A STUDY ON IMPROVING THE PERFORMANCE OF CONTROL CHARTS UNDER NON-NORMAL DISTRIBUTIONS

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SUMMARY

The control chart is a graphical tool that aids in the discovery of assignable causes of variability in these quality measurements. Shewhart type control charts are the most commonly used method to test whether or not a process is in-control. The basic idea is that given a quality measurement, which independently identically follows normal distribution, *k*-sigma limits would be use to detect an out-of-control signal. Usually *k* is set as 3 to achieve very desirable *ARL* properties. However, the assumption of having *iid* normal population is invalid in many cases, especially encountered frequently in real-application. Thus, the traditional 3-sigma limits for the Shewhart charts may not be appropriate in certain situations.

Exact probability limits are good alternatives to traditional 3-sigma control limits. The deduction of exact probability control limits of *R*- and *S*- charts has shown better properties in sense of signals at both sides of the limits in this thesis. This results in revised values for control chart construction constants D_3 and D_4. The new values of the constants provide a positive lower control limit for the process when the sampling subgroup size is lee than 6. Thus, the decrease of the process deviation can be detected at earlier stage.

The theoretical achievements in normalizing transformations provide another way to deal with the non-normality problem in constructing control charts with broader area of applications. In this paper, after being transformed to a normal distribution, the quality characteristic of traditional control charts can be simply monitored by a traditional Shewhart type individual chart. Although the transformed chart has its intrinsic defects, such as the extreme difficulty in interpretation and uncertainty in approximation, a valuable trade-off between the accuracy of normalizing and the simplicity of application is obtained. We illustrate that normalizing transformation could improve the performance of control limits in the sense that it achieves more desirable *ARL* performance, such as faster signals to process deterioration and symmetric responding. Moreover, sometimes, the control charts based on normalized data performs better than the exact probability charts as well. In this thesis we recommend some good forms of transformations to use and propose some simplifies forms for particular cases.

This thesis consists of 6 chapters. Chapter 1 is the brief introduction of this study. Chapter 2 is literature review of the related topics, non-normality problems in traditional control charting scheme and normalizing transformations. Chapter 3 focuses on the application of modifying the traditional control limits in *R*- and *S*-charts, which is probability limits related. Chapter 4 discusses more general method by applying various normalizing transformations on traditional Shewhart type control charts. Chapter 5 discusses the normalizing transformations, in particular, on multivariate control charts. Simplified forms have been raised. At the end of the thesis, the conclusion is given in Chapter 6.

NOMENCLATURE

AD	Anderson-Darling test
ARL	average run length
CL	center line
CuSum	cumulative sum
EWMA	exponentially weighted moving average
iid	independently identically distributed
LCL	lower control limit
LPL	lower probability limit
MBB	moving blocks bootstrap
MEWMA	multivariate exponentially weighted moving average control chart
pdf	probability density function
RL	run length
SD	standard deviation
SPC	statistical process control
UCL	upper control limit
UPL	upper probability limit

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Chapter 1

Introduction

1.1. Research Background and Motivation

Quality control schemes are widely used to improve the quality of a manufacturing process. It is often the case that some aspects of the quality of the output of a process can be described in terms of one or more parameters of the distribution of a quality measurement. The control chart is a graphical tool that aids in the discovery of assignable causes of variability in these quality measurements. It is used to monitor a process for the purpose of detecting special causes of process variation that may result in lower-quality process output.

Shewhart type control charts are the most commonly used method to test whether or not a process is in-control. The basic idea is that given a quality measurement, a Shewhart chart with 3-sigma control limits can be constructed as

$$\begin{aligned} UCL &= \mu + k\sigma \\ CL &= \mu \\ LCL &= \mu - k\sigma \end{aligned} \tag{1.1}$$

where μ and σ denote the mean and the standard deviation of the quality measurement, respectively, k=3, usually. Define E_i to be the event that i-th sample measurement X_i is either above UCL or below LCL. Then the events $\{E_i\}$ are independent and for all $i \geq 1$,

$$P(E_i) = P(X_i > UCL \cup X_i < LCL) = 1 - \Phi(3) + \Phi(-3) = 0.0027 \qquad (1.2)$$

where Φ is the distribution function of a $N(0,1)$ random variable. If we define U to be the number of samples until the first E_i occurs, then U is known as the run length of the chart and has a geometric distribution with parameter $p = P(E_i) = 0.0027$. It follows that the average run length (ARL) defined as the mean (E) of U and the standard deviation (SD) of U are given by

$$E(U) = \frac{1}{p} = 370.4$$

and

$$SD(U) = \frac{\sqrt{1-p}}{p} = 369.9 \qquad (1.3)$$

All the above calculations are often based on two assumptions: that sample observations are statistically independent, and that the monitoring statistic follows a normal distribution. These are the two assumptions for constructing both Shewhart type variable charts and attribute charts. However these assumptions are invalid in many cases and subsequently become debatable. Especially, non-normality is encountered frequently in real-application. Winterbottom (1993) mentioned this problem as very often the exact distribution is positively skewed, and this means that control limits are set significantly lower than they should be in order to come reasonably close to giving false alarm probabilities that correspond to those on $\overline{X}$ charts. Moreover, even if the quality characteristic monitored follows normal distribution some variables we plot on the control charts, i.e. the range R and the sample deviation S, differ from normal distribution, so that the traditional 3-sigma limits of R-chart and S-chart do not perform as well as desired.

The validity of normal distribution has been questioned in some control charting applications. Some authors had studied the problem. Since the statistic used for monitoring with attribute data usually have underlying distributions which are skewed to the right, the traditional 3-sigma limits for the Shewhart charts may not be appropriate as pointed out by Woodall (1997) and Xie et al. (2002). For geometric distribution, Xie et al. (1997) suggested that the traditional 3-sigma limits should not be used in this case because the geometric distribution is always skewed and normal approximation is not valid. Xie et al. (1992) calculated the exact probability limits that have been adopted in most of the publications. Probability-based methods for determining control limits have been discussed by Ryan and Schwertman (1997). They provided tables producing optimal control limits for u and c charts. Shore (2000) developed a methodology to construct control charts for attributes data. One of the features is to use far-tail quantile values to determine probability limits, control limits, or other performance measures. These quantile values are derived from a fitted distribution that preserves all first three moments of the plotted statistic.

However, few literatures have addressed the issues on solving non-normality problem in traditional Shewhart control charting scheme. Therefore, it would be very useful and contributive if any approach on that could be raised. Moreover, the approach should be simple and friendly-to-use without losing much of the accuracy and precision.

1.2. Objective of the Thesis

We have raised two approaches in this research to solve the non-normality problem we introduced in the section 1.1. The first approach is based on the exact distribution for the sample ranges and sample standard deviations. Control limits, especially the lower control limits are derived based on a fixed false alarm probability. The new control limits will always be positive and hence enable the chart users to detect process shift in terms of reduction in the process variability. The second approach is to make use of transformations. This solution succeeds in making a balanced control limits so that the *ARL* is large when the process behaves normally and smaller when the process deviates (Yang and Xie (2000)). However, this issue has not gained much attention in control charting scheme though many statisticians examined the transformation forms for various kinds of distributions to normalize them. To summarize those existing transforming formulas could contribute to the work dealing with non-normal data in process monitoring. Extensive simulation has been done to prove that using normalizing transformation results in satisfactory control chart performance in the sense of some desirable properties with *ARL* achieved under some circumstances. Moreover, simplified transformations are also proposed for more convenient use of certain distributions. A valuable trade-off between the accuracy of normalizing transformation and simplicity of applications has been obtained, which would benefit the industrial applications.

1.3. Organization of the Thesis

The remainder of this thesis is organized as follows:

Chapter 2 reviews the relative literature mainly focused on non-normality problems in traditional Shewhart type control charts and multivariate charts, and summarizes the papers looking into normalizing transformations.

Chapter 3 studies the application of modifying the traditional control limits in *R*- and *S*-charts. The procedures are presented and the tables for modification are provided as well.

Chapter 4 discusses more general method by applying various normalizing transformations on traditional Shewhart type control charts, including transformation selection and performance improvement comparison.

Chapter 5 contains the mathematical background and simplified approach of using transformations in multivariate control charts; the procedures to apply the chart are presented followed by an example.

Chapter 6 concludes the study. Further study and limitation are also discussed.

Chapter 2

Literature Review

2.1. Introduction

Non-normality problem has a vital influence on the performance of control charting schemes. Therefore, volumes of research have been carried out on this issue. The research ideas on applying control charts for non-normal populations can be divided into three categories. The first category has concentrated on the robustness of various control charts' performance to departures from the normality assumption so that the traditional charts can be employed within a reasonable scope. Borror et al (1999) even studied the issue of robustness on exponential weighted moving average (*EWMA*) control charts. However, this limits our intention in implementing the control charting scheme in more cases. Stoumbos and Sullivan (2002), The second category of research effort has attempted to develop control charts that either may be generally applied to non-normal populations after certain adjustment, like Xie et al (2000) to solve the so-called *ARL*-biased phenomenon or control charts that explicitly specify an underlying non-normal population; that is exact probability limits. The third category of endeavors to address the non-normality of process distribution has focused on transforming to normality the given data, so that traditional Shewhart control charting schemes could be employed with desirable average run length.

In the following section, we will review the literatures on non-normality problem in traditional Shewhart type charts, including the emergence of the problem, its effect, the solutions proposed as well as the related research trend. In section 2.2.1, section 2.2.2 and section 2.2.3, we will review the research work, in particular on attribute charts, variable charts and individual and multivariate charts. We will generalize some of the achievements that have been made by mathematicians and statisticians in the area of normalizing transformations in section 2.3. Section 2.3.1 and section 2.3.2 will discuss the transformation forms for generic distributions, some specific distributions and statistic families, respectively.

2.2. Review on Non-normality Issue on Control Charts

2.2.1. Attribute Charts

Control charting methods based on attribute data were first proposed by Shewhart in 1926. The *p* and *np* charts are widely used, primarily to monitor the fraction of non-conforming products. The *c* chart and *u* chart, on the other hand, can be used to monitor the number of non-conformities. The *p* and *np* chart control limits and performance measures are typically based on the binomial distribution whereas those of the *c* and *u* charts are based on the Poisson distribution. Woodall (1997) pointed out that, since the statistics used for monitoring with attribute data usually have underlying distributions which are skewed to the right, the traditional *k*-sigma limits for the Shewhart charts may be inappropriate.

Among others Ryan and Schwertman (1997) proposed some probability-based methods for determining control limits in order to improve the control charts' performance, They pointed out that control chart properties are determined by the reciprocals of the tail areas, but most approximations, including normal approximation, perform the poorest in the tails of the distributions. Normal approximations can also be poor when the binomial and Poisson parameters are small, as will occur frequently in applications. They, therefore, provided tables that can be used to produce optimal control limits for *u* and *c* charts. They used regression to extrapolate the optimal limits between the tabular values. The regression equations are then suitably adapted for use with *p* and *np* charts, for which a complete set of tables of optimal control limits would not be practical to construct.

Ryan (1998) indicates the contradiction in employing approximations. He showed that most approximations, including the normal approximations to the binomial and Poisson distributions, generally perform the poorest in the tails of a distribution. But control chart properties are determined by the reciprocals of the actual tail areas. These problems have resulted in new methods being proposed for determining the control limits for attribute charts.

Actually, Winterbottom (1997) talked about this problem as very often the exact distribution is positively skew, and this means that control limits are set significantly lower than they should be in order to come reasonably close to giving false alarm probabilities that correspond to those on $\overline{X}$ charts. He also mentioned that for attribute charts one can always calculate exact control limits in the sense that the false alarm probabilities are as close as possible, but do not exceed, designate levels. The inequality is due to the discrete nature of attribute data. Winterbottom, thus, improved the probabilistic accuracy of control limits for all of the previously mentioned attribute charts by determining the adjustment which makes use of the Cornish-Fisher-expansions (Cornish and Fisher (1937)). These adjustments given by Cornish and Fisher, or corrections, depend in simple ways on sample sizes, process parameter values and the standard normal value used as a multiplier of sigma in the unmodified formula.

H. Shore (2000) developed a new methodology to construct control charts for attributes data. Let Y be a measured attribute with known mean μ, standard deviation σ and skewness sk. The general expressions for the probability limits of a general control chart for attribute is developed as

$$UPL = \mu + z_{1-q_2}(A+C)\sigma - 0.7978C\sigma - \frac{1}{2}$$

$$CL = \mu$$

$$LPL = \mu + z_{1-q_2}(A-C)\sigma - 0.7978C\sigma + \frac{1}{2}, \qquad (2.1)$$

where A and C are the solution to

$$A^2 = 1 - 0.3635C^2$$

and

$$sk = 2.3940C - 0.6523C^3$$

For highly skewed distribution, modifications were made by Shore (2000). He also developed simplified limits based on some approximation assumptions and extended them into probability limits for the binomial, the Poisson, the geometric and the negative binomial distributions.

For p-chart, more specifically, Simon (1995) suggested using probability limits such that it would be equally likely for a false alarm to happen on either side of the control chart. Ryan (1997) studied the arcsine transformation, as given by Freeman and Tukey (1996) to construct a chart for monitoring p. Chen (1997) developed a variant of Ryan's chart. Another approach to the problem is to use a Q chart as developed by Quesenberry (1999). The Q chart provides a better approximation to the nominal upper tail area than arsine approach. Acosta-Mejia (1999) proposed an alternative to replace the lower control limit by a simple runs rule.

For c-chart, the modified 3-sigma control limits has been proposed by Winterbottom (1997) using Cornish and Fisher expansions for c chart. Q charts, proposed by Quesenberry (1992) can also be employed as an alternative to replace a c chart or u chart, and also as an alternative to standardized versions of these charts. Ryan (1995) indicated that it seems preferable to seek closeness to the reciprocals of the nominal tail areas rather than closeness to those tail areas. For u chart, Ryan also used a similar method.

As pointed out by Woodall (1997), more recently, it has been recommended that it is useful to base control charts on the number of conforming items found between nonconforming items. Here the underlying distribution is typically assumed to be the geometric distribution. Control charts based on geometric distribution have shown to be very useful in the monitoring of high yield manufacturing process and other applications. The traditional control limits have been given in Kaminsky (1992). It is pointed out in Xie and Goh (1994) that the traditional 3-sigma limits should not be used in this case because the geometric distribution is always skewed and normal approximation is not valid. Xie and Goh (1997) suggested using the exact probability limits. Further studied had been carried out by Xie, et al (2002). A new procedure for determination of control limits is developed, which provides maximum *ARL* when the process is in control. Moreover, a simple adjustment factor was suggested so that the probability limits can be used after the adjustment and compensate for the shortcoming that the control limits given above do not have a direct probabilistic interpretation.

In many situations which are characterized by a burn-in process, it seems appropriate to use the inverse Gaussian process to model the failure rate function. R. L. Edgeman proposed a Shewhart control charting scheme for the inverse Gaussian distribution which is a member of the exponential family. D. H. Olwell improved R. L. Edgeman's scheme and proposed a second scheme based on symmetric probability limits. The Cusum scheme for any distribution belonging to the exponential family was proposed by Bruyn (1968) and Hawkins (1992). Based on these, Hawkins and Olwell (1997) developed optimal decision interval Cusum schemes for both the location and the shape of the inverse Gaussian distribution. Hawkins and Olwell also sketched computational routines to complete the design process of the Cusum by determining their in-control and out-of-control *ARL*s.

2.2.2. Variable Charts

Recall that traditional Shewhart type variable control charting methods are often based on two assumptions: that sample observations are statistically independent, and that the monitoring statistic follows a normal distribution. However, these assumptions are invalid in many cases. When the distribution of the monitoring statistic used in process control is non-normal, traditional Shewhart variable charts may not be applicable. If the performance of the *SPC* scheme is adversely affected by a violation of the assumptions, modification of the methods used is essential to guarantee the required performance.

The most common approach nowadays to deal with non-normal data in quality-related applications involves the use of the Box-Cox transformation. The basis for this transformation, as articulated by Box and Cox (1964), is the empirical observation that a power transformation is equivalent to finding the right scale for given data.

Shore (2000) generalized the log term by presenting it as a Box-Cox transformation and obtained:

$$x = M \cdot EXP\left\{b\left[(1+az)^{c/a} - 1\right] + dz\right\},\ z > -\frac{1}{\alpha} \tag{2.2}$$

He examined four modified versions of this transformation, which result in inverse normalizing transformation with a reduced number of parameters. The four versions are as

$$x \approx M \cdot EXP\left\{B\left[EXP\{Cz\} - 1\right] + Dz\right\} \tag{2.3}$$

$$x \approx M \cdot EXP\{B[EXP\{Cz\} - 1]\} \tag{2.4}$$

$$x \approx M \cdot EXP\{B[(1 + Az)^C - 1]/C\},\ x > -\frac{1}{A} \tag{2.5}$$

$$x \approx EXP\{LogM + ABz/(1 + Az)\} \tag{2.6}$$

Shore (2000a), then, introduced a unified approach that applies to both normal and non-normal populations. Thus the traditional distinction between statistical process control (*SPC*) for normal and for non-normal processes is eliminated. The procedures for constructing the control charts are suggested as

- Draw K independent samples of n observations each. Calculate from the j-th (j=1,2,…,K) sample the median, $\hat{M}_j$, the mean, $\hat{\mu}_j$ and the mean of the log of the original observations, $\hat{\mu}_j(LT)$.
- Given the K values of $\{\hat{M}_j\}$, estimate the median, the mean and the mean of the log of the distribution of the sample-median. Repeat these same calculations for the distributions of the sample-mean and the sample mean-of-the-log.
- Based on procedures for fitting the distribution and the given sample estimates, find the approximations for the distributions of the three monitoring statistics; namely, the sample median, the sample mean and the sample mean of the log. From the fitted transformations, calculate the control limits.

Liu and Tang (1996) also developed some valid control charts for independent data that are not necessarily nearly normal. They derived the proposed charts from the standard bootstrap methods of which the constructions are completely nonparametric and no distributional assumptions are required. Let $\{X_1,...,X_N\}$ be an iid sample following the distribution f with mean μ and variance σ^2. The standard bootstrap procedure is to draw with replacement a random sample of size N from $\{X_1,...,X_N\}$. Denote the bootstrap sample by $\{X_1^*,...,X_N^*\}$ and denote their mean and standard deviation by $\overline{\overline{X}}_N^*$ and S_N^*. To construct an $\overline{X}$ chart for iid observations – to repeat the bootstrap procedure say K times and form a histogram of the resulting K terms of $\sqrt{n}\left(\overline{X}_n^* - \overline{\overline{X}}_N\right)$, and then locate the $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles, and denote them by $\tau_{\frac{\alpha}{2}}$ and $\tau_{1-\frac{\alpha}{2}}$. Thus they obtained the control limits for the $\overline{X}$ chart as:

$$UCL = \overline{\overline{X}}_N + \tau_{\frac{\alpha}{2}} / \sqrt{n}$$

$$LCL = \overline{\overline{X}}_N - \tau_{1-\frac{\alpha}{2}} / \sqrt{n} \tag{2.7}$$

Moreover, Liu and Tang (1996) studied the control charts for dependent data making use of the moving blocks bootstrap (*MBB*) method which was introduced by Kunsch (1989). They obtained the control limits for the $\overline{X}$ chart which will still give correct results when the data are independent as shown in simulations.

Bai and Choi (1995) proposed a heuristic method for controlling the mean of the skewed distribution based on a Weighted Variance method. The control limits of their $\overline{X}$ chart are:

$$UCL = \overline{\overline{X}} - \frac{3\overline{R}}{d_2' \sqrt{n}} \sqrt{2\hat{P}_X} = \overline{\overline{X}} + W_U \overline{R}$$

$$LCL = \overline{\overline{X}} - \frac{3\overline{R}}{d_2' \sqrt{n}} \sqrt{2(1 - \hat{P}_X)} = \overline{\overline{X}} - W_L \overline{R} \tag{2.8}$$

where $\overline{R}$ is the mean of sample ranges, and

$$\hat{P}_X = \frac{\sum_{i=1}^{k} \sum_{j=1}^{n} \delta\left(\overline{\overline{X}} - X_{ij}\right)}{n \times k}$$

$$\text{with } \delta(x) = \begin{cases} 0, x < 0 \\ 1, x \geq 0 \end{cases},$$

where n is the sample size, k is the number of pre-run samples. $\hat{P}_X$ is the proportion that X will be less than or equal to the estimated process mean $\overline{\overline{X}}$ and it can be calculated in the process pre-run stage.

Dou and Sa (2002) designed a new approach to construct the one-sided $\overline{X}$ chart for positively skewed distributions. This method is based on the Edgeworth expansion to adjust the *t*-statistic for the non-normality of the process. It can preserve an appropriate in-control *ARL* and also shows reasonably good power. When one has very little knowledge about the process except the positively skewed shape of the distribution, the proposed $\overline{X}$ control charting method is recommended.

2.2.3. Individual and Multivariate Charts

There are many process monitoring problems where application of the rational subgrouping principal leads to a sample size of n=1. The traditional method of dealing with the case is to use the Shewhart individuals control chart to monitor the process mean. The individual control chart, although, as indicated by Borror et al (1999), is easily implemented and can assist in identifying shifts and drifts in the process over time, one of its two widely cited disadvantages is that the performance of the chart can be adversely affected if the observations are not normally distributed. Thus, the individuals chart is not robust at all to the normality assumption if false alarms are a concern. To enhance the traditional chart, the main purpose of which is to have a quicker signal, Kittlitz (1999) made the long-tailed, positively skewed exponential distribution into an almost symmetric distribution by taking the fourth root of the data. The transformed data thus can be plotted conveniently on an individual charts, an *EWMA* chart, or a Cusum chart for statistical process control. The rationale for the use fourth-root transformation of the exponential distribution is that it produces essentially a bell-shaped distribution and can be obtained by depressing the square-root key twice on a pocket. The usual interpretations can then be easily made for prompt attention if a deterioration occurs or captured quickly for an improvement. Borror et al (1999) showed that the *ARL* performance of the Shewhart individuals control chart when the process is in control is very sensitive to the assumption of normality. They, therefore, suggested the *EWMA* control chart as an alternative to the individuals chart for non-normal data. They showed that, in the non-normal case, a properly designed *EWMA* control chart will have an in-control *ARL* that is reasonably close to the value of 370.4 for the individuals chart for normally distributed data.

With the rapid growth of data-acquisition technology and the use of online computers for process monitoring, more and more authors cast their lights on multivariate process control so that many advances in this have been proposed. Alt (1984) reviewed the topics on the use of Hotelling T^2-control chart to monitor mean and charts for the process variability for Phase I and Phase II. Detailed explanations were given to distinguish between the uses of charts for retrospectively testing whether the process was in control when the first subgroups were being drawn versus testing whether the process was in control when the future subgroups are drawn. Jackson (1985) discussed the Hotelling T^2-control chart, the use of principal components for control charts and multivariate analogs of Cusum charts, Andrew plots and multivariate acceptance sampling. Lowry (1994) gave a review of the literatures on control charts for multivariate quality control with a concentration on developments occurring during the mid-1980s. Basic issues concerned with T^2-control chart, have been discussed besides the topics on multivariate Cusum procedures and multivariate exponentially weighted moving average control chart. In the mean time, many authors began studying the sensitivity of the T^2-control chart with regard to the orthogonal decomposition of the statistic. This particular T^2- decomposition is shown to encompass most of the research findings on the interpretation of T^2 signals by many literatures, such as Wade (1993), Mason (1995), Manson (1997) and Manson (1999). They showed that by improving model specification at the time that the historical data set is constructed, it may be possible to increase the sensitivity of the T^2-statistic to signal detection. The resulting regression residuals also can be used to improve the sensitivity of the T^2-statistic to small but consistent process shifts. Some other authors examined the effects of using estimated parameters to construct the control limits of multivariate control charts, such as Quesenberry (1993) and Nedumaran (1999). They considered the issue of the minimum number of subgroups necessary for the control chart constructed using estimated parameters to perform similar to the control chart constructed using true parameters during the on-line monitoring stage. Implementation procedures were suggested so that on-line monitoring with T^2-control charts can begin at the crucial start-up stages of the process.

2.3. Literature Review on Normalizing Transformations

2.3.1. Transformations on Generic Distributions

Generally, it is not always the case that we know the exact underlying distribution. Moreover, most of the time, we do not have any method to determine which distribution it is. Thus, normalizing transformations for general distributions are in need.

Box-Cox Transformation

The most common approach nowadays to deal with non-normal data in quality-related applications involves the use of the Box-Cox transformation. The basis for this transformation, as articulated by Box and Cox (1964), is the empirical observation that a power transformation is equivalent to finding the right scale for given data. It is defined by

$$X^{(\lambda)} = \begin{cases} \dfrac{X^{\lambda}-1}{\lambda}, \lambda \neq 0 \\ \log X, \lambda = 0 \end{cases},$$

where λ is determined by maximizing the likelihood function $L_{\max}(\lambda)$ defined by:

$$L_{\max}(\lambda) = -\frac{1}{2} n \log \hat{\sigma}^2(\lambda) + \log \prod_{i=1}^{n} \left| \frac{dX_i^{(\lambda)}}{dX_i} \right| \tag{2.9}$$

This transformation has the advantage that almost all of the popular statistical and mathematical application softwares have the built-in feature to realize the function.

Therefore, it is easy and convenient to use. Moreover, this transforming form is proved to have good approximation, which, when the accuracy is not tightly required, provides moderate good results. However, the well-known power normal family has a serious defect, i.e. the correlation structure of the maximum-likelihood estimates of the parameters is not preserved under a scale transformation of the response variables (Isogai, 1999).

Johnson Curve Fitting

Johnson (1949) provided an alternative to the Pearson system of curves for modeling non-normal distributions. This approach was to start with a small set of curves capable of approximating the shape of a wide spectrum of probability distributions and then to find simple transformations that would convert these curves into the standard normal distribution. The three functional forms used in the Johnson system are

$$S_U:\ k_1(x,\lambda,\varepsilon)=\sinh^{-1}\left(\frac{x-\varepsilon}{\lambda}\right)$$

$$S_B:\ k_2(x,\lambda,\varepsilon)=\ln\left(\frac{x-\varepsilon}{\lambda+\varepsilon-x}\right)$$

$$S_L:\ k_3(x,\lambda,\varepsilon)=\ln\left(\frac{x-\varepsilon}{\lambda}\right) \tag{2.10}$$

each of which can be transformed into a standard normal distribution by the proper choice of the parameters η,γ,λ and ε in the formula

$$z=\gamma+\eta k_i(x,\lambda,\varepsilon)\ \text{for } i=1,2 \text{ or } 3 \tag{2.11}$$

The procedures in fitting a Johnson curve are described in Johnson (1949). After determining a positive value of *z* (a good compromise is the choice *z*=0.524), we can find the cumulative probabilities P_{-3z} , P_{-z} , P_z and P_{3z} . Slifker and Shapiro (1980) suggested using those percentiles to calculate $m = x_{3z} - x_z$, $n = x_{-z} - x_{-3z}$, $p = x_z - x_{-z}$. Thus, we may choose the appropriate Johnson curve according to the value of $\frac{mn}{p^2}$. In particular, $\frac{mn}{p^2}$ >1 is for S_U curves, $\frac{mn}{p^2}$ <1 is for S_B curves and $\frac{mn}{p^2}$ =1 is for S_L curves. Compared to Box-Cox transformation, Johnson-curve fitting has the drawback that it is not widely included in the application softwares. People have to design the codes themselves or download from some websites, such as Matlab^{TM} central file exchange, etc.

2.3.2. Transformations for Specific Distributions and Statistic Families

When the underlying distribution is known or can be specified through some statistical methods, an appropriate transforming formula may be used to seek an accurate approximation to normal distribution. Heading for this aim, many mathematicians and statisticians examined the transformation forms for various kinds of distributions and statistic families.

t-distribution

Prescott (1974) examined five normalizing transformations of a *t*-distribution with *v* degrees of freedom. The five transformations included in his comparative study are listed below.

$$\text{Quenouille: } z_1 = \pm\sqrt{v-1}\sinh^{-1}\left(\sqrt{\frac{t^2}{v}}\right) \tag{2.12}$$

$$\text{Anscombe: } z_2 = \pm\sqrt{\frac{2v-1}{3}}\sinh^{-1}\left(\sqrt{\frac{3t^2}{2v}}\right) \tag{2.13}$$

$$\text{Chu: } z_3 = \pm\sqrt{v\log\left(1+\frac{t^2}{v}\right)} \tag{2.14}$$

$$\text{Wallace: } z_4 = \pm\frac{8v+1}{8v+3}\sqrt{v\log\left(1+\frac{t^2}{v}\right)} \tag{2.15}$$

$$\text{Scott and Smith: } z_5 = t\left(1-\frac{z_\alpha^2+1}{4v}\right) \tag{2.16}$$

Comparisons showed that for small v, Wallace's transformation is more accurate than the others. For large v, Anscombe's form of the transformation may be expected to be very accurate, and Wallace's form is accurate as well.

Bailey (1980) derived a transformation which is uniformly more accurate than any previously given. He generalized from the forms proposed by Wallace and Micky and gave a general class of transformations as

$$z = \pm\frac{v+b}{v+c}\sqrt{(v-a)\log\left[1+\frac{t^2}{v+h}\right]} \tag{2.17}$$

where *a, b, c* and *h* are constants. He suggested using a simplest choice of the constants which gave the transformation

$$z = \pm\frac{8v+1}{8v+9}\sqrt{\left(v+\frac{19}{12}\right)\log\left[1+\frac{t^2}{v+\frac{1}{12}}\right]} \tag{2.18}$$

In case only a few critical values of the normal distribution to be stored; there is a particular need for a transformation that is accurate locally at a prescribed deviate of the standard normal distribution. Therefore, Bailey (1980) also derived such a transformation. For any chosen value $z = z_c$, at which the transformation is very accurate locally

$$z = \pm \frac{4v^2 + \frac{5(2z_c^2 + 3)}{24}}{4v^2 + v + \frac{4z_c^2 + 9}{12}} \sqrt{v \log\left[1 + \frac{t^2}{v}\right]} \tag{2.19}$$

It is useful for transforming an observed value of *t* into a value of *z* for comparison with a critical value z_c when testing a hypothesis. Balancing the convenience of use and the performance of transforming, the recommended the formula to use for *t*-distribution is expression (2.18) above.

F-distribution

Isogai (1999) introduced two types of formula for power transformation of the *F* variable to transform the *F* distribution to a normal distribution. One formula is an extension of the Wilson-Hilferty transformation for the χ^2 variable, which is

$$T_1(F) = \frac{sign(h)\left(F^h - E\left[F^h\right]\right)}{\left(Var\left[F^h\right]\right)^{\frac{1}{2}}} \tag{2.20}$$

where *F* is distributed as $F_{m,n}$, $sign(\cdot)$ is a function that gives the sign of its argument, and $h = -\frac{1}{3}\frac{m-n}{m+n}$. The other type is based on the median of the *F* distribution, which is

$$T_2(F) = \frac{sign(h)\left\{X^h - \left[\widetilde{F}(0.5)\right]^h\right\}}{\left(Var\left[F^h\right]\right)^{\frac{1}{2}}} \tag{2.21}$$

where $\widetilde{F}(0.5)$ denotes the median of $F_{m,n}$. Isogai combined those two formulas and derived a simple formula for the median of the *F* distribution, which leads to a power normal family from the generalized *F* distribution. This transformation is expressed as:

$$T_3(F) = \frac{sign(h)\left\{X^h - \left[\widetilde{F}(0.5)\right]^h\right\}}{\left[2h^2\left(\frac{1}{m} + \frac{1}{n}\right)\right]^{\frac{1}{2}}} \tag{2.22}$$

When *m* and *n* have the same degrees of freedom, the limiting form of $T_3(F)$ is:

$$\frac{\ln F}{(4m)^{1/2}}$$

where we put *m*=*n*.

Non-central *t*-distribution

If *U* and *V* are independent random variables and *U* is N(0,1), *V* is $\frac{\chi^2}{n}$ (where χ^2 denotes the central chi-square variable with n degrees of freedom), and δ is a real umber, then the random variable defined by $t = \frac{U + \delta}{\sqrt{V}}$ is known as the non-central *t* variable with *n* degrees of freedom and with non-centrality parameter δ. Assume throughout that $n \geq 4$. The first moment about zero, the second and third central moments of non-central *t* were obtained, respectively

$$\mu = \delta\sqrt{\frac{n}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$\mu_2 = a^2 + b^2\mu^2$$

$$\mu_3 = \mu\left[\frac{n(\delta^2 + 2n - 3)}{(n-2)(n-3)} - 2\mu_2\right], \tag{2.23}$$

where

$$a = \sqrt{\frac{n}{n-2}}$$

$$b = \sqrt{\frac{2\Gamma^2\left(\frac{n}{2}\right)}{(n-2)\Gamma^2\left(\frac{n-1}{2}\right)} - 1}$$

which is a positive number for $n \geq 4$. The variance-stabilizing transformation is

$$\xi(t) = \int_0^t \frac{d\mu}{a^2 + b^2\mu^2} = \alpha \sinh^{-1}(\beta t), \tag{2.24}$$

where $\alpha = \frac{1}{b}$ and $\beta = \frac{b}{a}$.

Based on a corollary from a theorem stated by Rao, Laubscher (1960) proposed three transformations for non-central t-distribution as

$$\xi_1(t) = \xi(t) - \alpha \sinh^{-1}(\beta\mu)$$

$$\xi_2(t) = \xi_1(t) + \frac{b^2\mu}{2\sqrt{\mu_2}}$$

$$\xi_3(t) = \xi_2(t) - \frac{b^4\mu_3\left(2\mu^2 - \frac{a^2}{b^2}\right)}{\sqrt{\mu_2^5}} \tag{2.25}$$

These three transformations have approximately, mean value zero and unit variance. ξ_1 is not a very good approximation for simultaneous small values of n and large values of δ.

When the values of *n* are large, ξ_2 and ξ_3 are seen to be very close to normality. Numerical work showed that ξ_1 is the most suitable transformation when $\alpha = 0.05$ and $\alpha = 0.01$. The selection, thus can be made based on the values of *n*, δ and α .

Non-central *F* distribution

If $X_1, \ldots, X_m$ are independently distributed and X_i is $N(\mu_i, 1)$, then the random variable $\chi'^2 = X_1^2 + \ldots + X_m^2$ is called a non-central chi-square variable with *m* degrees of freedom and non-centrality parameter $\lambda = \mu_1^2 + \ldots + \mu_m^2$. If χ'^2 has the non-central chi-square distribution with *m* degrees of freedom and non-centrality parameter λ , and if χ^2 , independently of χ'^2 , follows the central chi-square distribution with n degrees of freedom, then the ratio

$$F = \frac{\chi'^2 / m}{\chi^2 / n}$$

has the topside non-central *F* distribution with *m* and *n* degrees of freedom respectively, and with non-centrality parameter λ . It is well known that $\sqrt{2\chi^2}$ is approximately normal with mean $\sqrt{2n-1}$ and unit variance. Also, $\sqrt{2\chi'^2}$ is approximately normal with mean $\sqrt{2(m+\lambda) - \frac{m+2\lambda}{m+\lambda}}$, and variance $\frac{m+2\lambda}{m+\lambda}$.

From a theorem due to Fieller, if *X* and *Y* are normally and independently distributed with means m_x and m_y and standard deviations σ_x and σ_y respectively, then the function

$$R = \frac{m_x V - m_y}{\sqrt{\sigma_x^2 V^2 + \sigma_y^2}} \tag{2.26}$$

where $V = \dfrac{Y}{X}$, will be nearly normally distributed with zero mean and unit variance, provided the probability of X being negative is small. Applying this theorem, Laubscher (1960) proposed two normal transformations as

$$\tau_1 = \tau_1(F) = \frac{\sqrt{\dfrac{m(2n-1)F}{n}} - \sqrt{2(m+\lambda) - \dfrac{m+2\lambda}{m+\lambda}}}{\sqrt{\dfrac{mF}{n} + \dfrac{m+2\lambda}{m+\lambda}}}$$

$$\tau_2 = \tau_2(F) = \frac{\left(1 - \dfrac{2}{9n}\right)\sqrt{\dfrac{mF}{m+\lambda}} - 1 + \dfrac{2(m+2\lambda)}{9(m+\lambda)^2}}{\sqrt{\dfrac{2}{9n}\left(\dfrac{mF}{m+\lambda}\right)^{\frac{2}{3}} + \dfrac{2(m+2\lambda)}{9(m+\lambda)^2}}} \tag{2.27}$$

The closeness of approximation of two transformations is quite satisfactory. The first form is slightly simpler, as it only involves the square root calculation.

χ^2-distribution

In multivariate analysis, chi-square distribution is mostly made used of. Konish (1981) gave a general procedure for finding normalizing transformations of statistics including some previous forms of transformation of chi-square data and presented results for some statistics in multivariate analysis. Konish (1981) also derived a differential equation for making a normalizing transformation under certain assumptions of moments which is

$$g_1(T_n) = \frac{\sqrt{n}}{\sigma}\left\{\frac{1}{\xi}\left(e^{\xi(T_n - \mu)} - 1\right) - \frac{1}{n}\left(\mu_1 + \frac{1}{2}\sigma^2\xi\right)\right\} \tag{2.28}$$

Taneichi et al (1998) examined the normalizing transformation for multinomial populations and extended the results with chi-square distribution. In order to propose a power approximation of the test of homogeneity, they consider the distribution of the approximation D^{α}, which was the test statistics. The mean and first three moments about the mean of D^{α} under the reject hypothesis are evaluated as

$$E\left(D^{\alpha}\right)=\mu+\frac{1}{n_{..}}\mu_1+o\left(\frac{1}{n_{..}\sqrt{n_{..}}}\right)$$

$$V\left(D^{\alpha}\right)=\frac{1}{n_{..}}\mu_2+o\left(\frac{1}{n_{..}\sqrt{n_{..}}}\right)$$

$$E\left[\left\{D^{\alpha}-E\left(D^{\alpha}\right)\right\}^3\right]=\frac{1}{n_{..}^{\;2}}\mu_3+o\left(\frac{1}{n_{..}^{\;2}\sqrt{n_{..}}}\right) \tag{2.29}$$

with $n_{..}$ is the total number of responses in the test, so that the general method to find the normalizing transformation discussed by Konish can be applied. Therefore, for sufficient large $n_{..}$, the distribution of $g\left(D^{\alpha}\right)$under the reject hypothesis is approximated well to $N(0,1)$, where

$$g\left(D^{\alpha}\right)=\sqrt{\frac{n_{..}}{\mu_2}}\left\{\frac{1}{f'(\mu)}\left(f\left(D^{\alpha}\right)-f(\mu)\right)-\frac{1}{n_{..}}\left(\mu_1+\frac{\mu_2 f''(\mu)}{2f'(\mu)}\right)\right\} \tag{2.30}$$

with $f\left(D^{\alpha}\right)$is a strictly monotone function of D^{α}, and twice continuously differentiable at $D^{\alpha}=\mu$. From the numerical comparisons, the approximation proposed based on the transformation above was found to be very effective for the statistics under α=-1,-2,-0.5 and 0. Moreover, it is easy to calculate the approximation. Therefore, it is practically useful.

On the basis of Konish's study, Taneichi *et al*. (2002) derived a concrete normalizing transformation. They assumed that the mean, variance and third moment about the mean are expanded as

$$E(T_n) = \mu + \frac{1}{2}\mu_1 + o\left(\frac{1}{n}\right)$$

$$V(T_n) = \frac{1}{n}\sigma^2 + o\left(\frac{1}{n}\right)$$

and

$$E\left[(T_n - E(T_n))^3\right] = \frac{1}{n^2}v + o\left(\frac{1}{n^2}\right) \tag{2.31}$$

They obtained a transformation form as

$$g_2(T_n) = \begin{cases} \dfrac{\sqrt{n}}{\sigma}\left[\dfrac{\mu}{\eta}\left\{\left(\dfrac{T_n}{\mu}\right)^{\eta} - 1\right\} - \dfrac{1}{n}\left(\mu_1 + \dfrac{1}{2}\sigma^2\xi\right)\right], \eta \neq 0 \\ \dfrac{\sqrt{n}}{\sigma}\left[\mu \log\dfrac{T_n}{\mu} - \dfrac{1}{n}\left(\mu_1 + \dfrac{1}{2}\sigma^2\xi\right)\right], \eta = 0 \end{cases} \tag{2.32}$$

Let X_v be distributed as chi-square distribution with *v* degrees of freedom. Let $T_v = \dfrac{X_v}{v}$, then

$$g_2(T_v) = \sqrt{\frac{v}{2}}\left[3\left\{(T_v)^{1/3} - 1\right\} + \frac{2}{3v}\right]$$

$$g_1(T_v) = \sqrt{\frac{v}{2}}\left[-\frac{3}{2}\left\{e^{-\frac{2}{3(T_v - 1)}} - 1\right\} + \frac{2}{3v}\right] \tag{2.33}$$

Comparisons showed that the proposed transformation g_2 really improves the approximation to the distributions of the statistics, and the proposed transformation

g_2 performs better than the exponential-type transformation g_1 discussed by Kornish (1981) in almost all of the simulation studies. Therefore, we recommend g_2 to be used.

Exponential distribution

When control chart is used to monitor time-between-event data, the distribution is exponential. When the quantity characteristic follows exponential, the following transformations can be used. The common way for transforming exponential distribution is to use log transformation. This approach is more suitably used for proper analysis of variance calculations since it stabilizes the variance. It produces however a very negatively skewed distribution that is not suitable for *SPC* applications. Kittlitz (1999) proposed a method to make the long-tailed, positively skewed exponential distribution into an almost symmetric distribution by taking the fourth root of the data, i.e.

$$f(x) = \sqrt[4]{x}\ ,\ x \geq 0 \tag{2.34}$$

The observations for the fourth root can be easily calculated with a pocket calculator by depressing the square root key twice. The transformed data can then be plotted conveniently on an individual chart, an exponentially weighted moving average chart, or a cumulative sum chart. Yang and Xie (2000) discussed the best transformation within the power family, and investigated the properties of the control charts for exponentially distributed data. Denote by $Y = X^{\lambda}$ the power transformed observation and $f(y;\theta,\lambda)$ its probability density function. Minimizing the so-called Kullback-Leibler information number, they derived the best power transformation for exponential variables. The resulted optimal parameter values are:

$$\lambda_0 = 0.2654$$

$$\mu_0 = 0.9034\theta^{\lambda_0}$$

$$\sigma_0 = 0.2675\theta^{\lambda_0} \qquad (2.35)$$

When the parameter is unknown, the *Y* chart was constructed by using the mean of past data as an estimator of μ_0 and the mean of the moving ranges as an estimator of σ_0. Simulation results showed that the power transformation-based chart will be able to detect a process shift at a comparably fast speed to the original *X* chart. This study indicated that it is easy to use and possesses a number of interesting statistical properties; in particular, it is of great advantage that the transformation to normality does not depend on the specific parameter value of the exponential distribution. For simpler calculation, Kittlitz's expression (2.34) is better to use.

U-statistics

For a symmetric kernel $h(x_1,...,x_r)$, the *U*-statistic with degree *r* is given by

$$U_n = \frac{1}{\binom{n}{r}} \sum_{1 \le i_1 < ... < i_r \le n} h\left(X_{i_2},..., X_{i_r}\right) \qquad (2.36)$$

Fujioka and Maesono (2000) studied the normalizing transformations of asymptotic *U*-statistics. They firstly defined

$$\hat{S}_n = \frac{\sqrt{n}\left(\hat{\theta}-\theta\right)}{\hat{\sigma}} + \frac{\hat{p}}{\sqrt{n}} \frac{n\left(\hat{\theta}-\theta\right)^2}{\hat{\sigma}^2} + \frac{\hat{q}}{\sqrt{n}} \qquad (2.37)$$

which is the transformation that removes the bias and the skewness. The improved monotone transformation which was proposed by Hall (1992), is quoted in Fujioka and Maesono (2000).

$$\hat{S}_n^* = \pi_1\left(\frac{\sqrt{n}\left(\hat{\theta}-\theta\right)}{\hat{\sigma}}\right), \tag{2.38}$$

where

$$\pi_1(s) = s + \frac{\hat{p}}{\sqrt{n}}s^2 + \frac{\hat{q}}{\sqrt{n}} + \frac{\hat{p}^2}{3n}s^3$$

Based on the transformation $\hat{S}_n^*$, they discussed the transformations which remove the bias, skewness and kurtosis. Let $\hat{u}$ and $\hat{v}$ be the consistent estimators of u and v, respectively. The normalizing transformation which removes the bias, skewness and kurtosis is

$$\hat{T}_n = \hat{S}_n^* + \frac{\hat{u}}{n}\left(\hat{S}_n^*\right)^3 + \frac{\hat{v}}{n}\hat{S}_n^* \tag{2.39}$$

A monotone transformation was constructed as well which is given below.

$$\hat{T}_n^* = \pi_2\left(\hat{S}_n^*\right) = \pi_2\left(\pi_1\left(\frac{\sqrt{n}\left(\hat{\theta}-\theta\right)}{\hat{\sigma}}\right)\right), \tag{2.40}$$

where

$$\pi_2\left(s^*\right) = s^* + \frac{\hat{u}}{n}\left(s^*\right)^3 + \frac{\hat{v}}{n}s^* + \frac{3\hat{u}^2}{5n^2}\left(s^*\right)^5 + \frac{\hat{v}^2}{n^2}s^*$$

The simulation results showed that the normalizing transformation $\hat{T}_n^*$ is better than $\hat{S}_n^*$ and both transformations are better than the normal approximation. To improve the approximations based on these transformations, Fujioka and Maesono (2000) suggested making good estimators of p, q, u and v. These transforming forms suit sample standard deviation very well. They can be especially applied on S-chart.

Multinomial Populations

Taneichi at al (1998) examined the normalizing transformation for multinomial populations. In order to propose a power approximation of the test of homogeneity, they made use of the general normalizing transformation given by Konish. Consider the distribution of the approximation D^{α}, which was the test statistics. The mean and first three moments about the mean of D^{α} under the reject hypothesis are evaluated as

$$E\left(D^{\alpha}\right)=\mu+\frac{1}{n_{..}}\mu_1+o\left(\frac{1}{n_{..}\sqrt{n_{..}}}\right)$$

$$V\left(D^{\alpha}\right)=\frac{1}{n_{..}}\mu_2+o\left(\frac{1}{n_{..}\sqrt{n_{..}}}\right)$$

$$E\left[\left\{D^{\alpha}-E\left(D^{\alpha}\right)\right\}^3\right]=\frac{1}{n_{..}^{\,2}}\mu_3+o\left(\frac{1}{n_{..}^{\,2}\sqrt{n_{..}}}\right) \tag{2.41}$$

which $n_{..}$ is the total number of responses in the test, so that the general method to find the normalizing transformation discussed by Konish can be applied. Therefore, for sufficient large $n_{..}$, the distribution of $g\left(D^{\alpha}\right)$under the reject hypothesis is approximated well to $N(0,1)$, where

$$g\left(D^{\alpha}\right)=\sqrt{\frac{n_{..}}{\mu_2}}\left\{\frac{1}{f'(\mu)}\left(f\left(D^{\alpha}\right)-f(\mu)\right)-\frac{1}{n_{..}}\left(\mu_1+\frac{\mu_2 f''(\mu)}{2f'(\mu)}\right)\right\} \tag{2.42}$$

with $f\left(D^{\alpha}\right)$ is a strictly monotone function of D^{α}, and twice continuously differentiable at $D^{\alpha}=\mu$. From the numerical comparisons, the approximation proposed based on the transformation above was found to be very effective for the statistics under α=-1,-2,-0.5 and 0. Moreover, it is easy to calculate the approximation. Therefore, it is practically useful.

Unifying Density

Ross S.M. presented an improved unifying density function in 1980 which was shown that the unifying density function is the parent of the Weibull, gamma, Erlang, chi-square, exponential, and Rayleigh distribution. The improved unifying density function has the following form:

$$f(y)=\begin{cases} \dfrac{\alpha}{\beta^{\alpha^{\varphi}}\Gamma\left(\alpha^{\varphi}+1\right)} y^{\alpha-1} e^{-\beta^{-1} y^{-\left(\alpha^{\varphi}-\alpha-1\right)}}, & y>0 \\ 0, & y \le 0 \end{cases} \tag{2.43}$$

Let the random variable Y have the improved unifying distribution $f(y)$, Waissi (1993) find the distributions of

$$Z_1 = -\sqrt{Y}$$

$$Z_2 = \sqrt{Y} \tag{2.44}$$

Combine the two distributions into one, Waissi demonstrated that the last expression has the format of the standard normal distribution multiplied by a constant. Thus the improved unifying density function has been extended to serve as a parent of the normal and standard normal distributions.

2.3.3. Summary

From the literature, despite the substantial number of normalizing transformations proposed, there is no prefect form for all the distributions. Generally speaking, the more accurate transformations tend to have more complexity for application. Therefore, we recommend some transformations below which achieve a good trade-off between accuracy and simplicity. These proposed transformations are summarized in the Table 2.1 below.

Category	Distribution	Proposed Normalizing Transformation	Transformation Formula
Generic distributions		Box-Cox transformation	(2.9)
		Johnson curve fitting	(2.10)
Common specific distributions	*t*-distribution	Bailey's transformation	(2.18)
	F-distribution	Isogai's transformation	(2.22)
	Non-central *t*-distribution	Laubscher's transformation	(2.25)
	Non-central *F*-distribution	Laubscher's transformation	(2.27)
	χ^2-distribution	Taneichi's transformation	(2.33)
	Exponential distribution	Yang and Xie's transformation	(2.35)
U-statistics		Fujioka and Maesono's transformation	(2.40)
Multinomial populations		Taneichi's transformation	(2.42)
Unifying density		Waissi's transformation	(2.44)

Table 2.1. Summary table of recommend normalizing transformations

Chapter 3

Investigation of the Probability Limits in Traditional Shewhart *R*-Charts

3.1. Introduction

The $\overline{X}$ -*R* charts are widely used in industry. For each sample of size *n*, the mean $\overline{X}$ and range *R* can be computed, and control limits can be obtained. We first introduce the formulas commonly used for the calculation of the *UCL* and *LCL* here

$$\begin{aligned} UCL_{\overline{X}} &= \overline{X} + A_2 R \\ LCL_{\overline{X}} &= \overline{X} - A_2 R \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} UCL_R &= D_4 R \\ LCL_R &= D_3 R \end{aligned} \tag{3.2}$$

In the above, $\overline{X}$ is the mean of all observations and *R* is the average of the ranges. The values of A_2, D_3 and D_4 are taken from standard tables (see, e.g., Grant and Leavenworth, 1986 and Montgomery, 2001).

It can be noted that the control limits are 3-sigma limits equal to the mean plus and minus three times the standard deviation. Since under the assumption of normality, the process mean is also normally distributed, the control limits lead to a false alarm probability of 0.0027, a standard values in control chart analysis and interpretation. We can use the constant factors table in standard statistical process control texts to find the multiple of sigma associated with 0.0027 probabilities.

The distribution of $\overline{X}$, which is normal or approximately normal in general, is usually symmetric. The probability limits on an $\overline{X}$ chart such as 3-sigma limits is thus, equidistant from the central line on the chart. However, the subgroup range does not follow normal distribution and is not symmetrical. It is easily proved because the range is always positive. Hence, to let the *LCL* for *R*-chart to remain positive, we have to set D_3=0 when the subgroup size is small. From the standard tables, D_3 equals to zero for subgroup size of 6 or smaller. At the same time, it is not advisable to use a large subgroup size as the sampling interval might be too long or the sampling cost might go up. Hence, we usually face a zero *LCL* for *R*-chart.

On the other hand, it is clear that when the range value is zero or even only very small, there is a high probability that something has happened. This chapter investigates the interesting question of variability monitoring with probability limits. Because the range is a continuous positive random variable and the probability for it to be zero is zero, the lower probability limit will always be positive. Hence with probability limit, it is always possible to detect a reduction in the process variability, which is important in continuous improvement framework.

The validity of normal distribution has been questioned in some control charting applications. Some authors had investigated the problem. Woodall (1997) and Xie et al. (2002) pointed out that the traditional 3-sigma limits for the Shewhart charts may not be appropriate since the statistic used for monitoring with attribute data usually have underlying distribution which are skewed to the right. Several authors have discussed the probability-based methods for determining control limits, such as Ryan and Schwertman (1997), Shore (2000). However, few literatures have discussed the non-normality problem in variable control charts, especially for chart monitoring variability such as s-chart and R-chart.

The need for and the possibility of a positive *LCL* for the *R*-chart will be further discussed in section 3.2 and the probability limits for the *R*-chart derived. We first give the exact formula for calculating the probability limits, and then some approximate formulas are discussed. A numerical example is presented to illustrate the simplicity and usefulness of the study following the discussion in section 3.3. Conclusions are drawn in section 3.4.

3.2. Positive LCL of R-chart for Improvement Detection

3.2.1. The need of a positive LCL for *R-/S*-chart

For the case of standard normal distribution, as mentioned in Tippett (1925), the mean of R is given by

$$E[R] = \int_{-\infty}^{+\infty} \left[1 - \left(1 - \Phi(x)^n\right) - \Phi(x)^n\right] dx, \tag{3.3}$$

where $\Phi(x)$ is the cumulative standard normal distribution. Here, $E[R]$ is a function of the sample size n, which gives the definition of the coefficient. $E[R^2]$ can also be obtained (Tippett, 1925) and let

$$d_3 = \sqrt{E[R^2]}$$

The three sigma limits for R-chart are then:

$$\overline{R} \pm 3d_3 \frac{\overline{R}}{d_2} = (D_3\overline{R}, D_4\overline{R}) \tag{3.4}$$

where $\overline{R}/d_2$ is an estimate of population standard deviation and it is used here to adjust for the case of general normal distribution with standard deviation different from that of standard normal. That is we have,

$$D_3 = 1 - \frac{3d_3}{d_2} \tag{3.5}$$

Since R cannot be negative, D_3 is set to be zero whenever $1 - \frac{3d_3}{d_2}$ is negative.

Modern quality control philosophy and techniques are built upon the idea of continuous improvement and variability reduction. To be able to reduce the variability, a mean to detect a reduction in the variability would be very useful; it gives us positive signal and enables us to look for positive assignable cause. Hence, it is very important to be able to have positive *LCL* for *R*-chart. Fortunately, the *LCL* for *R*-chart can be positive and not zero as the standard tables provide. This is even the case for small subgroup size and can be seen by studying the quantity that is plotted on the *R*-chart.

The *R*-chart plots the difference between the largest and smallest values. Assuming normally distributed characteristic, the largest and smallest values should be quite different. Even for a subgroup of size 2, say, if the range is zero, it means that the two values are exactly the same. This will happen with probability zero. In fact, a small range should indicate that there are assignable causes that have lead to a small variability. If the cause can be retained, then a reduction in variability has been achieved.

It is traditional to use $D_3 = 0$ for small n. Grant and Leavenworth (1980) suggested constructing *R* charts using probability limits. Due to the asymmetrical distribution of the range, it is necessary to have separate factors for the upper and lower control limits if the probabilities of extreme variations are to be made equal. The probabilities given for these limits are strictly accurate when sampling from a normal universe and all positive.

It can be noted that it is possible to incorporate run rules or Cusum /EWMA scheme for *R*-chart (Acosta-Mejia, 1998; Srivastava, 1997; Crowder and Hamilton, 1992; and Nelson, 1990) to detect variability reduction. However, these advanced charts are not commonly used and the interpretation of out-of-control signals can be different from that of $\overline{X}$ –

chart. In what follows, the use of probability limits will be studied in details, and it provides a simple, but statistically justified solution to the problem of detecting variability reduction with simple *R*-chart that are commonly used together with $\overline{X}$ -chart.

The distribution of the range for a sample taken can be obtained by the convolution of the distribution of the extremes. The problem of the range for the normal distribution is of most importance in statistical process control. It will be discussed in details here.

3.2.2. The Distribution of the Range

Consider observations $X_1,\ldots, X_n$, where n is the subgroup size. The order statistics are $X_{(1)},\ldots, X_{(n)}$. Let R denote the range defined as $X_{(n)}$ - $X_{(1)}$, the distribution of $R = X_{(n)}$ - $X_{(1)}$ can be obtained as follows:

$$\begin{aligned} f_R(u) &= \int_{-\infty}^{\infty} f_{X_{(n)}-X_{(1)},X_{(n)}}(u,v)dv \\ &= \int_{-\infty}^{\infty} \frac{n!}{(n-2)!}\left[F(u+v)-F(v)\right]^{n-2} f(u+v)f(u)dv, \end{aligned} \tag{3.6}$$

where $f(x)$ is the probability density function of individual measurement X_i and $F(x)$ is the cumulative distribution function of X_i. The probability distribution function of *R* can be shown to be:

$$f_R(u) = \int_{-\infty}^{\infty} \frac{n!}{(n-2)!}\left[\frac{1}{\sigma\sqrt{2\pi}}\int_v^{u+v} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx\right]^{n-2} f(u+v)f(u)dv, \tag{3.7}$$

where

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

provided X_i is normally distributed.

The distribution of *R* was a widely discussed topic in statistics in the middle of twentieth century. The focus was on the computational issues which was a difficult problem when computers were not available. Several papers also deal with the expression of the mean and variance for the range. In fact, this has lead to the numerical values of D_3 and D_4 in the traditional *R*-chart.

The probability distribution function of R can be shown to be (Gumbel, 1947):

$$F_R(u) = n\int_{-0}^{1}\left[\frac{1}{\sigma\sqrt{2\pi}}\int_{v}^{u+v}\exp(-\frac{(x-\mu)^2}{2\sigma^2})dx\right]^{n-1} f(v)dv \tag{3.8}$$

McKay and Pearson (1993) showed a general expression of $f_R(u)$ as:

$$f_R(u) = n(n-1)\int_{-\infty}^{\infty} f\left(t+\frac{1}{2}u\right)f\left(t-\frac{1}{2}u\right)\left(\int_{t-\frac{1}{2}u}^{t+\frac{1}{2}u} f(x)dx\right)^{n-2} dt \tag{3.9}$$

McKay and Pearson (1993) inferred that when the *u* is small enough, the distribution function of range can be approximated as:

$$f_R(u) = n(n-1)u^{n-2}\int_{-\infty}^{\infty} f\left(t+\frac{1}{2}u\right)f\left(t-\frac{1}{2}u\right)[f(x)]^{n-2} dt\,, \tag{3.10}$$

while the *u* is large enough, the distribution function of range can be approximated as:

$$f_R(u) = n(n-1)\int_{-\infty}^{\infty} f\left(t+\frac{1}{2}u\right)f\left(t-\frac{1}{2}u\right)dt \tag{3.11}$$

This approximation provides a way to conduct the more reasonable control limits.

Besides the deduction of the range distribution formula, several authors have studied the computation of the moments, percentage points of the range distribution. Pearson and Hartley (1942) and (1943) have examined the probability integral of the range. Later, Harter (1960) has presented comprehensive results on the computation of the percentage points of the range distribution. Harter suggested that since the moments of the range distribution from a Normal population can be calculated, the use of probability limits as control limits for *R*-charts is possible.

3.2.3. The probability limits for R

For a subgroup of size *n*, given an acceptable level of false alarm probability α, the probability limits can be computed with the distribution of *R*. Note that the traditional 3-sigma limits correspond to the false alarm probability of $\alpha = 0.0027$. In general, the lower control limit and upper control limit for *R*, LCL_R and UCL_R can be determined by solving the following equations

$$F(UCL_R) = 1 - \frac{\alpha}{2}$$

and

$$F(LCL_R) = \frac{\alpha}{2}, \tag{3.12}$$

where $F(r)$ is the cumulative distribution function of *R*. Alwan (2000) has given an abridged version of some of the results of using probability limits for *R*-charts. The estimated probability limits are given as

$$UCL = D_{1-\frac{\alpha}{2}}\left(\frac{\overline{R}}{d_2}\right)$$

$$LCL = D_{\frac{\alpha}{2}}\left(\frac{\overline{R}}{d_2}\right). \tag{3.13}$$

Here we assume that the underlying distribution of each sample follows normal distribution. That is,

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} \tag{3.14}$$

where μ is the mean and σ^2 is the variance. For the simple case with n=2, we can generate the curve of the pdf of the range distribution as in Figure 3.1. As expected, it is highly skewed.

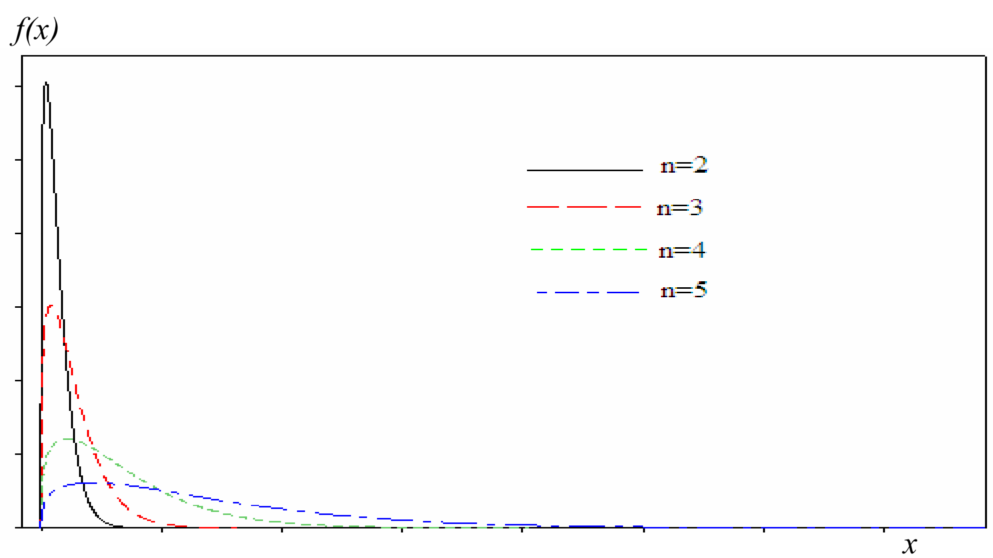


Figure 3.1. Probability density function of the range

From Figure 3.1, it is clear that the distribution is an asymmetric and very skewed normal approximation. D_3 and D_4 derived from this normal approximation are subsequently not appropriate. Actually, the distribution of the range flattens when the subgroup size increases, which is totally distinct from normal distribution. Therefore, simply using 0.0027 as the false alarm probability of the 3-sigma control limits of the range is somehow unreasonable. The exact false alarm probability can be computed with the exact distribution for the range.

Table 3.1 shows the *LCL/UCL* for the standard normal distribution computed by using Mathematica™. Note that the mean has no effect here. When the standard deviation is not 1, the control limits can be computed using the value multiplied with the standard deviation since $(X-\mu)/\sigma$ is standard normal.

			LCL	D_1^*	
α	n=2	n=3	n=4	n=5	n=6
0.0027	0.00239	0.0700	0.2206	0.3965	0.5690
0.01	0.00886	0.1348	0.3427	0.5549	0.7490
0.05	0.04432	0.3031	0.5946	0.8497	1.066
			UCL	D_2^*	
α	n=2	n=3	n=4	n=5	n=6
0.0027	4.5328	4.9503	5.1996	5.3775	5.5151
0.01	3.9697	4.4241	4.6941	4.8856	5.0335
0.05	3.1698	3.6823	3.9822	4.1970	4.3609

Table 3.1. Some probability *LCL/UCL* for *R*-chart ($\sigma = 1$) for different false alarm probability

Table 3.1 is actually the modification of traditional D_1 and $D_2\left(D_1^*, D_2^*\right)$. From the relationship between sample mean $\overline{R}$ and σ, which is:

$$d_2 = \frac{\overline{R}}{\sigma} \tag{3.15}$$

We can get the modified D_3 and $D_4\left(D_3^*, D_4^*\right)$ by simply computing the formula below:

$$D_3^* = \frac{D_1^*}{d_2}$$

and

$$D_4^* = \frac{D_2^*}{d_2} \tag{3.16}$$

The values of D_3^* and D_4^* are shown in Table 3.2.

			D_3^*		
α	n=2	n=3	n=4	N=5	n=6
0.0027	0.002119	0.041347	0.107139	0.170464	0.224546
0.01	0.007855	0.079622	0.16644	0.238564	0.29558
0.05	0.039291	0.179031	0.288781	0.365305	0.420679
			D_4^*		
α	n=2	n=3	n=4	N=5	n=6
0.0027	4.01844	2.923981	2.525304	2.311909	2.17644
0.01	3.519238	2.613172	2.279796	2.10043	1.986385
0.05	2.810106	2.175015	1.934046	1.804385	1.720955

Table 3.2. Some probability D_3^* / D_4^* for *R*-chart ($\sigma = 1$) for different false alarm probability.

3.2.4. False alarm probability and run length properties

With the exact distribution of the range, the run length properties can be studied. Recall that the *ARL* is the expected number of points plotted before an alarm is observed. For Shewhart-type control chart, this can be simply computed with

$$ARL_R = \frac{1}{P(R < LCL) + P(R > UCL)} \tag{3.17}$$

The denominator is the alarm probability which becomes false alarm probability when the process is in control. Usually, the *ARL* is simply computed as follows:

$$ARL_0 = \frac{1}{\alpha} = \frac{1}{0.0027} = 371,$$

where α is the probability of Type I error according to the 3-sigma limits of normal distribution. Since the range distribution is neither normal nor symmetrical, the probability of a point falls beyond the control limits cannot be expected to be equal to α.

Given the range distribution, we can compute the real *ARL* of the traditional control limits for the *R* charts. From Montgomery (2001), we can see the traditional *LCL* for the *R* chart is 0 for the subgroup with sample size less equal than 6. Therefore, the *ARL* may be revised as:

$$ARL = \frac{1}{P(R > UCL)} \tag{3.18}$$

Using the range distribution, the accurate *ARL* for the traditional control limits can be calculated for various subgroup sample size by:

$$ARL_0 = \frac{1}{\alpha} \tag{3.19}$$

In case of out-of-control, the *ARL* represents one over the probability of acceptance of a out-of-control point.

$$ARL_1 = \frac{1}{1-\beta} \tag{3.20}$$

To compare the performance of the traditional control limits and our proposed scheme, we have constructed the *ARL*-curves.

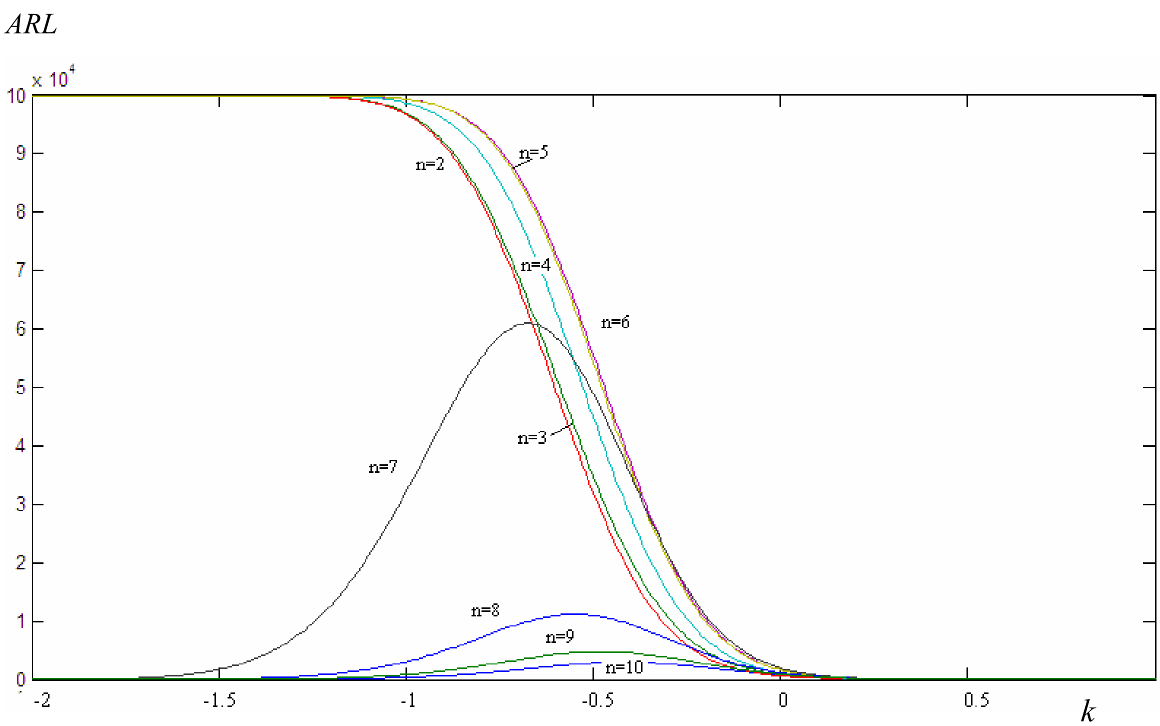


Figure 3.2. *ARL*-curve of traditional *R*-chart

Figure 3.2 shows the *ARL* for traditional *R*-chart when subgroup size varies from 2 to 10. The parameter on the vertical axis of these curves is the ARL_1, i.e. the *ARL* to detect a process shift. The parameter on the horizontal axis is $k = \frac{\delta}{\sigma}$, which is the process sample standard deviation shift (δ) in terms of the process target standard deviation (σ). From the plot we can see that it is almost impossible to detect the process variance decrease when subgroup size is lee than 7. When the process shift is about -0.5σ, the *ARL* is about 40,000. When the process variance shift is about -1.5σ, *ARL* increases dramatically to far more than 100,000. This results in almost no reaction of the control chart to great process variance improvement. Moreover, for any size of the subgroup, the *ARL* is severely biased.

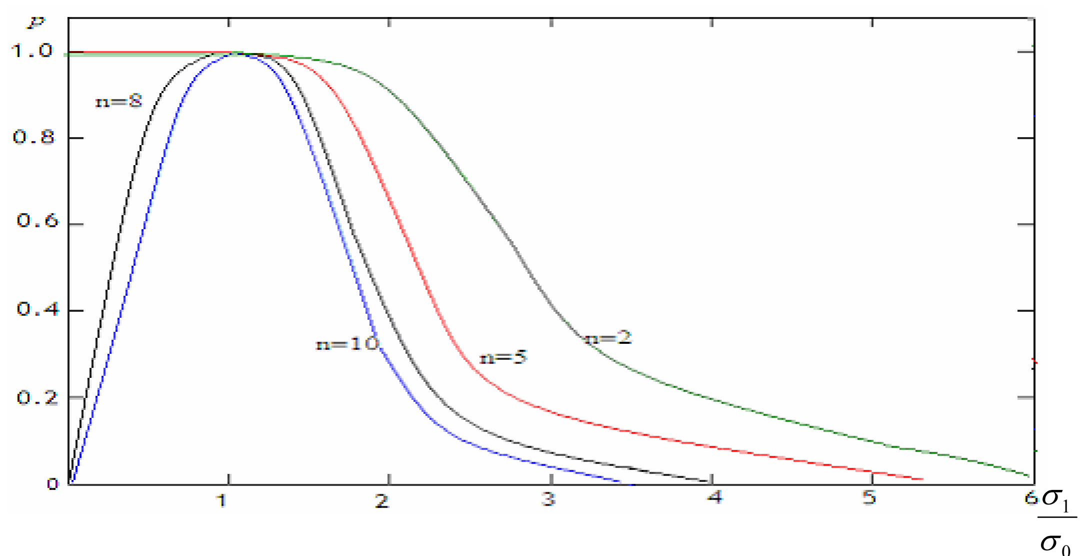


Figure 3.3. Full *OC*-curves for *R*-charts. (Adapted from Duncan(1979), pp445)

Figure 3.3 is the *OC*-curve of the traditional *R*-chart. The abscissa of the chart is the ratio of the new standard deviation σ_1 to the old σ_0, and the ordinate is the probability that the shift in σ_1 will not be caught by the *R*-chart on a single sample. The *OC*-curves for $n \leq 6$ do not drop to the left but continue getting closer to *P*=1, since in these cases there is no lower limit. For $n > 6$, an *R*-chart making use of 3σ limits does have a lower limit, and the *OC* curves for such charts are similar in shape to that for n=10. This result is corresponding to the properties of the *ARL* curves of the traditional *R*-chart.

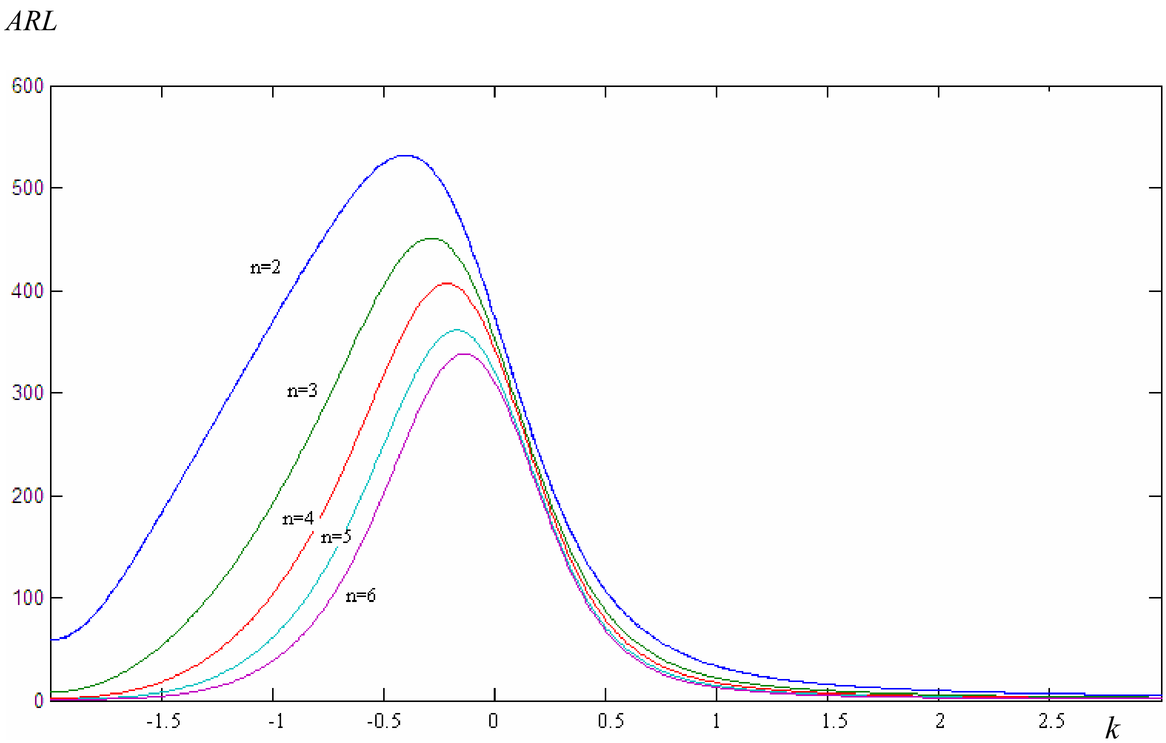


Figure 3.4. *ARL*-curve of modified *R*-chart

Figure 3.4 is the *ARL*-curve of modified *R*-chart by using the proposed D_3 and D_4. Compared to Figure 3.2, Figure 3.4 indicates the great improvement of the control limits performance. Although the *ARL* distribution is still skewed, it is much closer to the ideal shape. Another merit of the modified limits is that since it is derived from the exact distribution of range, it has the exact tail probability, which is 0.003 as used in the example. We may change this probability to any numbers we favor and derive the corresponding control limits. This is in line with the basic concept of traditional Shewhart type control charts.

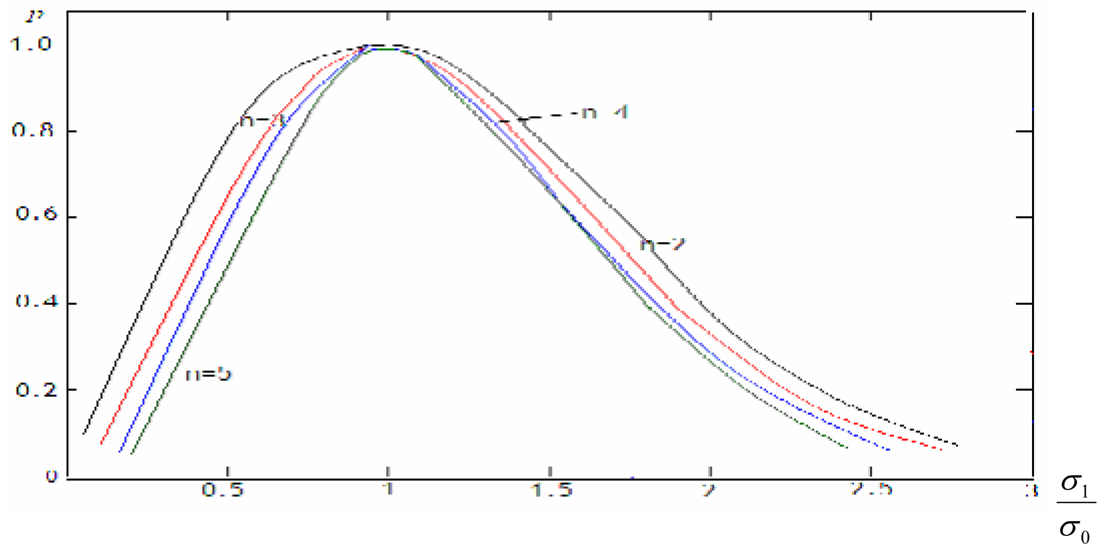


Figure 3.5. Full *OC*-curves for modified *R*-charts.

The *OC* curves of the modified *R*-chart indicate a similar property. Figure 3.5 is the *OC* curves of the modified *R*-charts when $n \le 6$. It is clear to see that after the modification, the *OC* curves turn to have a shape similar to that of $n > 6$. When the process deviation is decreasing, the probability of Type II error is decreasing, which translates to a faster signal of out-of-control. This is due to the fact that the positive lower limits have been assigned.

Using positive lower control limit enable us to observe variability reduction in manufacturing processes. This is also a way to ensure more accurate tail probability matched. Denote the tail probability at the lower control limit end as P_{tLCL} and the tail probability at the upper control limit end as P_{tUCL} when the process is in control. The tails probabilities can be computed by using the formulas below

$$P_{tLCL} = \int_{-\infty}^{LCL} f_R(x)dx = \int_{0}^{LCL} f_R(x)dx$$

$$P_{tUCL} = \int_{UCL}^{+\infty} f_R(x)dx \tag{3.21}$$

where $f_R(x)$ is the probability density function of Range distribution. Thus, the real tail probability of *R*-chart is obtained as in Table 3.3.

n	2	3	4	5	6
P_{tUCL}	0.00315	0.00553	0.00495	0.00486	0.00436
P_{tLCL}	0	0	0	0	0
n	7	8	9	10	
P_{tUCL}	0.00421	0.00427	0.00434	0.00449	
P_{tLCL}	0.00001	0.0001	0.0002	0.0003	

Table 3.3. Tail probability of traditional *R*-chart

The tail probability at the two ends is severely biased in Table 3.3. This is due to the biased traditional setting of D_3 and D_4. In contrast to that, the proposed setting of the parameters insures the ideal unbiased tail probability as 0.00135 at both ends since the lower and upper control limits are directly derived from the real range distribution.

3.3. Implementation example and Discussions

The $\overline{X}$ -chart can be set up in the same way as before. For *R*-chart, instead of using traditional D_3 and D_4, the probability limits can be used. In particular, this is the case when the subgroup size is less than 7 for which D_3 is zero and hence reduction in variability cannot be detected with the standard formula. Instead, the *LCL* limit can be determined in such a way that a different factor is used. Note that the factor depends on the subgroup size.

Table 3.4 is a set of simulated data. The first 30 observations are simulated with a mean of 100 and a standard deviation of 5 and the last 20 observations are simulated with a mean of 100 and a standard deviation of 2.

subgroup	X1	X2	X3	X4	X5	X-bar	R
1	90.45	106.31	108.4	97.25	100.09	100.5	17.96
2	107.97	97.2	112.52	90.71	96.94	101.07	21.8
3	101.33	99.05	99.46	101.98	92.46	98.86	9.52
4	106.35	103.96	99.17	106.07	94.3	101.97	12.05
5	101.54	101.81	92.35	101.74	105.87	100.66	13.53
6	93.06	109.38	97.09	102.84	103.74	101.22	16.32
7	111.81	97.5	100.9	98.81	95.42	100.89	16.39
8	98.29	103.5	108.19	105.05	98.09	102.62	10.09
9	99.71	94.34	103.69	105.74	106.66	102.03	12.32
10	111.48	100.39	99.18	101.57	101.4	102.8	12.3
11	95.06	103.06	92.12	100.49	93.35	96.82	10.94
12	107.7	97.74	104.53	99.82	97.14	101.39	10.56

13	108.92	97.09	91.97	92.81	98.82	97.92	16.95
14	102.98	100.83	99.42	100.05	97.64	100.18	5.34
15	99.59	96.8	96.5	92.91	96.46	96.45	6.67
16	106.06	97.01	103.69	96.82	102.83	101.28	9.23
17	97.51	92.99	98.41	99.16	112.44	100.1	19.45
18	95.82	101.96	107.54	105.81	97.24	101.67	11.72
19	100.52	93.11	103.03	101.2	86.16	96.8	16.87
20	110.54	105.44	92.35	96.47	97.66	100.49	18.19
21	96.69	103.61	98.46	99.34	90.12	97.64	13.5
22	103.47	108.51	94.41	101.56	101	101.79	14.1
23	104.78	93.85	95.97	97.32	103.6	99.1	10.93
24	93.63	104.26	93.71	101.18	99.65	98.49	10.62
25	102.88	92.79	108.83	98.41	105.41	101.66	16.03
26	105.8	97.04	94.25	97.21	98.95	98.65	11.56
27	107.48	103.62	102.91	101.58	109.47	105.01	7.88
28	96.11	97.63	95.66	100.44	105.05	98.98	9.39
29	113.42	98.29	103.27	103.18	102.01	104.03	15.13
30	106.01	101.8	98.74	90.86	99.84	99.45	15.15
31	102.394	101.201	98.079	98.149	101.616	100.2878	4.315
32	99.03	98.993	98.076	99.647	100.983	99.3458	2.907
33	102.21	99.664	97.975	98.366	102.148	100.0726	4.235
34	102.416	98.483	102.545	99.846	102.921	101.2422	4.438
35	103.162	101.766	97.163	99.416	100.717	100.4448	5.999
36	101.895	98.797	97.546	102.12	96.787	99.429	5.333
37	101.55	100.838	102.415	96.576	101.397	100.5552	5.839
38	98.209	99.347	95.685	98.829	103.371	99.0882	7.686
39	100.616	100.97	99.345	101.459	99.725	100.423	2.114
40	104.203	105.991	98.125	98.579	101.141	101.6078	7.866
41	98.362	102.188	98.688	98.399	99.76	99.4794	3.826
42	98.733	103.233	100.344	101.889	99.085	100.6568	4.5
43	100.003	97.741	99.196	101.847	95.888	98.935	5.959
44	100.26	104.245	95.065	99.184	102.452	100.2412	9.18
45	95.82	97.154	102.24	101.495	103.758	100.0934	7.938

46	98.83	100.158	99.933	99.803	99.979	99.7406	1.328
47	103.111	100.116	105.309	99.301	103.172	102.2018	6.008
48	102.299	101.411	97.747	101.943	99.51	100.582	4.552
49	97.816	100.697	103.834	99.803	102.634	100.9568	6.018
50	100.036	96.413	102.194	102.269	96.827	99.5478	5.856

Table 3.4. A set of simulated data with subgroup size of five.

Using the value of D_3 and D_4, we conduct the traditional *R* chart as:

$$UCL=24.59$$

$$CL=11.63$$

$$LCL=0.$$

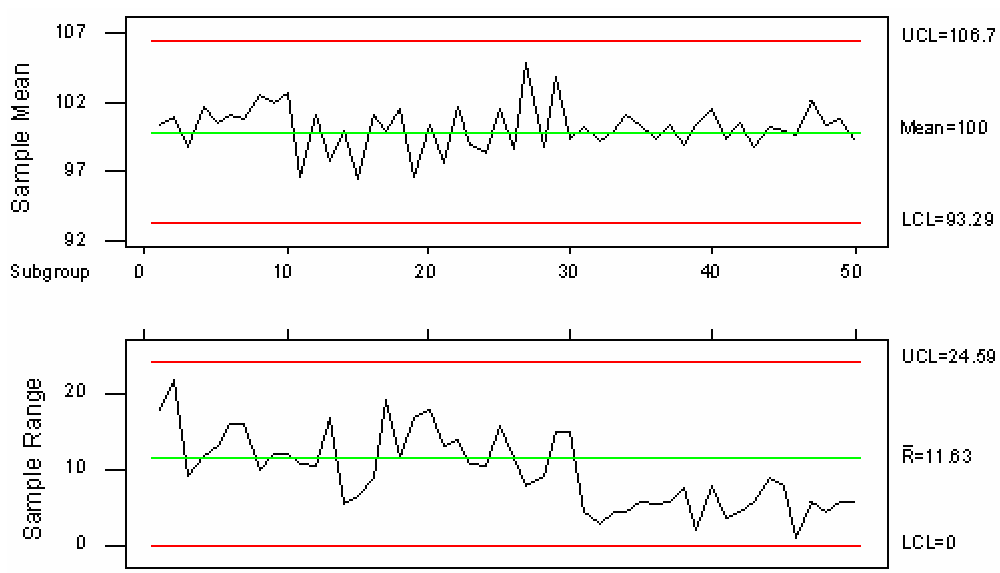


Figure 3.6 $\overline{X}$ and *R* charts for the data in Table 3.4.

Although it is clear from the plot that the variability is reduced, an *R*-chart with probability limit would have detected the change at an earlier stage (unless some run rules are used). Now consider applying the probability control limits. From the Table 3.1, as the square root of variance is 5 and sample size of rational subgroup is 5, the modified *LCL* for the *R*-chart is 1.98, while *UCL* is 26.94. Therefore, we conduct a new *R*-chart with the modified probability limits as:

$$UCL=26.94$$

$$CL=11.63$$

$$LCL=1.98.$$

The new control chart is shown in Figure 3.7.

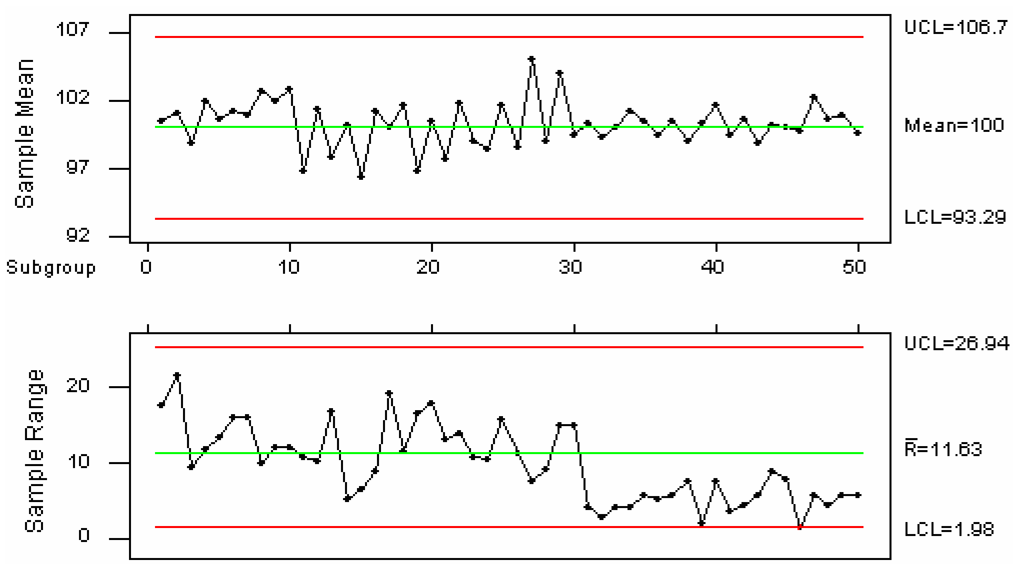


Figure 3.7 $\overline{X}$ and *R* charts for the example by using probability limits

The control charts indicated that the process is in control, until the range value from the forty-sixth sample is plotted. Due to the fact that the new control limits are derived accurately from the range distribution, the *ARL* may be calculated as follows:

$$ARL_0 = \frac{1}{\alpha} = \frac{1}{0.0027} = 371$$

when the process is in control. At the same time, if the process is out-of-control, with 0.5σ in range shift,

$$ARL_1 = \frac{1}{P(X < LCL, X > UCL)} = \frac{1}{0.068} = 15$$

Compared with the *ARL* in these two *R*-charts, the numerical example indicates that when the process is in control, the traditional *R*-charts tend to signal false alarm more frequently. On the other hand, when the process is out of control, the modified *R*-charts are able to detect the shift earlier.

This can be compared with the case when the process is in control. The *ARL* for *R*-chart with traditional limit is:

$$ARL_0 = \frac{1}{P(X > UCL)} = 196$$

Also, when the process is out of control, with 0.5σ in range shift, the *ARL* with traditional limits is:

$$ARL_1 = \frac{1}{P(X > UCL)} = 2.85 \times 10^{16}$$

This means, one out-of-control signal will be given every 28500 trillion points monitored. Therefore, there is no way to detect the out-of-control signal when *LCL*=0. In this case, as expected, no alarm will probably be raised and without additional rules, process changes will not be detected as the *ARL* increases significantly.

3.4. Conclusions

We have raised the question of whether a range of zero or even very small value should still be treated as out-of-control signal. When the subgroup size is small, between 2 to 6, the *LCL* for *R*-chart is usually set as zero. However, the exact distribution of the range can be used to derive the probability control limits which are always positive. In this chapter, exact formulas are given for the calculation of the probability limits, and we also discuss some approximate analytical results that can help with the calculation. Using positive lower control limit enable us to observe variability reduction in manufacturing processes. This is a widely discussed issue in many quality control texts.

A value below the *LCL* of an *R*-chart is probably because of positive assignable causes that have led to a reduction of the variability. Therefore, a positive *LCL* is always useful. Variability reduction is as important as, if not more than, targeting exactly at the mean. Hence, this study may help industrial statisticians to make better use of the statistical principles in developing process monitoring technique.

Chapter 4

Using Normalizing Transformations in Shewhart Charts

4.1. Introduction

As we discussed in the previous chapter, using the exact probability limits for *R*- and *S*-charts can have the control charts react faster to process deterioration and therefore improve their sensitivity. Mathematicians have investigated the distributions of the range and standard deviation of some well-known distribution families and have enabled us to calculate the probability limits based on those known probability density functions. This helps us to achieve the exact probability of our interest. Moreover, in this chapter, we will study the non-normality problem in traditional Shewhart type control chart in a more general way when the specific distribution is unkown.

Shewhart type control charts are the most commonly used method to test whether or not a process is in-control. As discussed in the Chapter 2, the normality assumption is invalid in many cases and very often the exact distribution is positively skewed (Winterbottom

(1993)). We recall that one of the consequences is that the control limits are set significantly lower than what they should be in order to come reasonably close to giving false alarm probabilities that correspond to those on $\overline{X}$ charts. Moreover, some quality measurements, such as the range R and the sample deviation S, differ from normal distribution, so that the performances of the traditional 3-sigma limits of traditional Shewhart type control charts are not satisfactory.

As reviewed in Chapter 2, the research on applying control charts for non-normal populations has been divided roughly into three categories. One of the categories of endeavors has focused on normalizing the given data, so that traditional Shewhart control charting schemes could be employed with desirable *ARL*. Transformation based approach succeeds in making a set of balanced control limits so that the *ARL* is large when the process behaves normally and smaller when the process deviates from that (Yang and Xie (2000)). However, this issue has not gained much attention in control charting scheme though many statisticians examined the transformation to normalize various kinds of distributions. We have summarized those existing transforming formulas in Chapter 2, which would contribute to the work dealing with non-normal data in process monitoring in this chapter.

In the coming section 4.2, we will discuss the transformation in traditional Shewhart type control charts in further depth and summarize the appropriate transformation forms for different type of control charts. Performance comparisons have been included based on the results of extensive simulation study. Detailed procedures on applying transformation in control chart constructing will be given in section 4.3, as well as numerical examples. In section 4.4, conclusions will be drawn.

4.2. Adaptation of Transformation in Control Charting Scheme

4.2.1. The need for a Transformation in Non-normal Process

As discussed in Chapter 2, the original purpose of constructing the traditional control limits as described in the introduction part is to apply the three-sigma philosophy directly under the normality assumption. Define E_i to be the event that *i*-th sample range *R* (or sample standard deviation *s*) is either above UCL or below LCL. The events $\{E_i\}$ are independent and for all $i \geq 1$,

$$P(E_i) = P(R_i > UCL \cup R_i < LCL) = 1 - \Phi(3) + \Phi(-3) = 0.0027,$$

where Φ is the cumulative distribution function of a $N(0,1)$ random variable. Define U to be the number of samples until the first E_i occurs, then U is known as the run length of the chart and has a geometric distribution with parameter $p = P(E_i) = 0.0027$. Care should be exercised that all the calculations above are based on the fundamental assumption that the underlying distribution of the monitored characteristic is normal. However, the validity of the assumption is questionable sometimes, especially in real-world applications. The underlying distribution of the monitored data can never be exactly independently identically normally distributed. Moreover, even when sampling from the normal distribution, some quality measurements are not normal. Thus, the standard interpretation of traditional control charts is not well applicable.

Alternatively, many authors have suggested the use of exact probability models to calculate the limits so that both ends are likely to have equal probability to decide the out of control state of a process. For instance, one possibility in constructing an *R*-chart would be to use $D_{0.001}$ and $D_{0.999}$ in place of D_3 and D_4. This would give equal tail areas of 0.001. The probability limits would then be obtained as

$$UCL = D_{0.999}\left(\frac{\overline{R}}{d_2}\right)$$

$$LCL = D_{0.001}\left(\frac{\overline{R}}{d_2}\right) \tag{4.1}$$

These limits obviously differ greatly from the 3-sigma limits. In particular (Ryan, 2000), the (0.001 and 0.999) probability limits will always be higher than the 3-sigma limits. The calculated limits are expected to have exact tail probabilities. However, it is not possible sometimes to meet the exact probabilities due to the discreteness of the models. In Chapter 3, we have also recommended to use the modified D_3 and D_4 to construct the *R*-chart, though it still has the flaw that the *ARL* is not symmetric. Let us recall *ARL* is a measure of the performance of a control chart which should be large when the process behaves normally and small when the process deviates from the target. Furthermore, for probability limits, it causes *ARL* to be biased which means the sensitivities of the control limits towards the increasing shift and decreasing one are different. It is common to see other control charts performing poorly under the non-normality data circumstance.

More generally, we study a right-skewed distribution without loss of universality. Figure 4.1 features the general shape of such distribution. For comparison, we draw the *PDF* of a standard normal distribution in the Figure 4.2.

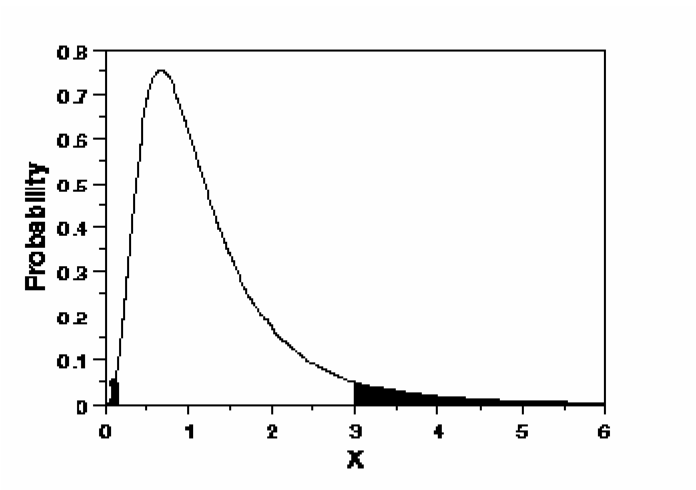


Figure 4.1. *PDF* of a right-skewed distribution

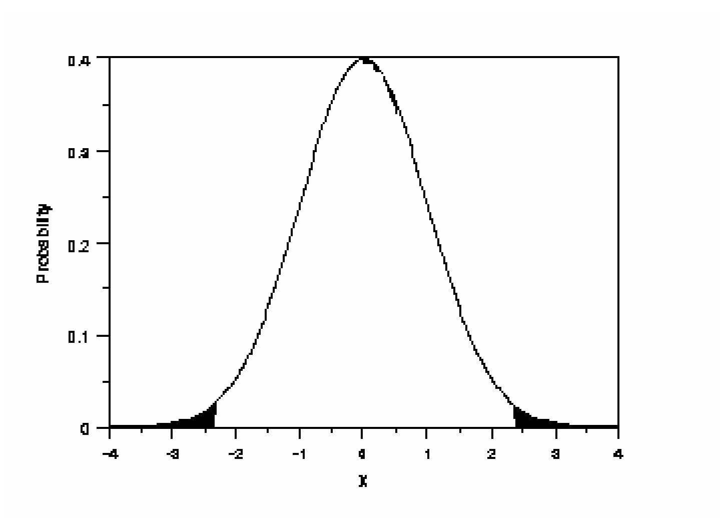


Figure 4.2 *PDF* of a standard normal distribution

From the two figures above, it is clear that in order to achieve same probability area beyond the control limits, the probability limits setting for skewed distribution and normal distribution should differ. For skewed distributions, we will only have asymmetric probability limits, which deteriorate the performance of the control charts in one of its sides.

This figure gives a strong indication of biased-*ARL* under the exact probability control limits. The curve shows an increasing-decreasing pattern. When the process variability increases, the *ARL* decreases steadily. However, when the process variability decreases, the *ARL* decreases faster than it does under up shift. Moreover, the maximum of the *ARL* is not located at shift equals 0. Instead, *ARL* achieves its peak around -0.2σ. All these features are not good to perform accurate process variability monitoring. However, our desired *ARL* is presented in the Figure 4.16 below for comparisons.

Moreover, sometimes the quality characteristic of concern does not follow normal distribution. In this case, it is almost impossible to derive the distributions for range or sample standard deviation. Therefore, we cannot calculate the exact probability limits. A method other than using real probability limits should be proposed to solve the problem. This results in our raising of normalizing transformation method. Our general idea here is to transform the distribution of the quality characteristics into normal distribution. The mapping is constructed as

$$X : f(x) \xrightarrow{T} Y : \phi(x)$$

where $f(x)$ is the original distribution and $\phi(x)$ is the normal distribution. Thereafter, traditional Shewhart type individual chart can by applied as

$$UCL = \overline{X} + 2.68\overline{MR}$$

$$CL = \overline{X}$$

$$LCL = \overline{X} - 2.68\overline{MR} \tag{4.2}$$

4.2.2. Performance Discussion on some Distributions

In the present section, we will discuss the transformation control limits on some most commonly met distributions in quality control area. For each distribution, extensive simulation has been conducted for traditional control limits, probability limits and normalized limits to compare the control limits performance. In each run, different parameter settings have been applied for 1,000,000 × 10,000 sample points. *ARL* curves are drawn based on the simulation results. Tail probability properties have also been investigated for comparison purpose.

***S*- Distribution**

S-chart is a popular alternative to *R*-chart. Traditional construction of *S*-chart is the same as *R*-chart; assume the sample standard deviation follow normal distribution and apply the 3-sigma limits. Unfortunately, the distribution of sample standard deviation is again far away from normal distribution. Therefore, traditional *S*-chart faces the same problem as *R*-chart does in process monitoring. For instance, the in-control *ARL* is supposed to be smaller than out-of-control *ARL* under the valid assumption. However, for the very skewed *S*-distribution, it is not the case. With very skewed distribution, the 3-sigma limits tend to deteriorate to be able to detect the shift in one side only, which cause the dramatically large *ARL* for out-of-control case, when sample size is less than 7. Figure 4.3 shows the *ARL*-curves of traditional *S*-chart with different subgroup sample sizes, knowing that the mean and variance of the sample distribution do not affect the *ARL* properties. The *y*-axis represents the *ARL*. The *x*-axis represents *k*, which is the shift of process standard deviation δ, in terms of target process standard deviation σ; that is

$$k = \frac{\delta}{\sigma}.$$

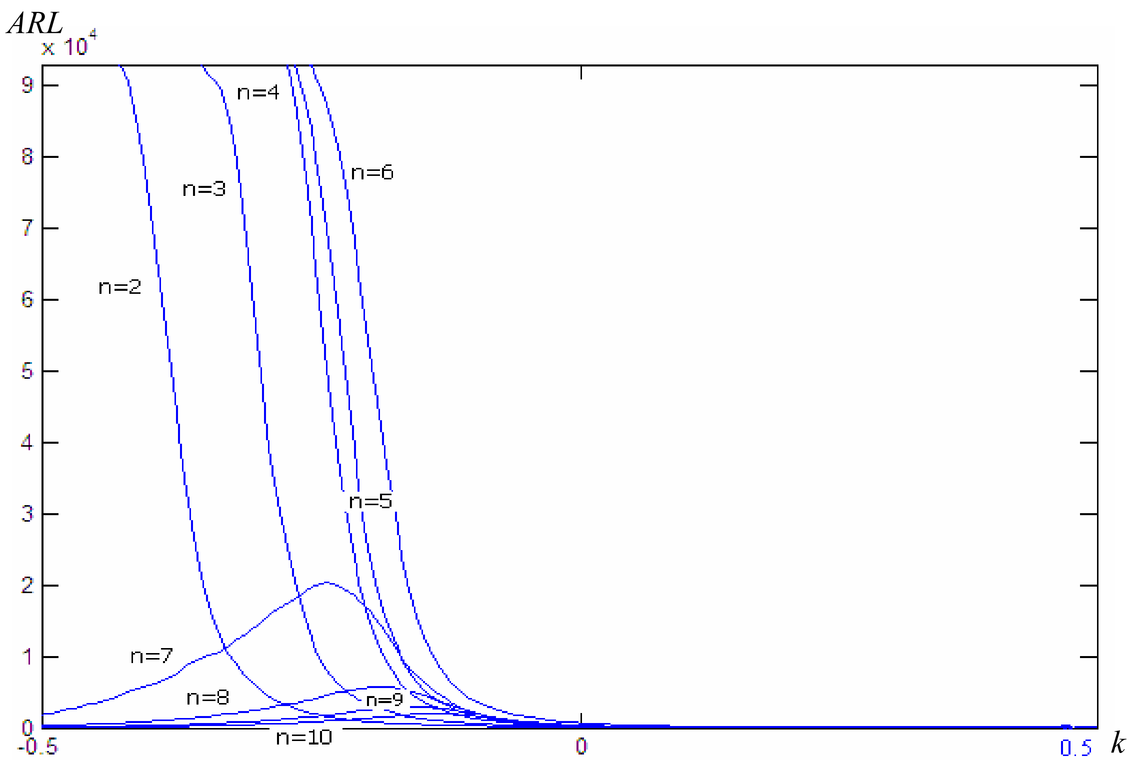


Figure 4.3. *ARL*-curves of *S*-chart with different sizes of subgroups

In the traditional parameter settings for *S*-chart, the *LCL* remains 0 when *n* varies from 2 to 6. Therefore, the control chart is not able to react to process deterioration (see the tail probability in the Table 4.1 for some subgroup sample size). This is one the most undesirable feature of using traditional Shewhart type control chart when the normal assumption is not valid. This phenomenon is quite common for skewed distribution. Other example includes R-chart.

N	2	3	4	5	6
P_{tUCL}	0.00932	0.00557	0.00145	0.00405	0.00353
P_{tLCL}	0	0	0	0	0
N	7	8	9	10	
P_{tUCL}	0.00316	0.00308	0.00301	0.00294	
P_{tLCL}	0.00002	0.00004	0.00013	0.00011	

Table 4.1. Tail probability of traditional *S*-chart

As discussed before, some authors have suggested using exact probability limits instead of the traditional control limits. The probability control limits can be set as follows

$$UCL = D_{0.99865}\left(\frac{\overline{S}}{c_4\sqrt{n}}\right)$$

$$LCL = D_{0.00135}\left(\frac{\overline{S}}{c_4\sqrt{n}}\right), \tag{4.3}$$

where $\overline{S}$ is the average sample standard deviation, n is the sample size, $D_{0.99865}$ and $D_{0.00135}$ are 0.99865 and 0.00135 percentiles, respectively. Figure 4.4 shows *ARL*-curves of the probability limit *S*-charts with different subgroup sample sizes. From the figure, one can see that the tail probability is achieved at 0.00135 for both ends; the biasness of *ARL* is not eliminated though. The smaller the subgroup sample size is, the more difficult it is to detect the process variance decrease. On the other hand, the larger subgroup sample size is, the more difficult it is to detect the process variance increase.

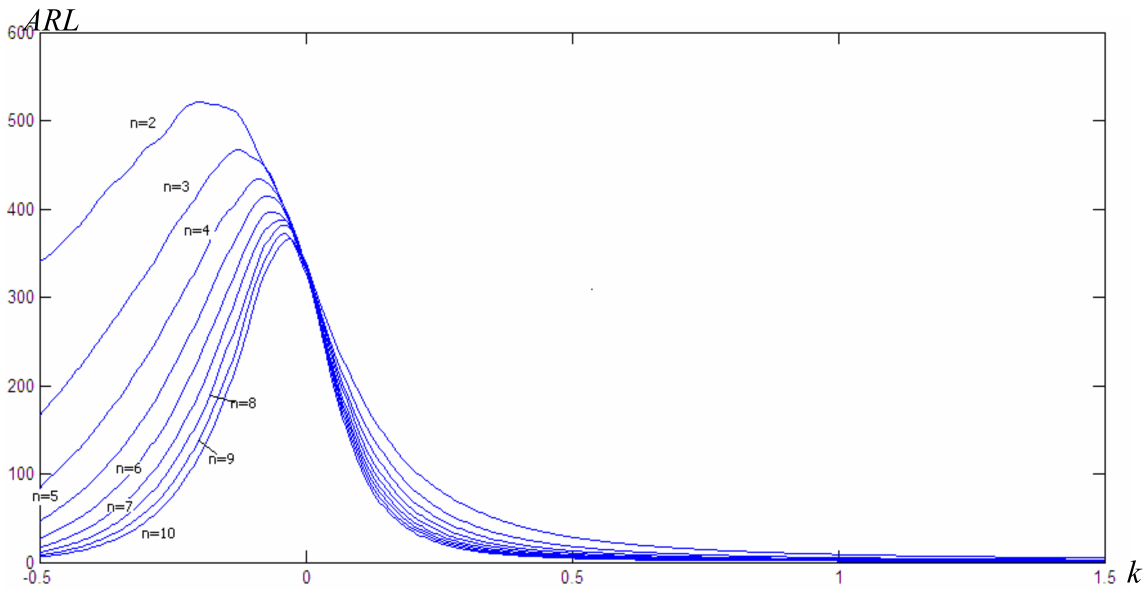


Figure 4.4. *ARL*-curves of the probability limit *S*-charts with different sizes of subgroups

By using the transformation in Fujioka and Maesono (2000) for *U*-statistics, we are able to normalize the sample standard deviations. Based on the transformed data, traditional 3-sigma limits can be applied and the new *ARL*-curves are drawn in Figure 4.5.

ARL

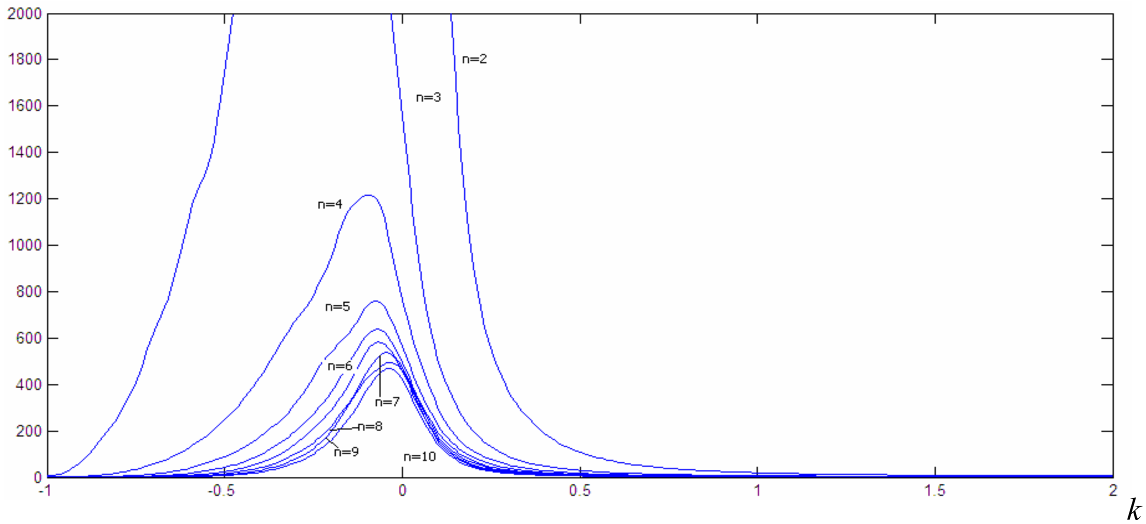


Figure 4.5. *ARL*-curves of the control chart on normalized *S* with different sizes of subgroups

In Figure 4.5, *ARL* is approximately symmetric with tail probability close to 0.0027. The sensitivity of the control chart improves dramatically when the subgroup sample size is larger than 3. The performance of such transformation is understandably bad when the subgroup size is only 2. This is due to the fact that sample standard deviation loses its accuracy greatly when the subgroup is too small to provide sufficient information. Adjustment multiplier can also be used to adjust the transformed data to standard Normal data, so that the standard properties of the control limits would be achieved. Range distribution has very similar properties as *S*-distribution, which we have discussed in Chapter 3. Normalization thus is a good application for Range distribution as well.

Exponential Distribution

Similar to the discussion above, the *ARL* distribution comparison for Exponential distribution can be seen the Figure 4.6 and Figure 4.7 below.

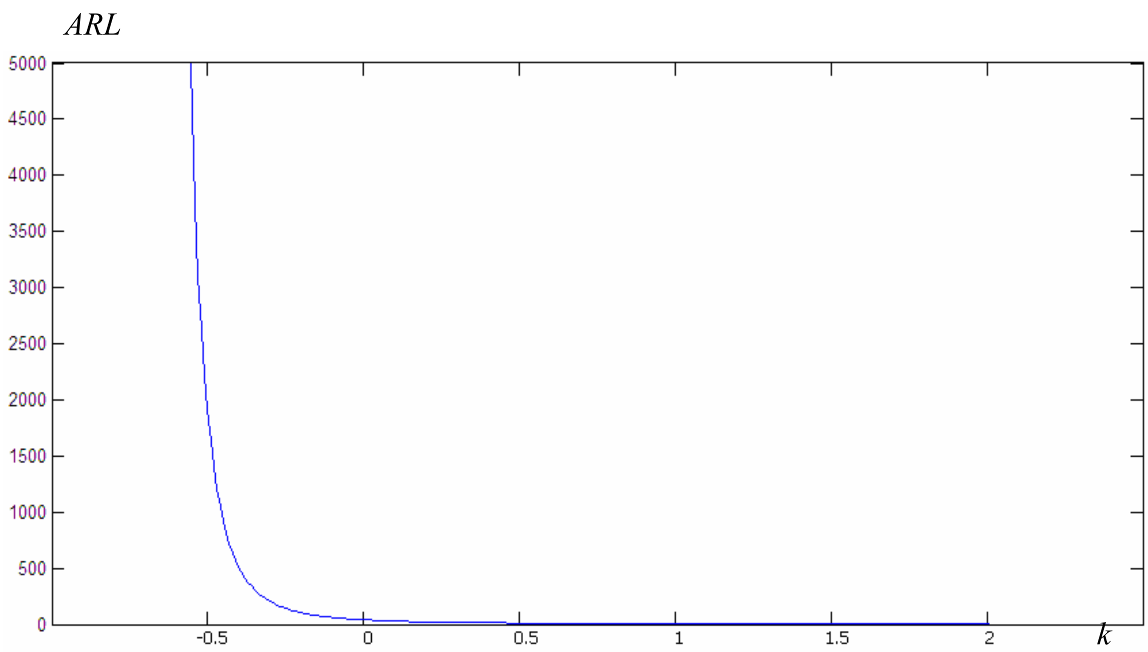


Figure 4.6. *ARL*-curve of traditional limits for Exponential-distribution

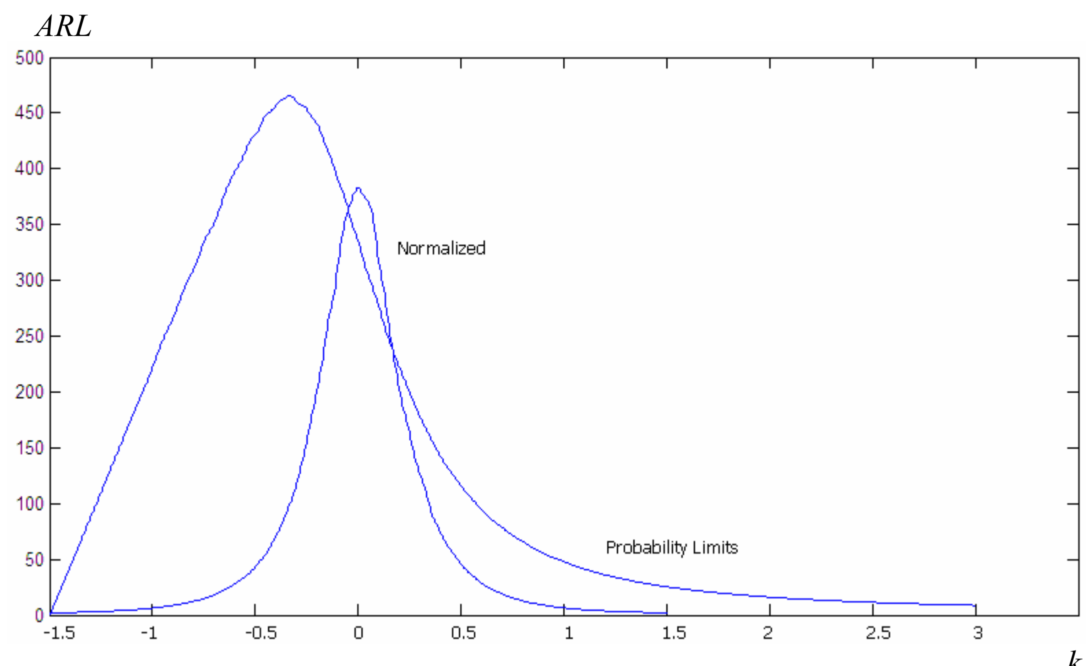


Figure 4.7. *ARL*-curves of probability limits and traditional limits on normalized data for Exponential-distribution

Figure 4.6 shows the *ARL* distribution for traditional control limits. Again, the phenomenon that out-of-control *ARL* is greater than in-control *ARL* happens. Under this circumstance, the control chart fails to signal at both sides. The tail probability for the traditional 3-sigma limits is approximately 0.0257, which is almost ten times the desired value of 0.0027 (indicated in Table 4.3).

λ	0.1	0.2	0.3	0.4	0.5	0.6
P_{tUCL}	0.025789	0.025933	0.0256	0.025525	0.025535	0.025847
P_{tLCL}	0	0	0	0	0	0
λ	0.7	0.8	0.9	1	2	3
P_{tUCL}	0.02555	0.025881	0.025526	0.025732	0.025551	0.025958
P_{tLCL}	0	0	0	0	0	0
λ	4	5	6	7	8	9
P_{tUCL}	0.025871	0.025525	0.025906	0.025613	0.025853	0.025814
P_{tLCL}	0	0	0	0	0	0
λ	10	11	12	13	14	15
P_{tUCL}	0.025732	0.025418	0.025553	0.025856	0.025674	0.025839
P_{tLCL}	0	0	0	0	0	0
λ	18	20	25	30	40	50
P_{tUCL}	0.025645	0.025877	0.025643	0.026033	0.026781	0.025676
P_{tLCL}	0	0	0	0	0	0

Table 4.3. Tail probability of traditional limits on Exponential distribution with some λ

Figure 4.7 has two curves that reflect the *ARL* distribution of the probability limits and the limits on normalized data. The normalizing transformation works very well. The tail-probability for transformed data is approximately 0.00262, which is very close to the target 0.0027. Moreover, the skewness has been reduced greatly compared to the probability limits. Although the two curves have very similar in-control *ARL*, the sensitivities to the process shift differ. The probability limits are much more biased.

t- distribution

t- distribution is another commonly seen distribution in quality control area. *ARL* distributions of the traditional limits and the limits on normalized data are drawn in the figures below with different sets of parameters. Denote the sample distribution as t_n. Figure 4.8 is the *ARL* distributions when *n* is set to 2. Figure 4.9 is the *ARL* distributions when *n* is fixed at 10. Figure 4.10 illustrates the *ARL* distributions when *n* set to 20.

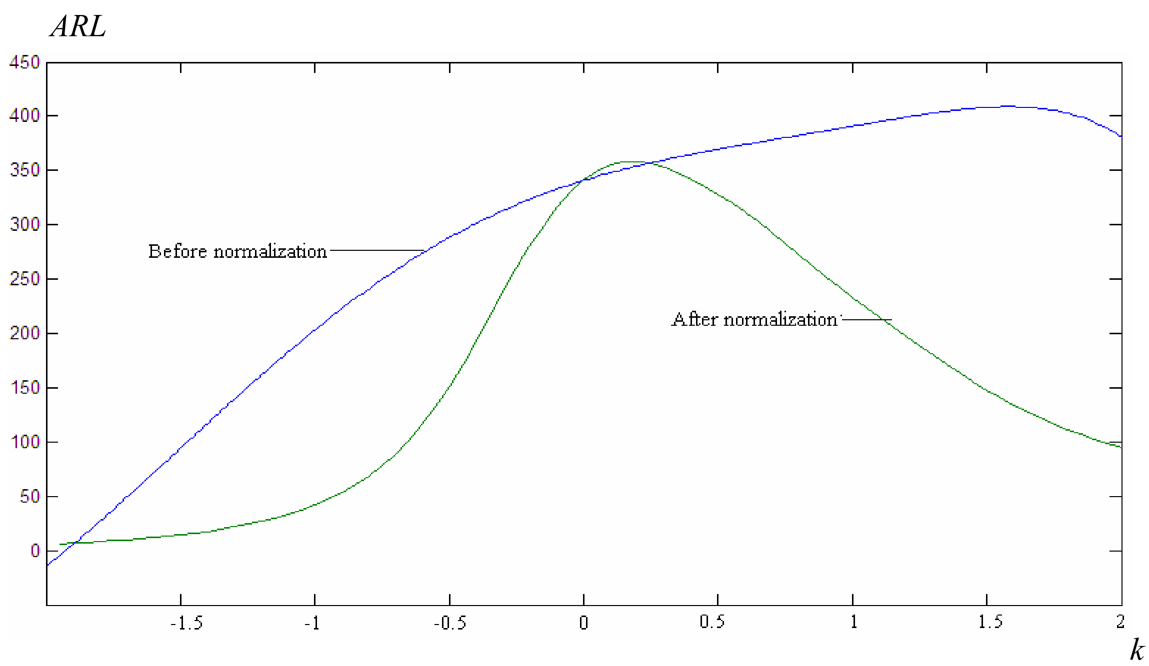


Figure 4.8. *ARL* distributions of control limits on *t*-distribution and normalized *t* for *n*=2

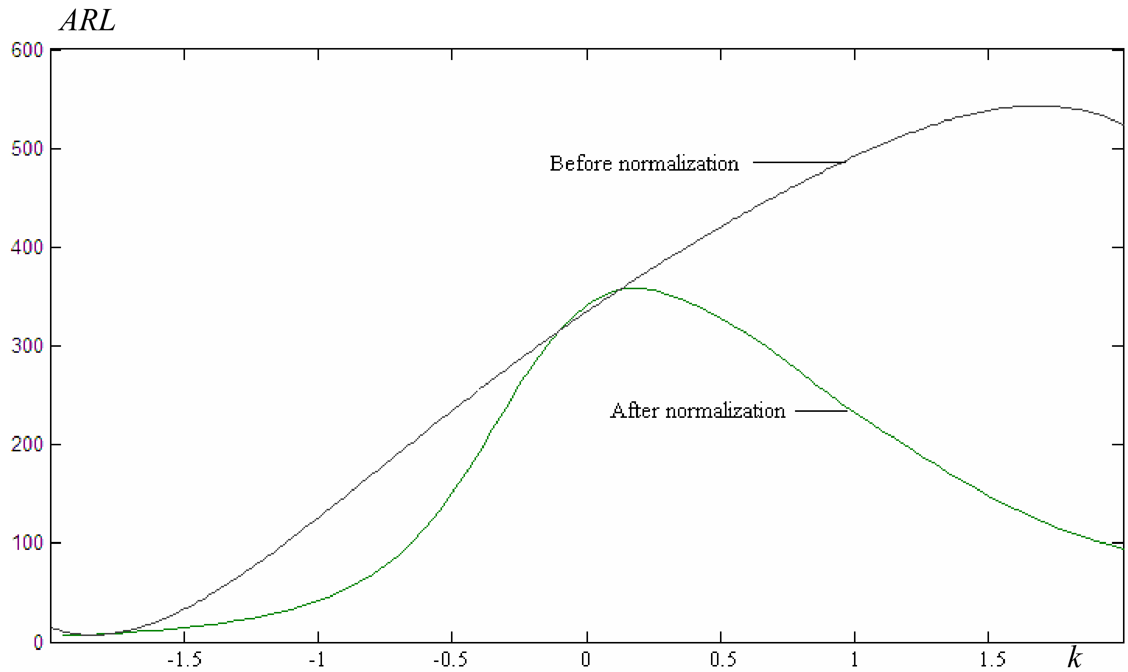


Figure 4.9. *ARL* distribution of control limits on *t*-distribution and normalized *t* for *n*=10

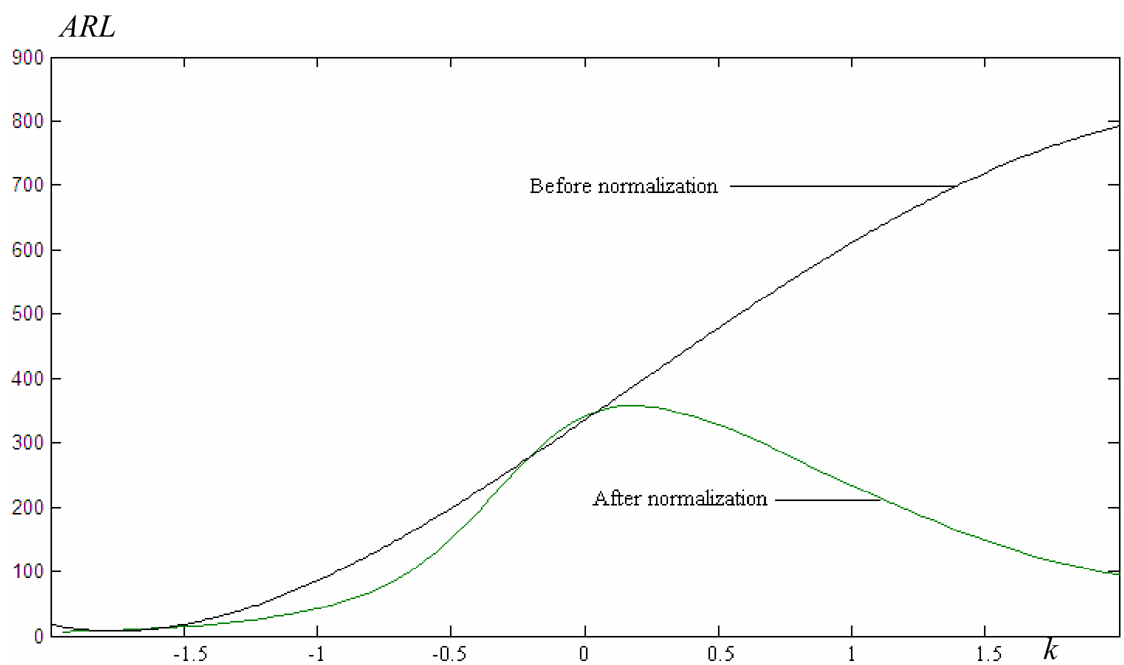


Figure 4.10. *ARL* distribution of control limits on *t*-distribution and normalized *t* for *n*=20

t-distribution has an opposite property to the distributions we have discussed before. It reacts very fast to the process deterioration when the shift is negative, but slow when the shift is positive. From Figure 4.8, 4.9 and 4.10, we can see that the higher the degree of freedom, the slower the reaction to positive deviation, but the faster the reaction to negative deviation. After transformation, the *ARL* distribution tends to be stable. Although the skewness does exist, the sensitivities to two sides shift are closer. The control chart performance improves greatly especially when the process tends to move far above the target. Another good point of the normalizing transformation used here is that the performance of the control chart does not change as much as before when the *t*-distribution parameter *n* varies.

4.2.3. Implementations

The application of normalizing transformation in control charting scheme involves six steps.

I. Specify the quality characteristic to be monitored

II. Specify the underlying distribution of the variable

III. Select the corresponding normalizing transformation form

IV. Transform the original data

V. Construct traditional Shewhart individual control chart based on the transformed data

VI. Monitor the process, detect the out-of-control signal and trace the assignable causes if necessary

One should be very careful about the selection of the transforming formulas. Although many accurate formulas have been given for certain distributions, our recommendation is those with simpler forms as it is of convenience in application. In general, we recommend Box-Cox transformation (Box and Cox, 1964) for their convenient applications when the underlying distribution is unknown. For more specific distributions, section 2.3.2 details the appropriate corresponding formulas.

4.3. Implementation Examples and Discussions

Two numerical examples are given below to help visualize the procedures described above.

4.3.1. Transformation on *t*-distribution: Application on individual chart

Suppose we have a group of 50 data, see Table 4.4. Say these data are representing the product yield from a process we are monitoring, which is denoted as X_i, i=1…50. We would like to know whether the process is in statistical control.

1.2621	0.2074	2.1287	7.2466	1.7631
0.5925	1.1198	6.8843	1.6392	0.7897
2.0059	1.4020	0.1103	1.3438	14.3413
1.2792	6.2193	2.8140	0.4515	0.7249
3.4651	2.5757	6.5523	1.3939	16.6878
1.1341	15.7454	0.2038	0.2832	5.4179

0.6581	4.1708	2.5775	3.4204	0.3771
3.5541	11.9934	6.3382	0.8586	2.8017
3.0380	0.7971	1.2738	2.1927	2.4429
3.7558	11.0146	4.7066	0.1788	1.2245

Table 4.4. Data from a Production Line

We follow the procedures to conduct the complete analysis.

I. Specify the quality characteristic on monitoring of concern

In this example, the variable of our concern is the product yield X_i shown in Table 4.4.

II. Specify the underlying distribution of the variable

There are numbers of approaches to check the distribution of the given data. First of all, we should check to determine whether the data the needs to be normalized. We plot the histogram and the Normal probability plot of the data as in Figure 4.11 and Figure 4.12 below.

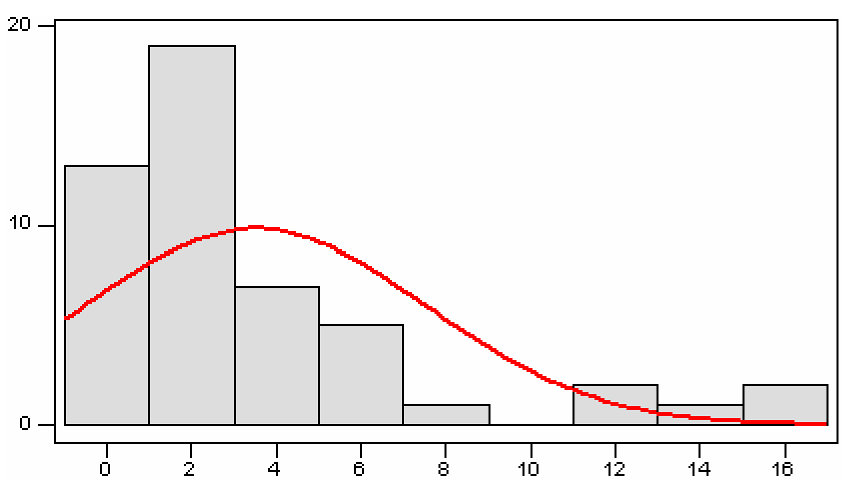


Figure 4.11. Histogram of the Data in Table 4.4

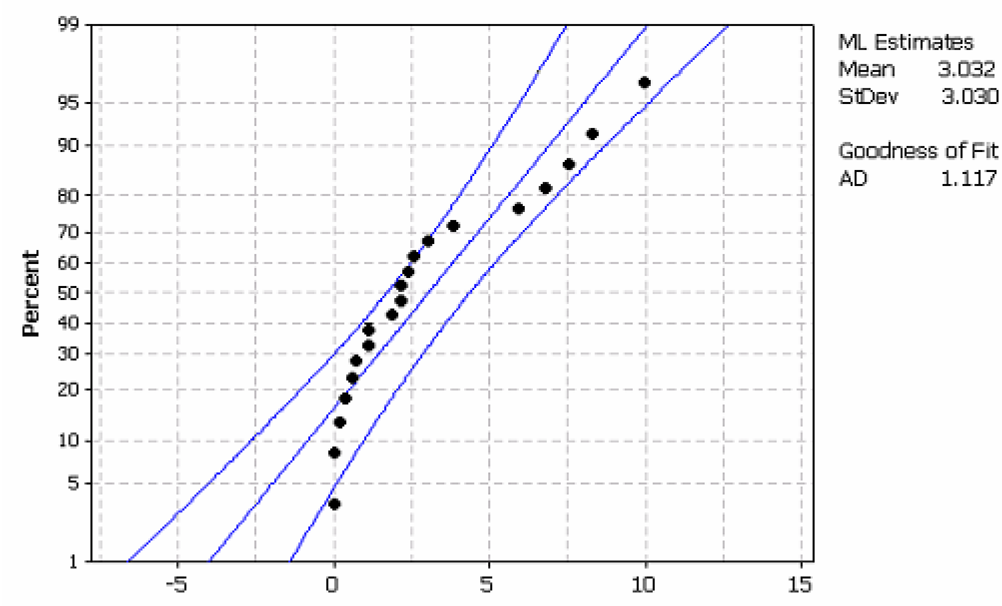


Figure 4.12 Normal probability plot of the data in Table 4.4

From the histogram, it is clearly seen that the underlying distribution is non-normal. Therefore, we use probability plot to estimate the distribution roughly. Care should be exercised that we may use normal probability plot to further investigate the normality if the histogram shows a bell-shaped curve. After several trials, we find out that the data fit exponential distribution well. The exponential probability plot is shown in Figure 4.13. The mean of the data is estimated as 3.503. Therefore, we conclude the underlying probability density function of the data is:

$$f(x)=\frac{1}{3.5}e^{-\frac{x}{3.5}},\ x>0 \tag{4.4}$$

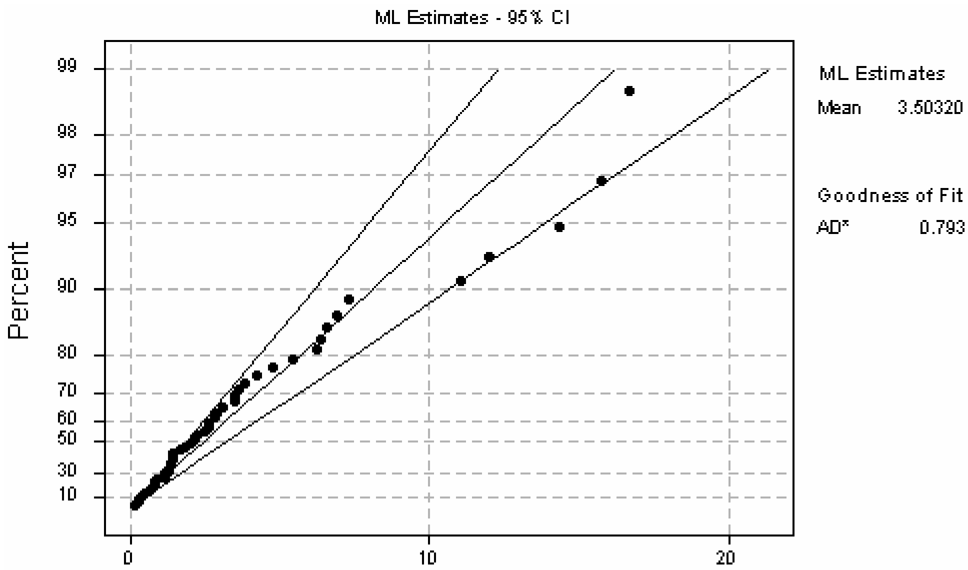


Figure 4.13. Exponential Probability Plot of the Data in Table 4.4

III. Select the corresponding normalizing transformation form

We refer to part2.1. and decide to use the transforming form in Kittlitz (1999). That is as follows,

$$Y = \sqrt[4]{X} \tag{4.5}$$

IV. Transform the original data

We use the Bailey transforming formula to calculate the transformed value. The transformed data are shown in Table 4.5.

1.05992	0.67487	1.20789	1.64072	1.15232
0.87737	1.02870	1.61981	1.13151	0.94268
1.19009	1.08815	0.57628	1.07667	1.94602
1.06350	1.57919	1.29519	0.81972	0.92271
1.36436	1.26684	1.59992	1.08658	2.02115
1.03196	1.99200	0.67188	0.72949	1.52566

0.90069	1.42907	1.26706	1.35994	0.78364
1.37304	1.86095	1.58669	0.96261	1.29377
1.32023	0.94487	1.06236	1.21687	1.25019
1.39212	1.82177	1.47291	0.65023	1.05193

Table 4.5. Kittlitz Transformation of Table 4.4 Data

V. Construct traditional Shewhart individual control chart based on the transformed data

We first plot the normal probability plot of the transformed data to see its normality. The histogram in Figure 4.14 shows a good normal fitting of the data.

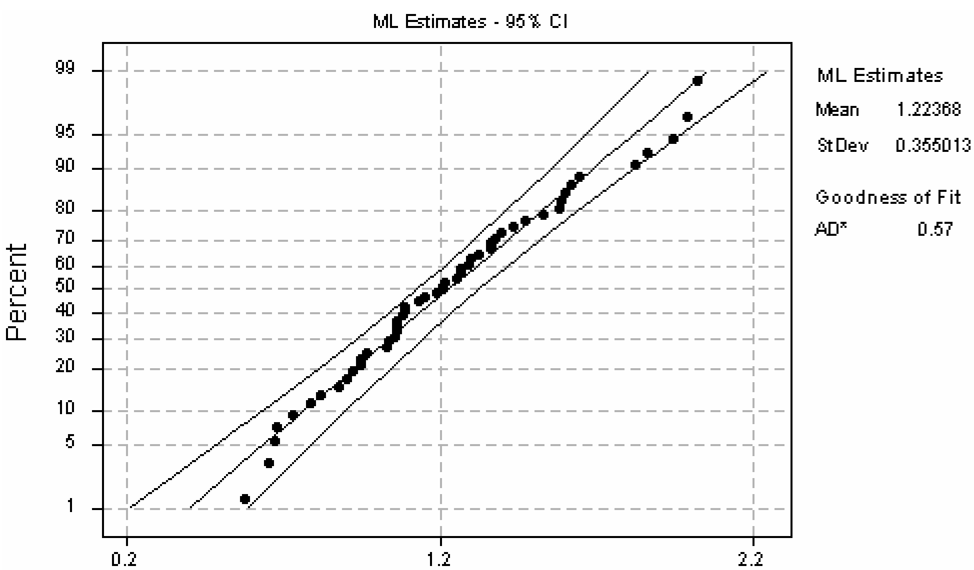


Figure 4.14. Normal Probability Plot of Data in Table 4.4

Then we directly apply the traditional Shewhart type individual chart on the transformed data, i.e. to use the formula for calculating control limits as below:

$$UCL = \overline{X} + 3\frac{\overline{MR}}{d_2}$$

$$CL = \overline{X}$$

$$LCL = \overline{X} - 3\frac{\overline{MR}}{d_2} \qquad (4.6)$$

From the control chart shown in Figure 4.15, the data indicate an in-control status. The features of both in-control and out-of-control *ARL*s are the same as the ones of Shewhart individual chart. Therefore, a false alarm will be given every 370 points when the process is in control and the speeds of signaling are expected to be the same when the process mean decreases or increases. Therefore, the individual chart based on normalizing transformed data excels the chart based on original data and performances as well as exact probability chart.

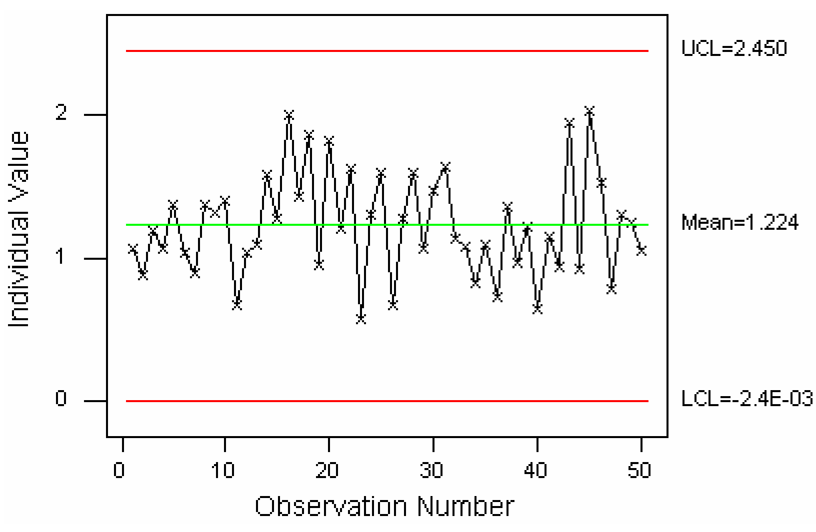


Figure 4.15. Traditional Individual Chart for Transformed Table 4.4 Data

VI. Monitor the process, detect the out-of-control signal and trace the assignable causes if necessary.

Further Discussions

For comparisons, the ideal out-of-control *ARL* for traditional Shewhart type control chart is plotted in Figure 4.16 below.

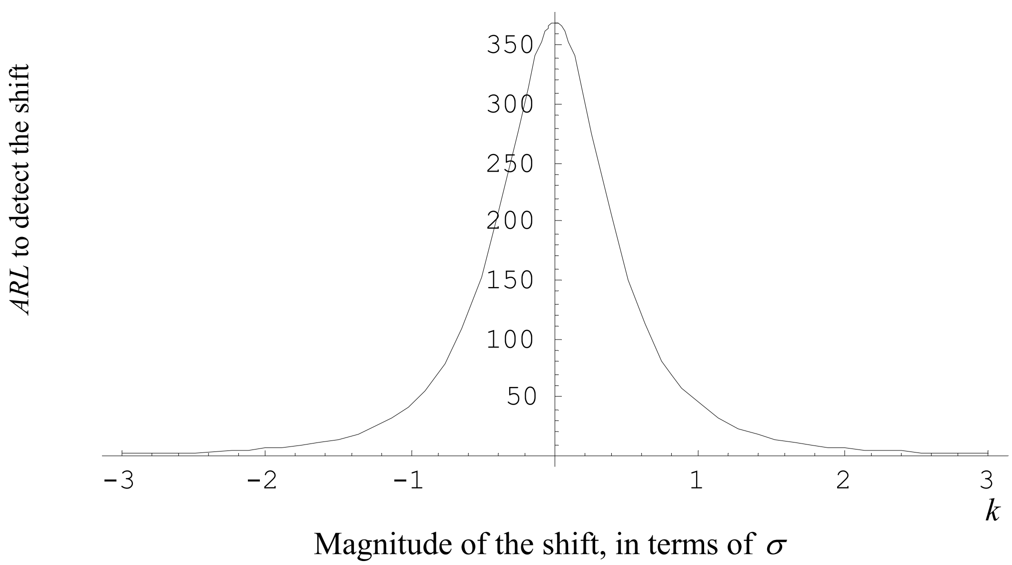


Figure 4.16. Ideal *ARL*-curve for Traditional Shewhart Type Control Chart

In order to show the effect of the transformation, we construct the traditional Shewhart individual chart on the original data first as shown in Figure 4.17. The control chart shows two points beyond the control limits though the data are actually in control. Calculating the in-control *ARL* as follows

$$ARL_1 = \frac{1}{P\{X_i < LCL, X_i > UCL\}} = 71$$

We find that under the traditional control limits, the data would arise a false alarms every 71 points. This is much smaller than the expected number of 370 which is achieved by the normalizing transformation. Therefore, control chart based on original data gives more frequent false alarm than that based on transformed data when the process is in statistical control.

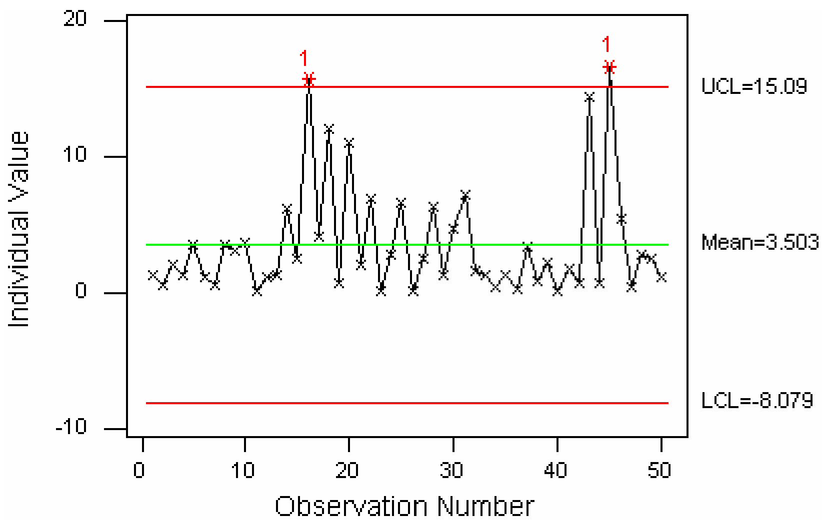


Figure 4.17. Traditional Individual Chart for *t*-distributed Data in Table 4.4

Then we construct the exact probability limits on the data as shown in Figure 4.18 by using 0.00135-percentile and 0.99865-percentile as upper and lower control limits, respectively.

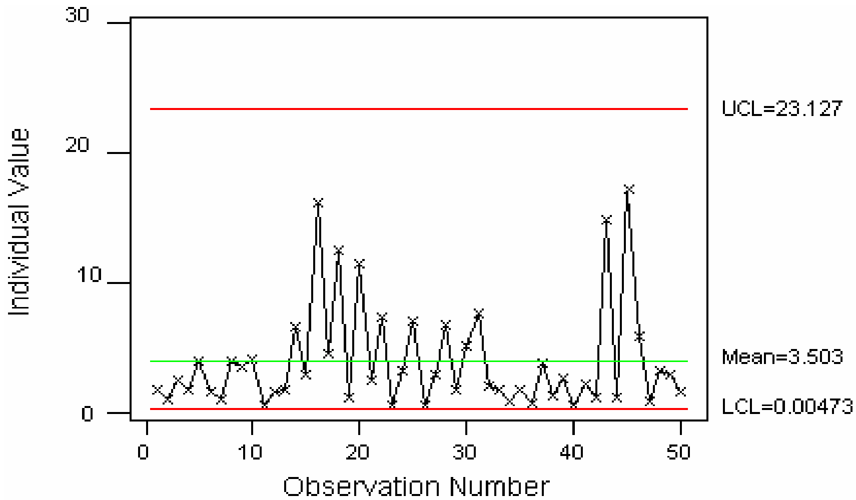
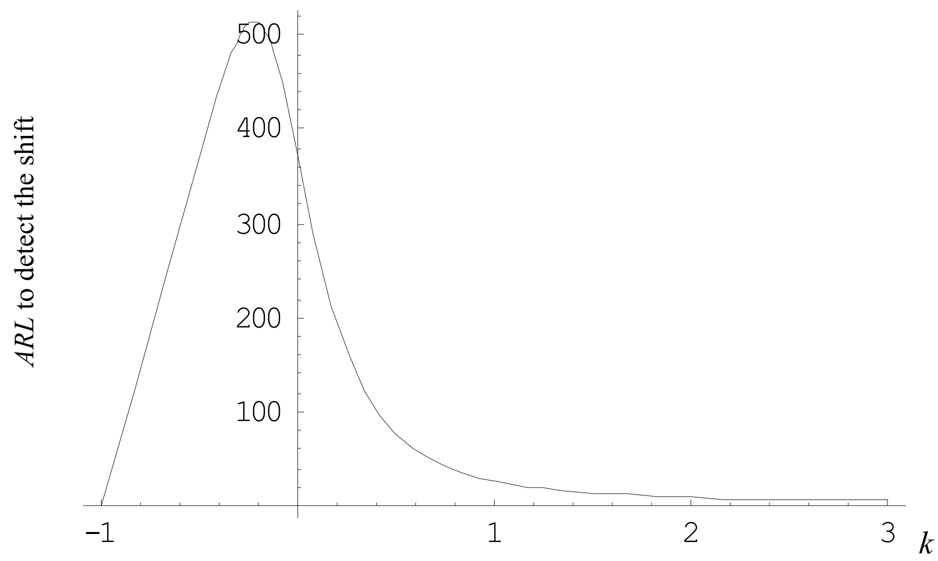


Figure 4.18. Exact Probability Limits for Exponential-distributed Data in Table 4.4

For these exact probability control limits, the in control *ARL* is 370 as desired. However, due to the asymmetric property of *t*-distribution, the out-of-control *ARL* is biased. Suppose the shift of the process mean is δ. We can simply compute the out-of-control *ARL* as:

$$ARL_1 = \frac{1}{P\{X < LCL, X > UCL\}} = \frac{1}{1 - e^{\frac{-0.00473}{\delta+3.5}} + e^{-\frac{23.127}{3.5+\delta}}}$$

The function is plotted below in Figure 4.19. It can be seen that the out-of-control *ARL* is biased for exact probability limits. When the process mean increases, it can detect the out-of-control signal quickly, especially when the shift is larger than one sigma (process standard deviation). However, when the process mean decreases, it takes even longer time to detect than to give a false alarm. When the decrease in the magnitude is larger than 0.2 σ approximately, the control chart signals faster. Therefore, the *ARL* is neither symmetric nor having its peak at in-control points. Thus, compared to Figure 4.16, transformation excels in the performance of the *ARL*.



Magnitude of the shift, in terms of σ

Figure 4.19. *ARL*-curve of Exact Probability Limits

4.3.2. Transformation on *U*-statistic: Application on *S*-chart

In this example, we have a set of 50 subgroups data each containing 5 observations generated from standard normal distribution. These data are the product yield from another process we are interested in, which is denoted as X_{ij}, i=1…50, j=1…5. We would like to monitor the process variability.

I. Specify the quality characteristic to be monitored

To monitor the process variability, we choose *S*-chart to use. Therefore, the concerned variable is sample standard deviation of each subgroup. We compute the values and fill in Table 4.6.

0.59763	1.03511	0.89162	1.22941	0.88331
0.96901	0.90043	1.56136	0.98573	0.77389
1.05484	1.11837	0.85740	0.75769	0.69058

0.98579	1.00698	0.85089	1.05963	0.62896
1.06666	1.37867	1.33374	0.77487	0.47154
0.56547	0.95779	0.79117	1.38009	0.84692
0.50424	0.77726	0.98545	1.15553	0.77087
1.41343	0.79711	0.57969	0.42040	1.43369
1.05562	0.52079	1.67372	1.03972	0.82874
1.73062	0.75216	0.57505	0.72048	0.54592

Table 4.6. Data of Sample Standard Deviation from N(0,1)

II. Specify the underlying distribution of the variable

The distribution of standard sample deviation for the dataset can be expressed as:

$$P\{S < x\} = \frac{1}{2^{\frac{5}{2}}\Gamma\left(\frac{5}{2}\right)} \int_0^{4x^2} e^{-\frac{r}{2}} r^{\frac{3}{2}} dr \tag{4.7}$$

III. Select the corresponding normalizing transformation form

As the distribution of sample standard deviation belongs to *U*-statistic family, we make use of the transformation in Fujioka and Maesono (2000). That is,

$$\hat{S}_n^* = \pi_1\left(\frac{\sqrt{n}\left(\hat{\theta} - \theta\right)}{\hat{\sigma}}\right), \tag{4.8}$$

where

$$\pi_1(s) = s + \frac{\hat{p}}{\sqrt{n}} s^2 + \frac{\hat{q}}{\sqrt{n}} + \frac{\hat{p}^2}{3n} s^3$$

We use the values of $\hat{p}$ and $\hat{q}$ selected in the example of Fujioka and Maesono (2000).

IV. Transform the original data

After calculating, the transformed data are shown in Table 4.7.

0.23360	-2.89568	1.49355	-1.03401	0.02050
1.06382	-0.51412	-0.86245	1.40172	0.48720
2.08847	-1.95010	0.27661	-0.25688	0.52153
-1.17305	0.29322	0.03460	-0.02832	1.98191
0.67464	-1.66449	-0.76999	-0.81813	0.49189
-2.68243	0.41715	0.56989	-0.40564	-1.21832
0.35335	1.48738	1.10730	0.90409	0.69515
0.19312	-0.14898	1.40447	-2.31095	-0.57037
0.77692	0.38390	0.53017	0.49277	-0.17997
1.47522	-0.47087	0.73191	1.35790	-0.34449

Table 4.7. Fujioka & Maesono Transformed Data in Table 4.6

Plot the histogram of the transformed data and it is roughly a bell-shape.

V. Construct traditional Shewhart individual control chart based on the transformed data

Now we can apply the traditional Shewhart individual chart on the transformed data. The control chart is constructed as follows in Figure 4.21.

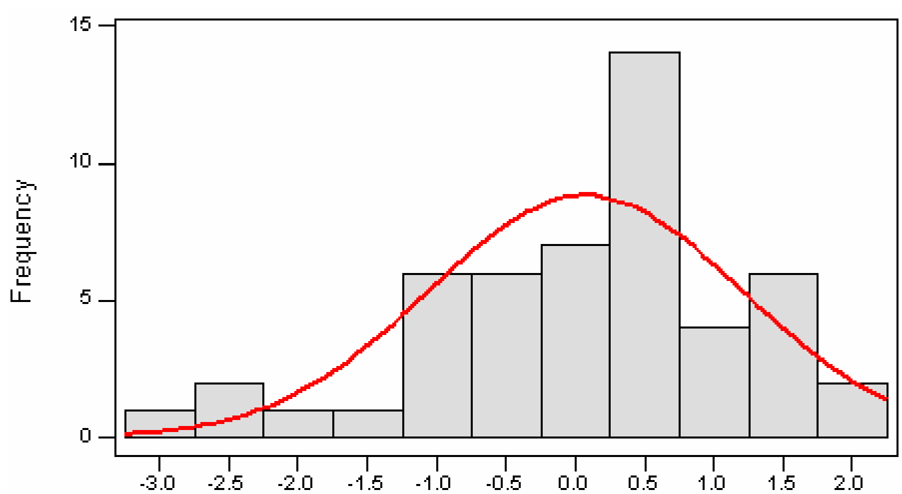


Figure 4.20. Histogram of Transformed Data in Table 4.7

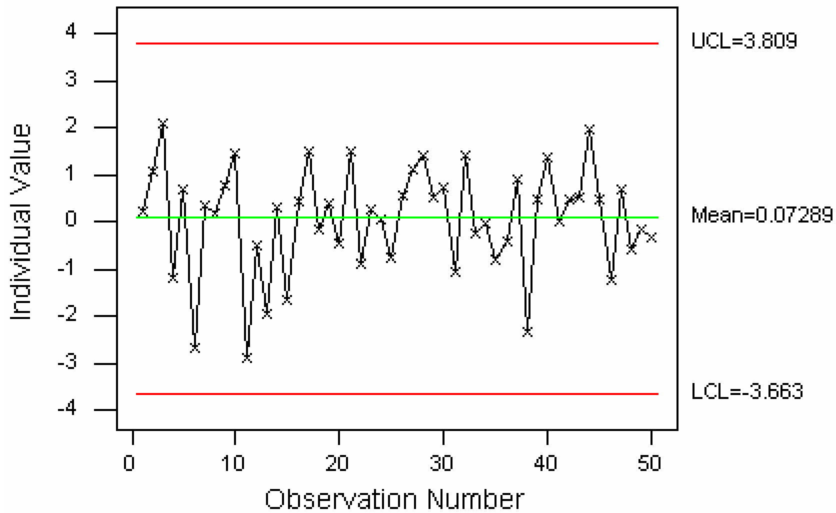


Figure 4.21. Traditional Individual Chart for Transformed Data in Table 4.7

As the data fulfill the normality assumption, the in-control average length is about 370. Moreover, the out-of-control *ARL* is drawn in Figure 4.22 against the shift in terms of σ. It is an unbiased-*ARL* and maximizes at 0.

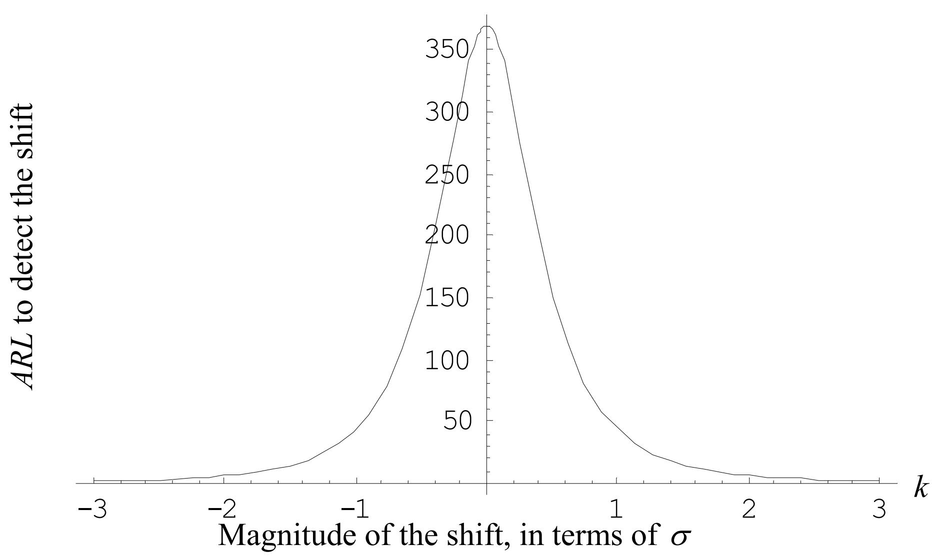


Figure 4.22. *ARL*-curve for Individual Chart on Transformed Table 4.7 Data

Therefore, the individual chart based on normalizing transformed data performs much better than traditional S-chart and exact probability limits as well.

VI. Monitor the process, detect the out-of-control signal and trace the assignable causes if necessary

Further Discussions

To compare the performance of the control limits, we first set up the traditional *S*-charts plotted in Figure 4.23. The in-control *ARL* can be calculated as:

$$ARL_1 = \frac{1}{P\{X_i < LCL, X_i > UCL\}} = 106$$

Similar to example 1, this in control *ARL* is shorter than the desired one as 370. This is due to the fact that the underlying distribution departs from normal.

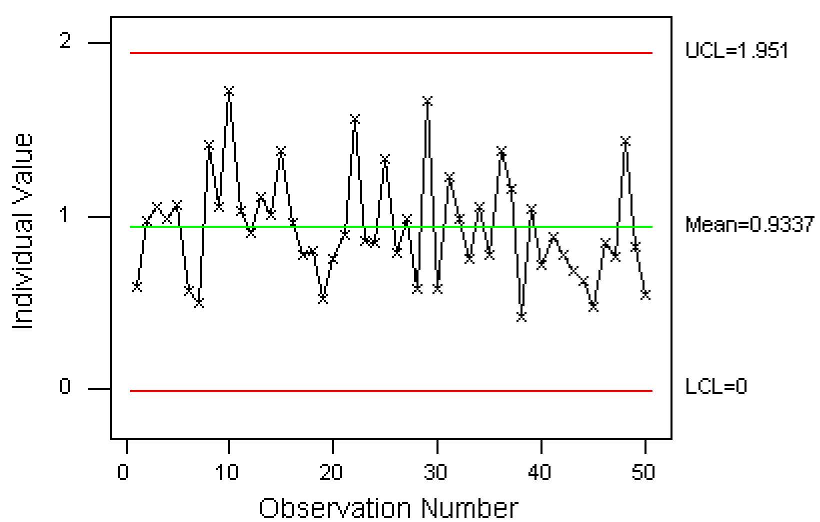


Figure 4.23. Traditional S chart for Data in Table 4.6

Now we look at the out-of-control *ARL* under these control limits. Figure 4.24 shows the curve of *ARL* versus the shift of the process variability from 0 to 3, in terms of σ .

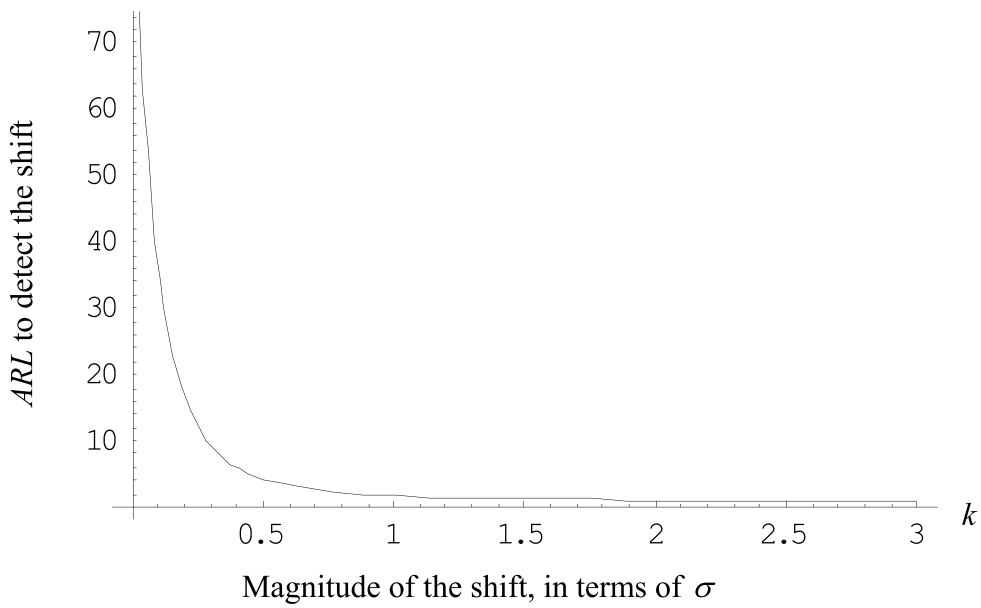


Figure 4.24. *ARL*-curve for control chart in Figure 4.21 under shift from 0 to 3 σ

From Figure 4.24, we can see that the curve falls within interval [0, 3] monotonically. This is a strong evidence of a biased-*ARL* phenomenon. It is subsequently easier to detect the upward process variability and the frequency of the signal does not change too much as the magnitude of the increase grows. Comparatively, it is almost impossible to detect the process variability deterioration. Therefore, traditional *S*-chart does not performance as well as individual chart based on normalizing transformation does.

Following the distribution of sample standard deviation, we can get the exact probability upper and lower control limits as:

$$UCL = \frac{\bar{s}}{c_4}\sqrt{\frac{\chi^2_{.99865}}{n-1}} = 2.09533$$

$$LCL = \frac{\bar{s}}{c_4}\sqrt{\frac{\chi^2_{.00135}}{n-1}} = 0.16152$$

These control limits ensure the in control *ARL* at the level of 370.

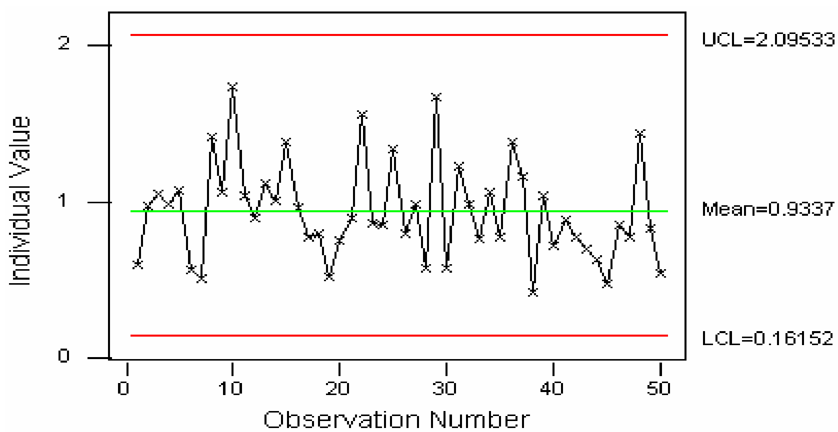


Figure 4.25. Exact Probability Limits for Data in Table 4.6

Now we study the performance of the *ARL* of the exact probability limits when the process is out of statistical control. Figure 4.26 below shows the behavior of the out-of-control *ARL* with magnitude of the process variability shift varying from -0.5 to 2.

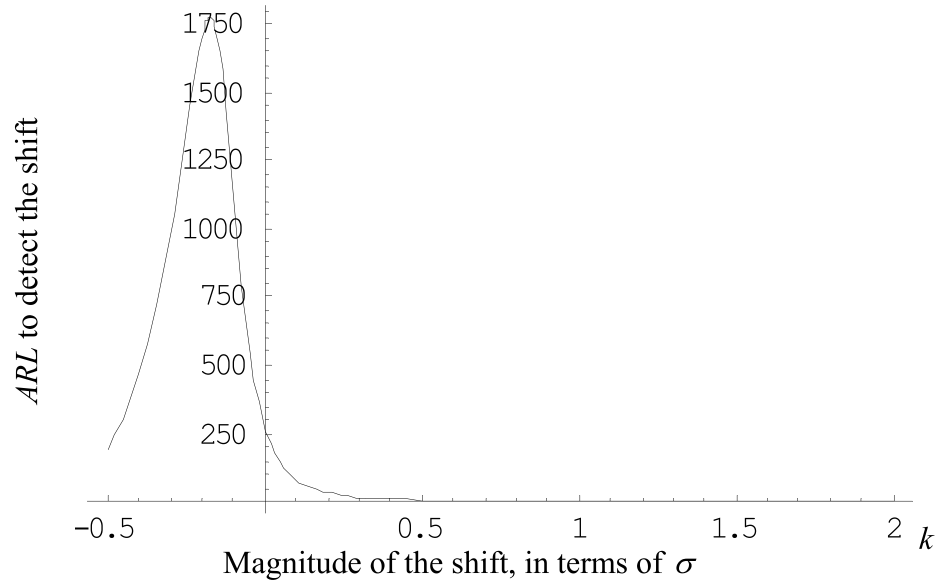


Figure 4.26. *ARL*-curve for *S*-chart under Exact Probability Limits on Data in Table 4.4 with shift from -0.5 σ to 2 σ

These figures show the strong indication of biased-*ARL* under the exact probability control limits. The curve shows an increasing-decreasing pattern. When the process variability increases, the *ARL* decreases steadily. However, when the process variability decreases, the *ARL* decreases faster than it does under up shift. Moreover, the maximum of the *ARL* is not located at shift equals 0. Instead, *ARL* achieves its peak around -0.2 σ. All these features are not good for conducting accurate process variability monitoring. Therefore, although exact limits chart ensures same in-control *ARL*, it is not as good as the chart based on transformation when it comes to detecting out-of-control signal quickly.

4.4. Conclusions

The theoretical achievements in normalizing transformations provide a way to monitor non-normal data more precisely. Very accurate transformation forms have been proposed in literatures on t, F, non-central t, non-central F, Exponential, Chi-square distributions. To widen the application area, transforming formulas have been presented on U-statistic and unifying density statistic as well. If the underlying distribution is detectable or approximately detectable, the appropriate transformations above could be done to apply the standard 3-sigma limits. Two numerical examples reveal that normalizing transformation could improve the performance of control limits greatly. Mostly, the control charts based on transformed data are better than the ones based on original data. Sometimes, the control charts based on normalized data excel the exact probability charts as well. This move would be of benefit to traditional Shewhart type control charts, for instance $\overline{X} - R$ chart and $\overline{X} - S$ chart. when non-normal or undesirably distributed data are encountered.

Chapter 5

Study on Normalization Transformation in Multivariate Charts

5.1. Introduction

In many situations, the simultaneous monitoring and control of two or more related quality characteristics are necessary. For example, physical dimensions of parts can be measured at several locations and various parameters of systems are typically derived simultaneously. In practice, one will frequently use a critical dimension as the one to record and track for purposes of quality control. In process-monitoring problems, several related variables are of interest; these problems are commonly called multivariate quality control. Harold Hotelling (1931) proposed a measure of distance that takes into account the covariance structure. Based on this measure, Hotelling T^2 chart, one of the most popular control charts for monitoring a multivariate normal process, has been developed. Hotelling T^2 chart is sometimes referred to as a multivariate Shewhart chart because it is a direct analog of the univariate Shewhart $\overline{X}$ chart. As long as the points plotted on the χ^2 or T^2 control chart fall below the UCL of the chart, the process is assumed to operate under a stable system of common causes, and hence, in a state of control. When one or more points exceed the *UCL*, the process is deemed out of control and an investigation is carried out to detect the special underlying causes.

The parameters of concern in multi-variate control charting are the process mean vector and covariance matrix. When these in-control parameters are known, the plotting statistics on a χ^2 control chart are directly derived from their values. The *UCL* of this chart is based on the chi-square distribution. Most of the time, however, these parameter values are unknown. In such case, they are estimated from some *m* initial subgroups of size *n* taken when the process is believed to be stable. The *UCL* of this chart is based on the *F* distribution. When a future subgroup is drawn from the process, Hotelling T^2 statistic is calculated using parameter estimates and is plotted on a T^2 control chart without changing the control limits.

Geometrically T^2 can be viewed as proportional to the squared distance of a multivariate observation from the target where equidistant points form ellipsoids surrounding the target. When the data are grouped, the chart displaying the distance of the group means from the targets can be accompanied by a chart depicting a measure of dispersion within the subgroups for all the analyzed characteristics. However, with only upper control limits, we can easily see that any Hotelling's multivariate type control procedure is not suitable to detect process deterioration or improvement. Moreover, the run rules for univariate Shewhart chart cannot be applied, so that the sensitivity of the Hotelling T^2 chart to the mean shift is not as good as desired. Employing *ARL* as the indicator, analysis and simulation runs show the merits of the simple usage of the transformation. Aware of that a control chart using an exact statistic can never be inferior to that using a transformed statistic, we develop a new methodology based on the normalizing approach which demonstrates good properties in certain cases. The subsequent simulation study shows that a valuable trade-off between the simplicity of use of the normalizing procedures and fair performances of the related charts can be achieved. A simple power transformation is recommended for the easy implementation of the method. However, one should be cautious when using our method since the transformed data do not have direct interpretations.

The present chapter is organized as follows. First, the multivariate process model and Hotelling T^2 control charts issues in particular are reviewed in Section 5.2. In section 5.3 we propose the simple power transformations used for T^2 statistics. The performance is evaluated by adopting Stephens (1974) modified *AD* test. Section 4 investigates the properties of the control charts based on transformation methods with vivid procedures on application. Section 5 studies the performance of the transformed control chart, where extensive simulation study results are presented. Conclusions are drawn in Section 6.

5.2. Multivariate Process Model and Control Limits Setting

5.2.1. Multivariate Process Model and control limits setting

Modern computers and data collection devices allow for more and more data to be gathered automatically. As a popular control statistic for monitoring multivariate processes, Hotelling T^2 has been extensively studied. We let $X_{ij} = \left(X_{ij1}, X_{ij2}, ..., X_{ijp}\right)'$ denote a $p \times 1$ vector that represents the p observations on the j-th component in the i-th subgroup, i=1,2,… and j=1,2,…,n. Assume that X_{ij}'s to be independent and identically distributed normal random variables with mean μ and covariance matrix Σ when the process is in control. Let $\overline{X}_i$ denote the average vector for the i-th subgroup, and let S_i denote the unbiased estimate of the covariance matrix for the i-th subgroup,

$$\overline{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$$

and

$$S_i = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij} - \overline{X}_i\right)\left(X_{ij} - \overline{X}_i\right)'$$

When the process is in control and the in-control process parameter values are known, the statistic plotted on the χ^2 control chart for the i-th subgroup is

$$\chi_i^2 = n\left(\overline{X}_i - \mu\right)' \Sigma^{-1}\left(\overline{X}_i - \mu\right) \tag{5.1}$$

When the process is in control this statistic has a chi-square distribution with p degrees of freedom (Montgomery, 1996). It is plotted on a χ^2 control chart with a UCL given by

$$UCL = \chi_{p,\alpha}^2 \tag{5.2}$$

where $\chi^2_{p,\alpha}$ is the $(1-\alpha)$th percentile point of chi-square distribution with p degrees of freedom and α is the probability of a false alarm for each subgroup plotted on the χ^2 control chart. The LCL is set to 0.

If the process parameter values are not known, data from m initial subgroups are collected when the process is believed to be in control. Then, pooling data from these m subgroups and assuming that the process was in control, unbiased estimates of the mean vector $\overline{\overline{X}}$ and the covariance matrix $\overline{S}$ are given by

$$\overline{\overline{X}} = \frac{1}{m}\sum_{i=1}^{m}\overline{X}_i$$

and $$\overline{S} = \frac{1}{m}\sum_{i=1}^{m}S_i$$

respectively. A T^2 control chart is constructed and the m initial subgroups at sample size n are tested retrospectively to ensure that the process was in control when these initial subgroups were drawn. When a future subgroup is drawn from the process for on-line monitoring purposes, the statistic plotted on the control chart is

$$T_i^2 = n\left(\overline{X}_i - \overline{\overline{X}}\right)' \overline{S}^{-1}\left(\overline{X}_i - \overline{\overline{X}}\right) \tag{5.3}$$

It follows theorem (iv) in Appendix II that

$$\frac{m(n-1)-p+1}{m(n-1)p}T_i^2 \sim F_{p,m(n-1)-p+1}$$

This statistic is plotted on a T^2 control chart. The UCL of this control chart is given as

$$UCL_{T^2} = \frac{p(m-1)(n-1)}{mn-m-p+1}F_{\alpha,p,mn-m-p+1} \tag{5.4}$$

where $F_{v_1,v_2,\alpha}$ is the $(1-\alpha)$th percentile point of the F distribution with v_1 and v_2 degrees of freedom, and α is the specified acceptable false alarm probability for each subgroup plotted on the T^2 control chart. If the process parameters are estimated from a reasonably large number of initial subgroups, the usual practice for constructing T^2 control chart is to use UCL_{χ^2} instead of the exact UCL_{T^2}. The LCL, same as χ^2 control chart, is set to 0. For Phase II, the control limits are:

$$UCL_{T^2} = \frac{p(m+1)(n-1)}{mn-m-p+1} F_{\alpha,p,mn-m-p+1}$$

$$LCL = 0 \tag{5.5}$$

When a point falls over upper control limits, the control chart indicates an out-of-control signal.

5.2.2. Non-normality Drawbacks on Hotelling T^2 chart

The drawback of Hotelling's multivariate type control charts lies in the original conception of any Hotelling's multivariate type control procedure. For instance, Hotelling T^2-control chart only has the upper control limits, which cannot be used to detect process deterioration or process improvement and the run rules are not applicable on the Hotelling T^2-control chart.

Let us elaborate on these limitations by considering the following example. Two quality characteristics are to be jointly monitored. We collected 20 preliminary data with sample size of 5 for each variable. We will consider this to be Phase I, establishing statistical control in the preliminary samples, and calculate the upper control limit from formula

(5.2). Assume that the process is in control in Phase I, we would proceed to Phase II. Phase II control limits could be calculated from formula (5.3). If $\alpha = 0.001$, the upper control limit is:

$$UCL_{T^2} = \frac{p(m+1)(n-1)}{mn-m-p+1} F_{\alpha,p,mn-m-p+1}$$

$$= \frac{2\times(20+1)(5-1)}{20\times5-20-2+1} F_{0.001,2,20\times5-20-2+1}$$

$$=16.0526$$

We simulate the data from a new process, of which one quality characteristic shifted from its original parameters. The T^2-statistics are calculated in Table 5.1.

4.6583	4.3236	5.6251	8.9789	2.7281
8.9812	7.1102	8.4883	2.5609	6.9204
15.8137	4.6497	2.6264	6.4381	1.9433
13.1988	2.3933	1.7526	2.4732	1.8584

Table 5.1 T^2-statistics for the shifted process 1

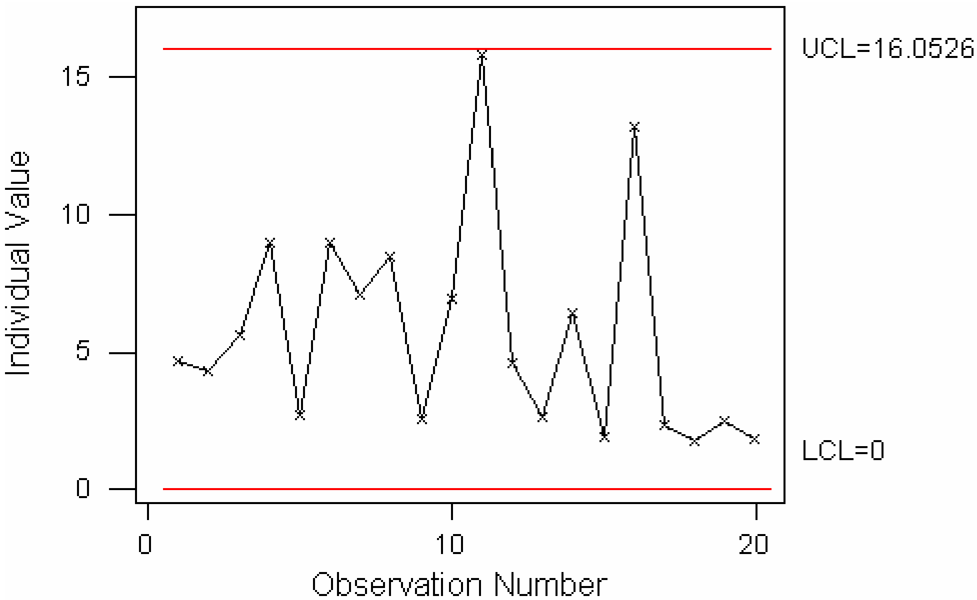


Figure 5.1. The Hotelling T^2 control chart for data in Table5.1 (Phase II)

Figure 5.1 shows an in-statistical-control state although the process level shifted. This is due to the fact that the process variance decreased. Since only the distance of the vectors of observed means from the target values are taken into account, the reduction of variance counterbalances the increase of the process average. Thus, the change of the process cannot be expressed by the T^2-statistics. Therefore, there is no out-of-control signal on the Hotelling T^2 chart.

We can study this problem by further investigation of the process *ARL* performance. Given the type II error β, the out-of-control *ARL* can formulated as,

$$ARL_1 = \frac{1}{1-\beta} = \frac{1}{\int_{UCL_0}^{\infty} f_{F_1}(x)dx}, \tag{5.6}$$

where β is type II error, UCL_0 is the UCL set when the process was in control, and $f_{F_1}(x)$ is the probability density function of T^2 computed from the shifted process. Assume we monitor two quality characteristics simultaneously, denoted by A and B. The upper control limit is set by using the 99.9-percentile. We vary the mean value of the distribution of variable A to detect the ARL of the T^2 chart. Then we decrease the variance of variable A by 20% while varying its mean. The pattern of the ARL is shown in Figure 5.2 below.

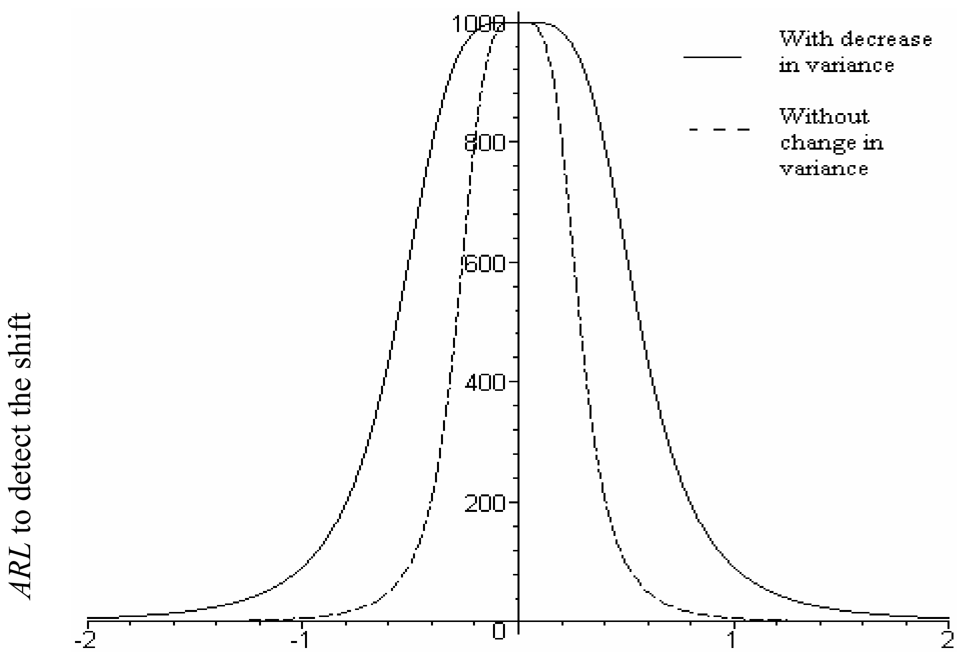


Figure 5.2. ARL performance of Hotelling T^2 chart
Magnitude of the shift, in terms of σ

The x-axis is the magnitude of the shift of the mean of variable A. The y-axis is the value of the ARL. The dash line represents the ARL without any change in variance. The solid line represents the ARL with 0.2-sigma decrease in the variance of variable A. The figure shows the properties of the Hotelling T^2 chart settings. When there is no change in the process's variance, it has a good responding rate of detection. This is very desirable, as we expect to have out-of-control signals sooner when the shift of the process is increasing. However, when there is the reduction of the process's variance, we have to wait for more sampling points until we can get a signal. Moreover, it would lead to the illusion that the process is improved. This results in misjudgments on the process performance.

Moreover, usually zone rules are considered to be applicable only for the $\overline{X}$ chart under the normality assumption. It is not considered to be useful for the Hotelling T^2 chart, although it is a direct analog of traditional univariate Shewhart type control chart. That is, we cannot divide the area within the control limits into zones. Thus, it is impossible to apply the other run rules though the idea of using information from more than one point should be applicable as well. It is known that in case of simultaneous shift of mean and decrease in variance, Hotelling T^2 chart has difficulty to signal. The privilege of applying the other run rules is not only to increase the sensitivity of the control chart, but also to detect the process deterioration as soon as possible. Thus, if these run rules could be applied, the Hotelling T^2 control chart would be able to detect the shift of both process mean and process variance.

0.46285	0.4685	0.70812	2.16781	2.19452
0.82333	1.50731	1.09594	0.02018	0.8372
0.31802	0.10281	1.2022	1.08346	4.63336
3.24242	3.46168	2.45687	7.0098	0.41252

Table 5.2. T^2-statistics for the shifted process 2

We change the process variables setting by reducing the variance of the two variables. Now the new data come from the shifted process. The T^2 statistics are calculated in Table 5.2 above. Figure 5.3 is the Hotelling T^2 control chart on the data below. The figure shows an un-random pattern which is that the majority of the points are close to the lower control limit. However, since run rules cannot be applied on this type of the charts, no out-of-control signal will be detected. If we assume there is a one-sigma limit, it can be considered as out-of-control state according to the run rules.

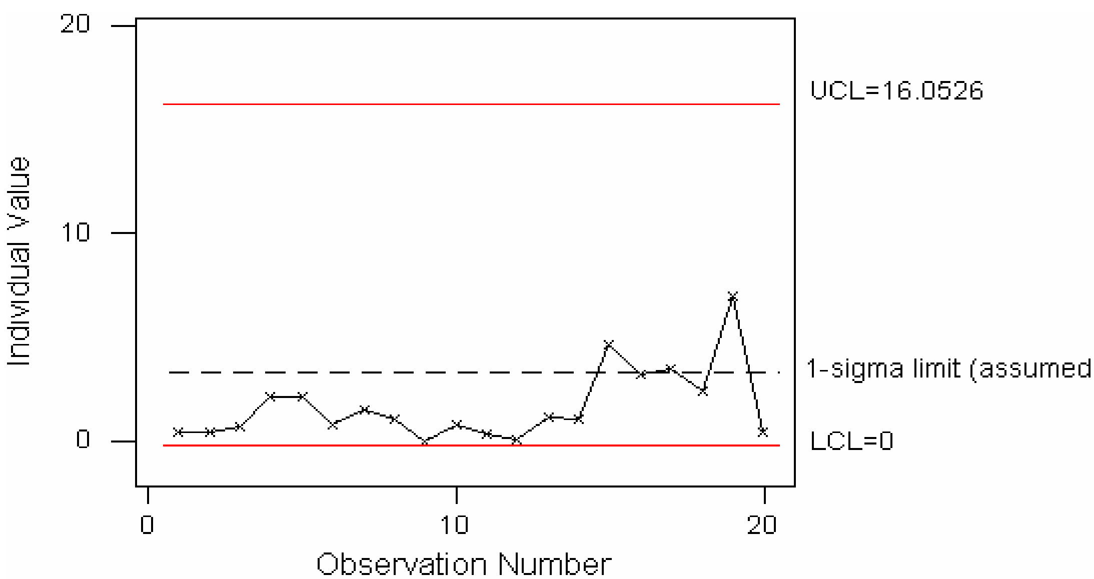


Figure 5.3. Hotelling T^2 chart for variance reduced process data

Another important drawback in direct application of Hotelling T^2 control charting method is that, the established control limits in Phase I is not the exact control limits for new observations in phase II. Indeed, when the new observations are collected and new plotting statistics are computed, the distributions of the new statistics are no longer the ones used to estimate the previous *UCL*. Thus, the *ARL* is not well applicable and the properties are not as we expected.

Following the approach in Chapter 4, we are thinking of normalizing the multi-variate charts. The advantages of this approach are well as the ones mentioned in Chapter 4, including expected *ARL* properties, feasible application of run-rules, etc. However, we have also noticed that this approach has its intrinsic defaults. Designed based on the real distribution of the original statistics, *Hotelling*-T^2 charts are explainable with the pattern they reflect. For normalized charts, since the plotting statistics are no more the original statistics, interpretation is difficult. Yet, Normalizing transformations does not intrinsically change the relative positioning of the data values. What they do is to re-express the data, while preserving the rank order, to a scale that allows the normal distribution to serve as a benchmark for interpretation and judgment (Alwan L., 2000). Therefore, normalized charts can be used to monitor the process of the original statistics. A good trade-off between the simplicity and the accuracy of the transformation is valuable for industry applications. Other approaches should be employed to investigate into the out-of-control interpretation, if a signal does happen.

5.3. The Best Normalizing Transformation

5.3.1. χ^2-distribution

Given known distribution parameters, the plotting statistics in Shewhart type multivariate control charts follow χ^2-distribution. We, therefore, first review the transforming methods for χ^2-distribution at first.

On the basis of Konish's (1981) study, Taneichi et al (2002) derived a concrete normalizing transformation. They assumed that the mean, variance and third moment about the mean of a χ_p^2-distribution are expanded as

$$E(T_p) = \mu + \frac{1}{2}\mu_1 + o\left(\frac{1}{p}\right)$$

$$V(T_p) = \frac{1}{p}\sigma^2 + o\left(\frac{1}{p}\right)$$

and

$$E\left[(T_p - E(T_p))^3\right] = \frac{1}{p^2}\nu + o\left(\frac{1}{p^2}\right) \tag{5.7}$$

They obtained the following transformation form as

$$g_1(T_p) = \begin{cases} \dfrac{\sqrt{p}}{\sigma}\left[\dfrac{\mu}{\eta}\left\{\left(\dfrac{T_p}{\mu}\right)^{\eta} - 1\right\} - \dfrac{1}{p}\left(\mu_1 + \dfrac{1}{2}\sigma^2\xi\right)\right], \eta \neq 0 \\ \dfrac{\sqrt{p}}{\sigma}\left[\mu \log\dfrac{T_p}{\mu} - \dfrac{1}{p}\left(\mu_1 + \dfrac{1}{2}\sigma^2\xi\right)\right], \eta = 0 \end{cases} \tag{5.8}$$

Let X_p be distributed as chi-square distribution with p degrees of freedom. Let $T_p = \dfrac{X_p}{p}$,

then

$$g_1(T_p) = \sqrt{\frac{p}{2}}\left[3\left\{(T_p)^{1/3} - 1\right\} + \frac{2}{3p}\right]. \tag{5.9}$$

Since the transforming formula proposed by Taneichi et al (2002) is shown to have the best performance under the comparative studies, we adopt formula (5.9) to apply normalizing transformation on χ^2 -distribution. For each given value $X_i \sim \chi_p^2$, the transformed values can be expressed as

$$Y_i = \sqrt{\frac{p}{2}}\left[3\left\{\left(\frac{X_i}{p}\right)^{1/3} - 1\right\} + \frac{2}{3p}\right] \tag{5.10}$$

This transforming formula is polynomial and hence can be also expressed in the form of

$$Y_i = a_1 X_i^{\ 1/3} - a_0 \tag{5.11}$$

where a_1 and a_0 are the coefficients. This formula transforms an *F*-distributed value to standard normal. To simplify, we just ignore the location parameter and scale parameter. That is, the expression (5.12) below can be used to transform an *F*-distributed value to a normally distributed value:

$$Y_i = X_i^{\ 1/3} \tag{5.12}$$

The power of the transformation is $\frac{1}{3}$, which is a very convenient number that can simply be handled by a pocket calculator. By using this simple power transformation, a control chart for the Hotelling χ^2 statistics can be constructed in the traditional way.

5.3.2 *F*-distribution

Normalizing transformation on *F*-distribution

When all of the parameters of the variates' distributions are unknown, it is proved that the

Hotelling T^2 statistics follow F-distribution. We now examine the normalizing transformations for F-distribution and find out the best one to use.

Isogai (1999) introduced two types of formula for power transformation of the F variable to transform the F distribution to a standard normal distribution. One formula is an extension of the Wilson-Hilferty transformation for the χ^2 variable, which is:

$$T_1(F) = \frac{sign(h)\left(F^h - E\left[F^h\right]\right)}{\left(Var\left[F^h\right]\right)^{\frac{1}{2}}} \tag{5.13}$$

where F is distributed as $F_{u,v}$, $sign(\cdot)$ is a function that gives the sign of its argument, and $h = -\frac{1}{3}\frac{u-v}{u+v}$. The other type is based on the median of the F distribution, which is:

$$T_2(F) = \frac{sign(h)\left\{X^h - \left[\tilde{F}(0.5)\right]^h\right\}}{\left(Var\left[F^h\right]\right)^{\frac{1}{2}}} \tag{5.14}$$

where $\tilde{F}(0.5)$ denotes the median of $F_{u,v}$. Isogai (1999) combined those two formulas and derived a simple formula for the median of the F distribution, which leads to a power normal family from the generalized F distribution. This transformation is defined by:

$$T_3(F) = \frac{sign(h)\left\{X^h - \left[\tilde{F}(0.5)\right]^h\right\}}{\left[2h^2\left(\frac{1}{u} + \frac{1}{v}\right)\right]^{\frac{1}{2}}} \tag{5.15}$$

When u and v have the same degrees of freedom, the limiting form of $T_3(F)$ is:

$$\frac{\ln F}{(4u)^{\frac{1}{2}}}, \tag{5.16}$$

where we have u=v.

From Isogai's study, the best transforming form is the formula (5.9) above. That is, given the data X_i from $F_{u,v}$, the transformed value is expressed as:

$$Y_i = \frac{sign(h)\left\{X_i^{\ h} - \left[\widetilde{F}(0.5)\right]^h\right\}}{\left[2h^2\left(\frac{1}{u}+\frac{1}{v}\right)\right]^{\frac{1}{2}}} \tag{5.17}$$

This formula transforms an *F*-distributed value to standard normal. To simplify, we just ignore the location parameter and scale parameter. That is, the expression (5.18) below can be used to transform an *F*-distributed value to a normally distributed value

$$Y_i = X_i^{\ h} \tag{5.18}$$

Thus, a power transformation is proposed for normalizing *F*-distribution with a transforming power parameter as:

$$h = -\frac{1}{3}\frac{u-v}{u+v} \tag{5.19}$$

It is shown that the power of the transformation *h* depends only on the degrees of freedom of the distribution.

Simple power transformation

We calculated the *h* values for some common settings of the triple (*p, n, m*) to assist the construction of the transformation Hotelling T^2 chart (some of the *h* values can be found in Appendix III). Here, *p* is the number of the quality characteristics being monitored simultaneously; *n* is the sample size of the observed subgroups; and *m* is the number of the preliminary subgroups. Recall the expressions for the numerator and denominator degree of freedom of an *F*-distribution, we can determine the two degrees of freedom by calculating $u = p$ and $v = mn - m - p + 1$, respectively. Therefore, the transforming parameter is computed as

$$h = -\frac{1}{3} \cdot \frac{2p + m - mn - 1}{mn - m + 1} \tag{5.20}$$

From Appendix III, we notice that the h values are very close to 0.3, which is a simple constant. Inspired by this, we consider that the transformation may have the possibility of being turned into simple constant power transformation. Thus a simple power transformation could be used as,

$$Y_i = X_i^{\,c} \tag{5.21}$$

where c is a constant. To find out the feasibility of using formula (5.21) as the transformation power for this simplicity, we conducted the simulation study.

A goodness of fit test

The Anderson-Darling test (Stephens, 1974) is used to test if a sample of data came from a population with a specific distribution. The test is defined as:

H_0 : The data follow a specified distribution.

H_1 : The data do not follow the specified distribution.

The test statistic, i.e. the quantitative measure of the goodness-of-fit used, is the Anderson-Darling test statistic, which is defined as:

$$A^2 = -N - S, \tag{5.22}$$

where

$$S = \sum_{i=1}^{N} \frac{(2i-1)}{N} \left[\ln F(Y_i) + \ln\left(1 - F(Y_{N+1-i})\right) \right]$$

N is the total number of the observations, F is the cumulative distribution function of the specified distribution, Y_i is the i-th observation among the ordered data. The AD^* statistic is based on a comparison of $F(x)$ with the empirical distribution function $F_n(x)$. By adopting modified test statistic (Stephens, 1974) with unknown distribution parameters, the test procedure is as follows:

1. Sort the given values in ascending order.
2. Calculate the required test statistic by applying the formula (5.21).
3. Test by using preceding statistic, calculate the modified statistic AD^* as:

$$AD^* = AD \cdot \left(1 + \frac{4}{N} - \frac{25}{N^2}\right) \tag{5.23}$$

4. Reject H_0 at a chosen level of significance if AD^* exceeds the significant point given.

With 99% significance level therefore, if the AD^* of using a constant as the transformation power is less than the critical value, we may use the constant to construct a simple power transformation. The critical value is 1.091.

Therefore, we construct a test to evaluate the performance of two different normalizing transformations for F-distribution. Calculated AD^* values under different degrees of freedom for both transformations. If the both of the below criteria are met, the conclusion can be drawn that the two different normalizing transformations do not have significant difference in terms of normalizing performance:

1. P_1 {AD^* values of transformation A is less than critical value| AD^* values of transformation B is less than critical value} ≥ p_1.

2. P_2 {AD^* values of transformation B is larger than critical value| AD^* values of transformation A is larger than critical value} $\geq p_2$.

p_1 and p_2 are the accepting confidence levels of performance comparison. The higher the p_1 and p_2 are, the more confident we are about the conclusion that the two transformations do not make different performances. In the simulation we use 95% for both p_1 and p_2. This probability can be approximately calculated as:

1. Compute the number of times that AD^* values of transformation A is less than critical value given that AD^* values of transformation B is less than critical value. Denoted by P_0 .
2. Compute the number of times that AD^* values of transformation B is less than critical value. Denoted by P_B.
3. $P_1 = \dfrac{P_0}{P_B}$.
4. Compute the number of times that AD^* values of transformation B is larger than critical value given that AD^* values of transformation A is larger than critical value. Denoted by P_{00} .
5. Compute the number of times that AD^* values of transformation A is larger than critical value. Denoted by P_A.
6. $P_2 = \dfrac{P_{00}}{P_A}$

If $P_1 \geq 0.95$ and $P_2 \geq 0.95$, we consider the two transformations do not have different performances.

Simulation and results

The simulation is conducted as follows. For each F-distribution with different degrees of freedom settings, 100,000 data are generated. The degrees of freedom (u, v) are calculated as $u = p$ and $v = mn - m - p + 1$. We make p vary from 2 to 50, m from 10 to 200 and n from 3 to 50. The process is repeated 1000 times. Then transformations by using constant power (specifically 0.3, $\frac{1}{3}$ and 0.35) and h-values as the power are applied onto the data to compute a AD^* value. The final AD^* value for each setting is the average of the 1000 trials.

Simulations show a satisfactory performance of using 0.3 as the constant power of the transformation. The data show an obvious domination; when the difference between the first degree of freedom and the second degree of freedom is large, using 0.3 as the power achieves smaller AD^* value than using h-value. This translates to that when the u-v is moderately large; using 0.3 as transforming power excels in normalization on F-distribution. More specifically, there are three different cases:

1. For u=2, 3, 4, 5, simple power transformation with parameter 0.3 has better performance than with parameter h when v is larger than 30.
2. For u=6, 7, 8, 9, 10, simple power transformation with parameter 0.3 has better performance than with parameter h when v is larger than 60.
3. For u larger than 10, simple power transformation with parameter 0.3 has better performance than with parameter h when v is larger than 80.

Special care has been focused on certain range of u and v in the study of simulation results. Recall the expression of u and v in terms of m, n and p, we have:

$$\begin{cases} u = p \\ v = mn - m - p + 1 \end{cases} \tag{5.24}$$

Substitute (*m, n, p*) with the settings that are commonly used in industry, we are able to draw more reliable and practical conclusions for real application. For this purpose, we have investigated the results with *m* varies from 15 to 50, *n* from 3 to 10 and *p* from 2 to 10. The calculated AD^* values for Box-Cox transformation, power transformation by using *h*-value and power transformation by using 0.3 are presented in Appendix III. In this simulation, 2592 different settings of degrees of freedom are tested. There are 1289 times that *h*-value power transformation has AD^* value smaller than the critical value, 1273 times that 0.3 power transformation has AD^* value larger than the critical value, 1241 times that both *h*-value and 0.3 power transformation has AD^* value smaller than the critical value, and 1212 times that both *h*-value and 0.3 power transformation has AD^* value larger than the critical value. Thus we have,

$$P_0 = 1241,\ P_{00} = 1212,\ P_A = 1273,\ P_B = 1289,$$

$$P_1 = \frac{P_0}{P_B} = 0.963,$$

$$P_2 = \frac{P_{00}}{P_A} = 0.952.$$

Therefore, we conclude that 0.3 can be used instead of real *h*-value for the power transformation to achieve same level of performance under common settings. Further investigation has shown that when we use reasonable sample size of the subgroups such as 3 to 5 or larger and moderate number of preliminary data such as 25 to 30 or larger, together with less than 10 simultaneously monitored variables, we can just use 0.3 as the transforming power. While for more than 10 variables, slightly more data need to be collected with larger sample size in order to apply the simple power transformation. The suggestion could be 30 to 40 data with sample size of 3 to 5. Therefore, our proposed transforming formula is:

$$Y_i = X_i^{0.3} \tag{5.25}$$

Care should be exercised that using 0.3 as the transforming power parameter value is for common proper settings mentioned above on transforming multi-variate chart only. Otherwise, if given very few samples at relatively large sample size, it is better to adopt h-value, that is:

$$Y_i = X_i^{h} \tag{5.26}$$

However, this will not be the common practice in industry as the setting is not recommended for constructing reliable control limits.

5.4. Implementation and Examples

5.4.1. Transformation selection and Implementing Procedures

When the parameters' values are known, the plotting statistics follow χ^2 distribution. Therefore, our corresponding transforming scheme is to normalize the χ^2-statistics. Having the χ^2-values, we can just simply use the formula (5.10) on each value.

When the parameters of the underlying distribution are unknown, the statistics plotted on multivariate chart follows F-distribution. Therefore, we need to know the number of quality characteristics p, the number of preliminary data m and the sample size n of each subgroup. For each set of (p,m,n) we can find out the corresponding transforming parameters in Appendix II. Substituting the values of the parameters into formula (5.16), we can get the transforming formula for the data. To transform the F-values, just simply use the formula on each value.

The procedures of constructing control charts for multivariate analysis with transformation involve the normalizing transformation and individual chart construction. The basic steps are as follows:

I. Find out the values of p, m, and n.

p is the number of quality characteristics that are monitored simultaneously. m is the number of preliminary data at sample size n.

II. Calculate T^2 (χ^2) statistics for subgroups.

When the distribution parameters of the variables being monitored are known, we use formula (5.1) to calculate the χ^2 statistic, which is:

$$\chi_i^2 = n\left(\overline{X}_i - \mu_0\right)' \Sigma_0^{-1}\left(\overline{X}_i - \mu_0\right),$$

where μ is the mean vector and Σ is the covariance matrix.

When the parameters are unknown, we use formula (5.3) to calculate the corresponding T^2 statistic, which is:

$$T_i^2 = n\left(\overline{X}_i - \overline{\overline{X}}\right)' \overline{S}^{-1}\left(\overline{X}_i - \overline{\overline{X}}\right),$$

where $\overline{\overline{X}} = \frac{1}{m}\sum_{i=1}^{m}\overline{X}_i$ and $\overline{S} = \frac{1}{m}\sum_{i=1}^{m} S_i$ are the unbiased estimates of the mean vector and the covariance matrix.

III. Specify the distribution of the T^2 (χ^2) statistics. Get the transforming parameters or transforming formula.

The distribution of χ^2-statistics is χ^2-distribution. We use formula (5.13) to perform the normalizing transformation. That is,

$$Y_i = X_i^{1/3}$$

The distribution of T^2-statistics is *F*-distribution. Based on the settings of (*m,n,p*) we can choose from using formula (5.25) or formula (5.26).

IV. Transform the T^2 (χ^2) statistics for subgroups.

By using the selected formula in step III, we substitute the calculated T^2 (χ^2) statistics for X_i. The Y_i is the plotted statistic.

V. Construct traditional individual chart on the transformed data.

Based on data Y_i, we just follow the procedures of constructing traditional Shewhart type chart to get the individual chart.

VI. Interpret the control chart.

The interpretation of the individual chart is similar to the traditional ones. The beyond-control-limits points indicate the out-of-control state. Special further investigations, i.e. the upward or downward shifts, are needed as the plotting statistics do not have the same meaning as the ones in traditional individual chart.

5.4.2. An implementation example

In this section, an example is used as an illustration of the control charts discussed in the previous section. It uses the data set in Montgomery (1996) for illustrating the parameters unknown case.

Example: Montgomery (1996)

The data are tensile strength and diameter of the textile fiber, which are to be jointly controlled. The engineer has decided to use n=10, 20 preliminary samples have been taken.

I. Find out the values of p, m, and n.

In the process, we have two quality characteristics to control simultaneously. Therefore, p=2. We collect 20 preliminary data of sample size equals 5. That is m=20, n=10.

II. Calculate T^2 statistics for subgroups. The data are shown in Table 5.3.

2.16	2.14	6.77	8.29	1.89
0.03	7.54	3.01	5.92	2.41
1.13	9.96	3.86	1.11	2.56
0.70	0.19	0.00	0.35	0.62

Table 5.3. T^2 values of the data in the example

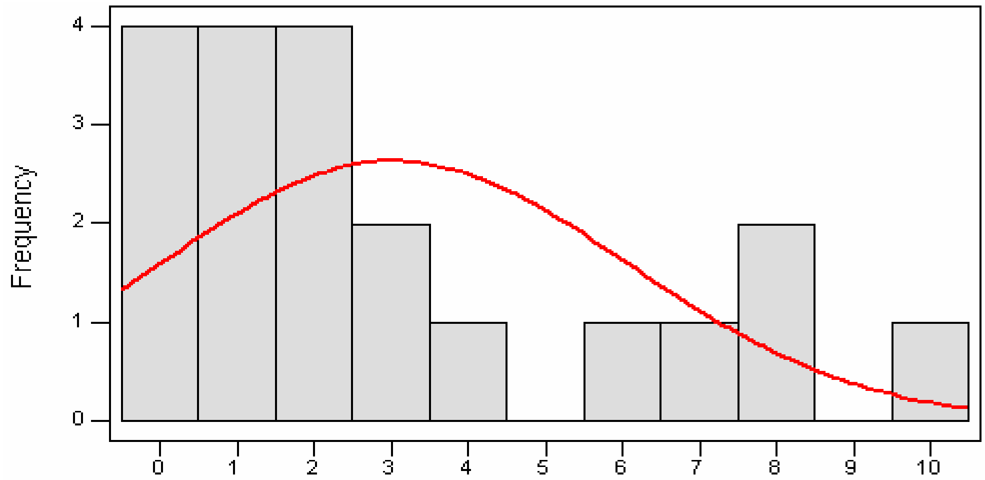


Figure 5.4. Histogram of T^2 statistics data in the example

III. Specify the distribution of the T^2 statistics. Get the transforming parameters.

In this example, the T^2 values follow F-distribution. p=2, m=20, n=10. Therefore, the degrees of freedom are 2 and 179. According to our decision rule, with p=2, since 179-2>>30, we can use 0.3 as the power for the transformation. That is, the transforming formula is:

$$Y_i = X_i^{0.3}.$$

IV. Transform the T^2 statistics for subgroups. The transformed data are shown in Table 5.4.

1.2599	1.256389	1.774911	1.886107	1.210427
1.037346	1.992865	1.499603	1.031803	1.325782
0.34925	1.833207	1.391778	1.704891	1.301982
0.898523	0.607612	0	0.729828	0.866398

Table 5.4. Transformed data in Table 5.3

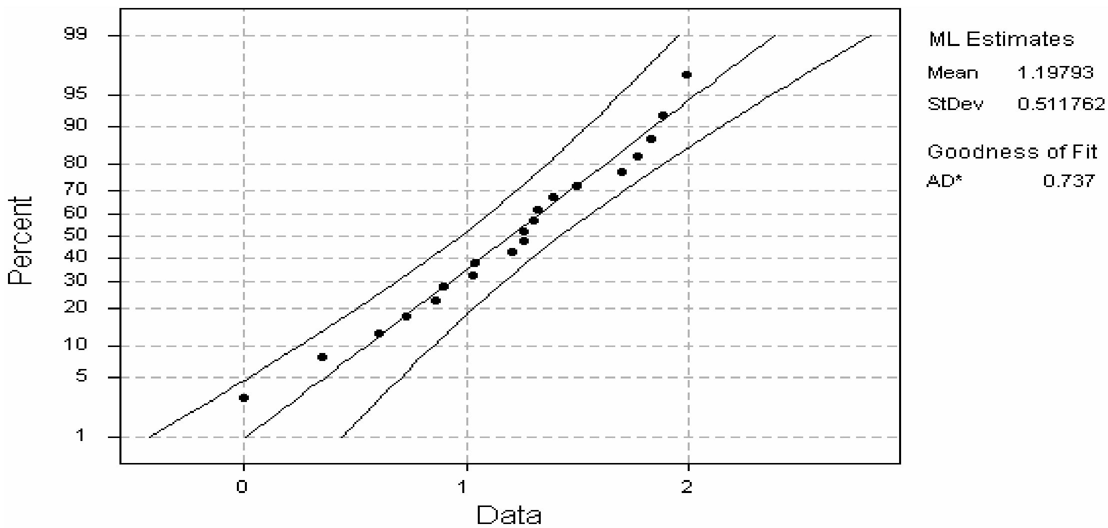


Figure 5.5. Normal probability plot of transformed data in Table 5.5

The probability plot and *AD* value above show a good approximation of normal distribution.

V. Construct traditional individual chart on the transformed data and analyze the control chart. After the transformation, the control chart in Phase I is:

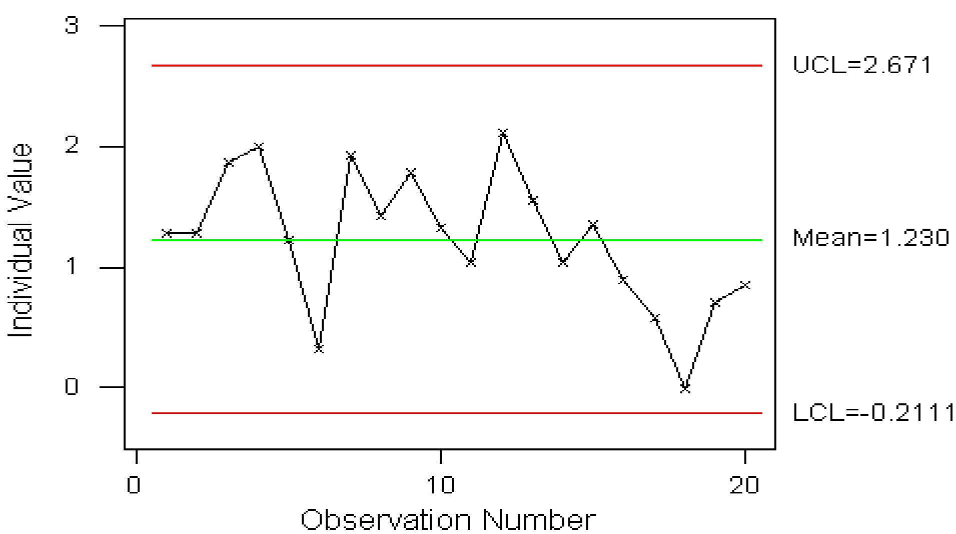


Figure 5.6. Individual chart for transformed data

All the points are lower than the upper control limit. Therefore, from the chart, we may draw the conclusion that the process is in statistical control. Thus, we may use the set of data to set the control limits for Phase II. In this example, we may use as well the control limits (-0.2111, 2.671) for future monitoring.

5.5. Performance Comparisons

5.5.1. An investigation of the *ARL* and discussions

As we discussed in the beginning of this chapter, traditional Hotelling T^2-control chart only has the upper control limit. The setting of this upper control limit is that of exact probability limits. Considering that the traditional 3-sigma limits provides the feasibility to apply the run rules and achieves the desired *ARL* properties, we think of using normalizing transformation approach as the alternative. Normalizing transformation is substantially helpful as for this aspect. The transformed data follow or approximately follow normal distribution, so that the ideal *ARL* properties could be achieved.

Recall that the desired features of traditional 3-sigma-limit control chart are that it detects the process improvement and deterioration at the targeted rate, and it signals both directions of process change at same sensitivity level. For multivariate chart, the exact probability limit can be used to achieve the desired run length when the process is in control, like *R*-chart, being discussed in Chapter 3. However, due to the skewed nature of *F*-/ Chi-square distribution, although traditional Hotelling-T^2 chart has symmetric *ARL*, it lacks ability to detect the process shift with desired probability. The proposed control chart scheme consists of introducing the characteristics in one chart, so that it has the same ability as the traditional 3-sigma-limit chart. This, indeed, is done by compromising part of the accuracy. For instance, in Phase II, the plotting statistics are following *F*-distribution. *ARL* distributions of the traditional limits and the limits on normalized data are drawn in the figures below with different sets of parameters. Denote the sample distribution as $F_{u,v}$. The mean and the variance of the *F*-distribution can be expressed as follows:

$$\mu = \frac{v}{v-2},$$

$$\sigma^2 = \frac{2v^2\left(v+u-2\right)}{u\left(v-2\right)^2\left(v-4\right)}.$$

Figure 5.7 is the *ARL* distributions when v is fixed at 400. Figure 5.8 is the *ARL* distributions when v is fixed at 100. Figure 5.9 is the *ARL* distributions when v is fixed at 20. In each figure, u varies from 2 to 10 and 20. In these figures, d represents the magnitude of the shift of T^2 in terms of the standard deviation of T^2. The control limits are set by using the exact probability limits of *F*-distribution. For the normalized curve, traditional *Shewhart* type control limits are set.

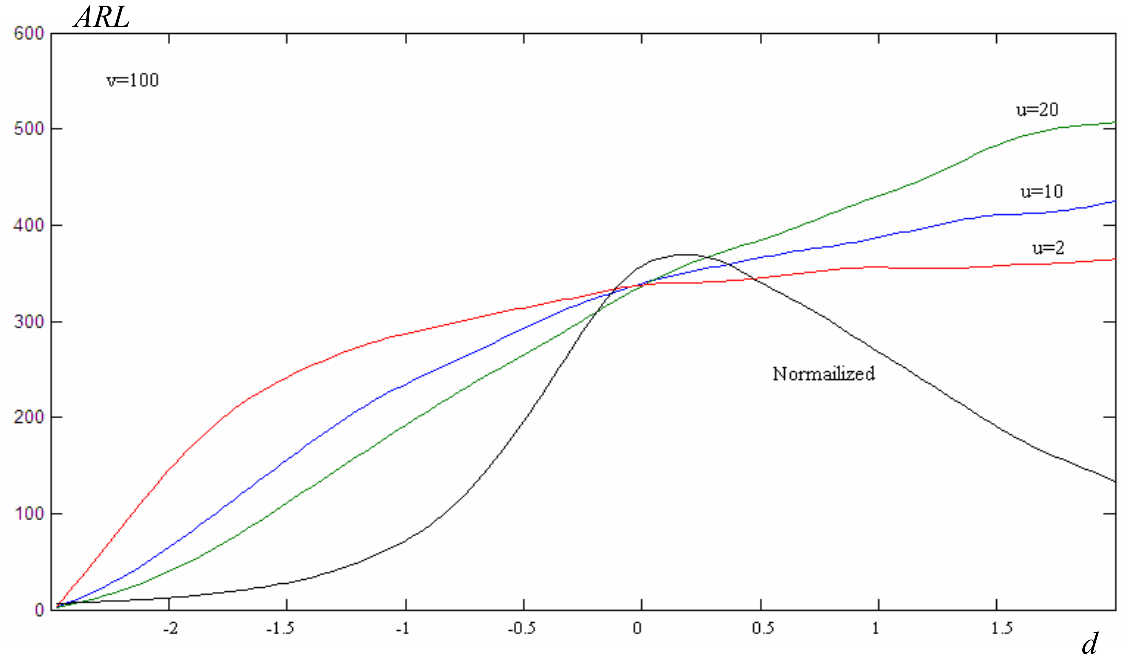


Figure 5.7. *ARL* distribution of probability limits and limits on normalized *F*-distribution (v=100)

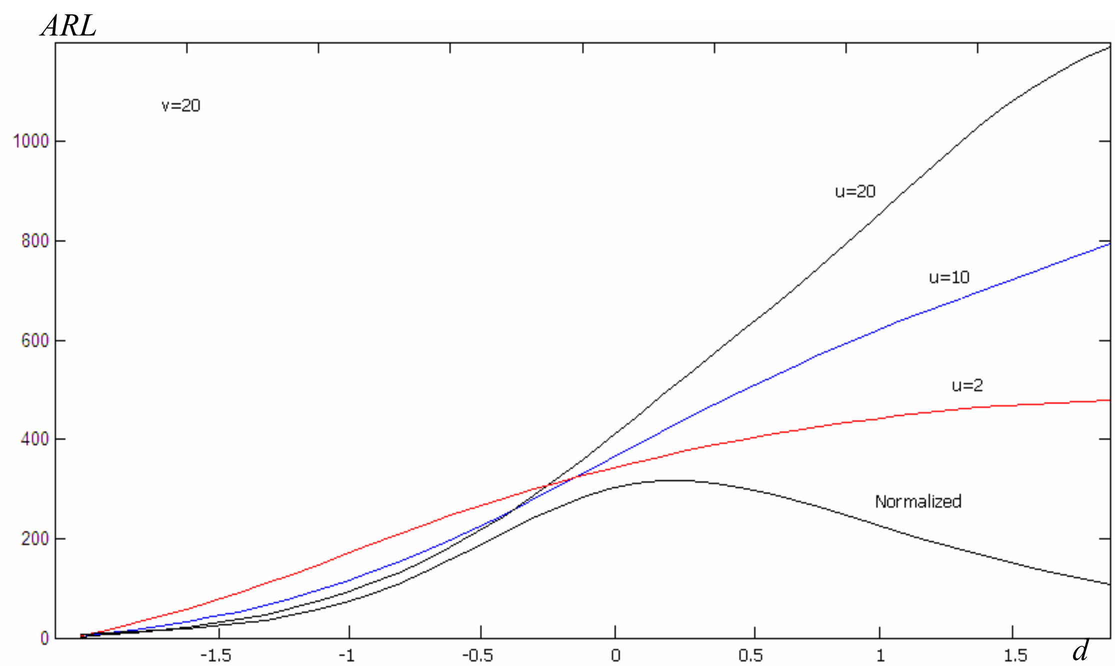


Figure 5.8. *ARL* distribution of probability limits and limits on normalized *F*-distribution

(v=20)

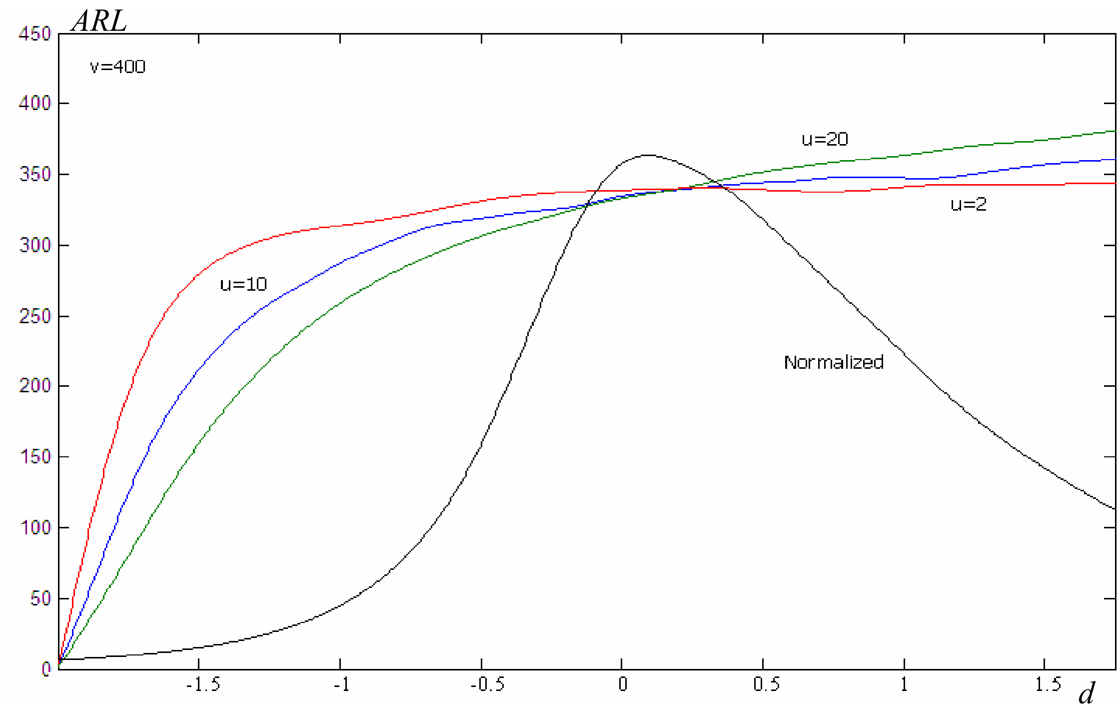


Figure 5.9. *ARL* distribution of probability limits and limits on normalized *F*-distribution (v=400)

Unlike Hotelling T^2-chart, exact probability limits tend to give slower signal when the process moves farer away from the target. Control chart on normalizing transformed data is acting to off-set traditional Hotelling T^2-chart and the exact probability limit chart. Although it is clear from the figures that the data after transformation are not perfect normal, in these cases the in-control *ARL* is close to the expected 370 and the out-of-control *ARL* perform more satisfactory compared to the exact probability limits do. When the process is far away from the target, the *ARL* is shorter at any point than the *ARL* of the probability limits. At the same time, when the process is close to the target, the *ARL* is shorter at any point than the *ARL* of the Hotelling T^2 limits. One should bear in mind that this comparison study has been done based on examining the Hotelling T^2 statistics themselves. It showcases the intrinsic properties of the the Hotelling T^2 statistics. However, it overlooks how the control charts perform when the change in the variables is of concern.

A more straightforward comparison between the two control charting scheme is presented in the simulation study below. We examine exactly how *ARL* moves when the means of the variables are changing. Without loss of generosity, we assume that the rest of all the *p* variables are dependent on one variable (say x_1) with a linear relationship. We monitor the *ARL* based on the shift of the mean of x_1. Denote the mean of x_1 as μ_0, the standard deviation of x_1 as σ_0, the shifted mean of this variable as μ'. *k* represents the magnitude of the variable's mean shift in terms of its standard deviation, that is $k = \dfrac{\mu' - \mu_0}{\sigma_0}$. For *n*=2,…,10, *p* =2,…,10, *m*=15,…,50, we generate the samples of multi-variables that follow standard normal distributions. *S* and μ is thus estimated to calculate T^2. To calculate an *ARL*, 1000,000 simulation runs are needed to ensure the good approximation. The *ARL's* for *k* values are computed by taking the average of 300 *ARL* values for each different *k*. By using normalizing transformation (5.25), that is $Y_i = X_i^{0.3}$, *ARL* are computed for normalized chart as well. We present selected simulation results in Figure 5.10, Figure 5.11, Figure 5.12, Figure 5.13, Figure 5.14 and Figure 5.15 below.

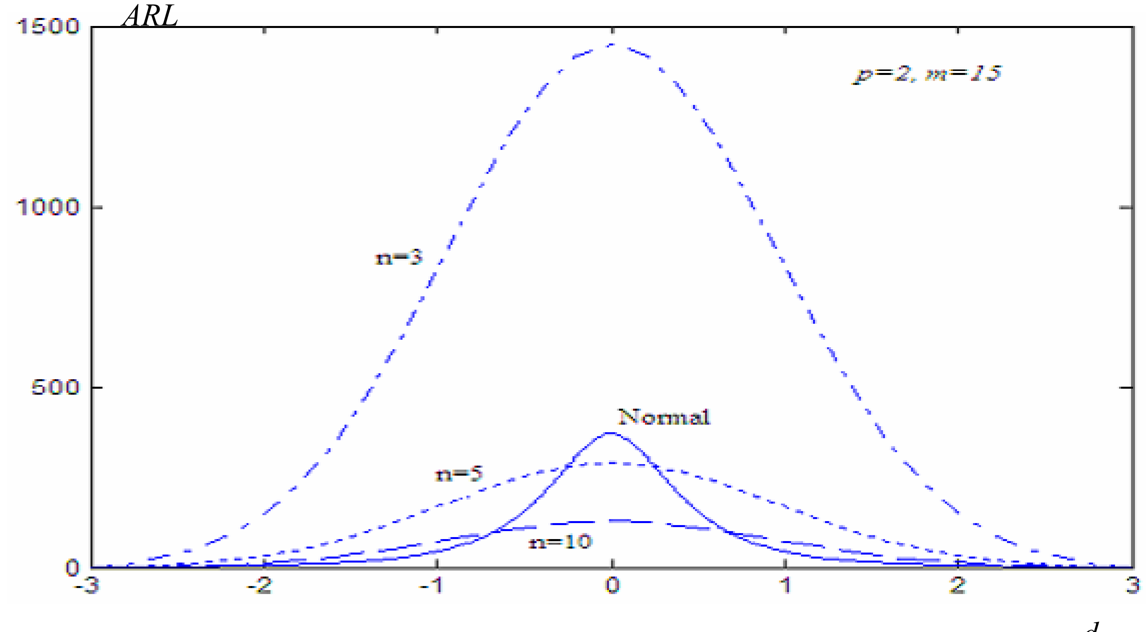


Figure 5.10. *ARL*-curves of *Hotelling*-T^2 chart and normalized chart (p=2, m=15)

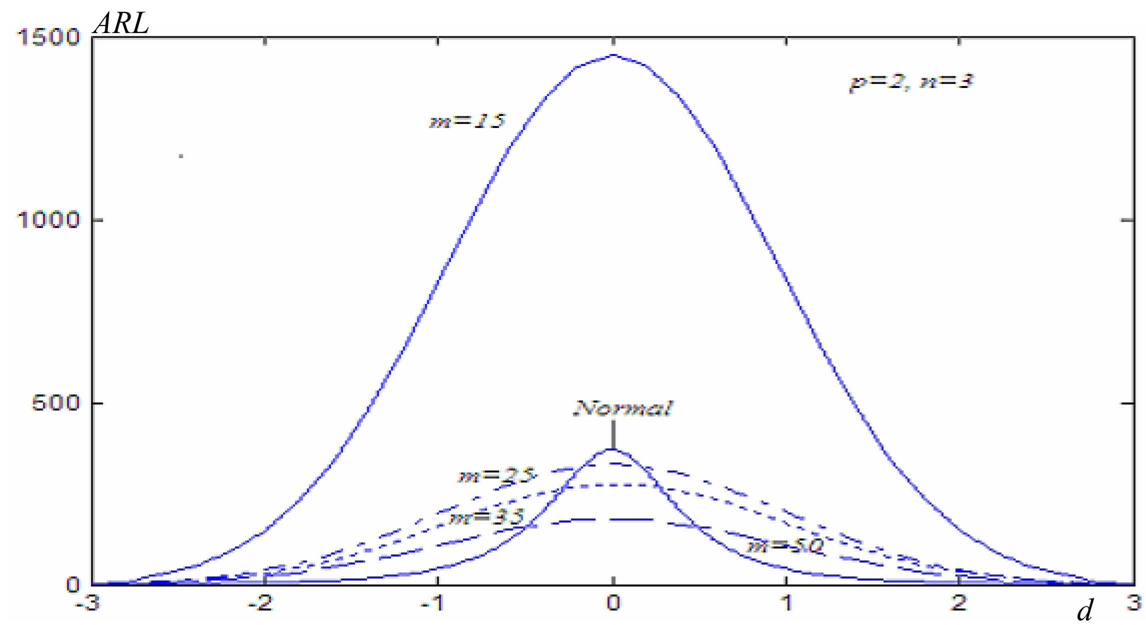


Figure 5.11. *ARL*-curves of *Hotelling*-T^2 chart and normalized chart (p=2, n=3)

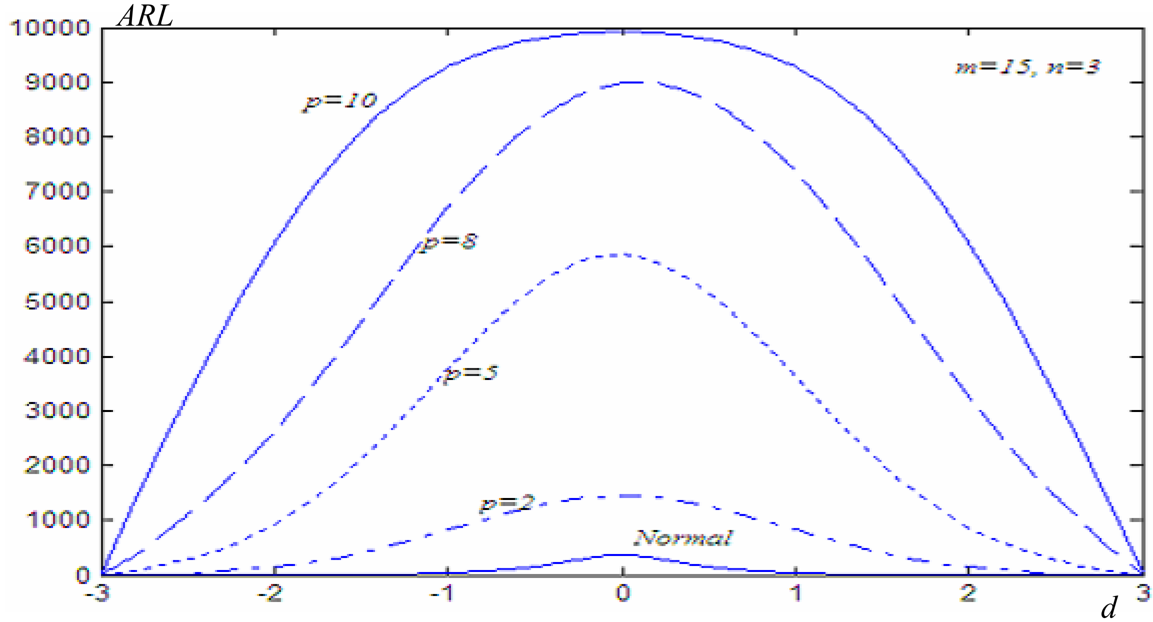


Figure 5.12. *ARL*-curves of *Hotelling*-T^2 chart and normalized chart (*m*=15, *n*=3)

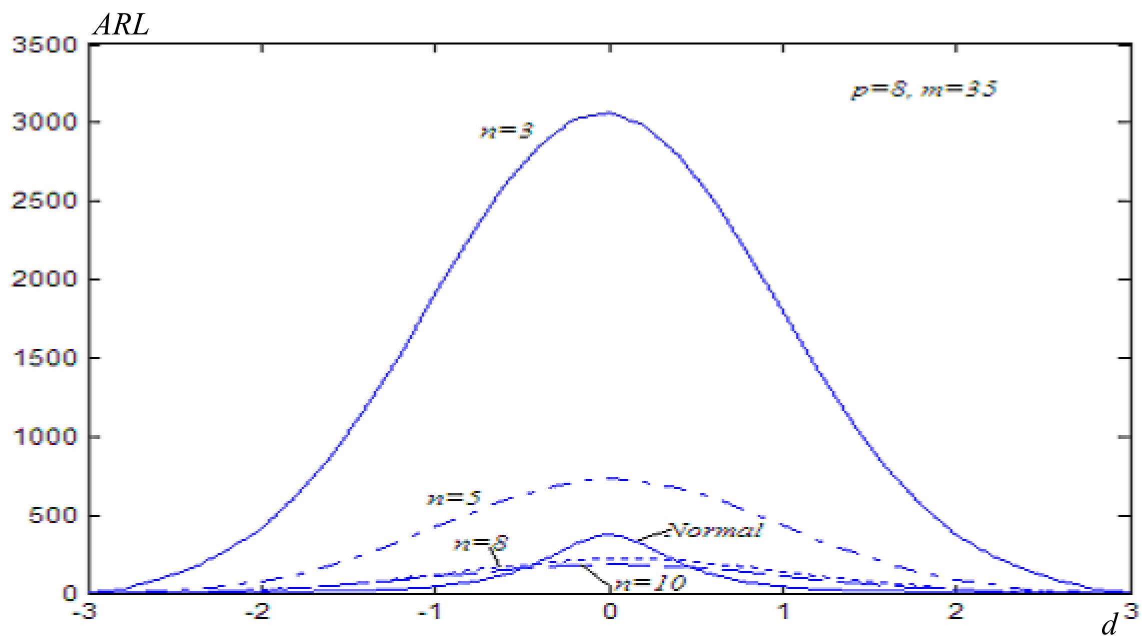


Figure 5.13. *ARL*-curves of *Hotelling*-T^2 chart and normalized chart (*p*=8, *m*=35)

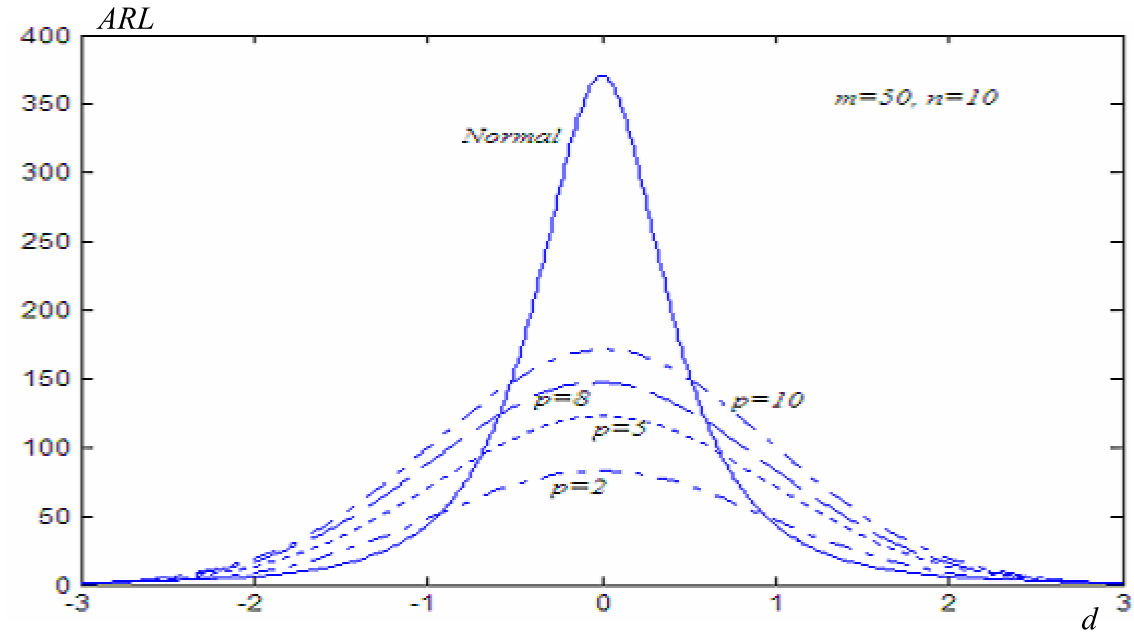


Figure 5.14. *ARL*-curves of *Hotelling*-T^2 chart and normalized chart (*m*=50, *n*=10)

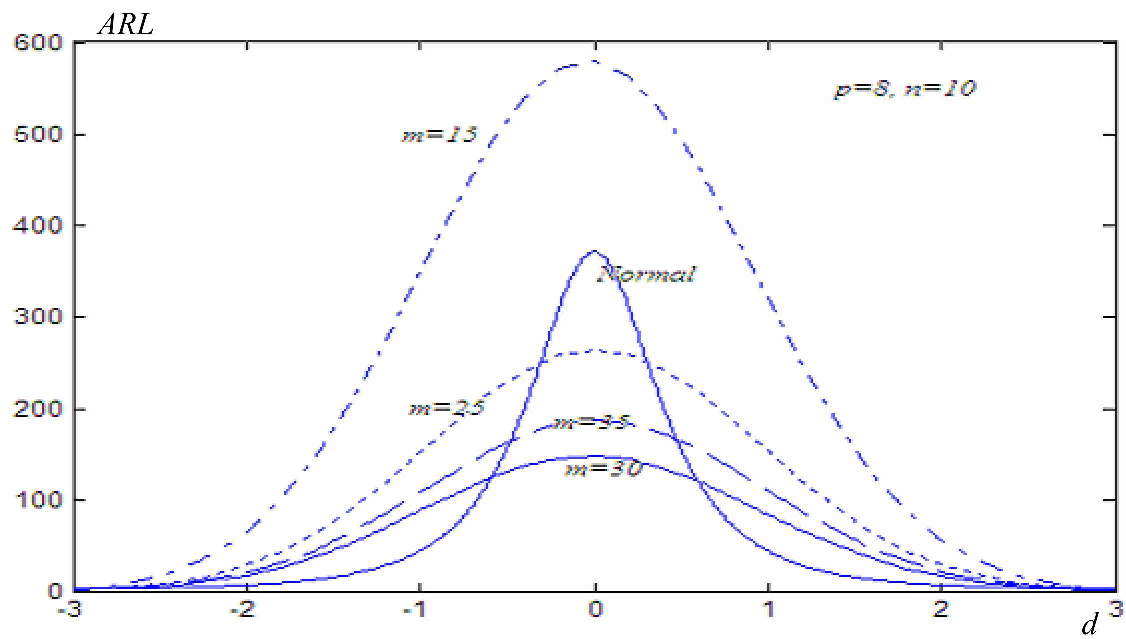


Figure 5.15. *ARL*-curves of *Hotelling*-T^2 chart and normalized chart (*p*=8, *n*=10)

It is clear from the figures above that the *ARL* curves of original multi-variate charts are symmetric, when it is dependent on the change of the variables. This meets one of the desired criteria of a good *ARL* curve, which is to be symmetric. However, the normalized chart shows a more reasonably sensitive response to mean shift. Comparing the shapes of *ARL* curves has revealed that the kurtosis of the curve is smaller for Hotelling-T^2 charts. Therefore, if the control limits are determined in the way that the in-control *ARLs* are set at same level, the out-of-control *ARL* of Hotelling-T^2 charts will be larger than that of normalized chart. In this case, normalized chart will detect a real out-of-control signal faster than the Hotelling-T^2 charts do. On the other hand, if the control limits are determined in the way that the out-of-control *ARLs* at one given point *k* are set at same level, the in- control *ARL* of Hotelling-T^2 charts will be smaller than that of normalized chart. In this case, normalized chart will give out a false out-of-control signal slower than the Hotelling-T^2 charts do. In a word, Normalized chart shows good sensitivity for both types of signals in this case.

Another one of the interesting results from the simulation is that despite the fact that the *UCL* for Hotelling-T^2 charts is set at 99.73% probability point, the in-control *ARL* are never anywhere close to the expected 370. This is actually due to small size of preliminary samples when lead to relatively rough estimation of S and μ. The trend is that the larger *m* is, the closer the in-control *ARL* gets to 370.

Comparing the tail probability also helps to determine the performance. We calculate the exact tail probability by using extensive simulation in Table 5.5 below.

u=2									
v	1	2	5	10	20	50	100	200	400
P_{T^2}	0.00058	0.01724	0.03324	0.03215	0.02909	0.0279	0.02601	0.02637	0.02531
P_N	0.00285	0.00287	0.00293	0.00298	0.00311	0.0313	0.00357	0.00384	0.00398
u=5									
v	1	2	5	10	20	50	100	200	400
P_{T^2}	0.00114	0.00924	0.03217	0.02772	0.02311	0.01774	0.01759	0.01613	0.01589
P_N	0.00257	0.00274	0.00293	0.00299	0.00310	0.0333	0.00337	0.00369	0.00392
u=10									
v	1	2	5	10	20	50	100	200	400
P_{T^2}	0.0004	0.01186	0.03107	0.02694	0.01958	0.01498	0.01365	0.01128	0.01101
P_N	0.00255	0.00268	0.00290	0.00289	0.00310	0.0313	0.00335	0.00378	0.00394
u=20									
v	1	2	5	10	20	50	100	200	400
P_{T^2}	0.00039	0.01618	0.03012	0.02579	0.01861	0.01286	0.0095	0.00885	0.00849
P_N	0.00252	0.00267	0.00269	0.00290	0.00301	0.0312	0.00327	0.00348	0.00378

Table 5.5. Tail probabilities for F-distribution for some of the parameters

P_{T^2} : tail probability of traditional Hotelling T^2 limits
P_N : tail probability of limits on transformed Hotelling T^2 statistics

To calibrate with the ideal probability, we can see from Table 5.5 that the limits on transformed data produce tail probability that is closer to the ideal tail probability than the traditional limits do. For example, with u=2 and v=2, the difference between the real tail probability and the ideal one decreases from 538.5% to 6.3%.

Moreover, in section 5.3, we provide very simple settings of the transformation parameters. When the new observations are collected and new plotting statistics are computed, the distributions of the new statistics are the very one used to estimated the previous *UCL*. Thus, we avoid the case that the control limits in Phase I and Phase II are different which could also lead to the problem with *ARL* performance. This ensures the consistency of the control limits used for one process, so that the limits for Phase I and II are same.

Although normalized charts can never be superior to the original Hotelling-T^2 charts, we have found a beneficial trade-off to make the normalized charts useful. Besides, the proposed methodology would be even more interesting for professionals accustomed to work with Shewhart type control charts, which are based on normal transformation. They will, to a certain extent, be given the opportunity to continue using the charts based on normal distribution and detecting signals as they are familiar with.

5.5.2. Simulation studies on Comparison to Box-Cox transformation

Nowadays, Box-Cox is one of the most popular transforming methods used to deal with non-normality problems as it provides very good normal approximation. We conduct the simulation work to compare the performance of the Box-Cox transformation with our simple power transformation.

The simulation is done by measuring the goodness-of-fit of the two transforming methods in the way of using Anderson-Darling test as described before. The whole simulation studies are divided into two parts. In the first part, we simply generate data from χ^2-distribution. Transform the data by using formula (5.11). Then, *AD* values are calculated and stored. The second part, we simulate data from *F*-distribution. Then the transformation of formula (5.26) is applied to calculate the *AD* values again. In each part, we generate 10,000 data as a set to perform the transformations in each run. For χ^2-distribution, we test the data with degrees of freedom from 2 to 30. For *F*-distribution, we test the data with setting as in Appendix III.

In the simulation for χ^2-distribution, 29 different settings of degrees of freedom are tested. There are 19 times that Box-Cox transformation has AD^* value smaller than the critical value, 12 times that 0.3 power transformation has AD^* value larger than the critical value, 14 times that both Box-Cox transformation and 0.3 power transformation has AD^* value smaller than the critical value, and 7 times that both Box-Cox transformation and 0.3 power transformation has AD^* value larger than the critical value. Thus we have,

$$P_0 = 14,\ P_{00} = 7,\ P_A = 12,\ P_B = 19,$$

$$P_1 = \frac{P_0}{P_B} = 0.737,$$

$$P_2 = \frac{P_{00}}{P_A} = 0.583.$$

Box-Cox transformation excels in the goodness-of-fit of normalization.

In the simulation for F-distribution, 2592 different settings of degrees of freedom are tested. There are 19 times that Box-Cox transformation has AD^* value smaller than the critical value, 1273 times that 0.3 power transformation has AD^* value larger than the critical value, 14 times that both Box-Cox transformation and 0.3 power transformation has AD^* value smaller than the critical value, and 1159 times that both Box-Cox transformation and 0.3 power transformation has AD^* value larger than the critical value. Thus we have,

$$P_0 = 1117,\ P_{00} = 1259,\ P_A = 1273,\ P_B = 1189,$$

$$P_1 = \frac{P_0}{P_B} = 0.939,$$

$$P_2 = \frac{P_{00}}{P_A} = 0.989.$$

Box-Cox transformation seems to be slightly better than 0.3 power transformation.

However, the difference of comparison is rather small. Besides, the Box-Cox power normal family has a serious defect, i.e. that the correlation structure of the maximum-likelihood estimates of the parameters is not preserved under a scale transformation of the response variable. Our simple power family is free from this defect. Moreover, the Box-Cox power transformation is not easy to apply while our simple power transformation can be done with Appendix III or a pocket calculator. Therefore, the simple power transformation is promising in industrial applications because of its ease of implementation.

5.6. Conclusions

Data non-normality is a common problem encountered in statistical process control. In multivariate process monitoring, Hotelling T^2-chart is widely used. However, this classic control chart is based on the normality assumption and thus fails to properly monitor an asymmetric distribution. Transformation is one of the convenient and easy ways to remedy the problem. Inspired by some existing normalizing transformation, we have proposed two simple power transforming on χ^2- and *F*-distributions to get some approximately normal data. Based on these data, the traditional Shewhart type 3-sigma individual control chart can be constructed. Thus a Hotelling T^2-chart is remodeled into a traditional control chart. Simulation study focuses on the selection of transforming parameters and comparison of the transformation performance which aids the simplification of the transforming formulas.

By developing simple power transformation, both numerical examples and analytical results show a satisfactory performance. Because of the non-negative property of the Hotelling T^2-chart, the run rules are not applicable due to the asymmetric property of both distributions of χ^2 and *F*. After transformations data tend to be normally distributed. Under the better fulfillment of the normality assumption, traditional individual chart is properly constructed and performs as expected.

Although this methodology has its intrinsic flaw that the transformed control charts will never be beyond the original charts with exact probability limits, it provides an easy and moderately satisfactory way in real applications. The physical interpretation of the variables may be lost through the transformation. Subsequently the transformed data may not be directly interpretable. This may be of concern in future studies.

Chapter 6

Conclusions

6.1. Concluding Remarks

The theoretical achievements in normalizing transformations provide a new approach to more precise monitoring on non-normal data. After being transformed to a normal distribution, the quality characteristic of traditional control charts can be simply monitored by a traditional Shewhart type individual chart.

Very accurate transformation forms have been proposed in literatures on *t, F*, non-central *t*, non-central *F*, Exponential, Chi-square distributions and some statistic families as well. If the underlying distribution is detectable or approximately detectable, the appropriate transformations above could be employed to further apply the standard 3-sigma limits. Otherwise, some formulas for generic distribution would be chosen as an alternative. One can follow the procedures described for implementation.

Performances analysis and numerical examples reveal that normalizing transformation could improve the performance of control limits in the sense that it achieves more desirable *ARL* performance. Moreover, the control charts based on normalized data outperforms the exact probability charts in certain situations on *ARL* properties when a proper normalizing transformation is employed. This approach would be beneficial when the underlying distribution of the monitored variable differs from normal and provide faster signals for multivariate charts and symmetrically responses for *R*-/*S*-charts.

6.2. Limitations and Recommendations for Future Research

As we have discussed in the previous chapter, a control chart using an exact statistic cannot be inferior to that using a transformed statistic which may or may not be normal after normal transformation. It is one of the most important limitations of this study. Another limitation is that transforming a non-normal statistic does not provide a valid physical interpretation of an out-of-control signal. Future research may address these two issues which are critical for in-depth analysis. Normalizing transformations do not intrinsically change the relative positioning of the data values, but re-express the data, while preserving the rank order, to a scale that allows the normal distribution to serve as a benchmark for interpretation and judgment (Alwan, 2000). Thus, we could conduct evaluation on the control chart performance based on data with different degrees of departure from normality, comparison on performances by using some indicators other than *ARL* properties or new methodology and investigation on the mapping of interpretation on original data and that of transformed data. In this way, a better solution could be brought up for the industry to enjoy using traditional 3-sigma limits for non-normal applications.

It is also critical to find out more reasonable and accurate estimators for the reason that the way of estimation affects greatly the results of process monitoring. An interesting continuation of this study is to embody the estimators and to assess their post-transformation accuracy. Involved in most of the control charts, assumption of non-normality is really questionable either in practice or in theory. Research has shown that more accurate results can be obtained if taking the non-normality into account when designing the 3-sigma control limits. Many methods have been employed to compute the process capability indices with non-normal data, such as non-parametric approach, probability plot, distribution-free tolerance method, weighted variance method, Box-Cox power transformation, Clements' method, Johnson transformation, etc. Using some of these methods to modify the 3-sigma limits for non-normal data is another solution, so that the real *ARL* could achieve the desirable value. It is especially relevant for *R-, S-* chart.

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Appendix I

Exact Distribution of the Range

Consider observations $X_1, \ldots, X_n$, where n is the subgroup size. The order statistics are $X_{(1)}, \ldots, X_{(n)}$. Let R denote the range defined as $X_{(n)}$ - $X_{(1)}$, and L denotes $X_{(n)}$, the distribution of $R = X_{(n)} - X_{(1)}$ can be obtained as follows:

$$f_R(u) = \int_{-\infty}^{\infty} f_{X_{(n)}-X_{(1)},X_{(n)}}(u,v)dv$$

$$= \int_{-\infty}^{\infty} \frac{n!}{(n-2)!}\left[F(u+v) - F(v)\right]^{n-2} f(u+v) f(u) dv$$

Here $f(x)$ is the probability density function of individual measurement X_i and $F(x)$ is the cumulative distribution function of X_i. The probability distribution function of R can be shown to be:

$$f_R(u) = \int_{-\infty}^{\infty} \frac{n!}{(n-2)!}\left[\frac{1}{\sigma\sqrt{2\pi}}\int_{v}^{u+v} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx\right]^{n-2} f(u+v) f(u) dv,$$

where

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

provided X_i is normally distributed.

The probability distribution function of R can be shown to be (Gumbel, 1947):

$$F_R(u) = n\int_{-0}^{1}\left[\frac{1}{\sigma\sqrt{2\pi}}\int_{v}^{u+v} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx\right]^{n-1} f(v) dv$$

McKay and Pearson (1993) showed a general expression of $f_R(u)$ as:

$$f_R(u) = n(n-1)\int_{-\infty}^{\infty} f\left(t+\frac{1}{2}u\right) f\left(t-\frac{1}{2}u\right)\left(\int_{t-\frac{1}{2}u}^{t+\frac{1}{2}u} f(x)dx\right)^{n-2} dt .$$

McKay and Pearson (1993) inferred that when the u is small enough, the distribution function of range can be approximated as:

$$f_R(u) = n(n-1)u^{n-2}\int_{-\infty}^{\infty} f\left(t+\frac{1}{2}u\right) f\left(t-\frac{1}{2}u\right)[f(x)]^{n-2} dt ,$$

while the u is large enough, the distribution function of range can be approximated as:

$$f_R(u) = n(n-1)\int_{-\infty}^{\infty} f\left(t+\frac{1}{2}u\right) f\left(t-\frac{1}{2}u\right) dt .$$

This approximation provides a way to conduct the more reasonable control limits.

For larger normal samples up to n=20, Pearson and Hartley (1954) calculated numerical tables of the probability of the range. Tippett (1925) also computed the mean, the standard deviation, and the k-th moment for the range of normal distribution up to n=1000. Several papers also deal with the expression of the mean and variance for the range. In fact, this has lead to the numerical values of D_3 and D_4 in the traditional R-chart based on the mean plus and minus three standard deviations.

Appendix II

Multivariate normal distribution analysis

In univariate statistical quality control, we generally use the normal distribution to describe the behavior of a continuous quality characteristic. This same approach can be used in the multivariate cases. Suppose that we have p variables, given by $X_1, \ X_2, \ldots, X_p$. Arrange these variables in a *p*-component vector $X' = [X_1, X_2, ..., X_p]$. Let $\mu' = [\mu_1, \mu_2, ..., \mu_p]$ be the vector of the means of the *X*'s, and let the variances and covariances of the random variables in *X* be contained in a $p \times p$ covariance matrix Σ. The main diagonal elements of Σ are the variances of the *X*'s and the off-diagonal elements are the covariances. Therefore, the multivariate normal probability density function is

$$f(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)' \Sigma^{-1} (x-\mu)}$$

where $-\infty < x_j < \infty, j = 1,2,..., p$. The χ^2-statistic is defined as

$$\chi_0{}^2 = n(X - \mu)' \Sigma^{-1} (X - \mu)$$

where $\mu' = \lfloor \mu_1, \mu_2, ..., \mu_p \rfloor$ is the vector of in-control means for each quality characteristic and Σ is the covariance matrix. Denote the sample mean vector as

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \ .$$

The Hotelling T^2 statistic for the vector is defined as:

$$T^2 = n\left(X - \overline{X}\right)' S^{-1}\left(X - \overline{X}\right)$$

where $\overline{X}$ and S are the common estimators of the mean vector and covariance matrix obtained from a historical data set. The theoretical properties of the estimates $\overline{X}$ and S have been extensively studied and were determined to be optimal under a variety of criteria. The distribution properties of those two estimates for a ransom normal sample $X_1, X_2, ..., X_p$ are given by the following theorems (Seber, 1984):

(i) $n\left(\overline{X} - \mu\right)' \Sigma^{-1}\left(\overline{X} - \mu\right) \sim \chi_p^2$

(iii) If the Hotelling T^2-statistic, denoted by T_M^2, is given by

$$T_M^2 = n\left(\overline{X} - \mu\right)' S^{-1}\left(\overline{X} - \mu\right).$$

The T_M^2-statistic is distributed as

$$\frac{n-p}{(n-1)p} T_M^2 \sim F_{p,n-p}.$$

(iv) When a sample of kn independent normal observations are grouped into *m* rational subgroups of size *n*, then the distance between the mean $\overline{X}_j$ of the *j*-th subgroup and the expected values μ is computed by T_M^2 defined as

$$T_M^2 = n\left(\overline{X}_j - \overline{\overline{X}}\right)' S_p^{-1}\left(\overline{X}_j - \overline{\overline{X}}\right).$$

It follows (iii) above that

$$\frac{m(n-1)-p+1}{m(n-1)p} T_M^2 \sim F_{p,m(n-1)-p+1}.$$

The distributional properties of these statistics provide the theoretical basis for deriving the distributions of the statistics used in the multivariate quality control procedures. These statistics assess the overall distance of a *p*-dimensional vector of observed means from the target values.

Appendix III

Simulation Results of Comparing Different Transforming Forms of *F*-distribution

p	*m*	*n*	*h*	*Box-Cox*	*h-value*	*0.3*	*p*	*m*	*n*	*h*	*Box-Cox*	*h-value*	*0.3*
2	15	3	0.2903	9.9116	8.4443	5.2844	2	33	3	0.3134	9.5874	8.1758	5.6811
2	15	4	0.3043	8.6608	7.2648	5.6957	2	33	4	0.3200	8.6842	7.3650	6.2152
2	15	5	0.3115	8.3363	7.0521	6.2756	2	33	5	0.3233	8.4723	7.1316	6.5498
2	15	6	0.3158	8.0357	6.7229	6.3240	2	33	6	0.3253	8.0266	6.7274	6.5231
2	15	7	0.3187	7.7380	6.4841	6.3653	2	33	7	0.3266	8.2956	7.0329	7.0685
2	15	8	0.3208	8.3319	7.1332	7.1985	2	33	8	0.3276	7.4746	6.2634	6.4578
2	15	9	0.3223	8.2779	6.9812	7.2015	2	33	9	0.3283	7.2231	6.0861	6.3798
2	15	10	0.3235	7.3974	6.1521	6.4722	2	33	10	0.3289	7.5601	6.3414	6.7574
2	16	3	0.2929	7.9058	6.6781	2.0654	2	34	3	0.3140	7.7162	6.4955	2.5565
2	16	4	0.3061	6.2100	5.0240	2.4493	2	34	4	0.3204	5.6909	4.5328	2.4857
2	16	5	0.3128	5.4884	4.3754	2.8741	2	34	5	0.3236	5.3414	4.2753	3.1422
2	16	6	0.3169	5.4393	4.3565	3.3722	2	34	6	0.3255	5.2122	4.1486	3.4115
2	16	7	0.3196	5.3084	4.1973	3.5066	2	34	7	0.3268	4.8910	3.8104	3.3417
2	16	8	0.3215	5.5412	4.4667	4.0393	2	34	8	0.3278	4.8488	3.8964	3.6789
2	16	9	0.3230	4.9283	3.9309	3.7079	2	34	9	0.3284	4.0040	3.0609	2.9805
2	16	10	0.3241	4.3033	3.4040	3.3215	2	34	10	0.3290	5.3172	4.2934	4.3190
2	17	3	0.2952	8.2485	7.1777	1.3799	2	35	3	0.3146	6.9641	5.9081	1.0257
2	17	4	0.3077	6.3258	5.2913	1.8842	2	35	4	0.3208	4.6494	3.6184	0.9543
2	17	5	0.3140	4.3694	3.3740	1.3335	2	35	5	0.3239	4.0562	3.0922	1.5139
2	17	6	0.3178	4.3152	3.3299	1.8173	2	35	6	0.3258	3.7704	2.8598	1.8392
2	17	7	0.3204	3.5139	2.6136	1.7181	2	35	7	0.3270	3.5411	2.6244	1.8605
2	17	8	0.3222	3.8232	2.8807	2.1030	2	35	8	0.3279	3.4738	2.6662	2.2226
2	17	9	0.3236	4.0283	3.1028	2.5352	2	35	9	0.3286	3.5212	2.6033	2.2087
2	17	10	0.3247	3.7425	2.8112	2.3964	2	35	10	0.3291	3.4151	2.6482	2.4534
2	18	3	0.2973	8.5159	7.5537	1.3223	2	36	3	0.3151	7.2449	6.2967	1.1223
2	18	4	0.3091	5.6301	4.6980	0.6702	2	36	4	0.3211	5.6845	4.7683	1.3466
2	18	5	0.3151	4.8688	3.9564	1.2103	2	36	5	0.3241	4.0419	3.1534	1.0789
2	18	6	0.3187	4.3468	3.4380	1.3908	2	36	6	0.3260	3.4681	2.5665	0.9110
2	18	7	0.3211	3.8729	3.0066	1.6008	2	36	7	0.3272	3.1341	2.2767	1.1948
2	18	8	0.3228	3.2857	2.4514	1.4573	2	36	8	0.3281	3.2852	2.4496	1.6191
2	18	9	0.3241	2.7364	1.9917	1.4112	2	36	9	0.3287	2.7854	1.9416	1.3057
2	18	10	0.3252	2.9909	2.2164	1.6657	2	36	10	0.3292	2.8263	2.0197	1.5687
2	19	3	0.2991	8.8801	7.9889	1.2582	2	37	3	0.3156	8.6066	7.7334	1.5984
2	19	4	0.3103	6.1002	5.2450	0.7047	2	37	4	0.3214	5.8438	4.9984	1.0284
2	19	5	0.3160	4.4485	3.6167	0.7856	2	37	5	0.3244	4.2241	3.3925	0.7386
2	19	6	0.3194	4.3509	3.5522	1.3846	2	37	6	0.3262	3.5717	2.7598	0.8578
2	19	7	0.3217	3.5498	2.7576	1.1125	2	37	7	0.3274	2.5264	1.7701	0.8642
2	19	8	0.3234	3.2394	2.4335	1.0289	2	37	8	0.3282	1.9371	1.2770	0.9800
2	19	9	0.3246	2.4389	1.6576	0.7036	2	37	9	0.3288	2.4922	1.6969	0.8235
2	19	10	0.3256	2.7454	2.0369	1.3437	2	37	10	0.3293	2.5868	1.8739	1.3092

2	20	3	0.3008	9.8333	8.9973	1.0905	2	38	3	0.3160	8.5078	7.6874	1.0722
2	20	4	0.3115	6.2330	5.4362	0.6432	2	38	4	0.3217	5.4102	4.6275	0.5397
2	20	5	0.3169	4.5311	3.7592	0.6974	2	38	5	0.3246	4.5185	3.7537	0.9147
2	20	6	0.3201	4.2206	3.4765	1.0620	2	38	6	0.3264	3.4454	2.6901	0.5953
2	20	7	0.3223	3.5075	2.7498	0.7054	2	38	7	0.3275	3.1202	2.3737	0.7108
2	20	8	0.3239	2.4863	1.7784	0.7683	2	38	8	0.3283	2.5384	1.8072	0.6215
2	20	9	0.3251	2.6612	1.9853	1.0464	2	38	9	0.3290	2.6296	1.9761	1.2070
2	20	10	0.3260	2.2268	1.5207	0.6731	2	38	10	0.3294	2.5109	1.8003	0.9837
2	21	3	0.3023	11.0130	10.2210	1.5845	2	39	3	0.3165	8.7562	7.9812	1.1089
2	21	4	0.3125	6.5418	5.7928	1.0798	2	39	4	0.3220	6.1680	5.4286	0.7200
2	21	5	0.3176	5.0099	4.2881	0.8545	2	39	5	0.3248	4.3816	3.6590	0.4643
2	21	6	0.3208	3.6741	2.9564	0.4052	2	39	6	0.3265	3.6923	2.9948	0.7825
2	21	7	0.3228	3.8833	3.1716	0.8286	2	39	7	0.3277	3.8453	3.1445	1.1819
2	21	8	0.3243	3.2806	2.5756	0.7027	2	39	8	0.3285	3.2579	2.5579	0.9656
2	21	9	0.3254	2.9310	2.2465	0.8457	2	39	9	0.3291	2.6444	1.9737	0.8541
2	21	10	0.3263	2.6737	2.0385	1.0642	2	39	10	0.3295	2.1211	1.4590	0.6330
2	22	3	0.3037	11.4970	10.7310	1.4560	2	40	3	0.3169	9.4387	8.6939	1.1414
2	22	4	0.3134	7.2257	6.5128	0.4772	2	40	4	0.3223	6.6823	5.9809	1.1288
2	22	5	0.3184	5.1748	4.4854	0.4997	2	40	5	0.3251	4.5272	3.8436	0.5738
2	22	6	0.3213	4.3218	3.6432	0.4882	2	40	6	0.3267	3.4850	2.8122	0.4478
2	22	7	0.3233	3.2992	2.6344	0.5845	2	40	7	0.3278	3.3296	2.6662	0.6526
2	22	8	0.3247	3.0042	2.3560	0.6609	2	40	8	0.3286	2.9103	2.2512	0.5426
2	22	9	0.3258	2.8118	2.1609	0.6197	2	40	9	0.3292	3.0935	2.4522	1.0789
2	22	10	0.3266	2.6010	1.9720	0.7720	2	40	10	0.3296	2.4777	1.8426	0.7635
2	23	3	0.3050	12.3000	11.5550	1.3048	2	41	3	0.3173	NaN + NaNi	NaN + NaNi	1.2935
2	23	4	0.3143	7.4882	6.8054	0.9123	2	41	4	0.3226	7.0176	6.3438	0.9188
2	23	5	0.3190	5.9504	5.2899	0.6863	2	41	5	0.3253	4.5480	3.8958	0.4902
2	23	6	0.3218	4.3159	3.6709	0.6160	2	41	6	0.3269	3.7666	3.1270	0.5816
2	23	7	0.3237	4.2357	3.5959	0.8025	2	41	7	0.3279	2.7739	2.1580	0.6905
2	23	8	0.3251	2.8645	2.2445	0.6001	2	41	8	0.3287	3.1059	2.4825	0.6427
2	23	9	0.3261	2.6446	2.0280	0.4911	2	41	9	0.3293	2.0496	1.4484	0.6317
2	23	10	0.3269	3.1641	2.5569	1.0544	2	41	10	0.3297	2.6994	2.0943	0.8838
2	24	3	0.3061	9.2651	7.8418	5.1135	2	42	3	0.3176	9.0383	7.6263	5.3477
2	24	4	0.3151	8.4963	7.1095	5.7259	2	42	4	0.3228	8.7848	7.4649	6.4370
2	24	5	0.3196	7.9143	6.6365	5.9884	2	42	5	0.3254	7.7709	6.4610	5.9976
2	24	6	0.3223	8.1791	6.9083	6.6227	2	42	6	0.3270	7.9207	6.6719	6.5536
2	24	7	0.3241	7.8567	6.6621	6.6269	2	42	7	0.3281	7.7553	6.5642	6.6581
2	24	8	0.3254	7.2019	5.9422	6.0820	2	42	8	0.3288	7.8221	6.6208	6.8674
2	24	9	0.3264	7.9567	6.6958	6.9722	2	42	9	0.3294	7.6262	6.4734	6.8184
2	24	10	0.3272	7.5826	6.3825	6.7436	2	42	10	0.3298	7.5548	6.3106	6.7807
2	25	3	0.3072	7.0618	5.8610	1.8888	2	43	3	0.3180	7.1878	5.9762	2.3827
2	25	4	0.3158	6.2689	5.0962	2.7803	2	43	4	0.3231	5.6865	4.5329	2.6667
2	25	5	0.3201	5.5835	4.4351	2.9906	2	43	5	0.3256	5.8190	4.7010	3.5745
2	25	6	0.3228	4.9893	3.9093	3.0587	2	43	6	0.3272	4.9166	3.8634	3.2305
2	25	7	0.3245	4.9854	3.9043	3.3449	2	43	7	0.3282	4.8292	3.8172	3.4649
2	25	8	0.3258	4.4530	3.3526	3.0080	2	43	8	0.3289	4.5727	3.5414	3.3659
2	25	9	0.3267	4.4313	3.4782	3.3370	2	43	9	0.3295	4.6313	3.6951	3.6654
2	25	10	0.3274	4.6937	3.7280	3.6969	2	43	10	0.3299	4.3535	3.4338	3.5015
2	26	3	0.3082	6.8624	5.8045	1.2281	2	44	3	0.3184	7.0320	5.9768	1.3534
2	26	4	0.3165	5.0898	4.0485	0.9600	2	44	4	0.3233	5.0537	4.0267	1.4016

2	26	5	0.3206	4.3107	3.3579	1.6727	2	44	5	0.3258	4.7271	3.7286	2.0522
2	26	6	0.3232	3.8607	2.9001	1.6285	2	44	6	0.3273	3.8438	2.9067	1.9037
2	26	7	0.3248	4.0843	3.1684	2.2863	2	44	7	0.3283	3.6930	2.7917	2.1174
2	26	8	0.3260	3.6698	2.6984	1.9846	2	44	8	0.3290	3.3070	2.4021	1.9327
2	26	9	0.3270	3.5620	2.6982	2.2751	2	44	9	0.3296	3.6752	2.7445	2.4089
2	26	10	0.3277	2.9253	2.0551	1.7667	2	44	10	0.3300	3.3069	2.4801	2.3123
2	27	3	0.3091	7.7118	6.7583	0.8868	2	45	3	0.3187	7.0057	6.0574	0.7430
2	27	4	0.3171	5.6669	4.7371	1.0211	2	45	4	0.3235	4.6682	3.7558	0.7987
2	27	5	0.3211	4.1943	3.2894	0.8714	2	45	5	0.3260	3.4619	2.5692	0.7316
2	27	6	0.3235	4.5039	3.6082	1.7885	2	45	6	0.3274	3.5625	2.6803	1.2395
2	27	7	0.3252	3.0151	2.1435	0.9499	2	45	7	0.3284	3.0611	2.2026	1.2040
2	27	8	0.3263	2.9321	2.1467	1.3916	2	45	8	0.3291	2.7984	2.0191	1.4044
2	27	9	0.3272	2.8426	2.0378	1.3893	2	45	9	0.3296	2.6980	1.9620	1.5362
2	27	10	0.3279	3.0004	2.2013	1.6839	2	45	10	0.3300	3.0932	2.2889	1.8934
2	28	3	0.3099	8.0646	7.1844	0.8160	2	46	3	0.3190	7.7930	6.9235	0.9813
2	28	4	0.3176	5.2552	4.4051	0.6454	2	46	4	0.3237	4.7718	3.9313	0.5314
2	28	5	0.3215	3.9386	3.1191	0.6734	2	46	5	0.3261	4.5934	3.7641	1.2276
2	28	6	0.3239	3.7585	2.9393	0.8330	2	46	6	0.3276	3.0694	2.2704	0.7593
2	28	7	0.3254	3.0901	2.3143	0.9955	2	46	7	0.3285	3.3253	2.5330	1.2358
2	28	8	0.3266	3.2551	2.4582	1.2107	2	46	8	0.3292	2.5715	1.8485	1.0958
2	28	9	0.3274	2.9314	2.1400	1.1590	2	46	9	0.3297	2.3541	1.5843	0.8627
2	28	10	0.3281	2.7339	2.0217	1.3851	2	46	10	0.3301	2.5419	1.8070	1.2646
2	29	3	0.3107	8.6060	7.7811	1.1114	2	47	3	0.3193	7.4714	6.6601	1.0180
2	29	4	0.3182	5.9764	5.1871	0.5963	2	47	4	0.3239	5.6675	4.8799	0.7653
2	29	5	0.3219	4.1368	3.3612	0.4780	2	47	5	0.3263	4.5104	3.7447	0.8874
2	29	6	0.3242	3.4761	2.7306	0.6629	2	47	6	0.3277	3.2928	2.5379	0.5829
2	29	7	0.3257	2.8001	2.0540	0.5123	2	47	7	0.3286	2.8224	2.0903	0.7141
2	29	8	0.3268	3.1388	2.4054	0.9926	2	47	8	0.3293	3.0051	2.2728	1.0625
2	29	9	0.3276	2.4586	1.7259	0.6272	2	47	9	0.3298	2.4844	1.9225	1.4478
2	29	10	0.3282	2.6513	1.9264	0.9729	2	47	10	0.3302	2.2904	1.5939	0.9074
2	30	3	0.3115	9.1367	8.3548	1.5096	2	48	3	0.3196	8.5853	7.8168	0.7148
2	30	4	0.3187	6.1609	5.4174	0.9602	2	48	4	0.3241	5.5892	4.8559	0.8487
2	30	5	0.3223	4.8501	4.1265	0.7435	2	48	5	0.3264	4.2044	3.4938	0.7170
2	30	6	0.3245	3.4151	2.7042	0.5026	2	48	6	0.3278	3.0454	2.3814	1.0084
2	30	7	0.3260	3.4253	2.7213	0.6831	2	48	7	0.3287	2.9553	2.2546	0.5661
2	30	8	0.3270	2.6774	1.9818	0.5219	2	48	8	0.3294	2.4559	1.8074	0.8187
2	30	9	0.3278	2.8553	2.1746	0.8725	2	48	9	0.3299	2.7556	2.0899	1.0420
2	30	10	0.3284	2.4476	1.7598	0.6520	2	48	10	0.3303	2.5180	1.8348	0.8932
2	31	3	0.3122	10.6330	9.8804	1.5706	2	49	3	0.3199	9.2716	8.5345	1.1818
2	31	4	0.3191	7.3227	6.6159	1.1111	2	49	4	0.3243	6.1855	5.4893	0.9429
2	31	5	0.3227	5.2228	4.5381	0.9195	2	49	5	0.3266	4.3386	3.6581	0.5589
2	31	6	0.3248	3.9653	3.2900	0.5598	2	49	6	0.3279	3.3750	2.7079	0.5656
2	31	7	0.3262	3.6398	2.9845	0.8454	2	49	7	0.3288	2.5646	1.9041	0.4677
2	31	8	0.3272	2.7914	2.1352	0.5343	2	49	8	0.3295	3.1901	2.5387	0.9631
2	31	9	0.3280	3.1609	2.5080	0.9734	2	49	9	0.3299	2.5385	1.8896	0.6473
2	31	10	0.3286	1.9274	1.2985	0.5621	2	49	10	0.3303	2.2036	1.6251	0.9152
2	32	3	0.3128	11.4540	10.7230	1.5605	2	50	3	0.3201	9.9718	9.2605	1.1846
2	32	4	0.3196	6.6068	5.9321	1.0635	2	50	4	0.3245	6.1801	5.5134	0.7093
2	32	5	0.3230	5.3265	4.6717	0.8162	2	50	5	0.3267	4.2064	3.5609	0.7580
2	32	6	0.3251	3.6989	3.0574	0.6249	2	50	6	0.3280	3.4543	2.8280	0.8100
2	32	7	0.3264	3.8177	3.1799	0.6039	2	50	7	0.3289	2.7708	2.1504	0.6010

2	32	8	0.3274	2.9388	2.3146	0.4908	2	50	8	0.3295	2.8458	2.2375	0.7090
2	32	9	0.3281	2.9803	2.3666	0.8149	2	50	9	0.3300	2.2803	1.6684	0.4746
2	32	10	0.3287	2.4520	1.8476	0.6052	2	50	10	0.3304	1.9228	1.3387	0.5903
3	15	3	0.2688	9.1155	7.7000	5.5799	3	33	3	0.3035	9.2389	7.8189	5.9988
3	15	4	0.2899	8.2629	6.9115	5.9730	3	33	4	0.3133	7.9383	6.6115	5.8998
3	15	5	0.3005	8.5291	7.1866	6.7965	3	33	5	0.3183	7.8034	6.5391	6.3257
3	15	6	0.3070	7.8454	6.6064	6.5568	3	33	6	0.3213	7.6057	6.4452	6.5076
3	15	7	0.3114	7.8107	6.5065	6.6727	3	33	7	0.3233	7.6429	6.3958	6.6531
3	15	8	0.3145	7.9194	6.6867	6.9910	3	33	8	0.3247	7.7188	6.4827	6.8734
3	15	9	0.3168	7.5565	6.3652	6.7656	3	33	9	0.3258	7.3456	6.1783	6.6405
3	15	10	0.3186	7.6017	6.4384	6.9117	3	33	10	0.3266	7.4790	6.2579	6.8244
3	16	3	0.2727	6.3900	5.1819	1.8876	3	34	3	0.3043	6.4258	5.2744	2.6494
3	16	4	0.2925	5.5115	4.3695	2.6796	3	34	4	0.3139	5.7127	4.5731	3.1151
3	16	5	0.3026	5.7472	4.6533	3.6707	3	34	5	0.3187	4.9503	3.8969	3.1576
3	16	6	0.3086	4.3348	3.3131	2.7984	3	34	6	0.3216	5.1609	4.1744	3.7972
3	16	7	0.3127	4.7440	3.7190	3.4275	3	34	7	0.3236	4.4427	3.4760	3.3165
3	16	8	0.3156	4.1187	3.1590	3.0551	3	34	8	0.3250	5.1734	4.1547	4.1415
3	16	9	0.3178	4.2498	3.3733	3.3887	3	34	9	0.3260	4.4066	3.4883	3.5840
3	16	10	0.3195	4.2474	3.4042	3.5013	3	34	10	0.3268	4.6943	3.7253	3.9197
3	17	3	0.2762	5.9335	4.8904	0.9028	3	35	3	0.3052	5.7191	4.6827	1.1548
3	17	4	0.2949	4.9194	3.8847	1.4032	3	35	4	0.3145	4.4951	3.5072	1.6144
3	17	5	0.3043	4.3298	3.3583	1.9280	3	35	5	0.3191	3.8767	2.9179	1.7406
3	17	6	0.3101	3.6417	2.7115	1.8130	3	35	6	0.3220	3.5629	2.6697	1.9808
3	17	7	0.3139	3.7648	2.7912	2.0992	3	35	7	0.3239	4.0771	3.1990	2.7338
3	17	8	0.3167	3.4757	2.6141	2.2347	3	35	8	0.3252	3.7572	2.8440	2.5399
3	17	9	0.3187	2.9667	2.1367	1.9172	3	35	9	0.3262	2.9123	2.1284	2.0069
3	17	10	0.3203	3.3925	2.5037	2.3682	3	35	10	0.3270	2.9963	2.2018	2.1649
3	18	3	0.2793	7.0296	6.0878	1.1093	3	36	3	0.3059	6.0730	5.1357	0.6989
3	18	4	0.2970	4.7197	3.8005	0.8470	3	36	4	0.3150	4.2469	3.3427	0.8988
3	18	5	0.3059	3.5852	2.7299	1.1642	3	36	5	0.3195	3.0454	2.1830	0.8910
3	18	6	0.3114	3.5115	2.6285	1.2966	3	36	6	0.3223	3.1098	2.2715	1.2896
3	18	7	0.3150	2.8642	2.0695	1.3284	3	36	7	0.3241	3.3485	2.5059	1.7436
3	18	8	0.3176	3.1605	2.3333	1.6723	3	36	8	0.3254	2.7478	1.9455	1.4590
3	18	9	0.3195	2.7736	1.9607	1.4990	3	36	9	0.3264	2.7774	2.0955	1.8338
3	18	10	0.3211	2.7306	1.9796	1.6851	3	36	10	0.3272	3.0182	2.3263	2.1380
3	19	3	0.2821	7.1269	6.2614	0.8014	3	37	3	0.3067	6.5717	5.7170	0.9228
3	19	4	0.2989	5.0040	4.1665	0.7339	3	37	4	0.3155	4.5004	3.6636	0.7586
3	19	5	0.3074	3.8568	3.0418	0.9073	3	37	5	0.3199	3.4492	2.6293	0.6867
3	19	6	0.3125	3.2591	2.4412	0.7873	3	37	6	0.3226	3.1461	2.3433	0.9656
3	19	7	0.3159	3.1322	2.3180	1.0449	3	37	7	0.3244	2.6199	1.9278	1.2414
3	19	8	0.3184	2.6907	1.9183	1.0610	3	37	8	0.3256	3.0045	2.2302	1.4817
3	19	9	0.3203	2.4986	1.7983	1.2614	3	37	9	0.3266	3.0212	2.2418	1.6577
3	19	10	0.3217	2.1659	1.5809	1.3143	3	37	10	0.3273	2.2614	1.5210	1.1373
3	20	3	0.2846	7.3677	6.5586	0.5807	3	38	3	0.3074	6.3840	5.5853	0.8164
3	20	4	0.3005	5.3615	4.5807	0.7736	3	38	4	0.3159	4.6103	3.8455	0.8503
3	20	5	0.3086	3.9438	3.1782	0.5913	3	38	5	0.3203	3.4938	2.7643	0.8856
3	20	6	0.3135	3.4324	2.6780	0.7425	3	38	6	0.3229	3.3900	2.6387	0.9690
3	20	7	0.3168	2.8761	2.1445	0.7996	3	38	7	0.3246	2.7224	1.9779	0.7566
3	20	8	0.3191	2.4116	1.6933	0.7349	3	38	8	0.3258	2.4475	1.7049	0.7390
3	20	9	0.3209	2.3369	1.6363	0.8694	3	38	9	0.3268	2.7314	2.0414	1.3719
3	20	10	0.3223	2.4008	1.7268	1.1289	3	38	10	0.3275	1.9965	1.3798	1.0056

3	21	3	0.2868	8.1158	7.3505	0.8194	3	39	3	0.3080	6.9364	6.1828	0.4985
3	21	4	0.3021	5.1261	4.3936	0.7210	3	39	4	0.3164	5.3711	4.6443	0.9376
3	21	5	0.3098	4.0882	3.3714	0.5394	3	39	5	0.3206	3.5167	2.8077	0.4626
3	21	6	0.3145	3.4035	2.7007	0.6513	3	39	6	0.3231	3.7618	3.0616	1.1847
3	21	7	0.3176	3.2682	2.5655	0.8777	3	39	7	0.3248	2.7950	2.1245	0.8735
3	21	8	0.3198	2.5905	1.9023	0.6564	3	39	8	0.3260	2.4089	1.7536	0.8373
3	21	9	0.3215	2.7315	2.0678	1.0946	3	39	9	0.3269	2.1088	1.4705	0.7882
3	21	10	0.3228	2.7596	2.1012	1.2906	3	39	10	0.3277	2.1584	1.4903	0.8209
3	22	3	0.2889	8.9928	8.2621	1.1671	3	40	3	0.3086	7.5933	6.8808	1.4952
3	22	4	0.3035	5.8009	5.1049	0.7091	3	40	4	0.3168	5.3544	4.6679	0.6800
3	22	5	0.3109	4.5820	3.9038	0.7084	3	40	5	0.3209	3.7493	3.0796	0.5426
3	22	6	0.3153	3.3959	2.7461	0.7897	3	40	6	0.3234	3.1551	2.5116	0.7757
3	22	7	0.3183	3.1825	2.5336	0.8239	3	40	7	0.3250	2.6234	1.9676	0.4724
3	22	8	0.3204	2.8914	2.2370	0.7670	3	40	8	0.3262	2.3634	1.7297	0.6445
3	22	9	0.3220	2.1343	1.5030	0.5470	3	40	9	0.3271	2.0064	1.3912	0.6090
3	22	10	0.3233	2.5557	1.9509	1.1049	3	40	10	0.3278	1.8419	1.2462	0.6676
3	23	3	0.2908	9.1328	8.4307	1.3118	3	41	3	0.3092	7.8055	7.1150	1.1339
3	23	4	0.3048	6.3999	5.7346	0.8878	3	41	4	0.3172	5.0257	4.3744	0.8415
3	23	5	0.3118	4.5956	3.9538	0.7230	3	41	5	0.3212	4.2114	3.5768	0.7510
3	23	6	0.3161	3.5280	2.9050	0.6937	3	41	6	0.3236	3.2431	2.6146	0.6178
3	23	7	0.3189	3.3045	2.6843	0.8484	3	41	7	0.3252	3.5275	2.9041	1.1217
3	23	8	0.3210	2.4201	1.8089	0.5240	3	41	8	0.3264	2.2769	1.6669	0.5056
3	23	9	0.3225	2.1499	1.5472	0.5476	3	41	9	0.3273	1.8453	1.2679	0.6079
3	23	10	0.3237	2.3550	1.7509	0.6946	3	41	10	0.3279	2.3522	1.8064	1.0995
3	24	3	0.2925	9.0023	7.6299	5.7614	3	42	3	0.3098	8.8420	7.4606	5.8509
3	24	4	0.3059	8.2084	6.8948	6.0963	3	42	4	0.3176	8.8414	7.4901	6.8447
3	24	5	0.3127	7.7435	6.4834	6.2023	3	42	5	0.3215	8.0735	6.8169	6.6679
3	24	6	0.3168	7.8258	6.5450	6.5573	3	42	6	0.3239	7.7441	6.5287	6.6445
3	24	7	0.3195	7.3363	6.1070	6.3127	3	42	7	0.3254	7.4457	6.2171	6.5122
3	24	8	0.3215	7.8124	6.5968	6.9400	3	42	8	0.3266	7.6723	6.4673	6.8830
3	24	9	0.3230	7.9242	6.7522	7.1837	3	42	9	0.3274	7.5944	6.3781	6.8950
3	24	10	0.3241	7.1733	6.0135	6.5170	3	42	10	0.3281	7.2575	6.1187	6.6694
3	25	3	0.2941	6.8116	5.6195	2.5656	3	43	3	0.3103	6.2234	5.0691	2.5882
3	25	4	0.3070	5.4388	4.3545	2.9324	3	43	4	0.3179	5.3866	4.2864	3.0220
3	25	5	0.3135	4.9506	3.9194	3.1325	3	43	5	0.3218	4.6298	3.5698	2.9056
3	25	6	0.3175	4.5434	3.5075	3.0468	3	43	6	0.3241	4.6773	3.7239	3.4227
3	25	7	0.3201	4.3945	3.4611	3.2619	3	43	7	0.3256	4.4754	3.5126	3.3995
3	25	8	0.3220	4.6443	3.7063	3.6511	3	43	8	0.3267	4.4197	3.4864	3.5142
3	25	9	0.3234	3.9705	3.0397	3.0977	3	43	9	0.3275	4.6120	3.6567	3.7943
3	25	10	0.3245	4.2250	3.2954	3.4438	3	43	10	0.3282	4.5868	3.5443	3.7935
3	26	3	0.2956	6.3867	5.3439	1.3979	3	44	3	0.3109	5.9499	4.9322	1.6873
3	26	4	0.3080	5.1915	4.1741	1.9007	3	44	4	0.3183	4.9556	3.9319	1.9266
3	26	5	0.3143	4.3283	3.3461	1.9732	3	44	5	0.3220	3.7616	2.8845	1.9868
3	26	6	0.3181	3.7338	2.7560	1.8532	3	44	6	0.3243	3.5640	2.6408	1.9645
3	26	7	0.3206	3.5919	2.7149	2.2059	3	44	7	0.3258	3.4577	2.5604	2.1469
3	26	8	0.3224	3.2078	2.3956	2.1059	3	44	8	0.3269	3.1369	2.2793	2.0575
3	26	9	0.3238	3.4365	2.5820	2.3930	3	44	9	0.3277	3.2630	2.5329	2.4525
3	26	10	0.3248	3.0596	2.2309	2.1527	3	44	10	0.3283	3.1555	2.3759	2.3724
3	27	3	0.2970	6.9979	6.0634	1.3715	3	45	3	0.3114	6.1467	5.2139	1.0057
3	27	4	0.3089	4.1073	3.2128	0.9311	3	45	4	0.3186	4.3123	3.4145	1.1411
3	27	5	0.3150	3.7972	2.9135	1.2158	3	45	5	0.3223	3.1765	2.2927	0.8545

3	27	6	0.3186	3.2162	2.3496	1.1824	3	45	6	0.3245	3.5308	2.6853	1.7053
3	27	7	0.3211	3.0823	2.2341	1.4054	3	45	7	0.3260	2.9907	2.1448	1.4590
3	27	8	0.3228	2.8702	2.0197	1.4052	3	45	8	0.3270	2.8696	2.1055	1.7003
3	27	9	0.3241	2.7850	2.0138	1.6433	3	45	9	0.3278	2.7408	2.0229	1.7742
3	27	10	0.3251	2.6630	1.9067	1.6550	3	45	10	0.3284	2.3714	1.6312	1.4694
3	28	3	0.2982	6.6887	5.8336	0.7637	3	46	3	0.3118	6.1313	5.2755	0.6143
3	28	4	0.3098	4.4837	3.6481	0.5599	3	46	4	0.3189	4.3249	3.4981	0.7407
3	28	5	0.3156	3.4805	2.6554	0.5711	3	46	5	0.3225	3.4143	2.6087	0.8693
3	28	6	0.3191	3.2853	2.4861	1.0317	3	46	6	0.3247	2.9814	2.2494	1.2716
3	28	7	0.3215	2.8604	2.0607	0.9607	3	46	7	0.3261	3.1063	2.3386	1.4488
3	28	8	0.3232	3.1292	2.3962	1.6601	3	46	8	0.3271	2.6935	1.9678	1.3789
3	28	9	0.3244	2.5825	1.8155	1.2002	3	46	9	0.3279	2.7492	1.9552	1.4201
3	28	10	0.3254	2.4439	1.6672	1.1842	3	46	10	0.3285	1.8575	1.4182	1.3361
3	29	3	0.2994	7.2180	6.4161	0.9149	3	47	3	0.3123	6.5075	5.7122	0.9761
3	29	4	0.3106	5.0213	4.2431	0.7456	3	47	4	0.3192	4.4651	3.6947	0.6246
3	29	5	0.3162	4.0129	3.2538	0.8981	3	47	5	0.3228	3.5170	2.7540	0.6296
3	29	6	0.3196	2.9548	2.2113	0.6088	3	47	6	0.3249	2.9374	2.1997	0.7676
3	29	7	0.3219	2.6821	1.9483	0.7295	3	47	7	0.3263	2.8333	2.0994	0.9741
3	29	8	0.3235	2.4079	1.7380	0.9878	3	47	8	0.3273	2.7387	2.0056	1.1186
3	29	9	0.3247	2.0190	1.3738	0.8742	3	47	9	0.3280	2.2987	1.6311	1.0876
3	29	10	0.3257	2.4325	1.7623	1.2214	3	47	10	0.3286	2.1573	1.4811	1.0374
3	30	3	0.3005	7.2136	6.4522	0.8182	3	48	3	0.3127	6.6258	5.8743	0.8486
3	30	4	0.3114	5.1398	4.4110	0.6784	3	48	4	0.3195	4.4388	3.7202	0.6361
3	30	5	0.3168	4.0591	3.3470	0.6588	3	48	5	0.3230	3.3034	2.5917	0.4104
3	30	6	0.3201	3.3387	2.6385	0.6872	3	48	6	0.3250	3.2080	2.5312	0.9764
3	30	7	0.3223	2.7962	2.1174	0.7651	3	48	7	0.3264	2.5833	1.9481	1.0040
3	30	8	0.3239	2.5985	1.9225	0.8158	3	48	8	0.3274	2.4802	1.8283	0.9487
3	30	9	0.3250	2.2742	1.6110	0.7600	3	48	9	0.3281	2.2001	1.5471	0.8330
3	30	10	0.3260	2.2007	1.5186	0.7447	3	48	10	0.3287	2.3644	1.7164	1.1215
3	31	3	0.3016	8.2685	7.5450	0.9392	3	49	3	0.3131	7.7211	7.0072	0.8968
3	31	4	0.3121	5.5961	4.9040	0.7521	3	49	4	0.3198	4.5963	3.9152	0.6414
3	31	5	0.3173	3.9690	3.2972	0.5905	3	49	5	0.3232	3.7601	3.0873	0.4784
3	31	6	0.3205	3.1446	2.4813	0.4763	3	49	6	0.3252	3.2986	2.6334	0.6616
3	31	7	0.3226	2.7229	2.0717	0.5313	3	49	7	0.3266	2.8098	2.1691	0.8097
3	31	8	0.3242	2.4352	1.7890	0.5522	3	49	8	0.3275	2.1889	1.5450	0.5059
3	31	9	0.3253	1.8837	1.2871	0.6732	3	49	9	0.3282	2.6326	1.9978	1.0724
3	31	10	0.3262	2.0225	1.3957	0.6012	3	49	10	0.3288	2.0145	1.4552	0.9467
3	32	3	0.3026	8.7404	8.0422	1.1189	3	50	3	0.3135	7.9549	7.2732	1.1692
3	32	4	0.3127	5.6206	4.9610	0.5978	3	50	4	0.3201	4.7720	4.1190	0.4796
3	32	5	0.3178	3.8901	3.2478	0.6291	3	50	5	0.3234	3.2629	2.6323	0.7734
3	32	6	0.3209	3.3917	2.7580	0.4859	3	50	6	0.3254	3.1768	2.5737	0.8694
3	32	7	0.3230	2.8175	2.1963	0.5072	3	50	7	0.3267	2.6390	2.0194	0.4533
3	32	8	0.3244	2.5965	1.9953	0.7149	3	50	8	0.3276	2.5161	1.9157	0.7781
3	32	9	0.3256	2.1654	1.5798	0.6511	3	50	9	0.3283	2.4179	1.8041	0.7477
3	32	10	0.3264	2.2157	1.6187	0.6888	3	50	10	0.3289	1.9125	1.3328	0.6378
4	15	3	0.2473	8.8164	7.4549	5.9950	4	33	3	0.2935	8.5477	7.1636	5.8848
4	15	4	0.2754	8.6574	7.3322	6.7836	4	33	4	0.3067	8.0261	6.7570	6.3746
4	15	5	0.2896	7.7331	6.5053	6.4171	4	33	5	0.3133	7.6721	6.3993	6.4114
4	15	6	0.2982	7.5972	6.3629	6.5271	4	33	6	0.3173	8.2052	6.9669	7.2154
4	15	7	0.3040	7.7972	6.5519	6.8926	4	33	7	0.3199	7.2568	6.0678	6.4559
4	15	8	0.3082	7.5929	6.4006	6.8443	4	33	8	0.3218	7.8161	6.6166	7.1224

4	15	9	0.3113	7.6315	6.4453	6.9774	4	33	9	0.3233	7.7265	6.4826	7.0968
4	15	10	0.3137	7.5508	6.3772	6.9727	4	33	10	0.3244	8.1542	6.9659	7.6174
4	16	3	0.2525	6.3512	5.1622	2.6469	4	34	3	0.2947	5.6222	4.4671	2.3979
4	16	4	0.2789	5.9333	4.7829	3.5049	4	34	4	0.3074	5.0250	3.9219	2.9232
4	16	5	0.2923	4.7201	3.7016	3.1364	4	34	5	0.3139	5.0016	3.9538	3.4775
4	16	6	0.3004	4.8294	3.8714	3.6154	4	34	6	0.3177	4.6881	3.6739	3.4931
4	16	7	0.3058	4.6863	3.7027	3.6299	4	34	7	0.3203	4.1640	3.1648	3.1710
4	16	8	0.3097	4.4046	3.4756	3.5393	4	34	8	0.3222	4.5590	3.6312	3.7596
4	16	9	0.3127	4.6363	3.6602	3.8362	4	34	9	0.3236	4.4751	3.5527	3.7731
4	16	10	0.3149	4.3481	3.5032	3.7208	4	34	10	0.3246	4.0905	3.1531	3.4518
4	17	3	0.2571	5.5908	4.5542	1.2694	4	35	3	0.2958	5.1908	4.1887	1.4994
4	17	4	0.2821	4.3618	3.3973	1.8031	4	35	4	0.3082	4.2863	3.3464	1.9875
4	17	5	0.2947	3.7705	2.8551	1.9310	4	35	5	0.3144	3.7166	2.7466	1.8501
4	17	6	0.3023	3.4134	2.5777	2.0741	4	35	6	0.3182	3.7648	2.9268	2.4958
4	17	7	0.3074	3.8486	2.9789	2.6283	4	35	7	0.3207	3.1969	2.3029	2.0368
4	17	8	0.3111	2.9160	2.1545	2.0079	4	35	8	0.3225	3.4111	2.5265	2.4132
4	17	9	0.3139	2.9629	2.1849	2.1303	4	35	9	0.3238	2.8258	2.1350	2.1378
4	17	10	0.3160	3.3030	2.5422	2.5684	4	35	10	0.3249	3.1575	2.4536	2.5243
4	18	3	0.2613	5.9932	5.0655	1.1111	4	36	3	0.2968	5.1679	4.2435	0.7921
4	18	4	0.2848	4.4021	3.5018	1.2921	4	36	4	0.3089	4.0261	3.1437	1.2697
4	18	5	0.2968	3.7515	2.8732	1.4594	4	36	5	0.3149	3.3432	2.5121	1.4418
4	18	6	0.3040	3.4153	2.5918	1.7366	4	36	6	0.3186	3.1942	2.3785	1.6577
4	18	7	0.3089	2.8158	2.0421	1.5251	4	36	7	0.3210	3.1334	2.2952	1.7768
4	18	8	0.3123	2.8243	2.0260	1.6340	4	36	8	0.3228	2.6844	1.9905	1.7528
4	18	9	0.3149	2.2652	1.5893	1.4114	4	36	9	0.3241	2.7150	2.0067	1.8574
4	18	10	0.3170	2.3430	1.5586	1.4166	4	36	10	0.3251	2.6003	1.8854	1.8187
4	19	3	0.2650	6.3451	5.4941	1.0055	4	37	3	0.2978	5.1241	4.2924	0.8831
4	19	4	0.2874	4.3448	3.5190	0.9227	4	37	4	0.3095	3.9410	3.1256	0.9164
4	19	5	0.2987	3.0514	2.2455	0.6850	4	37	5	0.3154	3.2459	2.4485	1.0209
4	19	6	0.3056	3.1450	2.3580	1.1940	4	37	6	0.3190	2.5912	1.8654	1.0997
4	19	7	0.3101	2.6835	1.9084	1.0859	4	37	7	0.3214	2.5054	1.7249	1.0146
4	19	8	0.3134	2.4692	1.7574	1.2506	4	37	8	0.3231	2.2497	1.5920	1.2420
4	19	9	0.3159	2.6348	1.8707	1.4211	4	37	9	0.3244	2.3877	1.7626	1.5197
4	19	10	0.3178	2.2872	1.5645	1.2797	4	37	10	0.3253	2.3787	1.7390	1.5633
4	20	3	0.2683	5.9347	5.1413	0.7675	4	38	3	0.2987	5.4929	4.7052	0.4982
4	20	4	0.2896	4.1972	3.4307	0.6310	4	38	4	0.3101	3.8098	3.0442	0.5043
4	20	5	0.3004	3.2390	2.4858	0.5613	4	38	5	0.3159	3.4437	2.6877	0.8768
4	20	6	0.3069	2.7682	2.0281	0.6737	4	38	6	0.3194	3.0513	2.3265	1.1280
4	20	7	0.3113	2.9276	2.2424	1.3433	4	38	7	0.3217	1.7925	1.1426	0.7333
4	20	8	0.3144	2.1516	1.4347	0.7040	4	38	8	0.3233	2.1764	1.5428	1.0713
4	20	9	0.3168	2.0359	1.4664	1.1485	4	38	9	0.3246	1.4401	0.7824	0.5057
4	20	10	0.3186	2.0278	1.3660	0.9874	4	38	10	0.3256	1.5228	0.8457	0.5587
4	21	3	0.2713	7.0838	6.3353	0.9760	4	39	3	0.2996	6.7006	5.9579	1.2013
4	21	4	0.2917	4.8749	4.1531	0.7972	4	39	4	0.3107	4.1869	3.4757	0.5737
4	21	5	0.3020	3.8455	3.1408	0.9394	4	39	5	0.3163	2.5154	1.8438	0.8608
4	21	6	0.3082	2.9637	2.2734	0.7460	4	39	6	0.3197	3.0404	2.3653	1.0313
4	21	7	0.3123	3.0082	2.3169	1.0599	4	39	7	0.3220	2.6984	2.0583	1.1747
4	21	8	0.3153	2.3447	1.6960	0.8920	4	39	8	0.3236	2.4759	1.8404	1.1403
4	21	9	0.3176	2.2224	1.5549	0.8562	4	39	9	0.3248	2.0341	1.4524	1.0262
4	21	10	0.3193	1.7222	1.2199	0.9883	4	39	10	0.3258	1.7716	1.1345	0.7224
4	22	3	0.2741	7.4887	6.7750	0.8577	4	40	3	0.3004	6.2376	5.5346	0.7705

4	22	4	0.2935	4.4038	3.7249	0.5754	4	40	4	0.3113	4.2995	3.6215	0.4605
4	22	5	0.3034	3.2471	2.5855	0.5960	4	40	5	0.3168	3.1476	2.5167	0.9745
4	22	6	0.3093	2.7172	2.0658	0.5346	4	40	6	0.3201	2.8869	2.2487	0.8262
4	22	7	0.3133	2.2348	1.5884	0.4288	4	40	7	0.3223	2.2616	1.6152	0.4792
4	22	8	0.3161	2.2996	1.6815	0.7712	4	40	8	0.3238	2.1199	1.4887	0.6351
4	22	9	0.3183	2.4250	1.7989	0.9759	4	40	9	0.3250	1.7611	1.1444	0.5379
4	22	10	0.3199	1.9463	1.3862	0.9081	4	40	10	0.3259	1.5236	0.9671	0.6378
4	23	3	0.2766	7.4538	6.7703	1.0490	4	41	3	0.3012	6.8121	6.1400	0.9656
4	23	4	0.2952	4.9074	4.2547	0.6133	4	41	4	0.3118	4.6202	3.9714	0.5515
4	23	5	0.3047	3.4737	2.8340	0.4449	4	41	5	0.3172	3.9634	3.3350	0.9641
4	23	6	0.3103	3.0146	2.3895	0.6026	4	41	6	0.3204	3.0149	2.3922	0.6888
4	23	7	0.3141	2.4451	1.8486	0.6728	4	41	7	0.3225	2.8827	2.2724	0.9732
4	23	8	0.3169	2.2524	1.6444	0.5521	4	41	8	0.3241	2.5291	1.9468	1.0158
4	23	9	0.3189	2.0393	1.4424	0.5801	4	41	9	0.3252	2.0301	1.4796	0.8642
4	23	10	0.3205	2.1136	1.6381	1.2095	4	41	10	0.3261	1.8858	1.3079	0.7046
4	24	3	0.2789	8.7120	7.3148	5.9150	4	42	3	0.3020	9.2916	7.8930	6.6866
4	24	4	0.2968	8.2355	6.9731	6.5288	4	42	4	0.3123	7.8494	6.5026	6.1554
4	24	5	0.3058	8.1792	6.8655	6.8259	4	42	5	0.3176	7.8783	6.6328	6.6901
4	24	6	0.3113	8.0093	6.7600	6.9706	4	42	6	0.3207	7.8123	6.5625	6.8505
4	24	7	0.3149	7.6017	6.4322	6.7828	4	42	7	0.3228	7.8950	6.6603	7.0981
4	24	8	0.3176	7.8294	6.5778	7.0787	4	42	8	0.3243	7.8863	6.6502	7.2003
4	24	9	0.3195	7.6255	6.4071	6.9822	4	42	9	0.3254	8.0649	6.8839	7.4880
4	24	10	0.3210	7.8274	6.7000	7.2936	4	42	10	0.3263	7.4337	6.2809	6.9287
4	25	3	0.2810	6.7867	5.6286	3.3389	4	43	3	0.3027	5.9436	4.7627	2.6692
4	25	4	0.2982	5.6246	4.5019	3.3691	4	43	4	0.3128	5.1188	4.0732	3.2170
4	25	5	0.3069	4.6236	3.6216	3.1298	4	43	5	0.3179	5.0788	3.9992	3.5579
4	25	6	0.3122	5.0922	4.0479	3.8095	4	43	6	0.3210	5.0014	4.0010	3.8642
4	25	7	0.3157	4.8694	3.9490	3.9196	4	43	7	0.3230	4.4822	3.5439	3.5828
4	25	8	0.3182	4.4800	3.4855	3.5916	4	43	8	0.3245	4.6762	3.7457	3.9036
4	25	9	0.3201	4.2986	3.4269	3.6062	4	43	9	0.3256	4.2946	3.3050	3.5731
4	25	10	0.3215	4.2683	3.3333	3.6066	4	43	10	0.3265	4.7758	3.8536	4.1729
4	26	3	0.2830	5.5881	4.5692	1.5701	4	44	3	0.3034	5.0846	4.0679	1.4410
4	26	4	0.2996	3.9425	2.9799	1.5229	4	44	4	0.3133	4.0592	3.0570	1.5894
4	26	5	0.3079	4.0605	3.0984	2.1379	4	44	5	0.3183	3.4704	2.4914	1.6519
4	26	6	0.3130	3.7197	2.7979	2.2372	4	44	6	0.3213	3.4537	2.5914	2.1901
4	26	7	0.3163	3.3199	2.5037	2.2356	4	44	7	0.3233	3.1476	2.4048	2.2371
4	26	8	0.3188	3.4943	2.6510	2.5096	4	44	8	0.3247	3.4834	2.6105	2.5346
4	26	9	0.3206	3.1399	2.4139	2.3914	4	44	9	0.3258	3.2104	2.3703	2.4038
4	26	10	0.3220	3.1007	2.2433	2.3060	4	44	10	0.3266	3.5877	2.6940	2.8225
4	27	3	0.2848	5.2674	4.3411	0.7746	4	45	3	0.3040	4.5768	3.6590	0.7321
4	27	4	0.3008	3.9168	3.0427	1.1999	4	45	4	0.3137	3.7270	2.8428	1.1143
4	27	5	0.3089	3.4361	2.5756	1.3400	4	45	5	0.3186	3.2061	2.3453	1.2704
4	27	6	0.3137	3.0001	2.1650	1.3679	4	45	6	0.3215	3.3818	2.5825	1.9309
4	27	7	0.3170	3.0708	2.2410	1.6853	4	45	7	0.3235	2.6291	1.8397	1.4330
4	27	8	0.3193	2.8610	2.1128	1.8000	4	45	8	0.3249	2.2021	1.4989	1.3009
4	27	9	0.3210	2.8882	2.0866	1.8659	4	45	9	0.3259	2.5736	1.8859	1.7718
4	27	10	0.3224	2.5247	1.8041	1.7086	4	45	10	0.3268	2.4310	1.7066	1.6659
4	28	3	0.2865	5.8531	5.0036	0.7257	4	46	3	0.3047	5.1606	4.3263	0.8759
4	28	4	0.3020	3.6669	2.8514	0.7112	4	46	4	0.3141	3.5743	2.7593	0.7077
4	28	5	0.3097	3.7123	2.9018	1.2654	4	46	5	0.3189	2.7808	1.9777	0.6312
4	28	6	0.3144	2.9011	2.1363	1.1463	4	46	6	0.3218	2.4261	1.6634	0.8513

4	28	7	0.3176	2.2168	1.5836	1.1783	4	46	7	0.3237	2.4897	1.7939	1.2878
4	28	8	0.3198	2.5585	1.8097	1.2820	4	46	8	0.3251	2.0690	1.4746	1.2305
4	28	9	0.3215	2.1606	1.4821	1.1782	4	46	9	0.3261	2.5149	1.7865	1.4954
4	28	10	0.3228	1.9302	1.4345	1.3276	4	46	10	0.3269	1.7570	1.0988	0.9593
4	29	3	0.2881	6.3701	5.5803	1.0927	4	47	3	0.3053	5.9229	5.1379	0.8589
4	29	4	0.3030	4.0850	3.3202	0.5972	4	47	4	0.3146	4.0870	3.3262	0.8542
4	29	5	0.3105	3.3670	2.6277	0.8640	4	47	5	0.3192	3.2417	2.4927	0.8440
4	29	6	0.3151	2.7169	2.0036	0.8791	4	47	6	0.3220	3.1629	2.4163	1.1908
4	29	7	0.3181	2.3079	1.6417	0.9504	4	47	7	0.3239	2.7745	2.1368	1.4998
4	29	8	0.3203	1.9153	1.2053	0.5791	4	47	8	0.3253	2.1312	1.4371	0.8861
4	29	9	0.3219	2.2166	1.5544	1.1003	4	47	9	0.3263	2.3423	1.6915	1.3089
4	29	10	0.3232	2.0332	1.3920	1.0689	4	47	10	0.3270	2.1911	1.5448	1.2672
4	30	3	0.2896	6.0339	5.2918	0.8979	4	48	3	0.3058	5.5957	4.8568	0.5089
4	30	4	0.3040	4.3218	3.6030	0.7282	4	48	4	0.3149	4.2871	3.5758	0.7928
4	30	5	0.3113	2.9136	2.2164	0.5939	4	48	5	0.3195	2.8337	2.1433	0.6052
4	30	6	0.3157	2.7255	2.0286	0.5552	4	48	6	0.3223	2.6046	1.9411	0.8164
4	30	7	0.3186	2.2837	1.6080	0.5995	4	48	7	0.3241	1.7308	1.1738	0.9070
4	30	8	0.3207	2.3463	1.6636	0.7892	4	48	8	0.3254	2.1299	1.4936	0.8763
4	30	9	0.3223	1.6419	0.9977	0.5123	4	48	9	0.3264	1.6904	1.0738	0.6680
4	30	10	0.3235	2.1749	1.5486	1.0861	4	48	10	0.3272	1.8841	1.2705	0.9127
4	31	3	0.2910	6.7705	6.0650	0.7155	4	49	3	0.3064	6.0589	5.3581	0.5984
4	31	4	0.3050	4.2370	3.5611	0.6735	4	49	4	0.3153	3.6004	2.9294	0.6144
4	31	5	0.3120	3.4443	2.7759	0.5262	4	49	5	0.3198	2.9492	2.2880	0.4719
4	31	6	0.3162	2.5611	1.9249	0.6523	4	49	6	0.3225	2.6594	2.0501	0.9314
4	31	7	0.3191	2.3096	1.6843	0.6548	4	49	7	0.3243	2.0099	1.3903	0.5627
4	31	8	0.3211	1.9914	1.4413	0.9183	4	49	8	0.3256	2.0434	1.4979	0.9820
4	31	9	0.3226	2.0253	1.4043	0.6878	4	49	9	0.3265	1.5047	0.9168	0.5183
4	31	10	0.3238	1.8754	1.2977	0.8167	4	49	10	0.3273	1.7488	1.1660	0.7435
4	32	3	0.2923	7.0375	6.3624	1.1481	4	50	3	0.3069	6.1988	5.5287	0.5234
4	32	4	0.3058	4.4864	3.8389	0.5936	4	50	4	0.3157	3.8622	3.2189	0.5118
4	32	5	0.3127	4.1811	3.5443	0.9708	4	50	5	0.3201	2.5285	1.9022	0.6955
4	32	6	0.3168	2.6076	2.0322	0.9341	4	50	6	0.3227	2.6002	2.0063	0.8065
4	32	7	0.3195	2.5626	1.9554	0.6807	4	50	7	0.3245	1.9546	1.3468	0.4363
4	32	8	0.3215	1.8931	1.2995	0.5286	4	50	8	0.3257	2.4291	1.8384	0.9248
4	32	9	0.3230	1.5462	1.0380	0.7736	4	50	9	0.3267	1.8749	1.2873	0.6036
4	32	10	0.3241	1.6106	1.1098	0.7786	4	50	10	0.3274	1.4758	0.9433	0.5951
5	15	3	0.2258	8.1387	6.7596	5.6770	5	33	3	0.2836	8.4967	7.1393	6.2325
5	15	4	0.2609	7.9488	6.6782	6.4111	5	33	4	0.3000	7.9117	6.6137	6.4428
5	15	5	0.2787	7.9183	6.6915	6.7891	5	33	5	0.3083	7.7450	6.5644	6.7290
5	15	6	0.2895	7.7463	6.5341	6.8451	5	33	6	0.3133	7.5523	6.3575	6.7223
5	15	7	0.2967	7.2072	6.0085	6.4584	5	33	7	0.3166	8.0069	6.8239	7.3187
5	15	8	0.3019	7.5413	6.3457	6.8993	5	33	8	0.3190	7.0994	5.9234	6.5076
5	15	9	0.3058	8.0321	6.8847	7.4900	5	33	9	0.3208	7.7339	6.4933	7.1913
5	15	10	0.3088	7.6470	6.4688	7.1514	5	33	10	0.3221	7.6931	6.5360	7.2389
5	16	3	0.2323	5.7512	4.6294	2.8497	5	34	3	0.2850	5.8128	4.7045	3.1304
5	16	4	0.2653	5.1883	4.1405	3.3471	5	34	4	0.3010	4.7119	3.6612	2.9854
5	16	5	0.2821	4.9435	3.8790	3.4993	5	34	5	0.3090	5.2884	4.2621	3.9882
5	16	6	0.2922	4.6167	3.6491	3.5568	5	34	6	0.3138	4.9167	3.9127	3.8866
5	16	7	0.2990	5.0556	3.9932	4.0756	5	34	7	0.3171	4.6196	3.7003	3.8231
5	16	8	0.3038	4.7493	3.8719	4.0445	5	34	8	0.3194	4.1987	3.2984	3.5206
5	16	9	0.3075	4.3444	3.4746	3.7252	5	34	9	0.3211	4.0644	3.2259	3.5026

5	16	10	0.3103	4.2595	3.3880	3.7046	5	34	10	0.3225	4.5498	3.6800	4.0331
5	17	3	0.2381	4.7787	3.7953	1.5299	5	35	3	0.2864	4.6120	3.5913	1.2934
5	17	4	0.2692	4.1098	3.1253	1.7901	5	35	4	0.3019	3.7152	2.8008	1.8128
5	17	5	0.2850	3.7151	2.7585	2.0047	5	35	5	0.3097	3.6357	2.7198	2.1251
5	17	6	0.2946	3.6933	2.8563	2.5058	5	35	6	0.3144	3.6917	2.7664	2.4421
5	17	7	0.3010	3.4750	2.6477	2.4781	5	35	7	0.3175	3.3922	2.5490	2.4387
5	17	8	0.3056	3.2322	2.3811	2.3398	5	35	8	0.3198	3.0707	2.4136	2.4242
5	17	9	0.3090	2.9414	2.1746	2.2278	5	35	9	0.3215	2.7501	2.1334	2.2034
5	17	10	0.3117	3.1521	2.3565	2.4887	5	35	10	0.3228	3.2781	2.6110	2.7506
5	18	3	0.2432	5.2726	4.3457	1.1183	5	36	3	0.2877	4.5573	3.6470	0.8968
5	18	4	0.2727	3.9141	3.0175	1.2409	5	36	4	0.3028	3.4793	2.6629	1.4708
5	18	5	0.2877	2.8774	2.0382	1.1250	5	36	5	0.3103	2.6751	1.9023	1.2602
5	18	6	0.2967	2.6969	1.8852	1.2988	5	36	6	0.3149	2.8135	2.0053	1.4932
5	18	7	0.3028	2.5078	1.7321	1.3808	5	36	7	0.3180	2.6031	1.8287	1.5357
5	18	8	0.3071	2.8227	1.9995	1.7541	5	36	8	0.3202	2.3324	1.6788	1.5641
5	18	9	0.3103	1.9835	1.4040	1.3462	5	36	9	0.3218	2.3568	1.6213	1.5740
5	18	10	0.3129	2.6252	1.9456	1.9310	5	36	10	0.3231	2.8020	2.0614	2.0928
5	19	3	0.2479	5.2647	4.4197	0.7453	5	37	3	0.2889	4.4898	3.6522	0.6088
5	19	4	0.2759	3.6561	2.8674	1.0619	5	37	4	0.3036	3.8595	3.0473	1.1814
5	19	5	0.2900	3.7613	2.9436	1.5345	5	37	5	0.3110	3.1621	2.3576	1.1563
5	19	6	0.2986	3.2781	2.5384	1.7328	5	37	6	0.3154	2.7088	1.9618	1.2657
5	19	7	0.3043	2.3841	1.6326	1.0766	5	37	7	0.3184	2.3651	1.6072	1.1116
5	19	8	0.3085	2.2068	1.5825	1.3160	5	37	8	0.3205	2.3806	1.6662	1.3686
5	19	9	0.3115	2.3438	1.5995	1.3350	5	37	9	0.3221	3.0089	2.2943	2.1006
5	19	10	0.3140	2.0408	1.4284	1.3176	5	37	10	0.3234	1.5072	1.0281	0.9953
5	20	3	0.2520	5.2478	4.4711	0.7995	5	38	3	0.2900	4.8228	4.0518	0.8991
5	20	4	0.2787	3.3155	2.5555	0.4772	5	38	4	0.3043	3.4594	2.7115	0.6882
5	20	5	0.2922	2.9248	2.1801	0.6425	5	38	5	0.3115	2.9488	2.2331	0.9805
5	20	6	0.3003	2.6055	1.8699	0.7974	5	38	6	0.3159	2.1766	1.4928	0.7974
5	20	7	0.3058	2.6738	1.9502	1.1624	5	38	7	0.3188	1.9528	1.2513	0.6705
5	20	8	0.3097	2.0449	1.3881	0.9507	5	38	8	0.3208	1.9938	1.3420	0.9660
5	20	9	0.3126	2.1760	1.5008	1.1301	5	38	9	0.3224	2.1891	1.5298	1.2356
5	20	10	0.3149	1.7847	1.3766	1.2948	5	38	10	0.3236	2.1522	1.4778	1.2685
5	21	3	0.2558	5.2287	4.4938	0.5234	5	39	3	0.2911	5.3264	4.5944	0.7201
5	21	4	0.2813	4.2386	3.5252	0.8185	5	39	4	0.3051	3.4492	2.7427	0.6101
5	21	5	0.2941	3.0285	2.3260	0.5828	5	39	5	0.3121	2.7769	2.0814	0.5667
5	21	6	0.3019	2.5761	1.9029	0.7790	5	39	6	0.3163	2.7232	2.0677	1.0650
5	21	7	0.3071	2.0716	1.4668	0.8900	5	39	7	0.3191	2.3443	1.6631	0.8419
5	21	8	0.3108	1.8898	1.2763	0.7778	5	39	8	0.3212	1.9120	1.2493	0.6913
5	21	9	0.3136	1.7838	1.1975	0.8429	5	39	9	0.3227	1.6203	0.9785	0.5975
5	21	10	0.3158	1.6215	1.0632	0.8207	5	39	10	0.3239	1.4778	0.9619	0.8084
5	22	3	0.2593	5.2048	4.5131	1.1303	5	40	3	0.2922	5.8892	5.1965	0.8209
5	22	4	0.2836	3.9553	3.2775	0.4587	5	40	4	0.3058	3.3743	2.7205	0.7433
5	22	5	0.2959	3.2928	2.6365	0.7592	5	40	5	0.3126	2.6886	2.0395	0.5320
5	22	6	0.3033	2.8946	2.2400	0.8024	5	40	6	0.3167	2.8618	2.2043	0.8692
5	22	7	0.3083	2.1282	1.5226	0.7446	5	40	7	0.3195	2.0628	1.5676	1.1646
5	22	8	0.3118	1.9046	1.3148	0.7272	5	40	8	0.3215	1.5408	0.9164	0.3747
5	22	9	0.3145	1.7893	1.1957	0.6969	5	40	9	0.3229	1.9363	1.3664	0.9346
5	22	10	0.3166	1.4333	0.8999	0.6552	5	40	10	0.3241	1.4438	0.8861	0.6140
5	23	3	0.2624	6.5536	5.8863	0.8858	5	41	3	0.2932	5.3350	4.6749	0.8564
5	23	4	0.2857	4.4097	3.7678	0.7581	5	41	4	0.3065	3.9803	3.3690	1.2479

5	23	5	0.2975	3.3538	2.7255	0.6287	5	41	5	0.3131	3.0556	2.4356	0.6227
5	23	6	0.3046	2.8301	2.2207	0.7642	5	41	6	0.3172	2.4656	1.8498	0.5304
5	23	7	0.3094	2.2065	1.6551	0.8906	5	41	7	0.3198	2.1456	1.5535	0.6399
5	23	8	0.3128	2.0453	1.4753	0.7591	5	41	8	0.3218	2.2831	1.7169	1.0114
5	23	9	0.3153	1.8716	1.3192	0.7684	5	41	9	0.3232	1.8060	1.2077	0.5861
5	23	10	0.3173	1.6362	1.0912	0.6791	5	41	10	0.3243	2.0992	1.5304	1.0620
5	24	3	0.2653	8.0446	6.7228	5.7789	5	42	3	0.2941	8.2760	6.9276	6.0991
5	24	4	0.2877	7.8335	6.4884	6.2577	5	42	4	0.3071	7.8548	6.5906	6.4704
5	24	5	0.2990	7.8747	6.5824	6.7272	5	42	5	0.3136	7.4004	6.1188	6.3350
5	24	6	0.3058	8.0853	6.9085	7.2401	5	42	6	0.3175	7.9601	6.7699	7.1604
5	24	7	0.3103	7.5269	6.3421	6.8127	5	42	7	0.3202	7.7459	6.5259	7.0601
5	24	8	0.3136	7.7187	6.5008	7.0883	5	42	8	0.3220	7.3906	6.1941	6.8102
5	24	9	0.3161	7.1884	6.0084	6.6497	5	42	9	0.3234	7.5029	6.2634	6.9777
5	24	10	0.3180	7.6762	6.4552	7.1826	5	42	10	0.3245	7.5759	6.4227	7.1370
5	25	3	0.2680	5.5933	4.4400	2.6665	5	43	3	0.2950	5.4865	4.3773	2.8961
5	25	4	0.2895	4.7281	3.7263	3.0550	5	43	4	0.3077	4.5186	3.4385	2.7892
5	25	5	0.3003	4.8963	3.8513	3.5288	5	43	5	0.3141	4.4866	3.4323	3.1943
5	25	6	0.3069	4.5540	3.5193	3.4565	5	43	6	0.3179	4.8481	3.8431	3.8495
5	25	7	0.3113	4.4425	3.5784	3.6669	5	43	7	0.3205	4.3388	3.3446	3.5064
5	25	8	0.3144	4.5249	3.5396	3.7647	5	43	8	0.3223	4.3767	3.4741	3.7194
5	25	9	0.3167	4.3753	3.4178	3.7229	5	43	9	0.3237	4.4282	3.5303	3.8521
5	25	10	0.3186	4.5181	3.6663	3.9934	5	43	10	0.3247	3.8918	3.1440	3.4486
5	26	3	0.2704	4.5673	3.5774	1.3914	5	44	3	0.2959	5.0872	4.0741	1.8389
5	26	4	0.2911	3.7122	2.7484	1.5664	5	44	4	0.3083	3.9908	3.0298	1.9633
5	26	5	0.3016	3.7736	2.9353	2.3818	5	44	5	0.3145	3.0846	2.2034	1.7115
5	26	6	0.3079	3.1474	2.2309	1.8772	5	44	6	0.3183	3.0747	2.1871	1.9266
5	26	7	0.3121	3.1821	2.3377	2.1981	5	44	7	0.3208	3.2408	2.4336	2.3577
5	26	8	0.3151	3.0969	2.3106	2.3000	5	44	8	0.3225	3.6561	2.8079	2.8497
5	26	9	0.3174	3.3879	2.6736	2.7442	5	44	9	0.3239	3.1199	2.3529	2.4715
5	26	10	0.3191	3.2453	2.5176	2.6556	5	44	10	0.3249	2.7719	1.9980	2.1828
5	27	3	0.2727	4.8240	3.9071	0.9038	5	45	3	0.2967	4.6702	3.7616	1.0649
5	27	4	0.2927	3.5391	2.7132	1.4138	5	45	4	0.3088	3.1939	2.3221	0.9728
5	27	5	0.3028	3.5344	2.7256	1.8534	5	45	5	0.3149	3.1799	2.3847	1.6545
5	27	6	0.3088	2.6315	1.8147	1.2624	5	45	6	0.3186	2.3655	1.7127	1.4349
5	27	7	0.3129	2.7864	1.9692	1.6088	5	45	7	0.3210	2.6297	1.9387	1.7248
5	27	8	0.3158	2.7203	2.0240	1.8648	5	45	8	0.3228	2.5288	1.8060	1.6899
5	27	9	0.3180	2.2973	1.5825	1.5148	5	45	9	0.3241	2.5710	1.8296	1.8046
5	27	10	0.3197	2.5607	1.7339	1.7433	5	45	10	0.3251	2.6145	1.9533	1.9984
5	28	3	0.2749	5.0116	4.1768	0.9057	5	46	3	0.2975	4.4010	3.5722	0.7085
5	28	4	0.2941	3.4525	2.6346	0.7139	5	46	4	0.3094	3.1803	2.4145	1.0439
5	28	5	0.3038	3.0571	2.2759	1.1134	5	46	5	0.3153	2.4358	1.6516	0.6886
5	28	6	0.3097	2.8613	2.0939	1.3050	5	46	6	0.3189	2.6465	1.8743	1.1664
5	28	7	0.3136	2.0749	1.4892	1.2268	5	46	7	0.3213	2.0428	1.4778	1.2613
5	28	8	0.3164	2.1711	1.4436	1.1113	5	46	8	0.3230	1.9506	1.2479	1.0033
5	28	9	0.3185	2.3396	1.6871	1.5033	5	46	9	0.3243	2.1606	1.5150	1.3828
5	28	10	0.3202	2.2979	1.5533	1.4229	5	46	10	0.3253	2.2471	1.5121	1.4380
5	29	3	0.2768	5.4039	4.6266	0.9238	5	47	3	0.2982	5.1970	4.4175	0.7988
5	29	4	0.2955	3.8077	3.0533	0.8509	5	47	4	0.3099	3.3517	2.5950	0.5917
5	29	5	0.3048	2.5061	1.7739	0.5728	5	47	5	0.3157	2.8276	2.1056	0.9072
5	29	6	0.3105	2.7988	2.0874	1.1363	5	47	6	0.3192	2.5168	1.7880	0.8996
5	29	7	0.3143	2.2277	1.5178	0.8360	5	47	7	0.3216	2.1560	1.5250	1.0773

5	29	8	0.3170	2.5853	1.9241	1.4666	5	47	8	0.3232	2.1044	1.4278	1.0354
5	29	9	0.3190	2.5207	1.8797	1.5622	5	47	9	0.3245	1.9813	1.3468	1.1057
5	29	10	0.3206	1.9077	1.3974	1.2633	5	47	10	0.3255	1.7988	1.2096	1.0752
5	30	3	0.2787	5.5517	4.8186	0.6654	5	48	3	0.2990	4.5731	3.8454	0.6051
5	30	4	0.2967	3.2807	2.5705	0.5010	5	48	4	0.3103	3.8560	3.1706	1.1073
5	30	5	0.3058	2.8834	2.1853	0.5803	5	48	5	0.3161	3.0838	2.3850	0.8379
5	30	6	0.3113	2.5248	1.9219	1.1413	5	48	6	0.3195	2.4155	1.7362	0.7205
5	30	7	0.3149	2.3980	1.7478	0.9669	5	48	7	0.3218	2.4505	1.7845	1.0411
5	30	8	0.3175	2.1080	1.4819	0.9463	5	48	8	0.3234	1.8540	1.1926	0.6653
5	30	9	0.3195	1.4962	0.8951	0.5982	5	48	9	0.3247	2.0947	1.4631	1.0887
5	30	10	0.3210	2.0732	1.4883	1.2014	5	48	10	0.3256	1.8774	1.3941	1.2352
5	31	3	0.2804	4.8437	4.1488	1.1251	5	49	3	0.2997	5.5138	4.8230	0.8275
5	31	4	0.2979	3.9971	3.3277	0.6024	5	49	4	0.3108	3.4161	2.7516	0.4767
5	31	5	0.3067	2.6122	1.9552	0.4424	5	49	5	0.3164	2.9650	2.3171	0.7489
5	31	6	0.3120	2.3207	1.6843	0.5634	5	49	6	0.3198	2.1263	1.5010	0.5767
5	31	7	0.3155	2.0068	1.3746	0.5200	5	49	7	0.3220	2.1913	1.6118	0.9547
5	31	8	0.3180	1.9317	1.3412	0.7651	5	49	8	0.3236	1.8844	1.2893	0.7530
5	31	9	0.3199	1.9650	1.3619	0.8389	5	49	9	0.3249	1.6295	1.0306	0.6121
5	31	10	0.3214	1.2825	0.7404	0.5419	5	49	10	0.3258	1.4325	0.8908	0.6549
5	32	3	0.2821	6.0806	5.4180	0.7024	5	50	3	0.3003	6.1704	5.5134	1.2669
5	32	4	0.2990	3.4244	2.7874	0.6759	5	50	4	0.3113	3.6046	2.9698	0.5693
5	32	5	0.3075	3.2007	2.5809	0.7013	5	50	5	0.3167	3.1417	2.5158	0.6806
5	32	6	0.3126	2.4319	1.8191	0.5034	5	50	6	0.3201	2.6297	2.0146	0.6976
5	32	7	0.3161	2.3314	1.7331	0.7171	5	50	7	0.3223	2.0192	1.4211	0.5433
5	32	8	0.3185	1.7831	1.1886	0.4762	5	50	8	0.3238	1.9825	1.3929	0.6993
5	32	9	0.3204	1.4596	0.9729	0.7291	5	50	9	0.3250	1.7180	1.3055	1.1033
5	32	10	0.3218	1.3985	0.8782	0.5961	5	50	10	0.3259	1.6642	1.1061	0.7183
6	15	3	0.2043	8.6662	7.2814	6.4896	6	33	3	0.2736	8.1170	6.7710	6.1327
6	15	4	0.2464	8.1515	6.8365	6.7548	6	33	4	0.2933	7.7044	6.4401	6.4401
6	15	5	0.2678	7.5370	6.3167	6.5524	6	33	5	0.3033	7.9397	6.6825	6.9867
6	15	6	0.2807	7.3518	6.1727	6.5826	6	33	6	0.3092	7.4335	6.2266	6.6939
6	15	7	0.2894	7.5371	6.3552	6.8915	6	33	7	0.3132	7.4122	6.1823	6.7822
6	15	8	0.2956	7.6617	6.4786	7.1062	6	33	8	0.3161	7.7682	6.5286	7.2238
6	15	9	0.3003	7.5116	6.2955	7.0114	6	33	9	0.3182	7.3233	6.1706	6.8745
6	15	10	0.3039	7.0877	5.9396	6.6633	6	33	10	0.3199	7.8302	6.6566	7.4249
6	16	3	0.2121	5.8775	4.7882	3.3980	6	34	3	0.2754	5.7771	4.6543	3.3303
6	16	4	0.2517	5.0010	3.9294	3.3294	6	34	4	0.2945	5.2339	4.1517	3.6354
6	16	5	0.2718	4.9981	4.0478	3.8727	6	34	5	0.3041	5.0139	4.0793	3.9691
6	16	6	0.2840	4.4410	3.4632	3.4985	6	34	6	0.3099	4.2410	3.3606	3.4379
6	16	7	0.2921	4.5234	3.5419	3.7272	6	34	7	0.3138	4.3606	3.6205	3.7828
6	16	8	0.2979	4.0312	3.1950	3.4345	6	34	8	0.3166	4.3609	3.3925	3.7213
6	16	9	0.3023	4.1939	3.3942	3.6884	6	34	9	0.3187	4.3658	3.4695	3.8401
6	16	10	0.3057	4.5317	3.5591	3.9970	6	34	10	0.3203	4.3231	3.4536	3.8652
6	17	3	0.2190	4.7214	3.7095	1.6128	6	35	3	0.2770	4.1392	3.1859	1.5866
6	17	4	0.2564	3.8954	2.9465	1.9593	6	35	4	0.2956	3.7480	2.8669	2.1084
6	17	5	0.2754	4.2298	3.2919	2.7542	6	35	5	0.3050	3.2992	2.4123	2.0115
6	17	6	0.2868	3.6717	2.8695	2.6677	6	35	6	0.3106	3.4787	2.5547	2.3740
6	17	7	0.2945	3.0815	2.2519	2.1997	6	35	7	0.3144	3.0503	2.3731	2.3703
6	17	8	0.3000	2.8431	2.0198	2.0815	6	35	8	0.3171	3.4399	2.7397	2.8256
6	17	9	0.3041	3.2523	2.5012	2.6359	6	35	9	0.3191	2.8785	2.1395	2.3023
6	17	10	0.3074	3.1912	2.4549	2.6475	6	35	10	0.3207	3.2127	2.4858	2.7059

6	18	3	0.2252	4.3328	3.4351	1.0045	6	36	3	0.2785	4.4916	3.5869	1.2049
6	18	4	0.2606	4.0368	3.1741	1.8332	6	36	4	0.2966	3.7234	2.8855	1.7473
6	18	5	0.2785	2.6202	1.8993	1.4026	6	36	5	0.3057	3.4107	2.5766	1.8896
6	18	6	0.2894	2.5529	1.7967	1.4234	6	36	6	0.3112	2.4140	1.6856	1.3943
6	18	7	0.2966	2.6464	1.8052	1.5355	6	36	7	0.3149	2.3454	1.7154	1.5950
6	18	8	0.3018	2.7116	2.0165	1.9278	6	36	8	0.3175	2.5691	1.9481	1.9086
6	18	9	0.3057	2.3460	1.6261	1.6235	6	36	9	0.3195	2.5128	1.7881	1.8253
6	18	10	0.3088	2.4591	1.7353	1.8059	6	36	10	0.3210	2.5464	1.8864	1.9806
6	19	3	0.2308	4.2538	3.4204	0.5623	6	37	3	0.2800	3.8759	3.0552	0.6765
6	19	4	0.2644	3.2594	2.4624	0.9248	6	37	4	0.2976	3.4708	2.6576	1.0823
6	19	5	0.2814	2.3537	1.5959	0.7836	6	37	5	0.3065	3.0681	2.3274	1.5070
6	19	6	0.2917	2.8189	2.0811	1.4717	6	37	6	0.3118	2.1807	1.5476	1.2035
6	19	7	0.2986	2.2475	1.5256	1.1477	6	37	7	0.3154	2.3404	1.6528	1.3555
6	19	8	0.3035	2.0811	1.3880	1.1659	6	37	8	0.3179	2.4489	1.7238	1.5276
6	19	9	0.3072	2.3791	1.7133	1.5928	6	37	9	0.3199	2.4322	1.7533	1.6719
6	19	10	0.3101	2.1099	1.4035	1.3573	6	37	10	0.3214	1.8212	1.4268	1.4253
6	20	3	0.2358	4.9572	4.1806	0.7411	6	38	3	0.2814	4.7454	3.9703	0.8412
6	20	4	0.2678	3.3465	2.6020	0.7545	6	38	4	0.2986	3.4084	2.6602	0.8715
6	20	5	0.2840	2.3635	1.6562	0.7154	6	38	5	0.3072	2.2743	1.5479	0.5719
6	20	6	0.2937	1.8601	1.2135	0.7586	6	38	6	0.3124	1.7841	1.2343	1.0061
6	20	7	0.3003	2.0727	1.3632	0.8070	6	38	7	0.3159	2.4796	1.8173	1.3723
6	20	8	0.3050	1.6149	0.9549	0.6541	6	38	8	0.3184	1.7398	1.0869	0.8163
6	20	9	0.3085	1.6467	1.1250	0.9982	6	38	9	0.3202	2.0652	1.4172	1.2324
6	20	10	0.3112	1.5759	0.9567	0.8352	6	38	10	0.3217	2.0228	1.3329	1.2189
6	21	3	0.2403	5.0680	4.3408	0.7053	6	39	3	0.2827	4.3686	3.6483	0.4986
6	21	4	0.2708	2.6130	1.9112	0.6464	6	39	4	0.2994	3.2554	2.5541	0.6579
6	21	5	0.2863	2.3624	1.7394	0.9525	6	39	5	0.3079	2.4105	1.7450	0.7126
6	21	6	0.2956	2.1240	1.4961	0.8068	6	39	6	0.3129	2.4963	1.8389	1.0110
6	21	7	0.3018	2.3020	1.7349	1.2512	6	39	7	0.3163	1.6375	0.9976	0.5354
6	21	8	0.3063	2.1523	1.5714	1.1828	6	39	8	0.3187	1.6140	0.9847	0.6271
6	21	9	0.3097	1.8402	1.2448	0.9526	6	39	9	0.3206	1.8448	1.2041	0.9088
6	21	10	0.3123	2.1042	1.6010	1.4325	6	39	10	0.3220	2.1635	1.5916	1.4201
6	22	3	0.2444	5.0935	4.4047	0.7134	6	40	3	0.2840	4.9562	4.2713	0.8058
6	22	4	0.2736	3.6931	3.0294	0.7051	6	40	4	0.3003	3.3313	2.6727	0.6626
6	22	5	0.2884	3.0444	2.4050	0.9107	6	40	5	0.3085	2.4748	1.8376	0.6241
6	22	6	0.2973	2.5738	1.9222	0.7317	6	40	6	0.3134	2.2479	1.6338	0.7757
6	22	7	0.3033	2.1731	1.5677	0.8596	6	40	7	0.3167	1.9754	1.3781	0.7624
6	22	8	0.3075	2.1602	1.5253	0.8860	6	40	8	0.3191	2.0739	1.4532	0.9155
6	22	9	0.3107	1.9063	1.3278	0.9345	6	40	9	0.3209	1.5833	0.9838	0.6334
6	22	10	0.3132	1.9089	1.3409	1.0450	6	40	10	0.3223	1.7221	1.1337	0.8684
6	23	3	0.2482	5.4847	4.8247	0.3940	6	41	3	0.2851	5.2758	4.6199	0.6800
6	23	4	0.2762	2.8679	2.2489	0.9350	6	41	4	0.3011	3.2374	2.6038	0.4137
6	23	5	0.2903	2.7595	2.1353	0.4518	6	41	5	0.3091	2.6273	2.0141	0.5464
6	23	6	0.2989	2.0619	1.4748	0.6094	6	41	6	0.3139	2.3231	1.7129	0.6200
6	23	7	0.3046	1.8410	1.2828	0.6783	6	41	7	0.3171	2.2074	1.6152	0.8159
6	23	8	0.3086	2.0777	1.4773	0.7583	6	41	8	0.3194	1.8650	1.4039	1.0839
6	23	9	0.3117	1.7712	1.2064	0.7309	6	41	9	0.3212	1.8615	1.3255	0.9338
6	23	10	0.3141	1.9917	1.4753	1.1413	6	41	10	0.3225	0.8002	0.6332	0.8210
6	24	3	0.2517	8.3120	7.0121	6.3447	6	42	3	0.2863	8.3531	7.0358	6.4667
6	24	4	0.2785	8.4922	7.2189	7.1807	6	42	4	0.3018	7.5059	6.3193	6.3521
6	24	5	0.2921	7.4939	6.3224	6.5753	6	42	5	0.3097	7.4527	6.3242	6.6159

6	24	6	0.3003	7.4023	6.2707	6.6838	6	42	6	0.3144	7.6342	6.4879	6.9494
6	24	7	0.3057	7.9732	6.7195	7.3141	6	42	7	0.3175	7.5416	6.3148	6.9316
6	24	8	0.3097	7.8485	6.6605	7.3082	6	42	8	0.3198	7.4716	6.2559	6.9512
6	24	9	0.3126	7.1756	6.0046	6.7059	6	42	9	0.3215	7.4635	6.3081	7.0268
6	24	10	0.3149	7.6412	6.4901	7.2301	6	42	10	0.3228	7.4186	6.1564	6.9984
6	25	3	0.2549	5.2128	4.1275	2.8404	6	43	3	0.2874	5.1188	4.0056	2.7977
6	25	4	0.2807	4.8541	3.7762	3.2207	6	43	4	0.3026	4.1919	3.2114	2.8174
6	25	5	0.2937	5.1284	4.1175	3.9617	6	43	5	0.3102	4.5527	3.6282	3.5483
6	25	6	0.3016	5.0297	3.9844	4.0536	6	43	6	0.3148	4.3122	3.4187	3.5195
6	25	7	0.3068	4.6738	3.7253	3.9259	6	43	7	0.3179	4.1306	3.2791	3.4887
6	25	8	0.3106	4.4807	3.5189	3.8261	6	43	8	0.3201	4.1053	3.1211	3.4729
6	25	9	0.3134	4.3379	3.3574	3.7515	6	43	9	0.3217	4.1894	3.2874	3.6748
6	25	10	0.3156	4.3178	3.3929	3.8207	6	43	10	0.3230	4.4751	3.6028	4.0297
6	26	3	0.2579	4.7150	3.7278	1.8168	6	44	3	0.2884	4.6223	3.6226	1.8190
6	26	4	0.2827	3.8659	2.9408	2.0542	6	44	4	0.3033	3.1745	2.2748	1.5720
6	26	5	0.2952	3.7084	2.8047	2.3452	6	44	5	0.3107	3.4387	2.5969	2.2575
6	26	6	0.3028	3.6872	2.8132	2.6157	6	44	6	0.3152	2.9986	2.1930	2.0746
6	26	7	0.3079	2.9835	2.1578	2.1308	6	44	7	0.3182	2.8539	2.0715	2.0883
6	26	8	0.3115	3.1389	2.3035	2.3899	6	44	8	0.3204	3.0401	2.2155	2.3406
6	26	9	0.3142	2.7862	2.1189	2.2456	6	44	9	0.3220	3.4123	2.5983	2.8039
6	26	10	0.3163	2.6709	1.9729	2.1621	6	44	10	0.3233	2.8931	2.0893	2.3521
6	27	3	0.2606	4.5957	3.6917	1.2027	6	45	3	0.2894	4.3828	3.4908	1.2896
6	27	4	0.2846	3.3108	2.4221	1.1036	6	45	4	0.3039	3.3848	2.5070	1.3199
6	27	5	0.2966	2.7830	1.9803	1.3466	6	45	5	0.3112	2.9365	2.1189	1.5137
6	27	6	0.3039	2.8718	2.0206	1.5743	6	45	6	0.3156	2.9571	2.1216	1.7581
6	27	7	0.3088	3.1395	2.5087	2.3541	6	45	7	0.3186	2.7099	2.0741	1.9612
6	27	8	0.3123	2.3676	1.6215	1.5482	6	45	8	0.3207	2.4351	1.7823	1.7572
6	27	9	0.3149	2.0351	1.4362	1.4485	6	45	9	0.3223	2.2707	1.4624	1.5267
6	27	10	0.3169	2.4021	1.6644	1.7556	6	45	10	0.3235	2.6829	1.9376	2.0678
6	28	3	0.2632	4.6785	3.8426	0.8977	6	46	3	0.2903	4.0060	3.1773	0.5775
6	28	4	0.2863	3.5680	2.7621	1.1538	6	46	4	0.3046	3.1277	2.3194	0.8750
6	28	5	0.2979	2.7004	1.9552	1.1384	6	46	5	0.3117	2.7626	1.9898	1.1555
6	28	6	0.3050	2.4385	1.7425	1.2578	6	46	6	0.3160	2.7211	1.9394	1.3757
6	28	7	0.3097	2.2281	1.5065	1.1599	6	46	7	0.3189	2.0743	1.4147	1.1766
6	28	8	0.3130	2.0870	1.4740	1.3172	6	46	8	0.3209	2.1973	1.4724	1.3037
6	28	9	0.3156	1.7977	1.2091	1.1373	6	46	9	0.3225	1.8432	1.3253	1.2901
6	28	10	0.3175	2.0457	1.4876	1.4708	6	46	10	0.3237	1.9814	1.3676	1.3794
6	29	3	0.2655	4.2083	3.4455	0.6789	6	47	3	0.2912	3.6842	2.9217	0.6248
6	29	4	0.2879	3.3489	2.5948	0.7437	6	47	4	0.3052	2.8287	2.0921	0.6410
6	29	5	0.2991	2.9675	2.2295	1.0143	6	47	5	0.3122	2.5195	1.7977	0.8000
6	29	6	0.3059	2.4541	1.7405	0.9773	6	47	6	0.3164	2.2154	1.4846	0.7686
6	29	7	0.3105	1.8817	1.2169	0.7974	6	47	7	0.3192	2.5771	1.9452	1.5582
6	29	8	0.3137	1.9178	1.3128	1.0562	6	47	8	0.3212	2.1194	1.4760	1.2177
6	29	9	0.3162	1.3850	0.9622	0.9180	6	47	9	0.3227	1.8750	1.2950	1.1662
6	29	10	0.3181	1.5778	0.9151	0.7938	6	47	10	0.3239	1.5459	0.9942	0.9384
6	30	3	0.2678	4.0185	3.3045	0.8867	6	48	3	0.2921	4.2150	3.4919	0.3797
6	30	4	0.2894	3.4624	2.7610	0.6916	6	48	4	0.3057	3.0143	2.3229	0.6768
6	30	5	0.3003	3.0575	2.3694	1.0011	6	48	5	0.3126	2.2894	1.6036	0.5018
6	30	6	0.3068	2.9459	2.2946	1.4104	6	48	6	0.3167	2.4930	1.8536	1.1172
6	30	7	0.3112	1.9880	1.3367	0.7496	6	48	7	0.3195	1.9287	1.3142	0.8572
6	30	8	0.3144	1.6027	1.1227	0.9531	6	48	8	0.3215	1.7015	1.0812	0.7461

6	30	9	0.3167	2.1974	1.6177	1.3419	6	48	9	0.3229	2.0089	1.3618	1.0803
6	30	10	0.3186	1.6213	1.1037	0.9655	6	48	10	0.3241	1.6921	1.2354	1.1448
6	31	3	0.2698	4.5196	3.8346	0.5749	6	49	3	0.2929	4.9628	4.2754	0.6216
6	31	4	0.2908	3.3395	2.6895	0.7361	6	49	4	0.3063	3.0717	2.4166	0.6775
6	31	5	0.3013	2.9820	2.3244	0.7495	6	49	5	0.3130	2.8643	2.2286	0.9084
6	31	6	0.3077	1.9640	1.3226	0.4292	6	49	6	0.3171	1.9936	1.3648	0.5401
6	31	7	0.3119	2.0822	1.4771	0.8024	6	49	7	0.3198	1.6721	1.1354	0.7795
6	31	8	0.3150	1.9456	1.3844	0.9512	6	49	8	0.3217	1.9401	1.3483	0.8968
6	31	9	0.3173	1.4142	0.8228	0.4916	6	49	9	0.3232	1.3981	0.9232	0.7710
6	31	10	0.3190	1.9045	1.3359	1.0599	6	49	10	0.3243	2.1866	1.6074	1.3591
6	32	3	0.2718	4.9801	4.3237	0.4150	6	50	3	0.2937	4.8379	4.1918	0.9532
6	32	4	0.2921	3.8061	3.1731	0.7581	6	50	4	0.3068	3.5550	2.9239	0.6050
6	32	5	0.3023	3.0633	2.4372	0.6930	6	50	5	0.3134	2.7019	2.0863	0.6056
6	32	6	0.3085	2.2421	1.6492	0.6602	6	50	6	0.3174	2.3132	1.6990	0.5979
6	32	7	0.3126	1.9663	1.3794	0.6216	6	50	7	0.3200	1.5646	1.0029	0.5463
6	32	8	0.3156	1.8236	1.2433	0.6420	6	50	8	0.3219	1.8160	1.2975	0.8869
6	32	9	0.3178	1.1588	0.7537	0.7328	6	50	9	0.3234	1.3776	0.9049	0.7076
6	32	10	0.3195	1.0094	0.6802	0.7373	6	50	10	0.3245	1.2235	0.8693	0.8260
7	15	3	0.1828	7.8426	6.5724	6.0814	7	33	3	0.2637	8.2350	6.9413	6.5279
7	15	4	0.2319	7.9476	6.6830	6.7523	7	33	4	0.2867	7.3576	6.1250	6.2522
7	15	5	0.2568	7.8440	6.6350	6.9762	7	33	5	0.2982	7.8643	6.6328	7.0277
7	15	6	0.2719	7.3466	6.1522	6.6542	7	33	6	0.3052	7.6437	6.4688	6.9982
7	15	7	0.2821	7.8666	6.6665	7.2860	7	33	7	0.3099	7.6873	6.5538	7.1649
7	15	8	0.2893	7.3913	6.1835	6.8881	7	33	8	0.3132	7.4363	6.2695	6.9742
7	15	9	0.2948	7.2308	6.0364	6.7935	7	33	9	0.3157	7.3921	6.2799	7.0034
7	15	10	0.2990	7.5151	6.3329	7.1297	7	33	10	0.3177	7.0656	5.8942	6.7029
7	16	3	0.1919	5.1818	4.0791	2.9374	7	34	3	0.2657	4.9495	3.8256	2.7648
7	16	4	0.2381	4.7278	3.7029	3.3090	7	34	4	0.2880	4.6585	3.6449	3.3276
7	16	5	0.2615	4.5537	3.5512	3.4909	7	34	5	0.2993	4.0882	3.1040	3.0979
7	16	6	0.2757	4.2764	3.3222	3.4543	7	34	6	0.3060	4.2210	3.3759	3.5247
7	16	7	0.2852	4.6061	3.6188	3.8933	7	34	7	0.3106	4.3434	3.4391	3.7186
7	16	8	0.2920	4.2860	3.3905	3.7214	7	34	8	0.3138	4.4800	3.5491	3.9270
7	16	9	0.2972	3.9456	3.0558	3.4492	7	34	9	0.3162	4.1088	3.2954	3.6730
7	16	10	0.3011	4.2613	3.3407	3.8067	7	34	10	0.3181	4.2926	3.3800	3.8655
7	17	3	0.2000	4.8853	3.8788	2.0965	7	35	3	0.2676	3.9732	3.0085	1.5861
7	17	4	0.2436	3.5896	2.6392	1.8601	7	35	4	0.2893	3.9040	2.9532	2.2463
7	17	5	0.2657	3.7294	2.8494	2.5104	7	35	5	0.3002	3.4472	2.4706	2.1453
7	17	6	0.2791	3.4217	2.5589	2.4496	7	35	6	0.3068	3.0330	2.2168	2.1619
7	17	7	0.2880	3.2869	2.5413	2.5758	7	35	7	0.3112	2.7801	2.0315	2.0986
7	17	8	0.2944	2.5107	1.7291	1.8583	7	35	8	0.3144	2.7739	2.0506	2.1977
7	17	9	0.2993	3.1741	2.4656	2.6509	7	35	9	0.3167	2.8937	2.0492	2.3102
7	17	10	0.3030	3.3047	2.4940	2.7774	7	35	10	0.3186	2.9435	2.2438	2.5000
7	18	3	0.2072	4.0895	3.2174	1.2723	7	36	3	0.2694	4.0198	3.1255	1.1451
7	18	4	0.2485	3.1439	2.3114	1.3443	7	36	4	0.2905	2.9017	2.0970	1.2901
7	18	5	0.2694	2.5373	1.7078	1.1465	7	36	5	0.3011	2.8929	2.0559	1.5273
7	18	6	0.2821	2.4760	1.7161	1.4426	7	36	6	0.3076	2.4407	1.6505	1.4080
7	18	7	0.2905	2.8619	2.1916	2.0856	7	36	7	0.3118	2.5841	1.9638	1.9033
7	18	8	0.2966	2.1522	1.3401	1.3278	7	36	8	0.3149	2.5980	1.9320	1.9544
7	18	9	0.3011	2.1363	1.3977	1.4710	7	36	9	0.3172	1.7941	1.0917	1.1856
7	18	10	0.3047	2.2902	1.7497	1.8407	7	36	10	0.3190	2.1792	1.6456	1.7529
7	19	3	0.2137	4.1051	3.2824	0.8197	7	37	3	0.2711	3.7792	2.9584	0.6369

7	19	4	0.2529	3.0565	2.2870	1.0940	7	37	4	0.2917	3.3070	2.5586	1.4690
7	19	5	0.2727	2.4973	1.7660	1.0937	7	37	5	0.3020	2.3317	1.6229	1.0671
7	19	6	0.2847	2.2487	1.5481	1.1456	7	37	6	0.3082	2.4473	1.6999	1.2804
7	19	7	0.2928	1.8061	1.1574	0.9597	7	37	7	0.3124	2.1002	1.5505	1.4186
7	19	8	0.2985	2.0650	1.4809	1.3830	7	37	8	0.3154	1.9647	1.2637	1.1685
7	19	9	0.3028	1.8868	1.2896	1.2579	7	37	9	0.3176	2.2641	1.6811	1.6762
7	19	10	0.3062	1.9120	1.3840	1.4044	7	37	10	0.3194	1.9823	1.3205	1.3810
7	20	3	0.2195	4.7629	3.9981	1.0496	7	38	3	0.2727	3.5030	2.7430	0.6176
7	20	4	0.2568	3.2060	2.4692	0.9296	7	38	4	0.2928	3.3009	2.5609	1.0777
7	20	5	0.2757	2.8042	2.0802	1.0936	7	38	5	0.3028	2.3514	1.6709	0.9433
7	20	6	0.2871	1.8289	1.2804	1.0317	7	38	6	0.3089	2.1158	1.4061	0.8410
7	20	7	0.2948	1.7623	1.1149	0.7905	7	38	7	0.3130	1.8635	1.3131	1.1048
7	20	8	0.3002	1.7980	1.2887	1.1525	7	38	8	0.3159	1.9569	1.3093	1.1135
7	20	9	0.3043	1.4040	0.9411	0.8888	7	38	9	0.3180	1.7993	1.2544	1.1798
7	20	10	0.3076	1.3808	0.7371	0.6809	7	38	10	0.3197	1.8149	1.2637	1.2422
7	21	3	0.2248	3.9473	3.2318	0.6422	7	39	3	0.2743	4.5137	3.7950	0.8431
7	21	4	0.2604	2.9901	2.2981	0.6383	7	39	4	0.2938	3.6296	2.9351	1.2063
7	21	5	0.2784	2.7943	2.1091	0.9191	7	39	5	0.3036	2.5583	1.8876	0.8758
7	21	6	0.2893	1.7229	1.0983	0.6257	7	39	6	0.3095	2.1959	1.5492	0.9011
7	21	7	0.2966	2.0213	1.3872	0.9011	7	39	7	0.3135	2.0309	1.4508	1.0913
7	21	8	0.3018	1.2930	0.7585	0.6233	7	39	8	0.3163	1.7285	1.1846	0.9727
7	21	9	0.3057	1.8245	1.1670	0.9056	7	39	9	0.3184	2.0239	1.3984	1.1989
7	21	10	0.3088	1.6718	1.0802	0.9469	7	39	10	0.3201	1.6621	1.0658	0.9634
7	22	3	0.2296	4.8296	4.1489	0.7118	7	40	3	0.2757	4.6115	3.9340	0.6772
7	22	4	0.2637	2.8064	2.1755	0.8444	7	40	4	0.2948	3.2428	2.5958	0.8512
7	22	5	0.2809	2.3762	1.7405	0.5953	7	40	5	0.3043	2.4128	1.8189	0.9401
7	22	6	0.2913	2.1531	1.5190	0.6409	7	40	6	0.3101	2.0025	1.4117	0.7732
7	22	7	0.2982	2.1525	1.5547	0.9716	7	40	7	0.3140	2.0806	1.4809	0.9515
7	22	8	0.3032	1.8286	1.2242	0.7850	7	40	8	0.3167	1.4367	0.8310	0.4809
7	22	9	0.3070	1.5453	1.1470	1.0386	7	40	9	0.3188	1.8625	1.3227	1.0927
7	22	10	0.3099	1.3999	0.9723	0.8782	7	40	10	0.3204	1.8616	1.2788	1.0938
7	23	3	0.2340	4.9850	4.3336	0.5813	7	41	3	0.2771	4.5374	3.8942	0.6263
7	23	4	0.2667	3.2137	2.5849	0.5367	7	41	4	0.2957	2.9259	2.3031	0.4789
7	23	5	0.2832	2.7391	2.1322	0.7571	7	41	5	0.3051	2.4046	1.7942	0.5488
7	23	6	0.2931	2.0640	1.4692	0.5750	7	41	6	0.3107	2.2094	1.6041	0.6804
7	23	7	0.2998	2.0100	1.4129	0.6742	7	41	7	0.3144	1.9314	1.3544	0.7401
7	23	8	0.3045	2.3268	1.7611	1.2298	7	41	8	0.3171	1.9161	1.3279	0.8304
7	23	9	0.3081	1.5355	1.1560	1.0309	7	41	9	0.3191	1.3434	0.8375	0.6183
7	23	10	0.3109	1.6892	1.0973	0.7708	7	41	10	0.3207	1.3426	0.8156	0.6192
7	24	3	0.2381	7.9595	6.6610	6.2020	7	42	3	0.2784	8.3987	7.0946	6.7192
7	24	4	0.2694	7.5970	6.4315	6.5231	7	42	4	0.2966	7.9628	6.7277	6.8836
7	24	5	0.2852	7.4086	6.2185	6.5757	7	42	5	0.3057	7.5304	6.3476	6.7448
7	24	6	0.2948	7.7876	6.6303	7.1341	7	42	6	0.3112	7.1532	5.9964	6.5321
7	24	7	0.3011	7.2123	6.0423	6.6581	7	42	7	0.3149	7.3485	6.1613	6.8162
7	24	8	0.3057	7.5413	6.3255	7.0493	7	42	8	0.3175	7.2271	6.0700	6.7799
7	24	9	0.3092	7.3567	6.2022	6.9441	7	42	9	0.3195	7.6236	6.4803	7.2364
7	24	10	0.3118	7.5156	6.3171	7.1362	7	42	10	0.3210	7.4747	6.3808	7.1426
7	25	3	0.2418	5.3662	4.2450	3.1231	7	43	3	0.2797	5.0748	4.0518	3.1923
7	25	4	0.2719	4.7578	3.7848	3.4533	7	43	4	0.2974	4.6705	3.6441	3.3529
7	25	5	0.2871	4.0855	3.0906	3.0587	7	43	5	0.3064	4.9109	3.9302	3.9484
7	25	6	0.2963	4.4503	3.5109	3.6621	7	43	6	0.3117	4.1897	3.3684	3.5276

7	25	7	0.3024	4.3472	3.4948	3.7395	7	43	7	0.3153	4.4409	3.6089	3.8757
7	25	8	0.3068	4.0941	3.2344	3.5630	7	43	8	0.3179	4.6890	3.7154	4.1290
7	25	9	0.3101	4.6609	3.6634	4.1317	7	43	9	0.3198	4.2420	3.2870	3.7567
7	25	10	0.3127	4.2962	3.4313	3.8771	7	43	10	0.3213	4.6168	3.6694	4.1889
7	26	3	0.2453	4.2064	3.2611	1.8372	7	44	3	0.2809	4.2296	3.2564	1.8299
7	26	4	0.2743	3.6123	2.7202	2.0634	7	44	4	0.2982	4.1404	3.2151	2.5724
7	26	5	0.2889	3.2835	2.4377	2.1540	7	44	5	0.3070	3.0066	2.1208	1.8750
7	26	6	0.2977	3.2815	2.4761	2.4002	7	44	6	0.3122	3.3833	2.5582	2.5225
7	26	7	0.3036	2.8983	2.1269	2.1804	7	44	7	0.3157	3.1411	2.2647	2.3699
7	26	8	0.3078	2.9135	2.1707	2.3097	7	44	8	0.3182	2.9293	2.2539	2.4021
7	26	9	0.3110	2.8910	2.1951	2.3861	7	44	9	0.3201	3.1340	2.3093	2.5788
7	26	10	0.3135	3.1052	2.4424	2.6725	7	44	10	0.3216	2.9550	2.2027	2.4950
7	27	3	0.2485	4.1247	3.2321	1.2020	7	45	3	0.2821	3.7043	2.8320	1.1479
7	27	4	0.2764	2.7367	1.8802	0.9294	7	45	4	0.2990	2.8699	2.1876	1.6758
7	27	5	0.2905	3.0708	2.2682	1.7466	7	45	5	0.3076	2.8686	2.1018	1.6817
7	27	6	0.2990	2.7882	1.9341	1.6219	7	45	6	0.3127	2.6124	1.8986	1.7117
7	27	7	0.3047	2.5139	1.8253	1.7363	7	45	7	0.3161	2.1840	1.6112	1.5736
7	27	8	0.3088	2.8025	2.1870	2.1932	7	45	8	0.3186	2.4541	1.7270	1.7688
7	27	9	0.3118	2.4538	1.7707	1.8526	7	45	9	0.3204	2.3617	1.7392	1.8352
7	27	10	0.3142	2.3729	1.7738	1.8924	7	45	10	0.3218	2.0008	1.3702	1.5160
7	28	3	0.2515	4.4237	3.5946	1.0964	7	46	3	0.2832	3.8595	3.0507	0.9564
7	28	4	0.2784	3.2976	2.4903	1.1368	7	46	4	0.2998	2.4288	1.7208	1.0241
7	28	5	0.2920	2.5875	1.8819	1.2775	7	46	5	0.3081	2.8521	2.0913	1.4188
7	28	6	0.3002	2.3104	1.5762	1.1533	7	46	6	0.3131	1.9482	1.2273	0.8923
7	28	7	0.3057	2.9261	2.1622	1.8803	7	46	7	0.3165	1.9575	1.2726	1.0964
7	28	8	0.3096	2.1140	1.4648	1.3628	7	46	8	0.3189	2.2625	1.5626	1.4838
7	28	9	0.3126	1.7351	1.0850	1.0646	7	46	9	0.3207	2.7116	2.0447	2.0548
7	28	10	0.3149	2.0096	1.3903	1.4313	7	46	10	0.3221	1.9104	1.4478	1.4901
7	29	3	0.2542	4.1024	3.3334	0.5548	7	47	3	0.2842	4.0952	3.3372	0.8113
7	29	4	0.2803	3.3460	2.6027	1.0459	7	47	4	0.3005	3.0959	2.3580	0.9595
7	29	5	0.2934	2.5238	1.8551	1.1099	7	47	5	0.3086	2.1630	1.4665	0.7689
7	29	6	0.3014	1.7986	1.2545	1.0183	7	47	6	0.3136	2.1791	1.5768	1.2050
7	29	7	0.3067	2.1856	1.5318	1.1884	7	47	7	0.3168	2.3750	1.6995	1.3702
7	29	8	0.3105	1.5308	0.8542	0.6384	7	47	8	0.3192	2.1013	1.4183	1.2209
7	29	9	0.3133	1.6145	1.0851	1.0089	7	47	9	0.3210	1.3916	0.7969	0.7329
7	29	10	0.3155	1.6531	1.0132	0.9704	7	47	10	0.3223	1.4903	0.8923	0.8824
7	30	3	0.2568	4.4932	3.7793	0.8540	7	48	3	0.2852	3.5394	2.8306	0.5729
7	30	4	0.2821	3.1168	2.4180	0.6682	7	48	4	0.3011	2.7952	2.1120	0.6776
7	30	5	0.2948	2.4374	1.7588	0.7005	7	48	5	0.3092	2.5975	1.9425	1.0206
7	30	6	0.3024	1.7454	1.1876	0.8636	7	48	6	0.3140	2.0344	1.4939	1.1182
7	30	7	0.3076	1.8934	1.3658	1.0773	7	48	7	0.3172	1.5721	1.0816	0.9094
7	30	8	0.3112	1.9150	1.3387	1.0698	7	48	8	0.3195	1.6761	1.0891	0.8568
7	30	9	0.3140	1.3090	0.7843	0.6708	7	48	9	0.3212	2.6365	1.9592	1.7512
7	30	10	0.3161	2.0769	1.5348	1.4275	7	48	10	0.3226	1.2914	0.6966	0.6182
7	31	3	0.2593	4.4901	3.8074	0.5276	7	49	3	0.2862	4.5912	3.9127	0.7665
7	31	4	0.2837	2.7789	2.1523	0.8499	7	49	4	0.3018	3.5932	2.9398	1.1165
7	31	5	0.2960	2.3698	1.7233	0.5509	7	49	5	0.3096	2.4541	1.8168	0.7082
7	31	6	0.3034	2.5420	1.8998	0.9904	7	49	6	0.3144	2.3280	1.7063	0.9403
7	31	7	0.3084	1.8889	1.3027	0.7974	7	49	7	0.3175	2.1115	1.5680	1.1502
7	31	8	0.3119	1.2289	0.7467	0.6432	7	49	8	0.3198	1.5612	1.1236	0.9747
7	31	9	0.3146	1.7540	1.1327	0.8075	7	49	9	0.3215	1.3857	0.9945	0.9170

7	31	10	0.3167	1.0939	0.7020	0.6695	7	49	10	0.3228	1.5951	1.0003	0.8318
7	32	3	0.2615	4.4710	3.8264	0.4030	7	50	3	0.2871	4.4824	3.8382	0.5933
7	32	4	0.2852	3.2977	2.6698	0.5848	7	50	4	0.3024	2.9911	2.3672	0.5500
7	32	5	0.2972	2.2978	1.6857	0.4750	7	50	5	0.3101	2.4963	1.8818	0.5905
7	32	6	0.3043	2.1005	1.5171	0.6938	7	50	6	0.3147	1.9795	1.3691	0.4802
7	32	7	0.3092	1.6765	1.0916	0.5039	7	50	7	0.3178	1.7939	1.2636	0.8191
7	32	8	0.3126	1.4138	0.9122	0.6546	7	50	8	0.3200	1.8733	1.2921	0.8313
7	32	9	0.3152	1.6097	1.0463	0.6979	7	50	9	0.3217	1.1784	0.6707	0.5028
7	32	10	0.3172	1.5038	1.0106	0.8178	7	50	10	0.3230	1.6509	1.1190	0.9101
8	15	3	0.1613	8.3439	7.0366	6.7012	8	33	3	0.2537	7.7447	6.4604	6.2081
8	15	4	0.2174	7.8764	6.6214	6.8072	8	33	4	0.2800	7.5136	6.2542	6.4910
8	15	5	0.2459	7.7271	6.4865	6.9267	8	33	5	0.2932	7.9300	6.7473	7.2012
8	15	6	0.2632	7.8819	6.6607	7.2475	8	33	6	0.3012	7.4582	6.3043	6.8832
8	15	7	0.2747	7.1279	5.9631	6.6173	8	33	7	0.3065	7.5733	6.3964	7.0839
8	15	8	0.2830	7.3764	6.2177	6.9400	8	33	8	0.3103	7.7211	6.5286	7.2960
8	15	9	0.2893	7.5312	6.3496	7.1421	8	33	9	0.3132	7.2740	6.2260	6.9401
8	15	10	0.2941	7.4221	6.1947	7.0643	8	33	10	0.3154	7.3055	6.1513	6.9820
8	16	3	0.1717	5.1093	4.0553	3.1908	8	34	3	0.2560	5.4987	4.3952	3.5453
8	16	4	0.2245	4.1290	3.2891	3.1003	8	34	4	0.2816	4.9267	3.9443	3.7533
8	16	5	0.2513	4.5568	3.5582	3.6004	8	34	5	0.2944	4.4194	3.6103	3.6758
8	16	6	0.2675	3.9686	3.0592	3.2566	8	34	6	0.3021	4.2128	3.2821	3.5202
8	16	7	0.2784	4.7326	3.7906	4.1177	8	34	7	0.3073	4.3180	3.5127	3.8061
8	16	8	0.2861	4.2173	3.2979	3.6959	8	34	8	0.3110	4.0226	3.2534	3.5949
8	16	9	0.2920	4.2018	3.2927	3.7475	8	34	9	0.3138	3.8414	2.9895	3.4286
8	16	10	0.2966	3.9234	3.1446	3.5647	8	34	10	0.3160	4.0085	3.1343	3.6335
8	17	3	0.1810	4.0106	3.0337	1.6465	8	35	3	0.2582	4.1345	3.1728	1.9212
8	17	4	0.2308	3.8795	2.9379	2.3217	8	35	4	0.2830	3.7083	2.8331	2.3516
8	17	5	0.2560	3.1170	2.2386	2.0207	8	35	5	0.2955	3.2855	2.3981	2.2232
8	17	6	0.2713	3.0185	2.2375	2.2235	8	35	6	0.3030	2.9925	2.2709	2.2889
8	17	7	0.2816	3.2902	2.4155	2.5395	8	35	7	0.3081	2.7703	2.0300	2.1543
8	17	8	0.2889	2.7075	2.0411	2.1958	8	35	8	0.3117	2.9666	2.2198	2.4280
8	17	9	0.2944	2.6830	1.9595	2.1952	8	35	9	0.3144	2.9164	2.1450	2.4274
8	17	10	0.2987	3.3027	2.4890	2.8260	8	35	10	0.3165	2.9061	2.2126	2.5052
8	18	3	0.1892	4.1312	3.2338	1.3630	8	36	3	0.2603	3.7561	2.8789	1.2629
8	18	4	0.2364	2.9199	2.0709	1.2201	8	36	4	0.2844	3.2031	2.3644	1.5811
8	18	5	0.2603	2.6423	1.9250	1.5875	8	36	5	0.2966	2.4954	1.7154	1.3706
8	18	6	0.2747	1.9041	1.1776	1.0271	8	36	6	0.3039	2.7237	2.0050	1.8710
8	18	7	0.2844	2.4321	1.7162	1.6762	8	36	7	0.3088	2.3034	1.6249	1.6179
8	18	8	0.2913	2.5167	1.8439	1.8958	8	36	8	0.3123	2.4723	1.6343	1.7418
8	18	9	0.2966	1.9141	1.1835	1.3144	8	36	9	0.3149	2.6655	2.0287	2.1663
8	18	10	0.3006	1.6950	1.3539	1.4042	8	36	10	0.3169	2.5439	1.8537	2.0609
8	19	3	0.1966	3.5003	2.6781	0.5448	8	37	3	0.2622	4.1861	3.3692	1.2906
8	19	4	0.2414	3.0203	2.2352	1.1228	8	37	4	0.2857	2.6222	1.8535	0.9183
8	19	5	0.2641	2.1073	1.4435	1.0506	8	37	5	0.2975	1.8473	1.1936	0.8895
8	19	6	0.2778	2.1401	1.4353	1.1219	8	37	6	0.3047	2.1946	1.4525	1.1447
8	19	7	0.2870	2.1399	1.5535	1.4302	8	37	7	0.3094	2.0937	1.4518	1.3378
8	19	8	0.2935	2.1237	1.5156	1.4678	8	37	8	0.3128	2.2469	1.6152	1.5923
8	19	9	0.2985	1.7879	1.1279	1.1516	8	37	9	0.3154	1.6731	1.2018	1.2295
8	19	10	0.3023	1.9255	1.3241	1.3999	8	37	10	0.3174	2.0515	1.4965	1.5854
8	20	3	0.2033	3.9791	3.2181	0.7078	8	38	3	0.2641	3.9718	3.2147	0.9269
8	20	4	0.2459	3.1992	2.4806	1.2013	8	38	4	0.2870	2.6344	1.9014	0.7029

8	20	5	0.2675	2.7720	2.0459	1.1904	8	38	5	0.2985	2.4130	1.7189	1.0383
8	20	6	0.2805	2.0135	1.4246	1.1043	8	38	6	0.3054	2.2700	1.5745	1.1262
8	20	7	0.2893	1.9287	1.3378	1.1174	8	38	7	0.3100	1.8958	1.4221	1.3012
8	20	8	0.2955	1.6946	1.1504	1.0454	8	38	8	0.3134	1.9531	1.3273	1.2101
8	20	9	0.3002	1.5977	1.0223	0.9697	8	38	9	0.3158	1.4896	1.1781	1.1714
8	20	10	0.3039	1.6618	1.2263	1.2280	8	38	10	0.3178	1.6241	1.0559	1.0812
8	21	3	0.2093	4.3842	3.6679	0.8903	8	39	3	0.2658	3.8440	3.1331	0.6503
8	21	4	0.2500	2.9934	2.3059	0.8208	8	39	4	0.2881	2.8075	2.1223	0.7442
8	21	5	0.2706	2.1235	1.4691	0.6817	8	39	5	0.2994	2.1553	1.4896	0.6879
8	21	6	0.2830	2.2908	1.6262	0.9854	8	39	6	0.3061	1.9218	1.3529	0.9790
8	21	7	0.2913	2.2003	1.6053	1.2495	8	39	7	0.3106	1.9962	1.3372	0.9556
8	21	8	0.2973	1.6185	1.0803	0.9062	8	39	8	0.3139	1.1959	0.6894	0.6037
8	21	9	0.3018	1.7516	1.2652	1.1673	8	39	9	0.3163	1.9141	1.3227	1.2088
8	21	10	0.3053	1.5825	1.1774	1.1433	8	39	10	0.3182	1.1106	0.6492	0.6309
8	22	3	0.2148	3.4416	2.7846	1.0944	8	40	3	0.2675	3.9460	3.2734	0.5601
8	22	4	0.2537	2.7669	2.1227	0.6457	8	40	4	0.2893	2.5594	1.9120	0.5356
8	22	5	0.2734	2.1926	1.5484	0.5110	8	40	5	0.3002	2.3516	1.7126	0.6990
8	22	6	0.2853	2.0240	1.4202	0.7742	8	40	6	0.3068	2.0229	1.3859	0.6884
8	22	7	0.2932	2.0305	1.4709	1.0566	8	40	7	0.3112	1.5764	1.0387	0.7524
8	22	8	0.2989	1.8542	1.2672	0.9358	8	40	8	0.3144	1.2880	0.7860	0.6373
8	22	9	0.3032	1.4588	1.0097	0.8947	8	40	9	0.3167	1.8663	1.3544	1.1996
8	22	10	0.3065	1.7923	1.2189	1.0748	8	40	10	0.3186	1.5318	1.1856	1.1455
8	23	3	0.2199	4.4898	3.8501	0.7475	8	41	3	0.2691	3.6121	2.9786	0.7608
8	23	4	0.2571	3.2854	2.6658	0.7794	8	41	4	0.2903	2.8386	2.2211	0.5931
8	23	5	0.2760	2.0404	1.4325	0.4171	8	41	5	0.3010	2.2352	1.6505	0.7381
8	23	6	0.2874	1.7772	1.2056	0.6041	8	41	6	0.3074	1.8016	1.2383	0.6515
8	23	7	0.2950	1.0067	0.6107	0.7661	8	41	7	0.3117	1.7490	1.1532	0.6029
8	23	8	0.3004	1.6782	1.1465	0.8092	8	41	8	0.3148	1.7460	1.1665	0.7804
8	23	9	0.3045	1.4422	0.9993	0.8439	8	41	9	0.3171	1.6020	1.1078	0.9117
8	23	10	0.3077	1.1507	0.6830	0.5780	8	41	10	0.3189	1.2091	0.9341	0.9188
8	24	3	0.2245	7.9384	6.6864	6.4063	8	42	3	0.2706	8.0331	6.7456	6.5270
8	24	4	0.2603	7.7029	6.4894	6.6928	8	42	4	0.2913	7.6571	6.4500	6.6976
8	24	5	0.2784	8.3129	7.0050	7.4937	8	42	5	0.3018	7.7600	6.4924	7.0008
8	24	6	0.2893	7.5779	6.3683	6.9636	8	42	6	0.3081	7.5575	6.3467	6.9714
8	24	7	0.2966	6.9569	5.7731	6.4513	8	42	7	0.3123	7.3621	6.1731	6.8790
8	24	8	0.3018	7.5436	6.4198	7.1298	8	42	8	0.3153	7.5671	6.4050	7.1608
8	24	9	0.3057	7.4337	6.2588	7.0560	8	42	9	0.3175	7.6149	6.4461	7.2572
8	24	10	0.3088	7.3573	6.2052	7.0266	8	42	10	0.3193	7.5763	6.3831	7.2521
8	25	3	0.2288	5.1517	4.0717	3.2149	8	43	3	0.2720	5.6559	4.5736	3.7884
8	25	4	0.2632	4.8492	3.7966	3.5565	8	43	4	0.2923	4.6853	3.7332	3.5757
8	25	5	0.2805	4.3033	3.4345	3.4881	8	43	5	0.3025	4.6811	3.7232	3.8248
8	25	6	0.2910	4.7496	3.8354	4.0550	8	43	6	0.3086	4.7390	3.7228	4.0076
8	25	7	0.2980	4.6680	3.6829	4.0428	8	43	7	0.3127	4.2396	3.3192	3.6764
8	25	8	0.3030	4.5742	3.6390	4.0592	8	43	8	0.3157	4.4066	3.5146	3.9344
8	25	9	0.3068	4.3132	3.4723	3.8993	8	43	9	0.3179	4.5311	3.5668	4.0884
8	25	10	0.3097	4.0389	3.1938	3.6658	8	43	10	0.3196	4.1122	3.2347	3.7461
8	26	3	0.2327	3.8398	2.8727	1.5858	8	44	3	0.2734	4.0434	3.0527	1.7771
8	26	4	0.2658	3.3985	2.5223	2.0223	8	44	4	0.2932	3.2773	2.4844	2.1147
8	26	5	0.2825	3.5906	2.6828	2.4749	8	44	5	0.3032	3.0810	2.2665	2.1354
8	26	6	0.2926	2.9073	2.1742	2.1773	8	44	6	0.3092	3.1759	2.3943	2.4310
8	26	7	0.2994	2.9356	2.2775	2.3728	8	44	7	0.3132	3.2738	2.6156	2.7364

8	26	8	0.3042	3.1710	2.3880	2.5982	8	44	8	0.3161	2.8936	2.1793	2.3866
8	26	9	0.3078	2.6910	2.0131	2.2400	8	44	9	0.3182	2.9522	2.0829	2.4238
8	26	10	0.3106	2.8092	2.2572	2.4684	8	44	10	0.3199	3.0044	2.3871	2.6505
8	27	3	0.2364	4.3507	3.4577	1.6641	8	45	3	0.2747	4.1500	3.2688	1.6418
8	27	4	0.2683	2.9378	2.0663	1.1924	8	45	4	0.2941	2.7628	2.0191	1.4790
8	27	5	0.2844	2.9641	2.2102	1.8478	8	45	5	0.3039	2.7246	2.0535	1.8006
8	27	6	0.2941	1.9145	1.2470	1.1340	8	45	6	0.3097	2.6556	1.8472	1.7075
8	27	7	0.3006	2.6792	2.0567	2.0373	8	45	7	0.3137	2.3612	1.7541	1.7599
8	27	8	0.3053	2.4514	1.7946	1.8576	8	45	8	0.3165	2.5834	1.9820	2.0605
8	27	9	0.3088	2.4903	1.7777	1.9228	8	45	9	0.3186	2.4634	1.9327	2.0466
8	27	10	0.3115	2.1493	1.7208	1.8175	8	45	10	0.3202	1.9318	1.4025	1.5457
8	28	3	0.2398	3.5659	2.7495	0.7342	8	46	3	0.2760	3.6974	2.9035	1.1130
8	28	4	0.2706	2.9304	2.1708	1.2039	8	46	4	0.2950	2.6836	1.9379	1.0873
8	28	5	0.2861	2.3118	1.5751	1.0377	8	46	5	0.3045	2.0545	1.3249	0.8677
8	28	6	0.2955	2.7125	1.9825	1.6497	8	46	6	0.3102	2.2383	1.5470	1.2965
8	28	7	0.3018	2.8500	2.1278	1.9612	8	46	7	0.3141	2.1621	1.4005	1.2656
8	28	8	0.3063	1.7179	1.1355	1.1063	8	46	8	0.3168	2.1532	1.4255	1.4139
8	28	9	0.3096	2.3291	1.6648	1.7045	8	46	9	0.3189	1.6363	1.2749	1.2953
8	28	10	0.3123	1.8436	1.3456	1.4102	8	46	10	0.3205	1.8131	1.2544	1.3504
8	29	3	0.2429	3.4570	2.7051	0.6357	8	47	3	0.2772	3.1208	2.4092	1.0049
8	29	4	0.2727	2.4154	1.7141	0.7672	8	47	4	0.2958	2.7747	2.0721	1.0195
8	29	5	0.2877	2.7304	2.0145	1.2235	8	47	5	0.3051	2.5878	1.9485	1.3878
8	29	6	0.2968	2.2699	1.6422	1.2562	8	47	6	0.3107	2.0277	1.3519	0.9721
8	29	7	0.3029	2.4952	1.8166	1.5304	8	47	7	0.3145	1.6882	1.2006	1.0934
8	29	8	0.3072	1.8558	1.2530	1.1308	8	47	8	0.3172	1.7443	1.0568	0.9384
8	29	9	0.3104	1.9118	1.3285	1.2841	8	47	9	0.3192	1.4728	1.1859	1.1828
8	29	10	0.3130	2.5810	1.9387	1.9573	8	47	10	0.3208	1.9850	1.4510	1.4857
8	30	3	0.2459	3.8719	3.1597	0.5891	8	48	3	0.2784	3.6334	2.9179	0.4578
8	30	4	0.2747	2.3808	1.7104	0.6423	8	48	4	0.2966	2.6455	1.9597	0.6821
8	30	5	0.2893	2.7166	2.0651	1.1966	8	48	5	0.3057	2.2040	1.5621	0.8447
8	30	6	0.2980	1.7124	1.1719	0.8854	8	48	6	0.3112	2.0028	1.4299	1.0481
8	30	7	0.3039	1.8556	1.2863	1.0020	8	48	7	0.3149	1.9227	1.2950	0.9739
8	30	8	0.3081	1.6255	1.0338	0.8322	8	48	8	0.3175	1.6547	1.0978	0.9438
8	30	9	0.3112	1.0263	0.6123	0.5964	8	48	9	0.3195	1.7038	1.2518	1.1931
8	30	10	0.3137	1.8269	1.1797	1.1129	8	48	10	0.3210	1.8618	1.3102	1.2835
8	31	3	0.2487	3.8230	3.1595	0.9052	8	49	3	0.2795	3.7771	3.1030	0.5433
8	31	4	0.2766	2.5615	1.9150	0.5335	8	49	4	0.2973	2.4088	1.7581	0.4720
8	31	5	0.2907	1.7984	1.1654	0.4649	8	49	5	0.3063	2.5269	1.9384	1.1323
8	31	6	0.2991	1.6697	1.0545	0.4918	8	49	6	0.3117	1.4198	0.8761	0.6269
8	31	7	0.3048	1.3151	0.7267	0.4276	8	49	7	0.3153	2.0610	1.4333	0.9695
8	31	8	0.3089	1.4564	1.0125	0.8803	8	49	8	0.3178	1.3609	0.8417	0.6710
8	31	9	0.3119	1.7833	1.2826	1.1254	8	49	9	0.3198	1.5056	1.0488	0.9481
8	31	10	0.3143	1.0913	0.6077	0.5451	8	49	10	0.3213	1.7785	1.1867	1.0835
8	32	3	0.2513	3.9986	3.3559	0.5918	8	50	3	0.2805	4.3553	3.7107	0.6607
8	32	4	0.2784	3.0384	2.4321	0.8297	8	50	4	0.2980	2.7524	2.1400	0.6477
8	32	5	0.2920	2.2690	1.6688	0.6107	8	50	5	0.3068	1.8411	1.2437	0.4489
8	32	6	0.3002	1.8533	1.3182	0.7966	8	50	6	0.3121	1.7928	1.2167	0.6075
8	32	7	0.3057	1.5294	0.9710	0.5632	8	50	7	0.3156	1.8929	1.3073	0.7884
8	32	8	0.3096	1.9154	1.3746	1.0162	8	50	8	0.3181	1.0659	0.6521	0.6197
8	32	9	0.3126	1.6811	1.0943	0.7880	8	50	9	0.3200	1.0841	0.6116	0.5113
8	32	10	0.3149	1.1797	0.8832	0.8638	8	50	10	0.3215	1.2530	0.8158	0.7320

9	15	3	0.1398	7.9121	6.6246	6.4395	9	33	3	0.2438	8.1319	6.8919	6.7742
9	15	4	0.2029	7.4044	6.1315	6.4176	9	33	4	0.2733	8.0105	6.7613	7.0839
9	15	5	0.2350	7.2395	6.0729	6.5498	9	33	5	0.2882	7.3929	6.1909	6.7158
9	15	6	0.2544	7.7925	6.6135	7.2341	9	33	6	0.2972	7.1509	5.9882	6.6215
9	15	7	0.2674	7.2204	6.1243	6.7811	9	33	7	0.3032	7.5976	6.3496	7.1275
9	15	8	0.2767	7.3411	6.1710	6.9413	9	33	8	0.3075	6.9816	5.8089	6.5976
9	15	9	0.2837	7.6452	6.4444	7.2872	9	33	9	0.3107	7.4433	6.2728	7.1081
9	15	10	0.2892	7.6725	6.5190	7.3650	9	33	10	0.3132	7.1618	6.0276	6.8709
9	16	3	0.1515	5.0119	3.9152	3.1664	9	34	3	0.2464	5.5203	4.4307	3.7543
9	16	4	0.2109	4.8194	3.8129	3.6673	9	34	4	0.2751	4.6794	3.7163	3.6247
9	16	5	0.2410	4.8238	3.8950	4.0111	9	34	5	0.2895	4.9119	3.9697	4.1230
9	16	6	0.2593	4.7140	3.7745	4.0488	9	34	6	0.2982	4.6048	3.7225	4.0026
9	16	7	0.2715	4.5171	3.7012	4.0233	9	34	7	0.3041	4.1984	3.4210	3.7429
9	16	8	0.2802	4.3530	3.4267	3.8762	9	34	8	0.3082	4.2483	3.4735	3.8556
9	16	9	0.2868	3.9982	3.2855	3.6596	9	34	9	0.3114	4.5230	3.6963	4.1595
9	16	10	0.2920	4.0666	3.2994	3.7451	9	34	10	0.3138	3.9824	3.2698	3.6920
9	17	3	0.1619	3.8703	2.9193	1.7991	9	35	3	0.2488	4.1677	3.2174	2.1610
9	17	4	0.2179	3.4811	2.6423	2.2495	9	35	4	0.2767	3.6997	2.7924	2.3926
9	17	5	0.2464	3.1967	2.4740	2.3794	9	35	5	0.2908	3.3306	2.5103	2.4323
9	17	6	0.2636	2.7791	1.9218	1.9803	9	35	6	0.2992	3.1715	2.2724	2.3712
9	17	7	0.2751	2.8954	2.2208	2.3541	9	35	7	0.3049	3.2251	2.4321	2.6261
9	17	8	0.2833	3.1777	2.4834	2.6969	9	35	8	0.3089	3.1863	2.3676	2.6536
9	17	9	0.2895	2.4332	1.7276	1.9927	9	35	9	0.3120	2.8895	2.1964	2.4794
9	17	10	0.2944	2.6250	1.9474	2.2455	9	35	10	0.3143	3.1370	2.4294	2.7658
9	18	3	0.1712	3.0618	2.2122	0.9788	9	36	3	0.2511	3.4445	2.5795	1.2227
9	18	4	0.2242	3.0983	2.2605	1.5512	9	36	4	0.2783	2.6689	1.8950	1.3715
9	18	5	0.2511	2.7626	2.0465	1.7795	9	36	5	0.2920	2.8028	2.0635	1.8176
9	18	6	0.2674	2.9605	2.1649	2.0468	9	36	6	0.3002	2.4970	1.7996	1.7360
9	18	7	0.2783	2.6706	2.0464	2.0650	9	36	7	0.3057	2.3027	1.6317	1.6764
9	18	8	0.2861	2.6527	1.9319	2.0467	9	36	8	0.3096	2.1026	1.4567	1.5719
9	18	9	0.2920	2.2855	1.7461	1.8683	9	36	9	0.3126	2.3029	1.5857	1.7898
9	18	10	0.2965	2.1834	1.5896	1.7731	9	36	10	0.3149	1.7500	1.2023	1.3710
9	19	3	0.1795	3.4614	2.6420	0.7557	9	37	3	0.2533	3.4861	2.6750	0.9237
9	19	4	0.2299	2.6213	1.8372	0.9040	9	37	4	0.2798	3.0988	2.3527	1.5437
9	19	5	0.2554	2.8862	2.1811	1.7218	9	37	5	0.2931	2.0774	1.3165	0.8741
9	19	6	0.2708	2.1022	1.3710	1.1161	9	37	6	0.3011	2.4773	1.7242	1.4900
9	19	7	0.2812	2.0603	1.3999	1.3104	9	37	7	0.3064	1.8413	1.1777	1.1184
9	19	8	0.2886	1.6545	1.2041	1.2058	9	37	8	0.3103	1.9422	1.2961	1.3246
9	19	9	0.2941	2.0636	1.5531	1.6064	9	37	9	0.3131	2.0232	1.3422	1.4487
9	19	10	0.2984	1.8941	1.3663	1.4640	9	37	10	0.3154	1.9898	1.5032	1.6059
9	20	3	0.1870	3.5464	2.8160	0.9679	9	38	3	0.2554	3.8610	3.1021	1.0243
9	20	4	0.2350	2.9235	2.1993	1.0726	9	38	4	0.2812	2.5356	1.8305	0.8982
9	20	5	0.2593	3.1314	2.4910	1.9134	9	38	5	0.2941	2.2302	1.5024	0.8596
9	20	6	0.2739	1.6374	0.9932	0.7019	9	38	6	0.3019	1.6429	0.9929	0.7221
9	20	7	0.2837	1.9651	1.2942	1.0805	9	38	7	0.3071	1.4755	0.9169	0.8137
9	20	8	0.2908	1.6672	1.2572	1.2210	9	38	8	0.3109	1.6952	1.1516	1.1069
9	20	9	0.2961	1.9598	1.2889	1.2798	9	38	9	0.3137	1.6513	1.2065	1.2159
9	20	10	0.3002	2.0256	1.4422	1.4933	9	38	10	0.3158	1.8078	1.1514	1.2362
9	21	3	0.1938	3.6522	2.9443	0.6023	9	39	3	0.2574	2.9387	2.2371	0.6446
9	21	4	0.2396	2.6213	1.9470	0.7407	9	39	4	0.2825	2.3004	1.6327	0.6262
9	21	5	0.2627	2.1919	1.5886	1.0024	9	39	5	0.2951	2.1458	1.5072	0.8650

9	21	6	0.2767	1.9655	1.3167	0.8330	9	39	6	0.3027	1.8189	1.1976	0.8094
9	21	7	0.2861	2.1176	1.5465	1.2862	9	39	7	0.3078	2.0067	1.4004	1.1471
9	21	8	0.2928	1.6126	1.0062	0.8391	9	39	8	0.3114	1.1052	0.6994	0.6758
9	21	9	0.2978	1.2540	0.9889	0.9857	9	39	9	0.3142	1.5843	0.9724	0.9131
9	21	10	0.3018	1.4079	1.0202	1.0133	9	39	10	0.3163	0.9576	0.6421	0.6424
9	22	3	0.2000	3.5395	2.8624	0.3859	9	40	3	0.2593	3.5580	2.8881	0.4848
9	22	4	0.2438	2.5994	1.9643	0.6931	9	40	4	0.2837	2.7729	2.1411	0.8716
9	22	5	0.2659	2.2239	1.5877	0.6631	9	40	5	0.2961	2.0650	1.4454	0.6915
9	22	6	0.2793	2.1578	1.5817	1.0639	9	40	6	0.3035	1.9715	1.3395	0.7568
9	22	7	0.2882	1.7600	1.1987	0.8675	9	40	7	0.3084	1.1522	0.7321	0.6936
9	22	8	0.2946	1.5024	1.0431	0.9078	9	40	8	0.3120	1.2974	0.9160	0.8589
9	22	9	0.2994	1.6272	1.0452	0.8882	9	40	9	0.3146	1.5226	1.0318	0.9412
9	22	10	0.3032	1.3585	0.7933	0.7167	9	40	10	0.3167	1.1939	0.6405	0.5925
9	23	3	0.2057	3.9042	3.2626	0.4610	9	41	3	0.2610	3.0258	2.3914	0.8053
9	23	4	0.2476	2.5969	1.9806	0.5177	9	41	4	0.2849	2.2302	1.6593	0.8207
9	23	5	0.2688	2.1624	1.5536	0.5243	9	41	5	0.2970	2.3754	1.7970	0.9352
9	23	6	0.2816	1.9517	1.4308	0.9480	9	41	6	0.3042	1.9389	1.3557	0.7370
9	23	7	0.2902	1.6581	1.1144	0.7261	9	41	7	0.3090	1.7295	1.1360	0.6734
9	23	8	0.2963	1.5403	1.0083	0.7339	9	41	8	0.3125	2.1011	1.5634	1.2806
9	23	9	0.3009	1.3434	1.0132	0.9612	9	41	9	0.3151	1.5862	0.9915	0.7722
9	23	10	0.3045	1.4325	0.8689	0.7229	9	41	10	0.3171	1.2812	0.9107	0.8658
9	24	3	0.2109	7.7370	6.4510	6.2980	9	42	3	0.2627	8.0579	6.7613	6.6667
9	24	4	0.2511	7.9000	6.6657	6.9635	9	42	4	0.2861	7.5536	6.3477	6.6753
9	24	5	0.2715	7.2840	6.1276	6.6158	9	42	5	0.2978	7.5966	6.3745	6.9240
9	24	6	0.2837	7.6482	6.4518	7.0945	9	42	6	0.3049	7.6498	6.5449	7.1564
9	24	7	0.2920	7.1906	6.0393	6.7416	9	42	7	0.3096	7.6425	6.4345	7.1952
9	24	8	0.2978	7.1930	6.0903	6.8216	9	42	8	0.3130	7.4084	6.2519	7.0384
9	24	9	0.3022	7.6116	6.4774	7.2786	9	42	9	0.3155	7.4006	6.2498	7.0776
9	24	10	0.3057	7.4488	6.2863	7.1456	9	42	10	0.3175	7.3515	6.2081	7.0654
9	25	3	0.2157	5.2931	4.2441	3.5749	9	43	3	0.2644	5.0728	4.0613	3.4951
9	25	4	0.2544	4.4964	3.5342	3.4210	9	43	4	0.2872	5.2032	4.1623	4.0830
9	25	5	0.2739	4.4521	3.4541	3.5990	9	43	5	0.2987	4.2836	3.3681	3.5295
9	25	6	0.2857	4.3644	3.4958	3.7567	9	43	6	0.3056	4.5935	3.7548	4.0303
9	25	7	0.2936	3.7914	2.9435	3.2863	9	43	7	0.3102	4.2546	3.4049	3.7728
9	25	8	0.2992	4.6162	3.6893	4.1508	9	43	8	0.3135	4.1086	3.3155	3.7149
9	25	9	0.3035	4.1004	3.2557	3.7193	9	43	9	0.3159	4.4319	3.5437	4.0535
9	25	10	0.3068	4.2677	3.3347	3.8993	9	43	10	0.3179	4.0534	3.2714	3.7474
9	26	3	0.2201	3.5372	2.5686	1.4844	9	44	3	0.2659	3.7055	2.7794	1.8445
9	26	4	0.2574	3.4955	2.6600	2.2937	9	44	4	0.2882	3.5919	2.7226	2.3759
9	26	5	0.2762	3.7023	2.8116	2.7021	9	44	5	0.2994	3.1098	2.4153	2.3688
9	26	6	0.2875	3.2405	2.5541	2.6106	9	44	6	0.3062	2.8469	2.0395	2.1379
9	26	7	0.2951	2.9031	2.0054	2.2164	9	44	7	0.3107	2.8754	2.3412	2.4603
9	26	8	0.3005	2.6773	1.9915	2.2058	9	44	8	0.3139	2.7713	2.0078	2.2766
9	26	9	0.3046	3.2013	2.4471	2.7549	9	44	9	0.3163	2.7367	1.9478	2.2881
9	26	10	0.3078	2.7129	1.9889	2.3225	9	44	10	0.3182	2.8967	2.0794	2.4859
9	27	3	0.2242	3.6614	2.7922	1.3538	9	45	3	0.2674	3.6283	2.7652	1.4319
9	27	4	0.2602	2.8967	2.0597	1.3908	9	45	4	0.2892	3.0342	2.1636	1.4950
9	27	5	0.2783	2.6494	1.9019	1.6372	9	45	5	0.3002	2.1091	1.3880	1.1933
9	27	6	0.2892	2.4339	1.7122	1.6302	9	45	6	0.3068	2.6907	1.9591	1.9049
9	27	7	0.2965	2.5769	1.8935	1.9277	9	45	7	0.3112	2.2397	1.4638	1.5342
9	27	8	0.3018	2.3675	1.6670	1.7881	9	45	8	0.3143	2.4536	1.7109	1.8675

9	27	9	0.3057	2.3655	1.5846	1.8019	9	45	9	0.3167	1.9968	1.3613	1.5391
9	27	10	0.3087	1.8743	1.3859	1.5251	9	45	10	0.3186	2.5057	1.8190	2.0742
9	28	3	0.2281	3.5011	2.6808	0.8117	9	46	3	0.2688	3.2927	2.4854	0.8429
9	28	4	0.2627	2.7745	2.0310	1.2373	9	46	4	0.2902	2.7057	2.0355	1.4535
9	28	5	0.2802	2.3697	1.6148	1.1415	9	46	5	0.3009	2.7136	1.9848	1.5715
9	28	6	0.2908	2.4399	1.7416	1.5184	9	46	6	0.3074	2.3936	1.6362	1.4215
9	28	7	0.2978	2.3420	1.6571	1.5756	9	46	7	0.3117	1.8861	1.3048	1.2676
9	28	8	0.3029	1.6580	0.9688	0.9862	9	46	8	0.3148	2.2556	1.5958	1.6385
9	28	9	0.3067	1.8793	1.2240	1.3116	9	46	9	0.3171	1.8068	1.2847	1.3617
9	28	10	0.3096	1.9660	1.4816	1.5763	9	46	10	0.3189	1.7080	1.3095	1.3851
9	29	3	0.2316	3.5883	2.8296	0.7340	9	47	3	0.2702	4.0138	3.2539	1.1995
9	29	4	0.2652	2.1599	1.4514	0.6480	9	47	4	0.2911	2.4796	1.7874	0.9339
9	29	5	0.2821	2.4256	1.6988	1.0207	9	47	5	0.3016	2.2103	1.5746	1.1332
9	29	6	0.2922	1.8729	1.2626	0.9949	9	47	6	0.3079	1.7811	1.1087	0.8148
9	29	7	0.2990	2.1880	1.5583	1.3802	9	47	7	0.3121	1.8331	1.2865	1.1791
9	29	8	0.3039	1.8565	1.2190	1.1442	9	47	8	0.3152	1.3265	0.7984	0.7706
9	29	9	0.3076	1.5088	1.0379	1.0406	9	47	9	0.3174	1.5898	1.1007	1.1190
9	29	10	0.3104	1.9441	1.3080	1.3777	9	47	10	0.3192	1.8632	1.3534	1.4200
9	30	3	0.2350	3.5724	2.8607	0.5931	9	48	3	0.2715	3.6825	2.9786	0.8204
9	30	4	0.2674	2.5533	1.8918	0.8055	9	48	4	0.2920	2.5816	1.9200	0.8696
9	30	5	0.2837	2.6177	2.0572	1.5129	9	48	5	0.3022	1.6664	1.0881	0.7469
9	30	6	0.2936	1.8053	1.2005	0.8250	9	48	6	0.3084	1.6016	1.0494	0.8013
9	30	7	0.3002	1.5218	0.8973	0.6414	9	48	7	0.3126	1.4920	0.9010	0.7009
9	30	8	0.3049	1.5792	1.0904	0.9934	9	48	8	0.3155	1.6431	1.1235	1.0301
9	30	9	0.3084	1.4012	0.9084	0.8634	9	48	9	0.3177	1.9068	1.3439	1.3022
9	30	10	0.3112	1.3446	0.9346	0.9328	9	48	10	0.3195	2.0570	1.5280	1.5450
9	31	3	0.2381	3.7355	3.0700	0.7875	9	49	3	0.2727	3.8462	3.1771	0.7506
9	31	4	0.2695	2.5192	1.8701	0.5249	9	49	4	0.2928	2.3059	1.6707	0.5668
9	31	5	0.2853	2.4330	1.8220	1.0040	9	49	5	0.3029	1.8118	1.2178	0.6454
9	31	6	0.2949	1.2599	0.6973	0.5069	9	49	6	0.3089	2.2046	1.5938	1.0604
9	31	7	0.3012	1.9105	1.3925	1.1146	9	49	7	0.3130	1.8556	1.2330	0.8692
9	31	8	0.3058	1.8400	1.3787	1.2255	9	49	8	0.3159	1.6948	1.1828	1.0254
9	31	9	0.3092	1.8177	1.2592	1.1232	9	49	9	0.3181	1.4988	0.9133	0.7991
9	31	10	0.3119	1.5334	1.0317	0.9780	9	49	10	0.3198	1.4171	0.9172	0.8840
9	32	3	0.2410	3.6985	3.0644	0.5882	9	50	3	0.2739	3.7426	3.1063	0.5463
9	32	4	0.2715	2.5112	1.9026	0.6123	9	50	4	0.2936	1.9830	1.3764	0.5608
9	32	5	0.2868	2.4964	1.8890	0.8627	9	50	5	0.3035	1.8523	1.3358	0.8705
9	32	6	0.2961	1.4070	0.8613	0.5337	9	50	6	0.3094	2.2571	1.6557	0.9911
9	32	7	0.3022	1.7538	1.3132	1.0784	9	50	7	0.3134	1.0216	0.6353	0.6684
9	32	8	0.3067	1.3334	0.9118	0.7918	9	50	8	0.3162	1.2620	0.8188	0.6972
9	32	9	0.3100	1.2406	0.7336	0.5888	9	50	9	0.3184	1.9875	1.4183	1.2269
9	32	10	0.3126	1.0814	0.5736	0.4848	9	50	10	0.3200	1.3070	0.8570	0.7922
10	15	3	0.1183	8.1860	6.8944	6.8285	10	33	3	0.2338	7.4799	6.1949	6.1824
10	15	4	0.1884	7.7365	6.4746	6.8386	10	33	4	0.2667	7.6830	6.4602	6.8461
10	15	5	0.2240	7.7611	6.5409	7.1040	10	33	5	0.2832	7.5035	6.2802	6.8707
10	15	6	0.2456	7.7331	6.5273	7.2105	10	33	6	0.2932	7.5222	6.3608	7.0367
10	15	7	0.2601	8.0735	6.9037	7.6489	10	33	7	0.2998	7.2404	6.0654	6.8295
10	15	8	0.2704	7.6208	6.4116	7.2445	10	33	8	0.3046	7.8579	6.6675	7.5020
10	15	9	0.2782	7.4644	6.2803	7.1402	10	33	9	0.3082	7.7390	6.4884	7.4134
10	15	10	0.2843	7.7295	6.5618	7.4449	10	33	10	0.3110	7.2534	6.1033	6.9829
10	16	3	0.1313	4.8583	3.8678	3.3533	10	34	3	0.2367	4.9237	3.8928	3.4005

10	16	4	0.1973	4.2837	3.3319	3.2827	10	34	4	0.2686	4.7332	3.7380	3.7253
10	16	5	0.2308	4.7043	3.7450	3.9332	10	34	5	0.2847	4.3059	3.4030	3.6040
10	16	6	0.2510	4.5406	3.7477	4.0153	10	34	6	0.2943	4.4008	3.4904	3.8288
10	16	7	0.2646	3.8800	3.0970	3.4372	10	34	7	0.3008	4.1869	3.3820	3.7536
10	16	8	0.2743	4.0695	3.2325	3.6658	10	34	8	0.3054	4.3931	3.4853	3.9804
10	16	9	0.2817	4.2161	3.3710	3.8597	10	34	9	0.3089	4.0675	3.2275	3.7259
10	16	10	0.2874	4.5591	3.6984	4.2378	10	34	10	0.3116	4.2280	3.3827	3.9225
10	17	3	0.1429	4.1245	3.1863	2.2262	10	35	3	0.2394	3.9106	3.0009	2.1737
10	17	4	0.2051	3.8406	2.9171	2.5594	10	35	4	0.2704	3.4661	2.5598	2.2626
10	17	5	0.2367	3.6875	2.8881	2.8456	10	35	5	0.2861	3.1612	2.3079	2.2959
10	17	6	0.2558	3.2119	2.5265	2.6178	10	35	6	0.2955	2.5778	1.8799	1.9904
10	17	7	0.2686	3.2614	2.5014	2.7079	10	35	7	0.3017	3.1661	2.2873	2.5597
10	17	8	0.2778	3.0039	2.3205	2.5647	10	35	8	0.3062	2.8312	2.2039	2.4363
10	17	9	0.2847	2.9031	2.1684	2.4902	10	35	9	0.3096	2.4528	1.8415	2.1040
10	17	10	0.2900	2.8627	2.0757	2.4720	10	35	10	0.3122	3.1575	2.4644	2.8209
10	18	3	0.1532	3.8144	2.9287	1.5330	10	36	3	0.2420	2.8272	2.0141	1.0890
10	18	4	0.2121	2.6936	1.9487	1.4970	10	36	4	0.2722	3.0495	2.3249	1.9104
10	18	5	0.2420	2.9975	2.2103	1.9760	10	36	5	0.2874	2.8481	2.1810	2.0304
10	18	6	0.2601	2.3036	1.5439	1.5035	10	36	6	0.2965	2.0884	1.5185	1.5118
10	18	7	0.2722	2.2400	1.6234	1.6826	10	36	7	0.3026	2.3564	1.8046	1.8724
10	18	8	0.2808	2.0774	1.5468	1.6476	10	36	8	0.3070	1.9063	1.4059	1.5071
10	18	9	0.2874	2.1756	1.5596	1.7410	10	36	9	0.3103	2.1316	1.5429	1.7276
10	18	10	0.2924	2.4527	1.8104	2.0525	10	36	10	0.3128	2.4041	1.7562	2.0155
10	19	3	0.1624	3.5063	2.6987	1.0709	10	37	3	0.2444	3.0517	2.2422	0.7387
10	19	4	0.2184	2.4139	1.7249	1.1691	10	37	4	0.2738	2.6152	1.9516	1.4372
10	19	5	0.2468	2.9510	2.2255	1.8303	10	37	5	0.2886	2.5275	1.7871	1.4315
10	19	6	0.2639	1.7441	1.2682	1.2026	10	37	6	0.2975	2.3450	1.7756	1.6765
10	19	7	0.2754	2.0215	1.5352	1.5147	10	37	7	0.3034	1.9539	1.2387	1.2276
10	19	8	0.2836	1.9723	1.4330	1.4722	10	37	8	0.3077	1.9792	1.3766	1.4427
10	19	9	0.2898	1.7248	1.2776	1.3424	10	37	9	0.3109	2.1774	1.5269	1.6687
10	19	10	0.2946	1.7392	1.2931	1.3914	10	37	10	0.3134	2.0406	1.4205	1.6027
10	20	3	0.1707	3.6135	2.8571	0.8969	10	38	3	0.2468	2.8966	2.1477	0.4695
10	20	4	0.2240	2.6842	1.9376	0.8777	10	38	4	0.2754	2.5568	1.8348	0.9353
10	20	5	0.2510	2.7823	2.0663	1.4800	10	38	5	0.2898	1.5963	0.9347	0.5915
10	20	6	0.2673	2.0631	1.4067	1.1305	10	38	6	0.2984	2.4952	1.8127	1.5409
10	20	7	0.2782	1.9709	1.5480	1.4849	10	38	7	0.3042	1.9650	1.3833	1.2929
10	20	8	0.2861	1.5185	0.8273	0.7885	10	38	8	0.3084	1.4698	0.8311	0.8216
10	20	9	0.2919	2.1801	1.5233	1.5679	10	38	9	0.3115	1.7005	1.1975	1.2411
10	20	10	0.2965	1.7182	1.2273	1.2963	10	38	10	0.3139	1.6694	1.1523	1.2413
10	21	3	0.1783	2.9010	2.2017	0.5362	10	39	3	0.2489	3.2480	2.5525	0.6716
10	21	4	0.2292	1.7438	1.0873	0.5888	10	39	4	0.2768	2.3145	1.6296	0.5728
10	21	5	0.2549	1.7975	1.2383	0.8917	10	39	5	0.2909	1.6729	1.0580	0.6407
10	21	6	0.2704	2.4110	1.7591	1.3329	10	39	6	0.2993	1.9431	1.3336	1.0090
10	21	7	0.2808	1.6541	1.0942	0.9176	10	39	7	0.3050	1.6656	1.0663	0.8859
10	21	8	0.2883	1.1633	0.6971	0.6526	10	39	8	0.3090	1.5829	1.0147	0.9381
10	21	9	0.2939	1.0418	0.8510	0.8560	10	39	9	0.3120	1.8541	1.2910	1.2821
10	21	10	0.2982	1.1689	0.8111	0.8200	10	39	10	0.3144	1.4054	0.9281	0.9625
10	22	3	0.1852	3.9516	3.2853	0.8832	10	40	3	0.2510	3.5911	2.9295	0.7547
10	22	4	0.2338	2.2341	1.5900	0.4759	10	40	4	0.2782	1.9488	1.3384	0.6268
10	22	5	0.2584	1.9360	1.3123	0.5861	10	40	5	0.2919	1.6911	1.0852	0.5388
10	22	6	0.2733	1.5786	1.0162	0.6717	10	40	6	0.3002	1.7538	1.2520	0.9814

10	22	7	0.2832	1.0802	0.6688	0.6545	10	40	7	0.3057	1.3546	0.9931	0.9374
10	22	8	0.2903	1.3372	0.7765	0.6185	10	40	8	0.3096	1.3560	0.8048	0.6721
10	22	9	0.2957	1.1624	0.7658	0.7324	10	40	9	0.3126	1.4313	0.9321	0.8761
10	22	10	0.2998	1.1958	0.7418	0.7237	10	40	10	0.3149	0.9267	0.5250	0.5225
10	23	3	0.1915	3.6806	3.0438	0.4857	10	41	3	0.2530	3.9590	3.3251	0.8096
10	23	4	0.2381	2.4974	1.9074	0.7849	10	41	4	0.2796	2.5462	1.9444	0.7112
10	23	5	0.2616	2.2084	1.6086	0.7037	10	41	5	0.2929	2.0140	1.4136	0.5850
10	23	6	0.2759	1.9971	1.4553	0.9649	10	41	6	0.3010	1.7823	1.2830	0.9273
10	23	7	0.2854	1.7871	1.2891	0.9988	10	41	7	0.3063	1.4669	1.0659	0.9365
10	23	8	0.2922	1.4954	1.0002	0.8124	10	41	8	0.3102	1.1029	0.9517	1.0070
10	23	9	0.2973	1.1388	0.7754	0.7321	10	41	9	0.3131	1.3051	0.8227	0.7267
10	23	10	0.3013	1.2527	0.8972	0.8645	10	41	10	0.3153	1.1648	0.8456	0.8283
10	24	3	0.1973	8.1822	6.9612	6.9250	10	42	3	0.2549	8.0933	6.8222	6.8344
10	24	4	0.2420	7.4980	6.2460	6.6248	10	42	4	0.2808	7.0939	5.9115	6.2972
10	24	5	0.2646	8.0186	6.8304	7.3906	10	42	5	0.2939	7.7544	6.5364	7.1368
10	24	6	0.2782	7.9666	6.7910	7.4671	10	42	6	0.3017	7.2689	6.0914	6.7864
10	24	7	0.2874	7.7308	6.4836	7.2900	10	42	7	0.3070	7.2711	6.1435	6.8825
10	24	8	0.2939	7.9517	6.7817	7.5940	10	42	8	0.3107	6.9828	5.7739	6.6277
10	24	9	0.2988	7.2591	6.1877	6.9672	10	42	9	0.3136	7.6831	6.5136	7.3818
10	24	10	0.3026	7.5746	6.4402	7.3025	10	42	10	0.3157	7.3922	6.2724	7.1332
10	25	3	0.2026	4.9793	4.0203	3.5541	10	43	3	0.2567	5.0047	3.9651	3.4936
10	25	4	0.2456	4.7353	3.7573	3.7256	10	43	4	0.2821	4.8555	3.8634	3.8696
10	25	5	0.2673	4.3873	3.4592	3.6536	10	43	5	0.2948	4.3494	3.4930	3.6939
10	25	6	0.2804	4.3029	3.4125	3.7300	10	43	6	0.3025	4.4526	3.4792	3.8570
10	25	7	0.2892	4.5426	3.6645	4.0681	10	43	7	0.3076	4.3484	3.3977	3.8589
10	25	8	0.2955	4.5048	3.6059	4.0876	10	43	8	0.3113	4.1980	3.2962	3.7947
10	25	9	0.3002	3.9536	3.1065	3.6015	10	43	9	0.3140	4.4710	3.6810	4.1544
10	25	10	0.3038	3.9096	3.1378	3.6177	10	43	10	0.3162	4.2788	3.4147	3.9741
10	26	3	0.2075	4.1746	3.2701	2.4104	10	44	3	0.2584	3.1539	2.1933	1.3468
10	26	4	0.2489	3.4062	2.5771	2.2999	10	44	4	0.2832	3.4175	2.5403	2.2799
10	26	5	0.2698	3.8323	3.0641	3.0388	10	44	5	0.2957	3.0018	2.1460	2.1499
10	26	6	0.2824	2.9922	2.1630	2.2940	10	44	6	0.3032	3.1738	2.5216	2.6368
10	26	7	0.2909	2.9468	2.2383	2.4346	10	44	7	0.3082	3.0789	2.3863	2.5962
10	26	8	0.2969	2.8636	2.1614	2.4213	10	44	8	0.3118	2.9570	2.3847	2.5998
10	26	9	0.3014	3.0517	2.3595	2.6677	10	44	9	0.3144	3.4144	2.6649	3.0231
10	26	10	0.3050	2.3919	1.8330	2.0929	10	44	10	0.3165	2.8957	2.1699	2.5506
10	27	3	0.2121	3.0054	2.1372	0.9464	10	45	3	0.2601	3.1735	2.2898	1.0692
10	27	4	0.2520	2.5279	1.7196	1.2191	10	45	4	0.2843	2.3501	1.4992	0.9908
10	27	5	0.2722	2.8894	2.0741	1.8475	10	45	5	0.2965	2.5097	1.8062	1.6630
10	27	6	0.2843	2.6047	1.9964	1.9778	10	45	6	0.3038	2.8054	2.1746	2.1768
10	27	7	0.2924	2.3133	1.8182	1.8684	10	45	7	0.3087	1.9291	1.3755	1.4459
10	27	8	0.2982	2.1861	1.5562	1.6973	10	45	8	0.3122	2.7214	2.0586	2.2337
10	27	9	0.3026	1.9713	1.3127	1.5166	10	45	9	0.3149	2.2867	1.6428	1.8621
10	27	10	0.3060	2.1280	1.6819	1.8319	10	45	10	0.3169	2.2680	1.6528	1.9010
10	28	3	0.2164	3.0535	2.2572	0.8043	10	46	3	0.2616	3.3219	2.5223	1.0688
10	28	4	0.2549	2.9755	2.2261	1.4972	10	46	4	0.2854	2.3451	1.6407	1.1031
10	28	5	0.2743	2.5750	1.8209	1.4305	10	46	5	0.2973	2.3404	1.7907	1.6014
10	28	6	0.2861	1.8663	1.4018	1.3387	10	46	6	0.3045	2.2823	1.5808	1.4569
10	28	7	0.2939	2.3626	1.7131	1.6910	10	46	7	0.3093	1.7459	1.1814	1.1828
10	28	8	0.2995	1.8930	1.4656	1.4974	10	46	8	0.3127	2.0829	1.4793	1.5558
10	28	9	0.3037	2.0338	1.4737	1.5792	10	46	9	0.3153	1.9791	1.4766	1.5797

10	28	10	0.3070	2.0089	1.3739	1.5521	10	46	10	0.3173	1.8846	1.4649	1.5729
10	29	3	0.2203	3.1049	2.3605	0.6626	10	47	3	0.2632	3.0839	2.3423	0.7104
10	29	4	0.2576	3.1421	2.4537	1.5711	10	47	4	0.2864	2.6892	1.9671	1.0795
10	29	5	0.2764	1.8431	1.1889	0.7954	10	47	5	0.2981	2.3322	1.6919	1.2916
10	29	6	0.2877	2.1074	1.5488	1.3539	10	47	6	0.3051	1.6365	1.0642	0.9087
10	29	7	0.2952	1.7572	1.0647	0.9296	10	47	7	0.3098	1.2942	0.9205	0.9042
10	29	8	0.3007	1.5110	0.9758	0.9607	10	47	8	0.3131	1.5748	1.1156	1.1165
10	29	9	0.3047	1.7478	1.2089	1.2495	10	47	9	0.3156	1.6034	1.0730	1.1272
10	29	10	0.3079	1.7215	1.2133	1.2938	10	47	10	0.3176	1.5464	1.1585	1.2176
10	30	3	0.2240	3.2558	2.5586	0.6388	10	48	3	0.2646	3.1678	2.4650	0.5542
10	30	4	0.2601	1.8854	1.3120	0.9341	10	48	4	0.2874	1.6472	1.0199	0.6558
10	30	5	0.2782	2.1712	1.5699	1.0613	10	48	5	0.2988	1.7356	1.1385	0.7439
10	30	6	0.2892	1.3965	0.8895	0.7442	10	48	6	0.3057	1.2978	0.7609	0.6189
10	30	7	0.2965	1.3133	0.8560	0.7809	10	48	7	0.3103	1.7180	1.0902	0.9065
10	30	8	0.3017	1.3650	0.8713	0.8121	10	48	8	0.3136	1.3770	1.0299	1.0095
10	30	9	0.3057	1.1166	0.7070	0.7002	10	48	9	0.3160	1.6686	1.1393	1.1404
10	30	10	0.3087	1.3878	0.9543	0.9771	10	48	10	0.3179	2.2390	1.6111	1.6798
10	31	3	0.2275	3.1267	2.4639	0.5066	10	49	3	0.2660	2.9121	2.2602	0.6740
10	31	4	0.2624	2.5193	1.8816	0.6952	10	49	4	0.2883	2.2372	1.6035	0.6022
10	31	5	0.2800	1.8957	1.2901	0.6567	10	49	5	0.2995	1.9478	1.3497	0.7600
10	31	6	0.2906	1.7849	1.2107	0.8172	10	49	6	0.3062	1.4479	0.9389	0.7264
10	31	7	0.2977	1.6331	1.0831	0.8422	10	49	7	0.3107	1.5343	1.0497	0.8914
10	31	8	0.3028	1.1428	0.8124	0.7959	10	49	8	0.3140	1.5749	1.0163	0.8817
10	31	9	0.3066	1.2653	0.9644	0.9485	10	49	9	0.3164	1.6522	1.2641	1.2314
10	31	10	0.3095	1.4282	1.1645	1.1602	10	49	10	0.3183	1.4707	1.1006	1.1011
10	32	3	0.2308	3.3293	2.7105	0.7404	10	50	3	0.2673	3.7244	3.0935	0.6751
10	32	4	0.2646	2.4218	1.8271	0.6755	10	50	4	0.2892	2.4248	1.8177	0.6037
10	32	5	0.2817	2.0696	1.4772	0.6696	10	50	5	0.3002	1.5680	1.0371	0.6787
10	32	6	0.2919	1.7513	1.1973	0.7286	10	50	6	0.3068	1.2137	0.6837	0.4814
10	32	7	0.2988	1.3652	0.8335	0.5718	10	50	7	0.3112	1.7315	1.1584	0.8161
10	32	8	0.3037	1.6586	1.0960	0.8529	10	50	8	0.3143	1.6515	1.1155	0.9205
10	32	9	0.3074	1.2127	0.7914	0.7227	10	50	9	0.3167	1.2691	0.8084	0.7299
10	32	10	0.3103	0.9356	0.7521	0.7649	10	50	10	0.3186	1.5226	1.0552	1.0139