EXISTENCE OF NASH EQUILIBRIUM IN ATOMLESS GAMES

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Summary

This thesis focuses on atomless games in game theory.

In Chapter 1, we review the development of game theory in history and introduce the main results of this paper. Chapter 2 consists of the mathematical preliminaries needed in this thesis. Then, in Chapter 3, we introduce some basic elements of game theory, and provide the classical proof of the existence of Nash equilibrium in mixed-strategies. Also atomless games are introduced.

The new results of this thesis are included in Chapter 4 and Chapter 5, in which we discuss certain atomless games in details. Chapter 4 deals with games with private information. Based on our mathematical results on the set of distributions induced by the measurable selections of a correspondence with a countable range, we provide the purification results and also prove the existence of a pure strategy equilibrium for a finite game when the action space is countable but not necessarily compact.

Chapter 5 focuses on large games. We show the existence of equilibrium for a game with continuum of players with finitely many types, and with countable actions, where a player's payoff depends on the action distributions of all the players with the same type. We also consider another kind of large games with a continuum of small players and a compact action space, where the players' payoffs depend on their own actions and the mean of the transformed strategy profiles. Part of the results in Chapter 5 has been written into a journal paper [41] with Zhu Wei, which is to be published in an international journal – "Economic Theory".

Chapter 1

Introduction

1.1 History of Game Theory

Game theory is the study of multi-person decision problems. Generally, it can be divided into two kinds: cooperative games and non-cooperative games. The usual distinction between these two theories of game is whether there is some binding agreement. If yes, the game is cooperative; Otherwise, non-cooperative. The Nobel Prize of Economic Sciences in 1994 was awarded to three experts of game theory: Nash, Selten and Harsanyi. Their main contributions to game theory are the insightful studies in non-cooperative game. This paper also focuses on non-cooperative games.

Historically speaking, the study of game theory began with the publication of *The Theory of Games and Economic Behavior* by Von Neumann and Morgenstern in 1944. The 1950s was a period filled with excitement in game theory. During that time, cooperative game had developed some crucial concepts, for instance, bargaining models by Nash [24], core in cooperative games by Gillies [13] and Shapley [36], Shapley value by Shapley [37], etc. Around the same period when cooperative game research peaked in 1950s, non-cooperative game began to develop. For example, Tucker [40] defined prisoner's dilemma; Nash published two of his most important papers of non-cooperative games – [25] in 1950 and [26] in 1951. Their works laid the foundation for non-cooperative game theory. The sixties and seventies in last century were decades of growth in game theory. Extensions such as games of incomplete information (see, for example, Harsanyi [14], [15], [16]), the concept of subgame perfect Nash equilibrium (see, for example, Selten [34], [35]), etc. made the theory more widely applicable. Since 1980s, the concepts and models have become more specified and formulated. For example, Kreps, Milgrom, Roberts and Wilson [20] on incomplete information in repeated games, Radner and Rosenthal [27] on private information and existence of pure -strategy equilibria, Milgrom and Weber [23] on distributional strategies for games with incomplete information, Khan and Sun [18] on pure strategies in games with private information with countable compact action space.

Most models of game theory in economics were developed after 1970s. Since 1980s in last century, game theory has gradually become one part of mainstream economics, even forming the basis of micro-economics. Here, I would like to quote the words in *Games and Information* by Eric Rasmusen [28] to sum up the position of game theory in economics. He said:

Not so long ago, the scoffer could say the econometrics and game theory were like Japan and Argentina. In the late 1940s both disciplines and both economies were full of promise, poised for rapid growth and ready to make a profound impact on the world. We all know what happened to the economies of Japan and Argentina. Of the disciplines, econometrics became an inseparable part of economics, while game theory languished as a subdiscipline, interesting to its specialists but ignored by the profession as a whole. The specialists in game theory were generally mathematicians, who cared about definitions and proofs rather than applying the methods to *economics problems. Game theorists took pride in the diversity of disciplines to which their theory could be applied, but in none had it become indispensable.*

In the 1970s, the analogy with Argentina broke down. At the same time the Argentina was inviting back Juan Peron, economists were beginning to discover what they could achieve by combining game theory with the structure of complex economic situations. Innovation in theory and application was especially useful for situations with asymmetric information and a temporal sequence of actions,$\cdots$. During the 1980s, game theory became dramatically more important to mainstream economics. Indeed, it seemed to be swallowing up microeconomics just as econometrics had swallowed up empirical economics.

1.2 Main Results

The main purpose of my thesis work is to focus on some aspects in the recent development of game theory. The main contents include two parts–one deals with game with private information and countable action spaces, and the other focuses on large games.

Chapter 4 deals with games with private information. It is based on an article [18] by Khan and Sun. We show that in the game with diffuse and independent private information, purification of mixed-strategy equilibrium as well as pure-strategy equilibrium exists when the action spaces are countable but not necessarily compact. To prove the results, we also develop the distribution theory of correspondences taking values in a countable complete metric space.

Radner and Rosenthal pointed out in [27] that randomized strategies have limited appeal in many practical situations, and thus it is important to ask under what general conditions, pure strategy equilibrium exists. They showed both the purification of mixed-strategy equilibrium and the existence of pure strategy equilibrium for a game with finitely many players, finite action spaces, and diffuse and independent private information. However, as shown by an example in Khan, Rath and Sun [17] that there exists a two-player game with diffuse and independent private information and with the interval $[-1, 1]$ as their action space that has no equilibrium in pure strategies. This means that the result of the existence of pure strategy equilibrium of Radner and Rosenthal cannot be extended to general action spaces.

On the other hand, it has been shown in Khan and Sun [18] that the purification of mixed-strategy equilibriums together with a pure strategy equilibrium does exist in a finite game with diffuse and independent private information and with countable compact metric spaces as their action spaces. However, the requirement of compactness for a countable action space excludes some interesting cases, including the most commonly used countable space, the space of natural numbers.

It was suggested in the section of concluding remarks in [18] that one can work with compact-valued correspondences taking values in countable metric action spaces and tie in with the setting studied in Meister [22] to generalize Theorem 3 in [18] to the case of general countable metric action spaces. However, we notice that the proof of Theorem 2.1 in [22] has some problems.[1] This also motivates us to consider how the compactness assumption on the action spaces in Theorem 3 of [18] can be relaxed. As we look into the problem more carefully, we realize that it may not be

[1]Meister [22] applied Theorem 3.1 (DWW theorem) in Dvoretzky et al [10] incorrectly. The DWW theorem was used to purify a mixed-strategy whose values are probability measures with finite supports that may change with respect to the sample information points and are not contained in a common finite set. The latter condition, however, is a crucial condition in the DWW theorem.

so obvious to generalize Theorem 3 of [18] to the case of general countable metric action spaces. In fact, we need to work with countable *complete* metric action spaces (which clearly include the space of natural numbers) to show the existence of pure strategy equilibrium. With such settings, we also show the purification results. Without the completeness assumption or other related assumptions, we do not know whether the result still holds.

In Chapter 5, we work with large games. After introducing a simple large game model developed by Rath[29], we show the existence of equilibrium for a game with continuum of players with finitely many types, and with countable actions, where a player's payoff depends on the action distributions of all the players with the same type in Section 5.2.
The similar result with finite action spaces has been studied in Radner and Rosenthal [27], and that with countable metric action space has been shown in Khan and Sun [18]. However, as we mention above, it would be more general and applicable to take an infinite action space but not necessarily compact. Based on the results of the set of distributions induced by the measurable selections of a correspondence, we show the action spaces can set to be countable complete metric action spaces, which extends the similar results shown before.

Then we discuss large games with transformed summary statistics. Non-cooperative games with a continuum of small players and a compact action space in a finite dimensional space have been used in the study of monopolistic competitions (see, for example, Rauh [32] and Vives [42]). It is often assumed that the players' payoffs depend on their own actions and the summary statistics of the aggregate strategy profiles in terms of the moments of the distributions of players' actions. The existence of pure-strategy Nash equilibrium for such kind of games is shown in Rauh [31] under some restrictions.

In last section of Chapter 5, we reformulate the above model so that the players' payoffs depend on their own actions and the mean of the strategy profiles under a general transformation. The existence of pure-strategy Nash equilibrium is then shown. Our result covers the case when the payoffs depend on players' own actions and finitely many summary statistics as considered in Rauh [31]. It is more general than that of Rauh [31] in several aspects. First, our action space is a general compact metric space while the formulation in Rauh [31] requires the action space to be a compact set in a finite dimensional space. Second, we work with a general transformation rather than the special functions obtained by taking the composition of some univariate vector functions with projections. Third, we do not need the unnatural assumption on the strict monotonicity of some component of the univariate vector functions as in Rauh [31].

The existence of pure-strategy Nash equilibrium is shown in Rath [29] for large games with a compact action space in a finite dimensional space, where the payoffs depend on players' own actions and the mean of the aggregate strategy profiles.[2] This result does not extend to infinite-dimensional spaces (see Khan, Rath and Sun [17]) when the unit interval with Lebesgue measure is used to represent the space of players; such an extension is possible if the space of players is an atomless hyperfinite Loeb measure space (see Khan and Sun [19]). It is claimed in Rauh [31] that "All these results involve the mean and hence do not apply to monopolistic competition models with summary statistics different from the mean or several summary statistics". However, our formulation shows that monopolistic competition models can indeed be studied via the mean under some transformation.

[2]The case of a finite action space is discussed in Schmeidler [33].

Chapter 2

Mathematical Background

The main purpose of this chapter is to study some mathematical preliminaries which will be used in the following parts. After giving some notations and definitions, we study some properties of correspondence, fixed points, etc., and provide some basic theorems needed in game theory, or, at least in this thesis.

2.1 Some Definitions

2.1.1 Notation

$\mathbb{R}^n$ denotes the $n-$fold Cartesian product of the set of real numbers $\mathbb{R}$.
2^A denotes the set of all nonempty subsets of the set A.
$conA$ denotes the convex hull of the set A.
$proj$ denotes projection.
$\varnothing$ denotes the empty set.
$\bigotimes$ denotes product $\sigma-$algebra.
meas(X, Y) denotes the space of $(\mathcal{X}, \mathcal{Y})-$measurable functions for any two measurable spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$.

$A_\infty = A \cup \{\infty\}$ is a compactification of A.

If X is a linear topological space, its dual is the space X^* of all continuous linear functionals on X. If $q \in X^*$ and $x \in X$ the value of q at x is denoted by $q \cdot x$.

2.1.2 Definitions

The first term we want to emphasize is the concept of correspondence. Simply speaking, a correspondence is a set-valued function. That is, it associates to each point in one set a set of points in another set. The discussion to the correspondence arises naturally here since this paper is dedicated to discuss game theory. For instance, when we deal with non-cooperative games, the best-reply correspondence is one of the most important tools.

Now, we start with a formal definition of correspondence, then followed by the continuity of it.

Definition 1. Let X and Y be sets. A *correspondence* ϕ from X into Y assigns to each x in X a subset $\phi(x)$ of Y. Let $\phi : X \twoheadrightarrow Y$[1] be a correspondence. The *graph* of ϕ is denoted by $G_\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}$.

Just as functions have inverses, each correspondence $\phi : X \twoheadrightarrow 2^Y$ has two natural inverses:

- the *upper inverse* ϕ^u defined by $\phi^u(A) = \{x \in X : \phi(x) \subset A\}$;
- the *lower inverse* ϕ^l defined by $\phi^l(A) = \{x \in X : \phi(x) \bigcap A \neq \varnothing\}$.

Now, we can give the definition of different continuity of correspondences.

Definition 2. A correspondence $\phi : X \twoheadrightarrow Y$ between topological spaces is:

[1] ϕ can also be viewed as a function from X into the power set 2^Y of Y. For this reason, we also denote a correspondence from X to Y as $\phi : X \to 2^Y$.

Also, here we note that in this thesis we use notation "$\twoheadrightarrow$" instead of notation "$\to$" to differ correspondences with common functions.

• *upper hemicontinuous(or, upper semicontinuous)* at the point x if for every open neighborhood U of $\phi(x)$, the upper inverse image $\phi^u(U)$ is a neighborhood of $x \in X$.

• *lower hemicontinuous(or, lower semicontinuous)* at the point x if for every open set U satisfying $\phi(x) \bigcap U \neq \varnothing$, the lower inverse image $\phi^l(U)$ is a neighborhood of x.

• *continuous* if ϕ is both upper and lower hemicontinuous.

We now turn to the definition of measurable correspondences.

Definition 3. Let (S, Σ) be a measurable space and X a toplogical space (usually metrizable). A correspondence $\phi : S \twoheadrightarrow X$ is:

• *weakly measurable* if $\phi^l(G) \in \Sigma$ for each open subset G of X.

• *measurable* if $\phi^l(F) \in \Sigma$ for each closed subset F of X.

Commonly, let (T, τ, μ) be a complete, finite measure space, and X be a separable Banach space. We say the correspondence $\phi : X \to 2^Y$ has a *measurable graph* if $G_\phi \in \tau \otimes \beta(X)$, where $\beta(X)$ denotes the Borel $\sigma-$algebra on X.

Now, let G be a correspondence from a probability space $(T, \mathcal{T}, \nu)$ to a Polish space X. We say that G is a tight correspondence if for every $\varepsilon > 0$, there is a compact set K_ε in X such that the set $\{t \in T : G(t) \subset K_\varepsilon\}$ is measurable and its measure is greater than $1 - \varepsilon$.

We say that the collection $\{G_\lambda : \lambda \in \Lambda\}$ of correspondences is uniformly tight if for every $\varepsilon > 0$, there is a compact set K_ε in X such that the set $\{t \in T : \text{for all } \lambda \in \Lambda, G_\lambda(t) \subset K_\varepsilon\}$ is measurable and its measure is greater than $1 - \varepsilon$.

After giving these definitions of correspondences, we now introduce the definition of selector (or, selection) of a correspondence. A selector from a relation $R \subset X \times Y$ is a subset S of Y such that for every $x \in X$, there exists a unique $y_x \in S$ satisfying $(x, y_x) \in R$. We first give the formal definition of it.

Definition 4. A *selector* from a correspondence $\phi : X \twoheadrightarrow Y$ is a function $f : X \to Y$ that satisfies $f(x) \in \phi(x)$ for each $x \in X$.

Another important item related to the game we discuss here is the concept of fixed-point. When we deal with non-cooperative games, one way to prove the existence of an equilibrium is to prove the existence of the fixed point of a best-reply correspondence. We now give the definition of fixed point.

Definition 5. Let A be subset of a set X. The point x in A is called a *fixed point* of a function $f : A \to X$ if $f(x) = x$. Similarly, A *fixed point* of a correspondence $\phi : A \twoheadrightarrow X$ is a point x in A satisfying $x \in \phi(x)$.

2.2 Known Facts

We have developed the definition of correspondence and some related items already. Now we present some classical results which we will use later. Note that we do not give specific proofs and just state these known facts. For the details about proofs, one can refer any related book(see, for example, [1]). The reason we present them here without proofs is to make the main theorems and proofs in this paper more self-contained.

The first needed result is about the equivalence of compactness and sequential compactness of a metric space.

Theorem 2.2.1. *For a metric space the following are equivalent:*

1.The space is compact.

2.The space is sequentially compact. That is, every sequence has a convergent subsequence.

The next set of theorems are concerned with the properties of correspondence.

Lemma 2.2.2. *(Uhc Image of a Compact Set) The image of a compact set under a compact-valued upper hemicontinuous correspondence is compact.*

When we deal with upper hemicontinuity of a correspondence, we can often transfer to prove it to be closed graph providing the following theorem.

Theorem 2.2.3. *(Closed Graph Theorem) A closed-valued correspondence with compact Hausdorff range space is closed if and only if it is upper hemicontinuous.*

From the definition of upper hemicontinuity, we can have some other ways to assert the upper hemicontinuity of a correspondence. The next theorem characterize upper hemicontinuity of correspondences.

Theorem 2.2.4. *(Upper Hemicontinuity) For $\phi : X \twoheadrightarrow Y$, the following statements are equivalent.*

1. ϕ is upper hemicontinuous.

2. $\phi^u(O)$ is open for each open subset O of Y.

3. $\phi^l(V)$ is closed for each closed subset V of Y.

The next theorem states that the set of solutions to a well behaved constrained maximization problem is upper hemicontinuous in its parameters and that the value function is continuous.

Theorem 2.2.5. *(Berge's Maximum Theorem) Let $\phi : X \twoheadrightarrow Y$ be a continuous correspondence with nonempty compact values, and suppose $f : X \times Y \to \mathbb{R}$ is continuous,. Define the "value function" $m : X \to \mathbb{R}$ by*

$$m(x) = \max_{y \in \phi(x)} f(x, y),$$

and the correspondence $\mu(x) : X \twoheadrightarrow Y$ of maximizers by

$$\mu(x) = \{y \in \phi(x) : f(x, y) = m(x)\}.$$

Then the value function m is continuous, and the "$\arg\max$" correspondence μ is upper hemicontinuous with compact values.

Now, we come to the measurability of a correspondence. We have given the definition of both measurability and weak measurability. In fact, for metric spaces, weak measurability is weaker than measurability, but not so much weaker. The next theorem shows that for compact-valued correspondences the two definitions coincide.

Theorem 2.2.6. *(Measurability VS Weak Measurability) For a correspondence $\phi : (S, \Sigma) \twoheadrightarrow X$ from a measurable space into a metrizable space:*

1. If ϕ is measurable, then ϕ is also weakly measurable.

2. If ϕ has compact values, then ϕ is measurable if and only if it is weakly measurable.

Another theorem is used to assert a measurable correspondence as follows.

Theorem 2.2.7. *Let $(T, \mathcal{T})$ be a measurable space, X a separable metrizable space, U a metrizable space and $\phi : T \times X \twoheadrightarrow U$. We suppose that ϕ is measurable in t and continuous in x. Then ϕ is measurable.*

Viewing relations as correspondences, we know that only nonempty-valued correspondences can admit selectors, and nonempty-valued correspondences always admit selectors. Recall the definition of selector. Similarly to that definition, a measurable selector from a correspondence $\phi : S \twoheadrightarrow X$ between measurable spaces is a measurable function $f : S \in X$ satisfying $f(s) \in \phi(s)$. We now state the main selection theorem for measurable correspondences.

Theorem 2.2.8. *(Kuratowski-Ryll-Nardzewski Selection Theorem) A weakly measurable correspondence with nonempty closed values form a measurable space into a Polish space admits a measurable selector.*

When we deal with the existence of equilibrium, one of those most basic way is to use fixed-point theorem to assert that. As long as the game theory begins to develop, the Brouwer fixed-point theorem is used by Von Neumann to prove the basic theorem in the theory of zero-sum, two-person games. Nash also used Kakutani fixed-point theorem to prove the existence of so called Nash equilibrium.[2] In some infinite dimensional cases, we may refer to Fan-Glicksberg fixed-point theorem to prove needed existence results.[3] And when we deal with the existence of equilibrium in this thesis, we also make quite lots of use of these fixed-point theorems. So, we would like to end this chapter with the following set of different versions of the fixed-point theorem.

Theorem 2.2.9. *(Brouwer Fixed-point Theorem)Let $f(x)$ be a continuous function defined in the $N-$dimensional unit ball $|x| \leq 1$. Let $f(x)$ map the ball into itself: $|f(x)| \leq 1$ for $|x| \leq 1$. Then some point in the ball is mapped into itself: $f(x_0) = x_0$.*

Theorem 2.2.10. *(Kakutani Fixed-point Theorem)Let X be a closed, bounded, convex set in the real $N-$dimensional space $\mathbb{R}^N$. Let the correspondence $\phi : X \twoheadrightarrow X$ be upper semicontinuous and have nonempty convex values. Then the set of fixed points of ϕ is nonempty, that is, some points $x^* \in \phi(x^*)$.*

The following theorem is just a infinite dimensional version of Kakutani fixed-point theorem.

Theorem 2.2.11. *(Fan-Glicksberg Fixed-point Theorem) Let K be a nonempty compact convex subset of a locally convex Hausdorff space, and let the correspondence $\phi : K \twoheadrightarrow K$ have closed graph and nonempty convex values. Then the set of fixed points of ϕ is compact and nonempty.*

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[2] One can refer to Nash [25].

[3] See, for example, Khan and Sun [18].

Chapter 3

Basic Game Theory

We start by describing a finite game[1] in Section 3.1. Section 3.2 is devoted to reviewing the theory of Nash equilibrium and the basic existence result. Section 3.3 discusses briefly state the setting of atomless games, which will be discussed with more details in Chapter 4 and Chapter 5.

3.1 Description of a Game

When we talk about a game, the essential elements of a game are players, actions, payoffs, and information. These elements are often called the *rules of the game*. In a game, each player is assumed to try maximize his payoffs, so he will take some plans known as strategies that make actions depending on the information faced

[1]In game theory, a game can be expressed into two different ways: normal (or strategic) form representation and extensive form representation. Although theoretically, these two representations are almost equivalent, the former one is more convenient for us to discuss static games, and last one is more useful in dynamic games. To enable a self-contained and yet concise treatment, we only present the game in normal form and discuss the properties of such expression in this thesis since we restrict our discussion to static games.

to him. The combination of strategies chosen by each player is known as the equilibrium. And that will lead to a particular result, which is called the outcome of a game. So, the basic concepts of game include *player, action, information, strategy, payoff, outcome and equilibrium.* In the following, we first describe these elements of a simple finite game(i.e., both the number of players and their actions set are finite and there is no other restrictions such as private information, etc., which will be discussed later). Note again that the analysis in this paper is restricted to games in normal form.

1. *Players* are the individuals that make decisions. In game, the goal of each player is to maximize his payoff by choosing his own action. We assume the number of the players is n and denote each player as i, $(i = 1, \cdots, n)$ and the set of players as I.

2. An *Action (or move)* of player i, say, a_i is a choice the player can make. Then, player i's *action set* A_i is the set of all actions available to him. And an *action combination* is an $n-$vector $a = (a_1, \cdots, a_n)$, of one action for each of the players in the game.

3. *Information* is the players' knowledge of the game. We will give more specific definition of it in the following chapters. Here, we use T_i to denote the information set of player i.

4. The *strategy* of player i, denoted by s_i, is a rule for player i to choose his action. Player i's *strategy space* $S_i = s_{i1}, \cdots, s_{iK}$ is the set of strategies available to him. And a vector $s = (s_1, \cdots, s_n)$ is called *strategy profile.* The set of these strategy profiles in the game is thus the cartesian product $S = \times_i S_i$, which is called

the *strategy space* of the game.

5. The *Payoff* of player i, denoted by $U(s_1, \cdots, s_n)$, is the expected utility he gets as a function of the strategies chosen by himself and the other players.[2]

6. The *outcome* of the game is a set of elements that one picks from the values of actions, payoffs, and other variables after the game is played out.

7. An *equilibrium* $s^* = (s_1^*, \cdots, s_n^*)$ is a strategy combination consisting of a best strategy for each player in the game.

8 The *best response* of player i to strategies s_{-i}[3] chosen by the other players is the strategy s_i^* the maximize his payoff; that is,

$$U_i(s_i^*, s_{-i}) \geq U_i(s_i', s_{-i}), \forall s_i' \neq s_i^*.$$

Till now, we have outlined most elements of a game. Normally, the analysis of games involves different types of strategies:

(1) A *pure strategy* is for each player i, to choose his action $s_i \in S_i$ for sure given the information he learns. More specifically, a pure strategy can be expressed as a measurable function $p_i : T_i \to A_i$[4]. In this case, the payoff function U_i of player i

[2] In economics, the payoffs are usually firms' profits or consumer's utility.

[3] Here, it means "all the other players' strategies", which follows usual shorthand notation in game theory. For any vector $x = (x_1, \cdots, x_n)$, we denote the vector $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ by x_{-i}.

[4] Strategy and action are two different concepts: strategy is the rule of action but not action itself. But, in static games, strategy is just the same as action. Thus, The pure strategy space is just A in our discussion. So, in the following discussions in this chapter, we do not distinguish s_i with a_i or S_i with A_i

is a function of $s(a)$; and for any given s, the value of U_i is fixed.

(2) A *behavior strategy*[5]of player i is when player i observe some information, he selects a action $a_i \in A_i$ randomly. More specifically, a behavior strategy strategy for player i is a function $\beta_i : A_i \times T_i \to [0, 1]$ with two properties: (a) For every $B \in A_i$, the function $\beta_i(B, \cdot) : T_i \to [0, 1]$ is measurable; (b) For every $t_i \in T_i$, the function $\beta_i(\cdot, t_i) : A_i \to [0, 1]$ is a probability measure.

(3) A *mixed strategy*[6]for player i is a probability distribution over his pure strategy set S_i of pure strategies given certain information. To differ from pure strategies, we now denote mixed strategies for player i as $sigma_i$ rather than s_i. More specifically, a mixed strategy σ_i for player i is a measurable function $\sigma_i : [0, 1] \times T_i \to A_i$. Thus, the mixed strategy of player i can be expressed as $\sigma_i = (\sigma_{i1}, \cdots, \sigma_{iK})$, where $\sigma_{ik} = \sigma(s_{ik})$ is the probability for player i to choose strategy s_{ik}, $\forall k = 1, \cdots, K, 0 \leq \sigma_{ik} \leq 1, \sum_1^K \sigma_{ik} = 1$. We use Σ_i to denote the mixed strategy space for player i(that is, $\sigma_i \in \Sigma_i$, where σ_i is one of the mixed strategies of player i). The vector $\sigma = (\sigma_1, \cdots, \sigma_n)$ is called a *mixed strategy profile* and cartesian product $\Sigma = \times_i \Sigma_i$ represents *mixed strategy space*$(\sigma \in \Sigma)$. The support of a mixed strategy σ_i is the set of pure strategies to which σ_i assigns positive probability. In finite case, for a mixed strategy profile σ, player i's payoff is $\sum_{s \in S}(\prod_{j=1}^I \sigma_j(s_j))U_i(s)$,

[5]Here, we only give a simple description of behavior strategy, since it is used more in dynamic games. In fact, although behavior strategy and mixed strategy are two different concepts, Kuhn(1953) proves that in games of perfect recall, both are equivalent. More details about the equivalence between mixed and behavior strategies under perfect recall is discussed in page 87-90 of Fudenberg and Tirole [11]

[6]From these definitions, we can see that pure strategy can be understood as the special case of mixed strategy. For instant, pure strategy s'_i is equivalent to the mixed strategy $\sigma_i(1, 0, \cdots, 0)$, which means, for player i, the probability of choosing s'_i is 1, probabilities of choosing any other pure strategies is 0.

which is still denoted as $U_i(\sigma)$ in a slight abuse of notation.[7]

As we talk about a game, one of the most important concepts is the notion of Nash equilibrium. And we will discuss such equilibrium in the next section with more details.

3.2 Nash Equilibrium

In essence, Nash equilibrium requires that a strategy profile $\sigma \in \Sigma$[8] should not only be such that each component strategy σ_i be optimal under some behalf of player i about the others' strategies σ_{-i}, but also should be optimal under the belief that σ itself will be played.

In terms of best response, a (mixed) strategy profile $\sigma \in \Sigma$ is a *Nash equilibrium* a best response to itself. More specifically, $\sigma^* = (\sigma_1^*, \cdots, \sigma_n^*)$ is a Nash equilibrium if for any player i, $i = 1, 2, \cdots, n$), one have ,

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*), \forall \sigma_i \in \Sigma_i.$$

The existence of Nash equilibrium was first established by Nash [25]. The progresses about the existence of equilibrium in different games after Nash's work are often still based on the techniques that Nash attempts. So we provide here both the theorem and the proof of the existence of Nash equilibrium which are stated in Nash [25]. The idea of the proof is to apply Kakutani's fixed-point theorem to the players' "reaction correspondences" which are defined in proof.

Theorem 3.2.1. *(Nash, 1950) There exists at least one Nash equilibrium(pure or mixed) for any finite game.*

[7] Note that the payoff $U_i(\sigma)$ of player i is linear function of player i's mixing probability σ_i.

[8] We keep the notation consistently with the last section. Note again that σ means a mixed strategy profile.

Proof: We use $r_i(\sigma)$ to represent the "reaction correspondences" of i, which maps each strategy profile σ to the set of mixed strategies that maximize player i's payoff when others play σ_{-i}. Define the correspondence $r : \Sigma \twoheadrightarrow \Sigma$ to be the Cartesian product of the r_i. If there exists a fixed point $\sigma^* \in \Sigma$ such that $\sigma^* \in r(\sigma^*)$ and for each i, $\sigma_i^* \in r_i(\sigma^*)$, then this fixed point is a Nash equilibrium by the construction. So, our task now is to show all the conditions of Kakutani fixed-point are satisfied.

First note that each Σ_i is a probability space, so it is a simplex of dimension $(J-1)$, where J is the number of pure strategies of player i. This means, Σ_i (so is Σ) is compact, convex and nonempty.

Second, as we noted before, each player's payoff is linear, and therefore continuous in his own mixed strategy. So $r_i(\sigma)$ is non-empty since continuous functions on compacts always can attain maxima.

Moreover the linearity of payoff function means: if $\sigma' \in r(\sigma)$ and $\sigma'' \in r(\sigma)$, then $\lambda\sigma' + (1-\lambda)\sigma'' \in r(\sigma)$, where $\lambda \in (0,1)$(that just means, if both σ_i' and σ_i'' are best responses to σ_{-i}, then so is their weighted average). So, $r(\sigma)$ is convex.

Finally, to show $r(\sigma)$ is upper hemi-continuous we need to show that $r(\sigma)$ has closed graph, i.e., if $(\sigma^m, \tilde{\sigma}^m) \to (\sigma, \tilde{\sigma}), \tilde{\sigma}^m \in r(\sigma^m)$, then $\tilde{\sigma} \in r(\sigma)$. Assume there is a sequence $(\sigma^m, \tilde{\sigma}^m) \to (\sigma, \tilde{\sigma}), \tilde{\sigma}^m \in r(\sigma^m)$, but $\tilde{\sigma} \notin r(\sigma)$. Then, $\tilde{\sigma}_i \notin r_i(\sigma)$ for some i. Thus, there is a $\varepsilon > 0$ and a σ_i' such that $U_i(\sigma_i', \sigma_{-i}) > U_i(\tilde{\sigma}_i, \sigma_{-i}) + 3\varepsilon$. And since U_i is continuous, and $(\sigma^m, \tilde{\sigma}^m) \to (\sigma, \tilde{\sigma})$, when m is large enough, we have

$$U_i(\sigma_i', \tilde{\sigma}_{-i}^m) > U_i(\sigma_i', \tilde{\sigma}_{-i}) - \varepsilon > U_i(\tilde{\sigma}_i, \sigma_{-i}) + 2\varepsilon > U_i(\tilde{\sigma}_i^m, \sigma_{-i}^m) + \varepsilon.$$

Hence, $\tilde{\sigma}_i^m \notin r_i(\sigma^m)$, which contradicts the assumption we made. So, $r(\sigma)$ is upper hemi-continuous.

Since all the conditions of Kakutani fixed-point theorem are satisfied, the result follows. $\square$

3.3 Atomless Games

On one hand, when we apply $n-$person game theory to economic analysis, it often becomes a problem that small games (i.e., games with a small number players) are hardly adequate to represent free-market situations. In this attempt, games with such a large number of players that any single player have a negligible effect on the payoffs to the other players are set to be atomless player space. For example, we can use the number of points on a line (for example, the unit interval, $[0, 1]$.) On the other hand, when we deal with finite games with infinite (countable) actions and private information, as we do in Chapter 4, the setting is also tied with atomless measure as a model of diffuse information. We call the games which are set with atomless property as atomless games. So far, we still need the following definitions.

Definition 6. A measurable set S is a null set for the measure μ if $\mu(S') = 0$ for every measurable $S' \subset S$. An atom of the measure μ is a measurable non-null set S such that, for every measurable $S' \subset S$ we have either S' is a null set or $\mu(S') = \mu(S)$.

Definition 7. If the measure μ has no atom, it is called atomless.

In the following chapters, we will discuss atomless games with more details. In Chapter 4, we discuss finite player games with countable action set and with informational constraints, where we also make the assumption of diffuse information with atomless measure. In Chapter 5, we deal with large games, where we assume I be the set of players, $\mathcal{I}$ be a $\sigma-$algebra of subsets of I, and λ be an atomless probability measure on I (Chapter 5).

Chapter 4

Games with Private Information and Countable Actions

Games with private information (or imperfect information,incomplete information) attracts a lot of attention in recent decades. It seems to be more practical to give a appropriate situation that players make their decisions depending on the observation of a certain information variable. As to such games, some interests concern with the problem that whether there exists an equilibrium point or at least an approximate equilibrium in pure strategies, if the game has an equilibrium in mixed strategies. Radner and Rosenthal [27] and Milgrom and Weber [23] deal with such problem together with the purification of a mixed strategy equilibrium under the assumption of finite action spaces, with diffuseness and independence of information, suitably formalized; The results with finite action sets also see those in Aumann et al [6].

However, it would be more general and applicable to take a infinite action space. Khan and Sun [18] extends the result into the case that action space can be chosen as a countable compact metric space. In this paper, we show that the compactness can be removed in our case. In fact, the idea of setting the action space without compact restriction is mentioned in the concluding remarks of Khan and Sun [18]. As we show in the introduction, we realize that it may not be so obvious to generalize the model and results in Khan and Sun [18] to the case of general countable metric action spaces. In fact, we need to work with countable *complete* metric action spaces (which clearly include the space of natural numbers) to show the existence of pure strategy equilibrium. With such settings, we also show the purification results. Without the completeness assumption or other related assumptions, we do not know whether the result still holds.

The organization of this chapter is as follows. In Section 4.1, we provide our mathematical results. More specifically, we work on the set of distributions induced by the measurable selections of a correspondence with a countable range by using the Bollobás and Varopoulos extension of the marriage lemma. In section 4.2, we discuss a typical kind of games with a finite number of players, a countable action set, and private information constraints . And we prove the purification results of behavior strategy equilibria and the existence of a pure strategy equilibrium in such games.

4.1 Distribution of an Atomless Correspondence

This section introduces some results that lead to a fairly general treatment to the games that we discuss later.

A denotes a countable complete metric space; $(T, \mathcal{T}, \lambda)$ denotes an atomless probability space. Let $\{a_i : i \in \mathbb{N}\}$ be a list of all the elements of A. Let F be a correspondence from T to A, where F is measurable if for each $a \in A$, $F^{-1}(a) = \{t \in T : a \in F(t)\}$ is measurable. For any F, let

$$\mathcal{D}_F = \{\lambda f^{-1} : f \textit{ is a measurable selection of } F\}.$$

We now state first a special case of the continuous version of the marriage lemma offered by Bollobás and Varopoulos [9].

We present it with our own notation. Let $(T_\alpha)_{\alpha\in I}$ be a family of sets in $\mathcal{T}$, and $\Lambda = (\tau_\alpha)_{\alpha\in I}$ be a family of non-negative numbers, I a countable index set. We call $(T_\alpha)_{\alpha\in I}$ is $\Lambda-$representable[1], if there is a family $(S_\alpha)_{\alpha\in I}$ of sets in $\mathcal{T}$ such that for all $\alpha, \beta \in I, \alpha \neq \beta$, $S_\alpha \subseteq T_\alpha$, $\lambda(S_\alpha) = \tau_\alpha, S_\alpha \cap S_\beta = \emptyset$.

Theorem 4.1.1. *$(T_\alpha)_{\alpha\in I}$ is $\Lambda-$representable if and only if*

$$\lambda(\cup_{\alpha\in I_F} T_\alpha) \geq \Sigma_{\alpha\in I_F}\tau_\alpha$$

for all finite subsets I_F of I.

We first state our main selection theorem for countable vectors.

Theorem 4.1.2. *Let $(T, \mathcal{T}, \lambda)$ be a atomless probability space; and $f_\alpha \in Meas(T, \mathbb{R}_+), \alpha \in I$, where I is a countable index set, such that for all $t \in T$, $\Sigma_{\alpha\in I} f_\alpha(t) = 1$. Then, there exist measurable functions $f^*_\alpha \in Meas(T, \{0, 1\}), \alpha \in I$, such that $\Sigma_{\alpha\in I} f^*_\alpha(t) = 1$ for all $t \in T$ and*

$$\int_T f_\alpha(t) d\lambda(t) = \int_T f^*_\alpha(t) d\lambda(t) \quad for \quad all \quad \alpha \in I.$$

Proof: First, we take $(T_\alpha)_{\alpha\in I}$ in Theorem 4.1.1 by choosing $T_\alpha = T$ for all $\alpha \in I$. Then we take $\tau_\alpha = \int_T f_\alpha(t) d\lambda(t)$ for all $\alpha \in I$. I is countable. Let $\Lambda = (\tau_\alpha)_{\alpha\in I}$. Clearly, we have

$$\lambda(\cup_{\alpha\in I_F} T_\alpha) = \lambda(T) = 1,$$

which is always bigger or equal to $\Sigma_{\alpha\in I_F}\tau_\alpha$ for all finite subsets I_F of I.

Then, we can apply Theorem 4.1.1 to assert that $(T_\alpha)_{\alpha\in I}$ is $\Lambda-$representable. That is, there is a set of sets $(S_\alpha)_{\alpha\in I}$ in $\mathcal{T}$ such that for all $\alpha, \beta \in I, \alpha \neq \beta$, $S_\alpha \subseteq T$, $\lambda(S_\alpha) = \tau_\alpha, S_\alpha \cap S_\beta = \emptyset$.

[1] As to certain examples that are $\Lambda-$representable, one can refer to the constructions used in the proofs of Theorem 4.1.2 and 4.1.4

That is,

$$\int_T f_\alpha(t)d\lambda(t) = \lambda(S_\alpha) = \int_T f^*_\alpha(t)d\lambda(t) \quad for\ all\ \alpha \in I,$$

where $f^*_\alpha(t)$ is characteristic function of S_α. $\square$

We now present another theorem which can be viewed as a corollary of the above theorem to cover some purification results for atomless games that we used later.

Theorem 4.1.3. *Let $(T, \mathcal{T}, \lambda)$ be a atomless probability space; A a countable metric space represented as $\{a_1, a_2, ...\}$; I a countable index set; and $g \in Meas(T, \mathcal{M}(A))$. Let $g(t; B)$ represent the value of the probability measure $g(t)$ at $B \subseteq A$ and $g(t; da)$ the integration with respect to it. Then there exists $g^* \in Meas(T, A)$ such that,*

(1) for all $B \subseteq A$, $\int_T g(t; B)d\lambda(t) = \lambda g^{*-1}(B)$;

(2) $g^*(t) \in \{a_i \in A : g(t; \{a_i\} > 0\} \equiv supp\ \ g(t)$ for $\lambda-$almost all $t \in T$.

Proof: By applying Theorem 4.1.2 to $g_i(t) = g(t, \{a_i\})$, we can assert the existence of functions $g^{*'}_i(t)$, such that

$$\int_T g_i(t)d\lambda(t) = \lambda(S_i) = \int_T g^{*'}_i(t)d\lambda(t) \quad for\ all\ \imath \in I, \tag{4.1}$$

where $(S_i)_{i\in I}$ is a family of countable partitions of T, that is, $(S_i)_{i\in I}, S_\alpha \cap S_\beta = \emptyset$, for all $\alpha, \beta \in I, \alpha \neq \beta$; and $g^{*'}_i(t)$ is the characteristic function of S_i.

Now we define $g^*(t, a_i) = a_i g^{*'}(t, a_i)$. Then,we assert this $g^* \in Meas(T, A)$ is just what we need. Note that $g^{*'}(t, a_i) = 1_{\{a_i\}}(g^*(t, a_i))$.

(1) Since A is countable, B is a subset of A,

$$\int_T g(t; B)d\lambda(t) = \sum_{a_i\in B} \int_T g(t; a_i)d\lambda(t) = \sum_{a_i\in B} \lambda(S_i).$$

And according to the definition of g^*, we can get

$$g^{*-1}(t; a_i) = a_i g^{*'-1}(t; a_i) = \lambda(S_i).$$

Now one can see that,

$$\int_T g(t;B)d\lambda(t) = \sum_{a_i \in B} g^{*-1}(t;a_i) = \lambda g^{*-1}(B);$$

(2)From the equation (4.1.1) and the definition of g^*, we can assert the conclusion directly. $\square$

The next result is about the convexity of the distribution of an atomless correspondence.

Theorem 4.1.4. *For any F, $\mathcal{D}_F$ is convex in the space $\mathcal{M}(A)$.*

Proof: One can pick up ι_1, ι_2 from $\mathcal{D}_F$ and $\alpha \in [0,1]$. According to the definition of $\mathcal{D}_F$, there are measurable selections f_1 and f_2 of F satisfying $\lambda f_1^{-1} = \iota_1$ and $\lambda f_2^{-1} = \iota_2$. Define $\tau_i = \tau(\{a_i\}) = \alpha\lambda f_1^{-1}(a_i) + (1-\alpha)\lambda f_2^{-1}(a_i)$, where $a_i \in A$ for any $i \in I$. I is countable. Let $\Lambda = (\tau_\alpha)_{\alpha \in I}$.

We can easily obtain that τ is a probability on T: $1 = \lambda(T) = \sum_{i\in I}\tau_i$. Take $T_i = f_1^{-1}(a_i) \cup f_2^{-1}(a_i)$. Then for any finite subset I_F of I,

$$\bigcup_{i\in I_F} T_i = (\cup_{i\in I_F} f_1^{-1}(a_i)) \bigcup (\cup_{i\in I_F} f_2^{-1}(a_i)).$$

Therefore

$$\lambda(\bigcup_{i\in I_F} T_i) \geq \max\{\lambda(\cup_{i\in I_F} f_1^{-1}(a_i)), \lambda(\cup_{i\in I_F} f_2^{-1}(a_i))\}$$

which implies

$$\lambda(\bigcup_{i\in I_F} T_i) \geq \alpha\lambda(\cup_{i\in I_F} f_1^{-1}(a_i)) + (1-\alpha)\lambda(\cup_{i\in I_F} f_2^{-1}(a_i)) = \sum_{i\in I_F}\tau_i.$$

Applying Theorem 4.1.1, we know that $(T_\alpha)_{\alpha\in I}$ is $\Lambda-$representable. That is, we can get a family $(S_i)_{i\in I}$ of subsets of T such that for $i, j \in I$ with $i \neq j$, $S_i \subset T_i$,

$\lambda(S_i) = \tau_i$, and $S_i \cap S_j = \emptyset$.[2]

Now we define $f(t) = \sum_{i \in I} a_i 1_{S_i}(t)$. Clearly, it is also a selection of F. Therefore $\alpha\iota_1 + (1-\alpha)\iota_2 \in \mathcal{D}_F$, and we reach the conclusion. □

The following lemma is a modification of Lemma 1 in [18]. Instead of the compactness condition on the countable action space, we only require a non-emptiness condition.

Lemma 4.1.5. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions from T to a countable metric space A such that $\tau_n = \lambda f_n^{-1}$ converges weakly to a probability measure τ on A as $n \to \infty$. Let $F(t) \equiv$ cl-Lim$\{f_n(t)\}$, the set of all limit points of the sequence $\{f_n(t)\}_{n=1}^{\infty}$. If $F(t)$ is nonempty for all t, then, there exists a measurable selection f of F such that $\lambda f^{-1} = \tau$ for each $t \in T$.*

Although we drop the compactness condition on the action space A, the proof of this lemma is the same as that of Lemma 1 in Khan and Sun [18], provided that $F(t) \neq \emptyset, \forall t$. Thus we skip it. Note that the proof in Khan and Sun [18] also uses Lemma 2.1.

Theorem 4.1.6. *If F is compact valued, then $\mathcal{D}_F$ is compact in $\mathcal{M}(A)$.*

Proof: Let A_∞ be a compactification of A. Note that $\mathcal{M}(A_\infty)$ is a compact metric space. So for any sequence $\{\mu_n\}_{n=1}^{\infty}$ from $\mathcal{D}_F \subset \mathcal{M}(A_\infty)$, there is a convergent subsequence. Without loss of generality, we assume $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to a probability measure μ on A_∞. From the definition of $\mathcal{D}_F$, one can pick up a sequence $\{f_n\}_{n=1}^{\infty}$ of measurable selections of F such that $\lambda f_n^{-1} = \mu_n$ for each $n \geq 1$.

[2] In fact, here we can another proof using a little trick similar to the proof of Theorem 4.1.2. The idea is as follows: First observe $1 = \lambda(T) = \sum_{i \in I} \tau_i$. Then one can follow the steps attempted in theorem 4.1.2, and get $\lambda(S_i) = \tau_i$, where $(S_i)_{i \in I}$ is a family of subsets of T satisfying $\alpha, \beta \in I, \alpha \neq \beta$, $S_\alpha \subseteq T$, $\lambda(S_\alpha) = \tau_\alpha, S_\alpha \cap S_\beta = \emptyset$.

According to Proposition 3.8 in Sun [38], $\{\mu_n : n = 1, 2, \dots\}$ is tight. That is , for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset A$, such that $\mu_n(K_\varepsilon) \geq 1 - \varepsilon$ for all n. Since K_ε is also compact in A_∞, the weak convergence of $\{\mu_n\}$ to μ implies that $\mu(A) \geq \mu(K_\varepsilon) \geq 1-\varepsilon$. Let ε tends to zero yielding $\mu(A) \geq 1$. So μ is concentrated on A, i.e., $\mu \in \mathcal{M}(A)$. Now that all the μ_n and μ_0 are concentrated on A, the weak convergence of μ_n to μ_0 in $\mathcal{M}(A_\infty)$ is equivalent to the weak convergence in $\mathcal{M}(A)$.[3]

Define G to be $G(t) = cl\text{-Lim}\{f_n(t) :\}$, which is nonvoid and included in $F(t)$, because all the sequence $f_n(t)$ is from the compact set $F(t)$ for each t. The preceding lemma yields that there exists a measurable selection f from G such that $\mu = \lambda f^{-1}$, in other words, $\mu \in \mathcal{D}_G \subset \mathcal{D}_F$. Therefore $\mathcal{D}_F$ is compact. □

Now we turn to investigate the upper semicontinuity of the distribution of a correspondence depending on a parameter.

Theorem 4.1.7. *Assume that for each fixed y in Y, a metric space, $G(\cdot, y)$ (which is also denoted by G_y) is a measurable correspondence from T to A, and for each fixed $t \in T$, $G(t, \cdot)$ is upper semicontinuous on Y. Also, assume that there exists a compact valued correspondence H from T to A such that $G(t, y) \subset H(t)$ for all t and y. Then $\mathcal{D}_{G_y}$ is upper semicontinuous on Y.*

Proof: By Theorem 2.2.4[4], in order to show $\mathcal{D}_{G_y}$ is upper semicontinuous on Y, it suffices to show that $\mathcal{D}_G^{-1}(V) \equiv \{y : \mathcal{D}_{G_y} \cap V \neq \emptyset\}$ is closed in Y for each closed subset V of $\mathcal{M}(A)$. Towards this end, suppose $\{y_n\}_{n\geq 1}$ is a sequence from $\mathcal{D}_G^{-1}(V)$ which converges to $y_0 \in Y$. By the definition, for each $n \geq 1$, there exist a measures

[3]In fact, for each open subset O_∞ of A_∞, $\limsup_n \mu_n(O_\infty) \geq \mu_0(O_\infty)$. So $\limsup_n \mu_n(O_\infty \cap A) \geq \mu_0(O_\infty \cap A)$, i.e., for each open subset $O \subset A$, $\limsup_n \mu_n(O) \geq \mu_0(O)$. Hence we get the weak convergence of μ_n to μ_0 in $\mathcal{M}(A)$.

[4]Also, see Lemma 14.4 in Aliprantis and Border [1]

$\mu_n \in V$ and a measurable selection g_n of $G(\cdot, y_n)$, such that $\mu_n = \lambda g_n^{-1}$. Note that $\mathcal{M}(A_\infty)$ is compact and $\mu_n \in \mathcal{M}(A_\infty)$. So there is a subsequence of $\{\mu_n\}$, say itself without loss of generality, converging weakly to some $\mu_0 \in \mathcal{M}(A_\infty)$. Since a compact valued correspondence H includes $G(\cdot, y)$ for all y, as in the proof of the preceding theorem, we have $\mu_0 \in \mathcal{M}(A)$ and μ_n converges weakly to μ_0 in $\mathcal{M}(A)$. Therefore $\mu_0 \in V$ since V is closed in $\mathcal{M}(A)$. Also Lemma 4.1.5 implies that there exists a measurable selection g of the correspondence $F \equiv cl\text{-Lim}\{g_m : m \geq 1\} \subset H$ such that $\lambda g^{-1} = \mu_0$. From the upper semicontinuity of $G(t, \cdot)$ for $t \in T$, we obtain $cl\text{-Lim}G(t, y_n) \subseteq G(t, y_0)$ for all $t \in T$. So for each $t \in T$, $F(t) \subseteq G(t, y_0)$. Thus $g(\cdot)$ is a measurable selection of $G(\cdot, y_0)$. Therefore $\mu_0 \in \mathcal{D}_{G_{y_0}}$. So $\mathcal{D}_{G_{y_0}} \cap V \neq \emptyset$ for it contains μ_0. This means that $y_0 \in \mathcal{D}_G^{-1}(V)$. Therefore $\mathcal{D}_G^{-1}(V)$ is indeed closed and we obtain the expected results. $\square$

4.2 Games with Private Information

Consider a game Γ consisting of a finite set I of l players. Suppose for each i, $(Z_i, \mathcal{Z}_i)$ and $(X_i, \mathcal{X}_i)$ are measurable spaces. Let $(\Omega, \mathcal{F})$ be the measurable space $(\prod_{i\in I}(Z_i \times X_i), \prod_{i\in I}(\mathcal{Z}_i \times \mathcal{X}_i))$, the product space with the product σ-algebra and μ a probability measure on $(\Omega, \mathcal{F})$. For a point $\omega = (z_1, x_1, ..., z_l, x_l) \in \Omega$, define the coordinate projections

$$\zeta_i(\omega) = z_i,$$

$$\chi_i(\omega) = x_i.$$

he random mappings $\zeta_i(\omega)$ and $\chi_i(\omega)$ are interpreted respectively as player i's private information related to his action and payoff.

Each player i in I first observes the realization, say $z_i \in Z_i$, of the random element $\zeta_i(\omega)$, then chooses his own action from a nonempty compact subset $D_i(z_i)$ of a countable complete metric space A_i.[5] The payoff of player i is given by utility function $u_i : A \times X_i \to \mathbb{R}$, where $A = \prod_{j \in I} A_j$ is the set of all combination of all players' moves. We also assume the following uniform integrability condition (**UI**):

> **(UI)** *For every $i \in I$, there is a real-valued integrable function h_i on $(\Omega, \mathcal{F}, \mu)$ such that for μ-almost all $\omega \in \Omega$, $|u_i(a, \chi_i(\omega))| \leq h_i(\omega)$ holds for $a \in A$.*

We can thus describe a finite game with private information as

$$\Gamma = (I, ((Z_i, \mathcal{Z}_i), (X_i, \mathcal{X}_i), (A_i, D_i), u_i)_{i \in I}, \mu).$$

For any player i, let $\mathrm{meas}(Z_i, D_i)$ be the set of measurable mappings f from $(Z_i, \mathcal{Z}_i)$ to A_i such that $f(z_i) \in D_i(z_i)$ for each $z_i \in Z_i$. An element g_i of $\mathrm{meas}(Z_i, D_i)$ is called a *pure strategy* for player i. A pure strategy profile g is an l-vector function $(g_1, ..., g_l)$ that specifies a pure strategy for each player. For a pure strategy profile $g = (g_1, ..., g_l)$, the expected payoff for player i is

$$U_i(g) = \int_{\omega \in \Omega} u_i\left(g_1(\zeta_1(\omega)), ..., g_l(\zeta_l(\omega)), \chi_i(\omega)\right) \mu(d\omega).$$

An *(Nash) equilibrium in pure strategies* is defined as a pure strategy profile $g^* = (g_1^*, ..., g_l^*)$ such that for each player i,[6]

$$U_i(g^*) \geq U_i(g_i, g_{-i}^*) \text{ for all } g_i \in \ \mathrm{meas}(Z_i, D_i).$$

Let $\mathcal{M}(A)$ be the space of probability measures on A_i endowed with the weak topology. Note that such topology is metrizable by the Prohorov metric since the space A_i is metrizable. A *behavioral strategy* for the player i, say g_i[7] is an element

[5] A mapping F from a set C to the set of nonepmty subsets of a set E is called a correspondence from C to E. Thus, D_i is a correspondence from Z_i to A_i that takes compact subsets of A_i as its values; such a correspondence is called a compact-valued correspondence.

[6] Note that g_{-i}^* is an $(l-1)$-vector function given by g^* with its ith component deleted, and (g_i, g_{-i}^*) is the l-vector obtained from g^* with its ith component replaced by g_i.

[7] Note that there is no inconsistency with our notation of pure strategy above, since every pure strategy can also be though of as a behavioral strategy with point measures.

of meas$(Z_i, \mathcal{M}(A_i))$, where $\mathcal{M}(A)$ is equipped with its Borel $\sigma-$algebra. Given the players play the behavioral the strategies $\{g_i\}_{i\in I}$, the resulting expected payoff to i is

$$U_i(g) = \int_{\omega\in\Omega}\int_{a_l\in A_l}\cdots\int_{a_1\in A_1} u_i(a_1,\cdots,a_l,\chi_i(\omega))g_1(\zeta_1(\omega);da_1),$$
$$\cdots, g_l(\zeta_l(\omega);da_l)\mu(d\omega),$$

where again g is an $l-$vector function given by $(g_i, ..., g_l)$. An *equilibrium (Nash) in behavioral strategies* is defined similarly to that in pure strategies. More formally, we say, $g^* = (g_1^*, ..., g_l^*) \in \prod_{i=1}^{l}$ meas$(Z_i, \mathcal{M}(A_i))$ is an equilibrium in behavioral strategies if for each player i, $U_i(g^*) \geq U_i(g_i, g_{-i}^*)$ for all $g_i \in$ meas(Z_i, A_i).

We say an equilibrium b^* in pure strategies is a purification of an equilibrium b in behavioral strategies if, for every player i, $U_i(b) = U_i(b^*)$, and for all $z_i \in Z_i$, $b_i^*(z_i) \in$ supp $b_i(z_i)$.

That means, a purification b^* of an equilibrium b is an equilibrium that gives every player the same expected payoff that b does. In the following, we first prove two results that under certain hypotheses about the random variables $\zeta_1, \chi_1, \cdots, \zeta_l, \chi_l$, every equilibrium has a purification. And we prove the existence of equilibrium of pure strategies under certain conditions.

We now provide our first two results concerning with the purification of mixed strategies.

Theorem 4.2.1. *If, for every player i,*
(a) the distribution of ζ_i is atomless,
(b) the random variables $\{\zeta_j : j \neq i\}$ together with the random variable $\xi_i \equiv (\zeta_j, \chi_j)$ form a mutually independent set,
then every equilibrium has a purification.

Proof: Let $g = (g_1, \cdots, g_l) \in \prod_{i=1}^{l}$meas$(Z_i, (A_i))$ be an equilibrium in behavioral strategies. Fix any player $i = 1, \cdots, l$. Apply Theorem 4.1.3 to the collection

$$\{(Z_i, \mathcal{Z}_i), \mu\zeta_i^{-1}, A_i, g_i\},$$

where $\mu\zeta_i^{-1}$ is defined as the measure induced on the measurable space $(Z_i, \mathcal{Z}_i)$[8]to obtain a pure strategy $g_i^* \in \text{meas}(Z_i, A_i)$ such that for each i,

(1) for all $B \subseteq A_i$, $\int_{z_i \in Z_i} g_i(z_i; B) d\mu\zeta_i^{-1}(t) = \mu\zeta_i^{-1}(g_i^{*-1}(B))$;

(2) $g_i^*(z_i) \in \{a_i \in A_i : g(z_i; \{a_i\} > 0\} \equiv \mathit{supp}\ \ g(z_i)$ for $\mu\zeta_i^{-1}-$almost all $z_i \in Z_i$.

Let $g^* = (g_1^*, \cdots, g_l^*)$. We should show now that g^* is a purification of g.

To see this, we now focus on player i and let ζ_{-i} be the random variable $(\zeta_1, \cdots, \zeta_{i-1}, \zeta_{i+1}, \cdots, \zeta_l)$, and (ξ_i, ζ_{-i}) be the random variable form Ω to the space $(Z_i \times X_i, \prod_{j \neq i} Z_j)$, with $\mu(\xi_i, \zeta_{-i})^{-1}$ the corresponding measure induced on that space. Hypothesis (b) in the theorem ensures that $\mu(\xi_i, \zeta_{-i})^{-1} = (\mu\xi_i^{-1}(\prod_{j \neq i} \mu\zeta_j^{-1})$.[9]

Then, since u_i is a $\mu-$integrable function on Ω for any $a \in A$, we can assert the existence of a function $z_i \to E\{u_i(a, \chi_i) : \zeta_i = z_i\}$ such that for any measurable $W \in \mathcal{Z}_i$,

$$\int_{\{\omega \in \Omega : \zeta_i(\omega) \in W\}} u_i(a, \chi_i(\omega)) d\mu(\omega) = \int_{z_i \in W} E\{u_i(a, \chi_i) : \chi_i = z_i\} d\mu\zeta_i^{-1}(z_i).$$

We know obtain

$$\begin{aligned}
U_i(g) &= \int_{\omega \in \Omega} \Sigma_{a \in A} u_i(a, \chi_i(\omega)) \Pi_{i=1}^l g_i(\zeta_j(\omega); \{a_j\}) d\mu(\omega) \\
&= \Sigma_{a \in A} \int_{\omega \in \Omega} u_i(a, \chi_i(\omega)) g_i(\zeta_i(\omega); \{a_j\}) \times \Pi_{i \neq j} g_i(\zeta_j(\omega); \{a_j\}) d\mu(\omega) \\
&= \Sigma_{a \in A} \int_{z_i \in Z_i} E\{u_i(a, \chi_i) : \zeta_i = z_i\} g_i(z_i : a_i) d\mu\zeta_i^{-1}(z) \times \Pi_{i \neq j} \int g_j(z_j : \{a_j\}) d\mu\zeta_j^{-1}(z_j) \\
&= \Sigma_{a \in A} \int_{z_i \in Z_j} E\{u_i(a, \chi_i) : \zeta_i = z_i\} g_i(z_i; \{a_i\}) d\mu\zeta_j^{-1}(z_i) \Pi_{i \neq j} \tau_j(\{a_j\}) \\
&= \int_{z_i \in Z_i} \Sigma_{a_i \in A_i} [\Sigma_{a_{-1} \in A_{-1}} E\{u_i(a, \chi_i) : \chi_i = z_i\} \times \Pi_{i \neq j} \tau_j(\{a_j\})] g_i(z_i; \{a_i\}) d\mu\zeta_i^{-1}(z_i).
\end{aligned}$$

[8]Hypothesis (a) ensures that the measure $\mu\zeta_i^{-1}$ is atomless. So we can apply our theorem 4.1.3.

[9]See, for example, Ash [2], pp. 213-214.

The first equality uses the fact that expectations taken over a countable space can be written as summations instead of integrals; the second equality relies on u_i being a uniformly summable function; the third invokes the "change of variable" formula;[10] and the independence hypothesis; the fourth is true just by definition; and the fifth appeals to the conditional expectation still being a uniformly summable function.

This computation brings out the fact that the payoff to the i^{th} player depends on the distribution of the other players' strategies, namely on $\tau_j, j \neq i$. Since we purified the other players' mixed strategies in the way that this distribution does not change, all we need to check is that g_i^* gives the same payoff to the i^{th} player as does g_i. Towards this end, let

$$F^{'}(z_j) = argmax_{a_i \in A_i} G_i(z_i, a_i), \quad where$$

$$G_i(z_i, a_i) = [\Sigma_{a_{-1} \in A_{-1}} E\{u_i(a, \chi_i) : \zeta_i = z_i\} \Pi_{i \neq j} \tau_j(\{a_i\})]$$

We now claim that

$$supp \;\; g_i(z_i; \cdot) \subset F^{'}(z_j) \;\; for \;\; \mu\zeta_i^{-1} \;\; a.e. \;\; z_i \in Z_i$$

If not, there must exist measurable function f_i, h_i from Z_i to A_i such that $g_i(z_i)(\{f_i(z_i)\}) > 0$ and $\{z_i : G_i(z_i, f_i(z_i)) < G_i(z_i, h_i(z_i))\}$ is not $\mu\zeta_i^{-1}$−null. Define a new mixed-strategy g_i' satisfied, $g_i'(z_i)$ equal $g_i(z_i)$, if $G_i(z_i, f_i(z_i)) \geq G_i(z_i, h_i(z_i))$, otherwise let it be equal to $g_i(z_i) - g_i(z_i)(\{f_i(z_i)\})\delta_{f_i(z_i)} + g_i(z_i)(\{f_i(z_i)\})\delta_{h_i(z_i)}$. But that means that $U_i(g) < U_i(g_i', g_{-i})$. That is a contradiction to the maximality of g.

Now, we clearly have that $U_i(g_i^*, g_{-i}) \geq U_i(g)$. And from the beginning of the induction, we know that for $j \neq i$, g_j^* and g_j induce the same distribution τ_j on A. So we have $U_i(g^*) = U_i(g_i^*, g_{-i}) = U_i(g)$, which completes the proof. $\square$

[10]See, for example, Billingsley [7], pp.222-223.

Theorem 4.2.2. *If, for every player i,*
(a') the distribution of ζ_i' is atomless,
(a'') the set Z_i'' is finite,
(b) the random variables $\{\zeta_j' : j \neq i\}$ together with the random variable $\xi_i \equiv (\zeta_i', \zeta_i'', \chi_j)$ form a mutually independent set,
then every equilibrium has a purification.

Sketch of the proof: Given Theorem 4.2.1, we can follow the idea in Khan and Sun [18].[11] The little trick is to replace in Theorem 4.2.1, for each player i, his action space A_i with the new space $\tilde{A}_i \equiv \prod_{z_i'' \in Z_i''} A_i$. For each i, A_i is a countable metric space and Z_i'' is finite, so $\tilde{A}_i$ is clearly a countable metric space. For any player i, his behavioral strategy $g_i \in \text{meas}((Z_i', Z_i''); \mathcal{M}(A_i))$. Now, define the behavioral strategy $\tilde{g}_i \in \text{meas}(Z_i'; \mathcal{M}(\tilde{A}_i))$ as:

$$\tilde{g}_i(z_i'; \{\tilde{a}_i\}) = \prod_{z_i'' \in Z_i''} g_i((z_i', z_i''), \{\tilde{a}_i(z_i'')\}), \forall z_i' \in Z_i', \forall \tilde{a}_i \in \tilde{A}_i,$$

where $\tilde{a}_i(z_i'')$ is the $z_i''-$th coordinate of $\tilde{a}_i$.

Thus the following is clear. Given a behavioral strategy $g = (g_1, \ldots, d_l)$ in the game, we define another strategy $\tilde{g}$ as above for another game with $Z' = Z_1' \times \cdots \times Z_l'$ as the space of relevant private information, and with $\tilde{A}_i$ the action space for player i. Hypotheses (a') and (b) guarantee that hypotheses (a) and (b) of Theorem 4.2.1 are satisfied. Thus, Theorem 4.2.1 yields a pure strategy $\tilde{g}^* = (\tilde{g}_1^*, \cdots, \tilde{g}_l^*)$ with $\tilde{g}_i^* \in \text{meas}(Z_o'; \tilde{A}_i)$ in this new game. Then, a pure strategy equilibrium $g^* = (g_1^*, \cdots, g_l^*)$ with $g^* \in$meas$((Z_i', Z_i''); \mathcal{M}(A_i))$ in original game can be obtained. $\square$

In our statement of Theorems 4.2.1 and 4.2.2, we have made an effort to keep the similar structure with the corresponding theorems in Khan and Sun [18] and

[11] Or, one can refer to earlier paper like Radner and Rosenthal [27].

those in Radner and Rosenthal. Note again that Radner and Rosenthal focus on finite actions, and Khan and Sun base their theorem on countable compact metric action set. Our cases only need action set to be countable metric space with compact-valued correspondence. Therefore, although the statement of theorem and the techniques that dealt with in proof are similar as works before, both model and its applications are new and more general. The next theorem is to assert the existence of an equilibrium in pure strategies.

Theorem 4.2.3. *Under the hypotheses of Theorem 4.2.2, and under the condition that for every player $i \in I$, $u_i(\cdot, \chi_i(\omega))$ is a bounded continuous function on A for $\mu-$almost all $\omega \in \Omega$, there exists an equilibrium in pure strategies.*

Proof: We use the Kakutani-Fan-Glicksberg fixed point theorem to prove the existence of Nash equilibrium in pure strategies. We shall present the proof for the special case that $Z_i = Z_i'$ for all i, i.e., there is no atom component for private information variable ζ_i. One can check, by following the sketch of proof of Theorem 2 in Khan and Sun [18], to get the same conclusion under the hypotheses of Theorem 4.2.3.

Let us consider a single player i. Since $u_i(a, \chi_i(\cdot))$ is uniformly μ-integrable function on Ω, we can assert[12] that there exists a function $V_i : A \times Z_i \to \mathbb{R}$, such that $V_i(a, \zeta_i(\omega))$ is the regular conditional expectation of $u_i(a, \chi_i(\omega))$ under the sub-σ-algebra of $\mathcal{F}$ generated by ζ_i. That is, for any measurable set $W \in \mathcal{Z}_i$, we have

$$\int_{\{\omega \in \Omega : \zeta_i(\omega) \in W\}} u_i(a, \chi_i(\omega)) d\mu(\omega) = \int_{z_i \in W} V_i(a, z_i) d\mu\zeta_i^{-1}(z_i).$$

Moreover, by Theorem 2.2 in Dynkin and Evstigneev [**?**], we know that for $\mu\zeta_i^{-1}$-almost all $z_i \in Z_i$, $V(\cdot, z_i)$ is continuous and bounded on A. Without loss of generality, we can assume for all $z_i \in Z_i$, $V(\cdot, z_i)$ is continuous and bounded on

[12]For example, one can refer to Theorem 2.1 in Dynkin an Evstigneev [**?**].

A.[13]

Denote $\mathcal{D}_{D_i} = \{(\mu\zeta_i^{-1})g_i^{-1} : g_i$ is a measurable selection of $D_i\}$. Construct a mapping from $Z_i \times A_i \times \prod_{j=1}^{l} \mathcal{D}_{D_j}$ into $\mathbb{R}$ defined by

$$(z_i, a_i, \lambda_1, \cdots, \lambda_l) \to G_i(z_i, a_i, \lambda_1, \cdots, \lambda_l) = \int_{a_{-i} \in A_{-i}} V_i(a, z_i) d\lambda_{-i}.$$

In fact, G_i is simply the payoff to player i when information z_i is revealed to him and he takes the action a_i, while all other players, generically indexed by $j \neq i$, play the mixed action $\lambda_j, j \neq i$.[14] It is obvious that, for any fixed $z_i \in Z_i$, G_i is a continuous real valued function on $A_i \times \prod_{i=1}^{l} \mathcal{D}_{D_i}$; and for any fixed $(a_i, \lambda_1, \cdots, \lambda_l)$, it is measurable on Z_i. Therefore G_i is jointly measurable, in particular, measurable on $Z_i \times \prod_{i=1}^{l} \mathcal{D}_{D_i}$ for each fixed $a_i \in A_i$.

Then, consider the set-valued mapping, from $Z_i \times \prod_{i=1}^{l} \mathcal{D}_{D_i}$ into A_i given by

$$(z_i, \lambda_1, \cdots, \lambda_l) \twoheadrightarrow F^i(z_i, \lambda_1, \cdots, \lambda_l) = \arg\max_{a_i \in D_i(z_i)} G_i(z_i, a_i, \lambda_1, \cdots, \lambda_l).$$

The joint continuity of G_i on A and the compactness of each $D_i(z_i)$ imply that $F^i(z_i, \lambda_1, \cdots, \lambda_l)$ is compact, measurable with respect to z_i, and upper semicontinuous with respect to $(\lambda_1, \cdot, \lambda_l) \in \prod_{i=1}^{l} \mathcal{D}_{D_i}$. The latter is guaranteed by Berge's maximum theorem. Furthermore, for each l-tuple $(\lambda_1, \cdots, \lambda_l) \in \prod_{i=1}^{l} \mathcal{D}_{D_i}$, there exists a measurable selection from the correspondence F^i by Kuratowski-Ryll-Nardzewski Selection Theorem.[15]

We now consider the object $\mathcal{D}_{F^i_{(\lambda_1, \cdots, \lambda_l)}} = \{(\mu\zeta_i^{-1})g_i^{-1} : g_i$ is a measurable selection of $F^i(\cdot, \lambda_1, \cdots, \lambda_l)\}$. By the assertion of the existence of a measurable selection, it is nonempty. Then, applying Theorem 4.1.4, Theorem 4.1.6 and Theorem 4.1.7,

[13] In fact, $V(\cdot, z_i) \leq \tilde{h}(z_i)$, with $\tilde{h}(\zeta_i) = E[h|\zeta_i]$. Recall that for μ-almost all $\omega \in \Omega$, $|u_i(a, \chi_i(\omega))| \leq h_i(\omega)$ holds for $a \in A$.

[14] Note that under the assumption on action choice (i.e., compact-valued property of D_j for any $j \in I$) of our models, $\lambda_j \in \mathcal{D}_{D_j}$.

[15] See, for example, Aliprantis and Border [1].

we know that it is convex, compact and upper semicontinuous with respect to $(\lambda_1, \cdots, \lambda_l) \in \prod_{i=1}^{l} \mathcal{D}_{D_i}$. Let Φ be the correspondence from $\prod_{i=1}^{l} \mathcal{D}_{D_i}$ to $\prod_{i=1}^{l} \mathcal{D}_{D_i}$ such that for any tuple $(\lambda_1, \cdots, \lambda_l) \in \prod_{i=1}^{l} \mathcal{D}_{D_i}$,

$$\Phi(\lambda_1, \cdots, \lambda_l) = \prod_{i=1}^{l} \mathcal{D}_{F^i_{(\lambda_1, \cdots, \lambda_l)}}.$$

Thus Φ is nonempty, compact, convex valued, and upper semicontinuous with respect to $(\lambda_1, \cdots, \lambda_l)$. And from Theorem 4.1.4 and Theorem 4.1.6, one can get $\mathcal{D}_{D_i}$ is nonempty, compact and convex. Applying the Kakutani-Fan-Glicksgerg fixed-point theorem, we know that there exists a fixed-point

$$(\lambda_1^*, \cdots, \lambda_l^*) \in \Phi(\lambda_1^*, \cdots, \lambda_l^*),$$

and for each player i, $\lambda_i^* \in \mathcal{D}_{F^i_{(\lambda_1^*, \cdots, \lambda_l^*)}}$. So there exists $g_i^* \in \text{meas}(Z_i, A_i)$ such that g_i^* is a selection of $F^i_{(\lambda_1^*, \cdots, \lambda_l^*)}$, and $\mu g_i^{*-1} = \lambda_i^*$. It is clear that $g^* = (g_1^*, \cdots, g_l^*)$ is an equilibrium in pure strategy. $\square$

Chapter 5

Large Games

In this chapter, we show the existence of pure-strategy Nash equilibrium for non-cooperative games with a continuum of small players. Such games are often so called as large games. The organization of this chapter is as follows:

Section 5.1 describes a typical large game model. Section 5.2 relies on the mathematical results developed in Chapter 4 and asserts the existence of equilibrium for a game with continuum of players that are divided into finite types, and with countable actions. Section 5.3 deals with a non-cooperative game with a continuum of small players and a compact action space.

5.1 A Simple Large Game

In 1973, Schmeidler [33] showed that a large game with an atomless space of players and finite actions has a Nash equilibrium in mixed strategies and if the payoffs are restricted so as to depend only on the average response of others then there is a pure strategy equilibrium. A simpler proof of the result is showed in Rath [29]. Also Rath [29] shows that when the analysis is restricted to pure strategies, it not only allows for a much simpler proof, but also extends to the case where the space of actions is a compact subset of n-dimensional Euclidean space.

We now restate the settings and the result in Rath [29].

Let $I = [0,1]$ endowed with Lebesgue measure λ be the set of players, P the space of actions where P is a compact subset of $\mathcal{R}^n$. A *strategy profile* is a measurable function from I to P. Let F_P denote the space of all strategy profiles and for any $f \in F_P$ let $s(f) = \int_I f d\lambda$, and $S_P = \{s(f) | f \in F_P\}$.

Now, let $\mathcal{U}_P$ denote the set of real-valued continuous functions defined on $P \times S_P$ endowed with sup norm topology.

Then, we say, a *game* is a measurable function $g : I \to \mathcal{U}_P$. And a *Nash equilibrium* of a game g is a $f \in F_P$ such that for almost all t, $g(t)(f(t), s(f)) \geq g(t)(x, s(f))$, $\forall x \in P$.

Theorem 5.1.1. *Every game described above has a Nash equilibrium.*

The argument of the proof also makes use of Kakutani's fixed point theorem as what is done in classical proof in Nash [26]. For details, one can refer to Rath [29].

5.2 Large Games with Finite Types and Countable Actions

This section is a generalization of Theorem 10 in Khan and Sun [18]. We consider the game here as a game with a continuum of players and with a countable action set, where the players are divided into finite different types. With the mathematical results developed in the last chapter, we can assert the existence of equilibrium in pure strategies of such games.

First, we give the game model as follows.

Let I be the set of players, $(I, \mathcal{I}, \lambda)$ an atomless probability space representing the space of player names, and A a countable metric space which represents the action space. Each player i, $i \in I$ choose his own actions $D : I \twoheadrightarrow A$ in A, where the correspondence D is compact-valued. The players are divided into l types. So let $I_1, \cdots, I_l$ be a partition of I according to the player's type, where the partition with positive λ-measures $c_1, \cdots c_l$. For each $1 \leq j \leq l$, we denote λ_j to be the probability measure on I_j such that for any measurable set $B \subseteq I_j$, $\lambda_j(B) = \lambda(B)/c_j$. Let $\mathcal{U}_A$ be the space of real-valued continuous functions on $A \times \mathcal{M}(A)^l$, endowed with its sup-norm topology and with $\mathcal{B}(\mathcal{U}_A)$ its Borel σ-algebra. A *strategy profile* is a measurable function $f : I \to A$ satisfying $f(i) \in D(i), i \in I$, which specifies a strategy for each player.

Definition 8. A game Γ is a function from I to $\mathcal{U}_A$. And an equilibrium (Nash) of a game Γ is a $f : I \to A$ with $f(i) \in D(i)$ for each $i \in I$, such that for λ-almost all $i \in I$,

$$u_i(f(j), \lambda_1 f_1^{-1}, \ldots, \lambda_l f_l^{-1}) \geq u_i(a, \lambda_1 f_1^{-1}, \ldots, \lambda_l f_l^{-1})$$

for all $a \in D(i)$, where $u_i = \Gamma(i)$ and f_j is the restriction of f to I_j.

We now apply our results on the distribution of atomless correspondence to prove the existence of Nash equilibrium in such a game. Before the main theorem, we define D_j as the restriction of D to I_j, and $\mathcal{D}_{D_j} = \{\lambda_j g_j^{-1}, g_j$ is a measurable selection of $D_j\}$, for $j = 1, \cdots, l$.

Theorem 5.2.1. *Every game described above has a Nash equilibrium.*

Proof: Consider the set-valued mapping, from $I \times \prod_{j=1}^{l} \mathcal{D}_{D_j}$ into A given by

$$(i, \mu_1, \cdots, \mu_l) \to F(i, \mu_1, \cdots, \mu_l) = \arg\max_{a \in D(i)} u_i(a, \mu_1, \cdots, \mu_l),$$

where $(\mu_1, \cdots, \mu_l) \in \prod_{j=1}^{l} \mathcal{D}_{D_j}$. It is obvious that for given $(\mu_1, \cdots, \mu_l)$, $F(\cdot, \mu_1, \cdots, \mu_l)$ is a compact-valued correspondence from I to A. Berge's maximum theorem and the joint continuity of u_i on $A \times \prod_{j=1}^{l} \mathcal{D}_{D_j}$ imply that for each $i \in I$, $F(i, \mu_1, \cdots, \mu_l)$

is upper semicontinuity of on $\prod_{j=1}^{l} \mathcal{D}_{D_j}$. Moreover, for each l-tuple $(\mu_1, \cdots, \mu_l) \in \mathcal{M}(A)^l$, u_i is measurable in $I \times A$,[1] since $u(\cdot, \cdot, \mu_1, \cdots, \mu_l)$ is a measurable function on I, and a continuous function on A. Therefore, there exists a measurable selection from the correspondence $F_{(\mu_1,\cdots,\mu_l)}$[2] by Kuratowski-Ryll-Nardzewski Selection theorem.

For each $1 \leq j \leq l$, let F^j be the restriction of correspondence F on $I_j \times \prod_{j=1}^{l} \mathcal{D}_{D_j}$. Now we consider the object $\mathcal{D}_{F^j_{(\mu_1,\cdots,\mu_l)}}$. By the assertion of the existence of a measurable selection, it is nonempty. Then, applying Theorem **??**, Theorem 4.1.6 and Theorem 4.1.7, we know that it is convex, compact and upper semicontinuous with respect to $(\mu_1, \cdots, \mu_l) \in \prod_{j=1}^{l} \mathcal{D}_{D_j}$. As we do in the proof of Theorem 2.1, let G be the correspondence from $\prod_{j=1}^{l} \mathcal{D}_{D_j}$ to $\prod_{j=1}^{l} \mathcal{D}_{D_j}$ such that for any tuple $(\mu_1, \cdots, \mu_l) \in \prod_{j=1}^{l} \mathcal{D}_{D_j}$,

$$G(\mu_1, \cdots, \mu_l) = \prod_{j=1}^{l} \mathcal{D}_{F^j_{(\mu_1,\cdots,\mu_l)}}.$$

The correspondence G is compact and convex valued, upper semicontinuous with respect to $(\mu_1, \cdots, \mu_l) \in \prod_{j=1}^{l} \mathcal{D}_{D_j}$. So, Kakutani-Fan-Glicksgerg fixed-point theorem implies the existence of a fixed-point $(\mu_1^*, \cdots, \mu_l^*) \in G(\mu_1^*, \cdots, \mu_l^*)$, and for each j, a measurable selection f_j^* of $F^j(\cdot, \mu_1^*, \cdots, \mu_l^*)$ such that $\lambda_j f_j^{*-1} = \mu_j^*$. Finally, let f^* be the mapping from T to A such that for each $i \in I_j$, $f^*(i) = f_j^*(i)$. It is clear that f^* is an equilibrium. □

[1] We can still apply Theorem 3.14 in Castaing and Valadier [12] to assert this declaration.

[2] As before, $F_{(\mu_1,\cdots,\mu_l)}$ is a shorthand notation of $F(\cdot, \mu_1, \cdots, \mu_l)$.

5.3 Large Games with Transformed Summary Statistics

Non-cooperative games with a continuum of small players and a compact action space in a finite dimensional space have been used in the study of monopolistic competitions (see, for example, Rauh [32] and Vives [42]). It is often assumed that the players' payoffs depend on their own actions and the summary statistics of the aggregate strategy profiles in terms of the moments of the distributions of players' actions. Rauh [31] takes into consideration of such games with some restrictions and shows the existence of pure-strategy Nash equilibrium for such kind of games. However, as we showed in the introduction, some restrictions are not natural. We show here the existence of pure-strategy Nash equilibrium for such games but with less constraints than others. We reformulate the above model so that the players' payoffs depend on their own actions and the mean of the strategy profiles under a general transformation.

And we discuss in Section 5.1, the existence of pure-strategy Nash equilibrium is shown in Rath [29] for large games with a compact action space in a finite dimensional space, where the payoffs depend on players' own actions and the mean of the aggregate strategy profiles. We note that this result does not extend to infinite-dimensional spaces (see Khan, Rath and Sun [17]) when the unit interval with Lebesgue measure is used to represent the space of players; such an extension is possible if the space of players is an atomless hyperfinite Loeb measure space (see Khan and Sun [19]). It is claimed in Rauh [31] that "All these results involve the mean and hence do not apply to monopolistic competition models with summary statistics different from the mean or several summary statistics". However, our formulation shows that monopolistic competition models can indeed be studied via the mean under some transformation.

In the following, we first provide the main theorem and two kinds of proofs of it. Then we state some specific examples and give remarks.

5.3.1 The Model and Result

Let I be the set of players, $\mathcal{I}$ be a $\sigma-$algebra of subsets of I, and λ be an atomless probability measure on I. We use $(I, \mathcal{I}, \lambda)$ to represent the space of player names. For example, one can take $(I, \mathcal{I}, \lambda)$ as the unit interval $[0, 1]$ with Lebesgue measure.

Let P denote a nonempty, compact and metric space such that each player $i \in I$ chooses a pure strategy from P. For instance, P might be the set of possible prices an individual firm can set for its product. A *strategy profile* is a measurable function $f : I \to P$, which specifies a strategy for each player.

Let s be a continuous function from P to the n-dimensional Euclidean space $\mathbb{R}^n$, and C the range of s.[3] The continuity of s and compactness of P imply that C is also compact. Let Σ^4 be a convex and compact subset of $\mathbb{R}^n$, which contains C. It is clear that for any strategy profile f, $\sigma_f = \int_I (s \circ f) d\lambda \in \Sigma$. The mean σ_f of $s \circ f$ is a summary statistics of the society which the players can observe. A payoff function for a player is a real-valued continuous function defined on $P \times \Sigma$, which means that it depends on her own action $p \in P$ and the vector $\sigma \in \Sigma$ of summary statistics. Let $\mathcal{P}$ denote the space of all continuous payoff functions with the supremum norm.

Now, we define a *game* to be a measurable function $\Gamma : I \to \mathcal{P}$, which assigns each player $i \in I$ a continuous payoff function $\Gamma(i)(\cdot, \cdot)$. An *equilibrium* (in pure

[3] A special case can be considered as: Let $s : R \to R^n$ by $s(x) = (x, x^2, ..., x^n)$ then the first n moments of the price profile $f : I \to P$ are given by $\int_I (s \circ f) d\lambda$. In the discussion in Vives[42] (1999, 167-176) the set of firms is $[0, N]$ with Lebesgue measure and the summary statistic is $\tilde{q} = \int_0^N s(q(i)) di$ where $q(i)$ is firm is output and $s : R \to R$ is a strictly increasing continuous function.

[4] For example, we can set $\Sigma = convC$.

strategies) for such a game is a strategy profile $f : I \to P$ such that each player plays a best response against the induced vector of summary statistics; i.e.,

$$\Gamma(i)(f(i), \sigma_f) \geq \Gamma(i)(p, \sigma_f)$$

for all $i \in I$ and $p \in P$ where $\sigma_f = \int_I (s \circ f) d\lambda$.

In the following theorem, we present a general result on the existence of equilibrium for the game Γ.

Theorem 5.3.1. *Let $(I, \mathcal{I}, \lambda)$ be an atomless probability space, P a nonempty, compact metric space, s a continuous function from P onto a compact subset C of $\mathbb{R}^n$, and Σ a compact, convex subset of $\mathbb{R}^n$ containing C. Let $\mathcal{P}$ denote the space of real-valued continuous functions on $P \times \Sigma$ with the supremum norm. Then every game $\Gamma : I \to \mathcal{P}$ has an equilibrium in pure strategies.*

Proof: First, define the best-response correspondence $B : I \times \Sigma \to P$ as

$$B(i, \sigma) = argmax_{p \in P} \Gamma(i)(p, \sigma),$$

which is the set of maximum points for the continuous function $\Gamma(i)(\cdot, \sigma)$ on P. By standard arguments (see, for example, Rath [29]), we can obtain that for each $\sigma \in \Sigma$, $B(\cdot, \sigma)$ is a closed-valued, measurable correspondence from I to P; and for each $i \in I$, $B(i, \cdot)$ is an upper semicontinuous correspondence from Σ to P.

Let $F : I \times \Sigma \to \Sigma$ be the correspondence defined by $F(i, \sigma) = s(B(i, \sigma))$, and $\Phi : \Sigma \to \Sigma$, a correspondence defined by $\Phi(\sigma) = \int_I F(i, \sigma) d\lambda$. We shall show that Φ is (a) nonempty-valued, (b) convex-valued, (c) upper semicontinuous.

(a)Let $\sigma \in \Sigma$. By the standard measurable selection theorem (see, for example, Theorem 8.1.3 in Aubin and Frankowska [3], there exists a measurable function $f : I \to P$ such that $f(i) \in B(i, \sigma)$ for all $i \in T$. Then the measurable function $g : I \to \Sigma$ defined by $g = s \circ f$ satisfies $g(i) \in F(i, \sigma)$ for all $i \in I$. Thus, (a) is proved.

(b) Since λ is atomless, Φ is convex-valued by Theorem 8.6.3 in Aubin and Frankowska [3], which is a simple consequence of the classical Lyapunov theorem.

(c)Since B is upper semicontinuous and s is continuous, Theorem 14.22 in Aliprantis and Border [1] implies that F is upper semicontinuous on Σ for each $i \in I$. A classical result of Aumann on the preservation of upper semicontinuity via integration (see, Aumann [4, 5]) says that Φ is also upper semicontinous.

By the Kakutani fixed-point theorem, there exists a $\sigma^* \in \Phi(\sigma^*)$. That is, there exists a measurable function $g : I \to \Sigma$ such that $\sigma^* = \int_I g d\lambda$ and $g(i) \in F(i, \sigma^*)$. Note that $F(i, \sigma^*) = s(B(i, \sigma^*))$, which is a subset of C. Thus, the measurable function g takes values in C.

Since s is a function from P onto C, we can define a correspondence s^{-1} from C to P such that $s^{-1}(c) = \{p \in P : s(p) = c\}$. Since s is continuous, it is obvious that s^{-1} is a weakly measurable correspondence with nonempty closed values from the measurable space C with Borel σ-algebra to the compact metric space P. Hence, the Kuratowski-Ryll-Nardzewski Selection Theorem in Aliprantis and Border [1], implies that we can find a Borel measurable selector h of s^{-1}. Then it is clear that the strategy profile $f : I \to P$ defined by $f = h \circ g$ is an equilibrium in pure strategies for the game Γ. $\square$

5.3.2 Remarks and Examples

(1) A continuum of firms, represented by $[0, 1]$, is considered in Vives [42]: the price p_i of firm i's product is given by $p_i = P_i(q_i, \tilde{q})$, where q_i is firm i's output, and $\tilde{q}$ is a vector of summary statistics which characterizes the output distribution of firms (e.g., $\tilde{q} = \int s(q_i) di$, here, when s is the identity function then $\tilde{q}$ is the average quantity). The profits of firm $i, i \in [0, 1]$, is given by $\pi_i = (P(q_i, \tilde{q}) - m)q_i - F$,

where F is a fixed cost and m is a constant marginal cost of production. By taking first-order condition, a Nash equilibrium can be obtained, characterized by $(p_i - m)/p_i = \epsilon_i$, where $\epsilon_i = -(q_i/p_i)(\partial P_i/\partial q_i)$ is the quantity elasticity of inverse demand. The existence of Nash equilibrium can be deduced in Rauh's model by viewing $[0, 1]$ as the set of players, the quantities that firms can maintain as their actions—elements in set P, and $\tilde{q}$ as a vector of summary statistics in Σ by taking $s : \mathbb{R} \to \mathbb{R}$ to satisfy one consumption-strict monotonicity. Clearly, it can also be obtained naturally by ours by taking similar constructions but without other constraints.

(2)The function s in Rauh [31] is defined by taking the composition of the univariate vector functions $s_1, \ldots, s_m$ with projections $proj_1, \ldots, proj_m$. Let C be the range of s. It is obviously contained in the set Σ, which is the product of the intervals between the minimum and maximum of the functions s_{rq} as in Rauh [31]. In our paper, we define s as any continuous function,[5] and target space Σ as any convex and compact subset of $\mathbb{R}^n$, which contains C, and also contains that Σ defined in Rauh [31]. Thus both the model and the main theorem in Rauh [31] are special cases of ours.

(3) The action set P is often set to be a subset of Euclidean space. So a natural question arises whether the action set can be a generic compact metric space. Our theorem gives an affirmative answer. Note that the action space in our model can be infinite dimensional. For example, we can take $P = \mathcal{M}(A)$, the space of probability measures on A endowed with the weak topology, where A is an infinite subset of an Euclidean space. We also consider another more specific example. Let the firms' payoffs depend on their own quantities (which are belonging to $\mathbb{R}$) along

[5]The type of assumption on the strict monotonicity of the functions s_{r1} as in Rauh [31] is not needed in our case.

the time and the summary statistics of the society. We formulate it as follows. We assume time set to be $[0,T]$. A continuum of firms $[0,1]$ take actions from action set P, where P is taken to be a bounded closed subset of $L^\infty([0,T],\mathbb{R})$ with topology $\sigma(L^\infty([0,T],\mathbb{R}), L^1([0,T],\mathbb{R}))$. Note that P is compact by Alaoglu Theorem. Let D be an upper bound for P. Let $s: P \to \mathbb{R}^n$ be a projection at n epoches: for $f \in P$, $s(f) = (f(\tau_1), \dots, f(\tau_n))$, where $(\tau_1, \dots, \tau_n)$ are n fixed sampling times. The set of summary statistics Σ can be taken as $[0,D]^n$. The payoff function for a firm is a real-valued continuous function defined on $P \times \Sigma$. Then, following our main model and theorem, we can claim the existence of Nash equilibrium in this example.

(4) The target space can only be finite-dimensional in general.[6] We now show that our model can adopt the target space to be any separable Banach space by choosing an atomless hyperfinite Loeb measure space $(I, \mathcal{I}, \lambda)$ as the space of players.[7] We will reserve all other notations discussed above except that Σ can be a weakly compact and convex subset of a separable Banach space$(X, \|\cdot\|)$ with weak topology instead of a subset of $\mathbb{R}^n$. Moreover, we see s as a weakly continuous function from P onto a weakly compact subset C of a separable Banach space$(X, \|\cdot\|)$. Our main theorem is still valid in this setting when the integral in the definition of σ_f is the Bochner integral. To prove this result, we can simply use Theorems 1 and 6 in Sun [39] to claim the convexity and upper semicontinuity as in (b) and (c) above; we can then use the Fan-Glicksberg fixed point theorem instead of the Kakutani fixed-point theorem to prove the existence of Nash equilibrium.

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[6]For instance, we just assume that I is the closed unit interval with Lebesgue measure, then an equilibrium may not exist as shown in Khan, Rath and Sun [17] and Rath, Sun and Yamashige [30].

[7]See the theory of correspondences on Loeb spaces developed in Sun [39].

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Thesis Title: Existence of Nash Equilibrium in Atomless Games

Abstract

In this thesis we first discuss games with private information. Based on our mathematical results on the set of distributions induced by the measurable selections of a correspondence with a countable range, we provide the purification results and also prove the existence of a pure strategy equilibrium for a finite game when the action space is countable but not necessarily compact.

Another aspect of this thesis focuses on large games. We show the existence of equilibrium for a game with continuum of players that are divided into finite types, and with countable actions. Also, the existence of pure-strategy Nash equilibrium is shown for a non-cooperative game with a continuum of small players and a compact action space. The players' payoffs depend on their own actions and the mean of the transformed strategy profiles. This covers the case when the payoffs depend on players own actions and finitely many summary statistics.

Keywords:

Non-cooperative Game; Nash Equilibrium; Atomless; Private information; Purification; Pure Strategy; Summary Statistics

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