OVERBOOKING MODELS FOR AIR CARGO YIELD MANAGEMENT

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Summary

Yield management was initiated in the airline industry with a special emphasis on ticket booking systems (overbooking, pricing and allocation etc.). In the last forty years since the first publication on overbooking control, air ticket reservation systems have evolved from a low-level inventory control process to a major strategic information system. Yield management, alternatively known as revenue management, can be defined as the integrated management of price and inventory to maximize the profitability of a company (or minimize the expected total cost). In the case of combination air carriers, or airlines operating airplanes that carry passengers as well as cargo, it is the management of passenger fares and seats together with cargo rates and cargo space.

In both ticket booking and space booking processes, airlines usually adopt the strategy of overbooking. Overbooking is a practice of intentionally accepting an excess of tickets or cargo bookings than the corresponding capacity along the booking process to compensate for possible cancellations and no-shows. The purpose of this study is to formulate mathematical models to determine the optimal cargo overbooking level so as to minimize the expected total under-sales cost and spoilage cost.

The cargo booking process in this paper is modeled as a two-dimension Markov process in the overbooking model. It is a combination of homogeneous Poisson arrival process with constant arrival rate and non-homogeneous Poisson cancellation process with cancellation rate depending on the number of customers in the system. The two state variables are defined as the number of bookings in the system and the total amount of space that has been booked. Within each decision period, one and only one of three events will happen: arrival event, cancellation event and non-event. Limiting probability distribution is used to approximate the joint probability distribution of the final number of bookings and final amount of cargo coming for boarding. Since the resulting state-space of the model is large, the mathematical software Mathematica is used to solve the problem. Simulation results show that the static overbooking model produces very good approximations on real booking behaviors and ensures the minimum expected total cost.

The overbooking model constructed is useful in airline cargo space booking operations. The study provides an effective mathematical approach to solve the real problem in air cargo space booking. It improves operational efficiency of the cargo booking system and helps airlines to maximize the revenue from the cargo sector.

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Nomenclature

$A_{i,j}$	Sub-transition matrix describing the transition from $X_n = i$ to $X_{n+1} = j$, where $j = i-1, i, or\ i+1$;
C	Flight capacity for air cargo;
$Ca^{(X_n,Y_n)}$	Cancellation variable in period n with state variables (X_n, Y_n);
CS	The summation of underage cost and overage cost;
D	Individual demand, which is assumed to have discrete uniform distribution, taking on a value from $\{1, 2, ..., d\}$;
$E[\cdot]$	Expectation operator;
$f_{X_N,Y_N}(x, y)$	Joint probability mass function of X_N and Y_N;
$f_{X_N}(x)$	Marginal probability mass function of X_N;
$f_{Y_N}(y)$	Marginal probability mass function of Y_N;
$\{(j, Y_n(X_n = j))\}$	The collection of all states, given that j customers are in the system;
n and N	Decision periods with n denoting any period along the process and N denoting the time of departure;
$N_{X_n=x}$	The number of all states given $X_n = x$;
OL	Overbooking level, which is defined as the extra space beyond capacity that can be reserved in booking process;
OC	Overage cost in overbooking model;
p_O	Overage penalty;

p_U	Underage penalty;
UC	Underage cost in overbooking model;
X_n	State variable one, which is defined as the number of customers in system in period n;
Y_n	State variable two, which is defined as the amount of space reserved in system in period n;
$y_n^1, y_n^2, ...y_n^{X_n}$	The X_n bookings in the system in period n;
$Y_n(X_n = x)$	Space of Y_n when $X_n = x$, which is the collection of all possible values of Y_n given $X_n = x$;
λ	Arrival rate;
$\tau(X_n)$	Cancellation rate, which is assumed to be a function of X_n;
$\pi = (\pi_1, \pi_2, ...\pi_{N_{Matrix}})$	Limiting probability distribution of the transition matrix, where $N_{matrix} = \sum_{i=0}^{NC} N_{X_n = i}$;

1. Introduction

The invention of aircraft in the early twentieth century is one of the greatest breakthroughs in human. In fact, air transport industry started initially in 1919 soon after the First World War, but it was not until peace was restored after the Second World War that the era of major expansion really began.

1.1 Airline industry

Air transport plays an important role in modern society. Airplanes, an invention of 20th century, have developed extensively during the past half-century. No other form of transportation still has as much potential for further development in this century. “It is a big industry, becoming the key element in the world’s largest industry, travel and tourism, which generates $3400 billion a year in revenue, accounts for approximately 10 percent of world GDP, takes almost 11 percent of consumer spending, and employs over 200 million people, or roughly one in every nine people in the global labor force. Over the last 50 years the airline industry has consistently grown at a very fast rate, well above the growth rate in world GDP”. (Hanlon, P. “Global airlines: competition in transnational industry”, Oxford, 1996, page 1)

Nowadays the influence of airline industry permeates every corner of the world. Millions of passengers travel by air each day. Passengers can be divided into tourist or business travel. With the difference of traveling purposes in mind, airlines charge different prices to different customers at different time points along booking process. Behind this popularity of passenger traveling in airline industry, air industry also provides transportation for cargo. This part of air transport called air cargo transport grows fast. Though the revenue from cargo transport accounts for only a fraction of total airline revenue, more attention is being paid on air cargo in recent years in view of its steady growth. Nowadays manufacturers usually prefer air transport to ship finished products to customers so as to minimize inventory cost. If you want to send out letters or parcels quickly, airmail service is readily available at any post office. Alternatively, express mail service is another way to increase the speed of delivery, such as FedEx, UPS and OCS to name a few. The above gives some examples of applications of airline industry familiar to readers.

1.2 Introduction of yield management

This thesis is focused on cargo yield management with an emphasis on overbooking problem. However, the origin of yield management comes from passenger sector. It is necessary to trace the origin of yield management and study the techniques used in passenger yield management, which will help the modeling of cargo overbooking later on. A general idea of yield management in passenger and cargo sectors will be presented first in this chapter and a detailed review of yield management will be presented in the next chapter.

To fully and systematically maintain profit, most big airlines around the world are employing yield management, also known as revenue management, which is the science of profit maximization, to capture each possible chance of earning. Yield management is the integration of science, information technology, and business process to deliver the right product to the right customer at the right time at the right price. The origin of yield management comes from situation where perishable products exist. For example, the value of rooms in a hotel cannot be realized until on the day that it is occupied by lodgers. Before being booked out, unoccupied rooms means lose of money to the hotel. To take another example in car-rental industry, only after cars are rented out, can the company possibly earn money; otherwise, cars are staying at garage. Similarly, the empty seats on a flight become valueless once the flight takes off. Managers in these industries are faced with the question on how to effectively allocate perishable products at suitable prices. Yield management is the tool to handle this problem.

The techniques used in yield management are mainly from operations research. Etschmaier and Rothstein (1974) presented an introduction to the use of operations research (OR) in the international airline industry and demonstrated the scope and significance of airline OR activities.

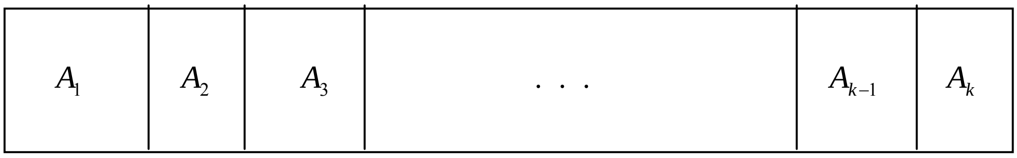
After 1978's deregulation in American, new airlines emerged to share the total profit. Intense competition ensued among them. Since customers were allowed to cancel their previous booked tickets, airlines started intentionally booking more tickets before take-off to counter against cancellations (overbooking).

At first, almost all-quantitative research in yield management is focused on controlling overbooking level, the amount of extra seats (more than flight fixed capacity) that can be intentionally booked out before departure. Airlines tend to overbook to counter against such factors as cancellations and no-shows. However, overbooking has its own downside in that airlines run the risk of not having enough seats for all ticket holders. When such a situation happens, airlines are forced to deny some boarding to the extra ticket holders and pay a penalty in the form of financial compensations to bumped passengers. The long-term effect is that airlines may lose of customer goodwill. On the other hand, if the overbooking level were set too low, there would be some un-occupied seats left on the day of departure. The tough decision remains how to set this level, later referred as overbooking level, so as to maximize total system revenue.

Later after acknowledging different economic conditions of two major groups of travelers, say leisure and business, airlines began to notice the importance of effectively controlling their perishable inventory—seats on a plane. That is, how many empty seats should be reserved for late-booking business travelers along the reservation process? One way to deal with this problem is to offer discounted tickets, which must be booked far earlier than other fare class tickets. By this policy the accept/deny decision becomes when is the best time to close the booking of discounted tickets.

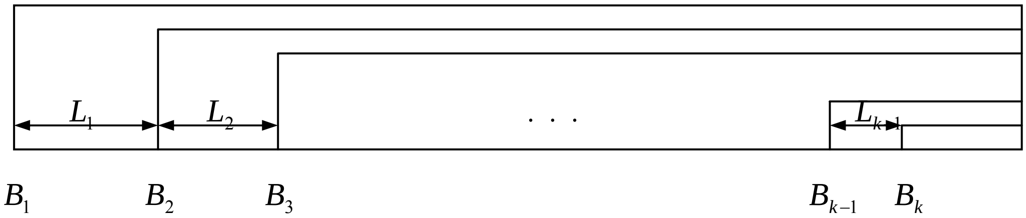
Generally speaking, there are two approaches towards this target. Non-nested allocation approach and nested booking limit approach. The latter—nested booking limit approach, in which a certain number of seats are "protected" from being sold to the booking classes of lower values at a certain time, is widely used nowadays. The figure (Lee and Hersh (1993)) below presents how the two approaches work. In the figure, A_i is defined as the allocation for booking class i, L_i is the number of seats that should be protected from being sold to the booking classes of lower values and B_i is the booking limit for class i.

(a) Non-nested Seat Allocation Approach



Booking Capacity= $A_1 + A_2 + A_3 + ... + A_{k-1} + A_k$

(b) Nested Booking Limit Approach



$$\text{Booking Capacity}= B_1 = L_1 + B_2$$
$$B_2 = L_2 + B_3$$
$$\cdots$$
$$B_{k-1} = L_{k-1} + B_k$$
$$B_k = L_k$$

Figure 1.1 Two Approaches to Seat Inventory Control

Another method solving accept/deny decision is based on EMSR (expected marginal seat revenue), where in the case of single-leg flight with two nested fare classes, a low-fare class demand is accepted as long as the revenue from this demand is greater than the expected marginal seat revenue for high-fare class got by reserving one more seat for this class. Usually this approach is used in conjunction with dynamic programming. There are a lot of papers on this subject. The mathematical models presented in these papers are constructed on different assumptions about arrival process, cancellations, no-shows behavior, demand distribution and segment control system.

Intentionally changing ticket price for identical seats on a flight is another tool used by airlines to differentiate products. Some researchers like Curry (1993b) suggested

Pricing $\approx$ Inventory Control and Inventory $\approx$ Pricing. Though historically pricing in revenue management is considered distinct form inventory control, the revenue management of the future must consider both pricing and inventory control simultaneously because of the inter-relationship between them.

These days another strategy named as "adjustable-curtain" strategy is practiced by some airlines. This practice means airline adjusts the size of the business-class section of the aircraft shortly before boarding takes place. This strategy enables the carrier to deny boarding of economy-class passengers in the event of a high show-up of business-class passengers by enlarging the business-class section at the expense of the economy-class section. Ringbom and Shy (2002) developed a method of computing a reservation policy that the above-mentioned strategy would be utilized prior to boarding. Other possible future research direction of revenue management is package bid price control (Weber, 2001), which consists of flight tickets, beds in hotels and hired car together.

Some difficulties in passenger yield management are concerned with: 1) multi-leg problem (networking); 2) group tickets booking. The amount of both analytical and mathematical efforts caused by the above difficulties is immense because it is not simply the question of rejecting lower fare class to cater for late-arriving business demand. Conceivably sometimes high fare class with only one leg itinerary would be rejected to reserve the seat for possible low fare class with two or more legs. Furthermore, any effective approach towards yield management must be operated in dynamic environment. So far only a handful of papers have delved into these areas.

Now yield management is finding its own ever wider applications in other industries that used to be off limits, manufacturing industry, tele-communication industry and energy industry to name a few. Metters and Vargas (1999) extended yield management concepts to the nonprofit sector, where profit maximization is no longer a goal. Secomandi, Abbott, Atan and Boyd (2002) presented the opportunity analysis study (OAS), designed by PROS research and design department, to determine the applicability of revenue management in new business situations. Bertsimas and Shioda (2003) developed optimizations models to maximize revenue in a restaurant. It is becoming increasingly difficult to categorically exclude any sector of the economy from its marketing magic. Yield management is a new way of doing business for many industries, the essence of success for others, and is increasingly at the heart of the fastest growing new e-business ventures.

1.3 Overview of cargo yield management

The above talked about the main features, difficulties, method used in passenger yield management and the perspectives of passenger yield management. Next attention will be turned to another major division of airline yield management — cargo yield management.

Concurrently, along with the popularity of passenger travel, manufacturers also turn to air transportation to ship their products to the perspective customers due to its speed, safety and reliability. And many companies are implementing just in time (*JIT*) strategy to decrease inventory-holding cost, which requires fast mobility of raw material and finished products from one place to another place. Air transportation is the best choice. Furthermore, airlines, equipped with advanced technology and under intense competition, are trying their best to reduce operational cost within airlines so as to compete with other means of transportation. As a result, the price charged for cargo is not as high as before.

Generally speaking, cargo will board the same flight as passengers, occupying the aircraft's cargo space. However, if the amount of cargo a customer wants to carry is big enough to take up all cargo space on a craft, the customer can turn to another type of cargo shipping service: freighter fleet. That is, airplanes are used exclusively for cargo. The need for freighter capacity will increase as pressure continues for improved air cargo service levels not easily satisfied with lower-hold capacity. Large freighters show the greatest proportionate increase.

Although the revenue from cargo transport accounts for only a fraction of total revenue of airline, more attention is being paid to air cargo in recent years in view of its steady growth. According to World Air Cargo Forecast (1995), air cargo growth is expected to average 6.6% per year until 2014 referring to the graph below.

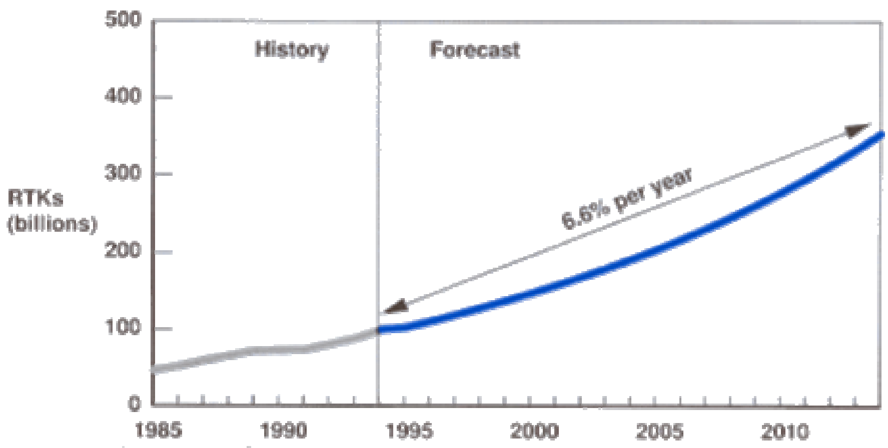


Figure 1.2 World Air Cargo Traffic Forecasting

Along with the development of yield management in passenger sector, the techniques of yield management are now being introduced into air cargo service, drawing on the passenger yield management experiences. Although both passenger and cargo yield management have very similar characteristics in terms of operation, problem description and ways of solving it, there are great differences between them. An obvious distinction is that air cargo travels only one way on a route. The majority of air passengers who fly out on a route will also return. Another problem is the cargo mix. Air cargo is extremely heterogeneous such as size, density, type, shape and fragility, which would greatly affect operational procedure, price level charged and handling process. All of these will affect the approach towards cargo yield management. Passengers are homogeneous in the sense that they each occupy a seat on a plane.

Cargo yield management process usually consists of the following four steps according to Kasilingam (1996). The first step is to forecast the space capacity available for sale, in terms of weight, volume, and position. Passenger ticket booking information is required at this stage. The second step is to allocate space for long-term contracts. The next step is to overbook the remaining capacity to compensate for the cargo booking behavior in terms of cancellations, no-shows and variable tendering. The final step is to allocate the overbooked capacity to different markets so as to maximize the whole cargo revenue.

Few articles can be found to deal with cargo overbooking problem. This paper is armed to bridge this gap and do some research in cargo overbooking problem.

Air cargo revenue management is an excellent research area with a high potential for new models and procedures to accurately represent the cargo business, and to provide the required decision support. Overbooking and allocation problems provide the most opportunity in terms of future research. There is a tremendous potential for modeling these problems under varying assumptions.

2. Literature Review

Yield management can be applied to industries with perishable inventories. The most successful application is in the airline industry. The management system for perishable inventories must be effective so that the revenue generated can be maximized.

In the forty years since the first publication on overbooking control, airline reservation systems have evolved from low-level inventory control processes to major strategic information systems. At the very beginning of the development of revenue management, almost all research work focused on controlled overbooking level. Since the overbooking calculations depended on the prediction of customers behavior, mainly on the probability distributions of the number of passengers who would finally show up for boarding on the day of departure, it has stimulated other useful research on forecasting of passenger cancellations, no-shows and go-shows.

Later, airlines began to offer discounted fare so as to stimulate leisure travelers to book earlier. The problem that airlines faced is how to set lower fare limit to ensure maximum overall revenue. This is the main theme behind a lot of papers. Along with the rapid development of computer and information technology over the last twenty years, it is technically possible for airlines to handle large amount of data within a relatively short time period. This advancement further boasted revenue management from single leg control, through segment control, to origin-destination control.

Kraft, Oum and Tretheway (1986) described the basic concept of airline seat management. Kimes (1989 a & b) discussed basic characteristics of yield management, classified different types of solution approaches and presented the managerial implications of yield management. Harris and Peacock (1995) summarized ten-step to yield management success. Brooks and Button (1994) examined the development of yield management in 1980s and 1990s and assessed its longer term viability for transport industries. A comprehensive taxonomy used in yield management can be found in Weatherford and Bodily (1992) and a more detailed account of the origins of yield management can be found in Belobaba (1987a & b), Smith, Leimkuhler and Darrow (1992) and Tang (1995). Mcgill and Ryzin (1999) provided a comprehensive and up-to-date overview of this area with a bibliography of over 190 references.

Weatherford and Bodily (1992) gave a detailed summary of taxonomy used in yield management. Belobaba and Wilson (1997) studied impacts of yield management in competitive environment. Pak and Piersma (2002) gave an overview of the solution methods presented throughout the literature. Barnhart, Belobaba and Odoni (2003) presented an overview of several important areas of operations research applications in the air transport industry. For each of these areas, the paper provided a historical perspective on OR contribution.

In airline industry, yield management first came from passenger ticket booking and later found its applications in cargo yield management. In the following sections, past research in passenger yield management will be reviewed in four key areas — forecasting, overbooking, inventory control and pricing. Emphasis is focused on overbooking, which is relevant to the thesis. Subsequently, the application of yield management in air cargo sector will be highlighted.

2.1 Passenger yield management

2.1.1 Forecasting

Forecasting is a particularly critical component in airline revenue management because of the direct impact on booking limits. Beckmann (1958 a & b) used Gamma distributions to model the components of show-ups and developed an approximate optimality condition for the overbooking level. Balobaba (1989) dealt with uncertain demand by assuming normal distribution for total demand for a flight. Weatherford, Bodily and Pfeifer (1993) constructed a mathematical model by using non-homogeneous Poisson process (NHPP) to model the timing of potential customer arrivals for each class.

Carpenter and Hanssens (1994) measured the inference of pricing strategy in airline. Proussaloglou and Koppelman (1995) developed a conceptual framework for analyzing carrier demand in a competitive. Curry (1994 a & b) pointed out that revenue management would more reply on the prediction of consumer behavior. Weatherford and Pfeifer (1994) analyzed the economic value of advance booking. Belobaba and Farkas (1999) incorporated yield management booking limits into the methodology used to estimate the number of spilled passengers. Zaki (2000b) analyzed the importance of forecasts in airline industry.

2.1.2 Overbooking

Overbooking has the longest research history of any of the components of the revenue management problem. The objective of most of the early technical research on airline overbooking was to control the probability of denied boarding within limits set by airline management. Overbooking generates a large portion of the revenue management benefits. One simple model is to assume the number of no-shows is a deterministic fraction of the number of bookings.

Beckmann (1958a) proposed a non-dynamic optimization model for overbooking by assuming that the probability distributions of cancellations and no-shows were known. Thompson (1961) analyzed the risk associated with static overbooking problem and studied revenue losses behind overbooking practice. Falkson (1968) expressed his viewpoints on airline overbooking problem: advantages and disadvantages. Rothstein (1971) classified the motivations of implementing overbooking in airline industry, which was not stated clearly in Falkson (1969). Vickrey (1972) discussed some problems related to overbooking. Rothstein (1975) attempted to set the problem of overbooking in proper perspective. Liberman and Yechiali (1978) studied an overbooking problem in hotel industry with stochastic cancellations, in which only single-day stay and one type of room were studied. Ruppenthal and Toh (1983) analyzed effects of airline deregulation on the booking policy (overbooking) to overcome no-shows.

Almost all modern models are probabilistic models. Rothstein (1985) gave a survey of the application of operations research to airline overbooking. This article analyzed the motivation of overbooking, discussed the relevant practices of the air carriers, and described significant contributions and implementations of operations research. Lau and Lau (1988) considered an extension of the classical newsboy problem where a stochastic price-demand relationship existed for the product. Alstrup (1986) presented an overbooking model for a single-leg flight with two types of passengers. Bodily and Pfeifer (1992) discussed the effect of overbooking decision rules under different assumption of passenger show-up behavior. Curry (1993a) considered common ideas behind overbooking models and discussed the effect of different factors to overbooking calculation. Chatwin (1998) considered a multi-period airline-overbooking problem that related to a single-leg flight and a single service class. In another paper of Chatwin (1999), he analyzed a model of airline overbooking in which customer cancellations and no-shows were explicitly considered. Coughlan (1999) used multi-dimensional search routines to find optimal overbooking level.

2.1.3 Seat inventory control

The problem of seat inventory control across multiple fare classes have been studied by many researchers since 1972. There has been significant progress from Littlewood's rule for two fare classes, to EMSR (expected marginal seat revenue) control, to optimal booking limits for single-leg flight, to segment control and, more recently, to origin-destination control. The simplest approach to controlling seat inventories is to deal with each flight leg independently, rather than trying to solve the whole network.

2.1.3.1 Single-leg seat inventory control

Alstrup, Boas, Madsen and Victor (1985) presented an overbooking model for a fixed nonstop flight with two types of passengers, taking cancellations, reservations prior to departure and no-shows into consideration. Pfeifer (1989) derived a decision rule as a function of the percentage difference in two fares and two defined probabilities.

Brumelle, Mcgill, Oum, Sawaki and Tretheway (1990) modeled that demands for classes were stochastically dependent. Wollmer (1992) constructed a model to two-fare-class problem. Bodily and Weatherford (1995) studies multi-price problem with overbooking and diversions. Zhao and Zheng (2001) illustrated a two-class dynamic seat allocation model with passenger diversion and no-shows. Belobaba (1987a) extended Littlewood's rule to multiple fare classes and introduced the term EMSR method for the general approach. Brumelle and Mcgill (1993) addressed the problem of determining optimal booking policies for multiple fare classes that shared the same seating pool on one leg.

Lautenbacher (1999) constructed a discrete-time, finite-horizon MDP model to solve the single-leg problem without cancellations, overbooking, or discounting. You (1999) formulated optimal decision policies for single-leg and two-leg with no multiple seat bookings.

Bitran and Mondschein (1995) studied optimal strategies for renting hotel rooms when there was a stochastic and dynamic arrival of customers with multiple day stays. Papastavrou, Rajagopalan and Kleywegy (1996) presented a model to determine the optimal policy for loading the knapsack within a fixed time horizon so as to maximize the expected accumulated reward. Van Slyke and Young (2000) studied a stochastic knapsack problem and found some application to yield management. Li and Oum (2002) provided the equivalence of the optimality conditions for three models that dealt with the seat allocation problem for a single-leg, multi-fare flight with independent fare class demands.

Dynamic programming treatments of the single leg problem were presented in Virtamo (1991), Lee and Hersh (1993), Robinson (1995) and Hamzaee, Vasigh (1997). Kleywegt and Papastavrou (1998 & 2001), Subramanian, Stidham Jr. and Lautenbacher (1999), Lauthenbacher and Stidham Jr. (1999) and Liang (1999). Other literatures dealing with single-leg and multiple fare classes can be found in Feng and Xiao (2000), Kuyumcu and Garcia-diaz (2000) and Gosavi, Bandla and Das (2002).

2.1.3.2 Segment and origin-destination control

The inventory control problem becomes even more complicated with the development of hub-and spoke route network by most large airlines. Ever since the 1980s, network effects in revenue management have become significant because the number of passengers taking more than one flight leg increased dramatically.

As early as 1977, Ladany and Bedi developed a decision model to determine an operating policy for allocation of seats to passengers flying full and partial spans. Dror, Trudeau and Ladany (1988) proposed a deterministic network minimum cost flow formulation that allowed for cancellations on arcs in the network. Other works on single fare network problem can be found in Curry (1990), Phillips (1994), Boyd and Grossman (1991) and Wong, Koppelman and Daskin (1993).

Soumis and Nagurney (1993) developed a stochastic, multi-class network equilibrium model of airline passenger transportation with an application to the national Air Canada airline.

In recent years, the most successful approaches to solving the airline yield management problem are bid pricing. Talluri and Ryzin (1999) proposed a randomized version of the deterministic linear programming (DLP) method for computing network bid price. Victoria Chen, Dirk and Johnson (1999 a & b) addressed bid-price policy in detail on how to determine bid prices for single-leg, two-leg and star network problems at different reading time period.

Other papers dealing with biding pricing can be found in Guenther (1998), Boyd (2000), Feng and Xiao (2001), Bertsimas and Popescu (2002), Bertsimas and Popescu (2000), Chen, Gunther and Johnson (2003), Brumelle and Walczak (2003) and Bertsimas and de Boer (2004).

2.1.4 Pricing

The existence of different prices for airline seats is the starting point for revenue management. Curry (1993b) emphasized that revenue management of the future must consider both pricing and inventory control simultaneously. Feng and Gallego (1995) addressed the problem of deciding the optimal timing of a single price change from a given initial price to either a given lower or higher second price. Treatments of single leg revenue management as a dynamic pricing problem can be found in Ladany and Arbel (1991), Gallego and Ryzin (1994, 1997), Feng and Gallego (1995, 2000), You (1999), Zhao and Zheng (2000) and Feng and Xiao (2000a).

Weatherford (1997) addressed optimal pricing problem of up to three-fare classes with diversion. Other related papers can be found in Li (2001), Jung and Weber (2001), Weatherford (2001), Chatwin (2000) and Chun (2003).

2.2 Cargo yield management

In view of the history of the development of revenue management, most references were devoted to passenger yield management. Research in air passenger problem has been far more advanced than that in air cargo sector. The reason behind this is partly because the profit portion of cargo in an airline is comparatively smaller than that of passenger sector, partly because passengers have higher priority than cargo and partly because the characteristics of cargo revenue management are more difficult to handle. Strictly speaking research in air cargo yield management is still in its infancy. Few works can be collected in air cargo yield management. More efforts need to be devoted in this area.

Hendricks and Kasilingam (1993) showed that three dimensional cargo capacity was difficult to forecast and affected by many factors: fuel weight, passenger weight, mail weight, belly volume, stacking loss, mail volume to name a few. In order to do a good forecasting, all factors must be considered together. A simple forecast model was presented in the paper. Kasilingam (1996) pointed out uncertain capacity added to the complexity of the cargo yield management in comparing with passenger yield management. Any approaches to cargo yield management problem depended on the precision of forecasting. Rao, Ivanov and Smith (1999) presented a forecasting model, which was shown to have gained an accuracy improvement in the range of 25%-60% over the exponential smoothing based model.

Hendricks and Kasilingam (1993) provided a brief discussion on the idea behind cargo-overbooking model that optimal solution was to maximize expected net revenue subjected to desired service constraints. Kasilingam (1997) presented two formulations of the air cargo-overbooking problem under discrete and continuous probability distribution for capacity respectively, and formulations were applicable for computing overbooking levels for any reading day. Irrgang (1999) presented a general idea on how to optimize fuel, cargo, and passenger payload on long haul flights.

Kasilingam (1996) analyzed characteristics of cargo revenue management. Cargo revenue management differed from passenger revenue management in several respects due to the specific characteristics of cargo inventory, cargo business and cargo booking behavior. A cargo-overbooking model was proposed by the paper. According to this model, the probability distributions of capacity and final show-up rate were assumed known and the overage cost and spoilage cost were assumed known. The expected underage cost and overage cost could be determined by comparing show-up with flight capacity. The expected total cost was the summation of the two. The optimal overbooking level could be determined by differentiating the expected total cost with respect to overbooking level and setting it equal to zero.

Karaesmen (2001) considered a simplified air cargo problem with one arrival requesting one item only with weight and volume two attributes. It showed that a linear programming based approach proposed for airline seat inventory control can be used for air cargo as well.

Mariana (2004) presented ten challenges in air cargo revenue management as networking, routing, clients, allotments, services, capacity, dimension, lumpiness, booking and IT business. Narayanan (2004) discussed the challenges in cargo systems with respect to forecasting and optimization and introduced archetypal science technique that would address challenges. Nielsen (2004) presented the advantages and disadvantages of applying Sebra software at Vergin Atlantic cargo and suggested some changes to the software. Couzy (2004) introduced the current cargo revenue management situation at KLM cargo. Froehlich (2004) summarized several key factors to the success of revenue management at Lufthansa cargo.

2.3 Conclusion

The review has presented some interesting characteristics of revenue management. It is an excellent research area with a high potential for new models and procedures to represent airline business, and to provide the required decision support. It is clear that revenue management will continues to generate applications and research questions for years to come.

3. Cargo Booking Process

3.1 Introduction

A major portion of air cargo travels through the hands of intermediaries known as freight forwarders. Please refer to figure 3.1 below. The essence of a forwarder's function is to consolidate many small shipments into one large shipment and then to offer the large shipment as one entity to the airline. Each freight forwarder may have its own customers and is able to book air space from different airlines. This is just like what is happening in air ticket-booking process. Passengers make reservations from travel agents because they can get discounts from airlines. Furthermore, travel agents can help to find the cheapest air-fare among different airlines.

Freight forwarders can be classified as big and small ones according to the amount of space they reserve from airlines. Big forwarders are preferred because they are able to book greater amount of space with higher load consistency than small ones do. Usually the price charged by forwarder for door-to-door service is lower than the rate that shipper would pay if they dealt directly with the airline. A forwarder with a steady volume of business is in a good position to negotiate a favored rate. Nowadays airlines and big forwarders are usually bounded by long-term contracts with specified terms and conditions. Signing contracts with forwarders are common practice in the airline industry.

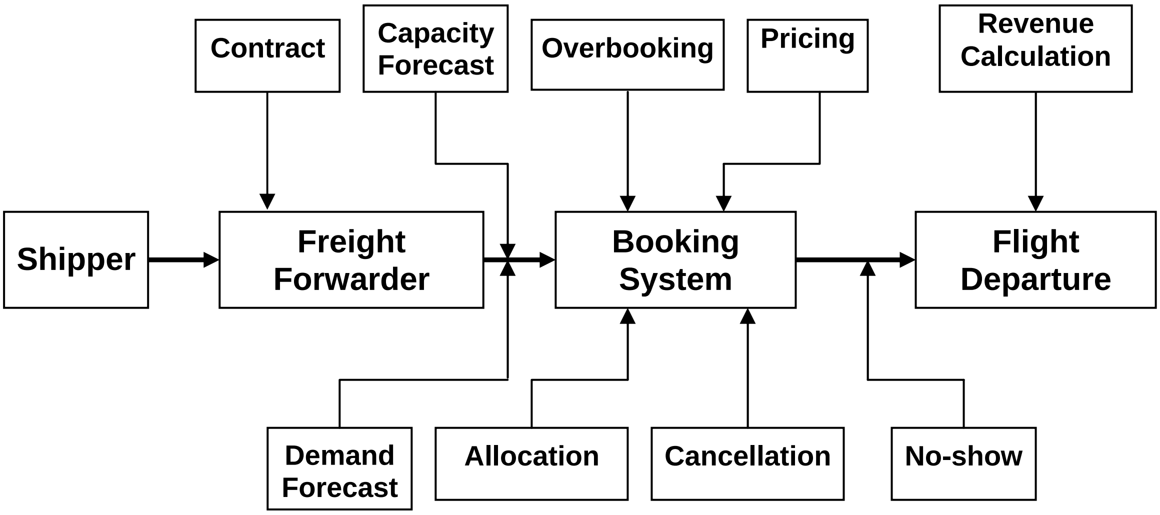


Figure 3.1 Space Booking Process

From the above figure, it can be seen that revenue management will come into place to help airline control booking before the process starts, such as demand forecast, capacity forecast and overbooking if overbooking level is set before process and will not change along the process. Space allocation will play a role along the process.

The cargo booking system opens a few weeks before the flight departs. Once the system opens, forwarders will come to make reservations. For those forwarders with contracts, they have to confirm their utilization. For others, they have to wait for the availability of space. Since the big forwarders are preferred with load consistency, space will usually be protected from selling to small ones. Then the rest is sold to other customers including small forwarders and ad-hoc customers. The purpose of the overbooking model that will be discussed later is to determine the optimal overbooking level for the capacity left for ad-hoc customers.

However, forwarders may cancel their previous partial or whole bookings. If cancellations are made earlier than a certain time before departure, which is set by airline, there will be no penalty for this kind of cancellations since the airline still has sufficient time to look for another customer to fill up the space. Otherwise, forwarders will be penalized. However, the long-term effect of such cancellations from the big forwarders is that they may not be able to sign the same amount of space again in the next year. As a result, big forwarders are cautious on the decision as to how much space to be obtained from contracts.

Since cancellations are allowed in the system, airline will overbook a certain amount of capacity to compensate for the effect of cancellations (overbooking). This extra space beyond flight capacity is defined as overbooking level in this thesis. The risk behind overbooking practice is that more cargo than the flight physical capacity may possibly show up on the day of departure. If such thing does happen, the airline will be penalized for not being able to provide space for reserved cargo. This is expensive and will affect the goodwill of an airline in a long run. The tough decision remains on how much the overbooking limit should be set so as to minimize the expected underage cost and overage cost.

Furthermore, any cargo has both volume and weight properties, which is different from ticket booking. The overall load of a flight is restricted by both volume and weight limits. When the space booking process is concerned, the lower of either volume or weight limit is set as the baseline of space allocation.

3.2 Assumptions

In order to simplify the problem, some assumptions are made. After the problem is simplified, a mathematical model will be formulated to find the optimal overbooking level with the minimum expected underage cost and overage cost. The assumptions made in this problem are listed in this section.

First, the capacity for cargo is supposed to be known. The carrier considered in this paper is a combination carrier in that passengers will board the same flight as the cargo. Passengers and their luggage will compete for the limited capacity of the flight in terms of both volume and weight. The capacity for cargo is to be determined by such factors as the shape of aircraft, the number of customers on board, the luggage weight, the mail weight and fuel weight. However the space booking system is parallel to the passenger ticket booking system. These factors cannot be determined in advance. Capacity forecasting is definitely needed in real time.

Secondly, demands will independently come for bookings. The number of bookings in the system and the total amount of reserved space etc. will not affect a customer's booking behavior. Furthermore, the demand pool is assumed to be big enough that demands would constantly arrive at the booking counter. The arrival process is supposed to be a Poisson process with a constant arrival rate.

Random demand for space is assumed to follow the same discrete uniform distribution. Cargo demand is multiple-unit in nature, which is different from ticket booking. Although each demand may vary from each other, they cannot have a wide range of demand size since the model is for the small forwarders and ad-hoc customers. They may be considered as having very similar characteristics in terms of booking size. So the probability distribution of each customer is assumed to be the same in this model. In the simplest form, the random demand is supposed to have a discrete uniform probability distribution, evenly taking a value from $1, 2, ..., d$.

Partial cancellation is supposed to be nonexistent in this system. Partial cancellation means that a forwarder, who is unable to take all of his previous booked space, wants to cancel part of the reserved space. This exercise is allowed in real practice, but it would pose a great difficulty in mathematic modeling. Partial cancellation, in modeling, will result in that a customer, who made a partial cancellation, will still remain in the system with a demand size different from his original one. In this case booking status is hard to capture. So this practice is not allowed in the model. That is, when a customer comes for cancellation, he will cancel his booking totally and leave the system.

The cancellation process is assumed to follow a Poisson process with a rate dependent on the number of customers in the system. At each period, customers who made reservations may want to cancel their previous bookings. The chance of a cancellation event in one period is directly related to the number of customers in the system. Intuitively the more customers in the system, the higher will be the chance that a customer would come to cancel. So the cancellation rate is assumed to be a function of the number of customers in the system. In the simplest form, assume that cancellation rate is a linear function of the number of customers in the system.

Finally the property of cargo is assumed to be compatible with each other. Some kind of materials cannot board on the same flight with others. For example, food cannot be on the same flight with chemicals. The characteristic of cargo is assumed to be the same in this model. This means all cargo from different customers can at least board the same flight. Furthermore, only single-leg flight will be considered in the model. The network effect will not be discussed.

3.3 Mathematical representation of the booking process

Notations:

X_n	State variable one, which is defined as the number of customers in the system at the end of period n;
Y_n	State variable two, which is defined as the amount of space reserved in the system at the end of period n;
C	Flight capacity for air cargo;
$Ca^{(X_n,Y_n)}$	Cancellation variable in period n with state variables (X_n, Y_n);
D	Individual demand, which is assumed to have discrete uniform distribution, taking on a value from $\{1, 2, ..., d\}$;
n and N	Decision period with n denoting any period along the process and N denoting the time of departure;
OL	Overbooking level, which is defined as the extra space beyond capacity that can be reserved in booking process;
λ	Constant arrival rate;
$\tau(X_n)$	Cancellation rate, which is assumed to be a function of X_n;

For a single-leg flight with a given departure date, the booking period starts from the opening to the departure. This whole booking period is divided into a set of stages, called decision periods, indexed by $1, 2, ..., N$ with the start point of period 1 (represented as 0) corresponding to the opening of the booking process and the end of period N corresponding to the time of departure. In each decision period, assume one and only one of the three events will occur: an arrival event that one customer comes for booking, a cancellation event that one of the customers in the system comes to cancel his previous booking and non-event that nothing happens.

The relation between booking period and the three events of arrival event, cancellation event and non-event can be shown in figure 3.2 below. Seen from Fig.

3.2, there is an arrival event in period 1, a cancellation event in period 3 and non-event in period 4. The numbers at the bottom indicate the decision periods.

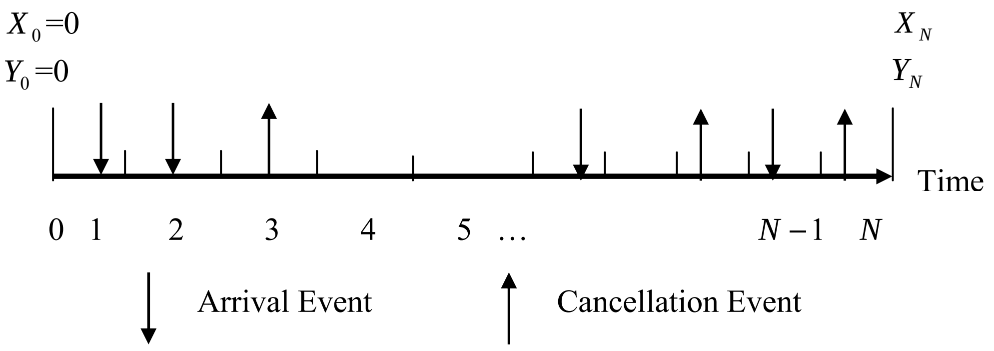


Figure 3.2 Decision Periods and Three Events

Since cancellations make it hard for the airline to predict the final boarding cargo on the flight, the airline usually adopts the strategy of overbooking. This means more space than the capacity will be booked before departure. High overbooking level will possibly result in bumped cargo on the day of departure, while low overbooking level will possibly result in the empty space after take-off. Both cases would result in loss of revenue to airline. The determination of the overbooking level is crucial to the airline. This thesis will mainly focus on the overbooking problem in air cargo revenue management.

In passenger yield management, one passenger is usually assumed to book only one ticket, so one state variable is enough to describe booking status. Two state variables are needed to describe the cargo-booking process. One is the number of customers in the system in period n denoted as X_n and another is the total booked space up to period n denoted as Y_n. At the beginning of the booking process, the two state variables are all equal to zero because no booking is accepted. That is, $X_0 = 0$ and

$Y_0 = 0$. And the values of X_N and Y_N represent the final number of customers and the final amount of cargo that come for boarding on the day of departure.

Along the booking process, suppose customers will independently come for reservations. And based on the assumption that the demand pool is big enough, arrivals follow a homogeneous Poisson process with a constant rate denoted as λ. Suppose λ equals to 0.3 bookings per day in the model.

If a random demand D comes for booking, an accept/deny decision is made according to the booking limit, which is defined as the summation of capacity C and overbooking level OL. Capacity C is supposed to be 20 units in the model. OL is defined as the extra space beyond flight capacity that can be booked before departure. This demand D will be accepted as long as there is enough empty space, and the state variables will transfer to $X_{n+1} = X_n + 1$ and $Y_{n+1} = Y_n + D$. Otherwise it will be rejected, and the state variables will transfer to $X_{n+1} = X_n$ and $Y_{n+1} = Y_n$. In the simplest form, assume the random variable D follows a discrete uniform probability distribution, evenly taking on a value from $\{1, 2, ..., d\}$. That is, D will take each value with probability $\frac{1}{d}$. In this paper, assume d equals to 10 units.

Meanwhile, customers who have made bookings before may come for cancellations. Unlike the arrival process, the cancellation events are directly related to bookings in the system:

1. Only those customers already in the system can cancel.
2. The cancellation size can only come from the bookings in the system.

Intuitively, the chance of a cancellation event in the next period would be higher when there are more customers in the system. Hence, it is appropriate to assume that the cancellation process follows a non-homogeneous Poisson process with a cancellation rate $\tau(X_n)$ dependent on the number of customers in the system at time n defined as X_n. Assume that cancellation rate is a 0.02 times of X_n, or $\tau(X_n) = \frac{X_n}{50}$. If a cancellation event happens in one period, one of the bookings will leave the system completely as no-partial cancellation has been assumed in the model. This transition corresponds to $X_{n+1} = X_n - 1$ and $Y_{n+1} = Y_n \text{-} Ca^{(X_n,Y_n)}$, where $Ca^{(X_n,Y_n)}$ is defined as the cancellation size, whose probability distribution will be discussed in the next chapter.

Transitions from period to period in connection with booking limit can be shown in figure 3.3 below. In Fig. 3.3, numbers below the time axis represent the decision period.

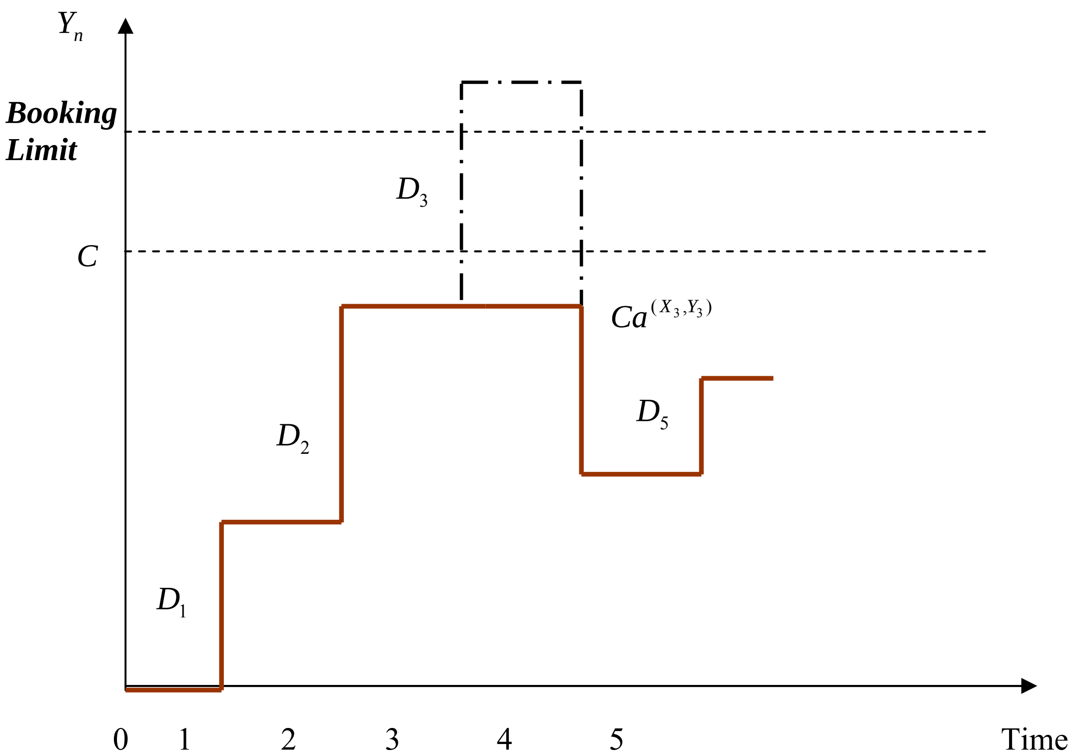


Figure 3.3 Transitions and Booking Limit

Seen from Fig. 3.3, the first two demands are accepted while the third one is rejected because accepting it will result in exceeding the booking level. In other words, the value of Y_n cannot exceed the booking limit.

Finally the booking process can be shown as a birth and death process in terms of the number of customers in the system X_n as presented in Fig. 3.4, where the numbers in the nodes indicate the value X_n. The node with a number 1 inside represents that only one booking is in the system. The three arrows starting from this node represent the three possible events in the next period. The arrow from node 1 to node 2 indicates an arrival event and is accepted. The arrow from node 1 to node 0 indicates a cancellation event. Also the arrow from node 1 to node 1 indicates a non-event. It is just like a birth-and-death process whereby the number of customers in the booking system will increase by one, or decrease by one, or remain the same in next period. The *NC* in the last node is the maximum number of customers that can possibly stay in the system. The value of NC is determined by the booking limit and the minimum demand size. In this case the minimum demand size is assumed to be one, so there may have at most $C+OL$ customers in the system.

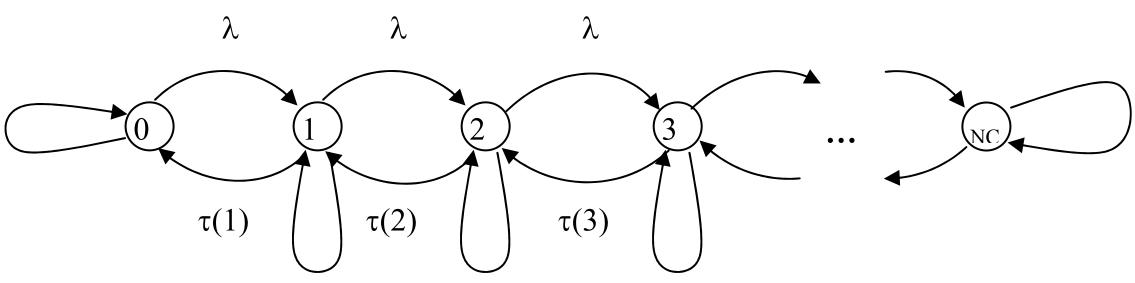


Figure 3.4 Booking Process in Terms of the Number of Customers in the System

Furthermore, the arrival events come with a constant rate λ and the cancellation events come with a rate $\tau(X_n)$, dependent on the current number of customers as shown in the Fig. 3.4.

The cargo booking process has been described. The booking status will change from period to period given different event happening. The booking process will be modeled having Markovian property in the next chapter and the limiting probability will be determined to calculate the expected cost.

4. Markovian Property and Transition Matrix

Booking process in this model will be formulated as having Markovian property. Any future behavior of the process can be fully determined by the current state. By using this property, we are able to see the long-run behavior of the process. How to formulate the space booking process as a Markov process will be discussed in this section first and the determination of the three transition probabilities will be addressed later. In the following, the derivation of cancellation probability distribution will be discussed and the structure of the transition matrix will be presented last.

4.1 Markovian property

When booking opens a few weeks before departure, the process will evolve from one period to the next, starting from no booking till the flight taking off. In each stage, not only the number of bookings (one customer has one booking) is recorded, the total booked space is also needed to describe the state of the system. As a result, two state variables (X_n, Y_n) are defined in this problem. The space of X_n and Y_n are $[0, NC]$ and $[0, C+OL]$ respectively as discussed before. Accordingly, the whole state space of (X_n, Y_n) is $(NC+1)\times(C+OL+1)$. The booking process is a two-dimensional process. Next it will be shown how to formulate it as a Markov process next.

Within each period, one and only one of three events may happen: an arrival event, a cancellation event and non-event. If an arrival comes for booking, the demand will be accepted as long as it does not exceed the remaining capacity (booking limit minus reservations on hand). In other words, the demand within the range $[0, C+OL-Y_n]$ will be accepted; otherwise, it will be rejected. If a cancellation event happens, one of the bookings in the system will leave the system completely and release a certain amount of space for future booking. Furthermore, in order to formulate the process having Markovian property, the amount of cancellation from one specific booking cannot be obtained. As a result, the probability distribution of the cancellation size must be determined by the current state of the system, which will be discussed in section 4.3. In this setting, the cancellation size is independent of the past events. Finally if nothing happens, the state will remain the same.

When the above three events are considered together, based only on the current state the system, transitions from one period to another period can be determined. Hence, the system is assumed to possess Markovian property because any particular future behavior of the system is dependent on the current state of the system when its current state is known exactly.

4.2 Transition probabilities

As stated above, state of the system at one period will transit to another state in the next period with a certain probability. The determination of the transition probabilities will be discussed in this section.

Suppose in one period, a demand, which is so big as to exceed booking limit $C+OL$, is rejected, resulting in the state variables being the same as before. In this case, the arrival but rejected event produces the same result as the non-event. The two events will result in the value of X_n and Y_n being unchanged. They are collectively called the no-change transition. Similarly, the arrival and accepted event and the cancellation event are referred to as the arrival and accepted transition and the cancellation transition respectively (refer to Fig. 4.1).

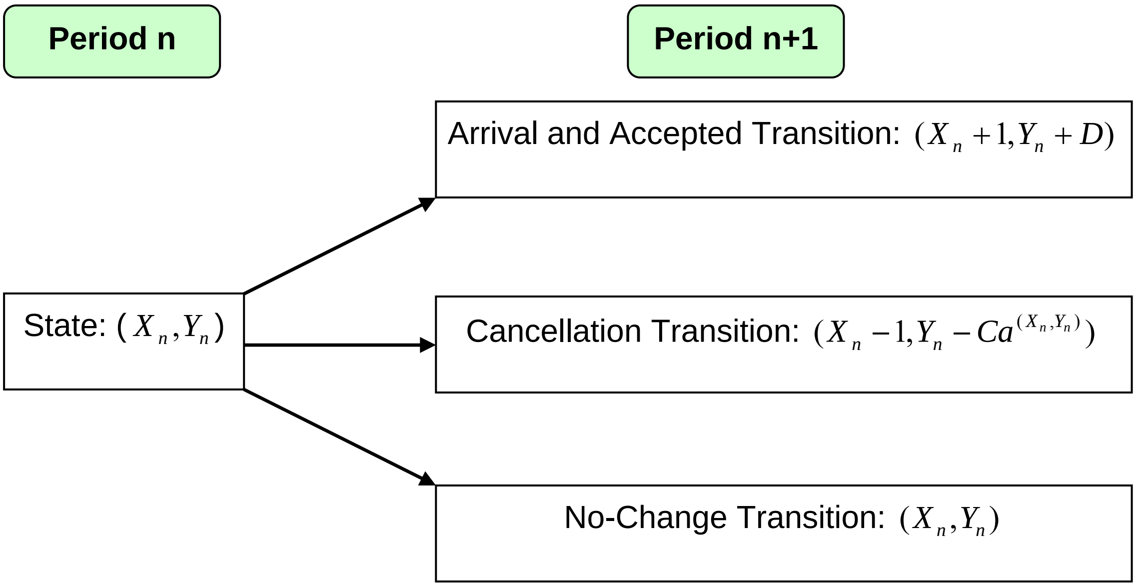


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Figure 4.1 Three Transitions in Booking Process

In period n, an arrival will come with a constant rate λ and a cancellation will come with a rate $\tau(X_n)$, depending on the current state. So the probability that either an arrival or a cancellation event will occur is calculated by $1 - e^{-(\lambda + \tau(X_n))}$ according to the exponential distribution of the waiting time between these two events. Complementarily, the probability of non-event within this period is $e^{-(\lambda + \tau(X_n))}$.

Next given an event (either an arrival or a cancellation) occurs, the probability that this event is an arrival event or a cancellation event is determined by the ratio of the corresponding rate. An arrival event comes with a probability $\frac{\lambda}{\lambda + \tau(X_n)}$, and a cancellation event comes with a probability $\frac{\tau(X_n)}{\lambda + \tau(X_n)}$. Furthermore, an arrival comes with a demand taking a value from $\{1, 2, ..., d\}$ with a probability $\frac{1}{d}$. If the demand is less than $C + OL - Y_n$ (the remaining capacity), it will be accepted; otherwise, it will be rejected. In the case of a cancellation event, the probability distribution was determined by the method discussed later, given known state (X_n, Y_n).

As a whole, the corresponding probabilities of three transitions within one period are summarized below:

- No-change transition:

$$P\{(X_{n+1}, Y_{n+1}) = (x, y) \mid (X_n, Y_n) = (x, y)\} = e^{-(\lambda+\tau(x))} + [1 - e^{-(\lambda+\tau(x))}] \cdot \frac{\lambda}{\lambda + \tau(x)} \cdot P\{D = i\}$$

$$i = \begin{cases} C + OL - y + 1, ..., d - 1, d & d > C + OL - y \\ 0 & otherwise \end{cases} \tag{4.1}$$

- Arrival and accepted transition:

$$P\{(X_{n+1}, Y_{n+1}) = (x+1, y+i) \mid (X_n, Y_n) = (x, y)\} = [1 - e^{-(\lambda+\tau(x))}] \cdot \frac{\lambda}{\lambda + \tau(x)} \cdot P\{D = i\}$$

$$i = 1, 2, ..., \min[d, C + OL - y] \tag{4.2}$$

- Cancellation transition:

$$P\{(X_{n+1}, Y_{n+1}) = (x-1, y-i) \mid (X_n, Y_n) = (x, y)\} = [1 - e^{-(\lambda+\tau(x))}] \cdot \frac{\tau(x)}{\lambda + \tau(x)} \cdot P\{Ca^{(X_n, Y_n)} = i\}$$

$$i = Ca_{\min}^{(X_n, Y_n)}, Ca_{\min}^{(X_n, Y_n)} + 1, ..., Ca_{\max}^{(X_n, Y_n)} \tag{4.3}$$

It can be shown that the summation of the above three kinds of transition probabilities within one period of time equals to one.

Where $P\{D=0\}=0$; $Ca_{\min}^{(X_n,Y_n)}$ and $Ca_{\max}^{(X_n,Y_n)}$ are the minimum and maximum value of $Ca^{(X_n,Y_n)}$ respectively given known state (X_n,Y_n). The determination of $Ca_{\min}^{(X_n,Y_n)}$ and $Ca_{\max}^{(X_n,Y_n)}$ will be discussed later.

Three transition probabilities have been discussed in this section. Each random demand is assumed to have the same discrete uniform distribution, and, based on the state of the system, the cancellation probability distribution can be determined as discussed in the next section. After that, a transition matrix will be constructed.

4.3 Derivation of cancellation probability distribution

As stated before, cancellation event is directly related to bookings in the system. If no one is in, it is impossible to have a cancellation event in the next period. Theoretically cancellation size can only be one of the existent bookings. In this section, the probability distribution of cancellation $Ca^{(X_n,Y_n)}$ will be shown to be derived only from current booking status so as to model the process as a Markov process.

4.3.1 Determination of cancellation size

Suppose the current state is (X_n,Y_n). If the next event is a cancellation event, it must be one of the X_n bookings in the system. However, according to the Markovian property, the conditional probability of any future event is independent of the past events and depends only upon the present state. As a result, no information about each individual booking is available. In other words, $Ca^{(X_n,Y_n)}$ must be derived from the current state (X_n,Y_n).

First the minimum and the maximum values of $Ca^{(X_n,Y_n)}$ should be determined, which are equal to the possible minimum and maximum demands already in the system. Define $Ca_{\min}^{(X_n,Y_n)}$ and $Ca_{\max}^{(X_n,Y_n)}$ as the minimum and the maximum values of the cancellation size respectively given that the current state is (X_n, Y_n). They can be determined by the following equations.

$$Ca_{\min}^{(X_n,Y_n)} = \max\{1, Y_n - (X_n - 1) \cdot d\} \tag{4.4a}$$

$$Ca_{\max}^{(X_n,Y_n)} = \min\{d, Y_n - (X_n - 1)\} \tag{4.4b}$$

Accordingly, the cancellation size $Ca^{(X_n,Y_n)}$ can take an integer from space $\Omega^{(X_n,Y_n)} = [Ca_{\min}^{(X_n,Y_n)}, Ca_{\max}^{(X_n,Y_n)}]$. The different combination of X_n and Y_n will determine the different values of $Ca_{\min}^{(X_n,Y_n)}$ and $Ca_{\max}^{(X_n,Y_n)}$, and in turn determine different scope of $\Omega^{(X_n,Y_n)}$. The superscription is used to denote these differences. Although $Ca^{(X_n,Y_n)}$ can take an integer from $Ca_{\min}^{(X_n,Y_n)}$ to $Ca_{\max}^{(X_n,Y_n)}$, the chance of taking each value is different, which will be discussed next.

4.3.2 Probability distribution of cancellation size

The method to obtain the probability distribution of $Ca^{(X_n,Y_n)}$, given state (X_n, Y_n), will be discussed in this section. After this distribution is obtained, the complete stationary transition matrix from period to period can be constructed in the next subsection.

Define $y_n^{\ 1}, y_n^{\ 2}, \ldots y_n^{\ X_n}$ as the X_n bookings in the system in period n. Because demand is independent, X_n bookings are exchangeable to each other. Without loss of generality, any $y_n{}^j$, $1 \le j \le X_n$ is considered as the jth booking in the system. Obviously, for any booking $y_n{}^j$, $1 \le j \le X_n$, we have $y_n{}^j \in \Omega^{(X_n,Y_n)}$, $1 \le j \le X_n$.

The next step is to determine the probability distribution of $Ca^{(X_n,Y_n)}$. The mathematical expression of this probability is as follows:

$$P\{Ca^{(X_n,Y_n)} = k^{(X_n,Y_n)} \mid (X_n, Y_n)\} \qquad k^{(X_n,Y_n)} \in \Omega^{(X_n,Y_n)} \tag{4.5}$$

where $k^{(X_n,Y_n)}$ represents a specific value from $\Omega^{(X_n,Y_n)}$. The relation between X_n and Y_n is as following.

$$\sum_{j=1}^{X_n} y_n{}^j = Y_n$$

The expression (4.5) can be re-written as the following:

$$P\{Ca^{(X_n,Y_n)} = k^{(X_n,Y_n)} \mid \sum_{j=1}^{X_n} y_n{}^j = Y_n\} \qquad k^{(X_n,Y_n)} \in \Omega^{(X_n,Y_n)} \text{ and } 1 \le j \le X_n$$

Furthermore, these demands are exchangeable to each other because the independent arrival of each demand is assumed in the modeling. Without loss of generality, assume $k^{(X_n,Y_n)}$ is the $X_n th$ demand in the system. So the above expression becomes

$$P\{Ca^{(X_n,Y_n)} = k^{(X_n,Y_n)} \mid \sum_{j=1}^{X_n} y_n{}^j = Y_n\}$$

$$= \frac{P\{Y_n{}^{X_n} = k^{(X_n,Y_n)}, \sum_{j=1}^{X_n} y_n{}^j = Y_n\}}{P\{\sum_{j=1}^{X_n} y_n{}^j = Y_n\}} \qquad k^{(X_n,Y_n)} \in \Omega^{(X_n,Y_n)} \tag{4.6}$$

$$= \frac{P\{Y_n{}^{X_n} = k^{(X_n,Y_n)}\} P\{\sum_{j=1}^{X_n - 1} y_n{}^j = Y_n - k^{(X_n,Y_n)}\}}{P\{\sum_{j=1}^{X_n} y_n{}^j = Y_n\}}$$

Seen from the above derivation, to calculate the probability distribution of $Ca^{(X_n,Y_n)}$, the probability of $P\{\sum_{j=1}^{X_n} y_n{}^j = Y_n\}$ and $P\{\sum_{j=1}^{X_n-1} y_n{}^j = Y_n - k^{(X_n,Y_n)}\}$ needs to be computed first. One way to do it is to recursively condition on the last item in the summation, referring to the formula below.

$$
\begin{aligned}
&P\{\sum_{j=1}^{X_n} y_n{}^j = Y_n\} \\
&= \sum_{\Omega^{(X_n,Y_n)}} P\{y_n{}^{X_n} = k^{(X_n,Y_n)}\} P\{\sum_{j=1}^{X_n-1} y_n{}^j = Y_n - k^{(X_n,Y_n)}\} \\
&= \sum_{\Omega^{(X_n,Y_n)}} P\{y_n{}^{X_n} = k^{(X_n,Y_n)}\} \\
&\quad [\sum_{\Omega^{(X_n-1,Y_n-k^{(X_n,Y_n)})}} P\{y_n{}^{X_n-1} = k^{(X_n-1,Y_n-k^{(X_n,Y_n)})}\} P\{\sum_{j=1}^{X_n-2} y_n{}^j = Y_n - k^{(X_n,Y_n)} - k^{(X_n-1,Y_n-k^{(X_n,Y_n)})}\}] \\
&\ldots
\end{aligned}
\tag{4.7}
$$

where $k^{(X_n,Y_n)} \in \Omega^{(X_n,Y_n)}$ and $k^{(X_n-1,Y_n-k^{(X_n,Y_n)})} \in \Omega^{(X_n-1,Y_n-k^{(X_n,Y_n)})}$.

Seen from the above derivation, it is able to calculate the probability of $P\{\sum_{j=1}^{X_n} y_n{}^j = Y_n\}$ and $P\{\sum_{j=1}^{X_n-1} y_n{}^j = Y_n - k^{(X_n,Y_n)}\}$ given known X_n and Y_n.

<u>Numerical example 1</u>

Suppose there is only one customer in the system with a demand i, $i \in \{1, 2, ..., d\}$. That is $X_n = 1$ and $Y_n = i$. The minimum cancellation size $Ca_{\min}^{(X_n,Y_n)}$ and the maximum cancellation size $Ca_{\max}^{(X_n,Y_n)}$ are obtained by

$$Ca_{\min}^{(1,i)} = \max\{1, i-(1-1)\cdot d\} = \max\{1, i\} = i$$

$$Ca_{\max}^{(1,i)} = \min\{d, i-(1-1)\} = \min\{d, i\} = i$$

So the probability distribution of the cancellation size is computed as follows:

$$P\{Ca^{(X_n,Y_n)} = k^{(X_n,Y_n)} \mid \sum_{j=1}^{X_n} y_n{}^j = Y_n\} = P\{Ca^{(1,i)} = i \mid \sum_{j=1}^{1} y_n{}^j = i\}$$
$$= \frac{P\{y_n{}^1 = i\}}{P\{\sum_{j=1}^{1} y_n{}^j = i\}} = 1 \tag{4.8}$$

Numerical example 2

Suppose the state of the booking system is $X_n = 3$, $Y_n = 5$ and $d = 10$. The minimum cancellation size $Ca_{\min}^{(X_n,Y_n)}$ and the maximum cancellation size $Ca_{\max}^{(X_n,Y_n)}$ are obtained by

$$Ca_{\min}^{(3,5)} = \max\{1, 5-(3-1)\cdot 10\} = \max\{1, -15\} = 1$$

$$Ca_{\max}^{(3,5)} = \min\{10, 5-(3-1)\} = \min\{10, 3\} = 3$$

So the probability distribution of the cancellation size is computed as follows:

$$P\{\sum_{j=1}^{2} y_n{}^j = 2\} = P\{y_n^2 = 1\}P\{y_n^1 = 1\} = \frac{1}{100}$$

$$P\{\sum_{j=1}^{2} y_n{}^j = 3\} = P\{y_n^2 = 1\}P\{y_n^1 = 2\} + P\{y_n^2 = 2\}P\{y_n^1 = 1\} = \frac{2}{100}$$

$$P\{\sum_{j=1}^{2} y_n{}^j = 4\} = P\{y_n^2 = 1\}P\{y_n^1 = 3\} + P\{y_n^2 = 2\}P\{y_n^1 = 2\} + P\{y_n^2 = 3\}P\{y_n^1 = 1\} = \frac{3}{100}$$

$$P\{\sum_{j=1}^{3} y_n{}^j = 5\} = P\{y_n^3 = 1\}P\{\sum_{j=1}^{2} y_n{}^j = 4\} + P\{y_n^3 = 2\}P\{\sum_{j=1}^{2} y_n{}^j = 3\} + P\{y_n^3 = 3\}P\{\sum_{j=1}^{2} y_n{}^j = 2\}$$
$$= \frac{1}{10}\frac{3}{100} + \frac{1}{10}\frac{2}{100} + \frac{1}{10}\frac{1}{100} = \frac{6}{1000}$$

We have

$$P\{Ca^{(3,5)}=1\mid\sum_{j=1}^{3}y_n{}^j=5\}=\frac{P\{y_n{}^3=1\}P\{\sum_{j=1}^{2}y_n{}^j=4\}}{P\{\sum_{j=1}^{3}y_n{}^j=5\}}=\frac{\frac{1}{10}\frac{3}{100}}{\frac{6}{1000}}=\frac{1}{2}$$

$$P\{Ca^{(3,5)}=2\mid\sum_{j=1}^{3}y_n{}^j=5\}=\frac{P\{y_n{}^3=2\}P\{\sum_{j=1}^{2}y_n{}^j=3\}}{P\{\sum_{j=1}^{3}y_n{}^j=5\}}=\frac{\frac{1}{10}\frac{2}{100}}{\frac{6}{1000}}=\frac{1}{3}$$

$$P\{Ca^{(3,5)}=3\mid\sum_{j=1}^{3}y_n{}^j=5\}=\frac{P\{y_n{}^3=3\}P\{\sum_{j=1}^{2}y_n{}^j=2\}}{P\{\sum_{j=1}^{3}y_n{}^j=5\}}=\frac{\frac{1}{10}\frac{1}{100}}{\frac{6}{1000}}=\frac{1}{6}$$

Based on the calculation procedure discussed above, the cancellation probability distribution in every period can be recursively computed, given the current state of the system (X_n, Y_n).

4.4 Formulation of transition matrix

The booking process is formulated as a two dimensional Markov process in this model. Given the state of the system, there are stationary transition probabilities in each period. These transition probabilities will transform the system from one state to another state. By summarizing all possible transition probabilities together in a matrix format, a transition matrix can be constructed. This is the main topic of this section.

4.4.1 Dimension of transition matrix

The dimension of the transition matrix is directly related to the value of each parameter assumed in the model and it will increase greatly if the value of parameter increases. If $X_n = 0$, Y_n can only be zero. So Y_n can only take one value given $X_n = 0$. Similarly, if $X_n = 1$, Y_n can take values from $\{1, 2, ..., d\}$, a total of d possible values. Define $Y_n(X_n = x)$ as the collection of all possible values of Y_n given that $X_n = x$. That is, $Y_n(X_n = x) = \{Y_n \mid X_n = x\}$. Define $N_{X_n = x}$ the size of $Y_n(X_n = x)$. The derivation of the value of $Y_n(X_n = x)$ and $N_{X_n = x}$ for a given X_n is illustrated below.

$X_n = x$	$Y_n(X_n = x)$	$N_{X_n = x}$
X_n=0	$\{Y_n \mid X_n = 0\} = \{0\}$	$N_{X_n=0}=1$
X_n=1	$\{Y_n \mid X_n = 1\} = \{1, 2, ..., d\}$	$N_{X_n=1}=d$
X_n=2	$\{Y_n \mid X_n = 2\} = \{2, 3, ..., 2 \cdot d\}$	$N_{X_n=2} = (d-1) \cdot 2 + 1$
X_n=3	$\{Y_n \mid X_n = 3\} = \{3, 4,..., \min\{C + OL, 3 \cdot d\}\}$	$N_{X_n=3} = \min\{C + OL - 2, (d-1)3 + 1\}$
…	…	…
$X_n = NC$	$\{Y_n \mid X_n = NC\} = \{C + OL\}$	$N_{X_n=NC} = 1$

The last line in the above derivation is obtained by the fact that the minimum demand size is 1 unit. So the maximum possible number of customer in the system is $NC = C + OL$.

In the general form, when $X_n = i$, $Y_n(X_n = i) = \{i, i+1, ..., \min\{C + OL, i \cdot d\}\}$ and

$$N_{X_n=i} = \begin{cases} i \cdot (d-1) + 1 & i \times d \le C + OL \\ Max\{0, C + OL - i + 1\} & otherwise \end{cases} \tag{4.9}$$

The value of Y_n must be bounded at $C + OL$, the booking limit, because the accept/reject decision is that any demand beyond the booking limit will be rejected.

The total number of states of the booking system is the summation of all $N_{X_n=i}$, $0 \le i \le NC$. Suppose N_{matrix} equal to $\sum_{i=0}^{NC} N_{X_n=i}$. So the dimension of the transition matrix is $N_{matrix} \times N_{matrix} = \sum_{i=0}^{NC} N_{X_n=i} \times \sum_{i=0}^{NC} N_{X_n=i}$. It could be very big if the value of OL and d are set high. Suppose $d = 10$ and $C + OL = 22$, then the dimension of the transition matrix becomes 240×240.

4.4.2 Break down of transition matrix

As stated above, the dimension of transition matrix is big even for a moderate problem. The complete matrix cannot be easily represented in this thesis due to the magnitude it involves. Instead, the whole matrix is broken down into a set of sub-matrices to ease the task of representation so that readers can have a global view of the format of the matrix. Three kinds of sub-matrices, each corresponding to one transition within one period, are defined as follows.

$A_{j,j}$ The sub-matrix representing the no-change transition, given $X_n = j$. The first j in the subscription indicates the value of X_n and the second j indicates the value of X_{n+1}.

$A_{j,j+1}$ The sub-matrix representing the accept transition, given $X_n = j$. j and $j+1$ in the subscription indicate the value of X_n and X_{n+1} respectively.

$A_{j,j-1}$ The sub-matrix representing the cancellation transition, given

$X_n = j$. j and $j-1$ in the subscription indicate the value of X_n and X_{n+1} respectively.

$\{(j, Y_n(X_n = j))\}$ The collection of all states, given that j customers are in the system. Because the space of $Y_n(X_n = j)$ depends on j, $\{(j, Y_n(X_n = j))\}$ represents the collection of all possible states given j customers in the system.

Suppose the current state (X_n, Y_n) comes from $\{(j, Y_n(X_n = j))\}$. When a demand is accepted, the state will transit to a state coming from $\{(j+1, Y_n(X_n = j+1))\}$. Similarly, when a cancellation and a non-event happens, the state will transit to a state coming from $\{(j-1, Y_n(X_n = j-1))\}$ and from $\{(j, Y_n(X_n = j))\}$ respectively. Transition probabilities can be determined by the method discussed previously. In the transition matrix, three sub-matrices, $A_{j,j}$, $A_{j,j+1}$, and $A_{j,j-1}$, are used to represent these transitions.

So the stationary transition matrix of booking process can be illustrated as follows in terms of sub-matrices.

$$P = \begin{array}{c} \\ \{0.Y_n(X_n = 0)\} \\ \{1, Y_n(X_n = 1)\} \\ \{2, Y_n(X_n = 2)\} \\ \cdots \\ \{NC, Y_n(X_n = NC)\} \end{array} \begin{array}{c} \begin{array}{ccccc} \{0, Y_n(X_n = 0)\} & \{1, Y_n(X_n = 1)\} & \{2, Y_n(X_n = 2)\} & \cdots & \{NC, Y_n(X_n = NC)\} \end{array} \\ \begin{bmatrix} A_{0,0} & A_{0,1} & 0 & \cdots & 0 \\ A_{1,0} & A_{1,1} & A_{1,2} & \cdots & 0 \\ 0 & A_{2,1} & A_{2,2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A_{NC,NC} \end{bmatrix} \end{array}$$

note: 0 in the above matrix represents zero matrix of corresponding dimension.

The dimension of each sub-matrix $A_{j,k}$, $k = j-1, j$ and $j+1$, is determined by the space of $Y_n(X_n = j)$ and $Y_n(X_n = k)$. That is, $N_{X_n=j}$ and $N_{X_n=k}$ to be specific. So the dimension of sub-matrix $A_{j,k}$ is $N_{X_n=j} \times N_{X_n=k}$. Similarly, the dimension of the whole transition matrix P is given by $N_{matrix} \times N_{matrix} = \sum_{i=0}^{NC} N_{X_n=i} \times \sum_{i=0}^{NC} N_{X_n=i}$.

Seen from the first row of the matrix, which represents transitions when no booking is in the system, only the no-change transition and the accept transition are possible because no booking is available to cancel. Similarly, when there are *NC* bookings in the system as shown in the last row of the matrix, there are only the no-change transition and the cancellation transition possible. In this case, demand will not be accepted any longer since booking level reaches booking limit already.

4.4.3 The input of probabilities in sub-matrices

The layout of the complete transition matrix has been discussed previously in terms of the sub-matrices. The detailed input of the sub-matrices $A_{j,j}$, $A_{j,j+1}$ and $A_{j,j-1}$ will be discussed one by one in this section.

The transition probability is defined as follows, which will be used in the sub-matrices.

- $p_{j,j}(y,y) = P\{(X_{n+1}, Y_{n+1}) = (j,y) \mid (X_n, Y_n) = (j,y)\} \quad , y \in Y_n(X_n = j);$
- $p_{j,j+1}(y_1,y_2) = P\{(X_{n+1}, Y_{n+1}) = (j+1,y_2) \mid (X_n, Y_n) = (j,y_1)\}$, $y_1 \in Y_n(X_n = j)$, $y_2 \in Y_n(X_n = j+1)$, and $y_1 < y_2$;
- $p_{j,j-1}(y_1,y_2) = P\{(X_{n+1}, Y_{n+1}) = (j-1,y_2) \mid (X_n, Y_n) = (j,y_1)\}$, $y_1 \in Y_n(X_n = j)$, $y_2 \in Y_n(X_n = j-1)$, and $y_1 > y_2$;

1. No-change transition matrix $A_{j,j}$:

Each probability in the sub-matrices $A_{j,j}$, $j = 0,1,2, ..., NC$ is composed of two parts. One is the probability that non-event will happen and another is the probability that a demand will be rejected. The layout of the matrix is shown below.

$$A_{j,j} = \begin{bmatrix} p_{j,j}(j,j) & 0 & 0 & \cdots & 0 \\ 0 & p_{j,j}(j+1,j+1) & 0 & .. & 0 \\ 0 & 0 & p_{j,j}(j+2,j+2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p_{j,j}(M_j, M_j) \end{bmatrix}$$

* M_j substitutes for $\min\{j \times d, C + OL\}$.

Seen from the above matrix, it is a diagonal matrix. The general expression of the entry $a_{kl}^{j,j}$ ($k = 1, 2, ..., N_{X_n} = j$ and $l = 1, 2, ..., N_{X_n} = j$) in the matrix $A_{j,j}$ can be summarized as below:

$$a_{kl}^{j,j} = \begin{cases} p_{k,k}(j+k-1, j+k-1) = e^{-(\lambda+\tau(j))} \\ \qquad + \dfrac{\lambda(1-e^{-(\lambda+\tau(j))})}{\lambda+\tau(j)} P\{D > C + OL - (j+k-1)\} & \text{for all } k = l \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

Seen from the above expression, $j + k - 1$ represents the value of Y_n. Only the diagonal items are non-zeros for the no-change transition sub-matrices, and the dimension of the matrix $A_{j,j}$ is $N_{X_n = j} \times N_{X_n = j}$.

2. Accept transition matrix $A_{j,j+1}$:

The item in the sub-matrices $A_{j,j+1}$, $j = 0,1, 2, ..., NC - 1$ represents the probability that a demand will be accepted. The layout of the sub-matrices is shown below.

$$A_{j,j+1} =$$
$$\begin{bmatrix} p_{j,j+1}(j,j+1) & p_{j,j+1}(j,j+2) & \ldots & p_{j,j+1}(j,j+d) & \ldots & 0 \\ 0 & p_{j,j+1}(j+1,j+2) & \ldots & p_{j,j+1}(j+1,j+d) & \ldots & 0 \\ 0 & 0 & \ldots & p_{j,j+1}(j+2,j+d) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & .. & \ldots \\ 0 & 0 & \ldots & 0 & \ldots & p_{j,j+1}(M_j, M_{j+1}) \end{bmatrix}$$

* M_j substitutes for $\min\{j \cdot d, C+OL\}$ and M_{j+1} substitutes for $\min\{(j+1)\cdot d, C+OL\}$.

The general expression of the entries $a_{kl}^{j,j+1}$ ($k = 1, 2, ..., N_{X_n=j}$ and $l = 1, 2, ..., N_{X_n=j+1}$) in the matrices $A_{j,j+1}, j = 0,1,2,...NC-1$ can be summarized as below:

$$a_{kl}^{j,j+1} =$$
$$\begin{cases} p_{j,j+1}[(j+k-1),(j+l)] = \dfrac{(1-e^{-(\lambda+\tau(j))})\lambda}{\lambda+\tau(j)} P\{D = l-k+1\} & 0 \le l-k < \min\{d, N_{X_n=j+1}-k\} \\ 0 & otherwise \end{cases} \tag{4.11}$$

Seen from the above expression, $j+k-1$ and $j+l$ represent the value of Y_n and Y_{n+1} respectively. As a result, the difference between the two, $l-k+1$ is the demand size. Similarly the dimension of $A_{j,j+1}$ is given by $N_{X_n=j} \times N_{X_n=j+1}$.

3. Cancellation transition matrix $A_{j,j-1}$:

The item in the sub-matrices $A_{j,j-1}, j = 0, 1, 2, ..., NC$ represents the probability that a cancellation event will happen. The layout of the sub-matrix is shown below.

$A_{j,j-1} =$

$$\begin{bmatrix} p_{j,j-1}(j,j-1) & 0 & 0 & \cdots & 0 \\ p_{j,j-1}(j+1,j-1) & p_{j,j-1}(j+1,j) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{j,j-1}(j+d-1,j-1) & p_{j,j-1}(j+d-1,j) & p_{j,j-1}(j+d-1,j+1) & \cdots & 0 \\ & p_{j,j-1}(j+d,j) & p_{j,j-1}(j+d,j+1) & \cdots & 0 \\ & & p_{j,j-1}(j+d+1,j+1) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p_{j,j-1}(M_j, M_{j-1}) \end{bmatrix}$$

* M_j substitutes for $\min\{j \cdot d, C+OL\}$ and M_{j-1} substitutes for $\min\{(j-1)\cdot d, C+OL\}$.

The general expression of the entries $a_{kl}^{j,j-1}$ ($k = 1, 2, ..., N_{X_n=j}$ and $l = 1, 2, ..., N_{X_n=j-1}$) in the matrices $A_{j,j-1}, j = 0, 1, 2, ..., NC$ can be summarized as below:

$a_{kl}^{j,j-1} =$

$$\begin{cases} p_{j,j+1}[(j+k-1),(j+l-2)] = \dfrac{(1-e^{-(\lambda+\tau(j))})\tau(j)}{\lambda+\tau(j)} P\{Ca^{(X_n,Y_n)} = k-l+1\} & 0 \le k-l < \min\{d,k\} \\ 0 & \textit{otherwise} \end{cases} \tag{4.12}$$

Seen from the above expression, $j+k-1$ and $j+l-2$ represent the value of Y_n and Y_{n+1}. And the difference between the two, $k-l+1$ represents the cancellation size. The dimension of $A_{j,j-1}$ is given by $N_{X_n=j} \times N_{X_n=j-1}$.

The determination of cancellation probability $P\{Ca^{(X_n,Y_n)} = k-l+1\}$ indicated in expression (4.12) has been discussed in the previous section.

Based on what has been discussed so far, the transition matrix can be determined once the value of OL, d, λ, $\tau(X_n)$ and capacity C are given. However the dimension of this matrix is too big to be presented manually. Instead, computer is used to do it.

Mathematica software is used to accomplish this task and each item in the transition matrix can be filled in by this software automatically and efficiently. The summation of each row in the matrix, which by probability theory must be equal to 1, can be used to test the correctness of the matrix.

Once the transition matrix is obtained, efforts will be spared to solve this matrix to obtain the limiting probability distribution. This limiting probability distribution is approximated as the joint distribution of the number of bookings and the amount of reservations on the day of departure. In turn, the expected total cost (underage cost and overage cost) can be determined.

5. Overbooking Model

A stationary transition matrix from period to period is obtained in the previous chapter. By solving this stationary transition matrix, a limiting probability distribution can be obtained. Limiting probability represents the probability that the system will be in a certain state when time goes to infinite. In the problem, limiting probability distribution is approximated as the joint distribution of two state variables on the day of departure. In other words, the distribution of final cargo coming for boarding can be obtained. In turn, the expected overage cost and underage cost can be determined to evaluate the control on the system.

5.1 Limiting probability distribution

After the complete transition matrix is obtained, the limiting probability distribution can be calculated accordingly. Theoretically, the limiting probability distribution in this problem means the joint probability distribution of the final number of customers and the final amount of cargo coming for boarding before flight takes off when the time goes to infinity. However, the space booking process cannot be opened forever in reality. Instead, the limiting probability distribution is used in this case as an approximation of the joint probability distribution of the final number of customers and the final show-up cargo on the day of departure. That is, it is approximated as a joint distribution of X_N and Y_N. If convergence is achieved fast, this approximation is acceptable; otherwise, it is unacceptable. The effect of time on the optimal result will be discussed later.

Define:

$\pi = (\pi_1, \pi_2, ...\pi_{N_{Matrix}})$ Limiting probability distribution of the transition matrix, where $N_{matrix} = \sum_{i=0}^{NC} N_{X_n = i}$;

$f_{X_N Y_N}(x, y)$ Joint probability distribution of X_N and Y_N ;

$f_{X_N}(x)$ Marginal probability mass function of the number of bookings on the day of departure;

$f_{Y_N}(y)$ Marginal probability mass function of the amount of reserved space on the day of departure;

Based on the theorem that the limiting probability distribution $\pi = (\pi_1, \pi_2, ...\pi_{N_{Matrix}})$ is the unique nonnegative solution of the equations

$$\pi_l = \sum_{k=1}^{N_{matrix}} \pi_k a_{kl} \qquad l = 1, 2, ..., N_{matrix} \qquad (5.1)$$

$$\sum_{k=1}^{N_{matrix}} \pi_k = 1 \qquad (5.2)$$

In the above formula, $N_{matrix} = \sum_{i=0}^{NC} N_{X_n = i}$ is the dimension of the transition matrix and a_{kl} is the (k, l) th item in the transition matrix, where $k = 1, 2, ..., N_{matrix}$ and $l = 1, 2, ..., N_{matrix}$. The total number of the equations in (5.1) equals N_{matrix}. To solve the system equations, one equation from (5.1) should be eliminated and equation (5.2) should be added to obtain the values of $\pi = (\pi_1, \pi_2, \pi_3, ...\pi_{N_{matrix}})$.

The dimension of the matrix will be very big even if the value of d and OL is assumed moderate. This matrix cannot be solved manually. Mathematica software is used to help solve the above systems of the linear equations.

As analyzed before, the solution to this system of linear equations is approximated as the joint probability distribution of X_N and Y_N, which is defined as $f_{X_N Y_N}(x,y) = P\{X_N = x, Y_N = y\}$, $0 \le X_N \le NC$ and $0 \le Y_N \le C + OL$. And consequently the marginal probability mass function of X_N and Y_N, which are defined by $f_{X_N}(x) = P\{X_N = x\}$ and $f_{Y_N}(y) = P\{Y_N = y\}$ respectively, can be determined by

$$f_{X_N}(x) = P\{X_N = x\} = \sum_{\phi_x} f_{X_N Y_N}(x,y) \tag{5.3}$$

$$f_{Y_N}(y) = P\{Y_N = y\} = \sum_{\phi_y} f_{X_N Y_N}(x,y) \tag{5.4}$$

where

ϕ_x denotes the set of all points in the range of (X_N, Y_N) for which $X_N = x$ and

ϕ_y denotes the set of all points in the range of (X_N, Y_N) for which $Y_N = y$.

By using the formula (5.4), the approximate probability distribution of the final cargo for boarding on the day of departure can be obtained. Consequently, the amount of over-load and the under-load can be determined after comparing it with the flight capacity. And finally the expected overage cost and the expected underage cost can be calculated.

5.2 Overbooking model

As stated before, higher overbooking level will possibly result in more bumped cargo on the day of departure, while lower overbooking level will leave some empty space on the flight after take-off. In either case, airline loses money. The objective is to find an optimal overbooking level to minimize the expected cost. In this section, it is shown how to formulate the overbooking model to find the optimal overbooking level with the minimum expected cost.

The model is basically a newsboy model. At the moment of take-off, if the cargo coming for boarding is below the capacity C (20 units), the aircraft will take off with empty space, which means a loss of money for airline called as underage cost. Otherwise, if the cargo for boarding exceeds the capacity C, some cargo cannot board the scheduled flight. The bumped cargo due to airline not being able to provide the capacity for the customer as promised will cause the overage cost. The cost used in this model to evaluate the performance of the system is defined as the summation of both underage cost and overage cost. This is different from the actual operational cost in practice. In the modeling, the underage cost and the overage cost are defined as p_U and p_O per unit of space respectively. The objective of this model is to find the optimal overbooking level to minimize the expected underage cost and overage cost.

Define:

p_O overage penalty, which is the cost incurred by airline not being able to provide the capacity for the customer as promised. It is assumed equal to 1.3 dollars per unit bumped space;

p_U underage penalty, which is the unit cost of empty space on an aircraft after flight take-off. It is assumed equal to 1.0 dollars per empty space.

The expected cost equals to the expected underage cost plus the expected overage cost. The formula of computing the expected cost must be obtained first. Then the optimal overbooking level can be determined by minimizing the expected cost. The algorithm will be shown below.

First assuming the value of Y_N is known, there are two cases concerning Y_N and capacity C. If $Y_N \le C$, spoilage occurs and the amount of spoilage is equal to $C - Y_N$; the corresponding underage cost is $p_U \times (C - Y_N)$. If $Y_N > C$, over-sale occurs and the amount of over-sale is equal to $Y_N - C$; the corresponding overage cost is equal to $p_O \times (Y_N - C)$.

Next the assumption that Y_N is deterministic is relaxed; then the expected underage cost (spoilage cost) denoted as $E[UC]$ and the expected overage cost denoted as $E[OC]$ given by equations (5.5) and (5.6) below.

$$E[UC] = p_U \times \sum_{y=0}^{C} (C - y) P(Y_N = y) \tag{5.5}$$

$$E[OC] = p_O \times \sum_{y=C+1}^{C+OL} (y - C) P(Y_N = y) \tag{5.6}$$

The expected cost denoted as $E[CS]$ can be computed as given below.

$$\begin{aligned} E[CS] &= E[UC] + E[OC] \\ &= p_U \sum_{y=0}^{C} (C - y) \cdot P(Y_N = y) + p_O \sum_{y=C+1}^{C+OL} (y - C) \cdot P(Y_N = y) \end{aligned} \tag{5.7}$$

Next the above cost function (5.7) will be shown to be a convex function of overbooking level.

Proof of convexity of the cost function

Seen from the equation (5.7), the first part on the right hand side (underage cost) decreases with the increase of booking limit, and the second part (overage cost), starting from zero, increases with the increase of booking limit. Since the overage penalty must be greater than the underage penalty, the increment from the second part will sooner or later overcome the decrease from the first part. The summation of the two parts is a convex function. Please refer to the figure 5.1 below.

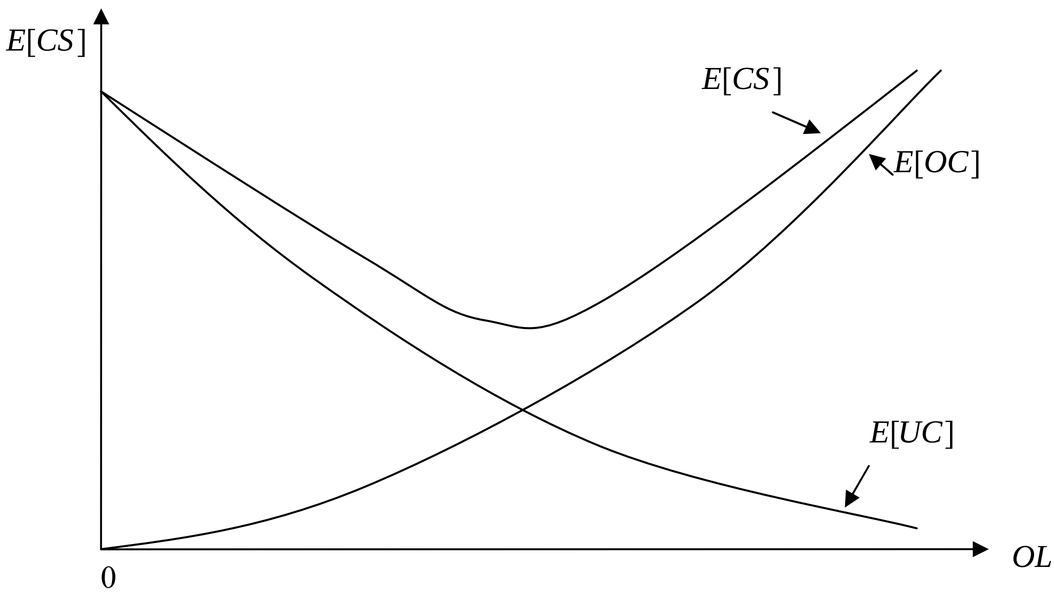

Figure 5.1 The Movement of Cost Function

The objective of the model is to find an optimal overbooking level OL^* such that the expected total cost is minimized. According to the algorithm discussed above the overbooking level OL must be provided first in order to obtain $E[CS]$. A searching method is used to find the optimal solution as follows:

Step 1: Initially set OL_a and OL_b equal to 1 and 2 respectively. So the booking limit is $C+1$ and $C+2$;

Step 2: Compute the corresponding expected total cost $E[CS_a]$ and $E[CS_b]$ respectively;

Step 3: Compare the two expected total cost $E[CS_a]$ and $E[CS_b]$. If $E[CS_a] > E[CS_b]$, set $OL_a = OL_b$, $OL_b = OL_b + 1$ and go to step 2; otherwise, the optimal overbooking level OL^* equals to OL_a.

Based on the algorithm discussed above, Mathematica software is used to calculate the optimal overbooking level. The program is composed of two parts. The first part is to input transition probabilities so as to construct transition matrix and the second part is to search the optimal overbooking level. The flowchart, as well as the Mathematica program, is appended at the end of this thesis. Equipped with this powerful software, the optimal overbooking level can be obtained, and the sensitivity of parameters in the model can also be checked. The time used in the calculation is very short and results will be presented and discussed in the following sections.

Numerical example

Suppose the flight capacity C is known as 20 units. Random demand D follows discrete uniform probability distribution, evenly taking a value from $\{1, 2, ..., d\}$, where d is assumed to be 10 units. Constant arrival rate λ is assumed to be 0.3 booking per day and cancellation rate $\tau(X_n)$ is assumed to be linear function of X_n, the number of bookings in period n. Assume $\tau(X_n) = \frac{X_n}{50}$ in the model. The underage cost p_U is assumed to be 1 and the overage cost p_O is assumed to be 1.3.

After running Mathematica program, the optimal result is as follows.

The optimal overbooking level OL^* is 2 and the corresponding expected total cost is 2.83459. The dimension of transition matrix is 240×240. It is big, but the calculation time is short, about several seconds by Mathematica. According to the program results, the expected load on the flight is 18.4424. And the probability that the load is below capacity is 0.641054, and the probability that the load is above capacity is 0.358964.

Next it will be shown numerically that the expected cost is a convex function of OL. Since OL is non-negative, the expected total cost will be determined given different OL. The results are shown in Fig. 5.2 below.

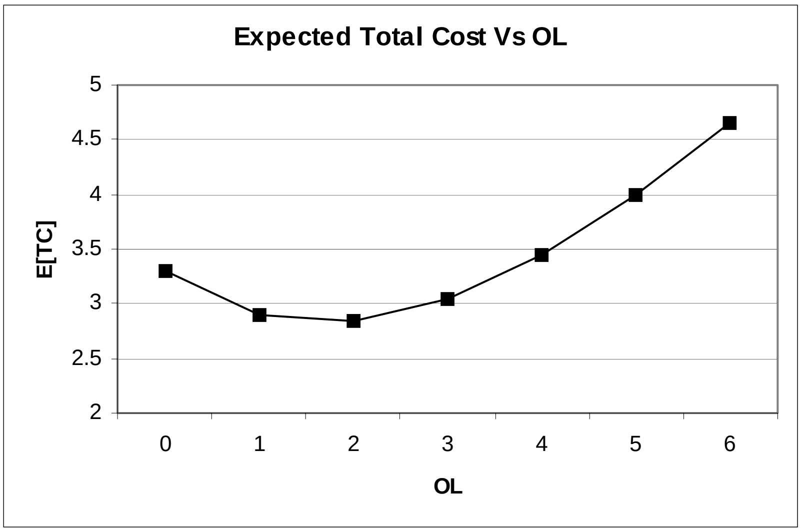

Figure 5.2 Cost Function for Overbooking Model

So far, the overbooking model has been constructed and a numerical example has been given. Recall the constructed overbooking model. Given a known demand (arrival rate, cancellation rate and individual demand size), a known capacity and a known pricing structure, the optimal overbooking level (OL^*) can be obtained. The relation between input variables and output will be analyzed in the next section, which may help to deepen the understanding of the model.

5.3 Sensitivity of input variables

The purpose of this section is to check the relationship between input parameter and output so that we can have a global view of the effect of one parameter on the optimal result. In addition, the obtained relationship between input and output may help check the correctness of the model. The combined effect of two parameters will be discussed in the subsequent section.

5.3.1 The effect of capacity (C)

In the example above, C is assumed to be 20 units. Given all other parameters the same, C is changed from 16 to 24 and the optimal overbooking level denoted as OL^* can be calculated. The results are shown in Fig. 5.3 and 5.4.

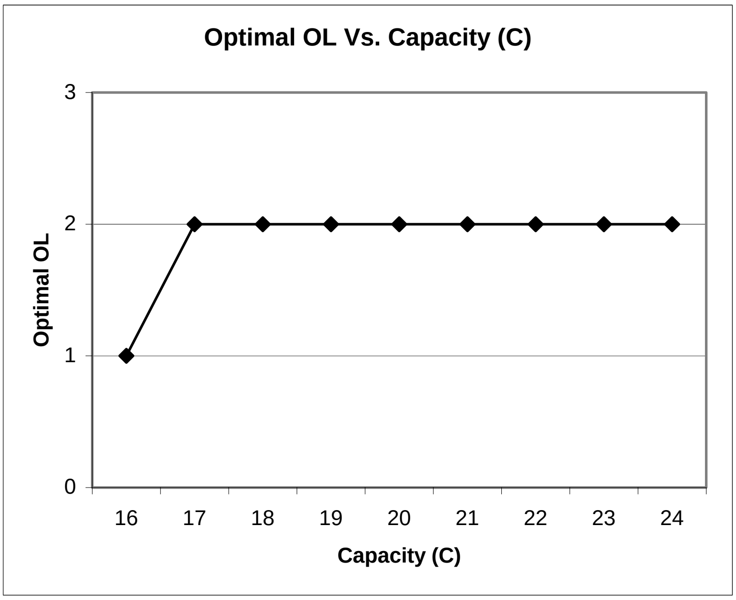

Figure 5.3 Optimal OL Vs. Capacity

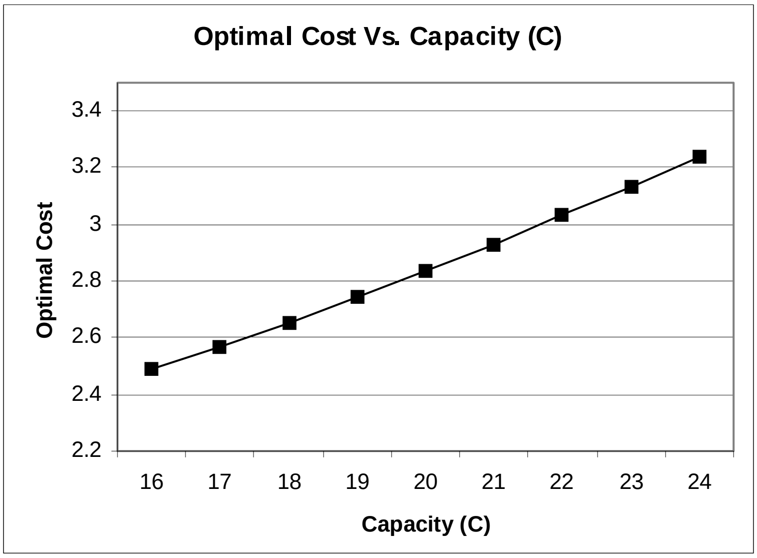

Figure 5.4 Optimal Cost Vs. Capacity

Seen from the above figures, with the increase of flight capacity, the OL^* increases slightly and the expected total cost increases also. It is reasonable in that when more capacity is available, more requests for bookings can be accepted and more revenue is generated. The overbooking level is insensitive to capacity given other parameters at the set values. Please refer to the curve in Fig. 5.2 from capacity 17 to 24. Although flight capacity increases, OL^* remains unchanged.

5.3.2 The effect of demand (D):

Random individual demand D is assumed to be independent and follow the same discrete uniform distribution with the maximum value $d = 10$. Given all other parameters the same, d is changed from 2 to 11 and the optimal overbooking level, as well as corresponding optimal cost, is determined by Mathematica program. The results are shown in figure 5.5 and 5.6 below.

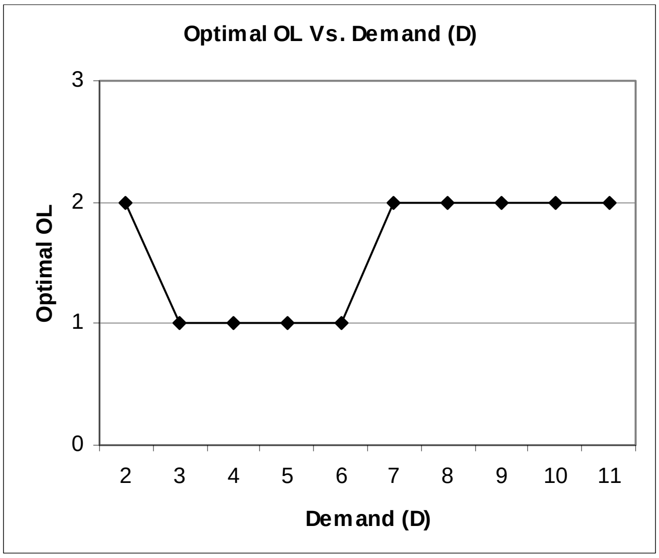

Figure 5.5 Optimal OL Vs. Demand

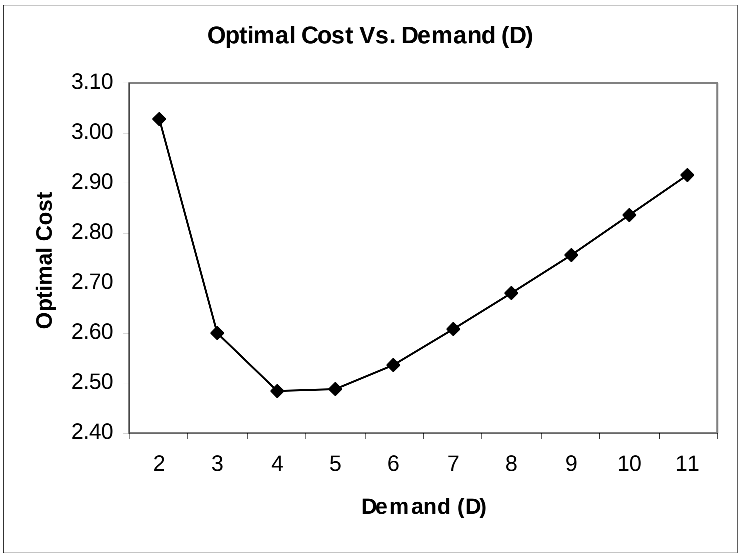

Figure 5.6 Optimal Cost Vs. Demand

Seen from the above two figures, the OL^* first decreases, then increases with the increase of maximum demand size d. The optimal cost first decreases, then increases with the increase of maximum demand size d. This is due to the combined effect of underage and overage penalty (p_U and p_O). At the low level of d (=2), the load is low, so the effect of underage is prominent. Underage cost plays a big role in the total cost, so the overall cost is big as seen at d =2 in Figure 5.6. At this point, overbooking level should be increased so as to absorb more demands. With the increase of d, load will increase accordingly, as a result the expected cost decreases while the overage cost is minor.

While at the high level of d (5 to 11), load tends to increase and more cargo is bumped. Overage cost is prominent in the total cost, while the underage cost is minor, so the total cost increases in this case. Overbooking level is slightly increased in this range of d so as to control the amount of overage. On the other hand, with the increase of d, the variance of demand increases greatly. The overbooking level should be increased to counter the effect of the great demand variance.

5.3.3 The effect of overage penalty (p_O)

In the example, the overage penalty is 1.3 dollar per unit of bumped cargo. Given all other parameters fixed, change overage penalty from 1.1 to 2.0 and determine the OL^* and the corresponding expected cost. Note here p_O must be greater than p_U, thus the minimum value is set at 1.1. The results are shown in figure 5.7 and 5.8 below.

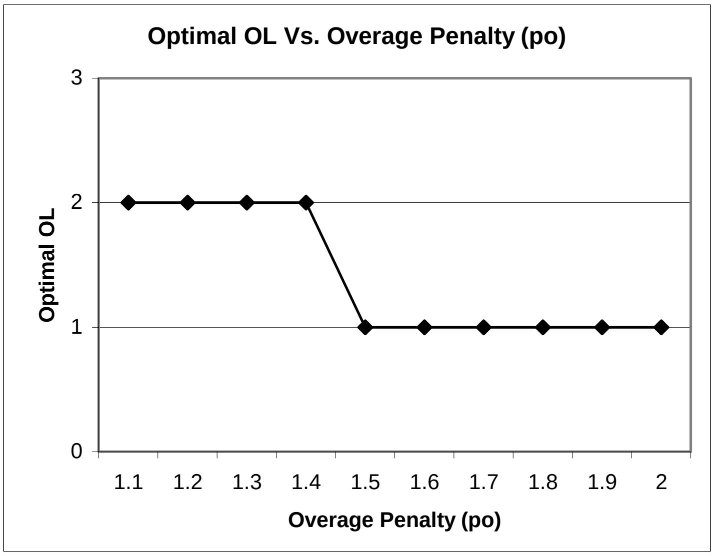

Figure 5.7 Optimal OL Vs. Overage Penalty

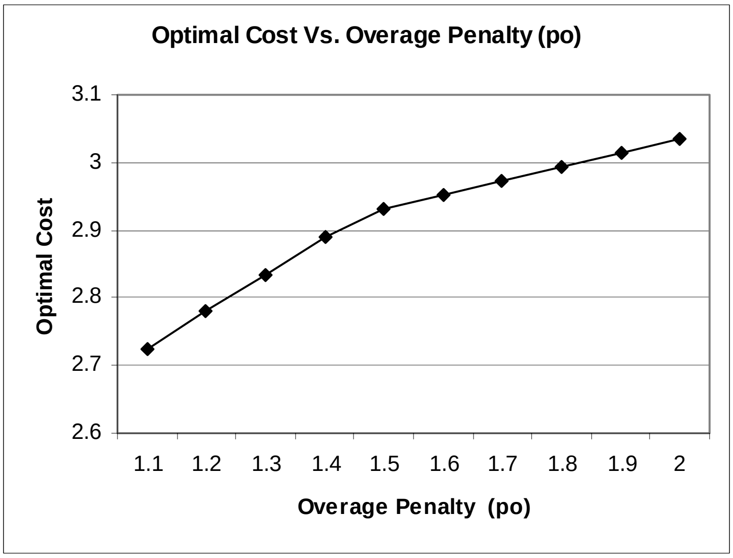

Figure 5.8 Optimal Cost Vs. Overage Penalty

With the increase of p_O ranging from 1.1 to 2.0, it is becoming more expensive to having the overage. As a result, OL^* decreases to decrease the chance of the occurrence of overage. The optimal cost increases with the increase of p_O. Suppose the overage cost is set very big, the overbooking level should be set at zero to avoid any possible overage.

5.3.4 The effect of underage penalty (p_U)

In the example, the underage penalty is 1.0 per unit of space. Underage penalty in this example represents perishable value of air space. Once flight takes off, the value of empty space on the flight is lost. Underage penalty helps airline take into consideration of load factor. The higher load, the lower underage cost is.

Given all other parameters fixed, change p_U from 0.4 to 1.2 and calculate optimal result for each case. The results are shown in figure 5.9 and 5.10 below.

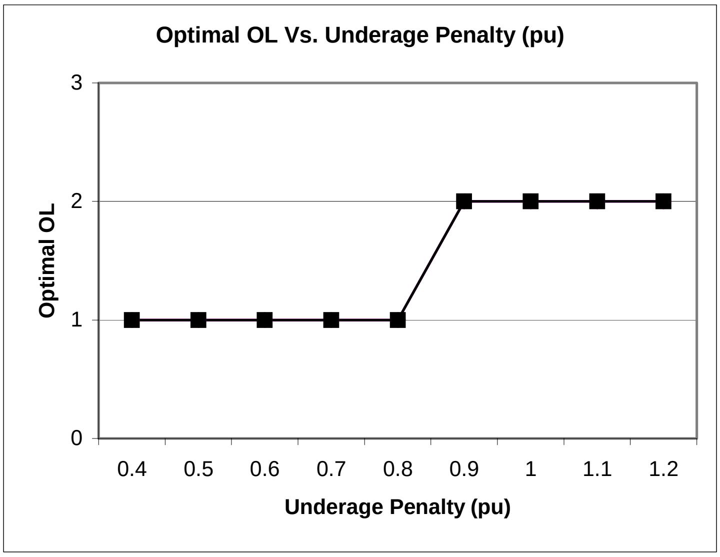

Figure 5.9 Optimal OL Vs. Underage Penalty

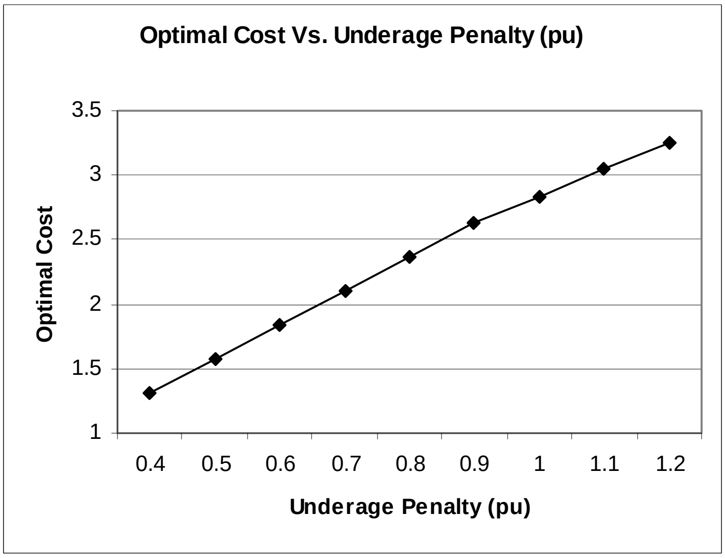

Figure 5.10 Optimal Cost Vs. Underage Penalty

Seen from the above two figures, with the increase of p_U ranging from 0.4 to 1.2, OL^* will slightly increase and optimal cost will increase. Overbooking level is forced to set higher so as to accept more bookings and to increase overall load. But with the increase of overbooking level, overage cost will increase, so the overall expected cost will increases.

5.3.5 The effect of arrival rate (λ)

It is assumed that requests have a constant arrival rate in the model. It reflects the intensity of demand. With the increase of arrival rate λ, more customers come for bookings. Since the cancellation rate is fixed, higher arrival rate means higher chance of arrival event in the next stage. In the model it is assumed to be 0.3 per day. It will be changed from 0.05 to 0.5 to see the effect on optimal result, given all other parameters fixed. The result is shown in Figure 5.11 and 5.12 below.

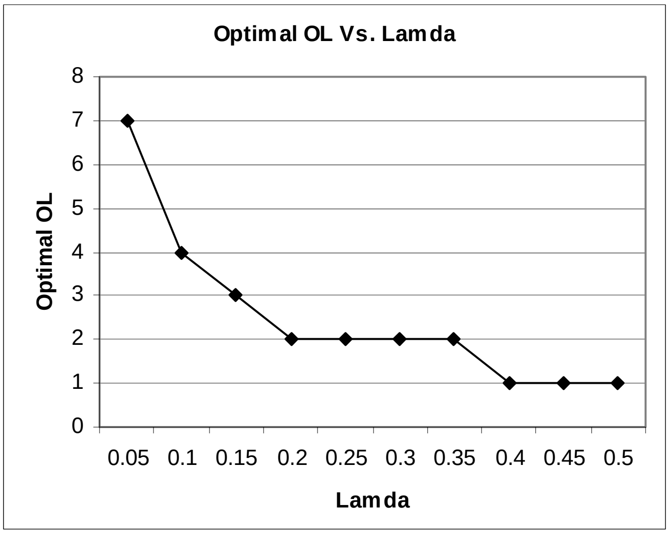

Figure 5.11 Optimal OL Vs. λ

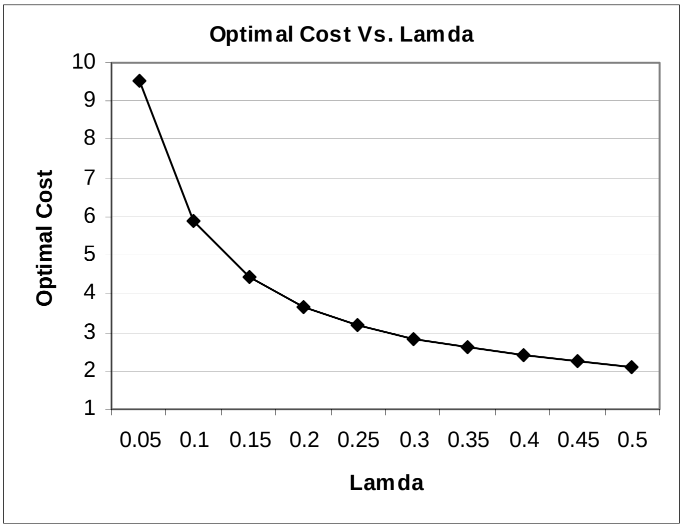

Figure 5.12 Optimal Cost Vs. λ

With the increase of λ, more demands will come for reservations, so higher chance of overage happens. OL^* should be decreased to decrease the chance of the occurrence of overage. Seen from figure 5.12, OL^* decreases greatly with the increase of λ. It is said OL^* is sensitive to λ. And with the decrease of OL^*, overage cost is decreased and more demand coming in will increase loading. As a result, the overall cost decreases with the increase of arrival rate.

In the meantime, a simulation model is constructed to visualize the booking process and to verify the robustness of the formulated overbooking model.

5.4 Combined effect of two input variables

In section 5.3, the effect of individual input variable on the optimal results has been discussed. But we do not know whether there is cross effect of two variables. In this section, the effect of two input variables will be analyzed to have better understanding of the model.

5.4.1 Effect of underage and overage penalty

The underage cost p_U is assumed as 1 dollar per unit of empty space on the flight and the overage cost p_O, which is must be greater than underage cost, is assumed as 1.3 dollar per unit of bumped cargo. It is shown that the optimal overbooking level and optimal cost will increase with the increase of p_U . It is also shown that the optimal overbooking level will decrease with the increase of p_O and the optimal cost will increase with the increase of p_O. In this sub-section, the value of p_U will be changed from 0.4 to 1.2 given two different values of p_O, say 1.3 and 1.8. The result is shown below.

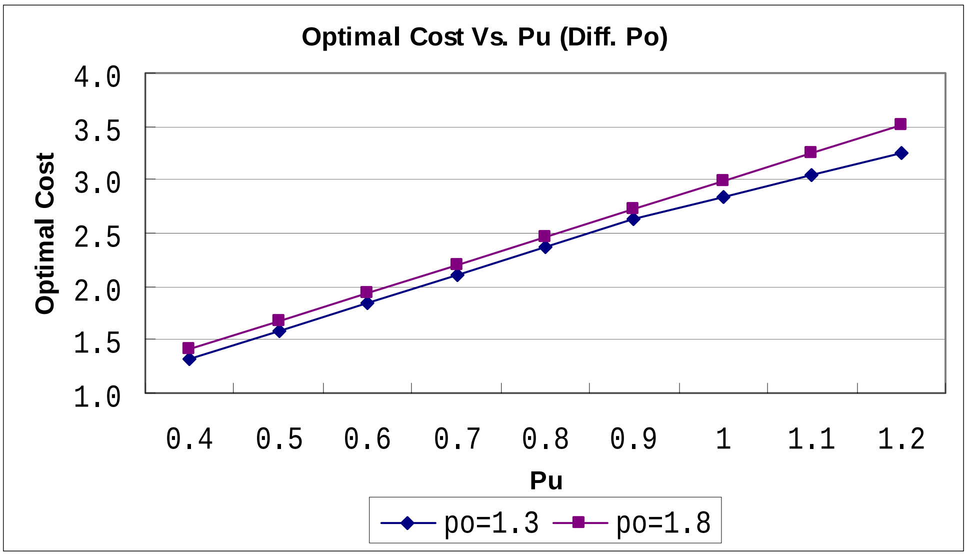

Figure 5.13 Optimal Cost Vs. p_U (Different p_O)

Seen from the above figure, the curve $p_O = 1.3$ is close to the curve $p_O = 1.8$. With the increase of p_U and p_O, the expected total cost will increase. But the amount of increment is different for two variables. The increment is significant for p_U as compared with the effect for p_O. For example, the expected cost increases about 2 dollars more when p_U changes from 0.5 to 1.0 for both given values of p_O. In other words, the optimal cost is more sensitive to p_U than p_O.

This effect can also be shown the other way around when p_O is changed from 1.1 to 2.0 given different values of p_U (0, 0.5 and 1.0 in the figure). Please refer to the figure 5.14 below.

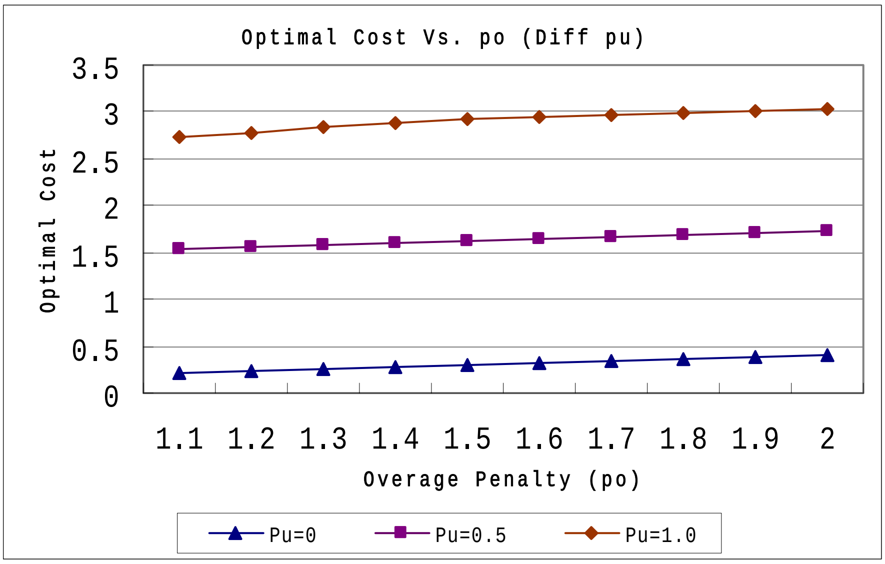

Figure 5.14 Optimal Cost Vs. p_O (Different p_U)

Seen from figure 5.14, optimal cost will slightly increase with the increase of p_O, but when p_U changes from 0 to 0.5, the increment in expected cost is significant. Please refer to the curve $p_U = 0$ and $p_U = 0.5$ in the figure.

5.4.2 Effect of capacity and arrival rate

Another interesting effect of the problem is the relation between flight capacity C and demand arrival rate λ. As discussed before, the optimal OL and optimal cost will increase with the increase of capacity and will decrease with the increase of arrival rate λ. Considered together, assume capacity C is changed from 16 to 24 given different arrival rate λ (0.1, 0.2 and 0.3). The results are shown in figures below.

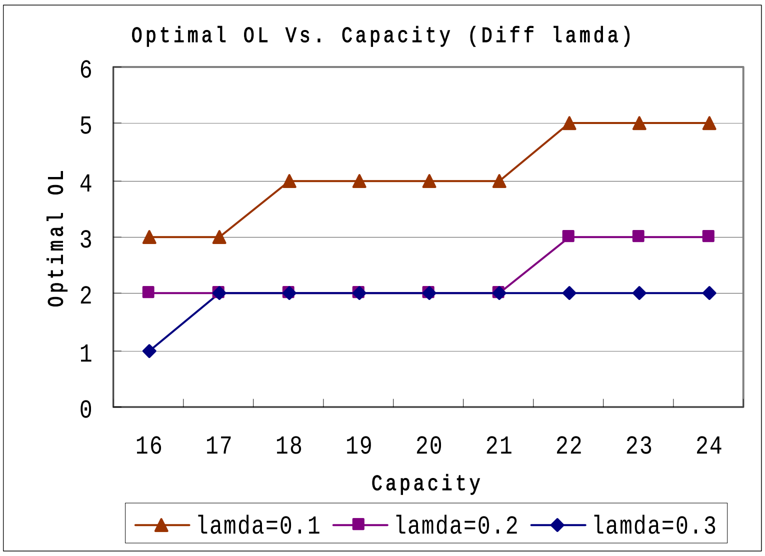

Figure 5.15 Optimal OL Vs. Capacity (Different λ)

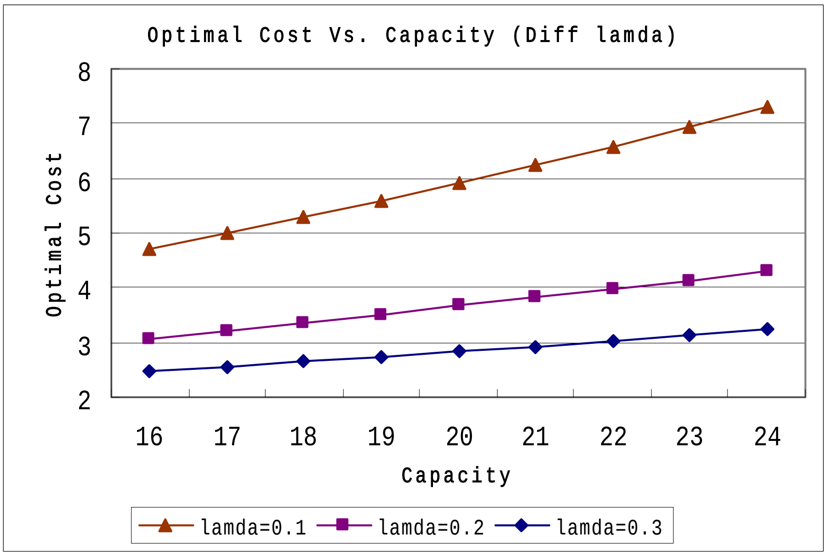

Figure 5.16 Optimal Cost Vs. Capacity (Different λ)

It is shown from the above two figures, when λ is at high level 0.2 and 0.3, the optimal results do not differ too much between two curves. But when λ is at low level at 0.1, the difference becomes significant. Please compare the curve $\lambda = 0.1$ with two others in the figures. And the difference in cost increases with the increase of C. It is due to the combined effect of C and λ. That is, the demand with an arrival rate $\lambda = 0.1$ is too weak and is unable to utilize capacity C ranging from 16 to 24. In this case, underage cost is significant and the overall cost increases. OL should be increased to try to accept more demands in the course of booking.

6. Result and Discussion

In the previous chapter, an overbooking model has been formulated to obtain optimal overbooking level so as to minimize the expected underage cost and overage cost. In the course of modeling, limiting probability distribution was used to approximate the joint probability distribution of the final number of customers and the final total amount of cargo coming for boarding in the overbooking model. Limiting probability distribution is the long-term behavior of the booking system, but the system for reservation opens only a few weeks (say four weeks in the example) before departure. Whether the limiting probability distribution is a suitable representation of final joint probability distribution needs to be checked. In other word, whether the transition matrix formulated in the model converges within four weeks needs to be checked. Furthermore, the model can help to find the optimal booking period for the airline.

In order to check the overbooking model formulated in the previous chapters, a simulation program is introduced first. The main purpose of this simulation program is to see the actual behavior of the final number of bookings and the final amount of cargo coming for boarding. In this chapter, the results obtained in these simulations will also be compared with previous overbooking model.

Firstly, the formulation of the simulation program is introduced. Then the optimal results obtained from overbooking model and simulation are compared with each other. The discussion on suitable range of capacity, the effect of time variable and the variance of optimal result will come next. And finally a conclusion is drawn at the end of this chapter.

6.1 Simulation program

The formulation of the simulation program will be addressed in this section. The main difference of simulation from the previous model is that time is no longer infinite. The booking period is set at four weeks as the example used in the previous chapter. Notations used in the program are listed below to ease the understanding of the program, which is appended at the end of this thesis.

6.1.1 Notations

xn	State variable one, which is represented by the number of customers in time n in the system;
yn	State variable two, which is represented by the total amount of space that has been reserved up to time n and takes a value from 0 to booking limit $c+ol$;
ol	Overbooking level, which is defined as the extra space that can be reserved before departure;
c	Flight physical capacity, which is assumed to be known in this model;
d	The random demand size of a customer, which is assumed to have discrete uniform probability distribution taking a value from 1, 2, ..., d ;
ca	The cancellation size, which is equal to the reservation size of one booking selected to leave the system;
λ	The constant arrival rate of booking process;
$\tau(xn)$	The cancellation rate of booking process at time n, which is assumed to be dependent on the value of xn ;
$timeclock$	The whole booking period from the opening to the departure;

waittime	The time period between two successive events;
uc	The underage cost per unit of empty space;
oc	The overage cost per unit cargo space;

6.1.2 Assumptions

The assumptions made for the simulation will be discussed in this section. Some are different from the previous model.

In comparing with the optimal result obtained in the previous model a fixed booking period (four weeks) is assumed. The result obtained from the previous model will be compared with simulation given four weeks of the booking period. The booking period will then be set longer to see the effect of time variable on the optimal result.

After the booking period is set, the next-event incrementing time advance mechanism is used to advance the simulation clock. Once the time is running out, the reservation system will be stopped. The waiting time between two successive events is determined by the arrival rate and the cancellation rate. No non-event occurs within waiting time.

Arrivals (requests for reservations) is assumed to come independently and to follow a Poisson process with a constant arrival rate λ. Booking status will not affect demand pattern. And also assume the demand distribution is the same for every individual demand, following discrete uniform probability distribution taking a value from $1, 2, ..., d$ with a probability $\frac{1}{d}$.

The cancellation rate $\tau(xn)$ is assumed to be a linear function of the number of bookings in the system. As the same as before, $\tau(xn)$ is assumed to be $\frac{xn}{50}$ in the simulation. When a cancellation event happens, a cancellation is uniformly selected from the xn bookings currently in the system. The cancellation size ca equals to the booking selected. Once a cancellation is selected, it will leave the system completely. This assumption is different from the previous model as the process is not formulated as a Markov process here. In the simulations, the Markovian restriction was not assumed. As a result, when a specific booking is selected to cancel, its corresponding size is the cancellation size. Partial cancellation is not allowed in the simulation.

Similar to the previous model, the criterion used to make an accept/deny decision in each period is the booking limit $c+ol$. The demand, which exceeds the remaining capacity $c+ol-yn$, will be rejected; otherwise, it will be accepted.

2000 replications are used to obtain the average result, which will be compared with the expected result obtained in the previous model. Furthermore, random number needs to be generated in the simulation. In order to repeat the process, a seed is assigned each time when running the simulation. The seed used to generate random number is assumed to be $10 \times ol$.

6.1.3 Description of simulation program

In running the program, time clock is set at four weeks. The time between two successive events is called as waiting time. The waiting time, which is determined by the arrival rate λ and the cancellation rate $\tau(xn)$ in period n, has an exponential distribution. In period n for one event to happen, the random waiting time has to be generated from an exponential distribution with the parameter equal to $\lambda+\tau(xn)$, referring to the equation (6.1) below. Then the time clock will be deducted by this waiting time to see whether the time is running out.

$$waiting\ time = -\frac{1}{\lambda + \tau(xn)} Ln(1 - Random) \tag{6.1}$$

In a period denoted as n, two state variables are defined similar to those defined in the previous model: the total number of customers (one customer one booking) in the system xn and the total booked space yn. Given an event, either a request for booking or a cancellation, comes, it has to be differentiated as an arrival event or a cancellation event by the corresponding probabilities:

$$P\{arrival\ event \mid one\ event\} = \frac{\lambda}{\lambda + \tau(xn)} \tag{6.2}$$

$$P\{cancellation\ event \mid one\ event\} = \frac{\tau(xn)}{\lambda + \tau(xn)} \tag{6.3}$$

Furthermore, if it is an arrival event, a decision must be made as to whether to accept it or not according to booking limit $c + ol$. However, if it is a cancellation event, one booking will be uniformly selected from the xn bookings in the system in period n and this booking is regarded as being canceled and will leave the system completely.

The booking process will proceed like what has been described above until the departure time is reached. At the end of the process, the final bookings on hand will be recorded. This final reservation on hand will be compared with flight capacity to compute the cost, either an underage cost or an overage cost.

After that, the whole booking process will repeat 2000 times. Each time a final bookings on hand and a cost can be determined. Hence, there are 2000 data of the final bookings on hand and 2000 data of the total cost. Finally the average final bookings on hand and the average total cost can be calculated by these data. The average final bookings on hand and the average total cost will be compared with the results obtained from the previous overbooking model.

A flowchart of the program and the Mathematica program can be found in the Appendix.

6.2 Optimal result

Take the same example as used in the previous model. Suppose capacity C=20 and individual demand D has a discrete uniform probability distribution, evenly taking a value from $1, 2, ..., 10$. That is, $d = 10$. Constant arrival rate λ =0.3, and the cancellation rate $\tau(X_n)$ is $\frac{X_n}{50}$, which is assumed to be a linear function of X_n. The overage penalty is p_O=1.3 and underage penalty is p_U=1.0.

The optimal overbooking level according to the overbooking model, is 2 with the expected cost 2.83459. The expected cargo coming for boarding on the day of departure is 18.4424. There is 0.641054 chance that the coming cargo is below flight physical capacity and 0.358946 chance that the coming cargo is above flight capacity. In the later case, overage occurs.

Given the same setting mentioned above, the simulation produced an average cost of 3.2761 after running 2000 times of replications if overbooking level is set at 2. The average amount of cargo coming for boarding is 17.9475. The probability that the final cargo for boarding is below the flight capacity is 0.657, and the probability that the final cargo for boarding is above the flight capacity is 0.343.

It was observed that the results from simulation were not far away from the results obtained in overbooking model. However, this single average cost given overbooking level equals to 2 cannot guarantee optimization at this moment. Hence, this simulation program is run several times given different overbooking level, each time the average cost and the average cargo coming for boarding are recorded and compared with other results. The point that produces the minimum cost is $ol = 2$, which is the same as overbooking model. Please refer to the table 6.1 below.

Table 6.1 Results from Simulation

OL	0	1	2	3	4	5	6	7	8	9
Ave_Cost	3.56	3.28	3.28	3.59	3.93	4.45	5.01	5.82	6.37	6.99
Ave_Cargo	16.4	17..2	18.0	18.7	19.6	20.7	20.9	21.7	22.4	22.9

Seen from the first row in table 6.1, the average cost first decreased then increased with the increase of overbooking level. At the point $OL = 2$, the average cost reached the minimum. The result confirmed with that obtained in overbooking model. The average cargo coming for boarding increases with the increase of overbooking level. Please refer to the second row in the table 6.1. As the randomness of the simulation is due to random numbers generated in the program, optimal result cannot be guaranteed every time when this program is run. However, the overall trend is obvious.

Recall that similar results are obtained in solving overbooking model in the last chapter. Plot the results obtained by two methods in one figure, we can have a clear view of the movement of the optimal results against overbooking level. Please refer to Figure 6.1 below, in which cost obtained from overbooking model and simulation is compared given different overbooking level (where the curve with marker ◊ represents results obtained from overbooking model and the curve with marker □ represents simulation).

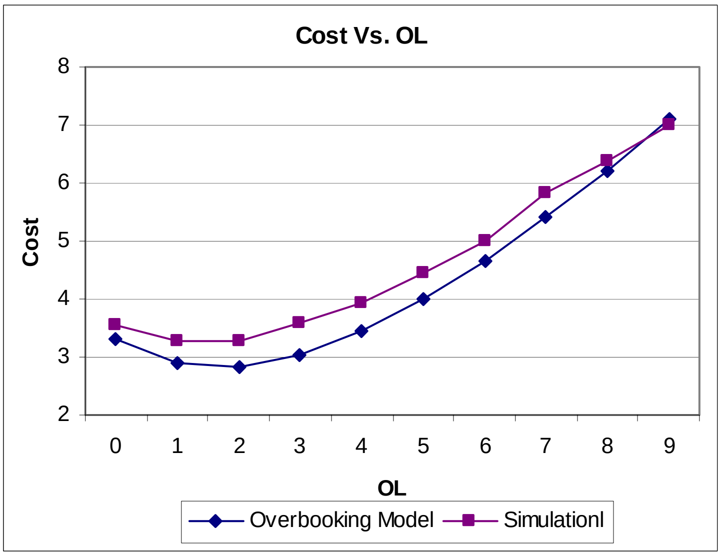

Figure 6.1 Cost Comparison between Overbooking Model and Simulation

Similarly, the final cargo coming for boarding of overbooking model and simulation is shown in Figure 6.2 below (again the curve with marker ◊ represents overbooking model and the curve with marker □ represents simulation).

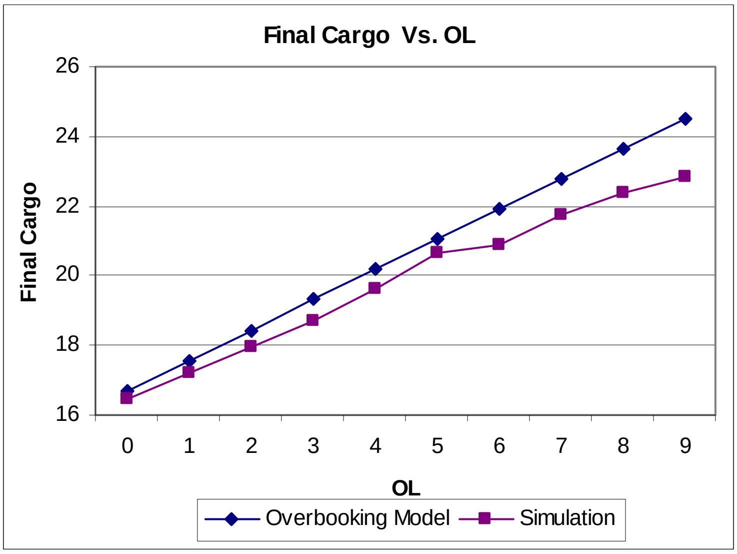

Figure 6.2 Final Load Comparison between Overbooking Model and Simulation

Seen from the above two figures, results from overbooking model and simulation are close either in terms of cost or final cargo. Overbooking model and simulation yield the same optimal result. That is, at point $OL = 2$, cost is the minimum. It is clear that cost is a convex function of OL. Furthermore, with the increase of OL, the final cargo coming for boarding increases.

Please refer to Figure 6.2, the difference in final cargo coming for boarding becomes bigger when the overbooking level is set at high value. It is due to the combined effect of flight capacity and booking time. Given a fixed setting of booking condition stated at the beginning of this section, the increase of overbooking level means the increase of booking limit. More demands can be accepted in the course of booking. Since the booking period is fixed in simulation, the booking system cannot fully utilize the extra space provided by airline within the set time span. In other word, 28 days is not long enough for the system to converge as in this example. As a result, the final coming cargo deviates greater in the area of bigger overbooking level than in the area of smaller overbooking level.

However, if the booking period increases, the difference between overbooking model and simulation, either in small overbooking level or in big overbooking level, will become almost identical. This would be discussed in section 6.4.

There comes another question: within what range of flight capacity is the overbooking model formulated in the previous chapter valid? This will be discussed in the next section.

6.3 Suitability check of the overbooking model

As stated towards the end of last section, the overbooking model formulated can only be applied to a certain range of flight capacity. Beyond this range, the result obtained by overbooking model cannot be a good estimation of actual booking behavior in airline since the limiting probability distribution is used to approximate joint probability distribution of final cargo coming for boarding and final number of bookings. In other word, if the system cannot converge in 28 days as set in the example, the optimal result obtained by overbooking model may not be convincing. The valid range of capacity will be discussed in this section. All other parameters remain the same as the previous numerical example.

In order to check the validity of capacity, the optimal results from overbooking model and simulation need to be compared, given different value of capacity and all other parameters unchanged. It is obvious that the smaller the flight capacity, the faster the booking system will converge to the limiting case. As a result, set capacity from 20 to 50. Under each value of the capacity, the optimal overbooking level and optimal cost by the overbooking model and simulation are determined. Results are shown in figure below.

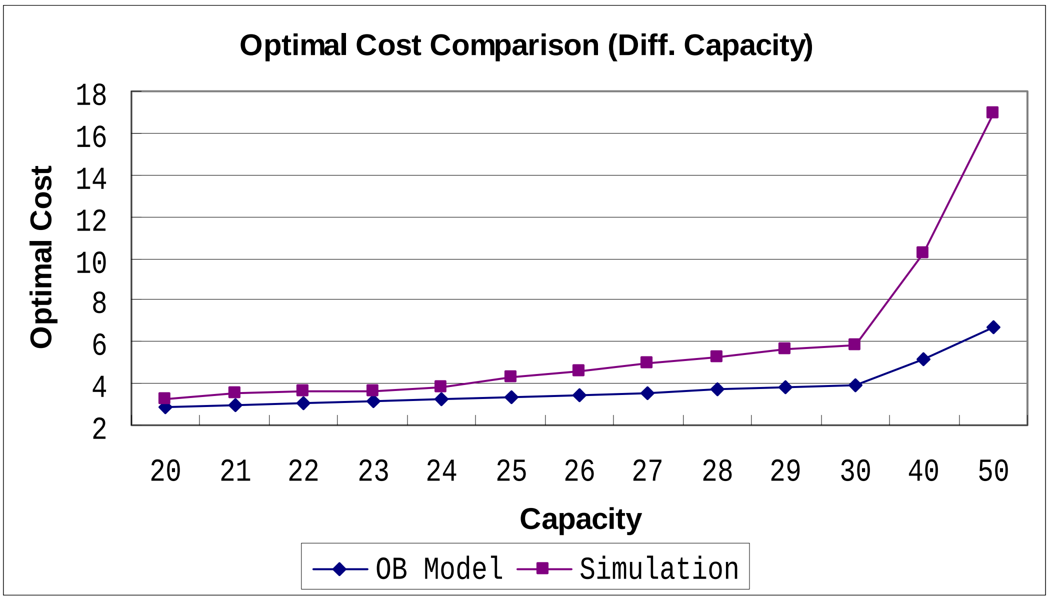

Figure 6.3 Optimal Cost Comparison Given Different Capacity

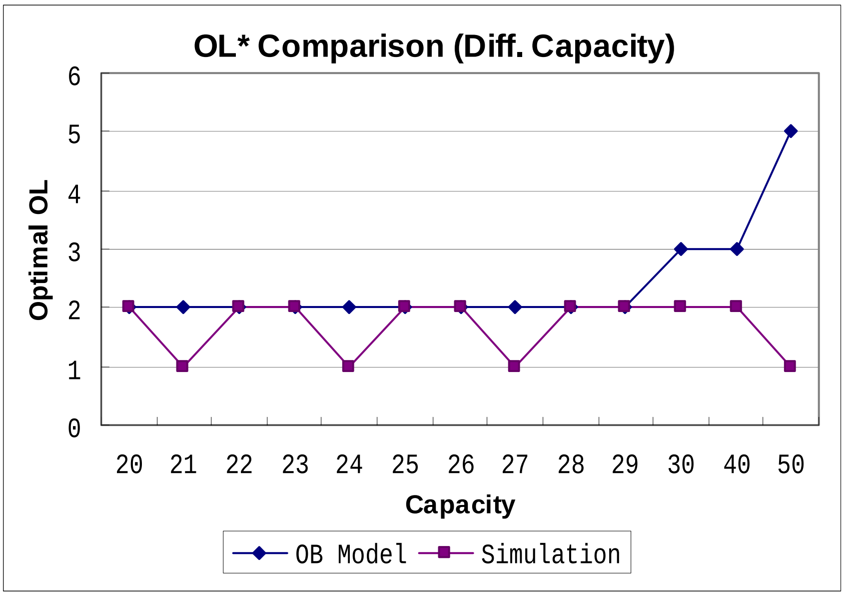

Figure 6.4 Optimal *OL* Comparison Given Different Capacity

In Figure 6.3, the difference in final cost between overbooking model and simulation is small when capacity is from 20 to 25, and increases when capacity is above 25. It implies that the actual booking system cannot converge fast to the limiting probability distribution within 28 days. When capacity increases, capacity becomes relatively big as compared with arrival rate and the intensity of demand. As a result, more time is needed to absorb the large demands. Hence, 28 days is not enough for the system to converge. With the increase of capacity, the deviation between overbooking model and simulation is becoming larger. In this case, overbooking model is not suitable for determining the optimal overbooking level.

In this section, only the capacity is checked to determine the validity of overbooking model. Other parameters can also be checked by the same method. The purpose of doing this is to bring the idea to reader that all the input parameters should match with each other in order to obtain valid result.

6.4 Effect of time variable

Recall that the final cargo probability distribution in the overbooking model is approximated from the limiting probability distribution. That is, the result from the infinite booking period is used to represent booking behavior of a finite period. It is not known for sure whether this approximation is good or not. In other words, whether the booking period (28 days) exercised is long enough for the system to converge. In this section, the effect of time variable will be discussed.

In the simulation as mentioned in section 6.2, booking period is supposed to be 28 days. 28 days may not be optimal from the airline's viewpoint. If the booking period is set too short, booking system cannot converge fast within the period, so the result obtained in the overbooking model is unsuitable. On the other hand, if the booking period is set too long, the operational expense will increase. There is an optimal booking period for airline to use.

To analyze the effect of time variable, the simulation is run several times by setting different booking period at 28, 40 and 50 days respectively. Under each situation, the final cargo coming for boarding and the corresponding cost are recorded for different overbooking level ranging from 1 to 10. The result is shown in figures below. For easy reference, the result from overbooking model is also included for comparison.

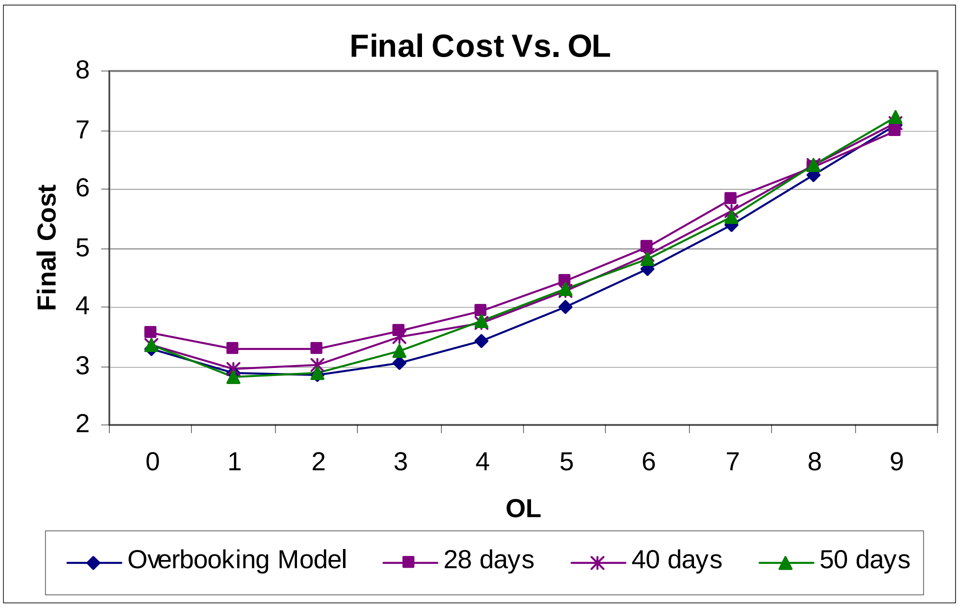

Figure 6.5 Final Cost Vs. OL Given Different Booking Period

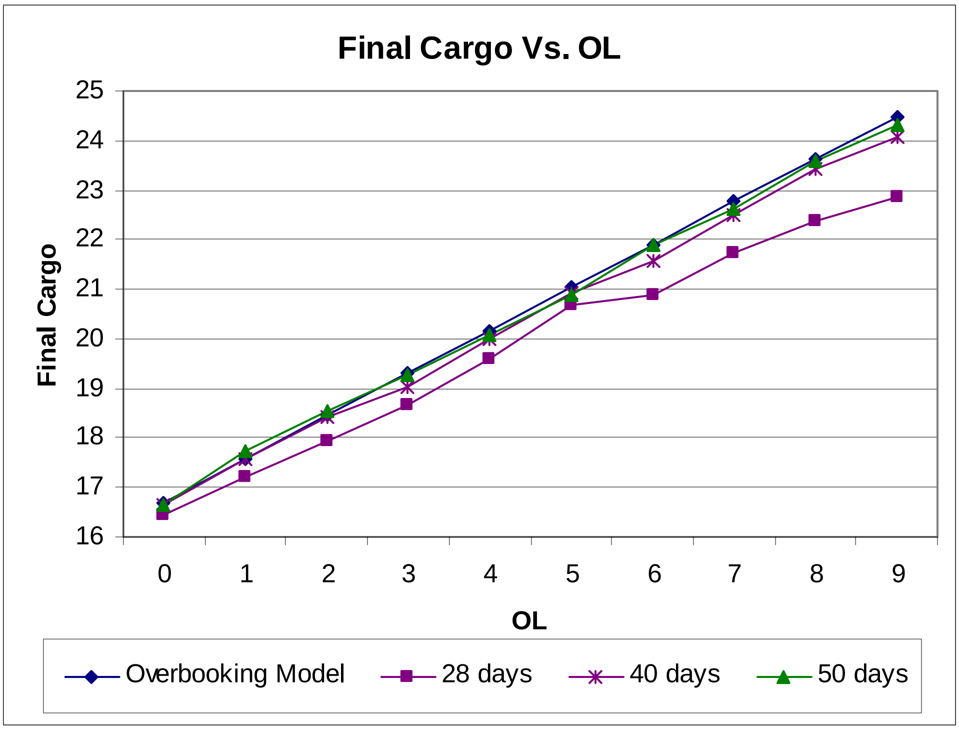

Figure 6.6 Final Cargo Vs. OL Given Different Booking Period

In the above two figures, the results from overbooking model are represented by the curve with marker ◊. It is shown that with the increase of booking period, the difference between overbooking model and simulation model decreases. When booking period is above 50 days, the difference between two models is almost negligible. Furthermore, except that 28 days curve is a little far away from that of overbooking model, all others are much closer to that of overbooking model. This means the system converges fast about 40 days later.

It should be noted that the system converges after 40 days does not mean that 40 days is better than 28 days or that the longer of booking period, the better the result which the airline will obtain. The determination of booking period depends on:

1. the operational complexity of a booking system;
2. the expense of implementing proposed booking system;
3. the stability of booking system.

The first two concerns are always related to the fact that complex system usually requires expensive operational cost. The third concern can be viewed as the robustness of a booking system. The above three concerns can be combined as an optimal booking period is the one which can balance operational complexity and robustness of the booking system.

Intuitively, the longer the booking period, the more complex a booking system is. The objective of selecting an optimal booking period becomes one of finding a shortest booking period, which can satisfy the stability requirement.

6.5 The variance of optimal result

The variance of the optimal results has not been considered. Since the limiting probability distribution used in overbooking model represents the booking behavior when time goes to infinity, the optimal results from the overbooking model have minimum variance as compared with the results from simulation. It does not matter the booking period is set at 28 days or 40 days. In simulation, the variance of optimal result (optimal final cost and optimal final cargo coming for boarding) will decrease with the increase of booking period. Hence, the variance of the optimal result can also be used to measure the convergence of simulation in the overbooking model.

In the overbooking model, limiting probability distribution is used to approximate the distribution of the final number of customers and the final amount of cargo for boarding, in which case the variances of the two parameters are the minimum. According to the overbooking model, the variance of the final cargo coming for boarding is 12.4895 and the variance of the final cost is 7.5349. These two values represent the baseline of the variance of final cargo and final cost for simulation given different booking period. The results are shown in figures below, in which the booking period is set at 10, 15, …, 45 and 50 days.

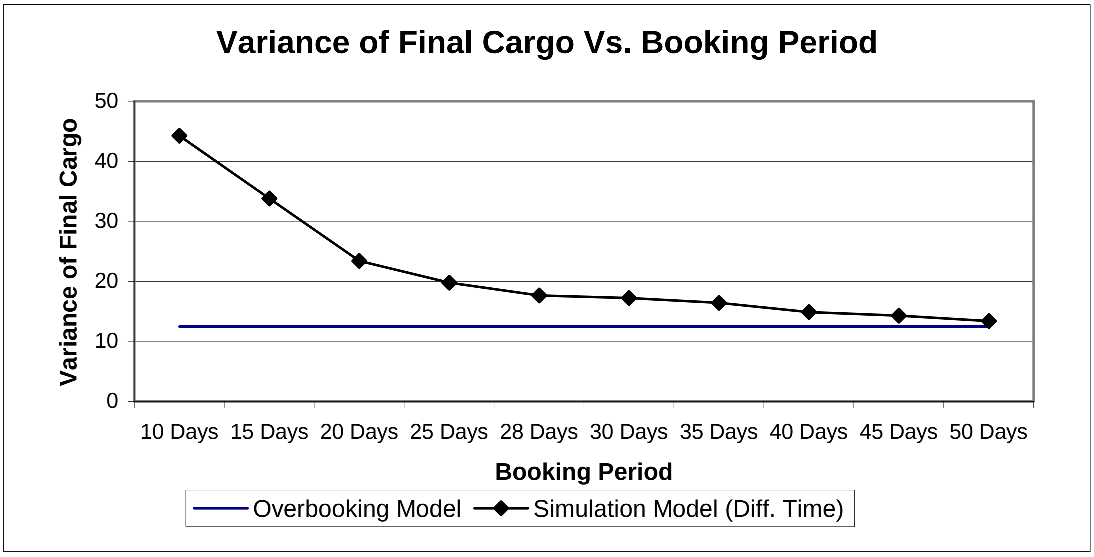

Figure 6.7 Variance of Final Cargo Vs. Booking Period

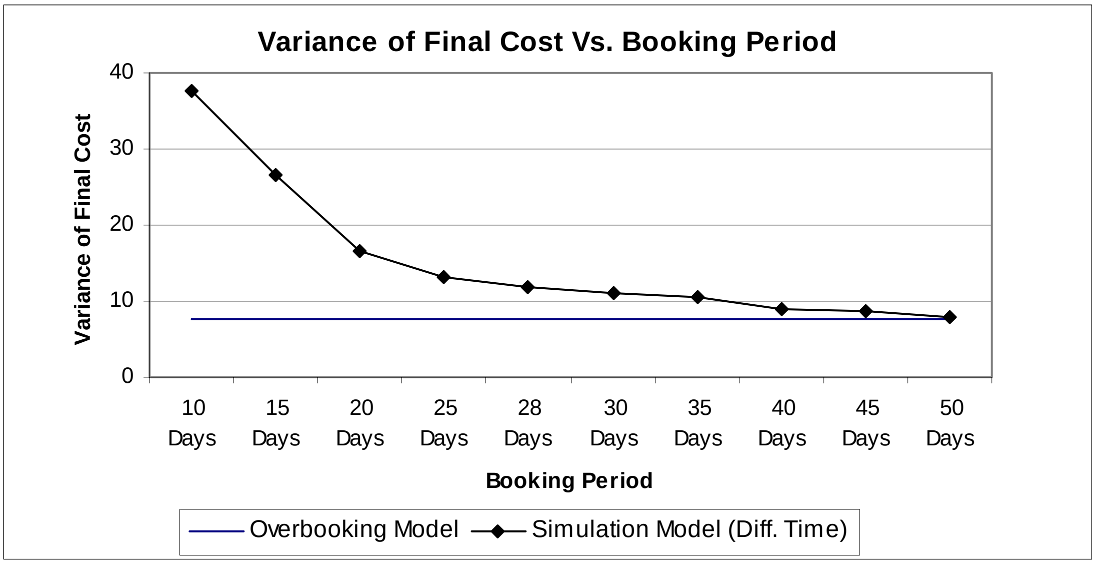

Figure 6.8 Variance of Final Cost Vs. Booking Period

Seen from the above two figures, the variance of final cargo and final cost decreases with the increase of booking period. The minimum is the one obtained in overbooking model. If airline wants to determine a reasonable booking period with a certain requirement on the variance of the optimal result, the above two figures are helpful.

6.6 Conclusion

From chapter 3 to chapter 6, the cargo booking process is assumed to be two-dimension Markov process. It is a combination of homogeneous Poisson arrival process with constant arrival rate and non-homogeneous Poisson cancellation process with cancellation rate depending on the number of bookings in the system. The two state variables are defined in the model. And the limiting probability distribution is used to approximate the joint probability distribution of the final number of customers and final amount of reservation coming for boarding on the day of departure. Mathematica was used to solve the problem.

The results from simulation confirm the same optimal results obtained by overbooking model, given the same setting of input variables. The simulation results show that the overbooking model produces good predictions on real booking behaviors and ensures the minimum expected total cost. Results show that the optimal overbooking level can be obtained with practical meaning as compared with simulation. When parameters are within certain range, the optimal result obtained is reliable. In applying the model, it is necessary to first check whether parameters are in the valid space.

Although the overbooking level obtained in this model is static in that it is determined before the booking opens and will remain constant along the process, the model helps understand the mechanism among arrivals, cancellations and overbooking control.

7. Conclusion

Most airlines around the world are employing yield management, also known as revenue management, to help maximize revenue. Yield management is the integration of science, information technology, and business processes to deliver the right product to the right customer at the right time at the right price. The origin of yield management comes from the situation where perishable products exist.

Along with the development of yield management in passenger sector, the techniques of yield management are now being introduced into air cargo service, drawing on the passenger experience. In both ticket booking and space booking processes, airlines usually adopt the strategy of overbooking. Overbooking is a practice of intentionally accept a certain more number of tickets or cargo bookings than the corresponding capacity along booking process to compensate for possible cancellations and no-shows. The purpose of this study is to formulate mathematical models to help determine optimal cargo overbooking level so as to minimize the expected total under-sale cost and spoilage cost (or maximize the expected revenue).

7.1 Concluding remarks

The cargo booking process is formulated as a two-dimension Markov process in the statistic overbooking model. It is a combination of homogeneous Poisson arrival process with a constant arrival rate and non-homogeneous Poisson cancellation process with cancellation rate depending on the number of customers in the system. The two state variables are defined in the model: one is the number of bookings in the system and the other is the total amount of space that has been booked.

The arrivals (request for bookings) were assumed to follow Poisson process with a constant arrival rate and each individual demand was assumed to have a discrete uniform probability distribution, evenly taking a value from $1, 2, ..., d$. Cancellation can only come from the bookings already in the system. The probability distribution of cancellation in one period can be purely derived from the booking status, so that the formulated process has Markovian property.

Within each decision period, one and only one of three events will happen: arrival event (a request for booking), cancellation event (a request for cancellation) and non-event (nothing happens). Limiting probability distribution is used to approximate the joint probability distribution of the final number of bookings and final amount of cargo coming for boarding on the day of departure. Since the resulting state-space of the model is large, the mathematical software Mathematica was used to solve the problem.

Since cargo booking starts only a few weeks before departure, the validity of using limiting probability distribution has to be checked. A simulation program was used to confirm this.

The simulation results showed that the proposed overbooking model converged within four weeks as used in the example. The overbooking model produced very good approximations on real booking behaviors and ensured the minimum expected total cost. Another point to be noted is that the optimal result is valid only when the model converges with booking period; otherwise, the result is useless.

The models constructed in the thesis would be useful in airline cargo space booking operations. The study provides an effective mathematical approach to solve the real problem in air cargo space booking. It would greatly improve operational efficiency of the cargo booking system and help airlines to maximize revenue from the cargo sector.

7.2 Limitations and future research

The overbooking model formulated is static in that the overbooking level is set at the beginning of booking process and it will not change over the whole process. In other words, the model is unable to capture the latest booking information. On the other hand, since the limiting probability distribution is used to approximate the joint distribution of final amount of cargo and final number of bookings coming for boarding, the overbooking model cannot be run several times along the process such that it can consider the latest booking information. This is because the limiting probability is in fact the behavior of the system when time goes to infinity. The optimal result is only valid when the transition matrix converges within booking period; otherwise, the model cannot be applied as discussed in the chapter 6.

One possible future research direction is to delve into dynamic overbooking model, which the author has been working for a long time. One possible structure is to consider discrete-time dynamic programming formulation. Suppose there is a known cargo capacity on a combined-carrier for cargo booking. The booking process is arbitrarily divided into several periods depending on the booking status, time left before departure and the capability of airline to update the demand/cancellation information. The status of the system is the amount of reservation on hand and the system will be reviewed in each period. The interval between two review points does not need to be equal. Mostly probably, the interval at the beginning of the process is longer than the interval at the end of process.

Requests for booking will come along the booking process. The individual arrival is not considered in the model. Instead, the demand that will come in the one period (from one review point to the next review point) is estimated and the cancellation from the current reservation that may possibly come in the one period is also estimated. The total demand in each period depends on the time left before departure and the number of cancellations in one period depends on the booked reservation in the system. Given that these two distributions can be obtained from the airline, the objective of the model is to determine how much reservation to accept in the one period, named as booking limit, so as to maximize the expected total revenue, based on the booking status, time left before departure and the demand and cancellation distribution pattern. Since these two probability distributions can be modified from period to period, the booking limit can also be modified from period to period till flight take-off.

Another possible research direction is to consider long-term allocation from airline's perspective. As stated in chapter 3, airline will sign contract with freight forwarders long before actual booking for a specific flight begins. Signing long-term contract will decrease the risk to airline for not being able to find enough demand to fill up space on flights when actual booking starts. Long-term contract will decrease the variance of final show-up cargo on a flight and simplify the operation in airline. This practice will also help airline keep a certain market share, which will increase an airline's competition in a long run. But freight forwarders signing contracts usually obtain a discounted price for cargo, which will in a sense decrease airline's revenue. There is a decision to be made by airline as to how much capacity should be allocated to long-term contracts. If less capacity is allocated to contract, airline may not be able to find enough ad-hoc demand to fill the capacity left. Or if more capacity is allocated to contract, airline may loss some revenue.

On the other hand, freight forwarders also need to consider how much capacity to obtain from airlines. If getting less, forwarders will loss money in the cast that more demand comes for reservation. If getting more, forwarders will loss of goodwill from airline if they cancel some contracted capacity, which will affect the competition in signing contract for the next season. The above mentioned problem can be modeled from airline's perspective, forwarder's perspective or both.

From the literature received, few people have been working these two problems in airline industry. Air cargo revenue management is an excellent research area with a high potential for new models and procedures to accurately represent the cargo business, and to provide the required decision support.

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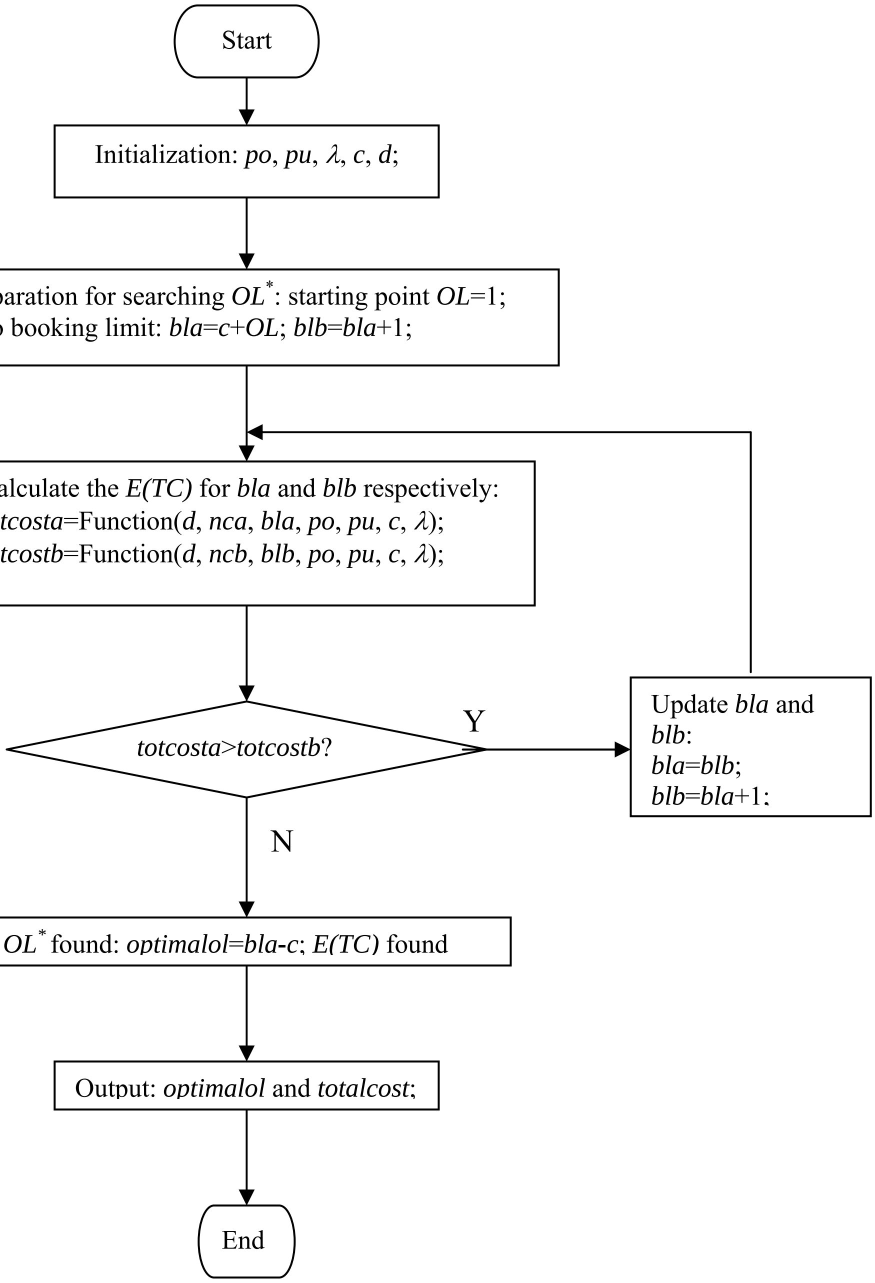
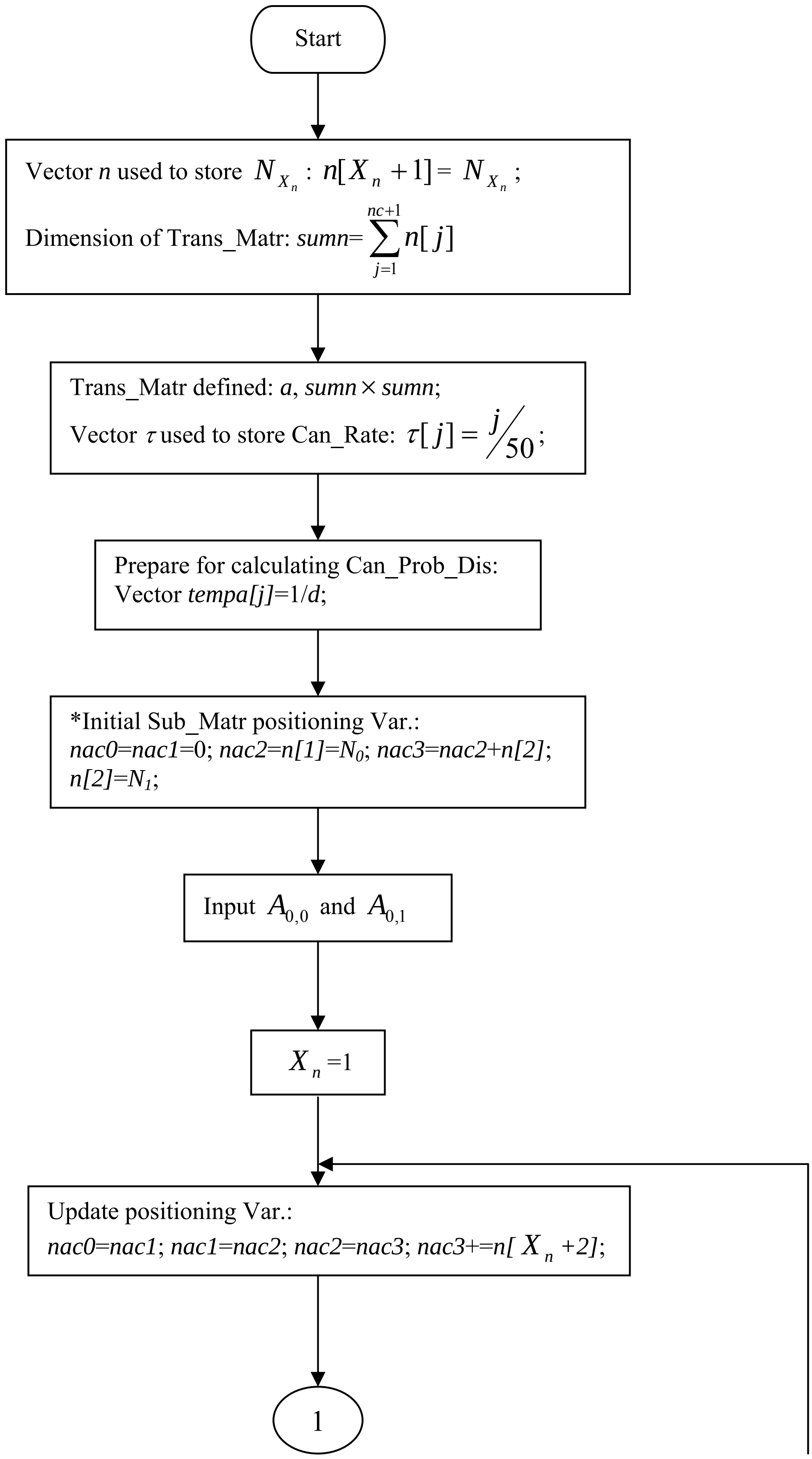
Appendix A Program Flowchart

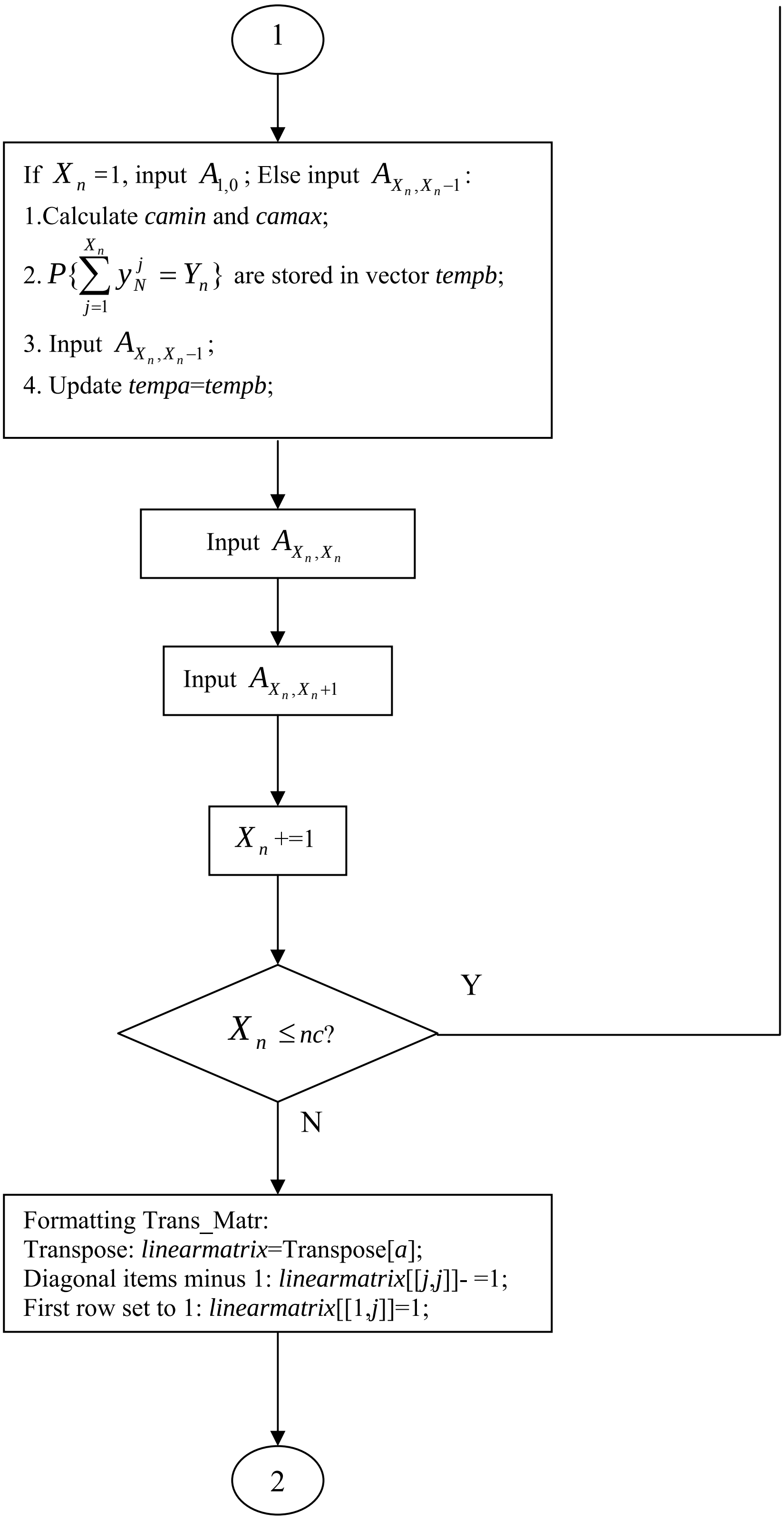
i. Flowchart of static overbooking model:

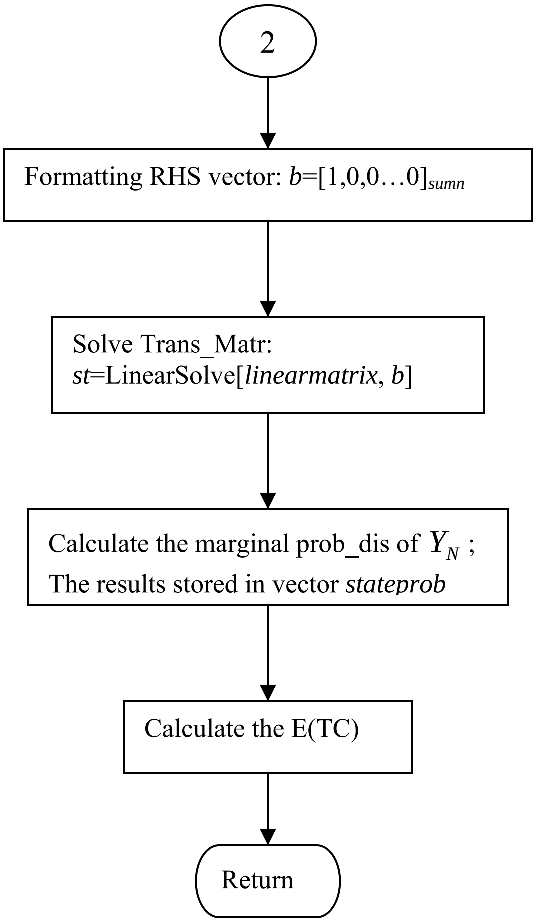
- *Main Program:*



- *Function (d, nc, ol, po, pu, c, λ):*

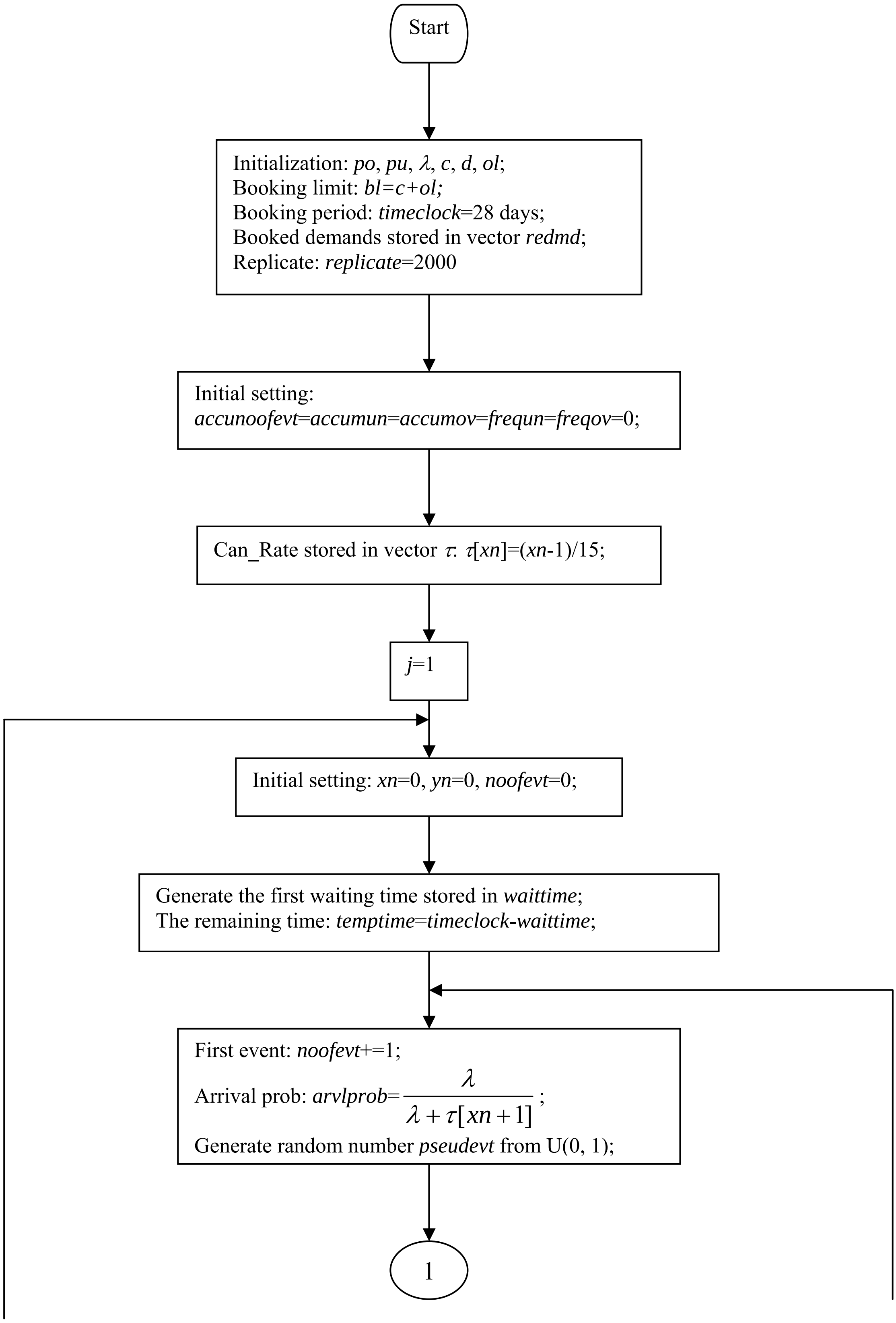


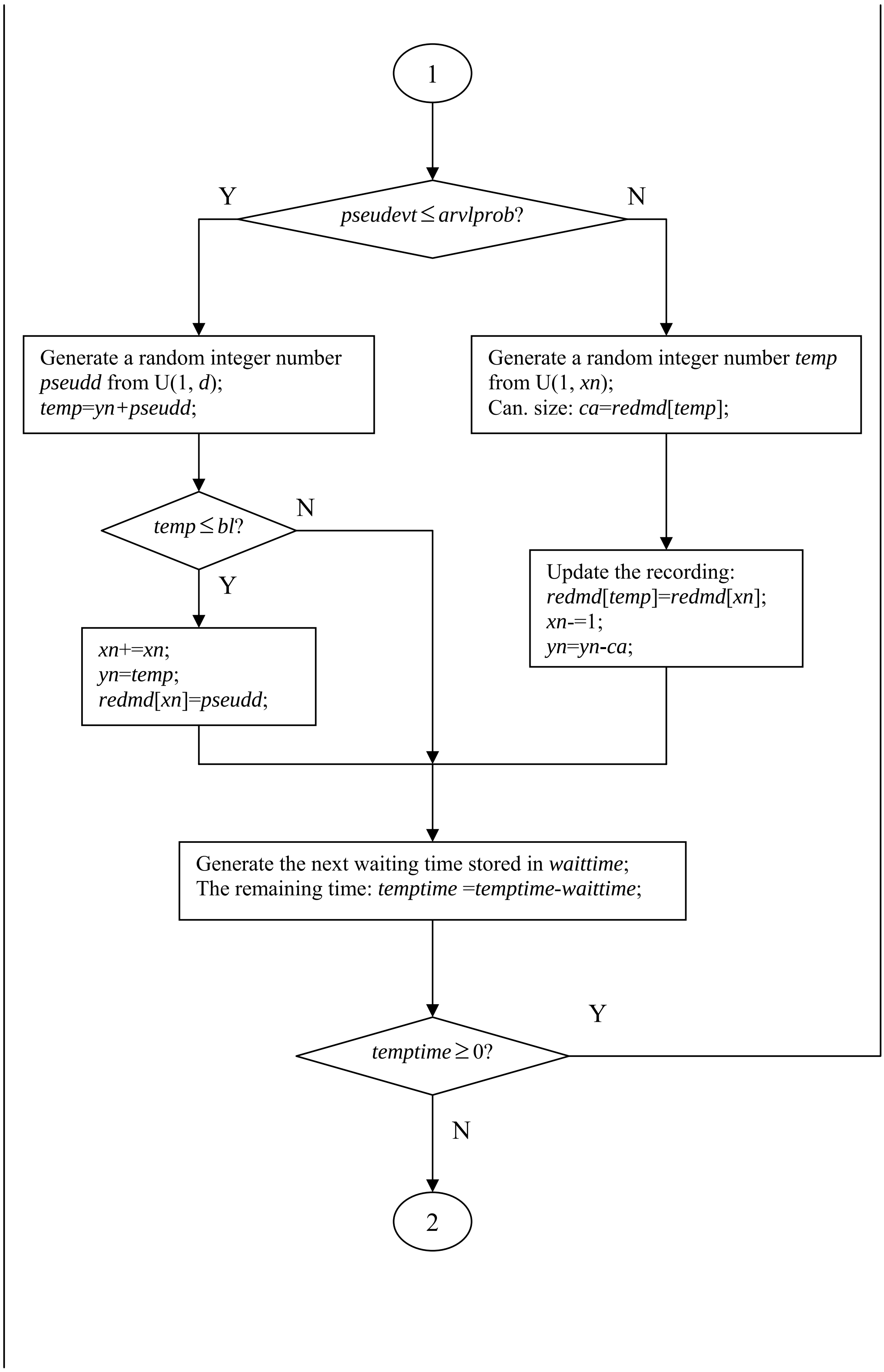

1
If X_n =1, input $A_{1,0}$; Else input A_{X_n,X_n-1}:
1.Calculate *camin* and *camax*;
2. $P\{\sum_{j=1}^{X_n} y_N^j = Y_n\}$ are stored in vector *tempb*;
3. Input A_{X_n,X_n-1};
4. Update *tempa*=*tempb*;
Input A_{X_n,X_n}
Input A_{X_n,X_n+1}
X_n +=1
$X_n \leq nc$?
Y
N
Formatting Trans_Matr:
Transpose: *linearmatrix*=Transpose[*a*];
Diagonal items minus 1: *linearmatrix*[[*j*,*j*]]- =1;
First row set to 1: *linearmatrix*[[1,*j*]]=1;
2

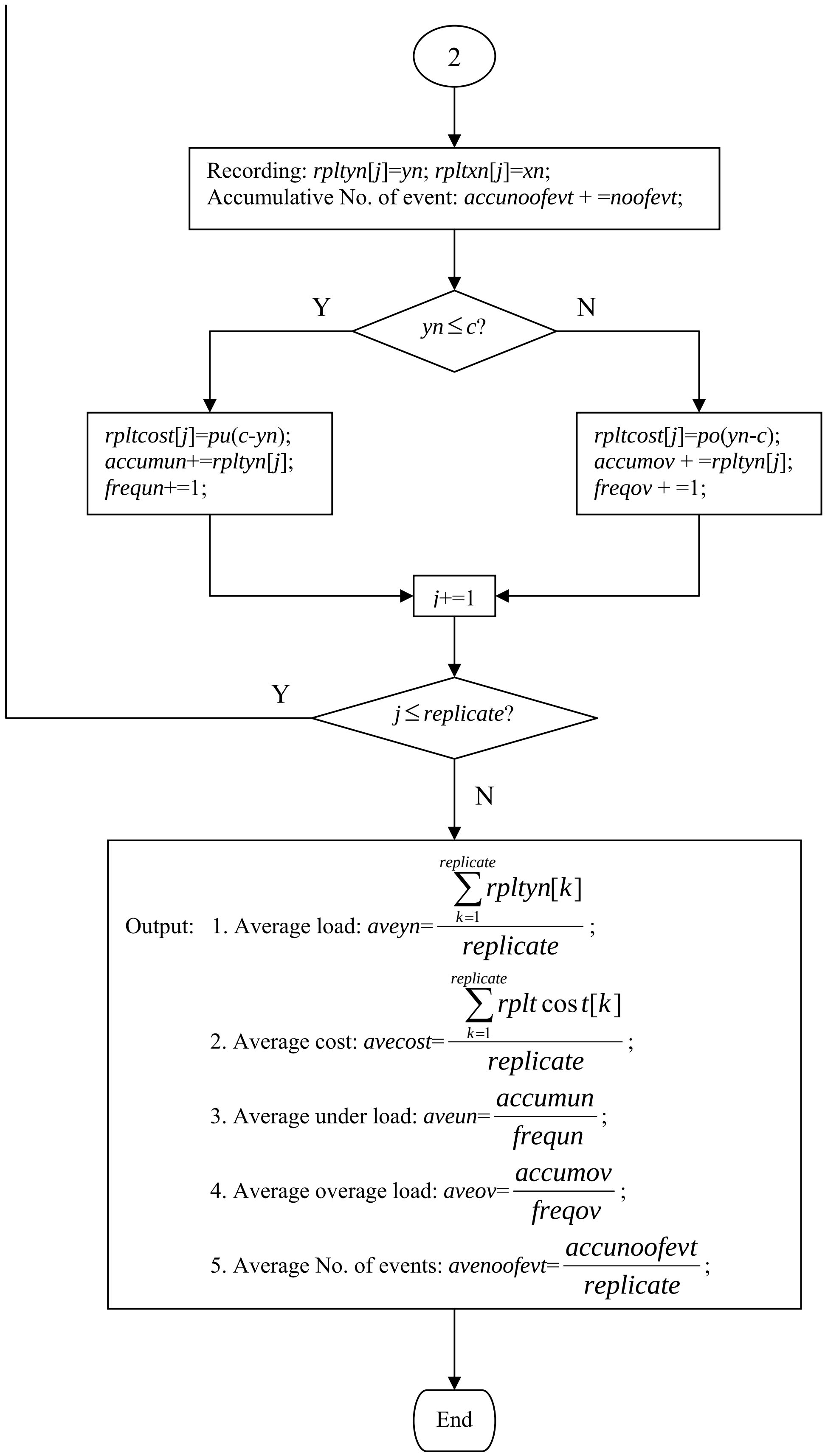


* The top-left items of the sub-matrix A_{X_n,X_n-1}, A_{X_n,X_n} and A_{X_n,X_n+1} are used to position them in the global transition matrix. Given X_n bookings in the system, the index of the top-left items of A_{X_n,X_n-1}, A_{X_n,X_n} and A_{X_n,X_n+1} are defined by (*nac1*+1, *nac0*+1), (*nac1*+1, *nac1*+1) and (*nac1*+1, *nac2*+1) in the program.

ii. Flowchart of the Simulation:




1
Y
N
pseudevt ≤ arvlprob?
Generate a random integer number pseudd from U(1, d);
temp=yn+pseudd;
Generate a random integer number temp from U(1, xn);
Can. size: ca=redmd[temp];
temp ≤ bl?
N
Y
xn+=xn;
yn=temp;
redmd[xn]=pseudd;
Update the recording:
redmd[temp]=redmd[xn];
xn-=1;
yn=yn-ca;
Generate the next waiting time stored in waittime;
The remaining time: temptime =temptime-waittime;
Y
temptime ≥ 0?
N
2


2
Recording: rpltyn[j]=yn; rpltxn[j]=xn;
Accumulative No. of event: accunoofevt + =noofevt;
Y
$yn \le c$?
N
rpltcost[j]=pu(c-yn);
accumun+=rpltyn[j];
frequn+=1;
rpltcost[j]=po(yn-c);
accumov + =rpltyn[j];
freqov + =1;
j+=1
Y
$j \le replicate$?
N
Output: 1. Average load: $aveyn = \frac{\sum_{k=1}^{replicate} rpltyn[k]}{replicate}$;
2. Average cost: $avecost = \frac{\sum_{k=1}^{replicate} rpltcost[k]}{replicate}$;
3. Average under load: $aveun = \frac{accumun}{frequn}$;
4. Average overage load: $aveov = \frac{accumov}{freqov}$;
5. Average No. of events: $avenoofevt = \frac{accunoofevt}{replicate}$;
End

Appendix B Mathematica Program

i. Optimal overbooking level program:

```
expectedtotalcost[d_, nc_, bl_, po_, pu_, c_, λ_] :=
(

  (*Determine the size of Yn(Xn)*)
  m = Array[n, nc + 2];
  For[j = 0, j d ≤ bl, j++, n[j + 1] = j (d - 1) + 1];
  breakn = j;
  For[j = breakn, j ≤ nc + 2, j++, n[j + 1] = Max[0, bl - j + 1]];

  (*Dimension of transition matrix*)
  sumn = Sum[n[j], {j, 1, nc+1}];

  (*a is transition matrix*)
  a = Table[0, {sumn}, {sumn}];

  (*Cancellation Rate*)
  tau = Array[τ, nc]; Do[τ[j] = j/50, {j, 1, nc}];
  recura = Array[tempa, bl]; recurb = Array[tempb, bl];
  Do[tempa[j] = 1/d, {j, bl}];
  arr[x_] := N[λ/(λ + τ[x])]; can[x_] := N[τ[x]/(λ + τ[x])]; expo[x_] := N[Exp[-λ - τ[x]]];

  (*Position of Submatrix*)
  nac0 = nac1 = 0; nac2 = n[1]; nac3 = n[2] + nac2;

  (*A0,0 and A0,1 *)
  a[[1, 1]] = N[Exp[-λ]];
  Do[a[[1, j]] = N[1/d (1 - Exp[-λ])], {j, nac2 + 1, nac3}];
```

```
For[i = 1, i ≤ nc, i++,
  nac0 = nac1; nac1 = nac2; nac2 = nac3; nac3 += n[i + 2];

    (*A1,0*)
   If[i == 1,
      Do[a[[j, 1]] = N[(1 - expo[i]) × can[i]], {j, nac1 + 1, nac2}],

    (*Determine cancellation distribution*)
     For[j = 1, j ≤ nac2 - nac1, j++,

    camin = Max[1, j + i - 1 - (i - 1) d]; camax = Min[d, j + i - 1 - (i - 1)];
    tempb[j + i - 1] = 1/d Sum[tempa[j + i - 1 - k], {k, camin, camax}];
    Do[a[[nac1 + j, nac0 + j - k + 1]] = tempa[j + i - 1 - k]/(d tempb[j + i - 1]) (1 - expo[i]) can[i],
                  {k, camin, camax}]
                                          ];
   Do[tempa[k] = tempb[k], {k, Min[i d, bl]}]
                                              ];
      (*Anc,nc*)
  If[i == nc,
     Do[a[[nac1 + j, nac1 + j]] = N[expo[i] + (1 - expo[i]) arr[i]], {j, n[i + 1]}],
      (*Ai,i*)
     Do[a[[nac1 + j, nac1 + j]] = N[expo[i]], {j, Min[n[i + 1], n[i + 2] - d + 1]}];
   Do[a[[nac1 + n[i + 2] - d + 1 + j, nac1 + n[i + 2] - d + 1 + j]] =
          N[expo[i] + j/d (1 - expo[i]) arr[i]], {j, n[i + 1] - n[i + 2] + d - 1}];
      (*Ai,i+1*)
    Do[a[[nac1 + j, nac2 + k]] = N[1/d (1 - expo[i]) arr[i]], {j, n[i + 1]},
       {k, j, Min[j + d - 1, n[i + 2]]}]
                                      ]
                                        ];

  (*Transfer a into A in AX=B*)
linearmatrix = Transpose[a];
Do[linearmatrix[[j, j]] -= 1., {j, 1, summ}];
Do[linearmatrix[[1, j]] = 1., {j, 1, summ}];
```

```
(*The format of vector B*)
b = Array[itemb, sumn];
itemb[1] = 1.; Do[itemb[j] = 0., {j, 2, sumn}];

(*Solve systems equation*)
st = LinearSolve[linearmatrix, b];
```

$$stsum = \sum_{j=1}^{sumn} st[[j]];$$

```
t3 = Table[0, {i, 1, nc}, {j, 1, Min[bl, nc*d]}];
k = 1;
For[ i = 1, i ≤ nc, i++, Do[ k++; t3[[i, i + j]] = st[[k]], {j, 0, n[i + 1] - 1}]];

(*Determine distribution of YN*)
stateprob = Array[tempd, Min[bl, nc*d] + 1];
stateprob[[1]] = st[[1]];
Do[stateprob[[j]] = Sum[t3[[k, j - 1]], {k, nc}], {j, 2, Min[bl, nc*d] + 1}];

(*Expted total cost*)
```

$$exptcost = po \sum_{k=Min[c,d*nc]+2}^{Min[bl,d*nc]+1} (k - 1 - c)\ stateprob[[k]] + pu \sum_{i=1}^{Min[c,d*nc]+1} (c - i + 1)\ stateprob[[i]]$$

(* Main Program *)

```
(*Input*)
c = 20; d = 10; po = 1.3; pu = 1.0; λ = 0.3;

(*Starting Point of OL*)
ola = 1;

(*bl stands for booking limit*)
bla = nca = c + ola;
blb = ncb = bla + 1;
totcosta = expectedtotalcost[d, nca, bla, po, pu, c, λ];
totcostb = expectedtotalcost[d, ncb, blb, po, pu, c, λ];

While[totcosta>totcostb,
    bla=blb;blb=ncb=bla+1; totcosta=totcostb;
    totcostb=expectedtotalcost[d,ncb,blb,po,pu,c,λ]];
```

```
   (*Optimal OL Found*)
optimalol=bla-c;

   (*Output*)
optimalol
totcosta
```

ii. **Simulation program**

```
d = 10; λ = 0.3; timeclock = 28; c = 20; pu = 1.0; po = 1.3;
replicate = 2000;

replicateyn = Array[rpltyn, replicate];
replicatexn = Array[rpltxn, replicate];
replicateCost = Array[rpltCost, replicate];
averageyn = Array[aveyn, 10];
averageCost = Array[aveCost, 10];
averageun = Array[aveun, 10];
averageov = Array[aveov, 10];
averagenoofevt = Array[avenoofevt, 10];

For[bl = 22, bl ≤ 22, bl++,

 SeedRandom[10 bl];
 accumun = accumov = accunoofevt = frequn = freqov = 0;
 tow = Array[τ, bl];
 Do[τ[j] = (j - 1)/50, {j, 1, bl}];
 recorddemand = Array[redmd, bl];
```

```
For[j = 1, j ≤ replicate, j++,
   xn = 0; yn = 0; noofevt = 0;
   waittime = - 1/(λ + τ[xn + 1]) Log[1 - Random[Real]];
   temptime = timeclock - waittime;

   While[temptime ≥ 0,
      noofevt = noofevt + 1;
      arvlprob = λ/(λ + τ[xn + 1]);
      pseudevt = Random[Real];

      If[pseudevt ≤ arvlprob,                  (*arrival event*)
         pseudd = Random[Integer, {1, d}];
         temp = yn + pseudd;
         If[temp ≤ bl ,
            xn = xn + 1; yn = temp;
            redmd[xn] = pseudd],
          temp = Random[Integer, {1, xn}];   (*cancellation event*)
          ca = redmd[temp];
          redmd[temp] = redmd[xn];
          redmd[xn] = 0;
          xn = xn - 1;
          yn = yn - ca];

      waittime = - 1/(λ + τ[xn + 1]) Log[1 - Random[Real]];
      temptime = temptime - waittime];
    accunoofevt = accunoofevt + noofevt;
   rpltyn[j] = yn;
   rpltxn[j] = xn;
   If[yn ≤ c,
      rpltCost[j] = pu (c - yn); accumun = accumun + rpltyn[j];
               frequn = frequn + 1,
      rpltCost[j] = po (yn - c); accumov = accumov + rpltyn[j];
               freqov = freqov + 1]
                                 ];
```

```
averageyn[[bl - 20]] = N[ (Σ_{k=1}^{replicate} replicateyn[[k]]) / replicate ];
  averageCost[[bl - 20]] = (Σ_{k=1}^{replicate} replicateCost[[k]]) / replicate ;
  aveun[bl - 20] = accumun / frequn ;
  aveov[bl - 20] = accumov / freqov ;
  avenoofevt[bl - 20] = accunoofevt / replicate
                                                        ];
averageyn
averageCost
averagenoofevt
averageun
averageov
frequn
freqov
```