A MARKOVIAN APPROACH TO THE ANALYSIS AND OPTIMIZATION OF A PORTFOLIO OF CREDIT CARD ACCOUNTS

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A mon grand-père Joseph et ma tante Marie-Thérèse

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Abstract

This thesis introduces a novel approach to the analysis and control of a portfolio of credit card accounts, based on a two dimensional Markov Decision Process (MDP). The state variables consist of the due status of the account and its unused credit limit. The reward function is thoroughly detailed to feature the specificities of the card industry. The objective is to find a collection policy that optimizes the profit of the card issuer. Sample MDPs are derived by approximating the transition probabilities via a dynamic program. In this approximation, the transitions are governed by the current states of the account, the monthly card usages and the stochastic repayments made by the cardholder. A characterization of the cardholders' rationality is proposed. Various rational profiles are then defined to generate reasonable repayments. The ensuing simulation results re-affirm the rationality of some of the current industrial practices. Two extensions are finally investigated. Firstly, a variance-penalized MDP is formulated to account for risk sensitivity in decision making. The need for a trade-off between the expected reward and the variability of the process is illustrated on a sample problem. Secondly, the MDP is transformed to embody the attrition phenomenon and the bankruptcy filings. The subsequent simulation studies tally with two industrial recommendations to retain cardholders and minimize bad debt losses.

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Nomenclature

MDP Markov Decision Process

ADP Approximate Dynamic Programming

$\delta(k)$ Discrete Dirac function defined by:

$$\delta : \mathbb{Z} \longrightarrow \{0,1\}$$
$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \in \mathbb{Z} - \{0\} \end{cases}$$

$H(k)$ Discrete Heaviside Unit Step function defined by:

$$\delta : \mathbb{Z} \longrightarrow \{0,1\}$$
$$H(k) = \sum_{n=0}^{\infty} \delta\left(n - k\right)$$

$B(a,b)$ Beta function defined by:

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \quad 0 < a,\, 0 < b$$

$F_\beta(x,a,b)$ Value in x of the Cdf associated to a Beta function with pair of parameters (a,b).

$$F_\beta(x,a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad 0 < a,\, 0 < b$$

APR Annual Percentage Rate

mrp rate for minimum required payment

Chapter 1

Introduction

1.1 Background

Since the introduction of the credit card in the 1960s, the banking industry in the field has been booming. Credit card banking has proven to be one of the most profitable consumer lending industries, which has been actively developing over the years. As in any lending activity, profit is yielded by running the risk of default or bankruptcy from the debtor side. Issuers, in order to handle the exploding demand, have no alternative but to rationalize and to automate their decision-making processes instead of using the classic judgemental analysis. Today, credit card institutions deal with substantial portfolios of accounts and a fierce competition is taking place to conquer new market shares. Credit card groups, eager to acquire new accounts, are thus led to take more risks and consequently suffer considerable overall debts and substantial write-offs due to bad debts. To remedy this situation, card issuers have been making intensive use of financial forecasting tools. With intensive data warehousing becoming a common place and steadily improving information systems, the sharpening competition has exacerbated growing needs for accurate predictive models of risk and for techniques to efficiently manage accounts. The credit granting decision has attracted considerable attention over the last four decades and has turned out to be one of the most lucrative applications of Management Science. Likewise behaviourial scoring, serving the purpose of assessing the risk of existing cardholders, has been the focus of intense research both in the academia and in the industry. On the other hand, relatively scant attention has been dedicated to the dynamic management of the approved applicants. The present study aims to develop an effective operational strategy to manage customers and, in particular, risky customers.

1.2 Impact of Delinquency and Default

Broadly speaking, the economic growth has, in recent years, generated a rise in per capita income that was accompanied by a rising consumption. These joint phenomenon together with an ever more widespread use of credit cards have resulted in an increasing consumer debt and in particular credit card debt. This growth in the credit card debt has been overall accompanied by raising charge-offs and delinquency. The following plot, reproduced from Ausubel [4], depicts such a trend for the American market.

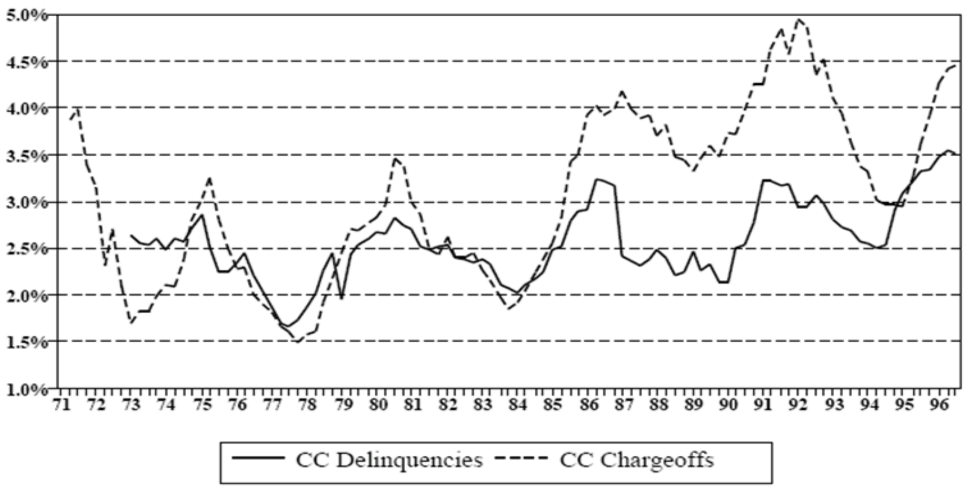

Figure 1.1: Credit Card Delinquencies and Charge-Offs from 1971 to 1996 (Reproduced from Ausubel [4])

The delinquency rates and charge-offs are substantial and thus prove the necessity of an appropriate management of the existing cardholders and in particular the need for an accurate collection policy. One such policy is crucial to the good evolution of the portfolio from month to month as well as the minimization of the amount of bad losses.

1.3 Characteristics of Credit Card Banking and Related Problems

Credit card banking is a consumer lending activity characterized by monthly periods of credit. It can be regarded as an open end loan featuring high interest rates and flexible monthly payments. The lifetime of a credit card account is bounded by its expiration date, after which the card will usually be reissued. Credit card banking is by nature a risky activity which leads the issuers to face two different types of problems: the credit granting problem and the cardholders management problem.

1.3.1 The Credit Granting Problem

Formally stated, the credit granting problem is to decide on whether to grant credit to an applicant and, in the case of approval, to accurately determine the credit lines. The credit lines should be set so as to fulfill the cardholder's needs of credit, be at low default risk and yield a maximum profit derived from the card usage. The problem consists then of optimizing the discriminative analysis amongst a population of applicants with respect to these objectives.

1.3.2 The Cardholders Management Problem

The second category of problems has a much wider scope as it is concerned with the management of a portfolio of existing accounts. The related objectives cover a wide variety of situations and the approaches to these problems may be very diverse. The card issuer may, for instance, aim to reduce attrition or seek to determine credit line changes that will increase the profitabilities of a qualified population of cardholders, with substantial usages and low risk profile. The minimization of default rate and charge-offs is yet another key problem. There are two different types of approaches to one such problem;

1. Statistical approaches using scorecards and behavioural scoring to estimate the risks of the applicants or the future profitabilities of the current customers.
2. Dynamic models of the customers' behaviours.

The literature review would be developed along these lines of distinctions between statistical and dynamic approaches. The statistical approaches would first be introduced in order to familiarize with the types of problems encountered and to understand their stakes. Emphasis shall then be put on the dynamic modeling as it constitutes the main focus of the present study.

1.4 Thesis Overview

1.4.1 Objectives

The objective of this research is to develop a general framework for the optimization and analysis of a portfolio of credit card accounts. The main focus is to work out collection policies which optimize the profitabilities of the accounts, minimize the credit losses and charge-offs, reduce the operating costs incurred by the undertaken collection strategies. A Markov decision process is so developed to capture the dynamic characteristics of the problem with consideration to the stochastic nature of the cardholders' repayments first and secondly to the attrition of accounts and to the possible bankruptcy filings. Finally an approach unifying the risk sensitivity and the expectation of profitability is formalized and computationally solved.

1.4.2 Research Scopes

A two dimensional Markov decision process with an absorbing state, accounting for the written-off accounts, is first defined. It is solved for both the finite horizon to derive value forecasts and for the infinite horizon to derive stationary collection policies.

A variance penalized Markov decision process is then proposed to model the risk variability.

As for the bankruptcy filings and the attrition phenomenon, the initial Markov decision process is modified so as to embody either of these stochastic components.

1.4.3 Methodology

The Markov decision process is first thoroughly specified. The rules defining the relation of the cardholder to the card issuer are precisely looked into and subsequently formalized. The different cash flows and the specificities of the credit card industry are thus accounted for in an implementable Markov decision process.

Owing to the difficulty of obtaining confidential data, a simulation approach is favored. To that end, an approximate dynamic programming approach is proposed to model the cardholders' behaviors. A criterion defining the rationality of the cardholders in their repayments is proposed and used to generate reasonable transition probabilities. Based on the credit card agreement of a major issuer in Singapore, a simulation study is conducted and the results are interpreted in the light of some industrial recommendations.

The variance penalized Markov decision process is adapted from Filar and Kallenberg [14]. Developing on their theoretical work, a scheme is proposed to computationally solve the related problem. A case sample shows that the different Pareto optimums for the expected total reward and the associated variability are worked out by increasing the penalization factor.

The novel approach to include either the attrition phenomenon or the bankruptcy filings is based on the embodiment of either of these stochastic variables in the original Markov decision process. Making use of the structural property of the initial Markov decision process featuring an absorbing state, additional transitions and their corresponding rewards are defined to account for the attrition of the accounts or the bankruptcy filings. Assuming these two phenomena to be one-step Markovian processes, the resulting problem is proven to be a proper Markov decision process.

Chapter 2

Literature Survey

2.1 Introduction

Credit scoring, behavioural scoring, models of repayment and usage behaviour are techniques used by financial institutions to make decisions in the risky environment of consumer and credit card lending. The objective of credit scoring is to decide on whether to grant credit to a new applicant, to determine the amount and the limits (lines) of the credit [see 1.3.1]. It aims to distinguish potentially "good" cardholders from "bad" [1] ones among the population of credit card applicants where limited information is available. On the other hand, behavioural scoring and behavioural models of usage provide a help in managing existing clients [see 1.3.2]. They allow financial institutions to forecast probability of default, expected profit and subsequently to manage their risky clients. These tools can be used to reduce the risk of cardholders defaulting, to minimize credit losses as well as costs, involved in debt collection. Scoring has been the focus of extensive commercial research and

[1] The definition of "good" and "bad" cardholders is somewhat arbitrary since it requires choosing some criteria to assess the quality of an account. However, a large consensus prevails in the industry [see 40]: "bad" cardholders are customers who, within the time window of consideration, either default or miss at least three consecutive payments (often referred to as "Ever 3 down"). The "good" cardholders are the complementary part of the population qualifying for the separation.

is widely used in the banking industry. Surveys can be found in [26, 35, 40]. Scoring techniques do not consider the stochastic and dynamic aspects of managing existing clients. They are, nevertheless, the most widespread decision systems in the industry for their efficient predictive powers and their abilities to handle and aggregate numerous characteristics of each cardholder. The literature review would first provide an overview of scoring. Secondly, the focus would be put on the behavioural modeling and particularly on stochastic modeling using Markov Chains. There has been a considerable amount of work done in the area, however some publications may suffer from a lack of clarity for confidentiality of data is a highly sensitive issue in the banking industry.

2.2 Predictive Models of Risk

2.2.1 Credit Scoring

2.2.1.1 Introduction

Durand [13] was a precursor in applying statistical methods to problems in corporate finance. In 1941, his study for the US National Bureau of Economic Research paved the way of using objective and rational techniques to discriminate good and bad loans. Henry Wells of Spiegel Inc. further pursued investigations in the field in order to build a predictive model. It is generally recognised that Wells elaborated the first credit model in the late 1940s. Predictive models, however, were sparsely used until Bill Fair and Earl Isaac completed their first works in the early 1950s. Later on, the successful introduction of credit cards and the consecutive high demand of credits resulted in numerous developments of credit scoring techniques. Thomas [40] and Baesens, Gestel, Viaene, Stepanova, Suykens, and Vanthienen [5] provided extensive academic insights of the different scoring techniques and algorithms in use today, while Mester [30] and Lucas [26] offer interesting approaches from a business perspective.

Credit scoring comprises methods of evaluating the risk of credit card applications. In particular, credit scoring aims to discriminate applicants that are likely to be "good" and profitable cardholders from applicants that are likely to be "bad" cardholders over a finite period of time. For accuracy reasons, the time horizon considered is usually limited to twelve months.

Originally, credit scoring produces a score for each applicant that measures how likely the applicant is to default or to miss three consecutive payments. Its computation makes use of inputs such as credit information reported through application form and Credit Bureau data concerning the cardholder credit history. The characteristics that have a predictive power are detected after thorough analysis of the historical data. Most scoring systems have a threshold score called the cutoff score above (below) which the applicant is believed to become a "good" ("bad") cardholder.

The definition of credit scoring has progressively been broadened. Nowadays, it refers to the class of problem of discriminating "good" from "bad" applicants when the only information available comprises answers provided on the application form and a possible check of the applicant's credit history with some external credit bureaus. Application scoring is mainly based on statistical techniques, neural networks and other operational research methods. Saunders [36] presented a discussion of these different methods.

2.2.1.2 Statistical Techniques

Statistical techniques can be divided into two categories, namely parametric and nonparametric approaches. Parametric approaches were the first to be developed.

The most commonly used techniques of this kind comprise linear regression, logistic regression, probit model and discriminant analysis. Later on, investigations of nonparametric approaches have led to the elaboration of techniques such as classification trees or k-nearest neighbors. The present review would first introduce the different parametric approaches and further give an overview of the nonparametric ones. The description of the parametric approaches is restricted to logistic and probit regressions for the linear one actually falls in the same vein.

2.2.1.3 Parametric Approaches

Logistic regression is currently the most widespread credit scoring technique. This approach assumes the logarithm of the ratio, between the probability of a cardholder being "good" given his application characteristics and the probability of a cardholder being "bad" given his application characteristics, to be a linear combination of the characteristic variables. Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ be the vector of application characteristics comprising, for each applicant, of information from application form and possible data from external credit bureau [5]. Let $\mathbf{w} = (w_1, w_2, ..., w_n)$ be the weight or importance granted to each characteristic of the vector $\mathbf{x}$. Let $p(good|\mathbf{x})$, $p(bad|\mathbf{x})$ be the probability that the applicant turns out to be a good (bad) cardholder given its application characteristics $\mathbf{x}$, respectively.

$$\ln(\frac{p(good|\mathbf{x})}{p(bad|\mathbf{x})}) = \ln(\frac{p(good|\mathbf{x})}{1 - p(good|\mathbf{x})}) = w_0 + \mathbf{w}^T\mathbf{x} \tag{2.1}$$

The parameters $w_0, \mathbf{w}$ are derived by applying maximum likelihood estimators to the samples reported from the historical data. The logistic regression can be connected to the scoring technique. Let $s(\mathbf{x})$ be the score of the applicant calculated as follows $s(\mathbf{x}) = w_0 + \mathbf{w}^T\mathbf{x}$. Equation 2.1 is hence equivalent to,

$$p(good|\mathbf{x}) = \frac{1}{1 + exp(-s(\mathbf{x}))} \tag{2.2}$$

The probability of an applicant being "good" given his characteristic is an increasing function of his score. This consideration is naturally consistent with the definition of a cutoff score above (below) which the application is approved (rejected).
Likewise, probit models aim to fit, as accurately as possible, a linear score of the application characteristics to the reported data. Whereas logistic regression postulates the logarithm of the odds of conditional probabilities of being "good" against being "bad" to be a linear combination of the application characteristics, probit models assume the probability $p(good|\mathbf{x})$ to be distributed according to a cumulative normal distribution of the score of the applicant $N(s(\mathbf{x}))$.

$$s(\mathbf{x}) = w_0 + \mathbf{w}^T\mathbf{x} \tag{2.3}$$

$$p(\text{good}|\mathbf{x}) = N(s(\mathbf{x})) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{s(x)} \exp(\frac{-s^2}{2}) ds \tag{2.4}$$

The probit model objective is, given the reported data, to find $w_0, \mathbf{w}$ for which the latter normality condition best holds.

Discriminant analysis differs from the above for it aims to divide applicants into high and low default-risk rather than estimating probability of default. To that effect, a classification rule is defined: an applicant is considered to be "good" if his probability of being "good" given his application characteristics is greater than his probability of being "bad". One should postulate a prior class of distributions for the conditional probabilities $p(\mathbf{x}|good)$, $p(\mathbf{x}|bad)$ of a cardholder having application characteristics $\mathbf{x}$ given that he is "good", "bad" respectively. It is commonly assumed that the latter probabilities belong to the class of multivariate Gaussian distributions. The decision rule is then a quadratic expression of $\mathbf{x}$, called quadratic discriminant analysis (QDA). The outputs of the discriminant analysis are the estimations of the parameters of the two normal multivariate distributions that best match the reported data.

In the special case, where the covariance matrices for $p(\mathbf{x}|good)$, $p(\mathbf{x}|bad)$ are equal, the rule simplifies to a linear rule. Such discriminant analysis, known as linear discriminant analysis (LDA), features two standard results;

- Fisher [17] elaborated a method called Fisher's Linear Classification Function (LCF) that, in this special case, can be used to find the parameters $\mathbf{w}$ defining a score that best separates the two groups.

- Beranek and Taylor [6] suggested a profit oriented decision rule in this particular case. The classes of "good" and "bad" cardholders are defined so as to minimize the expected losses due to the misclassification of "bad" cardholders into the "good" category and due to the misclassification of "good" cardholders into the "bad" category. The latter misclassification is actually a lost opportunity of making profit since applicants that would have turned out to be profitable are in this case rejected. In the present special case, this decision rule simplifies as well to a linear combination of the application characteristics weighted by their $\mathbf{w}$.

The previous parametric statistical techniques have two major flaws. Firstly, some difficulties arise when dealing with categorical information. Many questions in the application forms, such as "does the applicant own his residence?", typically generate yes/no answers that are called categorical answers. One way to overcome this difficulty [see 40] is to consider the answer to the question as binary variables. However, it often leads to a large number of variables even with a few questions of the kind. Another way to solve the problem is first to do prior grouping according to the answers of such questions, and then in each yes (no) category to compute the ratio between the probability of being "good" and the probability of being "bad". Such ratio is then the value of the variable associated to the categorical answer.

Secondly, the preceding parametric statistical approaches have strong hypotheses concerning the score and its linearity. They are subsequently sensitive to correlations of variables that are bound to happen in real cases.

2.2.1.4 Nonparametric Approaches

One of the most common nonparametric statistical approaches is the k-nearest neighbor classifier. This technique will divide the new applicants into two categories or labels; "goods" and "bads". Any existing or past individual is beforehand assigned to any of these labels depending on his reported results. In order to perform the classification of the new applicants, a metric defined on the space of application data and a decision rule are needed. The metric measures how similar new applicants and existing (or past) cardholders are. The Euclidian distance is commonly used. The decision rule should be defined so as to assign as accurately as possible new applicants to one of the two class labels. For instance, a rule frequently applied is that a new applicant belongs to the class that contains the majority of his k-nearest neighbors (in terms of the metric defined). Such a system can easily be updated. The choice of the metric together with the decision rule is a highly sensitive issue in this kind of model.

Classification trees were first developed in the 1960s. This type of classifiers aims to segment cardholders into groups of rather similar or homogeneous credit risk. Different algorithms exist to build such trees and to decide how to split the nodes. Nevertheless, they all split iteratively the sample of reported data into two subsamples. At each step, the criterion used in node splitting is to maximize the discrimination of the risk of default between the two resulting subsamples. Such a criterion allows one to point out which variable of the application characteristics best splits the subsamples and also allows one to decide when to stop. A terminal node is then assigned to the category of “goods” (“bads”) if the majority of its applicants is “goods” (“bads”). To predict the outcome of a new applicant, one just needs to scan down the tree according to his application characteristics. The new applicant will be considered “good” (“bad”) if his terminal node is “good” (“bad”).

2.2.1.5 Neural Networks

In the 1990s, neural networks started to be applied to discriminate “good” from “bad” applicants. They are artificial intelligence algorithms that are able to learn through experience and to discern the relationships existing between application characteristics and probability of the applicant to default. West [43] proposes a benchmarking approach that compares neural networks of increasing level of complexity to the traditional statistical approaches. The main feature of neural networks is their ability to model non-linear relationships between application characteristics and default risk. The type of networks commonly used for credit scoring is the multilayer perceptron which comprises of an input layer, some hidden layers and one output layer. The present description, solely aiming at the understanding of the concepts of neural networks, restricts to the introduction of a multilayer perceptron comprising of only one input layer of n entries, a single one hidden layer of m neurons and a unique output neuron. The input layer consists of the application characteristics $\big(x_i\big), i = 1, \ldots, n$. The output is a single neuron which eventually estimates the conditional probability of the applicant being “good” given his characteristics. Let $(\lambda_{i,\,j}), i = 1, \ldots, n, j = 1, \ldots, m$ be the weight to connect input i to hidden neuron j. The sum of the weighted inputs and of a bias term b_j is used to compute the output of each neuron j of the single hidden layer via a first transfer function φ^1. This function is identical for each neuron of the hidden layer. The transfer function is not necessarily linear and therefore allows modeling of non-linearity. The outputs of all the neurons j of the hidden layer are then used, in an identical manner. Let μ_j be the weight to connect the hidden neuron j to the unique output neuron. The sum of their weighted outputs and of a bias term c is used as the input of the final transfer function φ^2 to compute the output of the unique output neuron. This output is the conditional probability of default. The logistic transfer function is frequently used as the final transfer function for it takes values in $[0, 1]$.

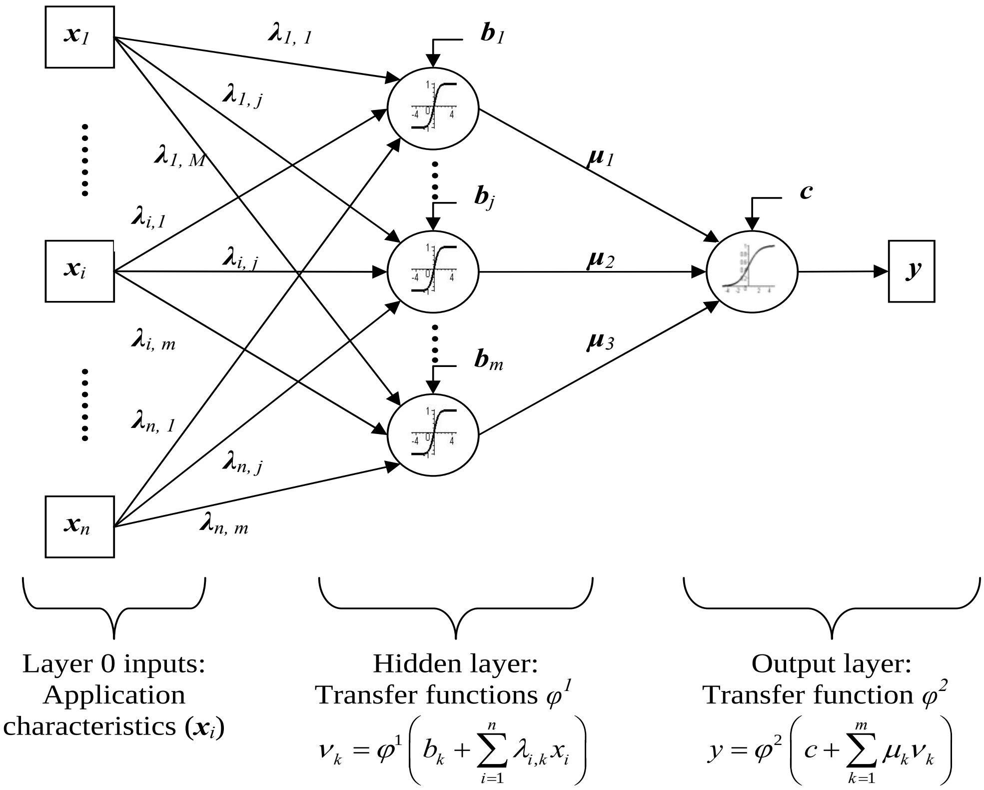

Figure 2.1: Multilayer Perceptron

The neural network is trained with the reported set of data. The training mainly consists of estimating as accurately as possible the weight parameters $(\lambda_{i,\,j})$, μ_j. After that, the neural network can be used as an updatable predictive model.

The features of the neural networks are obviously attractive. Nevertheless, they have not clearly proven, so far, to be superior to other approaches in the field. Rather, they are used for fraud detection for instance.

2.2.1.6 Operational Research Techniques

Some operational research techniques are frequently used in the industry. They mainly consist of linear programming (LP) and support vector machines.

LP is based on the assumption that an accurate score can be obtained as the sum of the weighted characteristic variables. A cutoff score c is a priori set. The latter defines a hyperplane that separates the categories of the "goods" from the "bads". The constraints of this LP are then defined as follows: the "goods" ("bads") are supposed to have a score higher (lower) than the cutoff score c. One should account however for possible misclassifications by introducing slack variables in the constraints. A "good" ("bad") account, for instance, may have a score slightly lower (higher) than the cutoff. The slack variables allows the constraint of the cutoff to be respected without misclassification. Solving this LP, according to the reported data, eventually builds a linear scorecard assigning the weights to the application characteristics that minimizes the misclassification errors. Joachimsthaler and Stam [23] provided an excellent review of this class of problems. Following their general presentation, an LP formulation is introduced to model the application scoring.

Let i be the index of the past applications. The range of i covers the whole training set of applications derived from past data. Applicant i can either belong to the class of "goods" or to the class of "bads". Let $\mathbf{x}_i = (x_{1,\, i}, x_{2,\, i}, ..., x_{n,\, i})$ be the vector of application characteristics of applicant i and $\mathbf{w} = (w_1, w_2, ..., w_n)$ be the vector of weights associated to each of these characteristics. $\mathbf{w} = (w_1, w_2, ..., w_n)$ is common to all the applicants in the training set. Introduce $d_{i,\, G}$, $d_{i,\, B}$ to be positive slack variables to model the misclassification errors in each category. The decision variables of the present LP are then $\mathbf{w}$, $d_{i,\, G}$, $d_{i,\, B}$. The l_1 - norm is used to measure the misclassification.

$$\min_{\mathbf{w},\, d_{i,\, G},\, d_{i,\, G}} \sum_{i \in goods} d_{i,\, G} + \sum_{i \in bads} d_{i,\, B} \tag{2.5}$$

subject to,

$$\begin{cases} \mathbf{w}^T \mathbf{x}_i + d_{i,G} \geq c & \text{if } i \in \text{``goods''} \\ \mathbf{w}^T \mathbf{x}_i + d_{i,B} \leq c & \text{if } i \in \text{``bads''} \\ d_{i,G} \geq 0 \\ d_{i,B} \geq 0 \\ \mathbf{w} \text{ unrestricted} \end{cases} \tag{2.6}$$

Solving this LP will provide the decision variables that minimize misclassification errors. In particular, the optimal $\mathbf{w}$ will define the scorecard to be applied to future applicants.

Other similar approaches to solve this kind of problem exist. They include mixed-integer programming formulation and hybrid model. The latter, for instance, does not require setting a prior cutoff score c. This task is a sensitive issue which is usually handled by experienced analysts. The hybrid model instead considers c as a decision variable of the optimization problem associated.

Recently support vector machines models have been developed to solve the preceding classification problem. The approach is similar to the LP formulation. The constraints are still linear but now include a featuring space, and the objective function differs by introducing a quadratic term $\mathbf{w^T w}$ representing the margin that separates the two classes of "good" and "bad" applicants. This constraint optimization problem belongs to the class of convex programming models that can be solved using Lagrangian multipliers.

2.2.2 Behavioural Scoring

Behavioural scoring aims to improve the management of cardholders so as to increase their profitabilities to the bank. The behavioural scorecards incorporate credit scores from external bureaus, data from application forms and data related to repayment histories and usages. The latter are extra information that is not available when performing the credit scoring. Thus, the building of the related scorecards requires a sample history of each existing cardholder that is referred to as performance period. The performance period can range from 12 to 18 months before the actual date of consideration. Likewise, the scorecards require a time horizon that sets an outcome date for the current account; 12 months after the end of the performance period is commonly used. Many characteristics related to usages made by cardholders are continuously reported and recorded in data warehouses. Behavioural scoring techniques thus include many variables describing cardholders' behaviour such as payment history, various installment balances, and outstanding balance together with the application characteristics. Behavioural scoring also makes use of delinquency history. The latter reports the history of overdue periods as well as the corresponding outstanding balance. Again, different techniques such as linear, multiple, or logistic regressions, and discriminant analysis have been applied to pinpoint the most sensitive variables and to forecast the likelihood of a cardholder defaulting according to his individual credit score and credit card usage.

The classification-based behavioural scoring systems may divide the population into different clusters and apply to them different scorecards and forecasts. Moreover, cardholders can be split into two categories; new cardholders and established cardholders. A reduced weight is then granted to the performance period for the new cardholders category. The performance period of the latter category can be reduced to 6 months after which the definition of 'bad account' is updated according to the cardholder's usage. This definition is then applied for the future credit management of the cardholder.

The probability of a cardholder defaulting his future payment, his delinquency history as well as the probability of his switching account to a competitor are the main elements of the scorecards. They provide essential information in order to build a value model for the portfolio and to decide optimal credit control. For instance, based on these data, the financial institution can decide whether or not to take reminding or warning actions and can set the timing and scheduling of these actions.

2.3 Behavioural Models

One of the shortcomings of the scoring techniques is that by nature the dynamic evolution of the cardholder is not considered in the model. However, they have proven to be sufficiently accurate to become the dominant tools of screening as of today. The review will present firstly a chronological approach to the development of behavioural models and secondly the latest developments.

2.3.1 Genesis of the Behavioural Models

Cyert, Davidson, and Thompson [12] introduced the first dynamic model to describe the evolution of accounts receivables. Their article is considered as the classic basis and the initial reference in the field. Their model, hereafter referred to as CDT, makes use of Markov chains to estimate the amount of dollars of receivables in a retail establishment that will turn out to be uncollectible. The idea underlying this Markov chain approach is to define a state space together with its related transition probabilities to estimate the moves of the dollars of receivables of the whole portfolio between the different due status. The CDT model deals with accounts of a retail establishment and other businesses but not necessarily with credit card accounts.

However the scope of the article and the techniques developed are of interest. Consider the dollars of receivables of a balance of a retail account at time t. Define the following age category as follows:

$B_0[t]$ dollars of receivables that are 0 month past due

$B_1[t]$ dollars of receivables that are 1 month past due

...

$B_i[t]$ dollars of receivables that are i months past due

...

$B_n[t]$ dollars of receivables that are n months or more past due

Hence, i is the state variable. B_n corresponds to the bad 'debt category' for which the account balance can be repaid eventually or charged off i.e. the account is written-off as uncollectible. The acceptable period of delinquency in the credit card industry is usually limited to 90 consecutive days overdue from the contractual due date after which the account is usually classified as substandard. A substandard account is subject to more severe collection reminders and strategies since the cardholder who kept falling into arrears with repayments is less likely to pay back. Cardholders who eventually miss 8 consecutive payments will have their accounts charged off and their debts is a loss due to default.

Consider now the evolution of the dollars of receivables from month t to month $t+1$. Let $B_{i,k}[t]$ be equal to the amount of dollars in age category i as of month t that moves to age category k as of month $t+1$. It is necessary to add one age category denoted B_θ to those categories previously defined in order to account for the dollar of receivables that are fully paid as of month t. A $(n+2) \times (n+2)$ square matrix, whose entries are $B_{i,k}[t]$, can be used to describe the transitions of dollars between the different age categories. The model has some structural properties; the two age categories, θ fully paid and n bad debt, are absorbing. Any dollar entering either B_θ or B_n as of month t cannot transit any more. From this month, it will therefore stay trapped therein.

A matrix of transition probabilities $\left(P_{i,\,k}[t]\right)$ can be defined, with $P_{i,\,k}[t]$ being the probability that dollars in age category i in month t will move to age category k next month. It is assumed in the CDT model that the transition probabilities are constant.

$$P_{i,k} = \frac{B_{i,k}}{\sum\limits_{p=\theta}^{n} B_{k,p}}, \quad (i, k = \theta, 0, 1, ..., n) \tag{2.7}$$

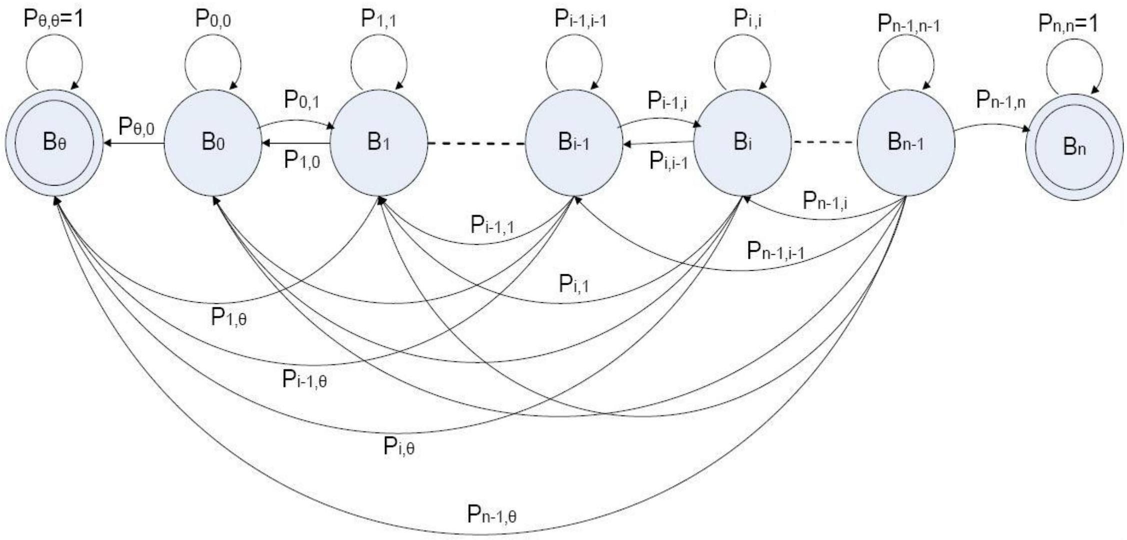

Figure 2.2: CDT State Transitions flowchart

The process previously defined consists of a Markov Chain process, having $n+2$ states, two of which are absorbing and a constant one step transition probabilities matrix denoted by $(P_{i,\,k})$. The latter is assumed to be independent of the initial age distribution of the accounts balances. The CDT model provides answers to the following three questions:

- Assuming the process has an infinite horizon, what proportion of the involved dollars will end up in the paid state and what proportion will end up as a bad debt B_n?
- Assuming c new dollars are distributed into the various age categories each month and assuming the way of distributing the c dollars to be constant, what is the steady state distribution of receivables by age category?
- Assuming c_i new dollars are received each month, the manner in which the dollars are distributed each month varies cyclically, and new charges grow geometrically over a period of length T with a factor α, what is the distribution of receivables by age category at the end of any period?

The answer to the first question provides an estimate of the loss due to credit loss. CDT defines the allowance for doubtful accounts at the point in time i as the dollar amount of accounts' receivables which will be uncollectible and thus bad debts in the future. Using the preceding estimation of default rate, CDT derives the allowance for doubtful accounts and its corresponding variance.

CDT is a useful tool in forecasting the evolution of a retail establishment portfolio and constitutes the very first step in building a net present value embodying a predictive model of risk that can be updated according to the cardholders' behaviour.

The authors discussed the assumption to the model of constant transition probabilities matrices for it is a restriction that does not allow changes of the economic conjuncture or seasonality in economic cycle to be taken into consideration.

Corcoran [10] developed a more refined model using exponentially-smoothed transition probabilities matrices to improve the stability of the model and the accuracy of the cash flow forecasts. Corcoran pinpointed a major issue of *CDT*. *CDT* assumes constant transition probabilities matrices to estimate the default rate and the steady state distribution of receivables. Corcoran made use of a simple exponential smoothing of the transition probabilities matrix applied to the same Markov chain as in *CDT* with the same state-space. The simple exponential smoothing provides reliable transient estimates that are useful in portfolio management. The existence of considerable variations in aging and monthly balance justifies the use of simple exponential moving rather than constant transition probabilities matrices. Simple exponential smoothing is appropriate to model variation around a mean value. Corcoran used Winter's triple exponential smoothing to model seasonality and found that introducing a seasonal factor clearly improves the forecast. The seasonal smoothing allows one to take into consideration the most recent behaviours and to reflect them as well as the seasonal effects in the transition probabilities matrices.

Besides, Cyert et al. [12] briefly suggested in their article that a possible extension of their model would be to consider the bank accounts themselves and their behaviours instead of the dollars. Therefore, Cyert and Thompson [11] developed a model called Credit Control Model, that in the remainder of the review will be referred to as *CCM*. Its scope is to study credit card accounts according to the risk profiles of the cardholders. Considering some risk categories, the model divides cardholders and applicants into k different risk categories. For each risk category, the model assesses the likelihood that a dollar of receivables from this certain category becomes uncollectible. Moreover, *CCM* allows credit managers to estimate potential expected net revenue of a credit application. For this purpose, each risk category has its own transition probabilities matrix. Similar to *CDT*, the transition probabilities matrix for each risk category gives the probabilities of dollars moving from state i to state j. Assumptions about the payment behaviours of the k-risk categories are embodied in the k different transition probabilities matrices. *CCM* requires first the k-risk categories to be populated. Cyert and Thompson suggested that a multiple regression of independent variables be used to develop a scoring function. The score can then be partitioned into k line segments dividing the whole sample of credit scores. The overall union of the k line segments naturally covers the whole credit score space and allows classification of each account into one of the k discrete risk categories. The model provides an estimate of the net present value which naturally can be used as a measure by credit managers to decide on whether to grant credit to a new applicant. *CCM* also provides a clear and easily computable present value including cases of bad debt losses. The present value is comprehensive for cases of bad debts, bookkeeping charges and the production costs of the loan themselves are taken into account. The variance of the present value can also be computed. An interesting feature of *CCM* is the introduction of a decision rule for application approval. From a fixed sample of applicants distributed into the k-risk categories, the decision rule consists of iteratively populating the portfolio starting from the less risky categories and aiming at the most risky ones until the coefficient of variation (ratio of the square root of the variance over the expected revenue) exceeds an arbitrary limit fixed by the financial institution according to its risk profile. *CCM* is very innovative since it offers a predictive model of risk taking into account heterogeneities of behaviours and risks among cardholders. These heterogeneities are embodied in the transition probabilities matrices themselves. The estimates of the net present value and its variance for each risk category are key indicators for managers to make a decision of acceptance.

2.3.2 Recent Developments

Similar approaches were adopted in most of the articles posterior to CDT and *CCM* from the 1980s through present. Kallberg and Saunders [24] introduced a Markov chain model with an account-focused perspective. Unlike the previous models, Kallberg and Saunders considered the due states of the accounts themselves to define different state spaces. The main difference between these two kinds of approaches lies in the nature of the data. *CDT*-like models rely on aggregated data derived from the aging methods that are applied to dollars of receivables. The transition probabilities in such models measure the likelihood of a dollar moving from one age category to another. Kallberg and Saunders instead focused on the age category of the account. The relevant probabilities measure the likelihood of an account moving from one age category to another. Their model subsequently makes the payment obligations become more influential for they govern the aging process of each account.

The main idea of their article is to introduce Markov chain models with three different types of state space; a first basic Markov model and then two refinements making use of relevant behaviour variables. The basic model defines $N+2$ states according to the number of payment(s) overdue. 'P' denotes "fully paid-up" state corresponding to an account without any outstanding balance in period t. '0' denotes current account state, that is to say the account has no payment overdue; at least the minimum required payment was paid in period $t-1$. Likewise, '1' denotes one-month overdue state; the repayment in period $t-1$ was less than the minimum required payment but the repayment in period $t-2$ was at least the corresponding minimum required payment[1]. The states are then defined iteratively with increasing overdue payment periods until 'N' which denotes bad debt state, that is account overdue for at least N consecutive periods. Another interesting variant of the *CDT* is to consider mover-stayer models. Frydman, Kallberg, and Kao [18] were the first to introduce such a model to credit behaviour. Prior to them, Blumen, Kogan, and McCarthy [9] initially developed a similar model to assess the mobility of labor. The mover-stayer model incorporates a simple form of heterogeneity. People who always

[1]Kallberg and Saunders noted that decreases in the age of an account in state $i, i = 1, \ldots, N-1$ are, with this definition, restricted to transition from i to either '0' or 'P' when the minimum required payment is met.

follow the same payment pattern and therefore who always stay in the same state are considered as stayers, while the others are considered as movers. The movers follow the prior Markov model. The mover-stayer model is thus the combination of the prior Markov model for movers and of the steady state behaviour for stayers. Algebraic manipulations and estimations of parameters show interesting results, particularly, that incorporating heterogeneity may be more important than modeling nonstationarity. Till and Hand [41] made an extensive review of behavioural models of credit card usage and were still using stationary Markov chain with straightforward estimations of the transition probabilities. The article presented a comparison of stationary, non stationary models together with the mover-stayer model. The authors concluded that the results are quite similar and the main trends are the same. Sojourn times are also derived from the one-step stationary transition probabilities matrix.

2.3.3 Managing Credit Card Delinquency

Liebman [25] pioneered the use of Markov decision model for selecting optimal credit control policies. His formal model defines a discrete three dimensional state space comprising the "age class" of the account (or due status), its charge volume and its previous credit experience. As for the reward function, it consists of the discounted total credit costs defined as the sum of the costs incurred by the undertaken actions, the interest carrying costs and an estimated bad loss per unit per account. The latter represents an approximation to the write-offs occurring each period. The formal Markov decision process, as defined, is transformed into an equivalent linear program in order to solve the infinite horizon problem.

One may argue that the model, as formulated, is too restrictive. The reward function solely includes costs. Therefore, neither the interest revenues nor the lines of income specific to the credit card banking (e.g. interchange revenue) are accounted for in the decision process. Moreover, the definition of the state space, though attractive, does not seem practical. For instance, the explanations of how to estimate the probabilities of transitions from a certain charge volume (or from a certain past credit experience) to another, are omitted. These two variables, being dynamic characteristics of an account, are however likely to change during the process all the more as an infinite horizon is considered in the model formulation. In the sample problem, the transitions are incidentally restricted to flows from one age class to another. The dimensions of the charge volume and credit experience should rather serve the purpose of prior partitioning of the portfolio of credit cards accounts. Liebman finally recommended further research in two areas,

1. Explicit consideration of the new account acceptance decision in the model
2. Extension of the formulation to include marketing policies within the model's framework

The present study would develop a model featuring a detailed value analysis of a credit card account under credit control. Such an approach would account for the different incomes derived from the credit card usages and repayments. It would consequently offer a tradeoff between the risk of bad debts and the expected revenues. Unlike Liebman [25], there would be explicit consideration of the bad debts and of the charge-off losses by defining an absorbing bad debt state.

Makuch, Dodge, Ecker, Granfors, and Hahn [27] created an automated system to manage GE Capital delinquent consumer credit. They developed a probabilistic account flow model of the stochastic delinquency processes of the accounts in the portfolio. The problem consists of finding the collection resource allocations that will optimize, for the whole portfolio, the sum of expected net collections over a specified number of monthly periods, under a limited availability of collection means. The optimal allocation is worked out as the solution of a linear program. An important assumption in the definition of the state space of the model is to consider that the accounts do not change their balance range within the period of consideration for the optimization. The assumption is justified by considering a time horizon limited to three months and large balance range categories.

One should notice three key points in the formulation of the model. Firstly, it requires a prior partition of the accounts according to the estimated risk profiles of the accounts. Such a partition is done by defining categories of performance scores, which segment the portfolio of opened accounts.

Secondly, the assumption that the balance ranges do not change over a three-month horizon, should be questioned. Although the model developed by Makuch et al. [27] aims to manage delinquent accounts, its definition excludes the possibility of having a delinquent cardholder making an important repayment so as to preserve his credit record and set his indebtedness. This situation does occur, as one may find in the delinquency state aggregated transition matrix reported in [41].

Thirdly, the study of the variance of the portfolio revenue is limited to the posterior checking that the implemented strategy has an admissible variability. The variance is a key factor to the card issuer. Its reduction would provide the issuer with more stable revenues and would increase the card issuer's protection against charge-offs. The reduction of the variance might moreover decrease the number of charge-offs and subsequently result in an increase of the volume of the portfolio as well as an improvement of the goodwills of the cardholders. The present study would investigate the study of the variance on a per account basis. To that end, a variance penalized Markov decision process would be formulated so as to work out a policy optimal in terms of trade-off between profitability and variability.

Chapter 3

Model Formulation

3.1 Background and Problem Introduction

The objective of the present research is to develop a general framework for the optimization and analysis of a portfolio of credit card accounts. In the present section, a novel Markov decision process is introduced so as to model credit card usages and repayments made by the cardholder depending on the collection actions initiated by the credit card issuer. The specific features of the credit card lending in terms of usage rules and profitabilities shall be quantified and embodied in the model. Its formulation requires the following issues to be addressed:

- How can the situation of an account be accurately described?
- What decisions and actions can be taken?
- What is the next course of actions?
- What are the criteria to be considered in making decisions?
- What are the immediate impacts of any decision?

- What resources are available to take actions?

These are necessary questions, one should answer to, in order to propose an appropriate dynamic programming model for the present problem. Indeed, their answers provide the definitions of the basic features of the dynamic programming model; the state space of the credit card accounts S, the control (or decision) set U, the set of decision epochs T, the objective (or reward) function g. The state space of the accounts S is such that, it is relevant to assume the Markov property to hold therein. The process of evolution of the accounts is then a Markov process. p refers to the probability distribution of the transitions of the account from one state to another. The collection of objects $\{T, S, U, p, g\}$ defines a Markov decision process, denoted *MDP* in the remainder of this thesis.

3.2 Preliminary Notions

3.2.1 Description of the delinquency process

Credit card banking is an open end loan based on monthly cycles of credit. It can be considered as a short term revolving loan with high interest rates and flexible repayments. These two features naturally raise the questions of:

- How is the interest calculated?
- What are the consequences of a cardholder defaulting on payments?

Particular time windows are defined in order to calculate the interest accruing on the outstanding balance. The common practice is to grant cardholders paying in full their bills a “grace period” (also called “free period” or “free-ride period”) which only applies to retail purchase transactions. On condition that a cardholder pays in full his current balance, he may benefit from a grace period (usually ranging from 20 to 30 days after issuance of the bill) which is the interest-free period granted by the lender between the date of purchase and the payment due date. There are two methods used in the industry to allot grace period:

Method 1: The cardholder has to pay in full his current balance by the due date so as to be eligible for a new grace period.

Method 2: The cardholder has to pay his current balance in full as well as to have fully paid the balances generated during the previous months by the due date; i.e. to benefit from a new grace period, the cardholder should not carry forward any balance from the previous and current months.

Each of the method features dynamic characteristics. The evolution of a credit card account depends on the cardholder's repayments and needs subsequently to be detailed. Consider first an account allotted a grace period which is the case for any new account or any paid up account. At the end of the billing period, one of the three following cases will occur:

Case 1: The cardholder pays in full, within the grace period, the outstanding balance reported on the bill. The cardholder is exempt from interest accruing on retail purchases and is allotted anew a grace period for the next billing cycle.

Case 2: The cardholder pays at least the minimum required payment by the due date which is a percentage of the outstanding balance (commonly ranging from 2% to 5%) or a minimum fixed amount (e.g. S\$50), whichever is greater. The cardholder is now revolving credit lines. His account is current and the balance roll-over in the next billing cycle is now charged with interest. The cardholder does not benefit from a grace period any more. Finance charges on new purchases will accrue the moment purchase transactions are made. The daily rate used to calculate the retail interest accruing on the purchases is derived from the annual percentage rate (APR) which commonly ranges between 15% to 25% a year. An average daily balance of the billing cycle is computed and is used as the basis to accrue interest. The cardholder may at any time pay in full his balance and therefore be allotted a new grace period.

Case 3: The cardholder either pays nothing or pays less than the minimum required payment. The cardholder is now delinquent (also called late payer). He is charged with delinquency (late payment) fee and is sometimes subject to higher APR. A delinquent cardholder may yet regularize his situation by paying at least the minimum required payment or may even be allotted a new grace period by paying his outstanding balance in full.

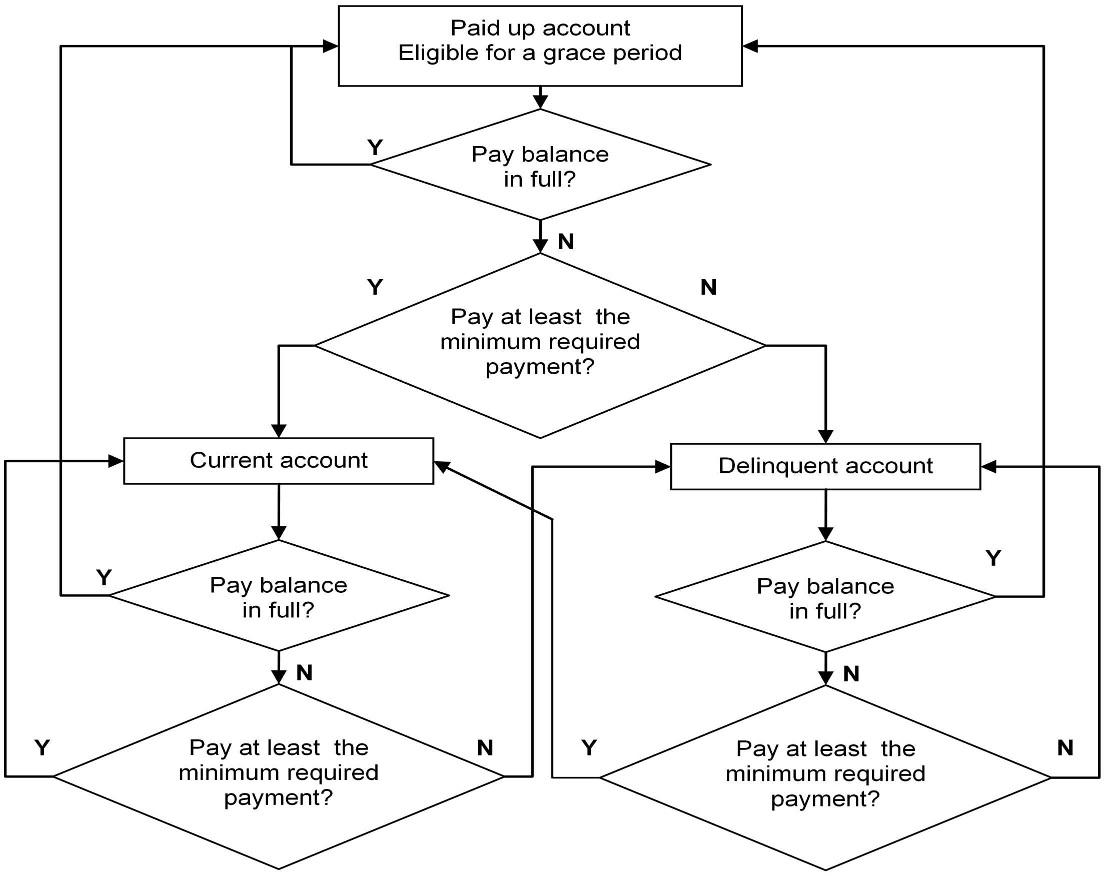

Figure 3.1: Delinquency cycle

The same process is then recursively applied month after month. Different fees and charges accrue according to the usage and repayments made by the cardholder. They will be detailed later in the value model of 3.4.3 on p58. The evolution of an account is an iterative process subject to the following constraints:

- The card is usually blocked to prevent further loss, if the payments are more than one month overdue.
- The delinquency should not exceed a threshold usually fixed to three months overdue after which the account becomes severely delinquent and faces more severe collection actions.

- If the cardholder becomes severely delinquent on a debt (usually at the point of six to seven months without payment), the creditor may declare the debt to be a charge-off.
- The outstanding balance is not to exceed either the credit limit or the credit limit augmented by a certain "fluff" (commonly 10% of the credit limit) depending on the card issuer's policy.

A charged-off account is considered to be "written-off as uncollectible". The charging-off practice, though usual, is questionable. One could argue that the creditor would rather keep a bad debt in the book, in hope of a later recovery. There are yet two major reasons motivating the charge-off. First, the severely delinquent accounts have demonstrated through experience high chances to turn out to be uncollectible. The second reason involves taxes; every year, each corporation files a *Profit And Loss Statement*. All of the year's bad debts (individual charged-off accounts) are added together as an item in the *Loss* section. The sum is then deducted from the corporation's tax return. Bad debts can then be considered as operating costs related to the risky activity of lending money.

3.2.2 Prior Segmentation of the Portfolio of Credit Card Accounts

A prior segmentation of the portfolio into homogenous risk groups should be made [see 10, 27]. In the present model, the partition should ideally be a trade-off between the accuracy of the classification and the homogeneity of default risk and card usage within each segment. Cardholders of the same segment should feature approximately same card usages, same credit limits and same credit risks. In the sequel, a probabilistic account flow model would be defined for each segment.

Till and Hand [41] investigated ways of segmenting[1] the behaviour profiles. They came out with the conclusion that, based on the application form variables, a reasonably accurate segmentation can be worked out. It is assumed in the present model that one such segmentation was beforehand performed. The model is then developed within each segment, ensuring the related characteristics to be constant for all the accounts of the same segment. The segmentation according to application variables has the advantage of defining steady segments which thus restricts the number of estimations necessary to build the model and hence eases its implementation.

One may argue that the segmentation should additionally include some selected behavioural variables. On the one hand, such additions are expected to yield a segmentation which better embodies the dynamic aspect of the process. On the other hand, any behavioural variable included in the segmentation would require watching the evolution of the accounts with respect to such a variable and thus account for the possibility of having some accounts moving from one segment to another. In the framework of Markov decision process, the implementation of such an approach greatly increases the number of possible transitions and is thus impractical for the low accuracy of the estimated transition probabilities. Makuch et al. [27] used a segmentation based on the score performance and on the balance range of the accounts. To remedy the afore mentioned problems, they had to define sufficiently large segments that can be considered as steady for up to 3 months. The distributions of accounts among the segments needs then to be updated.

Different data mining techniques are available to conduct such segmentations. Cluster analysis, neural networks [see 22, for application to credit card customers], logistic regression, classification trees, to name a few, can be used.

[1] Till and Hand adopted here the denomination used in Hand, Mannila, and Smyth [19]. A differentiation was instituted between the partitioning into "natural" components called *cluster analysis* and the partitioning into categories according to some predefined goals which is referred to as *segmentation analysis.*

3.2.3 State Space and State Information

The state space S of an account is defined as a two dimensional space that comprises the due status of the account together with its percentage of unused credit in relation to the credit limit. The due status of the account corresponds to the time elapsed since the last payment, greater or equal to the minimum payment, was made. For convenience, a cardholder may prefer to make several repayments in a month. He will not be a late payer so long as his payments are received before the due date. These two characteristics, due status and percentage of unused credit, provide a description of the account in terms of due status as well as in terms of level of debt. They will be used together with the repayments to forecast the profitability and the risk of the cardholder to the credit card issuer. The balance payments made by the cardholder are assumed to be conditioned by the state of the account at the beginning of one month and by the collection strategy undertaken by the issuer. The relevancy of the payment to the due status was established in the literature [see 24, 41]. The positive correlation[1] of the credit card debt with the delinquency confirms the relevancy of the delinquency to the credit card debt. Finally, the proven efficiencies of the different collection strategies in use today justify a posteriori the repayments to be relevant to the collection actions the cardholder is subject to. The set of monthly states is appropriate to describe the evolution of the accounts.

With regards to the state information, it is reasonably assumed that, whenever a decision is made, perfect information concerning the state of the account is available. Indeed, the delinquency state of the account and its unused credit limit are necessarily known information to the card issuer.

[1]The correlation coefficient between the credit card debt and the delinquency was found to be 0.4 in [39] based on the data from [3]. It is the biggest value compared with the correlations between other measures or ratios of debt (credit and total debt) and the delinquency.

3.2.4 Decisions

The collection practices are subject to legal restrictions, especially concerning the confidentiality of the communication between the collector and the debtor, who are respectively the card issuer and the cardholder in the present study. The Fair Debt Collection Practices Act [1], for instance, regulates the use of collection practices in the U.S.A. Let U be the set of front-end collection strategies currently used to recover debt from delinquent accounts. U is restricted to the practices available to the bank collection department in charge of managing delinquency. U does not account for the actions taken by external collection agencies to recover bad debt from written-off accounts. It is assumed that whenever the account is delinquent or over-limit, all of the collection strategies are available to the issuer. On the other hand, when the account is either paid up or current with a balance that is not over-limit, no collection strategy other than simple bill issuance is allowed. In other words, no pre-emptive strategy can be taken. In accordance with the legislation effective in the country under consideration, U may comprise:

1. simple bill issuance
2. mail reminder, sms, e-mail, fax
3. interactive and automated taped phone message with low level of severity; e.g. courteous reminder
4. interactive and automated taped phone message with mild level of severity
5. interactive and automated taped phone message with moderate level of severity
6. interactive and automated taped phone message with high level of severity; e.g. aggressive reminder urging the delinquent cardholder to pay

7. collection agent contacting the delinquent cardholder over the phone at a low frequency

8. collection agent intensively contacting the delinquent cardholder and urging him to pay or to at least make a clear promise of payment

9. charging off the delinquent account and reporting to the credit bureau

The collection strategies may as well comprise any combination of the basic strategies cited above, so as to reinforce the promises of repayments.

Additionally, the creditor reports to the credit bureau any delinquent cardholder. His credit score is then decreased in accordance to his state of delinquency. Any delinquent account falling behind thirty days overdue is blocked from further usage, until sufficient repayments (if any) are made. The delinquent cardholders, who fall even further behind in their payments, may either exceed the acceptable threshold of delinquency and thus be charged-off or file in for bankruptcy. The corresponding accounts are subsequently considered as losses by the bank and their debts are eventually either:

- worked on by a back-end collection department specialized in recovering bad debt

- sold to a collection agency for a price that depends on the amount of unrecovered debt

- simply discarded

Default accounts are listed as such on the debtor's credit bureau reports. The dates of defaults and the amount of the bad debts are also tracked.

3.2.5 Temporal Aspect

Four statements arise from the temporal analysis of the problem:

1. The collection strategy decision is to be taken monthly. At the end of each billing cycle, any account evolves to its next state and the proper collection strategy should then be initiated.
2. Most accounts have a finite lifetime usually ranging from one to five years.
3. Financial forecasts and results are expected after each semester. Annual forecasts are commonly used as a meaningful projection tool.
4. Discounted forecasts with infinite time horizon, though less accurate, will provide the lifetime value of an account and the corresponding stationary collection strategy.

Consequently, the decision epochs corresponding to the timing of the decisions for the collection strategies are formed by the beginning of each monthly billing cycle. The short term study will be illustrated by considering annual projection. The infinite horizon analysis, which provides the expected lifetime value of an account and the related stationary collection strategy, are needed in view of automatic collection implementation.

3.2.6 Costs and Expected Reward

The collection strategies and the corresponding policies will be assessed in terms of risk and profitability. To that end, a reward function will be defined to assess the profitability of the account at each stage. The expected cumulative cash flows per account per month are instituted as a comparative criterion defining a total order relation amid the different policies. It has been assumed in the present model that the costs are deterministic and constant over the whole time period. A monthly discount factor has been included in the model.

3.3 Definitions

3.3.1 Time Horizon and Decision Epochs

3.3.1.1 Definitions of the Finite Time Horizon and Decision Epochs

Definition 3.1. *Let N be the number of monthly billing cycles in consideration.*

Decisions are made monthly at points n in time, referred to as decision epochs.

Definition 3.2. *Let T be the set of decision epochs. T is the finite discrete set:*

$$T = \{0, 1, \ldots, N\}$$

3.3.1.2 Decomposition of Stages

Each monthly billing cycle can be divided into two different sub stages. Collection strategies are assumed to be adopted at the beginning of each month. Given the state of the account and the collection practice opted for, the cardholder has a certain transaction activity and e.g. a certain volume of purchases and cash advances. The time window that consists either of the grace period or of the period extending from statement issuance date until due date, is defined in the present model as *sub stage 1* of decision epoch n. The balance payments are to be made within *sub stage 1* in order for the cardholder not to become a late payer or delinquent. The period that extends from the end of *sub stage 1* to the issuance of the monthly credit card statement corresponds to *sub stage 2*. Cardholders making sufficient payment during *sub stage 2* will not fall further behind in their payments, but they may be liable to late fees. At the end of *sub stage 2*, the process evolves to a next state and a new decision is made at decision epoch *n+1*. Discrete time modeling imposes the assumption that potential balance payments are made at the end of the grace period (usually around 22 days after bill issuance) in case the latter applies, on the due date otherwise.

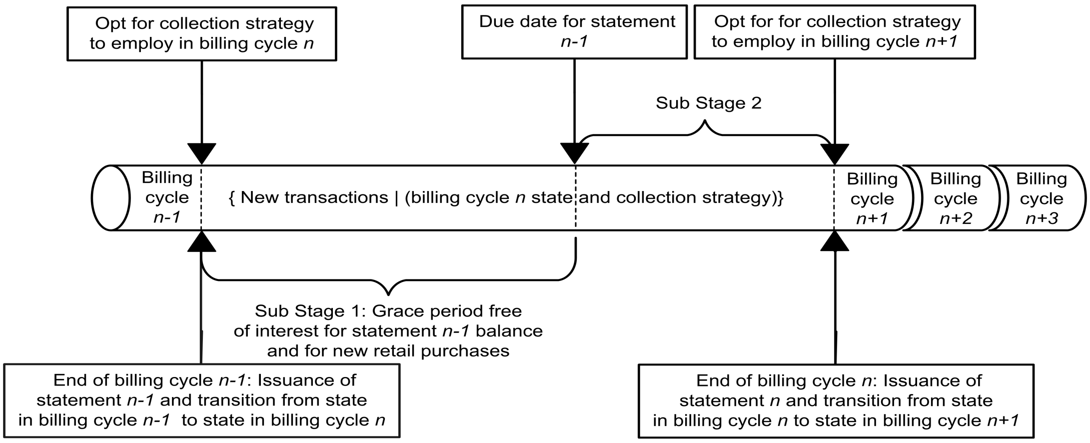

Figure 3.2: Timeline of an account eligible for a grace period

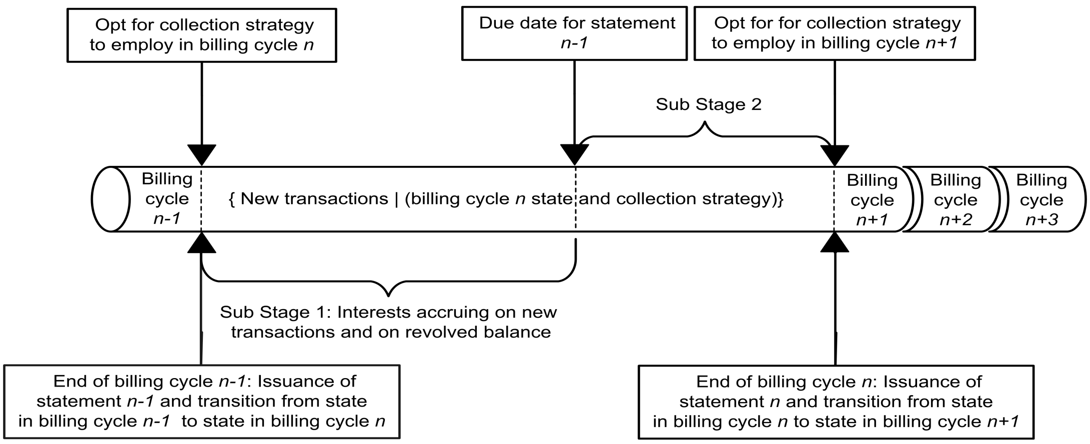

Figure 3.3: Timeline of an account non-eligible for a grace period

One should notice that the evolution of the account from its status in billing cycle n to its status in billing cycle $n+1$ is actually conditioned by the repayment of the balance generated up to the start of billing cycle n.

3.3.2 State Variables

As introduced in the preliminary notions, the two dimensional state space consists of the due status of the account and of the unused credit limit at the start of the billing cycle.

3.3.2.1 Due State

Definition 3.3. *Let i be the due status of the account at the beginning of the billing cycle.*

$$i \in \{0, 1, 2, \ldots, L-1, L\} \bigcup \{NA\}$$

So long as an account is not written-off, its due status i is an element of the constant discrete set $I = \{0, 1, 2, \ldots, L-1, L\}$, where L is the worst state of delinquency acceptable to the credit card issuer. The due status takes either value 0 when in paid up state or value $i \in \{1, 2, \ldots, L-1, L\}$ when the account is $i-1$ month(s) overdue. The account is current when $i = 1$, delinquent when $i = 2, \ldots, L$. If the due status of an account is such that $i = L$ and if the cardholder falls further behind, his account is then charged off. NA is used to denote the due status of an account has been written-off. In this case, the unused credit limit is not any more relevant and the pair of the due status and the unused credit limit is simply noted as (NA, NA).

3.3.2.2 Unused Credit Limit

With regards to the unused credit limit, it is necessary to use discretization since the unused credit limit is actually a real value, with infinite optional values. The credit lines are partitioned into a discrete number of line segments. $J = \{0, 1, 2, \ldots, M - 1, M\}$ is the discrete set of indices of the line segments that partition the credit limit. Depending on the type of account and cardholder's agreement, the balance either:

- should not exceed the credit limit; any transaction that may have the balance the credit limit is then rejected.
- may exceed the credit limit up to a certain threshold (commonly 10% of the credit limit), after which further transactions are rejected. Any cardholder, who exceeds the credit limit is usually liable to extra fees.

In the present model, the cardholder may exceed his credit limit and then be liable to over-limit fee. The amount due over the credit limit is limited and the overall balance is not to exceed a threshold defined as UB^M.

Definition 3.4. *Let Ξ be the discrete partition of the admissible balance range. Let LB^m, UB^m be the lower and upper bounds of each interval partitioning the whole admissible balance range*

$$\Xi \equiv \bigcup_{m=0}^{M-1} [LB^m, UB^m) \bigcup [LB^M, UB^M]$$

From the definition, the following equation holds:

$$UB^{m-1} = LB^m, \quad m = 1, \ldots, M$$

It can also be seen that two distinct cases occur, the case $j \leq M - 1$ where the balance is less than the credit limit and the case $j = M$ where the account is over-limit.

For each segment of the partition comprising the credit limit i.e $m = 0, \ldots, M-1$, a discrete value B^m is chosen as a discrete approximation of the balances falling within this segment. For the over-limit segment, the approximation of the balance is chosen to be the upper limit UB^M corresponding to the worst state of indebtedness.

$$\begin{cases} \forall B \in [LB^m, UB^m), \Rightarrow B \approx B^m, \quad m = 0, \ldots, M-1 \\ \forall B \in \left[LB^M, UB^M\right], \Rightarrow B \approx UB^M \end{cases}$$

Definition 3.5. *Denote by B the outstanding balance of the account at the beginning of the billing cycle. The index j of the discrete unused credit limit is defined by:*

$$j = \begin{cases} \{m \in \{0, \ldots, M-1\} | LB^m \leq B < UB^m\}, & \text{if } B < LB^M \\ M, & \text{otherwise} \end{cases}$$

Definition 3.6. *The corresponding unused credit limit UCL^j is defined by:*

$$UCL^j = CL - B^j$$

3.3.2.3 State Space

Definition 3.7. *Let S be the set of all possible states of the account at the beginning of each billing period.*

$$S = (I \times J) \bigcup \{(NA, NA)\}, I \equiv \{0, 1, \ldots, L\}, J \equiv \{0, 1, \ldots, M\}$$

S is assumed not to vary over the entire finite time horizon. S is a two dimensional discrete state space comprising the union of the singleton $\{(NA, NA)\}$ and the Cartesian product of the two discrete subsets I, J.

Definition 3.8. *Let S_{trans} be the set of all transient states*

$$S_{trans} = S - \left\{(NA, NA)\right\} = \{0, 1, \ldots, L\} \times \{0, 1, \ldots, M\}$$

3.3.3 Decision Variables and Decision Constraint

Definition 3.9. *Let the control space U be the set of controls ordered by increasing level of severity. The control space consists of the following finite discrete set of collection strategies.*

$$U = \{0, 1, \ldots, K\}$$

U does not vary over the entire finite time horizon. u_n is the generic notation which denotes the collection strategy adopted during billing cycle n. The ordering may appear somewhat subjective. However, considering the controls described in section 3.2.4, a clear classification emerges by increasing level of severity.

3.3.3.1 Constrained Control

The constraint C_{dec} imposes that pre-emptive collection actions are not to be undertaken against accounts that are not delinquent or over-limit. As a result, the set of available actions varies according to the state the system is in[1].

Definition 3.10. *For each state $x \in S$, let $U(x)$ be the set of available actions.*

$$U(x) \equiv \begin{cases} \{0\}, & \textit{if} \quad x = (i,j), \quad i = 0,1 \quad j = 0, \ldots, M-1 \\ U, & \textit{otherwise} \end{cases}$$

Definition 3.11. *Let κ be the set of state-actions pairs as follows:*

$$\kappa = \big\{(x,u) : x \in S, u \in U(x)\big\}$$

In the sequel, given any state $x \in S$ the control in consideration u will be an admissible control i.e. $u \in U(x)$.

[1]When the account is not delinquent the control, as defined, is restricted to the simple issuance of the bank statement. A further step to embody marketing strategies in such states is to include controls consisting of both the issuance of the bank statement and the use of incentives to promote card usage e.g. promotional offers, discounts, loyalty reward programs etc.

3.3.4 Stochastic Process

3.3.4.1 Probability Space

Definition 3.12. *Let X be the collection of the random states at the beginning of all billing periods in consideration:*

$$X = \{X_n, n = 0, \dots, N-1\}$$

For each n of the finite discrete set T, X_n is the discrete random variable associated to the state of the credit card account at the beginning of billing cycle n. The sample state S is assumed not to vary over the entire finite time horizon. If $X_n = (i_n, j_n)$, the process is said to be in state (i_n, j_n) in billing cycle n. x_n denotes the realization of X_n.

3.3.4.2 Causes of Evolution

In the present model, two causes of evolution of the account are to be distinguished:

1. $\Omega_n(x_n, u_n)$ stochastic repayment given the current state x_n and the admissible collection strategy applied u_n. Ω_n is a random "disturbance" of the system that takes values in the set D. Its realization is denoted ω_n. D is a priori defined as the real interval ranging from zero to the balance due.

2. $r_n(x_n, u_n, x_{n+1})$ estimated aggregate activity of the account during billing cycle n whose value depends on the present and next states x_n, x_{n+1} and on the admissible present collection strategy u_n applied to the account. $r_n(x_n, u_n, x_{n+1})$ comprises all the different expenses made by the cardholder together with the finance charges and operational costs accruing to the account and is subject to credit limit constraint.

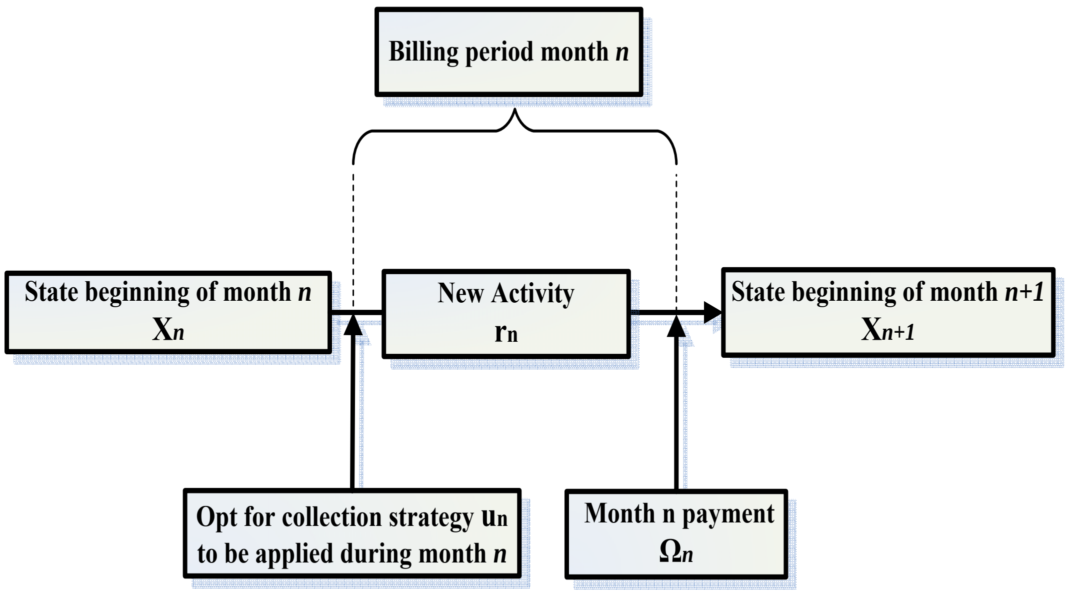

Figure 3.4: State transition

The equations that describe the evolution of the state of the credit card account, make natural use of $\Omega_n(x_n, u_n)$ and of $r_n(x_n, u_n, x_{n+1})$. They will be further specified in Section 3.5 p62.

Moreover, it is interesting to give here a first insight of a possible extension of the present model so as to embody the attrition phenomenon and the bankruptcy filings. The attrition, as stated before, corresponds to the "loss" of a cardholder by an issuer. This "loss" can be of three kinds; a substantial decrease in the credit usage, a definitive interruption of usage and finally a cancelation of the account. The reasons motivating such a loss are multiple. For instance, a cardholder can either be willing to replace his current account by a new one featuring an introductory offer, or simply be dissatisfied with the provided service. One may assume that the substantial decrease of usage and the definitive interruption produce the same effect, namely they have the account become completely inactive. The attrition can now be embodied in the present model. To that end, the extended model will keep the structure of the state space unchanged albeit some transitions will be added to model the process of repayment that accompanies the attrition. A random disturbance will naturally be added to the previous two elements that govern the evolution of the account. This random disturbance will represent the conditional risk of a cardholder having an account in state x_n and subject to the admissible collection strategy u_n to either close his/her credit card account next month or to start the process of voluntary writing-off. The bankruptcy filing risk can also be modeled as a disturbance representing the risk that a cardholder whose account is in state x_n and subject to collection strategy u_n files in for bankruptcy and have his outstanding debt fully uncollected.

3.3.4.3 Transition Probabilities

Property 3.1. *Markov Property: It is conjectured in the model that the conditional distribution of any future state of the account X_{n+1} given the past states $X_0, X_1, \ldots, X_{n-1}$, the past admissible collection actions $u_0, u_1, \ldots, u_{n-1}$, the present state X_n and the present admissible collection action u_n is independent of the past states and actions and depends only on the present state X_n and on the present admissible collection action u_n.*

Given that at decision epoch $n, n = 0, 1, \ldots, N-1$ the process is in state $X_n = x_n$ e.g. $x_n = (i, j)$ and the present admissible collection strategy is $u_n = u$, whatever the previous history may be, it follows from the Markov property that there is a fixed probability that the account will be in state $X_{n+1} = x_{n+1}$ e.g. $x_{n+1} = (k, l)$ next.
$\forall x_{n+1} \in S,\ \forall u_n \in U(x_n),\ n = 0, 1 \ldots, N-2$

$$P\{X_{n+1} = x_{n+1} | X_n = x_n, u_n, \ldots, X_0 = x_0, u_0\} = P\{X_{n+1} = x_{n+1} | X_n = x_n, u_n\}$$

The following notation is used in the further development;

$$p_n\big(x_{n+1} | x_n, u_n\big) = P\{X_{n+1} = x_{n+1} | X_n = x_n, u_n\}, \quad x_n, x_{n+1} \in S, u_n \in U(x)$$

so that,

$$\sum_{x_{n+1} \in S} p_n\big(x_{n+1}|x_n, u_n\big) = 1, \quad n = 0, 1, \ldots, N-1, x_n \in S, u_n \in U(x)$$

The *MDP* is characterized by its one step transition matrices $\mathbf{P}_{u_n}$.

$$\mathbf{P}_{u_n} = \begin{pmatrix} p_n\big((0,0)|(0,0), u_n\big) & & p_n\big((NA,NA)|(0,0), u_n\big) \\ \ddots & & \\ & p_n\big((k,l)|(i,j), u_n\big) & \\ & & \ddots \\ p_n\big((0,0)|(NA,NA), u_n\big) & & p_n\big((NA,NA)|(NA,NA), u_n\big) \end{pmatrix} \tag{3.1}$$

3.3.5 Motivations for the Markov Property Assumption

The assumption of Markov property for the process [see Property 3.1] is a clear simplification of the complexity of the credit card business. In particular, the state transitions are expected to be correlated to the whole history of the account and not only to its present state. The present study does not claim the process to a perfect Markov process. However, there are good reasons which motivate the use of such an assumption. They are twofold:

- The state variables carry, by essence, part of the history of the account. The due status clearly reflects the number of missed or incomplete repayments since the last minimum required payment was made. The novel inclusion of the unused credit limit in the definition of the state variables contributes to a better account of the history of usage and repayments. Moreover, the soundness of the approximation is further confirmed by a certain monotony of the delinquency process that has been empirically observed. Such a monotony thus confines most of the predictive power in the current month. Section 4.2.1 p73 looks more in details into this question.

- The one step Markov chain approach, although it is a simplification, has been widely used in the literature [see 10, 11, 12, 18, 24, 25, 27, 18, 42]. Such models have proven to be good approximations that are commonly used in the industry. Till and Hand [41] have pointed out that most of the predictive power lies in the last billing cycle variables.

These reasons justify the assumption of the Markov property which is being made in the present study. One could suggest to increase the dimensionality of the state space so as to mimic more accurately the process and its history. For instance, the due status could include both the current due state and the due state during the previous billing cycle. In the first place, such a modification looks very appealing. However, it is not viable for practical reasons. One of the difficulty of the present problem is to arrive with accurate transition matrices. Increasing the dimensionality of the state space will not only square the number of estimations necessary to obtained the desired transition probabilities but also worsen the accuracy of the estimates by reducing the sample size available to estimate the probability of each such transition.

3.3.6 Discrete Time Dynamic System

One feature of the present model is that the repayments and activities can be considered as disturbances of the system. Processes influenced by stochastic perturbations, which are independent of the state and control, are usually described in terms of trajectories. The *control theory* approach [33] describes the evolution of the system in terms of sample paths and system equation rather than in terms of transition probabilities. To that end, such an approach defines a “law of motion”, which relates the next state of the system to the present state, present control and disturbance. The present problem differs from the orthodox *control theory* in that the perturbations are indeed dependent on the present state and control. The "law of motion" can be written as;

$$X_{n+1} \approx h_n\big(X_n, u_n, \Omega_n(X_n, u_n)\big), \quad n = 0, 1, \ldots, N-1$$

The equivalence with the transition probabilities is made as follows:

$$\begin{aligned} p_n(x_{n+1}|x_n, u_n) &= P\Big[\Omega(x_n, u_n) \in \big\{\omega \in D:\ h_n(x_n, u_n, \omega) = x_{n+1}\big\}\Big] \\ &= \sum_{\{\omega \in D:\ x_{n+1} = h(x_n, u_n, \omega)\}} P\big(\Omega(x_n, u_n) = \omega | x_n, u_n\big) \end{aligned}$$

A one step *control theory* approach will be formulated in Chapter 4. It actually models the cardholder's perspective and serves the purpose of understanding his obligations of repayments. This approximate model would be introduced and discussed in Chapter 4. It shall then be used to generate and simulate reasonable processes.

3.3.7 Policy

Definition 3.13. *A deterministic admissible decision rule μ is a mapping of the state space S into the control space U for which the constraint C_{dec} holds.*

$$\mu : S \longrightarrow U$$

$$\forall x \in S, \big(x, \mu(x)\big) \in \kappa$$

Definition 3.14. *A deterministic admissible policy π is a sequence of admissible decision rules as follows:*

$$\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$$

$$\mu_n : S \longrightarrow U, \mu(x_n) \in U(x_n), \quad n = 0, 1, \ldots, N-1, x_n \in S$$

π associates to each x_n, a unique collection strategy that is feasible. Π will denote the set of all deterministic admissible policies in the remainder of this thesis.

3.3.8 Objective Function

3.3.8.1 Cost and Income Components

Recall $r_n(x_n, u_n, x_{n+1})$ is the activity of the account during billing cycle $n, n \leq N-1$. Suppose the bank account at the beginning of decision stage n is in state $x_n \in S$ and subject to the admissible collection strategy $u_n = \mu_n(x_n) \in U(x_n)$. The cardholder will have a certain activity during billing cycle n that depends on the state x_n of the account at the beginning of the cycle and naturally on the collection strategy u_n that is adopted. The latter is measured in terms of percentage of the credit limit CL. In the present model, the cash flows of the credit card account that are considered comprise the two following components:

1. positive cash flows: interchange revenue on new retail purchases, finance charges; retail purchases interest, cash advances interest, delinquency fees, withdrawal fees, partial (or full) debt recovery as a result of legal proceedings

2. negative cash flows: operating costs, cost of the collection strategy adopted, cost of lending money to the cardholder, credit loss in the case of writing off.

Definition 3.15. *Let $I_n(x_n, u_n, x_{n+1})$, $\quad n = 0, 1, \ldots, N-1$ be the consolidated income collected during stage n, when;*

• *the account, starting billing cycle n in state $X_n = x_n$, is subject to collection strategy u_n*

• *the repayment made by the cardholder $\Omega_n = \omega_n$ is such that the account will be in state $X_{n+1} = x_{n+1}$ at the beginning of the next billing cycle.*

Let $I_N(x_N)$ be the final (salvage) aggregate income of the forecasting period when the final state of the account is $X_N = x_N$.

With regards to $I_N(x_N)$, it consists of the income collected during the final stage N. Further decisions and states are not considered in the analysis. Hence, $I_N(\cdot)$ is a function on the last state X_N solely.

Definition 3.16. *Let $C_n(x_n, u_n, x_{n+1}), \quad n = 0, 1, \ldots, N-1$ be the consolidated cost incurred during stage n, when;*

- *the account, starting billing cycle n in state $X_n = x_n$, is subject to collection strategy u_n*
- *the repayment made by the cardholder $\Omega_n = \omega_n$ is such that the account will be in state $X_{n+1} = x_{n+1}$ at the beginning of next billing cycle.*

Let $C_N(x_N)$ be the final (salvage) cost incurred, when the final state of the account is x_N.

3.3.8.2 Reward Functions

Let β be the discount factor per month, $0 < \beta \leq 1$. The special case $\beta = 1$ corresponds to the non-discounted case.

Definition 3.17. *Let g be the sequence of reward functions associated to the process and to the admissible policy $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$*

$$g_n = \begin{cases} I_n(x_n, \mu_n(x_n), x_{n+1}) - C_n(x_n, \mu_n(x_n), x_{n+1}), & n = 0, \ldots, N-1 \\ I_N(x_N) - C_N(x_N), & n = N \end{cases}$$

Each $g_n, n = 0, \ldots, N-1$ measures the profit (positive or negative) derived from a credit card account transiting from state x_n to x_{n+1} and subject to the collection strategy $\mu_n(x_n)$. $g_N = I_N(x_N) - C_N(x_N)$, the salvage (or terminal) reward, represents the estimated worthiness of an account terminating the forecasting period in state x_N.

Definition 3.18. *Let $J_{\pi,\, g,\, N}(X_0)$ be the expected total discounted reward associated to the set of reward functions $g_n, n = 0, \ldots, N$ when the process starts in state $X_0 = x_0$ and when the admissible policy $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$ is applied*

$$J_{\pi,\, g,\, N} : \Pi \times S \longrightarrow \mathbb{R}$$

$$J_{\pi,\, g,\, N}(X_0) = \underset{\substack{X_n \\ n=1,\ldots,N-1}}{E} \Big\{ \beta^N g_N\big(X_N\big) + \sum_{n=0}^{N-1} \beta^n g_n\big(X_n, \mu_n(X_n), X_{n+1}\big) \Big\}$$

The objective of the card issuer is naturally to maximize the expected reward derived from the stochastic evolution of an account during the whole forecasting period.

Definition 3.19. *Given a finite time horizon N, the policy π^* associated to the set of reward functions g_n is optimal if and only if,*

$$J_{\pi^*,\, g,\, N}(X_0) = \sup_{\pi \in \Pi} J_{\pi,\, g,\, N}(X_0)$$

or equivalently

$$\pi^* = \arg\max_{\pi \in \Pi} J_{\pi,\, g,\, N}(X_0)$$

3.4 Value Analysis of the Credit Card Account

The present section shall introduce a value model of the credit card account. For clarity of exposition, notations are first introduced. The subscript $_n$ is used to refer to the realization during the billing cycle n. x_n or (i_n, j_n) as well as x_{n+1} or (i_{n+1}, j_{n+1}) are equivalently used to describe the current and next state of the account respectively.

3.4.1 Notations

3.4.1.1 General Characteristics

CL	credit limit
APR	annual percentage rate
mrp	minimum required payment rate
min_{pay}	minimum monthly payment amount
LB^j	lower bound of the j^{th} segment of the partition of CL
UB^j	upper bound of the j^{th} segment of the partition of CL
B^j	discrete outstanding balance associated to accounts whose balances fall into the j^{th} segment of the partition of CL
UCL^j	discrete unused credit limit associated to accounts whose balances fall into the j^{th} segment of the partition of CL
$r_n(x_n, u_n, x_{n+1})$	overall receivable activity of the account
$B_n^{use}(x_n, u_n, x_{n+1})$	sum of revolved balance and of card usage

Table 3.1: General Characteristics of a credit card account

3.4.1.2 Retail Purchases

icr	interchange rate
$PV_n(x_n, u_n, x_{n+1})$	average retail purchases volume
$PVOL(x_n, u_n, x_{n+1})$	retail purchases volume subject to credit limit constraint
$VOL_n(x_n, u_n, x_{n+1})$	overall volume of retail purchases and cash advances

Table 3.2: Retail Purchase/Order activity

3.4.1.3 Cash Advances

$CV_n(x_n, u_n, x_{n+1})$	average volume of cash advances
$CVOL(x_n, u_n, x_{n+1})$	cash advances' volume subject to credit limit constraint
$nb_{ca}(x_n, u_n, x_{n+1})$	average number of cash advances
ca	financial charge rate on cash advances
C_{thr}	maximum amount of a transaction such that any advance less than C_{thr} is charged at the minimum rate (ca_1) only
ca_1	fee rate on the first $S\$\, C_{thr}$ of any cash advance
ca_2	fee rate on the remaining (if any) balance the advance
min_{cf}	minimum fee for cash advances

Table 3.3: Cash Activity

3.4.1.4 Delinquency and Over-limit

ovr	allowable percentage of credit over-limit
OL	over-limit fee
$LF(UCL^j)$	tiered late payment fee when unused credit limit is UCL^j

Table 3.4: Delinquency and Over-limit

3.4.1.5 Operating Costs

fcr	cost of funding per dollar per billing cycle
PC	writing off penalty cost
$OC_n(x_n, u_n, x_{n+1})$	aggregate operating cost under collection strategy u_n
$DR(UCL^j)$	debt recovery when selling default account, with final unused credit limit UCL^j, to collection institution

Table 3.5: Operating Costs

3.4.1.6 Cash Flows

	Incomes
$CF_n(x_n, u_n, x_{n+1})$	fee charged on new cash advances
$AF_n(x_n, u_n, x_{n+1})$	aggregate fees (cash advances, delinquency and over-limit)
$IF_n(x_n, u_n, x_{n+1})$	gross fee income
$CI_n(x_n, u_n, x_{n+1})$	interest accruing on cash advances
$PI_n(x_n, u_n, x_{n+1})$	interest accruing on retail purchases
$BI_n(x_n, u_n)$	interest accruing on the balance revolved
$IR_n(x_n, u_n, x_{n+1})$	gross interest income
$AC_n(x_n, u_n, x_{n+1})$	aggregate financial charges
$IC_n(x_n, u_n, x_{n+1})$	interchange revenue
$DR_n(x_n, u_n, x_{n+1})$	debt recovery
$I_n(x_n, u_n, x_{n+1})$	consolidated income
	Costs and Losses
$FC_n(x_n, u_n, x_{n+1})$	cost of lending money
$OC_n(x_n, u_n, x_{n+1})$	aggregate operating cost
$DL_n(x_n, u_n, x_{n+1})$	debt loss
$C_n(x_n, u_n, x_{n+1})$	consolidated costs and losses

Table 3.6: Cash Flows

3.4.2 Cash Flows

The following flow chart in Figure 3.5 p58 illustrates the different cash flows generated by the card usage and the profitability (possibly loss) of the credit card account to the card issuer. These flows are conditioned by the account state, the cardholder repayments and the collection strategy adopted by the card issuer. The assumptions related to the computation of the different values are further detailed in 3.5 p62. One should notice that there are two types of incomes:

• an "internal" one derived directly from the cardholder paying interests, charges and fees

• an "external" one comprising interchange revenue derived from the card usage and comprising also partial bad debt recovery when selling the default account to an external collection agency. Interchange revenue is of essential importance since it is recognized as the card issuers' second largest income line item [see 2].

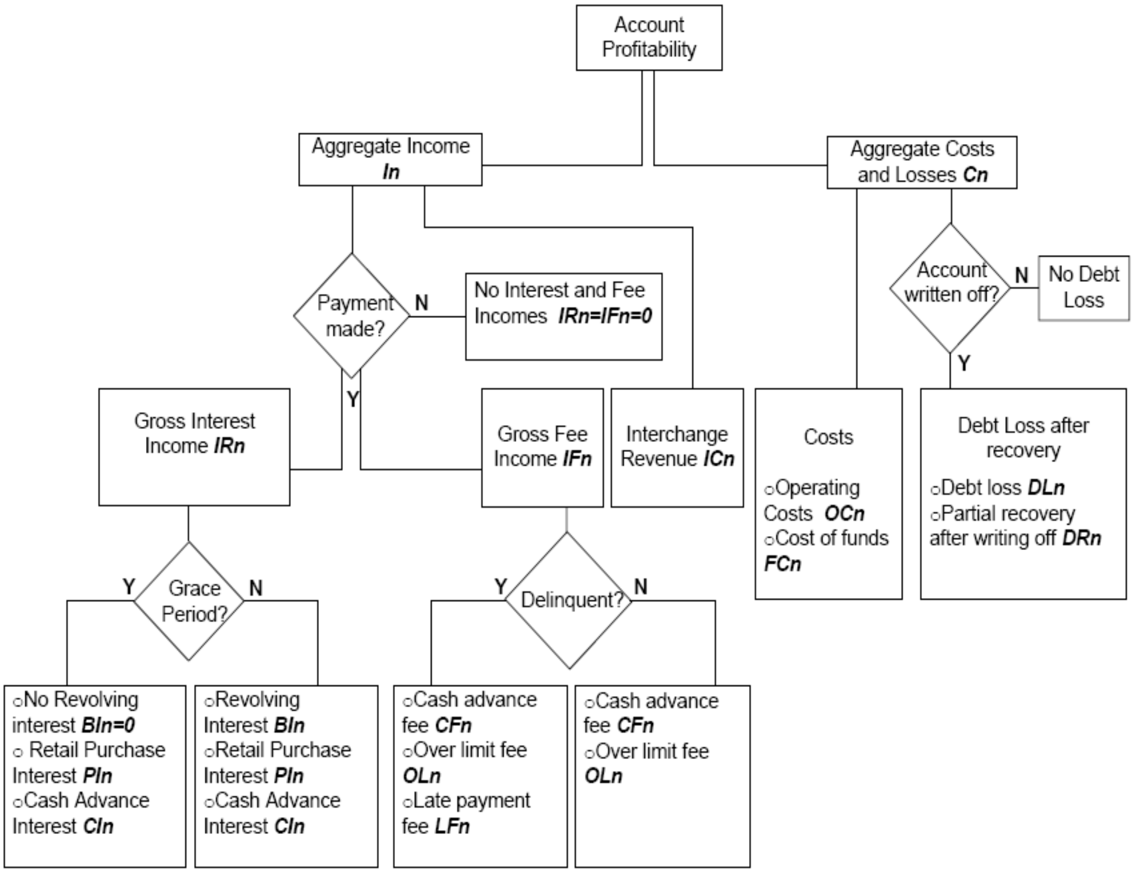

Figure 3.5: Credit Card Account Cash Flows

3.4.3 Delinquency Process

The minimum required payment is defined as a minimum amount min_{pay} (e.g. $S\$50$) or a fixed percentage mrp (e.g. 3%) of the balance newly reported on the statement, whichever is greater, plus any overdue amounts. For accounts that are over-limit, the minimum monthly repayment is mrp of the newly reported balance plus the excess over the credit limit, plus any overdue amounts. Overdue amounts are here defined as the sum of all the minimum monthly payments missed (or partially paid) until the last sufficient payment was made.

A cardholder paying at least the minimum payment every month remains current. Delinquency occurs whenever the cardholder during a billing cycle repays less than the minimum monthly payment.

Definition 3.20. *Given an account which is more than k months delinquent in cycle n, its k* month delinquent balance *is defined as the maximum between min_{pay} and the product of mrp and the balance reported on the statement issued at the end of billing cycle $n-k-1$. The k* month delinquent balance *is exclusive of the previous overdue amounts.*

Accordingly, card issuers classify their customers into delinquency buckets. The k^{th} delinquency bucket comprises all the accounts k months overdue. For instance, such an account will:

- move to the next delinquency bucket (i.e. be $k+1$ months overdue or charged-off if $k = L$) if the cardholder repayments during the cycle are strictly less than the k *month delinquent balance*.
- stay in the same delinquency bucket if the cardholder repayments are sufficient to settle the k *month delinquent balance* only.
- move to the previous delinquency bucket if the repayments made are sufficient to settle the k *month delinquent balance* and the $k-1$ *month delinquent balance* but not $k-2$ *month delinquent balance*.
- move to the $k-m$ delinquency bucket with $m \leq k-1$ if the repayments made are sufficient to settle the overdue balances starting from the k month delinquent balance up to the $k-m$ month delinquent balance.

- move to current or paid up if the repayments are sufficient to repay the minimum monthly payment or the whole balance, respectively.

Flow chart 3.6 depicts the conditional evolution of an account k months delinquent within the different delinquency buckets conditional to the repayment requirements.

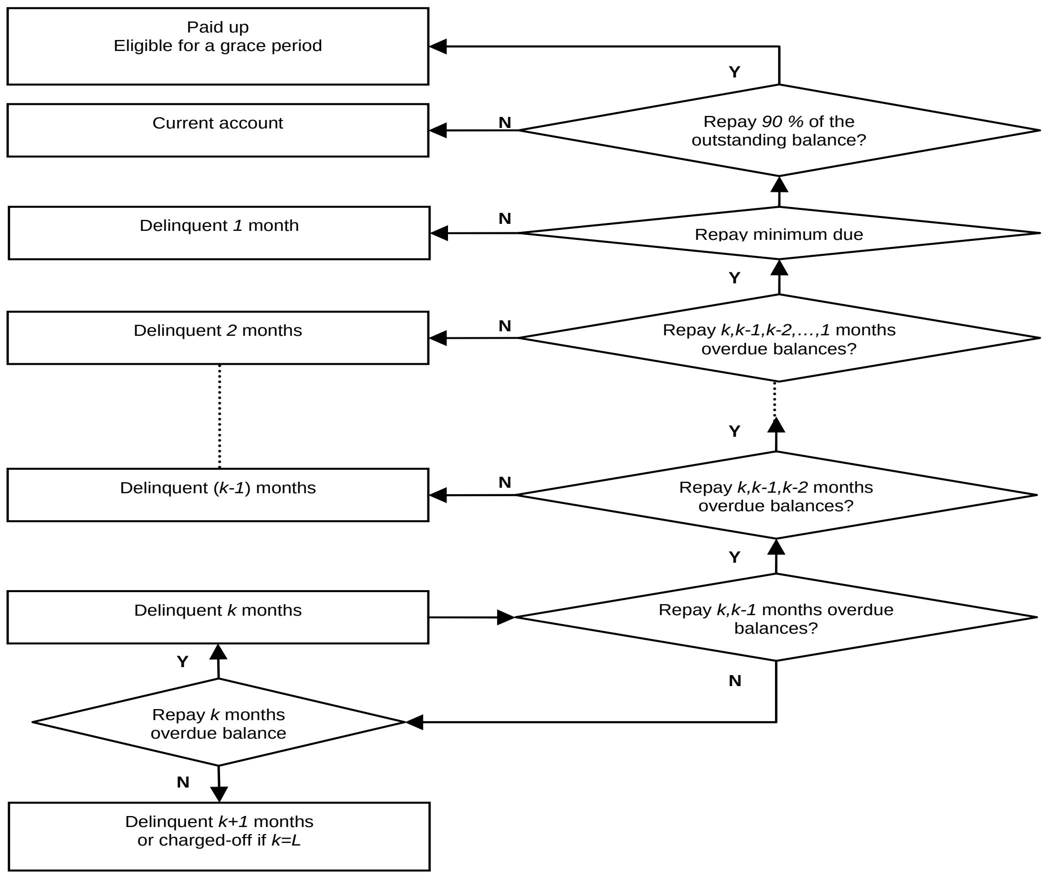

Figure 3.6: State of delinquency flow chart for an account k months delinquent

The exact forecast of the delinquency process requires keeping the history of the previous overdue amounts for each billing cycle included in the whole continuous delinquency period. In other words, the states of the account should be traced to when the account was last current ($i_n = 0, 1$). The trajectory of delinquency cannot be explicitly and exactly defined by a one step dynamic equation given the present state space. It would require otherwise redefining the state space so as to concatenate the previous delinquency states until the account was last current. The formidable enlargement of the state space would result firstly in an increase of the complexity and of the computational effort needed to solve the problem, and secondly and more importantly, in poor estimations of the transition probabilities. A one step Markov chain approach is preferred here since it has been widely demonstrated in the literature that such models are good approximations and that, by far, most of the predictive power lies in the last billing cycle variables [see 41].

3.4.4 Assumptions

An approximate value model would be presented in the present section. The value model is defined to follow the cardmembers' Agreement of a major credit card issuer in Singapore. It relies on estimated characteristics of usage and on the following assumptions:

- Annual fee, insurance fee and lost/stolen fee are omitted without loss of generality.
- "Method 2" (3.2.1 p31) is applied to allot grace period.
- Installment plans are not accounted for in the present value model
- Any account more than one month delinquent is blocked from further usage. By repaying the minimum monthly payment, any delinquent cardholder can unblock his lines and use his card again. PV_n, CV_n are functions of the inputs (x_n, u_n, x_{n+1}) so as to embody these two constraints.
- A payment hierarchy has been instituted. The payment ω_n is first used to refund any finance charges and fees, including the charges incurred during the cycle of payment. The remaining amount of ω_n (if any) is then used to settle the outstanding balance.
- The repayments ω_n are all supposed to be made on the due state. ω_n are assumed not to exceed the outstanding balance and hence, are bounded. For an account k

months delinquent, any repayment ω_n strictly lower than the k *month delinquent balance* is considered as a missed payment.

• The Markovian structure of the model requires some approximations concerning the evolution of the due state of the account when using the payment *control* description. The minimum repayment is approximated using solely the current activity (if any) and the balance outstanding at the start of the billing cycle.

• The value model does not include the possibility of having a cardholder transferring balance from the credit card account in consideration to another one. The assumption is relevant since the focus is put on delinquent cardholders that are less likely to use their account in due state of delinquency to pay up the balance on another account.

3.5 Equations

3.5.1 General Characteristics

$VOL_n(x_n, u_n, x_{n+1})$ Overall volume spent during billing cycle n subject to the two following constraints;

• credit limit constraint: the balance is not to exceed an arbitrarily fixed maximum amount UB^M equal to the credit limit increased by 10%. The volume spent during billing cycle n must therefore be less or equal to $UB^M - B^{j_n}$, which according to definition 3.4 is equal to $UCL^{j_n} - UCL^M$.

• Any over-limit account is charged with over-limit fee OL.

• any account more than 1 month delinquent (i.e. $i_n > 2$) is blocked from further usage until sufficient repayment is made (i.e. $i_{n+1} \leq 1$).

VOL_n is then defined by:

$$VOL_n(x_n, u_n, x_{n+1}) = \begin{cases} 0, & \text{if } (i_n > 2, i_{n+1} > 1) \\ \\ min\Big\{ \big(PV_n + CV_n\big)(x_n, u_n, x_{n+1}), \\ \qquad\qquad UCL^{j_n} - UCL^M \Big\}, & \text{otherwise} \end{cases} \tag{3.2}$$

$CVOL_n\,(x_n, u_n, x_{n+1})$, $PVOL_n\,(x_n, u_n, x_{n+1})$ Volumes of cash advances and retail purchases during cycle n subject to credit limit constraint, respectively. In case of over-limit account, $CVOL_n$ and $PVOL_n$ are assumed to be proportionally distributed within the admissible credit lines in accordance to their respective average volumes CV_n, PV_n when there is no over-limit.

$$CVOL_n(x_n, u_n, x_{n+1}) = \begin{cases} 0, & \text{if } (i_n > 2, i_{n+1} > 1) \\ \\ \frac{CV_n}{PV_n + CV_n} VOL_n(x_n, u_n, x_{n+1}), & \text{otherwise} \end{cases} \tag{3.3}$$

$$PVOL_n(x_n, u_n, x_{n+1}) = \begin{cases} 0, & \text{if } (i_n > 2, i_{n+1} > 1) \\ \\ \frac{PV_n}{PV_n + CV_n} VOL_n(x_n, u_n, x_{n+1}), & \text{otherwise} \end{cases} \tag{3.4}$$

$r_n(x_n, u_n, x_{n+1})$ Overall receivable activity during month n inclusive of the different financial charges accruing.

$$r_n\,(x_n, u_n, x_{n+1}) = \big(VOL_n + AC_n\big)\,(x_n, u_n, x_{n+1})\,, \tag{3.5}$$

where $AC_n\,(x_n, u_n, x_{n+1})$ is the aggregate financial charges. $AC_n\,(x_n, u_n, x_{n+1})$ is further defined in 3.5.2.

$B_n^{use}\,(x_n, u_n, x_{n+1})$ balance usage. It is the sum of the balance revolved in billing cycle n and of the card usage made during this cycle. $B_n^{use}(x_n, u_n, x_{n+1})$ features the credit funding effort made by the card issuer to satisfy the cardholder's demand.

$$B_n^{use}\,(x_n, u_n, x_{n+1}) = B^{j_n} + r_n\,(x_n, u_n, x_{n+1}) \tag{3.6}$$

3.5.2 Incomes

$PI_n\left(x_n, u_n, x_{n+1}\right)$ Interest accruing on new retail purchases. If the cardholder does enjoy a grace period, his retail purchases are subject to a finance charge calculated at the rate of APR (e.g. $APR = 24\%$) per annum or equivalently at the rate of $\frac{APR}{365}$ (e.g. 0.067% resp.) per day. The activity is assumed to be uniformly distributed within billing cycle n. That is, all cash advances are approximately assumed to be of the same amount. Using the discrete Heaviside function H as defined in the Nomenclature pvi, PI_n can be obtained as follows:

$$PI_n\left(x_n, u_n, x_{n+1}\right) = \frac{1}{2} \times \frac{APR}{12} \times H(i_n - 1) \times PVOL_n\left(x_n, u_n, x_{n+1}\right) \tag{3.7}$$

$CI_n\left(x_n, u_n, x_{n+1}\right)$ Interest accruing on new cash advances. Cash advances are always subject to a finance charge calculated at the rate of ca. The cash advance activity is also assumed to be uniformly distributed within billing cycle n.

$$CI_n(x_n, u_n, x_{n+1}) = \frac{1}{2} \times \frac{ca}{12} \times CVOL_n\left(x_n, u_n, x_{n+1}\right) \tag{3.8}$$

$CF_n(x_n, u_n, x_{n+1})$ Fee charged on new cash advances. The fee per transaction is charged as follows: at the rate of ca_1 for the first $S\$\,C_{thr}$ of the cash advance and at the rate of ca_2 on the remaining (if any) amount of the cash advance. The cash advance fee per transaction is then the maximum between the previous fee and a minimum fee of $S\$min_{cf}$. In order to simplify the calculation of the fee CF_n, the amount of each cash advance is set equal to the average cash advance per transaction. Each transaction is then assumed to be charged with the same fee and $CF_n(x_n, u_n, x_{n+1})$ is consolidated as the sum of all the fees charged for cash transactions done during billing cycle n.

$$\begin{aligned} CF_n(x_n, u_n, x_{n+1}) = \Bigg(& max\Big\{ca_1 \times min\big\{\frac{CVOL_n\left(x_n, u_n, x_{n+1}\right)}{nb_{ca}\left(x_n, u_n, x_{n+1}\right)}, C_{thr}\big\}, min_{cf}\Big\} \\ & + ca_2 \times max\big\{\frac{CVOL_n\left(x_n, u_n, x_{n+1}\right)}{nb_{ca}\left(x_n, u_n, x_{n+1}\right)} - C_{thr}, 0\big\}\Bigg) \times nb_{ca}\left(x_n, u_n, x_{n+1}\right) \end{aligned} \tag{3.9}$$

$BI_n(x_n, u_n)$ Interest accruing on the balance revolved in cycle n. It is assumed that the revolved balance is charged with the same finance charge rate APR without distinction between the cash advance and retail purchase indebtedness.

$$BI_n(x_n, u_n) = \frac{APR}{12} \times H(i_n - 1) \times B^{j_n} \tag{3.10}$$

$AF_n\left(x_n, u_n, x_{n+1}\right)$ Aggregate fees comprising the cash advances fee $CF_n(x_n, u_n, x_{n+1})$ the late payment fee $LF(UCL^j)$ and the over-limit fee OL. Tiered late payment fees are charged whenever the minimum payment is not received by the due date. Over-limit fee OL is charged if during billing cycle n there occurs a transaction resulting in the balance being over the limit; $B^{j_n} + VOL_n > CL$.

$$\begin{aligned} AF_n\left(x_n, u_n, x_{n+1}\right) = & H\left(VOL_n\left(x_n, u_n, x_{n+1}\right) - UCL^{j_n}\right) \times OL \\ & + CF_n\left(x_n, u_n, x_{n+1}\right) + H\left(i_n - 2\right) \times LF(UCL^{j_n}) \end{aligned} \tag{3.11}$$

$AC_n\left(x_n, u_n, x_{n+1}\right)$ Aggregate financial charges comprising cash advances interest, retail purchases interest, interest on revolving balance and aggregate fees.

$$AC_n\left(x_n, u_n, x_{n+1}\right) = \left(CI_n + PI_n + AF_n\right)\left(x_n, u_n, x_{n+1}\right) + BI_n\left(x_n, u_n\right) \tag{3.12}$$

$IC_n\left(x_n, u_n, x_{n+1}\right)$ Interchange revenue during billing cycle n is derived from a fixed percentage accruing on any card transaction. The percentage is paid by the merchant acquirer accepting the merchant's sales draft to the card issuer so as to cover handling costs and credit risk in a credit card transaction. It is assumed that retail purchases only generate interchange revenue and that the charges flowing in the opposite direction (e.g. point of sales transactions POS / electronic funds transfers EFT) are paid by the cardholder for being directly impacted on his balance. Hence, there are not considered thereafter.

$$IC_n\left(x_n, u_n, x_{n+1}\right) = icr \times PVOL_n\left(x_n, u_n, x_{n+1}\right) \tag{3.13}$$

$IF_n(x_n, u_n, x_{n+1})$ Gross fee income accounts for fees accruing during billing cycle n increased by a cumulative late payment fee and a cumulative over-limit fee. In accordance to the repayment hierarchy 3.4.4 p61, fee income is fully collected whenever the cardholder makes a repayment allowing him not to increase his delinquency.

$$IF_n(x_n, u_n, x_{n+1}) = \Big(\delta(i_n)H(1-i_{n+1})+\big(1-\delta(i_n)\big)H(i_n-i_{n+1})\Big)\times AF_n(x_n, u_n, x_{n+1}) + max\big\{i_n - i_{n+1}, 0\big\} \times H(i_n - 3) \times \Big(LF(UCL^{j_n}) + \delta(j_n - M)OL\Big) \quad (3.14)$$

$IR_n(x_n, u_n, x_{n+1})$ Gross interest income collected during month n comprising the cash advances interest CI_n, the retail purchases interest PI_n and the revolving balance interest BI_n. The interest income for billing cycle n is fully collected whenever the repayment allows the cardholder not to increase his delinquency. The previous unpaid interest are approximately calculated using the revolving balance. The approximation is reasonable since any account more than 30 days overdue is blocked from further usage.

$$IR_n(x_n, u_n, x_{n+1}) = \Big(\delta(i_n)H(1-i_{n+1})+\big(1-\delta(i_n)\big)H(i_n-i_{n+1})\Big)\times\Big(CI_n(x_n, u_n, x_{n+1}) + PI_n(x_n, u_n, x_{n+1})\Big) + max\big\{i_n - i_{n+1} + 1, 0\big\} \times BI_n(x_n, u_n) \quad (3.15)$$

$DR_n(x_n, u_n, x_{n+1})$ Debt Recovery after writing off in cycle $n+1$. The debt recovery of a default account is assumed to depend on the amount of bad debt. The card issuer can either work the bad debt internally with a back end collection department, sell the default account to external collection agencies for tiered amount according to the total debt outstanding, or simply write off the account with no further collection effort. In the present model, the default account is assumed to be sold to an external collection agency.

$$DR_n(x_n, u_n, x_{n+1}) = \delta(i_{n+1} - NA)\big(1 - \delta(i_n - NA)\big) \times DR\big(UCL^{j_{n+1}}\big) \quad (3.16)$$

$I_n(x_n, u_n, x_{n+1})$ consolidated income collected during billing cycle n comprising the sum of all the conditional positive cash flows from the cardholder IF_n, IR_n, DR_n to the card issuer and from the merchants IC_n to the card issuer.

$$I_n(x_n, u_n, x_{n+1}) = \big(IR_n + IC_n + IF_n + DR_n\big)(x_n, u_n, x_{n+1}) \tag{3.17}$$

3.5.3 Costs and Losses

$FC_n(x_n, u_n, x_{n+1})$ Cost of lending money to the cardholder is assumed to be proportional to B_n^{use} with a rate fcr.

$$FC_n(x_n, u_n, x_{n+1}) = fcr \times B_n^{use}(x_n, u_n, x_{n+1}) \tag{3.18}$$

$DL_n(x_n, u_n, x_{n+1})$ Debt loss occurs when the account first reached the absorbing state (NA, NA) and is equal to the final indebtedness increased by a penalty cost for writing off.

$$DL_n(x_n, u_n, x_{n+1}) = \delta(i_{n+1} - NA)\big(1 - \delta(NA - i_n)\big)\times \Big(B_n^{use}\big(x_n, u_n, x_{n+1}\big) + PC_n\Big) \tag{3.19}$$

$C_n(x_n, u_n, x_{n+1})$ Consolidate operating cost comprises FC_n, DL_n, and the aggregate operating cost OC_n incurred by the collection strategy adopted during billing cycle n.

$$C_n(x_n, u_n, x_{n+1}) = \big(OC_n + DL_n + FC_n\big)(x_n, u_n, x_{n+1}) \tag{3.20}$$

3.5.4 Expected Rewards and Optimal Policies

Any value function, say $d(x_n, u_n, x_{n+1})$ introduced in the previous section defines a random value received or incurred in period n as $D_n \equiv d(X_n, u_n, X_{n+1})$.

Definition 3.21. *The random reward function associated to the reward function g_n previously defined in Definition 3.17 can then be consolidated as follows:*

$$G_n = \begin{cases} \big(IR_n + IC_n + IF_n + DR_n \\ \quad -OC_n - DL_n - FC_n\big)\,(X_n, u_n, X_{n+1})\,, & n = 0, \ldots, N-1 \\ I_N(X_N) - C_N(X_N), & n = N \end{cases} \tag{3.21}$$

The expected total discounted reward associated to the set of reward functions $g_n, n = 0, \ldots, N$ when the process starts in state x_0 and when policy π is applied, can be obtained as follows:

$$\begin{aligned} J_{\pi,\,g,\,N}(x_0) &= \underset{\substack{G_n \\ n=0,\ldots,N}}{E} \Big\{ \sum_{n=0,\ldots,N} \beta^n G_n \Big\} \\ &= \underset{\substack{X_n \\ n=0,\ldots,N-1}}{E} \Big\{ \beta^N \big(I_N(X_N) - C_N(X_N)\big) \\ &+ \sum_{n=0}^{N-1} \beta^n \big(IR_n + IC_n + IF_n + DR_n\big)(X_n, \mu_n(X_n), X_{n+1}) \\ &- \sum_{n=0}^{N-1} \beta^n \big(OC_n + DL_n + FC_n\big)(X_n, \mu_n(X_n), X_{n+1}) \Big\} \end{aligned} \tag{3.22}$$

3.5.4.1 Optimal Policies for the Finite Horizon Problem

Puterman [33] showed that the set of deterministic policies are optimal under the expected total reward criteria. There will be therefore no necessity for randomized policies in the calculation of an/the optimal policy, all the more as randomized collection strategies are meaningless to the card issuer. Chapter 4 will present two resolutions: a finite horizon problem solved with the *finite-horizon policy evaluation algorithm* also known as the *backward induction algorithm* and an infinite horizon problem which will be solved by the *policy iteration algorithm*.

One should note that the expected reward can be consolidated as a dynamic equation. The *backward induction algorithm* intrinsically presents the proof of existence of an optimal policy which is independent from the starting state X_0.

3.5.4.2 Optimal Policies for the Infinite Horizon Problem when $\beta = 1$

As for the infinite horizon *MDP*, the present model features one absorbing state (NA, NA) which is assumed to communicate with all the transient state. This absorbing state is actually a cost-free termination state. Once the system reaches that state, it remains there at no further cost, that is,
$\forall x_{n+1} \in S, u_n \in U\big((NA, NA)\big)$

$$p_n\big(x_{n+1}|(NA, NA), u_n\big) = 1, \quad g_n\big((NA, NA), u_n, x_{n+1}\big) = 0$$

Under the latter assumption, reaching the termination state (NA, NA) is inevitable. The problem of interest is then clearly equivalent to the stochastic shortest path problem.

Definition 3.22. *A stationary policy π is said to be proper if, under this policy, there is strictly positive probability that the absorbing state (NA, NA) will be reached after at most n_π stages, regardless of the initial state x_0.*

$$\exists n_\pi \in \mathbb{N}^*, \quad \max_{x_0 \in S} P\big(X_{n_\pi} \neq (NA, NA)|x_0, \pi\big) < 1$$

The assumptions guarantees that all the deterministic policies are proper policies. Bertsekas [7] demonstrated that given the existence of proper policies on a discrete set, there exists a deterministic policy optimizing the expected reward over an infinite horizon.

3.5.4.3 Optimal Policies for the Infinite Horizon Problem when $0 < \beta < 1$

More generally, it is common practice in finance to discount cash flows (with $0 < \beta < 1$) so to calculate the net present value reflecting the time value of money. In this case, it is not necessary to assume that all the transient states communicate with the absorbing state (NA, NA). Using stationary discounted cash flows, the present model falls in the category of finite state space, control space *MDP* for which it is well know that there exist optimal deterministic policies that are stationary.

3.6 Summary

The features of the credit card control problem were identified under clearly defined assumptions. The model was developed based on the following characterizations:

- due status of the account at the beginning of each monthly billing cycle
- outstanding balance so as to account accurately for the indebtedness and the different cash flows of the credit card account
- available estimated transition probabilities
- no resource constraints for the bank undertaking collection strategies

The model, as introduced here, is easily implementable given that accurate estimates of transition probabilities are available. The prior data mining requires much work, as Trench, Pederson, Lau, Ma, Wang, and Nair [42] underlined. Partitioning and clustering of the historical data followed by estimations of the transition probabilities[1] are difficult tasks to achieve. Chapter 4 will introduce a simulation study of the present model so as to generate reasonable and sensible transition probabilities based on the assumption that the cardholder has a rational attitude towards his

[1]This is usually done with maximum likelihood estimations on the reported sample of transitions. Some models though embody heterogeneity by using mover-stayer estimation procedures [see 18]

debt and required payments. The generated *MDP* will then be used to investigate different risk aspects involved e.g. infinite horizon forecasts, explicit consideration of variance in the choice of the optimizing policy.

Chapter 4

Approximate Dynamic Programming and Simulation Study

4.1 Introduction

The methodology used to generate and simulate *MDP*s related to the model discussed in Chapter 3 would be specified in this chapter. A one step dynamic system, which approximates the state transitions of the accounts, would first be introduced. Different cardholder profiles governing the conditional distribution of repayments would then be defined. The generated models would be solved for finite time horizon using *finite-horizon policy evaluation algorithm* and for infinite time horizon using *policy iteration algorithm*. Some common industrial situations would be found again via the simulation study and the rationality of some practices re-affirmed. A discussion of the adequacy of the approximation concludes the chapter.

4.2 Approximate Dynamic Programming

The approximate dynamic program *ADP*, introduced in Chapter 3, 3.3.6 p50, would be presented here so as to mimic the stochastic process and therefore to generate sensible transition probabilities. There are two natural factors determining the transitions of the account:

- $\Omega_n(x_n, u_n)$ discrete stochastic repayment, given the current state x_n and the applied admissible collection strategy u_n. Its realization $\omega_n(x_n, u_n)$ takes values in D
- $r_n(x_n, u_n, x_{n+1})$ estimated aggregate activity of the account during billing cycle n

Figure 3.6 p60 shows the limitation of a one step approach since the exact computation of the minimum payment requires keeping the full history of the account transitions. Nevertheless, the fact that an account is blocked from further usage after 30 days past due allows one to approximate the evolution based on the sole current state X_n.

4.2.1 Approximate Minimum Repayments

In this section, the payment requirements according to the current state of the account would first be approximated. The continuous random variable of payment Ω would subsequently be discretized given the present state of the bank account.

Definition 4.1. *Given the due state of an account $x_n \in S_{trans}$, let $rp(x_n, k)$ be the approximate minimum repayment to be made by the due date of billing cycle n to decrease delinquency state by k months.*

In the previous definition, k is always positive except when $x_n = (0, j_n)$ and the cardholder repays the minimum payment to have a current account next period. This situation is noted by $k = -1$.

These approximate repayments rp consist thus of a Markovian approximation of the different *delinquent balances* introduced in 3.4.3 which govern the evolution of any account within the different due status buckets.

•$x_n = (0, j_n)$ the cardholder benefits from a grace period. He may repay 90% of the statement balance to enjoy anew a grace period (case $k = 0$), repay the minimum required payment to have a current account during next billing cycle (case $k = 1$), or repay nothing and become one month delinquent (case $k = 2$).

$$rp(x_n, k) = \begin{cases} max\big\{min_{pay}; mrp \times (B^{j_n})\big\} + max\{B^{j_n} - CL, 0\}, & k = -1 \\ max\{min_{pay}; \frac{90}{100}B^{j_n}\}, & k = 0 \end{cases} \tag{4.1}$$

•$x_n = (i_n, j_n)$, $i_n = 1$ the account is current. The cardholder may repay 90% of the statement balance to enjoy anew a grace period (case $k = 1$), repay the minimum required payment to have a current account accrued with interest in the next billing cycle (case $k = 0$) or make no repayment and become one month delinquent (case $k = 2$).

$$rp(x_n, k) = \begin{cases} max\{min_{pay}; mrp \times B^{j_n}\} + max\{B^{j_n} - CL, 0\}, & k = 0 \\ max\{min_{pay}; \frac{90}{100}B^{j_n}\}, & k = 1 \end{cases} \tag{4.2}$$

•$x_n = (i_n, j_n), 2 \leq i_n \leq L$ the account is delinquent. The card is blocked to prevent further risk when $i_n \geq 3$. One needs a way, based on the current state to approximate the previous balances. For instance, the first missed payment is based on the balance of the account at the opening of the billing cycle during which the cardholder ceased paying. Let r_{pra} be the ratio between the number of days separating the statement issuance from the due state and the number of days in the billing cycle. Let $\overline{VOL}$ be the mean on all the unblocked states of the overall volume spent. The two oldest *delinquent balances* are supposed to be the difference between the present balance B^{j_n} and a possibly prorated mean volume depending on whether the account was blocked i.e. $i_n \geq 3$ during the delinquency process. If $i_n \geq 3$, the remaining *delinquent balances* are related to the billing cycles during which the account was blocked. The latter are approximatively equal to B^{j_n} since during these periods only fees and interests will accrue. Therefore, $B^{j_n} \approx B^{j_{n-k}}$, $k \leq i_n - 3$. The approximate balances $\widetilde{B}^{j_{n-k}}$ are therefore calculated as follows:

$$\widetilde{B}^{j_{n-k}} \approx \begin{cases} B^{j_n} - (H(i_n-3)r_{pra} + \delta(k-1))\overline{VOL}, & k = i_n-2, i_n-1 \\ B^{j_n}, & k = 0, \ldots, i_n-3 \end{cases} \tag{4.3}$$

$rp(x_n, k)$ is thus defined by:

$$rp(x_n,k) = \begin{cases} \sum\limits_{q=0,\ldots,k} max\{min_{pay}; mrp\widetilde{B}^{j_{n-q}}\} + max\{\widetilde{B}^{j_{n-q}} - CL, 0\}, & 0 \leq k \leq i_n - 1 \\ max\{min_{pay}; \frac{90}{100}B^{j_n}\}, & k = i_n \end{cases} \tag{4.4}$$

The numerical error due to the approximation of the required payments $rp(x_n, k)$ is bounded by $mrp \times \big(\sum\limits_{q=0,\ldots,k} |\widetilde{B}^{j_{n-q}} - B^{j_n}|\big)$, with $mrp \ll 1$ (usually set to 3%) which justifies the preceding approximation. The calculation of $\widetilde{B}^{j_{n-k}}$ further assumes that the delinquency mainly follows a strictly worsening process until sufficient repayment is made. The latter assumption is empirically justified by the examination of the data reported in [41]. More specifically, delinquent accounts mainly revert to paid up states and current states i.e. $i_{n+1} = 0, 1$ respectively, or transit to the next worse states of delinquency i.e. $i_{n+1} = i_n + 1$[1].

Definition 4.2. *Given the due state of an account $x_n = (i_n, j_n) \in S_{trans}$, the minimum required payment $MRP(x_n)$ is defined as the minimum repayment to be made by the cardholder so as to remain or become non-delinquent. $MRP(x_n)$ is calculated as follows:*

$$MRP(x_n) = \begin{cases} rp(x_n, -1), & x_n = (0, j_n), \quad j_n = 0, \ldots, M \\ rp(x_n, i_n - 1), & x_n = (i_n, j_n), \quad i_n = 1, \ldots, L, j_n = 0, \ldots, M \end{cases} \tag{4.5}$$

[1]Such transitions were found, in the data reported in [41], to account for 79% of all the possible moves when in such states. The transitions originating from delinquent states represent 12% of the total number of transactions. The approximation concerning the monotonicity of the transitions should therefore be correct in 98% of the cases.

Without loss of generality, it is assumed in the present model that $MRP(x) < \frac{B^j}{2}$ which otherwise would mean that $B^j \leq 2min_{pay}$. Such an account is considered inactive and thus excluded from the present analysis.

4.2.2 Discrete Partition of the Payments

The preceding conditions define a necessary partition of the payments to accurately reflect the evolution of the due state of an account.

Definition 4.3. *Let $\Phi(x_n)$ be the discrete partition of the payments Ω_n given the account is in state $x_n \in S_{trans}$.*

•$x_n = (0, j_n), \quad j_n = 0, \ldots, M+1$

$$\Phi(0, j_n) = \big[0, rp(x_n, -1)\big) \bigcup \Psi\Big(\big[rp(x_n, -1), rp(x_n, 0)\big)\Big) \bigcup \big[rp(x_n, 0), B^{j_n}\big]$$

•$x_n = (i_n, j_n), \quad i_n = 1, \ldots, L, j_n = 0, \ldots, M$

$$\begin{aligned}\Phi(i_n, j_n) = \big[0, rp(x_n, 0)\big) &\bigcup_{k=0}^{i_n-2} \big[rp(x_n, k), rp(x_n, k+1)\big) \\ &\bigcup \Psi\Big(\big[rp(x_n, i_n - 1), rp(x_n, i_n)\big)\Big) \bigcup \big[rp(x_n, i_n), B^{j_n}\big],\end{aligned}$$

•$x_n = (NA, NA)$

$$\Phi\big((NA, NA)\big) = \{0\}$$

where $\Psi\big([a, b)\big)$ represents any arbitrary discrete partition[1] *of the line segment $[a, b)$.*

For any other line segment above, the repayments are assumed to be equal to the lower bound of the line segment that they belong to. In what follows, $\Psi\big([a, b)\big)$

[1] It is preferable for the discrete partition Ψ to be fine enough so as to exhaustively scan the possible transitions from the present due status to next state where the account is current.

is chosen so as to partition $[a, b)$ into line segments that are smaller than the line segments partitioning the balance $B^{j+1} - B^{j}$. To that end, the ratio Ψ_{ratio} is introduced and set so as to divide the balance into line segments that are Ψ_{ratio} times smaller than the ones defining the discrete unused balance status j.
D, previously introduced as the set of payments [see 3.3.4.2], is therefore the finite discrete set comprising all the discrete values defining $\Phi\big(x = (i, j)\big), \quad x \in S$.

4.2.3 Dynamic Programming System

The partition of the repayments enables, given x_n and ω_n, the derivation of the evolution of the due status of an account in the next period i.e. i_{n+1}. The explicit relations between $\big(x_n, \omega_n\big)$ and i_{n+1} are detailed in Tables 4.1, 4.2, 4.3 p80. To complete the definition of the dynamic system, the evolution of the unused credit limit has to be specified.

4.2.3.1 Evolution of the Unused Credit Limit j_n

The present section details the evolution of the j_n from one billing cycle to another. Let B^{bill} denote the outstanding balance reported on the statement from which j_{n+1} will be derived. The evolution of the balance is determined by the repayments Ω_n and by the use of the card during the current billing cycle $B_n^{use} = B^{j_n} + r_n(x_n, u_n, x_{n+1})$. Except for the volumes $CVOL_n(x_n, u_n, x_{n+1})$ and $PVOL_n(x_n, u_n, x_{n+1})$, all the other values needed to compute r_n do not depend on x_{n+1}. The sequel would establish an approximate equivalence between $CVOL_n(x_n, u_n, x_{n+1})$, $PVOL_n(x_n, u_n, x_{n+1})$ and $CVOL_n(x_n, u_n, \omega_n)$, $PVOL_n(x_n, u_n, \omega_n)$, respectively. The evolution of the unused credit limit j_n would therefore be derived from the triplet

(x_n, u_n, ω_n).

$\bullet x_n = (i_n, j_n) \in S_{trans}, 0 \leq i_n \leq 1$. The account is either paid up or current. It is reasonable to assume that $CVOL_n$, $PVOL_n$ depend only on the present state and collection strategy. Hence B^{bill} can be written as:

$$B^{bill} = B^{j_n} + r_n(x_n, u_n) - \omega_n(x_n, u_n) \tag{4.6}$$

$\bullet x_n = (i_n, j_n) \in S_{trans}, i_n = 2$. The payments are one month overdue. One should distinguish two cases depending on whether (a) the account was previously in an equal or better due status or (b) the account was previously in a strictly worse due state. These two situations are formally defined as follows:

a) $\exists k, \begin{cases} i_{n-k} \in \{0, 1\} \\ i_{n-k+q} = 2, q = 1, \ldots, k \end{cases}$

b) $i_{n-1} = 3, \ldots, L$

Situation (a) implies the cardholder, despite his overdue payments, can still make transactions whereas in situation (b) the card is blocked. Based on [41], one finds that situation (b) accounts for 4% of all the transitions to due state $i_n = 2$. Such transitions are thus neglected in the sequel and an account one month overdue is not yet blocked. From there, two situations can occur:

1. $rp(x_n, 0) \leq \omega_n$ The cardholder's repayment allows the due status of the account not to worsen. The account will, at worst, be one month overdue in the next billing cycle i.e. $i_{n+1} \leq 2$. The cardholder thus continues to enjoy the usage of his card. Similar to Equation (4.6), $CVOL_n$ and $PVOL_n$ are assumed to depend only on the present state and collection strategy which leads to the same calculation of B^{bill} as in (4.6).

2. $\omega_n < rp(x_n, 0)$ The cardholder does not make any payment. His repayments fall further behind and will be two months overdue. The card is blocked from the due date onwards. $CVOL_n(x_n, u_n, \omega_n)$, $PVOL_n(x_n, u_n, \omega_n)$ are therefore assumed to be the prorated amounts with factor r_{pra} of the volumes $CVOL_n(x_n, u_n)$, $PVOL_n(x_n, u_n)$ introduced in the previous case when the account is one month overdue and when the repayments are sufficient for the cardholder to keep usage of his card during the whole billing cycle i.e. $(i_n = 2,\ i_{n+1} \leq 2)$. $r_n^{pra}(x_n, u_n)$ is the corresponding activity when the volumes are $r_{pra}CVOL_n(x_n, u_n)$ and $r_{pra}PVOL_n(x_n, u_n)$. B^{bill} is then calculated as:

$$B^{bill} = B^{j_n} + r_n^{pra}(x_n, u_n) - \omega_n(x_n, u_n), \quad i_n = 2,\ rp(x_n, 0) \leq \omega_n \tag{4.7}$$

•$x_n = (i_n, j_n) \in S_{trans}, 3 \leq i_n,\ \omega_n < MRP(x_n)$. The account is more than one month overdue and hence blocked. Since the repayment is lower than the minimum required payment, the usage of the card is not recovered. Hence, $PVOL_n = CVOL_n = 0$

$$B^{bill} = B^{j_n} + AC_n(x_n, u_n) - \omega_n(x_n, u_n) \tag{4.8}$$

•$x_n = (i_n, j_n) \in S_{trans}, 3 \leq i_n,\ MRP(x_n) \leq \omega_n$. The account is more than one month overdue and hence blocked until sufficient repayment is made so as to decrease the due status to either current or paid up state. It assumes that the cardholder recovers the usage of his card on his payment date i.e on the due date. The volumes $CVOL_n$, $PVOL_n$ are therefore assumed to be prorated amounts with factor $(1 - r_{pra})$ of the quantities introduced in Equation (4.6) when $0 \leq i_n \leq 1$. Given that $MRP(x_n) \leq \omega_n$, these two quantities are calculated as follows:

$$CVOL_n(x_n, u_n, \omega_n) = \begin{cases} (1 - r_{pra})CVOL_n((1, j_n), u_n), & \text{if } \omega_n < rp(x_n, i_n) \\ (1 - r_{pra})CVOL_n((0, j_n), u_n), & \text{if } rp(x_n, i_n) \leq \omega_n \end{cases} \tag{4.9}$$

$$PVOL_n(x_n,u_n,\omega_n) = \begin{cases} (1-r_{pra})PVOL_n((1,j_n),u_n), & \text{if } \omega_n < rp(x_n,i_n) \\ (1-r_{pra})PVOL_n((0,j_n),u_n), & \text{if } rp(x_n,i_n) \leq \omega_n \end{cases} \tag{4.10}$$

$r_n^{\overline{pra}}(x_n,u_n)$ is the activity derived from the preceding volumes. B^{bill} can then be calculated as a function of x_n, u_n and ω_n only.

$$B^{bill} = B^{j_n} + r_n^{\overline{pra}}(x_n,u_n) - \omega_n(x_n,u_n) \tag{4.11}$$

The collection of the previous cases describes exhaustively the evolution of the balance for any payment and in any given state different from the written-off state (NA,NA). It hence follows that j_{n+1} is the unique solution of the following system ϕ defined by:

$$\phi(x_n,u_n,\omega_n) = \begin{cases} NA, & \text{if } x_n = (NA,NA) \\ NA, & \text{if } x_n = (L,j_n),\ \omega(x_n,u_n) < rp(x_n,0) \\ k|\ k \in J', LB^k < B_n^{bill} - \omega_n \leq UB^k, & \text{otherwise} \end{cases} \tag{4.12}$$

The existence and uniqueness of k, in the third case, is ensured by the consistent partition of the unused credit limit into disjoint discrete line segments.

4.2.3.2 Detailed Dynamic Evolution of (i_n,j_n)

Let $h(\cdot)$ be the discrete-time dynamic system, that governs the conditional state transitions as follows $x_{n+1} = h(x_n,u_n,\omega_n)$. $h(\cdot)$ is defined by:

ω_n	i_{n+1}	j_{n+1}
0	2	$\phi\big((i_n,j_n),u_n,0\big)$
$\Psi\Big(\big[rp(x_n,-1),rp(x_n,0)\big)\Big)$	1	$\phi\big((i_n,j_n),u_n,\omega_n\big)$
$rp\ (x_n,0)$	0	$\phi\big((i_n,j_n),u_n,B^{j_n}\big)$

Table 4.1: One step transitions. Case: $x_n = (i_n,j_n), i_n = 0, j_n = 0,\dots,M$

ω_n	i_{n+1}	j_{n+1}
0	$i_n + 1$	$\phi(x_n, u_n, 0)$
$rp(x_n, k)$	$i_n - k$	$\phi\big(x_n, u_n, rp(x_n, k)\big)$
$\Psi\Big([rp\ (x_n, i_n - 1), rp\ (x_n, i_n))\Big)$	1	$\phi(x_n, u_n, \omega_n)$
$rp\ (x_n, i_n)$	0	$\phi(x_n, u_n, B^{j_n})$

Table 4.2: One step transitions. Case: $x_n = (i_n, j_n), i_n = 1, \ldots, L, j_n = 0, \ldots, M$

ω_n	i_{n+1}	j_{n+1}
0	NA	NA

Table 4.3: One step transitions. Case: $x_n = (NA, NA)$

The present *MDP* has been approximated as a discrete-time dynamic system for which the random disturbance $\Omega(x_n, u_n)$ takes values in the finite discrete set D. The next section would characterize the distribution of the random repayments.

4.3 Cardholder's Profiles

This section would introduce different profiles of cardholder's behaviours. The profiles would ideally model the ability of the cardholder to repay his debt together with his willingness to do so, given a present due state and a collection strategy undertaken by the card issuer. Based on varied conjectures about the cardholder's repayments, two categories of rational profiles would be defined so as to model a wide variety of situations. The probability distributions would be derived from each profile.

For each of these two categories of rational profiles, the rationality assumption would then be relaxed in two different ways and the subsequent irrational and random profiles would be obtained.

The notations used in the sequel for the discrete probabilities and excess probability distributions of repayment are first introduced.

Definition 4.4. *Let $p_\Omega(\omega|x,u)$ be the discrete probability that the cardholder makes a repayment equal to $\omega \in \Phi(x)$ when the account is in state x and subject to collection action u.*

Definition 4.5. *The excess probability distribution function $G_\Omega(\omega)$ of the repayments Ω given a due state $x \in S_{trans}$ and a control $u \in U(x)$, is the probability that the repayment Ω would be strictly greater than ω;*

$$G_\Omega(\omega) = p(\Omega > \omega|x,u), \quad x \in S_{trans}, u \in U(x)$$

4.3.1 Rational profiles

4.3.1.1 Severity of Collection Actions

Recall the discrete control set U comprises all the collection strategies ordered by increasing level of severity.

4.3.1.2 Rational Repayment

Definition 4.6. *A rational payment behaviour is such that the excess distributions of repayments G_Ω are increasing functions of the collection strategy u.*

$$\frac{\partial G_\Omega}{\partial u}(\omega|x,u) \geq 0, \qquad 0 < \omega, x \in S_{trans}, u \in U(x)$$

In other words, for any two controls $u, v \in U(x)$ such that u is more severe than v, the excess probability distribution function of repayments $G_\Omega(\omega|x,u)$ stochastically dominates first order the excess probability distribution function of repayments $G_\Omega(\omega|x,v)$.

4.3.2 *Trimodal* Profile with Separable Willingness and Ability to repay

Recall that a prior segmentation of the cardholders into rather homogenous risk segments is required. The payment distributions of the cardholders belonging to the same segment are supposed to be independently and identically distributed and to depend on two characteristics:

- An ability to repay φ which measures the extent to which a cardholder, given a due payment, is able to repay.
- A willingness to repay ν which measures, the determination of a cardholder to repay.

This approach was adopted in papers related to credit risk and repayments such as Rhind et al. [34]. Let define a category of rational profiles, namely the *Trimodal* profile, to depict the approach. Each payment distribution would consist of a discrete distribution with three modes corresponding to the three following situations: no payment is made, the minimum required payment is repaid and the outstanding balance reported on the statement is fully settled. It is postulated in the *Trimodal* profile that the resulting probabilities of payment are calculated as the product of the willingness to repay ν and of the ability to pay φ. The rationale of this approach is to separate the causal factors of a phenomenon by assuming their resulting joint influence is a product of functions. The ability to repay is assumed to depend only on the due state of the account whereas the willingness is assumed to depend on both the collection strategy and the due state.
Following conventional financial analysis e.g. Scherr [37], the ability to repay is illustrated using the logistic function. The latter is relevant for its decreasing trend between an initial probability P_0 and 0.

Definition 4.7. *The ability to repay $\varphi(\omega)$ is defined as the probability that the cardholder is financially able, but not necessarily willing, to repay more than ω. It is chosen in the* Trimodal *profile to be calculated as follows:*

$$\varphi(\omega) = \left(1 - \frac{1}{1 + \exp\left(a_\varphi + b_\varphi \omega\right)}\right) P_0, \qquad 0 \leq \omega,$$

where $a_\varphi, b_\varphi, P_0$ are set arbitrarily to run simulations with the following conditions holding; $\quad b_\varphi \leq 0, \quad 0 \leq \frac{\exp(a_\varphi)}{1+\exp(a_\varphi)} P_0 \leq 1$

The probability that the cardholder is able to make any strictly positive payment (no payment) is chosen to be equal to $\frac{exp(a_\varphi)}{1+\exp(a_\varphi)} P_0$, $\left(\frac{1}{1+\exp(a_\varphi)} P_0\right)$ and is a priori supposed not to be equal to one, (zero) respectively.

Definition 4.8. *Given a due state $x \in S - \big\{(NA, NA)\big\}$ and a control $u \in U(x)$, the willingness to repay $\nu(x, u)$ measures the determination of a cardholder to make payment. $\nu(\cdot)$ is arbitrarily chosen in the* Trimodal *profile with the following properties holding:*

- $0 \leq \nu(x, u) \leq 1, \quad 0 \leq \frac{\partial \nu}{\partial u}(x, u), \quad x \in S_{trans}, u \in U(x)$

The bounds of $\nu(\cdot)$ are set to ensure a proper definition of probabilities. The dependence of $\nu(\cdot)$ in x accounts for the possibility of having a willingness to repay that differs from one due status to another. Finally, the increasing trend of $\nu(\cdot)$ with respect to the control u accounts for the rationality of the behaviour i.e. the more severe collection strategy u, the more determined the cardholder is to pay .
It is assumed in the *Trimodal* profile that the repayments made by the cardholder are of three types only: null repayment, minimum required payment and full balance payment. The three modes are subsequently defined by:

Definition 4.9. FullPay *mode corresponds to the repayment made by the cardholder to enjoy anew a grace period at the beginning of the next billing cycle. This mode is hence located at* $\Omega = rp(x,i)$ *and, given a due state* $x = (i,j) \in S_{trans}$ *and a collection strategy* $u \in U(x)$*, is such that:*

$$p_\Omega\big(rp(x,i)|x,u\big) = \varphi\big(rp(x,i)\big)\nu(x,u)$$

Definition 4.10. MRP *mode corresponds to the repayment of the minimum required amount. This mode is hence located at* $\Omega = MRP(x)$ *and, given a due state* $x = (i,j) \in S_{trans}$ *and a collection strategy* $u \in U(x)$*, is such that:*

$$p_\Omega\big(MRP(x)|x,u\big) = \varphi\big(MRP(x)\big)\nu(x,u) - p_\Omega\big(rp(x,i)|x,u\big)$$

The decreasing trend of the logistic function together with the properties of the chosen willingness ensure the *MRP* mode to have a probability between 0 and 1 and hence to be properly defined.

Definition 4.11. NullPay *mode corresponds to no repayment made. This mode of the distribution is hence located at* $\Omega = 0$ *and given a due state* x *and a collection strategy* $u \in U(x)$*, is such that;*

$$\begin{cases} p_\Omega\big(0|x,u\big) = 1, & \text{if } x = \big\{(NA,NA)\big\} \\ p_\Omega\big(0|x,u\big) = 1 - p_\Omega\big(MRP(x)|x,u\big) - p_\Omega\big(rp(x,i)|x,u\big), & \text{otherwise} \end{cases}$$

Definition 4.12. *The set* $\Upsilon_{Tri} = \big\{p_\Omega(\cdot|x,u), (x,u) \in \kappa\big\}$ *of payment distributions is then derived as:*

$$\Upsilon_{Tri} = \begin{cases} \bullet x = (i,j) \in S_{trans}, u \in U(x) \\ p_\Omega\big(0|x,u\big) = 1 - p_\Omega\big(MRP(x)|x,u\big) - p_\Omega\big(rp(x,i)|x,u\big) \\ p_\Omega\big(MRP(x)|x,u\big) = \varphi\big(MRP(x)\big)\nu(x,u) - p_\Omega\big(rp(x,i)|x,u\big) \\ p_\Omega\big(rp(x,i)|x,u\big) = \varphi\big(rp(x,i)\big)\nu(x,u) \\ \\ \bullet x = (NA,NA), u \in U(x) \\ p_\Omega\big(0|(NA,NA),u\big) = 1 \end{cases}$$

A *Trimodal* profile is exhaustively defined by its related ability to repay $\varphi(\cdot)$ and cardholder willingness to repay $\nu(\cdot)$.

Property 4.1. *The* Trimodal *profile, defined by the previous choice of $\varphi(\cdot)$ and $\nu(\cdot)$, is a rational profile.*

Proof. The repayments when the account is in state $\big\{(NA, NA)\big\}$ are all null. Hence, it is necessary and sufficient to show that for $x \in S_{trans}$ the excess probabilities at the modes *FullPay* and *MRP* are increasing functions of the control $u \in U$. To that effect, the partial derivatives of these probabilities with respect to the control $u \in U(x)$ are calculated. From the separation of φ and ν and from the increasing (decreasing) trend of ν (φ) in u (ω) respectively, it follows:

$$\frac{\partial p_\Omega}{\partial u}\big(rp(x,i)|x,u\big) = \varphi\big(rp(x,i)\big)\frac{\partial \nu}{\partial u}(x,u) \geq 0,$$
$$\frac{\partial p_\Omega}{\partial u}\big(MRP(x)|x,u\big) = \Big(\varphi\big(MRP(x)\big) - \varphi\big(rp(x,i)\big)\Big)\frac{\partial \nu}{\partial u}(x,u) \geq 0,$$

□

Some irrational *Trimodal* profiles can also be defined. They would comprise the same modes and would also be calculated as the product of the ability to repay $\varphi(\cdot)$ and of the cardholder willingness to repay $\nu(\cdot)$. The latter would be chosen so that the rationality property would not hold. This approach will be used in the simulation study to illustrate the impact of the collection strategies on a rational cardholder.

The *Trimodal* profile can be extended to include more than three types of repayments. To that end, additional probabilities should be introduced so as to account for the probability of making other repayments. For instance, the cardholder may either repay partially the minimum required payment or repay more than the minimum required payment yet without settling the debt. Nevertheless, considering the nature of the debt and the repayment rules, it is reasonable to let the two modes *FullPay* and *MRP* dominate the other repayments and to consider these probabilities to be comparatively small.

4.3.3 *Unimodal* Profile with Beta Distributions of Repayments

This section introduces a rational profile based on Beta distributions of repayments. Two parameters are necessary and sufficient to define each Beta distribution. The first moment and the mode (if any exists) or the shape of each Beta distribution would be characterized so as to derive the two relevant parameters.

Definition 4.13. *The Beta probability density function $B(z, c, d)$ for the normalized random variable $z, 0 \leq z \leq 1$ is defined by:*

$$B(z, c, d) = \frac{z^{c-1}(1-z)^{d-1}}{B(c, d)}, \qquad 0 < c, \quad 0 < d,$$

where $B(c, d)$ is the beta function of (c, d).

The two parameters c and d are necessary and sufficient to define a Beta distribution. Given a due state $x = (i, j) \in S_{trans}$ and a control $u \in U(x)$, their determination is based on the following points:

- The definition of a normalized variable of repayment Z as the ratio between the payments Ω and the outstanding balance B^j when the account is in state $x \in S_{trans}$:

$$Z = \frac{\Omega}{B^j}, \quad 0 \leq \Omega \leq B^j$$

- The calculation of the corresponding normalized expected repayment. The cardholder is assumed to have a prior conditional distribution of repayments $G_{\Omega}^{prior}(\omega|x,u)$ defined by a logistic function. This distribution, for which the rational property will be proven to hold [see 4.3.3.1], is so chosen to embody both the cardholder's willingness and financial ability to repay given x and u. An ad'hoc expected utility will be defined and assumed to be equal to the first moment of the related Beta distribution.

- The following conjecture concerning the probability density function of the related Beta distribution; depending on the minimum due, the cardholder is reasonably assumed, in decreasing order of capability to repay, to:

  1. preferably settle his debt, if he is "capable" of doing so.
  2. be most likely to repay the minimum due, if he is "capable" of paying the minimum due.
  3. be most likely to repay nothing, if the minimum due is beyond his capacity of repayment.

This conjecture, which illustrates both the rules of repayments and the rationality of the cardholder, will be detailed in 4.3.3.2 and in particular the meaning of "capable" will be specified.

The present procedure should be regarded as a two-step approach. The first step consists of the definition of a prior conditional distribution of repayments, for which the rational property holds. This distribution illustrates the overall payment made by a cardholder in a given state and under a given collection strategy. Its normalized expected utility is therefore solely retained. The second step aims to populate the range of the conjectured Beta distribution, the first moment of which is equal to the previous expected utility. To that end, the assumption concerning its probability density function is proposed. It enables one to complete the definition of the Beta distribution by deriving the two parameters c and d.

4.3.3.1 Characterization of the Normalized Expected Payments

This section shall detail the approach used to define the first moment of each Beta distribution.
Let first define the prior distribution of repayments $G_{\Omega}^{prior}(\omega|x,u)$.

Definition 4.14. *Given a due state $x \in S_{trans}$ and a control $u \in U$, the prior logistic excess probability distribution of repayments $G_{\Omega}^{prior}(\omega|x,u)$ is defined by:*

$$G_{\Omega}^{prior}(\omega|x,u) = \left(1 - \frac{1}{1+\exp\big(a_G(x,u)+b_G(x,u)\omega\big)}\right)P_0(x,u), \quad 0 \leq \omega$$

where $a_G(x,u), b_G(x,u), P_0(x,u)$ are chosen arbitrarily to run simulations with the following conditions holding:

- $a_G(x,u) \leq 0, \quad 0 \leq \frac{\partial a_G}{\partial u}(x,u)$
- $b_G(x,u) \leq 0, \quad 0 \leq \frac{\partial b_G}{\partial u}(x,u)$
- $0 \leq \frac{\exp\big(a_G(x,u)\big)}{1+\exp\big(a_G(x,u)\big)}P_0(x,u) \leq 1, \quad 0 \leq \frac{\partial P_0}{\partial u}(x,u)$

The sign of $b_G(x,u)$ is imposed to ensure the decreasing trend of the excess probability. The signs of the partial derivatives of $b_G(x,u)$ and $P_0(x,u)$ with respect to u are chosen to be positive for the rationality property to hold.
Similar to the *Trimodal* profile, the probability that the cardholder is able to make any strictly positive repayment (no repayment) is chosen to be equal to $\frac{exp\big(a_G(x,u)\big)}{1+\exp\big(a_G(x,u)\big)}P_0(x,u)$, $\left(\frac{1}{1+\exp\left(a_G(x,u)\right)}P_0(x,u)\right)$ and is a priori supposed not to be equal to one , (zero) respectively. Nevertheless, the influences of the due state x and of the control u are now, in a more general manner, assumed not to be separable.

Property 4.2. *The excess probability distribution* $G_{\Omega}^{prior}(\cdot)$ *defines a rational repayment behaviour.*

Proof. The repayments when in state $x = \{(NA, NA)\}$ are null. Hence, it is necessary and sufficient to show that for $x \in S_{trans}, u \in U(x), \omega > 0$, $G_{\Omega}^{prior}(\omega|x,u)$ are increasing functions of u. Similarly, let calculate their partial derivatives with respect to the control u.

$$\frac{\partial G_{\Omega}^{prior}}{\partial u}(\omega|x,u) = \frac{\exp\big(a_G(x,u) + b_G(x,u)\omega\big)}{1+\exp\big(a_G(x,u) + b_G(x,u)\omega\big)}\Bigg(\frac{\partial P_0}{\partial u}(x,u) + \frac{P_0(x,u)}{1+\exp\big(a_G(x,u) + b_G(x,u)\omega\big)}\Big(\omega\frac{\partial b_G}{\partial u}(x,u) + \frac{\partial a_G}{\partial u}(x,u)\Big)\Bigg)$$

From the conditions on $a_G(x,u)$, $b_G(x,u)$ and $P_0(x,u)$ [see Definition 4.14], it follows that:

$$\frac{\partial G_{\Omega}^{prior}}{\partial u}(\omega|x,u) \geq 0, \quad x \in S_{trans}, u \in U$$

□

The calculation of the normalized expected repayment should now be detailed. The prior distribution of repayments has $\mathbb{R}^+$ for support whereas the realizations of the payments range between zero and the outstanding balance B^j. The possibility of a cardholder settling his indebtedness B^j is accounted for by considering that the cardholder repays the outstanding balance with the excess probability $G_{\Omega}^{prior}(B^j|x,u)$. A related payment utility function ut_{norm}, embodying this truncation of the repayments at B^j shall consequently be defined. The normalized expected repayment would then be equal to the expected value of ut_{norm}.

Definition 4.15. *Given a due state $x = (i, j) \in S_{trans}$, the utility function ut_{norm} associated to the cardholder repayments is defined as follows:*

$$ut_{norm}(\omega|x) = \begin{cases} \frac{\omega}{B^j} & 0 \leq \omega \leq B^j \\ 1 & B^j < \omega \end{cases}$$

Definition 4.16. *Given a due state $x = (i, j) \in S_{trans}$, the normalized expected repayment $E_{(x,\,u)}\{Z\}$ is defined as follows:*

$$E_{(x,\,u)}\{Z\} = \underset{\Omega}{E}\{ut_{norm}(\Omega|x)|u\} = \int_0^{\infty} ut_{norm}(\omega|x)\left(-\frac{\partial G_{\Omega}^{prior}}{\partial \omega}\right)(\omega|x, u)d\omega$$
$$= \int_0^{B^j} \frac{\omega}{B^j}\left(-\frac{\partial G_{\Omega}^{prior}}{\partial \omega}\right)(\omega|x, u)d\omega + G_{\Omega}^{prior}(B^j|x, u)$$

Property 4.3. *The normalized expected repayment is an increasing function of the severity of the control u, and hence consistent with the concept of rational repayment.*

Proof. Given a due state $x = (i, j) \in S_{trans}$, let $u, v \in U(x)$ be two controls such that u is more severe than v, i.e. with respect to the ordered control set $U(x)$, $u > v$. The repayment Ω_u associated to alternative u stochastically dominates, first order, the repayments Ω_v of alternative v since the set of excess distributions associated $G_{\Omega}^{prior}(\cdot)$ defines a rational behaviour. The utility function $ut_{norm}(\Omega|x)$ is clearly increasing in Ω. The following result [see 31] will conclude the proof:
Given two random variables X_1, X_2 such that X_1 first-order dominates X_2 and given an increasing function ς

$$\underset{X_1}{E}\{\varsigma(X_1)\} \geq \underset{X_2}{E}\{\varsigma(X_2)\}$$

Hence,

$$\underset{\Omega_u}{E}\{ut_{norm}(\Omega_u|x)|u\} \geq \underset{\Omega_v}{E}\{ut_{norm}(\Omega_v|x)|u\}$$
$$E_{(x,\,u)}\{Z\} \geq E_{(x,v)}\{Z\}$$

$\square$

4.3.3.2 Characterization of the Probability Density Function of the Beta Distribution

Consider now the assumption concerning the distribution of the repayments. It has been assumed that if the cardholder has a sufficient capability to repay, he will preferably repay as much as he can in the range of the balance due. Otherwise, if his capacity is sufficient to repay the minimum due, he will concentrate his payment effort on such an amount and thereby define a mode for the probability density function at $MRP(x)$. Finally, if his capacity to repay is small compared to the minimum, he will most likely repay nothing.
This section shall firstly introduce a formal distinction of these three different cases and secondly complete the definition of the set of Beta distributions.
Let first state classic results of the Beta distribution that will be used in the sequel.

- The sufficient and necessary condition of existence of a mode in [0,1] is $1 \leq c$, $1 \leq d$
- In such a case, the mode is located at $z_0 = \frac{c-1}{c+d-2}$
- A "width" parameter N is be defined by $N = c + d$

Several cases should be distinguished depending on the values of the normalized expected payment $E_{(x,\,u)}\{Z\}$ and of the normalized minimum required payment $\frac{MRP(x)}{B^j}$:

a) $E_{(x,\,u)}\{Z\} > \frac{MRP(x)}{B^j}, \quad E_{(x,\,u)}\{Z\} > \frac{1}{2}$

b) $E_{(x,\,u)}\{Z\} > \frac{MRP(x)}{B^j}, \quad E_{(x,\,u)}\{Z\} \leq \frac{1}{2}$

c) $E_{(x,\,u)}\{Z\} \leq \frac{MRP(x)}{B^j}$

4.3.3.3 Cases (a) and (b): $E_{(x,\,u)}\{Z\} > \frac{MRP(x)}{B^j}$

The normalized expected repayment is strictly greater than the value of the mode $\frac{MRP(x)}{B^j}$, if any exists.

Theorem 4.1. *Given a due state $x = (i, j) \in S_{trans}$ a control $u \in U(x)$ and an expected payment $E_{(x,\,u)}\{Z\}$ such that $E_{(x,\,u)}\{Z\} > \frac{MRP(x)}{B^j}$, there exists a mode to the corresponding beta distribution if and only if $E_{(x,\,u)}\{Z\} \leq \frac{1}{2}$ (i.e. case (b)).*

Proof. The necessary condition shall be first proven by contradiction. The sufficient condition would be demonstrated by direct proof.

• Suppose that a mode exists and that $E_{(x,\,u)}(z) > \frac{1}{2}$.

Given the conjecture concerning the location of the mode, the following equations should hold:

$$E_{(x,\,u)}\{Z\} = \frac{c}{N}, \quad z_0 = \frac{c-1}{N-2} = \frac{MRP(x)}{B^j}$$

Hence,

$$N = \frac{B^j - 2MRP(x)}{B^j\, E_{(x,\,u)}\{Z\} - MRP(x)}$$

Definition 4.2 and the assumption that $E_{(x,\,u)}\{Z\} > \frac{1}{2}$ lead to:

$$N < 2 \Rightarrow \text{Contradiction with the existence of a mode.}$$

• Now suppose $E_{(x,\,u)}\{Z\} \leq \frac{1}{2}$,

$$N = \frac{B^j - 2MRP(x)}{B^j\, E_{(x,\,u)}\{Z\} - MRP(x)}, c = (N-2)\frac{MRP(x)}{B^j} + 1$$

N and c are such that,

$$N \geq 2, c \geq 1, \frac{N}{2} \geq c \text{ and thus, } d \geq 1$$

Hence, (c, d) uniquely define a beta distribution having a mode in $z_0 = \frac{MRP(x)}{B^j}$. □

In case (a) $E_{(x,\,u)}\{Z\} > \frac{MRP(x)}{B^j}$ and $E_{(x,\,u)}\{Z\} > \frac{1}{2}$, Theorem 4.1 implies that no mode exists. The cardholder has a substantial capability to repay more than the minimum required payment and is indeed expected to repay more than half of the outstanding balance. Thus, it is naturally assumed that the cardholder is likely to pay as much as he can and that the probability density function is an increasing function on the normalized range $[0, 1]$. The corresponding Beta distribution has no infinite branch in zero. A maximum weight is given to its tail (i.e $z \to 1$). Hence, the cardholder will preferably settle his debt. The parameters are defined by:

$$c = 1, d = \frac{1 - E_{(x,\,u)}\{Z\}}{E_{(x,\,u)}\{Z\}}$$

In case (b) $E_{(x,\,u)}\{Z\} > \frac{MRP(x)}{B^j}$ and $E_{(x,\,u)}\{Z\} \leq \frac{1}{2}$, Theorem 4.1 implies that a mode exists. The cardholder has no sufficient capability to be most likely to settle the whole debt but still his capability is sufficient to expect his payment to be most likely equal to the minimum required payment. The parameters are thus derived from Theorem 4.1.

4.3.3.4 Case (c): $E_{(x,\,u)}\{Z\} < \frac{MRP(x)}{B^j}$

In such a case, the expected repayment is considerably small compared to the minimum due. Despite the rational behaviour of the cardholder, one should not expect a mode at $\frac{MRP(x)}{B^j}$ since the cardholder is most likely to be unable to pay such an amount. Similarly, it is naturally assumed that the probability density function is a decreasing function on the normalized range $[0, 1]$. The corresponding Beta distribution has no infinite branch in one. A maximum weight is given to its branch in zero. The parameters are hence:

$$c = \frac{E_{(x,\,u)}\{Z\}}{1 - E_{(x,\,u)}\{Z\}}, d = 1$$

Definition 4.17. *The set* $\Upsilon_{Uni} = \{p_\Omega(\cdot|x,u), (x,u) \in \kappa\}$ *of payment distributions for the* Unimodal *profile is then derived from the following algorithm:*

Algorithm 1 Generation of Υ_{Uni} set of beta distributions of repayment for the *Unimodal* profile

---

Data: prior distribution of repayments G_Ω^{prior} and partition of repayments ϕ
Output: Set Υ_{Uni} of beta distributions of repayments.

for all $x = (i,j) \in S_{trans}, u \in U(x)$ **do**
  $E_{(x,u)}\{Z\} \leftarrow$ **Computeexpectpayment**(x, u, G_Ω^{prior}) /* According to Definition 4.16 */
  $(c,d) \leftarrow$ **Betaparameters**$(E_{(x,u)}\{Z\})$ /* According to Theorem 4.1 */
  for all $\omega_k, \omega_{k+1} \in \phi(x)$ **do**
    $p_\Omega(\omega_k|x,u) = F_\beta(\frac{\omega_{k+1}}{B^j}, c, d) - F_\beta(\frac{\omega_k}{B^j}, c, d)$ /* F_β: Cdf Beta distribution */
  end for
  $p_\Omega(rp(x,i)|x,u) = 1 - \sum_{\omega \in \phi(x)} p_\Omega(\omega|x,u)$
end for

$x = (NA, NA), u \in U(x)$
$p_\Omega(0|(NA,NA),u) = 1$

---

where **Computeexpectpayment**(x, u, G_Ω^{prior}) and **Betaparameters**$(E_{(x,u)}\{Z\})$ are the subroutines which compute the expected repayment $E_{(x,u)}\{Z\}$ and the pair of parameters of the beta distribution (c,d) according to Definition 4.16 and to Theorem 4.1, respectively.

4.3.4 Irrational Profiles: Relaxing the Sensitivity to the Collection Strategies

The category of irrational profiles comprises, in its broad generality, any profile for which the rationality criterion does not hold.

Consider the type of irrational profiles derived simply by relaxing the assumption that a rational cardholder is increasingly sensitive to firmer reminders. Two categories of irrational profiles are then defined by assuming the repayments distributions to be equivalent to the corresponding rational *Trimodal* and rational *Unimodal* profiles when the least strict strategy $u = 0$ is undertaken. In the perspective of the following computational study, these two categories of irrational *Trimodal* and *Unimodal* are of interest, for they would be used to measure the impact of the conjectured sensitivity to the collection reminders.

4.3.5 Random Profiles: Relaxing the Rational Distribution of Repayments

The profiles defined in this section comprise cardholders, who have a given ability to repay derived from either the *Trimodal* or the *Unimodal* profile and who distribute identically their payments within the range of their outstanding balances. These profiles would be used as a benchmark. Indeed, they consist of a relaxation of the rational way the cardholders distribute their repayments in the *Trimodal* or the *Unimodal* profile.

4.3.5.1 Random Profile Associated to a Rational *Trimodal* Profile

Starting from any *Trimodal* profile with a given ability to repay, the equivalent *Random* profile is derived by assuming the *Random* cardholder, when in state $x = (i, j) \in S_{trans}$ to:

- settle his outstanding balance B^j with a probability $\varphi(B^j)\nu(0)$
- repay nothing with a probability $1 - \varphi(MRP(x))\nu(0)$

- otherwise make repayments following a uniform distribution on the range $\big[0, rp(x,i)\big)$.

The rationale for the second assumption is that a cardholder who is not able to repay $MRP(x)$, would thus repay nothing. The third point illustrates the randomness of the cardholder. Given a certain repayment ability, his distribution of payment is uniform in the range $\big[0, rp(x,i)\big)$ and hence independent of the repayment obligations. The *Random* cardholder is furthermore insensitive to the collection strategy he is undergoing, in terms of repayments.

Property 4.4. *In this* Random *profile, the repayments uniformly distributed on* $\big[0; rp(x_n, 0)\big)$ *are independent of the undertaken collection strategy* u_n*.*

Proof.

$$\frac{\partial \varphi(\omega)\nu(0)}{\partial u} = 0 \Rightarrow \frac{\partial p_\Omega}{\partial u}(\omega|x,u) = 0, \qquad x \in S_{trans}, u \in U(x)$$
$$\frac{\partial^{(2)} p_\Omega}{\partial \omega^2}(\omega|x,u) = 0$$

□

Definition 4.18. *The set* $\Upsilon_{Rd} = \big\{p_\Omega(\cdot|x,u), (x,u) \in \kappa\big\}$ *of payment distributions for the* Random *profile associated to the rational* Trimodal *profile, with ability to repay* (φ, ν)*, is defined by:*

$$\Upsilon_{Rd} = \begin{cases} \bullet x = (i,j) \in S_{trans}, u \in U(x) \\ p_\Omega(0|x) = 1 - \varphi\big(MRP(x)\big)\nu(0) \\ p_\Omega\big(rp(x,i)|x\big) = \varphi(rp(x,i))\nu(0) \\ f_{Rd}\big(w|x\big) = \frac{1 - p_\Omega\big(0|x\big) - p_\Omega\big(rp(x,i)|x\big)}{rp(x,i)}\omega, \quad 0 \le \omega < rp(x,i) \\ \\ \bullet x = (NA, NA), u \in U(x) \\ p_\Omega\big(0|(NA,NA)\big) = 1 \end{cases}$$

where $f(\cdot|x,u)$ *is the probability density function of the payments on the range* $[0, rp(x,i))$ *when the account is in state* $x = (i,j) \in S - \{(NA, NA)\}$ *.*

The uniform distribution $f(\cdot|x,u)$ should afterwards be discretized according to the repayment partitions ϕ.

4.3.5.2 Random Profile Associated to a Rational *Unimodal* Profile

Starting from any *Unimodal* profile with a given ability to repay, the equivalent *Random* profile is derived by assuming that the associated *Random* cardholder, when in state $x = (i,j) \in S_{trans}$ will:

- identically distribute his repayments within the range of the outstanding balance B^j such that the expected repayment will be equal to $\frac{B^j}{2}$ if $E_{(x,u)}\{Z\} > \frac{1}{2}$ for the associated rational *Unimodal* profile.
- repay nothing with a probability of $1 - E_{(x,u)}\{Z\}$ and identically distribute within the range of B^j if $E_{(x,u)}\{Z\} \leq \frac{1}{2}$

The first assumption simply states that a random cardholder with a high ability of repayment identically distributes his payment within the range of the indebtedness regardless of the issuer's requirements. The second point corresponds to a cardholder with a lower ability to repay. He is assumed to identically distribute his payments such that his expected repayment is the same as the expected repayment of the associated rational *Unimodal* profile when $u = 0$. In that case, his probability of repaying nothing is simply the complement of one of the previous uniform distributions.

Property 4.5. *In this* Random *profile, the repayments uniformly distributed on* $\left[0, B^j\right)$ *are independent of the undertaken collection strategy* u_n*.*

The proof, in the same vein as the previous one, is omitted here.

Definition 4.19. *The set* $\Upsilon_{Rd} = \{p_\Omega(\cdot|x,u), (x,u) \in \kappa\}$ *of payment distributions for the* Random *profile associated to the rational* Unimodal *profile with prior ability to repay* G_Ω^{prior} *is defined by:*

Algorithm 2 Generation of Υ_{Rd} set of random distributions of repayment of a *Random* profile obtained by relaxing the rational distribution of repayments in the *Unimodal* profile

---

Data: prior distribution of repayments G_Ω^{prior} and partition of repayments ϕ
Output: Set Υ_{Rd} of random distributions of repayments.

for all $x = (i,j) \in S_{trans}, u \in U(x)$ **do**
  $E_{(x,u)}\{Z\} \leftarrow$ **Computeexpectpayment**(x, u, G_Ω^{prior})
  if $E_{(x,u)}\{Z\} > \frac{1}{2}$ **then**
    $f_{Rd}(\omega|x) = \frac{1}{B^j}$ /* Uniform repayments when high ability to repay */
  else
    $p_\Omega(0|x) = 1 - \frac{E_{(x,u)}\{Z\}}{B^j}$
    $f_{Rd}(\omega|x) = \frac{E_{(x,u)}\{Z\}}{B^j}$ /* Uniform repayments otherwise */
  end if
end for

$x = (NA, NA), u \in U(x)$
$p_\Omega(0|(NA,NA),u) = 1$

---

where **Computeexpectpayment**(x, u, G_Ω^{prior}) is the subroutine which computes the expected repayment $E_{(x,u)}\{Z\}$ according to Definition 4.16.
f_{Rd} should also be discretized according to the partition of repayments ϕ.

4.4 Computational Study

Owing to the confidentiality of real-life data, the approximate dynamic programming approach is adopted. The simulation study examines first the impact of the rationality assumption by comparing any rational profile to its equivalent irrational and random profiles which are derived by relaxing the rationality assumption in two different manners.

From there, the simulation work restricted to the two categories of rational profiles focuses on giving industrial insights and corroborating some of the current practices. The trends of the expected total discounted rewards are discussed and two profitable segments of cardholders are subsequently pointed out. A sensitivity analysis to the variations in the minimum required payment rate mrp and in the annual percentage rate APR is also conducted.

Finally, a posterior discussion of the approximate dynamic programming approach concludes the section. The accuracy of the approximation is assessed via Monte Carlo simulation on the selection of rational and profitable cardholders.

The procedure to solve a single problem is first detailed to show how the different inputs, necessary to conduct the following simulations, are generated.

4.4.1 Generation of the Simulation Inputs

4.4.2 Single problem solving

The following procedure is used to solve a single problem.

1. Define a value model consisting of the values defined by the cardholder's agreement together with the different estimated volumes defined in Tables 3.1, 3.3, 3.2, 3.4 and 3.5 of Chapter 3.

2. In accordance with the value model, generate the following outputs:

   - **R**: 3 dimensional matrix, the components (x, y, u) of which are the consolidated cash flows when the account transits from state x to state y

under control u. The constraint C_{dec} [see Definition 3.10], prohibiting the use of preemptive collection actions, is embodied in $\mathbf{R}$ by assigning dummy operating costs $M_{OC}, M_{OC} \gg 1$ to the related infeasible actions.

- $\mathbf{A}$: Nested matrix. Its components link each state x and collection action u to firstly the related partition $\Phi(x)$ of discrete repayments and secondly for each repayment of the given partition $\Phi(x)$ to the subsequent next state x_{n+1} derived from the resolution of the *ADP* which is detailed in Tables 4.1, 4.2, 4.3.

- $\mathbf{R}_N$: vector of terminal reward. Its components (k) are assessments of the worthiness of an account in state k at the end of the finite horizon i.e. after N billing cycles.

3. Define a cardholder payment profile. Given $\mathbf{A}$ and the chosen profile, generate accordingly the set Υ_{prof} of repayment distributions. Derive $\mathbf{P}$: 3 dimensional matrix, the components (x, y, u) of which are the transition probabilities $p(y|x, u)$

4. Solve the properly defined β-discounted finite horizon *MDP*: $\big(\mathbf{P}, \mathbf{R}, \mathbf{R}_N, \beta\big)$ using *finite-horizon policy evaluation algorithm*

5. Solve otherwise the β-discounted infinite horizon *MDP* $\big(\mathbf{P}, \mathbf{R}, \beta\big)$ by the *policy iteration algorithm.*

Steps 1 and 2 are carried out using the further detailed *Value Model Module.* Steps 3, 4 and 5 form the core of the optimization. They are executed via the *Optimization Module* detailed in 4.4.4.

4.4.3 *Value Model Module*

A Spreadsheet that gathers all the business rules introduced in Chapter 3 was defined under Microsoft Excel^{TM}. The value model is therefore easy to use and provides a clear visualization. The general value characteristics defined by the credit card agreement as well as the different parameters structuring the model can be directly changed onto the spreadsheet. For instance, the minimum required payment rate mrp, the number of admissible states of delinquency as well as the number of segments partitioning the balance are in this way easily modifiable. Different functions and macros were coded in VBA and gathered into an Add-In. The Spreadsheet featuring this Add-In is then directly used to generate the desirable outputs $\mathbf{R}$, $\mathbf{A}$ and $\mathbf{R}_n$. The inputs comprise the value characteristics, the structural parameters of the model and the estimated values of cash advance and retail purchase. In what follows, this main module, for clarity of exposition and understanding, is broken down into two distinct algorithms generating $\mathbf{R}$ and $\mathbf{A}$, respectively. $\mathbf{R}_n$ is directly exported from the spreadsheet data to the *Optimization Module* under a text-file format.

Algorithm 3 Generate consolidated Cash Flows Matrix $\mathbf{R}$

Data: Estimated usage values of the corresponding segment of accounts
Output: Cash Flows $\mathbf{R}$ matrix embodying constraints

for all $x, y \in S, u \in U$ **do**
  if $(x, u) \in \kappa$ **then**
    $\mathbf{R}(x, y, u) \leftarrow$ **Computecashflow**(x, u, y) /* compute consolidated cash flow */
  else
    $\mathbf{R}(x, y, u) \leftarrow -M_{OC}$ /* Infeasible collection strategy u when in state x */
  end if
end for

where **Computecashflow**(x, u, y) is the subroutine which computes the consolidated cash flow associated to a transition from state x to y under the feasible action u in accordance with the value model defined in Chapter 3 [see Equations (3.2) to (3.21)].

Otherwise, setting M_{OC} much greater than the bounded set of feasible reward functions ensures the infeasible collection strategies never to be chosen during either the optimization of the finite horizon *MDP* or the optimization of the infinite horizon discounted *MDP*.

Algorithm 4 Generate $\mathbf{A}$

Data: Estimated usage values of the corresponding segment of accounts, Φ partition of repayments

Output: $\mathbf{A}$ matrix embodying repayments and the corresponding conditional transitions to next state according to the *ADP*

```
for all x ∈ S, u ∈ U do
    Φ(x) ← Computepaymentpartition(x, Ψ_ratio)
    A(x, u) ← Φ(x)   /* A nested matrix linking first the state-action pair (x, u)
    to the related partition of payment Φ(x) */
    for all ω ∈ Φ(x) do
        Compute h(x, u, ω)   /* Solve ADP, Equations (4.6) to (4.12) */
        A(x, u, ω) ← h(x, u, ω)   /* Store next state reached from state x under
        action u when ω is repaid */
    end for
end for
```

where **Computepaymentpartition**(x, Ψ_{ratio}) is the subroutine which computes the partition of payments associated to state x and action u and to the partitioning ratio Ψ_{ration} in accordance with definition 4.3 p76.

4.4.4 *Optimization Module*

This module implemented under Matlab^{TM} can also be broken down into two algorithms. The first submodule transforms the data: $\mathbf{R}$, $\mathbf{A}$ and $\mathbf{R}_n$ generated via the *Value Model Module* into the standard form of discrete *MDP*s. The second submodule optimizes the previously formed *MDP*s for both finite and infinite horizon processes.

Algorithm 5 Prepare *MDP*

Data: Cardholder payment profile distribution Υ_{prof}, $\mathbf{A}$
Output: Transition probabilities $\mathbf{P}$
Initialize $\mathbf{P} = (0)$
for all $x \in S, u \in U$ **do**
  for all $\omega \in \mathbf{A}(x, u)$ **do**
    Compute $p_\Omega(\omega|x, u, \Upsilon_{prof})$ /* Conditional probability, Definition 4.17 to 4.18 */
    $y \leftarrow \mathbf{A}(x, u, \omega)$ /* Given x, u, ω, get next state y */
    $P(x, y, u) \leftarrow P(x, y, u) + p_\Omega(\omega|x, u, \Upsilon_{prof})$ /* Update the probability */
  end for
end for

One should remember here that in the nested matrix $\mathbf{A}$, $\mathbf{A}(x, u)$ is the vector containing the discrete partition of payments associated to the state-action pair (x, u).

The *MDP* is now in a standard form $(\beta, \mathbf{P}, \mathbf{R}, \mathbf{R}_N)$ and ready for optimization which is carried out via the following submodule.

Algorithm 6 Optimize *MDP*

Data: $(\beta, \mathbf{P}, \mathbf{R}, \mathbf{R}_N)$
Output: Optimal policy π^*, π^*_∞ and expected total discounted reward $\boldsymbol{J}_{\pi^*, g, N}$, $\boldsymbol{J}_{\pi^*_\infty, g}$ for the N billing cycles and infinite horizon, resp.
$(\pi^*, \boldsymbol{J}_{\pi^*, g, N}) \leftarrow$ **BackwardInduction**$(\beta, \mathbf{P}, \mathbf{R}, \mathbf{R}_N)$ /* Apply the backward induction algorithm to solve the N period problem */
$(\pi^*_\infty, \boldsymbol{J}_{\pi^*_\infty, g}) \leftarrow$ **PolicyIteration**$(\beta, \mathbf{P}, \mathbf{R})$ /* Apply the policy iteration algorithm to solve the infinite horizon problem */

where **BackwardInduction**$(\beta, \mathbf{P}, \mathbf{R}, \mathbf{R}_N)$ and **PolicyIteration**$(\beta, \mathbf{P}, \mathbf{R})$ are the subroutines which solve the N-period and infinite horizon MDPs by applying the backward induction algorithm and the policy iteration algorithm, respectively. These two algorithms are detailed in Appendix A.

The optimization for the two types of horizons provides short and long term forecasts. A steady collection policy is also worked out when optimizing the infinite horizon problem.

4.4.5 Sample Problem

A case example is now presented. The complete value model can be found in Appendix A. The cardholders' agreement of a major credit card issuer in Singapore was used to set the values of the model defined therein. The main features of the accounts, except for CL arbitrarily set, are in this way directly derived from a real case of the industry in Singapore. They are as follows:

State space $S = I \times J \bigcup \{(NA, NA)\}$				Control space U	
I		J		K	
7		20		8	
Main Characteristics					
CL	APR	mrp	min_{pay}	ca	C_{thr}
$S\$10,000$	24.455%	3%	$S\$50$	24.455%	$S\$1,000$
ca_1	ca_2	min_{cf}	ovr	OL	LF
3%	5%	$S\$15$	10%	$S\$15,00$	$S\$35,00$

The grace period is set to 22 days from the billing date. The decision $u = 8$ corresponds to the intentional and premature writing off of the account by the issuer. The case example is solved with and without such a decision.
The present case example model features a state space comprising overall $(7+1) \times (20+1) + 1 = 169$ states, the last one being the absorbing state $\{(NA, NA)\}$ when the account has been written-off.

4.4.6 Preliminary analysis of the results: bounding the parameters

The problem was first analyzed to point out reasonable values for the different types of profiles. The set of problems was constructed by varying a and b for each profile. Broadly speaking, the changes in a generate a translation along the abscissa axis of the related logit function whereas the changes in b affect its speed of decrease.
The parameters relevant to the simulation work were chosen as follows:

rational *Trimodal* profile							
P_0	a_φ			b_φ			$\nu(x,u)$
	min	*max*	*step*	*min*	*max*	*step*	
1	1.5	4.5	1	-0.01	0	$5e-4$	$min\left\{1+0.2\left(exp\left(\frac{u-7}{7}\right)-1\right),1\right\}$

Table 4.4: Parameters for the category of rational *Trimodal* profiles

The willingness to repay is chosen to increase exponentially with the severity of the collection action u. A complete willingness to repay is reached when $u = 7$[1]. The irrational *Trimodal* profile was constructed using the same parameters for P_0, a_φ, b_φ. The willingness to repay was set to a constant equal to the minimum willingness to repay in the rational *Trimodal* profile i.e. when $u = 0$. For such a profile, it is obvious that the rationality property does not hold.

rational *Unimodal* profile								
$x =$	P_0	a_G			b_G^0			$b_G(x,u)$
(i,j)		*min*	*max*	*step*	*min*	*max*	*step*	
$i < 3$	1	1.5	4.5	1	-0.01	0	$5e-4$	$min\left\{-b_G^0+0.0005\frac{u+1}{8},0\right\}$
$i \geq 3$	0.9	1.5	4.5	1	-0.01	0	$5e-4$	$min\left\{-b_G^0+0.001\frac{u+1}{8},0\right\}$

Table 4.5: Parameters for the category of rational *Unimodal* profile

[1]Recall here that $u = 8$ corresponds to the premature write-off of a delinquent account.

In the *Unimodal* profile, the sensitivity to the collection action is illustrated by choosing the coefficient b_G of the prior distribution of repayments $G_\Omega^{prior}(\omega|x,u)$ [see 4.14] to decrease linearly with respect to the undertaken collection strategy. The increase is supposed to be more substantial when the cardholder is severely delinquent i.e. $i \geq 3$. The rationale for such a choice is that a cardholder in arrears of at least two months would be more sensitive to the collection strategies so as not to fall further behind.
The irrational *Unimodal* profile was constructed using the same parameters for P_0, a_G, b_G^0 and $b_G(x,u)$ was set to be equal to $b_G(x,0)$ as defined for the rational *Unimodal*. Such a $b_G(x,u)$ is independent of u and the corresponding profile is obviously irrational.

4.4.7 Scenarios

Here, different scenarios are simulated. Each of them consists of a particular repayment profile associated to the value model of the corresponding segment of cardholders. For instance, such a scenario would be high risk cardholders having low monthly purchase and cash volumes.
Any pair of a value model and a repayment profile defines uniquely an *MDP* that is later solved for a finite horizon of 12 billing cycles and for the infinite horizon. Twelve value models were defined with increasing monthly purchase and cash volumes ranging from $S\$0.3K$ to $S\$5K$ and from $S\$0K$ to $S\$1.5K$, respectively. For each of the six category of profiles, 84 repayment profiles were generated by tuning the ability to repay and/or the sensitivity to the collection actions. Therefore a total of $6,048$ *MDP*s were generated and solved according to the simulation flowchart of Figure 4.1.

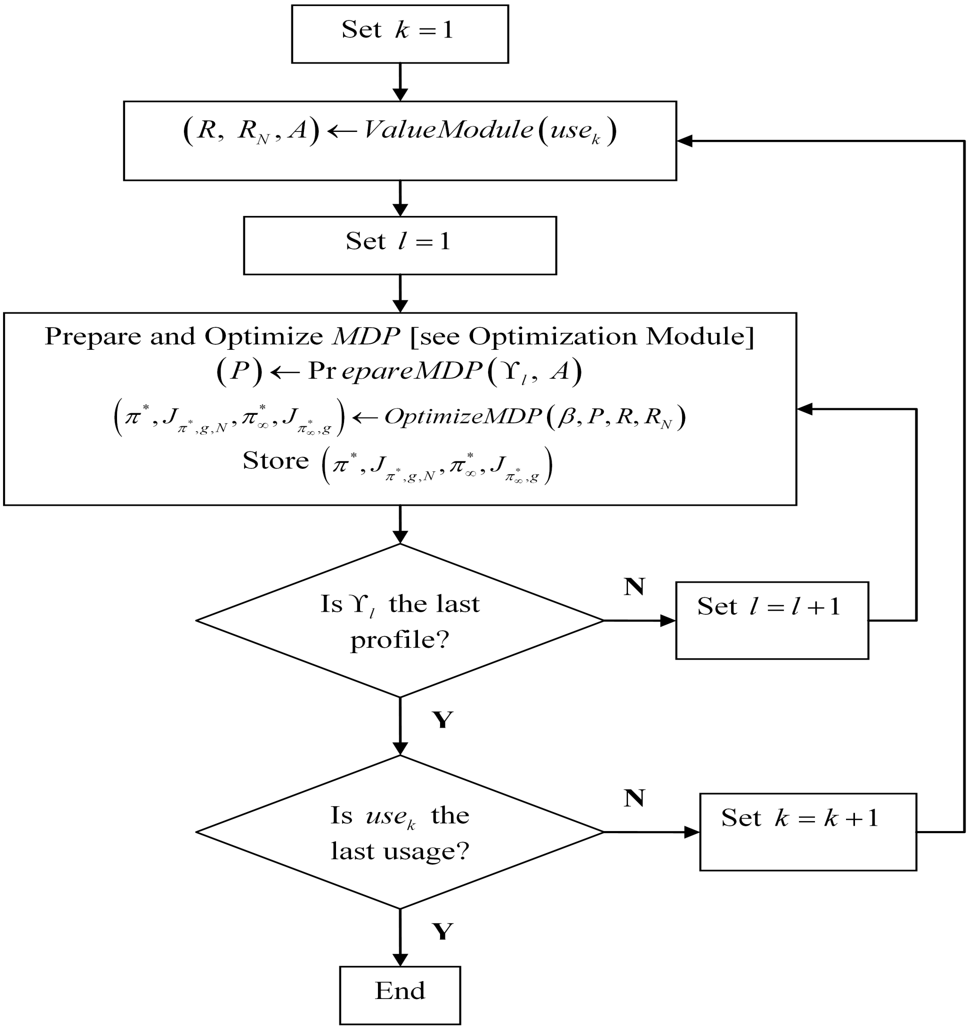

Figure 4.1: Flowchart for the simulation of a set of scenarios (use, Υ)

4.4.8 Relaxation of the Rationality Assumption

The numerical impact of the rationality assumptions is specified in the present section. To that effect, the rationality conjecture is relaxed for both the *Unimodal* and *Trimodal* profiles in two different ways via their comparisons with the irrational *Unimodal* and *Trimodal* [see 4.3.4] as well as with the two corresponding *Random* profiles [see 4.3.5].

Pairwise comparisons are made between the different profiles according to the following chart:

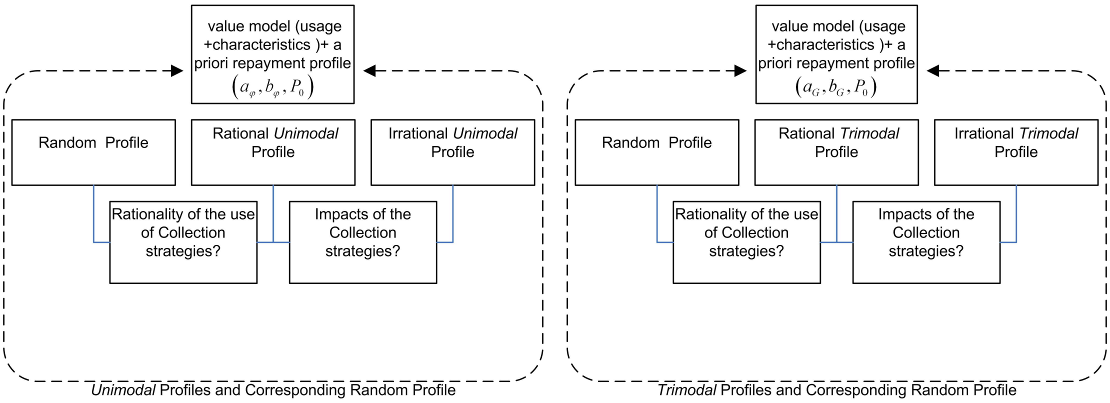

Figure 4.2: Comparison chart for the rationality conjecture

The comparisons are made between cardholders sharing the same value model and the same prior ability to repay. The sensitivities to the collections strategies and the impacts of the ways cardholders distribute their repayments are therefore measured.

The comparisons between any rational profile and its corresponding random profile aim to re-affirm the rationality in imposing repayment obligations. Likewise, the comparisons between a rational profile and a corresponding irrational profile aim to re-affirm and measure the efficiencies of the collection strategies that are used by the card issuer.

Given a scenario i.e. given a_φ (a_G), b_φ (b_G) and given a particular usage, the signed relative difference between the expected total discounted reward of each profile is calculated in each state (i, j). The arithmetic mean of the previous relative differences is then computed over the whole state space and used as the matching criterion. The comparisons between the rational *Unimodal* profile and irrational *Unimodal* as well as the comparisons between the rational *Unimodal* profile and its corresponding random profile are presented hereafter in the infinite horizon case.

4.4.8.1 Comparisons of Rational/Random profiles: Relaxing the Rational Distribution of Repayments

The comparisons between any rational profile and its corresponding random profile are first presented. Figure 4.3 plots the relative differences in the arithmetic average, over the entire state space, of the maximal expected total discounted reward between the rational profiles and their associated random profiles when the mean monthly purchase and cash volumes are $S\$1.5K$ and $S\$0.5K$, respectively.

Consider a pair which comprises a rational profile (either rational *Unimodal* or rational *Trimodal*) governed by the payment distributions Υ_{rt} (either Υ_{Uni} or Υ_{Tri}) and its associated random profile governed by payment distributions Υ_{Rd}. Any such pair, in accordance to the previous definitions and to Figure 4.2, is uniquely defined by the pair (a_G, b_G) for the rational *Unimodal*/*Random* pair and by (a_φ, b_φ) for the rational *Trimodal* /*Random* pair.

Definition 4.20. *Denote by* $J_{\pi^*_\infty,g}(use_k, \Upsilon)$ *the arithmetic average, over the entire state space, of the maximal expected total discounted reward for the infinite horizon problem when the monthly usage is defined by the* k^{th} *value model and the cardholder profile is governed by the payment distributions* Υ*.* $J_{\pi^*_\infty,g}(use_k, \Upsilon)$ *is calculated as follows:*

$$J_{\pi^*_\infty,g}(use_k, \Upsilon) = \frac{1}{\#(S)} \sum_{x \in S} J_{\pi^*_\infty,g}(x),$$

where $J_{\pi^*_\infty,g}(x)$ *is calculated with a monthly use equal to* use_k*.*

Definition 4.21. *The relative difference* Δ_{rt-Rd} *in* $J_{\pi^*_\infty,g}(use_k, \Upsilon)$ *between the rational profile governed by* Υ_{rt} *and its associated random profile governed by* Υ_{Rd} *is defined by:*

$$\Delta_{rt-Rd} = \frac{J_{\pi^*_\infty,g}(use_k, \Upsilon_{rt}) - J_{\pi^*_\infty,g}(use_k, \Upsilon_{Rd})}{min\{|J_{\pi^*_\infty,g}(use_k, \Upsilon_{rt})|, |J_{\pi^*_\infty,g}(use_k, \Upsilon_{Rd})|\}}$$

Figure 4.3 plots Δ_{rt-Rd} for the pairs of *Unimodal*/Random profiles and for the pairs of *Trimodal*/Random profiles within the selected range of values for the pairs

(a_G, b_G) and (a_φ, b_φ), respectively. The mean monthly purchase and cash volumes are set to $S\$1.5K$ and $S\$0.5K$, respectively.

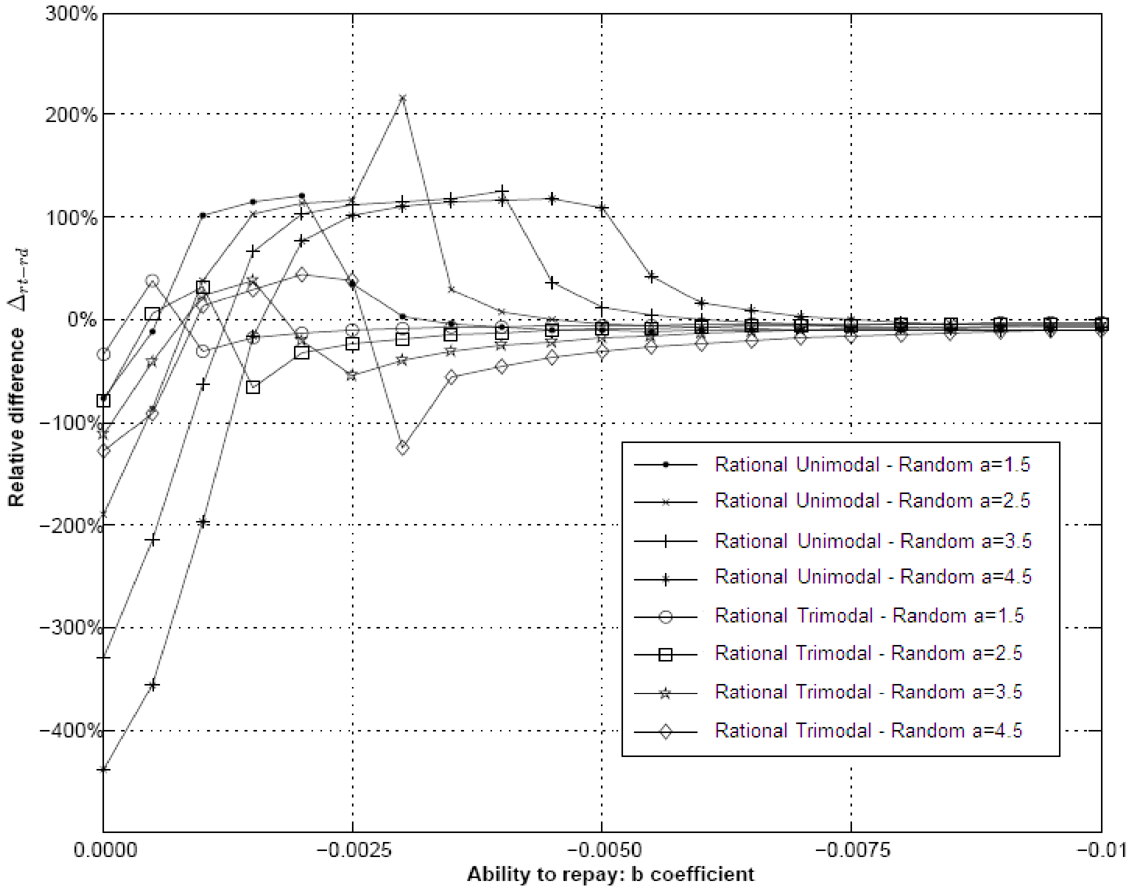

Figure 4.3: Relative difference in expected total discounted between the rational and random profiles for mean monthly purchase of $S\$1.5K$ and mean monthly cash advances of $S\$0.5K$

The analysis of the computed data provided the following conclusions. There is no absolute domination of the rational profile over the random profile when the ability to repay varies. However given any usage and any $a_\varphi(a_G)$, the following common pattern of domination is observed when $b_\varphi(b_G)$ decreases from 0 to its minimal value -0.01:

1. a first domination of the random profile over the rational profile when the

ability to repay is very high

2. a substantial domination of the rational profile over the random profile when the ability to repay is in the neighborhood of the maximum expected total reward of the rational profile

3. a relative difference converging to zero when the ability to repay worsens further

The first domination is relevant to the irrationality of a cardholder, who has a sufficient ability to repay his debt in full. According to the present definition of a random profile, such a cardholder distributes identically his payment within the balance range without prioritizing the repayment of the whole indebtedness. He is therefore more likely to revolve balance over the next billing cycle than a rational cardholder who has the same ability to repay. The surplus of balance interest generated thus explains the first point.

The second point corresponds to the most realistic situations where cardholders have a sufficient ability to support their card usage without necessarily settling their debts in full at the end of each billing cycle. They are expected to generate substantial revenues. In these situations, the rationality of the cardholders conditioned by the industry practices results in fewer charge-offs and higher revenues. When the ability of the cardholder worsens further, the rationality of the cardholder allows in the first place the limitation of the bad losses compared and therefore reaffirms the necessity of collection actions.

The third point corresponds to "bad" cardholders with high defaulting rates. The subsequent bad debt losses are not expected to be compensated by the revenues in both the random and rational profiles. The relative difference tends to zero when the ability worsens further. In any of the two cases, such accounts will rapidly be charged-off and the expected total reward therefore does not differ much.

To conclude one should notice that, despite the rather restrictive assumptions defining the rationality, there is no absolute domination. This is actually explainable by the very structure of the problem. The rationality in the repayments solely govern the way a cardholder will distribute his repayments. It therefore only impacts on the transition probabilities whereas the related reward functions do not feature any monotonic properties. For instance, a cardholder who is occasionally a late payer will repay late fees and generate extra profit that a cardholder with an account current and the same outstanding balance will not pay.

4.4.8.2 Comparisons of Rational/Irrational profiles: Relaxing the Sensitivity to the Collection Strategies

The comparisons between any rational profile and its corresponding irrational profile are now presented. Similar to 4.4.8.1, Figure 4.4 plots the relative differences in $J_{\pi^*_\infty,g}(use_k,\Upsilon)$ [see Definition 4.20] between the rational profiles and their associated irrational profiles when the mean monthly purchase and cash volumes are $S\$1.5K$ and $S\$0.5K$, respectively. Consider a pair which comprises a rational profile (either *Unimodal* or *Trimodal*) governed by the payment distributions Υ_{rt} (either Υ_{Uni} or Υ_{Tri}) and its associated irrational profile governed by the payment distribution Υ_{irt}. Any such pair, in accordance to the previous definitions and to Figure 4.2, is uniquely defined by the pair (a_G, b_G) for the rational *Unimodal*/irrational *Unimodal* pair and by (a_φ, b_φ) for the rational *Trimodal* - irrational *Unimodal* pair.

Definition 4.22. *The relative difference Δ_{rt-irt} between the reward functions of the rational profile governed by Υ_{rt} and its associated irrational profile governed by Υ_{irt}*

is defined by:

$$\Delta_{rt-irt} = \frac{J_{\pi^*_\infty,g}(use_k, \Upsilon_{rt}) - J_{\pi^*_\infty,g}(use_k, \Upsilon_{irt})}{min\{|J_{\pi^*_\infty,g}(use_k, \Upsilon_{rt})|, |J_{\pi^*_\infty,g}(use_k, \Upsilon_{irt})|\}}$$

Figure 4.4 plots Δ_{rt-irt} for the pairs of rational *Unimodal*- irrational *Unimodal* profiles and for the pairs of rational *Trimodal*/irrational *Trimodal* profiles within the selected range of values for the pairs (a_G, b_G) and (a_φ, b_φ), respectively. The mean monthly purchase and cash volumes are set to $S\$1.5K$ and $S\$0.5K$, respectively.

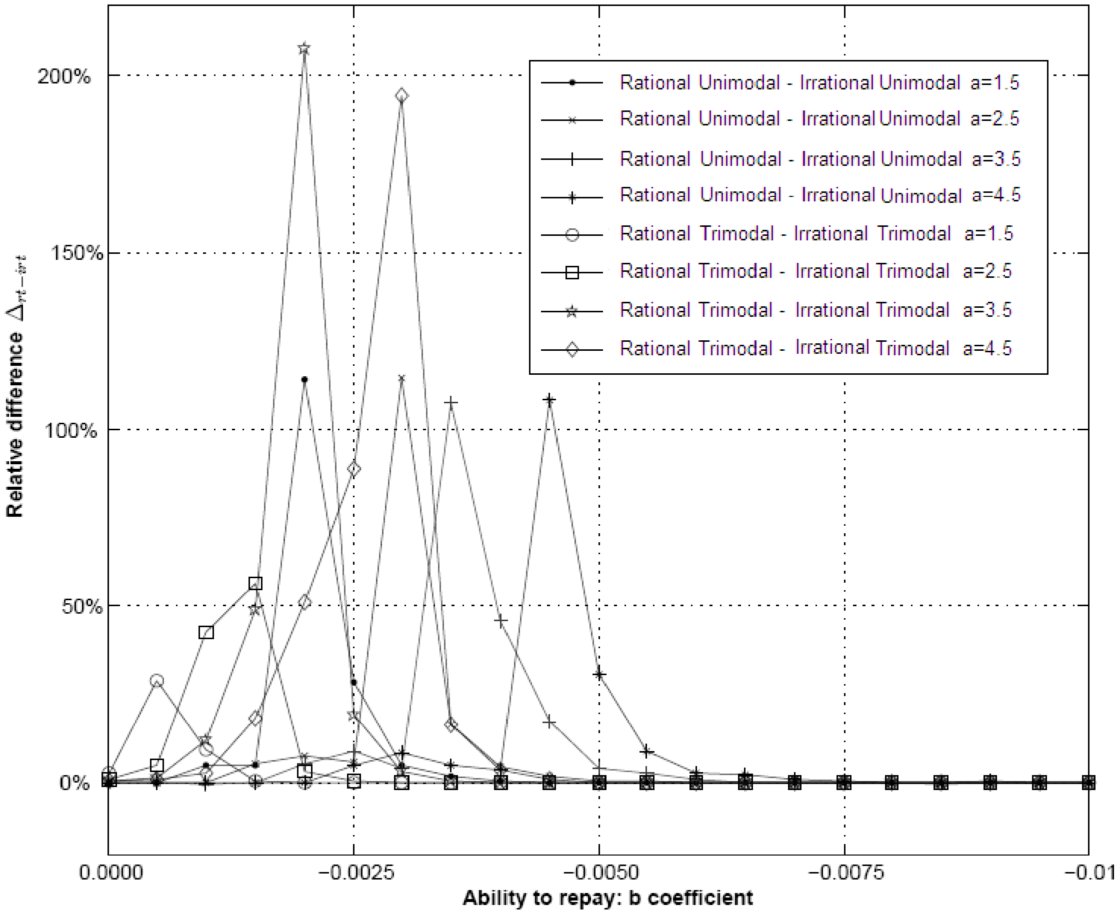

Figure 4.4: Relative difference in the reward functions between the rational and irrational profiles for mean monthly purchase of $S\$1.5K$ and mean monthly cash advances of $S\$0.5K$

The analysis of the computed data brought to light the following conclusions. There is an absolute domination of the rational profile over the random profile when the ability to repay varies. The conjectured sensitivity to collection actions showed that substantial improvements in collection can be achieved with a sound collection policy. For instance, the relative difference can easily be greater than 50% [see *Trimodal* profile $a_\varphi = 2.5,\ 3.5,\ 4.5$] when it is simply assumed that the willingness to repay varies exponentially from 0.8 to 1 within the set of possible collection actions.

The domination of rational/irrational reaches its maximum in the neighborhood of the maximum of the expected total reward $\boldsymbol{J}_{\pi^*,\,g,\,N}$ ($\boldsymbol{J}_{\pi^*_\infty,\,g}$) associated to the rational profile. Similar to the comparisons of Rational/Random, the rationality continues to ensure a substantial domination when the ability to repay worsens a bit beyond the reward. The domination over this range is particularly relevant since it puts in evidence the sensitivity of the delinquent cardholders to the collection actions. It re-affirms therefore that an issuer can better protect itself against reasonably risky cardholders by an appropriate use of collection strategies.

4.4.9 Trends of the expected total discounted rewards: $\boldsymbol{J}_{\pi^*,\,g,\,N}$ and $\boldsymbol{J}_{\pi^*_\infty,\,g}$

The trends of $\boldsymbol{J}_{\pi^*,\,g,\,N}$ and $\boldsymbol{J}_{\pi^*_\infty,\,g}$ were investigated over the ranges of a_φ, (a_G) and b_φ, (b_G) for the rational *Trimodal* (*Unimodal*) profile. Given a fixed a_φ (a_G), $\boldsymbol{J}_{\pi^*,\,g,\,N}$ and $\boldsymbol{J}_{\pi^*_\infty,\,g}$ were found to vary in the same way when the repayment ability of the cardholder deteriorates, namely when b_φ (b_G) decreases from 0 to its minimum value. The values defining the changes of trends were found to be the same for $\boldsymbol{J}_{\pi^*,\,g,\,N}$ and $\boldsymbol{J}_{\pi^*_\infty,\,g}$.

Figure 4.5 depicts the variations of the expected total discounted reward over 12 billing cycles when the mean monthly purchase and cash volumes are $S\$1.5K$ and $S\$0.5K$, respectively.

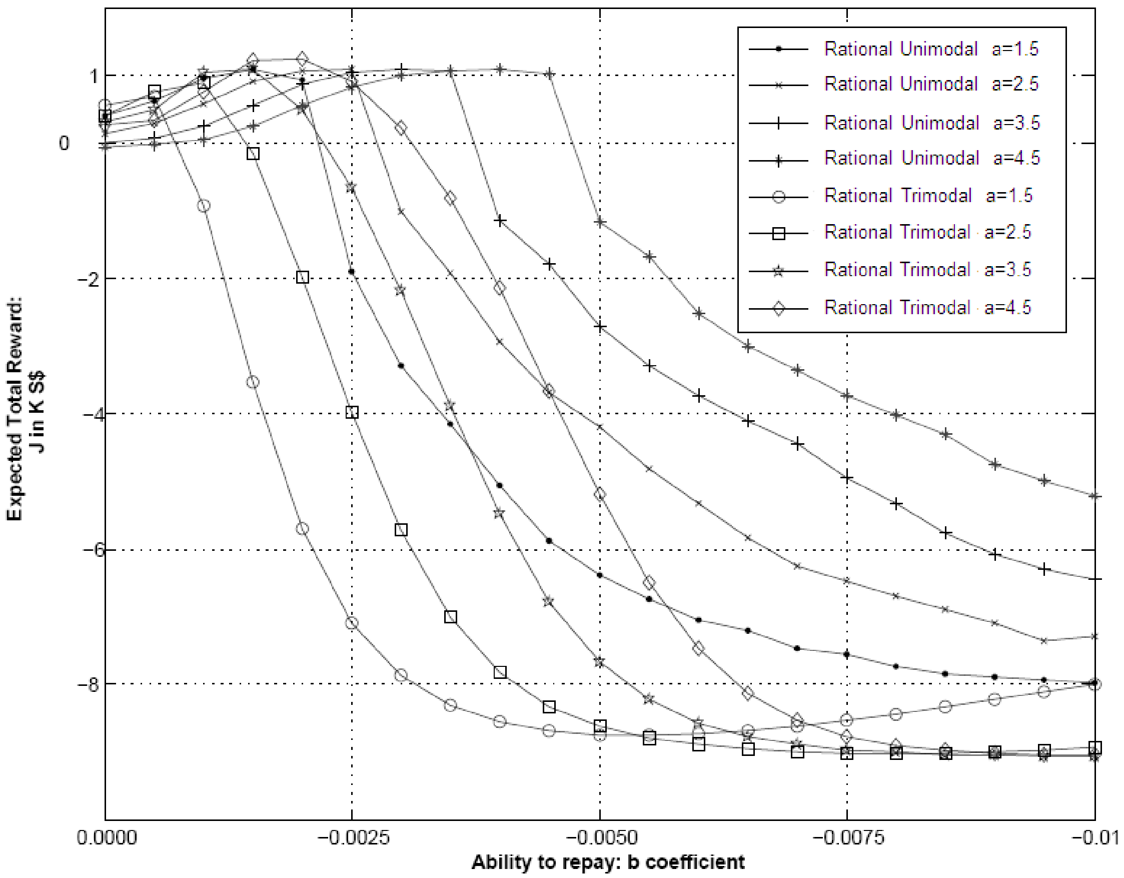

Figure 4.5: $\boldsymbol{J}_{\pi^*,g,12}$ for mean monthly purchase of $S\$1.5K$ and mean monthly cash advances of $S\$0.5K$

More specifically, the following common variations of the maximal expected total discounted reward are observed:

1. an increasing trend until a maximum revenue is reached
2. a substantial decreasing trend turning rapidly the expected total discounted reward into a loss for both the infinite and finite horizon problems.
3. a stabilization to a steady negative value. This value can be strictly superior to $-CL$ when the early write-off decision $u = 8$ was included in the set of controls.

These observations are, in what follows, analyzed in the light of common concepts of the industry.

Firstly one should notice that the cardholders, who have the highest ability to repay (i.e. b_φ (b_G) in the neighborhood of 0), are not the most profitable to the issuer. Such cardholders, known in the industry as "transactors", are most likely to repay in full their balances by the due date of each billing cycle. Interest charges (except for cash advances) are waived and the transactors fully benefit from the use of their credit cards as a form of revolving credit. The transactors still generate profit via the interchange revenue and the cash advances fees. They are the least risky cardholders and represent therefore a source of "secured" revenue that is yet limited by the fact that they do not revolve balance from one billing cycle to another. The issuer may attempt to have a transactor increase his usage and eventually revolve credit. Lowering the APR, sending usage incentives and promotional offers or simply raising the credit limit are means to achieve the subsequent increase. These means should yet be used with care and consideration of the additional risk of charge-off involved.

Secondly, an increasing trend of the expected total discounted reward is observed. This increasing trend until a maximum is reached, clearly brings to light a profitable segment of cardholders. Such a segment comprises those whose abilities to repay fall within the range of the increase. It is actually a subset of the set of cardholders known in the industry as "revolvers". Revolvers typically do not pay in full the outstanding balance and therefore pay interests. The subset defined by the increasing trend corresponds to low risk revolvers who are thus the most profitable holders. The issuer should naturally be willing to increase the number of such low-risk revolvers in its portfolio. This is part of the daunting challenge the credit

scoring has to take up by selecting the appropriate applicants. The selected applicants would ideally be granted adapted credit limits that would make them revolve balance with an acceptable risk of default to the issuer.
It is interesting to notice that for the *Unimodal* profile, the increasing trend is then followed by a plateau. The existence of such a plateau is directly related to the conjecture of a *Unimodal* profile for no such steady trend was observed for the *Trimodal* profile. Within the range of this plateau, revolvers who have sufficient but yet different abilities to repay, are generating approximately the same revenue. This steady revenue stems from the rationality of the cardholder and his subsequent prioritization of the minimum repayment. Such cardholders can be considered as “good” revolvers who carry balance from one cycle to another and repay enough not to fall arrear into payments. The trade-off between their expected risk of charge-off and the profit they generate is exceedingly profitable to the issuer.

Thirdly, a clear threshold emerges in the range of variations of b_φ and b_G for the *Trimodal* and *Unimodal* profiles, respectively. The former corresponds to the maximum expected total discounted reward whereas the latter is the right limit of the aforementioned plateau. In both cases, the decrease that follows the threshold is substantial. These limiting points actually define a maximal risk the issuer is willing to take when lending money to the cardholders. Beyond such thresholds, cardholders with worse abilities to repay will not generate sufficient revenue to compensate the bad debt losses due to the charged-off accounts. The observation of such risk limits reaffirms the need of an accurate application scoring and a proper allowance of credit limits.

4.4.10 Sensitivity Analysis

Following the analysis developed from 4.4.6 to 4.4.8, the rational profiles generating positive expected total discounted rewards are considered as being the most realistic ones. Indeed they are rational cardholders "good" enough to be expected to generate overall profit. They should therefore be the cardholders populating the portfolio of the issuer.

The sensitivity of the expected total reward to changes in the minimum required payment rate mrp as well as in the annual percentage rate APR are investigated in the following two subsections.

4.4.10.1 Sensitivity to the minimum required payment rate mrp

One should recall mrp governs the cardholder's obligation of repayment according to his outstanding balance. It has to be a trade-off between the flexibility of payment granted to the cardholders and a minimum obligation of repayment protecting the debtor against default and excessive bad debt losses. The study of its sensitivity can therefore provide useful forecasts and indications as to how to modify the mrp. An issuer may for instance lower its mrp so as to attract new customers interested in a less restrictive repayment scheme. Conversely, one may assume that an increase in the mrp would have the cardholders repay higher amounts and therefore reduce the amount of charged-off receivables.

For each of the twelve previously defined value models, which feature increasing monthly usages, the relevant $\mathbf{A}$ and $\mathbf{R}$ [see 4.4.3] were computed for $mrp = 2.0\%$, 2.5%, 3.5%, 4%, 4.5% (the previously used mrp being 3.0%). Therefore 60 new value models were generated. Each of them was then associated with each of the 84 rational *Unimodal* profiles and each of the 84 rational *Trimodal* profiles. A total of $10,080$ *MDP*s were thus solved so as to assess the sensitivity to mrp.

The simulation flowchart is therefore the same as in Figure 4.1 except for an additional higher loop on the different values of mrp is being performed.

In Table 4.6, the results are presented for 2 different usages:

1. use_1: mean monthly purchase of $S\$1.5K$ and mean monthly cash advances of $S\$0.5K$
2. use_2: mean monthly purchase of $S\$3.0K$ and mean monthly cash advances of $S\$1.0K$

The following table provides, for each value of the mrp, the mean value of the signed relative differences between the corresponding expected total discounted reward and the reference expected total discounted reward when $mrp = 3\%$. The rewards considered here are derived from the infinite horizon problem. For each of the 2 usages, two sets of profiles are defined by selecting among the 84 rational *Trimodal* and among the 84 rational *Unimodal*, those which generate positive expected total discounted reward when the mrp is set to its base value of 3%. The mean value is then calculated for each of the two sets of profiles.

Definition 4.23. *For* $k = 1, 2$ *denote by* $J_{\pi^*_\infty, g}(use_k, \Upsilon, mrp)$ *the maximal expected total discounted reward for the infinite horizon problem when the minimum required payment rate is set to mrp, the cardholder profile is* Υ *and the monthly usage is* use_k*. Let* $\Gamma^{Uni}_{use_k}$ $\left(\Gamma^{Uni}_{use_k}\right)$ *be the set of payment distributions governing the rational* Unimodal (*rational* Trimodal) *profiles which generate positive expected total discounted reward when mrp is set to* 3% *and the monthly usage is* use_k*:*

$$\Gamma^{Uni}_{use_k} = \left\{ \Upsilon_{Uni} | J_{\pi^*_\infty, g}(use_k, \Upsilon_{Uni}, 3\%) > 0 \right\}, \quad k = 1, 2$$

$$\Gamma^{Tri}_{use_k} = \left\{ \Upsilon_{Tri} | J_{\pi^*_\infty, g}(use_k, \Upsilon_{Tri}, 3\%) > 0 \right\}, \quad k = 1, 2$$

Definition 4.24. *The mean value* Δ_{mrp} *of the signed relative differences in* $J_{\pi^*_\infty, g}(use_k, \Upsilon, mrp)$ *between the two problems, whose minimum required payment* *rates are respectively set to mrp and to 3% is defined by:*

$$\Delta_{mrp} = \frac{1}{\#(\Gamma)} \sum_{\Upsilon \in \Gamma} \frac{J_{\pi^*_\infty,g}(use_k, \Upsilon, mrp) - J_{\pi^*_\infty,g}(use_k, \Upsilon, 3\%)}{J_{\pi^*_\infty,g}(use_k, \Upsilon, 3\%)}, \ \Gamma = \Gamma^{Uni}_{use_k}, \Gamma^{Tri}_{use_k}, \ k = 1, 2$$

$\Delta_{mrp}\%$		rational *Trimodal* use 1	rational *Trimodal* use 2	rational *Unimodal* use 1	rational *Unimodal* use 2
	2.00 %	101.96 %	49.97 %	0.20 %	0.91 %
	2.50 %	50.96 %	26.05 %	0.57 %	0.76 %
mrp	3.50 %	-38.92 %	-21.38 %	-10.76 %	-12.68 %
	4.00 %	-68.53 %	-38.67 %	-14.04 %	-19.09 %
	4.50 %	-90.61 %	-52.31 %	-16.98 %	-30.32 %

Table 4.6: Signed Relative Differences between $\boldsymbol{J}_{\pi^*_\infty,g}$ when $mrp = 2.00\%, \ldots, 4.50\%$ and the the reference reward when $mrp = 3\%$

The selection of profitable profiles is consistent with the prior process of selection of "good" applicants. The profiles considered in Table 4.6 are those of "good" cardholders.

For the *Trimodal* profiles, the lower the mrp the higher the profit. The substantial differences are due to the schematic nature of the profile. Whenever a repayment is made, the corresponding cardholder either repays the full balance or repays the minimum if he does not have sufficient ability to pay the balance in full. Such a cardholder is therefore a revolver who brings forward substantial balances and therefore generates high revenue. A decrease in the mrp leads these low risk cardholders to revolve bigger balances and hence, pay more interest. The differences of usages and particularly of cash advances, which generate high revenue independent of the mrp, explain the decrease in the relative differences between usages 1 and 2.

As for the rational *Unimodal* profile, a small domination of the revenue is observed when $mrp = 2.00\%, 2.50\%$. The domination of the total reward when $mrp = 3.00\%$ compared to when $mrp = 3.50\%, 4.00\%, 4.50\%$ is noticeable. For the selection of profitable rational *Unimodal* profiles, $mrp = 3.00\%$, as fixed by the cardholders' agreement , appears to be a good trade-off between the protection against default and the interest revenue generated by revolvers. This conclusion is naturally restricted to the present assumptions governing the rational repayment behaviors of the cardholders. Within this framework however, the industrial practice is confirmed.

The relative differences were also analyzed for "bad" cardholders' profiles that do not belong to the previous two sets. When the ability to repay worsens and the cardholders are expected to be unprofitable and to be most likely to default, it was observed that the highest $mrp = 4.5\%$ ultimately dominates the other mrp values by forcing the "bad" cardholders to make higher repayments before they default. The basic intuition is then confirmed. Yet for a portfolio populated by "good" cardholders, a higher mrp is not more profitable nor protective overall.

4.4.10.2 Sensitivity to the Annual Percentage Rate APR

The APR value governs the interest revenue. On the one hand, a higher APR will generate higher revenues through cardholders' revolving balances. On the other hand, such an APR may deter potential cardholders to open an account.

For the twelve previously defined value models, which feature increasing monthly usages, the relevant $\mathbf{A}$ and $\mathbf{R}$ [see 4.4.3] were computed for APR ranging from 10.455%, to 30.455% by steps of 2% (the reference APR being 24.455%). Therefore 120 new value models were generated. Each of them was then associated with each of the 84 rational *Unimodal* profiles and each of the 84 rational *Trimodal* profiles. A total of $20,160$ MDPs were therefore solved so as to assess the sensitivity to APR. The simulation flowchart is therefore the same as in Figure 4.1 except for an additional higher loop on the different values of APR is being performed.

The analysis is also restricted to the selection of profitable profiles. It is observed that for each such profile, the sensitivity to the APR follows an increasing linear trend.

For the non selected profiles, the total expected reward, though negative, is still an increasing function of the APR until a certain threshold in the ability to repay. Beyond this threshold, cardholders with a worse ability to repay generate very negative total expected rewards which are decreasing functions of the APR. These cardholders are most likely to default in the short term with a substantial outstanding balance. The worst cardholders barely repay anything and are charged with increasing revolving interests since, in accordance with the sensitivity analysis, the APR increases. The decrease in the expected total reward thus accounts for the increasing shortfall in interest revenue collection.

As a conclusion, one can see that the APR may prove to be a good means to secure a portfolio of reasonably “good” cardholders by increasing the interest revenue line and the subsequent overall revenue. Many issuers make use of this solution by offering a low introductory APR that is usually increased after 6 months. Likewise, delinquent cardholders may see their APR substantially increased. The setting of the APR should be soundly decided since it can be used at the same time as a marketing point to attract new customers.

4.5 Discussion of the Approximation

The approximation, presented in 4.2.1 and concerned with the computations of the repayments, is further discussed in this section. Other than the numerical argument of a small mrp and a card blocked after 30 days past due, a comparative simulation study would be detailed so as to justify the previous calculations. The simulated trajectories of an account, subject to either its exact payment requirements or its approximate ones, will therefore be matched.

4.5.1 Simulated Trajectories

A chosen repayment profile, an initial state and a chosen policy are necessary inputs to simulate trajectories of an account over a finite horizon. They define a basic input problem in this section.
Given this basic input problem, a properly defined *MDP* $\left(\mathbf{P}, \mathbf{R}, \mathbf{R}_N, \beta\right)$ is derived. The latter is solved over a finite horizon of monthly billing cycles and the resulting optimal policy π^* is used as the policy undertaken when running the simulations.

4.5.1.1 Exact Trajectories

While describing any trajectory, an account will be assigned, at each new billing cycle, a minimum required repayment that directly depends on its opening balance. In order to compute exactly the repayments, a vector $\mathbf{RP}_{vect}$, used to store the overdue payments, is updated along the process according to the repayments made. A conditional repayment is then generated according to the payment due, the balance outstanding and the due state of the account. Subsequently, the next state is derived and the vector $\mathbf{RP}_{vect}$ of overdue payment updated. The algorithm to generate a sample exact trajectory is detailed as follows:

Algorithm 7 Generate exact sample path $\mathbf{SP}_{exact} = generate_{exact}(x_0, T, \pi, \Upsilon)$

Data: initial state x_0, policy $\pi = \{\mu_0, \ldots, \mu_{T-1}\}$, repayment profile Υ, time horizon T

Output: sample path vector **traj** containing the generated exact sample paths $(x_0, \ldots, x_T)$

for $t = 0$ *to* T **do**
  if $x_t = (0, \cdot)$ **then**
    $\mathbf{RP}_{vect}(t) \leftarrow rp(x_t, -1)$ /* Compute the minimum repayment when the grace period applies */
  else
    $\mathbf{RP}_{vect}(t) \leftarrow rp(x_t, 0)$ /* Compute the exact minimum repayment incurred by the balance outstanding as of t */
  end if
  Generate $\Omega_t(x_t, \mu_t(x_t), \Upsilon)$ /* Conditional random payment */
  Update $\mathbf{RP}_{vect}(\mathbf{RP}_{vect}, \Omega_t)$ /* Update the debt according to Ω_t and overdue repayments as of t */
  Compute $x_{t+1}(x_t, \mu_t(x_t), \Omega_t, \mathbf{RP}_{vect})$ /* Move to next state */
  if $x_{t+1} = (NA, NA)$ **then**
    $x_{t+2}, \ldots, x_T \leftarrow (NA, NA)$
    break
  end if
end for
traj $\leftarrow (x_0, \ldots, x_T)$ /* generated exact sample Path */

For any set of basic input problems, the Monte Carlo simulations for the exact trajectories are then conducted according to the flowchart 4.6.

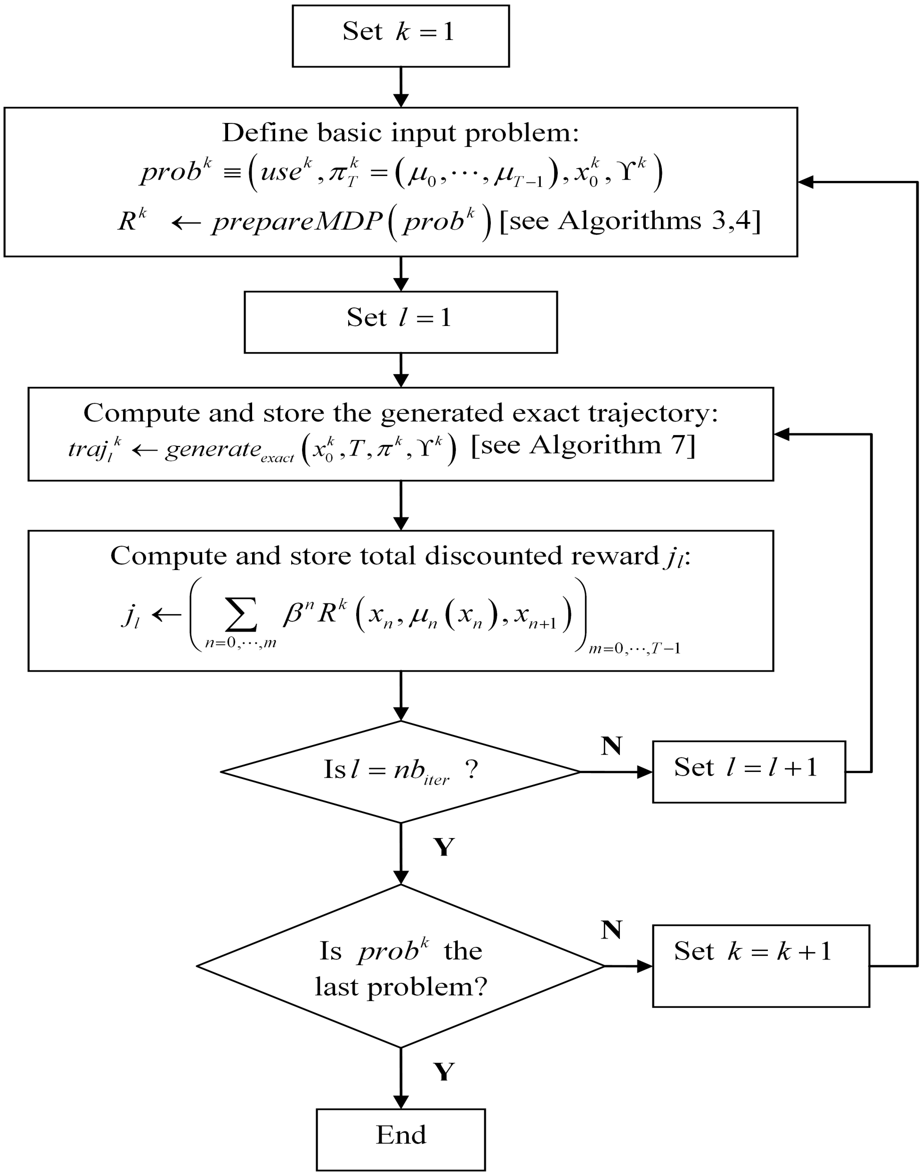

Figure 4.6: Flow Chart for the Monte Carlo simulation of the exact trajectories

4.5.1.2 Approximate Trajectories

For each basic input problem, the simulated approximate trajectories derive simply from the simulation of a Markov chain, the transition probabilities of which are defined by the undertaken collection policy $\pi = \{\mu_0, \ldots, \mu_{T-1}\}$. For each stage t, the one step transition probabilities matrix $\mathbf{P}_{\mu_t}$ is derived as $\mathbf{P}_{\mu_t} = \Big(p\big(x_{t+1}|x_t, \mu_t(x_t)\big)\Big)$.

The next state for any given x_t is then simply obtained by a random number generation and comparison with the distribution defined by the x_t^{th} row vector of $\mathbf{P}_{\mu_t}$. This generation will be referred to as $generate_{approx}\left(x_{_t}, P^k, \mu_{_t}^k\right)$ in the sequel.

The Markov chain of interest was derived from the *ADP* according to Step 2 of the simulation process i.e. "prepareMDP" [see Figure 4.7]. It thus features the transition probabilities that embody the approximation of the required repayments [see 4.2.1]. Starting from an initial state x_0, the account is moved from state to state using simple random number generation. At each stage t, the generated random number provides, according to the transition probabilities $p(y|x_t, \mu_t(x_t))$, the next state x_{t+1}. For any set of basic input problems, the Monte Carlo simulations for the approximate trajectories are conducted according to the flowchart in Figure 4.7.

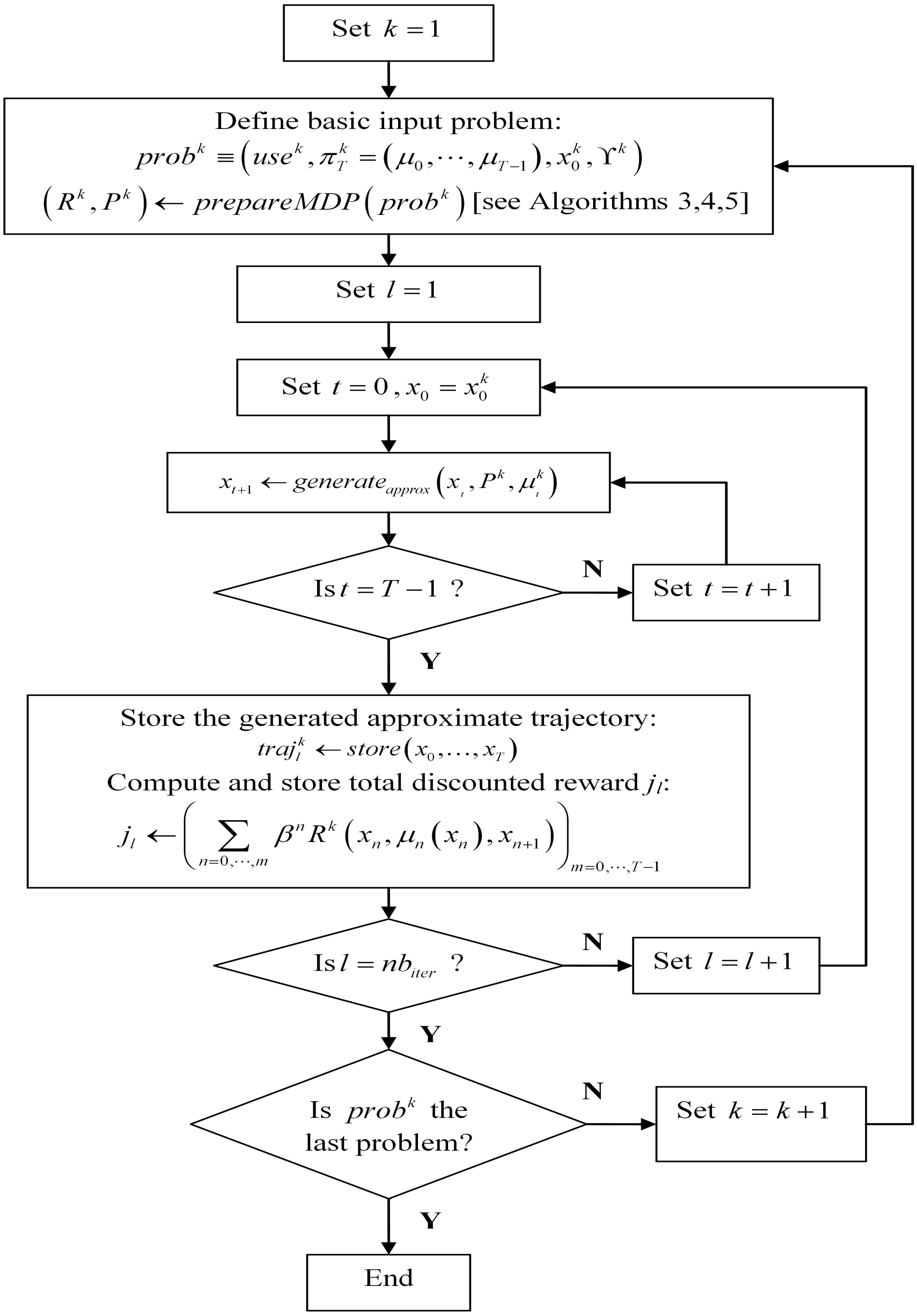

Figure 4.7: Flow Chart for the Monte Carlo simulation of the approximate trajectories

4.5.1.3 Results

The simulations were run for 100,000 sample paths i.e. $l = 1, \ldots, 100,000$ [see Figures 4.6 and 4.7]. The initial states are set such that $x_0 = (0, \cdot)$, which would otherwise require the definition of an initial non-null vector of required payments $\mathbf{RP}_{vect}(t)$. The rationale is to consider that the other feasible due states will be visited during the process if the number of iterations is sufficient.
Two sets of 30 problems were defined i.e. $k = 1, \ldots, 30$ [see Figures 4.6 and 4.7]. One set embodies the rational *Unimodal* profile only, likewise with the other set and the rational *Trimodal* profile. Each problem was constructed by choosing an arbitrary value model (characteristics and usage) and a particular repayment profile of either rational *Unimodal* or rational *Trimodal* for which the expected total reward $\boldsymbol{J}_{\pi^*,\, g,\, 12}$ is in the neighborhood of its maximal value.
The number of possible trajectories is considerable. Therefore, the resulting trajectories were compared by matching for each problem the two resulting distributions of the final state as well as the two resulting distributions of the total discounted reward. The comparisons were made by computing the average over the whole set of problems of the empirical mean, variance and skewness of each distribution.

Definition 4.25. *Denote by* $\boldsymbol{traj\text{-}exact}_l^k(12)$, $\left(\boldsymbol{traj\text{-}approx}_l^k(12)\right)$ *the final state generated at the* l^{th} *iteration of the exact (approximate) trajectories of the* k^{th} *problem. Likewise denote by* $\boldsymbol{j\text{-}exact}_l^k(12)$ $\left(\boldsymbol{j\text{-}approx}_l^k(12)\right)$ *the total discounted reward associated to the* l^{th} *iteration of the exact (approximate) trajectories of the* k^{th} *problem.*

Definition 4.26. *For each problem i.e.* $k = 1, ..., 30$ *and each type of profile either rational* Unimodal *or rational* Trimodal*, the empirical mean* $\mu_k(\boldsymbol{traj}_{12})$ *and variance*

$\sigma_k^2(\boldsymbol{traj}_{12})$ *of the final state distribution are computed as follows:*

$$\mu_k(\boldsymbol{traj}_{12}) = \frac{1}{100,000} \sum_{l=1,\ldots,100,000} \boldsymbol{traj}_l^k(12)$$
$$\sigma_k^2(\boldsymbol{traj}_{12}) = \frac{1}{100,000-1} \sum_{l=1,\ldots,100,000} \big(\boldsymbol{traj}_l^k(12) - \mu_k(\boldsymbol{traj}_{12})\big)^2,$$

where $\boldsymbol{traj}_{12} = \boldsymbol{traj\text{-}exact}_{12}, \boldsymbol{traj\text{-}approx}_{12}$.

Definition 4.27. *The relative difference in the empirical mean* $\Delta_{\mu,k}(\boldsymbol{traj}_{12})$ *and variance* $\Delta_{\sigma^2,k}(\boldsymbol{traj}_{12})$ *of the final state distribution are then defined by:*

$$\Delta_{\mu,k}(\boldsymbol{traj}_{12}) = \frac{\mu_k(\boldsymbol{traj\text{-}exact}_{12}) - \mu_k(\boldsymbol{traj\text{-}approx}_{12})}{min\big\{\mu_k(\boldsymbol{traj\text{-}exact}_{12}), \mu_k(\boldsymbol{traj\text{-}approx}_{12})\big\}}$$
$$\Delta_{\sigma^2,k}(\boldsymbol{traj}_{12}) = \frac{\sigma_k^2(\boldsymbol{traj\text{-}exact}_{12}) - \sigma_k^2(\boldsymbol{traj\text{-}approx}_{12})}{min\big\{\sigma_k^2(\boldsymbol{traj\text{-}exact}_{12}), \sigma_k^2(\boldsymbol{traj\text{-}approx}_{12})\big\}}$$

The empirical mean $\mu_k(\boldsymbol{j}_{12})$ and variance $\sigma_k^2(\boldsymbol{j}_{12})$ of the distribution of the total discounted reward as well as their respective relative differences $\Delta_{\mu,k}(\boldsymbol{J}_{\pi^*,g,12})$ and $\Delta_{\sigma^2,k}(\boldsymbol{J}_{\pi^*,g,12})$ are calculated in a similar fashion by using $\boldsymbol{j\text{-}exact}_l^k(12)$ and $\boldsymbol{j\text{-}approx}_l^k(12)$ instead of $\boldsymbol{traj\text{-}exact}_l^k(12)$ and $\boldsymbol{traj\text{-}approx}_l^k(12)$, respectively. The empirical skewness allows one to measure the degree of asymmetry of a distribution. A distribution with a longer tail less (greater) than its mode has a negative (positive) skewness. The signs of the skewness of each distribution are compared so as to check the similarity in the asymmetry (if any) between the two resulting distributions of the final state and between the two resulting distributions of the expected total reward.

Definition 4.28. *For each problem i.e.* $k = 1, ..., 30$ *and each type of profile either rational* Unimodal *or rational* Trimodal*, the empirical skewness of the final state distribution* $\gamma_k(\boldsymbol{traj}_{12})$ *and the skewness of the total discounted reward distribution*

$\gamma_k(\boldsymbol{J}_{\pi^*,g,12})$ *for the exact (approximate) trajectories are computed as follows:*

$$\gamma_k(\boldsymbol{traj}_{12}) = \frac{1}{100,000-1} \frac{\sum_{l=1,\dots,100,000} \left(\boldsymbol{traj}_l^k(12) - \mu_k(\boldsymbol{traj}_{12})\right)^3}{\left(\sigma_k^2(\boldsymbol{traj}_{12})\right)^{\frac{3}{2}}}$$

$$\gamma_k(\boldsymbol{J}_{\pi^*,g,12}) = \frac{1}{100,000-1} \frac{\sum_{l=1,\dots,100,000} \left(\boldsymbol{j}_l^k(12) - \mu_k(\boldsymbol{j}_{12})\right)^3}{\left(\sigma_k^2(\boldsymbol{j}_{12})\right)^{\frac{3}{2}}},$$

where $\boldsymbol{traj}_l^k(12) = \boldsymbol{traj\text{-}exact}_l^k(12), \boldsymbol{traj\text{-}approx}_l^k(12)$ *and* $\boldsymbol{j}_l^k(12) = \boldsymbol{j\text{-}exact}_l^k(12), \boldsymbol{j\text{-}approx}_l^k(12)$.

The averages of $\Delta_{\mu,k}(\boldsymbol{traj}_{12})$ $\left(\Delta_{\mu,k}(\boldsymbol{J}_{\pi^*,g,12})\right)$ and $\Delta_{\sigma^2,k}(\boldsymbol{traj}_{12})$ $\left(\Delta_{\sigma^2,k}(\boldsymbol{J}_{\pi^*,g,12})\right)$ over the whole whole set of problems were computed for both the rational *Unimodal* and rational *Trimodal* profile. They are denoted by $\Delta_\mu(\mathbf{traj}_{12})$ $\left(\Delta_\mu(\boldsymbol{J}_{\pi^*,g,12})\right)$ and $\Delta_{\sigma^2}(\mathbf{traj}_{12})$ $\left(\Delta_{\sigma^2}(\boldsymbol{J}_{\pi^*,g,12})\right)$, respectively. For each problem i.e. $k = 1, \dots, 30$, the signs of the skewness were compared. Table 4.5.1.3 summarizes the findings.

	rational *Unimodal*	rational *Trimodal*
$\Delta_\mu(\mathbf{traj}_{12})$	0.05	0.02
$\Delta_{\sigma^2}(\mathbf{traj}_{12})$	0.35	0.20
Skewness of same sign?	YES	YES
$\Delta_\mu(\boldsymbol{J}_{\pi^*,g,12})$	0.05	0.03
$\Delta_{\sigma^2}(\boldsymbol{J}_{\pi^*,g,12})$	0.38	0.25
Skewness of same sign?	YES	YES

Table 4.7: Comparisons between the exact and the approximate trajectories

The approximation is found to be reasonably accurate for the prescribed profiles. Nevertheless profiles with rather poor repayment potentials will have the accuracy of the approximation decrease dramatically. They clearly set a limit to the present method. This situation should not occur in the portfolio of an issuer practicing an efficient application scoring.

4.6 Summary

An approximate dynamic programming approach was developed so as to simulate, with reasonable transition probabilities, the problem introduced in Chapter 3. To that effect, a criterion of rationality was specified and used to define different cardholder profiles.

The subsequent simulation study allowed one to recognize common problems of the industry. The rationale for the use of collection actions was also illustrated.

Finally, the accuracy of the approximate dynamic program was assessed.

Chapter 5

Extensions: Risk Analysis, Bankruptcy and Attrition Phenomenon

The present section would develop on two natural extensions of the previous lifetime value model. The first one is concerned with the reduction of the variance and the second one provides a simple way to include either the attrition phenomenon or the bankruptcy filings in the value analysis of the credit card account. Both extensions are of key importance to the issuer. Indeed, the former would stabilize the revenues and to a certain extent reduce the charge-offs while the latter could improve the retention and the satisfaction of the cardholders.

5.1 Variance Analysis

5.1.1 Introduction

In the literature concerning credit card control, the orthodox approach consists of maximizing the expected profit first and of ensuring a posteriori that for such an optimal solution the induced variance remains below an acceptable threshold [see 42]. The present section would introduce a novel approach to the problem for the influence of the variance would be directly embodied in the optimization. To this end, an infinite horizon variance-penalized *MDP* would be formulated. As a measure of risk, the latter would incorporate the weighted variance lifetime values of an account induced by a given stationary collection policy. It would allow the decision makers to consider a tradeoff between risk and return when deciding an optimal collection policy. For instance, the management could be willing to forego an optimal level of expected lifetime values so as to reduce risk and variability. In each segment of cardholders [see 3.2.2], the variance of the lifetime value of a single account is therefore instituted as a measure of risk induced by the undertaken collection policy. This criteria is all the more relevant as it accounts for the expected amount of bad debts involved in the process.

A brief introduction of the literature about the variance-penalized *MDP*s would be first discussed. The choice of a type of variance relevant to the credit card control would follow. An intuitive insight of the necessity of considering a mean-variance tradeoff would then be given on a sample problem. The formal model and its related optimization shall follow.

5.1.2 Variance-Penalized *MDP*

The vast majority of the work in the area of *MDP* [see 20, 33] defines the objective function to be the expected value of the reward. Such approaches are risk-neutral and hence, objective. They may however be too restrictive and overlook the variability of the process.

Following this, two categories of *MDP*s were formulated so as to embody variability. The first one, pioneered by Howard and Matheson [21], consists of defining and solving *MDP*s, the objectives of which are risk-sensitive utility functions. Marcus, Fernandez-Gaucherand, and Hernandez-Hernandez [29] provides a complete survey of this category of *MDP*s. The second category [see 38, 15, 14] features an embodiment of risk-sensitive constraints. This is achieved either by including in the objective function a certain type of variance or by constraining the sample path and the costs. This second category features a more general way of modeling the risk-sensitivity. For instance, one is not required to define nor assume a certain risk utility of the credit card issuer. The decision makers would instead be able to work out the desired tradeoff between the expected revenue and their variability by assigning a proper weight to the variance. Such a weight would ideally measure the relative risk the issuer is willing to bear in order to ensure high revenue. Therefore, the objective is to minimize the difference between the expected total discounted reward and its variance.

5.1.3 Variance of the Discounted Total Reward

The most natural way of considering the variability induced by a policy would be to consider the variance of the discounted total reward. However, the calculation of this variance is difficult and results in the non-tractability of any optimization criteria associated to it. This non-tractability is first detailed for the problem of interest. The definition of an alternative long run criterion and its related optimization procedure shall follow.

Definition 5.1. *Let* $CG\big(X_0, \mu_0, \ldots, X_{n-1}, \mu_{n-1}, X_n\big)$ *be the stochastic total discounted reward received by the card issuer when using collection policy* $\pi = \{\mu_0, \ldots, \mu_{n-1}\}$. *In short, it is noted as* CG_n *and is calculated as follows:*

$$CG_n = \sum_{t=0}^{n-1} \beta^t G_t(X_t, \mu_t(X_t), X_{t+1}) \tag{5.1}$$

The following stochastic dynamic relation can be derived from the above:

$$CG_{n+1} = CG_n + \beta^n g_n(X_n, \mu_n(X_n), X_{n+1}) \tag{5.2}$$

Definition 5.2. *Let $V_n(\pi)$ be the variance of the total discounted reward over n periods induced by the application of the policy π when the initial distribution is X_0.*

$$V_n(X_0, \pi) = \underset{X_1,\ldots,X_n}{V} \{CG_n(X_0)\} \tag{5.3}$$

The relation (5.2) can be used to compute the variance. Apply the following relation on conditional variances $V\{Y\} = E\{V\{Y|X\}\} + V\{E\{Y|X\}\}$ to the calculation of $V_n(X_0, \pi)$. Conditioning on $Y = (X_0, \ldots, X_n)$ leads to:

$$\begin{aligned} V_{n+1}(\mathbf{X}_0, \pi) = E\Big\{V\Big\{CG_n(X_0) + \beta^n g_n(X_n, \mu_n(X_n), X_{n+1})|X_0, \cdots, X_n\Big\}\Big\} \\ + V\Big\{E\Big\{CG_n(\mathbf{X}_0) + \beta^n g_n(X_n, \mu_n(X_n))\Big\}\Big\} \end{aligned} \tag{5.4}$$

When $X_0, \ldots, X_n$ are known, $CG_n(X_0)$ is constant. The previous equation, using matrix notations, is hence simplified to:

$$\begin{aligned} V_{n+1}(\mathbf{X}_0, \pi) = E\Big\{V\Big\{\beta^n \mathbf{X}_n^{\mathbf{T}} \mathbf{R}_{\mu_n} \mathbf{X}_{n+1} \,|(\mathbf{X}_0, \ldots, \mathbf{X}_n)\Big\}\Big\} \\ + V\Big\{CG_n(\mathbf{X}_0) + (\mathbf{X}_n^{\mathbf{T}} \mathbf{R}_{\mu_n})(\mathbf{P}_{\mu_n} \mathbf{X}_n)\Big\}, \end{aligned} \tag{5.5}$$

where for each stage t, $\mathbf{X}_t$ is the random vector of states, $\mathbf{P}_{\mu_t}, \mathbf{R}_{\mu_t}$ are the transition probabilities and costs associated to collection policy π, respectively.

This relation does not feature the key property of iterative optimization that is applied to solve the *MDP* with total discounted reward criterion. Any attempt to minimize such variance would result in formidable mathematical complexity and non-tractability.

One may show [see A.3] that given a discount factor $0 \leq \beta < 1$, the previous variance defines a Cauchy sequence which is hence convergent in the long run. This naturally suggests looking into the infinite horizon *MDP*s with discounted total reward criterion and proposing an alternative variance criterion, which is tractable. *MDP*.

5.1.4 Discount Normalized Variance

The problem considered now is that of optimizing an infinite horizon *MDP*, whose state space and control space are both discrete and finite and whose objective is risk-sensitive. The reward functions G_t are assumed to be stationary and are noted as G.

The most natural and expressive measure of variance would be the limit of the variance defined in Chapter 5.2 when the time horizon tends to infinity. However, Sobel [38] showed that such a variance lacks the monotonicity property of dynamic programming. Likewise, the problem of the maximization of the difference between the expected reward and a weighted “stagewise-variance” introduced by Filar and Lee [16] cannot be formulated as a dynamic program. These two approaches would result in formidable mathematical difficulties and non tractable optimization problems.

Following this, the approach developed in Filar and Kallenberg [14] is known for its tractable formulation and its generality featuring a unified approach for both *MDP*s with average reward criterion and *MDP*s with total discounted reward. It is the approach that is adopted in the sequel.

The state space S and the control space U are finite. The reward functions are unchanged. However, any policy can now be considered. Therefore, an admissible decision rule is a mapping of the state space S into the state of control U which assigns a probability[1] to the event that action u is chosen at time t. The class of admissible decision rules defined in 3.13 is indeed a special case of the present admission rules for which the probability of choosing u is either 0 or 1.

Definition 5.3. *Let $J_{\pi^*,\,g}(\xi)$ be the expected discounted reward, for any policy π and initial distribution $\xi = \{\xi_{(0,0)}, \ldots, \xi_{(L,M)}, \xi_{(NA,NA)}\}$, over the infinite horizon.*

$$J_{\pi^*,\,g}(\xi) = \sum_{t=0}^{\infty} \beta^t \sum_{(i,j)\in S} \xi_{(i,j)} \sum_{y\in\, S} \sum_{u\in\, U} p_\pi\big(X_t = y, W_t = u | X_0 = (i,j)\big) g(y,u), \tag{5.6}$$

where $p_\pi\big(X_t = y, W_t = u|X_0\big)$ is the conditional probability that at time t the state is y and the action taken is u, given the policy π and the initial state X_0. $g(y,u)$ is the expected one step reward received for such a decision and state:

$$g(y,u) = \sum_{z\in\, S} p(z|y,u) g(y,u,x) \tag{5.7}$$

Definition 5.4. *Let $V^{\mathrm{nor}}\big(\xi,\pi\big)$ be the "discount normalized variance" induced by a policy π when the initial distribution is ξ. It is calculated as follows:*

$$V^{\mathrm{nor}}\big(\xi,\pi\big) = \sum_{t=0}^{\infty} \beta^t\, E\Big\{\big(G - (1-\beta) J_{\pi^*,\,g}(\xi)\big)^2\Big\} \tag{5.8}$$

The weightings of the variability in V^{nor} are geometrically decreasing with the ages of the stages. This is consistent with the concept of discounted cash flows used in the evaluation of the present worthiness of an account.

5.1.5 Expected Total Reward-"Discount Normalized Variance" Example

This section would introduce a sample set of pairs formed by the expected total discounted reward and the corresponding "discount normalized variance" so as to

[1]For complete generality, the policies are not assumed to have any particular properties. They are a priori neither stationary nor Markovian.

illustrate the necessity of embodying a variance criterion.

The rational *Unimodal* profile with $a_G = 2.5$ and $b_G = -0.0025$ is chosen. The monthly purchase volume is set to $S\$3.5K$ and the monthly cash volume to $S\$1.5K$. The space of policies is restricted to the stationary deterministic policies. This restriction together with the calculation of V^{nor} are detailed in the next section. $100,000$ feasible deterministic stationary policies are then generated. For each of them, the expected total discounted reward and the corresponding "discount normalized variance" are calculated. Figure 5.1 plots this variance against the expected reward in the neighborhood of the maximum expected total reward. It shows that a trade-off can be found between the expected total reward and the "discount normalized variance". For instance, a reduction of 8.3% of the maximum expected reward (from $S\$3.96K$ to $S\$3.63K$) can allow the variance to be reduced by 27.1% (from $S\$2.36\ 10^6$ to $S\$1.72\ 10^6$).

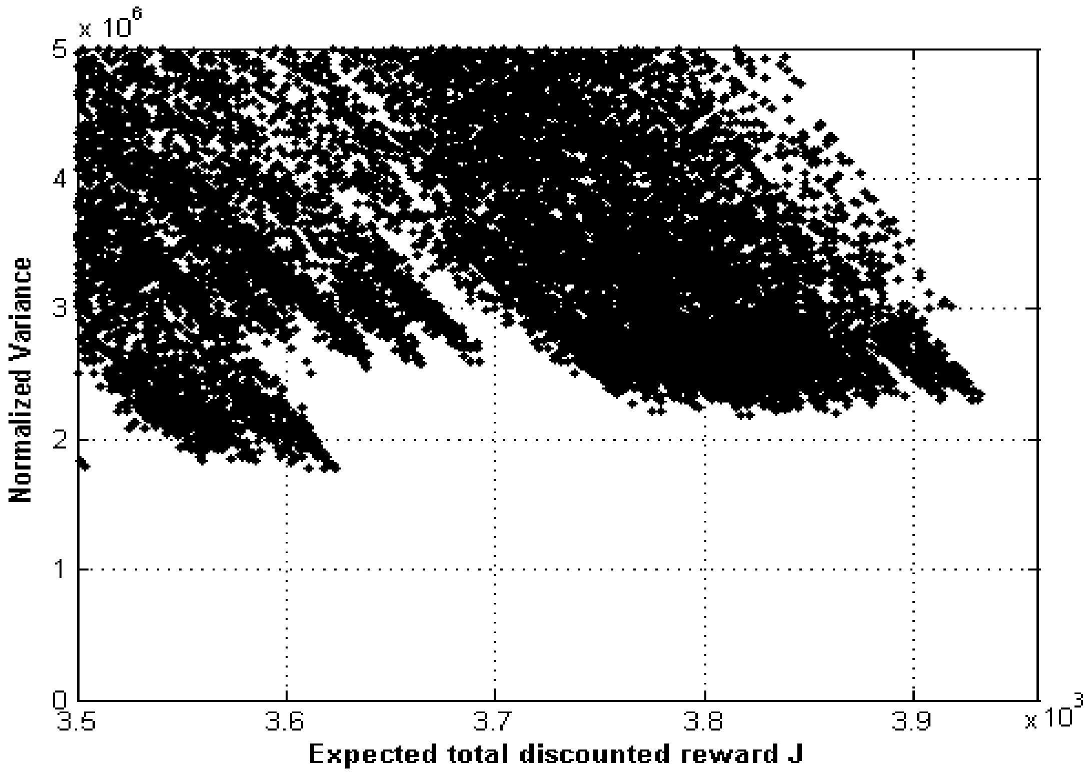

Figure 5.1: Sample set of the pairs Expected Total Reward-"Discount Normalized Variance"

5.1.6 Discount Normalized λ variance-penalized *MDP*

Definition 5.5. *The discount normalized λ variance-penalized* MDP *is the optimization problem defined by:*

$$\max_{\pi}\left\{J_{\pi^*,\,g}(\xi)-\lambda\,V^{\mathrm{nor}}\big(\xi,\pi\big)\right\},\quad 0\leq\lambda \tag{5.9}$$

Definition 5.6. *For any policy π and initial distribution ξ, let $x_{(i,j),\,u}(\pi)$ be the "discounted, expected, state action frequencies".*

$$x_{(i,j),\,u}(\pi)=\sum_{(k,\,l)\in\,S}\xi_{(k,\,l)}\sum_{t=0}^{\infty}\beta^t p_\pi\big(X_t=(i,j),W_t=u|X_0=(k,l)\big) \tag{5.10}$$

Filar and Kallenberg applied the state action frequency approach to transform the λ variance-penalized *MDP* into the following equivalent quadratic programming problem.

$$\max_{x_{(i,j),u}}\sum_{(i,j)}\sum_{u}g\big((i,j),u\big)x_{(i,j)u}-\lambda\sum_{(i,j)}g\big((i,j),u\big)^2x_{(i,j)u}+\lambda(1-\beta)\left(\sum_{(i,j)}g\big((i,j),u\big)x_{(i,j)u}\right)^2 \tag{5.11}$$

subject to

$$\begin{cases}\sum\limits_{(k,\,l)\in S}\ \sum\limits_{u\in U(k,\,l)}\Big(\delta(k-l)\delta(i-j)-\beta p\big((k,\,l)|(i,\,j),u\big)\Big)x_{(k,\,l)u}=\xi_{(i,\,j)}, & (i,j)\in S\\[2em] x_{(k,\,l)u}\geq 0, \qquad \big((k,l),u\big)\in\kappa\end{cases}$$

An optimal stationary policy is then derived from the optimal solution x^* as follows:

$$\pi_{(i,j)u}(x^*)=\begin{cases}\frac{x^*_{(i,j)u}}{x^*_{(i,j)}}, & x^*_{(i,j)}=\sum\limits_{u\in U(i,j)}x^*_{(i,j)u}>0\\ \text{arbitrary}, & x^*_{(i,j)}=0\end{cases} \tag{5.12}$$

The feasible region is a bounded polyhedron of $\mathbb{R}^{|\kappa|}$, the extreme points of which correspond to stationary policies that are furthermore deterministic. Filar and Kallenberg moreover proved that, if the penalization factor λ is strictly positive, there exists a deterministic optimal policy which is Pareto optimal for the two objectives $J_{\pi^*,\,g}(\xi)$ and $V^{nor}\big(\xi,\pi\big)$.

Using matrix notations, the previous quadratic programming problem can be rewritten as follows:

$$\max_{\mathbf{X}_{freq}} \mathbf{X}_{freq}^{\mathbf{T}}(\mathbf{P}\mathbf{R}^{\mathbf{T}})_{diag} - \lambda\mathbf{X}_{freq}^{\mathbf{T}}(\ \mathbf{P}\overline{\mathbf{R}}^{\mathbf{T}})_{diag} + \lambda(1-\beta)(\mathbf{X}_{freq}^{\mathbf{T}}(\mathbf{P}\mathbf{R}^{\mathbf{T}})_{diag})^2$$

subject to, (5.13)

$$\mathbf{I} - \beta\mathbf{P}\mathbf{X}_{freq} = \xi,$$

where $\mathbf{X}_{freq}$, ξ are the vector of components $x_{(i,j)u}$, $\xi_{(i,j)u}$, respectively. $\mathbf{P}$ and $\mathbf{R}$ are respectively the matrices of transition probabilities and cash flows previously defined.The latter are reduced to two dimensional matrices in accordance with the order imposed by the $x_{(i,j)u}$. For any square matrix $\mathbf{M}$, $(\mathbf{M})_{diag}$ is the vector consisting of the diagonal terms of $\mathbf{M}$. $\overline{\mathbf{R}}$ is the matrix of which the components are $\overline{r}_{(i,\,j)u} = r^2_{(i,\,j)u}$.

5.1.7 Convexity of the Objective Function

The objective function can also be rewritten as:

$$\max_{\mathbf{X}_{freq}} \frac{1}{2} \times 2\lambda(1-\beta)\mathbf{X}_{freq}^{\mathbf{T}}\Big((\mathbf{P}\mathbf{R}^{\mathbf{T}})_{diag}\big((\mathbf{P}\mathbf{R}^{\mathbf{T}})_{diag}\big)^{\mathbf{T}}\Big)\mathbf{X}_{freq} + \mathbf{X}_{freq}^{\mathbf{T}}\big((\mathbf{P}\mathbf{R}^{\mathbf{T}})_{diag} - \lambda(\ \mathbf{P}\overline{\mathbf{R}}^{\mathbf{T}})_{diag}\big) \quad (5.14)$$

The problem is that of optimizing under constraints a function which has the following hessian matrix $H = 2\lambda(1-\beta)\Big((\mathbf{P}\mathbf{R}^{\mathbf{T}})_{diag}\big((\mathbf{P}\mathbf{R}^{\mathbf{T}})_{diag}\big)^{\mathbf{T}}\Big)$. H is positive semidefinite and its kernel consists of the orthogonal of the vector $(\mathbf{P}\mathbf{R}^{\mathbf{T}})_{diag}$. If $(\mathbf{P}\mathbf{R}^{\mathbf{T}})_{diag} \neq \mathbf{0}$, H is of rank 1. The objective is otherwise simplified to a linear function.

In the former case, the optimization is difficult for it consists of maximizing a convex function on a bounded polyhedron. Given the convexity, there is necessarily at least one maximum in the set of extreme feasible points. Based on this property, Filar and Kallenberg demonstrated the existence of a deterministic optimal policy. They did not however detail on how to find such a solution.

5.1.8 Construction of a Stationary Deterministic Policy

This section presents an approach to systematically work out an optimal deterministic policy given that any solution has been found. For clarity, the convex quadratic objective function defined in (5.1.6) is denoted as f_{obj}.

Property 5.1. *If a feasible non extreme point $\boldsymbol{X}_M$ maximizes the convex objective function f_{obj}, then there exists an extreme point that also maximizes f_{obj}, the associated policy of which is stationary and deterministic.*

Proof. Recall that the convex set of feasible solutions is a bounded polyhedron of $\mathbb{R}^{\#(\kappa)}$ over which the continuous convex function f_{obj} reaches a maximum.
Suppose first that a maximum is found in $\mathbf{X}_M$, an interior point of the feasible set. It can easily be shown, using the convexity of f_{obj} and the consequent increasing trend of its slopes, that f_{obj} is constant over the whole convex set of feasible solutions. Any stationary deterministic and admissible policy is then a solution of this "degenerate" problem.
The search of the solutions otherwise is, similar to that of the simplex method, restricted to the extreme feasible points defining the deterministic policies and to the line segments joining any two adjacent extreme points. Now suppose that a maximum $\mathbf{X}_M$ is found to lie strictly inside the line segment joining the pair of

adjacent extreme points $(\mathbf{X}^1, \mathbf{X}^2)$.

$$\exists! \alpha, 0 < \alpha < 1, \mathbf{X}_M = \alpha \mathbf{X}^1 + (1-\alpha)\mathbf{X}^2$$

From the convexity of f_{obj} it follows that,

$$f_{obj}(\mathbf{X}_M) = f_{obj}(\alpha \mathbf{X}^1 + (1-\alpha)\mathbf{X}^2) \leq \alpha f_{obj}(\mathbf{X}^1) + (1-\alpha) f_{obj}(\mathbf{X}^2)$$

Given the maximality of f_{obj} in $\mathbf{X}_M$, this leads to:

$$f_{obj}(\mathbf{X}_M) \leq \alpha f_{obj}(\mathbf{X}^1) + (1-\alpha) f_{obj}(\mathbf{X}^2) \leq (\alpha + (1-\alpha)) f_{obj}(\mathbf{X}_M) = f_{obj}(\mathbf{X}_M)$$

Hence,

$$f_{obj}(\mathbf{X}^1) = f_{obj}(\mathbf{X}^2) = f_{obj}(\mathbf{X}_M)$$

The two extreme points $\mathbf{X}^1$ and $\mathbf{X}^2$ also maximize f_{obj}. Their associated stationary deterministic policies are then relevant solutions to the problem of interest.

A constructive method is given hereafter so as to derive the corresponding policy. Recall $\mathbf{X}^1$ and $\mathbf{X}^2$ are adjacent extreme points, and so their bases only differ by one component. Following the derivation of the stationary policy detailed in (5.12), the previously found $\mathbf{X}_M$ then defines a stationary policy which is randomized between two policies u_1 and u_2 for only one state (i_0, j_0). In this state, forcing the policy to be deterministic by either setting $\pi_{(i_0,j_0),u_1} = 1$ or $\pi_{(i_0,j_0),u_2} = 1$ defines a policy that is equal to the policy derived from $\mathbf{X}^1$ or $\mathbf{X}^2$, respectively and which therefore is an optimal stationary deterministic policy to the variance penalized *MDP*. $\square$

5.1.9 Numerical Experiments

The optimization problem of interest is, by its nature, difficult. The maximization of a convex function confines the search to the border. Classic optimization algorithms making use of the smoothness and of the derivability of the objective function are de facto excluded. The Matlab optimization toolbox was unable to solve the problem

although it offers a dedicated function to find a maximum of a linearly constrained nonlinear multivariable function. Smaller problems were tried and eventually solved with Matlab. Different optimization packages were experimented on the full scale problem and Tomlab Optimization Environment^{TM} has been singled out. It provides a Matlab interface with the SOL/UCSD Optimization Software developed by Systems Optimization Laboratory (SOL) of Stanford University. It proved to be successful in optimizing the objective of the full-scale problem. The quadratic programming problem comprises 1521 variables and 1690 constraints.
The same rational *Unimodal* profile with $a_G = 2.5$ and $b_G = 0.0025$ as in 5.1.5 is used to give an illustration. The initial distribution is chosen to be equally distributed between all the non delinquent states. In order to explore the different Pareto optimal solutions, the variance penalization coefficient λ is increased from 0 to 10 by steps of increasing magnitude. λ was assigned a total of 200 different values. The Pareto optimal pair was found to be constant beyond a threshold value for λ which means that the minimum variance was actually found. Figure 5.2 depicts the efficient frontier together with the previously generated pairs [see 5.1.5].

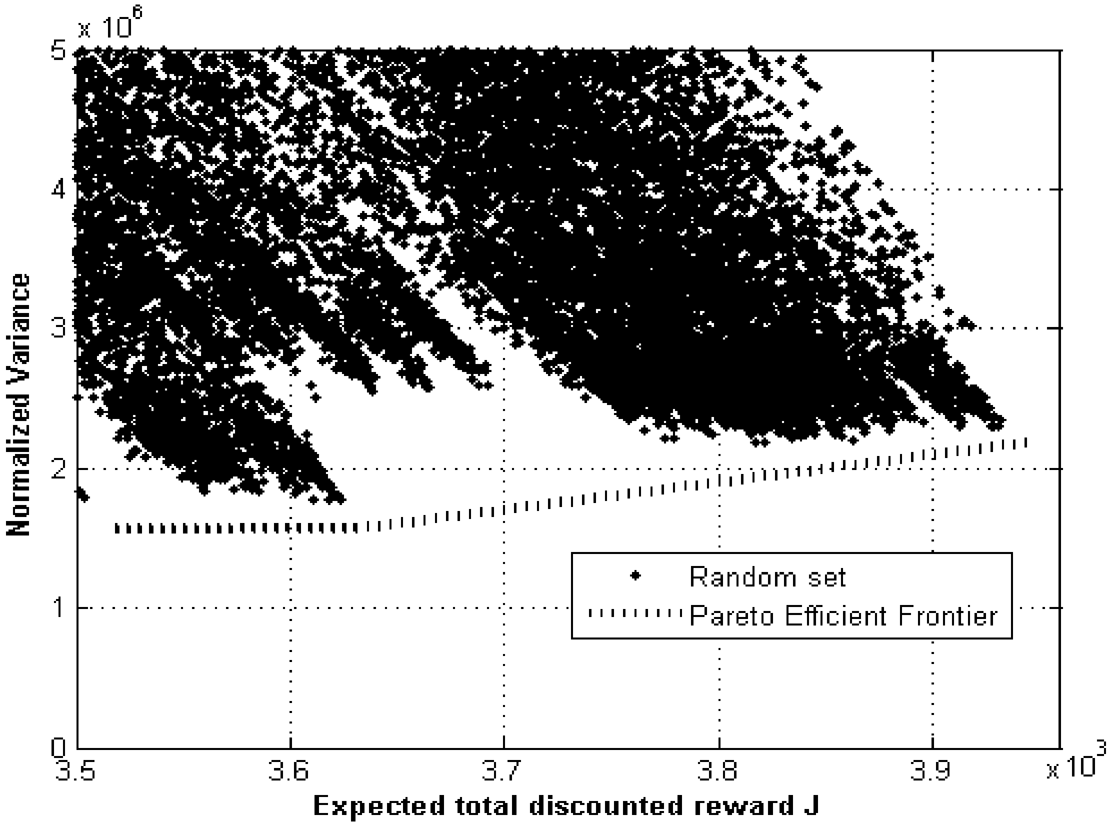

Figure 5.2: Pareto Efficient Frontier between J and V^{nor}

The corresponding policies were found to use firmer reminders as λ increases. The present example illustrates the fact that the issuer may reduce the variability, given it is willing to undertake more efficient and hence more costly collection strategies. The proper trade-off has to be found according to the objectives of the issuer and according to the availability of the collection resources.

5.2 Embodiment of the Attrition Phenomenon and of the Bankruptcy Filings

This section will introduce two structural modifications of the general *MDP* defined in Chapter 3, so as to account for the attrition phenomenon and the possibility for delinquent cardholders to file for bankruptcy.

5.2.1 The Attrition Phenomenon

The attrition corresponds to the “loss” of an active cardholder by the issuer. This “loss” can factually consist of a substantial decrease in the card usage, of a definitive interruption of usage or of a cancelation of the credit card account by the cardholder.

Definition 5.7. *In the present model, the attrition phenomenon is defined by the transition, initiated by the cardholder, of an account from a due status* $x \in S_{trans}$ *to the absorbing state* (NA, NA) *accompanied by the full repayment of the indebtedness.*

This definition accounts obviously for the cancelation of the account or the definitive interruption of usage as types of attrition. It furthermore assumes that the substantial decrease in usage can be approximated by a definitive interruption of usage.

Property 5.2. *Markov Property: It is conjectured in the model that the attrition phenomenon is a Markov process, the conditional distribution of which depends only on the present state of the account and on the present admissible collection action.*

One may argue that the Markov property is too restrictive to account properly for the attrition phenomenon. Indeed, attrition is mainly caused by cardholders' dissatisfactions or by cancelations of accounts by cardholders who decide to use competitors' products instead.

The attrition due to cardholders' dissatisfactions questions by its nature the relevancy of the one step Markov conjecture. However, one should agree that the principal source of dissatisfaction lies within the current billing cycle. This is especially true for delinquent cardholders undergoing a collection strategy which makes them highly discontented. Provided such delinquent cardholders have the ability to settle their indebtedness at once, they might fully repay and afterwards either close their accounts or simply stop using their cards.

As for the source of attrition due to the competitive business environment, it appears to be a stochastic process governed by the evolution of the industry and by factors exterior to the account into consideration. The Markov conjecture does not restrict the generality of such a process.

Definition 5.8. *Let Θ be the collection of random attrition events occurring during the set of billing periods in consideration*

$$\Theta = \{\Theta_n, \quad n = 0, \dots, N-1\}$$

For each n, Θ_n is a discrete random variable with two possible outcomes: 1 if an attrition event occurs, 0 otherwise. Following the Markovian conjecture, the probabilities are defined by:

$$\begin{aligned} & p^{\Theta}(x_n, u_n) = P(\Theta_n = 1 | X_n = x_n, u_n), \quad x_n \in S_{trans},\ u_n \in U(x_n) \\ \textit{and} \quad & \overline{p^{\Theta}(x_n, u_n)} = P(\Theta_n = 0 | X_n = x_n, u_n) = 1 - p^{\Theta}(x_n, u_n) \end{aligned}$$

Definition 5.9. *Define p^{att} for the model embodying the attrition phenomenon by:*

•$x_n \in S_{trans}, u_n \in U(x_n)$

$$\begin{aligned} & p^{att}(x_{n+1} | x_n, u_n) = \overline{p^{\Theta}(x_n, u_n)} p(x_{n+1} | x_n, u_n), \quad x_{n+1} \in S_{trans} \\ \textit{and} \quad & p^{att}\big((NA, NA) | x_n, u_n\big) = p^{\Theta}(x_n, u_n) + \overline{p^{\Theta}(x_n, u_n)} p\big((NA, NA) | x_n, u_n\big) \end{aligned}$$

•$x_n = (NA, NA), u_n \in U(x_n)$

$$\begin{aligned} & p^{att}\big(x_{n+1} | (NA, NA), u_n\big) = 0, \qquad x_{n+1} \in S_{trans} \\ \textit{and} \quad & p^{att}\big((NA, NA) | (NA, NA), u_n\big) = 1 \end{aligned}$$

Property 5.3. *p^{att} defines a proper one step Markov chain*

Proof. The Markov property directly derives from the Markovian property of the original chain together with the Markovian conjecture made for the attrition phenomenon.

The demonstration of the stochasticity of the matrix defined by p^{att} will conclude the proof.

$$\forall n, \forall x_n, x_{n+1} \in S, \forall u_n \in U(x_n), \quad 0 \leq p^{att}(x_{n+1}|x_n, u_n) \leq 1$$

- $\forall x_n \in S_{trans}, \forall u_n \in U(x_n)$

$$\sum_{y \in S} p^{att}(y|x_n, u_n) = \overline{p^{\Theta}(x_n, u_n)} \sum_{y \in S_{trans}} p(y|x_n, u_n) + p^{att}\big((NA, NA)|x_n, u_n\big)$$

From the stochasticity of the original Markov chain, it follows:

$$\begin{aligned}\sum_{y \in S} p^{att}(y|x_n, u_n) &= \overline{p^{\Theta}(x_n, u_n)}(1 - p\big((NA, NA)|x_n, u_n\big) + p^{att}\big((NA, NA)|x_n, u_n\big) \\ &= 1\end{aligned}$$

- $x_n = (NA, NA), \forall u_n \in U$

$$\sum_{y \in S} p^{att}(y|(NA, NA), u_n) = p^{att}\big((NA, NA)|(NA, NA), u_n\big) = 1$$

□

Figure 5.3 depicts the additional possible transitions due to the attrition.

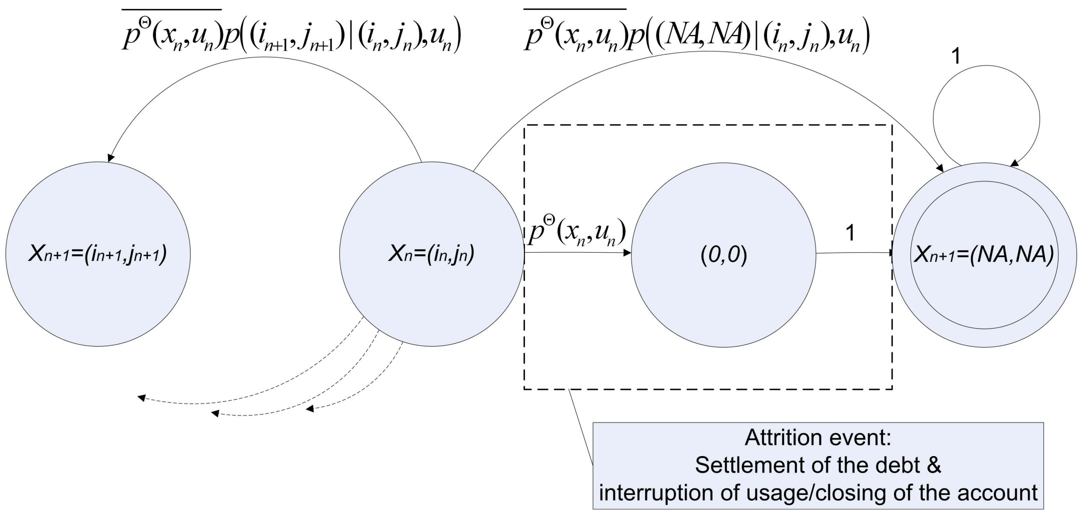

Figure 5.3: Equivalent transitions for the attrition phenomenon

The one step reward g_n^{att} should also be defined in order to embody the attrition possibility.

It is clear that the reward function is identical to the previously defined g_n so long as the transitions are made from and to transient states.

As for the transitions from a transient state to the absorbing state (NA, NA), they are either due to the attrition or to the writing off of the account initiated by the card issuer. The one step reward from any transient state to the absorbing state is then the expected value of the rewards generated along these two possible paths. In the event of attrition, the one step reward derives from the full repayment of the indebtedness followed by the closing or the full inactivity of the account. The case of account cancelation is solely considered in what follows so as to simplify the exposition.

Definition 5.10. *The aggregate one step reward g_n^{att} is defined by:*

- $x_n, x_{n+1} \in S_{trans}, u_n \in U(x_n)$

$$g_n^{att}(x_n, u_n, x_{n+1}) = g_n(x_n, u_n, x_{n+1})$$

- $\forall x_n \in S_{trans}, \forall u_n \in U(x_n)$

$$g_n^{att} = \begin{cases} 0, & if\ p^{att}\big((NA,NA)|x_n,u_n\big) = 0 \\ \frac{1}{p^{att}((NA,NA)|x_n,u_n)}\Big(p^{\Theta}(x_n,u_n)g\big(x_n,u_n,(0,0)\big)+ \\ p\big((NA,NA)|x_n,u_n\big)g\big(x_n,u_n,(NA,NA)\big)\overline{p^{\Theta}(x_n,u_n)}\Big), & otherwise \end{cases}$$

- $x_n = (NA,NA), x_{n+1} \in S, u_n \in U(x_n)$

$$g_n^{att}(x_n,u_n,x_{n+1}) = 0$$

5.2.1.1 Simulation Results

Similar to the methodology used in Chapter 4, scenarios featuring different attrition rates were run. To that end, a matrix of probabilities of attrition $\big(p^{\Theta}(x,u)\big)$ was first generated. Given each of the twelve previous value models, the *MDP*s embodying attrition were solved for each of the 84 rational *Unimodal* profiles and for each of the 84 rational *Trimodal* profiles.

The generation of each attrition probability $p^{\Theta}(x,u)$ assumes this probability to be equal to the product of a constant rate and of the probability of a cardholder in state x undergoing strategy u to repay in full his balance. The rationale for the weighting by the probability of repayment is that a cardholder willing to cancel his account or interrupt his usage must first repay his debt. This weighting moreover accounts for the differences existing between the collection strategies. Indeed, the more severe the reminder, the more likely the cardholder is to repay. Stricter strategies then generate higher probabilities of attrition. As for the constant rate, it is therefore the monthly rate at which accounts, whose balances are fully repaid, are subsequently attrited.

The simulation run consists of 41 different values of attrition equally distributed within $[0, 0.1]$ by steps of 0.0025. The maximum of 0.1 corresponds to a monthly attrition of 10% among the accounts that are paid up. It is sufficiently high to include most of the real situations since the attrition is frequently found, in the industry, to lie within a monthly rate of 0.5% to 1.5% (i.e. 5.8% to 16.6% annually). A total of 82,656 *MDP*s embodying increasing attrition phenomena were therefore generated and solved.

The corresponding simulation flowchart is therefore the same as in Figure 4.1 p108 except for an additional higher loop on the 41 increasing values of attrition rate is being performed.

For each *MDP*, the arithmetic mean of the expected total reward over the state space was used as a criterion of comparison.

Figure 5.4 illustrates the decreasing trends of the expected total rewards for the infinite horizon when the attrition increases.

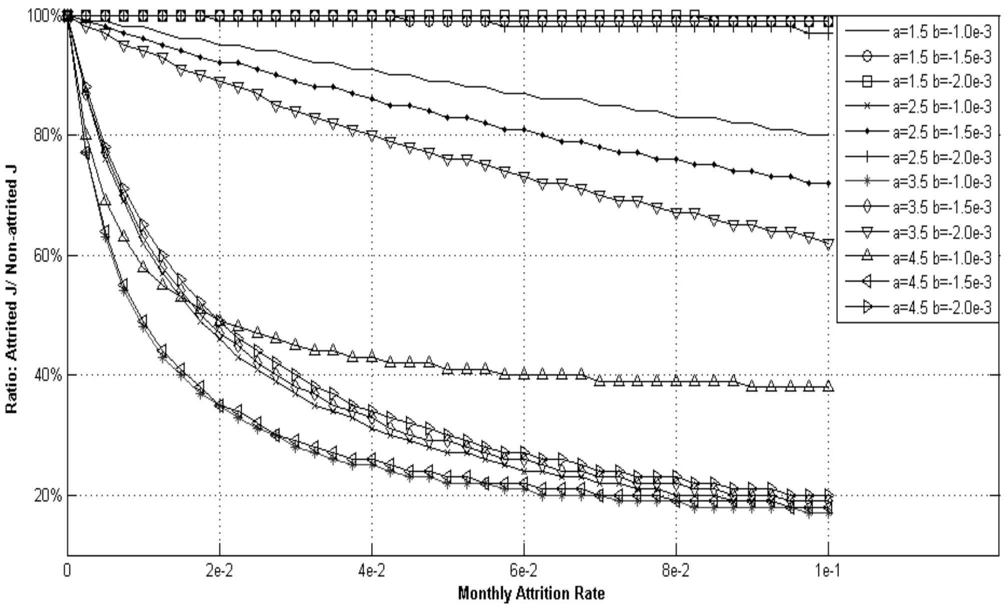

Figure 5.4: Ratio Attrited J/ Non-attrited J for some "good" repayers with rational *Unimodal* profiles, monthly purchase of $S\$1.5K$ and mean monthly cash advances of $S\$0.5K$

Two natural conclusions were derived from the data. For the profitable card-holders, the attrition can have a high impact on their lifetime values. For the "bad" cardholders, their expected lifetime values are actually increased by the attrition. For these "bad" accounts, the related account cancelations and interruptions of usage reduce the overall rate of default.

More interestingly, one can observe that the influence of the attrition on the lifetime value of the transactors (the most profitable holders; they are not too risky holders who carry forward balances and pay interests) is much more significant than for the transactors (the least risky holders who fully settle their debts every cycle). This is explainable by the present assumption that the probabilities of attrition are related to the abilities of the cardholders to repay in full their debts. Large outstanding balances are then a means to retain "good" cardholders. This situation is known to some issuers and they therefore increase, in a very selective manner, the credit lines of the revolvers who have the suitable low default risk profiles. These issuers put forward that, in this way, they respond to the increasing cardholders' needs of credit and improve hence their satisfactions [see 32]. At the same time, such issuers would eventually reduce the attrition since the cardholders, who are willing to cancel their accounts or to interrupt their usages, are forced to make an extra effort to repay in full their larger outstanding balances.

Figure 5.5 illustrates this recommendation to reduce the attrition for a "good" rational *Unimodal* profile with $a_G = 3.5$ $b_G = -0.002$ and increasing monthly usages.

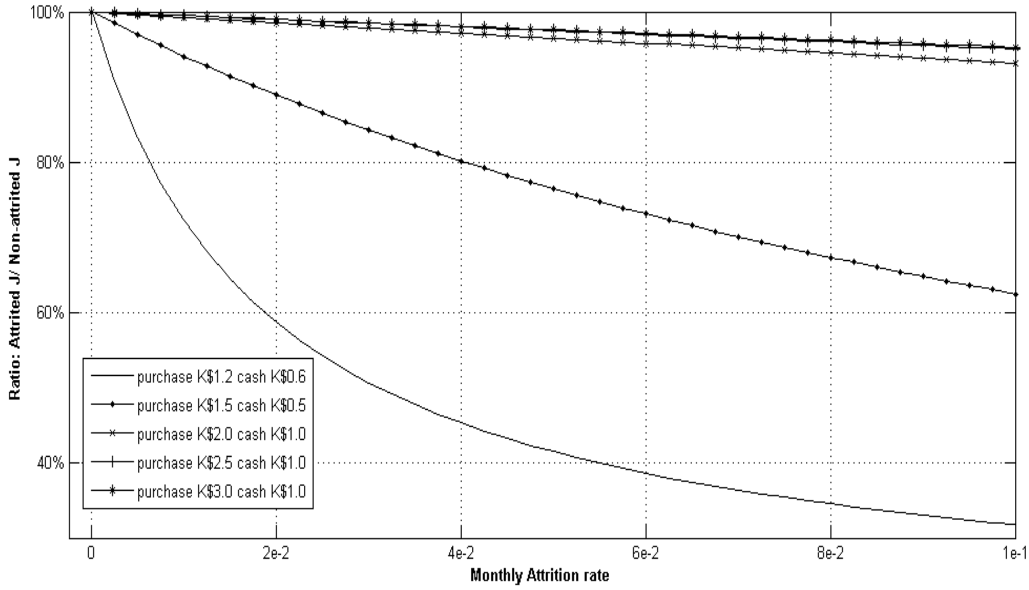

Figure 5.5: Ratio Attrited J/ Non-attrited J repayers for the rational *Unimodal* profile $a_G = 3.5$ $b_G = -0.002$ with increasing monthly usages

5.2.2 Bankruptcy Filings

The present section shall specify how the original *MDP* should be modified so as to account for the possible bankruptcy filings. The bankruptcy filings are regulated by the legislation of the country in consideration and the modes of the filings may thus differ. The analysis is subsequently restricted to the following definition.

Definition 5.11. *A bankruptcy filing is defined by the transition, initiated by the cardholder, of an account from a due status $x \in S_{trans}$ to the absorbing state (NA, NA). The card issuer neither collects any repayment nor recovers debt during such a transition.*

This definition is actually the worst case for the card issuer since the whole cardholder's indebtedness is then a loss that is not challenged by the issuer.

Property 5.4. *Markov Property: It is conjectured in the model that the stochastic bankruptcy filing events is a Markov process, the conditional distribution of which depends only on the present state of the account and on the present admissible collection action.*

One may also question this conjecture since the causes of bankruptcy filings may date back to several months. However, it is reasonable to assume that most of the predictive power lies in the current billing period. Similar to the attrition case, there are a lot of external factors influencing the probability of filing for bankruptcy. For instance, the other contracted debts and obligations of repayment influence the likelihood of a cardholder going bankrupt. The prior segmentation of the cardholders [see 3.2.2] allows the differentiation of the likelihood of cardholders going bankrupt according to their respective segments. Within each segment, the Markov conjecture then considers the relevancy of the random bankruptcy filing event to the current state of the account and the collection action the cardholder is subject to.

Similar to 5.2.1, the stochastic bankruptcy filing variable will first be defined. The transition probabilities and reward functions are consequently derived. The proof of the proper definition of the one step Markov chain, similar to the attrition case, is not repeated here.

Definition 5.12. *Let Λ be the collection of random bankruptcy filing events occurring during the set of billing periods in consideration*

$$\Lambda = \{\Lambda_n, \quad n = 0, \dots, N-1\}$$

For each n, Λ is a discrete random variable with two possible outcomes: 1 if the bankruptcy filing event occurs, 0 otherwise. Following the Markovian conjecture, the probabilities are defined by:

$$p^{\Lambda}(x_n, u_n) = P(\Lambda_n = 1 | X_n = x_n, u_n), \quad x_n \in S_{trans},\ u_n \in U(x_n)$$

$$\text{and} \quad \overline{p^{\Lambda}(x_n, u_n)} = P(\Lambda_n = 0 | X_n = x_n, u_n) = 1 - p^{\Lambda}(x_n, u_n)$$

Definition 5.13. *The transition probabilities p^{bky} for the model embodying the bankruptcy filing event are defined by:*

•$x_n \in S_{trans}, u_n \in U(x_n)$

$$p^{bky}(x_{n+1}|x_n,u_n) = \overline{p^{\Lambda}(x_n,u_n)}p(x_{n+1}|x_n,u_n), \quad x_{n+1} \in S_{trans}$$

$$and \quad p^{bky}\big((NA,NA)|x_n,u_n\big) = p^{\Lambda}(x_n,u_n) + \overline{p^{\Lambda}(x_n,u_n)}p\big((NA,NA)|x_n,u_n\big)$$

•$x_n = (NA,NA), u_n \in U(x_n)$

$$p^{bky}\big(x_{n+1}|(NA,NA),u_n\big) = 0, \qquad x_{n+1} \in S_{trans}$$

$$and \quad p^{bky}\big((NA,NA)|(NA,NA),u_n\big) = 1$$

The one step reward g_n^{bky} should be defined so as to embody the possible bankruptcy filings. Similar to the attrition case, the reward function is equal to the previously defined g_n so long as the transitions are made from and to transient states.
As for the transitions from a transient state to the absorbing state (NA,NA), they are due either to the cardholder filing for bankruptcy or to the writing off of the account initiated by the card issuer. The one step reward from any transient state to the absorbing state is then the expected value of the immediate rewards generated along these two possible paths. In the event of bankruptcy filing, the debt is not challenged by the issuer and the final indebtedness is fully lost.
Let $l^{bky}(x_n,u_n)$ be the aggregate loss associated to the event of bankruptcy when the account is in state x_n and subject to collection strategy u_n. l^{bky} is the sum of this final indebtedness and of the different cash flows accruing during billing cycle n [see Chapter 3 3.5] given that no repayment is made.

Definition 5.14. *The aggregate one step reward g_n^{att} is defined by:*

- $x_n, x_{n+1} \in S_{trans}, u_n \in U(x_n)$

$$g_n^{bky}(x_n,u_n,x_{n+1}) = g_n(x_n,u_n,x_{n+1})$$

- $\forall x_n \in S_{trans}, \forall u_n \in U(x_n)$

$$g_n^{bky} = \begin{cases} 0, & \textit{if } p^{bky}\big((NA, NA)|x_n, u_n\big) = 0 \\ \frac{1}{p^{bky}((NA,NA)|x_n,u_n)}\Big(p^{\Lambda}(x_n, u_n) l^{bky}(x_n, u_n) + \\ p\big((NA, NA)|x_n, u_n\big) g\big(x_n, u_n, (NA, NA)\big) \overline{p^{\Lambda}(x_n, u_n)}\Big), & \textit{otherwise} \end{cases}$$

- $x_n = (NA, NA), x_{n+1} \in S, u_n \in U(x_n)$

$$g_n^{bky}(x_n, u_n, x_{n+1}) = 0$$

5.2.2.1 Simulation Results

The approach used to simulate is identical to 5.2.1.1. Scenarios featuring different bankruptcy rates were run.

The matrix of probabilities of bankruptcy filings $\big(p^{bky}(x, u)\big)$ was generated by assuming the probability of a cardholder filing for bankruptcy to be equal to the product between a constant rate and the probability that this cardholder in state x undergoing strategy u makes no repayment at all. The rationale for the weighting by this probability is that any cardholder would not file for bankruptcy if he is able and willing to make a repayment even as a partial one. This weighting moreover accounts for the increasing severity of the collection strategies. Indeed, the more severe the reminder, the more likely the cardholder is to repay. Stricter strategies then generate lower probabilities of bankruptcy. It is additionally assumed that cardholders, whose states of delinquency are less than 3 months overdue, are not to file for bankruptcy. As for the constant rate, it is then the monthly rate at which cardholders, whose payments are at least two months overdue, fall further into arrears and subsequently file for bankruptcy.

The scenario comprises 41 different values of attrition equally distributed within $[0, 0.1]$ by steps of 0.0025. The maximum of 0.1 corresponds to a monthly bankruptcy filing of 10% among the delinquent cardholders that do not repay anything during this period.

A total of $82,656$ *MDP*s embodying increasing bankruptcy rates were generated and for each of them the arithmetic mean of the expected total reward over the state space was derived.

The corresponding simulation flowchart is therefore the same as in Figure 4.1 p108 except for an additional higher loop on the 41 increasing values of bankruptcy rate is being performed.

The total expected discounted total rewards are found to be almost linearly decreasing when the bankruptcy rate is increasing. For the set of profitable cardholders investigated in 5.2.1.1, the decrease can be as high as 27% of the initial value when the bankruptcy filings are not accounted for.

The influence of the bankruptcy filings is measured by comparing the expected discounted total rewards derived with and without the possibility of prematurely writing-off ($u = 8$). An issuer might be willing to write-off a delinquent account earlier and to proceed to further back-office collections. In this way, the risk of bankruptcy filing is reduced and the issuer still has a chance to challenge the debt. The relative differences between the two situations were found to be almost linear. Three situations have to be distinguished according to the cardholders' abilities to repay.

1. For the "good" cardholders, the relative difference is almost null. An issuer would be better off keeping such accounts even as delinquent ones since repayments can be expected in the short run.

2. For the cardholders with insufficient abilities to repay, the need of proceeding to write-off prematurely ($u = 8$) is clearly brought to light since the relative differences between the two strategies can be as high as 35%.

3. For the cardholders with very low abilities to repay, the relative differences between the two strategies are slightly lower than case 2 above. The issuer's

expected collection is very low for these cardholders, whenever they enter delinquency. The difference between prematurely writing-off or keeping the account is then reduced. Such cardholders should be found in limited numbers in the portfolio if the issuer practices a good application scoring.

5.3 Summary

Three extensions of the problem introduced in Chapter 3 were proposed.
The issue of the variability of the reward was addressed and computationally solved using a variance penalized Markov process.
Two novel approaches to embody the attrition phenomenon and the bankruptcy filings were specified. A classic recommendation to improve "good" cardholders' retention was reaffirmed by the simulation study. It consists of allowing for some selected "good" cardholders to increase their credit lines. Premature write-offs can be used as a preemptive measure against unchallenged bad debt due to bankruptcy. Their efficiency was lastly discussed.

Chapter 6

Conclusion

6.1 Summary of Results

This thesis deals with a Markovian approach to the analysis and optimization of a portfolio of credit card accounts. A general framework is presented and some of the industrial practices are reaffirmed via simulations.

A review of the literature concerning the credit scoring and the behavioural models is presented in Chapter 2. From the literature survey, it is observed that the scoring techniques, despite their proven efficiencies, conceptually overlook the dynamic aspects involved in the life of a credit card account. One step Markov approaches have been developed to model such dynamic aspects. No model unifying a detailed value analysis of the account and the possibility of accounting for monthly changes in the outstanding balance has been proposed.

This present work develops such a refinement. Chapter 3 formalizes the approach and designs a model implementable, as such, by the credit card issuer so as to control and optimize the profit derived from a portfolio of credit card accounts.

On account of the difficulty of obtaining real-life data, a simulation study, based on the credit card agreement of a major issuer in Singapore, is singled out in Chapter 4. To that effect, an approximate dynamic program is formulated so as to relate the cardholder's repayments and the evolution of his account from one due status to another. Subsequently, the notion of rationality in the repayments is also formalized to generate reasonable repayment distributions. From there, two categories of rational profiles as well as their random and irrational counterparts are defined and used within the approximate dynamic program to generate meaningful Markov decision processes.

Computational results re-affirm the rationale of some of the industrial practices. The necessity to differentiate between three essential segments of cardholders, namely transactors, revolvers and "bad" cardholders, is found again. The potential substantial revenue that can be derived from "good" revolvers are brought to light. The sensitivities to the minimum required payment rate mrp and to the annual percentage rate APR are discussed. The simulations confirm to a lesser extent the soundness of the values set by the major issuer of interest. The adequacy of the dynamic program approximation is a posteriori validated by Monte Carlo simulations.

Finally, a theoretical framework that comprises two extensions of the model, is proposed in Chapter 5. The first extension aims to account for the risk sensitivity in decision making. The second extension aims to embody in the model defined in Chapter 3, either the impacts of the attrition phenomenon or of the bankruptcy filings.

As for the risk sensitivity, a variance penalized Markov decision process is first adapted from Filar and Kallenberg [14]. A scheme to computationally derive stationary deterministic policies is proposed and applied to computationally solve some of the examples developed in Chapter 3. This approach is a first step towards the direct embodiment of the variability of the process in the decision making which differs from the usual approach consisting of checking a posteriori the variability. The extended model provides the optimal trade-off between profitability and risk according to the weight the management is willing to grant to the risk factor.

As for the attrition and the bankruptcy filings, a structural modification of the

Markov decision process defined in Chapter 3, provides a way to account for the attrition phenomenon or the possibility for the cardholders to file in for bankruptcy. The related simulation study, based on sensible assumptions concerning the trends of the attrition and of the bankruptcy filings, justifies quantitatively some industrial practices. The increase in the credit lines for selected cardholders as a means to retain profitable cardholders is re-affirmed. The use of premature write-offs and further challenge of the debt is found to protect the issuer against larger losses occurring when the cardholders file in for bankruptcy.

6.1.1 Future Work

The quality of the prior segmentation of the cardholders and the accurate estimations of the transition probabilities are crucial to obtain reliable forecasts. The techniques to segment as well as to estimate the transition probabilities are not discussed here. The basic approach to estimate the transition probabilities under different collection strategies consists of applying the champion/challenger approach [see 27, 42] and to make use of greedy heuristics to extrapolate the transition probabilities to strategies that are unlikely to be undertaken by the issuer. A possible extension of the present work is to account for the errors when estimating such transition probabilities. The interested reader may refer to the work of Mannor, Simester, Sun, and Tsitsiklis [28] for a treatment of this question. It is believed though that their approach is not suitable to the present problem for their assumption of multinomial distributions would not reflect the reality of the transitions of the present problem. The latter are usually skewed with a most likely reversion to a current due state.

Another possible extension is to improve cardholders' retention by including marketing strategies in the set of controls. It is believed that this would be particularly relevant with the extension modeling the attrition phenomenon. This approach, developed in [42], was limited to the improvement of retention and conversely to the present study, does not deal with delinquency management. Such an extension would then unify the two approaches and could provide interesting management insights.

Finally the inclusion of constraints or limitation in the availability of the collection resources seems to be a natural extension. The present lifetime value analysis actually represents the best scenario when the resources are not limited. Given this lifetime value, the problem could be formulated as a constrained dynamic program over the whole portfolio and, in particular, among the different segments of cardholders for which the transition probabilities were estimated. It would be interesting to develop a heuristic in the vein of the work of Bitran and Mondschein [8]. An adapted constrained dynamic programming approach could then account for both the dynamics of the problem by including this ideal lifetime value in its formulation and the limitation of the collection resources.

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Appendix A

The material for the *backward induction algorithm* and the *policy iteration algorithm* are derived from Puterman [33].

A.1 The Backward Induction Algorithm

1. Set $t = N$ and
$$J_N^*(x_N) = g_N(x_N), \quad \text{for all } x_N \in S,$$
2. Substitute $t-1$ for t and compute $J_t^*(x_t)$ for each $x_t \in S$ by
$$J_t^*(x_t) = \max_{u \in U(x_t)} \Big\{ \sum_{j \in S} p_t(j|x_t, u)\big(g_t(x_t, u, j) + J_{t+1}^*(j)\big) \Big\} \tag{A.1}$$
Set
$$U_{x_t,t}^* = \operatorname*{argmax}_{u \in U(x_t)} \Big\{ \sum_{j \in S} p_t(j|x_t, u)\big(g_t(x_t, u, j) + J_{t+1}^*(j)\big) \Big\} \tag{A.2}$$
3. If $t = 0$, stop. Otherwise return to step 2.

Theorem A.1. *Suppose* $J_t^*,\ t = 1, \ldots, N$ *and* $U_{x_t,t}^*,\ t = 1, \ldots, N-1$ *satisfy* (A.1), (A.2)*; then,*

- *for* $t = 1, \ldots, N$ *and* $\mu_t = (\mu_{t-1}, u_{t-1}, x_t)$
$$J_t^*(x_t) = \sup_{\pi \in \Pi^{HR}} J_t(\mu_t), \quad x_t \in S,$$
where Π^{HR} *is the general set of history dependent and randomized policies*

- *Let* $\mu_t^*(x_t) \in U_{x_t,t}^*$ *for all* $x_t \in S,\ t = 0, \ldots, N-1$, *and let* $\pi^* = (\mu_0^*, \ldots, \mu_{N-1}^*)$. *Then* $\pi^* \in \Pi$, *the set of Markovian deterministic policies.* π^* *is optimal and satisfies:*

$$J_{\Pi,g,N}^*(x) = \underset{\pi \in \Pi^{HR}}{sup}\ J_t(\mu_t) = \underset{\pi \in \Pi^{HR}}{sup}\ J_{\Pi,g,N}^*(x), \quad x \in S,$$

and

$$J_{\Pi,g,N}^*(x_t) = J_t^*(x_t), \quad x_t \in S \textit{ for } t = 0, \ldots, N$$

The backward induction algorithm provides a Markovian deterministic policy over the general set of history dependent and randomized policies. It computes iteratively the optimal expected total reward.

A.2 The Policy Iteration Algorithm

The *Policy Iteration Algorithm* detailed here is efficient for the discounted *MDP*s with a factor $\beta < 1$. The policies belong to Π^D, the set of stationary deterministic policies.

1. Set $k = 0$ and select an arbitrary decision rule $\pi_0 \in \Pi$

2. Policy Evaluation: Obtain J^k by solving

$$\big(\mathbf{I} - \beta \mathbf{P}_{\pi_k}\big) J = g_{\pi_k}$$

3. Policy Improvement: Choose π_{k+1} to satisfy

$$\mu_{k+1} \in \underset{\pi \in \Pi^D}{argmax}\{g_\pi + \beta \mathbf{P}_\pi J^k\},$$

   setting $\pi_{k+1} = \pi_k$ if possible.

4. if $\pi_{k+1} = \pi_k$, stop and set $\pi^* = \pi_k$. Otherwise increment k by 1 and return to step 2.

The *Policy Evaluation* step provides the expected total discounted reward for the infinite horizon problem.
The *Policy Improvement* step consists of a componentwise maximization. The optimal decisions are found for each state independently. The computing effort needed to realize this step is thus substantially reduced.

Theorem A.2. *For finite state space and control space* MDP*s, the policy iteration algorithm terminates in a finite number of iterations. It provides a solution to the optimality equation as well as the related optimal policy.*

The interested reader is referred to [33] for a complete proof. It mainly relies on the contracting properties (with the discounted factor $0 \leq \beta < 1$) of an operator defined as the upper bound on all the decisions of the expected total discounted reward. In a Banach space, such an operator has a fixed point which by construction is the optimum of the problem of interest.

A.3 Convergence of the Variance of the Discounted Total Reward

The variance of the total discounted reward is governed by the following recursive equation:

$$CG_{n+1} = CG_n + \beta^n g_n(X_n, \mu_n(X_n), X_{n+1})$$

Property A.1. $(CG_n)_{n \in \mathbb{N}}$ *is a Cauchy sequence.*

Proof. Taking the expectation of the squared values leads to:

$$E\{CG_{n+1}^2\} = E\{CG_n^2\} + \beta^{2n}\, E\big\{g_n(X_n, \mu_n(X_n), X_{n+1})^2\big\} +$$
$$2\,E\{\sum_{X_k \in S, k<n} \beta^{k+n} g_n(X_n, \mu_n(X_n), X_{n+1}) g_k(X_k, \mu_k(X_k), X_{k+1})\big\}$$

In the proof that the sequence of variances is Cauchy, the square of the expected value $E\big\{CG_n\big\}^2$ is omitted for the expected values and thus its squares are convergent and hence Cauchy. Since the sum of two Cauchy sequences is also Cauchy, it suffices to prove that the previous sequence is Cauchy to conclude that the sequence of variances is Cauchy.

By definition $|CG|$ is bounded, say by M_{UB}. From there, it follows that:

$$|\, E\{CG_{n+1}^2\} - E\{CG_n^2\}| \leq 2\beta^n \frac{1}{1-\beta} M_{UB}^2$$

The absolute value of the difference majorized by a convergent geometric sequence is clearly Cauchy. □

Appendix B

B.1 Parameter Interactions

B.1.1 Regression Analysis

The interactions between the parameters a_G (a_φ), b_G (b_φ), (i, j), $PVOL$, $CVOL$ and the expected total discounted reward for both long run and short run were investigated. These parameters are chosen for they respectively define a profile of repayment, a state of the account and a mean usage.

The following table summarizes the results of the linear regression analysis conducted over all the defined rational *Unimodal* profiles. The response to the regression was chosen to be the expected total discounted reward over an infinite time horizon (the decision of premature write-off is not included here in the set of controls) and the predictors were chosen to be the previously mentioned parameters, the range of which were defined in 4.4.6 and 4.4.7. The regression is then conducted on 170352 sextuple of predictors.

Predictors	Coef	SE Coef	T	P
Constant	7301.0	141.2	51.72	0.000
a_G	2085.81	29.23	71.36	0.000
b_G	3480597	10793	-322.47	0.000
i	114.49	14.15	8.09	0.000
j	-60.177	5.371	-11.20	0.000
$PVOL$	371.56	56.29	6.60	0.000
$CVOL$	8348.4	163.2	51.17	0.000

Table B.1: Regression analysis for the category of rational *Unimodal* profile

The S and R values are as follows $S = 13487.7$ $R - Sq = 43.5\%$ $R - Sq(adj) = 43.5\%$.

The sign of the coefficients for a_G and b_G confirms the intuition that overall the higher the ability to repay, the bigger the profit. This trend is not yet monotonous as was pointed out in the previous section with the difference between revolvers and transactors.

The values of the T statistics reveal the high relevance of the parameters to the expected total discounted reward. However, the quality of the linear fitting is poor, which leads one to think that the relationship between the parameters and the expected total discounted reward is not linear.

B.2 Value Model Spreadsheet

STATE SPACE		
Maximum number of overdue payment allowed Imax	7	
Number of segments partitioning the credit limit J	21	
	in	*jn*
Current state of the account: *(in,jn)*	1	4
Next state of the account	0	10
Payment *n*		
Payment partition ratio	4	
DECISION VARIABLES		
Number of collection strategies : *U_max*	9	
	un	
Which collection strategy to opt for; *u* € [0;U_max-1]	0	
Dummy cost: no preemptive strategy when paid up or current	$1.0E+8	
GENERAL CHARACTERISTICS		
Credit Limit: *CL*	$10,000.00	
Available credit limit: *UC*	73.42%	
Maximum unused lines allowed (negative value authorized fluff)	-5%	
Due date from the billing cycle in days	22	
Prorata: r_pra	0.29032258	
Next available credit limit: *UC*	45%	
Mean Usage	$385.30	
Annual percentage rate: *APR*	24.455%	
Minimum required payment: *mrp*	3%	
Minimum payment amount	$50.00	
interchange rate: *int_ch*	1.50%	
Cash advance rate: *ca*	24.455%	
Cash Threshold	$1,000.00	
Cash advances fee rate below Cthreshold	3%	
Cash advance fee rate above Cthreshold	5%	
Minimum cash advance fee min_cashfee	$15.00	
Penalty cost due to writing off: *PC*	$200	
Over-limit fee: *O_L*	$30	
Late payment fee LF*(UCj)*	$45	
Cost of lending money	0.00085189	

ESTIMATED DATA	
Cash & Retail Activity	
Volume of cash advance per month: *VCn(Xn, un)*	$0.00
Withdrawal fees per month: *WRn(Xn, un)*	$0.00
Volume of purchases per month: *VPn(Xn, un)*	$323.54
Operating costs	
Operating cost associated to collection strategy u: *OCn(Xn, un)*	$50.00
Debt recovery after having sold default accounts: *R(UCj)*	$526.26
INTERMEDIATE VALUES	
Overall volume subject to credit limit constraint VOL(Xn,un)	$323.54
volume of retail purchases in month *n* **st.** credit limit constraint PVOL(	$323.54
volume of cash advances in month *n* **st.** credit limit constraint CVOL(	$0.00
Interest accruing on new retail purchases: *PIn(Xn, un)*	$3.30
Interest accruing on new cash advances: *CIn(Xn, un)*	$0.00
Fees on new cash advances	$0.00
Interest accruing on the revolving balance *BIn*	$54.17
Aggregate Fees AFn*(Xn, un)*	$0.00
Aggregate financial charges ACn(Xn,un)	$57.46
Overall activity after interests accrue: *rn (Xn, un)*	$381.00
Balance reported on the statement *Bbill* inclusive of fees and charges	$3,038.90
Incomes	
Interchange revenue during month *n*: *ICn(Xn, un)*	$4.85
Gross Fee Income *IFn(Xn, Xn+1, un)*	$0.00
Financial charges/Interest revenue during month *n* : *IRn(Xn, Xn+1, un)*	$111.63
Debt recovery after sales of a default account: *DRn*	$0.00
Costs & Losses	
Cost of lending money	$2.59
Bad Debt loss cost during month *n*: *Ln*	$0.00
Operating cost associated to collection strategy un: *OCn(Xn, un)*	$50.00
CASH FLOWS	
Average income collected during month *n* : *In (Xn, Xn+1, un)*	$116.48
Overall operating cost: *Cn (Xn, Xn+1, un)*	$52.59
Overall profitability of the account: *Profit_n(Xn, Xn+1, un)*	$63.89

Figure B.1: Sample Value Model Spreadsheet