MULTI-PROJECT INTERACTIVE EFFECT ON OPTIMAL DEVELOPMENT TIMING STRATEGY

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Summary

Optimal timing and investment strategies are critical in markets when project demand is uncertain. When the two projects are developed jointly by the developer, positive interactive effects can be created by integrating the two projects to collectively enhance the values of the two projects. The synergetic effects of two projects are known as portfolio effects, in this context, which refer to the spill-over benefits generated by the second project when two projects located in close proximity enjoy positive externality. The spill-over effects may include higher revenue or lower cost for the second development project vis-à-vis the case when the two projects are developed as an integrated project. In contrary, the completion of one project may also create negative externality on the neighboring project owned by another developer, if the two competing developers engage in "unfriendly" and "combating" development strategies.

Our study aims to develop a real options model to examine multi-project interactive effects on developer's development timing strategies. The model will also evaluate how investment strategies change under different market situation and for different project type, either homogeneous or heterogeneous. We first set up a deterministic framework under the constant demand and cost functions to examine the portfolio effects of multiple projects and investment strategies under different market conditions. Then we extend to a stochastic framework with one developer who has development options on two different but contiguous land parcels, the developer will have the options to develop the two projects simultaneously or sequentially, and to develop the two land parcels into two

homogeneous or heterogeneous projects. The model evaluates whether the developer will make simultaneous development or sequential development under different market situations, and how the portfolio effects will impact the optimal development timing of the second project. Besides the close-form solution, we also use Least Square Monte Carlo Simulation to compare the different scenarios when the two projects have positive or negative correlation.

We find that the positive interactive effects between the projects will push the developer to trigger the development options on the two projects earlier. The developer will make simultaneous development, if the portfolio effects are strong enough to offset the opportunity costs of not waiting for one more period. In other words, the portfolio effects lower the trigger value of investment for the second project. He will otherwise be better off by delaying the development of the second project, which results in a sequential development process. On the other side, the positive correlation of two projects makes the developer to defer the second project because the portfolio is more sensitive with the future uncertainty. Also, developer will make different investment strategies under different demand conditions. The developer will abort the project when the demand is weak, and choose to develop single project when the demand curve is steep, while in a market with flat demand curve, he will prefer to invest in the both projects.

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Chapter 1 Introduction

1.1 Background

The traditional investment rules of the discounted-cash-flow (DCF) approach, such as net-present-value (NPV), are widely used to evaluating feasibility of investment projects on the assumption that investment is perfectly reversible. However, investment decision in the real world is irreversible, and the DCF valuation may underestimate the investment value of a project. The DCF model is also limited in its ability to capture the management flexibility, where decision can be revised in time of economic uncertainty. In the real market, new information will arrive over time, and uncertainty about the market condition as well as the interaction between different participants will change in time. Management flexibility is thus very important, which is analogous to financial options. A financial option is a derivative security that gives the option holder a right to buy or sell an asset at a pre-specified price in a pre-determined future date. Black and Scholes (1973) developed the first financial option pricing model in 1973, which has led to significant revolution in financial economic research with a flourish of research on different aspects of option pricing theory.

The financial option theory was subsequently extended to capital budgeting in investment making decision. Using the same analogy of the financial options, the opportunities to acquire real assets are called "real options". The real options approach can be used to conceptualize and quantify the option values for flexible management and strategic interactions. There are different kinds of real options, such as option to defer, option to alter operating scale (e.g. to expand; to contract; to shut down and restart), option to abandon, option to switch use and time-to-build option in a sequential investment. In the sequential option model, the option holder can choose the optimal timing to exercise a series of options when new information comes in stages. Researchers like Titman (1985), McDonald and Siegel (1986), William (1993) and Grenadier (1996) have developed real options models for valuing options quantitatively. Our idea in this thesis was developed on the Majd and Pindyck (1987) sequential option framework, which allowed stopping and restarting of a real estate project during the construction process.

To model investment decisions and quantify real options, two techniques are usually used: dynamic programming and contingent claims analysis. Dynamic programming breaks a whole sequence of decisions into two components: the immediate decision and a valuation function of all subsequent decisions, and solves it backward. Contingent claims consider an investment project as a stream of costs and benefits that vary through time and depend on the uncertainty events. Although the two methods have their own advantages and disadvantages, there is no fundamental difference between the two under the risk-neutral assumption.

The early real options papers analyzed each option in isolation, although some of them derived analytic closed-form solutions. Pricing real options in isolation lacks practical value since real-life projects are more complex with a collection of different real options. The interactive effect between options makes the value of a collection of options to be more valuable than the sum of individual options. The option on option, i.e. compound embedded option, makes the pricing of the options more complex and difficult. An early example of pricing multiple options was proposed by Brennan and Schwartz (1985), who combined values of options to shut down a mine and abandon it for its salvage value.

Our story begins with the following scenario. When a developer has two projects located in close proximity, interactive effects between the projects will have significant impact on the investment decision. When the two projects are developed jointly by the developer, positive interactive effects, i.e. positive externality or portfolio effects, can be achieved by integrating the two projects collectively. The portfolio effects may include higher revenue or lower cost for the second development project vis-à-vis the case when the two projects are developed as an integrated project. On the contrary, the completion of one project may also create negative externality on the neighboring project owned by another developer, if the two competing developers engage in "unfriendly" and "combating" development strategies.

Interactive effects between two projects increase the collective profits of the two projects. Demand uncertainty makes the development strategies more complex and increases the timing option value. In this thesis, the optimal development timing strategy will be firstly discussed in a deterministic framework, in which the demand function will be fixed without impact of economic shocks. After that, the fixed demand assumption will be relaxed to incorporate economic shocks in a stochastic model. The model evaluates simultaneous development or sequential development strategies of the developer under different market situations, and how portfolio effects are created through interaction of the two projects, and how these effects affect the optimal development timing of projects. The model is further extended in a game theoretic option framework by allowing the demand functions of the two projects to follow different stochastic processes with drift.

1.2 Motivation of the Study

Differentiating two identical projects with respective options to wait to develop, these two projects, if held by one developer and both located in close proximity, will have some interactive effects which will bring price premiums or cost savings for the two projects when considered as a whole. Instead of maximizing the value of each single project held by different developers, the single developer will maximize the collective value of the two projects. To achieve this object, the developer should decide:

1) When to develop the first project and the second one?
2) Which strategies will he choose: simultaneous development or sequential development?
3) Since the positive externality will bring the developer extra profit, will it impact on the optimal development timing?
4) Will developer defer or expedite investment in the projects?

Market structure and economic situation will also impact the investment strategy of developers. In a boom market, the developer will prefer to invest in a project early, while in a down market, he will wait till the market has recovered. The market demand curve will restrict the quantity and price of products. How will the developer change the investment strategy under different market situations?

Compared to the prevailing researches on real options and investment strategy, this study hopes to examine the interactive effect of multiple projects on the optimal development timing strategy. The objectives of this study are as follows:

a) To examine multi-project interactive effects on developer’s investment and development strategies in a deterministic framework;

b) To examine development timing strategies under different market conditions in a deterministic framework;

c) To explore the development options of a single developer on two different but contiguous land parcels in a stochastic framework.

d) To examine the portfolio effect on the optimal development timing of projects in a stochastic framework.

e) To analyze the complex real options problem by allowing the demand functions of the two projects to follow different stochastic processes with drift and numerically analyze the interactive effects between the two projects.

1.3 Scope of Research

This study assumes that in a monopoly market a developer holds two projects with options to wait to develop, and prices of projects depend on both economic shocks and demand elasticity. The developer is risk-neutral and rational, and he wants to maximize the collective profit of the two projects. The risk-neutral assumption implies that the option values are independent of individual's risk preference. The same valuation will be obtained independent of risk, perception of investors. Given this assumption, calculations of risk premiums for investment options can be determined without the need to determine the actual discount rate of investors. Investor will be compensated at risk-free rate of return when no arbitrage is possible in the market. To simplify this problem, a further assumption is that the cost of project is constant. In order to capture the interactive effect, we introduce an identical interactive effect multiplier for the first project θ_{12} and the second project θ_{21}, after the completion of the first project.

The deterministic part of this study uses an improved DCF model to compute the profits at different development timing and discusses the relationships between development strategy and market conditions. The stochastic part of this study assumes that the demand shock follows a Geometric Brownian Motion process and uses a two-stage option model to construct the development timing for the two projects. The sensitivity analysis is performed for a range inputs for the interactive effect multiplier.

The numerical analysis integrates optimal strategies of the developer into a game option framework by allowing the demand functions of the two projects to follow different stochastic processes with drift. The optimal development timing option values are simulated using the Least Square Monte Carlo Simulation based on pre-defined stochastic paths. Considering both statistical effect and computing speed, we choose 150 stages and 1000 paths in the simulation for the two projects.

1.4 Hypothesis

To examine the interactive effects of multiple projects on optimal development timing strategy, we propose theoretical development timing models both in a deterministic framework and a stochastic framework, and extend them to game-theoretic framework. Furthermore, we numerically test the interactive effect on the development timing in a range of parameters that are representative of the actual market. The basic concept framework is shown as Figure 1.1.

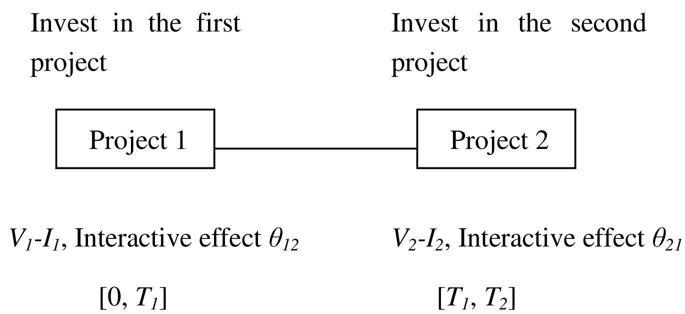

Figure 1.1 Conceptual framework for interactive effect of multiple-projects

The following hypotheses are proposed in this study:

a) The developer will prefer a simultaneous development strategy in a boom market, and prefer a sequential development strategy in a down market.

b) The positive interactive effect will make the developer to invest in the first project earlier.

c) The positive interactive effect will also impact on the development timing of second project, and the project will be developed earlier under the same market condition.

d) Market demand curve, investment cost and volatility will also impact the development timing, after incorporating the interactive effects.

e) The correlation of the two projects will also impact the development timing on the risk and return consideration.

1.5 Research Design and Methodology

The structure of our research design is shown in Figure 1.2.

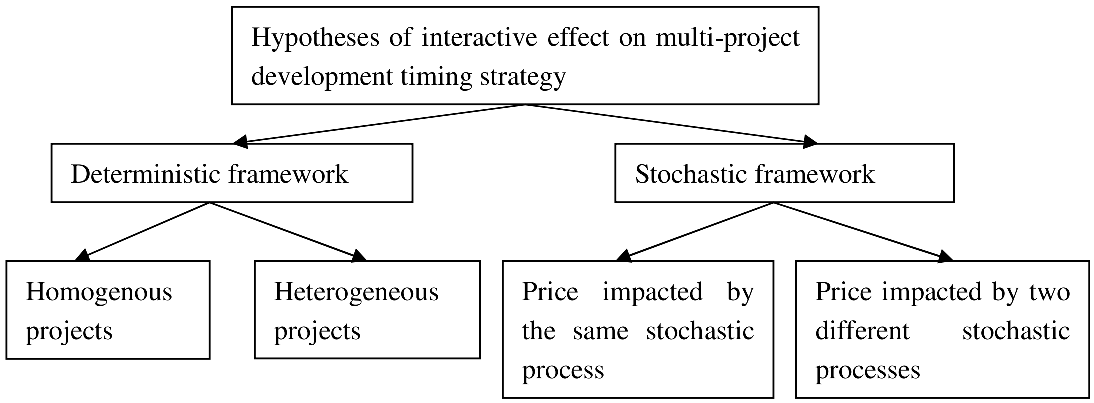

Figure 1.2 Research design of this study

We firstly build a multi-project investment strategy model with interactive effect in a deterministic framework. This part sets a basic optimal development timing model with only one developer, who will develop two homogenous projects. We further extend the model to incorporate two heterogeneous projects. We use the optimization of non-linear programming method and propose different investment strategies under different market conditions.

Secondly, based on the preliminary results in the deterministic framework, we develop a two-stage optimal development timing option model in a stochastic framework. A contingent claim analysis is used here and the two projects are assumed to be impacted by the same economic shock. We get the analytic solution in an Ordinary Differential Equation (ODE) and evaluate whether the developer will make simultaneous development or sequential development under different market situations, and how the portfolio effect will impact the optimal development timing of the two projects.

The third part of the study extends the stochastic framework into a complex game option model by allowing the demand functions of the two projects to follow different stochastic processes with drift. We derive a Partial Differential Equation (PDE) model, and perform numerical analysis using the Least Square Monte Carlo Simulation.

1.5.1 Optimization of non-linear programming

In the deterministic part, an improved Discount Cash Flow (DCF) model is used and we maximize the collective profits of both projects subject to some conditions. This is a non-linear programming model, since the objective function is the aggregation of DCF from the two projects after they are developed, which is a non-linear differential function, subject to two linear conditions that the development timing is greater than zero. This two dimensional problem has two independent variables: the development timing of project 1 and project 2.

The difficulty in the model is to find the "Global Optimum". The "Local Optimum" in the non-linear problem is a spurious solution that merely satisfies the requirements on the derivatives of the functions, and it should be eliminated. We compute the first derivatives of the objective function and reach the point (t_1*, t_2*) where the first derivatives are equal to 0. Our problem becomes easy because we prove that our objective function is decreasing monotonously before t_1* and increasing monotonously after t_1* for all t_2, and at the same time the objective function is decreasing monotonously before t_2* and increasing monotonously after t_2* for all t_1.

1.5.2 Stochastic Calculus and Ito Lemma

In the stochastic framework, we build a two-stage real option model to examine the interactive effects of the two projects. The basic assumption is that the price of the project is affected by economic shock, which follows a Geometric Brownian Motion process. The equation of Geometric Brownian Motion (Dixit & Pindyck, 1994) is given as:

$$dx = \mu \cdot x \cdot dt + \sigma \cdot x \cdot dz \qquad (1.1)$$

where μ is called the drift parameter and σ is the variance parameter, both of which are constant. Any change in *z, dz* in a time interval *dt*, satisfies the following conditions:

1. The relationship between *dz* and *dt* is given by

$$dz = \varepsilon_t \sqrt{dt}$$

where ε_t is a variable which follows a standard normal distribution.

2. The random variable ε_t is serially uncorrelated; therefore the values of *dz* for any two different time intervals are independent.

So $\varepsilon(dz) = 0$ and $v(dz) = \varepsilon(dz^2) = dt$

We use Ito's lemma to integrate the stochastic calculus. It is easier to explain by using a Taylor series expansion. Considering a function *F(x, t)* that is at least twice differentiable in *x* and in *t*, the Ito's lemma gives the differential *dF* as

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} dx + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} (dx)^2 \qquad (1.2)$$

Combine Equation (1.1) and (1.2), we get Equation (1.3):

$$dF = [\frac{\partial F}{\partial t} + \mu F \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 F}{\partial x^2}] dt + \sigma F \frac{\partial F}{\partial x} dz \qquad (1.3)$$

Under the risk-neutral assumption, we could use the stochastic calculus to construct the option model. Two kinds of boundary conditions are considered in this model: value-matching condition and smooth-pasting condition.

1) Value-matching condition matches the values of the unknown function *F(x, t)* to those of the known termination payoff function $\Omega(x,t)$.

$F(x^*(t),t) = \Omega(x^*(t),t)$ *for all t.*

2) Smooth-pasting condition requires not only the values but also the derivatives or slopes of the two functions to match at the boundary.

$F_x(x^*(t),t) = \Omega_x(x^*(t),t)$ *for all t.*

1.5.3 Dynamic Programming

There are two kinds of mathematical techniques employed to model investment decisions under uncertainty, one is contingent claims analysis and the other is dynamic programming. Contingent claims analysis has its roots in financial economics, where an implicit value of investment opportunity is computed by relating it to other assets that are traded as market price. In our study, it is assumed that the developer is risk-neutral, and the dynamic programming is used in the numerical analysis in the third part of our study.

The basic idea of dynamic programming technique is to divide the decision sequence into two parts, the immediate period and the whole continuation periods. Suppose that the current stage is t and the state is X_t. Let us denote the expected net present value of the project cash flows by $F_t(X_t)$ when we make all decisions optimally from this stage onwards. When we choose the control variables u_t, it gives an immediate profit flow $\pi_t(X_t, \mu_t)$. At the next stage ($t+1$), the state variable will be X_{t+1}. Optimal decisions thereafter will yield $F_{t+1}(X_{t+1})$. This is a random variable from the perspective of period t, so we must take its expected value $\varepsilon_t[F_{t+1}(X_{t+1})]$, which is called the continuation value. Discounting back to period t, $F_t(X_t)$ is the maximal sum of the immediate payoff and the continuation value:

$$F_t(X_t) = \max_{\mu_t}\left\{\pi_t(X_t, \mu_t) + \frac{1}{1+\rho}\varepsilon_t[F_{t+1}(X_{t+1})]\right\} \quad (1.4)$$

The idea behind the decomposition is formally stated in Bellman's Principle of Optimality (Trigeorgis, 1996). Advances in computing have made the backward calculation usable and the numerical simulation in the Chapter 6 is based on this method.

1.5.4 Least Square Monte Carlo Simulation

In complex real option problems, it is hardly possible to get a closed-form solution. Some methods are proposed for real option pricing. The finite difference method dealing directly with PDEs is difficult to implement when it meets an interdependent multiple option problem. Although the binomial lattice is very flexible for capital pricing problems with many embedded options, it suffers the curse of dimensionality. Monte Carlo simulation is a powerful and suitable numerical technique for real options for a long time ever since it was first proposed by Boyle (1977). Unfortunately the traditional Monte Carlo simulation is a forward-looking technique, while dynamic programming applies backward recursion. So in this study we use an improved algorithm, the Least Square Monte Carlo simulation proposed by Longstaff and Schwartz (2001), in solving the optimal development timing problems. This improved algorithm is based on the traditional Monte Carlo simulation and uses the least squares linear regression technique to determine the optimal stopping time in the problem.

In the simulation, we decompose the continual investment problem into discrete process consisting of many small steps. Firstly, we generate two stochastic variables, the impact of economic shocks on project 1 and project 2, which follow a Geometric Brownian Motion process in a finite time T.

$$Y_1(t+\Delta t) = Y_1(t) \cdot \exp[(u_1 - \frac{1}{2}\sigma_1^2) \cdot \Delta t + \sigma_1 \cdot \sqrt{\Delta t} \cdot \xi_1] \quad (1.5)$$

$$Y_2(t+\Delta t) = Y_2(t) \cdot \exp[(u_2 - \frac{1}{2}\sigma_2^2) \cdot \Delta t + \sigma_2 \cdot \sqrt{\Delta t} \cdot \xi_2] \quad (1.6)$$

$$\xi_1 = z_1$$

$$\xi_2 = \rho z_1 + z_2\sqrt{1-\rho^2}$$

where u_1, u_2 are the growth rate of the impact of economic shocks on project 1 and project 2 respectively; σ_1 and σ_2 are the standard deviation of impact of economic shock on project 1 and project 2; z_1 and z_2 are two standard normal distribution variables with mean 0 and variance 1; and ρ is the correlation parameter.

The simulation begins from the terminal stage and goes backward. We will exercise the option if it is in the money at stage *T*. Then we will move backward to the stage *T-1*. Based on the information at this stage, we could compute immediately the exercise value and the expected cash flow for continuation. We compare these two values and exercise the option if the immediate exercise option is more valuable. Since the immediate exercise value is easy to compute, the key point is to compute the expected cash flow for continuation. Because of the stochastic process of the economic shock, the cash flow for continuation follows a random distribution. In this method the expected project value is computed by regressing the subsequent cash flows on a set of basic functions of recent state variable. Using the same technique, we go on to compute stage *T-2, T-3*… etc. until the beginning stage.

This approach is path-dependent and it uses 1000 paths in this study to avoid the random factor of the result. To simplify this problem, we use the ordinary least squares to estimate the conditional expected function. We choose two polynomials, Y_1, Y_1^2, Y_2, and Y_2^2 as the basis functions. Considering the computing speed, we set the limitation of 1000 paths; alternatively we could use parallel computation.

1.6 Software Used

For the sensitivity analysis and the Monte Carlo Simulation, MATLAB 6.5 released by The MathWorks, Inc. is used, which is a powerful software on mathematics and scientific computing, especially for matrix. For the statistics, Microsoft Excel 2002 released by Microsoft Corporation is used, which is user-friendly and helpful for data process and statistics.

1.7 Organization of the Study

The study consists of six chapters. Chapter 1 gives a brief introduction and objective of the study. Chapter 2 covers the literature review, which sets up the background for this study. The next three chapters are written as three separate but related papers[1], and the last chapter is the

[1] The first one is accepted and presented at the Cambridge-Maastricht Symposium 2005.

conclusion and extensions for future research.

Chapter 1 sets up the background of real option theory, time-to-build theory and multi-project interactive effect. It introduces the motivation of our study, research questions, research objectives, research scope, hypothesis, research design and methodology.

Chapter 2 reviews the prevailing literatures on real option theory, time-to-build strategy, as well as investment strategy. It also provides literature review on externality and neighborhood effect. Selected literatures on game theory are also mentioned.

Chapter 3 develops a deterministic model to examine multi-project interactive effects on developer's investment and development strategies. Firstly, we build up a basic optimal development timing model with only one developer, who will develop two homogenous projects. We further extend this model to incorporate heterogeneous projects. We evaluate how investment strategies would change under different market situations and for different project types, either homogeneous or heterogeneous.

Chapter 4 develops a two-stage real option framework with one developer who has development options on two different, but contiguous land parcels. The model evaluates whether the developer will make simultaneous development or sequential development under different market situations, and how the portfolio effect will impact the optimal development timing of the projects.

Chapter 5 extends the model into a game-theoretic option model and solves the complex problem. With two stochastic variables, there are two PDEs with boundary conditions. After that we use the Least Squares Monte Carlo Approach proposed by Longstaff and Schwartz (2001) to simulate the complex multi-option problem.

Chapter 6 summarizes the main findings of this study and conclusions. We also point out some limitation and extensions for future research.

Chapter 2 Literature Review

2.1 Standard Investment Analyses and Uncertainty

Investment strategy includes not only decisions of whether to invest in a project, it would also include decisions on how much to invest and when to invest. The “how much” question is concerned with capital allocation, whereas “when” is a question of investment timing. Other strategies such as entry, exit, acquisition, competition, and cooperation, are considered together with the characteristics of different industries. The net present value criterions are accepted methodology for evaluating the feasibility of an investment decision.

Taggart (1991) reviewed various approaches to calculate the discount rates used in the standard NPV model. Three different methods were used to value firms and other assets, which included adjusted present value, adjusted discounted rate and flow to equity method. She also analyzed these three methods and how they could lead to identical valuation considering corporate and personal tax.

Myer & Ruback (1992) derived a simple but robust rule for calculating a discount rate for a risky asset in the NPV framework. The rule treated all projects as combinations of two assets: Treasury bills and the market portfolio. The discount rate can be treated as a weighted average of the after tax risk-free interest rate and the expected rate of return on a portfolio of risky securities. The weight on the portfolio's return was the risk of cash flow relative to the portfolio.

The traditional NPV rule has a problem with fixed discount rate, when applied to pricing property in real markets that are uncertain. It failed to take into consideration the future demand uncertainty, supply uncertainty and microeconomic variance. The naïve NPV method can not accommodate management flexibility and future uncertainty in the analysis. Researchers have attempted to expand the NPV framework by incorporating growth opportunity into future cash flows in the standard NPV to capture future uncertainties.

Nygard, Mai and Razaire (1999) proposed a new discounted cash flow method with a range of values over a probability curve. This approach overcomes the shortcoming of the traditional single-point valuation. They set up probability distributions within a range of the lowest and the highest possible values. A simulation program is then run hundreds of times to derive many values in short time. They constructed a curve using the range of possible appraised values simulated in the earlier stage.

Mallinson & French (2000) indicated that uncertainty was usual and a real phenomenon. The source of uncertainty can be identified in the valuation process. They identified various sources of uncertainty which included liquidity, depth of market, intrinsic uncertainty of the asset, and dynamic of market.

O'Brien (2003) developed a simple and flexible discounted cash flow formula for valuation based on the traditional DCF model. The model included some fundamental drivers of a firm's growth opportunities. The formulation of an asset's value was the sum of two present values. The first was the present value of the expected future earnings stream generated by the assets, while the second was the present value of growth opportunities. The formula assumed the expected growth rate and the expected convergence of their ROE to the firm's cost of equity were constant, which avoided the need to estimate a “continuing value” over an unknown horizon.

French & Gabrielli (2005) discussed various ways in which uncertainty can be incorporated into the DCF model. The paper utilized a probability-based valuation model by recognizing that input variables and their corresponding probability distributions. These input uncertainties would translate into an output uncertainty figure. The results showed that the central tendency of this distribution was very close to the single point estimate of the static model. The upside and downside risk could help to understand the uncertainty.

Besides the adjusted DCF models as discussed above, another approach to investment uncertainty - real options theory, has been an effective and efficient method in property pricing. Howell & Axel J. Jägle (1998) did a survey on 82 experienced managers from various functions, business levels, and industries; and they obtained high levels of agreement on various statements with respects to the application of the real options model.

2.2 Real Options Theory

2.2.1 Real Options Concept and Theory Development

After Black and Scholes (1973) classical paper on the financial option pricing method, researchers have been able to extend the pricing model to capital budgeting analysis in investment decision. The real options concept is widely applied to value growth options in real assets and investment projects.

Titman (1985) was the first to apply the model of Black and Scholes (1973) to determine the explicit value of a vacant land in a city centre. The paper presented a simple two-state model to determine the optimal time to develop a vacant land on the assumptions that the future price of the building units and the size of the building to be constructed were uncertain. The assumptions were further relaxed to allow unit price and cost to be specified at each date and each state. Illustrated with a numerical example, the analysis demonstrated that over a range of possible building size in the future, the timing option is valuable to the land owner. The value of the timing option increases in the future price uncertainty.

Brennan and Schwartz (1985) used a continuous stochastic model to value natural resource projects and derived the optimal policies for developing, managing and abandoning them. Compared with the valuation method under certainty such as DCF model, the approach replicated a self-financing portfolio under the assumption that the convenience yield was a function of the output price given that the interesting rate was fixed. It provided optimal policies for managing the natural resource and made empirical prediction. Some model specific limitations included the resource was homogeneous and of a known amount, the cost was known and interest rate was fixed.

Dixit (1989) modeled a firm's entry and exit decisions when the output price followed a Brownian motion process. It was a model of single firm decision, not including the externality. The analysis indicated that the reason for hysteresis that the entry trigger exceeded the variable cost plus the interest, in the entry decision, while the exit trigger was less than the variable cost minus the interest in the exit decision. Some extensions such as kinds of uncertainty, scale of output, scale of the project, and risk aversion, can be made.

Williams (1991) confirmed Titman (1985)'s results and expanded the model to analyze the effects of an option to abandon a project. He used an analytic model with two stochastic variables, namely development cost and project value, in the problem. Williams (1993) valued the real options under conditions of finite elasticity of demand, finite capacities of developers, limited supply of options, and the degree of concentration among developers. In the model, the option was the undeveloped property, while the underlying asset was the developed asset. The heterogeneous characteristics of real options were discussed in the paper, which include time to build, stochastic cost of construction, controlled quantity or density development and the option to abandon developed or under developed property. The model gave the equilibrium of an optimal exercise policy for developers and estimated the values of both developed and underdeveloped assets.

Extensive literature on the optimal irreversible investment strategy of a solitary firm was found since Pindyck (1991). Leahy (1993) addressed the relationships between the myopic firm and competitive equilibrium. His paper divided firms into two types. One could correctly anticipate the strategies of other firms in the market, which would interact with prices and exogenous shocks. The second type of firms ignored the effect of other firms on the price process, and they were called myopic firms. Results showed that the two types of firms were identical in prices that triggered investment in a setting with homogeneous, constant-returns-to-scale firms, linear costs of investment, and downward sloping demand, although the optimization problems of these two firms were different. The intuition was that competition reduced the value of actual and potential capital, at the same time, trade-off between the two firms was unaffected.

Sing (2002) used a one-factor contingent claim model to demonstrate the optimal development timing strategy and examine the irreversibility implications. While McDonald & Siegel (1986) assumed the project value was exogenous. They used numerical analysis to examine the effect of rent yield, risk-free interest rate, and volatility. The results showed that the value of option to wait to invest increased correspondingly to increases in future return uncertainty. The high option value reduced current investments in a highly volatile market.

After the development of single option model, researcher found the results were sometimes different from what had been observed in the actual market, where investment decisions involved multiple real options which were interdependent. The interactions of complex options depend on the type and the order of the option involved. Literature on how to divide the multiple options into separate ones, and how to include the interactive effect were abundant (Cortaza & Schwartz (1993); Panayi & Trigeogis (1998); Damodanra (2000)).

Herath and Park (2002) developed a compound-option model to value a multi-stage project, where each investment opportunity derived revenues from different markets but with common technological resource. A binomial lattice method was used because it was more flexible to analyze the complex real option, which could overcome the shortcoming of standard option pricing model that only considered one source of uncertainty. The model considered multiple underlying variables and multiple source of uncertainty. The paper illustrated the underlying asset volatility of each investment opportunity by developing return distributions using Monte Carlo Simulation.

Gamba (2003) developed a new approach based on Monte Carlo simulation for valuing a wide range of capital budgeting problems with many embedded real options. It divided the complex option into three kinds: sum of independent options, options on options, and mutually exclusive options. Firstly, the paper decomposed a complex real option problem with many options into a set of simple options, taking into account interaction and interdependence of the embedded real options. Then it used the Least Square Monte Carlo Simulation by Longstaff and Schwartz (2001) to implement the decomposition approach.

Some applications of real options in other industry such as R&D, patents and software development were also proposed (Cortazar & Schwartz (1993); Myers (1996); Damodaran (2000)). Miltersen and Schwartz (2004) developed a model to analyze patent-protected R&D investment projects, when there was competition in the development and marketing. Each of the duopolists had to take into account not only the factors affecting its own decisions, but also the factors affecting the decisions of the other duopolist.

2.2.2 Empirical Research on Real Option

Along with development of theoretic real options modeling, empirical tests of option pricing methods are lacking. Restricted by the unavailability of data in actual markets, there are only few empirical researches on real option among several countries, most of which are done in developed real estate markets.

Quigg (1993) was one of the first to empirically estimate the premium for the option of waiting to develop using data from 2,700 land sales in Seattle in US. She found a mean option premium of 6% of the theoretic land value and proposed that option model had more power over and above the intrinsic value for predicting transaction prices. She also estimated the standard deviation of property prices, which ranged from 18% to 28% per year.

Sing & Patel (2001) modified Quigg's methodology and applied it to estimate the premium for the option of waiting to develop based on a sample of 2,286 property transactions in the UK over a 14-year sample period from 1984 to 1997. Based on a one-factor contingent claim valuation model, they found that average premiums for the timing options were 28.78% for office sector, 25.75% for industrial sector and 16.06% for retail sector.

Yamazaki (2001) examined the uncertainty in land prices based on 4,368 land price data in Japan from 1985 through 2000. Both cross-sectional and time-series variables including two uncertainty variables were arithmetically combined and the OLS method was conducted. The results from the option-based models favored the application of the real option theory in land prices. The total uncertainty with respect to built asset return had a substantial effect on increasing land prices, which implied that an increase in uncertainty led to an increase in land prices.

Bulan, Mayer, and Somerville (2002) examined 1,214 condominium developments in Vancouver, Canada between 1979 and 1998 to identify the extent to which uncertainty delayed investment. The empirical result showed that a one-standard deviation increase in the return volatility reduced the hazard rate of investment by 13%, which was equivalent to a 9% decline in real prices. They indicated that idiosyncratic and systematic risks led developers to delay new real estate investments, but the increases in the number of potential competitors located near a project reduced the negative relationship between idiosyncratic risk and development, which supported the hypothesis that competition erodes option values.

2.3 Time to Build

Time to build is an important factor in a real estate venture, which involves optimal timing strategy. In the optimal stopping rule, the objective is to maximize the expected profit of a firm or a project by including embedded options. Several researchers focus on modeling the real options and analyze the optimal investment timing.

McDonald & Siegel (1986) studied the optimal timing of investment in an irreversible project, where both the benefits and the investment cost followed continuous stochastic processes.

Their paper built an investment option model that assumed a stochastic cost function. From the basic model, they derived optimal stopping rule and explicit formula for the value of the option to wait to invest for a risk-averse investor. Besides the analytic solutions, their simulation results showed that the option value was significant and it was better to wait until the benefit from the project was twice of the cost given reasonable parameters. Although the modeling was plausible, the paper confirmed the option value and proposed optimal investment timing.

Mayd & Pindyck (1987) used the contingent claims analysis to derive optimal decision rules of irreversible investment under uncertainty with a maximal rate of investment. Its basic idea was that investment decision was sequential and the construction process could be flexibly adjusted to wait for new information, so that there was an option value to postpone an irreversible investment. They considered the sequential investment as a compound option and built a two-stage option model. Because of the complexity of the model, they gave numerical analysis and the results presented high sensitivity to the perceived risk.

The earlier literature was developed on a single firm's perspective without considering the market competition. Trigeorgis (1991) used the option-based valuation to determine the optimal time to invest, when the timing and value of the opportunity may be affected by competition. Incorporating both optimal timing and competition, he built the model of deferrable investment opportunity with anticipated competitive arrivals. The model is analogous to an American call option with known dividends. The paper presented five scenarios: (i) Exercise Immediately and Pre-empt Competition, (ii) Exercise Immediately and Competition Enters, (iii) Proprietary Option but No Competition, (iv) Wait and Competition Enters, (v) Exercise Just Before t_j and Pre-empt Subsequent Competitors, and the relationships between them were discussed. The numerical results suggested that management may be justified in investing relatively early when the anticipated competitive loss was large and anticipated competitive entry was frequent.

Pindyck (1991) reviewed some basic models of irreversible investment to illustrate the option value of investment opportunity and surveyed recent applications of this methodology. He proposed that investment was irreversible because of industry or firm specific and lemon effect, so it was better to model this like a financial option by option pricing or dynamic programming than the traditional NPV. He used a simple two-period example, and made sensitivity analysis using various parameters. Then he used the continuous-time model examined by McDonald & Siegel (1986) to value the option and choose the investment decision timing. Extensions in cost functions, sequential investment and capacity choice and their policy implications could be examined.

Capozza & Li (1994) extended the real option model of durable-capital-investment decisions (McDonald & Siegel (1986)) to include intensity and analyze land-use decision. They provided an optimal-stopping framework to model the decision, which included land-redevelopment decisions and capital-replacement decisions, when intensity was a variable. The results showed that developed property value not only contained irreversible premium, but also an intensive premium. Development timing was affected by growth rate, uncertainty, output elasticity and interest rate.

Milne & Whalley (2000) expanded the time to build model by Majd & Pindyck (1987), which omitted an optimality condition: the marginal benefit to investing should equal to the marginal cost at the trigger time. They revealed that the time to build reduced the effects of increased project value volatility, compared to the standard real option models where investment was instantaneous. They suggested that when the time to build was long and the opportunity cost was high, the naive NPV calculation was an adequate guide for the initial decision to begin investment even for high levels of uncertainty. Milne & Whalley (2001) solved two models of time to build. One was that production cannot be suspended once started and the second was with costless suspension of production.

Sing (2001) recast the land development problems of Williams (1991) and Quigg (1993) by addressing the effects of scale elasticity of unit rental and unit construction cost in a real estate project. He built an option model with two different diseconomies of scale constraints on rental and cost variables. The comparative statistics simulated positive relationships between the option premium of waiting to develop and the volatilities of the unit rental and unit construction cost. Sing (2002b) made a time to build option model consisting of a stochastic rate of completion and a stochastic net project payoff in a sequential construction process of a large scale construction project. The results of the sensitivity analysis showed that the trigger payoff value increased positively with increases in cash flow volatility, input cost uncertainty, excess asset return per unit risk and maximum rate of investment. However, it had a negative relationship with the rental yield.

2.4 Game Theory

The real option model on a single project omits the interaction between projects and market competition. In recent years, researchers begin to adopt the game theory with real options to explore the interactive strategies between developers under different market structure.

Grenadier (1996) used strategic option exercise games to understand real estate development. He developed an equilibrium framework of symmetric duopoly using stochastic stopping-time game. Two building owners leased their existing properties in a local market, and each held the option to develop a new building. The exercise of development option by one owner would affect the values of both options. The first to build would pay the construction earlier, but benefit without competition. The other developer would see the value of existing building affected. If the follower exercised the development option, the leader would lose monopoly. The model described the interaction between the leader and the follower under different market condition and explained the building booms even in a declining market.

Grenadier (1999) analyzed strategic exercise equilibrium under asymmetric information over the underlying option parameters. He formulated a model of option exercise policies and information revelation with private signals and indicated strategy exercise patterns in realistic economic settings. With diffuse and imperfect information, equilibrium exercise timing would always deviate from the full-information optimum. Those with informative signals would exercise the earliest, which would be earlier than the optimal timing. Market observers may take action of overbuilding in real estate development. The model was also extended to include both information and payoff externalities.

Grenadier & Wang (2005) provided a model of optimal investment timing under conditions of principal-agency conflicts and asymmetric information. The principal-agent model decomposed the underlying option to invest into two components: a manager's option and an owner's option, which existed concurrently with hidden action and hidden information. Owners would design contracts to encourage managers to extend effort and truthfully reveal their private information. On the other hand, managers would take the opportunity to enhance their management in investment decisions, such that they would undertake actions and make decisions to enhance their personal utility or reputation instead of the owner's. The results showed that the investment behavior differed significantly from that of the first-best no-agency solution. In particular, greater inertia occurred in investment, because manager would have a more valuable option to wait than the owner.

Ong, Cheng, Boon and Sing (2003) present economic experiments in oligopolistic environment when there was an option to market pre-completed units to examine how developers priced their properties. The basic experimental design was to group the players into two: developers and investors. Developers acquired land, developed and sold the completed and homogeneous property units to investors and the objective of developers was to maximize profits. The results indicated that developer relied heavily on some forecasts of market price and competitor actions were important considerations in pricing decisions. The experiments further revealed that aggressive pricing strategies were not necessarily the most profitable.

2.5 Externality and Neighborhood Effect

Because of the heterogeneity of property, property pricing varies location externality. The interaction between projects may affect the developer's investment decision and development timing. An externality exists in economics when a decision creates costs or benefits to individuals or groups other than the person who makes the decision. Externalities are also called "neighborhood effects" or "spill-over". The classic example of a negative externality is pollution, which is generated by some productive enterprises, and affects others. An example of a positive externality is purchasing a car of a certain model increases demand and thus availability for mechanics who know that kind of car, which in turn improves the situation for others to own that car model.

Kauko (2003) used the combination of qualitative and quantitative methods to develop analytic hierarchy process instead of the single hedonic model. He firstly reviewed the traditional hedonic model technology and extended to the flexible regression and spatial model, and then proposed the hierarchical structure to demonstrate an improvement of the demand analysis. The results indicated that location externality was one of the important factors.

Hilber (2005) used the American Housing Survey to examine the importance of neighborhood externality for the homeownership rates in cities. Potential owners would like to think about changing neighborhood amenity risk when they made investment decision. He built a model to measure the externality risk after controlling for housing type, numerous location and household specific characteristics. The empirical results showed that neighborhood externality risk significantly reduced the probability of a housing unit owner-occupied.

Liebowitz & Margolis (1995) suggested that network externalities may not be the source of risks causing market failure and the importance of externality may be overstated. They defined the network externalities as the concept that a product's value to a consumer changed along with the number of users of the product changes. This paper considered two kinds of externalities, direct and indirect effect, and proposed to include the new technology in the externality model.

Caplin & Leahy (1998) developed a search theory with information externalities and used it to explain the rapid recession of the Sixth Avenue in the 1990’s. Their paper modeled the vacant building on Sixth Avenue as options for alternative use with information spill over. Owners waited for the development of others to get more information, which would help the sequential decision making. Based on the simple model, they got solution and discussed the effect of market structure and search technology.

Dong (2003) examined whether the neighborhood externalities would influence optimal development strategies in a competitive markets. She sets up models in a deterministic and a stochastic framework, and extended the two players into N players. The paper found that high positive externality would make the project to be developed earlier than in the case with low externality.

Deng (2004) incorporated both inter-temporal externality and "public good" externality to explore the relationships between market structure and urban institutions. His paper developed a two-period model, which analyzed four institution settings: bundle rental, separate rental, bundle sale and separate sale, and he found that the inter-temporal externality was important in land use.

2.6 Summary

This chapter reviewed the traditional investment strategy using NPV rules and the types of uncertainty faced in actual investments. Literature on real option, time to build option and game theory was also reviewed. Researches on the externality in real estate market were covered. From the prevailing literature, we find that researches on multiple options, which are interdependent and the externality of projects on investment strategy were lacking. In the following chapters, our study will incorporate the inter-project externality and explore the complex interdependent options that one developer faces when making investment strategy in a monopoly market.

Chapter 3 Multi-Projects Interactive Effects and Investment Strategy in Deterministic Framework

3.1 Introduction

In typical studies of investment strategies in property market, such as enter and exit decisions, a common assumption is that a developer only has one land parcel, and he will decide whether to develop the land according to the NPV rule, or choose an optimal development time as proposed by the real option theory. After Black and Scholes (1973) theory on financial option pricing, researchers have extended it to capital pricing in investment decision making. Real options theory has been widely accepted and applied to value growth option in real assets and investment projects. The real option concept is related to future uncertainty and used to determine the optimal investment timing strategy. Titman (1985) applied a binomial discrete time real option valuation model for pricing vacant land in a city center. He showed that the owner would delay building when price of developed property is uncertain in the future. Williams (1993) valued the real options under conditions of finite elasticity of demand, finite capacities of developers, limited supply of options and the ownership. He determined the equilibrium of optimal exercise strategies for developers and calculated the values of both developed and underdeveloped assets under uncertainty. Dixit (1989) modeled a firm's entry and exit decisions when the output price followed a random walk process. His model consists only of a single project, and the externality effects found in a multiple-projects case was not considered. Some papers integrated game theory, ownership structure, and equilibrium market assumptions into real option models. Grenadier (1996) developed a strategic equilibrium option exercise game model to describe the interaction between a leader and a follower under different market conditions. He focused on the timing of real estate development under different market conditions, i.e. competitive or oligopolistic assumptions.

The earlier studies are developed on the assumption that there exists only a single project owned by a single developer. The decision will be complicated when a developer possesses two or more vacant developable lands at any one time. He will not only have to make sure that limited resource is optimally allocated among different real estate projects, but also to evaluate possible project interactive effects when development timing strategy, either to develop projects concurrently or sequentially, is considered. The market structure will also affect the behavior of the developer. Developers with different market power will take different strategies, which means that one developer’s strategy will affect others and vice versa. In the real property market, a developer often has more than one land waiting to be developed at the same time or sequentially over different time periods. In a case involving two projects located in close proximity, positive interactive effects can be created either when they are developed jointly and simultaneously by a single developer as an integrated project, or they are developed sequentially as two independent but complimentary projects. The positive externality or synergetic benefits spilled-over from one project to another project will collectively enhance the values of the two projects. Similarly, if two adjacent projects were developed by two competing developers, who have no obligation to engage in “friendly” and synergetic development strategies, negative externality can be created as a result of non-cooperative strategies by the two developers. The completion of a neighboring project owned and developed by competing developer, will diminish project payoffs and increase the running costs for both developers in long run.

To examine the interactive effect of multiple projects and developer's behavior on investment strategies, this chapter attempts to set up a theoretic framework to find when is the optimal time to develop lands and how much to invest, given that a developer has more than one developable land. To begin with a simple scenario, we develop a deterministic model to examine the multi-projects interactive effects on developer's investment and development strategies. The model will also evaluate how investment strategies change under different market situations and for different project types, either homogeneous or heterogeneous. Under the constant demand and cost functions, portfolio effects of multiple projects are examined. The portfolio effects, in this context, refer to the spill-over benefits generated by the second project when two projects located in close proximity enjoy positive externality. The spill-over effects include higher revenue or lower cost accrued to the second development project vis-à-vis the case when the two projects are developed as if they are independent or by two independent developers.

This study examines the externalities between different projects from the perspective of a single developer, and how the portfolio effect is on the developer's investment strategy. Portfolio effects mean the intra/inter project externality where the developer will get spill over profit from projects in proximity sites as a result of economics of scale. In a monopolistic market, developers could also adjust their pricing strategies to increase profit via an increase in the quantity of projects. In addition, scale effects of multiple projects can reduce the total costs, which include financing cost, material cost, and construction cost.

In this chapter, we set a basic optimal development timing strategy model with only one developer, who will develop two projects in a deterministic framework. We explore how the portfolio effect will affect the developer's investment strategy. We make some basic assumptions. Firstly, we assume there is only one developer in a monopolistic market setting. The developer can choose a quantity at a given price such that he maximizes the profits of the two projects collectively. These two projects are identical. Secondly, we assume a deterministic demand and a deterministic cost function, which will be further relaxed in Chapter 4. The identical project assumption is then extended to include heterogeneous projects for the purposes of analyzing the portfolio effect. In the deterministic demand and cost framework, some intuitive results are derived. We found that the developer will abort the projects when the demand is weak, and he would choose to develop a single project when the demand curve is steep. In a market with flat demand curve, it will be optimal for the developer to develop the two projects simultaneously. The positive portfolio effects shorten the time to wait to develop for the two projects, and the developer will prefer to undertake the two projects simultaneously.

This chapter is organized as follows. Section 3.1 provides a general background of investment strategy, real option theory, and economics externality. Section 3.2 specifies the multi-projects investment strategy model in a deterministic framework with necessary assumptions. Section 3.3 and Section 3.4 analyze the investment strategies when one developer has two projects without and with portfolio effects. Section 3.5 concludes the findings with implications for investment strategies under different market situations.

3.2 Model Specification

In a simple deterministic framework, we assume that a developer has two projects at time t. The two projects are labeled project 1 and project 2 respectively, [$i = (1, 2)$]. Given that a unit project operating cost of k_i, and a unit project revenue of r_i, the unit project payoff upon completion at time t_i can be estimated as [$f_i = r_i - k_i$]. t_1 and t_2 respectively represent the development commencement time for project 1 and project 2. The two projects are assumed to have the same completion time of δ, and for simplification of the derivation, the time to build is reduced zero ($\delta=0$). This completion time assumption could be relaxed to allow $\delta>0$, and the optimal development time will then have a lead time of δ in the projects, and the optimal exercising time will not be significantly changed. The interactive effects between two projects is denoted by θ, where θ_{ij} denotes the positive externality spilled-over to project i upon the completion of project j. The developer is assumed to be risk-neutral. The portfolio return for the two projects expected by the developer is assumed to be a constant risk-adjusted rate of ρ, which reflect a systematic risk premium of [ρ - r], where r is the risk-free rate of return.

The scale of projects is measured in term of gross built-up areas of the property, q_i, and for N

number of projects completed at time *t,* the total quantity can be computed as [$Q_t = \sum_{i=1}^{N} q_i$].

Assume that supply of new real estate space can be quickly absorbed by in a perfectly elastic market in equilibrium; the gross revenue for a project can be written as a demand function as follows:

$$[R_i = y.q_i.D(Q_t)] \tag{3.1}$$

where y is systematic demand shocks, and $D(Q_t)$ is an inverse differentiable demand function, which decreases in the aggregate real estate space at time t, Q_t, such that [$D'(Q_t) < 0$].

In a deterministic market with no systematic shocks to demand for new space, [$y = 1$], the gross project values for projects 1 and 2, where the project completion time is given as [$t_1 < t_2$][2], can be written as follows,

$$\begin{aligned} R_1 = & \int_{t_1}^{t_2} D(q_1) \cdot q_1 \cdot \exp(-\rho \cdot t) \cdot dt + \\ & \int_{t_2}^{\infty} D(q_1 + q_2) \cdot q_1 \cdot (1 + \theta_{12}) \cdot \exp(-\rho \cdot t) \cdot dt \end{aligned} \tag{3.2}$$

$$R_2 = \int_{t_2}^{\infty} D(q_1 + q_2) \cdot q_2 \cdot (1 + \theta_{21}) \cdot \exp(-\rho \cdot t) \cdot dt \tag{3.3}$$

We assume that the developer will firstly invest in the project 1 and then invest in project 2. All the results will be the same when projects 1 and 2 are identical and symmetric, and the assumptions are unchanged.

On the cost side, the unit operation cost, k_i, is defined as a differentiable function of the production inputs, which include material and labor, and the aggregate quantity of the inputs

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2 The time-to-build is assumed to be zero, which implies that the project can be built instantly upon exercising the option to develop, an assumption that is consistent with the real options literature (Williams, 1991; Quigg, 1993; Sing, 2001).

is denoted as $S(Q_M)$. When projects commence either sequentially, or simultaneously, demand for input resources increases the unit cost of project at a positive and increasing rate, such that the first differential condition is positive, [$S'(Q_M) > 0$]. In addition, we also assume that the unit operation cost k_i is fixed at the initial time period. The cumulative cost functions for projects 1 and 2 over the entire project lifespan, which has an infinite tenure, can be respectively written as follows:

$$K_1 = \int_{t_1}^{\infty} S(Q_M) \cdot q_1 \cdot \exp(-r \cdot t) \cdot dt \tag{3.4}$$

$$K_2 = \int_{t_2}^{\infty} S(Q_M) \cdot q_2 \cdot \exp(-r \cdot t) \cdot dt \tag{3.5}$$

Based on the above assumption, the unit operation cost k_i is only fixed at time zero, and the cost may increase or decrease following the Q_M function. For a portfolio of N projects, the project payoffs will be computed as the summation of the net present values of the N projects at completion t_i. The objective function for the developer, who owns the N-project portfolio, can be written as follows:

$$F_t = \sum_{i=1}^{N} (R_i - K_i - C_i).\exp(-r \cdot t_i) \tag{3.6}$$

where C_i is the construction costs (inclusive of land cost) for project i, and the cumulative construction costs are given as [$\overline{C} = \sum_{i=1}^{N} C_i$]. If the two projects were developed simultaneously at time, [$t_i = t_1 = t_2 = 0$], the project payoffs for the projects is simply given as,

$$F = (R_1 - K_1) + (R_2 - K_2) - (C_1 + C_2) \tag{3.7}$$

3.3 Investment Strategies for Independent Projects

In this section, we assume the two projects are independent, but identical. Let us define that the project quantity equals to q_1 and q_2. There are no portfolio or interactive effects in this section, which means that $\theta_{12} = 0$, $\theta_{21} = 0$. Under the assumption that one developer has two homogenous and identical projects, when he makes a decision to invest, he has the following options, either to:

1. develop only one project
2. develop two projects simultaneously
3. develop two projects sequentially
4. wait until future market uncertainty is low

3.3.1 Single Project Development

If the developer invests in only one project, we assume that the developer invests in project 1. In equilibrium, the aggregate demand quantity is represented by Q, and the new supply from the project will add to aggregate demand incrementally, $[Q + \Delta_i]$ at time t_i, where $[\Delta_1 = q_1]$ at time t_1, and $[\Delta_2 = q_1+q_2]$ at time t_2. Based on the above assumptions, we could get the profit of the single project development, as denoted by F_{sig}:

$$F_{sig} = \frac{D(Q+\Delta_1)\cdot q_1}{\rho}\exp(-\rho\cdot t_1) - \frac{S(Q_{\mathrm{M}})\cdot q_1}{r}\exp(-r\cdot t_1) - C \qquad (3.8)$$

The profit from the single project development is the present value of the total perpetual income from all units supplied (discounted at the growth rate) minus total perpetual operation costs encountered during the infinite life of the investment and the construction cost.

To get the optimal payoff, we take the F.O.C $\frac{\partial F_{sig}}{\partial t_1} = 0$

$$t_1^* = \frac{\ln \frac{D(Q+\Delta_1)}{S(Q_M)}}{\rho - r} \qquad \text{(Local Minimum)}$$

F_s reaches the minimum at t_1^*, because when $t_1 \rightarrow t_1^{*-}$, $\frac{\partial F_{sig}}{\partial t_1} < 0$, and when $t_1 \rightarrow t_1^{*+}$, $\frac{\partial F_{sig}}{\partial t_1} > 0$. It implies that F_{sig} decreases as t_1 increases on the left side of t_1^*, and on the opposite side, F_{sig} increases as t_1 increases on the right side of t_1^*. When $t \rightarrow +\infty$, $F_{sig} \rightarrow -C$, the developer will decide whether to invest in t=0. The condition of the developer's investment decision at $t = 0$ is $F_{sig} > 0$.

The method is consistent with the conventional literature on real options. The project is evaluated by its discounted cash flows and is triggered based on the rule that the NPV is positive. In our model, demand shock is a stochastic process, and uncertainty in the decision making process is added in the model. The volatility created by the demand shocks implies that the timing to wait becomes significant to the investors.

Proposition 3.1

The profit of single project will decrease first, and then increase under the deterministic demand market as the time goes. So the developer will determine whether to invest immediately.

If the developer has one project, under the market situation

$$[\frac{D(Q+\Delta_1)}{\rho} - \frac{S(Q_M)}{r}] \cdot q_1 > C$$

He will develop the single project when t = 0, and get the profit

$$F_{sig} = [\frac{D(Q+\Delta_1)}{\rho} - \frac{S(Q_M)}{r}] \cdot q_1 - C$$

Otherwise, the developer will abort the project.

3.3.2 Simultaneous Development Strategy

If the developer would like to invest in two projects simultaneously, it equals to invest in one large project since the two projects are homogenous and identical. We assume that the developer invests in project 1 and project 2 simultaneously. Based on the above assumptions, we could derive the collective profit of both projects denoted as F_{sim} as follows:

$$F_{sim} = \frac{D(Q+\Delta_2)(q_1+q_2)}{\rho}\exp(-\rho \cdot t_1) - \frac{S(Q_M)(q_1+q_2)}{r}\exp(-r \cdot t_1) - 2C \qquad (3.9)$$

To obtain the optimal payoff, we take the F.O.C $\frac{\partial F_{\text{sim}}}{\partial t_1} = 0$

$$t_1^* = \frac{\ln \frac{D(Q+\Delta_2)}{S(Q_M)}}{\rho - r} \text{ (Local Minimum)}$$

Like in the single development case, F_{sim} decreases as t_1 increases to the left of t_1^*, and on the opposite side, F_{sim} increases as t_1 increases to the right of t_1^*. When $t \to +\infty$, $F_{\text{sim}} \to -C$. The collective profit of the simultaneous development of two projects will change as the development timing changes, the shape of which is similar to the single development case. The developer should decide whether to invest in the two project at $t=0$. The condition of the developer's investment decision at $t=0$ is $F_{\text{sim}} > 0$.

Proposition 3.2

The collective profit of two projects developed simultaneously will decrease first, and then increase under the deterministic demand market. If the investment time is infinite, the collective profit will be close to -2C. So the developer will determine whether to invest in the two projects immediately. That is to say, two homogenous projects developed simultaneously *are indifferent from a single project of which the quantity is the sum of two projects.*

If the developer have two projects, under the market situation

$$[\frac{D(Q+\Delta_2)}{\rho}-\frac{S(Q_M)}{r}](q_1+q_2)>2C$$

He will develop the two projects simultaneously at t=0, and get the collective profit

$$F_{sim} = [\frac{D(Q+\Delta_2)}{\rho}-\frac{S(Q_M)}{r}](q_1+q_2)-2C$$

Otherwise, the developer will abort the both projects.

3.3.3 Sequential Development Strategy

In the third development option, when the developer invests in two projects sequentially. Although the two projects are homogenous and identical, we assume that the developer invests in project 1 at t_1 and then in project 2 at t_2. According to the assumptions above, we could get the collective profit of both projects, denoted as F_{seq}.

$$F_{seq} = \left[\frac{D(Q+\Delta_1)q_1[\exp(-\rho\cdot t_1)-\exp(-\rho\cdot t_2)]+D(Q+\Delta_2)\cdot(q_1+q_2)\exp(-\rho\cdot t_2)}{\rho}\right] - \left[\frac{S(Q_M)q_1\exp(-r\cdot t_1)+S(Q_M)q_2\exp(-r\cdot t_2)]}{r}+2C\right] \tag{3.10}$$

To optimize payoff, we take the F. O. C $\frac{\partial F_{\text{seq}}}{\partial t_1}=0$, $\frac{\partial F_{\text{seq}}}{\partial t_2}=0$

$$t_1^* = \frac{\ln\frac{D(Q+\Delta_1)}{S(Q_M)}}{\rho-r} \qquad \text{(Local Minimum)}$$

$$t_2^* = \frac{\ln\frac{[D(Q+\Delta_2)\cdot(q_1+q_2)-D(Q+\Delta_1)\cdot q_1]}{S(Q_M)\cdot q_2}}{(\rho-r)} \qquad \text{(Local Minimum)}$$

$t_1^* < t_2^*$ is not a necessary outcome. t_1^* is a time at which the profit of project 1 is minimum, whereas t_2^* is the time at which the profit of project 2 is minimum. Both t_1^* and t_2^* are not the optimal development time, and therefore, the condition $t_1^* < t_2^*$ is not binding.

Provided that t_2 is fixed, F_{seq} reaches the minimum at t_1^*, because when $t_1 \to t_1^{*-}$, $\frac{\partial F_{\text{seq}}}{\partial t_1} < 0$, and when $t_1 \to t_1^{*+}$, $\frac{\partial F_{\text{seq}}}{\partial t_1} > 0$. It implies that F_{seq} decreases as t_1 increases to the left of t_1^*, and on the opposite side, F_{seq} increases as t_1 increases to the right of t_1^*. When $t_1 \to +\infty$, $F_{\text{seq}} \to -C$. For project 2, F reaches the minimum at t_2^*, because when $t_2 \to t_2^{*-}$, $\frac{\partial F_{\text{seq}}}{\partial t_2} < 0$, and when $t_2 \to t_2^{*+}$, $\frac{\partial F_{\text{seq}}}{\partial t_2} > 0$. It implies that F_{seq} decreases as t_2 increases to the left of t_2^*, and on the opposite side, F_{seq} increases as t_2 increases to the right of t_2^*. When $t_2 \to +\infty$, $F_{\text{seq}} \to -2C$. The developer should decide whether to invest in the projects at $t=0$. The situation will turn up to be the same as the development of both projects simultaneously at $t=0$

Proposition 3.3

If both projects are homogenous and identical under the deterministic market scenario, the developer will develop both projects simultaneously, instead of sequentially. The suboptimal strategy may be adopted for reasons other than profit maximization. The developer may be limited by capital or manpower resources to simultaneously develop both projects, or he will prefer to hold the development option of one project for future consideration.

3.3.4 Comparison of the Strategies between a Single Project Development and a Simultaneous Development

Compared to development of both projects simultaneously, the sequential development is not a less optimal strategy, if the developer has two independent projects given the assumptions above. Under the objective of maximizing the collective profit of the developer, we should compare the profit according to the two different strategies as follows, single project development and simultaneous development.

The profit of developing a single project at *t=0* is given in equation (8), while the profit of developing both projects simultaneously at *t=0* is given in equation (9)

If the profit of simultaneous development is larger than the single development, $F_{\text{sim}} > F_{\text{sig}}$,

the market condition is:

$$\frac{D(Q+\Delta_2)(q_1+q_2)-D(Q+\Delta_1)\cdot q_1}{\rho}\exp(-\rho\cdot t_1)-\frac{S(Q_M)\cdot q_2}{r}\exp(-r\cdot t_1)>C$$

If the above is not true, then the profit of simultaneous development is smaller than the single development.

Proposition 3.4

If one developer has two homogeneous and identical projects for development, the profit will decrease first, and then increase in time under the deterministic demand market. But if the investment time is infinite, the profit will be negative. So the developer will make investment decision at time zero.

If under the market situation that simultaneous development strategy gives more profit, and *the profit is positive as follows*

$$\frac{D(Q+\Delta_2)(q_1+q_2)-D(Q+\Delta_1)\cdot q_1}{\rho}-\frac{S(Q_M)\cdot q_2}{r}>C$$

and $[\frac{D(Q+\Delta_2)}{\rho}-\frac{S(Q_M)}{r}](q_1+q_2)>2C$

the developer will develop both projects simultaneously at t=0.

However, when

$$\frac{D(Q+\Delta_2)(q_1+q_2)-D(Q+\Delta_1)\cdot q_1}{\rho}-\frac{S(Q_M)\cdot q_2}{r}<C,$$

and $[\frac{D(Q+\Delta_1)}{\rho}-\frac{S(Q_M)}{r}]\cdot q_1>C$

the developer will develop only a single project at t=0. Otherwise, the developer will abort the projects.

The decision to develop one or both projects depends on the market situation as represented by the demand curve. When the slope of demand curve is very steep, the developer will prefer to develop only one project at a time, otherwise, he will prefer to develop both projects simultaneously.

3.4 Investment Strategy with Portfolio Effects

3.4.1 Basic Model

The earlier deterministic models are extended to incorporate portfolio effects, so that the opportunity cost of suboptimal waiting in the sequential options can be analyzed on the assumption that the two projects are inter-dependent. Project 1 and project 2 are different in quantity, such that different revenue, cost, and different profit function are defined.

We introduce the externality effect, as denoted by θ, to the model. The value of θ_{ij} measures the strength of the interactive effects from project j to project i. We assume that the developer invests in project 1 first and then followed by project 2. The sequence of development of project 1 and project 2 is inter-changeable, and the result is just the opposite. Based on the above assumptions, the profits for project 1 and project 2, and their collective profit, are given below as F_1, F_2, and F respectively:

$$F_1 = \frac{D(Q+\Delta_2)\cdot(1+\theta_{12})\cdot q_1\cdot\exp(-\rho\cdot t_2)+D(Q+\Delta_1)\cdot q_1\cdot[\exp(-\rho\cdot t_1)-\exp(-\rho\cdot t_2)]}{\rho}$$
$$-\frac{S(Q_M)\cdot q_1\cdot\exp(-r\cdot t_1)}{r}-C \tag{3.11}$$

$$F_2 = \frac{D(Q+\Delta_2)\cdot q_2\cdot(1+\theta_{21})]\cdot\exp(-\rho\cdot t_2)}{\rho}-\frac{S(Q_M)\cdot q_2\cdot\exp(-r\cdot t_2)}{r}-C \tag{3.12}$$

$$\begin{aligned} F \quad &= \frac{D(Q+\Delta_1)\cdot q_1[\exp(-\rho\cdot t_1)-\exp(-\rho\cdot t_2)]}{\rho} \\ &+\frac{D(Q+\Delta_2)\cdot[q_1\cdot(1+\theta_{12})+q_2\cdot(1+\theta_{21})]\cdot\exp(-\rho\cdot t_2)}{\rho} \\ &-\frac{S(Q_M)[q_1\cdot\exp(-r\cdot t_1)+q_2\cdot\exp(-r\cdot t_2)]}{r}-2C \end{aligned} \tag{3.13}$$

By taking the first order derivations of Equations (3.13) with respect to the interactive factor, θ_{ij}, the incremental effects by integrating the two projects as a portfolio can be represented as follows:

$$\frac{dF}{d\theta_{ij}} = \frac{D(Q+\Delta_2).q_i.\exp(-\rho.t_2)}{\rho} \tag{3.14}$$

where [$i = (1, 2)$]; the externality effects are only created upon the completion of the second project at t_2.

Proposition 3.5:

In the presence of positive externality effects, a single developer who possesses development options on two contiguous land parcels could optimize the project payoffs by integrating the two projects, such that there are positive spill-over effects from one project to another project, whether the two projects are developed simultaneously or sequentially. The incremental project payoff created on project i as a result of positive externality effects from project j is dependent on the scale of the development i, the market structure and also the risk-adjusted rate of return (Equation 3.14).

As one developer holds the two projects, he hopes to maximize the collective profit of the two projects, and the optimal payoff can be derived by taking F.O.C, $\frac{\partial F}{\partial t_1} = 0 \quad \frac{\partial F}{\partial t_2} = 0$

$$t_1^* = \frac{\ln \frac{D(Q+\Delta_1)}{S(Q_M)}}{\rho - r} \qquad \text{(Local Minimum)}$$

$$t_2^* = \frac{\ln \frac{[D(Q+\Delta_2)\cdot[q_1(1+\theta_{12}) + q_2(1+\theta_{21})] - D(Q+\Delta_1)\cdot q_1]}{S(Q_M)\cdot q_2}}{(\rho - r)} \qquad \text{(Local Minimum)}$$

The minimum point is similar to that in section 3.3. Provided that t_2 is fixed, F reaches the minimum at t_1^*. F decreases as t_1 increases to the left of t_1^*, and on the opposite side, F increases as t_1 increases to the right of t_1^*. When $t_1 \to +\infty$, $F \to -2C$. Given that t_1 is fixed, F reaches the minimum at t_2^*. F decreases as t_2 increases to the left of t_2^*, and on the opposite side, F increases as t_2 increases to the right of t_2^*. When $t_2 \to +\infty$, $F \to -2C$.

To explore how fast the profit increases or decreases as the time goes, we compute S.O.C,

$$\frac{d^2F}{dt_1^2}=0\ t_1^{'}>t_1^{*},\ \ t_1^{'}=\frac{\ln\frac{\rho\cdot D(Q+\Delta_1)}{r\cdot S(Q_M)}}{\rho-r}$$

$$\frac{d^2F}{dt_2^2}=0\ t_2^{'}>t_2^{*},\ \ t_2^{'}=\frac{\ln\frac{\rho\{[D(Q+\Delta_2)\cdot[q_1(1+\theta_{12})+q_2(1+\theta_{21})]-D(Q+\Delta_1)\cdot q_1]\}}{r\cdot S(Q_M)\cdot q_2}}{(\rho-r)}$$

Figure 3.1 below shows that the changes in collective profit of both projects with respects to changes in the development timing. Given that t_2 is fixed, F decreases as t_1 increases to the left of t_1^*, but at a decreasing rate. Then, between t_1^* and $t_1^{'}$, F increases as t_1 increases, and at an increasing rate. On the right side of the $t_1^{'}$, F increases as t_1 increases, but at a decreasing rate. F's changes in t_2, and the rate of change is similar to that in t_1. Given that t_1 is fixed, F decreases as t_2 increases to the left of t_2^*, but at a decreasing rate. Between the t_2^* and $t_2^{'}$, F increases as t_2 increases, and at an increasing rate. On the right side of the $t_2^{'}$, F increases as t_2 increases, but at a decreasing rate.

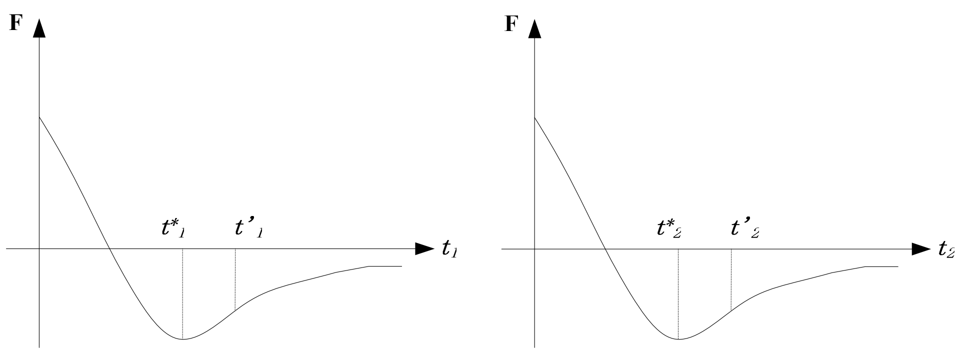

Figure 3.1 Collective profit of projects changes as the investment time changes

Given that t_2 is fixed and $t_1>t_1^*$, F increases as t_1 increases and $t_1\rightarrow+\infty,\ \ F\rightarrow-2C$. Project 1 will not be developed after t_1^*. F however decreases as t_1 shifts to the left of t_1^*, such that the development decision will be made at $t=0$. The change of F in t_2 is similar to that in t_1, so the developer will make his decision at $t=0$.

If $F>0$ at $t=0$, the developer will invest in project 1 at $t=0$. Otherwise, the developer will abort both projects. Considering project 2 after the investment in project 1, the developer will invest in the following project at $t=0$ if $F>0$ at $t=0$. The market condition is given as below:

$$\frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})]}{\rho}-\frac{S(Q_M)(q_1+q_2)}{r}>2C$$

Proposition 3.6

If the revenue is large enough, that is

$$\frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})]}{\rho}-\frac{S(Q_M)(q_1+q_2)}{r}>2C$$

The developer will invest in both projects simultaneously at time t=0.

The construction cost, construction period, and market demand will affect the optimal development time. The shorter the construction period is, and the lower the construction cost is, the higher will be the probability of immediately development. Obviously, the developer will prefer to develop the projects in an upward increasing market when it can obtain higher revenues.

3.4.2 Investment Strategies

Based on the model above, we compare the strategies of a single project development and a simultaneous development.

Given the following market demand condition:

$$\frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})]}{\rho}-\frac{S(Q_M)(q_1+q_2)}{r}>2C$$

the developer makes simultaneous development at $t=0$, and he will get the following collective profit:

$$F \quad = \frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})]}{\rho} - \frac{S(Q_M)(q_1+q_2)}{r} - 2C \tag{3.15}$$

When the developer only develops project 1 at *t=0*, the profit is given below

$$F_1 = \frac{D(Q+q_1).q_1}{\rho} - \frac{S(Q_M)\cdot q_1}{r} - C \tag{3.16}$$

When the developer only develops project 2 at *t=0*, the profit is given below

$$F_2 = \frac{D(Q+q_2).q_2}{\rho} - \frac{S(Q_M)\cdot q_2}{r} - C \tag{3.17}$$

If $F > F_1$ and $F > F_2$ the developer will choose the simultaneous development strategy.

Simultaneous development strategy is better than development of Project 1 alone, if the following condition is complied with: $F > F_1$

$$\frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})] - D(Q+q_1)\cdot q_1}{\rho} - \frac{S(Q_M)\cdot q_2}{r} > C$$

The simultaneous development strategy is better than developing Project 2 alone, if the following condition is satisfied: $F > F_2$

$$\frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})] - D(Q+q_2)\cdot q_2}{\rho} - \frac{S(Q_M)\cdot q_1}{r} > C$$

Otherwise, the developer will choose to develop only a single project.

The developer will invest in project 1, if $F_1 > F_2$. Project 1 has a higher payoff than Project 2 under the following condition:

$$\frac{D(Q+q_1).q_1 - D(Q+q_2).q_2}{\rho} > \frac{S(Q_M)\cdot(q_1-q_2)}{r}$$

If we exchange the subscript 1 for 2, the results will be the opposite.

Proposition 3.6

The developer will invest in both projects simultaneously at t=0 if it satisfies all three conditions below:

Market Condition of Simultaneous Development:

$$\frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})]}{\rho}-\frac{S(Q_M)(q_1+q_2)}{r}>2C$$

Simultaneous development is better than Project 1 under the following condition:

$$\frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})]-D(Q+q_1)\cdot q_1}{\rho}-\frac{S(Q_M)\cdot q_2}{r}>C$$

Simultaneous Development is better than Project 2 if and only if :

$$\frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})]-D(Q+q_2)\cdot q_2}{\rho}-\frac{S(Q_M)\cdot q_1}{r}>C$$

The developer will invest in project 1 at t=0 if it satisfies all three conditions below.

Market Condition for Single Development of Project 1is give as:

$$\frac{D(Q+q_1).q_1}{\rho}-\frac{S(Q_M)\cdot q_1}{r}>C$$

The simultaneous development strategy is worse than Project 1 under the following condition:

$$\frac{D(Q+\Delta_2).[q_1.(1+\theta_{12})+q_2.(1+\theta_{21})]-D(Q+q_1)\cdot q_1}{\rho}-\frac{S(Q_M)\cdot q_2}{r}<C$$

Project 1 is better than Project 2 if and only if:

$$\frac{D(Q+q_1).q_1-D(Q+q_2).q_2}{\rho}>\frac{S(Q_M)\cdot(q_1-q_2)}{r}$$

3.4.3 Effects of Market Demand Elasticity

Equation (3.14) indicates that the incremental portfolio value created by inter-project externalities is a function of the inverse market demand. Let [$g(Q)= dF/d\theta_{ij}$]. By taking the first order derivation of $g(Q)$ with respect to the aggregate demand, Q, the effects of demand elasticity on value associated with positive externalities can be explained as follows:

$$\frac{dg}{dQ} = \frac{q_i.\exp(-\rho.t_2)}{\rho}.D'(Q) \tag{3.18}$$

Proposition 3.7:

If the two projects are of homogenous type, the shift along the same market demand curve, [$D_1(Q)$]*, will change the demand elasticity of the project. The externality effects on the project payoffs will be affected by different market demand condition. In a weak market (point A), where quantity of demand is small, the externality effects on project payoff will be greater compared to a boom market (point B) where demand elasticity is relatively flatter* (see Figure 3.2).

Lemma 1:

For two heterogeneous projects represented by two different market demand curves, [$D_1(Q)$, $D_2(Q)$] the effects of inter-project externality in a portfolio with projects having steeper demand curve (point B) will be larger than those projects with demand elasticity represented by point C (see Figure 3.2).

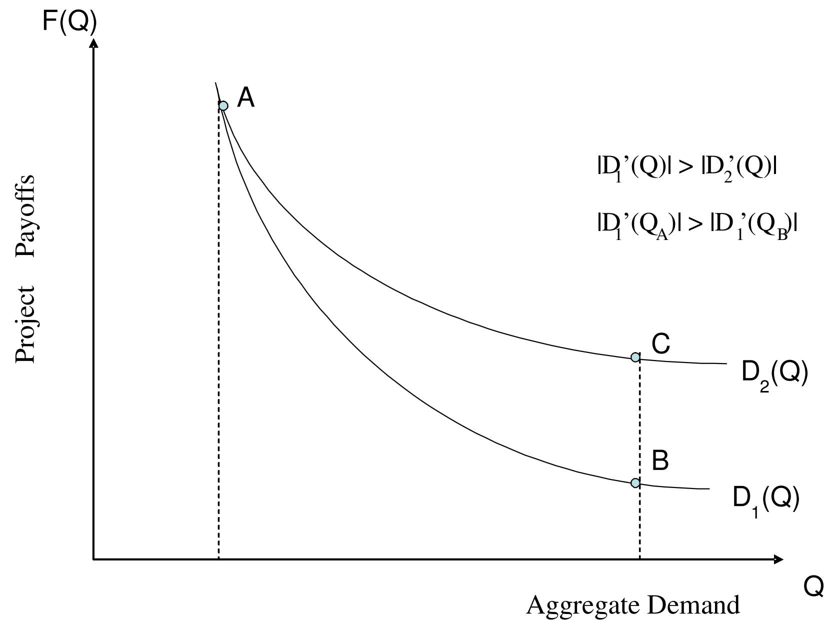

Figure 3.2: Effects of Market Demand Elasticity on Project Externalities

Given that the demand elasticity is negative, the effects of inter-project externality in a portfolio will be larger for projects having steep demand curve compared to those with flat demand curve. Price decreases at a faster rate in a steep demand curve. The developer will prefer simultaneous development in a flat demand curve to that with a steep demand curve.

3.4.4 Market-Induced Externality Effects

In the project payoff functions in Equation (3.13), inter-project externality effects are represented by a non-negative exogenous variable, [$\theta_{ij} \geq 0$]. An alternative way to represent the project interactive effects is through market adjustment that is endogenous to the demand curve. If the two projects could be well integrated to maximize positive externality benefits, project payoffs are enhanced collectively, which is represented by an upward shift in the demand curve from point A to point B (Figure 3.3).

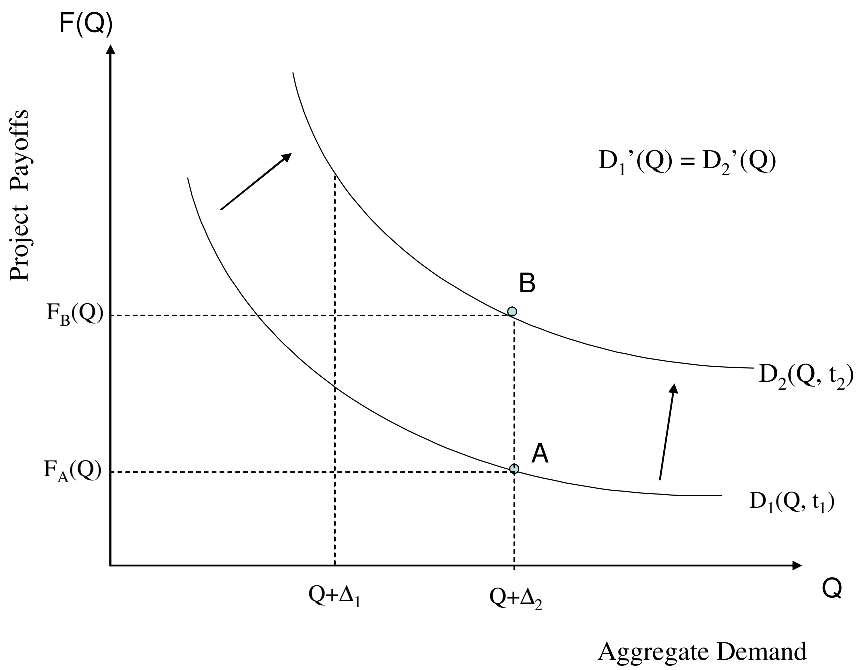

Figure 3.3: Structural Shift in Market Curve with Externality Effects

The developer's decision to invest in either one project or both projects depends on the project profits in the two strategies. Construction cost, construction period, market demand, and the portfolio effect θ will affect the development strategy. The lower the construction cost is, the higher is the probability of simultaneous development. Developer will prefer to invest in an upward market where he can increase project revenues. The larger is the portfolio effect, the higher will be the probability of simultaneous development. Figure 3.3 describes the portfolio effect. D_1 is the demand curve when the developer makes the decision to invest in the first project. After investing in the second project, the demand curve shifts from D_1 to D_2 because of the interactive effect of multi-project. This positive externality effect increases developer's revenue to $F_B(Q)$, compared to the revenue at $F_A(Q)$ under the demand curve D_1.

In the portfolio effects, the developer is assumed to maximize the portfolio values by creating positive externalities between the two projects. If the externality is negative, the model's assumptions will be changed, and the collective benefits would not be maximized. This will not be in line with the objective of a profit maximizing developers. However, the negative externality case could be extended in a case involving two competing developers, who would try to pre-empt the competitor through exercising the options earlier. The interactive effects between the two competing developers in a game theoretical framework are not within the scope of this study.

3.5 Implications and Conclusion

A developer's investment decision is affected by the demand elasticity in the market. In a deterministic framework, we assume the project cost is constant, and the demand curve will not change unless new project enters the market. Under the strict assumptions of independent projects, it is obviously that the developer will make the investment decision according to the market demand at the beginning. If the demand is low and the revenue from the project is not enough to cover the cost, the developer will abort the projects and wait for the market to recover. In a boom market, the developer will not give up the opportunity to develop. If the demand curve is steep, which means that when the demand quantity increases, the price will decrease very quickly, the developer will choose to only develop single project and keep the second investment opportunity until the market expands further. If the demand curve is flat, the developer will choose to develop the projects simultaneously to maximize his profit. In the deterministic framework, all information is known, and the value of option to wait is negligible in the market. The developer will give up the sequential development, which is less optimal compared to the simultaneous development strategy.

When we expand the model to include heterogeneous projects with portfolio effect, the basic model propositions remain unchanged. In a boom market, the developer will choose a single project development when the demand curve is steep, and develop both projects simultaneously when the demand curve is flat. The selection of project 1 or project 2 depends on which one could bring him a higher profit. Another important factor affecting the decision is multi-projects externality. The developer will prefer to invest in both projects simultaneously, if there is a high multi-projects externality. Positive externality reflects benefits as cost saving, sharing of management overheads and price overspills are obtained. For the same reason in the homogenous model, the developer will give up the sequential development option under the deterministic framework.

One of the testable hypotheses that can be developed is to examine development strategies in large projects that are undertaken in phases. If positive externalities are created, we should expect prices of subsequent phases of development to decrease, such that the developer can preempt other competitors from entering a competitive market. However, if a market is tightly controlled by only few large developers with monopolist power, he may maximize their profits by levying externality premiums on existing tenants that are benefited from spill-over effects from development in the later phases. It would also be challenging to observe the development strategies adopted by developers who own development options on multiple projects in a portfolio at any one time, whether they would pursue simultaneous strategies for projects that establish positive externalities, and defer projects that are mutually exclusive and independent.

Table 3.1 summarizes different strategies under different market situations for projects with and without portfolio effect in the deterministic framework.

	single	Simultaneous	Sequential	Abort
Without Portfolio Effect	Boom market Steep demand curve	Boom market flat demand curve	Instead by simultaneous	Low market
With Portfolio Effect	Boom market Steep demand curve Low externality	Boom market flat demand curve High externality	Instead by simultaneous	Low market

Table 3.1 Strategies of development with and without portfolio effect

As the real demand is not deterministic, the demand changes as time goes. One extension is that we could consider the stochastic demand. We could assume the demand is in a Brownian motion Process (as in Grenadier 1996). This will be explored in the next chapter. The cost of project changes over times, and is not constant. We will also think about the effect of stochastic cost function. Another extension is that we could expand two projects into n projects case, so that the multi-projects externality can be analyzed. The third extension is that we could consider the competitive strategies in a game theoretic framework, if two developers have two portfolios.

Chapter 4 Inter-Project Externality in Optimal Development Timing Strategies in Stochastic Framework

4.1 Introduction

There are two investment decisions faced by investors, one is the capital allocation, and the other is the investment timing. In the option timing literature, researchers build a real options model to find the optimal development timing in an uncertain future. McDonald & Siegel (1986) studied the optimal timing of investment of an irreversible project under the assumptions that project payoff and investment cost follow continuous time stochastic processes. Majd & Pindyck (1987) used a contingent claims analysis to derive optimal decision rules for irreversible investments under uncertainty. Milne & Whalley (2001) solved two models of time to build, one where production cannot be suspended once started, and another with costless suspension of production. They used numerical method to analyze the dynamics of work-in-progress.

The previous literature assumes a single project and the value of one project is determined at a local maximum point. In the actual property market, however, a developer always manages more than one project at any one time, these projects are located close to each other, and they share common resources, such that there exist interactive effects between the two projects. When developer makes a decision, he always maximizes the profit of the company, instead of the projects, which is a global maximum. Our study attempts to find the optimal timing to invest in projects, if one developer has more than one project waiting to be developed. We explore a case of multi-projects owned by the same developer and focus on the externalities between different projects. We consider how the portfolio effect will affect a developer's decision on optimal development timing. Will the developer expedite or defer the development when future is uncertain?

Portfolio effect means the intra/inters project externality, which will create spill-over benefits to a developer of the projects through economies of scale. Developers could use pricing and quantity strategies to maximize profits for projects under different market conditions: competitive, monopoly or duopoly. The multiple-projects externalities can also reduce the total costs, which include financing cost, material cost, and construction cost.

In a stochastic framework, the real demand market changes over time. We could assume that the demand follows a geometric Brownian motion process (as in Grenadier (1996)). For simplification of model, we assume that the cost of project is constant. We set up a stochastic framework with one developer who has development options on two different but contiguous land parcels, and the developer will have the options to develop the two projects simultaneously or sequentially. The two land parcels can be developed either as a homogeneous or two heterogeneous projects. In our model, we explore how the portfolio effect will affect the developer's optimal development timing decision, while the demand follows a stochastic process. We try to maximize the collective profit of the two projects owned by the developer when the embedded timing option value is valuable.

The model evaluates whether the developer will make simultaneous development or sequential development under different market situation, and how the portfolio effect will impact the optimal development timing of the two projects. The positive interactive effects between the projects will push the developer to trigger the development options on the two projects earlier. The developer will make simultaneous development, if the portfolio effect is strong enough to offset the opportunity costs of not waiting for one more period. In other words, the portfolio effect lowers the trigger threshold value of investment for the second project. He will otherwise be better off by delaying the development of the second project, which results in a sequential development process. Next, we would also evaluate the development strategies of the developer under different market demand conditions. The developer will choose to develop a single project when the demand curve is steep, while in a market with flat demand curve, he will prefer to invest in both projects.

We then extend the model by including two developers, each develops the land separately. This model allows us to analyze negative externality effects, in which both developers will undertake non-cooperative strategies, and the completion of one project will "destroy" and reduce the value of the competing project. The negative externality is expected on the neighboring projects, when the competing developer pre-empts his competitor by exercising the development option earlier. We will compare different timing strategies from co-operative development by two developers to separate development by two developers. Other extensions we will think about include the effect of stochastic cost function and to expand two projects into N projects where multi-projects externalities are considered. The third extension in the model involves integrating sub-game optimal strategies of the developer in a real options framework by allowing the demand functions for the two projects to follow different stochastic processes with drift. Compared to the earlier findings in Grenadier (1996), who assumes only one stochastic demand process for the project in his model, the results of this proposed model will shed new lights on how optimal strategies of one developer will be changed under different game dynamic scenario and different demand conditions. This extension will be discussed in the following Chapter 5.

This section is organized as follows. Section 4.1 provides a general background of real option theory, project externality and game theory. Section 4.2 specifies the multi-projects optimal development timing model and also assumptions. Section 4.3 and Section 4.4 give the analytical solutions of the model and the numerical results. Section 4.5 shows the comparative statistics and sensitivity analysis. The last Section includes conclusion and future extensions.

4.2 Stochastic Model Specification

In the stochastic framework, we start with one developer, who has two projects, project 1 and project 2. We track the development time by denoting t_i [$i = (1, 2)$]. We also assume that the construction time for a project is $\delta = 0$. To consider the portfolio effect of the two projects, an interactive factor on project *i* from project *j*, θ_{ij}, is included. As in the traditional model, we assume that the developer is risk-neutral. We use the inverse demand function, denoted as $D(Q_t)$. On the other side, the construction cost and management cost are constant, I_i [$i = (1, 2)$]. *r* stands for risk-free rate, and ρ is the required return rate.

After the completion of a project, we could get future cash flows denoted by P_i. We use α, which means the comparable advantage, to differentiate the two projects. We assume $\alpha_1 > \alpha_2$, that is project 1 has more comparable advantage than project 2. The price will be impacted by the exogenous economic shocks denoted by *y*:

$$P_{it} = \alpha_i \cdot y_t \cdot D(Q_t) \qquad [i = (1, 2)] \tag{4.1}$$

The economic shock *y* is exogenous and follows a Geometric Brownian Motion Process as follows

$$dy = \mu \cdot y \cdot dt + \sigma \cdot y \cdot dz \tag{4.2}$$

μ is the instantaneous expected growth rate

σ is the instantaneous standard deviation

dz is increment of stand Wiener Process

From Equation (4.1) and (4.2), we could derive the price as a Geometric Brownian Motion Process, as follows

$$dP = \mu \cdot P \cdot dt + \sigma \cdot P \cdot dz \tag{4.3}$$

There will be portfolio effect from project *j* on project *i*, denoted as $(1+\theta_{ij})$. The revenue functions for project 1 and project 2 are given as follows:

$$R_1 = \int_{t_1}^{t_2} P_{1t} \cdot q_1 \cdot \exp(-\rho \cdot t) \cdot dt + \int_{t_2}^{\infty} P_{1t} \cdot q_1 \cdot (1+\theta_{12}) \cdot \exp(-\rho \cdot t) \cdot dt \tag{4.4}$$

$$R_2 = \int_{t_2}^{\infty} P_{2t} \cdot q_2 \cdot (1+\theta_{21}) \cdot \exp(-\rho \cdot t) \cdot dt \tag{4.5}$$

At first, the developer has two projects waiting for investment, and he has options to invest in project 1 and project 2. The completion of project 1 will have portfolio effect on project 2, such that the portfolio effect will impact on the development timing of project 2. This means that there are embedded options in the decision of investment in project 1, because of the portfolio effect. We could consider the problem as a two-stage real option valuation. At the first stage, we will decide the development timing of project 1, and then at the second stage, we make the decision of investment in project 2. We will use the back-forward method to solve the problem.

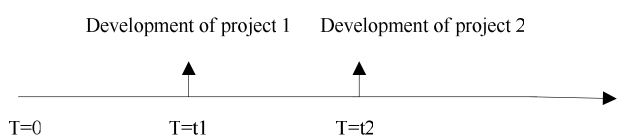

Figure 4.1 Two-stage options

After investment in Project 1, the total value of option of investment in project 2 is denoted by $G_2(Y)$, where the developer will receive the future cash flow P_1 per time under the assumption of risk neutrality.

The Ordinary Differential Equation is derived as follows:

$$\frac{1}{2}\sigma^2 Y^2 G_2^{''}(\mathrm{Y}) + \mu Y G_2^{'}(Y) + \alpha_1 D(Q+\Delta_1)Y - rG_2 = 0 \tag{4.6}$$

We define the boundary conditions: value matching, smooth pasting and initial condition, as follows:

$$G_2(Y_2^*) = \frac{\alpha_1(1+\theta_{12})D(Q+\Delta_2)}{r-u}Y_2^* - \frac{\alpha_1 D(Q+\Delta_1)}{r-u}Y_2^* + \frac{\alpha_2(1+\theta_{21})D(Q+\Delta_2)}{r-u}Y_2^* - I_2 \tag{4.7}$$

$$G_2^{'}(Y_2^*) = \frac{\alpha_1(1+\theta_{12})D(Q+\Delta_2)}{r-u} - \frac{\alpha_1 D(Q+\Delta_1)}{r-u} + \frac{\alpha_2(1+\theta_{21})D(Q+\Delta_2)}{r-u} \tag{4.8}$$

$$G_2(0) = 0 \tag{4.9}$$

Before the investment in Project 1, the total value of option of the developer is denoted as $G_1(Y)$. The developer has the option to invest in project 1 and project 2, and these two projects have portfolio effect, which reflects the embedded option value. $G_1(Y)$ is the total option, which can be defined in the Ordinary Differential Equation below:

$$\frac{1}{2}\sigma^2 Y^2 G_1^{''}(Y) + \mu Y G_1^{'}(Y) - rG_1 = 0 \tag{4.10}$$

The boundary conditions, which include value matching, smooth pasting and initial condition, are defined below:

$$G_1(Y_1^*) = \frac{\alpha_1 D(Q+\Delta_1)}{r-u}Y_1^* + G_2(Y_1^*) - I_1 \tag{4.11}$$

$$G_1^{'}(Y_1^*) = \frac{\alpha_1 D(Q+\Delta_1)}{r-u} + G_2^{'}(Y_1^*) \tag{4.12}$$

$$G_1(0) = 0 \tag{4.13}$$

4.3 Solution for Optimal Development Timing

Using the backward method, we first solve the Equation (4.6) subject to the boundary conditions of (4.7) and (4.8) and the initial condition of (4.9).

The general solution is given below:

$$G_2(Y) = A_1 Y^{\beta_1} + A_2 Y^{\beta_2} + H_1 Y \tag{4.14}$$

$$H_1 = \frac{\alpha_1 D(Q + \Delta_1)}{r - u} \tag{4.15}$$

where β is the root of fundamental quadratic equation given below:

$$\frac{1}{2}\sigma^2\beta^2 - (\frac{1}{2}\sigma^2 - \mu)\beta - r = 0 \tag{4.16}$$

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} > 1$$

$$\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} < 0$$

According to initial condition (4.9), when $Y \to 0,\ G_2(Y) \to 0$, we could omit the second component of Equation (4.14)

So $G_2(Y) = A_1 Y^{\beta_1} + H_1 Y$ (4.17)

Substituting into boundary conditions (4.7) and (4.8)

$$Y_2^* = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I_2}{H_2 - H_1} \tag{4.18}$$

$$H_2 = \frac{\alpha_1(1+\theta_{12})D(Q+\Delta_2)}{r-u} - \frac{\alpha_1 D(Q+\Delta_1)}{r-u} + \frac{\alpha_2(1+\theta_{21})D(Q+\Delta_2)}{r-u} \tag{4.19}$$

$$A_1 = (\frac{H_2 - H_1}{\beta_1})^{\beta_1} (\frac{I_2}{\beta_1 - 1})^{1-\beta_1}$$

After we get the option value of $G_2(Y)$, we solve backward to obtain the solution for Equation (4.10) subject to the boundary conditions of (4.11) and (4.12) and the initial condition of

(4.13).

Equation (4.10) is the homogenous equation and the general solution is given as follows:

$$G_1(Y) = B_1 Y^{\beta_1} + B_2 Y^{\beta_2} \tag{4.20}$$

According to initial condition (4.13), when $Y \to 0,\ G_1(Y) \to 0$, we could omit the second component of equation (4.20)

So $G_1(Y) = B_1 Y^{\beta_1}$

Substituting into boundary conditions (4.11) and (4.12),

$$Y_1^* = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I_1}{2H_1} \tag{4.21}$$

$$B_1 = \left(\frac{H_2 - H_1}{\beta_1}\right)^{\beta_1} \left(\frac{I_2}{\beta_1 - 1}\right)^{1-\beta_1} + \left(\frac{2H_1}{\beta_1}\right)^{\beta_1} \left(\frac{I_1}{\beta_1 - 1}\right)^{1-\beta_1}$$

Now we get the trigger value of project 1 and project 2. As soon as the economic shock increases to Y_1^*, the developer will invest in the project 1 and make the development of project 2 if the economic shock increases to Y_2^*.

Proposition 4.1

The developer will invest in project 1 as soon as the economic shock increases to Y_1^* and make the investment in project 2 once the economic shock increases to Y_2^*. The trigger time is defined by Equations (4.21) and (4.18).

4.4 Results Analysis

4.4.1 Initial Results

Let's compare $Y_1^* \& Y_2^*$ and consider different investment strategies. If $Y_1^* < Y_2^*$, which means the trigger time of the first project is lower than the second, the developer will choose a sequential development strategy. To simplify, we assume $I_1 = I_2$ here.

Under the condition $4\alpha_1 D(Q+\Delta_1) > [\alpha_1(1+\theta_{12}) + \alpha_2(1+\theta_{21})]D(Q+\Delta_2)$ (4.22)

The developer will choose a sequential development strategy.

Otherwise, if $Y_1^* \geq Y_2^*$, the developer will choose a simultaneous development strategy. In other words, if the portfolio effect is large, the developer will move the development of project 2 earlier in a simultaneous development strategy.

On the other hands, the demand curve will also impact the development decision. If the demand curve is steep, the developer will wait for a longer time and make a sequential development.

Proposition 4.2

Under the condition: $4\alpha_1 D(Q+\Delta_1) > [\alpha_1(1+\theta_{12}) + \alpha_2(1+\theta_{21})]D(Q+\Delta_2)$

The developer will choose a sequential development strategy. Otherwise, the developer will choose a simultaneous development strategy. Portfolio effect, if it is large, will make the developer invest in project 2 earlier. The demand curve also impacts the development decision. If the demand curve is steep, the developer will wait for a longer time and make a sequential development. If it is flat, the developer will choose a simultaneous development strategy.

Now we relax the assumption that $I_1 = I_2$. Under the condition that

$$4\alpha_1 D(Q+\Delta_1) > [\alpha_1(1+\theta_{12})+\alpha_2(1+\theta_{21})]D(Q+\Delta_2)$$

If $I_1 > I_2$, the trigger time Y_1^* will be close to Y_2^*. That means that larger investment cost in project 1 compared to project 2 will shorten the waiting time of investment in project 2.

Otherwise, $I_1 < I_2$, the trigger time Y_1^* will be far away from Y_2^*. In that situation, developer will wait more time for the investment in project 2.

On the other side, under the condition that

$$4\alpha_1 D(Q+\Delta_1) < [\alpha_1(1+\theta_{12})+\alpha_2(1+\theta_{21})]D(Q+\Delta_2)$$

if $I_1 > I_2$, the developer will make simultaneous development. On the other hand, if $I_1 < I_2$, the trigger time Y_1^* will be close to Y_2^*. Developer will wait to invest in project 2, if I_1 is smaller than I_2. One explanation is that the decision maker is exposed to higher risk when the scale of investment increases. However, the enlarged scale of investment will bring more positive portfolio effects.

Proposition 4.3

The investment costs for project 1 and project 2 will impact the trigger time of the two heterogeneous projects. If investment cost of project 1 is large, the time to wait for the investment in project 2 is shortened. The developer will prefer to develop simultaneously. Otherwise, if the investment cost of project 2 is larger, the developer will wait for longer time before making investment in project 2.

4.4.2 Effects of Externality and Demand Elasticity

From Equation (4.21), the trigger value for investing in project 1 depends only on the inverse demand curve, and comparable advantage of project 1. It has no relationship with the portfolio effects on project 1 and project 2.

Proposition 4.4

Although the portfolio effect makes developer invest in the project 1 earlier, the magnitude of portfolio effect will not impact the trigger value of project 1. However, the magnitude of the portfolio effect will impact the development timing of project 2.

By taking the first order derivations of Equations (4.21) and (4.18) with respect to the interactive factor, θ_{ij}, the effects on development timing by integrating the two projects as a portfolio can be represented as follows:

$$\frac{dY_1^*}{d\theta_{ij}} = 0$$

$$\frac{dY_2^*}{d\theta_{ij}} = -\frac{\beta_1}{\beta_1 - 1} \cdot \frac{I_2}{(H_2 - H_1)^2} \frac{\alpha_i D(Q + \Delta_2)}{r - u} \tag{4.23}$$

where [$i = (1, 2)$]. There is no externality on project 1 when the trigger time is Y_1^*. The externality effects are only created upon the completion of the second project at t_2. The interactive effect makes the development of the second project to take place earlier.

Let [$g(Q) = dY_2^*/d\theta_{ij}$]. By taking the first order derivation of $g(Q)$ with respect to the aggregate demand, Q, the effects of demand elasticity on trigger time given positive externalities can be explained as follows:

$$\frac{dg}{dQ} = -\frac{\beta_1}{\beta_1 - 1} \cdot \frac{I_2}{(H_2 - H_1)^2} \frac{\alpha_i D^{'}(Q + \Delta_2)}{r - u} \tag{4.24}$$

Proposition 4.5

Shift along the demand curve will change the demand elasticity. The negative effect from a portfolio on the trigger time of the second project will also be affected by the demand condition. In a weak market where the market elasticity is high, the portfolio effect on the trigger time of the second project is larger, compared to a boom market with high quantity of products.

4.4.3 Model for Independent Development

In a base case, where two developers have identical project, no portfolio effects exist. Then we can compare the difference for cases with and without portfolio effects.

The option value of project 1, $F_1(Y)$ is given in the Ordinary Differential Equation

$$\frac{1}{2}\sigma^2 Y^2 F_1^{''}(Y) + \mu Y F_1^{'}(Y) - rF_1 = 0 \tag{4.25}$$

Subject to the following boundary conditions and initial condition:

$$F_1(Y_1^*) = \frac{\alpha_1 D(Q + \Delta_1)}{r - u} Y_1^* - I_1 \tag{4.26}$$

$$F_1^{'}(Y_1^*) = \frac{\alpha_1 D(Q + \Delta_1)}{r - u} \tag{4.27}$$

$$F_1(0) = 0 \tag{4.28}$$

The option value of project 2, $F_2(Y)$ is defined in the following Ordinary Differential Equation

$$\frac{1}{2}\sigma^2 Y^2 F_2''(Y) + \mu Y F_2'(Y) - rF_2 = 0 \quad (4.29)$$

where the boundary conditions and initial condition are given below:

$$F_2(Y_2^*) = \frac{\alpha_2 D(Q+\Delta_2)}{r-u} Y_2^* - I_2 \quad (4.30)$$

$$F_2'(Y_2^*) = \frac{\alpha_2 D(Q+\Delta_2)}{r-u} \quad (4.31)$$

$$F_2(0) = 0 \quad (4.32)$$

The model for independent development is a one-stage option model while in the previous sections the model for two projects is a two-stage option model. The boundary conditions defined for both settings are therefore different.

The solution for Equation (4.25) can be solved subject to the Boundary conditions (4.26), (4.27), and initial condition (4.28):

$$F_1(Y) = C_1 Y^{\beta_1}$$

Substituting into Equations (4.26) and (4.27)

$$Y_1^* = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I_1}{H_1} \quad (4.33)$$

The solution of Equation (4.29) is obtained subject to the boundary conditions (4.30) and (4.31), and the initial condition (4.32):

$$Y_2^* = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I_2}{\frac{\alpha_2 D(Q+\Delta_2)}{r-u}} \quad (4.34)$$

Compare the Equations (4.21) and (4.33), we could determine the portfolio effects that push the developer to make investment of project 1 earlier.

By comparing Equations (4.18) and (4.34), the decision of whether the portfolio effect will push the developer to make investment of project 2 earlier or defer the second investment of project 2 depends on the demand curve and the magnitude of portfolio effects.

Under the condition: $\alpha_1(1+\theta_{12})D(Q+\Delta_2)+\alpha_2\theta_{21}D(Q+\Delta_2)>2\alpha_1 D(Q+\Delta_1)$

if the portfolio effect is large, investment in project 2 will take place earlier. The developer will prefer to invest in project 2, when the demand curve is flat.

Proposition 4.6

The portfolio effect will push the developer to make investment in project 1 earlier. Whether the portfolio effect will push the developer to make investment in project 2 earlier or defer the second investment of project 2 depends on the demand curve and the magnitude of portfolio effect. If the portfolio effect is large, the investment in project 2 will occur earlier. The developer will prefer to invest in project 2 when the demand curve is flat. Under the condition: $\alpha_1(1+\theta_{12})D(Q+\Delta_2)+\alpha_2\theta_{21}D(Q+\Delta_2)>2\alpha_1 D(Q+\Delta_1)$ *, the developer will make investment of project 2 earlier.*

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4.5 Comparative Static and Sensitive Analysis

In the last section, we derive the optimal investment timing and option values under various assumptions. With reasonable parameters for input variables, such as demand function, risk free return, growth rate, standard deviation, investment cost and comparable advantage of the two projects, sensitive analyses under different market conditions are conducted to derive comparative statistics.

a) Initial Assumptions

- $D(Q+\Delta_1)$ and $D(Q+\Delta_2)$ are inverse demand functions. The elasticity of the demand curve is analyzed. A flat demand curve is defined as $D(Q+\Delta_1)=100$ and $D(Q+\Delta_2)=80$. On the other hands, a steep demand curve is assumed to take the following values: $D(Q+\Delta_1)=100$ and $D(Q+\Delta_2)=50$. The flat demand curve is used in the base case scenario.
- r is the risk free return, we assume $r=10\%$
- μ is the instantaneous expected growth rate, we assume $\mu=4\%$
- σ is the instantaneous standard deviation, we assume $\sigma=20\%$
- The investment capital I_1 and I_2, both we assume 1 unit
- α means the comparable advantage. We assume $\alpha_1=1, \alpha_2=0.8$

b) Results from the models

$$H_1=\frac{\alpha_1 D(Q+\Delta_1)}{r-u}$$

$$H_2=\frac{\alpha_1(1+\theta_{12})D(Q+\Delta_2)}{r-u}-\frac{\alpha_1 D(Q+\Delta_1)}{r-u}+\frac{\alpha_2(1+\theta_{21})D(Q+\Delta_2)}{r-u}$$

$$Y_1^*=\frac{\beta_1}{\beta_1-1}\cdot\frac{I_1}{2H_1}$$

$$Y_2^*=\frac{\beta_1}{\beta_1-1}\cdot\frac{I_2}{H_2-H_1}$$

$$G_1(Y)=[(\frac{H_2-H_1}{\beta_1})^{\beta_1}(\frac{I_2}{\beta_1-1})^{1-\beta_1}+(\frac{2H_1}{\beta_1})^{\beta_1}(\frac{I_1}{\beta_1-1})^{1-\beta_1}]Y^{\beta_1}$$

$$G_2(Y)=(\frac{H_2-H_1}{\beta_1})^{\beta_1}(\frac{I_2}{\beta_1-1})^{1-\beta_1}Y^{\beta_1}+H_1Y$$

Under the initial assumptions, we will simulate investment trigger value of project 2 when the portfolio effect changes. In Figure 4.2, we show that when the portfolio effect on project 1 and project 2 changes from 0 to 1, with each incremental step of 0.02, the trigger value of development timing of project changes continuously. One significant point is that there is a linear section, around which the trigger value increases to a positive infinite or decreases down to a negative infinite. The points are found around the node that (θ_{12}) is 0.4 and (θ_{21}) is 0.4.We divide the graph into two parts. One is when the portfolio effect is small, when both θ_{12} and θ_{21} are from 0 to 0.35. The other is when the portfolio effect is quite large, both θ_{12} and θ_{21} are from 0.45 to 1.5. The details are found in Figure 4.3 and Figure 4.4.

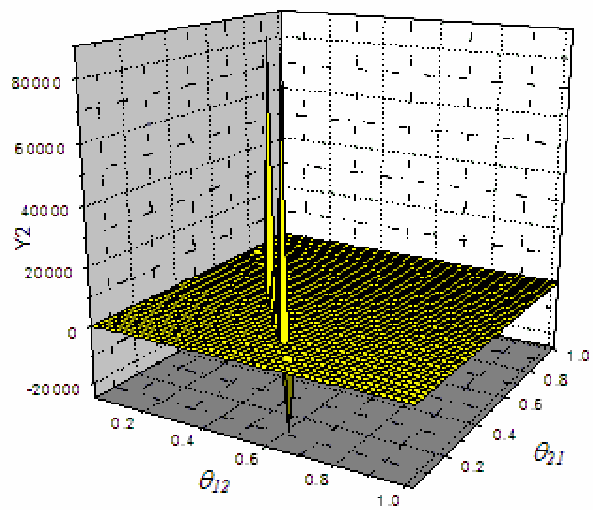

Figure 4.2 Trigger value changing as the θ_{12} and θ_{21} from 0 to 1

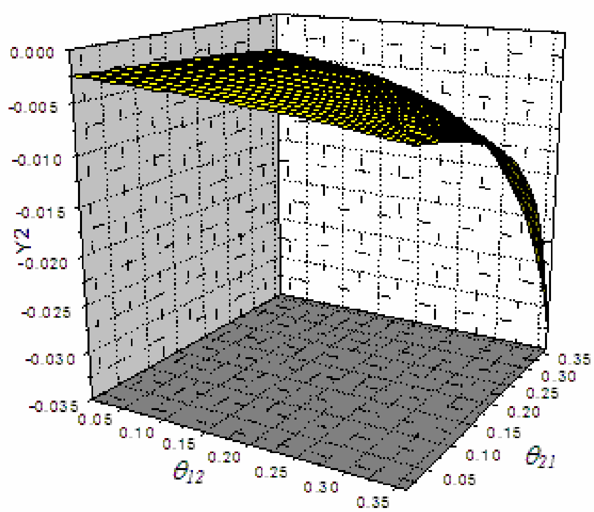

Figure 4.3 Trigger value changing as the θ_{12} and θ_{21} from 0 to 0.35

In Figure 4.3, as θ_{12} and θ_{21} increase from 0 to 0.35, by a step of 0.01, the trigger value of development timing is always negative and drops into negative infinite when both θ_{12} and θ_{21} are closed to 0.35. In this scenario when the portfolio effect is weak, the trigger value of development of project 2 is negative, which implies that the profit of project 2 is less than the profit decreasing in project 1. In that situation, the developer will defer the investment in project 2 infinitely.

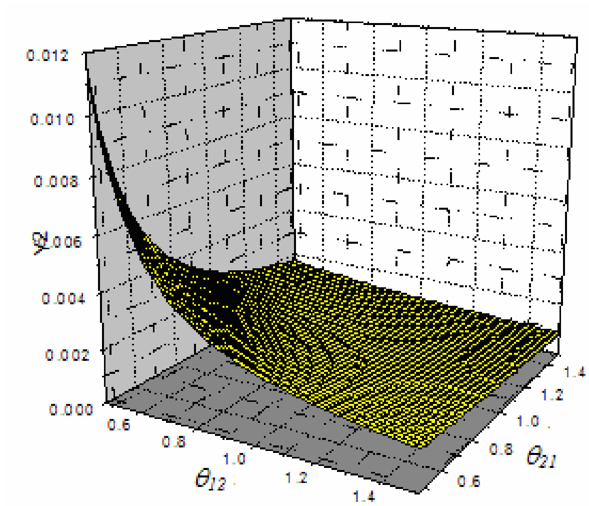

Figure 4.4 Trigger value changing as the θ_{12} and θ_{21} from 0.45 to 1.5

In Figure 4.4, when θ_{12} and θ_{21} change from 0.45 to 1.5, by a step of 0.02 each, the situation is the opposite of that in Figure 4.3. The trigger value of development timing is always positive and decreases as the θ_{12} increases, and also decreases as θ_{21} increases. It is close to a positive infinite when both θ_{12} and θ_{21} are close to 0.45. In this scenario, when the portfolio effect is strong, the trigger value of development of project 2 is positive, but decreases as the portfolio effect grows. In this situation, the developer will make the investment of project 2 as soon as the economic shock exceeds the trigger value. The stronger the portfolio effect is, the lower is the trigger value.

Proposition 4.7

The portfolio effect will impact the trigger value of development timing of project 2. When the portfolio effect is small, the developer will abort the development of project 2, because the profit of project 2 will be less than the losses that will be suffered by project 1. On the other hand, when the portfolio effect is large, the developer will make investment in the project 2 as soon as the economic shock exceeds the trigger value. If the magnitude of portfolio effect is large, the trigger value is lower. In other words, the large portfolio effect will make the developer to invest in project 2 earlier. The decision on project 2 depends on the trade-off between profit from project2 and losses in project 1.

In the above scenario, we assume a flat demand curve and examine changes in trigger timing value as the interactive effect changes. In the following scenario, we change the flat demand curve into a steep demand curve. A flat demand curve means the price will decrease at a lower rate as the quantity increases. While with a steep demand curve, $D(Q+\Delta_1)=100$ and $D(Q+\Delta_2)=50$, the price is expected to decrease at a faster rate as the quantity increases. All the other parameters are the same as the above scenario. $r=10\%$, $\mu=4\%$, $\sigma=20\%$, $I_1=I_2=1$, $\alpha_1=1, \alpha_2=0.8$

In Figure 4.5, θ_{12} and θ_{21} changes from 0 to 1, and each step is 0.05. The trigger value for investment in project 2 is always negative, which means that the profit from project 2 is not sufficient to compensate the decline in profit in project 1. The developer will not invest in project 2.

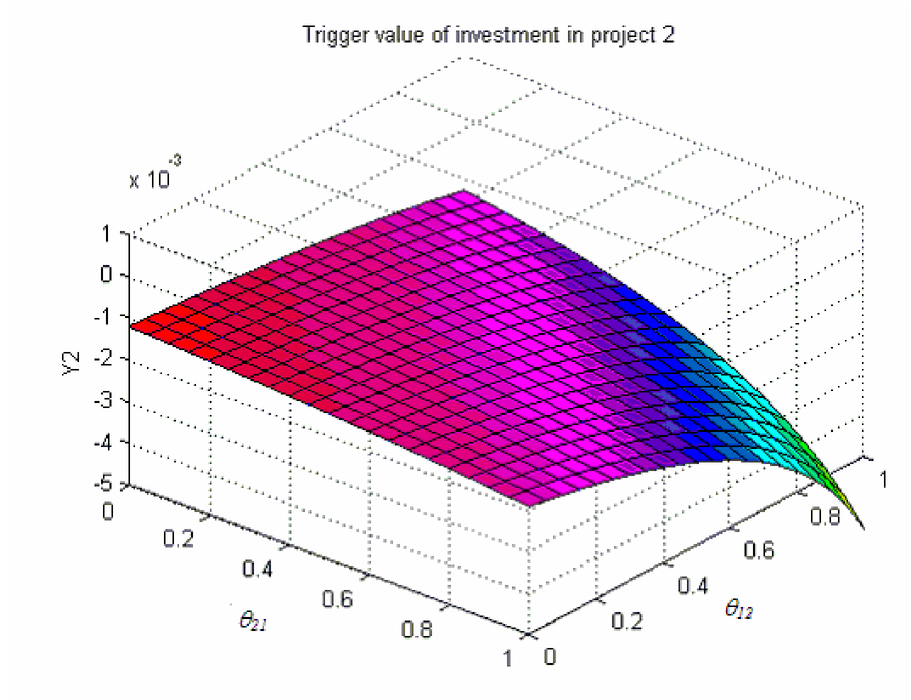

Figure 4.5 Trigger value of investment in project 2 under steep demand curve

Proposition 4.8

Under a steep demand curve, the developer will abort the development of project 2 because the profit from project will not be sufficient to compensation the loss in profit in project 1. The developer will defer the second project under a steep demand curve.

Comparable advantage is another important parameter that will impact the investment trigger values. By changing the comparable advantage: α_1 *and* α_2, we would explore the changes in trigger value of project 1 and project 2. The trigger value for development of project 1 will decrease as the comparable advantage α_1 increases. In other words, the developer will invest in project 1 earlier, if project 1 has larger comparable advantage.

Figure 6.5 and Figure 6.6 show the trigger values for investment in project 2. All the assumptions are the same as the basic scenario, except for α_2. *r=10%,* μ *=4%,* σ *=20%,* $I_1=I_2=1$, and $\alpha_1 = 1$. We assume a flat demand curve. In Figure 6.5, $\alpha_2 = 0.8$, while in Figure 4.6, $\alpha_2 = 1$. θ_{12} and θ_{21} change from 0.45 to 1.5, by 0.02 in each step. It is obvious that the trigger value for investment in project 2 is lower, when the comparable advantage of project 2 is high.

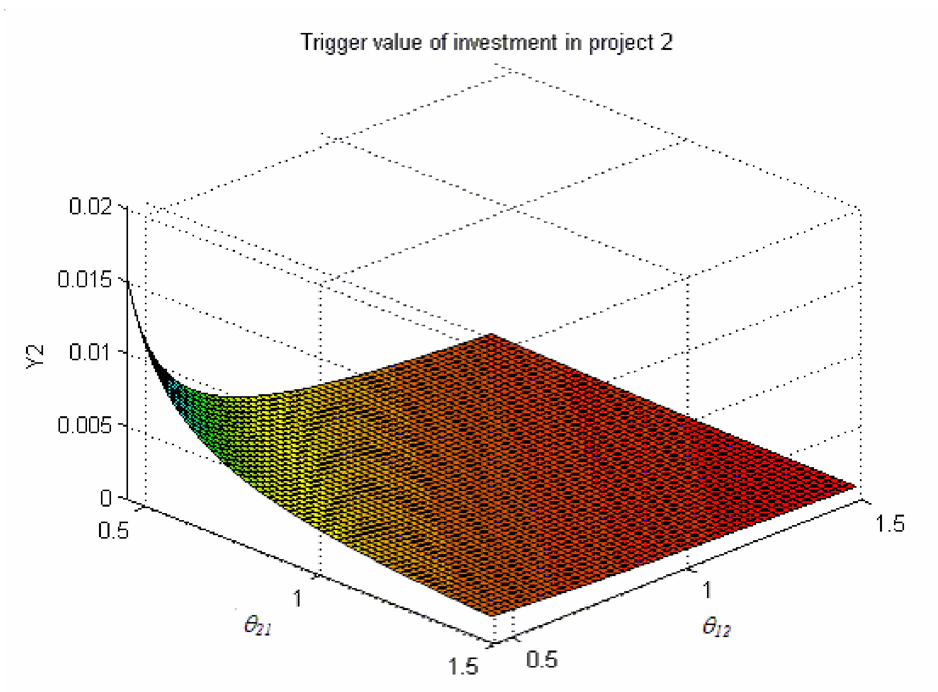

Figure 4.6 Trigger value of project 2 as the comparable advantage of project 2 is low

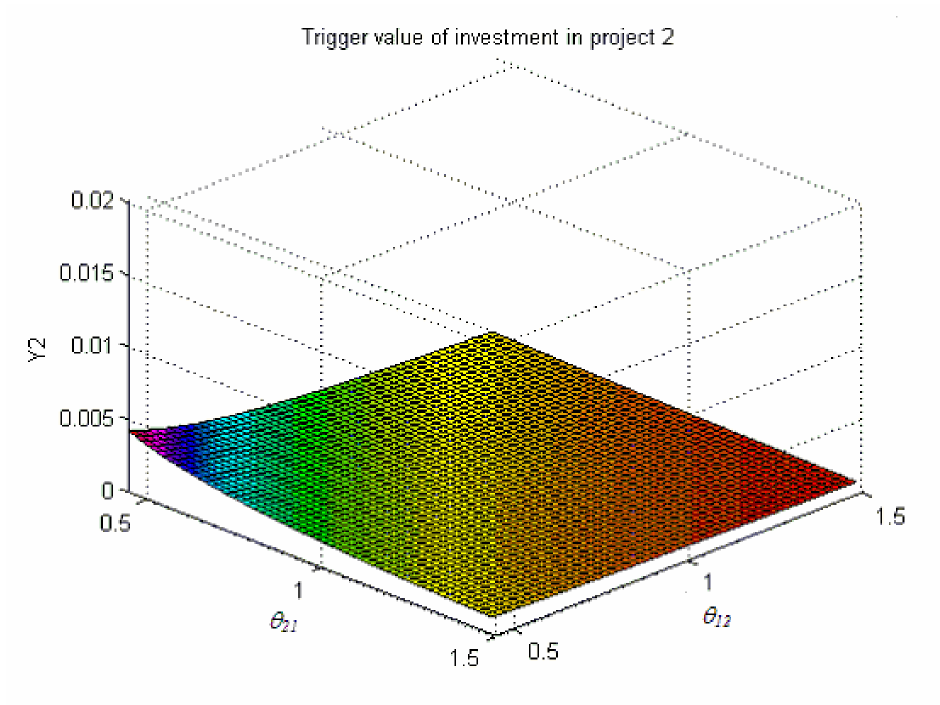

Figure 4.7 Trigger value of project 2 as the comparable advantage of project 2 is high

Proposition 4.9

The trigger value for development in project 1 will decrease as the comparable advantage α_1 increases. In other words, the developer will invest in project 1 earlier, if project 1 has higher comparable advantage. The trigger value for development of project 2 will also change along with α_1 and α_2, although the shape of graph remains unchanged. The comparable advantage will impact the decision to invest in the project 2. The larger the comparable advantage of project 2, the lower is the trigger value. The developer will invest in the project 2 earlier, if the project 2 has a higher comparable advantage.

4.6 Conclusion

Previous papers such as Majd & Pindyck (1987), Pindyck (1993) on "time to build options" explored the price uncertainty, cost uncertainty, construction delay, and project suspension and built option model on a single project. This study explores the interactive effect between different projects owned by the same developer. The portfolio effect between projects will bring spill-over effects on price and cost saving, which will also impact the development timing of different projects. This study presented the portfolio effect between projects by deriving the global optimization in the model.

We build an option model, which integrates the future demand uncertainty, market structure and portfolio effect between projects, to examine the optimal timing of development decision. We assume there are two projects waiting for development, and the real option model is developed as a two-stage option model. We use the backward process to solve the problem, and find that the portfolio effect will impact the optimal development timing of the two projects. The developer will make simultaneous development when the portfolio effect is large and he will prefer a sequential development strategy, when the portfolio effect is weak.

By using reasonable parameters for input variables in the model, comparative statistics and sensitivity analysis are conducted. The portfolio effect will make the investment in project 1 to happen earlier. The portfolio effect will also push the developer to make investment in project 2 to take place earlier, or to defer investment in project 2 depending on the demand curve and the portfolio effect. As the portfolio effect increases the price by 40%, the development of project 2 will be considered under the assumed parameters. Increases in the portfolio effect will lower the trigger value of investment in project 2. The developer will prefer to invest in project 2, when the demand curve is flat. Lastly, the comparable advantage of a project will also have significant effects on optimal development timing. A project with higher comparable advantage will be developed earlier by the developer.

The above models exclude the cost uncertainty, and it only assumes that the price is impacted by an economics shock, which follows a Geometric Brownian Motion Process. There are several extensions to be made in the future research. One is that we could consider the cost uncertainty, which is more practical though it makes the model more complex. Another extension is that we will consider the price interactive effect between two projects. In this section, we assume that the two projects are impacted by the same economic shock and the same price. We may further explore different price impact on the two projects that are driven by the same economic shock. The proposed extensions will be explored in the next chapter.

Chapter 5 Multi-Project Optimal Timing Strategy Using Least Square Monte Carlo Simulation

5.1 Introduction

In the last chapter, we set up a stochastic framework of one developer with two projects, in which the decision to develop is impacted by an exogenous economic shock. The economic shock follows a Geometric Brownian Motion Process. In a market with heterogeneous product types, the prices are different because of different location, quality, infrastructure and neighborhood. Some projects are more risky and impacted more seriously by the economic shock than others. In this chapter, we allow different economic shocks to affect the two projects. We assume that one project has a higher growth rate and a higher volatility, while another has a relatively lower growth rate and lower volatility. We assume that the economic shocks on the two projects follow two different Geometric Brownian Motion Processes, although these two processes may be correlated.

In our model, one developer has two heterogeneous projects, and he will decide when to invest in the two projects so as to maximize the collective profits. In our basic model, the complex problem is solved as a sequential option problem. The developer will invest in one project, and then in the second project. At first, the developer has the option to invest in the two projects. After the completion of the first project, the developer only has the option to invest in the second projects. It is a two-stage option problem as in models in chapter 4. Given the same input assumptions, the trigger value of project with a high growth rate is relatively lower. The developer will first invest in the project with a high growth rate and high volatility, followed by the project with a low growth rate and a low volatility.

In a two-stage real option model with two stochastic variables, there are two PDE equations with different boundary conditions that include value matching and smooth pasting conditions. There are three numerical algorithms commonly used to price options: lattice method that uses the binomial tree (firstly proposed by the Cox, Ross and Rubinstein (1979)), finite difference method that is the traditional PDE solution method (first introduced by Brennan and Schwartz (1977)), and Monte Carlo Simulation methods (introduced by Boyle (1977)). Binomial tree is useful for a discrete option model, but it has disadvantage when applied to a model with high dimensional state variables. Although the finite difference method could be used to solve the PDEs in our model, the process is complex and computationally intensive. The Monte Carlo simulation method is used in this study to solve the option timing problem.

Boyle (1977) firstly used the Monte Carlo simulation as a numerical method to obtain solutions for option valuation problems. She simulated the process generating returns on the underlying asset and derived the option value. The early Monte Carlo simulation is a forward-looking technique, while the dynamic programming is backward recursive process. Many approaches have been proposed to match simulation and dynamic programming, and one well-accepted approach is the Least Squares Monte Carlo Approach proposed by

Longstaff and Schwartz (2001), which was used to estimate the American option price. This approach used the least square method to estimate the conditional expected payoff of the option holder for continuation. The option holder would exercise the option, if the payoff of immediately exercising was higher than the conditional expected payoff. It was path-dependent and could be applicable in a multifactor situation. However, there are still problems with the technique, such as efficiency of LS method, choice of basic function, and computation speed.

The Monte Carlo simulation method has been expanded from pricing financial option to real options. Gamba (2003) used the Monte Carlo simulation to value a wide range of capital budgeting problems with embedded real options that depend on state variables and a related valuation algorithm. The valuation approach decomposed a complex real option problem with multiple options into a set of simple options taking into account interaction and interdependence of the embedded real options. The Longstaff and Schwartz (2001) least square Monte Carlo simulation method was used and it was extended to the decomposition process. We adopt the method proposed by Gamba (2003) and use the algorithm based on Least Squares Monte Carlo Approach by Longstaff and Schwartz (2001) to solve the complex multi-option problem.

Based on the Least Square Monte Carlo simulation technique, we simulate the basic scenarios by changing the correlation between economic impacts on project 1 and project 2, from a negative to a positive range. The distribution of the exercise time for project 1 is a normal distribution for the range of correlation between the two projects. The exercise time is longer when the correlation is positive. The developer will wait for new information when the two projects are positively correlated. The distribution of the exercise time for project 2 is a discrete process with a flat distribution. The exercise time for project 2 becomes significant when the correlation changes from a negative to a positive value. When the developer has a portfolio of projects with positive correlations, he is more sensitivity to the market volatility and will be more careful when making investment decisions. The developer will defer development, especially the second project, when the portfolio's correlation is positive. If the two projects have positive interactive effects, the exercise time for project 1 will be earlier, while the exercise time of project 2 will be much earlier. On the contrary, when two identical projects are owned by two independent developers, the developer will wait for the action of the other developer and defer the decision on his project. When the two projects are owned by a single developer, the portfolio effect will likely to encourage an earlier development for the projects.

This chapter is organized as follows. Section 5.1 gives a general background of the case with heterogeneous property, real option timing problem, and the Monte Carlo Simulation techniques. Section 5.2 specifies the multi-option investment timing model with two heterogeneous projects and decomposes the options into a sequential two-stage option problem. Section 5.3 discusses the solutions for the multi-option problem and introduces the Least Square Monte Carlo Simulation Method. Section 5.4 analyzes the investment timing results in a case where one developer owns two heterogeneous projects, using Least Square

Monte Carlo Simulation Method. Section 5.5 concludes the findings with extensions for further studies.

5.2 Interactive effect of Heterogeneous Projects on Development Timing Strategy Model

In the proposed stochastic framework, we assume that there is a single developer, who has two projects, project 1 and project 2, waiting for development. The development time is denoted by t_i [$i = (1, 2)$]. We assume that the time to build is zero, $\delta = 0$. These two projects are heterogeneous, and they are driven by different economic shocks that follow different stochastic processes. We assume that the developer is risk-neutral. As denoted below, r stands for risk-free rate and ρ is for the return required discount rate.

We use the inverse demand function, denoted by $D(Q_t)$, to determine the underlying asset prices. The prices for project 1 and project 2 are different, because they are heterogonous products that are driven by two different Geometric Brownian Motion processes. We assume that the two projects will generate after the completion future cash flows as denoted by P_i. We use two different economic shocks to differentiate the two projects. Prices of the two projects are defined as a function of different economic shocks, y_i, *as follows:*

$$P_{\mathrm{i}} = y_i \cdot D(Q_t) \qquad [i = (1, 2)] \qquad (5.1)$$

The economic shocks on different projects, y_i, are exogenous and follow two different Geometric Brownian motion processes given below:

$$dy_i = \mu_i \cdot y_i \cdot dt + \sigma_i \cdot y_i \cdot dz \quad (5.2)$$

μ_i is the instantaneous expected growth rate

σ_i is the instantaneous standard deviation

dz is increment of stand Wiener Process

From Equation (5.1) and (5.2), we could derive the prices for project 1 and project 2, which also follow the following Geometric Brownian Process:

$$dP_i = \mu_i \cdot P_i \cdot dt + \sigma_i \cdot P_i \cdot dz \quad (5.3)$$

The construction cost is assumed to be constant, and the construction cost is denoted by I_i [$i = (1, 2)$]. The revenue of project 1 as denoted by R_1, consists of two parts: one is before the investment in project 2, and the second part is after the investment in project 2, where interactive effects are considered. The revenue for project 2 as denoted by R_2, is also impacted by the interactive effect. The costs for project 1 and project 2 are computed on a constant unit cost.

$$R_1 = \int_{t_1}^{t_2} P_t \cdot q_1 \cdot \exp(-\rho \cdot t) \cdot dt + \int_{t_2}^{\infty} P_t \cdot q_1 \cdot (1 + \theta_{12}) \cdot \exp(-\rho \cdot t) \cdot dt \quad (5.4)$$

$$R_2 = \int_{t_2}^{\infty} P_t \cdot q_2 \cdot (1 + \theta_{21}) \cdot \exp(-\rho \cdot t) \cdot dt \quad (5.5)$$

At the beginning, the developer has the development options for two projects. After the investment in project 1, the developer has only one option to invest in project 2. The completion of project 1 will have interactive effects on both project 1 and project 2, such that the portfolio effect will impact the development timing of project 2. We will use θ_{ij} for the interactive effect. We could consider the problem as a two-stage real option valuation (see Figure 5.1). In the first stage, we will decide the development timing for project 1, and then at the second stage, we make the investment decision for project 2. The backward method is used to solve the problem.

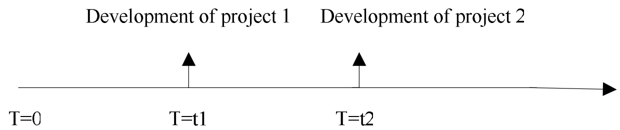

Figure 5.1 Two-stage options

After the completion of project 1, the developer will receive the future cash flow P_1 per unit time under the risk neutral assumption. After investing in Project 1, the option value for project 2 as denoted by $F(Y_1, Y_2)$, can be computed using the following Partial Differential Equation:

$$\frac{1}{2}\sigma_1^{\ 2}Y_1^{\ 2}F_{Y_1Y_1}(Y_1,Y_2)+\rho\sigma_1\sigma_2Y_1Y_2F_{Y_1Y_2}+\frac{1}{2}\sigma_2^{\ 2}Y_2^{\ 2}F_{Y_2Y_2}(Y_1,Y_2) + \mu_1Y_1F_{Y_1}+\mu_2Y_2F_{Y_2}+D(Q+\Delta_1)Y_1 - rF=0 \quad (5.6)$$

Subject to the following boundary condition: value matching, smooth pasting and initial condition:

$$F(Y_1^*,Y_2^*)=\frac{(1+\theta_{12})D(Q+\Delta_2)}{r-u}Y_1^*-\frac{D(Q+\Delta_1)}{r-u}Y_1^*+\frac{(1+\theta_{21})D(Q+\Delta_2)}{r-u}Y_2^*-I_2 \quad (5.7)$$

$$F_{Y_1}(Y_1^*,Y_2^*)=\frac{(1+\theta_{12})D(Q+\Delta_2)}{r-u}-\frac{D(Q+\Delta_1)}{r-u} \quad (5.8)$$

$$F_{Y_2}(Y_1^*,Y_2^*)=\frac{(1+\theta_{21})D(Q+\Delta_2)}{r-u} \quad (5.9)$$

$$F(0,0)=0 \quad (5.10)$$

Before investing in project 1, the total option value of the developer is denoted by $G(Y_1,Y_2)$. The developer has the options to either invest in project 1 or project 2, and these two projects have interactive effects. *G (Y_1, Y_2)* is determined in the following Partial Differential Equation:

$$\frac{1}{2}\sigma_1{}^2Y_1{}^2G_{Y_1Y_1}(Y_1,Y_2)+\rho\sigma_1\sigma_2Y_1Y_2G_{Y_1Y_2}+\frac{1}{2}\sigma_2{}^2Y_2{}^2G_{Y_2Y_2}(Y_1,Y_2)+\mu_1Y_1G_{Y_1}+\mu_2Y_2G_{Y_2}\text{ - }rG=0 \tag{5.11}$$

Subject to the boundary condition: value matching, smooth pasting and initial condition:

$$G(Y_1^{\#},Y_2^{\#})=\frac{D(Q+\Delta_1)}{r-u}Y_1^{\#}+F(Y_1^{\#},Y_2^{\#})-I_1 \tag{5.12}$$

$$G_{Y_1}(Y_1^{\#},Y_2^{\#})=\frac{D(Q+\Delta_1)}{r-u}+F_{Y_1}(Y_1^{\#},Y_2^{\#}) \tag{5.13}$$

$$G_{Y_2}(Y_1^{\#},Y_2^{\#})=F_{Y_2}(Y_1^{\#},Y_2^{\#}) \tag{5.14}$$

$$G(0,0)=0 \tag{5.15}$$

5.3 Least Square Monte Carlo Simulation

The boundary conditions are non-homogenous; therefore, it is difficult to reduce the Partial Differential Equation to an Ordinary Differential Equation. The closed form analytical solution is not available. We will use the Monte Carlo Simulation to derive the numerical solution. Based on the Least Square Monte Carlo Simulation proposed by Longstaff and Schwartz (2001), we decompose the complex option into a collection of simple options. Our model is an option on option problem. As a developer owns two heterogeneous projects, the following options are available:

Option to invest in project 1: the payoff until maturity as if the option exists in isolation is given as $\Pi_1(t,V_{1t}) = max \{V_{1t} - I_1, 0\}$, where the project value is denoted by V_1 and the cost outlay is denoted by I_1. The maturity of the option is T_1 years. F_1 denotes the option value.

Option to invest in project 2: the payoff until maturity is defined as $\Pi_2 (t, V_{2t}) = max \{V_{2t} - I_2, 0\}$ with the collective project value V_2 and a capital expenditure I_2. The maturity is T_2 years, and the option value is F_2.

Since the interactive effect depends on the completion of the first project, the payoff for the first project is given as max $\{V_{1t} - I_1 + F_2 (t, V_t), 0\}$. Although the second option can be exercised in an interval [$0, T_2$], the time interval for the second option is from the time the first option is exercised (a stopping time) to T_2. A graphical representation of the problem is shown in Figure 5.2.

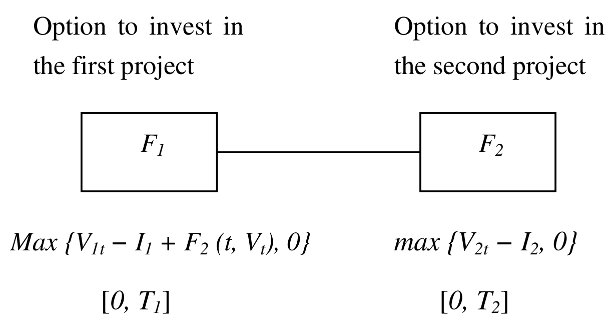

Figure 5.2 Option problem representations

The value F_1 depends on F_2, and both are American options. The option algorithm could be represented by the following formula.

$$F_2(t,Y_1,Y_2) = \max_{\tau\in(t,T_2)} \left\{ e^{-r(\tau-t)} E_t[\Pi_2(\tau,Y_1,Y_2)] \right\} \tag{5.16}$$

$$F_1(t,Y_1,Y_2) = \max_{\tau\in(t,T_1)} \left\{ e^{-r(\tau-t)} E_t[\Pi_1(\tau,Y_1,Y_2) + \mathrm{F}_2(t,Y_1,Y_2)] \right\} \tag{5.17}$$

Based on the above option algorithm, we use Matlab 6.5.0 to do the numerical simulation. Two projects, project 1 with a high growth rate, a high volatility and a high investment cost, while project 2 with a low growth rate, a low volatility and a low investment cost.

We set assumptions for the basic parameters. The life of the real options in our development case is limited by the development timing that is permitted for a project, i.e. the project must be developed in *n* years. The infinite life case though is a more generalized case, which however, will not change the results significantly. In our simulation, the maturity of development option for project 1, T_1 is 150, and the maturity of development option for project 2, T_2 is 150. To keep the assumption that the developer would invest in project 1 first and then project 2, we would examine the results and make adjustment after the simulation. With a time interval of 1 per period, a total 150 stages will be generated in the simulation process. Since the algorithm is path dependent, 5000 paths are generated. For project 1, the growth rate is 4%, the volatility is 20%, whereas for project 2, the growth rate is 2% and the volatility is 10%. The investment cost for project 1 is 5000, and for project 2 is 2000. The inverse demand curve is set at 100 if one project was developed, and it decreases to 80 when two projects were developed. The initial parameter for economic shocks on project 1 and project 2 is set at 1. The shocks on project 1 and project 2 will follow two different Geometric Brownian Motion Processes, and the correlation between two is given by ρ. We will change the value of ρ to explore different correlations of two shocks covering a positive, a negative and a zero value, on the option value. The interactive effect on project 1 is θ_{12} and on project 2 is θ_{21}. We would examine how the development timing changes with respects to changes in interactive effects.

Firstly, we generate economic shock on project 1 and project 2 using the following Geometric Brownian Motion Processes denoted by Y_1 and Y_2.

$$Y_1(t+\Delta t) = Y_1(t)\cdot \exp[(u_1 - \frac{1}{2}\sigma_1^2)\cdot \Delta t + \sigma_1 \cdot \sqrt{\Delta t}\cdot \xi_1] \tag{5.18}$$

$$Y_2(t+\Delta t) = Y_2(t)\cdot \exp[(u_2 - \frac{1}{2}\sigma_2^2)\cdot \Delta t + \sigma_2 \cdot \sqrt{\Delta t}\cdot \xi_2] \tag{5.19}$$

$$\xi_1 = \mathrm{z}_1$$

$$\xi_2 = \rho \mathrm{z}_1 + \mathrm{z}_2\sqrt{1-\rho^2}$$

μ_1, *and* μ_2 are the growth rate for project 1 and project 2. σ_1 and σ_2 are the volatility for project 1 and project 2. z_1 and z_2 are Winer Processes.

The decision making rule for project 2 is defined as:

$$\max\left\{\frac{(1+\theta_{12})D(Q+\Delta_2)}{r-u}Y_1^* - \frac{D(Q+\Delta_1)}{r-u}Y_1^* + \frac{(1+\theta_{21})D(Q+\Delta_2)}{r-u}Y_2^* - I_2\right\}$$

The decision making rule for project 1 is defined as:

$$\max\left\{\frac{D(Q+\Delta_1)}{r-u}Y_1^{\#} + F_2 - I_1\right\}$$

Using the backward algorithms at each stage, we compare the payoff for immediately exercising the options, and the conditional expected payoff for deferring the exercise decision. To get the expected payoff, we regress the discounted future cash flow on the economic shocks in the recent stage: Y_1, Y_2, Y_1^2, and Y_2^2. Repeating the process from the last stage to the first stage, we could get the exercise time for project 1 and project 2.

Two projects are either developed sequentially or simultaneously. If the exercise time t_1 is larger than t_2, we would adjust the investment timing and option value according to the following rules. If the collective value of the two projects is larger than zero at t_2, we exercise the two options simultaneously at t_2. It implies that the developer will make investment early because of the positive interactive effect. If the collective value of two projects at t_2 is less than zero but the option value of project 2 is positive at t_1, we choose to exercise both projects at t_1. Otherwise we will abort the project 2 at t_2.

5.4 Analysis of Results

According to the above assumptions, we use the least square Monte-Carlo (LSMC) algorithm to simulate the basic scenario when the correlations between the economic shocks on project 1 and project 2 are negative, zero and positive. First, by generating the economic shocks on project 1, the following assumptions are made: the growth rate u_1 is 0.04, the standard deviation sd_1 is 0.2, the decision time T is 150 periods, and the total path is 5000. Similarly, when generating the economic shock on project 2, the following assumptions are made: the growth rate u_2 is 0.02, the standard deviation sd_2 is 0.1, the correlation between impact on project 1 and project 2 is -0.6. The inverse demand curve when one project in the market is 100, while two projects in the market, it changes to 80. The investment cost for project 1 I_1 is 5000, and the cost for project 2 I_2 is 2000. The interactive effect on project 1, $\theta_{12} = 0$, and on project 2, $\theta_{21} = 0$. We could see the distributions of exercise time of project 1 and project 2 in Figure 5.3 and Figure 5.4.

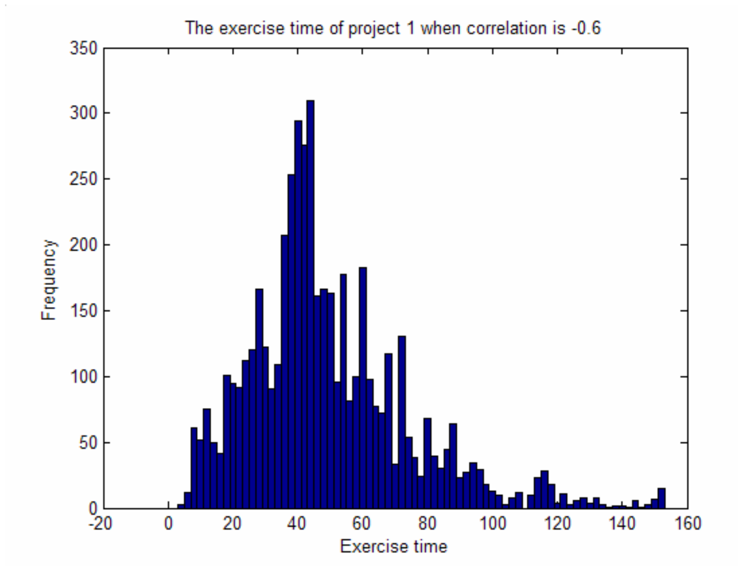

Figure 5.3 The exercise time of project 1 when correlation is -0.6

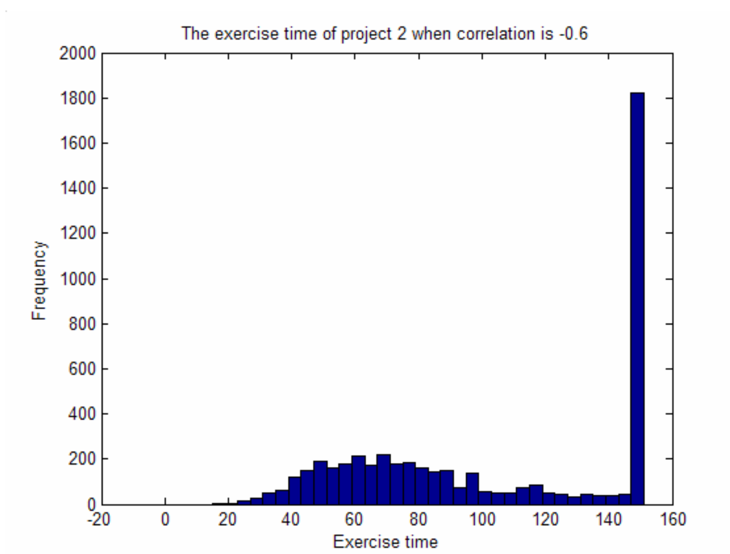

Figure 5.4 The exercise time of project 2 when correlation is -0.6

By setting the correlation between project 1 and project 2 at zero, the distributions of exercise times for project 1 and project 2 are shown in Figure 5.5 and 5.6.

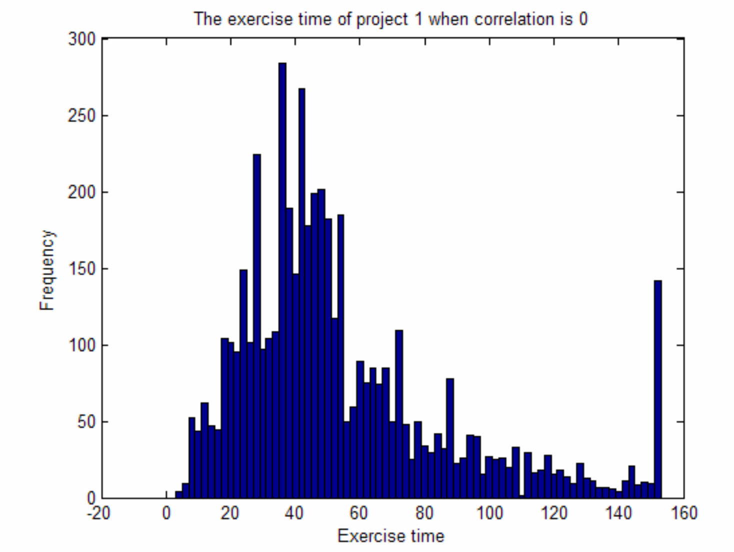

Figure 5.5 The exercise time of project 1 when correlation is 0

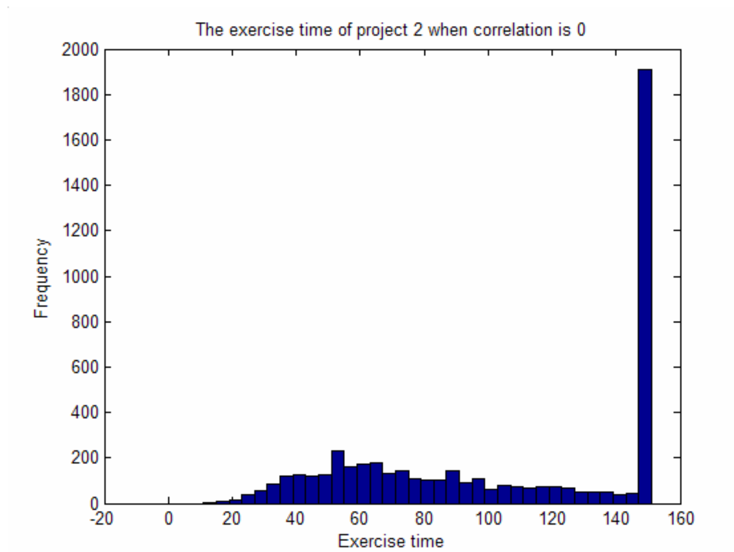

Figure 5.6 The exercise time of project 2 when correlation is 0

When the correlation between shocks on project 1 and project 2 increases to 0.6, the distributions of exercise time of project 1 and project 2 are shown in Figure 5.7 and Figure 5.8.

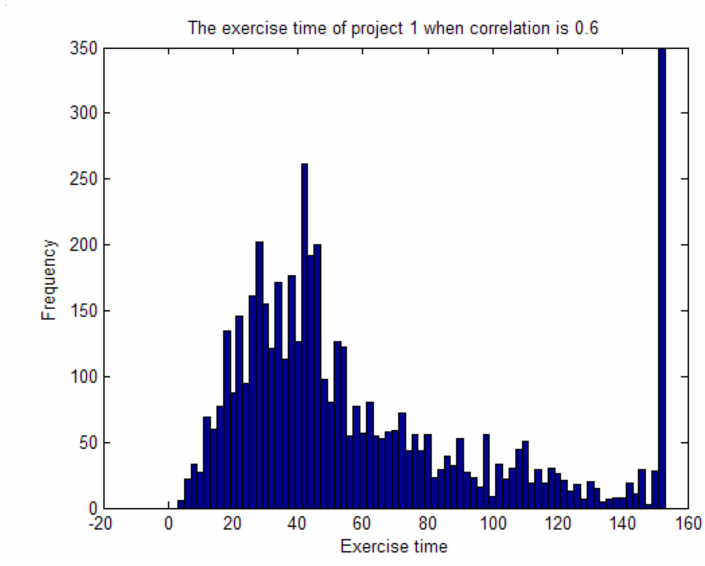

Figure 5.7 The exercise time of project 1 when correlation is 0.6

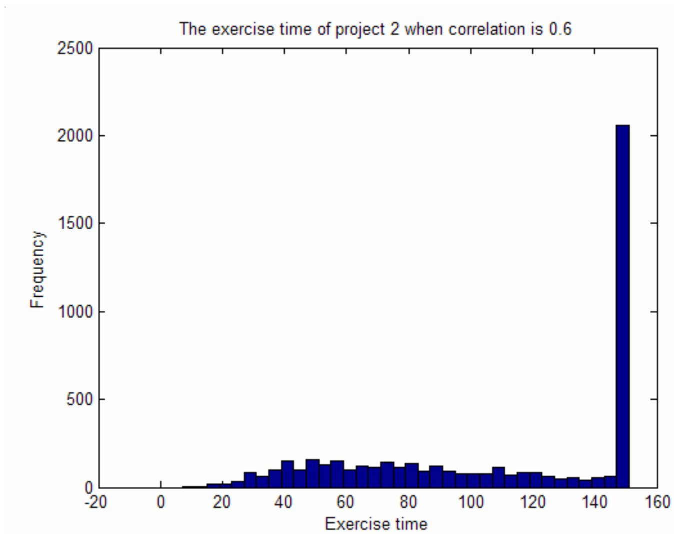

Figure 5.8 The exercise time of project 2 when correlation is 0.6

Figure 5.9 summarizes the frequency of exercise time of project 1 when the correlations between shocks on project 1 and project 2 are positive, zero and negative. Table 5.1 shows the basic comparative static, the option values, the average exercise time, and the median exercise time for project 1 and project 2. From the results, it seems like the correlations between shocks on project 1 and project 2 have no influence on the distribution of exercise times of project 1, which follows a normal distribution. When the correlation changes from negative to positive, the average and the median exercise time will increase. It implies that when the shocks on two projects are positively correlated, the portfolio effects are more critical and the developer will wait for more information before making the investment decision. One significant implication is that when the correlation changes from negative to positive, the developer will likely to wait until the end of the option period. The option value for project 1 is higher if the correlation between the shocks on the two projects changes from negative to positive, which means that the embedded option is more valuable if two projects have positive correlations.

	ρ=-0.6	ρ=0	ρ=0.6
Option Value of Project 1	231.6	233.5	233.8
Option Value of Project 2	11.5	11.7	7.3
Average Exercise Time of project 1	50	55	60
Average Exercise Time of project 2	104	105	109
Median Exercise Time of project 1	44	46	46
Median Exercise Time of project 2	97	107	118

Table 5.1 Statistic of option value and exercise time of project 1 and project 2

The distribution of exercise time for project 2 is discrete and flat when the correlation between the shocks on the two projects is positive. It looks like a uniform distribution from 40 to 120. When the correlation changes from negative to positive, the average and the median exercise time for project 2 increase. It implies that the developer will wait for longer time for new information before making his investment decision, when he has a portfolio with positive correlations. The option value of project 2 is small compared to the option value of project 1.

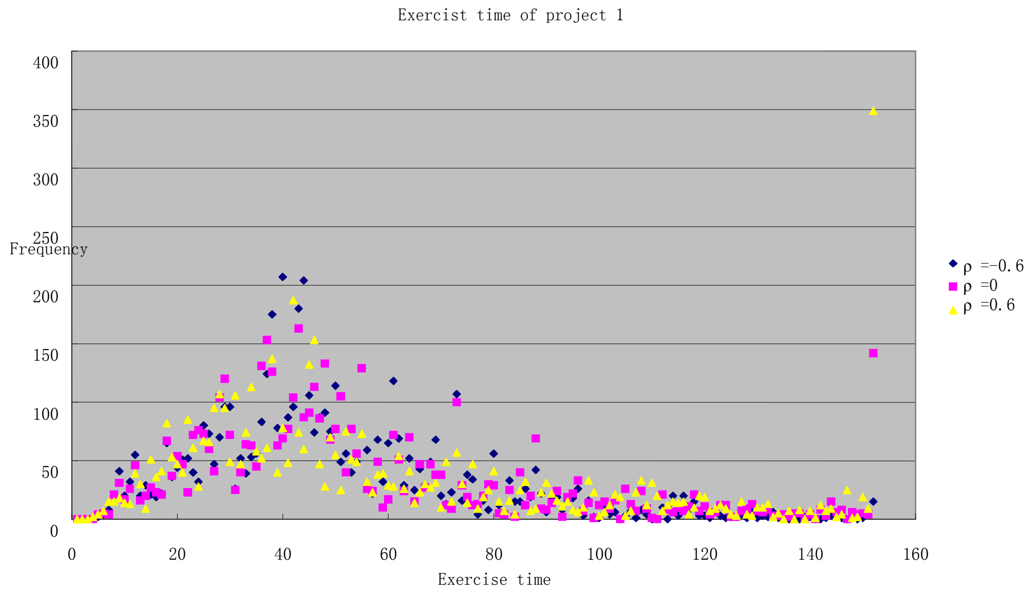

Figure 5.9 Frequency of exercise time of project 1

By keeping the correlation of shocks for project 1 and project2 at 0.6, and we set the interactive effect on project 1 and project 2, θ_{12} is 0.3, and θ_{21} is 0.3. The distributions of exercise times for project 1 and project 2 are shown in Figure 5.10 and 5.11. The average exercise time of project 1 is 54 and the median exercise time is 46. The distribution shows that the developer will likely make an early development decision. The average exercise time of project 2 is 71 and the median exercise time is 54. The exercise time of project 2 is much earlier than that in the base case scenario. The distribution looks like a normal distribution rather than a flat uniform distribution. The possibility of waiting until the end of decision time decreases for project 2. The present option value of project 1 is 161.6 while the present option value of project 2 is 61.5. Compared the two option values with that in the base case scenario, the option value of project 1 decreases and the option value of project 2 increases, which is due to the positive interactive effect. In conclusion, the positive interactive effect will impact on the development of project 1, but the impact on the development option for project 2 is larger.

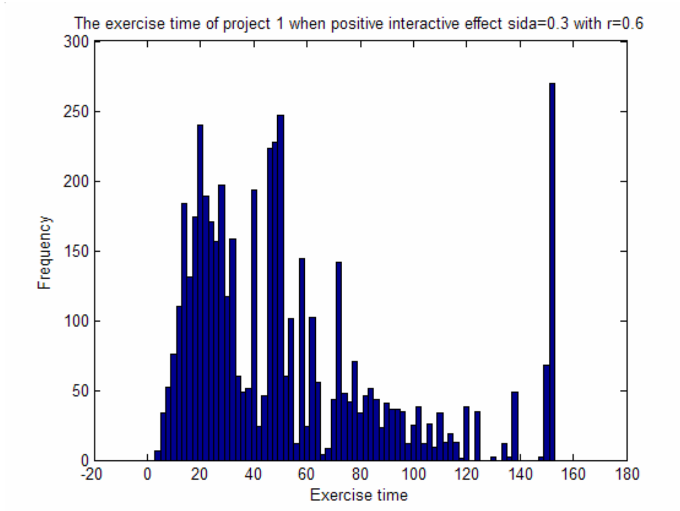

Figure 5.10 The exercise time of project 1 with positive interactive effect

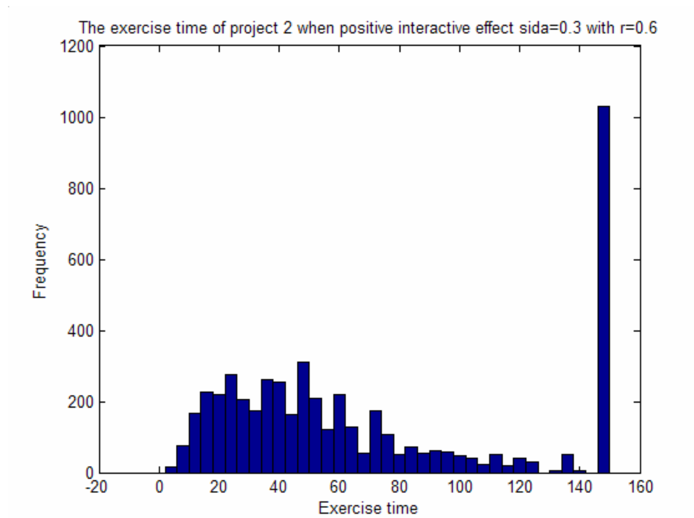

Figure 5.11 The exercise time of project 2 with positive interactive effect

If two projects are developed by two independent developers, they will make the investment decision separately by only considering the maximization of the value of a single project. Based on a correlation of 0.6 and assuming a sequential development option, where the developer will develop the project 1 with a higher growth rate, and a higher volatility, and then followed by project 2. The distributions of exercise times for project 1 and project 2 are flat, as shown in Figures 5.12 and 5.13. The average exercise time of project 1 is 100 and the average exercise time of project 2 is 114. At a possibility of 50%, the developer will wait until the end of decision time for project 1 and project 2. Compared to the case where the two projects are owned by the same developer, the development timing is longer. The option value of project 1 is 400 and the option value of project 2 is 18.4. The volatility and future risk is larger when the two projects are owned by two independent developers, and thus both option values for project 1 and project 2 will increase. Both developers will wait longer time for new information before making their investment decisions.

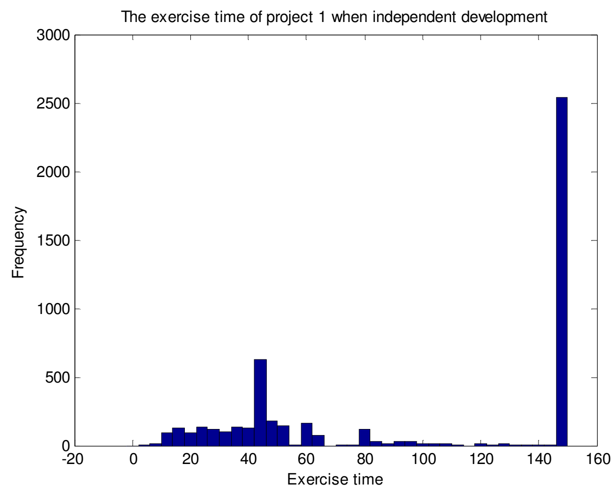

Figure 5.12 The exercise time of project 1 when independent development

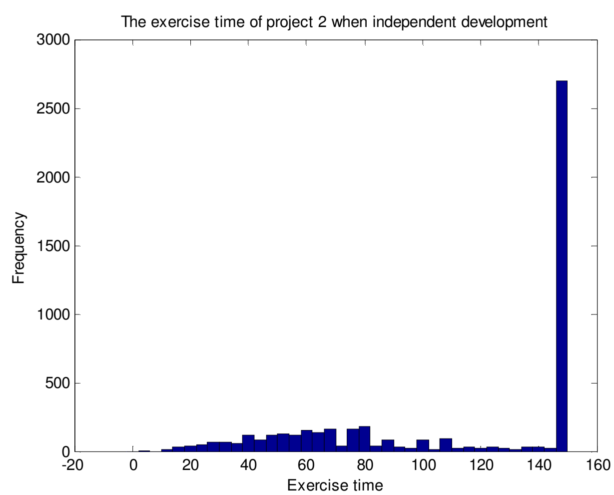

Figure 5.13 The exercise time of project 2 when independent development

Obviously, if one developer has two projects instead of separate developments, he will maximize the collective profit of both projects. Because of the embedded options and interactive effect, the developer will invest in the first project much earlier. The investment decision for project 2 depends on the interactive effect and the market volatility. He will invest in project 2 earlier in a boom market, when the two projects have positive interactive effect, whereas he will wait in a down market.

5.5 Conclusion

In the past literature on real options, the optimal development timing problem is analyzed on the basic of a single project. In an actual market, developers will have a portfolio of projects that have interactive effects between projects. They want to maximize the collective profit of the portfolio instead of a single project. The interactive effect between the projects impacts the development timing when the developer makes his investment decision. Heterogeneity is unique in real estate products, which can be differentiated by location and asset types, and they are also affected by different economic shocks. This chapter differentiates the two projects by different economic shocks through two pricing processes. We also consider the correlation between the prices of two projects and discuss whether the development timing changes with the different correlations. These two projects by the same developer may have some portfolio effect and we examine how the positive interactive effect impacts the development timing of the two projects. We also compare the results with the case when the two projects are owned by two independent developers.

We use the Least Square Monte Carlo Simulation to numerically determine the options values and optimal timing of development under different portfolio scenarios. When the two projects have positive correlations, the portfolio is more sensitive to the market volatility. The developer would likely defer the development, especially the second project, when these two projects have positive correlations. The option value is higher when the two projects have positive correlations. The portfolio effect makes the developer, who has more market power, to invest early in the property market. In our model, the preemptive strategies where developers will interact with each other will not be modeled. The first mover advantage as suggested will require a more complex game theoretical option framework, and it is not within the scope of the current study. In our case, developers wait for the market recovery because the profit is negative in the recent market condition. The option to wait is valuable because of the market uncertainty.

The option on option problem is complex. The price of project includes not only the intrinsic value, but also embedded option values. These options depend on the market volatility and future uncertainty. This chapter tries to decompose the interdependent options into a group of simple options, and uses a two-stage sequential option model to solve the problem. In this chapter, we use the improved Monte Carlo Simulation to estimate the option values given the assumptions of the initial parameters and paths. Following Grenadier (1996), we could use the game theory framework to discuss the interactive strategy of two developers in the future research.

Chapter 6 Conclusion and Extension

6.1 Summary of Main Findings

This study develops a real option model to examine multi-project interactive effects on developer's development timing strategies. The proposed model also evaluates how investment strategies change under different market situation and for different project type, either homogeneous or heterogeneous. When the two projects are developed jointly by the developer, positive interactive effects can be created by integrating the two projects to collectively enhance the values of the two projects. If the projects are developed by two competing developers, development strategy may be adopted such that the completion of one project may create negative externality on the neighboring project owned by another developer. The correlation between the two heterogeneous projects also impacts the development timing.

We first derive the development timing strategies under different market situation in a deterministic framework, where one developer has two projects on conditions that both the demand and the cost are constant. We find that the developer will abort the project when the demand is weak, and he will choose to develop a single project when the demand curve is steep. In a market with a flat demand curve, it will be economically optimal for the developer to develop the two projects simultaneously. The positive portfolio effects shorten the time to wait to develop for the two projects, and the developer will prefer to undertake the two projects simultaneously. Follow the objectives (a) and (b) in Chapter 1, the findings correspond to our hypothesis (a), (b) and (c) in Chapter 1. The positive interactive effect pushes the developer to make early development of both project 1 and project 2. As in the actual market, developer would likely to adjust strategies in respond to the market situation.

We extended the models in a stochastic framework with one developer, who has development options on two different but contiguous land parcels. The developer will have the options to develop the two projects simultaneously or sequentially, and to develop the two land parcels into two homogeneous or heterogeneous projects. We build a two-stage sequential option model and sensitive analyses were conducted on the model. The positive interactive effects between the projects will push the developer to trigger the development options on the two projects earlier. The developer will make simultaneous development, if the portfolio effect is strong enough to offset the opportunity costs of not waiting for one more period. In other words, the portfolio effect lowers the trigger value of investment for the second project. The developer will otherwise be better off by delaying the development of the second project, which results in a sequential development process. We also evaluate the development strategies of the developer under different market demand conditions. The developer will choose to develop a single project when the demand curve is steep, while in a market with a flat demand curve, he will prefer to invest in the both projects. The sensitive results show that as the portfolio effect increases the price by 40%, the development of project 2 will be considered under the assumed parameters. Interactive effect, demand curve, as well as the comparable advantage of the two projects will impact the development timing. All the findings are in line with the objectives (c) and (d) and hypotheses (a), (b), (c) and (d) in

Chapter 1.

We further extended the model to integrate the sub-game optimal strategies in a real options framework by allowing the demand functions for the two projects to follow different stochastic processes with drift. This assumption fit well to the model with heterogeneity in property types. Because of the complexity of the options model, we use the Least Square Monte Carlo Simulation proposed by Longstaff &Schwartz (2001) to solve the optimal timing option values. The numerical results show that positive correlations between the economic shocks on project 1 and project 2 will defer the development, especially for the second project. As the risk of portfolio increases with positive correlations, the portfolio will be more sensitive when the future uncertainty increases. The positive interactive effects will kick-start the development of both projects earlier. Compared with the case of two projects developed by a single developer, the results show that the developer, who has a portfolio of two projects, has more market power and would invest in the projects earlier. This part accomplishes the objective (e) and confirms the hypotheses (b), (c), (d) and (e) in Chapter 1.

6.2 Contribution of the Study

The real options theory was developed and extended from the financial option theory of Black & Scholes (1973). Past real options models explore the embedded option in the development and provide useful pricing framework for properties with embedded options. These option models were always developed from the perspective of a single project. In the actual property market, however, one developer always manages a portfolio consisting of two or more projects at the same time or develops the projects in a sequential process. These projects are not identical, they have interactive effects, which increase revenue, or save cost or reduce management resource. From the perspective of a developer, he would like to maximize the collective profit, and the portfolio effect impacts the development timing strategies. Our study, hinging on the special angle from developer's decision making, explores how the multi-project interactive effect impacts development timing strategies. We fill up the gap by examining the portfolio effects between multiple projects and their effects on developer's investment timing strategies.

Market volatility and future uncertainty are the sources of real option. Developers always make different strategies under different market situation. They would like to make investment in a boom market and defer the investment when the market is down. One phenomenon pointed out by Grenadier (1996) is that over-investment occurs even when the property market is declining. Our study also explores how the developer adjusts his investment timing strategies when he faces different market conditions. We differentiate the demand market by different curvature of demand functions. We also discuss the correlations between the economic impacts on the two projects. The relationship of the two projects would affect the risk of the portfolio, and the developer is more sensitive with the market volatility when the two projects have positive correlations. The positive interactive effect of the two projects is the main driver of the developer's optimal timing strategy. In a declining property market, the interactive effect will shorten the developer's option to wait to invest, which could be one of the reasons for in an over-investment phenomenon.

Our study begins with a simple framework which assumes that both demand and cost are constant. We then build the option timing model in a stochastic framework when one developer has two projects, whose prices are impacted by the same economic shocks. We decompose the complex options model into a two-stage option model and solve the ODE to get a close-form solution. We also do sensitivity analysis to explore the factors such as demand curve, comparable advantage and interactive effect and their effects on development timing option premiums. In the end, we extend the model into a game theoretic framework, in which the economic shocks impact on two projects that follow two different Brownian motion processes. The option on option model becomes more complex and we solve the PDE numerically using the Least Square Monte Carlo Simulation proposed by Longstaff & Schwartz (2001). Our research design is from deterministic to stochastic, and combines both analytic and numerical methods to test various hypotheses.

6.3 Limitation and Recommendation

One of the limitations of our study is the assumption of a constant cost. When we focus on the stochastic demand, we assume that the project cost is unchanged. While in the actual market, the material cost, construction cost and management cost may always vary over time. Minimizing the cost of a project is an important part of the developer's business goal in profit maximization. As the stochastic demand, we could assume that the unit cost of project follows a Geometric Brownian Motion process. The interaction between stochastic demand and stochastic cost makes the investment decision more complex. We could consider the ratio of revenue per cost in the model. Several studies have considered the stochastic cost function

(William (1991), Sing (2001)). In the future research, we could include the stochastic cost to better reflect the actual market condition.

In our study, both the deterministic or stochastic models are developed on the premise of a monopoly market with only one developer. The monopolistic property market is uncommon, and the markets can be better modeled as a duopoly or a competitive market. Market structure is one of the interesting factors which would impact the investment decision. We could extend the model into a duopoly market, with two developers, which have their own portfolios. These two developers would consider not only their own investment decision but also the impact from their rival's behavior. The interaction between the two developers would affect both developer's investment behaviors and development decisions. We could follow Grenadier (1996) to extend the model in a game-theoretic duopoly framework, where the two developers have their own portfolios and discuss the developer's behavior with interaction.

In our study, the portfolio consists of only two projects of either homogenous or heterogeneous types. We could extend the two projects into a n-project case, which may increase the complexity of the interactive effects. We could adjust the model to explore n-project portfolio effect on developer's decision making.

As the model in Chapter 5, when the two projects follow two different stochastic price processes, and there are no analytical solutions for the PDEs. Instead, we use the Monte Carlo Simulation technique to derive at the optimal solutions. In the simulation process, the initial parameters and the paths of the simulation would have significant impact on the results. In the future we could increase the simulation path using the parallel computation to reduce the processing time. The choice of basic regression function as well as the least square regression could also be improved.

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Appendix

GenerateShock1.m

```
clc
clear all
disp(['Now Input the Required Parameters as following:']);
disp(['                              ']);

%Determine the points of time, T1, T2, ti%
t0=input('    Real Option Start Time =');
disp(['                          ']);
T1=input('    The Decision Time Length of project1 is (Quarter) ');
disp(['                          ']);
T2=input('    The Decision Time Length of project2 are (Quarter) ');
disp(['                          ']);
ti=input('    Interval Time =');
disp(['                          ']);
t=t0:ti:T;

%Get other parameters from keyboard%
Y0=input('    Market uniform economic shock at the Start Time =');
disp(['                          ']);

u1=input('    The expected growth rate of economic shock on project1 is ');
disp(['                          ']);

sd1=input('    the standard deviation of the economic shock on project1 is ');
disp(['                          ']);

%Determine the number of normally distributed random variables need to be generated%
m=length(t);
disp(['The number of intervals is ', num2str(m)]);
disp(['                          ']);
disp(['    --------Simulation Started.    Please Wait a While for Results--------']);

np=input('    Please input the number of paths needed    ');

cput=cputime;
tic;
```

```
err=randn(np,m-1);

for inp=1:np
    Y1(inp,1)=Y0;
    for j=2:m
    Y1(inp,j)=Y1(inp,j-1)*exp((u1*ti-sd1^2/2)+err(inp,j-1)*sd1);
    end
end

disp(['---------economic shock on project1 path simulating accomplished----------']);
disp(['                    ']);

cput=cputime-cput;
disp(['    The CPU time for the above computation is', num2str(cput)]);
disp(['        ']);

save EconomicShock1
```

GenerateShock2.m

```
clear all
load EconomicShock1

%Start Generate the Path of project2
err1=randn(np,m-1);
corr=input('    Please input the correlation of the project1 and project2 ');
errcon=err.*corr+err1.*(1-corr^2)^0.5;
disp(['    The correlated random error has been generated ']);
disp(['               ']);

u2=input('    The expected growth rate of economic shock on project2 is ');
disp(['                         ']);

sd2=input('    the standard deviation of the economic shock on project2 is ');
disp(['                         ']);

disp(['    The correlated random error has been generated ']);
disp(['               ']);

concput=cputime;
tic;
```

```
for inp=1:np
    Y2(inp,1)=Y0;
  for j=2:m
    Y2(inp,j)=Y2(inp,j-1)*exp((u2*ti-sd2^2/2)+errcon(inp,j-1)*sd2);
  end
end

disp(['---------path simulating accomplished----------']);
disp(['                    ']);

concput=cputime-concput;
disp(['    The CPU time for the above computation is', num2str(concput)]);
disp(['        ']);

save EconomicShock2
```

ComputeOption.m

```
clear all
load EconomicShock2
%define risk free rate, discount rate, demand curve and investment cost%
rfrate=0.1;
demand1=100;
demand2=80;
I1=5000;
I2=2000;
sida1=input('    the interactive effect on project1 is ');
disp(['                          ']);
sida2=input('    the interactive effect on project2 is ');
disp(['                          ']);

%computer initial payoff of project 1 and 2%
payoff2=Y1.*(1+sida1)*demand2/(rfrate-u1)-Y1.*demand1/(rfrate-u1)+Y2.*(1+sida2)*dema
nd2/(rfrate-u2)-I2;
payoff1=Y1.*demand1/(rfrate-u1)-I1;

disp(['      -----------current task "paths of payoff" accomplished----------']);
disp(['        ']);

%start to compute the present option value and generate exercise node
```

```
lgthpf=length(payoff2(1,:));

%compute the last stage of payoff1 and payoff2%
for i=1:np
        if payoff2(i,lgthpf)<0
                payoff2(i,lgthpf)=0;
        else
        end
        payoff1(i,lgthpf)=payoff1(i,lgthpf)+payoff2(i,lgthpf)
        if payoff1(i,lgthpf)<0
              payoff1(i,lgthpf)=0;
        else
        end
end

%computer the whole stage of payoff1 and payoff2%
for j=1:lgthpf-1
      clear payoff1x1 payoff1x2 payoff1X payoff1y payoff1yh a nobe
      nr=0;
      for i=1:np
            if payoff2(i,lgthpf-j)<=0
                  payoff2(i,lgthpf-j)=0;
            else
                  nr=nr+1;
                  payoff1y(nr,1)=0;
                  for k=lgthpf-j+1:lgthpf
                  if payoff2(i,k)>0
                  payoff1y(nr,1)=payoff2(i,k)/(1+rfrate)^(k-lgthpf+j);
                  end
                  end
                  payoff1x1(nr,1)=Y1(i,lgthpf-j);
                  payoff1x2(nr,1)=Y2(i,lgthpf-j);
                  nobe(1,nr)=i;
            end
      end
      %less than 0, then 0, otherwise discount the further payoff into now%
      %stage for payoff2%

  if nr>0
          payoff1X=[ones(size(payoff1x1)) payoff1x1 payoff1x2 payoff1x1.^2 payoff1x2.^2];
          a=payoff1X\payoff1y;

      for iyh=1:nr
```

```
payoff1yh(iyh,1)=a(1,1)+a(2,1)*payoff1x1(iyh,1)+a(3,1)*payoff1x2(iyh,1)+a(4,1)*payoff1x1
(iyh,1).^2+a(5,1)*payoff1x2(iyh,1).^2;

        if payoff1yh(iyh,1)>payoff2(nobe(1,iyh),lgthpf-j)
            payoff2(nobe(1,iyh),lgthpf-j)=0;
        else
            for nj=0:j-1
                payoff2(nobe(1,iyh),lgthpf-nj)=0;
            end
        end
    end
 end
 % regression, if >0, now stage payoff=0 otherwise future stage=0%

  %computer payoff1, <0, then=0, otherwise, discount further stage%
  clear payoff1x1 payoff1x2 payoff1X payoff1y payoff1yh a nobe
  nr=0;
    for i=1:np
        for k=lgthpf-j:lgthpf
            if payoff2(i,k)>0
            payoff1(i,lgthpf-j)=payoff1(i,lgthpf-j)+payoff2(i,k)/(1+rfrate)^(k-lgthpf+j);
            else
            end
        end
        if payoff1(i,lgthpf-j)<=0
            payoff1(i,lgthpf-j)=0;
        else
            nr=nr+1;
            payoff1y(nr,1)=0;
            for k=lgthpf-j+1:lgthpf
            if payoff1(i,k)>0
            payoff1y(nr,1)=payoff1(i,k)/(1+rfrate)^(k-lgthpf+j);
            end
            end
            payoff1x1(nr,1)=Y1(i,lgthpf-j);
            payoff1x2(nr,1)=Y2(i,lgthpf-j);
            nobe(1,nr)=i;
        end
    end

  %regression, if >0, now stage payoff=0 otherwise future stage=0%
  if nr>0
        payoff1X=[ones(size(payoff1x1))    payoff1x1    payoff1x2    payoff1x1.^2
payoff1x2.^2];
```

```
        a=payoff1X\payoff1y;

    for iyh=1:nr

payoff1yh(iyh,1)=a(1,1)+a(2,1)*payoff1x1(iyh,1)+a(3,1)*payoff1x2(iyh,1)+a(4,1)*payoff1x1
(iyh,1).^2+a(5,1)*payoff1x2(iyh,1).^2;
        if payoff1yh(iyh,1)>payoff1(nobe(1,iyh),lgthpf-j)
            payoff1(nobe(1,iyh),lgthpf-j)=0;
        else
            for nj=0:j-1
                payoff1(nobe(1,iyh),lgthpf-nj)=0;
            end
        end
    end
  end

end

  disp(['--------The matrix of option value has been generated---------']);
  disp(['        ']);
  disp(['    >>>> Now start to compute the present value of the option value <<<< ']);
  disp(['        ']);

for i=1:np
    nobe1(i,1)=lgthpf+1;
    nobe2(i,1)=lgthpf+1;
end

cumwrong=0

for i=1:np
   for j=1:lgthpf
       if payoff1(i,j)>0
       nobe1(i,1)=j;
       end
       if payoff2(i,j)>0
       nobe2(i,1)=j;
       end
   end
   if nobe1(i,1)>nobe2(i,1)
       wrong(i,1)=1;
       cumwrong=cumwrong+1;
       sumpayoff2=payoff2(i,nobe2(i,1))+Y1(i,nobe2(i,1))*demand1/(rfrate-u1)-I1;
       if sumpayoff2>0
```

```
                if nobe1(i,1)<lgthpf+1
                payoff1(i,nobe1(i,1))=0;
                end
                payoff1(i,nobe2(i,1))=Y1(i,nobe2(i,1))*demand1/(rfrate-u1)-I1;
                nobe1(i,1)=nobe2(i,1);
            else
                if
nobe1(i,1)<(lgthpf+1)&(Y1(i,nobe1(i,1))*(demand2-demand1)/(rfrate-u1)+Y2(i,nobe1(i,1))*
demand2/(rfrate-u2)-I2>0)
                    payoff2(i,nobe2(i,1))=0;

payoff2(i,nobe1(i,1))=Y1(i,nobe1(i,1))*(demand2-demand1)/(rfrate-u1)+Y2(i,nobe1(i,1))*de
mand2/(rfrate-u2)-I2;
                    nobe2(i,1)=nobe1(i,1);
                else
                    payoff2(i,nobe2(i,1))=0;
                    nobe2(i,1)=lgthpf+1;
                end

            end
        end

end

undevelop1=0
undevelop2=0
for i=1:np
        if nobe1(i,1)==lgthpf+1
                nobe1(i,1)=0;
                undevelop1=undevelop1+1;
        end
        if nobe2(i,1)==lgthpf+1
                nobe2(i,1)=0;
                undevelop2=undevelop2+1;
        end
end

    sumprev1=0;
    sumprev2=0;
    for i=1:np
            cumprev1=0;
            cumprev2=0;
            for j=1:lgthpf
                cumprev2=cumprev2+payoff2(i,j)/(1+rfrate)^(j-1);
```

```
            cumprev1=cumprev1+payoff1(i,j)/(1+rfrate)^(j-1);
        end
        prev2(i,1)=cumprev2;
        prev1(i,1)=cumprev1;
        sumprev2=sumprev2+prev2(i,1);
        sumprev1=sumprev1+prev1(i,1);
    end

avprev2=sumprev2/np;
avprev1=sumprev1/np;
disp(['    The present option value of project 1 is ',num2str(avprev1)]);
disp(['    The present option value of project 2 is ',num2str(avprev2)]);
disp(['          ']);

    sumexe1=0;
    sumexe2=0;
    np1=np;
    np2=np;
    for i=1:np
        if nobe2(i,1)==0
            np2=np2-1;
        end
        if nobe1(i,1)==0
            np1=np1-1;
        end
        sumexe2=sumexe2+nobe2(i,1);
        sumexe1=sumexe1+nobe1(i,1);
    end

avexe2=sumexe2/np2;
avexe1=sumexe1/np1;
disp(['    The exercise time of project 1 is ',num2str(avexe1)]);
disp(['    The exercise time of project 2 is ',num2str(avexe2)]);
disp(['          ']);
disp(['          ']);

save OptionValue
```

IndependentOption.m

```
clear all
```

```
load EconomicShock2
%define risk free rate, discount rate, demand curve and investment cost%
rfrate=0.1;
demand1=100;
demand2=80;
I1=5000;
I2=2000;
%computer initial payoff of project 1 and 2%
Spayoff2=Y2.*demand2/(rfrate-u2)-I2;
Spayoff1=Y1.*demand1/(rfrate-u1)-I1;

disp(['      -----------current task "paths of payoff" accomplished----------']);
disp(['        ']);

%start to compute the present option value and generate exercise node
lgthpf=length(Spayoff2(1,:));

%compute the last stage of payoff1 and payoff2%
for i=1:np
        if Spayoff2(i,lgthpf)<0
              Spayoff2(i,lgthpf)=0;
        else
        end
        if Spayoff1(i,lgthpf)<0
              Spayoff1(i,lgthpf)=0;
        else
        end
end

%computer the whole stage of payoff1 and payoff2%
for j=1:lgthpf-1
clear payoff1x1 payoff1x2 payoff1X payoff2X payoff1y1 payoff1y2 payoff1yh1 payoff1yh2
a1 a2 nobex1 nobex2
      nr1=0;
      nr2=0;
      for i=1:np
            if Spayoff1(i,lgthpf-j)<=0
                  Spayoff1(i,lgthpf-j)=0;
            else
                  nr1=nr1+1;
                  payoff1y1(nr1,1)=0;
                  for k=lgthpf-j+1:lgthpf
                  if Spayoff1(i,k)>0
                  payoff1y1(nr1,1)=Spayoff1(i,k)/(1+rfrate)^(k-lgthpf+j);
```

```
                    end
                    end
                    payoff1x1(nr1,1)=Y1(i,lgthpf-j);
                    nobex1(1,nr1)=i;
                end
                if Spayoff2(i,lgthpf-j)<=0
                    Spayoff2(i,lgthpf-j)=0;
                else
                    nr2=nr2+1;
                    payoff1y2(nr2,1)=0;
                    for k=lgthpf-j+1:lgthpf
                    if Spayoff2(i,k)>0
                    payoff1y2(nr2,1)=Spayoff2(i,k)/(1+rfrate)^(k-lgthpf+j);
                    end
                    end
                    payoff1x2(nr2,1)=Y2(i,lgthpf-j);
                    nobex2(1,nr2)=i;
                end

            end
            %less than 0, then 0, otherwise discount the further payoff into now%
            %stage for payoff2%

        if nr1>0
                payoff1X=[ones(size(payoff1x1)) payoff1x1 payoff1x1.^2];
                a1=payoff1X\payoff1y1;
        end

        if nr2>0
                payoff2X=[ones(size(payoff1x2)) payoff1x2 payoff1x2.^2];
                a2=payoff2X\payoff1y2;
        end

            for iyh=1:nr1
                payoff1yh1(iyh,1)=a1(1,1)+a1(2,1)*payoff1x1(iyh,1)+a1(3,1)*payoff1x1(iyh,1)^2;

                if payoff1yh1(iyh,1)>Spayoff1(nobex1(1,iyh),lgthpf-j)
                    Spayoff1(nobex1(1,iyh),lgthpf-j)=0;
                else
                    for nj=0:j-1
                        Spayoff1(nobex1(1,iyh),lgthpf-nj)=0;
                    end
                end
            end
```

```
    for iyh=1:nr2
        payoff1yh2(iyh,1)=a2(1,1)+a2(2,1)*payoff1x2(iyh,1)+a2(3,1)*payoff1x2(iyh,1)^2;

        if payoff1yh2(iyh,1)>Spayoff2(nobex2(1,iyh),lgthpf-j)
            Spayoff2(nobex2(1,iyh),lgthpf-j)=0;
        else
            for nj=0:j-1
                Spayoff2(nobex2(1,iyh),lgthpf-nj)=0;
            end
        end
    end

end

for i=1:np
    nobe1(i,1)=lgthpf+1;
    nobe2(i,1)=lgthpf+1;
end

cumwrong=0
sumprev1=0;
sumprev2=0;
undevelop1=0
undevelop2=0
sumexe1=0;
sumexe2=0;

for i=1:np
   for j=1:lgthpf
       if Spayoff1(i,j)>0
       nobe1(i,1)=j;
       end
       if Spayoff2(i,j)>0
       nobe2(i,1)=j;
       end
   end

    if nobe1(i,1)>nobe2(i,1)
       wrong(i,1)=1;
       cumwrong=cumwrong+1;
       nobe1(i,1)=lgthpf+1;
       nobe2(i,1)=lgthpf+1;
   end
```

```
    if nobe1(i,1)==lgthpf+1
             undevelop1=undevelop1+1;
        else
             prev1(i,1)=Spayoff1(i,nobe1(i,1))/(1+rfrate)^(nobe1(i,1)-1)
             sumprev1=sumprev1+prev1(i,1);
             sumexe1=sumexe1+1;
    end

    if nobe2(i,1)==lgthpf+1
             undevelop2=undevelop2+1;
        else
             prev2(i,1)=Spayoff2(i,nobe2(i,1))/(1+rfrate)^(nobe2(i,1)-1)
             sumprev2=sumprev2+prev2(i,1);
             sumexe2=sumexe2+1;
    end

end

avprev2=sumprev2/sumexe1;
avprev1=sumprev1/sumexe2;
disp(['    The present option value of project 1 is ',num2str(avprev1)]);
disp(['    The present option value of project 2 is ',num2str(avprev2)]);
disp(['         ']);

%regression, if >0, now stage payoff=0 otherwise future stage=0%

save IndepOptionValue
```