LOOSE ENTRY FORMULAS AND THE REDUCTION OF DIXON DETERMINANT ENTRIES

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Abstract

Recently there has been much effort in using the Dixon method to construct sparse resultants. In this thesis, we present new loose entry formulas for the Dixon matrix and introduce the concept of exposed points for bidegree monomial supports. They combine to produce important results: the rows and columns associated with exposed points have a very simple description, and rows and columns near exposed points can be greatly simplified. These results provide useful information for the determination of maximal minors (numerators of the quotient sparse resultant) and exact information for the identification of extraneous factors (denominators of the quotient sparse resultant). In particular, for most corners with three exposed points, the thesis pinpoints the rows or columns generating the expected extraneous factors.

Keywords:

Dixon Matrix, Loose Entry Formulas, Exposed Points, Extraneous Factors, Reduction of Dixon Determinant Entries, Sparse Resultant

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Contents

Summary

The thesis consists of seven chapters.

Chapter 1 introduces the thesis. After giving the motivations for constructing sparse resultants using the Dixon method, it lists the main contributions of the thesis. These contributions can be attributed to three important findings and concepts: loose entry formulas, exposed points, and reductions of rows and columns. These are needed to identify extraneous factors for corners with three exposed points.

Chapter 2 lists the mathematical notations, the Dixon method, and presentation conventions used throughout this thesis. In addition, it proves a basic but very important theorem. This theorem gives the exact row and column supports after the removal of some monomial points from the corners of a rectangle monomial support.

Chapter 3 presents four loose entry formulas for the Dixon matrix and the corner-specific simplified formulas derived from them. All these entry formulas have uniform summation bounds, this property is indispensable in investigating the properties of Dixon matrix. We end the chapter by comparing these four new loose entry formulas with an existing concise entry formula.

Chapter 4 defines exterior points, exposed points (singular or otherwise), and corner cutting. These concepts lead to five important properties of the Dixon matrix: (1) inheritance of exterior points; (2) inheritance of non-singular exposed points; (3) non-inheritance of singular exposed points; (4) bracket factors from three exposed-point rows or columns; (5) linear dependence of four or more exposed-point rows or columns. In addition, based on the number of exposed points and current research results, we are able to classify the corner cut monomial supports into three categories: (1) at most two exposed points at a corner; (2) at most three exposed points at a corner; (3) any number of exposed points at a corner. For each category, one or more theorems are proved and some conjectures are proposed.

Chapter 5 examines a special corner cutting situation in which there are exactly three exposed points at a corner. Under this condition, there are six cases covering about 72% of the possibilities. The major result in this chapter is that rows and columns near exposed points can be reduced using basic row and column determinant operations. For the first two cases, we are able to identify rows or columns producing the expected extraneous factors after the reduction. For the remaining four cases, the extraneous factors are generated from the rows and columns that intersect at zero entries after the reduction.

Chapter 6 proposes two conjectures. To deal with the remaining 28% of the possibilities of a corner with three exposed points, an algorithm is proposed to identify the rows or columns generating the expected extraneous factors. The other conjecture speculates the linear independence of the rows or the columns proposed by the first conjecture. This is needed to ensure that the Dixon matrix is indeed the only maximal minor.

Chapter 7 concludes this thesis and states two more open problems. The resolution of the conjectures in Chapter 6 and the two problems here would completely solve the sparse resultant problem for corners having exactly three exposed points. The first open problem concerns the generation of the extraneous factors from the rows and columns specified in Conjecture 3. The second open problem is on the validity of the results when degeneracy occurs.

Chapter 1

Introduction

Background. Polynomial systems are widely used in many areas like geometric reasoning, implicitization, computer vision, robotics and kinematics. Elimination is an important approach in polynomial system solving [9, 22]. Among the various elimination techniques, the method of resultants stands out for its computational efficiency and its explicit formulation in matrix form [13]. The Dixon bracket method is a well-known technique for constructing resultants [12]. Recent research in Dixon resultants include [19, 16, 7].

Contributions. The research of this thesis aims to better understand the construction of sparse resultants using the Dixon method for three polynomial equations with an unmixed bidegree monomial support. To this end the contributions are

1. the discovery of four loose entry formulas for the Dixon matrix on which the rest of the results in the thesis depend; (see Theorems 2 and 3)

2. the formalization of the concept of corner cutting;

3. the formalization of the concept of exterior points and their simplification effects on the Dixon matrix; (see Theorem 1)

4. the introduction of exposed points and their simplification effects on the Dixon matrix; (see Theorems 4 and 5)

5. the simplification effects of exposed points on the Dixon determinants in terms of reduction to rows and columns near the exposed points; (see Theorems 14 and 15)

6. the consequences of corner cutting on the maximal minors of the Dixon matrix; (see Theorem 10)

7. the consequences of corner cutting on generation of the extraneous factors; (see Theorems 9 and 16)

8. the above results lead to a partial proof of a conjecture concerning unmixed bidegree supports with at most three exposed points at each corner. (see Chapter 5)

In the course of the research many observations have been made and these are formulated as conjectures in Chapter 6.

The above contributions can be attributed to the following three main discoveries and findings:

Loose Entry Formulas. An entry formula allows the Dixon matrix to be computed efficiently [5, 3] and is indispensable in deriving properties of the Dixon matrix [14, 15, 16]. While the concise entry formula given in [1] is good for computing the Dixon matrix, it is not as well suited for theoretical exploration because to be concise each entry has distinct and complicated summation bounds and this obscures rather than reveals useful information. It would greatly simplify derivation if the summation bounds can be the same for the entire matrix or at least for some rows or columns of the matrix. The thesis answers this need by presenting four loose entry formulas. These entry formulas have uniform summation bounds for the entire matrix for a canonical, or uncut, bidegree monomial support. For corner-cut monomial supports, each of these entry formulas become even simpler for some rows and columns of a particular corner but still maintains the uniform summation bounds. The tradeoff is that these formulas are loose rather than concise [1] because they may produce redundant brackets — a bracket that vanishes due to out of range indices or brackets that cancel mutually. It is gratifying that these loose entry formulas can be obtained quite easily, all we have to do is simply expand a formal power series a little differently.

Exposed Points. We are interested in finding explicit sparse resultant expressions, as quotients of determinants in brackets, for three generic bivariate polynomials over an unmixed monomial support. This motivates the adaptation of Dixon's method [12] to what we call corner-cut monomial supports [23, 2]. The classes of monomial supports for which bracket quotient formulas have been obtained are rectangular corner cutting, corner edge cutting, corner point pasting, and six-point isosceles triangular corner cutting [2, 14, 15, 16]. It turns out that instead of characterizing a monomial support by some geometric properties to deduce what the sparse resultant should be like, a better and simpler indicator is what we call exposed points.

Exposed points are significant in the formulation of sparse resultants because the entries of rows and columns of the Dixon matrix associated with exposed points can be described by a simple loose entry formula. These loose entry formulas show that, with respect to the same monomial support corner, three exposed point rows or columns will produce a bracket factor and four or more exposed point rows or columns are linearly dependent. This knowledge is valuable in finding a maximal minor of the Dixon matrix and determining the corresponding extraneous factors. Unlike the previous approach in [2, 14, 15, 16], the new approach is more unifying and revealing.

Exposed points also provide a quantitative classification of corner-cut unmixed monomial supports for the purpose of deducing the sparse resultant quotient formulas. The classification is based on the number of exposed points at each corner of the monomial support. This quantitative classification is much more encompassing and illuminating than the previous shape approach of [2, 14, 15, 16] that relied on the geometrical peculiarities of the monomial supports in deducing the sparse resultant formulas.

Reduction of Dixon Determinant Entries. It seems too much to hope for a non-hybrid determinant form sparse resultant [20]. Currently the best that can be done is to have a quotient determinant form. Thus the determination of extraneous factors (the denominator in the quotient form) is an open problem in many situations [8]. In particular, the conjecture of [17] deals with the extraneous factors for the bottom-left corner having three exposed points. This thesis extends the conjecture to all four corners and proves, in a sense 72% of the conjecture. This is done by reducing the entries of rows and columns near the exposed points by basic determinant operations. For some cases only rows or columns are sufficient to generate the expected extraneous factors, while for other cases both rows and columns that intersect at zero entries are needed to generate the expected extraneous factors.

Chapter 2

Preliminaries

2.1 Sets

Let $a..b$ denote the set of consecutive integers from a to b inclusively and let $a..b \times c..d$ denote the cartesian product of two sets of consecutive integers. We further abbreviate the product as $a \times c..d$ if $a = b$ and $a..b \times c$ if $c = d$.

For example,

$$0..1 \times 0..1 = \{0, 1\} \times \{0, 1\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}. \tag{2.1}$$

The Minkowski sum between two sets is denoted as:

$$\{(a, b), \cdots\} \oplus \{(c, d), \cdots\} = \{(a + c, b + d), \cdots\} \tag{2.2}$$

and we write $(a, b) \oplus \{(c, d), \cdots\}$ instead of $\{(a, b)\} \oplus \{(c, d), \cdots\}$.

$\mathbf{Z}$ denotes the set of integers. And $\mathbf{Z}_{\geq 0}$ denotes the set of non-negative integers.

2.2 Bi-degree Polynomials, Monomial Supports

A polynomial $f(s, t)$ is bi-degree (m, n) in the variables (s, t) if its degree in s and t are m and n respectively. That is, $f(s, t) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} s^i t^j$.

The monomial support of a polynomial $f(s, t)$ is the set of exponents (i, j) where the coefficients of the monomial $s^i t^j$ in f is non-zero. The monomial support of a general bi-degree (m, n) polynomial in (s, t) is thus

$$\mathcal{A}_{m,n} = \{0, 1, \cdots, m\} \times \{0, 1, \cdots, n\} = 0..m \times 0..n. \tag{2.3}$$

2.3 The Dixon Quotient, the Dixon Polynomial and the Dixon Matrix

In this section we describe the Dixon method of constructing resultants and along the way explain the notation used in the thesis.

Let f, g, h be bivariate polynomials on the monomial support $\mathcal{A} \subseteq \mathcal{A}_{m,n}$:

$$f(s,t) = \sum_{(i,j)\in\mathcal{A}} f_{i,j}s^i t^j,\ g(s,t) = \sum_{(i,j)\in\mathcal{A}} g_{i,j}s^i t^j,\ h(s,t) = \sum_{(i,j)\in\mathcal{A}} h_{i,j}s^i t^j. \tag{2.4}$$

Their Dixon quotient is

$$\Delta(f(s,t), g(\alpha,t), h(\alpha,\beta)) = \frac{\begin{vmatrix} f(s,t) & g(s,t) & h(s,t) \\ f(\alpha,t) & g(\alpha,t) & h(\alpha,t) \\ f(\alpha,\beta) & g(\alpha,\beta) & h(\alpha,\beta) \end{vmatrix}}{(s-\alpha)(t-\beta)}. \tag{2.5}$$

The quotient actually divides completely and becomes the Dixon polynomial

$$\Delta(f(s,t), g(\alpha,t), h(\alpha,\beta)) = \sum_{\sigma,\tau,a,b} \Delta_{\sigma,\tau,a,b} s^\sigma t^\tau \alpha^a \beta^b. \tag{2.6}$$

By writing the Dixon polynomial in the matrix form

$$\Delta = \begin{bmatrix} \cdots & s^\sigma t^\tau & \cdots \end{bmatrix} D \begin{bmatrix} \cdots & \alpha^a \beta^b & \cdots \end{bmatrix}^T, \tag{2.7}$$

we obtain the Dixon matrix for f, g, h

$$D = (\Delta_{\sigma,\tau,a,b}). \tag{2.8}$$

The monomials $s^\sigma t^\tau$ and $\alpha^a \beta^b$ that appear in the Dixon polynomial are called respectively the row and column indices of D. Furthermore, the monomial support of Δ considered as a polynomial in s, t or α, β is called the row support $\mathcal{R}$ or column support $\mathcal{C}$ of D respectively.

The classical Dixon resultant is the determinant $|D|$ when $\mathcal{A} = \mathcal{A}_{m,n}$. The row and column supports of the classical Dixon matrix are

$$\mathcal{R}_{m,n} = 0..m-1 \times 0..2n-1, \quad \mathcal{C}_{m,n} = 0..2m-1 \times 0..n-1. \tag{2.9}$$

Since the set cardinalities $|\mathcal{R}_{m,n}| = |\mathcal{C}_{m,n}| = 2mn$, the order of the classical Dixon matrix D is $2mn$.

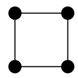
The coefficient $\Delta_{\sigma,\tau,a,b}$ is a sum of brackets, which are 3×3 determinants whose entries are coefficients of f, g, h. To be concise brackets are denoted by 6-tuples:

$$(i,j,k,l,p,q) = \begin{vmatrix} f_{i,j} & g_{i,j} & h_{i,j} \\ f_{k,l} & g_{k,l} & h_{k,l} \\ f_{p,q} & g_{p,q} & h_{p,q} \end{vmatrix} = \begin{vmatrix} f_{i,j} & f_{k,l} & f_{p,q} \\ g_{i,j} & g_{k,l} & g_{p,q} \\ h_{i,j} & h_{k,l} & h_{p,q} \end{vmatrix}. \tag{2.10}$$

To conserve space in examples we ignore punctuations; for example, we shorthand

$$(1,2,4,3,0,5) = 124305. \tag{2.11}$$

Example 1 *The Dixon polynomial for* $\mathcal{A}_{1,1} = 0..1 \times 0..1$ *is:*



$$\Delta = \mathcal{R}D\mathcal{C} = \begin{bmatrix} 1 & t \end{bmatrix} \begin{bmatrix} 001001 & 001011 \\ 001101 & 011011 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \end{bmatrix}. \tag{2.12}$$

$\square$

In expressions involving matrix multiplication, we let

$$(i,j) = \begin{bmatrix} f_{i,j} & g_{i,j} & h_{i,j} \end{bmatrix} \text{ or } \begin{bmatrix} f_{i,j} & g_{i,j} & h_{i,j} \end{bmatrix}^T. \tag{2.13}$$

Similarly the vector cross product $(k,l) \times (p,q)$ is treated either as a row or a column and is denoted with 4-tuples:

$$(k,l,p,q) = \left[\begin{vmatrix} g_{k,l} & h_{k,l} \\ g_{p,q} & h_{p,q} \end{vmatrix} \quad \begin{vmatrix} h_{k,l} & f_{k,l} \\ h_{p,q} & f_{p,q} \end{vmatrix} \quad \begin{vmatrix} f_{k,l} & g_{k,l} \\ f_{p,q} & g_{p,q} \end{vmatrix} \right] \text{ or its transpose.} \tag{2.14}$$

This notational polymorphism is very helpful because now we can express a bracket as a product of matrices in two ways:

$$(i,j,k,l,p,q) = (i,j)(k,l,p,q) = (k,l,p,q)(i,j). \tag{2.15}$$

This flexibility will greatly facilitate derivations involving brackets.

2.4 Presentation Convention

We want to state similar conditions at all the four corner $(0,0),(m,0),(m,n),(0,n)$ of $\mathcal{A}_{m,n}$ and then derive similar results in parallel for all the four corners. To do so concisely and make clear at a glance regarding the applicable corner, the following convention is adopted — we write

$$\text{If} \begin{array}{|c|c|}\hline P_{0,n} & P_{m,n} \\ \hline P_{0,0} & P_{m,0} \\ \hline \end{array} \text{then} \begin{array}{|c|c|}\hline Q_{0,n} & Q_{m,n} \\ \hline Q_{0,0} & Q_{m,0} \\ \hline \end{array}$$

to mean if $P_{i,j}$ then $Q_{i,j}$ where (i,j), $i=0,m$ and $j=0,n$.

To relate the points in $\mathcal{R}$ and $\mathcal{C}$ to the points in $\mathcal{A}$, we define the following convention:

Given the monomial point $(x,y)\in\mathcal{A}$, then (x',y') represents its corresponding monomial point in $\mathcal{R}$ given by

$$\begin{array}{|c|c|}\hline (0,n-1)\oplus(x,y) & (-1,n-1)\oplus(x,y) \\ \hline (0,0)\oplus(x,y) & (-1,0)\oplus(x,y) \\ \hline \end{array} \tag{2.16}$$

and (x'',y'') represents the corresponding monomial point in $\mathcal{C}$ given by

$$\begin{array}{|c|c|}\hline (0,-1)\oplus(x,y) & (m-1,-1)\oplus(x,y) \\ \hline (0,0)\oplus(x,y) & (m-1,0)\oplus(x,y) \\ \hline \end{array} \tag{2.17}$$

Let $S=a..b$ where a,b are two non-negative integers, then $S^*=-b..-a$ and

$$(S_1\times S_2)^* = \begin{array}{|c|c|}\hline S_1\times S_2^* & S_1^*\times S_2^* \\ \hline S_1\times S_2 & S_1^*\times S_2 \\ \hline \end{array}. \tag{2.18}$$

2.5 The Row/Column Supports of (i,j,k,l,p,q)

The following proposition gives the rows and columns in which the bracket (i,j,k,l,p,q) appears. In the proposition,

$$\begin{aligned}\mathcal{R}(i,j,k,l,p,q) &= \{(\sigma,\tau)|(i,j,k,l,p,q) \text{ is a term of } \Delta_{\sigma,\tau,a,b} \text{ for some } (a,b)\},\\ \mathcal{C}(i,j,k,l,p,q) &= \{(a,b)|(i,j,k,l,p,q) \text{ is a term of } \Delta_{\sigma,\tau,a,b} \text{ for some } (\sigma,\tau)\}.\end{aligned}$$

Proposition 1 *Let $i\le k\le p$. The row and column supports of the bracket (i,j,k,l,p,q) in D are*

$$\mathcal{R}(i,j,k,l,p,q) = (0,j)\oplus Q\cup(0,q)\oplus J \tag{2.19}$$

and

$$\mathcal{C}(i,j,k,l,p,q) = (p,0)\oplus Q\cup(i,0)\oplus J \tag{2.20}$$

where

$$\begin{aligned} Q &= i..k-1 \times \min(q,l)..\max(q,l)-1, \\ J &= k..p-1 \times \min(j,l)..\max(j,l)-1. \end{aligned}$$

2.6 The Row/Column Supports of D for when $\mathcal{A} = \mathcal{A}_{m,n} \backslash \cup_{i=0,m;j=0,n} E_{i,j}$

For the proofs of most of the theorems presented in Chapters 4 and 5, it is necessary to know exactly what the row and column supports are like after corner cutting has been applied to a monomial support $\mathcal{A}_{m,n}$. The following theorem describes the exact simplification effects of exterior points to the Dixon matrix.

Theorem 1 *Let the set of monomial supports removed at the four corners* $(0,0), (m,0), (m,n), (0,n)$ *be* $E_{0,0}, E_{m,0}, E_{m,n}, E_{0,n}$ *respectively with:*

$$\begin{array}{|c|c|} \hline E_{0,n} = 0..t \times l..m \setminus \cup_{i=1}^{N} t_i..t \times l..l_i & E_{m,n} = t..m \times r..n \setminus \cup_{i=1}^{N} t..t_i \times r..r_i \\ \hline E_{0,0} = 0..b \times 0..l \setminus \cup_{i=1}^{N} b_i..b \times l_i..l & E_{m,0} = b..m \times 0..r \setminus \cup_{i=1}^{N} b..b_i \times r_i..r \\ \hline \end{array} \tag{2.21}$$

Note the notations for a corner are not related to the same notations in other corners and

$$\begin{array}{|c|c|} \hline 0 < b_1 < \ldots < b_N < b, 0 < l_1 < \ldots < l_N < l & b > b_1 > \ldots > b_N > 0, 0 < l_1 < \ldots < l_N < l \\ \hline 0 < b_1 < \ldots < b_N < b, l > l_1 > \ldots > l_N > 0 & b > b_1 > \ldots > b_N > 0, l > l_1 > \ldots > l_N > 0 \\ \hline \end{array} \tag{2.22}$$

The row support $\mathcal{R}$ *of* D *is*

$$\mathcal{R} = \mathcal{R}_{m,n} \setminus \begin{array}{|ccc|ccc|} \hline (0,n-1) & \oplus & E_{0,n} & (-1,n-1) & \oplus & E_{m,n} \\ \hline (0,0) & \oplus & E_{0,0} & (-1,0) & \oplus & E_{m,0} \\ \hline \end{array} \tag{2.23}$$

and the column support $\mathcal{C}$ *of* D *is*

$$\mathcal{C} = \mathcal{C}_{m,n} \setminus \begin{array}{|ccc|ccc|} \hline (0,-1) & \oplus & E_{0,n} & (m-1,-1) & \oplus & E_{m,n} \\ \hline (0,0) & \oplus & E_{0,0} & (m-1,0) & \oplus & E_{m,0} \\ \hline \end{array}. \tag{2.24}$$

<u>Proof</u>

Consider the points:

$$\begin{array}{|c|c|} \hline (0,l-1),\cdots,(t_i,l_i),\cdots,(t+1,n) & (m,r-1),\cdots,(t_i,r_i),\cdots,(t-1,n) \\ \hline (0,l+1),\cdots,(b_i,l_i),\cdots,(b+1,0) & (m,r+1),\cdots,(b_i,r_i),\cdots,(b-1,0) \\ \hline \end{array}. \tag{2.25}$$

Let (i,j,k,l,p,q) be the bracket

$$\begin{array}{|c|c|} \hline (0,l-1,t_i,l_i,t+1,n) & (m,r-1,t_i,r_i,t-1,n) \\ \hline (0,l+1,b_i,l_i,b+1,0) & (m,r+1,b_i,r_i,b-1,0) \\ \hline \end{array} \tag{2.26}$$

then by Proposition 1, row support $\mathcal{R}(i,j,k,l,p,q)$ contains the following subset:

$$\begin{array}{|ccc|ccc|} \hline (0,n-1) & \oplus & t_i..t \times l..l_i & (-1,n-1) & \oplus & t..t_i \times r..r_i \\ \hline (0,0) & \oplus & b_i..b \times l_i..l & (-1,0) & \oplus & b..b_i \times r_i..r \\ \hline \end{array} \tag{2.27}$$

and the column support $\mathcal{C}(i,j,k,l,p,q)$ contains the subset:

$$\begin{array}{|ccc|ccc|}\hline (0,-1) & \oplus & t_i..t\times l..l_i & (m-1,-1) & \oplus & t..t_i\times r..r_i \\ \hline (0,0) & \oplus & b_i..b\times l_i..l & (m-1,0) & \oplus & b..b_i\times r_i..r \\ \hline \end{array}\,. \tag{2.28}$$

So the rows indexed by

$$\mathcal{R}_e = \begin{array}{|ccc|ccc|}\hline (0,n-1) & \oplus & \cup_{i=1}^N t_i..t\times l..l_i & (-1,n-1) & \oplus & \cup_{i=1}^N t..t_i\times r..r_i \\ \hline (0,0) & \oplus & \cup_{i=1}^N b_i..b\times l_i..l & (-1,0) & \oplus & \cup_{i=1}^N b..b_i\times r_i..r \\ \hline \end{array} \tag{2.29}$$

and the columns indexed by

$$\mathcal{C}_e = \begin{array}{|ccc|ccc|}\hline (0,-1) & \oplus & \cup_{i=1}^N t_i..t\times l..l_i & (m-1,-1) & \oplus & \cup_{i=1}^N t..t_i\times r..r_i \\ \hline (0,0) & \oplus & \cup_{i=1}^N b_i..b\times l_i..l & (m-1,0) & \oplus & \cup_{i=1}^N b..b_i\times r_i..r \\ \hline \end{array} \tag{2.30}$$

are nonzero.

Furthermore, from [2] it is already known that the rows of D indexed by

$$\mathcal{R}_r = \mathcal{R}_{m,n} \setminus \begin{array}{|ccc|ccc|}\hline (0,n-1) & \oplus & 0..t\times l..n & (-1,n-1) & \oplus & t..m\times r..n \\ \hline (0,0) & \oplus & 0..b\times 0..l & (-1,0) & \oplus & b..m\times 0..r \\ \hline \end{array} \tag{2.31}$$

and the columns D indexed by

$$\mathcal{C}_r = \mathcal{C}_{m,n} \setminus \begin{array}{|ccc|ccc|}\hline (0,-1) & \oplus & 0..t\times l..n & (m-1,-1) & \oplus & t..m\times r..n \\ \hline (0,0) & \oplus & 0..b\times 0..l & (m-1,0) & \oplus & b..m\times 0..r \\ \hline \end{array} \tag{2.32}$$

are non-zero.

According to the proofs in [2], it can easily deduced that the corner cutting given in (2.21) is inherited by both row and column supports. Since

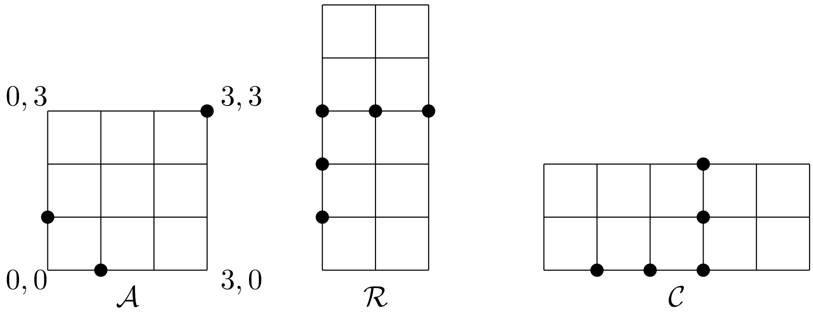
$$\mathcal{R} = \mathcal{R}_e \cup \mathcal{R}_r, \quad \mathcal{C} = \mathcal{C}_e \cup \mathcal{C}_r. \tag{2.33}$$

Thus $\mathcal{R}$ is the row support of D and $\mathcal{C}$ is the column support of D. This finishes the proof.

Q.E.D

The following example uses the above theorem to find the row and column support after the corner cutting:

Example 2 *Given the monomial support $\mathcal{A}$, we can find the row support $\mathcal{R}$ and column support $\mathcal{C}$ are:*



0, 3
3, 3
0, 0
3, 0
$\mathcal{A}$
$\mathcal{R}$
$\mathcal{C}$

□

Chapter 3

Loose Entry Formulas

Formal power series are used to derive four entry formulas for the Dixon matrix. With an uncut monomial support, these entry formulas have uniform summation bounds for the entire Dixon matrix. With a corner-cut monomial support, each of the four loose entry formulas simplifies greatly for some rows and columns associated with a particular corner but still maintains uniform summation bounds. Uniform summation bounds make the entry formulas loose because redundant brackets that eventually vanish are produced. But uniform summation bounds reveal valuable information about the properties of the Dixon matrix for a corner-cut monomial support.

This chapter consists of three sections. Section 3.1 presents the four loose entry formulas for the Dixon matrix in two theorems. Section 3.2 customizes the entry formulas for some rows and columns when the monomial support undergoes corner cutting. Section 3.3 gives a comparison between the concise entry formula in [1] and the loose entry formulas.

3.1 Four Loose Entry Formulas for Uncut Monomial Supports

The denominator of the Dixon quotient (2.5) can be regarded as a formal power series. Four ways of expanding this formal power series lead to four equivalent loose entry formulas that are different in form.

Theorem 2 *The Dixon matrix entry indexed by* $(s^\sigma t^\tau, \alpha^a \beta^b)$ *is*

$$\Delta_{\sigma,\tau,a,b} = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{k=0}^{m} \sum_{l=0}^{n} B \tag{3.1}$$

where

$$B = (\sigma + u + 1, \tau + v + 1 - l, k, l, a - u - k, b - v), \text{ or} \tag{3.2}$$

$$B = -(\sigma - u, \tau + v + 1 - l, k, l, a + u + 1 - k, b - v), \text{ or} \tag{3.3}$$

$$\begin{aligned} B &= (\sigma - u, \tau - v - l, k, l, a + u + 1 - k, b + v + 1), \ \ or & (3.4) \\ B &= -(\sigma + u + 1, \tau - v - l, k, l, a - u - k, b + v + 1) & (3.5) \end{aligned}$$

Proof

The entry formulas can be derived in parallel simply by expanding the quotient in the Dixon quotient four ways in terms of $\frac{s}{\alpha}$ or $\frac{\alpha}{s}$ and $\frac{t}{\beta}$ or $\frac{\beta}{t}$:

$$\begin{aligned} \frac{1}{(s-\alpha)(t-\beta)} &= -\sum_{u=0}^{\infty} \frac{\alpha^u}{s^{u+1}} \sum_{v=0}^{\infty} \frac{t^v}{\beta^{v+1}} &= \sum_{u=0}^{\infty} \frac{s^u}{\alpha^{u+1}} \sum_{v=0}^{\infty} \frac{t^v}{\beta^{v+1}} \\ &= \sum_{u=0}^{\infty} \frac{\alpha^u}{s^{u+1}} \sum_{v=0}^{\infty} \frac{\beta^v}{t^{v+1}} &= -\sum_{u=0}^{\infty} \frac{s^u}{\alpha^{u+1}} \sum_{v=0}^{\infty} \frac{\beta^v}{t^{v+1}} \quad . \end{aligned} \tag{3.6}$$

But

$$\begin{vmatrix} f(s,t) & g(s,t) & h(s,t) \\ f(\alpha,t) & g(\alpha,t) & h(\alpha,t) \\ f(\alpha,\beta) & g(\alpha,\beta) & h(\alpha,\beta) \end{vmatrix} = \sum_{i,j,k,l,p,q} (i,j,k,l,p,q) s^i t^{j+l} \alpha^{k+p} \beta^q \tag{3.7}$$

Thus, the Dixon polynomial can be written in any of the four expansions:

$$\begin{aligned} \Delta = &= \sum_{i,j,k,l,p,q} \sum_{u,v} (i,j,k,l,p,q) s^{i-u-1} t^{j+l-v-1} \alpha^{k+p+u} \beta^{q+v} \\ &= -\sum_{i,j,k,l,p,q} \sum_{u,v} (i,j,k,l,p,q) s^{i+u} t^{j+l-v-1} \alpha^{k+p-u-1} \beta^{q+v} \\ &= \sum_{i,j,k,l,p,q} \sum_{u,v} (i,j,k,l,p,q) s^{i+u} t^{j+l+v} \alpha^{k+p-u-1} \beta^{q-v-1} \\ &= -\sum_{i,j,k,l,p,q} \sum_{u,v} (i,j,k,l,p,q) s^{i-u-1} t^{j+l+v} \alpha^{k+p+u} \beta^{q-v-1} \quad . \end{aligned}$$

By comparing the coefficients of

$$\Delta = \sum_{\sigma,\tau,a,b} \Delta_{\sigma,\tau,a,b} s^\sigma t^\tau \alpha^a \beta^b \tag{3.8}$$

to each of the four expansions, we obtain respectively the four equations given by (3.2), (3.3), (3.4), (3.5).

Q.E.D

The above entry formulas will generate three types of redundant brackets:

1. self vanishing brackets such as (i,j,i,j,p,q),

2. mutually canceled brackets such as $(i,j,k,l,p,q) + (i,j,p,q,k,l)$, and

3. out of range brackets (i, j, k, l, p, q) with (i, j), (k, l), or $(p, q) \notin \mathcal{A}_{m,n}$. A bracket involving out of range indices is zero since $(i, j) \notin \mathcal{A}_{m,n}$ means $f_{i,j} = g_{i,j} = h_{i,j} = 0$.

For practical computation, we have to shrink the ranges of u and v in the above entry formulas:

Theorem 3 *The Dixon matrix entry indexed by* $(s^\sigma t^\tau, \alpha^a \beta^b)$ *is*

$$\Delta_{\sigma,\tau,a,b} = \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} \sum_{k=0}^{m} \sum_{l=0}^{n} B \tag{3.9}$$

where either $B = (i, j, k, l, p, q)$ *which can be (3.2) or (3.4), or* $B = -(i, j, k, l, p, q)$ *which can be (3.3) or (3.5).*

Proof

When $i = \sigma + u + 1$, to have $(i, j) \in \mathcal{A}_{m,n}$ we need

$$\sigma + u + 1 \leq m \Rightarrow u \leq m - 1 \ . \tag{3.10}$$

When $i = \sigma - u$, to have $(i, j) \in \mathcal{A}_{m,n}$, we need

$$\sigma - u \geq 0 \Rightarrow u \leq \sigma \leq m - 1 \ . \tag{3.11}$$

Consequently, the upper bound of u is reduced from ∞ to $m - 1$.

Similarly, when $q = b + v + 1$, to have $(p, q) \in \mathcal{A}_{m,n}$ we need

$$b + v + 1 \leq n \Rightarrow v \leq n - 1 \ . \tag{3.12}$$

When $q = b - v$, to have $(p, q) \in \mathcal{A}_{m,n}$ we need

$$b - v \geq 0 \Rightarrow v \leq b \leq n - 1 \ . \tag{3.13}$$

Consequently, the upper bound of v is reduced from ∞ to $n - 1$.

Q.E.D

Example 3 *We use Theorem 3 to compute the Dixon matrix* D *in Example 1 in four ways. Using (3.2), (3.3), (3.4), (3.5), we have respectively*

$$\begin{aligned} D_1 &= \begin{bmatrix} \sum_{k=0}^{1} \sum_{l=0}^{1} (1, 1-l, k, l, -k, 0) & \sum_{k=0}^{1} \sum_{l=0}^{1} (1, 1-l, k, l, 1-k, 0) \\ \sum_{k=0}^{1} \sum_{l=0}^{1} (1, 2-l, k, l, -k, 0) & \sum_{k=0}^{1} \sum_{l=0}^{1} (1, 2-l, k, l, 1-k, 0) \end{bmatrix} \\ &= \begin{bmatrix} 100100 & 110010 \\ 110100 & 110110 \end{bmatrix} , \end{aligned} \tag{3.14}$$

$$\begin{aligned} D_2 &= \begin{bmatrix} -\sum_{k=0}^{1}\sum_{l=0}^{1}(0,1-l,k,l,1-k,0) & -\sum_{k=0}^{1}\sum_{l=0}^{1}(0,1-l,k,l,2-k,0) \\ -\sum_{k=0}^{1}\sum_{l=0}^{1}(0,2-l,k,l,1-k,0) & -\sum_{k=0}^{1}\sum_{l=0}^{1}(0,2-l,k,l,2-k,0) \end{bmatrix} \\ &= \begin{bmatrix} -011000 & -001110 \\ -011100 & -011110 \end{bmatrix}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} D_3 &= \begin{bmatrix} \sum_{k=0}^{1}\sum_{l=0}^{1}(0,-l,k,l,1-k,1) & \sum_{k=0}^{1}\sum_{l=0}^{1}(0,-l,k,l,2-k,1) \\ \sum_{k=0}^{1}\sum_{l=0}^{1}(0,1-l,k,l,1-k,1) & \sum_{k=0}^{1}\sum_{l=0}^{1}(0,1-l,k,l,2-k,1) \end{bmatrix} \\ &= \begin{bmatrix} 001001 & 001011 \\ 001101 & 011011 \end{bmatrix}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} D_4 &= \begin{bmatrix} \sum_{k=0}^{1}\sum_{l=0}^{1}-(1,-l,k,l,-k,1) & \sum_{k=0}^{1}\sum_{l=0}^{1}-(1,-l,k,l,1-k,1) \\ \sum_{k=0}^{1}\sum_{l=0}^{1}-(1,1-l,k,l,-k,1) & \sum_{k=0}^{1}\sum_{l=0}^{1}-(1,1-l,k,l,1-k,1) \end{bmatrix} \\ &= \begin{bmatrix} -100001 & -100011 \\ -110001 & -100111 \end{bmatrix}. \end{aligned} \tag{3.17}$$

From the properties of determinants, it is obvious that $D = D_1 = D_2 = D_3 = D_4$. □

3.2 Corner-Specific Simplification

The following theorems describes the corner-specific simplification effects for the entries of some rows and columns when corner cutting is applied.

Theorem 4 *Let* $(x,y) \in \mathcal{A}_{m,n}$ *and cutting* C *be applied to* $\mathcal{A}_{m,n}$ *such that* $(x,y) \in \mathcal{A} \subseteq \mathcal{A}_{m,n} \setminus C$ *with* C *given by*

$0..x \times y..n \setminus \{(x,y)\}$	$x..m \times y..n \setminus \{(x,y)\}$
$0..x \times 0..y \setminus \{(x,y)\}$	$x..m \times 0..y \setminus \{(x,y)\}$

. (3.18)

Then the entries of the rows indexed by (x',y') *given by*

$(x,y) \oplus (0,n-1)$	$(x,y) \oplus (-1,n-1)$
$(x,y) \oplus (0,0)$	$(x,y) \oplus (-1,0)$

(3.19)

are $\Delta_{x',y',a,b}$ *given by*

$-\sum_{k=0}^{m}(x,y,k,n,a+1-k,b)$	$\sum_{k=0}^{m}(x,y,k,n,a-k,b)$
$\sum_{k=0}^{m}(x,y,k,0,a+1-k,b+1)$	$-\sum_{k=0}^{m}(x,y,k,0,a-k,b+1)$

. (3.20)

Proof

Apply the entry formula in Theorem 3 and choose the bracket B in the formula to be

Equation (3.3)	Equation (3.2)
Equation (3.4)	Equation (3.5)

(3.21)

After substituting $(\sigma, \tau) = (x', y')$ into B, the first ordered pair of B becomes

$$\begin{array}{|c|c|}\hline (x-u, y+n+v-l) & (x+u, y+n+v-l) \\ \hline (x-u, y-v-l) & (x+u, y-v-l) \\ \hline \end{array}\ . \tag{3.22}$$

To have this ordered pair not in C, we need

$$\begin{array}{|c|c|}\hline -u \geq 0, n+v-l \leq 0 & u \leq 0, n+v-l \leq 0 \\ \hline -u \geq 0, -v-l \geq 0 & u \leq 0, -v-l \geq 0 \\ \hline \end{array} \tag{3.23}$$

Thus,

$$\begin{array}{|c|c|}\hline u=v=0, l=n & u=v=0, l=n \\ \hline u=v=l=0 & u=v=l=0 \\ \hline \end{array} \tag{3.24}$$

Substituting the values of u, v, l into the entry formula we obtain the row entry formulas (3.20).

Q.E.D

Theorem 5 *Let $(x, y) \in \mathcal{A}_{m,n}$. If corner cutting C (3.18) is applied, then the entries of the column indexed by (x'', y'') given as*

$$\begin{array}{|c|c|}\hline (x,y) \oplus (0,-1) & (x,y) \oplus (m-1,-1) \\ \hline (x,y) \oplus (0,0) & (x,y) \oplus (m-1,0) \\ \hline \end{array} \tag{3.25}$$

are $\Delta_{\sigma,\tau,x'',y''}$ given as

$$\begin{array}{|c|c|}\hline -\sum_{l=0}^{n}(\sigma+1, \tau-l, 0, l, x, y) & \sum_{l=0}^{n}(\sigma, \tau-l, m, l, x, y) \\ \hline \sum_{l=0}^{n}(\sigma+1, \tau+1-l, 0, l, x, y) & -\sum_{l=0}^{n}(\sigma, \tau+1-l, m, l, x, y) \\ \hline \end{array} \tag{3.26}$$

Proof

Apply the entry formula in Theorem 3 and choose the bracket B in the formula to be

$$\begin{array}{|c|c|}\hline \text{Equation (3.5)} & \text{Equation (3.4)} \\ \hline \text{Equation (3.2)} & \text{Equation (3.3)} \\ \hline \end{array} \tag{3.27}$$

After substituting $(a, b) = (x'', y'')$ into B, the last ordered pair of B becomes:

$$\begin{array}{|c|c|}\hline (x-u-k, y+v) & (x+u+m-k, y+v) \\ \hline (x-u-k, y-v) & (x+u+m-k, y-v) \\ \hline \end{array} \tag{3.28}$$

To have this ordered pair not in C, we need

$$\begin{array}{|c|c|}\hline -u-k \geq 0, v \leq 0 & u+m-k \leq 0, v \leq 0 \\ \hline -u-k \geq 0, -v \geq 0 & u+m-k \leq 0, -v \geq 0 \\ \hline \end{array} \tag{3.29}$$

Thus,

$$\begin{array}{|c|c|}\hline u=v=k=0 & u=v=0,k=m \\ \hline u=v=k=0 & u=v=0,k=m \\ \hline \end{array} \tag{3.30}$$

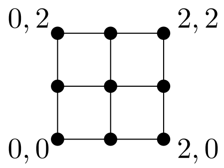
Substituting the values of u, v, k into the entry formula, we obtain the column entry formulas (3.26).

Q.E.D

Remark 1 *Monomial points like* (x, y) *discussed in Theorems 4 and 5 are called exposed points. They will be defined formally in Chapter 4. Their significance, besides the simplification effects mentioned in the theorems, will be discussed in Chapter 4.*

The following examples illustrate the row and column loose entry formulas (3.20), (3.26) for a corner-cut monomial support.

Example 4 *Consider the monomial support* $\mathcal{A} = \mathcal{A}_{2,2}$*:*



Take $(x, y) = (2, 2)$*. It can be easily seen that* $(2, 2)$ *satisfies the cutting condition (3.18) for the top right corner with* $C = \emptyset$*.*

Using formulas (3.20) and (3.26) we can easily calculate the row indexed by $(1, 3) = (2, 2) \oplus (-1, 1)$ *and the column indexed by* $(3, 1) = (2, 2) \oplus (1, -1)$ *respectively:*

$$D(s^1t^3, \alpha^a\beta^b) = \sum_{k=0}^{2}(2, 2, k, 2, a-k, b) \tag{3.31}$$

$$D(s^\sigma t^\tau, \alpha^3\beta^1) = \sum_{l=0}^{2}(\sigma, \tau - l, 2, l, 2, 2). \tag{3.32}$$

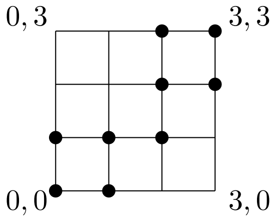
Substituting $(a, b) = (3, 1)$ *into Equation (3.31) and* $(\sigma, \tau) = (1, 3)$ *into Equation (3.32), we obtain the value of the entry* $D(s^1t^3, \alpha^3\beta^1)$ *in two ways:*

$$\sum_{k=0}^{2}(2, 2, k, 2, 3-k, 1) = 220231 + 221221 + 222211 = 221221, \tag{3.33}$$

$$\sum_{l=0}^{2}(1, 3-l, 2, l, 2, 2) = 132022 + 122122 + 112222 = 122122. \tag{3.34}$$

Note that brackets $220231 = 132022 = 0$ *since the indices* $(3, 1) \notin \mathcal{A}$*,* $(1, 3) \notin \mathcal{A}$*; and the brackets* $222211 = 112222 = 0$ *because they have two identical rows (or columns).* □

Example 5 *Consider the following monomial support* $\mathcal{A} \subseteq \mathcal{A}_{3,3}$*:*



It is easy to check that the sets $\{(1,0),(2,1),(3,2)\}$ *and* $\{(0,1),(2,3)\}$ *satisfy the cutting conditions (3.18) for the bottom right and top left corners respectively. Thus the* 3×2 *sub-matrix formed with the rows indexed by*

$$\{(1,0),(2,1),(3,2)\} \oplus (-1,0) = \{(0,0),(1,1),(2,2)\} \tag{3.35}$$

and the the columns indexed by

$$\{(0,1),(2,3)\} \oplus (0,-1) = \{(0,0),(2,2)\}$$

can be computed in two ways.

Using the row entry formula (3.20) for the bottom right corner, we have:

$$S_1 = \begin{bmatrix} -\sum_{k=0}^{3}(1,0,k,0,-k,1) & -\sum_{k=0}^{3}(1,0,k,0,2-k,3) \\ -\sum_{k=0}^{3}(2,1,k,0,-k,1) & -\sum_{k=0}^{3}(2,1,k,0,2-k,3) \\ -\sum_{k=0}^{3}(3,2,k,0,-k,1) & -\sum_{k=0}^{3}(3,2,k,0,2-k,3) \end{bmatrix} \tag{3.36}$$

Using the column entry formula (3.26) for the top left corner, we have:

$$S_2 = \begin{bmatrix} -\sum_{l=0}^{3}(1,0-l,0,l,0,1) & -\sum_{l=0}^{3}(1,0-l,0,l,2,3) \\ -\sum_{l=0}^{3}(2,1-l,0,l,0,1) & -\sum_{l=0}^{3}(2,1-l,0,l,2,3) \\ -\sum_{l=0}^{3}(3,2-l,0,l,0,1) & -\sum_{l=0}^{3}(3,2-l,0,l,2,3) \end{bmatrix} \tag{3.37}$$

A simple calculation shows that

$$S_1 = \begin{bmatrix} -100001 & -100023 \\ -210001 & -210023 \\ -320001 & -320023 \end{bmatrix} = S_2. \tag{3.38}$$

□

3.3 Comparison of Loose Entry Formulas and Concise Entry Formula

The loose entry formulas presented in this chapter have very simple summation bounds:

$$\Delta_{\sigma,\tau,a,b} \quad = \quad \sum_{u=0}^{m-1}\sum_{v=0}^{n-1}\sum_{k=0}^{m}\sum_{l=0}^{n} B$$

where the bracket B can be any one of (3.2), (3.3), (3.4), (3.5), this is in sharp contrast with the complicated summation bounds in the concise entry formula given in [1]:

$$\begin{aligned}\Delta_{\sigma,\tau,a,b} &= \sum_{u=0}^{\min(a,m-1-\sigma)} \sum_{v=0}^{\min(b,2n-1-\tau)} \sum_{k=\max(0,a-u-\sigma)}^{\min(m,a-u)} \sum_{l=\max(b+1,\tau+1+v-b)}^{\min(n,\tau+1+v)} B \\ &+ \sum_{u=0}^{\min(a,m-1-\sigma)} \sum_{v=0}^{\min(b,2n-1-\tau)} \sum_{k=\max(0,a-u-m)}^{\min(\sigma,a-u)} \sum_{l=\max(b+1,\tau+1+v-n)}^{\min(n,\tau+v-b)} B\end{aligned}$$

where B is given by formula (3.2). Even more significant is that when cutting of the types in Theorems 4, 5 are applied the entry formulas for certain rows and columns further simplify such that only one summation bound, instead of four, remains.

This single-uniform summation bound form is very helpful for discovering properties of the Dixon matrix. For example, with Theorem 4, we immediately see that the row associated with the monomial point (x, y) defined by (3.18) contains the monomial point (x, y) in every bracket of the sum. This observation, together with a single-uniform summation bound, leads to important conclusions concerning the linear dependence of and the bracket factors produced by some rows and columns. More details will be discussed in Chapter 4.

The loose entry formulas are very convenient for deriving theoretical results, but when computing the Dixon entry it is better to use the concise entry formula as it produces no redundant brackets. The total number of brackets in the Dixon matrix for bidegree polynomial [1] is

$$\frac{m(m+1)^2(m+2)n(n+1)^2(n+2)}{36}$$

but with the loose entry formulas the total number of brackets produced is

$$4m^3(m+1)n^3(n+1) \ .$$

Thus it is almost one hundred times faster to compute the Dixon matrix using the concise entry formula than using a loose entry formula.

Chapter 4

Exposed Points

We introduce the concept of exposed points. They are crucial in the construction of explicit sparse resultant quotients using the Dixon method. This is because rows and columns of the Dixon matrix that are associated with exposed points possess simple matrix entries. Consequently we know exactly when these rows and columns will be linearly dependent — a knowledge helpful in finding a maximal minor of the Dixon matrix, and what factors these rows and columns will produce — a knowledge helpful in determining the extraneous factors corresponding to the chosen maximal minor. Furthermore, the number of exposed points with respect to a monomial support corner serves as a key for classifying corner-cut unmixed monomial supports for which explicit sparse resultant quotients are to be constructed.

This chapter is organized as follows. Section 4.1 defines exterior points, exposed points, and corner cutting. Section 4.2 proves the important properties of exposed points. Section 4.3 classifies corner cut monomial supports with several theorems and conjectures.

4.1 Exterior Points, Exposed Points, and Corner Cutting

In this section we introduce exterior points and describe how exterior points lead to exposed points and corner cutting.

4.1.1 Exterior Points

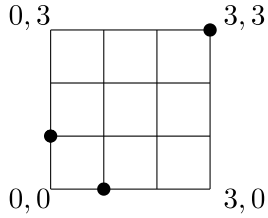
Exterior points are defined with respect to corners.

Definition 1 *A point $(x, y) \in \mathcal{A}_{m,n}$ is an exterior point of the monomial support $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ with respect to a corner if to that corner we have*

$$\begin{array}{|c|c|} \hline (0..x \times y..n) \cap \mathcal{A} = \emptyset & (x..m \times y..n) \cap \mathcal{A} = \emptyset \\ \hline (0..x \times 0..y) \cap \mathcal{A} = \emptyset & (x..m \times 0..y) \cap \mathcal{A} = \emptyset \\ \hline \end{array}. \tag{4.1}$$

Intuitively, a point $(x, y) \in \mathcal{A}_{m,n}$ is an exterior point with respect to a corner if every monomial point in the rectangle with (x, y) and the corner as diagonal is cut (removed from $\mathcal{A}_{m,n}$). [8] calls the set of exterior points as support complement of $\mathcal{A}$. Here we emphasize individual exterior points and the association of an exterior point to a corner.

Example 6 *Consider the monomial support* $\mathcal{A} = \{(1, 0), (0, 1), (3, 3)\} \subseteq \mathcal{A}_{3,3}$*:*



The sets of exterior points with respect to the four corners $(0, 0)$, $(3, 0)$, $(3, 3)$, $(0, 3)$ *are respectively:*

$0..2 \times 2..3$	$\emptyset$
$\{(0, 0)\}$	$2..3 \times 0..2$

(4.2)

□

4.1.2 Exposed Points

Next we define the concept of exposed points with respect to a corner.

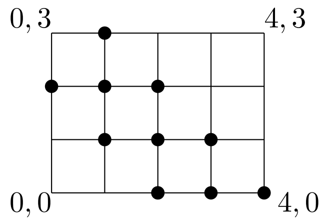
Definition 2 *A point* $(x, y) \in \mathcal{A}_{m,n}$ *is an exposed point of the monomial support* $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ *with respect to a corner if to that corner we have*

$(0..x \times y..n) \cap \mathcal{A} = \{(x, y)\}$	$(x..m \times y..n) \cap \mathcal{A} = \{(x, y)\}$
$(0..x \times 0..y) \cap \mathcal{A} = \{(x, y)\}$	$(x..m \times 0..y) \cap \mathcal{A} = \{(x, y)\}$

(4.3)

Intuitively, a point $(x, y) \in \mathcal{A}_{m,n}$ is an exposed point with respect to a corner if all monomial points (i, j) in the rectangle with (x, y) and the corner as diagonal are cut (removed from $\mathcal{A}_{m,n}$) except (x, y). Exposed points are called support hall vertices in [8]. Again here we stress the association of an exposed point to a corner.

Example 7 *Consider the monomial support* $\mathcal{A} \subseteq \mathcal{A}_{4,3}$, $|\mathcal{A}| = 10$*:*



The exposed points with respect to the corners $(0, 0)$, $(4, 0)$, $(4, 3)$, $(0, 3)$ *are respectively*

$(0, 2), (1, 3)$	$(1, 3), (2, 2), (3, 1), (4, 0)$
$(0, 2), (1, 1), (2, 0)$	$(4, 0)$

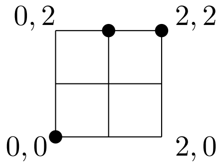
(4.4)

□

We also need to distinguish a class of exposed points called singular exposed points.

Definition 3 *An exposed point is singular if it is exposed to two adjacent corners. A singular exposed point with respect to the two bottom, two right, two top, two left corners are called respectively a bottom, right, top, or left singular exposed point.*

Example 8 *Consider the monomial support* $\mathcal{A} = \{(0,0),(1,2),(2,2)\} \subseteq \mathcal{A}_{2,2}$*:*



The exposed points at the four corners are respectively

$(0,0),(1,2)$	$(2,2)$
$(0,0)$	$(0,0),(2,2)$

(4.5)

□

Note that $(0,0)$ is both a bottom singular and a left singular exposed point; and $(2,2)$ is a right singular exposed point.

4.1.3 Corner Cutting

Definition 4 *Corner cutting refers to the introduction of exterior points to* $\mathcal{A}_{m,n}$ *to obtain a monomial support* $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ *without removing any edge of* $\mathcal{A}_{m,n}$ *entirely; that is,*

$$\begin{array}{ll} \mathcal{A} \cap (0..m \times 0) \neq \emptyset, & \mathcal{A} \cap (0..m \times n) \neq \emptyset; \\ \mathcal{A} \cap (0 \times 0..n) \neq \emptyset, & \mathcal{A} \cap (m \times 0..n) \neq \emptyset. \end{array} \tag{4.6}$$

The non-empty intersection with the edges condition (4.6) loses no generality; they either prevent unnecessarily high degrees due to zero coefficients or disallow trivial common factors of f, g, h of the form $s^u t^v$.

4.2 Effects of Exposed Points

All the results in the chapter come from the eight loose entry formulas given in Theorems 4 and 5. The loose entry formulas (3.20), (3.26) in Theorems 4, 5 have the following immediate consequences.

4.2.1 Inheritance of Exterior Points

[2] showed laboriously that exterior points are inherited in the row and column supports of D. This fact is obtained again here as a trivial consequence of the loose entry formulas.

Theorem 6 *Exterior points of the monomial Support $\mathcal{A}$ are inherited by the row support $\mathcal{R}$ and the column Support $\mathcal{C}$. That is, if $(x, y) \in \mathcal{A}_{m,n}$ is an exterior point with respect to a corner, then (x', y'), dependent on the corner and given by (3.19), is an exterior point of the row support $\mathcal{R}$ with respect to the same corner; and (x'', y''), dependent on the corner and given by (3.25), is an exterior point of the column support $\mathcal{C}$ with respect to the same corner.*

Proof

From the loose row and column entry formulas (3.20) and (3.26), we see that when an exposed point (x, y) is removed from $\mathcal{A}$ to become an exterior point, the formulas produce zero because every bracket in the formulas involves (x, y). That is, each time an exposed point is converted to an exterior point at a corner, a zero row and a zero column are introduced. In other words, the changes needed to convert $\mathcal{A}_{m,n}$ to $\mathcal{A}$ trigger similar changes at the row and column supports at the corresponding corners, and these changes convert $\mathcal{R}_{m,n}$ to $\mathcal{R}$ and $\mathcal{C}_{m,n}$ to $\mathcal{C}$.

Q.E.D

Example 9 *If $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ is a corner-cut monomial support and $|\mathcal{A}| = 2$, then the Dixon polynomial is zero. Due to Condition (4.6), there are only two possible cases:*

$$\mathcal{A} = \{(0, n), (m, 0)\}, \quad \mathcal{A} = \{(0, 0), (m, n)\} \tag{4.7}$$

and their diagrams are respectively:

In either case, there are $mn + mn = 2mn$ exterior points. By Theorem 6 and its remark we have $|\mathcal{R}| = |\mathcal{C}| = 2mn - 2mn = 0$. That is, all rows and columns are zero and the Dixon polynomial vanishes. This fact can also be established by brute force computation but the above derivation seems quite elegant. □

4.2.2 Inheritance of Non-Singular Exposed Points

Theorem 7 *Let $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ be a corner-cut monomial support. Let $\mathcal{R}$ and $\mathcal{C}$ be respectively the row and column support of D. If $(x, y) \in \mathcal{A}$ is a non-singular exposed point with respect to some corner, then the point $(x', y') \in \mathcal{R}$, dependent on the corner and given by (3.19), is an exposed point of $\mathcal{R}$ with respect to the same corner; the point $(x'', y'') \in \mathcal{C}$, dependent on the corner and given by (3.25), is an exposed point of $\mathcal{C}$ with respect to the same corner.*

Proof

Since (x, y) is a non-singular exposed point with respect to a corner and $\mathcal{A}$ is obtained by corner-cutting, there exist monomial points

$(0, l), (t, n) \in \mathcal{A}, t > x, l < y$	$(m, r), (t, n) \in \mathcal{A}, t < x, r < y$
$(0, l), (b, 0) \in \mathcal{A}, b > x, l > y$	$(m, r), (b, 0) \in \mathcal{A}, b < x, r > y$

(4.8)

It is routine to check, using Equation (3.20), that the row indexed by (x', y') contains the bracket

$-(x, y, t, n, 0, l)$	(x, y, t, n, m, r)
$(x, y, b, 0, 0, l)$	$-(x, y, b, 0, m, r)$

(4.9)

at the column indexed by

$(t-1, l)$	$(m+t, r)$
$(b-1, l-1)$	$(m+b, r-1)$

(4.10)

The above shows that (x', y') indexes a non-zero row, together with Theorem 6, we conclude that (x', y') is an exposed point of $\mathcal{R}$.

Again, it is routine to check, using Equation (3.26), that the columns indexed by (x'', y'') contains the bracket (4.9) at the row indexed by

$(t-1, n+l)$	$(t, n+r)$
$(b-1, l-1)$	$(b, r-1)$

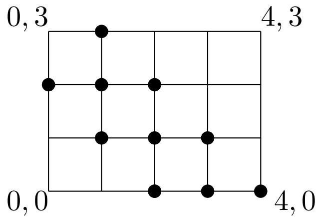
(4.11)

The above shows that (x'', y'') indexes a non-zero column, together with Theorem 6, we conclude that (x'', y'') is an exposed point of $\mathcal{R}$.

Q.E.D

The above theorem actually follows more or less from Theorem 1. But we proved it this way to illustrate the usefulness of the loose entry formulas.

Example 10 *Consider the monomial support given in Example 7:*

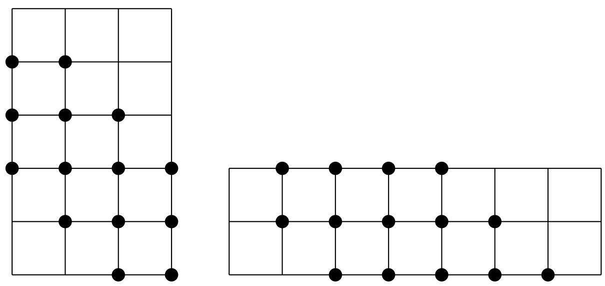


The set of non-singular exposed point with respect to the four corners of $\mathcal{A}$ are respectively:

$\emptyset$	$\{(2,2),(3,1)\}$
$\{(1,1),(2,0)\}$	$\emptyset$

(4.12)

Its corresponding row support $\mathcal{R}$ and column support $\mathcal{C}$ are:



It is easy to check that the set of points

$$\begin{array}{|c|c|}\hline \emptyset & \{(2,2),(3,1)\} \oplus (-1,2) \\ \hline \{(1,1),(2,0)\} & \emptyset \\ \hline \end{array} \tag{4.13}$$

and

$$\begin{array}{|c|c|}\hline \emptyset & \{(2,2),(3,1)\} \oplus (3,-1) \\ \hline \{(1,1),(2,0)\} & \emptyset \\ \hline \end{array} \tag{4.14}$$

are subsets of the set of the exposed points of $\mathcal{R}$ and $\mathcal{C}$ with respect to the four corners. □

4.2.3 Non-Inheritance of Singular Exposed Points

Due to Condition (4.6) of a corner-cut monomial support, clearly we have

- (x, y) is a bottom singular exposed point if and only if $\mathcal{A} \cap (0..m \times 0) = \{(x, y)\}$.
- (x, y) is a right singular exposed point if and only if $\mathcal{A} \cap (m \times 0..n) = \{(x, y)\}$.
- (x, y) is a top singular exposed point if and only if $\mathcal{A} \cap (0..m \times n) = \{(x, y)\}$.
- (x, y) is a left singular exposed point if and only if $\mathcal{A} \cap (0 \times 0..n) = \{(x, y)\}$.

Consequently, bottom/top singular exposed points are not inherited by the row support, and right/left singular exposed points are not inherited by the column support in the sense of the following theorem.

Theorem 8 *Let $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ be a corner-cut monomial support. Let $\mathcal{R}$ and $\mathcal{C}$ be respectively the row and column supports of D. Let $(x, y) \in \mathcal{A}$ be a singular exposed point. We have*

- *If (x, y) is a bottom singular exposed point then $\mathcal{R} \cap (0..m-1 \times 0) = \emptyset$. In particular, $(x, y) \notin \mathcal{R}$, $(x-1, y) \notin \mathcal{R}$.*
- *If (x, y) is a right singular exposed point then $\mathcal{C} \cap (2m-1 \times 0..n-1) = \emptyset$. In particular, $(x+m-1, y) \notin \mathcal{C}$, $(x+m-1, y-1) \notin \mathcal{C}$.*
- *If (x, y) is a top singular exposed point then $\mathcal{R} \cap (0..m-1 \times 2n-1) = \emptyset$. In particular, $(x, y+n-1) \notin \mathcal{R}$, $(x-1, y+n-1) \notin \mathcal{R}$.*
- *If (x, y) is a left singular exposed point then $\mathcal{C} \cap (0 \times 0..n-1) = \emptyset$. In particular, $(x, y) \notin \mathcal{C}$, $(x, y-1) \notin \mathcal{C}$.*

<u>Proof</u>

We prove the case for right singular exposed points. Other cases are similar.

Let $E_{i,j}$ be the set of exterior points with respect to the corner (i,j), $i = 0, m$ and $j = 0, n$. If (x, y) is a right singular exposed point, by Condition (4.6), we must have $\mathcal{A} \cap (m \times 0..n) = \{(x, y)\}$. Thus $(x + m - 1, y) \in E''_{m,n}$, $(x + m - 1, y - 1) \in E''_{m,0}$, and $\mathcal{C} \cap (2m - 1 \times 0..n - 1) = \emptyset$.

Q.E.D

Example 11 *Consider the monomial support and its row and column supports in Example 10.*

The point $(1, 3) \in \mathcal{A}$ *is a top singular exposed point, we have* $\mathcal{A} \cap (0..4 \times 3) = \{(1, 3)\}$ *and* $\mathcal{R} \cap (0..3 \times 5) = \emptyset$.

The point $(0, 2) \in \mathcal{A}$ *is a left singular exposed point, we have* $\mathcal{A} \cap (0 \times 0..3) = \{(0, 2)\}$ *and* $\mathcal{C} \cap (0 \times 0..2) = \emptyset$.

The point $(4, 0) \in \mathcal{A}$ *is a right singular exposed point, we have* $\mathcal{A} \cap (4 \times 0..3) = \{(4, 0)\}$ *and* $\mathcal{C} \cap (7 \times 0..2) = \emptyset$. □

4.2.4 Bracket Factors from Three Exposed-Point Rows or Columns

We say N rows (or columns) of the Dixon matrix D produce a factor F if the determinant of any $N \times N$ submatrix of these rows (or columns) has F as a factor. Here N is 2 or 3.

The following theorem sheds some light on extraneous factors when a maximal minor of the Dixon matrix is not an exact sparse resultant.

Theorem 9 *Let* $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ *be a corner-cut monomial support. Let* (x_1, y_1), (x_2, y_2), *and* (x_3, y_3) *be three exposed points at a corner. Let* (x'_i, y'_i), (x''_i, y''_i), *i=1,2,3, be given by (3.19), (3.25) respectively according to the corner. Then the rows indexed by*

$$\{(x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3)\} \cap \mathcal{R} \tag{4.15}$$

produce the bracket factor $(x_1, y_1, x_2, y_2, x_3, y_3)$, *and the columns indexed by*

$$\{(x''_1, y''_1), (x''_2, y''_2), (x''_3, y''_3)\} \cap \mathcal{C} \tag{4.16}$$

also produce the bracket factor $(x_1, y_1, x_2, y_2, x_3, y_3)$.

<u>Proof</u>

We prove the theorem for rows. The proof for columns is similar.

Case 1: (x_1, y_1), (x_2, y_2) and (x_3, y_3) are non-singular.

By Theorem 7, $\{(x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3)\} \cap \mathcal{R} = \{(x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3)\}$.

With Formula (3.20), any 3×3 submatrix of the three rows indexed by (x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3) can be written as a product of two 3×3 matrices $T'P'$ where

$$T' = \begin{bmatrix} (x_1, y_1) \\ (x_2, y_2) \\ (x_3, y_3) \end{bmatrix} = \begin{bmatrix} f_{x_1,y_1} & g_{x_1,y_1} & h_{x_1,y_1} \\ f_{x_2,y_2} & g_{x_2,y_2} & h_{x_2,y_2} \\ f_{x_3,y_3} & g_{x_3,y_3} & h_{x_3,y_3} \end{bmatrix} \tag{4.17}$$

and the columns of P' are

$$\begin{array}{|c|c|}\hline -\sum_k(k,n,a_j+1-k,b_j) & \sum_k(k,n,a_j-k,b_j) \\ \hline \sum_k(k,0,a_j+1-k,b_j+1) & -\sum_k(k,0,a_j-k,b_j+1) \\ \hline \end{array} \tag{4.18}$$

where each j denotes a column.

Thus, the three rows indexed by (x_1', y_1'), (x_2', y_2'), (x_3', y_3') produce a factor which is the bracket $(x_1, y_1, x_2, y_2, x_3, y_3)$.

Case 2: Some of (x_1, y_1), (x_2, y_2) and (x_3, y_3) are singular.

First we consider the row support $\mathcal{R}$. By Theorem 8, we need consider only bottom singular exposed points if K is the bottom left or bottom right corner and top singular exposed points if K is the top left or top right corner; otherwise, the situation is the same as Case 1 even with singular exposed points. Thus we may assume

$$\begin{array}{|c|c|}\hline \mathcal{A}\cap(0..m\times n)=\{(x_1,y_1)\} & \mathcal{A}\cap(0..m\times n)=\{(x_1,y_1)\} \\ \hline \mathcal{A}\cap(0..m\times 0)=\{(x_1,y_1)\} & \mathcal{A}\cap(0..m\times 0)=\{(x_1,y_1)\} \\ \hline \end{array} \tag{4.19}$$

and we have

$$\{(x_1', y_1'), (x_2', y_2'), (x_3', y_3')\} \cap \mathcal{R} = \{(x_2', y_2'), (x_3', y_3')\}. \tag{4.20}$$

But the bracket is zero when the summation index $k \neq x_1$, so the entries of the rows indexed by (x_i', y_i'), $i = 2, 3$, can be simplified to

$$(x_i, y_i, x_1, y_1, s_j, t_j) \tag{4.21}$$

where each j denotes a column and $(s_j, t_j), j = 1, 2$ is given by

$$\begin{array}{|c|c|}\hline -(a_j+1-x_1,b_j) & (a_j-x_1,b_j) \\ \hline (a_j+1-x_1,b_j+1) & -(a_j-x_1,b_j+1) \\ \hline \end{array}. \tag{4.22}$$

The determinant of any 2×2 submatrix of the two rows indexed by $(x_2'.y_2')$, (x_3', y_3') can be written as

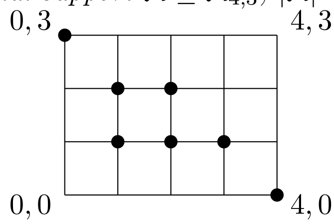
$$\begin{vmatrix} (x_2,y_2,x_1,y_1,s_1,t_1) & (x_2,y_2,x_1,y_1,s_2,t_2) \\ (x_3,y_3,x_1,y_1,s_1,t_1) & (x_3,y_3,x_1,y_1,s_2,t_2) \end{vmatrix} = (x_1,y_1,x_2,y_2,x_3,y_3)(x_1,y_1,s_1,t_1,s_2,t_2) \tag{4.23}$$

Thus, the rows indexed by (x_2', y_2'), (x_3', y_3') given by (3.19) produce the factor $(x_1, y_1, x_2, y_2, x_3, y_3)$.

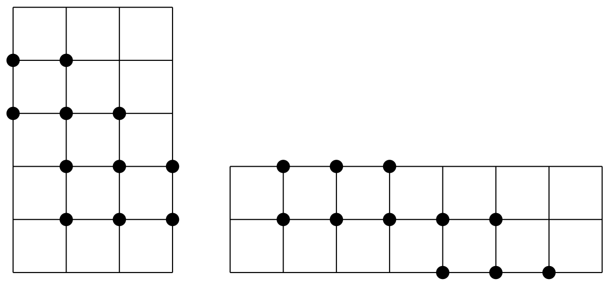
In both Case 1 and Case 2, the rows indexed by (4.15) produce a factor which is a bracket consisting of the three exposed points.

Q.E.D

Example 12 *Consider the monomial support $\mathcal{A} \subseteq \mathcal{A}_{4,3}$, $|\mathcal{A}| = 7$:*



Its row support $\mathcal{R}$ and column support $\mathcal{C}$ are respectively:



The exposed points are

$(0,3)_{lt}$	$(0,3)_{lt}, (2,2), (3,1), (4,0)_{br}$
$(0,3)_{lt}, (1,1), (4,0)_{br}$	$(4,0)_{br}$

(4.24)

where the subscripts b, r, t, l indicate a bottom, right, top, left singular exposed point respectively.
By Theorem 9, the rows and the bracket factors produced by them are as follows:

$\{(0,3),(1,1)\}$	031140
$\{(2,2),(3,1),(4,0)\} \oplus (-1,2)$	223140
$\{(2,2),(3,1)\} \oplus (-1,2)$	032231
$\{(2,2),(4,0)\} \oplus (-1,2)$	032240
$\{(3,1),(4,0)\} \oplus (-1,2)$	033140

. (4.25)

Again by Theorem 9, the columns and the bracket factors produced by them are as follows:

$\{(1,1),(4,0)\}$	031140
$\{(0,3),(2,2),(3,1)\} \oplus (3,-1)$	032231
$\{(0,3),(2,2)\} \oplus (3,-1)$	032240
$\{(0,3),(3,1)\} \oplus (3,-1)$	033140
$\{(2,2),(3,1)\} \oplus (3,-1)$	223140

. (4.26)

$\square$

4.2.5 Linear Dependence of Four or More Exposed-Point Rows or Columns

The following result follows almost immediately from the above theorem. But due to its importance we shall present it as a theorem and not a corollary.

Theorem 10 *Let $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ be a corner-cut monomial support. Let (x_i, y_i), $i = 1, 2, 3, 4$, be four exposed points at the same corner. Let (x'_i, y'_i), (x''_i, y''_i), i=1,2,3,4 be given by (3.19), (3.25) respectively according to the corner. Then the rows indexed by*

$$\{(x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3), (x'_4, y'_4)\} \cap \mathcal{R} \tag{4.27}$$

are linearly dependent; the columns indexed by

$$\{(x''_1, y''_1), (x''_2, y''_2), (x''_3, y''_3), (x''_4, y''_4)\} \cap \mathcal{C} \tag{4.28}$$

are also linearly dependent.

<u>**Proof**</u>

We prove the theorem for rows, the proof for columns are similar.

Case 1: The four monomial points (x_i, y_i), $i = 1, 2, 3, 4$, are non-singular. By Theorem 7, we have

$$\{(x'_i, y'_i) \mid i = 1, 2, 3, 4\} \cap \mathcal{R} = \{(x'_i, y'_i) \mid i = 1, 2, 3, 4\}. \tag{4.29}$$

Let the dimension of the Dixon matrix D be $N \times N$.

From the proof of Theorem 9, we see that the row indexed by the exposed point $(x'_i, y'_i), i = 1, 2, 3, 4$, can be written as a product of a 1×3 matrix (x_i, y_i) and a $3 \times N$ matrix P whose columns are given by (4.18). The entries of P are independent of (x_i, y_i). This means the row indexed by (x'_i, y'_i) is a linear combinations of the three rows of P. Since the four rows indexed by (x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3), (x'_4, y'_4) are generated from three rows, thus they are linearly dependent.

Case 2: Some of (x_i, y_i), $i = 1, 2, 3, 4$ are singular.

Applying the same argument and making the same assumption that leads to Equation (4.19), we have

$$\{(x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3), (x'_4, y'_4)\} \cap \mathcal{R} = \{(x'_2, y'_2), (x'_3, y'_3), (x'_4, y'_4)\} \tag{4.30}$$

and we can write the entry of the row indexed by $(x'_i, y'_i), i = 2, 3, 4$ as

$$(x_i, y_i, x_1, y_1, s_j, t_j). \tag{4.31}$$

By brute force computation or otherwise, it can be checked that the following identity holds:

$$\sum_{i=2}^{4} (x_{k_i}, y_{k_i}, x_{l_i}, y_{l_i}, x_1, y_1) \times (x_i, y_i, x_1, y_1, s_j, t_j) = 0 \tag{4.32}$$

where $k_2 = 3$, $l_2 = 4$; $k_3 = 4$, $l_3 = 2$; $k_4 = 2$, $l_4 = 3$. Since $(x_{k_i}, y_{k_i}, x_{l_i}, y_{l_i}, x_1, y_1)$ is nonzero, the rows indexed by (x_i', y_i'),$i = 2, 3, 4$, are linearly dependent.

Q.E.D

Example 13 *Consider the monomial support of Example 12. The four points* $(0, 3)$*,* $(2, 2)$*,* $(3, 1)$ *and* $(4, 0)$ *are exposed with respect to the top right corner in* $\mathcal{A}$*. By Theorem 10, we know the three rows indexed by*

$$\{(1, 4), (2, 3), (3, 2)\} = (\{(0, 3), (2, 2), (3, 1), (4, 0)\} \oplus (-1, 2)) \cap \mathcal{R} \tag{4.33}$$

and the three columns indexed by

$$\{(3, 2), (5, 1), (6, 0)\} = (\{(0, 3), (2, 2), (3, 1), (4, 0)\} \oplus (3, -1)) \cap \mathcal{C} \tag{4.34}$$

are linearly dependent. □

4.3 Classification of Corner Cut Monomial Supports for Sparse Resultant Expressions

Theorems 9 and 10 suggest that we can use the number of exposed points at each corner to classify the effects of corner cutting on the form of sparse resultants obtained by the Dixon method.

4.3.1 At Most 2 Exposed Points at Each Corner

It seems clearer to re-state the results of rectangular corner cutting in [2] as follows:

Theorem 11 *The Dixon determinant* $|D|$ *is the sparse resultant of the corner cut monomial support* $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ *if* $\mathcal{A}$ *has at most two exposed point with respect to each of the four corners of* $\mathcal{A}_{m,n}$*.*

The classical results of [12] becomes the special case in which there is exactly one exposed point at each corner.

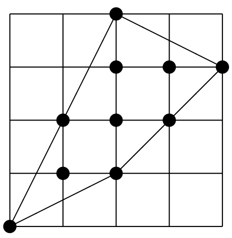
4.3.2 At Most 3 Exposed Points at Each Corner

The BKK degree bound [9] motivates the following definitions.

Definition 5 *The area deficiency at a corner of a corner-cut monomial support* $\mathcal{A}$ *is twice the area reduced at the corner between the convex hulls of* $\mathcal{A}$ *and* $\mathcal{A}_{m,n}$*.*

Definition 6 *The excess degree at a corner of a corner-cut monomial support* $\mathcal{A}$ *is the difference between the area deficiency and the number of exterior points at the corner.*

Example 14 *Consider the following monomial support* $\mathcal{A} \subseteq \mathcal{A}_{4,4}$*,* $|\mathcal{A}| = 10$*:*



The excess degree at each corner is:

$$\textit{area deficiency} \quad - \quad \textit{number of exterior points} \quad = \quad \textit{excess degree,}$$

$$\begin{array}{|c|c|}\hline 8 & 2 \\ \hline 0 & 10 \\ \hline \end{array} \quad - \quad \begin{array}{|c|c|}\hline 6 & 2 \\ \hline 0 & 7 \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|}\hline 2 & 0 \\ \hline 0 & 3 \\ \hline \end{array}$$

□

The excess degree at a corner with three exposed points has the following geometric interpretation.

Theorem 12 *If a corner K of a corner-cut monomial support $\mathcal{A} \subseteq \mathcal{A}_{m,n}$ has exactly three exposed points, the exposed points can be written as*

$$\begin{array}{|c|c|}\hline (0, y_1), (x_2, y_2), (x_3, n) & (m, y_1), (x_2, y_2), (x_3, n) \\ \hline (0, y_1), (x_2, y_2), (x_3, 0) & (m, y_1), (x_2, y_2), (x_3, 0) \\ \hline \end{array} \tag{4.35}$$

Note that the notations at a corner are independent of the same notations at other corners. As diagonals, the point (x_2, y_2) and the corner K determine a rectangle whose area is A, the point (x_2, y_2) and the point (x_3, y_1) determine a rectangle whose area is A'. The excess degree at corner K is

$$\epsilon = \min(A, A') \tag{4.36}$$

Figure 4.1 shows that how A, A' are computed by the three exposed points at the bottom left corner when $K = (0, 0)$.

Proof

Use brute force calculation.

Q.E.D

For the rest of discussion in this section, we recall the following important properties of the Dixon matrix and sparse resultants: (1) maximal minors of the Dixon matrix are multiples of the sparse resultant [13, 19], (2) sparse resultants are irreducible [9], and (3) the degree of the sparse resultant in the coefficients of each of the polynomials f, g, h is twice the area of the convex hull of the monomial support $\mathcal{A}$ of f, g, h. These properties and Theorem 9 suggest that the results of [14, 15] can be rephrased as follows:

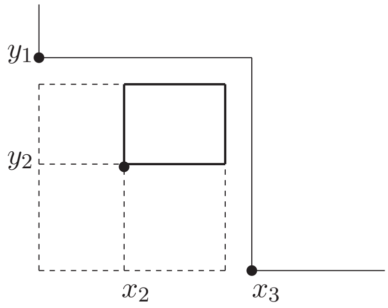
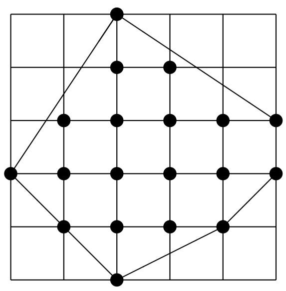


Figure 4.1: When $K = (0, 0), A = x_2y_2, A' = (x_3 - x_2)(y_1 - y_2)$.

Theorem 13 *If each corner of a corner-cut monomial support $\mathcal{A}$ has at most three exposed points and for each corner with three exposed points the excess degree is one, then the Dixon matrix is maximal and the sparse resultant is the Dixon determinant divided by a product of brackets, each bracket in the product consists of the three exposed points at a corner.*

Example 15 *Consider the following monomial support $\mathcal{A} \subseteq \mathcal{A}_{5,5}$, $|\mathcal{A}| = 19$:*



Since the excess degree at each corner is 1 and the set of the exposed points at each corner is respectively

$\{(0,2),(1,3),(2,5)\}$	$\{(2,5),(3,4),(5,3)\}$
$\{(0,2),(1,1),(2,0)\}$	$\{(2,0),(4,1),(5,2)\}$

(4.37)

by Theorem 13, we know that the Dixon matrix is maximal and its $\mathcal{A}$-resultant is

$$\frac{|D|}{021120 \cdot 204152 \cdot 253453 \cdot 021325}$$

□

A generalization of Theorem 13 is the following conjecture [17]:

Conjecture 1 *If each corner of a corner-cut monomial support $\mathcal{A}$ has at most three exposed points, then the sparse resultant is*

$$\frac{|D|}{\prod_{(i,j)\in K} B_{i,j}^{\epsilon_{i,j}}} \tag{4.38}$$

where $K \subseteq \{(0,0),(m,0),(m,n),(0,n)\}$ and $B_{i,j}$ is the bracket determined by the three exposed points at corner (i,j), $\epsilon_{i,j}$ is the excess degree at corner (i,j).

Some special cases of the conjecture will be proved in Chapter 5.

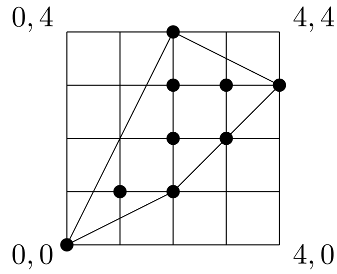
4.3.3 Any Number of Exposed Points at Each Corner

Theorems 9 and 10 also suggest that [16] proved a special case of the following conjecture.

Conjecture 2 *Suppose corner (i, j), $i = 0, m$ and $j = 0, n$, of a corner-cut monomial support $\mathcal{A}$ has $N_{i,j}$ exposed points, and when $N_{i,j} \geq 3$ the excess degree at corner (i, j) is $N_{i,j} - 1$. A maximal minor of the Dixon matrix can be obtained by dropping any $N_{i,j} - 3$ rows and any $N_{i,j} - 3$ columns associated with exposed points for each corner (i, j) with $N_{i,j} \geq 3$. The sparse resultant is the chosen maximal minor divided by a product of pairs of brackets; one bracket in the pair comes from the three remaining exposed points not involved in the dropping of rows and another bracket in the pair comes from the three remaining exposed points not involved in the dropping of columns.*

The following example illustrates Conjecture 2 at the bottom right corner, Theorem 11 at the top right corner, and Conjecture 1 at the top left corner.

Example 16 *Consider the following monomial support $\mathcal{A} \subseteq \mathcal{A}_{4,4}$, $|\mathcal{A}| = 9$:*



The excess degree at each corner and the set of exposed points at each corner are respectively:

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 3 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|}\hline \{(0,0),(1,1),(2,4)\} & \{(2,4),(4,3)\} \\ \hline \{(0,0)\} & \{(0,0),(2,1),(3,2),(4,3)\} \\ \hline \end{array}\ .$$

Since $(0, 0)$ is a bottom and left singular exposed point and $(4, 3)$ is a right singular exposed point, by Theorem 8, $(0, 0) \notin \mathcal{R}, (0, 0) \notin \mathcal{C}, (7, 3) \notin C$. So we have

$$\{(0,0),(2,1),(3,2),(4,3)\} \oplus (-1,0) \cap \mathcal{R} = \{(1,1),(2,2),(3,3)\}$$
$$\{(0,0),(2,1),(3,2),(4,3)\} \oplus (3,0) \cap \mathcal{C} = \{(3,0),(5,1),(6,2)\}$$

Thus, by Conjecture 2, the sparse resultant can be written as nine expressions:

$$\frac{|D(11,30)|}{001124 \cdot 003243 \cdot 213243}, \frac{|D(11,51)|}{001124 \cdot 003243 \cdot 003243}, \frac{|D(11,62)|}{001124 \cdot 003243 \cdot 002143},$$
$$\frac{|D(22,30)|}{001124 \cdot 002143 \cdot 213243}, \frac{|D(22,51)|}{001124 \cdot 002143 \cdot 003243}, \frac{|D(22,62)|}{001124 \cdot 002143 \cdot 002143},$$
$$\frac{|D(33,30)|}{001124 \cdot 002132 \cdot 213243}, \frac{|D(33,51)|}{001124 \cdot 002132 \cdot 003243}, \frac{|D(33,62)|}{001124 \cdot 002132 \cdot 002143}.$$

Here $D(\sigma\tau, ab)$ represents the maximal minor obtained by removing the row indexed by (σ, τ) and the column indexed by (a, b) from the Dixon matrix D. $\square$

Chapter 5

Corners with Three Exposed Points

This chapter examines an important corner cutting situation in which there are exactly three exposed points at a corner. Recall Conjecture 1:

> If each corner of a corner-cut monomial support $\mathcal{A}$ has at most three exposed points, then the sparse resultant is
> $$\frac{|D|}{\prod_{(i,j)\in K} B_{i,j}^{\epsilon_{i,j}}} \tag{5.1}$$
> where $K \subseteq \{(0,0),(m,0),(m,n),(0,n)\}$ and $B_{i,j}$ is the bracket determined by the three exposed points at corner (i,j), $\epsilon_{i,j}$ is the excess degree at corner (i,j).

Except for specific quantities, the following derivations and discussions are applicable to any of the four corners. We will thus put quantities peculiar to a corner in a box with four quadrants according to the convention of Section 2.4. In particular, we will simply write ϵ instead of $\epsilon_{i,j}$ in the discussion. Let T be the set of the three exposed points at the corner, that is, $T = \{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$. Note that actually T is

$$\begin{array}{|c|c|}\hline \{(0,y_1),(x_2,y_2),(x_3,n)\} & \{(m,y_1),(x_2,y_2),(x_3,n)\} \\ \hline \{(0,y_1),(x_2,y_2),(x_3,0)\} & \{(m,y_1),(x_2,y_2),(x_3,0)\} \\ \hline \end{array} \tag{5.2}$$

and we define w_1, h_1, w_2, h_2 to be

$$w_1 = \begin{array}{|c|c|}\hline x_2 & m-x_2 \\ \hline x_2 & m-x_2 \\ \hline \end{array} \quad h_1 = \begin{array}{|c|c|}\hline n-y_2 & n-y_2 \\ \hline y_2 & y_2 \\ \hline \end{array}, \tag{5.3}$$

$$w_2 = \begin{array}{|c|c|}\hline x_3-x_2 & x_2-x_3 \\ \hline x_3-x_2 & x_2-x_3 \\ \hline \end{array} \quad h_2 = \begin{array}{|c|c|}\hline y_2-y_1 & y_2-y_1 \\ \hline y_1-y_2 & y_1-y_2 \\ \hline \end{array}. \tag{5.4}$$

Section 5.1 shows for rows and columns near exposed points within a range $0..\min(w_1,w_2)-1 \times 0..\min(h_1,h_2)-1$, their entries can be greatly simplified in case they are considered as the rows or columns of a determinant. Section 5.2 presents the cases having been successfully proved to generate the expected denominator in (5.1).

5.1 Reducibility of Rows and Columns Near Exposed Points

Let (x,y) be an exposed point. Rows and columns indexed respectively by (σ,τ) "close" to (x',y') and (a,b) "close" to (x'',y'') can be simplified when they are treated as rows and columns of a determinant.

First we need the following lemma:

Lemma 1 *For any $P,Q,R,S,X,Y \in \mathcal{A}_{m,n}$, the following bracket identity holds:*

$$PQR \times XYS - QRS \times XYP + RSP \times XYQ - SPQ \times XYR = 0$$

Proof

This lemma can be easily verified with a computer algebra system like Maple.

Q.E.D

Lemma 2 *Let $(x,y) \in T$, $1 \le w \le \min(w_1,w_2)$ and $1 \le h \le \min(h_1,h_2)$. Consider the rows indexed by the points (σ,τ) given by*

$$\begin{array}{|c|c|}\hline (0,n-1)\oplus(x+p,y-q) & (-1,n-1)\oplus(x-p,y-q) \\ \hline (x+p,y+q) & (-1,0)\oplus(x-p,y+q) \\ \hline \end{array} \tag{5.5}$$

with $0 \le p \le w-1$ and $0 \le q \le h-1$. The row entry indexed by (σ,τ) can be represented as

$$\begin{array}{|c|c|}\hline \sum_{u=0}^{p}\sum_{l-v=n-q}^{n}\sum_{k=0}^{m} B_{0,n} & \sum_{u=0}^{p}\sum_{l-v=n-q}^{n}\sum_{k=0}^{m} B_{m,n} \\ \hline \sum_{u=0}^{p}\sum_{l+v=0}^{q}\sum_{k=0}^{m} B_{0,0} & \sum_{u=0}^{p}\sum_{l+v=0}^{q}\sum_{k=0}^{m} B_{m,0} \\ \hline \end{array} \tag{5.6}$$

where

$$\begin{aligned} B_{0,0} &= (x+p-u, y+q-v-l, k, l, a+u+1-k, b+v+1) \\ B_{m,0} &= -(x-p+u, y+q-v-l, k, l, a-u-k, b+v+1) \\ B_{m,n} &= (x-p+u, y+n+v-q-l, k, l, a-u-k, b-v) \\ B_{0,n} &= -(x+p-u, y+n+v-q-l, k, l, a+u+1-k, b-v) \end{aligned}$$

Proof

Apply the entry formula in Theorem 3 and choose the bracket B in the formula to be

$$\begin{array}{|c|c|}\hline \text{Equation (3.3)} & \text{Equation (3.2)} \\ \hline \text{Equation (3.4)} & \text{Equation (3.5)} \\ \hline \end{array} \tag{5.7}$$

After substituting the value of (σ, τ) in (5.5) into B, the first ordered pair of B becomes

$$\begin{array}{|c|c|}\hline (x+p-u, y+n+v-q-l) & (x-p+u, y+n+v-q-l) \\ \hline (x+p-u, y+q-v-l) & (x-p+u, y+q-v-l) \\ \hline \end{array} . \tag{5.8}$$

In order to let this ordered pair in $\mathcal{A}$, we need

$$\begin{array}{|c|c|}\hline 0 \le p-u, n+v-q-l \le 0 & -p+u \le 0, n+v-q-l \le 0 \\ \hline 0 \le p-u, 0 \le q-v-l & -p+u \le 0, 0 \le q-v-l \\ \hline \end{array} \tag{5.9}$$

Thus, together with $0 \le u, 0 \le v$ and $0 \le l \le n$ we have

$$\begin{array}{|c|c|}\hline 0 \le u \le p, \quad n-q \le l-v \le n & 0 \le u \le p, \quad n-q \le l-v \le n \\ \hline 0 \le u \le p, \quad 0 \le v+l \le q & 0 \le u \le p, \quad 0 \le v+l \le q \\ \hline \end{array} \tag{5.10}$$

Consequently we get the entry formula given in (5.6).

Q.E.D

Definition 7 *The set of the above rows close to (x', y') are called $(0..w-1 \times 0..h-1)$-near row block with respect to (x', y'). In particular, the row indexed by (5.5) is called (p, q)-near row with respect to (x', y').*

Lemma 3 *Let $(x, y) \in T$, $1 \le w \le \min(w_1, w_2)$ and $1 \le h \le \min(h_1, h_2)$.*
Consider the columns indexed by the points (a, b) given by

$$\begin{array}{|c|c|}\hline (0,-1) \oplus (x+p, y-q) & (m-1,-1) \oplus (x-p, y-q) \\ \hline (x+p, y+q) & (m-1, 0) \oplus (x-p, y+q) \\ \hline \end{array} \tag{5.11}$$

with $0 \le p \le w-1$ and $0 \le q \le h-1$. The column entry indexed by (a, b) can be represented as

$$\begin{array}{|c|c|}\hline \sum_{u+k=0}^{p} \sum_{v=0}^{q} \sum_{l=0}^{n} B_{0,n} & \sum_{k-u=m-p}^{m} \sum_{v=0}^{q} \sum_{l=0}^{n} B_{m,n} \\ \hline \sum_{u+k=0}^{p} \sum_{v=0}^{q} \sum_{l=0}^{n} B_{0,0} & \sum_{k-u=m-p}^{m} \sum_{v=0}^{q} \sum_{l=0}^{n} B_{m,0} \\ \hline \end{array} \tag{5.12}$$

where

$$\begin{aligned} B_{0,0} &= (\sigma+u+1, \tau+v+1-l, k, l, x+p-u-k, y+q-v) \\ B_{m,0} &= -(\sigma-u, \tau+v+1-l, k, l, x+m-p+u-k, y+q-v) \\ B_{m,n} &= (\sigma-u, \tau-v-l, k, l, x+m-p+u-k, y-q+v) \\ B_{0,n} &= -(\sigma+u+1, \tau-v-l, k, l, x+p-u-k, y-q+v) \end{aligned}$$

Proof

Apply the entry formula in Theorem 3 and choose the bracket B in the formula to be

Equation (3.5)	Equation (3.4)
Equation (3.2)	Equation (3.3)

(5.13)

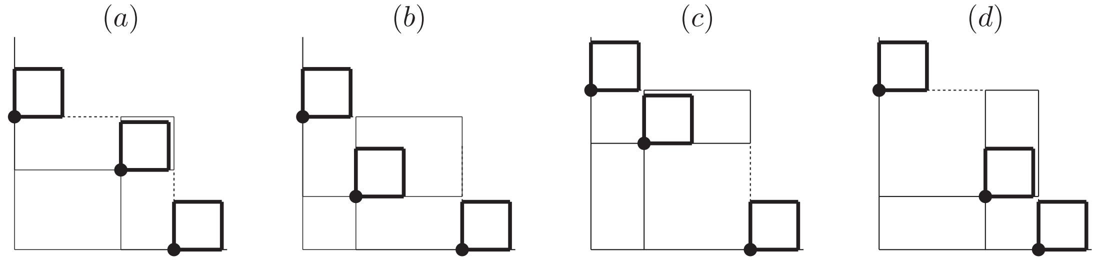


Figure 5.1: Bottom left corner. The small rectangles (with bold line) represent the rows and columns considered.

After substituting (a, b) into B, the last ordered pair of B becomes:

$$\begin{array}{|c|c|}\hline (x+p-u-k, y-q+v) & (x+m-p+u-k, y-q+v) \\ \hline (x+p-u-k, y+q-v) & (x+m-p+u-k, y+q-v) \\ \hline \end{array} \tag{5.14}$$

In order to let this ordered pair in $\mathcal{A}$, we need

$$\begin{array}{|c|c|}\hline 0 \le p-u-k, -q+v \le 0 & m-p+u-k \le 0, -q+v \le 0 \\ \hline 0 \le p-u-k, 0 \le q-v & m-p+u-k \le 0, 0 \le q-v \\ \hline \end{array} \tag{5.15}$$

Thus, together with $0 \le u, 0 \le v$ and $0 \le k \le m$ we have

$$\begin{array}{|c|c|}\hline 0 \le u+k \le p, \quad 0 \le v \le q & m-p \le k-u \le m, \quad 0 \le v \le q \\ \hline 0 \le u+k \le p, \quad 0 \le v \le q & m-p \le k-u \le m, \quad 0 \le v \le q \\ \hline \end{array} \tag{5.16}$$

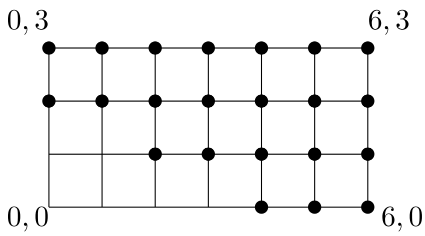
Consequently, we get the entry formula given in (5.12). This finishes the proof.

Q.E.D

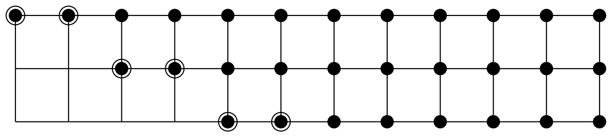
Definition 8 *The set of the above columns close to (x'', y'') is called $(0..w-1 \times 0..h-1)$-near column block with respect to (x'', y''). In particular, the column indexed by (5.11) is called (p, q)-near column.*

Figure 5.1 gives some examples at the bottom left corner of the rows and the columns we considered in Lemmas 2 and 3.

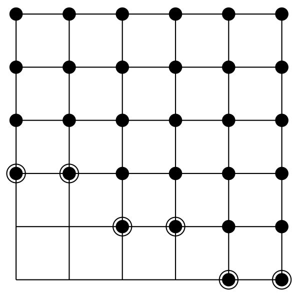
Example 17 *Consider the monomial support $\mathcal{A} \subseteq \mathcal{A}_{6,3}$:*



and its corresponding row and column support are



$\mathcal{R}$



$\mathcal{C}$

There are three exposed points at bottom left corner: $(0,2),(2,1),(4,0)$. *So by definition*

$$w_1 = 2, h_1 = 2, w_2 = 4-2 = 2, h_2 = 2-1 = 1.$$

Take $w = 2, h = 1$.

By Lemma 2, the rows indexed $\{(0,2),(2,1),(4,0)\} \oplus (p,q)$ *with* $0 \le p \le 1, q = 0$ *can be computed as follows:*

$p=0,q=0$	$(0,2)$	$\sum_{k=0}^{6}(0,2,k,0,a+1-k,b+1)$
	$(2,1)$	$\sum_{k=0}^{6}(2,1,k,0,a+1-k,b+1)$
	$(4,0)$	$\sum_{k=0}^{6}(4,0,k,0,a+1-k,b+1)$
$p=1,q=0$	$(1,2)$	$\sum_{u=0}^{1}\sum_{k=0}^{6}(1-u,2,k,0,a+u+1-k,b+1)$
	$(3,1)$	$\sum_{u=0}^{1}\sum_{k=0}^{6}(3-u,1,k,0,a+u+1-k,b+1)$
	$(5,0)$	$\sum_{u=0}^{1}\sum_{k=0}^{6}(5-u,0,k,0,a+u+1-k,b+1)$

(5.17)

By Lemma 3, the columns indexed by $\{(0,2),(2,1),(4,0)\} \oplus (p,q)$ *with* $0 \le p \le 1, q = 0$ *can be computed as*

$p=0,q=0$	$(0,2)$	$\sum_{l=0}^{3}(\sigma+1,\tau+1-l,0,l,0,2)$
	$(2,1)$	$\sum_{l=0}^{3}(\sigma+1,\tau+1-l,0,l,2,1)$
	$(4,0)$	$\sum_{l=0}^{3}(\sigma+1,\tau+1-l,0,l,4,0)$
$p=1,q=0$	$(1,2)$	$\sum_{u+k=0}^{1}\sum_{l=0}^{3}(\sigma+u+1,\tau+1-l,k,l,1-u-k,2)$
	$(3,1)$	$\sum_{u+k=0}^{1}\sum_{l=0}^{3}(\sigma+u+1,\tau+1-l,k,l,3-u-k,1)$
	$(5,0)$	$\sum_{u+k=0}^{1}\sum_{l=0}^{3}(\sigma+u+1,\tau+1-l,k,l,5-u-k,0)$

(5.18)

□

The above two lemmas lead to a chain reduction on $(0..w-1 \times 0..h-1)$-near row block and column block starting from $(x_i', y_i'), i = 1, 2, 3$ and $(x_i'', y_i''), i = 1, 2, 3$ respectively.

Theorem 14 *Let $(x, y) \in T$ and $(p, q) \in (0..w-1 \times 0..h-1)$, then for the (p, q)-near row with respect to (x', y'), its row entry formula (5.6) can be reduced to*

$\sum_{l-v=n-q} \sum_{k=0}^{m} -(x, y, k, l, a+p+1-k, b-v)$	$\sum_{l-v=n-q} \sum_{k=0}^{m} (x, y, k, l, a-p-k, b-v)$
$\sum_{v+l=q} \sum_{k=0}^{m} (x, y, k, l, a+p+1-k, b+v+1)$	$\sum_{v+l=q} \sum_{k=0}^{m} -(x, y, k, l, a-p-k, b+v+1)$

(5.19)

Proof

Base Case. $p = q = 0$.

The $(0, 0)$-near row is the row indexed by (x', y') itself whose formula has been given by (3.20). We substitute $p = q = 0$ into the formula in (5.19). The resulting formula is the same as the row entry formula given in (3.20). So the $(0, 0)$-near row entry can be reduced (5.19) trivially.

Induction. $p > 0$ or $q > 0$.

Step 1: The (p, q)-near row entry formula can be written as two parts.

To highlight that σ depends on x', p and τ depends on y', q, we denote (σ, τ) as $(\sigma(x', p), \tau(y', q))$. By Lemma 2, the (p, q)-near row entry formula (5.6) can be written as

$$\Delta_{\sigma(x',p),\tau(y',q),a,b} = \hat{\Delta}_{\sigma(x',p),\tau(y',q),a,b} + \tilde{\Delta}_{\sigma(x',p),\tau(y',q),a,b} \tag{5.20}$$

with $\hat{\Delta}_{\sigma(x',p),\tau(y',q),a,b}$ equal to

$\sum_{(u,l-v)=(p,n-q)} \sum_{k=0}^{m} B_{0,n}(\sigma(x', p), \tau(y', q))$	$\sum_{(u,l-v)=(p,n-q)} \sum_{k=0}^{m} B_{m,n}(\sigma(x', p), \tau(y', q))$
$\sum_{(u,v+l)=(p,q)} \sum_{k=0}^{m} B_{0,0}(\sigma(x', p), \tau(y', q))$	$\sum_{(u,v+l)=(p,q)} \sum_{k=0}^{m} B_{m,0}(\sigma(x', p), \tau(y', q))$

(5.21)

and $\tilde{\Delta}_{\sigma(x',p),\tau(y',q),a,b}$ equal to

$\sum_{(u,l-v)\in T'_{p,n-q}} \sum_{k=0}^{m} B_{0,n}(\sigma(x', p), \tau(y', q))$	$\sum_{(u,l-v)\in T'_{p,n-q}} \sum_{k=0}^{m} B_{m,n}(\sigma(x', p), \tau(y', q))$
$\sum_{(u,v+l)\in T'_{p,q}} \sum_{k=0}^{m} B_{0,0}(\sigma(x', p), \tau(y', q))$	$\sum_{(u,v+l)\in T'_{p,q}} \sum_{k=0}^{m} B_{m,0}(\sigma(x', p), \tau(y', q))$

(5.22)

where $T'_{p,n-q} = 0..p \times (n-q)..n - \{(p, n-q)\}$ and $T'_{p,q} = 0..p \times 0..q - \{(p, q)\}$.

Substituting $u = p$ into $B_{0,0}, B_{m,0}, B_{m,n}, B_{0,n}$ in (5.21) respectively, we get the formula in (5.19).

Step 2: $\tilde{\Delta}_{\sigma(x',p),\tau(y',q),a,b}$ can be canceled off by $\hat{\Delta}_{\sigma(x'_i,f),\tau(y'_i,g),a,b}$ with $i = 1,2,3$ and $(f,g) \in T'_{p,q}$.

Since $p > 0$ or $q > 0$, we have $T'_{p,q} \neq \emptyset$ and $T'_{p,n-q} \neq \emptyset$. Given $(f,g) \in T'_{p,q}$ for each corner, we get

$(f,g) \in T'_{p,q} \Rightarrow (f,n-g) \in T'_{p,n-q}$	$(f,g) \in T'_{p,q} \Rightarrow (f,n-g) \in T'_{p,n-q}$
$(f,g) \in T'_{p,q}$	$(f,g) \in T'_{p,q}$

(5.23)

By strong mathematical induction hypothesis, we know the row entry formula $\Delta_{\sigma(x'_i,f),\tau(y'_i,g),a,b}$ can be reduced to $\hat{\Delta}_{\sigma(x'_i,f),\tau(y'_i,g),a,b}$ which is

$\sum_{l-v=n-g}\sum_{k=0}^{m} -(x_i,y_i,k,l,a+f+1-k,b-v)$	$\sum_{l-v=n-g}\sum_{k=0}^{m}(x_i,y_i,k,l,a-f-k,b-v)$
$\sum_{v+l=g}\sum_{k=0}^{m}(x_i,y_i,k,l,a+f+1-k,b+v+1)$	$\sum_{v+l=g}\sum_{k=0}^{m} -(x_i,y_i,k,l,a-f-k,b+v+1)$

(5.24)

or

$\sum_{(u,l-v)=(f,n-g)}\sum_{k=0}^{m} B_{0,n}(\sigma(x'_i,f),\tau(y'_i,g))$	$\sum_{(u,l-v)=(f,n-g)}\sum_{k=0}^{m} B_{m,n}(\sigma(x'_i,f),\tau(y'_i,g))$
$\sum_{(u,v+l)=(f,g)}\sum_{k=0}^{m} B_{0,0}(\sigma(x'_i,f),\tau(y'_i,g))$	$\sum_{(u,v+l)=(f,g)}\sum_{k=0}^{m} B_{m,0}(\sigma(x'_i,f),\tau(y'_i,g))$

(5.25)

for $i = 1,2,3$. By Lemma 1 with $P = (x_1,y_1), Q = (x_2,y_2), R = (x_3,y_3)$ and

$S =$

$(x+p-f,y-q+g)$	$(x+f-p,y+g-q)$
$(x+p-f,y-g+q)$	$(x+f-p,y+q-g)$

(5.26)

it can easily checked that the sum

$\sum_{(u,l-v)=(f,n-g)}\sum_{k=0}^{m} B_{0,n}(\sigma(x',p),\tau(y',q))$	$\sum_{(u,l-v)=(f,n-g)}\sum_{k=0}^{m} B_{m,n}(\sigma(x',p),\tau(y',q))$
$\sum_{(u,v+l)=(f,g)}\sum_{k=0}^{m} B_{0,0}(\sigma(x',p),\tau(y',q))$	$\sum_{(u,v+l)=(f,g)}\sum_{k=0}^{m} B_{m,0}(\sigma(x',p),\tau(y',q))$

(5.27)

can be written as a linear combination of the three formulas (5.24) with coefficients independent of a,b for $(f,g) \in T'_{p,q}$.

In conclusion, $\Delta_{\sigma(x',p),\tau(y',q),a,b}$ can be reduced to the formula (5.19). This finishes the proof.

Q.E.D

Theorem 15 *Let $(x,y) \in T$ and $(p,q) \in (0..w-1 \times 0..h-1)$, then for the (p,q)-near column with respect to (x'',y''), its column entry formula (5.12) can be reduced to*

$\sum_{u+k=p}\sum_{l=0}^{n} -(\sigma+u+1,\tau-q-l,k,l,x,y)$	$\sum_{k-u=m-p}\sum_{l=0}^{n}(\sigma-u,\tau-q-l,k,l,x,y)$
$\sum_{u+k=p}\sum_{l=0}^{n}(\sigma+u+1,\tau+q+1-l,k,l,x,y)$	$\sum_{k-u=m-p}\sum_{l=0}^{n} -(\sigma-u,\tau+q+1-l,k,l,x,y)$

(5.28)

Proof

Base Case. $p = q = 0$.

The $(0,0)$-near column is the column indexed by (x'', y'') itself whose formula has been given by (3.26). We substitute $p = q = 0$ into the formula in (5.28). The resulting formula is the same as the column entry formula given in (3.26). So the $(0,0)$-near column entry can be reduced (5.28) trivially.

Induction. $p > 0$ or $q > 0$.

Step 1: The (p,q)-near column entry formula can be written as two parts.

To highlight that a depends on x'', p and b depends on y'', q, we denote (a,b) as $(a(x'',p), b(y'',q))$. By Lemma 3, the (p,q)-near column entry formula (5.12) can be written as

$$\Delta_{\sigma,\tau,a(x'',p),b(y'',q)} = \hat{\Delta}_{\sigma,\tau,a(x'',p),b(y'',q)} + \tilde{\Delta}_{\sigma,\tau,a(x'',p),b(y'',q)} \tag{5.29}$$

with $\hat{\Delta}_{\sigma,\tau,a(x'',p),b(y'',q)}$ equal to

$$\begin{array}{|c|c|}\hline \sum_{(u+k,v)=(p,q)} \sum_{l=0}^{n} B_{0,n}(a(x'',p), b(y'',q)) & \sum_{(k-u,v)=(m-p,q)} \sum_{l=0}^{n} B_{m,n}(a(x'',p), b(y'',q)) \\ \hline \sum_{(u+k,v)=(p,q)} \sum_{l=0}^{n} B_{0,0}(a(x'',p), b(y'',q)) & \sum_{(k-u,v)=(m-p,q)} \sum_{l=0}^{n} B_{m,0}(a(x'',p), b(y'',q)) \\ \hline \end{array} \tag{5.30}$$

and $\tilde{\Delta}_{\sigma,\tau,a(x'',p),b(y'',q)}$ equal to

$$\begin{array}{|c|c|}\hline \sum_{(u+k,v)\in T'_{p,q}} \sum_{l=0}^{n} B_{0,n}(a(x'',p), b(y'',q)) & \sum_{(k-u,v)\in T'_{m-p,q}} \sum_{l=0}^{n} B_{m,n}(a(x'',p), b(y'',q)) \\ \hline \sum_{(u+k,v)\in T'_{p,q}} \sum_{k=0}^{m} B_{0,0}(a(x'',p), b(y'',q)) & \sum_{(k-u,v)\in T'_{m-p,q}} \sum_{k=0}^{m} B_{m,0}(a(x'',p), b(y'',q)) \\ \hline \end{array} \tag{5.31}$$

where $T'_{m-p,q} = (m-p)..m \times 0..q - \{(m-p,q)\}$ and $T'_{p,q} = 0..p \times 0..q - \{(p,q)\}$.

Substituting $v = q$ into $B_{0,0}, B_{m,0}, B_{m,n}, B_{0,n}$ in (5.30) respectively, we get the formula in (5.28).

Step 2: $\tilde{\Delta}_{\sigma,\tau,a(x'',p),b(y'',q)}$ can be canceled off by $\hat{\Delta}_{\sigma,\tau,a(x''_i,f),b(y''_i,g)}$ with $i = 1,2,3$ and $(f,g) \in T'_{p,q}$.

Since $p > 0$ or $q > 0$, we have $T'_{p,q} \neq \emptyset$ and $T'_{m-p,q} \neq \emptyset$. Given $(f,g) \in T'_{p,q}$ for each corner, we get

$$\begin{array}{|c|c|}\hline (f,g) \in T'_{p,q} & (f,g) \in T'_{p,q} \Rightarrow (m-f,g) \in T'_{m-p,q} \\ \hline (f,g) \in T'_{p,q} & (f,g) \in T'_{p,q} \Rightarrow (m-f,g) \in T'_{m-p,q} \\ \hline \end{array} \tag{5.32}$$

By strong mathematical induction hypothesis, we know the row entry formula $\Delta_{\sigma,\tau,a(x''_i,f),b(y''_i,g)}$

can be reduced to $\hat{\Delta}_{\sigma,\tau,a(x_i'',f),b(y_i'',g)}$ which is

$\sum_{u+k=f}\sum_{l=0}^{n}-(\sigma+u+1,\tau-g-l,k,l,x_i,y_i)$	$\sum_{k-u=m-f}\sum_{l=0}^{n}(\sigma-u,\tau-g-l,k,l,x_i,y_i)$
$\sum_{u+k=f}\sum_{l=0}^{n}(\sigma+u+1,\tau+g+1-l,k,l,x_i,y_i)$	$\sum_{k-u=m-f}\sum_{l=0}^{n}-(\sigma-u,\tau+g+1-l,k,l,x_i,y_i)$

(5.33)

or

$\sum_{(u+k,v)=(f,g)}\sum_{l=0}^{n}B_{0,n}(a(x_i'',f),b(y_i'',g))$	$\sum_{(k-u,v)=(m-f,g)}\sum_{l=0}^{n}B_{m,n}(a(x_i'',f),b(y_i'',g))$
$\sum_{(u+k,v)=(f,g)}\sum_{l=0}^{n}B_{0,0}(a(x_i'',f),b(y_i'',g))$	$\sum_{(k-u,v)=(m-f,g)}\sum_{l=0}^{n}B_{m,0}(a(x_i'',f),b(y_i'',g))$

(5.34)

for $i=1,2,3$. Similarly by Lemma 1, it can easily checked that the sum

$\sum_{(u+k,v)=(f,g)}\sum_{l=0}^{n}B_{0,n}(a(x'',p),b(y'',q))$	$\sum_{(k-u,v)=(m-f,g)}\sum_{l=0}^{n}B_{m,n}(a(x'',p),b(y'',q))$
$\sum_{(u+k,v)=(f,g)}\sum_{l=0}^{n}B_{0,0}(a(x'',p),b(y'',q))$	$\sum_{(k-u,v)=(m-f,g)}\sum_{l=0}^{n}B_{m,0}(a(x'',p),b(y'',q))$

(5.35)

can be written as a linear combination of the three formulas (5.33) with coefficients independent of σ,τ with $(f,g)\in T'_{p,q}$.

In conclusion, $\Delta_{\sigma,\tau,a(x'',p),b(y'',q)}$ can be reduced to the formula (5.28). This finishes the proof.

Q.E.D

Example 18 *Consider the monomial support in Example 17.*
By Theorem 14, we can reduce the rows indexed by $(1,2),(3,1),(5,0)$ *to*

$\hat{\Delta}_{1,2,a,b}$	$\sum_{k=0}^{6}(0,2,k,0,a+2-k,b+1)$
$\hat{\Delta}_{3,1,a,b}$	$\sum_{k=0}^{6}(2,1,k,0,a+2-k,b+1)$
$\hat{\Delta}_{5,0,a,b}$	$\sum_{k=0}^{6}(4,0,k,0,a+2-k,b+1)$

(5.36)

By Theorem 15, we can reduce the columns indexed by $(1,2),(3,1),(5,0)$ *to*

$\hat{\Delta}_{\sigma,\tau,1,2}$	$\sum_{u+k=1}\sum_{l=0}^{3}(\sigma+u+1,\tau+1-l,k,1,0,2)$
$\hat{\Delta}_{\sigma,\tau,3,1}$	$\sum_{u+k=1}\sum_{l=0}^{3}(\sigma+u+1,\tau+1-l,k,l,2,1)$
$\hat{\Delta}_{\sigma,\tau,5,0}$	$\sum_{u+k=1}\sum_{l=0}^{3}(\sigma+u+1,\tau+1-l,k,l,4,0)$

(5.37)

□

5.2 Extraneous Factors Generation for Six Solved Cases

By Theorem 12, we have $\epsilon = \min(w_1h_1, w_2h_2)$.

When $\epsilon = 1$. It indicates $w_1 = h_1 = 1$ or $w_2 = h_2 = 1$. $w_1 = h_1 = 1$ corresponds to corner edge cutting in [14] and $w_2 = h_2 = 1$ corresponds to corner point pasting in [15]. The conjecture has been shown by Foo & Chionh to be true for these two special cases.

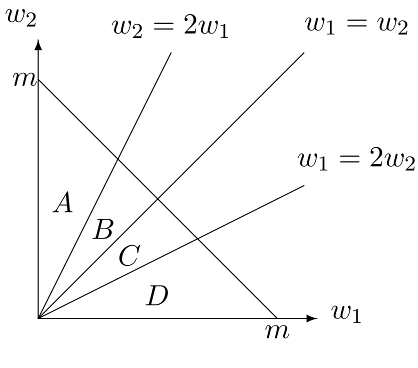
When $\epsilon > 1$. For most situations, we are able to show certain rows or columns of the Dixon matrix D does produce the denominator given in (5.1). The results are summarized systematically in the following table:

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="3">$w_1 > w_2$</td><td rowspan="2">$w_1 = w_2$</td><td colspan="3">$w_1 < w_2$</td></tr>
<tr><td>$w_1 > 2w_2$</td><td>$w_1 = 2w_2$</td><td>$w_1 < 2w_2$</td><td>$w_2 < 2w_1$</td><td>$w_2 = 2w_1$</td><td>$w_2 > 2w_1$</td></tr>
<tr><td rowspan="3">$h_1 > h_2$</td><td>$h_1 > 2h_2$</td><td colspan="3" rowspan="3">solved</td><td rowspan="3">solved</td><td>solved</td><td>solved</td><td>open</td></tr>
<tr><td>$h_1 = 2h_2$</td><td>solved</td><td>solved</td><td>solved</td></tr>
<tr><td>$h_1 < 2h_2$</td><td>open</td><td>solved</td><td>solved</td></tr>
<tr><td colspan="2">$h_1 = h_2$</td><td colspan="3">solved</td><td>solved</td><td colspan="3">solved</td></tr>
<tr><td rowspan="3">$h_1 < h_2$</td><td>$h_2 < 2h_1$</td><td>solved</td><td>solved</td><td>open</td><td rowspan="3">solved</td><td colspan="3" rowspan="3">solved</td></tr>
<tr><td>$h_2 = 2h_1$</td><td>solved</td><td>solved</td><td>solved</td></tr>
<tr><td>$h_2 > 2h_1$</td><td>open</td><td>solved</td><td>solved</td></tr>
</table>

Given a rectangle monomial support $\mathcal{A}_{m,n}$, by definition,the following restrictions are imposed on w_1, w_2 and h_1, h_2:

$$0 \le w_1, w_2 \le m, w_1 + w_2 \le m; \quad 0 \le h_1, h_2 \le n, h_1 + h_2 \le n \tag{5.38}$$

To estimate the percentages of the solved and unsolved cases, we draw the following figure:



Then we have:

the probability of $w_2 \ge 2w_1$ is $\frac{A}{A+B+C+D} = \frac{1}{3}$,

the probability of $w_1 \le w_2 < 2w_1$ is $\frac{B}{A+B+C+D} = \frac{1}{6}$,

the probability of $w_2 < w_1 < 2w_2$ is $\frac{C}{A+B+C+D} = \frac{1}{6}$,
the probability of $w_1 \geq 2w_2$ is $\frac{D}{A+B+C+D} = \frac{1}{3}$.

We can compute the probabilities of h_1, h_2 under the similar conditions in a similar way. As a result, the following probability table can be obtained:

		$Pr(w_1 \geq 2w_2)$ $\frac{1}{3}$	$Pr(w_2 < w_1 < 2w_2)$ $\frac{1}{6}$	$Pr(w_1 \leq w_2 < 2w_1)$ $\frac{1}{6}$	$Pr(w_2 \geq 2w_1)$ $\frac{1}{3}$
$Pr(h_1 \geq 2h_2)$	$\frac{1}{3}$	1/9	1/18	1/18	1/9
$Pr(h_2 < h_1 < 2h_2)$	$\frac{1}{6}$	1/18	1/36	1/36	1/18
$Pr(h_1 \leq h_2 < 2h_1)$	$\frac{1}{6}$	1/18	1/36	1/36	1/18
$Pr(h_2 \geq 2h_1)$	$\frac{1}{3}$	1/9	1/18	1/18	1/9

Thus the probability of unsolved cases is $\frac{1}{9} + \frac{1}{36} + \frac{1}{9} + \frac{1}{36} = \frac{5}{18} \doteq 28\%$.

And the probability of solved cases is $1 - 28\% \doteq 72\%$.

In fact, the above solved conditions can be grouped into six cases with some overlapping:

1. $w_1 \geq w_2$, $h_1 \geq h_2$
2. $w_1 \leq w_2$, $h_1 \leq h_2$
3. $w_1 \geq 2w_2$, $h_1 < h_2 \leq 2h_1$
4. $w_2 < w_1 \leq 2w_2$, $2h_1 \leq h_2$
5. $w_1 < w_2 \leq 2w_1$, $2h_2 \leq h_1$
6. $w_2 \geq 2w_1$, $h_2 < h_1 \leq 2h_2$.

We will show the proofs of how the rows and the columns of the Dixon matrix produce the extraneous factors given in the denominator of (5.1) step by step for the six cases. Section 5.2.1 proves the first two cases. And the proofs of the last four cases are given in Section 5.2.2.

5.2.1 The Two Cases: $w_1 \leq w_2, h_1 \leq h_2$ and $w_1 \geq w_2, h_1 \geq h_2$

By Theorem 12, $\epsilon = \min(w_1h_1, w_2h_2)$. Consequently, these two cases can be characterized as $\epsilon = \min(w_1, w_2)\min(h_1, h_2)$. Figure 5.2 shows the two conditions for the bottom left corner:

By Theorems 14, 15 and 9, we can conclude that

Theorem 16 *The three $(0..w-1 \times 0..h-1)$-near row blocks with respect to $(x_i', y_i'), i = 1, 2, 3$ produce a factor $(x_1, y_1, x_2, y_2, x_3, y_3)^{wh}$. The three $(0..w-1 \times 0..h-1)$-near column blocks with respect to $(x_i'', y_i''), i = 1, 2, 3$ also produce a factor $(x_1, y_1, x_2, y_2, x_3, y_3)^{wh}$.*

Proof

We prove the theorem for the rows. The proof of columns are similar.

By Theorem 14, we know that for each $(p,q) \in (0..w-1 \times 0..h-1)$, the (p,q)-near row with respect to $(x_i, y_i), i = 1,2,3$ can be reduced to

$\sum_{l-v=n-q} \sum_{k=0}^{m} -(x_i, y_i, k, l, a+p+1-k, b-v)$	$\sum_{l-v=n-q} \sum_{k=0}^{m} (x_i, y_i, k, l, a-p-k, b-v)$
$\sum_{v+l=q} \sum_{k=0}^{m} (x_i, y_i, k, l, a+p+1-k, b+v+1)$	$\sum_{v+l=q} \sum_{k=0}^{m} -(x_i, y_i, k, l, a-p-k, b+v+1)$

(5.39)

Similar to the proof of Theorem 9 Case 1, any 3×3 submatrix of the (p,q)-near rows with respect to $(x_i, y_i), i = 1,2,3$ can be written as a product of BP' where $B = (x_1, y_1, x_2, y_2, x_3, y_3)$ and P' equal to

$\sum_{l-v=n-q} \sum_{k=0}^{m} -(k, l, a_j+p+1-k, b_j-v)$	$\sum_{l-v=n-q} \sum_{k=0}^{m} (k, l, a_j-p-k, b_j-v)$
$\sum_{v+l=q} \sum_{k=0}^{m} (k, l, a_j+p+1-k, b_j+v+1)$	$\sum_{v+l=q} \sum_{k=0}^{m} -(k, l, a_j-p-k, b_j+v+1)$

(5.40)

where j indexes a column.

Thus, a group of (p,q)-near rows produces a factor $(x_1, y_1, x_2, y_2, x_3, y_3)$.

And there are wh groups in $(0..w-1 \times 0..h-1)$-near row blocks with respect to $(x_i, y_i), i = 1,2,3$. By Laplace Expansion these three $(0..w-1 \times 0..h-1)$-near row blocks produce a factor $(x_1, y_1, x_2, y_2, x_3, y_3)^{wh}$.

Q.E.D

Example 19 *Consider the monomial support in Example 17.*

By Theorem 16, the rows indexed by $\{(0,2),(2,1),(4,0)\} \oplus (0..1 \times 0)$ *generate the factor* 022140^2*. This also can checked from the results presented in Example 18.*

By Theorem 16, the columns indexed by $\{(0,2),(2,1),(4,0)\} \oplus (0..1 \times 0)$ *can also generate the factor* 022140^2*.* □

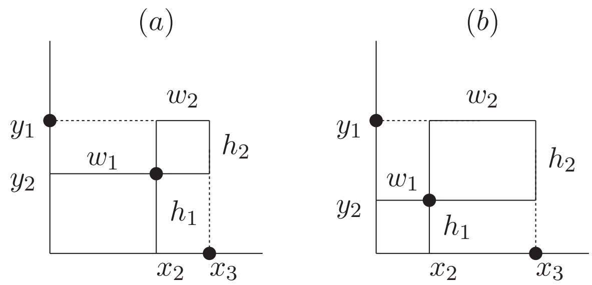
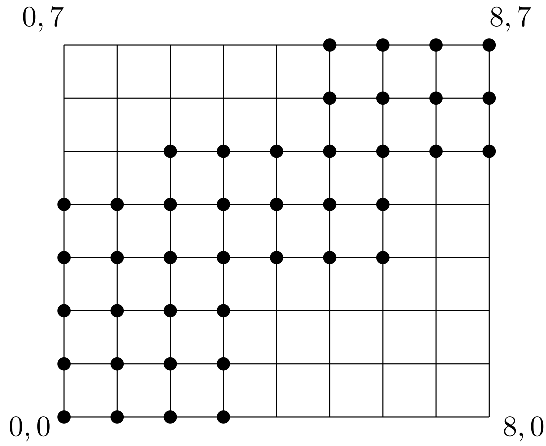


Figure 5.2: Bottom left corner. $(a) w_1 \geq w_2, h_1 \geq h_2$; $(b) w_1 \leq w_2, h_1 \leq h_2$.

Example 20 *Consider the following monomial support* $\mathcal{A} \subseteq \mathcal{A}_{8,7}$*:*



Bottom right corner. *The three exposed points at this corner are:* $(8,5),(6,3),(3,0)$*. Thus*

$$w_1 = 8-6 = 2, h_1 = 3, w_2 = 6-3 = 3, h_2 = 8-6 = 2.$$

Take $w = 2, h = 2$*. By Theorem 16, we have the rows indexed by*

$$(-1,0) \oplus \{(8,5),(6,3),(3,0)\} \oplus (-1..0 \times 0..1)$$

generate the factor 856330^4 *and the columns indexed by*

$$(7,0) \oplus \{(8,5),(6,3),(3,0)\} \oplus (-1..0 \times 0..1)$$

can also generate the factor 856330^4*.*

Top left corner. *The three exposed points at this corner are:* $(0,4),(2,5),(5,7)$*. Thus*

$$w_1 = 2, h_1 = 7-5 = 2, w_2 = 5-2 = 3, h_2 = 5-4 = 1.$$

Take $w = 2, h = 1$*. By Theorem 16, we have the rows indexed by*

$$(0,6) \oplus \{(0,4),(2,5),(5,7)\} \oplus (0..1 \times 0)$$

generate the factor 042557^2 *and the columns indexed by*

$$(0,-1) \oplus \{(0,4),(2,5),(5,7)\} \oplus (0..1 \times 0)$$

can also generate the factor 042557^2*.* □

Corollary 1 *Let* $w_1 \geq w_2$ *and* $h_1 \geq h_2$*.*

The $(0..w_2 - 1 \times 0..h_2 - 1)$*-near row blocks with respect to* $(x_i', y_i'), i = 1,2,3$ *generate a factor* $(x_1, y_1, x_2, y_2, x_3, y_3)^\epsilon$*.*

The $(0..w_2 - 1 \times 0..h_2 - 1)$*-near column blocks with respect to* $(x_i'', y_i''), i = 1,2,3$ *also generate a factor* $(x_1, y_1, x_2, y_2, x_3, y_3)^\epsilon$*.*

Proof

By definition, we have $\epsilon = \min(w_1h_1, w_2h_2) = w_2h_2$. Since $w_2 = \min(w_1, w_2)$ and $h_2 = \min(h_1, h_2)$, by Theorem 16, we have the $(0..w_2 - 1 \times 0..h_2 - 1)$-near row blocks generate the factor

$$(x_1, y_1, x_2, y_2, x_3, y_3)^{w_2h_2} = (x_1, y_1, x_2, y_2, x_3, y_3)^{\epsilon}. \tag{5.41}$$

The proof of the columns is similar.

Q.E.D

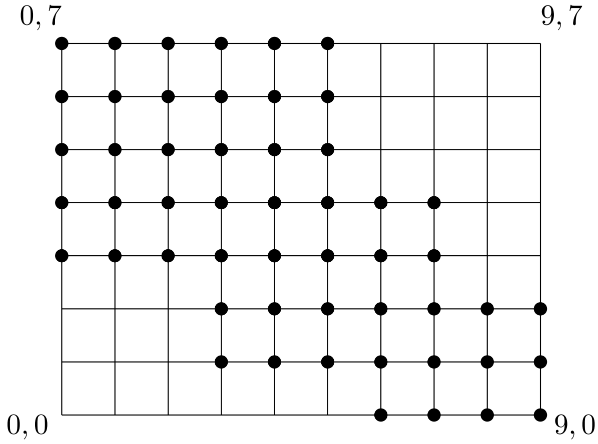
Corollary 2 *Let* $w_1 \le w_2$ *and* $h_1 \le h_2$.
The $(0..w_1 - 1 \times 0..h_1 - 1)$*-near row blocks with respect to* $(x'_i, y'_i), i = 1, 2, 3$ *generate a factor* $(x_1, y_1, x_2, y_2, x_3, y_3)^{\epsilon}$.
The $(0..w_1 - 1 \times 0..h_1 - 1)$*-near column blocks with respect to* $(x''_i, y''_i), i = 1, 2, 3$ *also generate a factor* $(x_1, y_1, x_2, y_2, x_3, y_3)^{\epsilon}$.

Proof

The proof is similar to Corollary 2 except that $w_1 = \min(w_1, w_2), h_1 = \min(h_1, h_2)$ and we substitute w, h in Theorem 16 by w_1, h_1.

Q.E.D

Example 21 *Consider the monomial support* $\mathcal{A} \subseteq \mathcal{A}_{9,7}$*:*



Top right corner. *The three exposed points are* $(9, 2), (7, 4), (5, 7)$. *So we have*

$$w_1 = 2, h_1 = 3, w_2 = 2, h_2 = 2 \Longrightarrow w_1 = w_2, h_1 \ge h_2, \epsilon_{9,7} = 4.$$

By Corollary 1, the rows indexed by

$$(-1, 6) \oplus \{(9, 2), (7, 4), (5, 7)\} \oplus (-1..0 \times -1..0)$$

generate the factor 927457^4 *and the columns indexed by*

$$(8, -1) \oplus \{(9, 2), (7, 4), (5, 7)\} \oplus (-1..0 \times -1..0)$$

can also generate the expected extraneous factor 927457^4.

Bottom left corner. *The three exposed points are* $(0,3),(3,1),(6,0)$. *So we have*

$$w_1 = 3, h_1 = 1, w_2 = 6-3 = 3, h_2 = 3-1 = 2 \Longrightarrow w_1 = w_2, h_1 \leq h_2, \epsilon_{0,0} = 3.$$

By Corollary 2, the rows indexed by

$$\{(0,3),(3,1),(6,0)\} \oplus (0..2 \times 0)$$

generate the factor 033160^3 *and the columns indexed by the same set of points can also generate the factor* 033160^3. □

5.2.2 The Other Four Cases

In this section, we will prove how extraneous factors are generated for the rest four cases by using Theorem 16 and row/column intersections. It is trivial to check the following cases satisfy the condition

$$\min(w_1,w_2)\min(h_1,h_2) < \epsilon \leq 2\min(w_1,w_2)\min(h_1,h_2). \tag{5.42}$$

Case	$\min(w_1,w_2)\min(h_1,h_2)$	ϵ	$2\min(w_1,w_2)\min(h_1,h_2)$
$w_1 \geq 2w_2, h_1 < h_2 \leq 2h_1$	w_2h_1	w_2h_2	$w_2 2h_1$
$w_2 < w_1 \leq 2w_2, h_2 \geq 2h_1$	w_2h_1	w_1h_1	$2w_2h_1$
$w_1 < w_2 \leq 2w_1, h_1 \geq 2h_2$	w_1h_2	w_2h_2	$2w_1h_2$
$w_2 \geq 2w_1, h_2 < h_1 \leq 2h_2$	w_1h_2	w_1h_1	$w_1 2h_2$

(5.43)

Let T be the set of the three exposed points, $T = \{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$ and

$$T' = \{(x_1',y_1'),(x_2',y_2'),(x_3',y_3')\}, \quad T'' = \{(x_1'',y_1''),(x_2'',y_2''),(x_3'',y_3'')\}. \tag{5.44}$$

Theorem 17 *If* $w_2 < w_1 \leq 2w_2, h_2 \geq 2h_1$, *then*

1. *when* w_1 *is even.*
   The intersecting entries of the rows indexed by

$$T' \oplus \left(0..\frac{w_1}{2} - 1 \times 0..h_1 - 1\right)^*$$

   and the columns indexed by

$$T'' \oplus \left(0..\frac{w_1}{2} - 1 \times 0..h_1 - 1\right)^*$$

   are zeros.

2. *when w_1 is odd but h_1 is even.*
   The intersecting entries of the rows indexed by

$$T' \oplus \left(0..\frac{w_1-1}{2}-1 \times 0..h_1-1\right)^* \cup T' \oplus \left(\frac{w_1-1}{2} \times 0..\frac{h_1}{2}-1\right)^*$$

   and the column indexed by

$$T'' \oplus \left(0..\frac{w_1-1}{2}-1 \times 0..h_1-1\right)^* \cup T'' \oplus \left(\frac{w_1-1}{2} \times 0..\frac{h_1}{2}-1\right)^*$$

   are zeros.

3. *when w_1 and h_1 are both odd.*
   The intersecting entries of the rows indexed by

$$T' \oplus \left(0..\frac{w_1-1}{2}-1 \times 0..h_1-1\right)^* \cup T' \oplus \left(\frac{w_1-1}{2} \times 0..\frac{h_1+1}{2}-1\right)^*$$

   and the columns indexed by

$$T'' \oplus \left(0..\frac{w_1-1}{2}-1 \times 0..h_1-1\right)^* \cup T'' \oplus \left(\frac{w_1-1}{2} \times 0..\frac{h_1-1}{2}-1\right)^*$$

   are zeros.

<u>Proof</u>

Let $B = (i, j, k, l, p, q)$ which can be (3.2) or (3.4), or $B = -(i, j, k, l, p, q)$ which can be (3.3) or (3.5). Apply the entry formula in Theorem 3 and choose the bracket B in the formula to be

$$\begin{array}{|c|c|} \hline \text{Equation (3.5)} & \text{Equation (3.4)} \\ \hline \text{Equation (3.2)} & \text{Equation (3.3)} \\ \hline \end{array} \tag{5.45}$$

Substituting

$$(\sigma, \tau) = \begin{array}{|c|c|} \hline x_i' + \Delta_1, y_i' - \Delta_2 & x_i' - \Delta_1, y_i' - \Delta_2 \\ \hline x_i' + \Delta_1, y_i' + \Delta_2 & x_i' - \Delta_1, y_i' + \Delta_2 \\ \hline \end{array} \tag{5.46}$$

and

$$(a, b) = \begin{array}{|c|c|} \hline x_j'' + \Delta_3, y_j'' - \Delta_4 & x_j'' - \Delta_3, y_j'' - \Delta_4 \\ \hline x_j'' + \Delta_3, y_j'' + \Delta_4 & x_j'' - \Delta_3, y_j'' + \Delta_4 \\ \hline \end{array} \tag{5.47}$$

into the bracket B, thus the first ordered pair (i, j) of B becomes

$$\begin{array}{|c|c|} \hline (x_i + \Delta_1 + u + 1, y_i - \Delta_2 + n - 1 - v - l) & (x_i - \Delta_1 - u - 1, y_i - \Delta_2 + n - 1 - v - l) \\ \hline (x_i + \Delta_1 + u + 1, y_i + \Delta_2 + v + 1 - l) & (x_i - \Delta_1 - u - 1, y_i + \Delta_2 + v + 1 - l) \\ \hline \end{array} \tag{5.48}$$

and the last ordered pair (p, q) of B becomes

$$\begin{array}{|c|c|}\hline (x_j + \Delta_3 - u - k, y_j - \Delta_4 + v) & (x_j - \Delta_3 + u + m - k, y_j - \Delta_4 + v) \\ \hline (x_j + \Delta_3 - u - k, y_j + \Delta_4 - v) & (x_j - \Delta_3 + u + m - k, y_j + \Delta_4 - v) \\ \hline \end{array} \tag{5.49}$$

So in order to ensure (i, j) in the monomial support $\mathcal{A}$, we need:

$$\begin{array}{|c|c|}\hline \Delta_1 + u + 1 \geq 0, -\Delta_2 + n - 1 - v - l \leq 0 & -\Delta_1 - u - 1 \leq 0, -\Delta_2 + n - 1 - v - l \leq 0 \\ \hline \Delta_1 + u + 1 \geq 0, \Delta_2 + v + 1 - l \geq 0 & -\Delta_1 - u - 1 \leq 0, \Delta_2 + v + 1 - l \geq 0 \\ \hline \end{array} \tag{5.50}$$

and (p, q) in $\mathcal{A}$ we need

$$\begin{array}{|c|c|}\hline \Delta_3 - u - k \geq 0, -\Delta_4 + v \leq 0 & -\Delta_3 + u + m - k \leq 0, -\Delta_4 + v \leq 0 \\ \hline \Delta_3 - u - k \geq 0, \Delta_4 - v \geq 0 & -\Delta_3 + u + m - k \leq 0, \Delta_4 - v \geq 0 \\ \hline \end{array} \tag{5.51}$$

These conditions can be combined to become

$$\begin{array}{|c|c|}\hline k \leq \Delta_1 + \Delta_3 + 1, l \geq n - 1 - \Delta_2 - \Delta_4 & k \geq m - 1 - \Delta_1 - \Delta_3, l \geq n - 1 - \Delta_2 - \Delta_4 \\ \hline k \leq \Delta_1 + \Delta_3 + 1, l \leq \Delta_2 + \Delta_4 + 1 & k \geq m - 1 - \Delta_1 - \Delta_3, l \leq \Delta_2 + \Delta_4 + 1 \\ \hline \end{array} \tag{5.52}$$

By the cases given, we have the possibilities of the sum pair of $\max(\Delta_1 + \Delta_3)$ and $\max(\Delta_2 + \Delta_4)$:

$$\begin{array}{c|c} \max(\Delta_1 + \Delta_3) & \max(\Delta_2 + \Delta_4) \\ \hline w_1 - 2 & 2h_1 - 2 \\ w_1 - 3 & 2h_1 - 2 \\ w_1 - 1 & h_1 - 2 \\ w_1 - 2 & \frac{3}{2}h_1 - 2 \\ w_1 - 2 & \frac{3}{2}h_1 - \frac{5}{2} \\ w_1 - 2 & \frac{3}{2}h_1 - \frac{3}{2} \end{array} \tag{5.53}$$

When $\max(\Delta_1 + \Delta_3) \leq w_1 - 2, \max(\Delta_2 + \Delta_4) \leq 2h_1 - 2,$

$$\begin{array}{|c|c|}\hline k \leq w_1 - 1, l \geq n + 1 - 2h_1 & k \geq m + 1 - w_1, l \geq n + 1 - 2h_1 \\ \hline k \leq w_1 - 1, l \leq 2h_1 - 1 & k \geq m + 1 - w_1, l \leq 2h_1 - 1 \\ \hline \end{array} \tag{5.54}$$

When $\max(\Delta_1 + \Delta_3) = w_1 - 1, \max(\Delta_2 + \Delta_4) = h_1 - 2,$

$$\begin{array}{|c|c|}\hline k \leq w_1, l \geq n + 1 - h_1 & k \geq m - w_1, l \geq n + 1 - h_1 \\ \hline k \leq w_1, l \leq h_1 - 1 & k \geq m - w_1, l \leq h_1 - 1 \\ \hline \end{array} \tag{5.55}$$

Both ranges of (k, l) are not in $\mathcal{A}$. So their intersecting entries are zeros.

Q.E.D

We can prove the other three cases holding the similar properties in a similar way. They are stated in the following three theorems:

Theorem 18 *If $w_1 \geq 2w_2, h_1 < h_2 \leq 2h_1$, then*

1. *when h_2 is even.*
   The intersecting entries of the rows indexed by
   $$T' \oplus \left(0..w_2 - 1 \times 0..\frac{h_2}{2} - 1\right)^*$$
   and the columns indexed by
   $$T'' \oplus \left(0..w_2 - 1 \times 0..\frac{h_2}{2} - 1\right)^*$$
   are zeros.

2. *when h_2 is odd but w_2 is even.*
   The intersecting entries of the rows indexed by
   $$T' \oplus \left(0..w_2 - 1 \times 0..\frac{h_2 - 1}{2} - 1\right)^* \cup T' \oplus \left(0..\frac{w_2}{2} - 1 \times \frac{h_2 - 1}{2}\right)^*$$
   and the column indexed by
   $$T'' \oplus \left(0..w_2 - 1 \times 0..\frac{h_2 - 1}{2} - 1\right)^* \cup T'' \oplus \left(0..\frac{w_2}{2} - 1 \times \frac{h_2 - 1}{2}\right)^*$$
   are zeros.

3. *when h_2 and w_2 are both odd.*
   The intersecting entries of the rows indexed by
   $$T' \oplus \left(0..w_2 - 1 \times 0..\frac{h_2 - 1}{2} - 1\right)^* \cup T' \oplus \left(0..\frac{w_2 + 1}{2} - 1 \times \frac{h_2 - 1}{2}\right)^*$$
   and the columns indexed by
   $$T'' \oplus \left(0..w_2 - 1 \times 0..\frac{h_2 - 1}{2} - 1\right)^* \cup T'' \oplus \left(0..\frac{w_2 - 1}{2} - 1 \times \frac{h_2 - 1}{2}\right)^*$$
   are zeros.

Theorem 19 *If $w_1 < w_2 \leq 2w_1, h_1 \geq 2h_2$, then*

1. *when w_2 is even.*
   The intersecting entries of the rows indexed by
   $$T' \oplus \left(0..\frac{w_2}{2} - 1 \times 0..h_2 - 1\right)^*$$
   and the columns indexed by
   $$T'' \oplus \left(0..\frac{w_2}{2} - 1 \times 0..h_2 - 1\right)^*$$
   are zeros.

2. *when w_2 is odd but h_2 is even.*
   The intersecting entries of the rows indexed by

$$T' \oplus \left(0..\frac{w_2-1}{2}-1 \times 0..h_2-1\right)^* \cup T' \oplus \left(\frac{w_2-1}{2} \times 0..\frac{h_2}{2}-1\right)^*$$

   and the column indexed by

$$T'' \oplus \left(0..\frac{w_2-1}{2}-1 \times 0..h_2-1\right)^* \cup T'' \oplus \left(\frac{w_2-1}{2} \times 0..\frac{h_2}{2}-1\right)^*$$

   are zeros.

3. *when w_2 and h_2 are both odd.*
   The intersecting entries of the rows indexed by

$$T' \oplus \left(0..\frac{w_2-1}{2}-1 \times 0..h_2-1\right)^* \cup T' \oplus \left(\frac{w_2-1}{2} \times 0..\frac{h_2+1}{2}-1\right)^*$$

   and the columns indexed by

$$T'' \oplus \left(0..\frac{w_2-1}{2}-1 \times 0..h_2-1\right)^* \cup T'' \oplus \left(\frac{w_2-1}{2} \times 0..\frac{h_2-1}{2}-1\right)^*$$

   are zeros.

Theorem 20 *If $w_2 \geq 2w_1, h_2 < h_1 \leq 2h_2$, then*

1. *when h_1 is even.*
   The intersecting entries of the rows indexed by

$$T' \oplus \left(0..w_1-1 \times 0..\frac{h_1}{2}-1\right)^*$$

   and the columns indexed by

$$T'' \oplus \left(0..w_1-1 \times 0..\frac{h_1}{2}-1\right)^*$$

   are zeros.

2. *when h_1 is odd but w_1 is even.*
   The intersecting entries of the rows indexed by

$$T' \oplus \left(0..w_1-1 \times 0..\frac{h_1-1}{2}-1\right)^* \cup T' \oplus \left(0..\frac{w_1}{2}-1 \times \frac{h_1-1}{2}\right)^*$$

   and the column indexed by

$$T'' \oplus \left(0..w_1-1 \times 0..\frac{h_1-1}{2}-1\right)^* \cup T'' \oplus \left(0..\frac{w_1}{2}-1 \times \frac{h_1-1}{2}\right)^*$$

   are zeros.

3. *when h_1 and w_1 are both odd.*
The intersecting entries of the rows indexed by

$$T' \oplus \left(0..w_1 - 1 \times 0..\frac{h_1-1}{2} - 1\right)^* \cup T' \oplus \left(0..\frac{w_1+1}{2} - 1 \times \frac{h_1-1}{2}\right)^*$$

and the columns indexed by

$$T'' \oplus \left(0..w_1 - 1 \times 0..\frac{h_1-1}{2} - 1\right)^* \cup T'' \oplus \left(0..\frac{w_1-1}{2} - 1 \times \frac{h_1-1}{2}\right)^*$$

are zeros.

Theorem 21 *If $w_2 < w_1 \leq 2w_2, h_2 \geq 2h_1$, then the rows together with the columns given in Theorem 17 generate the extraneous factor B^ϵ with $B = (x_1, y_1, x_2, y_2, x_3, y_3)$. The other three cases hold the similar properties.*

<u>Proof</u>

Since $w_2 < w_1 \leq 2w_2, h_2 \geq 2h_1$, we get $w_2h_2 \geq w_1h_1$. So $\epsilon = w_1h_1$.

When w_1 is even, since $\frac{w_1}{2} \leq w_2 = \min(w_1, w_2)$ and $h_1 \leq \min(h_1, h_2)$, by Theorem 16, we have the rows indexed by

$$T' \oplus \left(0..\frac{w_1}{2} - 1 \times 0..h_1 - 1\right)^* \tag{5.56}$$

generate a factor $B^{\frac{w_1h_1}{2}}$ and the columns indexed by

$$T'' \oplus \left(0..\frac{w_1}{2} - 1 \times 0..h_1 - 1\right)^* \tag{5.57}$$

also generate a factor $B^{\frac{w_1h_1}{2}}$. Moreover, by Theorem 17, we know that the intersecting entries of these rows and these columns are zero. So we can conclude that the the determinant of the Dixon matrix has a factor

$$B^{2\frac{w_1h_1}{2}} = B^{w_1h_1}. \tag{5.58}$$

When w_1 is odd and h_1 is even, by Theorems 16 and 17, similarly we can conclude that $|D|$ has a extraneous factor

$$B^{2(\frac{w_1-1}{2}h_1 + \frac{h_1}{2})} = B^{w_1h_1}. \tag{5.59}$$

When w_1 and h_1 are both odd, by Theorems 16 and 17, we have

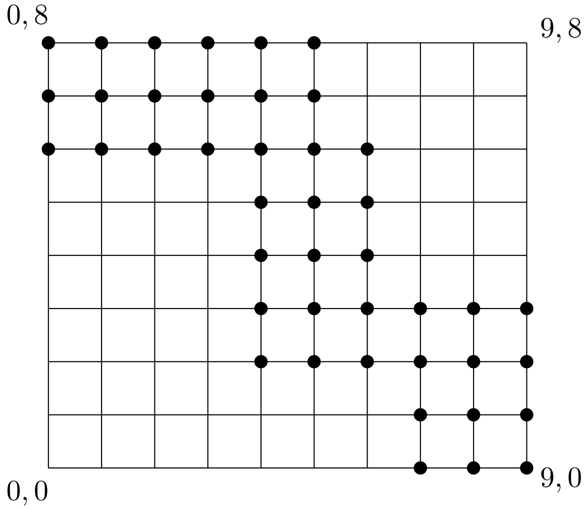
$$B^{2\frac{w_1-1}{2}h_1 + \frac{h_1-1}{2} + \frac{h_1+1}{2}} = B^{w_1h_1}. \tag{5.60}$$

In any one of three situations, we can get the expected extraneous factor $B^{w_1h_1} = B^\epsilon$.

Similar arguments apply for the rest three cases.

Q.E.D

Example 22 *Consider the monomial support $\mathcal{A} \subseteq \mathcal{A}_{9,8}$:*



There are three exposed points at the bottom left corner: $(0,6),(4,2),(7,0)$. *By definition, the* w_1, h_1, w_2, h_2 *associated with this corner can be calculated:*

$$w_1 = 4, h_1 = 2, w_2 = 7-4 = 3, h_2 = 6-2 = 4.$$

By Theorem 12, $\epsilon_{0,0} = \min(8,12) = 8$.

Since $h_2 = 2h_1, w_2 < w_1 \leq 2w_2$ *and* $w_1 = 4$ *is even, by Theorem 21, we know that the rows indexed by*

$$\{(0,6),(4,2),(7,0)\} \oplus (0..1 \times 0..1) \cap \mathcal{R}$$

together with the columns indexed by the same set points can generate the factor 064270^8.

Similarly, there are three exposed points at the top right corner: $(9,3),(6,6),(5,8)$. *By definition, the* w_1, h_1, w_2, h_2 *associated with this corner can be calculated:*

$$w_1 = 9-6 = 3, h_1 = 8-6 = 2, w_2 = 6-5 = 1, h_2 = 6-3 = 3.$$

By Theorem 12, $\epsilon_{9,8} = \min(6,3) = 3$.

Since $w_1 \geq 2w_2, h_1 < h_2 \leq 2h_1$ *and* w_2, h_2 *are both odd, by Theorem 21, we know that the rows indexed by*

$$\{(8,10),(5,13),(4,15),(8,9),(5,12),(4,14)\} = \{(9,3),(6,6),(5,8)\} \oplus (-1,7) \oplus (0 \times -1..0) \cap \mathcal{R}$$

together with the columns indexed by

$$\{(17,2),(14,5),(13,7)\} = \{(9,3),(6,6),(5,8)\} \oplus (8,-1) \cap \mathcal{C}$$

generate the factor 936658^3. □

Chapter 6

Conjectures

In this chapter, two conjectures are presented. We propose an algorithm for finding the rows or columns generating the desired extraneous factors when a corner has exactly three exposed points. The other conjecture concerns the linear independence of these rows or columns.

6.1 Algorithm for Finding the Rows or Columns Generating Expected Extraneous Factors for Corners with Three Exposed Points

Conjecture 3 *Let $w_{\min}$ and $h_{\min}$ be the corresponding width and height that produces extraneous factor degree $\epsilon = w_{\min}h_{\min}$.*

Suppose Q'_i be the first quadrant associated with the exposed points (x'_i, y'_i):

$$Q'_i = (x'_i, y'_i) \oplus (\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0})^*, \quad i = 1, 2, 3 \tag{6.1}$$

Initial Step: Let the sets of indexing points be:

$$T'_i = (x'_i, y'_i) \oplus (0..w_{\min} - 1 \times 0..h_{\min} - 1)^* \cap \mathcal{R}, \quad i = 1, 2, 3 \tag{6.2}$$

while true

if $(\sigma, \tau) \in T'_i \cap Q'_j$, $i \neq j$ then
$T'_k = T'_k \cup \{(\sigma, \tau) \ominus (x'_j, y'_j) \oplus (x'_k, y'_k)\} \cap \mathcal{R}$, $k = 1, 2, 3$;

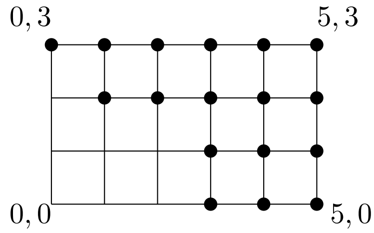
until no new points are added to T'_1, T'_2, T'_3.
As a result, the rows indexed by the points in $T'_1 \cup T'_2 \cup T'_3$ will generate the factor $(x_1, y_1, x_2, y_2, x_3, y_3)^\epsilon$.

A similar algorithm applies for finding the columns generating the expected extraneous factors.

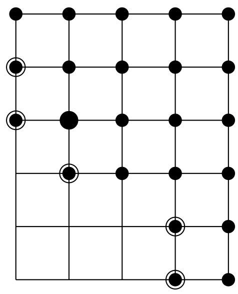
Remark 2 *The two cases examined in Section 5.2.1 do not execute the while loop. The other four cases given in Section 5.2.2 are the "expand-once" cases since for each condition it only occurs once that $T'_i \cap Q'_j \neq \emptyset$ for $i \neq j$.*

This theorem may look a little complicated. We use a simple example to illustrate the idea:

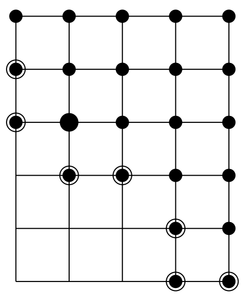
Example 23 *Consider the following monomial support $\mathcal{A} \subseteq \mathcal{A}_{5,3}$:*



By definition, we have $w_1 = 1, h_1 = 2, w_2 = 2, h_1 = 1$ *and* $(x_1, y_1) = (0, 3), (x_2, y_2) = (1, 2), (x_3, y_3) = (3, 0)$. *Since* $w_1 h_1 = w_2 h_2$. *Without loss of generality, we take* $w_{\min} = 1, h_{\min} = 2$. *So initially we have the following indexing points in the set* $\cup_{i=1,2,3} T'_i \subset \mathcal{R}$*:*



Since the point $(1, 3) \in Q'_1 \cap T'_2$, *and* $(1, 3) \ominus (0, 3) = (1, 0)$, *two more points* $(1, 0) \oplus (1, 2) = (2, 2)$ *and* $(1, 0) \oplus (3, 0) = (4, 0)$ *should be added to* T'_2 *and* T'_3 *respectively:*



Since $(2, 2)$ *and* $(4, 0)$ *are only lying in their own quadrants, thus these rows indexed by the eight points marked with bigger radius circle generate the factor* 031230^2. □

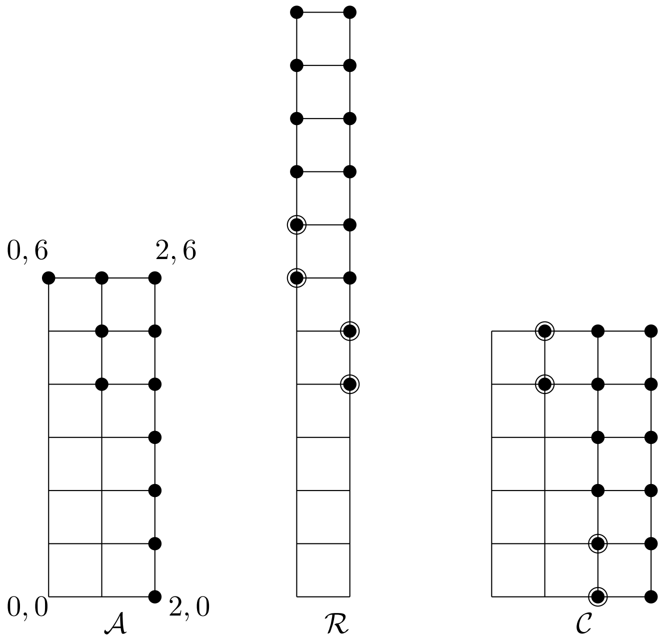
Note that the above example is a demonstration of "expand-once" conditions.

6.2 Maximal Minors

Conjecture 4 *The rows found to generate extraneous factors for the four corners given by Conjecture 3 are linearly independent. And the columns found to generate extraneous factors for the four corners given by Conjecture 3 are also linearly independent.*

Remark 3 *If these rows or columns are linearly independent, then it can be shown that the Dixon matrix is indeed the maximal minor using the irreducibility property of the sparse resultant and BKK bound.*

Example 24 *Consider the monomial support $\mathcal{A} \subseteq \mathcal{A}_{2,6}$:*



There are three exposed points at bottom left corner: (0,6),(1,4),(2,0).
By definition, we have

$$w_1 = 1, h_1 = 4, w_2 = 1, h_2 = 2. \tag{6.3}$$

Take $w = 1, h = 2$.
It can be checked that the rows indexed by

$$\{(0,6),(1,4),(2,0)\} \oplus (0 \times 0..1) \cap \mathcal{R} = (0,6),(0,7),(2,4),(2,5) \tag{6.4}$$

are linearly independent. Because the rows indexed by $(0,6),(0,7),(2,4),(2,5)$ and the columns indexed by $(1,4),(1,5),(2,2),(2,3)$ form a 4×4 a lower triangular submatrix with all diagonals equal to the bracket 061420.
Similarly, it can be checked that the columns indexed by

$$\{(0,6),(1,4),(2,0)\} \oplus (0 \times 0..1) \cap \mathcal{C} = (1,4),(1,5),(2,0),(2,1) \tag{6.5}$$

are linearly independent. Because the columns indexed by (1,4),(1,5),(2,0),(2,1) and the rows indexed by $(0,8),(0,9),(1,4),(1,5)$ form a 4×4 lower triangular submatrix with all diagonals equal to the bracket 061420. □

Chapter 7

Conclusion

In this thesis we used four loose entry formulas to explore the extraneous factors incurred when using the Dixon method to construct sparse resultants for bi-degree monomial supports with three exposed points at any of the four corners. The results are derived in parallel for the corners with some simple presentation conventions. By imposing certain constraints, we are able to identify the extraneous factor generating rows (only), columns (only), or both rows and columns intersecting at zero entries. The technique of reduction is used to explain why they produce these factors. These constraints account for at least 72% of all the possible cases.

To completely solve the sparse resultant problem with the Dixon method for bi-degree monomial supports with three exposed points at the corners, the following has to be achieved:

- To establish that the Dixon matrix is maximal. This is stated as Conjecture 4.
- To prove that the rows and columns identified in Conjecture 3 are responsible for the extraneous factors in the remaining 28% of the possibilities. This also raises an open problem: is the method of reduction or are other methods needed to show that these rows and columns indeed produce the expected extraneous factors?
- With the method of reduction, (1) a near exposed point row or column is reduced using all preceding reduced rows or reduced columns near the three exposed points, and (2) a bracket factor is generated from three reduced rows or reduced columns near the three exposed points. Thus another open problem is: when one of the three rows or columns near an exposed point degenerates to a zero row or column, will the remaining two rows or columns near the other two exposed points still generate the expected extraneous factor and how this affects the reduction of other rows and columns?

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