Q_{ij} = \sum_{m=0}^{1}\sum_{n=0}^{1}(-1)^{m+n}P(A_i = m, B_j = n). \tag{3.11}$$

For two qutrits, Ref. [49] gave a Bell inequality in terms of correlation functions, which reads

$$\mathrm{Re}[Q'_{11} + Q'_{12} - Q'_{21} + Q'_{22}] + \frac{1}{\sqrt{3}}\mathrm{Im}[Q'_{11} - Q'_{12} - Q'_{21} + Q'_{22}] \leq 2, \tag{3.12}$$

where the correlation functions Q'_{ij} are defined by ascribing α^{m+n} to each probability,

$$Q'_{ij} = \sum_{m=0}^{2}\sum_{n=0}^{2}\alpha^{m+n}P(A_i = m, B_j = n), \tag{3.13}$$

with $\alpha = e^{i2\pi/3}$. The inequality (3.12) was reformed in Ref. [48] as,

$$Q_{11} + Q_{12} - Q_{21} + Q_{22} \leq 2, \tag{3.14}$$

by defining $Q_{ij} = \mathrm{Re}[Q'_{ij}] + 1/\sqrt{3}\mathrm{Im}[Q'_{ij}]$ for $i \geq j$, and $Q_{12} = \mathrm{Re}[Q'_{12}] - 1/\sqrt{3}\mathrm{Im}[Q'_{12}]$. The author then showed that, for 2 quNits, the correlation functions Q_{ij} can be defined by ascribing $f^{ij}(m,n)$ to each probability as follows [48]:

$$Q_{ij} \equiv \frac{1}{S}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f^{ij}(m,n)P(A_i = m, B_j = n), \tag{3.15}$$

in which $S = \frac{N-1}{2}$, the spin of the particle for the N-dimensional system, $f^{ij}(m,n) = S - M[\varepsilon(i-j)(m+n), N]$, and $\varepsilon(x)$ is the sign function: $\varepsilon(x) = \{^{1\ x\geq 0}_{-1\ x\leq 0}$. $M[x,N]$ means x modulo N. Then he constructed the CHSH-like expression for arbitrarily dimensional systems which takes the same form as the CHSH inequality, namely

$$I'_N = Q_{11} + Q_{12} - Q_{21} + Q_{22}. \tag{3.16}$$

The author proved that the maximum value of I'_N for local hidden variable theories is 2, i.e., $I'_N \leq 2$ [48]. The maximum value that can be attained for I'_N for quantum measurement on an entangled state is the same as that obtained in [20]. The standard form of the CHSH inequality for arbitrarily high-dimensionality by introducing the general correlation functions is an equivalent form of the CGLMP inequalities.

3.3 Maximal Violation of the Collins-Gisin-Linden-Massar-Popescu Inequalities

As far as Bell inequality is concerned, the CHSH inequality plays a very important role. It is already known that the CHSH inequality is violated by all pure bipartite entangled states [18]. Its maximal violation is only obtained for the maximally entangled state of two qubits. Maximally entangled states are those that all its partial traces are maximally mixed. In 1990, Peres and Gisin [50] showed that for two particles of N level, there is a limit for the violation of the CHSH inequality when $N \to \infty$. The result was given under such a condition: dichotomic observables are applied to a 2-quNit entangled state. For the case of any dichotomic observables measured on two quNits the violation of the CHSH inequality does not exceed the value bounded by the Cirel'son's limit, or $2\sqrt{2}$. For general observables other than dichotomic observables, whether the violation of Bell inequalities increases or not with growing N has attracted much attention. In 2000, Kaszlikowski *et al* [51] showed that violations of local realism are stronger for two maximally entangled quNits ($3 \leq N \leq 9$) than for two qubits. Moreover, the violation increases with increasing N. In that paper, authors used a numerical linear optimization method to show violation of local realism since no Bell inequality for 2 quNit except $N = 2$ was presented at that time. One year later, Durt *et al* [52] used a simple method, in which certain experimental settings for maximal violation of local realism were given, to extend similar calculations to $N = 16$.

In 2002, the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequalities [20] were developed that generalize the CHSH inequality to two-particle systems of arbitrary dimensions. This offers a possibility of testing violation of local realism based on the inequalities given in Ref. [20] as those done for the CHSH inequality. It was shown that two maximally entangled quNits violate the CGLMP inequalities stronger than two maximally entangled qubits in Ref. [20]. The authors also showed that the violation of the CGLMP inequalities increases with growing N. It is tempting to achieve the limit of $N \to \infty$, 2.96981 [20]. Due to the considered N-outcome measurement, violation of maximally entangled state can exceed Cirel'son's bound. However, it seems that such a limit is not a maximal violation of the CGLMP inequalities.

Recently it was shown [53] that there exist non-maximally entangled states that lead to more robust violations of local realism than maximally entangled states. In Ref. [53], Acín *et al* investigated such problems for bipartite systems in low dimensional Hilbert space, or for the cases of 2 quNit with $N = 3, 4, 5, 6, 7, 8$. They showed that a larger violation exists for a non-maximally entangled state. It will be interesting to generalize these results to higher dimensional systems. In this section, we will extend the computations up to $N = 8000$ and try to find the limit of the maximal violation of the CGLMP inequalities. The maximal violation and corresponding entangled state will be given for different dimensional systems[3].

It has been shown that symmetric multiport beam splitter [54] can be used to test violation of local realism of two maximally entangled quNits. Symmetric N-port beam splitter has the following property: a photon entering at any input ports has equal chance ($\frac{1}{N}$) of exit the device at any output ports. Bell multiport [54] is a device consists of the symmetric multiport beam splitter and phase shifters. Bell multiport beam splitters play an important role in testing nonexistence of local realism. For dichotomic observables, the Bell-EPR experiment can be realized by using the 2×2 beam splitter. For general nondichotomic observables, the Bell-EPR experiment can be generalized by using the Bell multiport beam splitters.

Here we follow Ref. [54] to review beam splitters. Beam splitter is an important device in experiment in quantum optics. It consists of two input ports and two output ports [55] (see Figure 3.1). The action of the beam splitter can be described by a unitary transformation V_2 that transforms the two input modes into the two output modes

$$\left(\begin{array}{c} a_1' \\ a_2' \end{array} \right) = V_2 \left(\begin{array}{c} a_1 \\ a_2 \end{array} \right) = \left(\begin{array}{cc} \sin\omega & e^{i\phi}\cos\omega \\ \cos\omega & -e^{i\phi}\sin\omega \end{array} \right) \left(\begin{array}{c} a_1 \\ a_2 \end{array} \right), \tag{3.17}$$

where the phase ϕ is the relative phase between two inputs, and ω represents the property of a beam splitter. Two parameters of a beam splitter are determined by ω: reflectivity $R = \cos^2\omega$ and transmittance $T = \sin^2\omega$. Specifically, the action of a 50:50 beam splitter can be represented by the following transformation matrix [54]

$$\frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right), \tag{3.18}$$

[3] This work is submitted for publication, see [8] in the publication list in Appendix A.

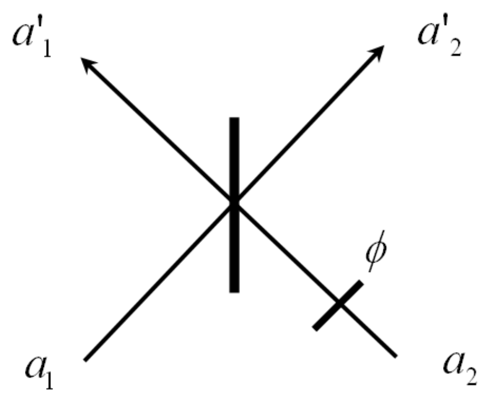

Figure 3.1: A beam splitter transforms (a_1, a_2) into (a_1', a_2') [54].

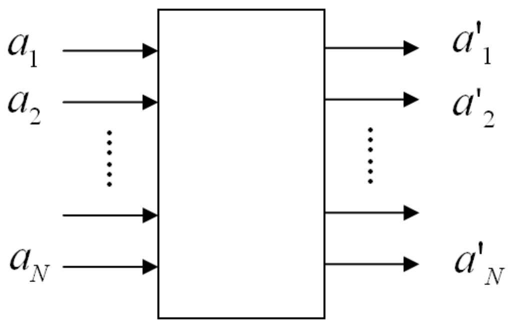

Figure 3.2: A multiport beam splitter transforms $(a_1, ..., a_N)$ into $(a_1', ..., a_N')$ [54].

in which the relative phase is chosen to be zero and $R = T = \frac{1}{2}$.

The concept of beam splitters can be generalized to multiport beam splitters for systems in N-dimensional Hilbert space [54]. Multiports transform N input modes into N output modes as shown in Figure 3.2. In Figure 3.2, a box was used to represent a multiport beam splitter which consists of beam splitters, mirrors and phase shifters. One type of unitary multiports is the symmetric multiport which is of interest in generalizing the test of violation of local realism to nondichotomic observables. The symmetric multiport beam splitter has an interesting property. The property is that the elements of its matrix are of the same modulus. Therefore, the action of the symmetric multiport is that one photon entering at one input port of a symmetric $N \times N$ multiport has the probability $\frac{1}{N}$ of being detected at any output port. A form for the symmetric multiport transformation is defined as V_N

with elements

$$V_N^{ij} = \frac{1}{\sqrt{N}} \alpha_N^{ij} \tag{3.19}$$

where $i, j = 0, ..., N-1$, by using the N-th root of unity $\alpha_N = \exp i\frac{2\pi}{N}$ [54]. In the following, two simple multiports (tritters and quarters) are explained in detail.

The 3×3 symmetric multiport is called a tritter which is a generalization of the 50:50 beam splitter with 3 input ports and 3 output ports. The form of the tritter matrix is [54]

$$V_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{2\pi}{3}} \end{pmatrix}. \tag{3.20}$$

The 4×4 symmetric multiport is called a quarter which is a generalization of the 50:50 beam splitter with 4 input ports and 4 output ports. One form of the quarter matrix is [54]

$$V_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{i\phi} & -1 & -e^{i\phi} \\ 1 & -1 & 1 & -1 \\ 1 & -e^{i\phi} & -1 & e^{i\phi} \end{pmatrix} \quad \text{with} \quad \phi = \pi/2. \tag{3.21}$$

By setting $\phi = \pi/2$, the matrix elements of the 8 ports beam splitter are powers of a root of unity.

The action of the multiport beam splitter can be described by a unitary transformation V with elements $V_{kl} = \frac{1}{\sqrt{N}} \alpha_N^{kl}$ where $\alpha_N = \exp(i2\pi/N)$. In front of i-th input port of the device a phase shifter is put to change the phase of the incoming photon by ϕ^i. One can denote the phase shifts as a N-dimensional vector $\hat{\phi} = (\phi^0, \phi^1, ..., \phi^{N-1})$ for convenience. These phase factors are the local parameters that can be changed by observers. The symmetric N-port beam splitter together with the N phase shifters perform the unitary transformation $U(\hat{\phi})$ with the elements $U_{kl} = V_{kl} \exp[i\phi^l]$. Devices with such a transformation matrix were called Bell multiports [54] as shown in Figure 3.3. One thing worth to note is that the N-th root of unity can also be chosen as $\alpha_N^* = \exp(-i2\pi/N)$ and hence $V'_{kl} = \frac{1}{\sqrt{N}} (\alpha_N^*)^{kl}$. In this way, the Bell multiports perform the transformation $U'(\hat{\phi}) = V' \exp[i\hat{\phi}]$.

Recall that the maximally entangled state of a bipartite system (3.5) which reads

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |jj\rangle,$$

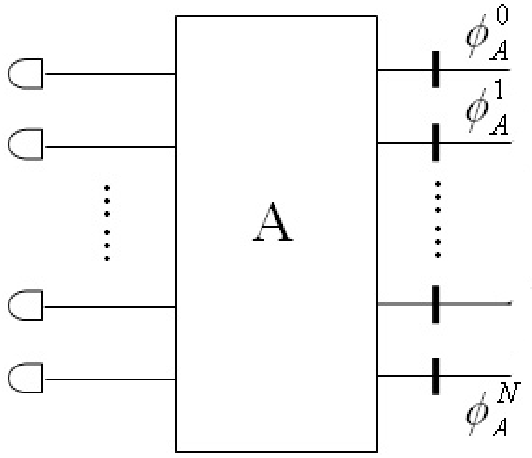

Figure 3.3: A Bell multiport consists of a symmetric multiport beam splitter and N phase shifters. N photon detectors are put at the output ports of the device [51].

where $|j\rangle$ is orthonormal base in each subsystem. The violation of the CGLMP inequalities by the maximally entangled state (3.5) can be analyzed in the following. The initial state is transformed by the Bell multiports into

$$|\psi'\rangle = U(\hat{\phi}_a) \otimes U'(\hat{\varphi}_b)|\psi\rangle \tag{3.22}$$

Thus the quantum joint probability $P(A_a = k, B_b = l)$ to detect a photon at the k-th output of A and another one at the l-th output of B is given by

$$\begin{aligned} P(A_a = k, B_b = l) &= |\langle kl|\psi'\rangle|^2 = |\langle kl|U(\hat{\phi}_a) \otimes U'(\hat{\varphi}_b)|\psi\rangle|^2 \\ &= \langle\psi|U(\hat{\phi}_a)^\dagger \otimes U'(\hat{\varphi}_b)^\dagger|kl\rangle\langle kl|U(\hat{\phi}_a) \otimes U'(\hat{\varphi}_b)|\psi\rangle \\ &= \mathrm{Tr}(U(\hat{\phi}_a)^\dagger \otimes U'(\hat{\varphi}_b)^\dagger\Pi_k \otimes \Pi_l U(\hat{\phi}_a) \otimes U'(\hat{\varphi}_b)|\psi\rangle\langle\psi|) \\ &= \frac{1}{N^3}\sum_{j,m=0}^{N-1} e^{i[\phi_a^j+\varphi_b^j+\frac{2\pi j}{N}(k-l)-\phi_a^m-\varphi_b^m-\frac{2\pi m}{N}(k-l)]}, \end{aligned} \tag{3.23}$$

where $\Pi_i (i = k, l)$ are projection operators. The quantum joint probabilities have one symmetry

$$P(A_a = k, B_b = l) = P(A_a = k + c, B_b = l + c), \tag{3.24}$$

for all integers c. The symmetry property leads to the following relation,

$$P(A_a = B_b + c) = \sum_{j=0}^{N-1} P(A_a = j + c, B_b = j) = NP(A_a = c, B_b = 0). \tag{3.25}$$

The values of the phases in the definition of joint probabilities can be chosen as [52]

$$\phi_1^j = 0, \quad \phi_2^j = \frac{j\pi}{N}, \quad \varphi_1^j = \frac{j\pi}{2N}, \quad \varphi_2^j = -\frac{j\pi}{2N}. \tag{3.26}$$

With the given experimental settings, quantum mechanics predicts that the Bell quantity I_N in Eq. (3.3) is given by

$$\begin{aligned} I_N &= 4N \sum_{k=0}^{[N/2]-1} (1 - \frac{2k}{N-1})(\frac{1}{2N^3 \sin^2[\pi(k+\frac{1}{4})/N]} \\ &\quad - \frac{1}{2N^3 \sin^2[\pi(-k-1+\frac{1}{4})/N]}), \end{aligned}$$

which is just Eq. (3.9). It was shown that the violation increases with growing N. When $N \to \infty$,

$$\lim_{N\to\infty} I_N = \frac{2}{\pi^2} \sum_{k=0}^{\infty} [\frac{1}{(k+1/4)^2} - \frac{1}{(k+3/4)^2}] \simeq 2.96981. \tag{3.27}$$

This is just the result given in Ref. [20].

However, it may happen that a larger value of the violation of the CGLMP inequalities can be found if a different initial state is considered [53]. Take an arbitrary entangled state of a bipartite system which reads

$$|\Phi\rangle = \sum_{j,j'=0}^{N-1} \alpha_{j,j'} |jj'\rangle, \tag{3.28}$$

as initial state. The quantum prediction of the joint probability can be written as

$$\begin{aligned} P(A_a = k, B_b = l) &= \mathrm{Tr}(U(\hat{\phi}_a)^\dagger \otimes U'(\hat{\varphi}_b)^\dagger \Pi_k \otimes \Pi_l U(\hat{\phi}_a) \otimes U'(\hat{\varphi}_b) |\Phi\rangle\langle\Phi|) \\ &= \frac{1}{N^2} \sum_{j,j',m,m'=0}^{N-1} \alpha_{j,j'} \alpha^*_{m,m'} \times \\ &\quad e^{i[\phi_a^j + \varphi_b^{j'} + \frac{2\pi}{N}(jk - j'l) - \phi_a^m - \varphi_b^{m'} - \frac{2\pi}{N}(mk - m'l)]}. \end{aligned} \tag{3.29}$$

To construct the Bell quantity $I_N(|\Phi\rangle)$, we have such a relation first,

$$P(A_a = B_b + c) = \sum_{j=0}^{N-1} P(A_a = j + c \pmod N, B_b = j). \tag{3.30}$$

Then,

$$
\begin{aligned}
I_N(|\Phi\rangle) = \quad & \frac{1}{N^2}\sum_{j,j',m,m'=0}^{N-1}\alpha_{j,j'}\alpha^*_{m,m'}\sum_{l=0}^{N-1}e^{i\frac{2\pi}{N}[(j-m)-(j'-m')]l}\times \\
& \{e^{i[\phi_1^j-\phi_1^m+\varphi_1^{j'}-\varphi_1^{m'}]}\times \\
& \sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{i\frac{2\pi}{N}k(j-m)}-e^{-i\frac{2\pi}{N}(k+1)(j'-m')})+ \\
& e^{i[\phi_1^j-\phi_1^m+\varphi_2^{j'}-\varphi_2^{m'}]}\times \\
& \sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{-i\frac{2\pi}{N}k(j'-m')}-e^{i\frac{2\pi}{N}(k+1)(j-m)})+ \\
& e^{i[\phi_2^j-\phi_2^m+\varphi_1^{j'}-\varphi_1^{m'}]}\times \\
& \sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{-i\frac{2\pi}{N}(k+1)(j'-m')}-e^{i\frac{2\pi}{N}k(j-m)})+ \\
& e^{i[\phi_2^j-\phi_2^m+\varphi_2^{j'}-\varphi_2^{m'}]}\times \\
& \sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{i\frac{2\pi}{N}k(j-m)}-e^{-i\frac{2\pi}{N}(k+1)(j'-m')})\}.
\end{aligned}
\tag{3.31}
$$

$I_N(|\Phi\rangle)$ can be expressed as $\langle\Phi|\hat{\mathcal{B}}|\Phi\rangle$ with $\hat{\mathcal{B}}$ the so called Bell operator [56]. As we know, a Bell quantity can be derived from the associated Bell operator. The expectation value of a Bell operator is the corresponding Bell quantity. In this case, the joint probability $P(A_a = k, B_b = l)$ when A_a and B_b are measured in the initial state $|\Phi\rangle$ is given by Eq. (3.29)

$$P(A_a = k, B_b = l) = \mathrm{Tr}(U(\hat{\phi}_a)^\dagger \otimes U'(\hat{\varphi}_b)^\dagger \Pi_k \otimes \Pi_l U(\hat{\phi}_a) \otimes U'(\hat{\varphi}_b)|\Phi\rangle\langle\Phi|).$$

From the formula, the Bell quantity is

$$I_N(|\Phi\rangle) = \mathrm{Tr}(\hat{\mathcal{B}}|\Phi\rangle\langle\Phi|) = \langle\hat{\mathcal{B}}\rangle_{|\Phi\rangle} = \mathcal{B}, \tag{3.32}$$

since the Bell quantity $I_N(|\Phi\rangle)$ is a linear combination of the joint probabilities.

One can write the Bell operator in a matrix form. Then the maximal eigenvalue is the highest quantum prediction of Bell quantity $I_N(|\Phi\rangle_{\mathrm{eig}})$ and the corresponding eigenfunction is the state $|\Phi\rangle_{\mathrm{eig}}$ that maximally violates the Bell inequality [53]. Starting from the CGLMP inequalities, we derive the corresponding Bell operator for the experimental settings (3.26) with elements,

$$\begin{aligned}
\hat{\mathcal{B}}_{(mm')(jj')} = \quad & \frac{1}{N^2}\sum_{l=0}^{N-1} e^{i\frac{2\pi}{N}[(j-m)-(j'-m')]l} \times \\
& \{e^{\frac{i\pi}{2N}(j'-m')}\sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{i\frac{2\pi}{N}k(j-m)} - e^{-i\frac{2\pi}{N}(k+1)(j'-m')}) + \\
& e^{-\frac{i\pi}{2N}(j'-m')}\sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{-i\frac{2\pi}{N}k(j'-m')} - e^{i\frac{2\pi}{N}(k+1)(j-m)}) + \\
& e^{\frac{i\pi}{N}(j-m)+\frac{i\pi}{2N}(j'-m')} \times \\
& \sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{-i\frac{2\pi}{N}(k+1)(j'-m')} - e^{i\frac{2\pi}{N}k(j-m)}) + \\
& e^{\frac{i\pi}{N}(j-m)-\frac{i\pi}{2N}(j'-m')} \times \\
& \sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{i\frac{2\pi}{N}k(j-m)} - e^{-i\frac{2\pi}{N}(k+1)(j'-m')})\}.
\end{aligned} \tag{3.33}$$

The maximum value of I_N is thus reached when $|\Phi\rangle$ is the eigenstate associated to the maximal eigenvalue of $\hat{\mathcal{B}}$, $|\Phi\rangle_{\rm eig}$. To determine what the eigenvalue is, note that $\sum_{l=0}^{N-1} e^{i\frac{2\pi}{N}[p-q]l} = N\delta_{pq}^{(N)}$, where $\delta_{pq}^{(N)} = 1$ when $p = q$ modulo N and 0 otherwise. So one can decompose the Bell operator $\hat{\mathcal{B}}$ into a sum of N reduced Bell operators that act individually inside the subspaces spanned by the following vectors $\{|00\rangle, |11\rangle, ..., |(N-1)(N-1)\rangle\}$, $\{|01\rangle, |12\rangle, ..., |(N-1)0\rangle\}$,...,$\{|0(N-1)\rangle, |10\rangle, ..., |(N-1)(N-2)\rangle\}$ respectively [53]. The problem is hence simplified. Inside the subspace spanned by the vectors $\{|00\rangle, |11\rangle, ..., |(N-1)(N-1)\rangle\}$, $j-m = j'-m'$ and the reduced Bell operator is given as

$$\begin{aligned}
\hat{\mathcal{B}}^{red1}_{mj} = \frac{1}{N} \quad & \{e^{\frac{i\pi}{2N}(j-m)}\sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{i\frac{2\pi}{N}k(j-m)} - e^{-i\frac{2\pi}{N}(k+1)(j-m)}) + \\
& e^{-\frac{i\pi}{2N}(j-m)}\sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{-i\frac{2\pi}{N}k(j-m)} - e^{i\frac{2\pi}{N}(k+1)(j-m)}) + \\
& e^{\frac{i3\pi}{2N}(j-m)}\sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{-i\frac{2\pi}{N}(k+1)(j-m)} - e^{i\frac{2\pi}{N}k(j-m)}) + \\
& e^{\frac{i\pi}{2N}(j-m)}\sum_{k=0}^{[N/2-1]}(1-\frac{2k}{N-1})(e^{i\frac{2\pi}{N}k(j-m)} - e^{-i\frac{2\pi}{N}(k+1)(j-m)})\}.
\end{aligned} \tag{3.34}$$

However, for the other subspaces, there is no general explicit form for the reduced Bell operators. The reason for this is due to the phase factors $e^{\frac{i\pi}{2N}(j'-m')}$, $e^{\frac{i\pi}{N}(j-m)+\frac{i\pi}{2N}(j'-m')}$ etc in Eq. (3.33). Take for example the second subspace spanned by the vectors $\{|01\rangle, |12\rangle, ..., |(N-1)0\rangle\}$, only when $j'-m' = (j-m)\text{mod}N$, the Bell operator does exist. In this case, these phase factors $e^{\frac{i\pi}{2N}(j'-m')}$, $e^{\frac{i\pi}{N}(j-m)+\frac{i\pi}{2N}(j'-m')}$ cannot be expressed only in terms of m, j. Thus, there is no simplified form for $\hat{\mathcal{B}}^{red2}$.

We proceed further to show how to find the maximal violation and the corresponding $|\Phi\rangle_{\text{eig}}$.

When $N = 3$, the reduced Bell operators can be written in the following matrices,

$$\hat{\mathcal{B}}^{red1} = \frac{2}{3}\begin{pmatrix} 0 & \sqrt{3} & 3 \\ \sqrt{3} & 0 & \sqrt{3} \\ 3 & \sqrt{3} & 0 \end{pmatrix},$$

$$\hat{\mathcal{B}}^{red2} = \frac{2}{3}\begin{pmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\mathcal{B}}^{red3} = \frac{2}{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \end{pmatrix}. \tag{3.35}$$

Now, the problem is to determine the maximal eigenvalues of these 3×3 matrices. The eigenvalues of $\hat{\mathcal{B}}^{red1}$ are equal to $-2, 1-\sqrt{11/3}$ and $1+\sqrt{11/3}$. For $\hat{\mathcal{B}}^{red2}$ and $\hat{\mathcal{B}}^{red3}$, we have $-2/\sqrt{3}, 0$ and $2/\sqrt{3}$. It is easy to check that the maximal violation is equal to $1+\sqrt{11/3} \simeq 2.9149$ and the corresponding eigenvector is $\frac{\sqrt{2}}{\sqrt{11-\sqrt{33}}}(|00\rangle + \frac{\sqrt{11}-\sqrt{3}}{2}|11\rangle + |22\rangle)$. These are the results shown in Ref. [53].

When $N = 4$, the reduced Bell operators can be written in the following matrices,

$$\hat{\mathcal{B}}^{red1} = \frac{2}{3}\begin{pmatrix} 0 & \sqrt{4-2\sqrt{2}} & \sqrt{2} & \sqrt{4+2\sqrt{2}} \\ \sqrt{4-2\sqrt{2}} & 0 & \sqrt{4-2\sqrt{2}} & \sqrt{2} \\ \sqrt{2} & \sqrt{4-2\sqrt{2}} & 0 & \sqrt{4-2\sqrt{2}} \\ \sqrt{4+2\sqrt{2}} & \sqrt{2} & \sqrt{4-2\sqrt{2}} & 0 \end{pmatrix},$$

$$\hat{\mathcal{B}}^{red2} = \frac{2}{3}\begin{pmatrix} 0 & \sqrt{4-2\sqrt{2}} & \sqrt{2} & 0 \\ \sqrt{4-2\sqrt{2}} & 0 & \sqrt{4-2\sqrt{2}} & 0 \\ \sqrt{2} & \sqrt{4-2\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\mathcal{B}}^{red3} = \frac{2}{3}\begin{pmatrix} 0 & \sqrt{4-2\sqrt{2}} & 0 & 0 \\ \sqrt{4-2\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4-2\sqrt{2}} \\ 0 & 0 & \sqrt{4-2\sqrt{2}} & 0 \end{pmatrix},$$

$$\hat{\mathcal{B}}^{red4} = \frac{2}{3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{4-2\sqrt{2}} & \sqrt{2} \\ 0 & \sqrt{4-2\sqrt{2}} & 0 & \sqrt{4-2\sqrt{2}} \\ 0 & \sqrt{2} & \sqrt{4-2\sqrt{2}} & 0 \end{pmatrix}. \quad (3.36)$$

Now, the problem is to determine the maximal eigenvalues of these 4×4 matrices. The eigenvalues of $\hat{\mathcal{B}}^{red1}$ are equal to
$\frac{1}{2}\sqrt{\frac{32}{9}+\frac{16\sqrt{2}}{9}}+\frac{2}{3\sqrt{3}}\sqrt{24-9\sqrt{2}+16\sqrt{\frac{2}{\frac{32}{9}+\frac{16\sqrt{2}}{9}}}}$,
$-\frac{1}{2}\sqrt{\frac{32}{9}+\frac{16\sqrt{2}}{9}}-\frac{2}{3\sqrt{3}}\sqrt{24-9\sqrt{2}-16\sqrt{\frac{2}{\frac{32}{9}+\frac{16\sqrt{2}}{9}}}}$,
$-\frac{1}{2}\sqrt{\frac{32}{9}+\frac{16\sqrt{2}}{9}}+\frac{2}{3\sqrt{3}}\sqrt{24-9\sqrt{2}-16\sqrt{\frac{2}{\frac{32}{9}+\frac{16\sqrt{2}}{9}}}}$ and
$\frac{1}{2}\sqrt{\frac{32}{9}+\frac{16\sqrt{2}}{9}}-\frac{2}{3\sqrt{3}}\sqrt{24-9\sqrt{2}+16\sqrt{\frac{2}{\frac{32}{9}+\frac{16\sqrt{2}}{9}}}}$.

For $\hat{\mathcal{B}}^{red2}$ and $\hat{\mathcal{B}}^{red4}$, we have $\frac{\sqrt{2}}{3}(1+\sqrt{17-8\sqrt{2}})$, $-\frac{2\sqrt{2}}{3}$, $\frac{\sqrt{2}}{3}(1-\sqrt{17-8\sqrt{2}})$ and 0.

For $\hat{\mathcal{B}}^{red3}$, we have $-\frac{2}{3}\sqrt{4-2\sqrt{2}}$ and $\frac{2}{3}\sqrt{4-2\sqrt{2}}$.

It is easy to check that the maximal violation is equal to $\frac{1}{2}\sqrt{\frac{32}{9}+\frac{16\sqrt{2}}{9}}$ $+\frac{2}{3\sqrt{3}}\sqrt{24-9\sqrt{2}+16\sqrt{\frac{2}{\frac{32}{9}+\frac{16\sqrt{2}}{9}}}} \simeq 2.9727$ and the corresponding eigenvector is $\frac{1}{\sqrt{2+2a^2}}$ $(|00\rangle + a|11\rangle + a|22\rangle + |33\rangle)$, with $a \simeq 0.739372$.

One more thing worthy to note is that experimental settings have no effect on the maximal violation achieved. The authors in Ref. [53] proved that there would not be a larger violation of the CGLMP inequalities by choosing different experimental settings. They took $N = 3$ as an example, by varing $\hat{\phi}_1$ and keeping the others fixed:

$$\begin{aligned} \phi_1^0 &= 0, \quad \phi_1^1 = \varepsilon, \\ \phi_1^2 &= \theta, \quad \phi_2^j = \frac{j\pi}{N}, \\ \varphi_1^j &= \frac{j\pi}{2N}, \quad \varphi_2^j = -\frac{j\pi}{2N}, \end{aligned} \quad (3.37)$$

the varied reduced Bell operators will be

$$\begin{aligned}
\hat{\mathcal{B}}^{red1} &= \frac{1}{3}\begin{pmatrix} 0 & \sqrt{3} & 3 \\ \sqrt{3} & 0 & \sqrt{3} \\ 3 & \sqrt{3} & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & \sqrt{3}e^{i\varepsilon} & 3e^{i\theta} \\ \sqrt{3}e^{-i\varepsilon} & 0 & \sqrt{3}e^{i(\theta-\varepsilon)} \\ 3e^{-i\theta} & \sqrt{3}e^{i(\varepsilon-\theta)} & 0 \end{pmatrix}, \\
\hat{\mathcal{B}}^{red2} &= \frac{1}{3}\begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & 0 & -3 \\ -\sqrt{3} & -3 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & \sqrt{3}e^{i\varepsilon} & \sqrt{3}e^{i\theta} \\ \sqrt{3}e^{-i\varepsilon} & 0 & 3e^{i(\theta-\varepsilon)} \\ \sqrt{3}e^{-i\theta} & 3e^{i(\varepsilon-\theta)} & 0 \end{pmatrix}, \\
\hat{\mathcal{B}}^{red3} &= \frac{1}{3}\begin{pmatrix} 0 & -3 & -\sqrt{3} \\ -3 & 0 & \sqrt{3} \\ -\sqrt{3} & \sqrt{3} & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 3e^{i\varepsilon} & \sqrt{3}e^{i\theta} \\ 3e^{-i\varepsilon} & 0 & \sqrt{3}e^{i(\theta-\varepsilon)} \\ \sqrt{3}e^{-i\theta} & \sqrt{3}e^{i(\varepsilon-\theta)} & 0 \end{pmatrix}.
\end{aligned} \tag{3.38}$$

The authors checked that the eigenvalues of the matrices given above are respectively $-1, (1-\sqrt{11/3})/2$ and $(1+\sqrt{11/3})/2$. These eigenvalues are not larger than $1+\sqrt{11/3}$ and similar results can be obtained by varing $\hat{\phi}_2, \hat{\varphi}_1$ or $\hat{\varphi}_2$. It is reasonable to draw a conclusion that the experimental settings defined in Eq. (3.26) are optimal for violation of the CGLMP inequalities [53].

Actually, an arbitrary state $|\Phi\rangle = \sum_{j,j'=0}^{N-1} \alpha_{jj'}|jj'\rangle$ can always be transformed into its Schmidt decomposition form $|\Phi\rangle = \sum_{j=0}^{N-1} a_j|jj\rangle$ through local unitary transformations, thus it is sufficient to study the maximal violation problem in the first subspace. By diagonalizing exactly the matrix $\hat{B}^{red1}$, we have extended the calculations of maximal violation of local realism $I_N(|\Phi\rangle_{\rm eig})$ of 2 quNits to a system of dimensions higher than 8. Tables 3.1 and 3.2 summarize these results. The higher the dimension, the more difficult to find a maximal violation is. The highest dimension that we have calculated is $N = 8000$. In Fig. 3.4, one may observe that $I_N(|\Phi\rangle_{\rm eig})$ increases with dimension N slowly. This means that there exists a limit for quantum violation when N goes to infinity. Until now, we do not have an exact value of the limit. Based on the data of $I_N(|\Phi\rangle_{\rm eig})$ from $N = 2$ to $N = 8000$, one has an empirical formula fitting $I_N(|\Phi\rangle_{\rm eig})$ numerically to the dimension N:

$$I_N^{\rm rough}(|\Phi\rangle_{\rm eig}) \simeq 3.9132 - 1.2891N^{-0.2226} \tag{3.39}$$

from which one can see that $I_N^{\rm rough}(|\Psi\rangle_{\rm eig}) \simeq 3.9132$ is a coarse-grained limit of the maximal violation for the CGLMP inequality when N tends to infinity.

Analysis of the eigenvectors $|\Phi\rangle_{\rm eig}$ shows that these eigenvectors numerically satisfy some general properties: for instance, $|\Phi\rangle_{\rm eig} = \sum_{j=0}^{N-1} a_j^{\rm eig}|jj\rangle$ with maximal

$N(Dimensions)$	3	4	5	6	7	8
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	2.9149	2.9727	3.0157	3.0497	3.0777	3.1013
$N(Dimensions)$	9	10	11	12	13	14
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.1217	3.1396	3.1555	3.1698	3.1827	3.1946
$N(Dimensions)$	15	16	17	18	19	20
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.2054	3.2155	3.2248	3.2335	3.2416	3.2492
$N(Dimensions)$	21	22	23	24	25	26
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.2564	3.2632	3.2696	3.2757	3.2815	3.287
$N(Dimensions)$	27	28	29	30	31	32
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.2923	3.2974	3.3022	3.3068	3.3113	3.3156
$N(Dimensions)$	33	34	35	36	37	38
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.3197	3.3237	3.3275	3.3312	3.3348	3.3383
$N(Dimensions)$	39	40	41	42	43	44
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.3416	3.3449	3.348	3.3511	3.3541	3.357
$N(Dimensions)$	45	46	47	48	49	50
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.3598	3.3625	3.3652	3.3678	3.3703	3.3728
$N(Dimensions)$	51	52	53	54	55	56
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.3752	3.3776	3.3799	3.3821	3.3843	3.3865
$N(Dimensions)$	57	58	59	60	61	62
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.3886	3.3906	3.3926	3.3946	3.3965	3.3984
$N(Dimensions)$	63	64	65	66	67	68
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.4003	3.4021	3.4039	3.4057	3.4074	3.4091
$N(Dimensions)$	69	70	71	72	73	74
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.4107	3.4124	3.414	3.4155	3.4171	3.4186
$N(Dimensions)$	75	76	77	78	79	80
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.4201	3.4216	3.423	3.4245	3.4259	3.4273
$N(Dimensions)$	81	82	83	84	85	86
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.4286	3.43	3.4313	3.4326	3.4339	3.4351
$N(Dimensions)$	87	88	89	90	91	92
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.4364	3.4376	3.4388	3.44	3.4412	3.4423
$N(Dimensions)$	93	94	95	96	97	98
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.4435	3.4446	3.4457	3.4468	3.4479	3.449
$N(Dimensions)$	99	100	110	120	130	140
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.4501	3.4511	3.4609	3.4697	3.4776	3.4848

Table 3.1: Maximal violation of the Collins-Gisin-Linden-Massar-Popescu inequalities for different dimensional systems (Part I).

$N(Dimensions)$	150	160	170	180	190	200
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.4914	3.4975	3.5031	3.5083	3.5132	3.5178
$N(Dimensions)$	210	220	230	240	250	260
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.5221	3.5261	3.5299	3.5336	3.537	3.5403
$N(Dimensions)$	270	280	290	300	310	320
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.5434	3.5464	3.5492	3.552	3.5546	3.5571
$N(Dimensions)$	330	340	350	360	370	380
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.5595	3.5619	3.5641	3.5663	3.5684	3.5704
$N(Dimensions)$	390	400	410	420	430	440
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.5724	3.5743	3.5761	3.5779	3.5797	3.5814
$N(Dimensions)$	450	460	470	480	490	500
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.583	3.5846	3.5861	3.5877	3.5891	3.5906
$N(Dimensions)$	510	520	530	540	550	560
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.592	3.5934	3.5947	3.596	3.5973	3.5985
$N(Dimensions)$	570	580	590	600	610	620
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.5997	3.6009	3.6021	3.6033	3.6044	3.6055
$N(Dimensions)$	630	640	650	660	670	680
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.6066	3.6076	3.6087	3.6097	3.6107	3.6117
$N(Dimensions)$	690	700	710	720	730	740
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.6126	3.6136	3.6145	3.6154	3.6163	3.6172
$N(Dimensions)$	750	760	770	780	790	800
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.6181	3.6189	3.6198	3.6206	3.6214	3.6222
$N(Dimensions)$	810	820	830	840	850	860
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.623	3.6238	3.6245	3.6253	3.626	3.6268
$N(Dimensions)$	870	880	890	900	910	920
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.6275	3.6282	3.6289	3.6296	3.6303	3.6309
$N(Dimensions)$	930	940	950	960	970	980
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.6316	3.6323	3.6329	3.6335	3.6342	3.6348
$N(Dimensions)$	990	1000	1100	1200	1300	1400
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.6354	3.636	3.6417	3.6468	3.6514	3.6556
$N(Dimensions)$	1500	1600	1700	1800	1900	2000
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.6594	3.6629	3.6662	3.6692	3.672	3.6747
$N(Dimensions)$	2250	2500	2750	3000	3500	4000
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.6807	3.6859	3.6905	3.6946	3.7017	3.7077
$N(Dimensions)$	5000	6000	7000	8000		
Maximal violation $I_N(\|\Phi\rangle_{\text{eig}})$	3.7174	3.7250	3.7311	3.7362		

Table 3.2: Maximal violation of the Collins-Gisin-Linden-Massar-Popescu inequalities for different dimensional systems (Part II).

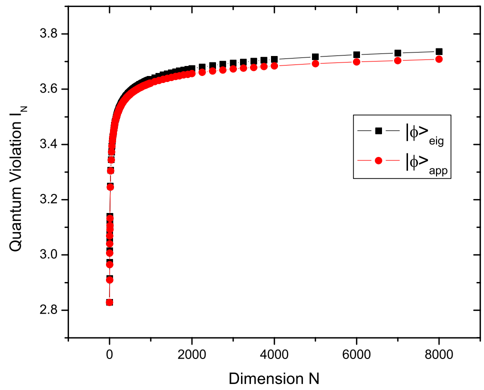

Figure 3.4: Variations of $I_N(|\Phi\rangle)$ with increasing dimension N ($2 \leq N \leq 8000$). The black line is the result for $|\Phi\rangle_{\text{eig}}$ and the red line is the result for $|\Phi\rangle_{\text{app}}$.

eigenvalue has the following symmetric properties for the coefficients: $a_j = a_{N-1-j}$; and $a_0 : a_1 : a_2 : a_3 : \cdots \simeq 1 : \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{3}} : \frac{1}{\sqrt{4}} : \cdots$ for large N. Thus, we may approximate a family of elegant entangled states

$$
\begin{aligned}
|\Phi\rangle_{\text{app}} &= \sum_{j=0}^{N-1} a_j^{\text{app}} |jj\rangle, \quad a_j^{\text{app}} = \frac{1}{\sqrt{\mathcal{N}}} \frac{1}{\sqrt{(j+1)(N-j)}}, \\
\mathcal{N} &= \sum_{j=0}^{N-1} \frac{1}{(j+1)(N-j)}
\end{aligned}
\tag{3.40}
$$

whose corresponding Bell expressions $I_N(|\Phi\rangle_{\text{app}})$ are closed to the actual ones $I_N(|\Phi\rangle_{\text{eig}})$. For example, for $N = 8000$, the error rate between $I_N(|\Phi\rangle_{\text{eig}})$ and $I_N(|\Phi\rangle_{\text{app}})$ is only about 0.745%. We have also plot $I_N(|\Phi\rangle_{\text{app}})$ versus dimension N in Fig. 3.4. It is clear that the Bell quantities $I_N(|\Phi\rangle_{\text{app}})$ and $I_N(|\Phi\rangle_{\text{eig}})$ show similar variation with increasing dimension. The Bell quantities $I_N(|\Phi\rangle_{\text{app}})$ are closed to the actual ones $I_N(|\Phi\rangle_{\text{eig}})$. As we know that nonlocal resource which is highly resistant to noise is needed in quantum information processing. It may be significant and interesting to apply the symmetric entangled states $|\Phi\rangle_{\text{app}}$ to quantum protocol of quantum information.

3.4 A New Set of Bell Inequalities Based on Multi-Component Correlation Functions

In this section, we propose a new set of Bell inequalities, which are based on multi-component correlation functions, for bipartite systems by utilizing N-outcome measurement. We then investigate violation of the inequalities for continuous-variable systems with finite value of squeezing parameter. The violation strength of continuous-variable state with finite squeezing parameter is stronger than that of the maximally entangled state[4].

3.4.1 Bell Inequalities for Multi-Component Correlation Functions

We consider a Bell-type scenario: two space-separated observers, denoted by Alice and Bob, measure two different local observables of N outcomes, labeled by $0, 1, ..., N-1$. We denote X_i the observable measured by party X and x_i the outcome with $X = A, B (x = a, b)$. If the observers decide to measure A_1, B_2, the result is $(0, 4)$ with probability $P(a_1 = 0, b_2 = 4)$. Now let us introduce N $(N-1)$-dimensional unit vectors

$$
\begin{aligned}
\mathbf{v}_0 &= (1,0,0,0,\cdots,0,0),\\
\mathbf{v}_1 &= \Big(-\frac{1}{N-1},\frac{\sqrt{(N-1)^2-1}}{N-1},0,0,\cdots,0,0\Big),\\
\mathbf{v}_2 &= \Big(-\frac{1}{N-1},-\frac{1}{N-1}\sqrt{\frac{N(N-1)}{(N-1)(N-2)}},\frac{N-3}{N-1}\sqrt{\frac{N(N-1)}{(N-2)(N-3)}},0,\\
&\quad\cdots,0,0\Big),\\
&\ \vdots\\
\mathbf{v}_{N-2} &= \Big(-\frac{1}{N-1},-\frac{1}{N-1}\sqrt{\frac{N(N-1)}{(N-1)(N-2)}},-\frac{1}{N-1}\sqrt{\frac{N(N-1)}{(N-2)(N-3)}},\\
&\quad\cdots,-\frac{1}{N-1}\sqrt{\frac{N(N-1)}{3\cdot 2}},\frac{1}{N-1}\sqrt{\frac{N)N-1)}{2\cdot 1}}\Big),\\
\mathbf{v}_{N-1} &= \Big(-\frac{1}{N-1},-\frac{1}{N-1}\sqrt{\frac{N(N-1)}{(N-1)(N-2)}},-\frac{1}{N-1}\sqrt{\frac{N(N-1)}{(N-2)(N-3)}},\\
&\quad\cdots,-\frac{1}{N-1}\sqrt{\frac{N(N-1)}{3\cdot 2}},-\frac{1}{N-1}\sqrt{\frac{N(N-1)}{2\cdot 1}}\Big).
\end{aligned}
\tag{3.41}
$$

[4]This work was published, see [4] in the publication list in Appendix A.

These N vectors satisfy following properties:

$$
\begin{aligned}
&(i) \quad \sum_{j=0}^{N-1} \mathbf{v}_j = 0, \\
&(ii) \quad \mathbf{v}_j \cdot \mathbf{v}_k \equiv -\frac{1}{N-1} \quad (j \neq k).
\end{aligned}
\tag{3.42}
$$

For $N = 2$, there are just two valued variables (i.e., $\mathbf{v}_0 = 1, \mathbf{v}_1 = -1$) obtained from a measurement. If the measured result of Alice is j, and Bob's result is k (where j and k are less than N), we then associate a vector $\mathbf{v}_{j+k}$ for the correlation between Alice and Bob [$\mathbf{v}_{j+k}$ understood as $\mathbf{v}_m$, where $m = (j + k), \text{modulo } N$]. Based on which, we now construct a multi-component correlation function:

$$
\begin{aligned}
\vec{Q}_{ij} &= \sum_{m,n} \mathbf{v}_{m+n} P(a_i = m, b_j = n) \\
&= \sum_{t=0}^{N-1} \mathbf{v}_t P(m + n = t),
\end{aligned}
\tag{3.43}
$$

where $P(a_i = m, b_j = n)$ is the joint probability of a_i obtaining outcome m and b_j obtaining outcome n. We ascribe a vector $\mathbf{v}_{m+n}$ to each probability $P(a_i = m, b_j = n)$ to define a new correlation function just as $(-1)^{m+n}$ has been ascribed to $P(a_i = m, b_j = n)$ to define usual correlation function for two qubits. $\vec{Q}_{ij} = (Q_{ij}^{(0)}, Q_{ij}^{(1)}, Q_{ij}^{(2)}, \cdots, Q_{ij}^{(N-2)})$, $Q_{ij}^{(k)}$ represents the k-th component of the vector correlation function $\vec{Q}_{ij}$.

We now define a Bell quantity involving the multi-component correlation functions,

$$
\begin{aligned}
\mathcal{B}_N &= \mathcal{B}^{(0)} + \sqrt{\frac{(N-1)(N-2)}{N(N-1)}} \mathcal{B}^{(1)} + \sqrt{\frac{(N-2)(N-3)}{N(N-1)}} \mathcal{B}^{(2)} \\
&\quad + \cdots + \sqrt{\frac{2 \cdot 1}{N(N-1)}} \mathcal{B}^{(N-2)} \\
&= \sum_{k=0}^{N-2} \sqrt{\frac{(N-k)(N-1-k)}{N(N-1)}} \mathcal{B}^{(k)},
\end{aligned}
\tag{3.44}
$$

where

$$
\begin{aligned}
\mathcal{B}^{(0)} &= Q_{11}^{(0)} + Q_{12}^{(0)} - Q_{21}^{(0)} + Q_{22}^{(0)}, \\
\mathcal{B}^{(k)} &= Q_{11}^{(k)} - Q_{12}^{(k)} - Q_{21}^{(k)} + Q_{22}^{(k)} \quad (k \neq 0).
\end{aligned}
\tag{3.45}
$$

Any local realistic description of the previous Gedanken experiment imposes the following inequality:

$$\mathcal{B}_N \leq 2. \tag{3.46}$$

To see that the above inequality is always satisfied in a local realistic description, let us reconstruct the expression of $\mathcal{B}_N$. First, we write

$$\begin{aligned}
Q_{11} &= \sum_{k=0}^{N-2} \sqrt{\frac{(N-k)(N-1-k)}{N(N-1)}} Q_{11}^{(k)}, \\
Q_{12} &= Q_{12}^{(0)} - \sum_{k=1}^{N-2} \sqrt{\frac{(N-k)(N-1-k)}{N(N-1)}} Q_{12}^{(k)}, \\
Q_{21} &= \sum_{k=0}^{N-2} \sqrt{\frac{(N-k)(N-1-k)}{N(N-1)}} Q_{21}^{(k)}, \\
Q_{22} &= \sum_{k=0}^{N-2} \sqrt{\frac{(N-k)(N-1-k)}{N(N-1)}} Q_{22}^{(k)}.
\end{aligned} \tag{3.47}$$

Then, the Bell quantity $\mathcal{B}_N$ can be written in a CHSH-type form, namely, $\mathcal{B}_N = Q_{11} + Q_{12} - Q_{21} + Q_{22}$. It is calculated that Q_{11}, Q_{21}, $Q_{22} = 1, \frac{N-3}{N-1}, \frac{N-5}{N-1}, ..., -\frac{N-5}{N-1}, -\frac{N-3}{N-1}, -1$, and $Q_{12} = 1, -1, -\frac{N-3}{N-1}, -\frac{N-5}{N-1}, ..., \frac{N-5}{N-1}, \frac{N-3}{N-1}$ under local realistic description. So the maximum value for each Q_{ij} is plus one and minimum value is minus one. As a result, it seems that the maximum value of $\mathcal{B}_N$ is 4. However, the bound of 4 is not achievable because a_1, b_1, a_2 and b_2 are correlated. To prove that the actual bound of $\mathcal{B}_N$ is 2, we define that $a_i + b_j = t_{ij}$ and the correlation of a_i, b_j gives

$$a_1 + b_1 + a_2 + b_2 = a_1 + b_2 + a_2 + b_1. \tag{3.48}$$

So we have

$$t_{11} + t_{22} = t_{12} + t_{21}. \tag{3.49}$$

The correlation functions Q_{ij} can be written in terms of t_{ij} as follows,

$$\begin{aligned} Q_{11} &= \frac{N-1-2M[t_{11},N]}{N-1}, \\ Q_{12} &= \frac{2M[t_{12},N]-N-1}{N-1}, \\ Q_{21} &= \frac{N-1-2M[t_{21},N]}{N-1}, \\ Q_{22} &= \frac{N-1-2M[t_{22},N]}{N-1}, \end{aligned} \tag{3.50}$$

for $t_{ij} > 0$ and where $M[t_{ij}, N]$ is defined as t_{ij} mod N. When $t_{ij} = 0$, $Q_{ij} = 1$. By using these Q_{ij}s, we can find that $\mathcal{B}_N$ is bounded by 2. We consider different cases of the values of t_{ij} to give the bound.

1. First we consider cases in which t_{12} is equal to zero because when $t_{12} = 0$, Q_{12} cannot be described by (3.50), while the other three Q_{ij} can be described by (3.50) even when $t_{ij} = 0$. So the cases in which $t_{12} = 0$ are considered specially.

   (a) When $t_{12} = 0$ and $0 \leq t_{11}, t_{21}, t_{22} < N$,

   $$\mathcal{B}_N = \frac{2t_{21} - N + 1 - 2t_{11} + N - 1 - 2t_{22} + N - 1}{N-1} + 1 = 2, \tag{3.51}$$

   because $t_{21} = t_{11} + t_{22}$.

   (b) When $t_{12} = 0$, $t_{21} \geq N$, and $0 \leq t_{11}, t_{22} < N$,

   $$\begin{aligned} \mathcal{B}_N &= \frac{2(t_{21} - N) - N + 1 - 2t_{11} + N - 1 - 2t_{22} + N - 1}{N-1} + 1 \\ &= -\frac{2}{N-1}, \end{aligned} \tag{3.52}$$

   because $t_{21} = t_{11} + t_{22}$.

   (c) When $t_{12} = 0$, $t_{11}/t_{22} \geq N$, and $0 \leq t_{21}, t_{22}/t_{11} < N$, these cases cannot exist since $t_{12} + t_{21} < N$ while $t_{11} + t_{22} > N$ which disobey constraint (3.49).

   (d) When $t_{12} = 0$, $t_{11}/t_{22}, t_{21} \geq N$, and $0 \leq t_{22}/t_{11} < N$,

   $$\begin{aligned} \mathcal{B}_N &= \frac{2(t_{21} - N) - N + 1 - 2t_{11} + N - 1 - 2t_{22} + N - 1 + 2N}{N-1} + 1 \\ &= 2, \end{aligned} \tag{3.53}$$

   because $t_{21} = t_{11} + t_{22}$.

(e) When $t_{12} = 0$, $t_{11}, t_{22} \geq N$, and $0 \leq t_{21} < N$, the case cannot exist since $t_{12} + t_{21} < N$ while $t_{11} + t_{22} > N$ which disobey constraint (3.49).

(f) When $t_{12} = 0$, $t_{11}, t_{22}, t_{21} \geq N$, the case cannot exist since $t_{12} + t_{21} < 2N$ while $t_{11} + t_{22} \geq 2N$ which disobey constraint (3.49).

2. In the following cases, t_{12} is larger than zero.

(a) For the case that all the t_{ij} are less than N,

$$\mathcal{B}_N = \frac{2(t_{12} + t_{21} - t_{11} - t_{22}) - 2}{N - 1} = -\frac{2}{N - 1}, \tag{3.54}$$

because of the constraint (3.49).

(b) For the case that t_{11} (or t_{22}) is larger than or equal to N, and t_{12}, t_{21}, t_{22} (or t_{11}) are less than N,

$$\mathcal{B}_N = \frac{2(t_{12} + t_{21} - t_{11} - t_{22} + N) - 2}{N - 1} = 2, \tag{3.55}$$

because of the constraint (3.49).

(c) For the case that t_{12} (or t_{21}) is larger than or equal to N, and t_{11}, t_{22}, t_{21} (or t_{12}) are less than N,

$$\mathcal{B}_N = \frac{2(t_{12} + t_{21} - t_{11} - t_{22} - N) - 2}{N - 1} = -\frac{2N + 2}{N - 1}, \tag{3.56}$$

because of the constraint (3.49).

(d) There are two special cases when two of the four t_{ij} are less than N and the other two are larger than or equal to N. One is that t_{11} and t_{22} are less than N, and t_{12} and t_{21} are larger than or equal to N; the other one is that t_{12} and t_{21} are less than N, and t_{11} and t_{22} are larger than or equal to N. The constraint (3.49) tells us these two cases cannot exist.

(e) For the case that $t_{11}, t_{12}/t_{21}$ are larger than or equal to N, and $t_{22}, t_{21}/t_{12}$ are less than N,

$$\mathcal{B}_N = \frac{2(t_{12} + t_{21} - N - t_{11} - t_{22} + N) - 2}{N - 1} = -\frac{2}{N - 1}. \tag{3.57}$$

(f) For the case that $t_{22}, t_{12}/t_{21}$ are larger than or equal to N, and $t_{11}, t_{21}/t_{12}$ are less than N,

$$\mathcal{B}_N = \frac{2(t_{12} + t_{21} - N - t_{11} - t_{22} + N) - 2}{N - 1} = -\frac{2}{N - 1}. \tag{3.58}$$

(g) For the case that t_{11} (or t_{22}) is less than N, and t_{12}, t_{21}, t_{22} (or t_{11}) are no less than N,

$$\mathcal{B}_N = \frac{2(t_{12}+t_{21}-N-N-t_{11}-t_{22}+N)-2}{N-1} = -\frac{2N+2}{N-1}. \tag{3.59}$$

(h) For the case that t_{12} (or t_{21}) is less than N, and t_{11}, t_{22}, t_{21} (or t_{12}) are no less than N,

$$\mathcal{B}_N = \frac{2(t_{12}+t_{21}-N-t_{11}-t_{22}+N+N)-2}{N-1} = 2, \tag{3.60}$$

because of the constraint (3.49).

(i) For the case that all the t_{ij} are larger than or equal to N,

$$\begin{aligned}\mathcal{B}_N &= \frac{2(t_{12}+t_{21}-N-N-t_{11}-t_{22}+N+N)-2}{N-1} \\ &= -\frac{2}{N-1},\end{aligned} \tag{3.61}$$

because of the constraint (3.49).

Therefore, for all choices of t_{ij}, the Bell quantity $\mathcal{B}_N$ is no larger than 2.

Obviously, the inequalities reduce to the usual CHSH inequality for $N=2$. In the case of $N=3$, a Bell inequality for two qutrits can be derived from inequality (3.46),

$$Q_{11}^{(0)}+Q_{12}^{(0)}-Q_{21}^{(0)}+Q_{22}^{(0)}+\frac{1}{\sqrt{3}}(Q_{11}^{(1)}+Q_{12}^{(1)}-Q_{21}^{(1)}+Q_{22}^{(1)}) \leq 2. \tag{3.62}$$

This is an equivalent version to the inequality for two qutrits given in Ref. [49].

The quantum prediction for the joint probability reads

$$P^{QM}(a_i=m,b_j=n) = \langle\psi|\hat{P}(a_i=m)\otimes\hat{P}(b_j=n)|\psi\rangle, \tag{3.63}$$

where $i,j=1,2$; $m,n=0,...,N-1$, $\hat{P}(a_i=m)=\mathcal{U}_A^\dagger|m\rangle\langle m|\mathcal{U}_A$ is the projector of Alice for the i-th measurement and similar definition for $\hat{P}(b_j=n)$. Then the quantum version of $\mathcal{B}_N$ can be calculated by using $P^{QM}(a_i=m,b_j=n)$. The violation of local realism for 2-quNit discrete-variable system has been investigated in Refs. [20] and [48]. We shall investigate violation of local realism for 2-quNit continuous-variable systems by using the inequalities (3.46) in next section.

3.4.2 Violation of the Bell Inequalities for Continuous-Variable Systems

Recently, Banaszek and Wódkiewicz [32] invoked the notion of parity as the measurement operator and interpreted the Wigner function as a correlation function for these parity measurements. They showed that the EPR state and the two-mode squeezed vacuum state do not have a local realistic description in the sense that they violate Bell inequalities such as the Clauser and Horne inequality [15] and the Clauser-Horne-Shimony-Holt (CHSH) [9] inequality. In the limit $r \to \infty$, when the original EPR state is recovered, an obvious violation of Bell inequality takes place, however, the violation is not very strong. To avoid the unsatisfactory feature, Chen *et al.* [23] introduced "pseudospin" operators based on parity, due to the fact that the degree of quantum nonlocality that we can uncover crucially depends not only on the given quantum state but also on the Bell operator [56]. To test quantum violation of the CHSH inequality for the two-mode squeezed vacuum states, the authors in Ref. [23] wrote the CHSH-Bell operator in terms of these "pseudospin" operators and obtained the maximum value of the expectation value of the CHSH-Bell operator as

$$\langle \mathrm{NOPA}|\hat{\mathcal{B}}_{\mathrm{CHSH}}|\mathrm{NOPA}\rangle_{\max} = 2\sqrt{1+\tanh^2 2\mathrm{r}}. \tag{3.64}$$

When the squeezing parameter r goes to infinity, the NOPA state becomes the original normalized EPR state for which $\langle \mathrm{NOPA}|\hat{\mathcal{B}}_{\mathrm{CHSH}}|\mathrm{NOPA}\rangle_{\max} = 2\sqrt{2}$. Thus the violation of the CHSH inequality for the original EPR states can reach the Cirel'son bound $2\sqrt{2}$.

Now we have Bell inequalities for 2 quNits. It will be interesting to extend two-outcome measurement to N-outcome measurement when testing quantum nonlocality of continuous-variable systems. The new inequalities involving correlation functions for 2 quNits are used to test violation of local realism for a general continuous-variable case in this section.

It is well known that the two-mode squeezed vacuum state can be generated in the nondegenerate optical parametric amplifier (NOPA) [22]

$$|\mathrm{NOPA}\rangle = e^{r(a_1^\dagger a_2^\dagger - a_1 a_2)}|00\rangle = \sum_{n=0}^{\infty} \frac{(\tanh r)^n}{\cosh r}|nn\rangle.$$

Following Brukner *et al.* [57], we can map the two-mode squeezed state onto a N-dimensional pure state:

$$|\psi_N\rangle = \frac{\mathrm{sech} r}{\sqrt{1-\tanh^{2N} r}} \sum_{n=0}^{N-1} (\tanh r)^n |nn\rangle. \tag{3.65}$$

If the measurement result of Alice is j, and Bob's result is k, we then ascribe a vector $\mathbf{v}_{j+k}$ for the correlation between Alice and Bob. $P(a_i = m, b_j = n)$ is the joint probability of a_i obtain m and b_j obtain n. More precisely, for the two-mode squeezed state one obtains following quantum joint probability

$$P^{QM}(a_i = m, b_j = n) = \langle\psi_N|\hat{P}(a_i = m) \otimes \hat{P}(b_j = n)|\psi_N\rangle. \tag{3.66}$$

For $N = 3$, we have three vectors $\mathbf{v}_0 = (1, 0)$, $\mathbf{v}_1 = (-1/2, \sqrt{3}/2)$, $\mathbf{v}_2 = (-1/2, -\sqrt{3}/2)$. Accordingly the NOPA state is divided into three groups, namely,

$$\begin{aligned} |\mathrm{NOPA}\rangle \;=\; & \frac{1}{\cosh r} \sum_{n=0}^{\infty} \Big(\tanh^{3n} r|3n\rangle|3n\rangle \\ & + \tanh^{3n+1} r|3n+1\rangle|3n+1\rangle \\ & + \tanh^{3n+2} r|3n+2\rangle|3n+2\rangle \Big). \end{aligned} \tag{3.67}$$

If we use Bell multiports to test the violation of local realism for the NOPA state, the projection operator can be written as

$$\begin{aligned} \hat{P}(a_i = m) &= U(\hat{\phi}_i)^\dagger \Pi_m U(\hat{\phi}_i), \\ \hat{P}(b_j = n) &= U(\hat{\varphi}_j)^\dagger \Pi_n U(\hat{\varphi}_j), \end{aligned} \tag{3.68}$$

where $U(\hat{\phi}_i)$ and $U(\hat{\varphi}_j)$ are the transformations performed by the Bell multiports. As a result, the probability defined in Eq. (3.66) is given as

$$\begin{aligned} P(a_i = m, b_j = n) &= \langle\psi_N|U(\hat{\phi}_i)^\dagger \otimes U(\hat{\varphi}_j)^\dagger \Pi_m \otimes \Pi_n U(\hat{\phi}_i) \otimes U(\hat{\varphi}_j)|\psi_N\rangle \\ &= \frac{1}{N^2 \cosh^2 r(1-\tanh^{2N} r)} \times \\ & \sum_{k,l=0}^{N-1} \cos(\phi_i^k + \varphi_j^k - \phi_i^l - \varphi_j^l \\ & + \frac{2\pi(k-l)(m+n)}{N}) \tanh^{(k+l)} r. \end{aligned} \tag{3.69}$$

When $N = 3$,

$$P(a_i = m, b_j = n) = \frac{1}{9\cosh^2 r(1-\tanh^6 r)} \times \sum_{k,l=0}^{2} \cos(\phi_i^k + \varphi_j^k - \phi_i^l - \varphi_j^l + \frac{2\pi(k-l)(m+n)}{3}) \tanh^{(k+l)} r. \quad (3.70)$$

Local realistic description imposes $\mathcal{B}_{N=3} \leq 2$. Numerical results show that $\mathcal{B}_{N=3}(r = 1.4068) \simeq 2.90638$; $\mathcal{B}_{N=3}(r \to \infty) = 4/(6\sqrt{3}-9) \simeq 2.87293$. It should be noted that the maximally entangled state is recovered when the squeezing parameter goes to infinity. So $\mathcal{B}_N(r \to \infty)$ is the quantum prediction for the Bell quantity constructed from maximally entangled state. For $\mathcal{B}_{N=3}(r \to \infty)$, the four optimal two-component quantum correlations read

$$\begin{aligned}\vec{Q}_{11} = \vec{Q}_{22} = \vec{Q}_{12}^* &= (\frac{2\sqrt{3}+1}{6}, -\frac{2-\sqrt{3}}{6}),\\ \vec{Q}_{21} &= (-\frac{1}{3}, -\frac{2}{3}),\\ |\vec{Q}_{ij}| &= \sqrt{(Q_{ij}^0)^2 + (Q_{ij}^1)^2} = \frac{\sqrt{5}}{3}.\end{aligned} \quad (3.71)$$

We can similarly get $\mathcal{B}_N(r = \text{finite value})$ and $\mathcal{B}_N(r \to \infty)$ with different N. We list them in Tables 3.3 and 3.4. Obviously, the degree of the violation increases with dimension N, and the violation strength of continuous-variable states with finite squeezing parameter is stronger than that of maximally entangled states. When squeezing parameter and dimension N tend to infinity, NOPA state gives the original EPR state. It is interesting to note that for maximally entangled state, the four optimal multi-component quantum correlations share the same module: $|\vec{Q}_{ij}| = \sqrt{\frac{2N-1}{3N}}$. When N tends to infinity, or when the original EPR state recovers, $|\vec{Q}_{ij}| = \sqrt{2/3}$. We calculate the maximal quantum violation for continuous-variable states with different N. The more the value of dimension, the more difficult to find a maximal violation is. Hence the violation strength points we get are for $N \leq 330$. With these values, it is easy to see that the violation increases slowly with increasing N. Which means that there exists a limit for quantum violation when N goes to infinity. However, we do not have an analytical way to find a bound for the violation with finite squeezing parameter. For this case, what we do is draw a graph to see the

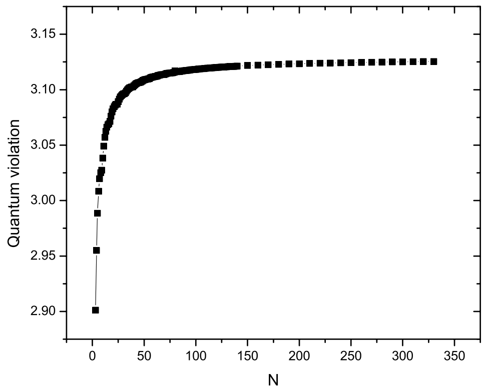

Figure 3.5: Violation of multi-component Bell inequalities for continuous-variable states with finite squeezing parameter for different dimension N.

variation of $\mathcal{B}_N(r = \text{finite value})$ with increasing dimension, see Figure 3.5. Until now, we do not have an exact value of the limit. We numerically find a expression that describes the curve in Figure 3.5,

$$\mathcal{B}_N = 3.12885 - 1.06535/N + 2.13122/N^2 - 2.19262e^{-N}. \tag{3.72}$$

When $N \to \infty$, quantum violation ($\mathcal{B}_N$), or the quantum predictions for the Bell quantity, goes to 3.12885. Hence, such a value can be thought as an approximate violation limit for continuous-variable states with finite squeezing parameter.

The correlation-Bell inequalities presented in the section are of importance in testing violation of local realism. We investigate violation of the Bell inequalities for continuous-variable cases. When the dimension increases, the violation of the inequalities increases slowly. The variation of the violation is similar to that for maximally entangled states. Numerical results show that the violation strength of continuous-variable state with finite squeezing parameter is stronger than that of maximally entangled state.

$\langle\mathcal{B}_N\rangle$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$
ME	2.87293	2.89624	2.91054	2.9202	2.92716	2.93241	2.93651
$\vert\psi_N\rangle$	2.9011	2.95502	2.9886	3.00848	3.01962	3.02524	3.02742
with r	1.49983	1.42954	1.44614	1.48829	1.54037	1.59655	1.6541
$\langle\mathcal{B}_N\rangle$	$N=10$	$N=11$	$N=12$	$N=13$	$N=14$	$N=15$	$N=16$
ME	2.9398	2.9425	2.94475	2.94666	2.9483	2.94973	2.95097
$\vert\psi_N\rangle$	3.03842	3.04917	3.05702	3.06254	3.06619	3.06836	3.06935
with r	1.72082	1.73827	1.7597	1.78375	1.80958	1.8366	1.86444
$\langle\mathcal{B}_N\rangle$	$N=17$	$N=18$	$N=19$	$N=20$	$N=21$	$N=22$	$N=23$
ME	2.95208	2.95306	2.95393	2.95473	2.95544	2.95609	2.95668
$\vert\psi_N\rangle$	3.07141	3.07621	3.07993	3.08273	3.08472	3.08602	3.08674
with r	1.91912	1.93354	1.94913	1.96562	1.98281	2.00056	2.01874
$\langle\mathcal{B}_N\rangle$	$N=24$	$N=25$	$N=26$	$N=27$	$N=28$	$N=29$	$N=30$
ME	2.95723	2.95773	2.95819	2.95862	2.95902	2.95939	2.95974
$\vert\psi_N\rangle$	3.08697	3.08932	3.09159	3.09336	3.09469	3.09562	3.0962
with r	2.03726	2.07377	2.08582	2.09833	2.11122	2.12441	2.13788
$\langle\mathcal{B}_N\rangle$	$N=31$	$N=32$	$N=33$	$N=34$	$N=35$	$N=36$	$N=37$
ME	2.96006	2.96036	2.96065	2.96092	2.96117	2.96141	2.96164
$\vert\psi_N\rangle$	3.09646	3.0971	3.09867	3.09993	3.10091	3.10163	3.10212
with r	2.15156	2.18315	2.19295	2.20301	2.21331	2.22381	2.23449
$\langle\mathcal{B}_N\rangle$	$N=38$	$N=39$	$N=40$	$N=41$	$N=42$	$N=43$	$N=44$
ME	2.96185	2.96206	2.96225	2.96243	2.96261	2.96278	2.96293
$\vert\psi_N\rangle$	3.1024	3.10248	3.10343	3.10439	3.10516	3.10575	3.10618
with r	2.24532	2.25629	2.28107	2.28949	2.29806	2.30678	2.31563
$\langle\mathcal{B}_N\rangle$	$N=45$	$N=46$	$N=47$	$N=48$	$N=49$	$N=50$	$N=51$
ME	2.96309	2.96323	2.96337	2.96351	2.96364	2.96376	2.96388
$\vert\psi_N\rangle$	3.10645	3.10658	3.10685	3.10762	3.10826	3.10876	3.10913
with r	2.32458	2.33364	2.35594	2.36318	2.37052	2.37797	2.38552
$\langle\mathcal{B}_N\rangle$	$N=52$	$N=53$	$N=54$	$N=55$	$N=56$	$N=57$	$N=58$
ME	2.96399	2.9641	2.96421	2.96431	2.96441	2.9645	2.96459
$\vert\psi_N\rangle$	3.10939	3.10954	3.10959	3.11007	3.11061	3.11104	3.11137
with r	2.39315	2.40086	2.40865	2.42738	2.4338	2.44031	2.44689
$\langle\mathcal{B}_N\rangle$	$N=59$	$N=60$	$N=61$	$N=62$	$N=63$	$N=64$	$N=65$
ME	2.96468	2.96477	2.96485	2.96493	2.96501	2.96508	2.96515
$\vert\psi_N\rangle$	3.11162	3.11178	3.11186	3.11199	3.11245	3.11283	3.11314
with r	2.45353	2.46024	2.46701	2.48426	2.48997	2.49574	2.50157

Table 3.3: Violation of multi-component Bell inequalities for $\vert\text{NOPA}\rangle$ (nondegenerate optical parametric amplifier) and $\vert\text{ME}\rangle$ (maximally entangle) states with different N (Part I).

$\langle \mathcal{B}_N \rangle$	$N=66$	$N=67$	$N=68$	$N=69$	$N=70$	$N=71$
ME	2.96522	2.96529	2.96536	2.96542	2.96549	2.96555
$\|\psi_N\rangle$	3.11337	3.11353	3.11362	3.11366	3.11394	3.11428
with r	2.50746	2.5134	2.51938	2.52541	2.54046	2.54565
$\langle \mathcal{B}_N \rangle$	$N=72$	$N=73$	$N=74$	$N=75$	$N=76$	$N=77$
ME	2.96561	2.96566	2.96572	2.96577	2.96583	2.96588
$\|\psi_N\rangle$	3.11456	3.11478	3.11494	3.11504	3.1151	3.11516
with r	2.55089	2.55617	2.56149	2.56686	2.57226	2.58632
$\langle \mathcal{B}_N \rangle$	$N=78$	$N=79$	$N=80$	$N=81$	$N=82$	$N=83$
ME	2.96593	2.96598	2.96603	2.96607	2.96612	2.96616
$\|\psi_N\rangle$	3.11547	3.11573	3.11593	3.11609	3.1162	3.11627
with r	2.59103	2.59578	2.60057	2.6054	2.61026	2.61515
$\langle \mathcal{B}_N \rangle$	$N=84$	$N=85$	$N=86$	$N=87$	$N=88$	$N=89$
ME	2.96621	2.96625	2.96629	2.96633	2.96637	2.96641
$\|\psi_N\rangle$	3.11629	3.11647	3.11671	3.1169	3.11706	3.11717
with r	2.62007	2.63264	2.63699	2.64137	2.64578	2.65022
$\langle \mathcal{B}_N \rangle$	$N=90$	$N=91$	$N=92$	$N=93$	$N=94$	$N=95$
ME	2.96645	2.96648	2.96652	2.96656	2.96659	2.96662
$\|\psi_N\rangle$	3.11725	3.11729	3.11732	3.11754	3.11773	3.11788
with r	2.65469	2.65918	2.67106	2.67506	2.6791	2.68316
$\langle \mathcal{B}_N \rangle$	$N=96$	$N=97$	$N=98$	$N=99$	$N=100$	$N=110$
ME	2.96666	2.96669	2.96672	2.96675	2.96678	2.96706
$\|\psi_N\rangle$	3.11799	3.11807	3.11812	3.11814	3.11826	3.1193
with r	2.68725	2.69136	2.69549	2.69965	2.71045	2.75402
$\langle \mathcal{B}_N \rangle$	$N=120$	$N=130$	$N=140$	$N=150$	$N=160$	$N=170$
ME	2.96729	2.96748	2.96765	2.96779	2.96792	2.96803
$\|\psi_N\rangle$	3.12004	3.12061	3.12126	3.12173	3.1221	3.12253
with r	2.79434	2.83708	2.8707	2.90307	2.93767	2.96527
$\langle \mathcal{B}_N \rangle$	$N=180$	$N=190$	$N=200$	$N=210$	$N=220$	$N=230$
ME	2.96813	2.96822	2.9683	2.96837	2.96844	2.9685
$\|\psi_N\rangle$	3.12286	3.12311	3.12343	3.12367	3.12385	3.12398
with r	2.99448	3.02549	3.04729	3.07007	3.09393	3.1063
$\langle \mathcal{B}_N \rangle$	$N=240$	$N=250$	$N=260$	$N=270$	$N=280$	$N=290$
ME	2.96855	2.9686	2.96865	2.96869	2.96873	2.96877
$\|\psi_N\rangle$	3.12428	3.12442	3.12462	3.12475	3.12486	3.12501
with r	3.132	3.15908	3.17319	3.18771	3.21808	3.23397
$\langle \mathcal{B}_N \rangle$	$N=300$	$N=310$	$N=320$	$N=330$		
ME	2.9688	2.96884	2.96887	2.9689		
$\|\psi_N\rangle$	3.12513	3.12524	3.12536	3.12546		
with r	3.25309	3.25881	3.27608	3.29396		

Table 3.4: Violation of multi-component Bell inequalities for $|\text{NOPA}\rangle$ (nondegenerate optical parametric amplifier) and $|\text{ME}\rangle$ (maximally entangled) states with different N (Part II).

Chapter 4

3-QuNit Bell Inequalities

Three quNits are quantum systems of three particles each in N-dimensional Hilbert space. Bell inequalities for 3 quNits are not so well formulated as those for 2 quNits. For a three N-dimensional system with an arbitrary value of N, new developments have been made in constructing the corresponding Bell inequalities recently. The first step came in 1990 with a paper of Mermin [33] in which he derived a Bell inequality for arbitrary N-qubit states; quantum mechanics violates this inequality by an amount that grows with N. This result clearly gave us a first Bell inequality for three qubits in a correlation form which is maximally violated by three-qubit GHZ state.

$$Q_{112} + Q_{121} + Q_{211} - Q_{222} \leq 2, \tag{4.1}$$

where $Q_{ijk}(i, j, k = 1, 2)$ is correlation function of measurements among three observables for the subsystems. The second step is due to Ref. [39]. The authors developed a three-qutrit Bell inequality which can be given in an probability form,

$$\begin{aligned} P(a_1 + b_1 + c_1 = 0) + P(a_1 + b_2 + c_2 = 1) + P(a_2 + b_1 + c_2 = 1) \quad &+ \\ P(a_2 + b_2 + c_1 = 1) + 2P(a_2 + b_2 + c_2 = 0) - P(a_2 + b_1 + c_1 = 2) \quad &- \\ P(a_1 + b_2 + c_1 = 2) - P(a_1 + b_1 + c_2 = 2) \quad &\leq \quad 3, \end{aligned} \tag{4.2}$$

where $P(a_i + b_j + c_k = r)$ with $i, j, k = 1, 2$ is joint probability which is defined in Eq. (4.40). This inequality imposes a necessary condition on the existence of a local realistic description for the correlations generated by three qutrits. For a system more composite than three qutrits, no Bell inequality has been found yet.

The inequality (4.1) gives one type of inequalities for three-qubit systems, and it is not satisfactory. Since the inequality (4.1) is not violated by all pure entangled states, it seems that the inequality can not be used to characterize entanglement of three qubits. To solve the problem, a new Bell inequality for three qubits is to be developed and is explained in next section.

4.1 Gisin's Theorem for Three Qubits

Characterizing entanglement based on Bell inequality is an important issue in quantum information theory. If a Bell inequality is violated by all pure entangled states, the Bell inequality can be used to characterize entanglement. In 1991, Gisin [18] demonstrated that every pure bipartite entangled state violates the CHSH inequality. This was known subsequently as Gisin's theorem and it was probably the first step towards characterizing entanglement. A few years later, the Horodecki family [58] and Werner [59] showed that the CHSH inequality was insufficient to characterize entanglement of mixed states. Bell inequalities for N qubits were first developed by Mermin-Ardehali-Belinskii-Klyshko (MABK) [33, 34, 35]. However, soon later, Gisin and Scarani [36] noticed that there exist pure states of N qubits that do not violate any of the inequalities. These states are the generalized Greenberger-Horne-Zeilinger (GHZ) states given by

$$|\psi\rangle_{\rm GHZ} = \cos\xi|0\cdots 0\rangle + \sin\xi|1\cdots 1\rangle, \tag{4.3}$$

with $0 \leq \xi \leq \pi/4$. The GHZ states [37] are for $\xi = \pi/4$. In 2001, Scarani and Gisin noticed that for $\sin 2\xi \leq 1/\sqrt{2^{N-1}}$ the states (4.3) do not even violate the MABK inequalities [33, 34, 35]. These results prompted Scarani and Gisin to note that "this analysis suggests that MK (MABK [38]) inequalities, and more generally the family of Bell's inequalities with two observables per qubit, may not be the 'natural' generalizations of the CHSH inequality to more than two qubits" [36]. This was confirmed subsequently by two independent teams [24], who proposed the more general Bell inequalities (in form of correlation functions), now known as Żukowski-Brukner (ŻB) inequalities, for N qubits with two settings per site. The ŻB inequalities include MABK inequalities as special cases. Ref. [38] showed that (a) For $N =$ even, although the generalized GHZ states (4.3) do not violate the MABK inequalities, the states violate the ŻB inequalities and (b) For $\sin 2\xi \leq 1/\sqrt{2^{N-1}}$ and $N =$ odd, the generalized GHZ states (4.3) satisfy all known Bell inequalities involving correlation functions, which involve two dichotomic observables per local measurement station.

It therefore appears that Gisin's theorem is not valid for N (odd numbers) qubits. Recently, we provide a further twist to the results. We construct a 3-qubit Bell inequality, thus the return of Gisin's theorem for 3-qubit systems[5].

4.1.1 Bell Inequalities Involving Probabilities for Three Qubits

In the investigation, we focus on three-qubit systems, whose corresponding generalized GHZ states read $|\psi\rangle_{GHZ} = \cos\xi|000\rangle + \sin\xi|111\rangle$. Up to now, there is no 3-qubit Bell inequality violated by the pure entangled states for the region $\xi \in (0, \pi/12]$ based on the standard Bell experiment. The region $\xi \in (0, \pi/12]$ is calculated from the result that 3-qubit ŻB inequality or MABK inequality is not violated by the generalized GHZ states for $\sin 2\xi \leq 1/\sqrt{2^{N-1}}$ with $(N = 3)$.

Can Gisin's theorem be generalized to 3-qubit pure entangled states? Can one find a Bell inequality that is violated by $|\psi\rangle_{GHZ}$ for the whole region? These questions are all answered in this section. In the following, we firstly present a Theorem that all generalized GHZ states of three-qubit systems violate a Bell inequality in terms of probabilities, secondly we will provide a universal Bell inequality involving probabilities which is violated by all pure entangled states of three qubits.

Theorem 1: All generalized GHZ states of three-qubit systems violate a Bell inequality involving probabilities.

Proof: Let us consider the following Bell-type scenario: three space-separated observers, denoted by A, B and C (or Alice, Bob and Charlie), can measure two different local observables of two outcomes, labeled by 0 and 1. We denote X_i the observable measured by party X and x_i the outcome with $X = A, B, C$ ($x = a, b, c$). If the observers decide to measure A_1, B_1 and C_2, the result is $(0, 1, 1)$ with probability $P(a_1 = 0, b_1 = 1, c_2 = 1)$. The set of these 8×8 probabilities gives a complete description of any statistical quantity that can be observed in this Gedanken experiment. One can easily see that, any local realistic (LR) description

[5] This work was published, see [1] in the publication list in Appendix A.

of the previous Gedanken experiment satisfies the following Bell inequality:

$$\begin{aligned}
&P(a_1+b_1+c_1=0)+P(a_1+b_1+c_1=3)+P(a_1+b_2+c_2=2)\\
&+P(a_2+b_1+c_1=0)+P(a_2+b_1+c_1=3)+P(a_2+b_2+c_2=1)\\
&-P(a_1+b_1+c_1=1)-P(a_1+b_2+c_2=1)\\
&-P(a_2+b_1+c_1=2)-P(a_2+b_2+c_2=2)\leq 2. \qquad (4.4)
\end{aligned}$$

where $P(a_i+b_j+c_k=r)$ is joint probability with $i,j,k=1,2$ and $r=0,1,2,3$. For instance, $P(a_i+b_j+c_k=1)=P(a_i=1,b_j=0,c_k=0)+P(a_i=0,b_j=1,c_k=0)+P(a_i=0,b_j=0,c_k=1)$. It will be shown that the above inequality is always satisfied in a local realistic model. According to a local realistic theory, any probability model can be transformed into a deterministic one by postulating some variables [60]. The value of $P(a_i+b_j+c_k=r)$ is either 1 or 0 in a local realistic theory. In order to beat the bound 2, one may take as many of the positive terms in inequality (4.4) as possible equal to one. However, local realistic constraints force some of the other terms with negative sign to be the value of one. For example, if the terms $P(a_1+b_1+c_1=0)$ and $P(a_2+b_2+c_2=1)$ are taken to be one, one will have $a_1+b_1+c_1+a_2+b_2+c_2=1$. Since $P(a_1+b_1+c_1=0)=1$, $P(a_1+b_1+c_1=3)$ and $P(a_1+b_1+c_1=1)$ should be zero. Similarly, $P(a_2+b_2+c_2=2)=0$ because $a_2+b_2+c_2=1$. Now there remain three terms with positive sign except the above three ones. It can be seen that $P(a_1+b_2+c_2=2)$ should be zero also, otherwise, $a_2+b_1+c_1=-1$ according to the constraint $a_1+b_1+c_1+a_2+b_2+c_2=1$. $a_2+b_1+c_1$ can not assume the value of minus one since all the measurement outcomes are 0 or 1. Next one can take $P(a_2+b_1+c_1=0)$ equal to one and as a result, $P(a_2+b_1+c_1=3)=0$ and $P(a_2+b_1+c_1=2)=0$. With $a_2+b_1+c_1=0$, $a_1+b_2+c_2$ is fixed by the value of one from the local realistic constraint, which means that $P(a_1+b_2+c_2=1)=1$. So the left hand side of inequality (4.4) is taken the value of $1+0+0+1+0+1-0-1-0-0=2$, which does not beat the bound. Similar calculations can be done for other choices, but the inequality is always bounded by 2 no matter which values a_i,b_j,c_k take.

However, quantum mechanics will violate the Bell inequality for any generalized

GHZ states. The quantum prediction for the joint probability reads

$$P^{QM}(a_i = m, b_j = n, c_k = l) = \langle\psi|\hat{P}(a_i = m) \otimes \hat{P}(b_j = n) \otimes \hat{P}(c_k = l)|\psi\rangle, \tag{4.5}$$

where $i, j, k = 1, 2$; $m, n, l = 0, 1$, and

$$\begin{aligned} \hat{P}(a_i = m) &= \frac{1 + (-1)^m \hat{n}_{a_i} \cdot \vec{\sigma}}{2} \\ &= \frac{1}{2} \begin{pmatrix} 1 + (-1)^m \cos\theta_{a_i} & (-1)^m \sin\theta_{a_i} e^{-i\phi_{a_i}} \\ (-1)^m \sin\theta_{a_i} e^{i\phi_{a_i}} & 1 - (-1)^m \cos\theta_{a_i} \end{pmatrix}, \end{aligned} \tag{4.6}$$

is the projector of Alice for the i-th measurement, and similar definitions for $\hat{P}(b_j = n)$, $\hat{P}(c_k = l)$. More precisely, for the generalized GHZ states one obtains

$$\begin{aligned} & P^{QM}(a_i = m, b_j = n, c_k = l) \\ & = \frac{1}{8} \cos^2 \xi [1 + (-1)^m \cos\theta_{a_i}][1 + (-1)^n \cos\theta_{b_j}][1 + (-1)^l \cos\theta_{c_k}] \\ & + \frac{1}{8} \sin^2 \xi [1 - (-1)^m \cos\theta_{a_i}][1 - (-1)^n \cos\theta_{b_j}][1 - (-1)^l \cos\theta_{c_k})] \\ & + \frac{1}{8} \sin(2\xi)(-1)^{m+n+l} \sin\theta_{a_i} \sin\theta_{b_j} \sin\theta_{c_k} \cos(\phi_{a_i} + \phi_{b_j} + \phi_{c_k}). \end{aligned} \tag{4.7}$$

For convenience, let us denote the left hand side of the Bell inequality (4.4) by $\mathcal{B}_{(4.4)}$, which represents the Bell quantity. For the following settings $\theta_{a_1} = \theta_{a_2} = \theta, \phi_{a_1} = -\pi/3, \phi_{a_2} = 2\pi/3, \theta_{b_1} = \theta_{c_1} = 0, \phi_{b_1} = \phi_{c_1} = 0, \theta_{b_2} = \theta_{c_2} = \pi/2, \phi_{b_2} = \phi_{c_2} = \pi/6$, the Bell quantity $\mathcal{B}_{(4.4)}$ is given as

$$\begin{aligned} \mathcal{B}_{(4.4)} &= \frac{1}{2} + \frac{3}{2}(\cos\theta + \sin(2\xi)\sin\theta) \\ &\leq \frac{1}{2} + \frac{3}{2}\sqrt{1 + \sin^2(2\xi)}, \end{aligned} \tag{4.8}$$

the equal sign occurs at $\theta = \tan^{-1}[\sin(2\xi)]$. Obviously the Bell inequality is violated for any $\xi \neq 0$ or $\pi/2$ when $\theta = \tan^{-1}[\sin(2\xi)]$. This ends the proof.

This Theorem indicates that it is possible for the Bell inequality in terms of probabilities to be violated by all pure entangled generalized GHZ states. Recently, classification of N-qubit entanglement via quadratic Bell inequality consisting of MABK inequalities has been presented in Ref. [61]. For $N = 3$, there are three types of 3-qubit states. One type is totally separable states denoted as (1_3); One type is 2-entangled states which are denoted as $(2, 1)$ and the other type is fully entangled states which are denoted as $(3) = \{\rho_{ABC}\}$. Ref. [61] has drawn an ancient Chinese coin (ACC) diagram for the classification of 3-qubit entanglement (see Figure 4.1). In this Figure, $\mathcal{B}_3$ and $\mathcal{B}'_3$ are the MABK-Bell quantities which are defined as

$$\begin{aligned}\mathcal{B}_3 &= Q(A_1B_1C_2) + Q(A_1B_2C_1) + Q(A_2B_1C_1) - Q(A_2B_2C_2),\\ \mathcal{B}'_3 &= Q(A_1B_2C_2) + Q(A_2B_2C_1) + Q(A_2B_1C_2) - Q(A_1B_1C_1),\end{aligned} \tag{4.9}$$

where $Q(A_iB_jC_k)$, $i,j,k = 1,2$ is the correlation function. Quantum mechanically,

$$Q(A_iB_jC_k) = \mathrm{Tr}[\rho\vec{\sigma}\cdot\hat{n}_{a_i}\otimes\vec{\sigma}\cdot\hat{n}_{b_j}\otimes\vec{\sigma}\cdot\hat{n}_{c_k}]. \tag{4.10}$$

For totally separable states ($\rho \in \{1_3\}$), the MABK-Bell quantities read

$$\max\{|\mathcal{B}_3|, |\mathcal{B}'_3|\} \leq 2. \tag{4.11}$$

Namely, separable states lie in the inner square. Two different entanglement classes of 3-qubit states: 2-entangled states and fully entangled states give rise to different violations of the MABK inequality [62].

$$\begin{aligned}\mathcal{B}_3^2 + \mathcal{B}_3'^2 &\leq 2^3 \qquad \mathrm{if}\rho \in (2,1),\\ \mathcal{B}_3^2 + \mathcal{B}_3'^2 &\leq 2^4 \qquad \mathrm{if}\rho \in (3).\end{aligned} \tag{4.12}$$

All the results are put into an ancient Chinese coin diagram as shown in Figure 4.1 [61]. However, for the four points located on the four corners of the square, some of the above three types of 3-qubit states coexist. For instance, the totally separable states and the generalized GHZ states for $\xi \in (0, \pi/12]$ coexist at these four corners, it looks somehow that these four points are "degenerate". The above Bell inequality for probabilities is useful, at least, it can distinguish the generalized GHZ states for $\xi \in (0, \pi/12]$ from the totally separable states.

There are two different entanglement classes for 3-qubit states, namely, 2-entangled states and fully entangled states. Why the MABK inequalities as well as the ŻB inequalities fail for the region $\xi \in (0, \pi/12]$ maybe due to the fact that their inequalities contain only fully 3-particle correlations. If one expands $\hat{P}(a_i = m) \otimes \hat{P}(b_j = n) \otimes \hat{P}(c_k = l)$ and substitutes them into the Bell quantity $\mathcal{B}_{(4.4)}$ constructed from Eq. (4.4), one will find that $\mathcal{B}_{(4.4)}$ contains not only the terms of fully 3-particle correlations, such as $\hat{n}_{a_i}\cdot\vec{\sigma}\otimes\hat{n}_{b_j}\cdot\vec{\sigma}\otimes\hat{n}_{c_k}\cdot\vec{\sigma}$, but also the terms of 2-particle correlations, such as $\hat{n}_{a_i}\cdot\vec{\sigma}\otimes\hat{n}_{b_j}\cdot\vec{\sigma}\otimes\mathbf{1}$, $\hat{n}_{a_i}\cdot\vec{\sigma}\otimes\mathbf{1}\otimes\hat{n}_{c_k}\cdot\vec{\sigma}$ and $\mathbf{1}\otimes\hat{n}_{b_j}\cdot\vec{\sigma}\otimes\hat{n}_{c_k}\cdot\vec{\sigma}$. The

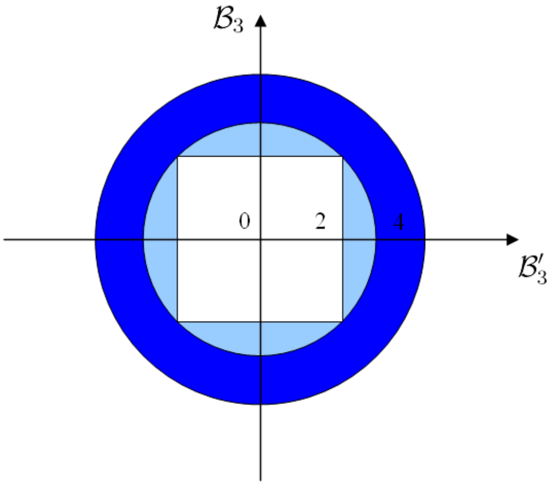

Figure 4.1: Classification of 3-qubit entanglement in an ancient Chinese coin diagram in Ref. [61] by regarding $\mathcal{B}_3$ and $\mathcal{B}_3'$ as two axes.

above theorem implies that 2-particle correlations may make a contribution to the quantum violation of Bell inequality.

The remarkable property of the Bell inequality (4.4) is that it is violated by all pure entangled generalized GHZ states. However, some of other pure entangled states do not violate it, such as the W state $|\psi\rangle_W = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$. One possible reason for this is that the Bell inequality (4.4) does not contain all the possible probabilities. This motivates us to introduce a Bell inequality with all possible probabilities:

$$\begin{aligned} & P(a_1+b_1+c_1=1)+2P(a_2+b_2+c_2=1) \\ & +P(a_1+b_2+c_2=2)+P(a_2+b_1+c_2=2)+P(a_2+b_2+c_1=2) \\ & -P(a_1+b_1+c_2=0)-P(a_1+b_2+c_1=0)-P(a_2+b_1+c_1=0) \\ & -P(a_1+b_1+c_2=3)-P(a_1+b_2+c_1=3)-P(a_2+b_1+c_1=3)\leq 3. \end{aligned} \tag{4.13}$$

This inequality is symmetric under the permutations of three observers Alice, Bob and Charlie. It can also be tested that the inequality (4.13) is always bounded by 3 using the method given before. Here one of the conditions is taken as an example. To beat the bound 3, terms $P(a_1+b_1+c_1=1)$ and $P(a_2+b_2+c_2=1)$ are taken equal to one first. This means that $a_1+b_1+c_1+a_2+b_2+c_2=2$. The remained three terms with positive sign are also taken equal to one to maximize the value of left hand side of the inequality (4.13). As a result, three of the last six terms should be zero, they are $P(a_1+b_1+c_2=3)$, $P(a_1+b_2+c_1=3)$ and $P(a_2+b_1+c_1=3)$. The other three are all equal to one according to the constraint $a_1+b_1+c_1+a_2+b_2+c_2=2$. Therefore one has $1+2+1+1+1-1-1-1-0-0-0=3$. After tedious yet straightforward calculations, it can be shown that the inequality (4.13) is always bounded by 3 in a local realistic model.

Quantum mechanically, the inequality (4.13) is shown numerically to be violated by all pure entangle states. Pure states of three qubits constitute a five-parameter family, with equivalence up to local unitary transformations. This family has the representation [63]

$$\begin{aligned} |\psi\rangle &= \sqrt{\mu_0}|000\rangle + \sqrt{\mu_1}e^{i\phi}|100\rangle + \sqrt{\mu_2}|101\rangle \\ &\quad + \sqrt{\mu_3}|110\rangle + \sqrt{\mu_4}|111\rangle, \end{aligned} \tag{4.14}$$

with $\mu_i \geq 0$, $\sum_i \mu_i = 1$ and $0 \leq \phi \leq \pi$. We follow Ref. [63] to give a simple proof that any pure state of 3 qubits can always be written as a linear superposition of five states. Write a pure three-qubit state as

$$|\psi\rangle = \sum_{ijk=0,1} \lambda_{ijk}|ijk\rangle, \tag{4.15}$$

and find two matrices Λ_0 and Λ_1 with elements

$$(\Lambda_i)_{jk} \equiv \lambda_{ijk}. \tag{4.16}$$

There always exists a unitary transformation performed on the first qubit

$$\Lambda_i' = \sum_j \alpha_{ij}\Lambda_j, \tag{4.17}$$

such that $\det\Lambda_0' = 0$. Since one can find a unitary matrix U which transforms Λ_0' to its diagonalized form Π_0',

$$U\Lambda_0' U^\dagger = \Pi_0', \tag{4.18}$$

one has

$$(\Pi_0')_{01} = (\Pi_0')_{10} = 0. \tag{4.19}$$

Due to the fact that $\det\Pi_0' = 0$, $(\Pi_0')_{00} = 0$ or $(\Pi_0')_{11} = 0$. This completes the proof.

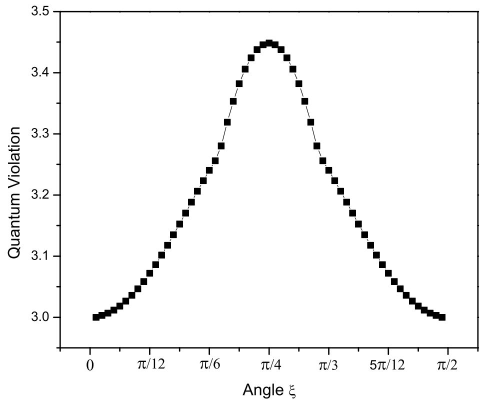

Figure 4.2: Numerical results for the generalized GHZ states $|\psi\rangle_{GHZ} = \cos\xi|000\rangle + \sin\xi|111\rangle$, which violate Bell inequality for probabilities (4.13) except $\xi = 0$ and $\pi/2$. For the GHZ state with $\xi = \pi/4$, the Bell quantity reaches its maximum value $\frac{3}{8}(4 + 3\sqrt{3})$ which is shown analytically .

Numerical results show that this Bell inequality for probabilities is violated by all pure entangled states of three-qubit systems. However, it is difficult to provide an analytic proof. In the following, some special cases will be given to show that the inequality (4.13) is violated by all pure entangled states.

In Figure 4.2, we show the numerical results for the generalized GHZ states $|\psi\rangle_{GHZ} = \cos\xi|000\rangle + \sin\xi|111\rangle$, which violate the above symmetric Bell inequality for probabilities except $\xi = 0$ and $\pi/2$. For the measuring angles $\theta_{a_1} = \theta_{a_2} = \theta_{b_1} = \theta_{b_2} = \theta_{c_1} = \theta_{c_2} = \pi/2, \phi_{a_1} = -5\pi/12, \phi_{a_2} = \pi/4, \phi_{b_1} = -5\pi/12, \phi_{b_2} = \pi/4, \phi_{c_1} = -\pi/3, \phi_{c_2} = \pi/3$, all the probability terms with positive signs in Bell inequality (4.13) are equal to $\frac{3}{16}(2 + \sqrt{3})$, while the terms with negative signs are equal to $\frac{1}{8}$, so the quantum prediction of Bell quantity for the GHZ state (where $\xi = \pi/4$) is obtained as $6 \times \frac{3}{16}(2 + \sqrt{3}) - 6 \times \frac{1}{8} = \frac{3}{8}(4 + 3\sqrt{3}) > 3$. In Figure 4.3, we show the numerical results for the family of generalized W states $|\psi\rangle_W = \sin\beta\cos\xi|100\rangle + \sin\beta\sin\xi|010\rangle + \cos\beta|001\rangle$ with the cases $\beta = \pi/12, \pi/6, \pi/4, \pi/3, 5\pi/12$ and $\pi/2$, which show the quantum violation of $|\psi\rangle_W$ except the product cases with $\beta = \pi/2, \xi = 0$ and $\pi/2$. For the standard W state $|\psi\rangle_W = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$, the quantum violation is 3.55153. We then proceed to present the second theorem.

Theorem 2: All pure 2-entangled states of three-qubit systems violate a Bell inequality involving probabilities.

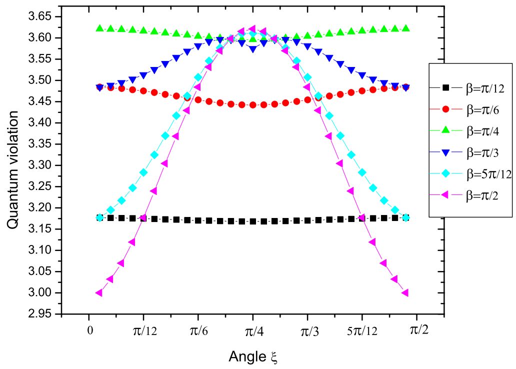

Figure 4.3: Numerical results for the family of generalized W states $|\psi\rangle_W = \sin\beta\cos\xi|100\rangle + \sin\beta\sin\xi|010\rangle + \cos\beta|001\rangle$ with the cases $\beta = \pi/12, \pi/6, \pi/4, \pi/3, 5\pi/12$ and $\pi/2$. These states violate the inequality (4.13).

Proof: By pure 2-entangled states of three-qubit systems, we mean $|\psi_{AB}\rangle\otimes|\psi_C\rangle$, $|\psi_{AC}\rangle\otimes|\psi_B\rangle$ and $|\psi_{BC}\rangle\otimes|\psi_A\rangle$. It is sufficient to consider one of them, say $|\psi_{AB}\rangle\otimes|\psi_C\rangle$, since Bell inequality (4.13) is symmetric under the permutations of A, B and C. Moreover, one can always have $|\psi_{AB}\rangle\otimes|\psi_C\rangle = (\cos\xi|00\rangle_{AB} + \sin\xi|11\rangle_{AB})\otimes|0\rangle_C$ due to local unitary transformations. For the measuring angles $\theta_{a_1} = \theta_{a_2} = \theta, \phi_{a_1} = 2\pi/3, \phi_{a_2} = -\pi/3, \theta_{b_1} = \theta_{c_1} = 0, \phi_{b_1} = \phi_{c_1} = 0, \theta_{b_2} = \pi/2, \theta_{c_2} = \pi, \phi_{b_2} = \pi/3, \phi_{c_2} = 0$, we obtain from the left-hand side of Bell inequality (4.13) that

$$\frac{3}{2}(1 - \cos\theta + \sin(2\xi)\sin\theta) \leq \tfrac{3}{2}(1 + \sqrt{1 + \sin^2(2\xi)}), \tag{4.20}$$

the equal sign occurs at $\theta = -\tan^{-1}[\sin(2\xi)]$. Obviously the Bell inequality is violated for any $\xi \neq 0$ or $\pi/2$ when $\theta = -\tan^{-1}[\sin(2\xi)]$. This ends the proof. Indeed, the quantum violation for the state $|\psi_{AB}\rangle\otimes|\psi_C\rangle$ corresponds to the curve with $\beta = \pi/2$ as shown in Figure 4.3, because $|\psi_{AB}\rangle\otimes|\psi_C\rangle$ is equivalent to $|\psi\rangle_W$ for $\beta = \pi/2$ up to a local unitary transformation.

There is a simpler and more intuitive way to prove Theorem 2, because the symmetric Bell inequality (4.13) can be reduced to a CHSH-like inequality for two qubits and then from Gisin's theorem for two qubits one easily has Theorem 2. By taking $c_1 = 0, c_2 = 1$, we have from Eq. (4.13) that

$$\begin{aligned} & P(a_1 + b_1 = 1) + 2P(a_2 + b_2 = 0) \\ & +P(a_1 + b_2 = 1) + P(a_2 + b_1 = 1) + P(a_2 + b_2 = 2) \\ & -P(a_1 + b_1 = -1) - P(a_1 + b_2 = 0) - P(a_2 + b_1 = 0) \\ & -P(a_1 + b_1 = 2) - P(a_1 + b_2 = 3) - P(a_2 + b_1 = 3) \leq 3. \end{aligned} \tag{4.21}$$

Since $a_1, a_2, b_1, b_2 = 0, 1$, the probabilities $P(a_1+b_1 = -1)$, $P(a_1+b_2 = 3)$ and$P(a_2+b_1 = 3)$ will be equal to zero, by using $P(a_2+b_2 = 0)+P(a_2+b_2 = 2) = 1-P(a_2+b_2 = 1)$, we arrive at the following Bell inequality for two qubits

$$\begin{aligned} & P(a_1 + b_1 = 1) + P(a_1 + b_2 = 1) + P(a_2 + b_1 = 1) + \\ & P(a_2 + b_2 = 0) - P(a_1 + b_1 = 2) - P(a_1 + b_2 = 0) - \\ & P(a_2 + b_1 = 0) - P(a_2 + b_2 = 1) \leq 2. \end{aligned} \tag{4.22}$$

This Bell inequality is symmetric under the permutations of Alice and Bob and it is an alternative form of the CHSH inequality for two qubits. For the two-qubit states $|\psi\rangle = \cos\xi|00\rangle + \sin\xi|11\rangle$ and the projector as shown in Eq. (4.6), one can have the quantum probability

$$\begin{aligned} & P^{QM}(a_i = m, b_j = n) \\ & = \frac{1}{4}\cos^2\xi[1 + (-1)^m \cos\theta_{a_i}][1 + (-1)^n \cos\theta_{b_j}] \\ & +\frac{1}{4}\sin^2\xi[1 - (-1)^m \cos\theta_{a_i}][1 - (-1)^n \cos\theta_{b_j}] \\ & +\frac{1}{4}\sin(2\xi)(-1)^{m+n}\sin\theta_{a_i}\sin\theta_{b_j}\cos(\phi_{a_i} + \phi_{b_j}). \end{aligned} \tag{4.23}$$

For the measuring angles $\theta_{a_1} = \theta_{a_2} = \theta, \phi_{a_1} = \pi - \phi, \phi_{a_2} = -\phi, \theta_{b_1} = 0, \phi_{b_1} = 0, \theta_{b_2} = \pi/2, \phi_{b_2} = \phi$, the left-hand side of Bell inequality (4.22) becomes $\frac{1}{2} + \frac{3}{2}(-\cos\theta + \sin(2\xi)\sin\theta) \leq \frac{1}{2}(1 + 3\sqrt{1 + \sin^2(2\xi)})$, the equal sign occurs at $\theta = -\tan^{-1}[\sin(2\xi)]$. Obviously the Bell inequality (4.22) is violated for any $\xi \neq 0$ or $\frac{\pi}{2}$ when $\theta = -\tan^{-1}[\sin(2\xi)]$, just the same as the CHSH inequality violated by the 2-qubit states $|\psi\rangle = \cos\xi|00\rangle + \sin\xi|11\rangle$.

As pointed in Section 2.4.3, the violation strength of a Bell inequality can be measured in terms of threshold visibility V_{thr} [47] which is the minimal amount of the given entangled state $|\psi\rangle$ that one has to add to pure noise so that the resulting state ρ violates local realism. Consider the violation strength by using threshold visibility, the entangled state is now described by the Werner state. The Werner state is $\rho_W = V|\psi\rangle\langle\psi| + (1-V)\rho_{\rm noise}$, where $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is the maximally entangled state. The critical value of V below which a local realism is still possible by this Bell inequality is $V_{thr} = 1/\sqrt{2}$, just the same as the case for the CHSH inequality. Actually, if one denotes the left-hand side of Bell inequality (4.22) by $\mathcal{B}_{(4.22)}$ and redefines a new Bell quantity $\mathcal{B}'_{(4.22)} = \frac{4}{3}(\mathcal{B}_{(4.22)} - \frac{1}{2})$, one still has the Bell inequality $\mathcal{B}'_{(4.22)} \leq 2$. For quantum mechanics, $\mathcal{B}'^{max}_{(4.22)} = 2\sqrt{1+\sin^2(2\xi)}$, which reaches $2\sqrt{2}$ and then $\mathcal{B}'_{(4.22)}$ recovers the usual CHSH inequality.

Theorem 2 is remarkable. If one knows that a pure state is a 2-entangled state of a three-qubit system, one can use the Bell inequality (4.13) to measure the degree of entanglement (or concurrence denoted by $\mathcal{C}$ [64]) of the state. Since the left hand side of (4.13) is $\frac{3}{2}(1+\sqrt{1+\sin^2(2\xi)}) = \frac{3}{2}(1+\sqrt{1+\mathcal{C}^2})$, thus one has the concurrence $\mathcal{C} = |\sin(2\xi)| \in [0,1]$, just the same as the case that the CHSH inequality measures the concurrence of pure states of two qubits [65]. But for a fully entangle state of three qubits, there is no one to one relation between the maximal violation of the 3-qubit Bell inequality and the entanglement measure found until now. In summary, (i) since all pure entangled states (including pure 2-entangled states) of three-qubit systems violate the Bell inequality (4.13), thus we have Gisin's theorem for 3-qubit system; (ii) the Bell inequality (4.13) can be reduced to an alternative form of the CHSH inequality (in terms of probabilities), thus it can be viewed as a good candidate for a"natural" generalization of the usual CHSH inequality. (iii) the MABK inequalities and the Żukowski-Brukner inequalities are binary correlation Bell inequalities. However, one may notice that Bell inequalities (4.4) and (4.13) are both ternary Bell inequalities, i.e., where the inequalities are "modulo 3". Most recently, a ternary Bell inequality in terms of probabilities for three qutrits was presented in Ref. [39] [or inequality (4.2)], this inequality can be connected to Bell inequality (4.13), which is for three qubits, if one restricts the initial three possible outcomes of each measurement to only two possible outcomes.

4.1.2 Bell Inequalities Involving Correlation Functions for Three Qubits

Our recent investigation shows that one set of Bell inequalities for 3 qubits can be derived in terms of correlation functions[6]. In this section, these Bell inequalities involving correlation functions for three qubits are developed. We show that the inequalities are violated by quantum mechanics. The violation is the same as that predicted in the previous section. So the inequalities are the correlation function version of the one (4.13) given in the previous section.

Consider 3 observers, Alice, Bob and Charlie. Suppose they are each allowed to choose between two dichotomic observables, parameterized by $\vec{n}_1$ and $\vec{n}_2$. Each observer can choose independently two arbitrary directions. The outcomes of observer X's measurement on the observable defined by $\vec{n}_1$ and $\vec{n}_2$ are represented by $X(\hat{n}_1)$ and $X(\hat{n}_2)$ (with $X = A, B, C$). Each outcome can take values +1 or -1 under the assumption of local realism. In a specific run of the experiment the correlations between all 3 observers can be represented by the product $A(\hat{n}_i)B(\hat{n}_j)C(\hat{n}_k)$, where $i, j, k = 1, 2$. For convenience, we write $A(\hat{n}_i)B(\hat{n}_j)C(\hat{n}_k)$ as $A_iB_jC_k$. In a local realistic theory, the correlation function of the measurements performed by the three observers is the average over many runs of the experiment

$$Q(A_i, B_j, C_k) = \langle A(\hat{n}_i)B(\hat{n}_j)C(\hat{n}_k)\rangle = \langle A_iB_jC_k\rangle. \tag{4.24}$$

Similarly, the correlation functions between any two observers can be given as follows

$$\begin{aligned} Q(A_i, B_j) &= \langle A(\hat{n}_i)B(\hat{n}_j)\rangle = \langle A_iB_j\rangle, \\ Q(A_i, C_k) &= \langle A(\hat{n}_i)C(\hat{n}_k)\rangle = \langle A_iC_k\rangle, \\ Q(B_j, C_k) &= \langle B(\hat{n}_j)C(\hat{n}_k)\rangle = \langle B_jC_k\rangle. \end{aligned} \tag{4.25}$$

The following inequality holds for the predetermined results:

$$\begin{aligned} &A_1B_1C_1 - A_1B_2C_2 - A_2B_1C_2 - A_2B_2C_1 + 2A_2B_2C_2 \\ &-QA_1B_1 - A_1B_2 - A_2B_1 - A_2B_2 + A_1C_1 + A_1C_2 \\ &+A_2C_1 + A_2C_2 + B_1C_1 + B_1C_2 + B_2C_1 + B_2C_2 \leq 4. \end{aligned} \tag{4.26}$$

[6]This work was published, see [5] in the publication list in Appendix A.

The proof of the above inequality consists of enumerating all the possible values of $A_i, B_j, C_k (i, j, k = 1, 2)$. This proof is easily seen by fixing values of A_1, B_1, C_1. We now consider different cases depending on the signs of A_1, B_1, C_1.

1. A_1, B_1, C_1 are all $+1$. The inequality (4.26) can be written as $(A_2B_2+1)(C_2-1) \leq 0$. Since $A_2, B_2, C_2 = \pm 1$, $A_2B_2+1 = 2$ or 0 and $C_2 - 1 = 0$ or -2, thus the inequality is satisfied regardless of the values of A_2, B_2, C_2.

2. A_1, B_1, C_1 are all -1. The inequality (4.26) can be written as $[B_2(A_2+1)+(A_2-1)]C_2 \leq 2$. If $A_2 = 1$, one finds $[B_2(A_2+1)+(A_2-1)]C_2 = 2B_2C_2$ which is not greater than 2 since $B_2, C_2 = \pm 1$. If $A_2 = -1$, one finds $[B_2(A_2+1)+(A_2-1)]C_2 = -2C_2$ which is not greater than 2 since $C_2 = \pm 1$.

3. $A_1 = 1, B_1 = C_1 = -1$. The inequality (4.26) can be written as $(A_2C_2-1)(B_2+1) \leq 0$. Since $A_2, B_2, C_2 = \pm 1$, $A_2C_2 - 1 = 0$ or -2 and $B_2 + 1 = 2$ or 0, thus the inequality is satisfied no matter which values A_2, B_2, C_2 take.

4. $B_1 = 1, A_1 = C_1 = -1$. The inequality (4.26) can be written as $(B_2C_2-1)(A_2+1) \leq 0$. Since $A_2, B_2, C_2 = \pm 1$, $B_2C_2 - 1 = 0$ or -2 and $A_2 + 1 = 2$ or 0, thus the inequality is satisfied no matter which values A_2, B_2, C_2 take.

5. $C_1 = 1, A_1 = B_1 = -1$. The inequality (4.26) can be written as $A_2B_2(C_2-1)+(A_2+B_2)(C_2+1)-C_2 \leq 3$. If $C_2 = 1$, one finds $A_2B_2(C_2-1)+(A_2+B_2)(C_2+1)-C_2 = 2(A_2+B_2)-1$ which is not greater than 3 since $A_2, B_2 = \pm 1$. If $C_2 = -1$, one finds $A_2B_2(C_2-1)+(A_2+B_2)(C_2+1)-C_2 = -2A_2B_2+1$ which is not greater than 3 since $A_2, B_2 = \pm 1$.

6. $A_1 = B_1 = 1, C_1 = -1,$. The inequality (4.26) can be written as $A_2B_2C_2 - A_2 - B_2 + C_2 \leq 4$. If $A_2 = 1$, one finds $A_2B_2C_2 - A_2 - B_2 + C_2 = (B_2+1)(C_2-1)$ which is less than 4 since $B_2, C_2 = \pm 1$. If $A_2 = -1$, one finds $A_2B_2C_2 - A_2 - B_2 + C_2 = (1-B_2)(C_2+1)$ which is not greater than 4 since $A_2, B_2 = \pm 1$.

7. $A_1 = C_1 = 1, B_1 = -1$. The inequality (4.26) can be written as $A_2B_2(C_2-1)+A_2(C_2+1) \leq 2$. If $C_2 = 1$, one finds $A_2B_2(C_2-1)+A_2(C_2+1) = 2A_2$ which is no greater than 2 since $A_2 = \pm 1$. If $C_2 = -1$, one finds $A_2B_2(C_2-1)+A_2(C_2+1) = -2A_2B_2$ which is not greater than 2 since $A_2, B_2 = \pm 1$.

8. $B_1 = C_1 = 1, A_1 = -1$. The inequality (4.26) can be written as $A_2B_2(C_2 - 1) + B_2(C_2 + 1) \leq 2$. If $C_2 = 1$, one finds $A_2B_2(C_2 - 1) + B_2(C_2 + 1) = 2B_2$ which is no greater than 2 since $B_2 = \pm 1$. If $C_2 = -1$, one finds $A_2B_2(C_2 - 1) + B_2(C_2 + 1) = -2A_2B_2$ which is not greater than 2 since $A_2, B_2 = \pm 1$.

Thus, in each case, the inequality is satisfied regardless which values A_i, B_j, C_k $(i, j, k = 1, 2)$ take. After many runs of experiment, one can use correlation functions to express the left hand side of inequality (4.26), so we have

$$\begin{aligned}
&Q(A_1B_1C_1) - Q(A_1B_2C_2) - Q(A_2B_1C_2) - Q(A_2B_2C_1) \\
&+2Q(A_2B_2C_2) - Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1) \\
&-Q(A_2B_2) + Q(A_1C_1) + Q(A_1C_2) + Q(A_2C_1) + Q(A_2C_2) \\
&+Q(B_1C_1) + Q(B_1C_2) + Q(B_2C_1) + Q(B_2C_2) \leq 4. \qquad (4.27)
\end{aligned}$$

The above inequality (4.27) does not include the terms of single-particle correlation function, it is symmetric under the permutation of A_j and B_j. Moreover, by setting appropriate values of C_1 and C_2, the inequality reduces directly to an equivalent form of the CHSH inequality for two qubits. When $C_1 = -1, C_2 = 1$, the inequality becomes

$$-Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1) + Q(A_2B_2) \leq 2. \qquad (4.28)$$

The inequality (4.27) is symmetric under permutation of A and B. We then show that another Bell inequality involving correlation functions for three qubits can be constructed in a similar way. The new inequality is symmetric under permutations of three observers. The inequality has the form,

$$\begin{aligned}
&-Q(A_1B_1C_1) + Q(A_1B_2C_2) + Q(A_2B_1C_2) + Q(A_2B_2C_1) \\
&-2Q(A_2B_2C_2) - Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1) \\
&-Q(A_2B_2) - Q(A_1C_1) - Q(A_1C_2) - Q(A_2C_1) - Q(A_2C_2) \\
&-Q(B_1C_1) - Q(B_1C_2) - Q(B_2C_1) - Q(B_2C_2) \leq 4. \qquad (4.29)
\end{aligned}$$

To prove the inequality (4.29), we first write the following inequality

$$\begin{aligned}
&-A_1B_1C_1 + A_1B_2C_2 + A_2B_1C_2 + A_2B_2C_1 - 2A_2B_2C_2 \\
&-A_1B_1 - A_1B_2 - A_2B_1 - A_2B_2 - A_1C_1 - A_1C_2 \\
&-A_2C_1 - A_2C_2 - B_1C_1 - B_1C_2 - B_2C_1 - B_2C_2 \leq 4, \qquad (4.30)
\end{aligned}$$

inequality (4.29) is given by averaging the left hand side of inequality (4.30). Proof of the above inequality (4.30) is given in the following:

1. A_1, B_1, C_1 are all $+1$. The inequality (4.30) can be written as $-2A_2B_2C_2 - 2A_2 - 2B_2 - 2C_2 \leq 8$. If $C_2 = 1$, one finds $-2A_2B_2C_2 - 2A_2 - 2B_2 - 2C_2 = -2(A_2+1)(B_2+1)$ which is not greater than 8 since $A_2, B_2 = \pm 1$. If $C_2 = -1$, one finds $-2A_2B_2C_2 - 2A_2 - 2B_2 - 2C_2 = 2(A_2 - 1)(B_2 - 1)$ which is not greater than 8 for the same reason.

2. $A_1 = B_1 = 1, C_1 = -1$. The inequality (4.30) can be written as $-(A_2B_2 + 1)(C_2 + 1) \leq 0$. Since $A_2, B_2, C_2 = \pm 1$, $A_2B_2 + 1 = 2$ or 0 and $C_2 + 1 = 2$ or 0, thus the inequality is satisfied regardless of the values of A_2, B_2, C_2. Since the inequality is symmetric under permutations of A, B and C, the results of the cases that $A_1 = C_1 = 1, B_1 = -1$ and $B_1 = C_1 = 1, A_1 = -1$ are the same as that of the case $A_1 = B_1 = 1, C_1 = -1$.

3. $A_1 = B_1 = -1, C_1 = 1$. The inequality (4.30) can be written as $-2A_2B_2C_2 - 2A_2C_2 - 2B_2C_2 + 2C_2 - 4 \leq 0$. If $C_2 = 1$, one finds $-2A_2B_2C_2 - 2A_2C_2 - 2B_2C_2 + 2C_2 - 4 = -2(A_2 + 1)(B_2 + 1)$ which is not greater than 0 since $A_2, B_2 = \pm 1$ and hence, $(A_2 + 1)/(B_2 + 1) = 2$,or 0. If $C_2 = -1$, one finds $-2A_2B_2C_2 - 2A_2C_2 - 2B_2C_2 + 2C_2 - 4 = 2(A_2 + 1)(B_2 + 1) - 8$ which is not greater than 0 for the same reason. Since the inequality is symmetric under permutations of A, B and C, the results of the cases that $A_1 = C_1 = -1, B_1 = 1$ and $B_1 = C_1 = -1, A_1 = 1$ are the same as that of the case $A_1 = B_1 = -1, C_1 = 1$.

4. A_1, B_1, C_1 are all -1. The inequality (4.30) can be written as $-2A_2B_2C_2 - 2A_2B_2 - 2A_2C_2 - 2B_2C_2 + 2A_2 + 2B_2 + 2C_2 - 6 \leq 0$. If $C_2 = 1$, one finds $-4(A_2B_2 + 1) \leq 0$ which is satisfied since $A_2, B_2, = \pm 1$ and $A_2B_2 + 1 = 2,\ 0$. If $C_2 = -1$, one finds $A_2 + B_2 \leq 2$ which is satisfied since $A_2 + B_2 = 2$, 0 or -2.

Thus, in each case, the inequality (4.30) is satisfied no matter which values A_i, B_j, C_k $(i, j, k = 1, 2)$ take and so is the inequality (4.29). The inequality (4.29) does not include the terms of single-particle correlation function, it is symmetric under the permutation of A_j, B_j and C_k. When $C_1 = 1, C_2 = -1$, the inequality is reduced to an equivalent form of the CHSH inequality for two qubits

$$-Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1) + Q(A_2B_2) \leq 2. \tag{4.31}$$

Similarly, five more Bell inequalities for three qubits can be constructed. All the correlation Bell inequalities are listed in Table 4.1. The first one and the fourth one are exactly the inequalities (4.27) and (4.29), the others can be proved to be satisfied under local realism by using similar methods. It is worth noting that these seven inequalities are equivalent to each other. By interchanging B and C, the second one is transformed to the first one. By interchanging A and C, the third one is transformed to the first one. By changing C_i to $-C_i$, the fourth one is transformed to the first one. By changing A_i to $-A_i$ and B_i to $-B_i$, the fifth one is transformed to the first one. By changing B_i to $-B_i$, the sixth one is transformed to the first one. By changing A_i to $-A_i$, the seventh one is transformed to the first one. So it is sufficient to consider only one of them when testing the quantum violation of local realism. We will take the first one, or inequality (4.27) as an example to show quantum mechanics violates local realism.

To test the quantum violation of any Bell inequalities, observables and quantum states should be specified. We consider the Bell type experiment in which three spatially separated observers Alice, Bob, and Charlie respectively measure two non-commuting observables, namely, $A_i = \hat{n}_{a_i} \cdot \vec{\sigma}(i = 1, 2)$ for Alice, $B_j = \hat{n}_{b_j} \cdot \vec{\sigma}(j = 1, 2)$ for Bob, and $C_k = \hat{n}_{c_k} \cdot \vec{\sigma}(k = 1, 2)$ for Charlie on the generalized GHZ states $|\psi\rangle$ of three qubits

$$|\psi\rangle_{GHZ} = \cos\xi|0\rangle_A|0\rangle_B|0\rangle_C + \sin\xi|1\rangle_A|1\rangle_B|1\rangle_C, \tag{4.32}$$

where $|k\rangle_i(k = 0, 1)$ describes k-th basis state of the qubit $i(i = A, B, C)$ respectively. The matrix forms for each set of observables A_i, B_j, and C_k are

$$\begin{aligned}
A_i = \hat{n}_{a_i} \cdot \vec{\sigma} &= \begin{pmatrix} \cos\theta_{a_i} & \sin\theta_{a_i} e^{-i\phi_{a_i}} \\ \sin\theta_{a_i} e^{i\phi_{a_i}} & -\cos\theta_{a_i} \end{pmatrix}, \\
B_j = \hat{n}_{b_j} \cdot \vec{\sigma} &= \begin{pmatrix} \cos\theta_{b_j} & \sin\theta_{b_j} e^{-i\phi_{b_j}} \\ \sin\theta_{b_j} e^{i\phi_{b_j}} & -\cos\theta_{b_j} \end{pmatrix}, \\
C_k = \hat{n}_{c_k} \cdot \vec{\sigma} &= \begin{pmatrix} \cos\theta_{c_k} & \sin\theta_{c_k} e^{-i\phi_{c_k}} \\ \sin\theta_{c_k} e^{i\phi_{c_k}} & -\cos\theta_{c_k} \end{pmatrix},
\end{aligned} \tag{4.33}$$

No.	Explicit Expression for the Bell Inequalities
1	$Q(A_1B_1C_1) - Q(A_1B_2C_2) - Q(A_2B_1C_2) - Q(A_2B_2C_1)$ $+2Q(A_2B_2C_2) - Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1) - Q(A_2B_2)$ $+Q(A_1C_1) + Q(A_1C_2) + Q(A_2C_1) + Q(A_2C_2) + Q(B_1C_1)$ $+Q(B_1C_2) + Q(B_2C_1) + Q(B_2C_2) \leq 4$
2	$Q(A_1B_1C_1) - Q(A_1B_2C_2) - Q(A_2B_1C_2) - Q(A_2B_2C_1)$ $+2Q(A_2B_2C_2) + Q(A_1B_1) + Q(A_1B_2) + Q(A_2B_1) + Q(A_2B_2)$ $-Q(A_1C_1) - Q(A_1C_2) - Q(A_2C_1) - Q(A_2C_2) + Q(B_1C_1)$ $+Q(B_1C_2) + Q(B_2C_1) + Q(B_2C_2) \leq 4$
3	$Q(A_1B_1C_1) - Q(A_1B_2C_2) - Q(A_2B_1C_2) - Q(A_2B_2C_1)$ $+2Q(A_2B_2C_2) + Q(A_1B_1) + Q(A_1B_2) + Q(A_2B_1) + Q(A_2B_2)$ $+Q(A_1C_1) + Q(A_1C_2) + Q(A_2C_1) + Q(A_2C_2) - Q(B_1C_1)$ $-Q(B_1C_2) - Q(B_2C_1) - Q(B_2C_2) \leq 4$
4	$-Q(A_1B_1C_1) + Q(A_1B_2C_2) + Q(A_2B_1C_2) + Q(A_2B_2C_1)$ $-2Q(A_2B_2C_2) - Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1) - Q(A_2B_2)$ $-Q(A_1C_1) - Q(A_1C_2) - Q(A_2C_1) - Q(A_2C_2) - Q(B_1C_1)$ $-Q(B_1C_2) - Q(B_2C_1) - Q(B_2C_2) \leq 4$
5	$Q(A_1B_1C_1) - Q(A_1B_2C_2) - Q(A_2B_1C_2) - Q(A_2B_2C_1)$ $+2Q(A_2B_2C_2) - Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1) - Q(A_2B_2)$ $-Q(A_1C_1) - Q(A_1C_2) - Q(A_2C_1) - Q(A_2C_2) - Q(B_1C_1)$ $-Q(B_1C_2) - Q(B_2C_1) - Q(B_2C_2) \leq 4$
6	$-Q(A_1B_1C_1) + Q(A_1B_2C_2) + Q(A_2B_1C_2) + Q(A_2B_2C_1)$ $-2Q(A_2B_2C_2) + Q(A_1B_1) + Q(A_1B_2) + Q(A_2B_1) + Q(A_2B_2)$ $+Q(A_1C_1) + Q(A_1C_2) + Q(A_2C_1) + Q(A_2C_2) - Q(B_1C_1)$ $-Q(B_1C_2) - Q(B_2C_1) - Q(B_2C_2) \leq 4$
7	$-Q(A_1B_1C_1) + Q(A_1B_2C_2) + Q(A_2B_1C_2) + Q(A_2B_2C_1)$ $-2Q(A_2B_2C_2) + Q(A_1B_1) + Q(A_1B_2) + Q(A_2B_1) + Q(A_2B_2)$ $-Q(A_1C_1) - Q(A_1C_2) - Q(A_2C_1) - Q(A_2C_2) + Q(B_1C_1)$ $+Q(B_1C_2) + Q(B_2C_1) + Q(B_2C_2) \leq 4$

Table 4.1: The explicit expression of a set of Bell inequalities for 3 qubits involving correlation functions.

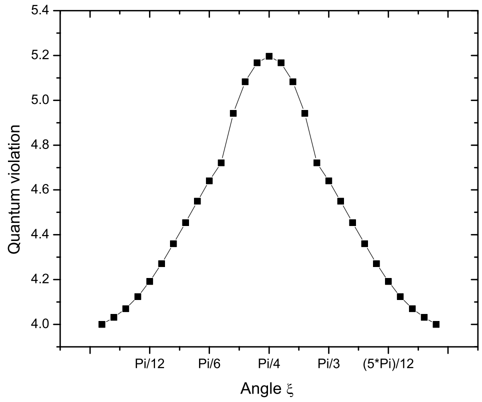

Figure 4.4: Numerical results for the generalized GHZ states $|\psi\rangle_{\rm GHZ} = \cos\xi|000\rangle + \sin\xi|111\rangle$, which violate a Bell inequality involving correlation functions (4.27) except $\xi = 0, \pi/2$. For the GHZ state with $\xi = \pi/4$, the quantum violation reaches its maximum value $3\sqrt{3} = 5.1965$.

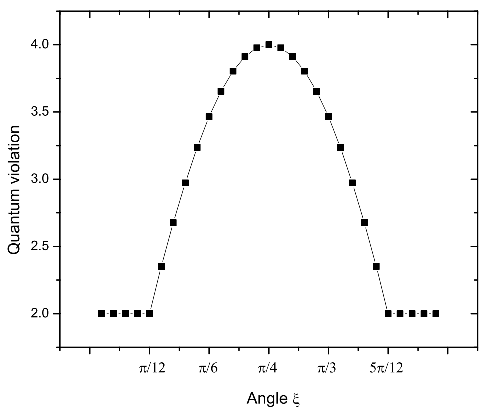

Figure 4.5: Numerical results for the generalized GHZ states $|\psi\rangle_{\rm GHZ} = \cos\xi|000\rangle + \sin\xi|111\rangle$, which violate 3-qubit $\dot{\rm Z}$B inequality except $(0, \pi/12]$, $[7\pi/12,\ \pi/2)$. For the GHZ state with $\xi = \pi/4$, the quantum violation reaches its maximum value 4.

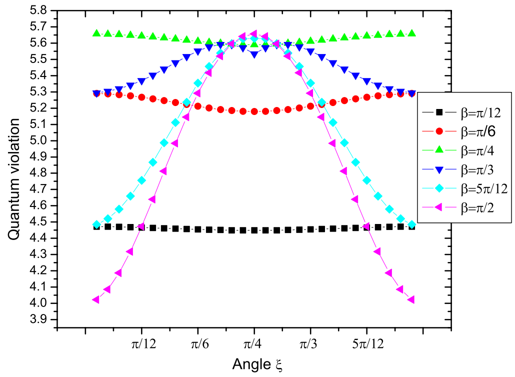

Figure 4.6: Numerical results for the family of the generalized W states $|\psi\rangle_W = \sin\beta\cos\xi|100\rangle + \sin\beta\sin\xi|010\rangle + \cos\beta|001\rangle$ which violate the inequality (4.27) for different ξ and β. Here the cases $\beta = \pi/12, \pi/6, \pi/4, \pi/3,\ 5\pi/12$ and $\pi/2$ are considered.

where $i, j, k = 1, 2$.

Using the following correlation functions,

$$\begin{aligned} Q^{QM}(A_i, B_j, C_k) &= \langle\psi|A_i \otimes B_j \otimes C_k|\psi\rangle, \\ Q^{QM}(A_i, B_j) &= \langle\psi|A_i \otimes B_j \otimes \mathbf{1}|\psi\rangle, \\ Q^{QM}(B_j, C_k) &= \langle\psi|\mathbf{1} \otimes B_j \otimes C_k|\psi\rangle, \\ Q^{QM}(A_i, C_k) &= \langle\psi|A_i \otimes \mathbf{1} \otimes C_k|\psi\rangle, \end{aligned} \tag{4.34}$$

for generalized GHZ states, we find

$$\begin{aligned} Q^{QM}_{GHZ}(A_i, B_j, C_k) &= \cos\theta_{a_i}\cos\theta_{b_j}\cos\theta_{c_k}\cos 2\xi + \\ &\quad \cos(\phi_{a_i} + \phi_{b_j} + \phi_{c_k})\sin\theta_{a_i}\sin\theta_{b_j}\sin\theta_{c_k}\sin 2\xi, \\ Q^{QM}_{GHZ}(A_i, B_j) &= \cos\theta_{a_i}\cos\theta_{b_j}, \\ Q^{QM}_{GHZ}(A_i, C_k) &= \cos\theta_{a_i}\cos\theta_{c_k}, \\ Q^{QM}_{GHZ}(B_j, C_k) &= \cos\theta_{b_j}\cos\theta_{c_k}. \end{aligned} \tag{4.35}$$

By using these correlation functions, quantum predictions of the Bell inequalities listed in Table 4.1 can be easily calculated for the generalized GHZ states. These seven inequalities are equivalent to each other, so we consider the quantum violation of inequality (4.27) without loss of generality. Numerical results show that the inequality (4.27) is violated by the generalized GHZ states for the whole region except $\xi = 0, \pi/2$, see Figure 4.4. When $\xi = \pi/4$, i.e., the GHZ state is given, the maximal quantum violation is $3\sqrt{3}$. For the purpose of comparability, we show the numerical results for 3-qubit Żukowski-Brukner inequality in Figure 4.5. It is clear that the 3-qubit ŻB inequality is not violated by the generalized GHZ states in two regions $(0, \pi/12], [7\pi/12, \pi/2)$. When $\xi = \pi/4$, the maximal violation is 4.

To measure strength of violation of the inequality (4.27), threshold visibility is calculated. Visibility V [47] is the amount of entangled state presented in the system with pure noise ρ_{noise}. In this case, the considered state is described by $\rho = V|\psi\rangle\langle\psi| + (1-V)\rho_{\text{noise}}$, where $\rho_{\text{noise}} = \frac{1}{8}$ for three qubits. The visibility V is bounded by 0 and 1, or $0 \leq V \leq 1$. For $V = 0$, no violation of local realism occurs and for $V = 1$, local realism description does not exist. Correspondingly, correlation functions are given as

$$\begin{aligned}
Q^{QM}_{\rho}(A_i, B_j, C_k) &= \text{Tr}[\rho A_i \otimes B_j \otimes C_k] = V\langle\psi|A_i \otimes B_j \otimes C_k|\psi\rangle \\
&= VQ^{QM}(A_i, B_j, C_k), \\
Q^{QM}_{\rho}(A_i, B_j) &= \text{Tr}[\rho A_i \otimes B_j \otimes \mathbf{1}] = V\langle\psi|A_i \otimes B_j \otimes \mathbf{1}|\psi\rangle \\
&= VQ^{QM}(A_i, B_j),
\end{aligned}$$

$$\begin{aligned}
Q^{QM}_{\rho}(B_j, C_k) &= \text{Tr}[\rho \mathbf{1} \otimes B_j \otimes C_k] = V\langle\psi|\mathbf{1} \otimes B_j \otimes C_k|\psi\rangle \\
&= VQ^{QM}(B_j, C_k), \\
Q^{QM}_{\rho}(A_i, C_k) &= \text{Tr}[\rho A_i \otimes \mathbf{1} \otimes C_k] = V\langle\psi|A_i \otimes \mathbf{1} \otimes C_k|\psi\rangle \\
&= VQ^{QM}(A_i, C_k). \qquad (4.36)
\end{aligned}$$

If we describe the left hand side of the inequality (4.27) as the Bell quantity,

$$\begin{aligned}
\mathcal{B}_{(4.27)} &= Q(A_1B_1C_1) - Q(A_1B_2C_2) - Q(A_2B_1C_2) - Q(A_2B_2C_1) \\
&\quad +2Q(A_2B_2C_2) - Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1) \\
&\quad -Q(A_2B_2) + Q(A_1C_1) + Q(A_1C_2) + Q(A_2C_1) + Q(A_2C_2) \\
&\quad +Q(B_1C_1) + Q(B_1C_2) + Q(B_2C_1) + Q(B_2C_2). \qquad (4.37)
\end{aligned}$$

Local realistic description requires $\mathcal{B}_{(4.27)} \leq 4$. In a quantum theory, results are different. For a pure entangled state, the maximal $\mathcal{B}^{QM}_{(4.27)}$ can be found. If the considered state is a mixed state defined by ρ, the quantum prediction of the Bell quantity is $\mathcal{B}_\rho^{QM} = V\mathcal{B}_{(4.27)}^{QM}$. To violate the Bell inequality (4.27), $\mathcal{B}_\rho^{QM}$ must be larger than 4, namely, $V\mathcal{B}_{(4.27)}^{QM} > 4$. Therefore, there exists a threshold visibility $V_{thr} = \frac{4}{\mathcal{B}_{(4.27)}^{QM}}$ above which the state cannot be described by local realism, and it is sometimes called the critical visibility. Take the GHZ state for example that the maximum value of $\mathcal{B}_{(4.27)}^{QM}$ is $3\sqrt{3}$, so we have the critical visibility is $V_{thr}^{GHZ} = \frac{4\sqrt{3}}{9}$. In other words, given inequality (4.27), the GHZ state cannot be described by local realism if and only if $V > 4\sqrt{3}/9$.

If we consider the W state $|\psi\rangle_W = (|100\rangle + |010\rangle + |001\rangle)\sqrt{3}$, correlation functions are given as

$$\begin{aligned}
Q_W^{QM}(A_i, B_j, C_k) &= -\cos\theta_{a_i}\cos\theta_{b_j}\cos\theta_{c_k} \\
&\quad +\frac{2}{3}\cos(\phi_{a_i}-\phi_{b_j})\sin\theta_{a_i}\sin\theta_{b_j}\cos\theta_{c_k} \\
&\quad +\frac{2}{3}\cos(\phi_{a_i}-\phi_{c_k})\sin\theta_{a_i}\cos\theta_{b_j}\sin\theta_{c_k} \\
&\quad +\frac{2}{3}\cos(\phi_{b_j}-\phi_{c_k})\cos\theta_{a_i}\sin\theta_{b_j}\sin\theta_{c_k}, \\
Q_W^{QM}(A_i, B_j) &= \frac{1}{3}[-\cos\theta_{a_i}\cos\theta_{b_j} + 2\cos(\phi_{a_i}-\phi_{b_j})\sin\theta_{a_i}\sin\theta_{b_j}], \\
Q_W^{QM}(A_i, C_k) &= \frac{1}{3}[-\cos\theta_{a_i}\cos\theta_{c_k} + 2\cos(\phi_{a_i}-\phi_{c_k})\sin\theta_{a_i}\sin\theta_{c_k}], \\
Q_W^{QM}(B_j, C_k) &= \frac{1}{3}[-\cos\theta_{b_j}\cos\theta_{c_k} + 2\cos(\phi_{b_j}-\phi_{c_k})\sin\theta_{b_j}\sin\theta_{c_k}].
\end{aligned} \tag{4.38}$$

The maximum value of the quantum violation of the inequality (4.27) by the W state is calculated as 5.471 based on these correlation functions. Then it is clear that the critical visibility is $V_{thr}^{W} = 0.7312$. It is also found numerically that the correlation Bell inequality (4.27) is violated by all pure entangled states as is the case for inequality (4.13) in section 4.1.1. Take generalized W states for example, the inequality (4.27) is violated when ξ and β take different values (see Figure 4.6). These results are the same as those results of inequality (4.13) given in section 4.1.1. Thus, the inequality (4.27) is an equivalent form of the one (4.13) in section 4.1.1. Although inequality (4.27) is violated by all pure entangled states of three qubits as shown in section 4.1.1, the visibility of the GHZ state is not optimal ($\frac{3\sqrt{3}}{9}$). The visibility of the 3-qubit ŻB inequality for the GHZ state is 0.5, which is optimal. It can be tested that all the seven inequalities given in this section yield the same results. The improvement of this work is that a set of Bell inequalities involving correlation functions, which are violated by the generalized GHZ states for the whole region, are constructed. However, there is no inequality which is not only maximally violated by the GHZ state, but also violated by all pure entangled states. The formulation of such a new Bell inequality for three qubits is still an open problem. In next section, a method will be proposed to solve this problem partly.

4.2 An Observation on Qutrit Inequalities and Qubit Inequalities

In this section, we analyze the relation between Bell inequalities for 3-level and 2-level systems. For two particles, it is shown that any inequality for higher dimensional systems can be reduced to one for lower dimensional systems. All the inequalities constructed are optimal ones in the sense that they are maximally violated by Bell states. For three particles, however, an inequality for 3 qubits reduced from the inequality for 3 qutrits is not maximally violated by the GHZ states, but it is violated by any pure entangled state of three qubits. This observation gives us a clue that we would derive an new inequality for 3 qubits if any inequalities for 3 particles in higher dimensional Hilbert space are given. It is anticipated that such an inequality for 3 qubits is not only violated by all pure entangle states, but also maximally violated by the GHZ state.

4.2.1 Two-Particle Systems

In 2002, Bell inequalities for two quNits were developed by Collins *et al* [20]. Recall that the Collins-Gisin-Linden-Massar-Popescu inequalities read

$$
\begin{aligned}
I_d \equiv \sum_{k=0}^{[d/2]-1} (1-\frac{2k}{d-1})\{ \quad + \quad & [P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) \\
& +P(A_2 = B_2 + k) + P(B_2 = A_1 + k)] \\
& -[P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) \\
& +P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1)]\}.
\end{aligned}
$$

For different dimensional systems, there are different numbers of measurement outcomes. For example, there are 0 and 1 for qubits; 0, 1 and 2 for qutrits. It is reasonable to think that inequalities for higher dimensional systems (for example, qutrits) can be reduced to ones for lower dimensional systems (for example, qubits) if one omits one or more outcomes for higher dimensional system. For the reduction from 2-qutrit inequalities to 2-qubit inequalities, there are many choices to do. More specifically, for the three possible measurement outcomes: 0,1,2, one can choose 0 and 1 as the two possible measurement outcomes for 2 qubits obviously, on the other hand, one can choose 1 and 2 also.

There exists a perfect rule which connects the inequalities for two particles in arbitrary d-dimensional Hilbert space. That is, an inequality for 2 quNits can be reduce to an inequality for 2 qudits, where $N > d$. The most interesting thing is that such an inequality for 2 qudits is a optimal one. By optimal inequality, we mean that the inequality is maximally violated by Bell states. Usually visibility (V) is used to measure the strength of quantum correlations for violating local realism, so one can check the visibility of reduced inequality for lower dimensional systems to see whether the inequality is optimal or not.

4.2.2 Three-Particle Systems

For three particles, the simplest system is 3 qubit. The inequality (4.1) for a three 2-level system is given by Mermin [33] or Żukowski *et al* [24] in terms of correlation functions,

$$Q_{112} + Q_{121} + Q_{211} - Q_{222} \leq 2,$$

which can rewritten in terms of probabilities,

$$\begin{aligned}
&P(a_1 + b_1 + c_2 = 0) + P(a_1 + b_2 + c_1 = 0) + P(a_2 + b_1 + c_2 = 0) + \\
&P(a_2 + b_2 + c_2 = 1) - P(a_1 + b_1 + c_2 = 1) - P(a_1 + b_2 + c_1 = 1) - \\
&P(a_2 + b_1 + c_2 = 1) - P(a_2 + b_2 + c_2 = 0) \leq 2,
\end{aligned} \tag{4.39}$$

where $P(a_i + b_j + c_k = r)$ is the joint probability with $i, j, k = 1, 2$; $m, n, l = 0, 1$. For instance, $P(a_i + b_j + c_k = 1) = P(a_i = 1, b_j = 0, c_k = 0) + P(a_i = 0, b_j = 1, c_k = 0) + P(a_i = 0, b_j = 0, c_k = 1)$. The joint probability $P(a_i + b_j + c_k = r)$ is defined as measurements A_i, B_j and C_k have outcomes that sum to r modulo 2. However, quantum mechanics will violate this Bell inequality for GHZ state of 3 qubits $|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Quantum violation of inequality (4.39) can be found by using quantum predictions for the joint probabilities defined in Eq. (4.5). For the following settings $\xi = \pi/4, \theta_{a_1} = \theta_{b_1} = \theta_{c_1} = \theta_{a_2} = \theta_{b_2} = \theta_{c_2} = \pi, \phi_{a_1} = \phi_{b_1} = \phi_{c_1} = -\pi/6, \phi_{a_2} = \phi_{b_2} = \phi_{c_2} = \pi/3$, the left hand side of inequality (4.39) is 4. As pointed by Żukowski et al, however, inequality (4.39) is not violated by generalized GHZ states when $\xi \in (0, \pi/12]$. It can be seen from Figure 4.5.

For a 3-level system, a Bell inequality in terms of probabilities for three-qutrit systems was found in Ref. [39]. The inequality given in [39] is carefully derived in a symmetric form with respect to any permutation of the subsystems. They presented the Bell inequality for the following scenario: three space separated observers (denoted by A, B and C) can measure two local observables with outcomes labeled by 0, 1, and 2. The observers will get (l, m, n) with probability $P(a_i = l, b_j = m, c_k = n)$ when measure A_i, B_j, and C_k, with $m, n, l = 0, 1, 2$ and $i, j, k = 1, 2$. The non-trivial inequality imposed by local realistic theories is inequality (4.2). Recall that inequality (4.2) reads

$$\begin{aligned} &P(a_1 + b_1 + c_1 = 0) + P(a_1 + b_2 + c_2 = 1) + P(a_2 + b_1 + c_2 = 1) + \\ &P(a_2 + b_2 + c_1 = 1) + 2P(a_2 + b_2 + c_2 = 0) - P(a_2 + b_1 + c_1 = 2) - \\ &P(a_1 + b_2 + c_1 = 2) - P(a_1 + b_1 + c_2 = 2) \leq 3, \end{aligned}$$

where $P(a_i + b_j + c_k = r)$ is the joint probability,

$$P(a_i + b_j + c_k = r) = \sum_{a,b=0,1,2} P(a_i = a, b_j = b, c_k = r - a - b). \tag{4.40}$$

The joint probability $P(a_i + b_j + c_k = r)$ is defined as measurements A_i, B_j and C_k have outcomes that sum to r modulo 3.

To test quantum violation of this inequality,

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle), \tag{4.41}$$

the above state was taken as initial state. The measurements were restricted to the tritter measurements, or unbiased symmetric six-port beamsplitter [54]. As shown in Figure 4.7, three down converted photons are fed into three identical separated tritters. The phase shifters are placed close to the input ports of the tritter. Recall that the matrix elements of an unbiased symmetric six-port beamsplitter are given

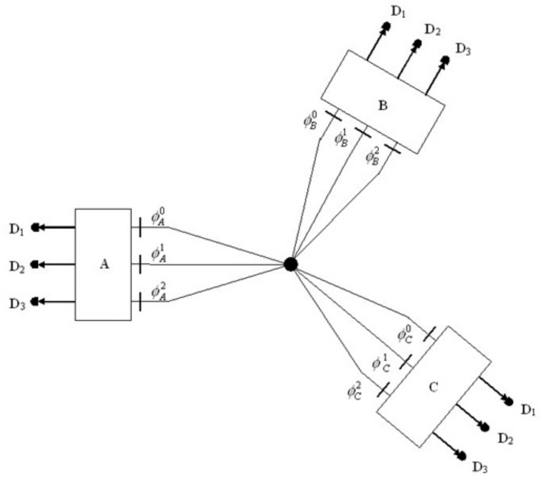

Figure 4.7: A three-tritter Bell-type experiment. This is an experiment of Alice, Bob and Charlie analyze entangled qutrits.

by $U_3^{kl}(\vec{\phi}) = \frac{1}{\sqrt{3}}\alpha^{kl}\exp(i\phi^l)$, where $\alpha = \exp(\frac{2i\pi}{3})$ and $\phi^l(l = 0, 1, 2)$ are the settings of the appropriate phase shifters, for convenience one can denote them as a three dimensional vector $\vec{\phi} = (\phi^0, \phi^1, \phi^2)$.

The quantum version of probability was given as [39]

$$P(a_i = k, b_j = l, c_k = m) = |\langle klm|U_3(\vec{\phi}_{A_i}) \otimes U_3(\vec{\phi}_{B_j}) \otimes U_3(\vec{\phi}_{C_k})|\psi_3\rangle|^2. \quad (4.42)$$

Thus the quantum analogue of the joint probability can be easily calculated [39]

$$\begin{aligned} P(a_i + b_j + c_k = r) = \frac{1}{9} \quad [\quad & 3 + 2\cos(\varphi^1_{ijk} - \varphi^0_{ijk} + 2r/3\pi) \\ & +2\cos(\varphi^2_{ijk} - \varphi^0_{ijk} + 4r/3\pi) \\ & +2\cos(\varphi^2_{ijk} - \varphi^1_{ijk} + 2r/3\pi)]. \end{aligned} \quad (4.43)$$

where $\varphi^i_{ijk} = \phi^i_{A_i} + \phi^i_{B_j} + \phi^i_{C_k}(i = 0, 1, 2)$. It was shown [39] that for the optimal settings,

$$\begin{aligned} \vec{\phi}_{A_1} &= (0, 0, 0), \\ \vec{\phi}_{A_2} &= (0, 0, \frac{2\pi}{3}), \\ \vec{\phi}_{B_1} &= (0, -\frac{2\pi}{3}, -\frac{\pi}{3}), \\ \vec{\phi}_{B_2} &= (0, -\frac{2\pi}{3}, \frac{\pi}{3}), \\ \vec{\phi}_{C_1} &= (0, \frac{\pi}{3}, 0), \\ \vec{\phi}_{C_2} &= (0, \frac{\pi}{3}, \frac{2\pi}{3}), \end{aligned} \quad (4.44)$$

the left hand side of inequality (4.2) gives $\frac{39}{9} > 3$, namely, this inequality is violated by quantum mechanics.

Consider these two inequalities (4.39) and (4.2), there is no direct connection between them. A new inequality for 3 qubits is given when inequality (4.2) is reduced to 2 dimensions,

$$\begin{aligned}
&P(a_1 + b_1 + c_1 = 1) + 2P(a_2 + b_2 + c_2 = 1) \\
&+P(a_1 + b_2 + c_2 = 2) + P(a_2 + b_1 + c_2 = 2) + P(a_2 + b_2 + c_1 = 2) \\
&-P(a_1 + b_1 + c_2 = 0) - P(a_1 + b_2 + c_1 = 0) - P(a_2 + b_1 + c_1 = 0) \\
&-P(a_1 + b_1 + c_2 = 3) - P(a_1 + b_2 + c_1 = 3) - P(a_2 + b_1 + c_1 = 3) \leq 3.
\end{aligned}$$

Which is just the inequality (4.13) given in section 4.1.1. It has been shown that Gisin's theorem can be generalized to three-qubit systems by using the inequality, i.e., all pure entangled states of a three-qubit system are numerically shown to violate the Bell inequality.

As shown in section 4.1.2, although inequality (4.27), which is an equivalent from of the inequality (4.13), is violated by any pure entangled states of three qubits, the visibility of the GHZ state is not optimal and $V_{thr} = 4\sqrt{3}/9 = 0.7698$. The visibility of the inequality (4.39) given by Żukowski *et al* for the GHZ state is 0.5. Such an interesting result gives us a clue that inequalities for 3 qubits may be improved by reducing from inequalities for $d(> 3)$ dimensional systems. By improved inequality, we mean that it is violated by all pure entangled states of three qubits, in the mean time, visibility given by the inequality is close to 0.5 (at least < 0.7698). It is anticipated that such an inequality for 3 qubits can be derived by reducing from an inequality for $d(> 3)$-dimemsional systems, if a new inequality for $d(> 3)$-dimensional systems can be given.

A new Bell inequality for 3 particles in four-level Hilbert space will be presented in next section. It will also be shown that when reduced to 2-dimensional systems, this Bell inequality results in an improved inequality which is violated by all pure entangled states and has a better visibility than inequality (4.13) has[7].

[7]This work is to be submitted for publication, see [7] in the publication list in Appendix A.

4.3 A Bell Inequality for Three Four-Level Systems

In this section, we present a new Bell inequality expressed by probabilities for tripartite four-dimensional systems. We show that the inequality imposes a necessary condition on the existence of a local realistic description for the correlations generated by three four-level systems. The new inequality is violated by maximally entangled states of three four-dimensional systems.

4.3.1 A Bell Inequality for 3 Four-Level Systems

Our approach to develop a new Bell inequality for tripartite four-dimensional systems is based on the Gedanken experiment. There are three separated observers, denoted by A, B, and C hereafter, each can carry out two possible local measurements, A_1 or A_2 for A, B_1 or B_2 for B and C_1 or C_2 for C respectively. Each measurement may have four possible outcomes, labeled by 0, 1, 2 and 3. We denote the observable X_i measured by party X and the outcome x_i with $X = A, B, C(x = a, b, c)$. If observers decide to measure observables A_1, B_2 and C_1, the result is $(0, 1, 2)$ with probability $P(a_1 = 0, b_2 = 1, c_1 = 2)$. A local realistic theory can be described by $2^3 \times 4^3$ probabilities $P(a_i = a, b_j = b, c_k = c)$ with $i, j, k = 1, 2$ and $a, b, c = 0, 1, 2, 3$. The total number of probabilities is $2^3 \times 4^3$ can be seen in the following way. 4^3 describes all possible probabilities for a specific observables chosen by the three observers (say A_1, B_2 and C_1) since each observer can have one of four possible results $(0, 1, 2, 3)$. There are 2^3 possible combinations of observables because $i, j, k = 1, 2$. Thus there are totally 8×56 probabilities. Here we denote the joint probability $P(a_i + b_j + c_k = r)$ that the measurements A_i, B_j and C_k have outcomes that sum, modulo four, to r:

$$P(a_i + b_j + c_k = r) = \sum_{a,b=0,1,2,3} P(a_i = a, b_j = b, c_k = r - a - b). \tag{4.45}$$

Some of the local realistic constraints are trivial, such as normalization and the no-signaling conditions which are not violated by quantum predictions. Only the non-trivial inequality, which is not true for quantum mechanics, is of use for checking whether we can describe quantum correlations by a classical model. The new Bell inequality for three four-dimensional systems has the form

$$\begin{aligned}
&-3P(a_1+b_1+c_1=0)+P(a_1+b_1+c_1=2)-5P(a_1+b_1+c_2=1)\\
&-P(a_1+b_1+c_2=3)-5P(a_1+b_2+c_1=1)-P(a_1+b_2+c_1=3)\\
&-5P(a_2+b_1+c_1=1)-P(a_2+b_1+c_1=3)+P(a_1+b_2+c_2=0)\\
&-3P(a_1+b_2+c_2=2)+P(a_2+b_1+c_2=0)-3P(a_2+b_1+c_2=2)\\
&+P(a_2+b_2+c_1=0)-3P(a_2+b_2+c_1=2)-P(a_2+b_2+c_2=1)\\
&+3P(a_2+b_2+c_2=3))\leq 0. \qquad (4.46)
\end{aligned}$$

Similar as those done for inequalities for three qubits, the maximum value of the left hand side of inequality (4.46) in local theories is shown to be 0. This can be explained as follows. To beat the bound 0, terms $P(a_1+b_1+c_1=2)$ and $P(a_2+b_2+c_2=3)$ are firstly taken to be equal to one. This means that $a_1+b_1+c_1+a_2+b_2+c_2=5$. Among the remaining terms, we take $P(a_1+b_1+c_2=3)$, $P(a_1+b_2+c_1=3)$ and $P(a_2+b_1+c_1=3)$ equal to one to maximize the value of left hand side of the inequality (4.46). As a result, $a_2+b_2+c_1=2$, $a_2+b_1+c_2=2$ and $a_1+b_2+c_2=2$ according to the constraint $a_1+b_1+c_1+a_2+b_2+c_2=5$. So $P(a_1+b_1+c_2=1)=P(a_1+b_2+c_1=1)=P(a_2+b_1+c_1=1)=0$, $P(a_2+b_2+c_1=2)=P(a_2+b_1+c_2=2)=P(a_1+b_2+c_2=2)=1$ and $P(a_2+b_2+c_1=0)=P(a_2+b_1+c_2=0)=P(a_1+b_2+c_2=0)=0$. Therefore we have $0+1-0-1-0-1-0-1+0-3+0-3+0-3-0+3=-8\leq 0$. If initially terms $P(a_1+b_1+c_1=0)$ and $P(a_2+b_2+c_2=3)$ are taken equal to one first, we have $a_1+b_1+c_1+a_2+b_2+c_2=3$. Among the remaining terms, we take $P(a_1+b_1+c_2=3)$, $P(a_1+b_2+c_1=3)$ and $P(a_2+b_1+c_1=3)$ equal to one to maximize the value of left hand side of the inequality (4.46). As a result, $a_2+b_2+c_1=0$, $a_2+b_1+c_2=0$ and $a_1+b_2+c_2=0$ according to the constraint $a_1+b_1+c_1+a_2+b_2+c_2=3$. So $P(a_1+b_1+c_2=1)=P(a_1+b_2+c_1=1)=P(a_2+b_1+c_1=1)=0$, $P(a_2+b_2+c_1=2)=P(a_2+b_1+c_2=2)=P(a_1+b_2+c_2=2)=0$ and $P(a_2+b_2+c_1=0)=P(a_2+b_1+c_2=0)=P(a_1+b_2+c_2=0)=1$. Therefore we have $-3+0-0-1-0-1-0-1+1-0+1-0+1-0-0+3=0\leq 0$. Thus, after some lengthy calculations, it can be shown that the inequality (4.46) is always bounded by 0 in a local realistic model.

Let us now consider the maximum value that can be attained for the inequality

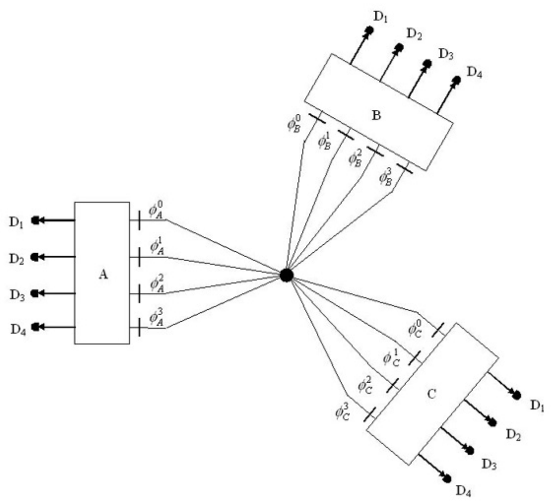

Figure 4.8: A three-quarter Bell-type experiment. This is an experiment of Alice, Bob and Charlie analyze entangled particles each in a four-dimensional Hilbert space.

(4.46) for quantum measurements on an entangled quantum state. First, we specify the quantum state and measurement. The initial state is the maximally entangled state of a three four-level system,

$$|\psi_4\rangle = \frac{1}{2}(|000\rangle + |111\rangle + |222\rangle + |333\rangle). \tag{4.47}$$

Consider a Gedanken experiment in which A, B and C measure observables defined by unbiased symmetric eight-port beamsplitters (quarter) [54] on $|\psi_4\rangle$. A three-quarter Bell-type experiment is plotted in Figure 4.8. Three down converted photons are fed into three identical separated quarters. The phase shifters are placed close to the input ports of the quarter. Recall that the matrix elements of an unbiased symmetric eight-port beamsplitter are given by $U_4^{kl}(\vec{\phi}) = \frac{1}{2}\alpha^{kl}\exp(i\phi^l)$, where $\alpha = \exp(\frac{2i\pi}{4})$ and $\phi^l (l = 0, 1, 2, 3)$ are the settings of the appropriate phase shifters, for convenience we denote them as a four dimensional vector $\vec{\phi} = (\phi^0, \phi^1, \phi^2, \phi^3)$.

The quantum prediction for the probability of obtaining the outcome (a, b, c) is then given as

$$P(a_i = a, b_j = b, c_k = c) = |\langle abc|U_4(\vec{\phi}_{A_i}) \otimes U_4(\vec{\phi}_{B_j}) \otimes U_4(\vec{\phi}_{C_k})|\psi_4\rangle|^2. \tag{4.48}$$

$p(a_1+b_1+c_1=0)$	$p(a_1+b_1+c_1=2)$
0	2/3
$p(a_1+b_1+c_2=1)$	$p(a_1+b_1+c_2=3)$
0	0
$p(a_1+b_2+c_1=1)$	$p(a_1+b_2+c_1=3)$
0	0
$p(a_2+b_1+c_1=1)$	$p(a_2+b_1+c_1=3)$
0	0
$p(a_1+b_2+c_2=0)$	$p(a_1+b_2+c_2=2)$
2/3	0
$p(a_2+b_1+c_2=0)$	$p(a_2+b_1+c_2=2)$
2/3	0
$p(a_2+b_2+c_1=0)$	$p(a_2+b_2+c_1=2)$
2/3	0
$p(a_2+b_2+c_2=1)$	$p(a_2+b_2+c_2=3)$
0	8/9

Table 4.2: The values of the probabilities in inequality (4.46) with appropriate angle settings.

Thus the quantum analogue of the joint probability can be easily calculated

$$\begin{aligned}
&P(a_i+b_j+c_k=r)\\
&=\frac{1}{16}[4+2\cos(\varphi^1_{ijk}-\varphi^0_{ijk}+\frac{\pi}{2}r)+2\cos(\varphi^2_{ijk}-\varphi^0_{ijk}+\pi r)\\
&+2\cos(\varphi^2_{ijk}-\varphi^1_{ijk}+\frac{\pi}{2}r)+2\cos(\varphi^3_{ijk}-\varphi^0_{ijk}+\frac{3\pi}{2}r)\\
&+2\cos(\varphi^3_{ijk}-\varphi^1_{ijk}+\pi r)+2\cos(\varphi^3_{ijk}-\varphi^2_{ijk}+\frac{\pi}{2}r)],
\end{aligned} \tag{4.49}$$

where $\varphi^i_{ijk}=\phi^i_{A_i}+\phi^i_{B_j}+\phi^i_{C_k}(i=0,1,2,3)$. In order to look for the maximal violation of the inequality, we choose the optimal settings as the following:

$$\begin{aligned}
&\vec{\phi}_{A_1}=\vec{\phi}_{B_1}=\vec{\phi}_{C_1}=(0,\frac{1}{3}\arccos(-\frac{1}{3}),\frac{1}{3}\arccos(-\frac{1}{3})-\frac{\pi}{3},\frac{\pi}{3}),\\
&\vec{\phi}_{A_2}=\vec{\phi}_{B_2}=\vec{\phi}_{C_2}=(0,\frac{1}{3}\arcsin\frac{7}{9},\frac{1}{3}\arcsin\frac{7}{9}+\frac{\pi}{6},-\frac{\pi}{6}).
\end{aligned} \tag{4.50}$$

Numerical results show that for this choice, all the probabilities terms have definite values as listed in Table 4.2. Putting them into the left hand side of the inequality (4.46), we arrive at $-3\times 0+\frac{2}{3}+3(-5\times 0-0)+3(\frac{2}{3}-3\times 0)-0+3\frac{8}{9}=\frac{16}{3}>0$.

In Ref. [51], a proposal was made to measure the strength of violation of local realism by the minimal amount of noise that must be added to the system in order to hide the non-classical character of the observed correlations. This is equivalent to a replacement of the pure state $|\psi\rangle\langle\psi|$ by the mixed state $\rho(F)$ of the form $\rho(F) = (1-F)|\psi\rangle\langle\psi| + \frac{F}{64} I\otimes I\otimes I$, where I is an identity matrix and $F(0 \leq F \leq 1)$, is the amount of noise present in the system. For $F = 0$, local realistic description does not exist, whereas it does for $F = 1$. Therefore, there exists some threshold value of F, denoted by F_{thr}, such that for every $F \leq F_{thr}$, local and realistic description does not exist. The threshold fidelity for three four-level system is $F_{thr} = \frac{8}{17} = 0.4706$. Obviously, the relation between F and visibility is $F = 1 - V$. So the threshold visibility is $V_{thr} = \frac{9}{17}$.

4.3.2 A New Bell Inequality for 3 Qubits

When restricted to 2-dimensional systems, the Bell inequality (4.46) reduces to a new Bell inequality for three qubits, which is violated by all pure entangled states of three-qubit systems. The visibilities above which GHZ state and W state can not be described by local realism are calculated. The visibilities are improved ones compared with those given in section 4.1. The fact is part of the reason that the new inequality for 3 qubits is better than previous ones. In this section, we present the new Bell inequality for three-qubit system which is reduced from inequality (4.46)

$$\begin{aligned}
&3P(a_1+b_1+c_1=0)+P(a_1+b_1+c_1=1)-5P(a_1+b_1+c_1=2)\\
&+P(a_1+b_1+c_1=3)+3P(a_1+b_1+c_2=0)+P(a_1+b_1+c_2=1)\\
&+3P(a_1+b_1+c_2=2)-7P(a_1+b_1+c_2=3)+3P(a_1+b_2+c_1=0)\\
&+P(a_1+b_2+c_1=1)+3P(a_1+b_2+c_1=2)-7P(a_1+b_2+c_1=3)\\
&+3P(a_2+b_1+c_1=0)+P(a_2+b_1+c_1=1)+3P(a_2+b_1+c_1=2)\\
&-7P(a_2+b_1+c_1=3)-5P(a_1+b_2+c_2=0)+P(a_1+b_2+c_2=1)\\
&+3P(a_1+b_2+c_2=2)+P(a_1+b_2+c_2=3)-5P(a_2+b_1+c_2=0)\\
&+P(a_2+b_1+c_2=1)+3P(a_2+b_1+c_2=2)+P(a_2+b_1+c_2=3)\\
&-5P(a_2+b_2+c_1=0)+P(a_2+b_2+c_1=1)+3P(a_2+b_2+c_1=2)\\
&+P(a_2+b_2+c_1=3)-P(a_2+b_2+c_2=0)+5P(a_2+b_2+c_2=1)\\
&-P(a_2+b_2+c_2=2)-3P(a_2+b_2+c_2=3))\leq 12,
\end{aligned}\tag{4.51}$$

which can be expressed in terms of correlation functions

$$
\begin{aligned}
&-Q(A_1B_1C_1)+Q(A_1B_1C_2)+Q(A_1B_2C_1)+Q(A_2B_1C_1)-Q(A_2B_2C_2)\\
&-Q(A_1B_2)-Q(A_2B_1)-Q(A_2B_2)-Q(A_1C_2)-Q(A_2C_1)-Q(A_2C_2)\\
&-Q(B_1C_2)-Q(B_2C_1)-Q(B_2C_2)+Q(A_1)+Q(B_1)+Q(C_1)\le 3. \qquad (4.52)
\end{aligned}
$$

The above inequality (4.52) includes the terms of single-particle correlation function, it is symmetric under the permutations of A_j, B_j and C_j. The inequality (4.52) can be derived by averaging the following inequality

$$
\begin{aligned}
&-A_1B_1C_1+A_1B_1C_2+A_1B_2C_1+A_2B_1C_1-A_2B_2C_2\\
&-A_1B_2-A_2B_1-A_2B_2-A_1C_2-A_2C_1-A_2C_2\\
&-B_1C_2-B_2C_1-B_2C_2+A_1+B_1+C_1\le 3. \qquad (4.53)
\end{aligned}
$$

By fixing the values of A_1, B_1 and C_1, as done in Section 4.1.2, the inequality (4.53) is shown to be always satisfied under a local realistic description and so is inequality (4.52).

1. For the case that A_1, B_1 and C_1 are all plus one, the inequality (4.53) becomes

$$-A_2B_2C_2-A_2B_2-A_2C_2-B_2C_2-A_2-B_2-C_2-1\le 0. \qquad (4.54)$$

   If $C_2=1$, we have $-2(A_2+1)(B_2+1)\le 0$ from inequality (4.54). Because A_2 and B_2 can be either plus one or minus one, $-2(A_2+1)(B_2+1)$ will be -8 or 0. These two values are no larger than 0. If $C_2=-1$, from inequality (4.54) we have $0\le 0$, which is obviously satisfied.

2. For the case that $A_1=B_1=1$ and $C_1=-1$, the inequality (4.53) becomes

$$-A_2B_2C_2-A_2B_2-A_2C_2-B_2C_2-A_2-B_2-C_2-1\le 0. \qquad (4.55)$$

   The inequality is the same as inequality (4.54). Seen from the first case, no matter which values A_2, B_2 and C_2 take, the inequality (4.55) is always correct. Similar conclusions can be drawn for the cases that $A_1=C_1=1$ and $B_1=-1$, and $B_1=C_1=1$ and $A_1=-1$ because the inequality (4.53) is symmetric under the permutations of A, B and C.

3. For the case that $A_1 = B_1 = -1$ and $C_1 = 1$, the inequality (4.53) becomes

$$-A_2B_2C_2 - A_2B_2 - A_2C_2 - B_2C_2 - A_2 - B_2 + 3C_2 - 5 \leq 0. \qquad (4.56)$$

   If $C_2 = 1$, we have $-2(A_2 + 1)(B_2 + 1) \leq 0$ from inequality (4.56). Because A_2 and B_2 can be either plus one or minus one, $-2(A_2 + 1)(B_2 + 1)$ will be -8 or 0. These two values are no larger than 0. If $C_2 = -1$, from inequality (4.56) we have $-3 \leq 5$, which is obviously correct whichever values A_2, B_2 and C_2 take. Similar conclusions can be drawn for the cases that $A_1 = C_1 = -1$ and $B_1 = 1$, and $B_1 = C_1 = -1$ and $A_1 = 1$ because the inequality (4.53) is symmetric under the permutations of A, B and C.

4. For the case that A_1, B_1 and C_1 are all minus one, the inequality (4.53) becomes

$$-A_2B_2C_2 - A_2B_2 - A_2C_2 - B_2C_2 + 3A_2 + 3B_2 + 3C_2 - 5 \leq 0. \qquad (4.57)$$

   If $C_2 = 1$, we have $-2(A_2 - 1)(B_2 - 1) \leq 0$ from inequality (4.57). Because A_2 and B_2 can be either plus one or minus one, $-2(A_2 - 1)(B_2 - 1)$ will be -8 or 0. These two values are no larger than 0. If $C_2 = -1$, from inequality (4.57) we have $4(A_2 + B_2 - 2) \leq 0$, which is satisfied because A_2 and B_2 can be either plus one or minus one, $4(A_2 + B_2 - 2)$ will be -16, -8 or 0.

Thus, the inequality (4.53) is always satisfied whichever values A_i, B_j and C_k take and hence inequality (4.52) is always correct under a local realistic description. When setting $C_1 = -1, C_2 = 1$, the inequality (4.52) reduces directly to an equivalent form of the CHSH inequality for two qubits

$$Q(A_1B_1) - Q(A_1B_2) - Q(A_2B_1) - Q(A_2B_2) \leq 2.$$

Similar to those done in the previous sections, by using the universal pure entangled states described by expression (4.14) and calculating the correlation functions, it is checked numerically that the above inequality (4.52) is violated by all pure entangled states of three qubits. However, no analytical proof of the conclusion can be constructed at this stage. In the following, some special cases will be given to show that the inequality (4.52) is violated by all pure entangled states. The

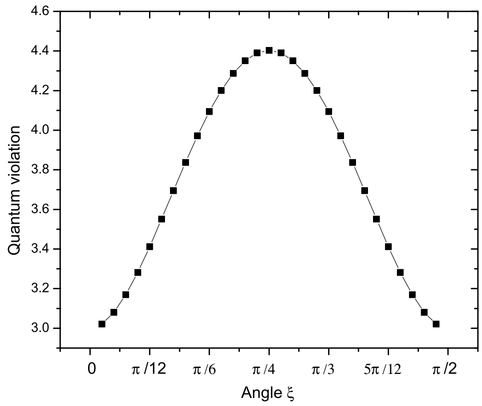

Figure 4.9: Numerical results for the generalized GHZ states $|\psi\rangle_{\mathrm{GHZ}} = \cos\xi|000\rangle + \sin\xi|111\rangle$, which violate the inequality (4.52) except $0, \pi/2$.

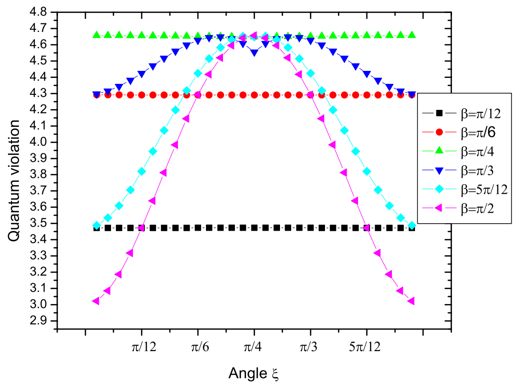

Figure 4.10: Numerical results for the family of generalized W states $|\psi\rangle_W = \sin\beta\cos\xi|100\rangle + \sin\beta\sin\xi|010\rangle + \cos\beta|001\rangle$ which violate the inequality (4.52) for different ξ and β. Here the cases $\beta = \pi/12, \pi/6, \pi/4, \pi/3$, $5\pi/12$ and $\pi/2$ are considered.

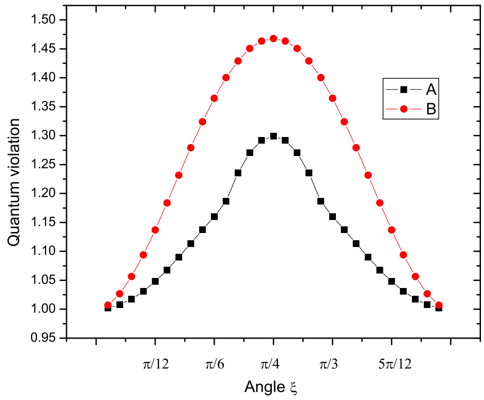

Figure 4.11: Violation of two three-qubit Bell inequalities by the generalized GHZ states with different values of ξ, where curve A is for inequality (4.27) given in section 4.1.2 and curve B is for inequality (4.52).

first family of quantum states considered is the family of generalized GHZ states $|\psi\rangle_{\text{GHZ}} = \cos\xi|000\rangle + \sin\xi|111\rangle$. The inequality (4.52) is violated by the generalized GHZ states for the whole region except $\xi = 0, \pi/2$. For the GHZ state with $\xi = \pi/4$, the quantum violation reaches its maximum value 4.40367. The variation of the violation with ξ is shown in Figure 4.9. Another family considered is the family of generalized W states $|\psi\rangle_W = \sin\beta\cos\xi|100\rangle + \sin\beta\sin\xi|010\rangle + \cos\beta|001\rangle$. By fixing the value of β, quantum violation of the inequality (4.52) varies with ξ (see Figure 4.10). The inequality (4.52) is violated by generalized W states except the cases with $\beta = \frac{\pi}{2}$, $\xi = 0/\xi = \frac{\pi}{2}$. The states in these cases are product states which do not violated any Bell inequality. For the standard W state, quantum violation of the inequality (4.52) approached 4.54086.

Hence inequality (4.52) is also one candidate to generalize the theorem of Gisin to three-qubit systems. One of the interests of the new inequality for three qubits is that it is more resistant to noise. The inequality (4.52) is violated by the GHZ state $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, the threshold visibility is $V_{thr}^{GHZ} = 0.68125$. The inequality (4.52) is also violated by the W state, the threshold visibility is $V_{thr}^{W} = 0.660668$. We plot the variation of quantum violation for the generalized GHZ states with angle ξ for inequality (4.52) and inequality (4.27) given in section 4.1.2, see Figure 4.11. In plotting the figure, we rewrite the expressions of inequalities (4.27) and (4.52) as

$$\begin{aligned}&\frac{1}{4}[Q(A_1B_1C_1)-Q(A_1B_2C_2)-Q(A_2B_1C_2)-Q(A_2B_2C_1)+2Q(A_2B_2C_2)\\&-Q(A_1B_1)-Q(A_1B_2)-Q(A_2B_1)-Q(A_2B_2)+Q(A_1C_1)+Q(A_1C_2)\\&+Q(A_2C_1)+Q(A_2C_2)+Q(B_1C_1)+Q(B_1C_2)+Q(B_2C_1)+Q(B_2C_2)]\leq 1\end{aligned} \tag{4.58}$$

$$\begin{aligned}&\frac{1}{3}[-Q(A_1B_1C_1)+Q(A_1B_1C_2)+Q(A_1B_2C_1)+Q(A_2B_1C_1)-Q(A_2B_2C_2)\\&-Q(A_1B_2)-Q(A_2B_1)-Q(A_2B_2)-Q(A_1C_2)-Q(A_2C_1)-Q(A_2C_2)\\&-Q(B_1C_2)-Q(B_2C_1)-Q(B_2C_2)+Q(A_1)+Q(B_1)+Q(C_1)]\leq 1\end{aligned} \tag{4.59}$$

respectively. In these forms, the violation degrees of the two inequalities can be compared directly. Comparing with the results of the inequalities given in section 4.1.2, the new inequality (4.52) is really more resistant to noise. It seems that we could derive some new 3-qubit Bell inequalities, which would be more highly resistant to noise, if we know other Bell inequalities of tripartite N($N > 4$)-dimensional quantum systems.

4.4 Proposed Experiment for Testing Quantum Nonlocality

Experimental verification on the conflict between quantum mechanics and local realism for two particles have been demonstrated in several experiments [17, 66, 67]. For a system more composite than two particles, for example three particles, experimental verification of nonexistence of local realism is generally more difficult. Recently, by exploiting the results of a fourth experiment constructed from three specific experiments, conflicts between quantum mechanics and local realism for three qubits and four qubits have also been verified [68, 69]. For N qubits, experimental observation for violation of Bell inequalities is still lacking. In this section, we propose an experimental scheme for testing quantum nonlocality of three qubits. The scheme can be generalized to N qubits[8].

[8]This work was published, see [6] in the publication list in Appendix A.

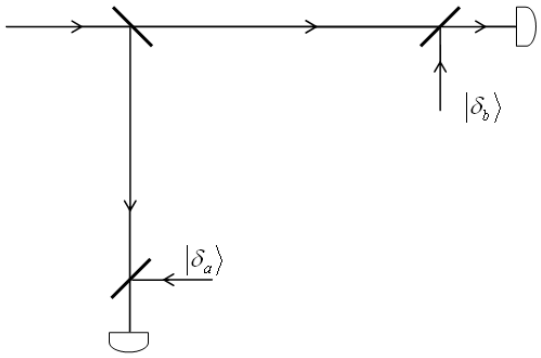

Figure 4.12: The optical setup proposed to test nonlocality of two qubits in Ref. [70].

In Ref. [70], the authors proposed an optical setup for testing quantum nonlocality in phase space for two qubits. The setup essentially demonstrated quantum nonlocality based on phase space measurement of the Wigner function using violation of the CHSH inequality. The source used in Ref. [70] is a single photon incident on a 50:50 beam splitter (see Figure 4.12). The generated state is $|\psi(2)\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle)$ written in terms of the exit ports a_1 and a_2. For example, $|10\rangle$ means one photon exits at port a_1 and no photon exits at port a_2. $\pm$ can be realized by one phase shifter to adjust the relative phase and the relative phase can be set to zero without loss of generality. As pointed in Ref. [70], the measuring apparatus consists of a beam splitter and photon counting detector. The power transmission of the beam splitter is T. The second input port of the beam splitter is fed with an excited coherent state $|\delta\rangle$. The action of the beam splitter is described by $\hat{D}(\sqrt{1-T}\delta)$ in the limit of $T \to 1$ and $\delta \to \infty$. The realistic measurement proposed in Ref. [70] is a test that the detectors can resolve the number of absorbed photons and $+1/-1$ is assigned to events in which an even/odd number of photons is registered. In this way, correlation function measured in the scheme is resulted by setting $\alpha = \sqrt{1-T}\delta$,

$$Q_{ab}(\alpha,\beta) = \langle\psi(2)|\hat{Q}_a(\alpha) \otimes \hat{Q}_b(\beta)|\psi(2)\rangle, \tag{4.60}$$

where α and β are coherent displacements for the modes a and b. $\hat{Q}_a(\alpha)$ is an operator defined as [70]

$$\begin{aligned}\hat{Q}_a(\alpha) &= \hat{D}_a(\alpha)\sum_{k=0}^{\infty}|2k\rangle\langle 2k|\hat{D}_a^{\dagger}(\alpha) \\ &\quad -\hat{D}_a(\alpha)\sum_{k=0}^{\infty}|2k+1\rangle\langle 2k+1|\hat{D}_a^{\dagger}(\alpha). \qquad (4.61)\end{aligned}$$

The operator $\hat{Q}_a(\alpha)$ is also of the form [70]

$$\hat{Q}_a(\alpha) = \hat{D}_a(\alpha)(-1)^{\hat{n}_a}\hat{D}_a^{\dagger}(\alpha), \qquad (4.62)$$

and similar definition for $\hat{Q}_b(\beta)$. For the state $|\psi(2)\rangle$, it was shown [70] that

$$Q_{ab}(\alpha,\beta) = (2|\alpha-\beta|^2-1)e^{-2|\alpha|^2-2|\beta|^2}. \qquad (4.63)$$

The experimental setting of the coherent displacements was chosen as 0 or α for observer a; 0 or β for observer b. Then from the CHSH inequality, the CHSH-Bell quantity was constructed in Ref. [70] as

$$\begin{aligned}\mathcal{B}_{CHSH} &= Q_{ab}(0,0)+Q_{ab}(\alpha,0)+Q_{ab}(0,\beta)-Q_{ab}(\alpha,\beta) \\ &= -1+(4r-2)e^{-2r}-(8r\sin^2\varphi-1)e^{-4r} \qquad (4.64)\end{aligned}$$

where $|\alpha|^2 = |\beta|^2 = r$ and $\beta = e^{2i\varphi}\alpha$. The minimum value of $\mathcal{B}_{CHSH}$ was shown to be about -2.2. As local realistic theories require $-2 \leq \mathcal{B}_{CHSH} \leq 2$, the violation of local realism is obvious.

Since some Bell inequalities for three qubits have been constructed in the previous sections. We would like to generalize the experimental scheme to three qubits. Similar to the two-qubit scheme, an optical setup to demonstrate quantum nonlocality of three qubits is shown in Figure 4.13. The source of nonclassical radiation is a single photon incident on a beam splitter with transmittance $T = 2/3$ and reflectivity $R = 1/3$ followed by a 50:50 beam splitter, which generates a three-qubit W state. The quantum state of the source is of the form $|\psi(3)\rangle = \frac{1}{\sqrt{3}}(|100\rangle+|010\rangle+|001\rangle)$. The measuring devices are photon counting detectors preceded by beam splitters. The beam splitters have the transmission coefficient T_i close to one and strong coherent states $|\delta_i\rangle$ injected into the auxiliary ports. In this limit, the beamsplitters effectively perform coherent displacements $\hat{D}_{a_1}(\alpha_1)$, $\hat{D}_{a_2}(\alpha_2)$ and $\hat{D}_{a_3}(\alpha_3)$ on the

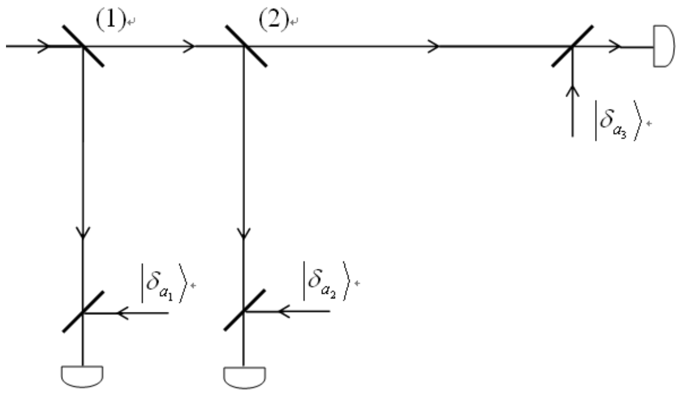

Figure 4.13: The optical setup for testing nonlocality of three qubits. The beam splitter (1) has a reflectivity $R = 1/3$ and the beam splitter (2) is a 50:50 beam splitter.

three ports (modes) of the input field with $\alpha_i = \sqrt{1 - T_i}\delta_i$. The correlation function measured is

$$\begin{aligned} Q_{a_1a_2a_3}(\alpha_1, \alpha_2, \alpha_3) &= \langle\psi(3)|\hat{Q}_{a_1}(\alpha_1) \otimes \hat{Q}_{a_2}(\alpha_2) \otimes \hat{Q}_{a_3}(\alpha_3)|\psi(3)\rangle \\ &= \frac{1}{3}e^{-2(|\alpha_1|^2+|\alpha_2|^2+|\alpha_3|^2)}(4|\alpha_1 + \alpha_2 + \alpha_3|^2 - 3). \end{aligned} \tag{4.65}$$

We next construct a Bell quantity from the 3-qubit ŻB inequality (4.1):

$$\begin{aligned} \mathcal{B}_{(4.1)} &= Q_{a_1a_2a_3}(\alpha_1^1, \alpha_2^1, \alpha_3^2) + Q_{a_1a_2a_3}(\alpha_1^1, \alpha_2^2, \alpha_3^1) \\ &+ Q_{a_1a_2a_3}(\alpha_1^2, \alpha_2^1, \alpha_3^1) - Q_{a_1a_2a_3}(\alpha_1^2, \alpha_2^2, \alpha_3^2). \end{aligned} \tag{4.66}$$

where α_j^is with $i = 1, 2$ and $j = 1, 2, 3$ are the two experimental settings of the coherent displacements for ports a_1, a_2 and a_3 respectively. For a local realistic theory, $\mathcal{B}_{(4.1)} \leq 2$. Unfortunately 3-qubit ŻB inequality does not reveal quantum nonlocality since a numerical calculation gives $\mathcal{B}_{(4.1)} \leq 2$. The possible reason is that the degree of quantum nonlocality depends not only on the given entangled state but also on the Bell operator [56]. Hence the result means that 3-qubit ŻB inequality may not reveal quantum nonlocality for displaced parity measurements on the system.

New Bell inequalities for three qubits have been proposed in the previous sections. The Bell inequalities are violated for any pure entangled state. We next show that quantum nonlocality of three qubits can be exhibited in the proposed experimental scheme using the correlation-form inequality (4.52).

Unlike the 3-qubit ŻB inequality where there are only three-site correlation functions, the recent Bell inequality (4.52) for three qubits contains three-site, two-site and one-site correlation functions. The two-site correlation functions and one-site correlation functions are given by

$$
\begin{aligned}
&Q_{a_1a_2}(\alpha_1,\alpha_2) = \langle\psi(3)|\hat{Q}_{a_1}(\alpha_1)\otimes\hat{Q}_{a_2}(\alpha_2)\otimes\mathbf{1}|\psi(3)\rangle,\\
&Q_{a_1a_3}(\alpha_1,\alpha_3) = \langle\psi(3)|\hat{Q}_{a_1}(\alpha_1)\otimes\mathbf{1}\otimes\hat{Q}_{a_3}(\alpha_3)|\psi(3)\rangle,\\
&Q_{a_2a_3}(\alpha_2,\alpha_3) = \langle\psi(3)|\mathbf{1}\otimes\hat{Q}_{a_2}(\alpha_2)\otimes\hat{Q}_{a_3}(\alpha_3)|\psi(3)\rangle,\\
&Q_{a_1}(\alpha_1) = \langle\psi(3)|\hat{Q}_{a_1}(\alpha_1)\otimes\mathbf{1}\otimes\mathbf{1}|\psi(3)\rangle,\\
&Q_{a_2}(\alpha_2) = \langle\psi(3)|\mathbf{1}\otimes\hat{Q}_{a_2}(\alpha_2)\otimes\mathbf{1}|\psi(3)\rangle,\\
&Q_{a_3}(\alpha_3) = \langle\psi(3)|\mathbf{1}\otimes\mathbf{1}\otimes\hat{Q}_{a_3}(\alpha_3)|\psi(3)\rangle.
\end{aligned}
\tag{4.67}
$$

The two-site correlation functions are measured when one of three observers does not perform any measurement on his detector. The one-site correlation functions are measured when two of three observers do not perform any measurements on their detectors. Similar calculations to the one for three-site correlation functions, we have

$$
\begin{aligned}
&Q_{a_ia_j}(\alpha_i,\alpha_j) = \frac{1}{3}e^{-2(|\alpha_i|^2+|\alpha_j|^2)}(4|\alpha_i+\alpha_j|^2-1),\\
&Q_{a_i}(\alpha_i) = \frac{1}{3}e^{-2(|\alpha_i|^2)}(4|\alpha_i|^2+1),
\end{aligned}
\tag{4.68}
$$

where $i,j=1,2,3$. Using the correlation form inequality (4.52) for three qubits, we construct a new Bell quantity for three qubits

$$\begin{aligned}
\mathcal{B}_{(4.52)} &= -Q_{a_1a_2a_3}(\alpha_1^1,\alpha_2^1,\alpha_3^1)+Q_{a_1a_2a_3}(\alpha_1^1,\alpha_2^1,\alpha_3^2)\\
&+ Q_{a_1a_2a_3}(\alpha_1^1,\alpha_2^2,\alpha_3^1)+Q_{a_1a_2a_3}(\alpha_1^2,\alpha_2^1,\alpha_3^1)\\
&- Q_{a_1a_2a_3}(\alpha_1^2,\alpha_2^2,\alpha_3^2)-Q_{a_1a_2}(\alpha_1^1,\alpha_2^2)\\
&- Q_{a_1a_2}(\alpha_1^2,\alpha_2^1)-Q_{a_1a_2}(\alpha_1^2,\alpha_2^2)\\
&- Q_{a_1a_3}(\alpha_1^1,\alpha_3^2)-Q_{a_1a_3}(\alpha_1^2,\alpha_3^1)\\
&- Q_{a_1a_3}(\alpha_1^2,\alpha_3^2)-Q_{a_2a_3}(\alpha_2^1,\alpha_3^2)\\
&- Q_{a_2a_3}(\alpha_2^2,\alpha_3^1)-Q_{a_2a_3}(\alpha_2^2,\alpha_3^2)\\
&+ Q_{a_1}(\alpha_1^1)+Q_{a_2}(\alpha_2^1)+Q_{a_3}(\alpha_3^1).
\end{aligned} \tag{4.69}$$

Local realism theories impose the upper bound value of 3 for the Bell quantity $\mathcal{B}_{(4.52)}$. By taking the coherent displacements as $\alpha_1^1=\alpha_2^1=\alpha_3^1=0.471669$, $\alpha_1^2=\alpha_2^2=\alpha_3^2=-0.0205849$, $\mathcal{B}_{(4.52)}=3.1605$ which is greater than 3. Thus one can detect quantum nonlocality of a three-qubit system in the proposed experiment.

It should be mentioned that other factors should be taken into account, such as detector inefficiencies, in practice. If one considers the inefficiency of detector, one has to modify the correlation slightly to account for the imperfections, which are characterized by the quantum efficiency of detectors η $(0\le\eta\le1)$. For ideal detectors, $\eta=1$ and the correlation is perfect. For non-ideal detectors, $\hat{a}^\dagger\hat{a}$ is changed to $\eta\hat{a}^\dagger\hat{a}$ the correlation is modified by $Q'_{a_1a_2a_3}(\alpha_1\alpha_2\alpha_3)$. If we assume that all the photon detectors have the same efficiencies η, we have

$$Q'_{a_1a_2a_3}(\alpha_1,\alpha_2,\alpha_3)=\langle\psi(3)|\hat{Q}'_{a_1}(\alpha_1)\otimes\hat{Q}'_{a_2}(\alpha_2)\otimes\hat{Q}'_{a_3}(\alpha_3)|\psi(3)\rangle, \tag{4.70}$$

and

$$\begin{aligned}
Q'_{a_1a_2}(\alpha_1,\alpha_2) &= \langle\psi(3)|\hat{Q}'_{a_1}(\alpha_1)\otimes\hat{Q}'_{a_2}(\alpha_2)\otimes\mathbf{1}|\psi(3)\rangle,\\
Q'_{a_1a_3}(\alpha_1,\alpha_3) &= \langle\psi(3)|\hat{Q}'_{a_1}(\alpha_1)\otimes\mathbf{1}\otimes\hat{Q}'_{a_3}(\alpha_3)|\psi(3)\rangle,\\
Q'_{a_2a_3}(\alpha_2,\alpha_3) &= \langle\psi(3)|\mathbf{1}\otimes\hat{Q}'_{a_2}(\alpha_2)\otimes\hat{Q}'_{a_3}(\alpha_3)|\psi(3)\rangle,
\end{aligned}$$

$$
\begin{aligned}
Q'_{a_1}(\alpha_1) &= \langle\psi(3)|\hat{Q}'_{a_1}(\alpha_1)\otimes\mathbf{1}\otimes\mathbf{1}|\psi(3)\rangle, \\
Q'_{a_2}(\alpha_2) &= \langle\psi(3)|\mathbf{1}\otimes\hat{Q}'_{a_2}(\alpha_2)\otimes\mathbf{1}|\psi(3)\rangle, \\
Q'_{a_3}(\alpha_3) &= \langle\psi(3)|\mathbf{1}\otimes\mathbf{1}\otimes\hat{Q}'_{a_3}(\alpha_3)|\psi(3)\rangle,
\end{aligned}
\tag{4.71}
$$

where $\hat{Q}'_{a_i}(\alpha_i)$ is an operator defined as

$$
\hat{Q}'_{a_i}(\alpha_i) = \hat{D}_{a_i}(\alpha_i)(1-2\eta)^{\hat{n}_{a_i}}\hat{D}^{\dagger}_{a_i}(\alpha_i). \tag{4.72}
$$

Straightforward calculations yield the modified correlation functions for the state $|\psi(3)\rangle$,

$$
\begin{aligned}
&Q'_{a_1a_2a_3}(\alpha_1,\alpha_2,\alpha_3) = \frac{1}{3}\{(-2\eta)^2|\sum_{i=1}^{3}\alpha_i|^2 + 3(1-2\eta)\}e^{-2\eta\sum_{i=1}^{3}|\alpha_i|^2} \\
&Q'_{a_ia_j}(\alpha_i,\alpha_j) = \frac{1}{3}\{(-2\eta)^2(|\alpha_i+\alpha_j|^2) + 3 - 4\eta\}e^{-2\eta(|\alpha_i|^2+|\alpha_j|^2)} \\
&Q'_{a_i}(\alpha_i) = \frac{1}{3}\{(-2\eta)^2(|\alpha_i|^2) + 3 - 2\eta\}e^{-2\eta(|\alpha_i|^2)}
\end{aligned}
\tag{4.73}
$$

where $i,j = 1,2,3$. A Bell quantity $\mathcal{B}^{\eta}_{(4.52)}$ is constructed from the Bell inequality (4.52),

$$
\begin{aligned}
\mathcal{B}^{\eta}_{(4.52)} \;=\;& -Q'_{a_1a_2a_3}(\alpha_1^1,\alpha_2^1,\alpha_3^1) + Q'_{a_1a_2a_3}(\alpha_1^1,\alpha_2^1,\alpha_3^2) \\
&+ Q'_{a_1a_2a_3}(\alpha_1^1,\alpha_2^2,\alpha_3^1) + Q'_{a_1a_2a_3}(\alpha_1^2,\alpha_2^1,\alpha_3^1) \\
&- Q'_{a_1a_2a_3}(\alpha_1^2,\alpha_2^2,\alpha_3^2) - Q'_{a_1a_2}(\alpha_1^1,\alpha_2^2) \\
&- Q'_{a_1a_2}(\alpha_1^2,\alpha_2^1) - Q'_{a_1a_2}(\alpha_1^2,\alpha_2^2) \\
&- Q'_{a_1a_3}(\alpha_1^1,\alpha_3^2) - Q'_{a_1a_3}(\alpha_1^2,\alpha_3^1) \\
&- Q'_{a_1a_3}(\alpha_1^2,\alpha_3^2) - Q'_{a_2a_3}(\alpha_2^1,\alpha_3^2) \\
&- Q'_{a_2a_3}(\alpha_2^2,\alpha_3^1) - Q'_{a_2a_3}(\alpha_2^2,\alpha_3^2) \\
&+ Q'_{a_1}(\alpha_1^1) + Q'_{a_2}(\alpha_2^1) + Q'_{a_3}(\alpha_3^1).
\end{aligned}
\tag{4.74}
$$

Since, for a local realistic description, $\mathcal{B}^{\eta}_{(4.52)} \leq 3$ and for quantum nonlocality, the Bell quantity $\mathcal{B}^{\eta}_{(4.52)}$ has a maximum value of 3.1605, which is greater than 3 when $\eta > 0.9804$. Thus, the nonlocality of a three-qubit system exhibits in the proposed experiment if $\eta > 0.9804$. Quantum nonlocality of N qubits (N is an arbitrary number) can be tested in the proposed experiment in a similar way.

Chapter 5

Conclusion and Outlook

The main objective of this thesis is to develop Bell inequalities for composite quantum systems and explore quantum entanglement of different systems by using such inequalities.

In the first part of the study, quantum entanglement of multipartite systems in two dimensional Hilbert space (N qubits) is investigated. The most common model of N qubits is quantum XX model (or spin chains). Two types of XX models are considered here. One is the model with site-dependent coupling strength; the other one is the model with constant coupling strength. By violating the Żukowski-Brukner inequalities, quantum XX models are shown to be quantum entangled. The effects of temperature and external magnetic field on the entanglement of quantum XX model are investigated explicitly. It is shown that the quantum entanglement could be controlled by adjusting temperature and magnetic field strength. The results provide exact conditions for experimental realization of quantum communication and computation, in which entanglement is needed, in spin chains.

Another N-qubit system investigated is the multipartite system with continuous variables. Based on parity measurement, correlation functions of multipartite continuous-variable states are derived in the Wigner presentation. By using these correlation functions, it is shown that multipartite continuous-variable states violate the Żukowski-Brukner inequalities. The degree of violation grows with increasing number of particles N of the system. This variation is consistent with those reported for discrete-variable quantum systems [38]. The variation of the degree of violation with N indicates that classical properties do not automatically emerge for large quantum systems with either discrete variables or continuous variables.

In the second part of this thesis, Bell inequalities for 2 quNits and their applications are discussed. Firstly, as it was shown that the Collins-Gisin-Linden-Massar-Popescu inequalities are maximally violated by non-maximally entangled states in Ref. [53], we extend the calculations to high dimensions of $N = 8000$. The degree of violation grows slowly with increasing dimensions. The results appear to confirm that the violation is asymptotically constant when N tends to infinity. An approximate value of the constant is found numerically to be 3.9132. However, it is possible that other approximation methods could lead to different values due to the virtue of each method. Therefore, it is necessary to present a more accurate limit for the violation. This could be achieved by adding more points in the calculations. Since the higher the dimension, the more difficult to find a maximal violation is, our results are for $N \leq 8000$. From the analysis of the eigenstates which maximally violate the CGLMP inequalities, we construct a set of entangled states $|\Phi\rangle_{\text{app}}$ whose corresponding Bell quantities are closed to the actual ones. As we know that non-local resource which is highly resistant to noise is needed in quantum information processing. It may be significant and interesting to apply the symmetric entangled states $|\Phi\rangle_{\text{app}}$ to quantum protocols of quantum information.

Secondly, new Bell inequalities involving correlation functions for 2 quNits are constructed. The Bell inequalities are derived based on multi-component correlation functions constructed from N-outcome measurements. Then 2-quNit systems with continuous variables are shown to be quantum entangled by violating the Bell inequalities. When the dimension increases, the violation of the inequalities grows slowly. The variation of the violation is similar to that for non-maximally entangled states given in Section 3.3. Numerical results show that the violation strength of continuous-variable states is weaker than that of non-maximally entangled states. The limit of the violation for the continuous-variable states is found numerically to be 3.129, which exceeds the Cirel'son bound $2\sqrt{2}$. The excess is due to the fact that we considered N (> 2)-outcome measurement, while Cirel'son considered 2-outcome measurement.

In the third part of this thesis, Bell inequalities for three qubits in terms of both probabilities and correlation functions are constructed. Numerical results show that these inequalities are violated by all pure entangled states of three qubits. An explanation for the violation may be that these inequalities contain both 3-particle correlation functions and 2-particle correlation functions. The results indicate that Gisin's theorem can be generalized to 3-qubit systems. However, these inequalities are not optimal Bell inequalities for 3 qubits since they are not maximally violated by the GHZ state.

It is found that the Bell inequality (4.13) for three qubits can be reduced from the Bell inequality (4.2) for three qutrits from an observation on Bell inequalities for 3 qubits and 3 qutrits. Contrary to the expectation, the Bell inequality (4.2) for three qutrits does not give the three-qubit Żukowski-Brukner inequality when it is restricted to two dimensional systems. So it is possible to assume that a new Bell inequality for three qubits can be constructed if Bell inequalities for 3 particles in higher level systems than qutrits could be found.

A new probability Bell inequality (4.46) for three four-level quantum systems is derived in Section 4.3.1. The inequality gives a necessary condition for the existence of a local realistic description of quantum correlations. The violation of the inequality for the maximally entangled state is shown. The strength of the violation is stronger than that for three qutrits or two four-level systems. As we know, the higher the degree of violation of Bell inequality, the more noise-tolerant the system is and noise is unavoidable in the practical implementations of quantum computing. Therefore, the result makes a new quantum protocol, which is more noise-tolerant than the ones proposed in [6], possible for realization in a system more composite than 3 qutrits. In addition, a new Bell inequality for three qubits is derived from the inequality for three four-level systems. Inequality (4.52) is violated by all pure entangled states. It is also more resistant to noise than the inequalities (4.13) and (4.27). But the inequality (4.52) is still not optimal because the degree of violation for the GHZ state is not maximal. This may be due to the fact that the inequality (4.52) is reduced from inequality (4.46) and the inequality (4.46) is only a necessary condition for a local realistic description. An experimental setup to test violation of local realism by using three-qubit Bell inequalities is proposed. The optical setup could also be generalized to a system more composite than three qubits.

By using the method of violating Bell inequalities given in this thesis, quantum entanglement of other types of XX models can be possibly determined. This may be achieved as long as we can find the solutions of the models.

Based on the results in this thesis, one possible direction for the future work is to further explore the formulations of new Bell inequalities for M d-level quantum systems (M and d are arbitrary integers). It is worth noting that generalization of Gisin's theorem to N qubits (with N is an odd number) is not explored comprehensively in this thesis. The problem is solved partly in the case of three qubits. For three qubits, although the Bell inequalities (4.13), (4.27) and (4.52) are violated by all pure entangled states, they are not optimal since they are not maximally violated by the GHZ state. But it is possible to construct new Bell inequalities for three qubits with improved threshold visibility by using the procedure given in this thesis if new Bell inequalities for three d ($d > 4$)-level systems are developed. Similarly, new Bell inequalities for N $(= 5, 7, 9, ...)$ qubits could be constructed if new Bell inequalities for N $(= 5, 7, 9, ...)$ d ($d > 2$)-level systems are developed by further investigation. It is anticipated that these inequalities will be violated by all pure entangled states. The application of the new Bell inequalities in quantum information processing, for example quantum cryptography, also deserve future investigation.

It should be noted that all the results in this thesis have been given under the assumption that two local settings are provided for each observer. The formulation of new Bell inequalities for composite quantum systems based on more local settings per site than two is another possible direction for the future study. It is anticipated that new Bell inequalities constructed under the new assumption may give stronger restrictions for the local realistic description than those inequalities derived under the assumption of two local settings per site.

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Appendix A

Publication List

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2. C. F. Wu, J. L. Chen, D. M. Tong, L. C. Kwek and C. H. Oh, "Quantum nonlocality of Heisenberg XX model with site-dependent coupling strength", J. Phys. A: Gen. Math. **37**, 11475 (2004).

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